The application of the theory of fibre bundles to differential geometry

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THE APPLICATION OF THE THEORY OF FIBRE BUNDLES TO DIFFERENTIAL GEOMETRY

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ERRATA

p44 §6.2 The second paragraph of the Proof should read:-

The unique inverse of γ is $d\mu(\gamma)$. For since $\mu: G \Longleftrightarrow G$ then $d\mu: T(G) \Longleftrightarrow T(G)$ [§4.4]. Also $\lambda \circ (\mu, \epsilon) \circ \delta$ is a constant map onto e and therefore so is $d\lambda \circ (d\mu, d\epsilon) \circ d\delta$ [§4.3 v)]. That is

$$d\lambda \circ (d\mu, d\epsilon) \circ d\delta(\gamma) = d\lambda \circ (d\mu, d\epsilon)(\gamma, \gamma) \qquad \text{[§4.3 ii)]}$$
$$= d\lambda(d\mu(\gamma), \gamma)$$
$$= d\mu(\gamma) \cdot \gamma = e$$

p104 §14.4 Line 8 should read:-

all $\phi_x \in \Phi$ since $I^4 = 1$ then $j \circ j \circ j \circ j = j^4 = \epsilon$.

p105 §14.5 The first paragraph should read:-

Since $j^4 = \epsilon$ then of course $dj^4 = \epsilon$ but $(\tau \circ dj)^4$ is not necessarily also the identity map.

INTRODUCTION

In a great number of brilliant papers and in a few books and memorials Élie Cartan exploited a new approach to differential geometry which he called the method of moving frames or "répères mobiles"[1,2,3]. Unfortunately the expositions he gave of his methods and ideas appear to be based to a large extent on intuition and many steps have been taken to describe his theory with greater precision. Of great value in this respect has been the theory of fibre bundles and in this thesis an attempt is made to introduce precise definitions of some intuitive concepts like "infinitesimal circuits", "parallel displacements" and "infinitesimal holonomy groups" using this theory and without sacrificing geometrical significance to the requirements of rigour. In doing this a number of well established ideas have been generalised and several new results have been proved.

An important step in the application of fibre bundle theory to differential geometry in the large was made in 1942, when Steenrod proved the existence of a positive definite Riemannian metric by, in fact, showing that the problem is equivalent to the existence of a substructure with group the orthogonal group [14]. Chern also developed this approach in his papers publishing in 1946 "Some new viewpoints in differential geometry in the large", which explained his ideas and methods [12]. In the same year Chevalley published the first part of his "Theory of Lie groups" which, although not using bundles explicitly, defined manifolds, vectors, forms and the Poisson bracket operation very rigorously [17]. This book greatly influenced subsequent workers, many of whom refer to his definitions, the differential map defined by him being particularly useful.

In 1947 at an international conference on topology Ehresmann described some of the applications of fibre bundles and stressed the importance of substructures or subordinated structures [9]. He discussed the effect of various reductions of the group on a tangent bundle among them that which he called a nearly complex structure.

This latter idea gave rise to an important field of research. In 1951 Eckmann and Frölicher gave integrability conditions for such a substructure using, however, an equivalent definition by means of tensors and the classical methods [18]. Later, at a conference in 1953, Eckmann gave a paper, still using classical methods, in which he showed that one could always find a connection in which a given nearly complex structure was, in a certain sense, parallel and that the integrability conditions were that a symmetric connection with this property existed [19]. It is interesting to compare this result with another result to be published soon in a paper by Willmore [27]. These two apparently unconnected papers have a number of affinities, in fact the first result stated above is a particular case of a general theorem proved here and possibly the second could be generalised similarly.

The first attempt to give a rigorous definition of a connection based on the methods of Cartan was published in 1947 by Ehresmann, who elaborated it later in a paper given at a congress in 1950 [8, 11]. This treats thoroughly with what he calls "infinitesimal connections" giving many examples and applications. Ehresmann defines, roughly speaking, an infinitesimal connection as a field of planes on a bundle space complementary to the tangent planes of the fibre. This infinitesimal connection was the first generalisation of the classical connection since

Cartan defined an affine connection in 1923 [1]. Besides infinitesimal connections Ehresmann introduces in this paper the "solder" a particular case of which is used here.

In 1951 Steenrod produced the first book on the theory of fibre bundles [15]. Since then a large number of papers have appeared applying the theory to differential geometry. Perhaps it had long been felt that some of the basic concepts needed to be redefined in a precise yet suggestive manner and that the work on fibre bundles might enable this to be done, for recently four papers have appeared almost simultaneously defining connections and other concepts each with either more geometrical significance than Eisenhart's definitions or more rigour than Cartan's [29, 3].

The first of these appeared in September 1953, when work for this thesis had already made good progress. This was a paper by Flanders which aimed to "set up an algebraic machinery for the theory of affine connections" [22]. It is based on the definition of a vector given by Chevalley and derives the formulae of Cartan and Chern in a very rigorous manner.

Another paper two months later by Ambrose and Singer contains a review of the theory, again using Chevalley's vectors, but including the infinitesimal connections of

Ehresmann [23]. This paper gives a definition of curvature and torsion and attempts a geometrical interpretation of these entities, whereas Ehresmann discussed neither [23 p434] .

Two other papers appeared in January 1954, one by Nomizu and the other by Kabayashi, whose work has evidently coincided to some extent with mine, for he states the results of two theorems I had already proved [25]. The paper by Nomizu gives the definition of a connection due to Kozul again using the tangent vectors as defined by Chevalley [24].

This thesis is divided into five chapters and fourteen sections. The first chapter deals with well established results and definitions that I will need. It is divided into three sections, the first being merely a resume of fibre bundle theory, while in the second definitions of manifolds, regular maps and other related ideas are given. The third section defines integrable structures and gives a few examples which show that this concept is a generalization of one, already well known. It might be possible to find integrability conditions for a general structure and I have given a conjecture on the problem. The idea is not original and has also been noted by Libermann [20]. Nevertheless it is quite useful and would become very interesting if general integrability conditions could be found.

The second chapter contains some more preliminaries but of a more original nature. The fourth section is on differentials and I have copied Chevalley in defining a differential as a map, although I define it in a different way. It is notwithstanding essentially the same idea and most of its properties have been derived by considering those given by Chevalley. The fifth section is on the solder and twist maps, the former having already been introduced by Ehresmann is not new. The twist map, however, is original and has some very interesting properties. I was led to this map by considering the operation of exterior differentiation and the Poisson bracket [3, 17]. Both these are defined only for vector fields or forms given over an open set of the manifold. Thus one has really to consider the space of local cross sections of the tangent bundle or principal bundle and it is difficult to make use of the topology of the bundle space. The aim here is to characterise such operations as these two by topological means and it is to this end that I have introduced these two maps. They certainly simplify some of the formulae that arise when using the methods of Eisenhart. The last section of the chapter proves some theorems on Lie groups that will be needed and is not really new.

The third chapter is completely original and contains the basic theorem of this thesis. I have divided the chapter into two sections - the seventh and the eighth. The seventh section contains the proof that the tangent bundle of a bundle space can be fibred in two different ways; this is one of the results stated by Kabayashi [25]. This theorem is then elaborated with some theorems on subbundles and substructures. The next section deals with the special case of this theorem that occurs in the classical theory and which has some interesting special properties, for instance, the twist map which is an isomorphism. I do not know whether any of these properties characterize it completely.

The fourth chapter deals with connections, curvature and torsion in three sections respectively. The definition of a connection as a substructure was discovered simultaneously by Kabayashi [25]. It follows simply and naturally from the preceding theory and a number of results and definitions is the logical consequence. An example is the rigorous definition of a "parallel displacement around an infinitesimal circuit".

The definition of curvature which is the subject of the next section is quite unusual. A number of properties are given but I have not discussed it in much detail. The torsion is defined in a manner very similar to that in which I defined curvature and the two concepts have not unnaturally a number of correspondingly similar properties. I have defined the infinitesimal holonomy group in this chapter using the properties of curvature. It is quite possible that this generates the holonomy group, for in the special case of a linear connection this has already been proved by Ambrose and Singer [23].

The last chapter is concerned with special connections. The first section of the chapter is on linear connections and it is proved that such a connection is associated uniquely with a general connection, thus proving the existence of this specialised connection. A linear connection is also a field of planes complementary to the tangent planes of the fibres of the principal bundle and is thus an "infinitesimal connection" in the sense used by Ehresmann. In fact it is also proved that the integrability of this field of planes is equivalent to the semi-integrability of the connection which is probably equivalent to flatness. It is also shown that for these special connections the torsion and curvature are isomorphisms which have "square roots". Using this it is shown that from a given general connection one can derive uniquely a linear and symmetric connection. The last two sections are devoted to the interpretation in this theory of work done by others, the only original section of the chapter being that on linear connections, although reinterpretation of existing results has its interests.

This work on the tangent bundles of fibre bundles and connections appears to offer a wide field for further investigations which I hope to continue. I give a number of conjectures with this in view.

I would like to express my thanks to Dr. T.J. Willmore for his guidance and for showing me some of his unpublished work.

The letters in square brackets refer to the bibliography.

§0 Notation

I have endeavoured to use a consistent, clear and suggestive notation in this thesis, but this has not always been possible without introducing too many symbols. Consequently some symbols are duplicated. I have also made use of some unusual conventions which are noted here.

§0.1 $\pi : M \longrightarrow N$ means "π is a map of M into N"

$\pi : M \Longrightarrow N$ means "π is a map of M onto N"

$\pi : M \longleftrightarrow N$ means "π is a one-one map of M into N"

and $\pi : M \Longleftrightarrow N$ means "π is a homeomorphism of M onto N"

If $U \subset M$ I will sometimes write $\pi : U \longrightarrow N$ instead of $\pi | U : U \longrightarrow N$.

§0.2 ϵ will be reserved for the identity map irrespective of the space on which it acts.

Similarly $\delta : M \longleftrightarrow M \times M$ which maps a space M onto the diagonal of $M \times M$ is defined by $\delta(x) = (x,x)$ and is the symbol for this map whatever the space M.

§0.3 In order to avoid the use of indexing sets the following convention has been made:- $\{U\}$ means the set of entities of which a typical entity is U. If I wish to consider two elements of $\{U\}$ they will be called U_1 and U_2.

§0.4 If $\mathfrak{F}$ and $\mathfrak{G}$ are two sets of maps then $\mathfrak{F} \circ \mathfrak{G}$ is the set of maps $\{f \circ g$ where $f \in \mathfrak{F}$ and $g \in \mathfrak{G}\}$.

If $\pi_1:M_1 \longrightarrow N_1$ and $\pi_2:M_2 \longrightarrow N_2$ then $(\pi_1,\pi_2):M_1 \times M_2 \longrightarrow N_1 \times N_2$ is defined by $(\pi_1,\pi_2)(x,y)=(\pi_1(x),\pi_2(y))$. $\mathfrak{F} \times \mathfrak{G}$ will then have the obvious meaning $\{(f,g)$ where $f \in \mathfrak{F}$ and $g \in \mathfrak{G}\}$.

§0.5 I will use Chevalley's notation for the classical groups and Euclidean number spaces with a few obvious moderations [17]. The identity of any group will always be denoted by e, although I will sometimes use $\mathbf{1}$ for the identity matrix.

§0.6 The notation defined in a section will be presumed to hold throughout that section, any modification being temporary unless the reverse is explicitly stated.

It will be found that the different notations in the various sections are just modifications of a basic notation for a general fibre bundle. These variations are generally due to the fact that we wish to consider a special case of a fibre bundle, such as the principle bundle or tangent bundle.

CHAPTER 1   FUNDAMENTAL DEFINITIONS

§1 Fibre bundles

I propose here to give a brief sketch of those parts of fibre bundle theory that will be used in this thesis because, although the theory of fibre bundles is well established, the notation and terminology vary with different authors and I think it necessary to introduce this section to establish those we are going to use.

§1.1 DEFINITION Let B, X and Y be topological spaces and G a transformation group of Y. Let $p:B \Rightarrow X$ [§0.1].

Suppose $\Phi_x = \phi_x \circ G$ where $\phi_x : Y \Leftrightarrow B_x = p^{-1}(x)$ is a set of maps defined for each $x \in X$. Then $\Phi = \bigcup_{x \in X} \Phi_x$ is the structure of the fibre bundle $B(X,Y,G,\Phi)$ if:-

a) for any $x_0 \in X$ there exists an open set $U \subset X$, $x_0 \in U$, and a map $\phi : U \times Y \Leftrightarrow B_U = p^{-1}(U)$ such that $\phi_x(y) = \phi(x, y)$ defines a map $\phi_x \in \Phi_x$ for every $x \in U$. The map ϕ is called a strip map.

b) if $\phi':U'\times Y \Longleftrightarrow B_{U'}$ is any other strip map then $\phi'_x=\phi_x\cdot g(x)$ where $g:U\cap U'\rightarrow G$. [15 p7, 4].

B is called the bundle space, X the base space and Y the type fibre of this fibre bundle. p is the bundle projection and B_x is the fibre over x.

We shall often use B as an abbreviation for $B(X,Y,G,\Phi)$ when no ambiguity is likely to arise.

The bundle space has a number of properties in common with $X\times Y$ and a few examples of these are given by Steenrod [15 p13]. I would like to mention one that he has omitted:- The restriction of p to any subset $W\subset B$ is an open map if $p(W)$ is open.

The topological product $X\times Y$ is a fibre bundle with group e [15 p14, 5]. Such a fibre bundle is called trivial and we make a convention that the first factor is always the base space, the second being the type fibre.

§1.2 We will always give Φ the topology such that for any strip map $\phi:U\times Y\Longleftrightarrow B_U$ then $\bar{\phi}(x,g)=\phi_x\cdot g$ defines a map $\bar{\phi}:U\times G\Longleftrightarrow \Phi_U=\bigcup_{x\in U}\Phi_x$. With this topology $\Phi(X,G,G,\bar{\Phi})$ is a fibre bundle called the principal bundle, the map $\bar{\phi}$ being a strip map of $\bar{\Phi}$ [15 p35, 6].

There is a lemma that is not out of place here and which we will often have to use. I give a fairly full proof because, although the lemma is really quite simple, I have not seen it stated or proved explicitly.

<u>LEMMA</u> Let a) $\psi : U \times Y \Leftrightarrow B_U$ be a strip map of $\mathfrak{B}$;

b) $\bar{\psi} : U \times G \Leftrightarrow \bar{\mathfrak{B}}_U$ be a strip map of $\bar{\mathfrak{B}}$;

c) $\tilde{\psi} : U \leftrightarrow \bar{\mathfrak{B}}_U$ be a local cross section of $\bar{\mathfrak{B}}$,

that is $\tilde{\psi}(x) \in \bar{\mathfrak{B}}_x$; then any one of these three maps defines the other two uniquely by the relations $\psi_x \circ g = \bar{\psi}(x,g) = \tilde{\psi}(x) \circ g$.

<u>PROOF</u> Given a) then b) follows by the definition of the topology on $\bar{\mathfrak{B}}$ and given b) then c) follows immediately. It is only necessary then to show that a) follows from c).

Let $\phi : U' \times Y \Leftrightarrow B_{U'}$ be a strip map of $\mathfrak{B}$ and $\bar{\phi}$ the strip map of $\bar{\mathfrak{B}}$ defined from it. Let $\nu : G \times Y \Rightarrow Y$ and $\mu : G \Leftrightarrow G$ be defined by $\nu(g,y) = g \cdot y$ and $\mu(g) = g^{-1}$.

Now suppose $\tilde{\psi}$ is given. Write $\bar{\phi}^{-1} \circ \tilde{\psi} = g : U'' = U \cap U' \to G$. Then

$$\phi \circ (\epsilon, \nu) \circ (\epsilon, g, \epsilon) \circ (\delta, \epsilon) : U'' \times Y \Rightarrow B_{U''} \qquad \text{a)}$$

is defined to be a restriction of ψ. This map is in fact a homeomorphism because it has the inverse

$$(\epsilon, \nu) \circ (\epsilon, \mu \circ g, \epsilon) \circ (\delta, \epsilon) \circ \phi^{-1}.$$

It is easy to see that both these maps are independent of the strip map ϕ and thus define ψ uniquely.

We will say that a set of strip maps or of the cross sections thus associated with them <u>covers</u> X if the union of the arguments of the cross sections if X.

<u>§1.3</u> Suppose $H \subset G$ is a topological subgroup and suppose $\Psi_x = \lambda_x \circ H \subset \Phi_x$ is given for each $x \in X$, $\lambda_x \in \Phi_x$. Then if $\Psi = \bigcup_{x \in X} \Psi_x$ is a structure for a fibre bundle $B(X,Y,H,\Psi)$ we say that Ψ is a <u>substructure</u> of Φ [15 p43, 6]. The fibre bundle $B(X,Y,H,\Psi)$ is said to be a <u>reduction</u> of $B(X,Y,G,\Phi)$.

The lemma we have just proved shows that to prove Ψ is a substructure of Φ it is only necessary to demonstrate a set of local cross sections of Φ with values in Ψ and which cover X.

<u>§1.4</u> Let $B'(X',Y',G',\Phi')$ be another fibre bundle with projection p' and let $\pi : B \rightarrow B'$. π is said to be <u>fibre-preserving</u> if $p' \circ \pi \circ p^{-1} = \bar{\pi} : X \rightarrow X'$. In this case $\bar{\pi}$ is called the <u>projection</u> of π.

A map $\pi : B \Leftrightarrow B'$ is an <u>isomorphism</u> between the fibre bundles if it is fibre-preserving, if its projection $\bar{\pi} : X \Leftrightarrow X'$ and further if there exists $\tilde{\pi} : Y' \Leftrightarrow Y$ such that $\pi \circ \Phi' \circ \tilde{\pi} = \Phi$ [5].

In this case the principal bundles Φ and Φ' are isomorphic, $\pi^*:\Phi \Leftrightarrow \Phi'$ being defined by $\pi^*(\phi_x)=\pi\circ\phi_x\circ\tilde{\pi}$ for any $\phi_x \in \Phi$. Conversely if we are given the isomorphism $\pi^*:\Phi \Leftrightarrow \Phi'$ and a map $\pi:Y' \Leftrightarrow Y$ such that $G\circ\pi=\pi\circ G'$ then the fibre bundles B and B' are isomorphic.

§1.5 If $\Phi(X,G,G,\bar{\Phi})$ is a principal bundle $\bar{\Phi}$ has a topology and is the bundle space of another principal bundle [§1.2]. However the map $\pi:\Phi \Leftrightarrow \bar{\Phi}$ defined by $\pi(\phi_x)(g)=\phi_x\circ g$ for $\phi_x \in \Phi$ and $g \in G$ is an isomorphism between these principal bundles.

If two fibre bundles have isomorphic principal bundles they are associated [15 p43, 6]. In this case we will use the same symbols for their structures, groups and base spaces.

This implies in the above case, for instance, that we use the same symbol for $\pi(\phi_x)$ and ϕ_x and write any principal bundle as $\Phi(X,G,G,\Phi)$.

§1.6 If $V\subset X$ is any subset we write $B_V=p^{-1}(V)$ and $\Phi_V=\bigcup_{x\in V}\Phi_x$ and it is easy to see that $B_V(V,Y,G,\Phi_V)$ is a fibre bundle called the restriction of $B(X,Y,G,\Phi)$ to the base space V.

If $Y' \subset Y$ is invariant under G, that is $G \cdot Y' = Y'$, then we may define a fibre bundle $B'(X,Y',G,\Phi)$ called a <u>subbundle</u> of B [15 p24].

We shall sometimes use an abuse of language and say that $B' \subset B$ is a subbundle when it is really only the subbundle of a reduction of B.

<u>§1.7</u> With the notation of §1.4 we see that $(B \times B')(X \times X', Y \times Y', G \times G', \Phi \times \Phi')$ is another fibre bundle called the <u>product bundle</u> of the fibre bundles $B(X,Y,G,\Phi)$ and $B'(X',Y',G',\Phi')$.

When I use $B \times B'$ as an abbreviation for this product bundle I shall be careful to say so in order to avoid confusion with the trivial bundle.

