Crossing and unitarity in a multichannel static model

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CROSSING AND UNITARITY
IN A MULTICHANNEL STATIC MODEL

by

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ABSTRACT

The two channel static model has been found to be exactly and simply soluble for a range of values of the crossing matrix parameter. This thesis rederives the known elastic solutions and obtains formal power series solutions for the inelastic elements. An unexpected connection between crossing and unitarity is discovered, and finally, equations are derived which indicate that if the two channel model is extended to a multichannel model incorporating inelasticity, then there will be an infinity of channels.

INTRODUCTION.

This thesis deals with the problem of determining the S matrix which describes meson nucleon scattering processes.

Chapter one commences with a brief outline of the problem, followed by a discussion of the work of previous authors, all of them concerned wholly with elastic processes. It is shown how the well known solutions for the diagonal elements are obtained.

In chapter two, still concerned with elastic processes, we rederive these solutions, and in the process indicate that isotopic spin invariance and unitarity are more closely linked than is usually appreciated.

Chapter three investigates the more challenging problem of the extension to inelastic processes. Off diagonal elements are not obtained in closed form, but only as formal power series.

We work in the multichannel static model, and it turns out that if we allow inelastic coupling between different channels involving specific mesons of given isospin, then automatically such coupling appears in every channel: we cannot assign arbitrarily the relative amounts of coupling between different channels.

CHAPTER I.

It is often found in strong interaction calculations that in any formulation of a problem in which it is easy to incorporate unitarity requirements, it is difficult to incorporate crossing symmetry, and vice versa. For example, partial wave dispersion relations incorporate unitarity without difficulty, but it is very difficult to put in crossing, whereas in the Mandelstam representation crossing symmetry is easily satisfied, while unitarity cannot be satisfied without great difficulty. In this thesis, we shall be making use both of crossing and of unitarity, in a very simple model, the static model; and indeed we shall give one example where results obtainable using crossing can alternatively be found by a calculation using unitarity. In that case, we shall rederive the known exact solutions for the elastic scattering of a meson with isotopic spin j off a nucleon, without any reference to the crossing matrix appropriate to two channel unitarity in Su(2). The work is concerned with the one meson approximation of the Low equations[1 to 9] The mathematical formulation is as follows: we let z be the energy of the meson and its momentum be q; then

$$q = (z^2-1)^{1/2} \qquad (1)$$

In the complex z plane, the branch cuts of q are chosen to run from $-\infty$ to -1 and from 1 to ∞, and q is defined to be real and positive just above the cut $(1, \infty)$. From this definition, we see that on the first Riemann sheet

$$\begin{gathered} q^*(z) = -q(z^*) \\ \operatorname{Im} q \geq 0 \end{gathered} \tag{2}$$

so that iq is a real analytic function in the cut z plane. Now we define the scattering amplitude $f_\alpha(z)$ in the channel α, where α refers to the isotopic spin (or spin) in that channel to be

$$f_\alpha(z) = \sin \delta_\alpha(z)\, e^{i\delta_\alpha(z)} \tag{3}$$

where $\delta_\alpha(z)$ is the phase shift.

Now we know that the S matrix satisfies certain dispersion relations, which represent a system of coupled non linear integral equations for the S matrix elements of various scattering channels, and that the particles are represented by poles in the S matrix elements in the appropriate channels. The scattering amplitudes must satisfy similar equations, so that $f_\alpha(z)$ satisfies the following equation[5], expressing the requirements of elastic unitarity, crossing symmetry, and analyticity, the usual basic postulates.

$$f_\alpha(z) = \frac{\lambda_\alpha g^2}{z} + \frac{1}{\pi} \lim_{\epsilon \to 0+} \int_1^\infty dz' \rho(z') \times$$

$$\times \left[\frac{|f_\alpha(z')|^2}{z' - z - i\epsilon} + \sum_\beta \frac{C_{\alpha\beta} |f_\beta(z')|^2}{z' + z} \right] \tag{4}$$

$$(\alpha, \beta = 1, 2)$$

where $C_{\alpha\beta}$ is a crossing matrix with the general property

$$\sum_\beta C_{\alpha\beta} C_{\beta\gamma} = \delta_{\alpha\gamma} \qquad \text{and} \tag{5}$$

$\rho(z)$ is the cut off function.

We define the S matrix element of the channel α by

$$S_\alpha(z) = 1 + 2i f_\alpha(z) \tag{6}$$

and from here on, we work exclusively with S matrix elements, rather than with amplitudes. All physical two particle thresholds are taken to occur at $z = 1$. Then $S_\alpha(z)$ satisfies the following conditions.

a. Analyticity: $S_\alpha(z)$ is real meromorphic in the cut z plane

b. Elastic unitarity; $S_\alpha(z)$ has only one branch point on the positive real axis, at the threshold point $z = 1$; the branch point is of the square (7) root type. The analytic continuation of $S_\alpha(z)$ onto the second Riemann sheet is given by

$$S_\alpha^{(2)}(z) = \frac{1}{S_\alpha(z)}$$

c. Crossing symmetry; $S_\alpha(-z) = \sum_\beta C_{\alpha\beta} S_\beta(z)$ where $C_{\alpha\beta}$ is the crossing matrix.

These conditions define the elements of our S matrix. Our problem is to find S matrix elements for meson nucleon scattering processes. For example, for such reactions as

$$\begin{aligned} &\pi N \rightarrow \pi N \\ &\pi N \rightarrow \omega N \qquad\qquad (8) \\ &\omega N \rightarrow \pi N \quad \text{etc.} \end{aligned}$$

noting that these reactions go via definite isotopic spin states (and via definite spin states). The first may go via $I = \frac{1}{2}$ or $I = \frac{3}{2}$, the other two only via $I = \frac{1}{2}$. The first is an elastic process: there is no change in the identity of the interacting particles. The other two are inelastic processes, in that the outgoing meson is not the same as the original meson.

A realistic model must, clearly, cope both with elastic and inelastic processes. It may, though, be valuable first to consider the less realistic case where we ignore all inelastic processes, and attempt to find the S matrix elements for the elastic processes under the assumption that all off diagonal elements are identically zero. This is the case which has already been extensively studied, notably by Martin and McGlinn:[4] we shall find that our results are entirely in agreement with theirs.

They extend the work of Wanders.[2] The essential part of the method is the factorisation of the S matrix elements into symmetric and anti-symmetric parts, though we note that this is not always possible.[7]

The fundamental properties of the elastic crossing matrix are

$$\begin{aligned} &\text{a.} \quad \sum_j C_{ij} = 1 \\ &\text{b.} \quad \sum_j C_{ij} C_{jk} = \delta_{ik} \end{aligned} \tag{9}$$

These properties are proved[4] from the basic requirement that the matrix C transforms projection operators for the irreducible representations in the Kronecker product decomposition in the S channel, into projection operators in the u channel. Equation (9a) expresses the completeness of the set of projection operators, i.e. expresses the property of conservation of probability. It has the consequence that the same crossing matrix relates S matrix elements as transition amplitudes (5). The second property (9b) is a consequence of the fact that two successive applications of the crossing matrix brings us back to the original position. For

$$C\,S(z) = S(-z)$$

so $C\,S(-z) = S(z)$ since the crossing matrix has a constant value, and is independent of z. (This form of crossing relation is what mainly typifies static models).

The general form of a 2 x 2 matrix with these properties is

$$C = \begin{pmatrix} c & 1-c \\ 1+c & -c \end{pmatrix} \tag{10}$$

Where the parameter c is arbitrary. The real analyticity of the S matrix elements constrains it to be real.

The solution of the crossing relation is easily found to be

$$\begin{aligned} S_1(z) &= s(z) - (1-c)a(z) \\ S_2(z) &= s(z) + (1+c)a(z) \end{aligned} \tag{11}$$

Where s(z) and a(z) are, respectively, symmetric and antisymmetric functions.

Martin and McGlinn here digress to discuss what they call two "trivial" solutions, trivial in that they have only 2 Riemann sheets, whereas the usual solutions have an infinity of Riemann sheets.

They are, firstly, the solution with $a(z) = o$. This solution is independent of c and is a single channel problem with trivial crossing symmetry, since the S matrix elements are identical and symmetric. It has been considered by Castillejo, Dalitz and Dyson[11], and by Wanders.[2] This is the crossing matrix of neutral scalar theory, which describes the scattering of a neutral scalar meson by a fixed baryon. Huang and Low[12] point out that mathematically it can be considered as a special case of the charged scalar theory.

The second is the solution with $s(z) = o$. Here unitarity can only be satisfied if $c = o$, when the crossing matrix becomes

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (12)$$

This is the static model problem solved by Castillejo, Dalitz and Dyson.[11] The general form for an antisymmetric S matrix is known. This is the charged scalar theory, where there are two scalar mesons of opposite charge, which are coupled in a charge symmetric manner to a charged baryon. Clearly the crossing symmetry of this problem merely gives us

$$S_1(-z) = S_2(z).$$

Going back to the general case, Martin and McGlinn re-write their factorisation as

$$\begin{aligned} S_1(z) &= A(z)\left[B(z) - (1-c)\right] \\ S_2(z) &= A(z)\left[B(z) + (1+c)\right] \end{aligned} \tag{13}$$

where both $A(z)$ and $B(z)$ are antisymmetric functions, with the properties of being real analytic and meromorphic in the cut z plane.

