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### *Conjugacy classes of parabolic subalgebras in complex semi-simple lie algebras*

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## ABSTRACT

For a complex semi-simple Lie algebra  $\mathfrak{g}$ , Richardson's dense orbit theorem gives a map between conjugacy classes of parabolic subalgebras in  $\mathfrak{g}$  and conjugacy classes of nilpotent elements. Unfortunately, this map is not surjective, in general, and hence does not give a direct classification of the nilpotent conjugacy classes in  $\mathfrak{g}$ . Despite this, the theorem is used by Bala and Carter to produce an indirect classification of the nilpotent conjugacy classes in  $\mathfrak{g}$ .

The map is not injective, either, and this thesis attempts to discover a necessary and sufficient condition for two parabolic subalgebras to give the same nilpotent conjugacy class under the above map. Springer conjectured that associated parabolics would give the same nilpotent conjugacy class. The problem was also raised in another form by Dixmier in his work concerning the distribution of nilpotent polarisable elements in  $\mathfrak{g}$ . He conjectured a generalisation of Kostant's results on the regular nilpotent elements. We prove both these conjectures correct, the method of proof being inspired by Dixmier's work.

Unfortunately, the necessary and sufficient condition is clearly more complicated than this and we give two examples (one trivial, one non-trivial) of non-associated parabolics giving the same nilpotent conjugacy class under Richardson's map

Conjugacy classes of parabolic subalgebras  
in complex semi-simple Lie algebras.

M Sc thesis by D.S Johnston.

To the memory of my parents.

24 NOV 1976

## INTRODUCTION

Let  $G$  be a simple algebraic group over an algebraically closed field  $k$ . Let  $P$  be a parabolic subgroup of  $G$  (i.e. a subgroup of  $G$  that contains a maximal connected solvable subgroup).  $P$  decomposes as a semi-direct product  $P = L.U$  where  $L$  is a reductive subgroup of  $G$  (called a Levi subgroup) and  $U$  is the maximal normal subgroup of  $P$  consisting of unipotent elements (the unipotent radical of  $P$ ).

One central theorem in the classification of the conjugacy classes of unipotent elements in  $G$  by Bala and Carter<sup>1</sup> is the theorem of Richardson<sup>12</sup> which states that there exists an element  $u$  in  $U$  such that the  $P$ -conjugacy class of  $u$  is open, dense in the unipotent radical  $U$ . This theorem gives a map between conjugacy classes of parabolic subgroups of  $G$  and conjugacy classes of unipotent elements in  $G$ , by associating to each parabolic subgroup  $P$  the conjugacy class of its dense orbit elements. In general, this map is neither injective nor surjective. All unipotent conjugacy classes in  $G$  do not arise as the dense orbit of a parabolic subgroup on its unipotent radical and two non-conjugate parabolic subgroups can give rise to the same unipotent conjugacy class. The problem we are concerned with, is to discover a condition for two non-conjugate parabolic subgroups to give the same unipotent conjugacy class.

Two parabolic subgroups  $P$  and  $Q$  are said to be associated if their Levi subgroups are conjugate. Springer conjectured that associated parabolic subgroups have conjugate dense orbits and this conjecture is proved

correct in a paper<sup>8</sup> produced in collaboration with R.W. Richardson. In  $SL(n, K)$ , it follows that the map introduced above is a bijection between classes of associated parabolic subgroups and unipotent conjugacy classes. Unfortunately, in general, there are non-associated parabolic subgroups which have conjugate dense orbits and two examples of this situation, one trivial and one non-trivial, are given in this thesis.

In this thesis, the original method of proof is used, in the case where  $\mathfrak{g}$  is a complex semi-simple Lie algebra and  $G$  its adjoint group. In the paper (attached as an appendix) the result is proved in general by a neater proof. The method of proof was inspired by Dixmier's work on polarisations<sup>4</sup>, in connection with the classification of infinite dimensional irreducible representations of finite dimensional Lie algebras.

Let  $\mathfrak{g}$  be a complex semi-simple Lie algebra with  $G$  its adjoint group. Then  $G$  acts on  $\mathfrak{g}$  via the adjoint action. Let  $P$  be a parabolic subgroup of  $G$ , with Lie algebra  $\mathfrak{p}$ .  $\mathfrak{p}$  decomposes as a direct sum  $\mathfrak{l} + \mathfrak{u}$ , corresponding to the Levi decomposition  $P = L.U$  of  $P$ . Richardson's theorem, in this case, states that there exists an element  $e$  in  $\mathfrak{u}$  such that the  $P$ -orbit of  $e$  (under the adjoint action) is open, dense in  $\mathfrak{u}$ .

Ozeki and Wakimoto<sup>11</sup> have proved firstly, that all polarisations are parabolic subalgebras (i.e. the Lie algebras of parabolic subgroups in  $G$ ), and secondly that any nilpotent element  $e$  which admits  $\mathfrak{p}$  as a polarisation, belongs to the

dense orbit of  $P$  acting on the nilpotent radical  $\underline{u}$ .

Dixmier<sup>4</sup> describes elements of  $\underline{g}$  which admit a polarisation as limits of certain semi-simple elements. In particular, this gives a way of describing a nilpotent element in the "dense orbit" as a limit of certain semi-simple elements. This approach enables one to prove Springer's conjecture and also to answer certain questions raised by Dixmier as to the distribution of those nilpotent elements, which admit a polarisation.

Finally I would like to thank my supervisor Professor R.W. Richardson for all his help over the past two years. I also acknowledge my source of finance, the Science Research Council.

## CHAPTER ONE

Let  $\underline{g}$  be a semi-simple Lie algebra over the complex field.

Let  $G$  be its adjoint group.  $G$  acts on  $\underline{g}$  via the adjoint representation.

### 1.1) Definition

A Borel subalgebra of  $\underline{g}$  is a maximal solvable subalgebra of  $\underline{g}$ .

### 1.2) Definition

A parabolic subalgebra of  $\underline{g}$  is any subalgebra containing a Borel subalgebra.

Let  $\underline{p}$  be a parabolic subalgebra of  $\underline{g}$ . It is well known that  $\underline{p}$  decomposes as a direct sum  $\underline{l} + \underline{n}$  where  $\underline{l}$  is a reductive subalgebra (a Levi subalgebra) and  $\underline{n}$  is an ideal in  $\underline{p}$  (the nilpotent radical).

### 1.3) Theorem (Richardson)

- i) There exists an  $e$  in  $\underline{n}$  such that  $e$  belongs to only a finite number of conjugates of  $\underline{n}$ .
- ii) Let  $G.\underline{n} = \{ g.e : \text{for all } g \text{ in } G, e \text{ in } \underline{n} \}$ . Then  $G.\underline{n}$  is a closed subvariety of  $\underline{g}$  of dimension equal to  $\dim \underline{g} - \dim \underline{p}/\underline{n}$ .
- iii)  $G.\underline{n}$  contains a unique dense class  $C$  of the same dimension as its own. In this case,  $C \cap \underline{n}$  is dense in  $\underline{n}$  and forms a single class under  $P$ .

### Proof

See<sup>12</sup> Prop 6 p. 23.

### Remark

Throughout this thesis, underlined lower case letters will denote subalgebras of  $\underline{g}$  and the corresponding upper

case letters will denote the corresponding subgroup of  $G$ .  
 e.g.  $\underline{p}$  and  $P$  above.

Choose a Cartan subalgebra  $\underline{h}$  of  $\underline{g}$  and let  $\overline{\Phi}$  denote the root system of  $\underline{g}$  with respect to  $\underline{h}$

Choose  $\Pi$  a fundamental system of  $\overline{\Phi}$  and let  $J$  be a subset of  $\Pi$ .