It is very useful to consider an original if simple concept here. If $B(X,Y,G,\Phi)$ and $B'(X,Y',G,\Phi)$ are associated fibre bundles we define $\bigsqcup_{x \in X} B_x \times B'_x = B \hat{\times} B'$ as the bundle space of the fibre bundle $(B \hat{\times} B')(X, Y \times Y', G, \Phi)$ which is isomorphic to $(B \times B')_{\overline{X}}(\overline{X}, Y \times Y', \overline{G}; \overline{\Phi})$ where $\overline{X} \subset X \times X$, $\overline{G} \subset G \times G$ and $\overline{\Phi} \subset \Phi \times \Phi$ are the diagonals and $\delta : \Phi \leftrightarrow \overline{\Phi} \subset \Phi \times \Phi$ is an isomorphism [§0.2, §1.4]. This fibre bundle is called the <u>inner product</u> of the two associated fibre bundles. It is also associated with them.

§1.8 The bundle structure theorem is the theorem which justifies the introduction of fibre bundles into differential geometry. I wish to use the proof of this theorem to define certain maps that I am interested in and I will therefore sketch it. A more rigorous and detailed proof may be found elsewhere [15 p14, 5].

THEOREM Let X and Y be topological spaces and G a topological transformation group of Y. Suppose $\{U\}$ is an open covering of X and for any ordered pair U_1, U_2 of this covering there is defined $g_{12}: U_1 \cap U_2 \rightarrow G$ [§0.3].

Then if for U_1, U_2 and $U_3 \in \{U\}$ $g_{12}(x) \cdot g_{23}(x) = g_{13}(x)$ for all $x \in U_1 \cap U_2 \cap U_3$ a fibre bundle $B(X,Y,G,\Phi)$ is uniquely defined.

PROOF Put $B^* = \sum (U \times Y)$ summation being taken over the sets U of the covering. Let $\mathcal{R}$ be the equivalence relation

$$(x \in U_1, y_1) \sim (x \in U_2, g_{21}(x) \cdot y_1).$$

Then the bundle space $B = B^*/\mathcal{R}$.

Let $q: B^* \Rightarrow B$ be the natural projection and $\phi^*: U \times Y \hookrightarrow B^*$ the inclusion map, so that we have a set of maps $\{\phi^*: U \times Y \hookrightarrow B^*\}$ corresponding to the covering $\{U\}$.

The bundle projection $p: B \Rightarrow X$ is then defined by $p \circ q(x \in U, y) = x$ and the maps

$$\{\phi = q \circ \phi^*: U \times Y \Leftrightarrow p^{-1}(U)\} \qquad a)$$

are strip maps which define the structure Φ.

§1.9 If H G is a subgroup the set of maps $\phi_x \cdot H$ where $\phi_x \in \Phi$ can be considered as a point in a space which we call Φ/H. Then $\Phi/H(X,G/H,G,\Phi)$ is a fibre bundle where G/H is the space of right cosets [6].

There is a very important theorem about this fibre bundle, a proof of which will be sketched here as some details of the proof will be referred to later. The theorem is well known [15 p44].

THEOREM Let $H \subset G$ be a subgroup and let $W \subset G/H$ be an open set with $d: W \hookrightarrow G$ such that $\tilde{q} \circ d = \iota$ where $\tilde{q}: G \Rightarrow G/H$ is the natural projection. Then the substructures of Φ with group H are in one-one correspondence with the cross sections of $\Phi/H(X,G/H,G,\Phi)$.

PROOF Let $q: \Phi \Rightarrow \Phi/H$ be the natural projection. Suppose $\Psi \subset \Phi$ is a substructure with group H. Then if we define $f(x) = q(\Psi_x)$, $f: X \hookrightarrow \Phi/H$ is a cross section.

On the other hand suppose $f: X \hookrightarrow \Phi/H$ is a cross section. We put $\Psi = q^{-1}\{f(X)\}$.

Let $\phi: U \times G \Leftrightarrow \Phi_U$ be a strip map of Φ then we may also write $\phi: U \times (G/H) \Leftrightarrow \Phi_U/H$ since Φ and Φ/H are associated [§1.5]. Then the map

$$\tilde{\mu} = \phi \circ (\iota, d) \circ \phi^{-1} \circ f : U' \hookrightarrow \Phi \qquad \text{a)}$$

where $U' \subset U$ is open is a local cross section of Φ and the set of such cross sections clearly covers X so that we have only to show that ψ has values in $\mathbb{M}$ [§1.3].

Now it is easy to see that $q \circ \phi = \phi \circ (\epsilon, \tilde{q})$ from which

$$q \circ \tilde{\psi} = \phi \circ (\epsilon, \tilde{q} \circ d) \circ \phi^{-1} \circ f$$

$$= \phi \circ (\epsilon, \epsilon) \circ \phi^{-1} \circ f = f.$$

So that $q \circ \tilde{\psi} = f$ and $\tilde{\psi}$ has values in $\mathbb{M}$ which completes the proof.

§2 Manifolds

I have made as few restrictions as possible in my definition of a manifold not requiring, for instance, a manifold to be connected or separable, although these extra hypothesis must of course be made for certain theorems.

"O-dimensional" manifolds are also allowed which are just discrete spaces. This is not done solely for the sake of completeness for it is often useful, for instance, to consider a discrete group as a Lie group.

§2.1 DEFINITION Let M be a topological space. Let $\mathfrak{F}_r$ be a set of maps such that to each map $f \in \mathfrak{F}_r$ there corresponds an open set $U \subset M$ and an open set $E \subset R^m$, m being a fixed integer $\geqslant 0$, such that $f: E \leftrightarrow U$.

If a) the set $\{U\}$ associated with $\mathfrak{G}_r$ covers M;

b) $f_1^{-1} \circ f_2$ is non-singular and of class r for any pair of maps $f_1, f_2 \in \mathfrak{G}_r$;

then $\mathfrak{G}_r$ is called an <u>atlas</u> of the <u>manifold</u> $M(R^m, \bar{\mathfrak{G}}_r)$ where $\bar{\mathfrak{G}}_r$ is the maximum atlas $\mathfrak{G}_r \subset \bar{\mathfrak{G}}_r$ [15 p21, 17 p68, 7].

Any atlas defines a unique manifold.

m is called the <u>dimension</u> of the manifold and r its class. If $r \geqslant 1$ we say the manifold is differentiable.

The notation $M(R^m, \bar{\mathfrak{G}}_r)$ is apt to be cumbersome and therefore it is usual to speak of the manifold M without specifying the atlas implied.

A number of properties of a manifold can be extended to a complex manifold $M(C^m, \mathfrak{A})$ which is defined in a similar manner but with the requirement that the maps in b) be analytic. However it will be assumed that the manifolds considered are real.

<u>§2.2</u> The product of two manifolds $M(R^m, \mathfrak{G}_r)$ and $N(R^n, \mathfrak{H}_s)$ is the manifold of dimension m+n and of class min(r,s) defined by the space $M \times N$ and the atlas $\mathfrak{G}_r \times \mathfrak{H}_s$ [17 p75].

If $\pi: M \to N$ we say that π is of class t if

$g^{-1}\circ\pi\circ f$ is of class $\geq t$ for all $f \in \mathcal{F}_r$ and $g \in \mathcal{G}_s$. We say that it is regular at $x \in M$ if π is differentiable, that is $t \geq 1$, and if $g^{-1}\circ\pi\circ f$ has a Jacobian matrix of maximum rank at $f^{-1}(x)$ [17 p80]. π is <u>regular</u> if it is regular at every point of M. This definition of regularity is consistent because of condition b) in the definition of a manifold.

If $f: E \Leftrightarrow U$ where $E \subset R^m$ and $U \subset M$ are open and if f is regular and of class r then $f \in \mathcal{F}_r$.

If $N \subset M$, N having the induced topology we say that N is a submanifold of class t if the inclusion map $i: N \leftrightarrow M$ is regular and of class t [17 p85]. In particular if N is open it is a submanifold $N(R^m, \mathcal{F}'_r)$ where $\mathcal{F}'_r \subset \mathcal{F}_r$.

<u>§2.3</u> <u>DEFINITION</u> Let $M(R^m, \mathcal{F})$ be a differentiable manifold. Suppose $f_1: E_1 \Leftrightarrow U_1$ and $f_2: E_2 \Leftrightarrow U_2$ are two maps of $\mathcal{F}$. Let $J_{12}(x)$ be the Jacobian matrix of $f_1^{-1}\circ f_2$ associated with $x \in U_1 \cap U_2$ so that $J_{12}: U_1 \cap U_2 \rightarrow L(m)$, the real linear group on m variables. Now if J_{23} and J_{13} are defined similarly $J_{12}(x)\cdot J_{23}(x) = J_{13}(x)$ from the theory of Jacobian matrices. Thus from the bundle structure theorem any atlas of the manifold defines a unique fibre bundle $T(M)(M, R^m, L(m), \Phi)$ since L(m) is a

transformation group of R^m [§1.8]. This fibre bundle is called the <u>tangent bundle</u> of M [15 p22, 7].

We shall use a symbol similar to T(M) for all tangent bundles, T being an operator associating with a differentiable manifold its tangent bundle.

We will write $T_V(M)$ for $T(M)_V$ for convenience. $T_x(M)$ is the tangent plane at x and $\in T_x(M)$ is called a <u>tangent</u> at x. $T_x(M)$ is a vector space and a map $\pi : T(M) \quad T(N)$ will be called <u>linear</u> if it is fibre-preserving and $\pi \; T_x(M)$ is linear for every $x \in M$ [15 p24].

<u>§2.4</u> We note that L(m) leaves $0 \in R^m$ invariant and thus there is in every tangent bundle a subbundle $o(M)(M,0,L(m),\Phi)$. The fibre being a point, o(M) is homeomorphic to M under the bundle projection. This subbundle is called the bundle of zero tangents and, following the usual practice, we identify it with M.

<u>§2.5</u> From the construction theorem to every $f \in \mathcal{F}$, $f : E \Leftrightarrow U$, there corresponds a strip map of the tangent bundle which we call

$$\partial f : U \times R^m \Leftrightarrow T_U(M) \quad [§1.8\ a)]$$

The set of such strip maps we call $\partial \mathcal{F}$.

If E is an open set in R^m it is a manifold, an atlas being just the inclusion map $i:E \hookrightarrow R^m$. Hence $\partial i: E \times R^m \Longleftrightarrow T(E)$ and the tangent bundle has a reduction which is trivial. In general we ignore i and identify $E \times R^m$ and $T(E)$ to avoid cumbersome notation. Later, however, a situation will arise where care has to be taken to avoid confusion due to this simplification.

If $U \subset M$ is open it is a submanifold $U(R^m, \mathcal{A}')$ where $\mathcal{A}' \subset \mathcal{A}$ [§2.2]. Now $\partial \mathcal{A}'$ is a set of strip maps of $T_U(M)$ covering U so that in a very natural way we identify $T_U(M)$ and $T(U)$.

$T(M)$ is itself a manifold of dimensions $2m$. The atlas is defined by the set of maps

$$d\mathcal{A} = \{df = \partial f \circ (f, \epsilon): E \times R^m \Longleftrightarrow T(U), \text{ where } f \in \mathcal{A}\}.$$

Finally I would like to make a few remarks on manifolds of dimension 0. In the definition R^0 is taken to consist of one point only and $L(0)$ is the identity. Such a manifold is clearly discrete and analytic.

A necessary and sufficient condition for a manifold M to be 0-dimensional is that, using the convention in §2.4, $T(M)=M$.

§3 Integrable Structures

An integrable structure is quite a useful idea and coordinates some well known concepts. I commenced my research by trying to develop some necessary or sufficient conditions for a given substructure of a tangent bundle to be integrable, but this seems very difficult and at present I can only make a few conjectures, the main work of this thesis being concerned instead with connections and fibre bundles over bundle spaces.

§3.1 DEFINITION Let $M(R^m, \mathfrak{G})$ be a differentiable manifold and $T(M)(M,R^m,L(m),\bar{\Phi})$ its tangent bundle. Consider $V \subset M$. A substructure $\bar{\Pi}_V \subset \bar{\Phi}_V$ is said to be integrable if there exists a set of maps $\mathfrak{G}^* \subset \mathfrak{G}$ such that the restrictions of $\partial\mathfrak{G}^*$ are a set of strip maps of $\bar{\Pi}_V$ covering V [21]. The reduction of $T_V(M)$ is called an integrable reduction.

A number of workers have used the idea of a substructure to define important properties of a manifold and a few examples will be given here to show what the condition of integrability means in each case. These examples will be referred to later.

I consider a differentiable manifold $M(R^m, \mathcal{A})$ and its tangent bundle $T(M)(M,R^m,L(m),\mathfrak{B})$. $\mathfrak{M} \subset \mathfrak{B}$ is a sub-structure with group $G \subset L(m)$. Of course $\mathfrak{M}_V$ may be integrable without $\mathfrak{M}$ being integrable ($\mathfrak{M}_M$ being the same as $\mathfrak{M}$).

<u>§3.2</u> <u>SCHOLIUM</u> If $G=SL(m)$, the special linear group, $\mathfrak{M}$ is said to <u>orientate</u> the manifold. In this case $\mathfrak{M}$ is always integrable.

<u>PROOF</u> Consider $f \in \mathcal{A}$, $f:E \Leftrightarrow U$ where U is connected. Let $q:M \times L(m) \Rightarrow L(m)$ be the natural projection and $\alpha : R^m \Leftrightarrow R^m$ the reflection in the origin. Then $f \circ \alpha \in \mathcal{A}$ and $\partial(f \circ \alpha) \circ \partial f^{-1}(x,g) = -g$ for $x \in U$ and $g \in L(m)$ [§2.2].

Now since U and $SL(m)$ are connected so is $\mathfrak{M}_U$ and hence, since $L(m)$ has only two components, $q \circ \partial f^{-1}(\mathfrak{M}_U)$ is either contained in $SL(m)$ when ∂f is a strip map of $\mathfrak{M}$ or not in which case $\partial(f \circ \alpha)$ is a strip map of $\mathfrak{M}$. Hence $\mathfrak{M}$ is integrable.

<u>§3.3</u> <u>SCHOLIUM</u> If G is the complex linear group $CL(k) \subset L(2k)$ where $m=2k$, then $\mathfrak{M}$ is said to be a nearly complex structure. If it is integrable M admits a complex atlas $\mathfrak{A}$ such that $\mathfrak{A} \circ \theta^{-1} \subset \mathcal{A}$ for some $\theta : C^k \Leftrightarrow R^m$. [15 p209, 10, 11].

PROOF Let $\mathcal{A}^*$ be the atlas of M such that $\partial\mathcal{A}^*$ is a set of strip maps of $\mathbb{I}$. Without loss of generality we suppose that CL(k) is represented by the partitioned matrices of the form $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$. In this case we put $\theta(a+ib)= \begin{bmatrix} a \\ b \end{bmatrix} \in R^{2k}$.

Then $\mathcal{A}^*\circ\theta$ is an atlas defining $\mathfrak{A}$ for the condition $\partial f_{1.x}^{-1}\circ \partial f_{2.x} \in CL(k)$, where $f_1, f_2 \in \mathcal{A}^*$, is just the Cauchy-Riemann conditions for $\theta^{-1}\circ f_1^{-1}\circ f_2\circ\theta$ to be complex analytic. Since $\mathcal{A}^*\circ\theta\subset\mathfrak{A}$ then $\mathfrak{A}\circ\theta^{-1}\subset\mathcal{A}$ since $\mathcal{A}$ is maximal.

§3.4 SCHOLIUM If G is the group which leaves a k-dimensional linear subspace of R^m invariant then T(M) has a subbundle B(M,Y',G,$\mathbb{I}$) called a field of k-planes. If $\mathbb{I}$ is integrable then the manifold has a laminated structure uniquely defined by $\mathbb{I}$ [16 p99].

PROOF Without loss of generality we may suppose $G\subset L(m)$ to be the set of partitioned matrices of the form $\begin{bmatrix} A & C \\ \cdot & B \end{bmatrix}$ where $A\in L(k)$ and $B\in L(m-k)$.

Let $\mathcal{A}^*\subset\mathcal{A}$ be an atlas such that $\partial\mathcal{A}^*$ is a set of strip maps of $\mathbb{I}$. If $f_1, f_2 \in \mathcal{A}^*$ then the condition $\partial f_{1.x}^{-1}\circ \partial f_{2.x}\in G$ implies, if $f_1^{-1}\circ f_2(a_2,b_2)=(a_1,b_1)\in R^k\times R^{m-k}$, that b_1 is independent of a_2. This property defines the laminated structure.

The field of k-planes is in fact integrable in the usual sinse of the word [27]. Later we will discuss the Poisson bracket operation and show how the usual definition of integrable k-planes may be interpreted.

It is possible to show that if M has a pair of fields complementary to each other and which are simultaneously integrable then M is an open set of the product of two other manifolds. This problem is equivalent to the integrability of a substructure with group $L(k)\times L(m-k)$. It is probably true that if the complementary fields are each integrable then they are simultaneously integrable.

<u>§3.5</u> The last example I wish to give of an integrable structure concerns a Riemannian metric. Since the manifolds we discuss are not necessarily separable the quadratic form does not always generate a true metric but it is a useful concept. Only positive definite metrics will be considered, although analogous results hold without this restriction. We use the fact that a positive definite quadratic form over a manifold is equivalent to a substructure of the tangent bundle with group the orthogonal group [7].

SCHOLIUM If $G=O(m)$ then the Riemannian metric associated with $\mathfrak{M}$ is locally Euclidean if and only if $\mathfrak{M}$ is integrable [8 p56].

PROOF Suppose $f \in \mathcal{A}$, $f:E \Longleftrightarrow U$. Let $\psi : U' \times R^m \Longleftrightarrow T(U')$, where $U' \subset U$ is open, be a strip map of $\mathfrak{M}$. Write $A(x,f)=\psi_x^{-1} \cdot \partial f_x \in L(m)$. Then $G(x,f)=A^t(x,f)\cdot A(x,f)$, where A^t is the transpose of the matrix A, is independent of ψ and is the symmetric matrix of the Riemannian metric at x with respect to the coordinate system f.

Clearly if the Riemannian metric is locally flat, that is, if there exists an atlas $\mathcal{A}^*$ such that $G(x,f)$ if the identity for all $f \in \mathcal{A}^*$ then $\mathfrak{M}$ is integrable.

Conversely if $\mathfrak{M}$ is integrable, $\partial\mathcal{A}^*$ being a set of strip maps of $\mathfrak{M}$ with $\mathcal{A}^* \subset \mathcal{A}$, then $G(x,f)$ is the identity for any $f \in \mathcal{A}^*$.

§3.6 To finish this chapter we will prove a theorem that is straightforward but essential later see 17 p82 .

THEOREM Let $M(R^m,\mathcal{A})$ and $N(R^n,\mathfrak{B})$ be two differentiable manifolds. Then the product bundle of their tangent bundles

$$(T(M)\times T(N))(M\times N, R^m \times R^n, L(m)\times L(n), \mathfrak{N}\times\mathfrak{M})$$

is isomorphic to an integrable reduction of the tangent bundle

$T(M\times N)(M\times N, R^{m+n}, L(m+n), \Theta^*)$.

PROOF An atlas of $M\times N$ is $\mathfrak{F}\times\mathfrak{K}$ therefore $\partial(\mathfrak{F}\times\mathfrak{K})$ is a set of strip maps of Θ^*. In fact $\partial(\mathfrak{F}\times\mathfrak{K})$ is also a set of strip maps of a substructure $\Theta\subset\Theta^*$ with group $L(m)\times L(n)$. Because if $f_1, f_2\in\mathfrak{F}$ and $k_1, k_2\in\mathfrak{K}$ then $(f_1,k_1)^{-1}\circ(f_2,k_2)=(f_1^{-1}\circ f_2,\ k_1^{-1}\circ k_2)$ and the Jacobian matrix

$$\partial(f_1,k_1)^{-1}_{(x,y)}\ \partial(f_2,k_2)_{(x,y)}$$

is the partitioned matrix

$$\begin{bmatrix} \partial f^{-1}_{1.x}\ \partial f_{2.x} & \cdot \\ \cdot & \partial k^{-1}_{1.y}\ \partial k_{2.y} \end{bmatrix} \in L(m+n)$$

Now since this group $L(m)\times L(n)$ acts on $R^{m+n}=R^m\times R^n$ in the same way as the group of the product bundle of the tangent bundles, it is only necessary to show that the product of the principal bundles $\mathfrak{M}\times\mathfrak{N}$ is isomorphic to Θ. This is clearly true, $\partial(f,k)\ (\partial f^{-1},\partial k^{-1})$ being a restriction of the isomorphism for all $f\in\mathfrak{F}$ and $k\in\mathfrak{K}$.

