Some two-dimensional markov processes

C.J. Ridler-Rowe

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C.J. Ridler-Rowe

Grey College, Durham

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This thesis is primarily concerned with the mathematical analysis of some Markov processes which take place on a two-dimensional lattice of points.

In the first two chapters, mathematical models of two biological phenomena are considered, namely the competition for survivial between two species, and the effect of an epidemic on a population. These models are obtained by a known method which permits certain random variations in the population sizes. For the model of the competition process, it is found that one of the species almost certainly becomes extinct, and the likelihood of the extinction of a given species is investigated. Also, the expectation of the the time at which extinction occurs is bounded, irrespective of the initial state, and an estimate is made of the total number of births and deaths that occur before this time. For the epidemic model, it is found that the epidemic almost certainly dies out, and the expectation of the time at which this event first occurs is estimated when the initial population is large. Various questions on the eventual state of the population are also considered.

In the third chapter, a class of recurrent two-dimensional random walks in discrete time is considered. A limiting law is found for the probability distribution of first passage times which is identical to the limiting law in the analogous situation for Brownian motion. The method is also applied to certain continuous time random walks and to certain random walks in three dimensions.

The last problem considered is the distribution of points at which a simple unsymmetric discrete time random walk makes its first passage through the boundaries of the half and quarter planes. The limiting distribution is found to be a form of either normal distribution or stable distribution of order half.

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Introduction

This thesis is chiefly concerned with the mathematical analysis of some Markov processes which have a two-dimensional lattice of points as their state space. The general background to this work may be found in the book 'Introduction to Probability Theory' by W. Feller [7], and reference is made to some well known properties of characteristic functions, which may be found, for instance, in the book 'Characteristic Functions' by E. Lukacs [16]. The background to the work on processes in continuous time may be found in the paper 'The calculation of the ergodic projection for Markov chains and processes with a countable infinity of states' by Kendall and Reuter [13].

In chapters 1 and 2, some probabilistic models of certain biological phenomena are considered, namely the competition for survival between two species, and the effect of an epidemic on a population. Models of these phenomena in continuous time may be obtained by specifying the birth and death rates for particular states of the populations, although a complete mathematical solution is then difficult. This approach, using theoretical or Monte Carlo methods, has received much attention recently, in particular in papers by

Bartlett [1], [2], Kendall [12], and Reuter [20]. Chapters 1 and 2 are a continuation of the work of the last mentioned paper, 'Competition Processes' by Reuter, and contain results on the limiting behaviour of the models.

In the case of the competition between two species, considered in chapter 1, suppose m and n denote the number of individuals of each species, and let αm, γmn be the birth and death rates of the species of size m, and βn, δmn the birth and death rates of the species of size n. In 'Competition Processes', Reuter has shown that it is almost certain that one of the species becomes extinct, and it is now found that the expected time for this to happen is bounded. The probability that a given species survives the other is investigated in the case when the total population is large, and it is shown that the behaviour of the process then approximates to that of a simple random walk with transition probabilities proportional to the death rates γmn, δmn. In fact if

$$n\gamma - m\delta \sim t(m\delta(\gamma + \delta))^{\frac{1}{2}} \qquad \text{as } m \text{ and } n \to \infty,$$

then the probability that the species of size n survives the other tends to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\, du.$$

The expected number of births and deaths that occur before one of the species becomes extinct is also investigated, and again this depends only on the death rates when the initial population is large. If m and n are the initial populations, and $E(m,n)$ is the expected number of births and deaths that occur, then

$$E(m,n) \sim (\gamma+\delta) \min(m/\gamma, n/\delta), \qquad \text{as } m \text{ and } n \to \infty.$$

All the functions whose limiting behaviour is considered here are functions of position which are characterised by being the least positive solutions of certain inequalities. By finding sufficiently good solutions of these inequalities, and by comparing the process with other simpler processes, the limiting behaviour of these functions is found when the initial population is large.