Equation (7b) implies that $A(z)$ and $B(z)$ can have only one branch point on the positive real axis, at $z = 1$, of type $(z-1)^{1/2}$. Further, it gives

$$\begin{aligned} B^{(2)}(z) &= -B(z) - 2c \\ A^{(2)}(z) &= \frac{-1}{A(z)\left[B(z)-(1-c)\right]\left[B(z)+(1+c)\right]} \end{aligned} \tag{14}$$

Martin and McGlinn, following Wanders' work, give the solution for $B(z)$ as

$$B(z) = -c - \frac{2ic}{\pi}\ln\left(z + q(z)\right) + iq(z)\beta(z) \tag{15}$$

where $\beta(z)$ is antisymmetric, real analytic and meromorphic in the whole z plane. We require now to find solutions for A(z). They attempt to do this by inspection. Having written the general solution for A(z) as

$$A(z) = A_0(z)\, D(z) \tag{16}$$

where $A_0(z)$ is any special solution and D(z) is an arbitrary symmetric "S matrix element" (i.e. it is real analytic in the cut z plane and obeys the continuation equation $D^{(2)}(z) = {}^1/D(z)$), they argue that it should be possible to express all special solutions for A(z) as rational functions of B(z). Here they are again following Wanders: he found that this was the case for his particular crossing metrix.

Now clearly, since both A(z) and B(z) are to be antisymmetric, B(z) must appear as a linear factor in an otherwise symmetric expression. Equation (14) also indicates that for large B(z), $A(z) \sim \frac{1}{B(z)}$ (17) and that the product $A^{(2)}(z)\, A(z)$ must reduce to an inverse quadratic in B(z). The simplest possibility is clearly

$$A(z) = \frac{1}{B(z)} \tag{18}$$

Then $A^{(2)}(z) = \dfrac{1}{B^{(2)}(z)} = \dfrac{-1}{B(z) + 2c}$

$$\therefore \big(B(z) - (1-c)\big)\big(B(z) + (1+c)\big) = B(z)\big(B(z) + 2c\big)$$

so $c = \pm 1$. (19)

The next simplest possibility is

$$A(z) = \frac{B(z)}{\big(B(z) + 2c\big)\big(B(z) - 2c\big)} \tag{20}$$

which by a similar analysis is shown to be a special solution for $c = \pm \frac{1}{3}$, while

$$A(z) = \frac{\big(B(z) + 2c\big)\big(B(z) - 2c\big)}{\big(B(z) + 4c\big)\,B(z)\,\big(B(z) - 4c\big)} \tag{21}$$

is a special solution for $c = \pm\frac{1}{5}$

This can clearly be continued, for increasingly complex expressions for A(z). Our conclusion is that special solutions for A(z) exist for

$$c = \pm \frac{1}{2n+1} \quad , n = 0, 1, 2 \; --- \tag{22}$$

No other such simple expressions for A(z) in terms of B(z) are obtainable. For expressions like

$$A(z) = \frac{B(z)}{\big(B(z) + c\big)\big(B(z) - c\big)} \tag{23}$$

lead to inconsistencies when substituted into (14): the resulting equation cannot be satisfied by a single value of c.

What do these "special" values of c represent?

Consider the scattering of spinless particles, isospin n (i.e. mesons) by particles isospin $\frac{1}{2}$. For this process, the crossing matrix[5] is

$$c = \frac{1}{2n+1}\begin{pmatrix} -1 & 2(n+1) \\ 2n & 1 \end{pmatrix} \qquad (24)$$

Here we have designated $S_1 = S(j = n - \frac{1}{2})$, $S_2 = S(j = n + \frac{1}{2})$. Alternatively this is the crossing matrix for the scattering, say, of spinless particles with orbital angular momentum n by particles spin $\frac{1}{2}$ (with, clearly, infinite mass for we are working on the static model. This interpretation is clearly meaningful only in this limit where the partial waves are not coupled together).

In any case, (24) is clearly the crossing matrix(10), with $c = \frac{-1}{2n+1}$, just the case for which we have obtained special solutions. Let us work out the S matrix elements which we obtain.

We have $B = -2c(\pi^{-1}\sin^{-1} z)$

if we write $w = \pi^{-1}\sin^{-1} z$ (25)

then $B = -2cw$.

For $c = -1$, $B = 2w$, $A = \frac{1}{B} = \frac{1}{2w}$, and $S_1 = \frac{w-1}{w}$, $S_2 = 1$.

For $c = -\frac{1}{3}$, $B = \frac{2}{3}w$, $A = \frac{3w}{2(w-1)(w+1)}$

and $S_1 = \frac{w(w-2)}{(w+1)(w-1)}$, $S_2 = \frac{w}{w-1}$

and clearly we can build up the complete elastic S matrix, obtaining, in this variable w, for the successive diagonal elements,

$$\frac{w-1}{w},\ 1,\ \frac{w(w-2)}{(w+1)(w-1)},\ \frac{w}{w-1},\ \frac{(w+1)(w-1)(w-3)}{(w+2)\,w\,(w-2)},$$

$$\frac{(w+1)(w-1)}{w\,(w-2)} \text{ and so on.} \tag{26}$$

These are clearly particularly simple forms for the S matrix elements, consisting as they do of ratios of finite polynomials in the variable w. We shall see that we obtain them again, later, by a different method (95).

Are these, then, the only solutions? Martin and McGlinn next attempt to construct further special solutions, for arbitrary values of c. For if these are the only solutions, we see that the possible crossing matrixes are considerably restricted, and therefore certain symmetries should be evident, coming only from the requirements of unitarity and crossing symmetry. They find, though, that this is not the case. For they manage to calculate special solutions for arbitrary c, as follows.

Since the crossing relations have already been solved, it is clearly sensible to use a mathematical formulation in which it is easy to satisfy the unitarity condition. Martin and McGlinn work with phase shift dispersion relations: one employs the logarithm of the S matrix element. Then the unitarity condition on the function

$$\Delta_\alpha(z) = \ln S_\alpha(z) \qquad (27)$$

is a linear one. Clearly $\Delta_\alpha(z)$ is intimately connected with the usual phase shift, $\delta_\alpha(z)$ for we have

$$S_\alpha(z) = e^{2i\delta_\alpha(z)} \qquad (28)$$

The disadvantage of the use of $\Delta_\alpha(z)$ lies in the introduction of additional branch points at the zeros and poles of the S matrix elements, but this is unimportant in the present case.

We know, from equation (16) that A(z) may be written in the form

$$A(z) = Ao(z)\ D(z)$$

where both Ao(z) and D(z) are real analytic, and meromorphic in the cut z plane, Ao(z) being antisymmetric, D(z) symmetric. Evidently, if we have any solution Ao(z) which is a special solution of equation (14), there exists a D(z) for which the product A(z) = Ao(z) D(z) has no zeros or poles (away from the two cuts) except a simple pole at the origin. We therefore have a simple solution of the form

$$A(z) = \frac{C(z)}{z} \tag{29}$$

where now C(z) is a symmetric, real analytic, non vanishing entire function in the cut z plane. Then, the factorisation of the S matrix elements equation (13) becomes

$$\begin{aligned} S_1(z) &= \frac{1}{z}\, C(z) \left[B(z) - (1-c)\right] \\ S_2(z) &= \frac{1}{z}\, C(z) \left[B(z) + (1+c)\right] \end{aligned} \tag{30}$$

To obtain the general solution from these expressions for the S matrix elements, it is merely necessary to put in the common arbitrary factor D(z). They must satisfy the unitarity condition on the physical cuts. Now, using equation (30), we may write

$$\begin{aligned} \Delta_1(z) &= -\ln z + \ln C(z) + \ln\left(B(z) - (1-c)\right) \\ \Delta_2(z) &= -\ln z + \ln C(z) + \ln\left(B(z) + (1+c)\right) \end{aligned} \tag{31}$$

and these phase shifts must satisfy

$$\Delta_\alpha^{(2)}(z) = -\Delta_\alpha(z) , \quad z \geq 1 , \qquad (32)$$

in consequence of equations (13) and (27).

We note that the branch cuts arising from zeros or poles of the S matrix elements are to be drawn away from the physical cut in such a way that we preserve the real analyticity of the phase shifts. (This is always possible for real analytic $S_\alpha(z)$). In particular, the cut arising from the $\ln z$ term in equation (31) is chosen to be $(-\infty, 0)$. This then gives the expression

$$\ln C(z) + \ln C^{(2)}(z) = 2\ln(z) - \ln\left[-(B(z) - (1-c))(B(z) + (1+c))\right].$$

Using the function

$$\chi(z) = \frac{\ln C(z)}{iq(z)} \qquad (34)$$

which is symmetric and obeys the usual conditions on the cut z plane, they obtain, finally, the special solution for A(z) as

$$A(z) = \frac{1 - iq(z)}{z} \exp\left[\frac{iq(z)}{\pi} \int_1^\infty \frac{\ln\left[1 + F^2(x)\right] x\,dx}{(x^2-1)^{1/2}(x^2 - z^2)} \right] \qquad (35)$$

where $F(z) = \frac{2c}{\pi} \ln(z + q(z)) - q(z)\beta(z)$ (36)

and is real for $z \geqslant 1$.

The integral in equation (35) can be evaluated for $c = \pm\frac{1}{2n+1}$ and we get results which agree, clearly, with the solutions obtained previously (26) within a factor of D(z). For other values of c, Martin and McGlinn found no way of carrying out the integration in an analytic fashion.