In future we will not hesitate to call $T(M)\times T(Y)$ the tangent bundle of $M\times N$, although this is really an abuse of language.

When considering $T(T(R^m))$ however we must be particularly careful as this may be considered as $T(R^m \times R^m)$ since $T(R^m)$ is trivial and thus again as the <u>trivial</u> bundle $(R^m \times R^m) \times R^{2m}$ or alternatively by this theorem as the <u>product</u> bundle $T(R^m) \times T(R^m)$ or $(R^m \times R^m) \times (R^m \times R^m)$. This distinction occurs repeatedly when every care is taken to avoid confusion.

CHAPTER 2 DIFFERENTIALS AND OTHER MAPS

§4 Differentials

The idea of defining a differential as a map is due to Chevalley [17 p78]. However the definitions of tangents and manifolds that he used were quite different from those given in the first chapter of this thesis and thus it is necessary to define and discuss this differential map here.

§4.1 DEFINITION This definition is in three parts.

a) Let $E \subset R^m$ and $F \subset R^n$ be open sets. If $\pi : E \longrightarrow F$ is a differentiable map then $d\pi : T(E) \longrightarrow T(F)$ is defined by

$$d\pi(a,y)=(\pi(a), \frac{\partial\pi}{\partial a}\cdot y)$$

where $\frac{\partial\pi}{\partial a}$ is the Jacobian matrix of π at a, so that $y \in R^m$ and $\frac{\partial\pi}{\partial a}\cdot y \in R^n$.

b) Let $M(R^m, \mathcal{F})$ be a differentiable manifold and consider $f \in \mathcal{F}$, $f:E \Longleftrightarrow U$. We define

$$df=\partial f\circ(f,\epsilon):E\times R^m=T(E)\Longrightarrow T(U).$$ [§2.5].

c) Finally let $N(R^n,\mathfrak{F})$ and $L(R^l,\mathfrak{A})$ be two other differentiable manifolds and let $\pi_1:M\to N$ and $\pi_2:L\Rightarrow M$ be differentiable. Then we define

$$d(\pi_1\circ\pi_2)=d\pi_1\circ d\pi_2 \ .$$

The map $d\pi$ thus associated with any differentiable map π is called the differential of π.

§4.2 The consistency of this definition is perhaps not immediately obvious and we must show that the condition in c) holds for the differentials in a) and b).

If L, M and N are open sets in R^l, R^m and R^n respectively this condition is the immediate interpretation of the theorem on the Jacobian matrices of functions of functions.

In the second case for $f_1,f_2\in\mathfrak{F}$ we must show that $(df_1)^{-1}\circ df_2=d(f_1^{-1}\circ f_2)$. Now

$$\begin{aligned}(df_1)^{-1}\circ df_2(a,y)&=(f_1^{-1},\epsilon)\circ\partial f_1^{-1}\circ\partial f_2(f_2(a),y)\\&=(f_1^{-1},\epsilon)(f_2(a),g_{12}(a)\cdot y)\\&=(f_1^{-1}\circ f_2(a),g_{12}(a)\cdot y)\end{aligned}$$

where $g_{12}(a)$ is the Jacobian matrix of $f_1^{-1}\circ f_2$ at a [§2.5]. This last term is by definition $d(f_1^{-1}\circ f_2)(a,y)$ which proves that $(df_1)^{-1}\circ df_2=d(f_1^{-1}\circ f_2)$.

Now by definition $(dg)^{-1}\circ d(g\circ\pi_1\circ f^{-1})\circ df$ for $g\in\mathfrak{G}$ and $f\in\mathfrak{F}$ is a restriction of $d\pi_1$ and it is easy to see from what I have just shown that such restrictions define $d\pi_1$ uniquely.

§4.3 A large number of very useful properties of the differential as defined in a) can be extended to the general case because the maps $d\mathscr{C}$ as defined in b) are bundle isomorphisms. We shall not prove all these results for they are quite straightforward, but will just state those which will be needed later.

i) The projection of $d\pi$ is π. That is $d\pi|M=\pi$ [§1.4, §2.4].

ii) $d\epsilon=\epsilon$ and $d\delta=\delta$ [§0.2].

iii) This last result implies that if π^{-1} is also differentiable then $(d\pi)^{-1}=d(\pi^{-1})$ because $d(\pi^{-1})\circ d\pi= d(\pi^{-1}\circ\pi)=d\epsilon=\epsilon$. We may therefore write $d\pi^{-1}$ for both $d(\pi^{-1})$ and $(d\pi)^{-1}$.

iv) If $\pi:M\times N\Longrightarrow M$ is the natural projection so is $d\pi:T(M)\times T(N)\Longrightarrow T(M)$ [§3.6].

v) If $\pi:M\longrightarrow N$ is a constant map so is $d\pi$. This has in fact a converse. If $d\pi:T(M)\longrightarrow N$ then π is constant on every connected component of M [17 p80]

vi) If (π_1,π_2): $M_1\times M_2\rightarrow N_1\times N_2$ then $d(\pi_1,\pi_2):T(M_1\times M_2)\rightarrow T(N_1\times N_2)$ corresponds to

$(d\pi_1, d\pi_2): T(M_1)\times T(M_2) \to T(N_1)\times T(N_2)$ [§3.6] .

vii) If $\pi: M\times N \to L$ and for $x\in M$ $\pi_x: N \to L$ is defined by $\pi_x(y)=\pi(x,y)$ then $d(\pi_x)=(d\pi)_x$ where $(d\pi)_x(\eta)=d\pi(x,\eta)$ so that we may write $d\pi_x$ for both $d(\pi_x)$ and $(d\pi)_x$.

viii) Any differentiable map is linear [§2.3]. In the case when $\pi: M\times N \to L$ this implies that $d\pi(\xi,\eta)=d\pi(x,\eta)+d\pi(\xi,y)$ where $\xi\in T_x(M)$ and $\eta\in T_y(N)$.

<u>§4.4</u> If the map π is regular it has certain properties that will be very important [§2.2]. These properties are again extensions of simple properties of the special case a) of the definition and so I will not give detailed proofs.

It will be seen from this that Chevalley's definition of regularity is slightly more restricted but otherwise equivalent to that given here [17 p80].

Consider a differentiable map $\pi: M \to N$ where M and N are differentiable manifolds of dimensions m and n respectively. Let $d\pi | T_x(M)=\sqcap : T_x(M) \to T_{\pi(x)}(N)$ so that $\sqcap$ is a linear map. The properties of such linear maps enables us to assert the following:-

a) $\sqcap$ is onto if and only if $m \geqslant n$ and π is regular at x;

b) Π is one-one if and only if $m \leqslant n$ and π is regular at x.

Further, the classical theorems on the independence and reversibility of functions of several variables can be interpreted in the following way:-

α) When the conditions a) are satisfied there exists an open set $U \subset M$ with $x \in U$ such that $\pi|U:U \Rightarrow \pi(U)$ is an open map [17 p80];

β) When b) is true then there exists an open set $U \subset M$ with $x \in U$ such that $\pi|U:U \Leftrightarrow \pi(U)$ and has a regular inverse [17 p79].

We will require these results most often in the following form.

<u>THEOREM</u> <u>If $M(R^m, \mathcal{F})$ and $N(R^n, \mathcal{G})$ are differentiable manifolds and $\pi:M \rightarrow N$ is regular then</u>:-

a) $\pi:M \Rightarrow N$ implies $m \geqslant n$, π is open and $d\pi:T(M) \Rightarrow T(N)$;

b) $\pi:M \hookrightarrow N$ implies $m \leqslant n$ and $d\pi:T(M) \hookrightarrow T(N)$.

c) $\pi:M \Leftrightarrow N$ implies $m=n$, π^{-1} is regular and $d\pi:T(M) \Leftrightarrow T(N)$.

§5 The twist and solder

The solder has been defined by Ehresmann but the twist is my own conception [11 p42]. Both these maps prove very useful because they charactize properties of differentials without having to refer to each individual coordinate system. Just how far these maps go towards characterizing the differential of a differential I do not know, but it is clear that if a substructure is integrable it will have some special relation to these maps and they might be used to characterize integrability conditions.

The twist has some relation to the "exterior differentiation of forms" which is used to such good purpose by Cartan [3, 17].

Both these maps could be generalised, indeed the solder is already a special case of the solder as defined by Ehresmann. We do not require such generalisations here, however, and so will not discuss them.

§5.1 Consider differentiable and linear map $\Pi: T(E) \longrightarrow T(F)$ where $E \subset R^m$ and $F \subset R^n$ are open. We can write $\Pi(a,b) = (\pi(a), \tilde{\Pi}(a)\cdot b)$ where $\tilde{\Pi}(a)$ is some $(m \times n)$ matrix.

Let $T(T(E))$ be written as the trivial bundle

$T(E)\times R^{2m}$ or $(E\times R^m)\times(R^m\times R^m)$ writing $T(T(F))$ similarly. Then

$$d\Pi((a,b),(c,d))=((\pi(a),\tilde{\Pi}(a)\cdot b),(\frac{\partial\pi}{\partial a}\cdot c,\frac{\partial\tilde{\Pi}}{\partial a}\cdot b\cdot c+\tilde{\Pi}(a)\cdot d))$$

where $\frac{\partial\pi}{\partial a}$ is the Jacobian matrix of π at a and $\frac{\partial\tilde{\Pi}}{\partial a}\cdot b\cdot c$ is linear in both b and c.

If Π were a differential so that $\tilde{\Pi}(a)=\frac{\partial\pi}{\partial a}$ then $\frac{\partial\tilde{\Pi}}{\partial a}\cdot b\cdot c$ is also symmetric in b and c. We can express these remarks in the following way:-

We denote $((R^m\times O)\times(O\times R^m))$ by $T^{\#}(R^m)$ and define

$$\tau : T(T(R^m))\Longleftrightarrow T(T(R^m))$$

by $$\tau((a,b),(c,d))=((a,c),(b,d))$$

and $$\sigma : T^{\#}(R^m)\Longleftrightarrow T(R^m)$$

by $$\sigma((a,O),(O,d))=(a,d).$$

We note immediately that

i) $dd\pi\circ\tau=\tau\circ dd\pi$ ii) $\Pi\circ\sigma=\sigma\circ d\Pi$

The map τ may be used in characterizing the differential of a differential and σ in characterizing a linear map. τ is especially useful because the property it represents is one of the most important in the treatment of integrability conditions.

§5.2 These simply defined maps may be extended to manifolds.

<u>DEFINITION</u> Let $M(R^m,\mathcal{F})$ be a manifold of class ≥ 2. If $\tau : T(T(R^m)) \Longleftrightarrow T(T(R^m))$ is as defined in §5.1 then we define $\tau \circ ddf \circ \tau \circ ddf^{-1}$ to be a restriction of the <u>twist</u> map $\tau : T(T(M)) \Longleftrightarrow T(T(M))$ for all $f \in \mathcal{F}$.

The consistency of this definition follows directly from formula i) of §5.1 and we use the symbol τ for the twist map on any manifold.

<u>§5.3</u> As with differentials we can generalise some of the properties of τ in §5.1, the most important being:-

a) $\tau \circ \tau = \epsilon$;

b) $\tau \circ dd\pi = dd\pi \circ \tau$ for any map π of class ≥ 2;

c) the twist $\tau : T(T(M \times N)) \Longleftrightarrow T(T(M \times N))$

corresponds to $(\tau,\tau) : T(T(M)) \times T(T(N)) \Longleftrightarrow T(T(M)) \times T(T(N))$.

<u>DEFINITION</u> If $W \subset T(T(M))$ is invariant under τ, that is, if $\tau(W) \subset W$ we say that it is <u>symmetric</u>. Similarly if $\sqcap : T(T(M)) \rightarrow T(T(N))$ is such that $\tau \circ \sqcap = \sqcap \circ \tau$ we say that $\sqcap$ is <u>symmetric</u>.

It is interesting to note that if a differential is symmetric it must be the differential of a differential. This may seem surprising but it is quite simple to prove.

However I shall not need this result and will not prove it in detail.

§5.4 The solder is a similar generalisation of σ in §5.1.

DEFINITION Let $M(R^m, \mathcal{F})$ be a manifold of class ≥ 2. We define $T^{\#}(M) \subset T(T(M))$ by $T^{\#}(M) = \bigcup_{f \in \mathcal{F}} ddf\{T^{\#}(E)\}$.

Then if $\sigma : T^{\#}(R^m) \Leftrightarrow T(R^m)$ is defined in §5.1, $df \circ \sigma \circ ddf^{-1}$ is defined to be a restriction of the solder map $\tilde{\sigma} : T^{\#}(M) \Leftrightarrow T(M)$ for all $f \in \mathcal{F}$.

The consistency of this definition is a consequence of the formula ii) of §5.1. In future we will use for every solder map irrespective of the manifold to which it refers.

$T^{\#}(M)$ is pointwise invariant under τ. That is, every point of T (M) is a symmetric set. In fact an alternative definition of $T^{\#}(M)$ is the set of symmetric points in $T_M(T(M))$ [§8.2].

As before we can generalise a number of properties of σ and $T^{\#}(M)$ as defined in §5.1. The most important are:-

a) the solder is linear;

b) $T^{\#}(M\times N)=T^{\#}(M)\times T^{\#}(N)$ and the solder $\sigma: T^{\#}(M\times N)\Longleftrightarrow T(M\times N)$ corresponds to the solders $(\sigma,\sigma): T^{\#}(M)\times T^{\#}(N)\Longleftrightarrow T(M)\times T(N)$;

c) If $\Pi: T(M)\longrightarrow T(N)$ is linear and differentiable then $d\Pi : T^{\#}(M)\longrightarrow T^{\#}(N)$ and $\Pi\circ\sigma=\sigma\circ d\Pi$.

<u>§5.5</u> <u>DEFINITION</u> Let $\Pi: T(M)\longrightarrow T(N)$ be a differentiable map such that $d\Pi : T^{\#}(M)\longrightarrow T^{\#}(N)$. Then

$$\sigma\circ d\Pi\circ\sigma^{-1}: T(M)\longrightarrow T(N)$$

is called the <u>linear part</u> of Π.

From the property c) of the previous paragraph it follows immediately that if Π is linear it is identical with its linear part. Since σ is linear, so also is the linear part, so the name is quite justified.

It must be noted that this definition is slightly vague as no conditions have been given on Π for the condition on $d\Pi$ to hold. As a matter of fact it is sufficient that Π be fibre-preserving and maps $M\subset T(M)$ into $N\subset T(N)$. That this is so will not appear until later, but this will not matter as this definition will not be used just yet [§8.2]. In fact the definition has been put here for the sake of completeness and could have been left till later.

Let us note a very important property of this linear part. If $\sqcap_1$ and $\sqcap_2$ satisfy the hypothesis and $\sqcap_1 \circ \sqcap_2$ is defined then if the linear parts of $\sqcap_1$ and $\sqcap_2$ are $\sqcap_1^*$ and $\sqcap_2^*$ respectively then $\sqcap_1^* \circ \sqcap_2^*$ is the linear part of $\sqcap_1 \circ \sqcap_2$.

§6 Lie Groups

The theory of Lie groups has been discussed in great detail elsewhere [17]. However this approach is slightly different from the usual one and we need some theorems which have not been proved elsewhere so I will give a brief discussion of the subject.

§6.1 DEFINITION Let G be a group and let $G(R^r, \mathscr{O})$ be an analytic manifold. G is then a Lie group if for $g_1, g_2 \in G$ $\lambda(g_1, g_2) = g_1 \cdot g_2$ and $\mu(g_1) = g_1^{-1}$ define two regular and analytic maps $\lambda: G \times G \Longrightarrow G$ and $\mu: G \Longleftrightarrow G$.

In this case λ is open and so $\lambda g: G \Longleftrightarrow G$ is defined by $\lambda g(g') = \lambda(g, g')$ [§4.4]. The definition implies also if $f \in \mathscr{O}$ then $\lambda g \circ f \in \mathscr{O}$ for all $g \in G$ and thus $\{\lambda g \circ f$ where $g \in G\}$

is an atlas of G [§2.1].

§6.2 The theorem to be proved now is essential to this work and has often been assumed but as far as I know never stated or proved explicitly.

THEOREM If G is a Lie group so is T(G), the maps λ and μ being replaced by $d\lambda$ and $d\mu$.

PROOF The identity is $e \in G \subset T(G)$ for λg is regular and $\lambda_e = \epsilon$ so that if $\gamma \in T(G)$ then

$$e\cdot\gamma = d\lambda(e,\gamma) = d\lambda_e(\gamma) = d\epsilon(\gamma) = \gamma$$ [§4.3 vii) and ii)].

The unique inverse of γ is $d\mu(\gamma)$. For since $\mu: G \Longrightarrow G$ then $d\mu: T(G) \Longrightarrow T(G)$ [§4.4]. Also $\lambda\circ(\epsilon,\mu)\circ\delta$ is a constant map onto e and therefore so is $d\lambda\circ(d\epsilon,d\mu)\circ d\delta$ [§4.3 v)]. That is

$$d\lambda\circ(d\epsilon,d\mu)\circ d\delta(\gamma) = d\lambda\circ(d\epsilon,d\mu)(\gamma,\gamma)$$ [§4.3 ii)]

$$= d\lambda(\gamma, d\mu(\gamma))$$

$$= \gamma\cdot d\mu(\gamma) = e$$

The associative law in G may be written

$$\lambda\circ(\lambda,\epsilon) = (\lambda,\epsilon)\circ\lambda : G\times G\times G \Longrightarrow G$$

Hence

$$d\lambda\circ(d\lambda,d\epsilon) = (d\lambda,d\epsilon)\circ d\lambda : T(G)\times T(G)\times T(G) \Longrightarrow T(G)$$

which is the associative law in T(G).

This completes the proof of the theorem.

§6.3 Two important subgroups of $T(G)$ are G and $T_e(G)$. That G is a subgroup follows from the relations $d\lambda | G\times G=\lambda$ and $d\mu | G=\mu$ [§4.3 1)].

$T_e(G)$ is a subgroup because if $p:T(G)\Rightarrow G$ is the bundle projection then $p\circ d\lambda=\lambda\circ(p,p)$ and $p\circ d\mu=\mu\circ p$. So that if $\gamma_1\in T_{g_1}(G)$ and $\gamma_2\in T_{g_2}(G)$ then $\gamma_1\cdot\gamma_2\in T_{g_1\cdot g_2}(G)$ and $\gamma_1^{-1}\in T_{g_1^{-1}}(G)$. Thus $T_e(G)$ is closed with respect to multiplication and inversion and is in fact a normal subgroup.

$T_e(G)$ is also abelian because if $\gamma_1,\gamma_2\in T_e(G)$ then $d\lambda(\gamma_1,\gamma_2)=d\lambda(\gamma_1,e)+d\lambda(e,\gamma_2)=\gamma_1+\gamma_2$ [§4.3 viii)].

Suppose $H^*\subset T(G)$ is a subgroup then $p(H^*)=H$ is a subgroup of G so also is $T_H(G)$ and $T(H)$ if the latter exists. This follows immediately from the result we have just used to show that $T_e(G)$ is a normal subgroup.

§6.4 $T(T(G))$ is also a Lie group, multiplication being defined by $dd\lambda$. These double differentials have special properties which lead to the following theorem.