Similar approximations are applied in chapter 2 to a problem on the epidemic process. For the given model, it is shown that, if m and n are the initial numbers of susceptibles and infectives respectively, then the expectation of the time when the infectives first die out completely is proportional to $\log(m+n)$ when $(m+n)$ is large, and $n > 0$. Also, if $p_{ij}(t)$ denotes the transition probability that the process, initially in state i, is in state j at time t, then, for the epidemic model, $p_{ij}(t)$ tends to a unique limit π_j as t tends to infinity. Various questions are examined on the rate of convergence, with respect to j, of the series $\sum_j \pi_j$.

At the beginning of chapter 3, the known result is noted that the simple one-dimensional random walk and one-dimensional Brownian motion possess the same limiting laws for their distributions of first passage times, i.e. the probability that the first passage through the origin, starting at a distance y, occurs before time ty^2 tends to

$$1 - (2/\pi)^{\frac{1}{2}}\int_0^{t^{\frac{1}{2}}} e^{-u^2/2}\, du, \qquad \text{as} \quad y \to \infty,$$

which is the positive stable distribution of order $\frac{1}{2}$. The analogous problems are then investigated in the two-dimensional case. Starting from a result due to F.Spitzer, it is shown that the probability that the two-dimensional Brownian motion reaches a disc about the origin before time r^α, starting at a distance r, tends to $1 - 2\alpha^{-1}$ as r tends to infinity, where $\alpha \geq 2$. The generating function for the distribution of times of first passage through the origin is then found for a general recurrent two-dimensional random walk. Then, by using a Tauberian argument of a type due to Karamata, it is found that the same limiting law holds for the random walk under suitable conditions. An interesting corollery to the last result is its application to the limiting behaviour of the distribution of first hits on an axis for a restricted class of recurrent three dimensional random walks. It follows that, when the random walk starts at a distance r from the axis, the probability that the size of the displacement parallel to the axis is less than r^{α}, when the first hit occurs, tends to $1 - \alpha^{-1}$ as r tends to infinity, where $\alpha \geq 1$.

The approximations used in chapters 1 and 2 involve examining the behaviour of a particularly simple type of random walk on a two-dimensional lattice, and in chapter 4 more extensive consideration is given to a completely unsymmetric simple random walk. McCrea and Whipple [17] investigated the case of a simple random walk on a rectangular region, and extended their result to various infinite regions. Later Henze [11] obtained the transition probabilities on the whole plane for the simple random walk, and from these obtained the transition probabilities on the half plane with an absorbing boundary by using a reflection argument. By employing a generalisation of a transformation used by Henze, and earlier by McCrea and Whipple, the characteristic function of the distribution of first hits on the boundary of the half plane is now found. Then, using methods given by Gnedenko and Kolmogorov [9], limiting laws with error terms are found for the distribution of first hits when the random walk starts at a large distance y, say, from the boundary. Thus, if the expected step of the random walk is not parallel to the boundary, the distribution of first hits obeys a type of central limit law, with mean and various proportional to y. However, if the expected step is parallel to the boundary, the probability that, when the first hit occurs, the displacement in the direction of the expected step is less than tcy^2 (where c is a positive constant) tends to

$$1 - (2/\pi)^{\frac{1}{2}} \int_0^{t^{-\frac{1}{2}}} e^{-u^2/2}\, du, \quad \text{as} \quad y \to \infty \quad \text{when} \quad t > 0,$$

$$\text{and} \quad 0, \qquad \text{as} \quad y \to \infty \quad \text{when} \quad t \leq 0.$$

By using a reflection argument, these results are applied to obtain similar results for the distribution of first hits on the boundaries of the quarter plane.

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Chapter 1.

Competition between Two Species

1.1. This chapter begins with the description of a continuous time stochastic process which was used by Reuter in 'Competition Processes' [20] as a model of the competition berween two species.