Let  $\overline{\Phi}^+$  denote the set of positive roots of  $\overline{\Phi}$  determined by the choice of  $\Pi$  and let  $\overline{\Phi}_J$  denote those roots which are integral linear combinations of the elements of  $J$ .

Let  $\underline{g}_r$  denote the 1-dimensional root subspace of  $\underline{g}$  corresponding to the root  $r$  in  $\overline{\Phi}$ .

$\underline{g}$  has a direct sum Cartan decomposition

$$\underline{g} = \underline{h} + \sum_{r \in \overline{\Phi}} \underline{g}_r$$

Let  $\underline{p}_J = \underline{h} + \sum_{r \in \overline{\Phi}_J} \underline{g}_r + \sum_{r \in \overline{\Phi}^+ \setminus \overline{\Phi}_J} \underline{g}_r$ . This is the standard

parabolic subalgebra associated to the subset  $J$ .

Every parabolic subalgebra of  $\underline{g}$  is conjugate to a unique standard parabolic subalgebra  $\underline{p}_K$ , for some subset  $K$  of  $\Pi$ .

In this case,  $\underline{n}_J = \sum_{r \in \overline{\Phi}^+ \setminus \overline{\Phi}_J} \underline{g}_r$  is the nilpotent radical and one

can take  $\underline{l}_J = \underline{h} + \sum_{r \in \overline{\Phi}_J} \underline{g}_r$  as the Levi subalgebra in the Levi decomposition.

We now define a partition of the semi-simple elements as follows:

Let  $\underline{h}_{\overline{\Phi}_J} = \left\{ \begin{array}{l} X \text{ in } \underline{l}_J : r(X) = 0 \text{ } r \text{ in } \overline{\Phi}_J \\ \neq 0 \text{ otherwise } \end{array} \right\}$

#### 1.4) Definition

The cone  $\underline{g}_{\overline{\Phi}_J}$  associated to  $J$ , is the  $G$ -orbit of the

subspace  $\mathfrak{h}_{\Phi_J}$

If  $Y \in \mathfrak{g}$ , let  $\mathfrak{g}^Y$  denote the centraliser of  $Y$  in  $\mathfrak{g}$

1.5) Definition

The extended cone  $\mathfrak{g}_{\Phi_J}^{\sim}$  equals

$\{ Y \in \mathfrak{g}_{\Phi_J} \mid \dim \mathfrak{g}^Y = \ell + |\Phi_J| \}$ , where  $\ell$  is the rank of  $\mathfrak{g}$ .

1.6 Remark

If  $Y \in \mathfrak{h}_{\Phi_J}$ , then  $\mathfrak{g}^Y = \mathfrak{h} + \sum_{r \in \Phi_J} \mathfrak{g}_r = \mathfrak{h}_J$

1.7 Lemma

The cones  $\mathfrak{g}_{\Phi_J}$  form a partition of the semi-simple elements of  $\mathfrak{g}$

Proof

Let  $s$  be a semi-simple element of  $\mathfrak{g}$ .

Up to conjugacy, one can assume that  $s \in \mathfrak{h}$

Let  $\bar{\Phi} = \{ r \in \Phi : r(s) = 0 \}$

$\bar{\Phi}$  is a root system. (see Bourbaki<sup>2</sup> cor. p 145).

By Bourbaki<sup>2</sup> p. 165 prop 24, one can choose a fundamental system  $\bar{\Pi}'$  of  $\bar{\Phi}$  such that there exists a subset  $J$  of  $\bar{\Pi}'$  which forms a fundamental system for  $\bar{\Phi}$ .

Up to conjugacy, one can assume that  $\bar{\Pi}' = \bar{\Pi}$  and that  $s \in \mathfrak{g}_{\bar{\Phi}_J}$

Two subsets  $J, K$  of the fundamental roots can give rise to the same cone  $\mathfrak{g}_{\bar{\Phi}_J}$ . We will now derive a condition for this to happen.

1.8) Definition

Two parabolic subalgebras  $\mathfrak{p}$  and  $\mathfrak{q}$  are associated if their Levi subalgebras are conjugate, under  $G$ .

### 1.9 Lemma

Let  $\underline{l}_1, \underline{l}_2$  be two Levi subalgebras of  $\underline{g}$  containing  $\underline{h}$ . Let their root systems be  $\overline{\Phi}_1$  and  $\overline{\Phi}_2$ . Then  $\underline{l}_1, \underline{l}_2$  are conjugate under  $G$  if and only if  $\overline{\Phi}_1$  and  $\overline{\Phi}_2$  are conjugate under the Weyl group  $W$ .

#### Proof

If  $\overline{\Phi}_1, \overline{\Phi}_2$  are conjugate under  $W$ , then clearly,  $\underline{l}_1, \underline{l}_2$  are conjugate under  $G$ .

Suppose that  $x.\underline{l}_1 = \underline{l}_2$  for some  $x$  in  $G$ . Then  $x.\underline{h}$  and  $\underline{h}$  are Cartan subalgebras in  $\underline{l}_2$ , hence there exists a  $y$  in  $L_2$  such that  $x.\underline{h} = y.\underline{h}$

So,  $y^{-1}x.\underline{h} = \underline{h}$  and  $y^{-1}x \in N_G(\underline{h})$ , the normaliser of  $\underline{h}$  in  $G$ .

Thus,  $y^{-1}x$  corresponds to an element  $w$  in  $W$  such that

$$w.\underline{l}_1 = y^{-1}x.\underline{l}_1 = y^{-1}\underline{l}_2 = \underline{l}_2$$

Let  $r \in \overline{\Phi}_1$  and let  $e_r \in \underline{l}_1$  be an element of the root space corresponding to  $r$ .

Then  $w.e_r = e_{w(r)}$ . So  $w(r) \in \overline{\Phi}_2$  i.e.  $w(\overline{\Phi}_1) \subseteq \overline{\Phi}_2$ .

Similarly,  $w^{-1}(\overline{\Phi}_2) \subseteq \overline{\Phi}_1$ , which proves the lemma.

### 1.10 Lemma

The following conditions are equivalent:

- 1)  $p_J$  is associated to  $p_k$
- 2)  $\mathfrak{g}_{\overline{\Phi}_J} = \mathfrak{g}_{\overline{\Phi}_k}$

#### Proof

Assume that  $p_J$  is associated to  $p_k$ .

By lemma 1.9, there exists a  $w$  in  $W$  such that  $w(\overline{\Phi}_J) = \overline{\Phi}_k$

Via the normal identification, one can get an action of the



## CHAPTER TWO

We will now digress to give a brief survey of Dixmier's work on the infinite dimensional irreducible representations of finite dimensional Lie algebras. The reference for this chapter is (3).

Let  $A$  be a ring.

### 2.1 Definition

An ideal  $P$  is primitive if  $P$  is the kernel of an irreducible representation of  $A$ .

Let  $\mathfrak{g}$  be any finite dimensional Lie algebra over an algebraically closed field. There is no satisfactory method of classifying the infinite dimensional irreducible representations of  $\mathfrak{g}$ .

Let  $U(\mathfrak{g})$  denote the universal enveloping algebra of  $\mathfrak{g}$ . Regard  $\mathfrak{g}$  as being contained in  $U(\mathfrak{g})$  via the Poincare-Birkhoff-Witt theorem.