This solution (35) for A(z) holds for an arbitrary value of the crossing matrix parameter c. This parameter enters only as a linear factor in F(z). For we note

$$1 + F^2(z) = |B(z) + (1+c)|^2 = |B(z) - (1-c)|^2$$
$$= -[B(z) - (1-c)][B(z) + (1+c)] \qquad (37)$$

Where we have taken $B^*(z) = B^{(2)}(z)$, given by equation (14), and have used equation (15). Thus A(z) appears to depend only trivially on the crossing matrix parameter c, and Martin and McGlinn point out that this would make it appear unlikely that any particular values of the parameter c should be singled out as of special significance. In any case, it is apparent that analyticity, unitarity and crossing symmetry are not of themselves sufficient to restrict solutions to specific values of the parameter c only. Rather, it appears that further conditions are required before any such symmetry prediction can be made.

One possible further condition is to demand that the exact solutions should satisfy the Huang-Low bootstrap criterion of self consistency.[12] This is to demand that solutions should satisfy Levinson's theorem[13], which states

$$\Delta\delta_\alpha(z) = \delta_\alpha(\infty) - \delta_\alpha(1) = -\pi b_\alpha \tag{38}$$

where $\delta_\alpha(z)$ is the phase shift and b_α is the number of bound states in the channel α. In other words, every bound state in the channel α is a zero of $D_\alpha(z)$, and vice versa, where $D_\alpha(z)$ is the denominator in the usual N/D formulation, and has only the right hand cut.

Cunningham[8] imposes this criterion. He finds that solutions are obtainable only for values of the parameter c corresponding to Su(2) symmetry. He uses the Rothleitner method, which we next discuss: we shall therefore leave over his work until later (equation (57) on). Both Rothleitner and Cunningham[7] make a transformation to the variable W, defined by

$$z = \cosh W \tag{39}$$

In this variable, the whole z plane is transferred to the strip $0 \leq \operatorname{Im} W \leq \pi i$. (40).

Now it is well known that, employing equation (7b), we may continue the function $S_\alpha(W)$ into the strip $-\pi i \leq \operatorname{Im} W \leq 0$ by

$$S_\alpha(-W) = \frac{1}{S_\alpha(W)} \tag{41}$$

If we now consider equations (7c) and (40), we see that the crossing relationship becomes

$$S_\alpha(W + i\pi) = \sum_\beta C_{\alpha\beta} \frac{1}{S_\beta(W)} \tag{42}$$

Equation (7) also gives us $S(-W^*) = S^*(W)$ (43)

Rothleitner first considers one of Martin and McGlinn's "trivial" cases, the one dimensional case. Here he mentions both the Martin and McGlinn case with the "crossing matrix" c = 1, and also the case where c = -1. In the case c = 1, equation (42) gives

$$S(W) = S(W + 2\pi i), \tag{44}$$

so that S(W) is periodic in W with period $2\pi i$.
To accomodate the various restrictions which come

from equation (7), this gives, in terms of the variable z,

$$S(z) = \frac{K(z^2) + iq(z)}{K(z^2) - iq(z)}, \tag{45}$$

a symmetric function of z, as required by Martin and McGlinn. Here $K(z^2)$ is a somewhat complicated expression, first calculated by Castillejo, Dalitz and Dyson.[11]

In the case of c = -1, equation (42) yields

$$B(W + \pi i) = B(W) - 1 \tag{46}$$

where $$B(W) = \frac{1}{1-S(W)} \tag{47}$$

which has the solution

$$B(W) = \frac{1}{2} + \frac{iW}{\pi} + i\,\gamma(W) \tag{48}$$

where $\gamma(z)$ is an antisymmetric real analytic meromorphic function of z in the whole z plane. In fact,

$$\gamma(z) = q(z)\,\beta(z) \tag{49}$$

where $\beta(z)$ is given by equation (15).

We now go on to the case in which we are particularly interested, with the two dimensional crossing matrix of equation (24). We have now, equation (42),

$$S_\alpha(\omega + \pi i) = \sum_{\beta=1,2} C_{\alpha\beta} \frac{1}{S_\beta(\omega)}$$

Rohrleitner solves this equation using the single valued substitution

$$B(W) = \frac{nS_1(W) + (n+1)\,S_2(W)}{S_2(W) - S_1(W)}\,D(W),\quad U(W) = S_2(W)\,D(W) \tag{50}$$

Then

$$B(W + \pi i) = B(W) - 1,\quad U(W + \pi i) = \frac{1}{U(W)}\,\frac{B(W) + n - 1}{B(W) - n - 1} \tag{51}$$

As in the one dimensional case of $c = -1$, we may write

$$B(W) = \frac{1}{2} + \frac{iW}{\pi} + i\gamma(W),$$

where $\gamma(W)$ is an arbitrary periodic function in W, with period πi,

while

$$U(W) = \frac{\tan \frac{1}{2}\pi B(W)}{\tan \frac{1}{2}\pi (B(W) + n)}\,U_0(W) \tag{53}$$

where

$$U_0(W) = \frac{\Gamma\left[\frac{1}{2}(B(W) + n + 1)\right]\Gamma\left[\frac{1}{2}(B(W) - n)\right]}{\Gamma\left[\frac{1}{2}(B(W) - n + 1)\right]\Gamma\left[\frac{1}{2}(B(W) + n)\right]} \tag{54}$$

and D(W) is an arbitrary periodic function in W, with period $2\pi i$.

Now if we compare the two ways of writing the crossing matrix, equations (10) and (24), we see that Martin and McGlinn's $c = \pm \frac{1}{2n+1}$ which is their condition for the existence of special solutions, is equivalent to the requirement here that n should be integral. If we consider equation (54) for integral n, we see that $U_0(W)$ reduces to a rational function of $B(W)$, in that

$$U_0(W) = \frac{B(W)}{B(W)-n} \prod_{k=1}^{k=m} \frac{[B(W)+n-(2k-1)][B(W)-n+(2k-1)]}{[B(W)+n-2k]\ [B(W)-n+2k]} \tag{55}$$

where $m = \frac{n}{2}$ for $n = 2x$

$m = \frac{n-1}{2}$ for $n = 2x + 1$

and $U_0(W) = 1$ when $n = 0$.

Thus Rothleitner has found particularly simple solutions in just those cases where Martin and McGlinn find them also.

To satisfy the analyticity properties of $S_\alpha(z)$, $D(z)$ must have zeros at the poles of $U_0(z)$. $D(z)$ can also have poles at the zeros of $U_0(z)$ on the physical sheet, as well as unlimited "extra zeros".

D(z) may then be represented in the form

$$D(z) = \prod_a \frac{1 - i r_a q(z)}{1 + i r_a q(z)} \prod_b \frac{1 - m_b q(z)}{1 + m_b q(z)} \quad \frac{1 + m_b^* q(z)}{1 - m_b^* q(z)} \tag{56}$$

where $\mathrm{Im}\, r_a = 0$, $\mathrm{Re}\, m_b > 0$

Clearly this is normalised to $D \longrightarrow 1$ as $q \longrightarrow 0$, so as to comply with the threshold condition.

For all non integral values of n, we see that we have an infinite product. We must therefore put in a factor to ensure convergence, thereby allowing extra arbitraryness to the solutions.

Rothleitner states that the simplest cases are the cases of $n = \infty$, i.e. $c = 0$, where we have the crossing matrix of equation (12), which case is discussed there, and the cases $n = 0$ and $n = 1$, i.e. $c = -1$ and $c = \frac{-1}{3}$. These have already been discussed. We shall come to them again later, when we try to build up the whole S matrix, using both unitarity and crossing. (See in particular equation (95)), which duplicates equation (26)).

Rothleitner does not attempt to discuss the case of $n = \frac{1}{2}$ or $c = \frac{-1}{2}$. This corresponds to the scattering of K mesons by spinless nucleons, and since n is not integral this is not one of the simple cases.

He next goes on to three dimensional and four dimensional cases, which do not concern us. We turn our attention rather to Cunningham and his attempts[8] to impose the Huang Low bookstrap criterion. In another paper[7], he comes to the conclusion that the degree of arbitrariness inherent in the model prevents any great restriction on the form of the solution by such a criterion, at any rate in the case where $N \geq 1$, where N is given by

$$c = \frac{1-N}{N(n+1)-1} \qquad (57)$$

where c is the crossing matrix parameter. Now if $N = 2$ and n is integral, we have again the crossing matrix of equation (24), corresponding to Su(2) symmetry. However, if $N = 2$ the arbitrariness is far less. He shows the restriction to Su(2) symmetry as follows.

For $z \geqslant 1$, following Huang and Low, write

$$S_\alpha(z) = e^{2i\delta_\alpha(z)} \qquad (28)$$

$$D(z) = e^{-2i\theta(z)} \qquad (58)$$

$$\frac{B(z)-(n+1)}{B(z)+n} = e^{-2i\phi(z)} \qquad (59)$$

$$u(z) = e^{2i\phi(z)} \tag{60}$$

where $\delta_\alpha(z)$, $\Theta(z)$, $\psi(z)$ and $\phi(z)$ are real. These equations represent valid expressions.