Let X_t be a time homogeneous Markovian random variable with a continuous time parameter t, let X_t take values on a countable set E, and let $\{p_{ij}(t)\}$ be the corresponding matrix of transition probabilities, i.e.

$$p_{ij}(t) = Pr\{X_{t_o+t} = j | X_{t_o} = i\}, \qquad i,j \in E,\ t_o, t \geq 0$$

where $Pr\{B|C\}$ is the conditional probability of event B, given that event C occurs. The competition process is determined by specifying the matrix of transition rates $Q = \{q_{ij}\}$, defined by

$$q_{ij} = p'_{ij}(0),$$

where

$$(1) \qquad \begin{aligned} q_{ij} &\geq 0, \qquad\qquad i \neq j, \\ q_i &= -q_{ii} \geq 0, \\ \sum_{j \neq i} q_{ij} &= q_i < \infty. \end{aligned}$$

Let the state space E consist of the elements (m,n), where $m,n = 0, 1, 2, \ldots$, but $(0,0)$ is excluded. Then for $i = (m,n)$, with $m,n > 0$, define the transition rates by

$$\begin{aligned} q_{ij} &= \alpha m, & j &= (m+1,n), \\ q_{ij} &= \beta n, & j &= (m,n+1), \\ q_{ij} &= \gamma mn, & j &= (m-1,n), \\ q_{ij} &= \delta mn, & j &= (m,n-1). \end{aligned}$$

where α, β, γ, δ are all positive constants, and otherwise put $q_{ij} = 0$. Then $q_{ij} = 0$ if $i = (m,n)$ lies on one of the axes $m = 0$, $n = 0$. The states i for which $q_i = 0$ are called absorbing. Let A denote the set of all such states, which, for the process just defined, consists of the positive parts of the m and n axes.

The following results were proved by Reuter in [20]. There is a unique matrix $\{p_{ij}(t)\}$ of transition probabilities corresponding to the given matrix $Q = \{q_{ij}\}$, and a process starting from any point (m,n), with $m,n > 0$, is almost certain to reach some state in A, and will remain there. Also the expected time to reach A is finite for every starting point.

It is demonstrated later that the proof of the last result may be extended to show that the expected time to reach A is bounded for all starting points. Most of this chapter is however concerned with examining by analytical methods the asymptotic behaviour as the starting point goes to infinity (i) for the probability that the process is absorbed in a given one of the m axis and n axis, and (ii) for the average number of transitions that occur before reaching A. In both cases it is found that the asymptotic behaviour is just as if the rates αm and βn were ignored.

In this model, if m and n denote the populations of species 1 and species 2 say, then αm and βn are the respective birth rates, and γmn and δmn the respective death rates. As soon as one of m and n becomes zero, the process is stopped at the point where one of the species first becomes extinct.

The results may be interpreted as follows. Whatever the initial populations are, it is almost certain that one of the species becomes extinct, and the expected time for this to happen is bounded. The asymptotic behaviour as the size of the initial populations goes to infinity is found (i) for the probability that a given species survives the other, and (ii) for the expected number of births and deaths that occur before one population becomes extinct.

1.2. In this section the behaviour of the probability that the process is absorbed in a given one of the m axis and n axis is examined. Firstly it is shown that the absorption probability of the continuous time process is identical with that of a certain discrete time process. Then the following lemmas are used to provide a crude estimate in Lemma 5 of the behaviour of the absorption probability, and, from this, the main result is proved.

Suppose a matrix Q with properties (1) defines a unique continuous time process $\{p_{ij}(t)\}$ on a countable set E which contains a subset C, each member of which is an absorbing state, i.e. $q_i = 0$ when $i \in C$. Kendall and Reuter in [13], Theorem 8 (iii), showed that the absorption probabilities $\{x_i\}$, where

(2) $x_i = \Pr\{\text{Process reaches } C \text{ starting from } i\}$, $i \in E$,

are the least non negative solution $\{y_i\}$ of

(3)
$$\sum_{j \in E} q_{ij}\, y_j = 0, \qquad i \in E,$$
$$y_i = 1, \qquad i \in C.$$

Now consider a second process on E in discrete time, defined by the one step transition probabilities

$$\bar{p}_{ij} = (1-\delta_{ij})q_{ij} / q_i \ , \qquad q_i > 0,$$
$$\bar{p}_{ij} = \delta_{ij} \ , \qquad q_i = 0,$$

where δ_{ij} is the Kronecker delta. Clearly C is also a set of absorbing points for this process. It is known (see Feller [7], Chapter XV, section 8) that the absorption probabilities, as in the definition (2), for the discrete time process are the least non negative solution $\{y_i\}$ of

(4)
$$\sum_{j \in E} \bar{p}_{ij}\, y_j = y_i \, , \qquad i \in E,$$
$$y_i = 1, \qquad i \in C.$$

Since conditions (3) and (4) are identical, the absorption probabilities for the continuous time process may be calculated by observing the corresponding discrete time process.