To study the representation theory of  $\mathfrak{g}$ , one can study the representation theory of  $U(\mathfrak{g})$ , as the following proposition shows:

### 2.2 Proposition

Let  $V$  be a vector space,  $R$  (resp  $R'$ ) be the set of representations of  $\mathfrak{g}$  (resp  $U(\mathfrak{g})$ ) in  $V$ . For all  $\rho$  in  $R$ , there exists a unique  $\rho'$  in  $R'$  which extends  $\rho$  and the map  $\rho \mapsto \rho'$  is a bijection of  $R$  into  $R'$ .

#### Proof

See (3) Cor. 2.2.2 p. 73.

Dixmier proposed that, instead of attempting to classify the infinite dimensional irreducible representation of  $\mathfrak{g}$ , one

should determine the set of primitive ideals in  $U(\underline{g})$  (denoted by  $\text{Prim } U(\underline{g})$ ).

In order to study  $\text{Prim } U(\underline{g})$ , one needs the idea of an induced representation.

Let  $\underline{h}$  be a Lie subalgebra of  $\underline{g}$  and let  $\rho$  be a representation of  $\underline{h}$  with representation space  $W$ .

Let  $U(\underline{h})$  denote the universal enveloping algebra of  $\underline{h}$ .

Form  $V = U(\underline{g}) \otimes_{U(\underline{h})} W$  where one considers

$U(\underline{h})$  as acting  $U(\underline{g})$  by right multiplication.

$U(\underline{g})$  acts on  $V$ , via left multiplication.

Let  $\pi$  be the representation of  $\underline{g}$  corresponding to the action of  $U(\underline{g})$  on  $V$ . Then  $\pi$  is the representation of  $\underline{g}$  induced by  $\rho$ , denoted by  $\text{Ind}_{\underline{h}}^{\underline{g}}(\rho)$ .

For the case of  $\underline{g}$  a nilpotent Lie algebra, the problem of determining the structure of  $\text{Prim } U(\underline{g})$  has been completely solved, using polarisations.

Let  $\underline{g}^*$  denote the dual of  $\underline{g}$ .

Let  $f$  belong to  $\underline{g}^*$  and let  $\underline{h}$  be a Lie subalgebra of  $\underline{g}$ .

### 2.3 Definition

$\underline{h}$  is subordinate to  $f$  if  $f|_{\underline{h}}$  is a 1-dimensional representation of  $\underline{h}$ .

### 2.4 Definition

$\underline{h}$  is a polarisation of  $f$  if the dimension of  $\underline{h}$  is maximal among the set of subalgebras of  $\underline{g}$  subordinate to  $f$ .

Kirillov has proved the following programme, for  $\underline{g}$  a nilpotent Lie algebra, over an algebraically closed field of characteristic zero.

- i) for all  $f$  in  $\underline{g}^*$ , there exists a polarisation  $\underline{h}$  of  $f$ .

ii) given  $f, \underline{h}$  as in (i) then  $\text{Ind}_{\underline{h}}^{\mathfrak{g}} (f|_{\underline{h}})$  is irreducible.

Regard  $\text{Ind}_{\underline{h}}^{\mathfrak{g}} (f|_{\underline{h}})$  as a representation of  $U(\underline{\mathfrak{g}})$ , via 2.2. Denote the kernel of  $\text{Ind}_{\underline{h}}^{\mathfrak{g}} (f|_{\underline{h}})$  in  $U(\underline{\mathfrak{g}})$  by  $P(f, \underline{h})$  \*

iii)  $P(f, \underline{h})$  is independent of  $\underline{h}$

Call it  $P(f)$ .

There is, therefore, a map  $\text{Dix}: \mathfrak{g}^* \longrightarrow \text{Prim } U(\underline{\mathfrak{g}})$   
 $f \longmapsto P(f)$

iv) Dix is surjective

v)  $P(f_1) = P(f_2)$  if and only if  $f_1, f_2$  are in the same orbit for the adjoint group  $G$  acting on  $\mathfrak{g}^*$ .

### CHAPTER THREE

Let  $\mathfrak{g}$  be a complex semi-simple Lie algebra,  $G$  its adjoint group and let  $\mathfrak{g}^*$  denote the dual

Let  $f \in \mathfrak{g}^*$

Set  $\mathfrak{g}^f = \{ Y \text{ in } \mathfrak{g} : f([Y, Z]) = 0 \text{ for all } Z \text{ in } \mathfrak{g} \}$

#### 3.1 Definition

A polarisation  $\mathfrak{p}$  of  $f$  is a Lie subalgebra  $\mathfrak{p}$  of  $\mathfrak{g}$  such that

$$i) \quad f|_{[\mathfrak{p}, \mathfrak{p}]} = 0$$

$$ii) \quad \dim \mathfrak{p} = \frac{1}{2} (\dim \mathfrak{g} + \dim \mathfrak{g}^f)$$

#### Remark

Condition (i) is clearly equivalent to definition 2.3 above.

Condition (ii) is equivalent to definition 2.4 by Dixmier (3) p. 54.

Let  $B$  denote the Killing form of  $\mathfrak{g}$ .

As  $B$  is non-degenerate, one can identify  $\mathfrak{g}$  and  $\mathfrak{g}^*$  via  $B$  to get the following definition:

#### 3.2 Definition

Let  $X \in \mathfrak{g}$ .

A polarisation  $\mathfrak{p}$  of  $X$  is a Lie subalgebra  $\mathfrak{p}$  of  $\mathfrak{g}$  such that

$$i) \quad B(X, [p, p]) = 0$$

$$ii) \quad \dim \mathfrak{p} = \frac{1}{2} (\dim \mathfrak{g} + \dim \mathfrak{g}^X).$$

#### 3.3 Definition

An element  $X$  in  $\mathfrak{g}$  is called polarisable if it admits a polarisation.

The following result gives the connection between nilpotent polarisable elements and Richardson's dense orbit theorem (1.1).

### 3.4 Theorem (Ozeki and Wakimoto)

Let  $X \in \mathfrak{g}$  and let  $\mathfrak{p}$  be a subalgebra of  $\mathfrak{g}$

The following are equivalent:

i)  $\mathfrak{p}$  is a polarisation of  $X$

ii)  $\mathfrak{p}$  is a parabolic subalgebra of  $\mathfrak{g}$  and the space  $[\overline{X}, \mathfrak{p}]$  coincides with the nilpotent radical of  $\mathfrak{p}$ .

#### Proof

See (11) Thm 2.2 p. 447.

### 3.5 Corollary

The nilpotent polarisable elements are precisely those elements which occur as the dense orbits of parabolic subgroups on the nilpotent radicals of their corresponding subalgebras.

#### Proof

Let  $X$  be a nilpotent element in  $\mathfrak{g}$  and let  $\mathfrak{p}$  be a polarisation of  $X$ .

By 3.4,  $\mathfrak{p}$  is a parabolic subalgebra of  $\mathfrak{g}$  and the space  $[\overline{X}, \mathfrak{p}]$  coincides with the nilpotent radical  $\mathfrak{n}$  of  $\mathfrak{p}$ .

As  $\mathfrak{p}$  is self-normalising,  $[\overline{X}, \mathfrak{p}] \subseteq \mathfrak{p}$  implies that  $X \in \mathfrak{p}$ .

Write  $X = \mathfrak{l} + \mathfrak{n}$  where  $\mathfrak{l} \in \mathfrak{l}$  and  $\mathfrak{n} \in \mathfrak{n}$   
and  $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$  is a Levi  
decomposition for  $\mathfrak{p}$ .

$[\overline{X}, \mathfrak{p}] \subseteq \mathfrak{n}$  implies that  $[\mathfrak{l}, \mathfrak{l}] = 0$

i.e. that  $\mathfrak{l}$  belongs to the centre of  $\mathfrak{l}$  (\*)

$\mathfrak{l}$  is a reductive subalgebra and, hence, its radical is its centre  $\mathfrak{z}(\mathfrak{l})$ .