Then

$$\begin{aligned} \delta_1(z) &= \phi(z) - \Theta(z) - \psi(z) \\ \delta_2(z) &= \phi(z) - \Theta(z) \end{aligned} \tag{61}$$

and

$$\begin{aligned} \Delta\delta_1(z) &= \Delta\phi(z) - \Delta\Theta(z) - \Delta\psi(z) \\ \Delta\delta_2(z) &= \Delta\phi(z) - \Delta\Theta(z) \end{aligned} \tag{62}$$

where $\Delta\phi(z)$, $\Delta\Theta(z)$, $\Delta\psi(z)$ are defined by analogy to equation (38). He now discusses the various possible values which are obtainable for $\Delta\delta_\alpha(z)$ considering separately the three cases $\Delta\phi=0$, $\Delta\phi=\Delta\psi$ and $\Delta\phi=\frac{1}{2}\Delta\psi$, which he first proves are the only possible values. His analysis indicates that only for values of the parameter c corresponding to an internal Su(2) symmetry are solutions possible. This indicates that the model is not applicable to, say, KN scattering; this was implicitly suggested by Rothleitner.

CHAPTER 2

In our calculation, we employ the substitution suggested by Meshcheryakov,[6] which is to write

$$w = \pi^{-1} \sin^{-1} z \quad . \tag{25}$$

This has been employed previously to facilitate the evaluation of the Martin-McGlinn solutions. It is similar to the substitution used by Rothleitner and Cunningham.

We now give the S matrix elements, explicitly, two labels i, j, to indicate the presence of off diagonal (inelastic) elements. Then, in the variable w, equation (7) reduces to the following restrictions:

a. $S_{ij}^{*}(w) = S_{ij}(1-w)$

b. $S_{ij}(w)$ is a meromorphic function in the complex w plane. (63)

c. $\sum_{k=1,2} S_{ik}(w) S_{kj}(1-w) = \delta_{ij}$ — unitarity.

d. $S_{ii}(-w) = \sum_{j=1,2} C_{ij}(w) S_{jj}(w)$ — crossing,

where C_{ij} is the matrix of equation (10), and $S_{ii}(w)$, $S_{jj}(w)$ describe reactions involving the same meson.

and e. $S_{ii}(w)$ has a pole at the origin, with residue $-\lambda_i$.

Clearly, if these conditions allow one solution, $S_{ij}(w)$, then there is a whole class of solutions, which may be written

$$S_{ij}(w+\beta(z))\,\mathcal{D}_i(z)\,\mathcal{D}_j(z) \qquad (64)$$

(not summing over i,j)

where $\mathcal{D}_i^*(z) = \mathcal{D}_i(z^*)$, $\mathcal{D}_i(z) = \mathcal{D}_i(-z)$

and $\beta^*(z) = \beta(z^*)$, $\beta(z) = -\beta(-z)$.

Our first step is to prove the following uniqueness theorem, which is fundamental to all succeeding work.

<u>Theorem</u> (65)

If Sij(w) (i,j = 2,3) satisfy the unitarity equation, equation (63c), and the single inelastic amplitude $S_{23}(w)$ (single since $S_{32}(w) = S_{23}(w)$, due to time reversal invariance) is odd or even in w (the reason for this restriction is given below) then we can determine $S_{23}(w)$ and $S_{33}(w)$ uniquely in terms of $S_{22}(w)$, except insofar as we allow possible arbitrariness by the introduction of $\beta(z)$ and $\mathcal{D}(z)$.

<u>Proof</u>.

Equation (63c) reduces to the four equations

a. $S_{22}(w)\,S_{22}(1-w) + S_{23}(w)\,S_{32}(1-w) = 1$

b. $S_{22}(w)\,S_{23}(1-w) + S_{23}(w)\,S_{33}(1-w) = 0$ (66)

$$\text{c.}\quad S_{32}(w)\, S_{22}(1-w) \;+\; S_{33}(w)\, S_{32}(1-w) = 0$$

$$\text{d.}\quad S_{32}(w)\, S_{23}(1-w) \;+\; S_{33}(w)\, S_{33}(1-w) = 1$$

Then equations (66a), (66d) yield us

$S_{22}(w) S_{22}(1-w) = S_{33}(w)\, S_{33}(1-w)$, which

means that we may write

$$S_{33}(w) = \frac{A(w)}{A(1-w)}\, S_{22}(w) \tag{67}$$

where $A(w)$ is an arbitrary function of w.

If we now substitute this expression for $S_{33}(w)$ into equation (66b) we get

$$S_{22}(w)\, S_{23}(1-w) \;+\; S_{23}(w)\, S_{22}(1-w)\, \frac{A(1-w)}{A(w)} = 0. \tag{68}$$

Multiply by $S_{23}(w)$

$$S_{22}(w)\, S_{23}(w)\, S_{23}(1-w) \;+\; S_{23}^2(w)\, S_{22}(1-w)\, \frac{A(1-w)}{A(w)} = 0$$

and substitute for $S_{23}(w)\, S_{23}(1-w)$ from (66a),

using

$S_{23}(w) = S_{32}(w)$; then

$$S_{22}(w) \left[1-S_{22}(w)\, S_{22}(1-w)\right] \;+\; S_{23}^2(w)\, S_{22}(1-w)\, \frac{A(1-w)}{A(w)} = 0$$

so

$$S_{23}^2(w) = S_{22}^2(w) \left[1-S_{22}^{-1}(w)\, S_{22}^{-1}(1-w)\right] \frac{A(w)}{A(1-w)} \tag{69}$$

Thus we have obtained solutions, equations (67), (69) for $S_{33}(w)$ and $S_{23}(w)$ in terms of $S_{22}(w)$. We must now prove their uniqueness. We see that imposition of the requirement that $S_{23}(w)$ should be odd or even in w fixes

$\frac{A(w)}{A(1-w)}$ uniquely up to an arbitrary $D(z)$. For suppose that $\frac{Ao(w)}{Ao(1-w)}$ is one such solution, then the most general solution is

$$\frac{Ao(w)}{Ao(1-w)} \quad \frac{B(w)}{B(1-w)} \quad \frac{B(-w)}{B(1+w)} \tag{70}$$

where $B(w) = B(2+w)$. Now, since $B(w)$ is periodic in w, it must be possible to write it as a Fourier series,

$$B(w) = \sum_n a_n \sin n\pi w \tag{71}$$

But $z = \sin \pi w$ (25). So this expression (71) is merely a power series in z. Thus, (71) just introduces a multiplicative factor of $D(z)$ type. We therefore have unique expressions for $S_{23}(w) = S_{32}(w)$, and $S_{33}(w)$, up to the arbitraryness allowed, as required.

We note that the theorem holds for any such 2x2 array of four S matrix elements.

Our task now is to find the simplest possible S matrix elements $S_{ij}(w)$; in other words, to find the solutions with the least number of poles. To do this, we follow Meshcheryakov's suggestion and write the S matrix elements as formal power series in $\frac{1}{w}$. In other words,

$$S_\alpha(w) = \alpha_0 + \frac{\alpha_1}{w} + \frac{\alpha_2}{w^2} + \dots = \sum_n \frac{\alpha_n}{w^n} \qquad (72)$$

where the subscript α labels any S matrix element. Meshcheryakov's work ignores off diagonal (inelastic) elements. In other words, we are considering equation (7), and not equation (63), and we may revert to labelling our S matrix elements with a single suffix. Then if we consider equation (7c), we find that it has an identical form in terms of the variable w, i.e.

$$\sum_\beta C_{\alpha\beta} S_\beta(-w) = S_\alpha(w) \qquad (73)$$

where the labels α and β refer to S matrix elements adjacent on the diagonal, and describing different isospin channels belonging to the same (elastic) process.

Using (72), this becomes

$$\sum_{\beta} C_{\alpha\beta}\beta_n = (-1)^n \alpha_n \tag{74}$$

Meshcheryakov's unitarity equation is clearly

$$S_\alpha(W)\, S_\alpha(1-W) = 1 \tag{75}$$

and involves one S matrix element only.

We see that

$$S_\alpha(1-W) = \alpha_0 + \frac{\alpha_1}{1-W} + \frac{\alpha_2}{(1-W)^2} + \cdots$$

$$= \alpha_0 - \frac{\alpha_1}{W} + \frac{\alpha_2 - \alpha_1}{W^2} - \frac{\alpha_3 - 2\alpha_2 + \alpha_1}{W^3} + \cdots$$

so that (75) becomes

$$\left[\alpha_0 + \frac{\alpha_1}{W} + \frac{\alpha_2}{W^2} + \cdots\right]\left[\alpha_0 - \frac{\alpha_1}{W} + \frac{\alpha_2 - \alpha_1}{W^2} - \cdots\right] = 1$$

and on equating coefficients of $\frac{1}{W}$, we obtain

$$\alpha_0^2 = 1 \qquad \text{So } \alpha_0 = \pm 1$$

$$2\alpha_2\alpha_0 - \alpha_1\alpha_0 - \alpha_1^2 = 0$$

$$\text{So } \alpha_2 = \tfrac{1}{2}\left[1 + \alpha_1\right]\alpha_1 \text{ or } \tfrac{1}{2}\left[1 - \alpha_1\right]\alpha_1,$$

according as we choose $\alpha_0 = \pm 1$

etc.

i.e.

$$\alpha_{2n} = f(\alpha_{2n-1}, \alpha_{2n-3}, \ldots \alpha_1) \tag{76}$$

Equations (74), (76) are an infinite set of coupled equations. The system may be solved by a finite number of steps, since the first N coefficients of the polynomial are linearly independent, where N is the number of poles of $S_\alpha(w)$. And when we have done so, we shall have connected any pair of adjacent S matrix elements which describe different isospin channels for the same process. At first sight, this appears not to give us any connection between the alternate adjacent pairs, those describing processes going via the same total isospin state, but involving different mesons. We note, however, that any element of the diagonalised S matrix can be thought of as a member of either of such types of pairs: we hope to obtain a general expression, such as the one sought by Meshcheryakov, which gives us a recipe for all the pairs of elements usually regarded as linked by crossing, and thus for all the elements of the elastic S matrix. We shall then have the complete, diagonalised, S matrix. Clearly this will be of infinite dimension, since the set of equations never terminates, there being no a priori reason why any S matrix element Sii(w) on the diagonal should be zero.