Lemma 1 is a simple generalisation of condition (4), and provides a useful characterisation of the absorption probabilities. Let p be a discrete time process with one step transition probabilities $\{p_{ij}\}$ on a countable set E, and let C be a subset of absorbing states in E. Let $\{x_i\}$ be the set of absorption probabilities

$$x_i = \Pr\{p \text{ reaches } C \text{ starting from } i\}, \qquad i \in E.$$

<u>Lemma 1.</u> The absorption probabilities $\{x_i\}$ are the least non negative solution of

$$y_i \geq \sum_j p_{ij}\, y_j, \qquad i \in E,$$

$$y_i \geq 1. \qquad i \in C,$$

and x_i satisfies these conditions with equality.

<u>Proof:</u> Let

$$x_i^s = \Pr\{p \text{ reaches } C \text{ within at most } s \text{ steps starting at } i\}, \qquad i \in E,\ s > 0,$$

(5) and $x_i^0 = 1$ or 0, according as $i \in C$ or not.

Clearly x_i^s satisfies

(6) $$x_i^{s+1} = \sum_j p_{ij}\, x_j^s, \qquad i \in E,\ s \geq 0,$$

$$x_i^{s+1} \geq x_i^s, \qquad i \in E,\ s \geq 0,$$

By definition, $x_i^0 \leq y_i$. Suppose that, for some $s \geq 0$, $x_i^s \leq y_i$ for all $i \in E$. Then

$$x_i^{s+1} = \sum_j p_{ij} x_j^s \leq \sum_j p_{ij} y_j \leq y_i , \qquad i \in E.$$

It follows by induction that

$$(7) \qquad x_i^s \leq y_i, \qquad i \in E,\ s \geq 0.$$

To complete the proof of the lemma, let $s \to \infty$ in (6) and (7).

The next lemma proves some simple monotonic properties, which are required later, for a certain type of process p. Suppose that p has the state space of all points (m,n), where $m, n = 0, 1, 2, \ldots,$ but $(0,0)$ is excluded, and suppose that each point of the positive parts of the m and n axes is absorbing. Let C be the positive part of the n axis, and let $x^s(m,n)$, $x(m,n)$ be the probabilities corresponding to x_i^s, x_i, defined in Lemma 1, with $i = (m,n)$. Suppose the one step transition probabilities of p are given, for $i = (m,n)$ with $m, n > 0$, by

$$p_{ij} = a, \qquad j = (m+1,n),$$

$$p_{ij} = b, \qquad j = (m,n+1),$$

$$p_{ij} = c, \qquad j = (m-1,n),$$

$$p_{ij} = d, \qquad j = (m,n-1),$$

where a, b, c, d are non negative constants and $a+b+c+d = 1$.

<u>Lemma 2.</u> For $m, n > 0$ and $s \geq 0$,

(8) $$x^s(m-1,n) \geq x^s(m,n) \geq x^s(m,n-1),$$

(9) $$x(m-1,n) \geq x(m,n) \geq x(m,n-1).$$

<u>Proof:</u> Clearly from (5), $x^0(m-1,n) \geq x^0(m,n) \geq x^0(m,n-1)$ when $m, n > 0$. Assuming (8) holds for all $m, n > 0$ for some fixed $s \geq 0$, and using (6), it follows that for $m > 1$, $n > 0$,

$$\begin{aligned} x^{s+1}(m,n) &= ax^s(m+1,n) + bx^s(m,n+1) + cx^s(m-1,n) + dx^s(m,n-1) \\ &\leq ax^s(m,n) + bx^s(m-1,n+1) + cx^s(m-2,n) + dx^s(m-1,n-1) \\ &= x^{s+1}(m-1,n), \end{aligned}$$

whilst for $m = 1$, $n > 0$, $s \geq 0$,

$$x^s(m,n) \leq x^s(m-1,n) = 1$$

Hence $x^{s+1}(m,n) \leq x^{s+1}(m-1,n)$ for all $m, n > 0$, and similarly $x^{s+1}(m,n) \geq x^{s+1}(m,n-1)$. Thus the first part of the lemma follows by induction, and the second follows on letting $s \to \infty$.