As  $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$  is a direct sum, the radical of  $\mathfrak{p} = \mathfrak{z}(\mathfrak{l}) + \mathfrak{n}$   
i.e. by (\*),  $X \in$  radical of  $\mathfrak{p}$ .

But  $X$  is nilpotent, so  $X \in \underline{n}$  ?  
 As  $[\underline{X}, \underline{p}] = \underline{n}$ ,  $\dim \underline{p}^X = \dim \underline{p} - \dim \underline{n}$   
 or  $\dim Z_{\underline{p}}(X) = \dim \underline{p} - \dim \underline{n}$   
 Where  $Z_{\underline{p}}(X) = \{ p \text{ in } P : p.X = X \}$   
 i.e.  $\dim P(X) = \dim \underline{n}$  or  $\overline{P(X)} = \underline{n}$

In order to study the polarisable elements of  $\underline{g}$ , Dixmier introduced the partition of the semi-simple elements as in Chapter I and proved the following result:

### 3.6 Theorem (Dixmier)

The polarisable elements in  $\underline{g}$  are precisely the union of the  $\underline{g}_{\underline{\Phi}_J}^{\sim}$  as  $J$  runs over all subsets of  $\Pi$ .

#### Proof

See (4) Prop 2.6

### 3.7 Definition

Let  $X \in \underline{g}$

$X$  is regular of  $\dim \underline{g}^X = l$  (the rank of  $\underline{g}$ ) ?

If  $J$  is the empty set,

then

- a) the cone  $\underline{g}_{\underline{\Phi}_J}$  is precisely the regular semi-simple elements
- b) the closed cone  $\overline{\underline{g}_{\underline{\Phi}_J}}$  is the whole of  $\underline{g}$
- and c) the extended cone  $\underline{g}_{\underline{\Phi}_J}^{\sim}$  is the regular elements of  $\underline{g}$

For this case, Kostant has proved the following:

### 3.8 Theorem (Kostant)

i) The set of regular nilpotent elements in  $\underline{g}$  is non-empty and forms a single orbit under the action of  $G$ .

ii) The set of all regular nilpotent elements is dense in the set of all nilpotent elements of  $\underline{g}$

#### Proof

See Kostant (9) Cor 5.5 p. 1000

Dixmier conjectured that this result holds for an arbitrary cone  $\mathfrak{g}_{\mathbb{F}_J}$ . We prove this conjecture correct.

## CHAPTER FOUR

Before proving Dixmier's conjecture which, as we shall see, also proves that associated parabolics have conjugate dense orbits, we will prove a technical lemma.

### 4.1 Lemma

Let  $\underline{p}$  be a parabolic subalgebra, let  $\underline{n} = \{ Y \text{ in } \underline{p} : B(Y, \underline{p}) = 0 \}$

then  $\underline{n}$  is the nilpotent radical of  $\underline{p}$ .

### Proof

As  $B$  is  $G$ -invariant, assume that

$$\underline{p} = \underline{h} \oplus \sum_{r \text{ in } \overline{\Phi}_J} \mathfrak{g}_r \oplus \sum_{r \text{ in } \overline{\Phi}^+ \setminus \overline{\Phi}_J} \mathfrak{g}_r$$

where  $J$  is a subset of  $\Pi$ , the fundamental roots.

Let  $e$  belong to  $\underline{n}$  and let

$$e = H + \sum_{r \text{ in } \overline{\Phi}_J} c_r e_r + \sum_{r \text{ in } \overline{\Phi}^+ \setminus \overline{\Phi}_J} c_r e_r$$

where

$$H \in \underline{h}, c_r \in \mathbb{C} \quad \text{and } e_r \in \mathfrak{g}_r$$

As  $e \in \underline{n}$ ,  $B(e, \mathfrak{g}_s) = B(e, \underline{h}) = B(e, \mathfrak{g}_r) = 0$   
for all  $s$  in  $\overline{\Phi}_J$ ,  $r$  in  $\overline{\Phi}^+ \setminus \overline{\Phi}_J$

Recall two orthogonality properties of the Cartan decomposition

- i)  $B(\mathfrak{g}_r, \mathfrak{g}_s) = 0$  if  $r + s \neq 0$
- ii)  $B(\mathfrak{g}_r, \underline{h}) = 0$  for all  $r$  in  $\overline{\Phi}$

See Humphrey's (7) p. 36.

By i) and ii) above

$$\begin{aligned} B(e, \mathfrak{g}_s) &= 0 && \text{if and only if } c_s = 0 \text{ for all } s \text{ in } \overline{\Phi}_J \\ B(e, \underline{h}) &= 0 && \text{if and only if } H = 0 \\ B(e, \mathfrak{g}_r) &= 0 && \text{for all } r \text{ in } \overline{\Phi}^+ \setminus \overline{\Phi}_J \end{aligned}$$

Hence  $e \in \underline{n}$  if and only if  $e$  belongs to the nilpotent radical of  $\underline{p}$ .

We are now in a position to prove our main result.

#### 4.2 Theorem

Let  $\mathfrak{g}_{\mathfrak{P}_J}$  be as above

- i) The nilpotent elements in  $\tilde{\mathfrak{g}}_{\mathfrak{P}_J}$  form a single  $G$ -orbit and this orbit is  $G(\Omega_J)$  where  $\Omega_J$  is the dense orbit of  $P_J$  acting on the nilpotent radical  $\underline{n}_J$ .
- ii) The nilpotent elements in  $\tilde{\mathfrak{g}}_{\mathfrak{P}_J}$  form a dense subset of the nilpotent elements in  $\bar{\mathfrak{g}}_{\mathfrak{P}_J}$ .

#### Remark

We will prove that the set of nilpotent elements in  $\bar{\mathfrak{g}}_{\mathfrak{P}_J}$  is precisely  $G(\underline{n}_J)$ .

#### Proof

Firstly, we will prove part i).

Let  $e$  be a nilpotent element of  $\tilde{\mathfrak{g}}_{\mathfrak{P}_J}$ .

Then there exists a sequence  $y_1, y_2, \dots, y_n, \dots$  tending to  $e$  ( $y_i$  in  $\mathfrak{g}_{\mathfrak{P}_J}$ )

As  $y_i \in \mathfrak{g}_{\mathfrak{P}_J}$ , set  $y_i = g_i(x_i)g_i$  in  $G$ ,  $x_i$  in  $\mathfrak{h}_{\mathfrak{P}_J}$

If  $X \in \mathfrak{h}_{\mathfrak{P}_J}$ , then  $[X, \mathfrak{p}_J] = \underline{n}_J$  and, by 3.4,  $\mathfrak{p}_J$  is a polarisation of  $X$ .

So,  $g_i(\mathfrak{p}_J)$  is a polarisation of  $g_i(x_i) = y_i$  for all  $i$ .

Consider the Grassmanian of  $[+\frac{1}{2}(|\mathfrak{P}_J| + |\bar{\mathfrak{P}}_J|)]$  planes in  $\mathfrak{g}$

As the Grassmanian is compact, one can consider a subsequence

$g_1(\mathfrak{p}_J), \dots, g_n(\mathfrak{p}_J), \dots$  which tends to a limit  $Q$ .

Clearly,  $Q$  is a polarisation of  $e$

3.1 ?