We must consider which elements of our S matrix describe which process. Let us start with the element $S_{11}(w)$, and take it to be the element which describes elastic scattering of the ω meson, since the ω meson has the lowest isospin, and we would therefore expect to start with it. But we must assign an isospin value to $S_{11}(w)$. In general, elastic scattering of a meson may go via one of two isospin channels, which are $I \pm \frac{1}{2}$, where I is the isospin of the particular meson. Thus each such elastic process has two S matrix elements associated with it, one for each isospin value, and, as mentioned above, they are connected by crossing.

However, the ω meson has isospin 0. Thus the I-$\frac{1}{2}$ channel is not physically attainable. Nevertheless, it is convenient to assign an S matrix element to this non physical process, to bring the ω meson into line with the other mesons. Thus we choose $S_{11}(w)$ to describe elastic scattering of the ω meson in the $I = -\frac{1}{2}$ channel.

Next we choose $S_{22}(w)$ to be the element which describes the elastic scattering of the ω meson via the other channel, i.e. the $I = \frac{1}{2}$ channel. Clearly $S_{11}(w)$ and $S_{22}(w)$ constitute one of the aforementioned pairs of elements, generally regarded as connected by crossing, which were considered by Meshcheryakov.

The next such pair is $S_{33}(w)$ and $S_{44}(w)$, which describe the elastic scattering of the π meson, via the $I = \frac{1}{2}$ and $I = \frac{3}{2}$ channels respectively. And so on.

Now in a paper with Fairlie[9], we showed that it was possible to connect such pairs of elements without explicit use of crossing. We do this by postulating elements at e.g. $S_{12}(w)$ and linking such elements with the adjacent pair of diagonal elements, using equation (66) and expanding all our elements in the form of equation (72). We then "switch off" the coupling, i.e. let the off diagonal elements go to zero. We find that this method is a way of rederiving the Martin-McGlinn solutions (26).

When we have obtained a form for the elastic S matrix, we shall turn our attention to the task of finding inelastic elements also: their presence will, of course, affect the values of the elastic elements; this is done in Chapter 3.

Consider any four elements of the S matrix, arranged in a 2 x 2 array on the diagonal. Label them with the suffices p, q, where we take $q = p + 1$. We wish to find some connection between them, using uintarity.

First, consider $S_{pq}(w)$ and $S_{qp}(w)$. They describe an inelastic process and the reverse process respectively.

So, clearly, $S_{pq}(w) = S_{qp}(w)$, (77)

by time reversal invariance.

Take, as an example, $p = 2$, $q = 3$, so that the elements in equation (77) describe inelastic scattering of an ω meson and of a π meson respectively.

Now consider the process

$$\bar{\pi} N \longrightarrow \bar{\omega} N \tag{78}$$

It is clearly connected with

$$\bar{\pi} N \longrightarrow \bar{\omega} N \tag{79}$$

by a "crossing matrix" whose value is ± 1, which implies that $S_{23}(w)$ is odd or even in w. We see that this will hold for any $Spq(w)$ by a similar argument, and we take it to apply, in particular, to elements such as $S_{12}(w)$ and $S_{34}(w)$, which we wish to make use of.

Then, if we expand the S matrix elements in formal power series in $\frac{1}{W}$, as in equation (72), and write

$$\begin{aligned} S_{pp}(w) &= f(w) = \sum_n \frac{f_n}{w^n} \\ S_{pq}(w) &= S_{qp}(w) = g(w) = \sum_n \frac{g_n}{w^n} \\ S_{qq}(w) &= h(w) = \sum_n \frac{h_n}{w^n} \end{aligned} \tag{80}$$

we see that the evenness or oddness of

$Spq(w)$ requires that

$$\begin{aligned} &\text{either} \quad g_{2n} = 0, \quad n = 0, 1, 2 \ldots \\ &\text{or} \quad g_{2n+1} = 0, \quad n = 0, 1, 2 \ldots \end{aligned} \tag{81}$$

This is the reason for the restriction on $S_{23}(w)$ in the theorem (65). It is clear that equation (81) produces a considerable simplification in the unitarity equations which we have to solve, (66).

Let us consider the case where the off diagonal elements are odd in w.

A convenient representation of the three S matrix elements is then

$$\begin{aligned} &\text{a.} \quad Spp(w) = f(w)\, g(w) \\ &\text{b.} \quad Spq(w) = Sqp(w) = \frac{co}{w}\, g(w) \\ &\text{c.} \quad Sqq(w) = h(w)\, g(w) \end{aligned} \tag{82}$$

where co is a constant, f(w) and h(w) are functions of w, and g(w) is an even function of w. We shall write f(w), g(w) and h(w) respectively as powere series in $\frac{1}{w}$. We note that equation (82) constitutes a different substitution to equation (80). The equation (66b) becomes

$$f(w)\cdot\frac{1}{1-w}+\frac{1}{w}h(1-w)=0 \qquad (83)$$

or

$$\left[f_0+\frac{f_1}{w}+\frac{f_2}{w^2}+\cdots\right]=\left[1-\frac{1}{w}\right]\left[h_0-\frac{h_1}{w}+\frac{h_2-h_1}{w^2}\cdots\right]$$

Equating coefficients of like powers of $\frac{1}{w}$, we find that the terminating solution for f(w) and h(w) is given by

$$f(w) = 1-\frac{1+ao}{w} \qquad h(w) = 1+\frac{ao}{w} \qquad (84)$$

Thus equation (82) becomes

$$\text{a.}\quad Spp(w) = \left(1-\frac{1+ao}{w}\right)g(w)$$

$$\text{b.}\quad Spq(w) = Sqp(w) = \frac{co}{w}g(w) \qquad (85)$$

$$\text{c.}\quad Sqq(w) = \left(1+\frac{ao}{w}\right)g(w)$$

where we must still evaluate the even function g(w) by means of the remaining unitarity equations (66).

Equation (66c) becomes

$$\left[\left(1+\frac{ao}{w}\right)\left(1+\frac{ao}{1-w}\right)+\frac{co^2}{w(1-w)}\right]g(w)\,g(1-w) = 1. \qquad (86)$$

or

$$\left[1 + \frac{a_1}{w(1-w)}\right] g(w)\, g(1-w) = 1 \qquad (87)$$

where we have written $a_1 = a_o^2 + a_o + c_o^2$. (88)

Equation (87) yields

$$g(w) = 1 + \frac{a_1}{2w^2} + \frac{3a_1^2 - 2a_1}{8w^4} + \cdots \qquad (89)$$

when we solve in the usual way by equating coefficients.

Therefore our expressions for the S matrix elements are given by equation (85), with g(w) given by equation (89). Next we must consider what happens when we switch off the coupling provided by Spq(w), or in other words, let co go to zero.

First, however, consider the expressions we have obtained. Will they yield us, as Spp(w) and Sqq(w), elements which can be connected, consistently, by the crossing matrix (10) with an appropriate value for the parameter c? Let us try to connect the expressions of equation (85a), (85c) using such a crossing matrix.

Then

$$\begin{pmatrix} 1 - \frac{1+ao}{W} \\ 1 + \frac{ao}{W} \end{pmatrix} = \begin{pmatrix} c & 1-c \\ 1+c & -c \end{pmatrix} \begin{pmatrix} 1 + \frac{1+ao}{W} \\ 1 - \frac{ao}{W} \end{pmatrix} \tag{90}$$

where we have used the fact that $g(w) = g(-w)$. The solution of equation (90) is

$$c = \frac{-1}{2ao+1} \tag{91}$$

Thus Spp(w) and Sqq(w) are linked together by a crossing matrix identical to that of equation (24), if we identify $n = ao$. They represent therefore a pair of elements such as S_{11} and S_{22} or S_{33} and S_{44}.

Let us see what the expressions (85a), (85c) reduce to when we set $c_o = o$. In this case, $a_1 = ao(1+ao)$, (92)

and (89) becomes

$$g(w) = 1 + \frac{ao(1+ao)}{2w^2} + \frac{3ao^2(1+ao)^2 - 2ao(1+ao)}{8w^4} + \tag{93}$$

so that,

$$Spp(w) = S_{-(1+ao)}(w) = \left(1 - \frac{1+ao}{w}\right) x$$

$$x \left(1 + \frac{ao(1+ao)}{2w^2} + \frac{3ao^2(1+ao)^2 - 2ao(1+ao)}{8w^4} + - - \right)$$

and $$Sqq(w) = S_{ao}(w) = (1 + \frac{ao}{w}) \quad x \qquad (94)$$

$$x \left(1 + \frac{ao(1+ao)}{2w^2} + \frac{3ao^2(1+ao)^2 - 2ao(1+ao)}{8w^4} + -- - \right)$$

where we have introduced the notation $S_{-(1+ao)}(w)$ and $S_{ao}(w)$ in place of $Spp(w)$ and $Sqq(w)$ to bring out the fact that the second expression can be obtained from the first by replacing $-(1+ao)$ by ao.