THEOREM <u>The twist $\tau:T(T(G))\Leftrightarrow T(T(G))$ is a group isomorphism. The set $T^{\#}(G)$ is a subgroup of $T(T(G))$ and $\sigma:T^{\#}(G)\Leftrightarrow T(G)$ is also a group isomorphism.</u>

PROOF That τ is a group isomorphism follows from the symmetry of $dd\lambda$ and $dd\mu$ [§5.3].

Also $dd\lambda: T^{\#}(G \times G) \Longrightarrow T^{\#}(G)$ and $dd\mu: T^{\#}(G) \Longrightarrow T^{\#}(G)$ and $T^{\#}(G \times G) = T^{\#}(G) \times T^{\#}(G)$ [§5.4]. So that $T^{\#}(G)$ is closed with respect to multiplication and inversion and is therefore a subgroup.

That σ is a group isomorphism follows from the relations $\sigma \circ dd\lambda = d\lambda \circ \sigma$ and $\sigma \circ dd\mu = d\mu \circ \sigma$.

<u>§6.5</u> I am now going to give a proof of a theorem that is fundamental in any application of the theory of Lie groups. It is well known, although the theorem has been written in a number of different ways [17 p103]. It is proposed to give a proof here because the methods and definitions differ from those normally used and also because we will refer to some of the details of the proof.

<u>THEOREM</u> <u>If G is a Lie group the manifold $G(R^r, \mathcal{F})$ is parallelisable, that is, the tangent bundle $T(G)(G, R^r, L(r), \Phi)$ has a substructure $\Psi \subset \Phi$ with group identity</u>.

<u>PROOF</u> Let λ and λ_g be defined as in §6.1. Consider $f \in \mathcal{F}, f: E \Longleftrightarrow U$, where $e \in U$. Then $\Psi = \{d\lambda_g \circ \partial f_e$ where $g \in G\}$ is the substructure required.

We note that $\{\lambda_g \circ f;\ g \in G\}$ is an atlas of G and hence

$\partial(\lambda_g \circ f)$ is a strip map of Φ for all $g \in G$. Now

$$d\lambda_g \circ \partial f \circ (f,\epsilon) = d\lambda_g \circ df$$
$$= d(\lambda_g \circ f)$$
$$= \partial(\lambda_g \circ f) \circ (\lambda_g \circ f, \epsilon)$$

so that $d\lambda_g \circ \partial f = \partial(\lambda_g \circ f) \circ (\lambda_g, \epsilon)$ and hence

$$d\lambda_g \circ \partial f_e = \partial(\lambda_g \circ f)_{\lambda_g(e)} = \partial(\lambda_g \circ f)_g$$

for all $g \in G$. Hence $\Phi \subset \Phi$.

We define

$$\mu = d\lambda \circ (\epsilon, \partial f_e) : G \times R^r \to T(G) \qquad a)$$

and note that $\mu_g = d\lambda_g \circ \partial f_e \in \Phi$ where $\mu_g(y) = \mu(g,y)$. This shows that μ_g is onto for each $g \in G$ and thus μ is onto. Also since $d\lambda$ is open and ∂f_e is a homeomorphism μ is open [§4.4]. Finally μ is one-one for if $\mu(g_1, \partial f_e(\gamma_1)) = \mu(g_2 \cdot \partial f_e(\gamma_2))$ where $\gamma_1, \gamma_2 \in T_e(G)$ then $g_1 \cdot \gamma_1 = g_2 \cdot \gamma_2$. But $g_1^{-1} \cdot g_2 \in G$ and $\gamma_1 \cdot \gamma_2^{-1} \in T_e(G)$ while $G \cap T_e(G) = e$ so that $g_1^{-1} \cdot g_2 = \gamma_1 \cdot \gamma_2^{-1} = e$ [§6.3].

Thus $\mu : G \times R^m \Longleftrightarrow T(G)$ is a strip map of Φ covering G so that Φ is in fact a substructure which clearly has group the identity.

<u>§6.6</u> <u>DEFINITION</u> We say that a Lie group G is a transformation group of the <u>manifold</u> $M(R^m, \mathfrak{F})$ if there is given a regular map $\nu : G \times M \Longrightarrow M$ of maximum class such that $\nu(e,x) = x$ and $\nu \circ (\epsilon, \nu) = \nu \circ (\lambda, \epsilon)$ [15 p7].

This definition implies that ν is an open map so that $\nu_g : M \Leftrightarrow M$ where $\nu_g(x)=\nu(g,x)$. It is a common abuse of language, which we will also use, to speak of the map g when ν_g is meant. In this case the map dg means $d\nu_g$ which may also be written as the map g.

<u>THEOREM</u> <u>If G is a transformation group of the differentiable manifold $M(R^m,\mathcal{A})$ then T(G) is a transformation group of the manifold T(M).</u>

<u>PROOF</u> The proof is quite trivial. In fact it is easy to see that $d\nu : T(G)\times T(M) \Rightarrow T(M)$ has the required properties.

<u>§6.7</u> Now the general linear group L(m) is a Lie group and is a transformation group of the manifold R^m.

<u>LEMMA</u> T(L(m)) is isomorphic to the normaliser of the partitioned matrix $\begin{bmatrix} 1 & \cdot \\ 1 & 1 \end{bmatrix}$ where 1 is the identity matrix in L(m). This group we denote by $D(m,m) \subset L(2m)$.

<u>PROOF</u> The space of L(m) is an open set in R^{m^2} so $T(L(m))=L(m)\times R^{m^2}$ [17 p16]. If we write an element of R^{m^2} in matricial form and denote this set of $(m\times m)$ matrices by $\mathfrak{M}$ then $T(L(m))=L(m)\times\mathfrak{M}$. Thus $\nu : L(m)\times R^m \Rightarrow R^m$ being defined by $\nu(g,y)=g\cdot y$ it is not difficult to see that

$$d\nu((g,\bar{g}),(y,\bar{y}))=(g\cdot y,g\cdot\bar{y}+\bar{g}\cdot y)$$

where $(g,\bar{g})\in L(m)\times\mathfrak{M}=T(L(m))$ so that $(g,\bar{g})$ may be represented by the partitioned matrix

$$\begin{bmatrix} g & \cdot \\ \bar{g} & g \end{bmatrix} \in D(m,m)$$

CHAPTER 3 TANGENTS TO BUNDLE SPACES

§7 The tangent bundle of a fibre bundle

In this section it is proved that the fibre bundle we discuss can be fibred in two different ways. This fundamental theorem gives rise to a discussion of the subbundles and maps connecting those two fibre bundles.

§7.1 We must first give the definition of a differentiable fibre bundle that we are going to use. Ehresmann has given one that is almost exactly the same [8].

DEFINITION Let B, M and Y be manifolds and let G be a transformation group of the manifold Y [§6.5]. The fibre bundle $B(M,Y,G,\mathfrak{S})$ is said to be a differentiable fibre bundle of class $r \geq 1$ if there exists a set of regular strip maps each of class r and covering M whose inverses are also of class r.

This implies of course that the manifolds B, M and Y are all of class $\geq$ r. It also implies that the bundle projection and every map of Φ is regular and of class r because the natural projections of M Y onto M and Y are regular and of class r.

It is easy to see that if $B(M,Y,G,\Phi)$ is a differentiable fibre bundle we may define the structure of a manifold on Φ so that the principal bundle Φ is also a differentiable fibre bundle and of the same class.

Also if M is of class $\geq$ r+1 then the tangent bundle T(M) is a differentiable fibre bundle of class r.

§7.2 If $\tilde{\phi}$ is a local cross section of Φ then it defines a strip map of Φ [§1.2 a)]. Suppose $\tilde{\phi}$ is regular and of class r then it can be seen from the construction that the strip map defined by it is regular and of class r as also is the inverse of the strip map.

An important theorem of Steenrod states that if M is separable and metric then any cross section of the differentiable fibre bundle can be "approximated to" by a cross section of class r unless $r=\omega$ [15 p25]. Thus in some sense any strip map of Φ can be approximated to by a strip map of class r.

§7.3 We are now in a position to prove one of the main theorems of this thesis.

THEOREM *If $B(M,Y,G,\Phi)$ is a differentiable fibre bundle then $T(B)$ is the bundle space not only of the tangent bundle to B but also of a well defined fibre bundle $T(B)(T(M),T(Y),T(G),T^*(\Phi))$. Further the space $T^*(\Phi)$ is homeomorphic to $T(\Phi)$.*

PROOF It is convenient to use the abbreviation $T^*(B)$ for the fibre bundle $T(B)(T(M),T(Y),T(G),T^*(\Phi))$ reserving $T(B)$ as an abbreviation for the tangent bundle. With this convention the space $T(B)$ may also be denoted by $T^*(B)$.

If $p:B \Rightarrow M$ is the bundle projection, it is regular and so $dp:T(B) \Rightarrow T(M)$ and dp is the bundle projection for $T^*(B)$ [§4.4].

If $\phi:U\times Y \Leftrightarrow B_U$ is a regular strip map of Φ then $d\phi:T(U)\times T(Y) \Leftrightarrow T(B_U)$ [§4.4]. We will see that we may define $d\phi$ to be a strip map of $T^*(\Phi)$. These strip maps will then define the fibre bundle $T^*(B)$.

First we note that $p\circ\phi:U\times Y \Rightarrow U$ is the natural projection, so therefore is $dp\circ d\phi:T(U)\times T(Y) \Rightarrow T(U)$ [§4.3 iv)]. Hence $dp^{-1}\{T(U)\}=d\phi\{T(U)\times T(Y)\}= T(B_U)$ so that

$$d\phi:T(U)\times T(Y) \Leftrightarrow T^*_{T(U)}(B)=dp^{-1}\{T(U)\}$$

and $$d\phi:\xi\times T(Y) \Leftrightarrow T^*_{\xi}(B)=dp^{-1}(\xi)$$

where $\xi \in T(U)$.

Now consider two regular strip maps

$\phi_1: U_1 \times Y \Leftrightarrow B_{U_1}$ and $\phi_2: U_2 \times Y \Leftrightarrow B_{U_2}$

defining $\phi_{1.x}$ and $\phi_{2.x}$ as usual [§1.1]. Let $g_{12}: U_1 \cap U_2 \to G$ be the map defined by $g_{12}(x) = \phi_{1.x}^{-1} \circ \phi_{2.x}$ so that g_{12} is differentiable and for $x \in U_1 \cap U_2$

$\phi_1^{-1} \circ \phi_2(x,y) = (x, g_{12}(x) \cdot y)$.

If $\nu: G \times Y \Rightarrow Y$ defines the transformation group G then

$$\phi_1^{-1} \circ \phi_2 = (\epsilon, \nu) \circ (\epsilon, g_{12}, \epsilon) \circ (\delta, \epsilon) \qquad [§6.6].$$

Thus $d\phi_1^{-1} \circ d\phi_2 = (d\epsilon, d\nu) \circ (d\epsilon, dg_{12}, d\epsilon) \circ (d\delta, d\epsilon)$. But $d\nu: T(G) \times T(Y) \Rightarrow T(Y)$ defines $T(G)$ as a transformation group of $T(Y)$ and $d\epsilon = \epsilon$, $d\delta = \delta$ [§4.3 ii)]. So that

$$d\phi_1^{-1} \circ d\phi_2(\xi, \eta) = (\xi, dg_{12}(\xi) \cdot \eta) \qquad a)$$

where $dg_{12}: T(U_1) \cap T(U_2) \to T(G)$ and hence $d\phi_1$ and $d\phi_2$ may be defined as strip maps of $T^*(\Phi)$.

Consider the principal bundle $\Phi(M, G, G, \bar{\Phi})$ and let $\bar{\phi}: U \times G \Leftrightarrow \bar{\Phi}_U$ be the strip map of $\bar{\Phi}$ associated with ϕ. Then $\bar{\Phi}$ is a differentiable fibre bundle and thus we may define $T(\bar{\Phi})(T(M), T(G), T(G), T^*(\bar{\Phi}))$, a strip map of $T^*(\bar{\Phi})$ being $d\bar{\phi}$.

On the other hand the principal bundle $T^*(\Phi)(T(M), T(G), T(G), \overline{T^*(\Phi)})$ has a strip map associated with $d\phi$ which we call $\overline{d\phi}$. These last two fibre bundles are clearly isomorphic the strip map $d\bar{\phi}$ corresponding to the strip map $\overline{d\phi}$. This follows because in the fibre bundle $T(\Phi)(T(M),T(G),T(G),T^*(\bar{\Phi}))$ $T(G)$ acts on $T(G)$ by left translations. Thus $T(\Phi)$ and $T^*(\Phi)$ are homeomorphic $\overline{d\phi}\circ d\bar{\phi}^{-1}$ being a restriction of the homeomorphism. This completes the proof of the theorem.

<u>§7.4</u> As an example of this theorem let $M(R^m, \mathcal{A})$ and $Y(R^n, \mathcal{B})$ be differentiable manifolds. It is easy to see that $T^*(M\times Y)$ is the <u>trivial</u> bundle $T(M)\times T(Y)$ whereas $T(M\times Y)$ is the <u>product</u> bundle $T(M)\times T(Y)$ [§3.6].

If ϕ is a regular strip map of Φ and $k\in\mathcal{A}\times\mathcal{B}$ then $\phi\circ k$ is a coordinate map of the manifold B. Therefore $\partial(\phi\circ k)$ is a strip map of the tangent bundle whereas $d\phi$ is a strip map of the fibre bundle $T^*(B)$. There is of course a relation between them. In fact

$$d\phi\circ\partial k\circ(k,\epsilon)=d\phi\circ dk \qquad [§4.1]$$
$$=d(\phi\circ k)$$
$$=\partial(\phi\circ k)\circ(\phi\circ k,\epsilon)$$

so that
$$d\phi\circ\partial k=\partial(\phi\circ k)\circ(\phi,\epsilon) \qquad a)$$

<u>§7.5</u> The last part of the theorem in §7.3 shows that we may continue the convention we established identifying

Φ and $\bar{\Phi}$ and also identify $T^*(\Phi)$, $T^*(\bar{\Phi})$ and $\overline{T^*(\Phi)}$. Thus we identify $\overline{d\phi_\xi}\circ\gamma$, $d\bar{\phi}(\xi,\gamma)$ and $d\phi_\xi\gamma$ for $\gamma\in T(G)$ and $\xi\in T(M)$.

This means that we identify $d\bar{\phi}(x,e)$ and $d\phi_x$. But $d\bar{\phi}|G\times Y=\bar{\phi}$ so that $d\bar{\phi}(x,e)=\bar{\phi}(x,e)=\phi_x\in\Phi\subset T(\Phi)$. Thus we do not distinguish between the maps $d\Phi\subset T^*(\Phi)$ and the zero tangents $\Phi\subset T(\Phi)$.

These identifications enable us to prove the following very useful lemma:-

<u>LEMMA</u> Let $\alpha:\Phi\times Y\Rightarrow B$ be defined by $\alpha(\phi_x,y)=\phi_x(y)$ then $d\alpha(\theta_\xi,\eta)=\theta_\xi(\eta)$ where $d\alpha:T^*(\Phi)\times T(Y)\Rightarrow T^*(B)$.

<u>PROOF</u> Let $\nu:G\times Y\Rightarrow Y$ be defined by $\nu(g,y)=g\cdot y$ then if ϕ is a regular strip map of Φ the map $\phi\circ(\epsilon,\nu)\circ(\phi^{-1},\epsilon)$ is a restriction of α. Thus $d\phi\circ(d\epsilon,d\nu)\circ(d\phi^{-1},d\epsilon)$ is a restriction of $d\alpha$.

Consider $\theta_\xi\in T^*_\xi(\Phi)$. Then we may write $\theta_\xi=d\phi_\xi\circ\gamma$ for some regular strip map ϕ and some $\gamma\in T(G)$. In this case

$$\begin{aligned}
d\alpha(\theta_\xi,\eta)&=d\alpha(d\phi_\xi\circ\gamma,\eta)\\
&=d\phi\circ(\epsilon,d\nu)\circ(d\phi^{-1},\epsilon)(d\phi_\xi\circ\gamma,\eta)\\
&=d\phi\circ(\epsilon,d\nu)(\xi,\gamma,\eta)\\
&=d\phi_\xi(\gamma\cdot\eta)\\
&=d\phi_\xi\circ\gamma(\eta)=\theta_\xi(\eta).
\end{aligned}$$

A very useful consequence of this lemma is, if $\bar{q}:T(\Phi)\Rightarrow\Phi$ and $q:T(B)\Rightarrow B$ and $\tilde{p}:T(Y)\Rightarrow Y$ are bundle projections, that $q\circ d\alpha=\alpha\circ(\bar{q},\tilde{p})$ [§4.3]. That is $q\{\theta_{\xi}(\eta)\}=\bar{q}(\theta_{\xi})\{\tilde{p}(\eta)\}$.

<u>§7.6</u> If $B'(M',Y',G',\Phi')$ is some other differentiable fibre bundle then the tangent bundle $T(B\times B')$ has a reduction to the product bundle $T(B)\times T(B')$ [§3.6]. In fact from a short scrutiny of the proof in §7.2 one sees that the fibre bundle $T^*(B\times B')(T(M\times M'),T(Y\times Y'),T(G\times G'),T^*(\Phi\times\Phi'))$ is reducible to the product bundle of $T^*(B)$ and $T^*(B')$. This is a simple consequence of a result on differentials [§4.3 vi)].

The same result also enables us to see that if B and B' are associated then $T^*(B\hat{\times}B')=T^*(B)\hat{\times}T^*(B')$.

<u>§7.7</u> Suppose $\pi:B\to B'$ is a differentiable and fibre-preserving map so that if $p:B\Rightarrow M$ and $p':B'\Rightarrow M'$ are the bundle projections $p'\circ\pi=\bar{\pi}\circ p$ where $\bar{\pi}$ is the projection of π [§1.3]. It is easy to see that $\bar{\pi}$ is differentiable so that $dp'\circ d\pi=d\bar{\pi}\circ dp$. Thus both $d\pi:T(B)\to T(B')$ and $d\pi:T^*(B)\to T^*(B')$ are fibre-preserving.

§7.8 The last of these short remarks is about subbundles. Suppose $B'(M,Y',G,\Phi)$ is a subbundle of B. Then $T^*(B')$ is a subbundle of $T^*(B)$ with fibre $T(Y')$. Before showing this it is necessary to prove that $T(Y')$ is invariant under $T(G)$. Now the fact that Y' is invariant under G may be written $\nu(G\times Y')\subset Y'$ and hence $d\nu\{T(G)\times T(Y')\}\subset T(Y')$ where ν is defined as in §7.5 . Hence $T^*(B)$ will have a subbundle with fibre $T(Y')$. That this is in fact $T^*(B')$ can be seen from the fact that if $\phi:U\times Y\Leftrightarrow B_U$ is a regular strip map of Φ then $\phi:U\times Y'\Leftrightarrow B'_U$ so that $d\phi:T(U)\times T(Y')\Leftrightarrow T^*_{T(U)}(B')$.

§7.9 The fibres of the fibre bundles are submanifolds and form a laminated structure of the manifold B [8, 16]. It is clear that the set of tangents to the fibre should be an integrable field of n-planes, Y being of dimension n. The next theorem describes this integrable field.

THEOREM $T^*_M(B)\subset T(B)$ is an integrable field of n-planes.

PROOF $T^*_M(B)=\bigcup_{\phi_x\in\Phi} d\phi_x\{T(Y)\}=\bigcup_{x\in M} T(B_x)$ since if $\phi_x\in\Phi_x$ $\phi_x:Y\Leftrightarrow B_x$ and hence $d\phi_x:T(Y)\Leftrightarrow T(B_x)$. That is $T^*_M(B)$ is just the set of the tangents to the fibres of B.