Consider the map  $\sigma : G/P_J \longrightarrow G(\mathfrak{p}_J)$

$$\bar{\mathfrak{g}} = \mathfrak{g}^{P_J} \longleftarrow \mathfrak{g}(\mathfrak{p}_J)$$

This is well defined and continuous

$P_J$  is a parabolic subgroup of  $G$ , so  $G/P_J$  is complete

By Mumford (10) Thm 2 p 114,

$G/P_J$  is compact in the Hausdorff topology.

So  $\bar{g}_1, \dots, \bar{g}_k, \dots$  has a convergent subsequence,

$\bar{g}_{i_1}, \dots, \bar{g}_{i_k}, \dots$  tending to a limit  $\bar{g}$ .

As  $\sigma$  is continuous,  $\sigma(\bar{g}_{i_1}), \dots, \sigma(\bar{g}_{i_k}), \dots$  tends to  $\sigma(\bar{g})$

i.e.  $Q = g(p_J)$  for some  $g$

By Cor. 3.5,  $e \in G(\Omega_J)$

To prove part ii) one proceeds as follows:

Let  $e$  be nilpotent in  $\bar{g}_{\mathfrak{g}_J}$

then

by similar arguments to part i) there exists a  $\underline{p}$ , conjugate to  $\underline{p}_J$  under  $G$ , such that

$$i) \quad B(e, [\underline{p}, \underline{p}]) = 0$$

and ii)  $e \in \underline{p}$

By lemma 4.1  $B(e, [\underline{p}, \underline{p}]) = B([\underline{e}, \underline{p}], \underline{p}) = 0$

implies that  $[\underline{e}, \underline{p}] \in \underline{n}$ , the nilpotent radical of  $\underline{p}$

By the proof of Cor 3.5,  $[\underline{e}, \underline{p}] \in \underline{n}$  and  $e$  nilpotent implies that  $e \in \underline{n}$ .

So, the nilpotent elements in  $\bar{g}_{\mathfrak{g}_J}$  are precisely  $G(\underline{n}_J)$ .

The result follows from Thm 1.3

#### 4.3 Corollary

Associated parabolic subalgebras have conjugate dense orbits.

#### Proof

Up to conjugacy, one can assume that the parabolic subalgebras are the "standard" subalgebras  $\underline{p}_J$  and  $\underline{p}_k$ .

The result follows by lemma 1.10 and thm 4.2

## CHAPTER FIVE

Unfortunately, non-associated parabolics can have conjugate dense orbits. The object of this chapter is to give two examples (one trivial and one non-trivial) of non-associated parabolics having conjugate dense orbits.

We will need to know when two parabolic subalgebras are associated in terms of their Dynkin diagrams of their root systems.

### 5.1 Lemma

Let  $\bar{\Phi}$  be an indecomposable root system. Two parabolic subsystems of  $\bar{\Phi}$  are equivalent under the Weyl group of  $\bar{\Phi}$  if and only if their Dynkin diagrams are the same, except that in  $D_{2m}$  ( $m \geq 2$ ) there are two non-conjugate systems of the type  $A_{l_1} + A_{l_2} + \dots + A_{l_k}$  where  $l_1, l_2, \dots, l_k$  are odd integers satisfying  $(l_1 + 1) + (l_2 + 1) + \dots + (l_k + 1) = 2m$ , and in  $E_7$  there are two non-conjugate systems of type  $A_3 + A_1, A_5$  and  $A_1$

(Note: it is understood that root lengths are taken into account when defining equality of Dynkin diagrams).

### Proof

See Dynkin (5) Thm 5.4 p. 146.

We will now consider the simple Lie algebra with root system of type  $B_l$ .

This can be represented by  $(2l + 1)$  by  $(2l + 1)$  complex matrices  $(a_{ij})$  such that

$$a_{ij} = -a_{2l+2-j, 2l+2-i}$$

On the antidiagonal (i.e. those  $a_{ij}$  such that

$l + j = 2l + 2$  the entries are zero.

Choose as a Cartan subalgebra the diagonal matrices in this model i.e. matrices of the form

$$\text{diag} \{ a_{11} \dots a_{ll} \ 0, \ -a_{11} \dots -a_{ll} \}$$

Choose the positive root vectors to be of the form

$$E_{ij} = E_{2l+2-j, 2l+2-i} \quad \text{where } j > 1, i + j = 2l + 2 \text{ and}$$

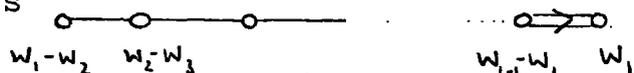
$E_{ij}$  is the matrix with 1 in the  $i^{\text{th}}$  row,  $j^{\text{th}}$  column and zeros elsewhere.

Given a diagonal matrix  $X$ , set  $w_i(X) = a_{ii}$   
 $1 \leq i \leq l$

Then the fundamental root system corresponding to the given positive root vectors is  $\{ w_1 - w_2, w_2 - w_3, \dots, w_{l-1} - w_l, w_l \}$

In this case,  $w_l$  is the short root and  $w_1 - w_2, w_2 - w_3, \dots, w_{l-1} - w_l$  are the long roots (all of equal length).

The Dynkin diagram is



Consider the case of the root system  $B_2$ . The two fundamental roots  $\{ \alpha_1, \alpha_2 \}$  are of unequal length. By lemma 5.1 the rank-1 parabolic subalgebras  $\mathfrak{p}_{\alpha_1}, \mathfrak{p}_{\alpha_2}$  determined by  $\alpha_1$  and  $\alpha_2$  are not associated.

However, it is well known that the rank-1 parabolics have a unique class of nilpotent elements as the conjugacy class of the dense orbits (the subregular elements). See Steinberg (13) p. 145. Thm 1.

To give a non-trivial example of non-associated parabolic sub-algebras which have conjugate dense orbits, we use a method of Gersterhaber (6) which gives, indirectly, a method of calculating the nilpotent conjugacy class which is the dense orbit of a given parabolic subalgebra.

In the model of  $B_L$  given above, parabolic subalgebras can be represented by square blocks positioned along the diagonal or, in effect, by partitions of  $2L+1$  of the form  $(\lambda_1, \dots, \lambda_r, \lambda_{r+1}, \lambda_r, \dots, \lambda_1)$ .

Example for  $B_2$

$$p_{\kappa_1} = \begin{array}{cccccc} * & * & | & * & * & \circ \\ * & * & | & * & \circ & * \\ \hline \circ & \circ & | & \circ & * & * \\ \circ & \circ & | & \circ & * & * \\ \circ & \circ & | & \circ & * & * \end{array}$$

The partition is  $(2, 1, 2)$ .

### 5.2 Lemma

If  $C, D$  are  $GL_{2l+1}$ -conjugate matrices contained in the model of  $B_l$  then they are conjugate under the adjoint group of  $B_l$ .

### Proof

See Gerstenhaber (6) Prop 2 p. 549.

Hence, conjugacy classes of nilpotent elements in  $B_l$  can be represented by partitions corresponding to the Jordan canonical form of their corresponding  $GL_{2l+1}$ -conjugacy class.

Gerstenhaber's method allows one to compute the partition corresponding to the dense orbit class from the partition corresponding to the parabolic subalgebra.

Let  $(p_1 \dots p_k)$  be a partition of  $(2l+1)$  corresponding to a parabolic subalgebra of  $B_l$ .

Reorder the partition to get  $(p'_1 \dots p'_k)$  such that  $p'_1 \geq p'_2 \geq \dots \geq p'_k$

Form the dual of this reordered partition to get  $Q = (q_1 \dots q_r)$ .