We note that the crossing relationship pointed out above holds still, as required, for the two expressions of equation (94).

It appears, therefore, that the case of $Spq(w)$ odd in w furnishes us with the desired pairs of elements, those usually regarded as connected by crossing. We then obtain the following result for the S matrix, when we set ao = o, 1, 2 _ _ _ in (94);

for $wN \rightarrow wN$ via $\bar{I} = -\frac{1}{2}$, $a_o = 0$,

$$S_1 = \frac{w-1}{w}.$$

for $\omega N \to \omega N$ via $I = \frac{1}{2}$,

$$a_0 = 0, \quad S_2 = 1.$$

for $\pi N \to \pi N$ via $I = \frac{1}{2}$,

$$a_0 = 1, \quad S_3 = \frac{w(w-2)}{(w+1)(w-1)}$$

for $\pi N \to \pi N$, via $I = \frac{3}{2}$,

$$a_0 = 1, \quad S_4 = \frac{w}{w-1} \tag{95}$$

for $f N \to f N$, via $I = \frac{3}{2}$,

$$a_0 = 2, \quad S_5 = \frac{(w+1)(w-1)(w-3)}{(w+2)\,w\,(w-2)}$$

for $f N \to f N$, via $I = 5/2$,

$$a_0 = 2, \quad S_6 = \frac{(w+1)(w-1)}{w(w-2)}.$$

and so on: we have obtained identical results to those of Martin and McGlinn[4] (26). As before, we note the remarkable way in which the S matrix elements reduce to ratios of finite polynomials in w. Notable, also, is the very close link which has been thrown up between crossing and unitarity. For at no stage in the above analysis have we explicity made use of crossing to obtain a result: we have merely identified the two S matrix elements obtained for the case where $S_{12}(w)$ is odd (94) with S matrix elements linked by the usual two channel static model crossing matrix (24). And we have considered, of course, only the limiting case where all the inelastic channels are switched off, which has been treated before.

CHAPTER 3

We wish now to consider how we might incorporate the inelastic channels into our scheme. We shall clearly use the information which has already been thrown up, to aid in finding our solution.

The first, and most obvious method, is as indicated by equation. (82). We know from our theorem, (65) that it is possible to obtain the other three elements of any 2 x 2 array of the S matrix in terms of the first. So, if we consider the "unitarity boxes" along the diagonal of the S matrix to lie thus:

$$\begin{array}{cccccc} S_1 & & & & & \\ & S_2 & S_{23} & & & \\ & S_{32} & S_3 & & & \\ & & & S_4 & S_{45} & \\ & & & S_{54} & S_5 & \\ & & & & & S_6 \end{array} \qquad (96)$$

etc, we may obtain $S_{23}(w) = S_{32}(w)$, and $S_3(w)$ in terms of $S_2(w)$. The above, inelastic, analysis indicates that the elements such as $S_{23}(w)$ and $S_{45}(w)$ will be even in w, so we will assume this is the case. We therefore work out the coefficients of the various powers of $\frac{1}{w}$ in the power series expansions of $S_{23}(w)$ and $S_3(w)$ using equation (66).

We then get $S_4(w)$ from $S_3(w)$ by crossing, using the appropriate value for the parameter of the crossing matrix. Next we again use equation (66), to get $S_{45}(w) = S_{54}(w)$ and $S_5(w)$ in terms of $S_4(w)$, and hence in terms of $S_2(w)$. This can, in principle, be continued to give all the coefficients of all the elements of the S matrix.

The details of the calculation are as follows. We use the expressions (82) for the S matrix elements of the 2 x 2 "unitarity box" array. Then the equations (66) become

a. $$\left(f_0 + \frac{f_1}{w} + \frac{f_2}{w^2} + \cdots\right)\left(f_0 - \frac{f_1}{w} + \frac{f_2 - f_1}{w^2} - \cdots\right) + \left(g_0 + \frac{g_2}{w^2} + \cdots\right)\left(g_0 + \frac{g_2}{w^2} + \frac{2g_2}{w^3} + \cdots\right) = 1 \tag{97}$$

b. $$\left(f_0 + \frac{f_1}{w} + \frac{f_2}{w^2} + \cdots\right)\left(g_0 + \frac{g_2}{w^2} + \frac{2g_2}{w^3} + \cdots\right) + \left(g_0 + \frac{g_2}{w^2} + \cdots\right)\left(h_0 - \frac{h_1}{w} + \frac{h_2 - h_1}{w^2} - \cdots\right) = 0$$

c. $$\left(g_0 + \frac{g_2}{w^2} + \text{---}\right)\left(f_0 - \frac{f_1}{w} + \frac{f_2 - f_1}{w^2} \text{ --}\right) + \left(h_0 + \frac{h_1}{w} + \frac{h_2}{w^2}\right)\left(g_0 + \frac{g_2}{w^2} + \frac{2g_2}{w^3} + \text{-}\right) = 0$$

d. $$\left(g_0 + \frac{g_2}{w^2} + \text{---}\right)\left(g_0 + \frac{g_2}{w^2} + \frac{2g_2}{w^3} + \text{---}\right) + \left(h_0 + \frac{h_1}{w} + \frac{h_2}{w^2} + \text{---}\right)\left(h_0 - \frac{h_1}{w} + \frac{h_2 - h_1}{w^2} + \text{---}\right) = 1$$

The first thing to notice is that equations (97b), (97c) are essentially the same equation. This is due to the symmetric nature of the S matrix which makes $S_{pq}(w) = S_{qp}(w)$. No information, then, is contained in equation (97c) that is not present in equation (97b) and we need therefore only consider one of them, say equation (97b). Following Meshcheryakov once again, we equate coefficients of powers of $\frac{1}{w}$. From equation (97b), coefficient of unity, we get

$$fogo + goho = o$$

so if $go \neq o$, $fo = -ho$.

Coefficient of unity of the other two equations yields

$$fo^2 + go^2 = 1$$
$$ho^2 + go^2 = 1$$

which fits the above result, and also gives a value for go.

We note that equations (97a), (97d) yield no equation for the coefficient of $\frac{1}{w}$. But (97b) yields

$$f_1 g_0 - h_1 g_0 = 0$$

so $$f_1 = h_1 .$$

(97a) coefficient of $\frac{1}{w^2}$ gives a value for g_2;

$$2f_2 f_0 - f_1 f_0 - f_1^2 + 2g_2 g_0 = 0$$

and then (97d) coefficient of $\frac{1}{w^2}$ gives us h_2:

$$2g_2 g_0 + 2h_2 h_0 - h_1 h_0 - h_1^2 = 0$$

from which $h_2 = f_1 - f_2$,

which we could instead have obtained from (97b) coefficient of $\frac{1}{w}2$. In fact, the three equations (97a), (97b), (97d) are probably most amenable to solution by taking in turn coefficients of $\frac{1}{w^{2n}}$ of (97a), then of (97d), then coefficient of $\frac{1}{w^{2n+1}}$ of (97b), then coefficient of $\frac{1}{w^{2n+2}}$ of (97a) then of (97d), and so on. For we see that the coefficients of odd powers of $\frac{1}{w}$ of equations (97a), (97d) yield no new information, and neither do the coefficients of even powers of $\frac{1}{w}$ of equation (97b). We tabulate the first few terms of h(w) and g(w) below, in terms of f(w).

They are:

$$h_0 = -f_0$$

$$h_1 = f_1$$

$$h_2 = f_1 - f_2 \tag{98}$$

$$h_3 = f_3 + \frac{1}{1-f_0^2}\left(f_1^2 f_0 + f_1 - 2f_2 \right)$$

$$h_4 = 3f_3 - f_4 + \frac{1}{1-f_0^2}\Big[\tfrac{1}{2} f_1^2 f_0 + f_1 - 3f_2 + \tfrac{1}{2} f_1 f_0^2 - f_1^3 + 2 f_0 f_1 f_2 \Big].$$

and $\quad g_{2n+1} = 0, \quad n = 0, 1, 2 \ldots$

$$g_0 = (1 - f_0^2)^{1/2}$$

$$g_2 = \tfrac{1}{2}(1-f_0^2)^{-\frac{1}{2}}\left(f_1^2 + f_1 f_0 - 2 f_2 f_0\right)$$

$$g_4 = \tfrac{1}{4}(1-f_0^2)^{-\frac{1}{2}}\big(-4f_4 f_0 + 6 f_3 f_0 - f_1 f_0 + 4 f_3 f_1 - 2 f_2 f_1\big)$$
$$+ \tfrac{1}{8}(1-f_0^2)^{-\frac{3}{2}}\big(-f_1^4 + f_1^2 f_0^2 - 4 f_2^2 - 2 f_1^2 - 2 f_1^3 f_0 + 4 f_2 f_1^2 f_0 + 4 f_2 f_1 f_0^2\big).$$

For interest, the first few terms of $g(w)$ and $h(w)$ in terms of $f(w)$, when $g(w) = -g(-w)$, are given in an appendix.