Let $\phi_1:U_1\times Y\Leftrightarrow B_{U_1}$ and $\phi_2:U_2\times Y\Leftrightarrow B_{U_2}$ be two regular strip maps of Φ and consider $x\in U_1\cap U_2$. Let

$k_1, k_2 \in \mathcal{O}\times\mathcal{O}$ be such that $k_1^{-1}(x,y_1)$ and $k_2^{-1}(x,y_2)$ are defined where $\phi_1(x,y_1)=\phi_2(x,y_2)=b$. Then

$$\begin{aligned}\partial(\phi_1\circ k_1)_b(O\times R^n) &= \partial(\phi_1\circ k_1)(\phi_1(x,y_1),(O\times R^n)) \\ &= d\phi_1\circ \partial k_1((x,y_1),(O\times R^n)) \qquad [\S 7.4\ a)] \\ &= d\phi_{1.x}\{T_{y_1}(Y)\} \\ &= d\phi_{2.x}\{T_{y_2}(Y)\} \\ &= \partial(\phi_2\circ k_2)_b(O\times R^n).\end{aligned}$$

Now the set of maps like $\phi_1\circ k_1$ is an atlas for B say $\mathcal{L}$. We have seen that the strip maps $\partial\mathcal{L}$ define an integrable substructure on T(B) leaving $(O\times R^n)$ invariant. Thus we have a subbundle which is an integrable field of n-planes. The subbundle is clearly $\bigcup_{x\in M} T(B_x)=T^*_M(B)$.

__§7.10__ Another subbundle of T(B) is the set of zero tangents $B\subset T(B)$ in fact it is a subbundle of the subbundle in the preceding paragraph. It will be proved that it is also a subbundle of $T^*_M(B)$.

Before doing this however we will define "semi-integrability" which extends the idea of an integrable structure a little further [§5.1].

__DEFINITION__ If $\Theta\subset T^*_W(\Phi)$ is a substructure where $W\subset T(M)$ it is said to be __semi-integrable__ if there exists a set of strip maps of Φ where restricted differentials are strip maps of Θ covering W.

The reduced fibre bundle with structure Θ is then said to be a semi-integrable reduction of $T^*_W(B)$. We are mainly interested in the case when W is the whole of T(M).

THEOREM $d\Phi \subset T^*_M(\Phi)$ is a semi-integrable substructure. The semi-integrable reduction of $T^*_M(B)$ with this substructure has a subbundle isomorphic to $B(M,Y,G,\Phi)$ [§7.5].

PROOF We use the notations of the previous paragraphs of this section. $g_{12}:U_1 \cap U_2 \to G$ is defined by $g_{12}(x)=\phi^{-1}_{1.x} \circ \phi_{2.x}$. Then $d\phi^{-1}_{1.\xi} \circ d\phi_{2.\xi}=dg_{12}(\xi)$ where $\xi \in T(U_1 \cap U_2)$ [§7.3 a)], so that $d\phi^{-1}_{1.x} \circ d\phi_{2.x}=dg_{12}(x)=g_{12}(x)$ since $dg_{12}|U_1 \cap U_2=g_{12}$. Thus $d\Phi \subset T^*_M(\Phi)$ is a semi-integrable substructure with group G and is clearly isomorphic to Φ.

Since G leaves $Y \subset T(Y)$ invariant there is a subbundle of the reduction of $T^*_M(B)$ with fibre Y. Its bundle space is $\bigcup_{\phi_x \in \Phi} d\phi_x(U \times Y)= \bigcup_{\phi_x \in \Phi} \phi_x(U \times Y)=B \subset T(B)$. This subbundle is isomorphic to $B(M,Y,G,\Phi)$.

This theorem shows that we may consider the fibre bundle B as being inbedded in $T^*(B)$.

<u>§7.11</u> Suppose the principal bundle $\mathfrak{B}(M,G,G,\mathfrak{B})$ is of class r. Consider a substructure $\mathfrak{H} \subset \mathfrak{B}$ with group $H \subset G$. Then the principal bundle $\mathfrak{H}(M,H,H,\mathfrak{H})$ may be of class $s \leq r$ or may not even be differentiable [§7.1].

If it is differentiable and of class s then $T^*(\mathfrak{H})$ is a substructure of $T^*(\mathfrak{B})$ as might easily be seen from the construction of the latter [§7.3]. In this case we will say that the substructure $\mathfrak{H}$ is of class s.

<u>THEOREM</u> <u>Let $f: M \leftrightarrow \mathfrak{B}/H$ be the cross section of $\mathfrak{B}/H$ defining $\mathfrak{H}$ [§1.9]. Then $\mathfrak{H}$ is of class s if and only if f is of class s.</u>

<u>PROOF</u> Let us consider the proof sketched in §1.9. If $\mathfrak{H}$ is of class s so then is f because the natural projection $q: \mathfrak{B} \Rightarrow \mathfrak{B}/H$ is of class $r \geq s$.

Conversely if the cross section f is of class s then since G is a Lie group we may choose $d: W \leftrightarrow G$ to be analytic [17 p110]. Then if ϕ is of class $r \geq s$ the map $\tilde{\mathfrak{V}} = \phi \circ (\epsilon, d) \circ \phi^{-1} \circ f$ is of class s. But if the local cross section $\tilde{\mathfrak{V}}$ is of class s so is the strip map which is defined by $\tilde{\mathfrak{V}}$ and so is the inverse of this strip map [§7.2]. Hence $\mathfrak{H}$ is of class s.

This theorem shows that besides approximating to strip maps by differentiable strip maps as in §7.2 we can, under similar conditions, "approximate to" substructures by others of higher class. This is very useful as substructures play an important part in the most recent theories of differential geometry.

<u>§8 The tangent bundle of a tangent bundle</u>

We will discuss here a special case of the fibre bundle defined in the last section which is that used in the classical theory of connections. It is shown that the twist is an isomorphism between $T^*(T(M))$ and a reduction of $T(T(M))$ and that there exists three inbedded fibre bundles isomorphic to $T(M)$.

I do not know whether any of these properties characterize $T^*(T(M))$, although it seems possible that a suitable selection of them will do so.

<u>§8.1</u> <u>THEOREM</u> <u>Let $M(R^m, \mathfrak{F})$ be a manifold of class $\geqslant 2$. The twist $\tau : T^*(T(M)) \Longleftrightarrow T(T(M))$ is an isomorphism between the fibre bundle $T^*(T(M))$ and an integrable reduction of the tangent bundle $T(T(M))$.</u>

<u>PROOF</u> Consider $f \in \mathfrak{F}$, $f : E \Longleftrightarrow U$. Let $k = (f, \epsilon)$ so that k

is a coordinate map of the manifold $M\times R^m$.

If we consider $T(T(E))$ as the product bundle $T(E)\times T(R^m)$ then $dk=d(f,\epsilon)=(df,d\epsilon)$ [§4.3 vi), §5.6]. So that writing $T(T(E))$ as the trivial bundle if $((a,b),(c,d))\in T(T(E))$ then

$$dk\circ\tau((a,b),c,d))=d(f,\epsilon)(a,c),(b,d))$$
$$=(df(a,b),d\epsilon(c,d))$$
$$=(df,\epsilon)((a,b),c,d)).$$

That is when $T(T(E))$ is considered as the trivial bundle $T(E)\times T(R^m)$ then $dk\circ\tau=(df,\epsilon)$.

Now still considering $T(T(E))$ as the trivial bundle $ddf=\partial df\circ(df,\epsilon)$ since $d\mathcal{A}$ is an atlas for $T(M)$ [§2.5, §4.1]. But also $df=\partial f\circ k$ so that

$$ddf=\tau\circ ddf\circ\tau \qquad [§5.3]$$
$$=\tau\circ d(\partial f\circ k)\circ\tau$$
$$=\tau\circ d\partial f\circ dk\circ\tau$$
$$=\tau\circ d\partial f\circ(df,\epsilon)$$

Consequently

$$\tau\circ d\partial f=\partial df:T(U)\times T(R^m)\Longleftrightarrow T(T(U)) \qquad a)$$

Let $p:T(M)\Longrightarrow M$ and $q:T(T(M))\Longrightarrow T(M)$ be bundle projections then $dp\circ d\partial f=q\circ\partial df:T(U)\times T(R^m)\Longrightarrow T(U)$ both being the natural projection. Hence $q\circ\tau\circ d\partial f=q\circ\partial df=dp\circ d\partial f$ for all $f\in\mathcal{A}$ and so

$$q\circ\tau=dp \qquad b)$$

That is $\tau : T^*(T(M)) \Longleftrightarrow T(T(M))$ is fibre-preserving, its projection being ϵ [§1.4].

We have already proved that $\tau \cdot d\partial\mathcal{O} = \partial d\mathcal{O}$. Therefore since $d\partial\mathcal{O}$ is a set of strip maps of $T^*(T(M))$ covering $T(M)$ and $\partial d\mathcal{O}$ is a set of strip maps of $T(T(M))$ covering $T(M)$, $d\mathcal{O}$ being an atlas of $T(M)$, τ is an isomorphism.

We have shown that representing $T(R^m)$ as $R^m \times R^m$ implies the representation of $T(L(m))$ as $D(m,m)$ [§6.7]. Hence $\partial d\mathcal{O}$ are the strip maps of a structure with group $D(m,m)$ which is therefore integrable. This completes the proof.

Note that since $D(m,m) \subset SL(2m)$ this theorem implies that $T(M)$ is always orientable [§3.2, 14 p23].

<u>§8.2 THEOREM $T^\#(M) = T^*_M(T(M)) \cap T_M(T(M))$ and thus $T^\#(M)$ is a subbundle of an integrable reduction of $T_M(T(M))$ [§7.9]. $T^\#(M)$ is thus a fibre bundle and the solder $\sigma : T^\#(M) \Longleftrightarrow T(M)$ is an isomorphism [§5.4].</u>

<u>PROOF</u> Let us first prove that $T^\#(M) = T^*_M(T(M)) \cap T_M(T(M))$. We write $T(T(R^m))$ as the trivial bundle, and hence $T^*(T(R^m))$ as the product bundle, $(R^m \times R^m) \times (R^m \times R^m)$. So that $T^*_{R^m}(T(R^m)) = (R^m \times R^m) \times (0 \times R^m)$ and $T_{R^m}(T(R^m)) = T(R^m \times 0) \times (R^m \times R^m)$ and thus

$$T^{\#}(R^m)=(R^m\times 0)\times(0\times R^m)=T^*_{R^m}(T(R^m))\cap T_{R^m}(T(R^m)).$$

Now consider $f\in\mathcal{O}$, $f:E\Longleftrightarrow U$. Both $ddf:T(T(E))\Longleftrightarrow T(T(U))$ and $ddf:T^*(T(E))\Longleftrightarrow T^*(T(U))$ are fibre-preserving [§7.7]. Therefore since $df(E\times 0)=U\subset T(U)$, $ddf\{T_E(T(E))\}=T_U(T(U))$ and $ddf\{T^*_E(T(E))\}=T^*_U(T(U))$. But by definition

$$\begin{aligned}T^{\#}(U)&=ddf\{T^{\#}(E)\}\\&=ddf\{T^*_E(T(E))\cap T_E(T(E))\}\\&=T^*_U(T(U))\cap T_U(T(U)).\end{aligned}$$

Therefore $T^{\#}(M)=\bigcup T^{\#}(U)=T^*_M(T(M))\cap T_M(T(M))$.

Now $df\circ\sigma=\sigma\circ ddf$ [§5.4], so that if $a\in E$, $f(a)=x$,

$$\begin{aligned}\sigma\circ\partial df(x,(0,y))&=\sigma\circ ddf((a,p),(0,y))\\&=df\circ\sigma((a,0),(0,y))\\&=df(a,y)=\partial f(x,y).\end{aligned}$$

Since $\partial\mathcal{O}$ is a set of strip maps for $T(M)$ covering M and $\partial d\mathcal{O}$ is a set of strip maps for the subbundle $T^*_M(T(M))\subset T(T(M))$ this proves that $\sigma:T^{\#}(M)\Longleftrightarrow T(M)$ is an isomorphism. This isomorphism maps the diagonal of $L(m)\times L(m)$, which is clearly in $D(m,m)$ onto $L(m)$.

<u>§8.3</u> We remember that M was identified with the zero tangents $o(M)\subset T(M)$ [§2.4]. Since $o(M)$ is a subbundle of $T(M)$ with fibre the point $0\in R^m$ then $T^*(o(M))$ or $T^*(M)$ is a subbundle of $T^*(T(M))$ with fibre $T(0)=0$. The subset $T^*(M)$ is not, however, $T(M)\subset T(T(M))$. In

fact the following theorem can be proved quite simply.

<u>THEOREM</u> <u>$\tau\{T^*(M)\}=T(M)$. Further if $o:M\longleftrightarrow T(M)$ is the cross section associated with the zero tangents $do(\xi)=\tau(\xi)\in T^*(M)$</u>.

<u>PROOF</u> $\tau:T^*(T(M))\Longleftrightarrow T(T(M))$ is an isomorphism [§8.1]. $T^*(M)$ is a subbundle of $T^*(T(M))$ with fibre 0 therefore $\tau\{T^*(M)\}$ is a subbundle of $T(T(M))$ with fibre 0, that is, the subbundle $T(M)$.

Further $do:T(M)\longleftrightarrow T^*(M)$ so that if $\xi\in T(M)$, $do(\xi)\in T^*(M)$ and thus $\tau\circ do(\xi)\in T(M)$. That is

$$\begin{aligned}\tau\circ do(\xi)&=q\circ\tau\circ do(\xi)\\ &=dp\circ do(\xi) && [\S 8.1\ b)]\\ &=d(p\circ o)(\xi) && [\S 4.3]\\ &=d\epsilon(\xi)=\xi\end{aligned}$$

Therefore $do(\xi)=\tau(\xi)\in T^*(M)$ since $\tau\circ\tau=\epsilon$ [§5.3].

It is a well established simplification to identify a manifold and its zero tangents and this convention does simplify appreciably the notation. In adopting it however, and especially when referring to differentials, one must be very careful to avoid writing ξ when $do(\xi)$ or $\tau(\xi)$ is meant.

<u>§8.4</u> We have now defined three subsets of $T(T(M))$ which

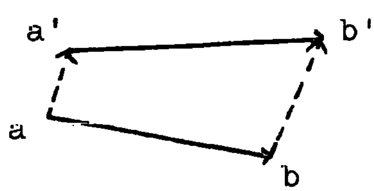
we have called $T(M)$, $T^*(M)$ and $T^{\#}(M)$, all three being homeomorphic to the base space $T(M)$.

$T^{\#}(M)$ is, as we have already seen, a subbundle of $T_M(T(M))$ which is isomorphic to $T(M)$. Since τ is an isomorphism and leaves $T^{\#}(M)$ pointwise invariant $T^{\#}(M)$ is also a subbundle of $T^*_M(T(M))$ which is again isomorphic to $T(M)$ [§5.4].

$T^*(M)$ is a subbundle of $T^*(T(M))$ with fibre a point. Now $\tau\{T^*(M)\}=T(M)$ and since $T(M)$ is a subbundle of $T^*_M(T(M))$ then $T^*(M)$ is also a subbundle of $T_M(T(M))$ which is isomorphic to $T(M)$ [§7.10].

Finally $T(M)$ is also a subbundle of both $T(T(M))$ and $T^*_M(T(M))$. In the former the fibre is a point and in the latter it is isomorphic to $T(M)$.

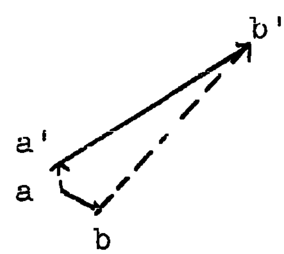
An intuitive description of these subbundles will be attempted here, but it must be emphasized that this has significance only when it stimulates the imagination.



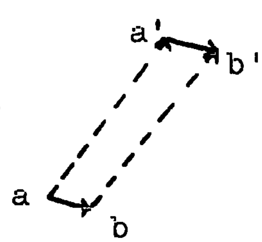
The diagram above is supposed to represent an element of T(T(M)) which we will call χ. It is the displacement of the tangent ξ represented by ab along the tangent ξ' represented by aa'. Thus $\chi \in T_{\xi}(T(M)) \cap T^*_{\xi'}(T(M))$. Now $T_{\xi}(T(M))$ and $T^*_{\xi'}(T(M))$ intersect only if $\xi, \xi' \in T_x(M)$ for some point $x \in M$ which is represented in the diagram by a.

The twist map τ sends this element into the displacement of ξ' along ξ by interchanging the broken and continuous lines.

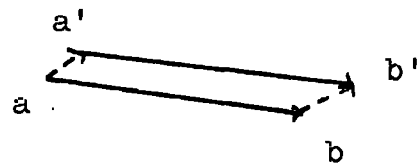
The set $T^{\#}(M)$ will then be those elements for which a, a' and b coincide;



T*(M) consists of those for which a and a' coincide with b and b' respectively;



T(M) being then represented by those elements for which a and b coincide with a' and b' respectively.



The properties discussed in this section can be represented by this model which I have found quite useful. The element λ can be thought of this way as an "infinitesimal circuit", a conception that will be useful when we discuss curvature and "parallel transport around infinitesimal circuits".

CHAPTER 4 CONNECTIONS, CURVATURE AND TORSION

§9 Connections

The idea of a connection has been treated in a number of different ways which vary greatly in complexity and character [3, 22, 23, 24, 29]. This important concept is treated in yet another way here and it is hoped that this viewpoint will perhaps prove clearer and simpler than some others. The definition is of course a generalization of the classical definition and of Ehresmann's which corresponds to a great extent to my definition of a linear connection [11 p30].

§9.1 DEFINITION Let $B(M,Y,G,\Phi)$ be a differentiable fibre bundle. Any substructure $D(\Phi) \subset T^*(\Phi)$ with group $G \subset T(G)$ such that $d\Phi \subset D(\Phi)$ is called a Φ-connection [25].

Since this definition only refers to the principal bundle $T^*(\Phi)$ a Φ-connection is common to any set of associated fibre bundles.

Let us make another simple observation. Since $d\Phi$ is a subbundle of $T^*_M(\Phi)$ with group G, the condition $d\Phi \subset D(\Phi)$ implies that $d\Phi = D_M(\Phi)$ where this last symbol has the obvious meaning [§7.10].

An important consideration is whether such a connection exists. We know that with certain restrictions this is always so, the proof following immediately from this definition. From the proof of the parallelisability of T(G) we see that the coset space T(G)/G is homeomorphic to $T_e(G)$ and hence to Euclidean number space [§6.5]. Therefore T(G)/G is solid. By a well known theorem of Steenrod if M is normal and is such that every open covering has a countable subcovering then T(M) also has these properties and the local cross section $d\Phi/G$ of $T^*(\Phi)/G$ may be extended to a full cross section [15 p55]. This is equivalent to saying that a connection exists [§1.9].

§9.2 A notation for elements of a connection which is quite suggestive will be used here. Before this can be done, however, we must prove the following theorem.

<u>THEOREM</u> <u>Each Φ-connection $D(\Phi)$ determines a unique map $\Delta: T(M)\hat{\times}\Phi \Longleftrightarrow D(\Phi)$</u>.

<u>PROOF</u> Let $p:\Phi \Longrightarrow M$, $\bar{p}:T(M)\Longrightarrow M$ and $q:T(\Phi)\Longrightarrow \Phi$ be bundle projections. Consider $(dp,q)\circ\delta : D(\Phi)\longrightarrow T(M)\hat{\times}\Phi$, noting that $p\circ q=\bar{p}\circ dp$ [§4.3 i)]. We will show that this map is open, one-one and onto and is thus a homeomorphism.

Consider $\theta_\xi \in D_\xi(\Phi)$ where $q(\theta_\xi)=\phi_x$. Then any other element of $D_\xi(\Phi)$ may be written uniquely as $\theta_\xi\circ g$ where $g\in G\subset T(G)$. But $q(\theta_\xi\circ g)=\phi_x\circ g$ so that since $\Phi_x=\phi_x\circ G$ the map $q:D_\xi(\Phi)\longrightarrow \Phi_x$ is one-one and onto [§7.5]. This being true for any $\xi\in T_x(M)$ and any $x\in M$ the map $(dp,q)\circ\delta|D(\Phi)$ is also one-one and onto.