It is now possible to show how the behaviour of a process with non constant transition probabilities may be compared with that of a process which has constant transition probabilities. Suppose P is a second discrete time process on the same state space as p, with the axes absorbing again, and let $X(m,n)$ be the probability that P is absorbed in the positive part of the n axis starting from (m,n). Suppose that, corresponding to the transition probabilities a, b, c, d to neighbouring states for the process p, P has transition probabilities A, B, C, D which are not necessarily constant functions of (m,n).

<u>Lemma 3.</u> Suppose the transition probabilities satisfy

$$(10) \qquad A \le a, \quad B \ge b, \quad C \ge c, \quad D \le d, \quad A+B+C+D = 1,$$

when $m, n > 0$. Then $X(m,n) \ge x(m,n)$ for all $m, n > 0$.

<u>Proof:</u> Using the convention in Lemma 1, (5), it follows that $X(m,n) \ge x^0(m,n)$. Suppose that for some $s \ge 0$, $X(m,n) \ge x^s(m,n)$. Then using the iterative construction (6), it follows that for $m, n > 0$,

$$x^{s+1}(m,n) = ax^s(m+1,n) + bx^s(m,n+1) + cx^s(m-1,n) + dx^s(m,n-1)$$

$$= Ax^s(m+1,n) + Bx^s(m,n+1) + Cx^s(m-1,n) + Dx^s(m,n-1)$$

$$+(a-A)x^s(m+1,n) + (b-B)x^s(m,n+1) + (c-C)x^s(m-1,n) + (d-D)x^s(m,n-1).$$

But from (8) in Lemma 2, and (10), $(a-A)x^s(m+1,n) \leq (a-A)x^s(m,n)$, and similarly $(b-B)x^s(m,n+1) \leq (b-B)x^s(m,n)$, etc. Hence

$$x^{s+1}(m,n) \leq AX(m+1,n) + BX(m,n+1) + CX(m-1,n) + DX(m,n-1)$$
$$+ \{(a-A) + (b-B) + (c-C) + (d-D)\} x^s(m,n)$$
$$= X(m,n),$$

using the last part of Lemma 1 applied to P. Clearly $x^{s+1}(m,n) \leq X(m,n)$ on the positive parts of the m and n axes. Hence, by induction, $x^s(m,n) \leq X(m,n)$ for all $s \geq 0$, and then $x(m,n) \leq X(m,n)$ on letting $s \to \infty$.

The following lemma gives the behaviour of $x(m,n)$ in a particularly simple case which is used in Lemma 5.

<u>Lemma 4.</u> Suppose $b = 0$ and $c > a > 0$. Then there is a constant k such that, given $\varepsilon > 0$,

$$x(m,n) \geq 1 - \varepsilon \qquad \text{when} \quad n \geq md/(c-a) + k\sqrt{\frac{m}{\varepsilon}},$$

where $k = \{d(c+a-4ac)/(c-a)^3\}^{\frac{1}{2}}$.

<u>Proof:</u> It is noted first that $x(m,n)$ is 0 and 1 on the positive parts of the m and n axes respectively. For θ real and $|\theta| < 1$, let

$$V_m(\theta) = \sum_{n=0}^{\infty} x(m,n)\, \theta^n, \qquad \text{when} \quad m > 0,$$

and let $V_0(\theta) = \sum_{n=1}^{\infty} x(0,n)\,\theta^n = \frac{\theta}{1-\theta}$. Then from the difference equation of the type (4) satisfied by $x(m,n)$, it follows that $V_m(\theta)$ satisfies