Then, perform an operation (orthogonalisation) on this partition to get a partition  $(Q)_0$

Orthogonalisation is defined recursively

If  $r = 1$  and  $q_1$  is odd, then  $(Q)_0 = (q_1)$

If  $r = 1$  and  $q_1$  is even, then  $(Q)_0 = (q_1 - 1, 1)$

For  $r > 1$  if  $q_1$  is odd then  $(Q)_0 = (q_1) + (q_2 \dots q_r)_0$

if  $q_1 = q_2$  then  $(Q)_0 = (q_1 q_2) + (q_3 \dots q_r)_0$

if  $q_1 > q_2$  and  $q_1$  even then  $(Q)_0 = (q_1 - 1) + (q_2 + 1, \dots, q_r)_0$

The partition obtained by this process gives the conjugacy class of the dense orbit of the parabolic subalgebra corresponding to the initial partition.

The example of non-associated parabolics having conjugate dense orbits is found in  $E_5$

Let  $\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5$  be the fundamental root system of  $E_5$  and consider the subsets  $\{\kappa_1, \kappa_2, \kappa_3, \kappa_5\}$  and  $\{\kappa_1, \kappa_2, \kappa_4, \kappa_5\}$ . One is of type  $A_3 + A_1$ , while the other is of type  $A_2 + B_2$ . Hence, by lemma 5.1 their corresponding parabolic subalgebras are not associated.

Consider first the subset of type  $A_3 + A_1$

This gives the partition  $(4, 3, 4)$ .

Reorder and form the dual to get  $(3, 3, 3, 2)$ .

Orthogonalise to get that the dense orbit partition is  $(3, 3, 3, 1, 1)$ .

Now consider the subset of type  $A_2 + B_2$

This gives the partition  $(3, 5, 3)$

Reorder and form the dual  $(3, 3, 3, 1, 1)$

Orthogonalise to get that the dense orbit partition is  
(3, 3, 3, 1, 1).

So  $\underline{p} \{ \kappa_1 \kappa_2 \kappa_3 \kappa_5 \}$  and  $\underline{p} \{ \kappa_1 \kappa_2 \kappa_4 \kappa_5 \}$  have conjugate  
dense orbits.

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APPENDIX

CONJUGACY CLASSES IN PARABOLIC SUBGROUPS OF  
SEMISIMPLE ALGEBRAIC GROUPS, II

By D.S. Johnston and R.W. Richardson

Let  $P$  be a parabolic subgroup of a connected semisimple algebraic group  $G$  and let  $U_P$  denote the unipotent radical of  $P$ . In another paper of the same title [7] it was shown that there exists  $u \in U_P$  such that  $C_P(u)$ , the  $P$ -conjugacy class of  $u$ , is an open subset of  $U_P$ ; the proof required the assumption that  $G$  has only a finite number of unipotent conjugacy classes, but G. Lusztig [5] has recently shown that this is always true. Let  $\Omega_P$  denote the open  $P$ -conjugacy class in  $U_P$ . Now let  $Q$  be another parabolic subgroup of  $G$  and let  $L$  (resp.  $M$ ) be a Levi subgroup of  $P$  (resp.  $Q$ ). The parabolic subgroups  $P$  and  $Q$  are associated if  $L$  and  $M$  are conjugate subgroups of  $G$ . In a conversation with one of the authors, T.A. Springer conjectured that if  $P$  and  $Q$  are associated parabolic subgroups, if  $u \in \Omega_P$  and  $v \in \Omega_Q$ , then  $u$  and  $v$  are conjugate in  $G$ . In this note we shall prove that this conjecture is correct. We shall also prove the analogous result for the Lie algebra  $\mathfrak{g}$  of  $G$ , subject to certain restrictions on the characteristic of the base field.

Our proof is based on an idea introduced by Dixmier [3] in connection with "polarizations" of an element in a complex semisimple Lie algebra. Roughly speaking, Dixmier's idea is to represent elements in  $\Omega_P$  as limits of  $P$ -conjugates of certain semisimple elements in the centre of the Levi subgroup  $L$ . Our main result allows us to answer several questions raised by Dixmier in [3]. The connection with Dixmier's paper is briefly discussed in the final section of the paper.

## 1. Proof of the Main Theorem

Our basic reference for algebraic groups and algebraic geometry is [1]. All algebraic groups and algebraic varieties are taken over an algebraically closed base field  $k$ .

Let  $G$  be a connected semisimple affine algebraic group, let  $T$  be a maximal torus of  $G$  and let  $B$  be a Borel subgroup of  $G$  which contains  $T$ . Let  $\Phi$  denote the set of roots of  $G$  with respect to  $T$ , let  $\Phi^+$  be the set of positive roots corresponding to  $B$  and let  $\Delta \subset \Phi^+$  be the corresponding set of simple roots. For each root  $\alpha \in \Phi$ , let  $U_\alpha$  denote the corresponding one-dimensional unipotent subgroup and let  $x_\alpha : k \rightarrow U_\alpha$  be an isomorphism of algebraic groups. If  $t \in T$  and  $c \in k$ , we have  $tx_\alpha(c)t^{-1} = x_\alpha(\alpha(t)c)$ .

If  $\theta$  is a subset of  $\Delta$ , we let  $[\theta]$  be the set of all roots which are linear combinations with integral coefficients of the elements of  $\theta$ . We set  $\pi_\theta = \Phi^+ \cup [\theta]$  and  $\tau_\theta = \{\alpha \in \Phi^+ \mid \alpha \notin [\theta]\}$ . Let  $P_\theta$  denote the "standard" parabolic subgroup corresponding to  $\theta$ ;  $P_\theta$  is generated by  $T$  and  $\{U_\alpha \mid \alpha \in \pi_\theta\}$ . Let  $L_\theta$  denote the "standard" Levi subgroup of  $P_\theta$  and let  $U_\theta$  denote the unipotent radical of  $P_\theta$ ;  $L_\theta$  (resp.  $U_\theta$ ) is generated by  $T$  and  $\{U_\alpha \mid \alpha \in [\theta]\}$  (resp. by  $\{U_\alpha \mid \alpha \in \tau_\theta\}$ ).  $P_\theta$  is the semi-direct product of  $L_\theta$  and  $U_\theta$ . Set  $Z_\theta = \{t \in T \mid \alpha(t) = 1 \text{ for every } \alpha \in [\theta]\}$ ;  $Z_\theta$  is the centre of  $L_\theta$ . We let  $Z_\theta'$  be the set of all  $z \in Z_\theta^0$  such that  $\alpha(z) \neq 1$  for every root  $\alpha \notin [\theta]$ ;  $Z_\theta'$  is a dense open subset of  $Z_\theta^0$ . For later use we note the following characterization of  $Z_\theta'$ :

$$1.1 \quad Z_\theta' = \{z \in Z_G(L_\theta)^0 \mid Z_G(z)^0 = L_\theta\}$$

The proof of 1.1 is elementary and will be omitted.

We let  $G$  act on  $G$  by conjugation. Thus, if  $x \in G$ , then  $G(x)$ , the  $G$ -orbit of  $x$ , is the conjugacy class  $C_G(x)$ . If  $X$  is a subset of  $G$  and  $H$  a subgroup of  $G$ , then  $H(X) = \{h x h^{-1} \mid h \in H \text{ and } x \in X\}$ .

LEMMA 1.2. Let  $z \in Z'_\theta$ . Then the centralizer of  $z$  in  $U_\theta$  is the trivial subgroup  $\{1\}$ .

Proof. Let  $v \in U_\theta$ . We may write  $v$  uniquely in the form  $v = \prod_{\alpha \in \tau_\theta} x_\alpha(c_\alpha)$ , where each  $c_\alpha \in k$  and the product is taken in a fixed (but arbitrary) order. We have  $z v z^{-1} = \prod_{\alpha \in \tau_\theta} x_\alpha(\alpha(z)c_\alpha)$ . If  $v \neq 1$ , then  $c_\alpha \neq 0$  for some  $\alpha$  and thus  $z v z^{-1} \neq v$ .