Now dp and q are bundle projections so that $dp|D(\Phi)$ and $q|D(\Phi)$ are open, being both onto [§1.1]. Hence $(dp,q)\circ\delta|D(\Phi)$ is open and thus a homeomorphism. Its inverse is the map $\Delta: T(M)\hat{\times}\Phi \Longleftrightarrow D(\Phi)$ which was required.

The element $\Delta(\phi_x,\xi)$ we will call $D\phi_\xi$. If ϕ is a strip map of Φ, not necessarily differentiable, and if γ is the local cross section of Φ corresponding to it then the map $\Delta\circ(\tilde{\phi}\circ p,\epsilon)\circ\delta$ is a local cross section of $D(\Phi)$ and thus defines a strip map of $D(\Phi)$ which we call $D\phi$ [§1.2]. We see that

$$D\phi(\xi,e)=\Delta(\tilde{\phi}\circ p(\xi),\xi)$$
$$=\Delta(\phi_x,\xi)=D\phi_\xi$$

for $\xi\in T_x(M)$.

This notation has been chosen so as to compare this strip map with a differential with which it shares a very useful property. This is $q\circ D\phi=\phi\circ(\bar{p},\tilde{p})$ where $\tilde{p}:T(Y)\Rightarrow Y$ is the bundle projection. To see this we notice that $q\circ D\phi(\xi,\eta)=q\circ D\phi_\xi(\eta)=q(D\phi_\xi)\circ\tilde{p}(\eta)$ and $q(D\phi_\xi)=\phi_{\bar{p}(\xi)}$ by definition [§7.5].

§9.3 Since $G\subset T(G)$ leaves $Y\subset T(Y)$ invariant the connection $D(\Phi)$ determines a subbundle of $T^*(B)$ with fibre Y. This is called D(B).

Now from the definition of $D\phi$ and the lemma in §7.5 $D\phi_\xi\circ g=D(\phi_x\circ g)_\xi$ and consequently $Db_\xi=D\phi_\xi\circ\phi_x^{-1}(b)\in D(B)$ is defined for any $(b,\xi)\in B\hat{\times}T(M)$ independently of $\phi_x\in\Phi_x$. In fact we can define a map $\Delta':B\hat{\times}T(M)\Rightarrow D(B)$ in this way.

It is then useful to think of Db_ξ as a parallel displacement of b along the tangent ξ but this phrase is not credited with any real meaning as far as this thesis is concerned.

<u>§9.4</u> Suppose $\mathbb{H} \subset \mathbb{B}$ is a substructure with group $H \subset G$ and not necessarily differentiable then it is easy to see that $\Delta(T(M)\hat{\times}\mathbb{H})$ is a substructure of $D(\mathbb{B})$ with group H. This substructure we call $D(\mathbb{H})$. If μ is any strip map of $\mathbb{H}$ then $D\mu$ is a strip map of $D(\mathbb{H})$.

$D(\mathbb{H})$ is a $\mathbb{H}$-connection if $D(\mathbb{H}) \subset T^*(\mathbb{H})$ which is of course not necessarily true in fact $T^*(\mathbb{H})$ may not even exist. If $D(\mathbb{H})$ is a $\mathbb{H}$-connection we sometimes say that $\mathbb{H}$ is parallel with regard to the connection $D(\mathbb{B})$.

Similarly if $B' \subset B$ is a subbundle of B we can define $D(B')$ and say that B' is <u>parallel</u> with respect to $D(\mathbb{B})$ if $D(B') \subset T^*(B')$ thus requiring the latter to exist.

This definition is a generalization of a concept appearing under various disguises in the classical theory. We will not stop to discuss this unification here but will discuss the point later [§14].

<u>§9.5</u> Suppose G has the discrete topology. Then $T(G)=G$ and so $T^*(\mathbb{B})$ is itself a connection [§2.5]. An

important example of this is when G is just the identity, that is when the fibre bundle is trivial.

§10 Curvature

The definition of curvature that is given here is a generalization of the classical one, although it is not easy to see this from the definition because the approach is rather different from that normally used. All other definitions of curvature have either followed Eisenhart or Cartan [3, 29].

This section depends upon a rather curious theorem.

§10.1 We will consider a differentiable fibre bundle $B(M,Y,G,\Phi)$ of class 2 [§7.1]. We are going to discuss $T^*(T^*(\Phi))$ and will need to introduce the following bundle projections:-

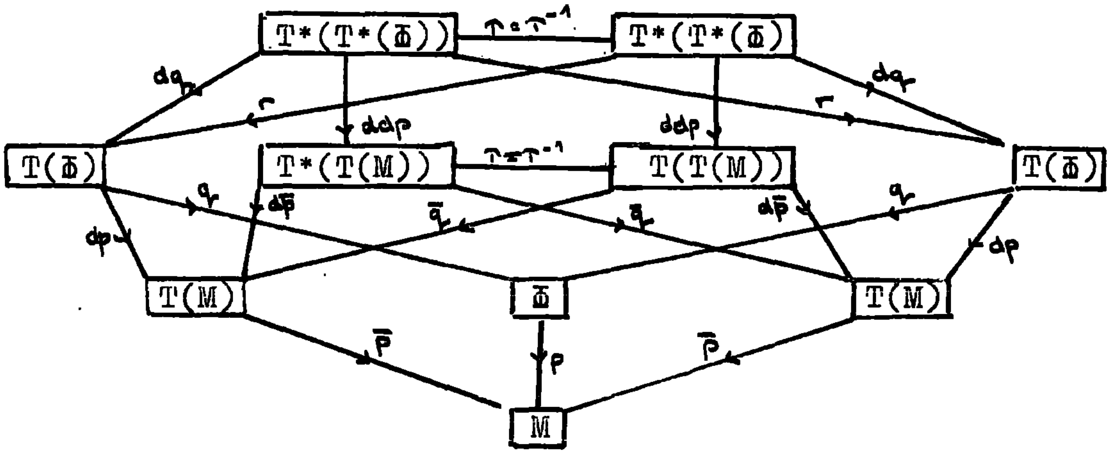
$$p:\Phi \Rightarrow M,\quad \bar{p}:T(M) \Rightarrow M,\quad \tilde{p}:T(G) \Rightarrow G$$

$$q:T(\Phi) \Rightarrow \Phi,\quad \bar{q}:T(T(M)) \Rightarrow T(M),\quad \tilde{q}:T(T(G)) \Rightarrow T(G)$$

$$r:T(T(\Phi)) \Rightarrow T(\Phi)$$

These have a number of relations with the twist and among themselves which follow easily from three simple properties [§4.3 i), §5.3 b), and §8.1 b)].

These properties are best described by the following commutative diagram.



<u>§10.2 THEOREM Consider a differentiable $\bar{\Phi}$-connection $D(\bar{\Phi}) \subset T^*(\bar{\Phi})$ then $D(D(\bar{\Phi}))=T^*(D(\bar{\Phi})) \cap T^*_{D(\bar{\Phi})}(T(\bar{\Phi}))$ is a $D(\bar{\Phi})$-connection [§7.11].</u>

<u>PROOF</u> Consider $\Delta : T(M)\hat{\times}\bar{\Phi} \Longleftrightarrow D(\bar{\Phi})$ as defined in §9.2 . Since Δ has a regular inverse it is also regular and thus $d\Delta : T^*(T(M))\hat{\times}T^*(\bar{\Phi}) \Longleftrightarrow T^*(D(\bar{\Phi}))$ [§7.6]. Now the inverse of $d\Delta$ is $(ddp,dq)\circ d\delta$ so that $dq\circ d\Delta : T^*(T(M))\hat{\times}T^*(\bar{\Phi}) \Longrightarrow T^*(\bar{\Phi})$ is just the natural projection. Hence

$$d\Delta : T^*(T(M))\hat{\times}D(\bar{\Phi}) \Longrightarrow T^*(D(\bar{\Phi})) \cap dq^{-1}\{D(\bar{\Phi})\}=D(D(\bar{\Phi})).$$

Now we have seen that $D\phi_\xi \circ g=D(\phi_x \circ g)_\xi$ for $(\xi,\phi_x) \in T(M)\hat{\times}\bar{\Phi}$ and $g \in G$. This may be written as

$$\Delta \circ (\epsilon, \alpha)=d\alpha \circ (\Delta, \sigma) : T(M)\hat{\times}\bar{\Phi}\times G \Longrightarrow D(\bar{\Phi})$$

where $\alpha : \Phi \times G \Rightarrow \Phi$ is defined as in §7.5 . Therefore

$$d\Delta \circ (d\epsilon, d\alpha) = dd\alpha \circ (d\Delta, d\epsilon)$$

that is to say

$$d\Delta(\chi, \theta_\xi) \circ \tau(\gamma) = d\Delta(\chi, \theta_\xi \circ \gamma)$$

for all $\gamma \in T(G)$ [§8.3]. Hence using an obvious notation

$$\begin{aligned} D_\chi(D(\Phi)) &= d\Delta(\chi, D_\xi(\Phi)) \\ &= d\Delta(\chi, D\phi_\xi \circ G) \\ &= d\Delta(\chi, D\phi_\xi) \circ G \end{aligned}$$

To complete the proof now we note that if $D\tilde{\phi}$ is any local cross section of $D(\Phi)$ then $d\Delta \circ (D\tilde{\phi} \circ dp, \epsilon) \circ \delta$ is a local cross section of $T^*(D(\Phi))$ with values in $D(D(\Phi))$ the set of such local cross sections covering $T(T(M))$ [§9.2]. Hence $D(D(\Phi))$ is a substructure with group G [§1.3]. This completes the proof.

If ϕ is any strip map of Φ then a strip map $D\phi$ of $D(\Phi)$ is uniquely defined so then is the strip map $DD\phi$ of $D(D(\Phi))$. This acts in a number of ways like a differential of a differential, although it is not necessarily symmetric [§5.3].

Of course as before $r \circ DD\phi = D\phi \circ (\bar{q}, \tilde{q})$ but also $dq \circ DD\phi = D\phi \circ (d\bar{p}, d\tilde{p})$. To see this we note that $dq \circ dd\alpha = d\alpha \circ (dq, d\tilde{p})$, that is, $dq(\theta_{\chi} \gamma\Gamma) = dq(\theta_\chi) \circ d\tilde{p}(\Gamma)$

where $\Theta_\chi \in T^*(T^*(\bar{\Phi}))$ and $\Gamma \in T(T(G))$. The proof then follows as in §9.2 [§7.5].

It is not difficult to see that $D(D(B))=T^*(D(B)) \cap T^*_{D(B)}(T(B))$. If $b \in B_x$ and $\chi \in T_\xi(T(M))$ where $\xi \in T_x(M)$ then DDb_χ is defined to be $D(Db_\xi)_\chi$. This element may be interpreted intuitively as a parallel displacement of b around the "infinitesimal circuit" χ [§8.4].

<u>§10.3</u> We are now in a position to define curvature.

<u>DEFINITION</u> Given a differentiable $\bar{\Phi}$-connection $D(\bar{\Phi})$ let $D(D(\bar{\Phi}))$ be defined as in the preceding paragraph. Then $_T \circ DD\phi \circ _T \circ DD\phi^{-1}$ is defined to be a restriction of the map $R: T(T(\bar{\Phi})) \Longleftrightarrow T(T(\bar{\Phi}))$ for any strip map ϕ of $\bar{\Phi}$. R is then called the <u>curvature</u> of the connection.

This definition is justified by the fact that $D(D(\bar{\Phi}))$ has group $G \subset T(T(G))$ which is pointwise symmetric [§5.4]. In fact any substructure of $T^*(T^*(\bar{\Phi}))$ with group a subgroup of $T^\#(G)$ would define such a map.

<u>§10.4</u> Certain properties of R follow directly from

definition. Only one or two of them will be mentioned here.

THEOREM $R:T^*(T^*(\Phi))\Longleftrightarrow T^*(T^*(\Phi))$ is an isomorphism.

PROOF Let $\phi:U\times G\Longleftrightarrow \Phi_U$ be a strip map of Φ. Then $ddp\circ DD\phi:T(T(U))\times T(T(G))\Longrightarrow T(T(U))$ is the natural projection and $\tau\circ ddp=ddp\circ\tau$. Hence

$$ddp\circ R=ddp\circ\tau\circ DD\phi\circ\tau\circ DD\phi^{-1}$$

$$=\tau\circ\tau\circ ddp=ddp \qquad a)$$

Thus R is fibre-preserving.

Also $R(\theta_\chi)=R\circ\theta_\chi$ for any $\theta_\chi\in T^*(T^*(\Phi))$ so that

$$R(\theta_\chi\circ\Gamma)=R\circ\theta_\chi\circ\Gamma=R(\theta_\chi)\circ\Gamma \qquad b)$$

so that R is an isomorphism with projection ϵ.

Another immediate property is

$$R\circ\tau\circ R=\tau \qquad c)$$

Now $dp\circ R=dq\circ\tau\circ DD\phi\circ\tau\circ DD\phi^{-1}$

$$=\tau\circ DD\phi\circ\tau\circ DD\phi^{-1} \qquad [\S 10.1]$$

$$=D\phi\circ(\bar{q},\tilde{q})\circ\tau\circ DD\phi^{-1} \qquad [\S 10.2]$$

$$=D\phi\circ(d\bar{p},d\tilde{p})\circ DD\phi^{-1}$$

$$=D\phi\circ D\phi^{-1}\circ dq=dq$$

so that

$$dq\circ R=dq \quad\text{and}\quad r\circ R=r \qquad d)$$

this equation being obtained similarly.

Finally by an abuse of language we may write also

$R:T^*(T^*(B)) \Leftrightarrow T^*(T^*(B))$ which is again an isomorphism.

<u>§10.5</u> The holonomy group has played an important role in differential geometry. It is not proposed however to discuss it here but a definition of the infinitesimal holonomy group will be given, which is a generalization of an important concept due to Cartan [2].

Consider $\theta_\chi \in T(T(\Phi))$ then $R(\theta_\chi)=\theta_\chi \circ \Gamma$ for some unique $\Gamma \in T(T(G))$ since R is an isomorphism. Then

$$r \circ R(\theta_\chi) = r(\theta_\chi \circ \Gamma)$$
$$= r(\theta_\chi) \circ \tilde{q}(\Gamma) = r(\theta_\chi) \qquad [§7.5]$$

since $r \circ R = r$.

Similarly $dq \circ R(\theta_\chi) = dq(\theta_\chi) \circ d\tilde{p}(\Gamma) = dq(\theta_\chi)$ [§10.2].

Hence $\tilde{q}(\Gamma) = d\tilde{p}(\Gamma) = e$ and therefore $\Gamma \in T^*_e(T(G)) \cap T_e(T(G)) = T^{\#}_e(G)$ [§5.4]. This enables us to make the following definition:-

<u>DEFINITION</u> The linear subspace of $T_e(G)$ generated by the set $\sigma\left[\bigcup_{\theta \in D(D(\Phi))} \{\theta_\chi^{-1} \circ R(\theta_\chi)\}\right]$ is called the <u>infinitesimal holonomy group</u>.

If $\theta_\chi \in D_\chi(D(\Phi))$ any other element in $D_\chi(D(\Phi))$ can be written as $\theta_\chi \circ g$ and since $R(\theta_\chi \circ g) = R(\theta_\chi) \circ g$

then $(\theta_\chi \circ g)^{-1} \circ R(\theta_\chi \circ g) = g^{-1} \circ \theta^{-1} \circ R(\theta_\chi) \circ g$. So that since σ is an isomorphism the elements of the infinitesimal holonomy group associated with the elements of $D_\chi(D(\Phi))$ vary only by the adjoint group [17 p123].

It should be possible to prove that the infinitesimal holonomy group is the same for any open submanifold of M with the induced connection, but I have not yet attempted this problem.

It may also be possible to prove that, given a connected subgroup $H \subset G$, there always exists a sub-structure of Φ with a group which has the component H if $T_e(H)$ contains the infinitesimal holonomy group due to some connection $D(\Phi)$ [23].

§10.6 If the curvature R is the identity map then the connection is said to be flat [22, 23, 29 p84]. In this case the infinitesimal holonomy group consists of the identity alone. If the last conjecture above is correct then this would imply that there existed a substructure of Φ with a discrete group. Let us compare this remark with the following.

If $D(\Phi)$ is semi-integrable then so is $D(D(\Phi))$ [§7.10].

In fact if $d\phi$ is a strip map of $D(\Phi)$ then $dd\phi$ is a strip map of $D(D(\Phi))$. Since $dd\phi$ is symmetric R is the identity and the connection is flat. A particular case of a semi-integrable connection is when G is discrete [§9.5].

§11 Torsion

In this section will be defined a map which we will call the torsion and which is associated in a unique manner with a connection on a tangent bundle. It is defined by means of the twist map in much the same way as the curvature. In fact there are a number of other comparisons that might be made between the torsion and curvature. Some writers have attempted to define the torsion as part of the curvature but here, although they are in many respects similar, there appears to be no direct relation between them [24].

§11.1 We will be considering here a manifold $M(R^m, \mathcal{G})$ of class ≥ 2. $T(M)(M, R^m, G, \Phi)$ is some differential reduction of the tangent bundle and $D(\Phi) \subset T^*(\Phi)$ is a Φ-connection. We let $p: T(M) \Rightarrow M$,

$q:T(T(M)) \Rightarrow T(M)$ and $\tilde{p}:T(G) \Rightarrow G$ be the bundle projections.

If $f \in \mathcal{F}$, $f:E \Leftrightarrow U$, and $T^*(T(E))$ is written as the trivial bundle $T(E) \times T(R^m)$ then we define

$$Ddf = D\partial f \circ (df, \epsilon): T(E) \times T(R^m) = T^*(T(E)) \Leftrightarrow T^*(T(U)).$$

Since $dp \circ D\partial f: T(U) \times T(R^m) \Rightarrow T(U)$ is the natural projection $Ddf: T^*(T(E)) \Leftrightarrow T^*(T(U))$ is fibre-preserving. Also $q \circ D\partial f = \partial f \circ (p, \tilde{p})$ so that $Ddf: T(T(E)) \Leftrightarrow T(T(U))$ is also fibre-preserving [§9.2].

Thus Ddf has some of the characteristics, as had $DD\phi$, of a differential of a differential. It is not however always symmetric and this leads to the torsion [compare §10.3].

<u>DEFINITION</u> The map $\tau \circ Ddf \circ \tau \circ Ddf^{-1}$ is defined to be a restriction of $S:T(T(M)) \Leftrightarrow T(T(M))$ for all $f \in \mathcal{F}$. The map S is called the <u>torsion</u> of the connection.

<u>§11.2</u> It is necessary to justify this definition by showing that $\tau \circ Ddf \circ \tau \circ Ddf^{-1}$ is independent of $f \in \mathcal{F}$. To do this consider f_1, $f_2 \in \mathcal{F}$ then we must show that

$$\tau \circ Ddf_1 \circ \tau \circ Ddf_1^{-1} = \tau \circ Ddf_2 \circ \tau \circ Ddf_2^{-1}$$

wherever both are defined, or, since $\tau \circ \tau = \epsilon$,

$$Ddf_1^{-1} \circ Ddf_2 = \tau \circ Ddf_1^{-1} \circ Ddf_2 \circ \tau .$$

Let $g_{12}(x) = \partial f_{1.x}^{-1} \circ \partial f_{2.x} = D\partial f_{1.\xi}^{-1} \circ D\partial f_{2.\xi}$ for all $\xi \in T_x(M)$ and suppose $f_1(a_1) = f_2(a_2) = x$. Then, still considering $T^*(T(E))$ as the trivial bundle,

$$\tau \circ Ddf_1^{-1} \circ Ddf_2 \circ \tau((a_2,b),(c,d))$$
$$= \tau \circ Ddf_1^{-1} \circ Ddf_2((a_2,c),(b,d)) \qquad [§5.1]$$
$$= \tau \circ (df_1^{-1}, \epsilon) \circ D\partial f_1^{-1} \circ D\partial f_2\ (df_2(a_2,c),(b,d))$$
$$= \tau \circ (df_1^{-1}, \epsilon)(df_2(a_2,c), g_{12}(x)\cdot(b,d))$$
$$= \tau \circ (df_1^{-1}, \epsilon)(df_2(a_2,c),(g_{12}(x)\cdot b, g_{12}(x)\cdot d)) [§6.7]$$
$$= \tau \circ ((a_1, g_{12}(x)\cdot c),(g_{12}(x)\cdot b, g_{12}(x)\cdot d))$$
$$= ((a_1, g_{12}(x)\cdot b),(g_{12}(x)\cdot c, g_{12}(x)\cdot d))$$
$$= Ddf_1^{-1} \circ Ddf_2((a_2,b),(c,d)).$$

Thus $\tau \circ Ddf \circ \tau \circ Ddf^{-1}$ is independent of $f \in \mathcal{F}$ and the definition is justified.