(11) $$aV_{m+1}(\theta) - (1-d\theta)V_m(\theta) + cV_{m-1}(\theta) = 0, \qquad m > 0,$$

with $V_m(\theta)$ bounded by $(1-|\theta|)^{-1}$. The general solution of (11) is

$$Hu_1^m + Ku_2^m,$$

where H and K are arbitrary constants, and u_1 and u_2 are the roots of

$$au^2 - (1-d\theta)u + c = 0.$$

Since $u_1u_2 = c/a > 1$, one of the roots has modulus > 1, and the corresponding term in the general solution is unbounded. Hence the bounded solution of (11), satisfying the boundary condition for $V_0(\theta)$, is

(12) $$V_m(\theta) = \left\{\frac{\theta}{1-\theta}\right\}\left\{\frac{(1-d\theta) - [(1-d\theta)^2 - 4ac]^{\frac{1}{2}}}{2a}\right\}^m$$

Suppose m is temporarily fixed, and, noting the inequality (9) in Lemma 2 and also that $x(m,0) = 0$, consider a random variable Z_m with

$$\Pr\{Z_m = n\} = x(m,n+1) - x(m,n), \qquad n \geq 0.$$

Then for $|\theta| \leq 1$, the generating function of the distribution of Z_m is

$$\sum_{n=0}^{\infty} \Pr\{Z_m = n\}\ \theta^n = \sum_{n=0}^{\infty} \{x(m,n+1) - x(m,n)\}\ \theta^n$$

$$= \left\{\frac{1-\theta}{\theta}\right\} V_m(\theta)$$

$$= \left\{\frac{(1-d\theta) - [(1-d\theta)^2 - 4ac]^{\frac{1}{2}}}{2a}\right\}^m$$

from(12),

$$= G_m(\theta) \text{ say.}$$

The form of $G_m(\theta)$ may be explained as follows. Let the random variable Z be the number of steps $(0,-1)$ taken before the displacement in the m direction reaches -1. Then Z_m is in fact the sum of m independent random variables, each with the same distribution as Z. Hence Z has the generating function $G_1(\theta) = \sum_{n=0}^{\infty} \Pr\{Z_1 = n\}\theta^n$ and Z_m has the generating function $G_m(\theta) = \{G_1(\theta)\}^m$.

Since $G_m(1) = 1$, it follows that Z_m is finite with probability 1. By successively differentiating $G_m(\theta)$ at $\theta = 1$, it is easily found that the distribution of Z_m has first moment $\mu = md/(c-a)$,

and variance $\sigma^2 = md(c+a-4ac)(c-a)^{-3} = k^2 m$ say, where k is independent of m. Also, by the Tchebychev inequality,

$$\Pr\{Z_m \geq \mu+h\} \leq \Pr\{|Z_m - \mu| \geq h\} \leq \frac{\sigma^2}{h^2}, \qquad h > 0.$$

To complete the proof, let $h = \frac{\sigma}{\sqrt{\varepsilon}} = k\sqrt{(m/\varepsilon)}$. Then if $n \geq md/(c-a) + k\sqrt{(m/\varepsilon)} = \mu+h$, it follows that

$$1 - x(m,n) = \Pr\{Z_m \geq n\} \leq \Pr\{Z_m \geq \mu+h\} \leq \varepsilon.$$

The process P defined in the discussion preceding Lemma 3 is now identified with the discrete time process corresponding to the competition process in the way described at the beginning of this section. A crude estimate is obtained for the probability $X(m,n)$ that P reaches the n axis, and it is interesting to compare this estimate with results described by Neyman, Park and Scott in [18], section 3.

Lemma 5. Suppose the process P is defined by

$A(m,n) = \alpha m/(\alpha m+\beta n+\gamma mn+\delta mn)$, $B(m,n) = \beta n/(\alpha m+\beta n+\gamma mn+\delta mn)$,

$C(m,n) = \gamma mn/(\alpha m+\beta n+\gamma mn+\delta mn)$, $D(m,n) = \delta mn/(\alpha m+\beta n+\gamma mn+\delta mn)$,

for $m,n > 0$. Then for every positive ε and η, there is a constant k such that

$$X(m,n) \geq 1 - \varepsilon, \qquad \text{when} \quad n \geq m(\delta\gamma^{-1}+\eta) + k,$$

$$\text{and} \quad X(m,n) \leq \varepsilon, \qquad \text{when} \quad n \leq m(\delta\gamma^{-1}-\eta) - k.$$