Next we must consider the connection between successive "unitarity boxes". Clearly the $S_{qq}(w)$ element of one is connected with the $S_{pp}(w)$ element of the next by crossing. If wemake the identification, say

$$S_{pp}(w) = \sum_n \frac{f'_n}{W^n} = f'(w) \qquad (99)$$

where $S_{pp}(w)$ is the first element of the next "unitarity box" succeeding the one which we have been discussing (in other words $p = 4$), then clearly

$$\begin{pmatrix} h(-w) \\ f^1(-w) \end{pmatrix} = \begin{pmatrix} c & 1-c \\ 1+c & -c \end{pmatrix} \begin{pmatrix} h(w) \\ f^1(w) \end{pmatrix} \qquad (100)$$

using the usual crossing matrix, equation (9), so

$$f^1(w) = \frac{h(-w) - ch(w)}{1 - c}$$

and clearly

$$\begin{aligned} f'_{2n} &= h_{2n} \\ f'_{2n+1} &= -\frac{1+c}{1-c} h_{2n+1}, \end{aligned} \qquad (101)$$

$$n = 0, 1, 2 \ldots$$

Now identifying

$$S_{45}(w) = S_{54}(w) = g^1(w) = \sum_n \frac{g_n^1}{w^n} \qquad (102)$$

and $$S_5(w) = h^1(w) = \sum_n \frac{h_n^1}{w^n},$$

We may do exactly the same calculation as that leading up to equation (98), giving $h^1(w)$ and $g^1(w)$ in terms of $f^1(w)$. Which we have in terms of $h(w)$ equation (101), which we know, by equation (98) in terms of $f(w)$. This gives us, then, values for the terms of $f^1(w)$, $h^1(w)$ and $g^1(w)$ in terms of $f(w)$ as follows:

$$f_0^1 = -f_0$$

$$f_1^1 = -\tfrac{1}{2} f_1$$

$$f_2^1 = f_1 - f_2$$

$$f_3^1 = -\tfrac{1}{2} f_3 + \tfrac{1}{2}(1 - f_0^2)^{-\frac{1}{2}}(2f_2 - f_1 - f_1^2 f_0)$$

$$f_4^1 = 3f_3 - f_4 + (1 - f_0^2)^{-1}\Big(\tfrac{1}{2} f_1^2 f_0 + f_1 - 3f_2 + \tfrac{1}{2} f_1 f_0^2 - f_1^3 + 2 f_0 f_1 f_2\Big)$$

$$h_0^1 = f_0 \qquad (103)$$

$$h_1^1 = -\tfrac{1}{2} f_1$$

$$h_2' = f_2 - \frac{3}{2} f_1$$

$$h_3' = -\frac{1}{2} f_3 + (1-f_0^2)^{-1}\left(\frac{1}{4} f_1^2 f_0^2 - 2 f_1 + f_2\right)$$

$$h_4' = f_4 - \frac{9}{2} f_3 + \frac{3}{4} f_1^2 + \frac{1}{4}(1-f_0^2)^{-1}\left(4 f_1^2 f_0 \right.$$

$$\left. + \frac{9}{2} f_1^3 - 9 f_1^2 - 24 f_1 + 36 f_2 - \frac{3}{2} f_1 f_0^2 - 12 f_0 f_1 f_2\right)$$

$$g_{2n+1}' = 0 \ , \quad n = 0, 1, 2 \ldots$$

$$g_0' = (1-f_0^2)^{1/2}$$

$$g_2' = \frac{1}{8}(1-f_0^2)^{-\frac{1}{2}}\left(f_1^2 + 10 f_1 f_0 - 8 f_2 f_0\right)$$

$$g_4' = \frac{1}{4}(1-f_0^2)^{-\frac{1}{2}}\left(15 f_3 f_0 - 4 f_4 f_0 - \frac{5}{2} f_1 f_0 \right.$$

$$\left. + f_3 f_1 - \frac{1}{4} f_1^2 - 10 f_2 f_1\right)$$

$$+ \frac{1}{8}(1-f_0^2)^{-\frac{3}{2}}\left(\frac{23}{4} f_1^2 f_0^2 + 18 f_1 f_0 - 36 f_2 f_0 \right.$$

$$\left. - \frac{29}{4} f_1^3 f_0 + 22 f_2 f_1 - 4 f_2^2 - \frac{1}{16} f_1^4 + f_2 f_1^2 f_0\right] .$$

Where we have used the fact that the crossing matrix between $S_3(w)$ and $S_4(w)$ has parameter $C = \frac{-1}{3}$.

Clearly we can now cross $h^1(w) = S_5(w)$ into $S_6(w)$, which we may label $f^{11}(w)$, using a crossing matrix with parameter $c = \frac{-1}{5}$. This process can be continued, to give us all terms in the power series expansion of every matrix element,

$$S_{ii}(w) \text{ and } S_{ij}(w) \text{ where } j = i \pm 1. \tag{104}$$

in terms of the f_ns, or coefficients of the matrix element $S_{22}(w)$.

We do not continue this process down to $S_{11}(w)$, preferring to work only in coefficients of elements describing physical processes.

As an alternative, it is possible to obtain the inelastic elements in a rather neater fashion than that just indicated, in that we obtain a single expression which can be used for all the elements. Clearly, incorporation of off diagonal elements modifies the diagonal ones, and this shows up in our treatment: it also brings out the interesting fact that if we allow some of the elements $S_{pq}(w)$ $(p = q \pm 1)$ to be non zero, then all such elements will become non zero; the inelasticities cannot be arbitrarily assigned. The method we use is a perturbation treatment; even with our simplifications we have not been able to obtain a closed solution.

First we make a significant change in our notation. For elements $S_{ii}(\omega)$, $S_{i+1,i+1}(\omega)$ we write instead

$$S_{\mp}\left(n(n+1), \omega\right) \tag{105}$$

where the subscripts $\mp$ refer respectively to the channels with total isospin $n \mp \frac{1}{2}$: the reason for labeling the element by $n(n+1)$ rather than by n will become apparent later. For the off diagonal elements $S_{ij}(\omega)$ we also adopt a new notation: we write the element describing the process

$$n + N \longrightarrow (n+1) + N \tag{106}$$

as
$$S(n+1, \omega) \tag{107}$$

this process going, of course, solely via the $n+1$ channel. The matrix elements still, of course, satisfy (7).

Following Wanders[2] and Martin and McGlinn[4], we rewrite the diagonal elements as

$$\begin{aligned} S_-(n(n+1),\omega) &= A(n(n+1),\omega)\left[B(n(n+1),\omega) - n - 1\right] \\ S_+(n(n+1),\omega) &= A(n(n+1),\omega)\left[B(n(n+1),\omega) + n\right]. \end{aligned} \tag{108}$$

where A and B are antisymmetric functions of w. This automatically satisfies the crossing requirement with the crossing matrix of (24). Note also that if we replace n by -(n + 1), we interchange S- and S+, with $C_{\alpha\beta} \Rightarrow C_{\beta\alpha}$: thus crossing is still satisfied. Hence the reason for labelling with n(n + 1) rather than with n.

The unitarity equations are now

$$\begin{aligned} A(n(n+1),\omega)\left[B(n(n+1),\omega)+n\right] A(n(n+1),1-\omega)\left[B(n(n+1),1-\omega)+n\right] \\ + S(n+1,\omega) S(n+1,1-\omega) = 1. \end{aligned} \tag{109}$$

and $A(n(n+1), w) \left(B\left(n(n+1), w\right) + n\right)$
$S(n+1, 1-w) + S(n+1, w)\, A\left((n+1)\right.$ (110)
$\left.(n+2), 1-w\right) \left(B\left((n+1)(n+2), 1-w\right) - n - 2\right)$
$= 0$

and we impose the restriction

$$S(n+1, w) = \pm S\left(-(n+1), w\right), \tag{111}$$

which ensures the compatibility of (109) and (110) when we replace n by $-(n+2)$. The three equations (109), (110), (111) come directly from (63c) and are the basic equations of this sechion, together with the symmetry requirements on A and B. We know the exact solution when $S(n+1, w) = 0$: it is

$$B_0 = w \tag{112}$$

where we introduce the subscript to indicate that this is the value at $S(n+1, w) = 0$. We see that B_0 has no explicit dependence on n. A_0 is obtained from the recurrence relation

$$A_0(n(n+1), w)\ A_0\left((n+1)(n+2), w\right) = \frac{-1}{w^2 - (n+1)^2} \tag{113}$$

with $A_0(o) = \frac{1}{w}$ (114)

This is merely a repeat of Martin and McGlinn's work in a slightly condensed form. We know by analogy with equation (93) $Ao(n(n+1),w)$ in power series form: it is

$$Ao\left(n(n+1),w\right) = (-1)^{(n+1)}\frac{1}{w}\left[1 + \frac{n(n+1)}{2w^2} + \frac{3n^2(n+1)^2 - 2n(n+1)}{8w^4} + - - \right] \quad (115)$$

where the dependence on $n(n+1)$ is explicit, which it is not in the form (113).