LEMMA 1.3. Let  $z \in Z'_\theta$ . Then  $P_\theta(z) = U_\theta z$ .

Note:  $U_\theta z$  denotes the set  $\{uz \mid u \in U_\theta\}$ , not the orbit  $U_\theta(z)$ .

Proof. Let  $x \in P_\theta$ . We may write  $x = vy$ , with  $v \in U_\theta$  and  $y \in L_\theta$ .

We have

$$x z x^{-1} z^{-1} = v y z y^{-1} v^{-1} z^{-1} = v z v^{-1} z^{-1} \in U_\theta.$$

Hence  $x z x^{-1} \in U_\theta z$  and  $P_\theta(z) = U_\theta(z) \subset U_\theta z$ . By Lemma 1.2, the orbit  $U_\theta(z)$  has dimension equal to  $\dim U_\theta z$ . By a theorem of Kostant and Rosenlicht (see [9, p. 35] for a proof) every orbit of a unipotent group on an affine variety is closed. Thus  $P_\theta(z) = U_\theta(z) = U_\theta z$ .

Remark. For  $k$  of characteristic zero, the Lie algebra analogue of Lemma 1.3 is proved by Dixmier in [3] and our proof is based on his.

We shall need the following elementary result...

1.4. Let the affine algebraic group  $H$  act morphically on the algebraic variety  $X$ , let  $P$  be a parabolic subgroup of  $H$  and let  $Y$  be a closed  $P$ -stable subset of  $X$ . Then  $H(Y)$  is a closed subset of  $X$ .

For a proof, see [9, p. 68].

LEMMA 1.5.  $G((Z'_\theta)^0 U_\theta)$  is the closure of  $G(Z'_\theta)$ .

Proof.  $(Z'_\theta)^0 U_\theta$  is the radical of  $P_\theta$ , in particular it is a normal subgroup of  $P_\theta$ . It follows from 1.4 that  $G((Z'_\theta)^0 U_\theta)$  is closed in  $G$ . By

1.3,  $Z_\theta' U_\theta \subset G(Z_\theta')$  and it is clear that  $Z_\theta' U_\theta$  is an open dense subset of  $(Z_\theta^0)U_\theta$ . This proves Lemma 1.5.

PROPOSITION 1.6. (i) Every unipotent element in  $G(Z_\theta U_\theta)$  is contained in  $G(U_\theta)$ . (ii)  $G(U_\theta)$  is closed and  $G(\Omega_{P_\theta})$  is open in  $G(U_\theta)$ .

Proof. Let  $x \in Z_\theta U_\theta$  and write  $x = zv$ ,  $z \in Z_\theta$  and  $v \in U_\theta$ . Let  $\pi : P_\theta \rightarrow P_\theta / U_\theta$  be the canonical homomorphism. If  $x$  is unipotent,  $\pi(x) = \pi(z)$  is both unipotent and semisimple, hence  $\pi(x) = e$  and  $x \in U_\theta$ . This proves (i). It follows from 1.4 that  $G(U_\theta)$  is closed. Since  $\Omega_{P_\theta}$  is an open subset of  $U_\theta$ ,  $G(\Omega_{P_\theta})$  is an open subset of  $G(U_\theta)$ .

THEOREM 1.7. Let  $P$  and  $Q$  be associated parabolic subgroups of  $G$  and let  $u \in \Omega_P$ ,  $v \in \Omega_Q$ . Then  $u$  and  $v$  are conjugate in  $G$ . Moreover  $G(U_P) = G(U_Q)$ .

Proof. After conjugating, we may assume that  $P = P_\theta$  and  $Q = P_\psi$ , where  $\theta$  and  $\psi$  are subsets of  $\Delta$ . The Levi subgroups  $L_\theta$  and  $L_\psi$  are conjugate in  $G$ . It follows from 1.1 that  $G(Z_\theta') = G(Z_\psi')$ . Hence, by Lemma 1.4,  $G((Z_\theta^0)U_\theta) = G((Z_\psi^0)U_\psi)$ . Proposition 1.6 (i) implies that  $G(U_\theta) = G(U_\psi)$  and 1.6 (ii) then shows that  $u$  and  $v$  are conjugate in  $G$ .

## 2. The analogue for Lie algebras

We denote the Lie algebra of an algebraic group  $G$ ,  $P_\theta$ ,  $U_P$ , etc., by the corresponding lower case German letter  $\underline{g}$ ,  $\underline{p}_\theta$ ,  $\underline{u}_P$ , etc.

Let  $G$  be a semisimple algebraic group of adjoint type. (In characteristic zero, one may equally well start with a semisimple Lie algebra  $\underline{g}$  and let  $G$  be the adjoint group of  $\underline{g}$ .) We assume that the characteristic of  $k$  is "good" for  $G$ . For definition and some consequences, see [8, pp. E-12 - E-19]; if  $\text{char}(k)$  is either 0 or  $>5$ , then it is good for every  $G$ . The assumption that  $\text{char}(k)$  is good for  $G$  implies in particular that  $\underline{g}$  has

only a finite number of nilpotent conjugacy classes [8, p.E-19]. Hence, by [7], if  $P$  is a parabolic subgroup of  $G$ , there exists an open  $P$ -orbit  $\Lambda_P$  on the Lie algebra  $\underline{u}_P$ .

Let the notation be as in section 1. If  $\alpha \in \Phi$ , then  $d\alpha : \underline{t} \rightarrow k$  denotes the differential of  $\alpha$ .

- 2.1. (i) The set  $\{d\alpha | \alpha \in \Delta\}$  is a basis of the dual space  $\underline{t}^*$ .  
(ii) If  $\theta \subset \Delta$  and if  $\gamma \in \tau_\theta$ , then the set  $\{d\alpha | \alpha \in \theta\} \cup \{d\gamma\}$  is a linearly independent subset of  $\underline{t}^*$ .

Proof. 2.1. (i) is an immediate consequence of the fact that  $G$  is of adjoint type. 2.1.(ii) follows from the definition of good characteristic and the following well-known result:

2.2. Let  $G$  be simple, let  $\Delta = \{\alpha_1, \dots, \alpha_r\}$ , let  $\alpha = \sum_{j=1}^r m_j \alpha_j$  be a positive root and let  $\beta = \sum_{j=1}^r n_j \alpha_j$  be the highest root. If a prime  $p$  divides one of the non-zero  $m_j$ 's, then  $p$  divides one of the  $n_j$ 's.

Let  $\underline{z}_\theta = \{x \in \underline{t} | d\alpha(x) = 0 \text{ for every } \alpha \in [\theta]\}$  and let  $\underline{z}'_\theta = \{z \in \underline{z}_\theta | d\alpha(x) \neq 0 \text{ for every } \alpha \in \tau_\theta\}$ .

- LEMMA 2.3. (i)  $\underline{z}_\theta$  is the Lie algebra of  $Z_\theta$  and is the centre of  $\underline{l}_\theta$ .  
(ii) If  $\theta \neq \Delta$ , then  $\underline{z}'_\theta \neq \{0\}$ . (iii)  $\underline{z}'_\theta = \{x \in \underline{l}_\theta | \underline{z}_g(x) = \underline{l}_\theta\}$ .

The proof of 2.3 follows easily from 2.1 and will be omitted. We remark that it does require both the assumption that  $G$  be of adjoint type and that the characteristic of  $k$  be good for  $G$ .