It is easy to see that $S \circ \tau \circ S = \tau$, $q \circ S = q$ and $dp \circ S = dp$ as for the curvature [§10.4 c), d)].

<u>§11.3</u> <u>THEOREM</u> <u>A necessary and sufficient condition for the torsion S to be the identity is that D(T(M)) be symmetric</u>.

<u>PROOF</u> If S is the identity then every map of $Dd\mathcal{F}$ is symmetric. Therefore if we use the notation of §11.1 since $Ddf: T(E) \times R^m \Longleftrightarrow D(T(U))$, then

$$Ddf \circ \tau = \tau \circ Ddf : T(E) \times R^m \Longleftrightarrow \tau\{D(T(U))\}$$

But $\tau\{T(E) \times R^m\} = T(E) \times R^m \subset T(E) \times T(R^m) = T^*(T(E))$ from the definition of τ so that $D(T(U)) = \tau\{D(T(U))\} = Ddf\{T(E) \times R^m\}$.

That is $D(T(U))$ is symmetric and so therefore is $D(T(M))$.

Conversely suppose $D(T(M))$ is symmetric. Then if $\xi, \xi' \in T_x(M)$ this implies that $\tau(D\xi_{\xi'}) \in D(T(M))$. But since $q \circ \tau = dp$ and $dp \circ \tau = q$ then $q \circ \tau(D\xi_{\xi'}) = \xi'$ and $dp \circ \tau(D\xi_{\xi'}) = \xi$ and hence we must have $\tau(D\xi_{\xi'}) = D\xi'_{\xi}$ [§9.2].

Suppose $\xi = df(a,b)$ and $\xi' = df(a,c)$ and that $Ddf((a,b),(c,d)) = ddf((a,b),(c,\gamma(b)\cdot c+d))$ where $\gamma(b)$ is some matrix really dependent upon both a and b [§6.7]. Then by hypothesis

$$
\begin{aligned}
ddf((a,b),(c,\gamma(c)\cdot b)) &= ddf\circ\tau((a,c),(b,\gamma(c)\cdot b)) \\
&= \tau\cdot ddf((a,c),(b,\gamma(c)\cdot b)) \\
&= \tau\circ Ddf((a,c),(b,0)) \\
&= \tau(D\xi_{\xi'}) \\
&= D\xi'_{\xi} \\
&= Ddf((a,b),(c,0)) \\
&= ddf((a,b),(c,\gamma(b)\cdot c)).
\end{aligned}
$$

So therefore $\gamma(c)\cdot b = \gamma(b)\cdot c$. Hence

$$
\begin{aligned}
\tau\circ Ddf((a,b),(c,d)) &= \tau\circ ddf((a,b),(c,\gamma(b)\cdot c+d)) \\
&= ddf\circ\tau((a,b),(c,\gamma(b)\cdot c+d)) \\
&= ddf((a,c),(b,\gamma(b)\cdot c+d)) \\
&= ddf((a,c),(b,\gamma(c)\cdot b+d)) \\
&= Ddf((a,c),(b,d) \\
&= Ddf\circ\tau((a,b),(c,d))
\end{aligned}
$$

That is, $\tau \circ Ddf = Ddf \circ \tau$ for all $f \in \mathfrak{F}$ and thus S is the identity.

If S is the identity we say the connection is symmetric. This theorem shows that this is equivalent to D(T(M)) being symmetric. This definition of the symmetry of a connection is in fact a generalization of the usual one. This is more easily seen from the other equivalent condition $\tau(D\xi_{\xi'}) = D\xi'_{\xi}$ for all $\xi, \xi' \in T_x(M)$ and $x \in M$ [22, 23].

CHAPTER 5 SPECIAL CONNECTIONS

§12 Linear Connections

The standard definitions of a connection usually require implicitly the connection to be linear. This is because any relaxation of this condition greatly increases the complexity of the notation and calculus of the classical theories. Here, although there is no simplification associated with the condition of linearity, linear connections have a number of useful special properties.

It is shown that there are several different but equivalent definitions of linearity. All the strip maps of a linear connection D(Ɱ) are linear and D(B) is a field of m-planes in T(B), the base space being a manifold of dimensions m. This fact is used by Ehresmann in his definition of an infinitesimal connection and he remarks that a non-linear connection might be considered as a field of "elementary m-cones"[11p36

There is a section discussing semi-integrable connections (which are automatically linear) and proving that the field of m-planes is then integrable.

It is shown that for a linear connection the curvature and torsion have the property of being isomorphisms and in this case we can define the "square root" of these isomorphisms. This leads to the unique derivation of a linear and symmetric connection from any given general connection.

__§12.1__ __DEFINITION__ Let $B(M,Y,G,\Phi)$ be a differentiable fibre bundle and $D(\Phi)\subset T^*(\Phi)$ a connection. We say that $D(B)$ is linear if $D(B)\cap T_b(B)$ is a linear subspace of $T_b(B)$ for every $b\in B$ [§2.3].

__THEOREM__ __D(B) is linear if and only if every strip map of D(Φ) is a linear map into T(B).__

__PROOF__ Suppose first that $D(B)$ is linear. Consider $\xi_1,\xi_2\in T_x(M)$ and $b\in B_x$. If $\alpha,\beta\in R$ we will write $\alpha\xi_1+\beta\xi_2=\xi$. Then since $D(B)$ is linear $\alpha Db_{\xi_1}+\beta Db_{\xi_2}\in D(B)$. But if $p:B\Rightarrow M$ and $q:T(B)\Rightarrow B$ are the bundle projections since dp is linear $dp(\alpha Db_{\xi_1}+\beta Db_{\xi_2})=\alpha\xi_1+\beta\xi_2=\xi$ and of course $q(\alpha Db_{\xi_1}+\beta Db_{\xi_2})=b$ so that we must have $\alpha Db_{\xi_1}+\beta Db_{\xi_2}=Db_{\xi}$.

Suppose $\phi: U\times Y \Longleftrightarrow B_U$ is a regular strip map of Φ where $x\in U$. We write $D\phi_\xi = d\phi_\xi\gamma(\xi)$ so that $\gamma: T(U) \to T_e(G)$. Then if $b=\phi(x,y)$

$$\begin{aligned}\alpha Db_{\xi_1}+\beta Db_{\xi_2} &= \alpha D\phi(\xi_1,y)+\beta D\phi(\xi_2,y)\\ &= \alpha d\phi(\xi_1,\gamma(\xi_1)\cdot y)+\beta d\phi(\xi_2,\gamma(\xi_2)\cdot y)\\ &= d\phi(\xi,\alpha\gamma(\xi_1)\cdot y+\beta\gamma(\xi_2)\cdot y)\end{aligned}$$

since $d\phi$ in linear. But $Db_\xi = D\phi(\xi,y)=d\phi(\xi,\gamma(\xi)\cdot y)$ therefore $\alpha\gamma(\xi_1)\cdot y+\beta\gamma(\xi_2)\cdot y=\gamma(\xi)\cdot y$ for all $y\in Y$.

Now consider $\eta_1,\eta_2\in T_y(Y)$ writing $\alpha\eta_1+\beta\eta_2=\eta$. Then $D(\alpha\xi_1+\beta\xi_2,\alpha\eta_1+\beta\eta_2)=d\phi(\xi,\gamma(\xi)\cdot\eta)=d\phi(\xi,\gamma(\xi)\cdot y+\eta)$ [§4.3 viii)]. But

$$\begin{aligned}\gamma(\xi)\cdot y+\eta &= \alpha\gamma(\xi_1)\cdot y+\beta\gamma(\xi_2)\cdot y+\alpha\eta_1+\beta\eta_2\\ &= \alpha\{\gamma(\xi_1)\cdot y+\eta_1\}+\beta\{\gamma(\xi_2)\cdot y+\eta_2\}\\ &= \alpha\gamma(\xi_1)\cdot\eta_1+\beta\gamma(\xi_2)\cdot\eta_2.\end{aligned}$$

Thus since $d\phi$ is linear

$$\begin{aligned}D\phi(\alpha\xi_1+\beta\xi_2,\alpha\eta_1+\beta\eta_2) &= \alpha d\phi(\xi_1,\gamma(\xi_1)\cdot\eta_1)+\beta d\phi(\xi_2,\gamma(\xi_2)\cdot\eta_2)\\ &= \alpha D\phi(\xi_1,\eta_1)+\beta D\phi(\xi_2,\eta_2)\end{aligned}$$

and hence $D\phi$ is linear.

It is clear that any strip map of $D(\Phi)$ can be written as $D\psi$ where ψ is a strip map of Φ which is not necessarily regular [§9.2]. But $D\phi^{-1}\circ D\psi$ is clearly linear because if $\psi_x=\phi_x\circ g(x)$ then $D\phi^{-1}\circ D\psi(\xi,\eta)=(\xi,g(x)\cdot\eta)$ for all $\xi\in T_x(M)$ and $g(x)\cdot\eta$ is linear in η[§4.3 viii), §6.2]. Since the set of regular strip maps of Φ covers

M every strip map of $D(\bar{\Phi})$ must be linear.

Conversely suppose every strip map of $D(\bar{\Phi})$ is linear. Since $T_x(M)\times y$ is a linear subspace of $T_x(M)\times T_y(M)$ then $D(B)\cap T_b(B)=D\phi(T_x(M)\times y)$ is a linear subspace of $T_b(B)=D\phi(T_x(M)\times T_y(M))$.

<u>§12.2</u> We suppose that $M(R^m,\mathfrak{O})$ and $Y(R^n,\mathfrak{G})$ are manifolds. Then B is an m+n dimensional manifold.

<u>THEOREM</u> <u>D(B) is linear if and only if D(B) is a field of m-planes in T(B) [§3.4, 11 p36]</u>.

<u>PROOF</u> Suppose D(B) is linear. Then, using the notation of the preceding paragraph, $D\phi$ is linear.

Consider $k\in\mathfrak{O}\times\mathfrak{G}$, $k:F\Longleftrightarrow V\subset U\times Y$ and let $\phi(V)=W\subset B_U$. Then the map

$$\Theta=D\phi\circ\partial k\circ(\phi^{-1},\epsilon):W\times R^{m+n}\Longleftrightarrow T(W) \qquad a)$$

has the property $q\circ\Theta(b,a)=b$ from the properties of $D\phi$ [§9.2]. Also if $a_1,a_2\in R^{m+n}$ and $\alpha,\beta\in R$ since $D\phi$ is linear $\Theta(b,\alpha a_1+\beta a_2)=\alpha\Theta(b,a_1)+\beta\Theta(b,a_2)$. Thus Θ is a strip map of the tangent bundle T(B) and further

$$\Theta:W\times(R^m\times 0)\Longleftrightarrow D(W).$$

Since such strip maps clearly cover B, D(B) is a field of m-planes.

The converse follows immediately from the definition of linearity.

§12.3 There are therefore several possible equivalent definitions of linearity of a connection. These last two theorems enable us to use whichever is most convenient.

If $D(\Phi)$ is linear then so is $D(B)$ where B is any differentiable fibre bundle associated with the principal bundle Φ. However the linearity of $D(B)$ only implies that of $D(\Phi)$ if $T(G)$ is effective on $T(Y)$ and I do not know any conditions for this to be so.

It follows immediately from the fact used in §10.2

$$d\Delta : T^*(T(M)) \hat{\times} D(B) \Longleftrightarrow D(D(B))$$

that if $D(B)$ is linear so is $D(D(B))$ because $d\Delta$, being a differential, is a linear map.

THEOREM The linear parts of the regular strip maps of a differentiable Φ-connection $D(\Phi)$ are the strip maps of a linear Φ-connection which is therefore called the linear part of $D(\Phi)$.

PROOF If ϕ is a strip map of Φ we will denote the linear part of $D\phi$ by $D^*\phi$.

Suppose ϕ_1 and ϕ_2 are two strip maps of Φ. Then

$D\phi_1^{-1} \circ D\phi_2$ is a linear map [§12.1]. But the linear part of $D\phi_1^{-1} \circ D\phi_2$ is $D^*\phi_1^{-1} \circ D^*\phi_2$ and therefore $D^*\phi_1^{-1} \circ D^*\phi_2 = D\phi_1^{-1} \circ D\phi_2$ [§5.5]. Hence the maps $D^*\phi$ are obviously the strip maps for a connection which we call $D^*(\Phi)$. To show that this is a Φ-connection we must prove that $D^*(\Phi) \subset T^*(\Phi)$.

Now the condition $D(\Phi) \subset T^*(\Phi)$ is equivalent to the condition that for any regular strip map $\phi : U \times G \Leftrightarrow \Phi$

$$d\phi^{-1} \circ D\phi : T(U) \times G \longleftarrow T(U) \times T(G) \qquad \text{a)}$$

but this is equivalent to

$$d\phi^{-1} \circ D\phi : T(U) \times T(G) \Leftrightarrow T(U) \times T(G) \qquad \text{b)}$$

because both $\gamma \cdot G \subset T(G)$ and $\gamma \cdot T(G) = T(G)$ are equivalent to the condition $\gamma \in T(G)$.

Now from b) the linear part of $d\phi^{-1} \circ D\phi$ which, since $d\phi$ is linear, is $d\phi^{-1} \circ D^*\phi$ has the property

$$d\phi^{-1} \circ D^*\phi : T(U) \times T(G) \Leftrightarrow T(U) \; T(G)$$

and that is equivalent as above to saying that $D^*(\Phi) \subset T^*(\Phi)$ and hence $D^*(\Phi)$ is a Φ-connection.

<u>§12.4</u> If $D(\Phi)$ is semi-integrable there exists a set of strip maps of $D(\Phi)$ covering $T(M)$ which are differentials and thus linear [§4.3]. Thus $D(\Phi)$ is linear and we can prove the following theorem, M

being m-dimensional as usual.

THEOREM $D(\Phi)$ is semi-integrable if and only if the field of m-planes $D(\Phi)$ $T(\Phi)$ is integrable [11 p37; 8].

PROOF Suppose first of all that $D(\Phi)$ is semi-integrable. We refer to §12.2. The map Θ is a strip map of the structure defining the field of m-planes. By hypothesis we may choose ϕ so that $D\phi = d\phi$ and $d\phi \circ \partial k = \partial(\phi \circ k) \circ (\phi, \epsilon)$ [§7.4 a)]. Therefore $\Theta = D\phi \circ \partial k \circ (\phi^{-1}, \epsilon) = d\phi \circ \partial k \circ (\phi^{-1}, \epsilon) = \partial(\phi \circ k)$ and the m-planes are integrable.

Now suppose instead that the field of m-planes $D(\Phi)$ is integrable. Let Φ^* be the set Φ with the new topology of the laminated structure associated with these integrable m-planes [§3.4]. If $i : \Phi^* \leftrightarrow \Phi$ is the identity correspondence it is easily seen to be regular and by definition of Φ^* $di : T(\Phi^*) \Rightarrow D(\Phi)$.

Now if $p : \Phi \Rightarrow M$ is the bundle projection then $dp : D(\Phi) \Rightarrow T(M)$ and hence if $p^* : p \circ i$, $dp^* : T(\Phi^*) \Rightarrow T(M)$. But Φ^* is m-dimensional as is M so that p^* is regular and therefore locally a homeomorphism [§4.4].

Suppose then that $p^* : W^* \Leftrightarrow U$ where $W^* \subset \Phi^*$ and $U \subset M$ are open. Then $\tilde{\phi} = i \circ (p^*|W^*)^{-1}$ is a local cross section of Φ, and since $di : T(\Phi^*) \Rightarrow D(\Phi)$ then $d\phi : T(U) \leftrightarrow D(\Phi)$. $d\tilde{\phi}$ is therefore a local cross section of $D(\Phi)$.

Now a glance at the lemma of §1.2 shows immediately that if the cross section γ defines the strip map ϕ of Φ then the local cross section $d\gamma$ defines the strip map $d\phi$ of $D(\Phi)$ [§7.2]. The set of such differentials covers T(M) and thus $D(\Phi)$ is semi-integrable. This completes the proof.

It is quite possible that this theorem could be extended to give conditions for a general substructure of $T^*(\Phi)$ to be semi-integrable.

The problem of the integrability of a field of m-planes has been widely studied and a large number of results are known. Later an interpretation of the integrability conditions for such a field will be given [§14.2]. In this particular case the integrability condition, which is by this theorem the condition for $D(\Phi)$ to be semi-integrable, is just that $D(D(\Phi))$ be symmetric, that is, that the connection be flat [§10.6].

<u>§12.5</u> Consider a manifold $M(R^m, \mathfrak{G})$ and its tangent bundle $T(M)(M, R^m, T(L(m)), \Phi)$. We will show that if $D(\Phi)$ is a linear connection then the torsion S

associated with $D(\Phi)$ has some important special properties.

<u>THEOREM</u> <u>The torsion $S:T(T(M)) \Leftrightarrow T(T(M))$ of the connection $D(\Phi)$ is an isomorphism if and only if $D(\Phi)$ is linear. In which case $S:T^*(T(M)) \Leftrightarrow T^*(T(M))$ is also an isomorphism.</u>

<u>PROOF</u> Consider $f \in \mathfrak{F}$, $f:E \Leftrightarrow U$. Then

$$Ddf:T^*(T(E)) \Leftrightarrow T^*(T(U))$$

is an isomorphism, the strip map $D\partial f$ being associated with the strip map $d\partial \epsilon$ [§11.1]. On the other hand

$$Ddf:T(T(E)) \Leftrightarrow T(T(U))$$

is clearly an isomorphism if and only if Ddf is linear, that is if $D\partial f$ is linear. The strip map $\partial d\epsilon$ is in this case associated with the strip map θ defined from $D\partial f$ by §12.2 a).

Now S is an isomorphism if and only if the restrictions $\tau \circ Ddf \circ \tau \circ Ddf^{-1}$ of S are isomorphisms for every $f \in \mathfrak{F}$. But since $\tau:T^*(T(U)) \Leftrightarrow T(T(U))$ and $Ddf:T^*(T(E)) \Leftrightarrow T^*(T(U))$ are isomorphisms and $\tau = \tau^{-1}$ then

$$\tau \circ Ddf \circ \tau : T(T(E)) \Leftrightarrow T(T(U))$$

is an isomorphism [§8.1]. Thus S is an isomorphism if and only if $Ddf:T(T(E)) \Leftrightarrow T(T(U))$ is an isomorphism for every $f \in \mathfrak{F}$. That is if every map in $D\partial\mathfrak{F}$ is linear which is equivalent to $D(T(M))$ being linear [§12.1] . Finally since $T(L(m))$ is effective on $T(R^m)$ this is equivalent to $D(\bar{\Phi})$ being linear.

The last part of the theorem follows from the fact that ${}_T : T^*(T(M)) \Longleftrightarrow T(T(M))$ is an isomorphism and $S = {}_T \circ S^{-1} \circ {}_T : T^*(T(M)) \Longleftrightarrow T^*(T(M))$ [§8.1].

A similar result can be proved analogously for the curvature.

<u>§12.6 THEOREM</u> <u>Let $D(\bar{\Phi})$ be a linear connection on the tangent bundle of the manifold $M(R^m, \mathfrak{G})$ and let S be the torsion. Then S^κ exists and is an isomorphism like S and satisfies the relation $S^\kappa \circ {}_T \circ S^\kappa = {}_T$ for every rational number κ.</u>

<u>PROOF</u> $S : T^*(T(M)) \Longleftrightarrow T^*(T(M))$ is an isomorphism with projection the identity [§11.2, §1.4] . Therefore we may write $S \circ \theta_\xi = \theta_\xi \circ \gamma$ where $\gamma \in T(L(m))$ is defined uniquely by each $\theta_\xi \in T^*(\bar{\Phi})$.