We now switch on the coupling of $S(n,w)$. Let it be small, and take it proportional to an inelasticity parameter ϵ. Then to first order in ϵ, equation (110) gives us

$$S(n,w) = \epsilon\,\frac{F(n^2)}{w^2 - n^2} \quad (116).$$

Now using equation (109), written both as above, and for $n \rightarrow -(n+1)$, and dividing the two forms, we have

$$\frac{\left[B\left(n(n+1),w\right) + n\right]\left[B\left(n(n+1),1-w\right) + n\right]}{\left[B\left(n(n+1),w\right) - n - 1\right]\left[B\left(n(n+1),1-w\right) - n - 1\right]} = \frac{1 - S(n+1,w)\,S(n+1,1-w)}{1 - S(n,w)\,S(n,1-w)} \quad (117)$$

since $S(-n,w) = S(n,w)$. Then if we write

$$B\left(n(n+1), w\right) = w + \epsilon^2 B_1\left(n(n+1), w\right) \tag{118}$$

noting that $B_1\left(n(n+1), w\right)$ is antisymmetric in w, (117) yields us

$$F(n^2) = \left(n^2 - \tfrac{1}{4}\right)^{1/2} \tag{119}$$

while
$$B_1\left(n(n+1), w\right) = \frac{w\left(w^2 - n(n+1) - \frac{1}{2}\right)}{(w^2 - n^2)\left(w^2 - (n+1)^2\right)} \tag{120}$$

Putting this into (109), we get the equation

$$A_1\left(n(n+1), w\right) + A_1\left(n(n+1), 1-w\right) = - \frac{B_1(w)}{w+n} - \frac{B_1(1-w)}{1-w+n} - \frac{(n+1)^2 - \frac{1}{4}}{\left(w^2-(n+1)^2\right)\left((1-w)^2-(n+1)^2\right)} \tag{121}$$

where A_1 is the correction to A_o to order ϵ^2;

$$A\left(n(n+1), w\right) = A_o\left(n(n+1), w\right)\left[1 + \epsilon^2 A_1\left(n(n+1), w\right)\right] \tag{122}$$

and clearly is symmetric in w.

Now the third term on the right of equation (121) may be split into $\dfrac{2w+1}{4\left(w^2-(n+1)^2\right)} + \dfrac{2(1-w)+1}{4\left((1-w)^2-(n+1)^2\right)}$.

Then the solution may be expressed as

$$A_1\left(n(n+1), w\right) = s(w) + 2\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, a(p) \sin(p-w)x \tan\frac{px}{2} \tag{123}$$

(we derive this solution in an appendix), where $S(w)$ and $a(w)$ are the parts of

$$-\frac{B_1(w)}{w+n} + \frac{2w+1}{4\left(w^2-(n+1)^2\right)}$$ which are even

and odd respectively in w.

We now have solutions for $S_{\mp}(n(n+1),w)$ and $S(n+1,w)$ up to order ϵ^2. In principle, this perturbative type of solution can be extended to higher orders of ϵ. One feature of the solution, however, is already clear; this is that the relative magnitudes of the couplings of the allowed inelastic processes are not arbitrarily assignable, but rather are determined by equations (116), (119). This is a feature which is ignored in the usual treatment of the coupled channel problem, in which the potentially infinite set of two body coupled scattering processes is arbitrarily truncated.

Appendix 1.

The case of $g(w) = -g(-w)$.

$$h_o = f_o = 1$$

$$h_1 = -(1 + f_1)$$

$$h_2 = f_2$$

$$h_3 = (2f_2 - f_1 - f_1^2)^{-1} \; (-2f_4 + 3f_3 - f_2 + 3f_3 f_1 - 2f_2 f_1 + f_2^2 - f_2 f_1^2 - 2f_3 f_2 + f_3 f_1^2)$$

$$h_4 = (2f_2 - f_1 - f_1^2)^{-1} \; (2f_4 f_2 + f_4 f_1 - f_4 f_1^2 - 4f_3 f_2 + \tfrac{3}{2} f_2^2 - \tfrac{1}{2} f_2 f_1 + f_2^2 f_1 - f_4 + \tfrac{3}{2} f_3 - \tfrac{1}{2} f_2)$$

$$g_o = g_2 = g_4 \dots = 0$$

$$g_1 = \left(2f_2 - f_1 - f_1^2 \right)^{1/2}$$

$$g_3 = (2f_2 - f_1 - f_1^2)^{-\frac{1}{2}} \; (f_4 - \tfrac{3}{2} f_3 + \tfrac{f_2}{2} - f_3 f_1 + \tfrac{1}{2} f_2 f_1 + \tfrac{1}{2} f_2^2)$$

Appendix 2.

Derivation of equation (123), and its solution. Our problem is to find the solution of the equation

$$s(w) + s(1-w) = f(w) + f(1-w) \qquad \text{(A1)}$$

where s(w) is an even function of w, and f(w) is arbitrary. Clearly, it is sufficient to consider the case where f(w) is odd, for an arbitrary f(w) may be split into odd and even parts, and the even part taken over to the left.

If, now, S(k) and F(k) are the Fourier transforms of s(w) and f(w) respectively, then (A1) yields

$$S(k)\,(1 + e^{-ikw}) = F(k)\,(1 - e^{-ikw}), \qquad \text{(A2)}$$

the different signs on left and right being, clearly, due to the evenness and oddness of s(w) and f(w) respectively. The solution of this equation is then immediately

$$s(w) = \frac{i}{2\pi}\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dk\; e^{i(w-p)k}\,\cancel{\cos kw \sin kp}\,\tan\frac{kp}{2}\, f(p) \qquad \text{(A3)}$$

This equation may be rewritten to show more clearly the evenness of s(w):

$$s(w) = \frac{2}{\pi}\int_{0}^{\infty} dp \int_{0}^{\infty} dk\, \cos kw \sin kp \tan\frac{kp}{2}\, f(p). \qquad \text{(A4)}$$

We must still add the general solution of the homogeneous equation for $s(w)$: this must be both symmetric about the origin and antisymmetric about the point $w = \frac{1}{2}$, so that we may write

$$S(w) = \sum_{n=0}^{\infty} a_n \cos(2n+1)\pi w \tag{A5}$$

where the a_n are arbitrary, as the solution.

The solution of equation (122) may now be evaluated explicity. Our solution is to be a function of n through the combination $n(n+1)$ only, so we consider the sum of (122) with the same equation with n replaced by $-(n+1)$. (Clearly the difference is automatically satisfied by our solution of (117)). Then the function $f(w)$ of equation (A1) is

$$\begin{aligned} f(w) = {} & \frac{1}{8}\left(\frac{1}{(w+n)^2} - \frac{1}{(w-n)^2}\right) + \frac{1}{8}\left(\frac{1}{(w-n-1)^2} - \frac{1}{(w+n+1)^2}\right) \\ & + \frac{1}{4(2n+1)}\left(\frac{1}{w+n} + \frac{1}{w-n}\right) + \frac{n+1}{4(2n+1)}\left(\frac{1}{w+n+1} + \frac{1}{w-n-1}\right) \end{aligned} \tag{A6}$$

If we make use of the identity

$$i\tan\frac{kp}{2} = 1 - 2e^{-ikp} + 2e^{+2ikp} + \_\_\_ \ \_\_\_\_\_\_$$

$$— — + (-1)^m e^{-mikp}$$

$$+ (-1)^m e^{-mikp} \, i \tan \frac{kp}{2} \qquad \text{(A7)}$$

we can peform the integrals in equation (A3) for the first two brackets of (A6), obtaining the finite sum

$$- \frac{1}{8} \left(\frac{1}{(w+n)^2} + \frac{1}{(w-n)^2} \right) + \frac{1}{2} \sum_{-n}^{+n} (-1)^m \frac{1}{(w+m)^2} \qquad \text{(A8)}$$

$$- \frac{1}{8} \left(\frac{1}{(w+n+1)^2} + \frac{1}{(w-n-1)^2} \right)$$

n being taken as integral.

A similar trick gives the integral for the last two brackets as an infinite series of poles, which may be rearranged as the derivative of the logarithm of a product of gamma functions. Alternatively, we can use

$$Ao\ (n(n+1),\ w)\ Ao\ (n(n+1),\ 1-W)\ (w+n)\ (1-w+n) = 1 \qquad \text{(A9)}$$

whose logaritum we may differentiate to give

$$\frac{d}{dn} \log Ao\ (w) + \frac{d}{dn} \log Ao\ (1-w) + \frac{1}{w+n} + \frac{1}{1-w+n} = o \qquad \text{(A10)}$$

Now $\frac{d}{dn}$ log Ao (w) is an even function of w,

so $\frac{d}{dn}$ log Ao (w) + $\frac{n}{w^2-n^2}$ is the solution of (A1)

with $f(w) = -\frac{w}{w^2-n^2} = -\frac{1}{2}\left[\frac{1}{w+n} + \frac{1}{w-n}\right]$ (A11)

Thus the solution for the two terms in (A6) may instead be written as

$$\frac{1}{4}\left(\frac{-n}{2n+1}\frac{d}{dn} + \frac{n+1}{2n+1}\frac{d}{d(n+1)}\right) \log Ao\,(w)$$

$$+ \frac{1}{4}\frac{1}{2n+1}\left(\frac{n^2}{w^2-n^2} - \frac{(n+1)^2}{w^2-(n+1)^2}\right) \qquad \text{(A12)}$$

If we collect together the various equations, we get as our solution to equation (123),

$$A_1(n(n+1),w) = \frac{-1}{4(2n+1)}\left(n\frac{d}{dn} - (n+1)\frac{d}{d(n+1)}\right)$$

$$\log Ao\,(n(n+1),\, w)$$

$$+ \frac{1}{2(2n+1)}\left(\frac{1}{w^2-n^2} - \frac{1}{w^2-(n+1)^2}\right)$$

$$- \frac{1}{8}\left(\frac{1}{(w+n)^2} + \frac{1}{(w-n)^2} + \frac{1}{(w+n+1)^2} + \frac{1}{(w-n-1)^2}\right)$$

$$+ \frac{1}{2}\sum_{-n}^{n}(-1)^m\frac{1}{(w+m)^2} \qquad \text{(A13).}$$

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