LEMMA 2.4. Let  $z \in \underline{z}'_\theta$ . Then  $P_\theta(z) = z + \underline{u}_\theta$ .

For  $k$  of characteristic zero, this is proved by Dixmier in [3]. With the assumptions made on  $G$  and  $\text{char}(k)$ , the same proof goes through in our case.

LEMMA 2.5.  $G(\underline{z}_\theta + \underline{u}_\theta)$  is the closure of  $G(\underline{z}'_\theta)$ .

PROPOSITION 2.5. (i) Every nilpotent element in  $G(\underline{z}_\theta + \underline{u}_\theta)$  is contained in  $G(\underline{u}_\theta)$ . (ii)  $G(\underline{u}_\theta)$  is closed in  $\underline{g}$  and  $G(\Lambda_{P_\theta})$  is open in  $G(\underline{u}_\theta)$ .

THEOREM 2.7. Let  $P$  and  $Q$  be associated parabolic subgroups of  $G$  and let  $u \in \Lambda_P$ ,  $v \in \Lambda_Q$ . Then  $u$  and  $v$  are conjugate in  $g$ . Moreover  $G(\underline{u}_P) = G(\underline{u}_Q)$ .

The proofs of Lemma 2.5, Proposition 2.6 and Theorem 2.7 are essentially the same as those of (respectively) Lemma 1.5, Proposition 1.6 and Theorem 1.7. We omit the details.

Now we can drop the assumption that  $G$  be of adjoint type.

THEOREM 2.8. Let  $H$  be a semisimple algebraic group and assume that the characteristic of the base field  $k$  is good for  $H$ . Let  $P$  and  $Q$  be associated parabolic subgroups of  $H$  and let  $u \in \Lambda_P$ ,  $v \in \Lambda_Q$ . Then  $u$  and  $v$  are conjugate in  $h$ . Moreover  $H(\underline{u}_P) = H(\underline{u}_Q)$ .

Proof. By Chevalley's theory of isogenies (see Exposes 19-24 of [2], in particular Expose 23, Théorème 1, pp. 23-04), there exists a semisimple group  $G$  of adjoint type and a central isogeny  $\eta : H \rightarrow G$ . Let  $N_{\underline{h}}$  (resp.  $N_{\underline{g}}$ ) denote the closed subvariety of all nilpotent elements of  $\underline{h}$  (resp.  $\underline{g}$ ). Since  $\eta$  is central, the kernel of  $d\eta : \underline{h} \rightarrow \underline{g}$  consists of semisimple elements. It follows easily from standard properties of the Jordan decomposition in  $\underline{h}$  and  $\underline{g}$  (see [1, p. 355]) that  $d\eta$  maps  $N_{\underline{h}}$  bijectively onto  $N_{\underline{g}}$ . Moreover, if  $V$  is a connected unipotent subgroup of  $H$ ,  $d\eta$  maps  $\underline{v}$  isomorphically onto the Lie algebra of  $\eta(V)$ .

Let  $P' = \eta(P)$  and  $Q' = \eta(Q)$ . One checks easily that  $P'$  and  $Q'$  are associated parabolic subgroups of  $G$  and  $U_{P'} = \eta(U_P)$ ,  $U_{Q'} = \eta(U_Q)$ . Since  $P'$  and  $Q'$  are associated, Proposition 2.7 gives that  $G(\underline{u}_{P'}) = G(\underline{u}_{Q'})$ . But

since  $(d\eta) \circ (\text{Ad } h) = (\text{Ad } \eta(h)) \circ (d\eta)$  for every  $h \in H$  and  $d\eta$  maps  $N_{\underline{h}}$  bijectively onto  $N_{\underline{g}}$ , we see immediately that  $H(\underline{u}_p) = H(\underline{u}_q)$ . Since  $H(u)$  (resp.  $H(v)$ ) is the only open orbit in  $H(\underline{u}_p)$  (resp.  $H(\underline{u}_q)$ ), we must have  $H(u) = H(v)$ . Thus  $u$  and  $v$  are conjugate.

Remarks. (a) For the groups  $SL_n(k)$ ,  $SO_n(k)$  and  $Sp_n(k)$  ( $\text{char}(k) \neq 2$  in the latter two cases), Theorems 1.7 and 2.8 are consequences of the results of Gerstenhaber in [4]. (b) For  $H$  of type  $A_n$  and not of adjoint type, 2.1 - 2.5 do not necessarily hold. For such groups a detour such as we have used (proving for the adjoint case and then "lifting" to  $h$  via a central isogeny) seems to be necessary for the proof of Theorem 2.8. However, one could probably give a (long and tedious) proof by using the result for  $SL_n(k)$  (proved by Gerstenhaber) and checking cases. If  $H$  does not have any normal subgroups of type  $A_n$ , then  $d\eta : \underline{h} \rightarrow \underline{g}$  is an isomorphism because of the restrictions on the characteristic of  $k$  and the detour is unnecessary.

### 3. Connections with the work of Dixmier

Let  $\underline{g}$  be a semisimple Lie algebra over an algebraically closed field of characteristic zero and let  $G$  be the adjoint group of  $\underline{g}$ . Let  $x \in \underline{g}$ . A polarization of  $x$  is a subalgebra  $\underline{p}$  of  $\underline{g}$  such that (i)  $2 \dim \underline{p} = \dim \underline{g} + \dim Z_{\underline{g}}(x)$  and (ii)  $(x, [\underline{p}, \underline{p}]) = 0$  (here  $(\ , \ )$  denotes the Cartan-Killing form of  $\underline{g}$ ). An element  $x \in \underline{g}$  is polarizable if it admits a polarization. Dixmier shows in [3] (see also [6]) that a nilpotent element  $x \in \underline{g}$  is polarizable if and only if there exists a parabolic subgroup  $P$  of  $G$  such that  $x \in \Lambda_P$ ; in this case  $\underline{p}$  is a polarization of  $x$ .

Now let the notation be as in the previous section and let  $X$  denote the closure of  $G(z_\theta')$ . One checks easily that if  $x \in X$ , then  $\dim G(x) \leq 2 \dim \underline{u}_\theta$ .

Let  $X^\sim = \{x \in X \mid \dim G(x) = 2 \dim \underline{u}_0\}$ . Then  $X^\sim$  is an open dense subset of  $X$  and  $G(\Lambda_{p_\theta}) \subset X^\sim$ . In connection with applying polarizations to some problems in the infinite-dimensional representation theory of  $\underline{g}$ , Dixmier asks the following questions:

- (a) Is  $G(\Lambda_{p_\theta})$  the only nilpotent conjugacy class in  $X^\sim$  ?
- (b) Is every nilpotent conjugacy class in  $X$  in the closure of  $G(\Lambda_{p_\theta})$ ?

It follows immediately from the results of section 2 that the answer to both of these questions is "yes". Let  $Y$  denote the set of nilpotent elements of  $X$  and let  $I(Y)$  denote the ideal of  $Y$  in  $k[X]$ , the algebra of polynomial functions on  $X$ . Let  $J = k[X]^G$  be the algebra of  $G$ -invariant polynomial functions on  $X$  and let  $J^+ = \{f \in J \mid f(0) = 0\}$ . Dixmier also asks the following question:

- (c) Is  $J^+$  a set of ideal generators for  $I(Y)$ ?

Let  $J$  be the ideal generated by  $J^+$ . By using the Nullstellensatz, it is not difficult to show that  $I(Y)$  is the radical of  $J$ . Thus (c) is equivalent to asking whether  $J$  is a radical ideal. We have no conjecture as to the answer. It seems to be a difficult question.

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