Now let $\bar{q} : T(\bar{\Phi}) \Longrightarrow \bar{\Phi}$, $q : T(T(M)) \Longrightarrow T(M)$ and $\tilde{p} : T(R^m) \Longrightarrow R^m$ be bundle projections. Then it follows from the conventions we have made that $\bar{q}(\theta_\xi) \circ \tilde{p} = q \circ \theta_\xi$ [§7.5] . Therefore

$$\bar{q}(S \circ \theta_\xi) \circ \tilde{p} = q \circ S \circ \theta_\xi$$

$$=q\circ\Theta_\xi \qquad [\S 11.2]$$

$$=\bar{q}(\Theta_\xi)\circ\tilde{p}$$

$$=\bar{q}(\Theta_\xi\circ\gamma)\circ\tilde{p}.$$

Therefore since L(m) is effective on R^m $\bar{q}(\Theta_\xi)=\bar{q}(\Theta_\xi\circ\gamma)$ and thus $\gamma\in T_e(L(m))$ [§7.5].

That is, we may represent γ by a partition matrix of the form $\begin{bmatrix} 1 & \cdot \\ \gamma & 1 \end{bmatrix}$ where 1 is the unit matrix in L(m) [§6.7].

Now it is obvious that the partitioned matrix $\begin{bmatrix} 1 & \cdot \\ \kappa\gamma & 1 \end{bmatrix}$ will represent γ^κ and thus $(\Theta_\xi^{-1}\circ S\circ\Theta_\xi)^\kappa=\gamma^\kappa$ exists whatever $\Theta_\xi\in T^*(\bar{\Phi})$. We define then $\Theta_\xi\circ(\Theta_\xi^{-1}\circ S\circ\Theta_\xi)^\kappa\circ\Theta^{-1}$ to be a restriction of S^κ for every $\Theta_\xi\in T^*(\bar{\Phi})$. It is easy to see that

$$S^\kappa:T^*(T(M))\Longleftrightarrow T^*(T(M))$$

is an isomorphism. If we prove that $S^\kappa\circ{}_T\circ S^\kappa={}_T$ then it will follow that

$$S^\kappa:T(T(M))\Longleftrightarrow T(T(M))$$

is also an isomorphism [§12.5].

Now consider $f\in\mathcal{O}$, $f:E\Longleftrightarrow U$. Let $T^*(T(E))$ be the trivial bundle $T(E)\times T(R^m)$. Then we may write

$$S\circ{}_T\circ S\circ ddf((a,b),(c,d))$$

$$=S\circ{}_T\circ ddf((a,b),(c,\gamma(b)\cdot c+d))$$

$$=S\circ ddf((a,c),(b,\gamma(b)\cdot c+d))$$

$$=ddf((a,c),(b,\gamma(b)\cdot c+\gamma(c)\cdot b+d))$$

$=\tau\circ ddf((a,b),(c,d))$ since $S\circ\tau\circ S=\tau$

$=ddf((a,c),(b,d))$

where $\gamma(b)$ and $\gamma(c)$ are defined as γ above. Therefore $\gamma(b)\cdot c+\gamma(c)\cdot b=0$ and consequently $\kappa\gamma(b)\cdot c+\kappa\gamma(c)\cdot b=0$.

Thus since $S^{\kappa}\circ ddf((a,b),(c,d))=ddf((a,b),(c,\kappa\gamma(b)\cdot c+d))$ this implies that $S^{\kappa}\circ\tau\circ S^{\kappa}=\tau$.

It follows from this theorem that $S^{\kappa}\circ D(\bar{m})$ is a connection for any rational number κ. Its torsion is defined by $\tau\circ S^{\kappa}\circ Ddf\circ\tau\circ Ddf^{-1}\circ S^{-\kappa}$ where $f\in\mathcal{G}$. But since $\tau\circ S^{\kappa}=S^{-\kappa}\circ\tau$ this is just a restriction of $S^{-\kappa}\circ S\circ S^{-\kappa}=S^{1-2\kappa}$. The torsion of $S^{\kappa}\circ D(\bar{m})$ is therefore $S^{1-2\kappa}$.

Thus given any connection on a tangent bundle (and we know one exists) we may define its linear part which is a linear connection having torsion S and another connection $S^{1/2}\circ D(\bar{m})$ which is symmetric. Thus a symmetric connection always exists and is linear.

§13 Lie groups

This is a very short section and contains only a few remarks and no theorems. The particular manifolds considered, however, give good examples of some of the definitions and theorems we have used.

<u>§13.1</u> Let $G(R^r,\mathfrak{G})$ be a Lie group and let $\lambda: G\times G \Rightarrow G$ be defined by $\lambda(g_1,g_2)=g_1\cdot g_2$. Then $G(R^r,\mathfrak{G})$ is parallelisable.

Let $\mu: G\times R^r \Leftrightarrow T(G)$ be the strip map of T(G) defined in §6.5 and $\mathbb{H}$ the trivial substructure defined from it. Then $T^*(\mathbb{H})$ is a $\mathbb{H}$-connection having some special properties. It is of course semi-integrable and therefore flat [§10.6].

<u>SCHOLIUM</u> Consider $\gamma_1,\gamma_2 \in T_g(G)$ then $D_{\gamma_1\gamma_2}=\tau(\gamma_2)\cdot g^{-1}\cdot\gamma_1$.

<u>PROOF</u> Let $o: M \hookrightarrow T(M)$ be the cross section of T(M) associated with the zero tangents. Then $do(\gamma)=\tau(\gamma)\in T^*(G)$ [§8.3].

Now by definition $\mu=d\lambda\circ(o,\mu_e)$ and therefore $d\mu=dd\lambda\circ(do,d\mu_e)$ [§6.5 a)]. Also $\mu_g=d\lambda_g\circ\mu_e$ and thus $\mu_g^{-1}(\gamma_1)=\mu_e^{-1}(g^{-1}\cdot\gamma_1)$. Hence

$$\begin{aligned} D_{\gamma_1\gamma_2} &= D\mu(\gamma_2,\mu_g^{-1}(\gamma_1)) \\ &= d\mu(\gamma_2,\mu_g^{-1}(\gamma_1)) \\ &= d\mu(\gamma_2,\mu_e^{-1}(g^{-1},\gamma_1)) \\ &= dd\lambda(do(\gamma_2),g^{-1}\cdot\gamma_1) \qquad [§7.5] \\ &= dd\lambda(\tau(\gamma_2),g^{-1}\cdot\gamma_1)=\tau(\gamma_2)\cdot g^{-1}\cdot\gamma_1. \end{aligned}$$

<u>§13.2</u> Now suppose $\gamma_1, \gamma_2 \in T_e(G)$ then let the cross sections ξ_1 and ξ_2 of $T(G)$ be defined by $\xi_1(g)=g\cdot\gamma_1$ and $\xi_2(g)=g\cdot\gamma_2$. That is $\xi_1=d\lambda_{\gamma_1}\circ o$ where $d\lambda_{\gamma_1}(\gamma)=\gamma\cdot\gamma_1$ and similarly $\xi_2=d\lambda_{\gamma_2}\circ o$.

Since $d\lambda_{\gamma_1}(\Gamma)=\Gamma\cdot\gamma_1$ and $do(\gamma)=\tau(\gamma)$ then $d\xi_1(\gamma)=\tau(\gamma)\cdot\gamma_1$ [§8.3]. Thus $d\xi_1\circ\xi_2(g)=g\cdot\tau(\gamma_2)\cdot\gamma_1$.

Now the "difference" between the maps $\tau\circ d\xi_1\circ\xi_2$ and $d\xi_2\circ\xi_1$ corresponds to the Poisson bracket operation. We will show this later in detail [§14.2]. Here it is better to express that "difference" by means of the group operation. In fact

$$\{d\xi_2\circ\xi_1(g)\}^{-1}\{\tau\circ d\xi_1\circ\xi_2(g)\}=\gamma_2^{-1}\cdot\tau(\gamma_1)^{-1}\cdot\gamma_2\cdot\tau(\gamma_1)=\Gamma$$

which is independent of $g\in G$. Since $\gamma_1, \gamma_2 \in T_e(G)$ it is not difficult to see that $\Gamma\in T_e^*(T(G))\cap T_e(T(G))=T_e^{\#}(G)$ [§8.2].

If $\sigma: T^{\#}(G)\Longleftrightarrow T(G)$ is the solder the element $\sigma(\Gamma)\in T_e(G)$ thus associated with γ_1 and $\gamma_2\in T_e(G)$ corresponds to the Poisson bracket of these elements [17 p102].

<u>§13.3</u> In the paper on holonomy groups Ambrose and Singer define an isomorphism between the Lie algebra of the group of a differentiable principal bundle $\mathfrak{B}(M,G,G,\mathfrak{H})$ and the Lie algebra of vertical vector

fields". This isomorphism they call q [23].

A "vertical vector" is just an element of the subbundle $T^*_M(\Phi) \subset T(\Phi)$, a "horizontal vector" being an element of $D(\Phi)$ [§7.9]. The isomorphism q maps the cross section ξ of T(G) defined by $\xi(g)=g\cdot\gamma$ into the cross section of $T(\Phi)$ defined by $\xi^*(\phi_x)=d\phi_x\circ\gamma$ where $\gamma \in T_e(G)$.

This example shows how simple some of the concepts of this subject might become using the definitions and theorems of this thesis.

§14 Integrability conditions

This section contains no original work being just a summary and interpretation of what is already known about integrability conditions of three particular substructures. All three have been mentioned before.

Although we speak of integrability conditions it is not meant that these conditions are sufficient for the substructure to be integrable, although they are necessary. This will be so if the manifold is analytic but not always in other cases [18].

Let $M(R^m, \mathcal{G})$ be a manifold of class $\geqslant 2$ and let

$T(M)(M,R^m,G,\Phi)$ be a reduction of its tangent bundle. We will consider three cases i) a Riemannian metric ii) a field of k-planes and iii) a nearly complex structure.

§14.1 If $G=O(m)$ then the structure Φ defines a Riemannian metric over M [§3.5]. A Φ-connection is clearly a connection that preserves this metric in the classical sense.

It is well known that there exists a unique symmetric Φ-connection and the integrability conditions are that this connection be semi-integrable [3 p48, 29 p31].

I do not know any very simple method of defining this symmetric Φ-connection using the methods of this thesis and thus these integrability conditions do not suggest any solution to the general problem.

§14.2 If G is the group that leaves $R^k \times 0 \subset R^m$ invariant then Φ defines a field of k-planes [§3.4]. Let us call this subbundle B. A Φ-connection is one in which these k-planes are parallel in the classical sense.

The well known integrability conditions are just that $T^*_B(B)$ be symmetric. The proof of this fact illustrates a number of useful points.

SCHOLIUM The well known conditions for the field of k-planes B to be integrable are equivalent to the condition that $T^*_B(B)$ be symmetric [17 p87].

PROOF Consider $f \in \mathcal{O}^t, f: E \Leftrightarrow U$, and a regular strip map ϕ of $\mathfrak{F}$, $\phi: U \times R^m \Leftrightarrow T(U)$. Let $\xi_1: U \leftrightarrow B_U$ and $\xi_2: U \leftrightarrow B_U$ be any two differentiable cross sections. Put $\xi_1(x) = df(a, X_1) = \phi(x, \omega \cdot X_1)$ defining X_2 similarly where $\omega \in L(m)$. Then $df^{-1} \circ \xi_1 \circ f: E \leftrightarrow T(E)$ is such that $df^{-1} \circ \xi_1 \circ f(a) = (a, X_1)$.

Therefore considering $T(T(E))$ as the product bundle $T(E) \times T(R^m)$ $ddf \circ d\xi_1 \circ df(a, X_2) = ((a, X_2), (X_1, \frac{\partial X}{\partial a}1 \cdot X_2))$ where $\frac{\partial X}{\partial a}1$ is a Jacobian matrix [§4.1]. Hence

$$d\xi_1 \circ \xi_2(x) = ddf((a, X_2), (X_1, \frac{\partial X}{\partial a}1 \cdot X_2))$$

and $d\xi_2 \circ \xi_1(x) = ddf((a, X_1), (X_2, \frac{\partial X}{\partial a}2 \cdot X_1))$.

so that

$$\tau \circ d\xi_1 \circ \xi_2(x) - d\xi_2 \circ \xi_1(x)$$
$$= ddf((a,0), (X_2, \frac{\partial X}{\partial a}1 \cdot X_2 - \frac{\partial X}{\partial a}2 \cdot X_1))$$
$$= ddf((a,0), (X_2, [X_1, X_2])) \qquad [17\ p83]$$
$$= d\phi(x, (\omega \cdot X_2, \omega \cdot [X_1, X_2])) \qquad [§6.7]$$

Now since $B_U = \phi(U \times (R^k \times 0))$ then $T(B_U) = d\phi(T(U) \times T(R^k \times 0))$ and $\tau \circ d\xi_1 \circ \xi_2(x) - d\xi_2 \circ \xi_1(x)$ is in $T(B)$ if and only if $\omega \cdot [X_1, X_2] \in R^k \times 0$. That is, if $[X_1, X_2] \in \omega^{-1}(R^k \times 0)$ and the classical integrability conditions are just that this is true for all local cross sections ξ_1 and ξ_2. Thus we have established that the integrability conditions are equivalent to $\tau \circ d\xi_1 \circ \xi_2(x) - d\xi_2 \circ \xi_1(x)$ being in T(B) for all $x \in U$ and all cross sections of B.

Now since $d\xi_2 \circ \xi_1 : U \longleftrightarrow T(B)$ anyway and since T(B) is linear, this is equivalent to $\tau \circ d\xi_1 \circ \xi_2 : U \longleftrightarrow T(B)$ for all differentiable local cross sections of B. But it is easy to see that the union of the images of such maps as $d\xi_1 \circ \xi_2$ is just $T^*_B(B)$ so that the condition is equivalent to $\tau\{T^*_B(B)\} \subset T(B)$.

Now $\tau\{T(B)\} = T^*_B(T(M))$ [§8.1 b]. So that we finally have that the integrability conditions are equivalent to $\tau\{T^*_B(B)\} \subset T^*_B(B)$ which is the condition for $T^*_B(B)$ to be symmetric.

If B'(M,Y,H,$\mathfrak{M}$) is any differentiable fibre bundle of class $\geq$ 2 and if D(Φ) is a differentiable linear connection this theorem says that the integrability conditions for the field of m-planes D(B') is that D(D(B')) be symmetric [§10.2]. Thus the integrability condition for D(Φ) is that D(D(Φ)) is symmetric, which implies immediately that the connection is flat [§12.4].

<u>§14.3</u> In a recent paper Willmore has given a proof that there exists a symmetric Φ-connection if and only if these integrability conditions are satisfied, that is, if $T^*_B(B)$ is symmetric [27]. It is in fact easy to verify that if there exists a symmetric Φ-connection $T^*_B(B)$ must be symmetric but I have not found a direct proof of the converse.

<u>§14.4</u> If $G=CL(k)\subset L(2k)=L(m)$ the structure Φ is called a nearly complex structure [§3.3]. Suppose the element $i\mathbf{1}\in CL(k)$, where $\mathbf{1}$ is the unit matrix, corresponds to $I\in L(2k)$. Then if $\phi:U\times R^m \Longrightarrow T(U)$ is a strip map of Φ, the map $\phi\circ(\epsilon,I)\circ\phi^{-1}$ is independent of ϕ and defines an isomorphism $j:\Phi\Longrightarrow\Phi$ or, by an abuse of language, $j:T(M)\Longrightarrow T(M)$. $j(\phi_x)=\phi_x\circ I$ for all $\phi_x\in\Phi$ since $I^2=\mathbf{1}$ then $j\circ j=j^2=\epsilon$.

This isomorphism j may be taken to define Φ. It corresponds to the tensor referred to by Eckmann and others [18, 19].

A Φ-connection is one for which the covariant derivative of this tensor vanishes. To see this we consider $T(L(m))$ as the group of matrices $D(\dot{m},m)$ [§6.7]. Since $CL(k)$ is the normaliser of I then $T(CL(k))$ is the normaliser of the partitioned matrix $\begin{bmatrix} I & \cdot \\ \cdot & I \end{bmatrix}$ which is just $I \in T(L(m))$ [§6.6].

Now the condition $D(\Phi) \subset T^*(\Phi)$ implies that for any regular strip map $\phi: U \times G \Longrightarrow \Phi$

$$d\phi^{-1} \circ D\phi : T(U) \times G \longrightarrow T(U) \times T(G) \qquad [§12.3\ a)]$$

that is $d\phi_\xi^{-1} \circ D\phi_\xi \in T(G)$ for all $\xi \in T_x(M)$ which again implies that

$$D\phi_\xi^{-1} \circ dj \circ D\phi_\xi = D\phi_\xi^{-1} \circ d\phi_\xi \circ dI \circ d\phi_\xi^{-1} \circ D\phi_\xi = dI.$$

Now let $f: E \Longrightarrow U$, $f \in \mathcal{U}$, then the matrix $\partial f_x^{-1} \circ j \circ \partial f_x = J(x)$ is the component of the tensor j with respect to the coordinate system f at x. $J: U \longrightarrow L(m)$. We put now $\Gamma(\xi) = D\partial f_\xi^{-1} \circ d\partial f_\xi$ and this corresponds to the Chrystoffel symbols.

Since $D\phi_\xi^{-1} \circ D\partial f_\xi = \phi_x^{-1} \circ \partial f_x$ for all $\xi \in T_x(M)$ the equation

$$D\phi_\xi^{-1} \circ dj \circ D\phi_\xi = dI$$

may be written $Ddf_\xi^{-1} \circ dj \circ D\partial f_\xi = D\partial f_\xi^{-1} \circ D\phi_\xi dI \circ D\phi_\xi^{-1} \circ D\partial f_\xi$

or $\Gamma(\xi) \circ d\partial f_\xi^{-1} \circ dj \circ d\partial f_\xi \circ \Gamma(\xi)^{-1} = \partial f_x^{-1} \circ \phi_x \circ I \circ \phi_x^{-1} \circ \partial f_x$

or $\Gamma(\xi) \circ dJ(\xi) \circ \Gamma(\xi)^{-1} = J(x)$

This last equation is obviously an immediate interpretation of the vanishing of the covariant derivative of the tensor.

<u>§14.5</u> Since $j^2 = \epsilon$ then of course $dj^2 = \epsilon$ but $({}_T \circ dj)^2$

cannot be the identity map unless $j=\epsilon$ because if $p:T(M)\Rightarrow M$ and $q:T(T(M))\Rightarrow T(M)$ are the bundle projections $p\circ j=p$. Thus

$$q\circ\tau\circ dj\circ\tau\circ dj=dp\circ dj\circ\tau\circ dj \qquad [\S 8.1\ b)]$$
$$=dp\circ\tau\circ dj$$
$$=q\circ dj=j\circ q$$

But it is possible nevertheless that $(\tau\circ dj)^4=\epsilon$. In fact using the same methods as in the Scholium of §14.2 it is not difficult to verify that the classical integrability conditions are equivalent to $(\tau\circ dj)^4=\epsilon$ [18, 19].

<u>§14.6</u> Eckmann has shown very recently that there exists a Φ-connection such that its torsion $S=(\tau\circ dj)^4$ and further that there exists a symmetric Φ-connection if and only if the integrability conditions are satisfied [19]. That is if $(\tau\circ dj)^4=\epsilon$.

<u>§14.7</u> These three cases do not generalize very easily. There is however one property common to all three and that is the existence of a symmetric Φ-connection. It might be possible then that if any substructure Φ is integrable there exists a symmetric Φ-connection, but a proof seems difficult.

The case when G is the quarternionic group has yet to be examined [20]. It would be interesting to know how a Φ-connection would be expressed in the classical theory in this case and it would not be surprising if, using this classical interpretation of a Φ-connection, it were shown that the integrability conditions were equivalent to the existence of a symmetric Φ-connection.

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