Proof: It is only necessary to prove the lower bound, since the upper bound follows on interchanging the roles of m and n, using the result of Reuter in [20] that the process is certain to reach the axes, and using the result obtained in the discussion at the beginning of this section. The proof is divided into two parts. In the first part, by comparing P with a second process p it is shown that, for large m and n, P behaves almost as if α and β were ignored, and P reaches a certain line $m = M$, $n \geq N$ with probability at least $1 - \varepsilon/2$, when the starting point (m_0, n_0) satisfies

$$(13) \qquad m_0 \geq M, \quad n_0 \geq m_0(\delta\gamma^{-1} + \eta) + k.$$

In the second part, a similar approximation shows that $X(m,n) \geq 1-\varepsilon/2$ when $0 < m \leq M$, $n \geq m(\delta\gamma^{-1} + \eta) + k$. Thus if N is large enough, P reaches the n axis with probability at least $1 - \varepsilon/2$, starting from any point of the line $m = M$, $n \geq N$. Combining this with the first part, the Strong Markov Theorem shows that $X(m_0,n_0) \geq 1 - \varepsilon$ when (13) holds.

(i) Consider the region $m \geq M$, $n \geq N$, where M and N are positive integers to be chosen later. An estimate is now found for the probability that P reaches the line $m = M$, $n > N$. A second process p is constructed on this region and it is supposed temporarily that the boundary points absorb both processes. For $m > M$, $n > N$, let

$$a = (\tfrac{\alpha}{N})/(\gamma+\delta) \geq (\tfrac{\alpha}{n})/(\tfrac{\alpha}{n} + \tfrac{\beta}{m} + \gamma + \delta) = A(m,n),$$

$$b = 0 \leq B(m,n),$$

$$c = \gamma/(\tfrac{\alpha}{N} + \tfrac{\beta}{M} + \gamma + \delta) \leq \gamma/(\tfrac{\alpha}{n} + \tfrac{\beta}{m} + \gamma + \delta) = C(m,n),$$

$$d = 1 - a - b - c,$$

where $d \geq D(m,n)$ if

$$1 - (\tfrac{\alpha}{N})/(\gamma+\delta) - \gamma/(\tfrac{\alpha}{N} + \tfrac{\beta}{M} + \gamma+\delta) \geq \delta/(\tfrac{\alpha}{n} + \tfrac{\beta}{m} + \gamma+\delta), \quad m > M, n > N.$$

The last condition holds if

$$\frac{\beta\gamma}{M} \geq \frac{\alpha}{N}(\alpha + \beta + \delta) \tag{14}$$

Also, since $a \to 0$, $c \to \gamma/(\gamma+\delta)$, $d \to \delta/(\gamma+\delta)$ as $M, N \to \infty$, there exist fixed integers M_0 and N_0 such that

$$d/(c-a) < \delta\gamma^{-1} + \eta/2, \qquad M \geq M_0,\ N \geq N_0. \tag{15}$$

Suppose

$$M = M_0 \quad \text{and} \quad N > \max\{N_0,\ M\alpha(\alpha + \beta + \delta)/(\beta\gamma)\}. \tag{16}$$

Condition (14) now holds, so that P and p now satisfy the conditions (10) of Lemma 3 applied to the region $m \geq M$, $n \geq N$. Also Lemma 4 may be applied to p. Hence there is a constant k_0 such that for all starting points (m_0, n_0) satisfying

$$m_0 \geq M,\ n_0 \geq N + (m_0 - M)d/(c-a) + k_0\{2(m_0 - M)/\varepsilon\}^{\frac{1}{2}}, \tag{17}$$

$$\begin{aligned} &\Pr\{P \text{ reaches the line } m = M,\ n > N\} \\ &\geq \Pr\{P \text{ reaches the line } m = M,\ n > N \\ &\qquad \text{before the line } m > M,\ n = N\} \\ &\geq \Pr\{p \text{ reaches the line } m = M,\ n > N \\ &\qquad \text{before the line } m > M,\ n = N\} \\ &\geq 1 - \varepsilon/2 \end{aligned}$$

(ii) A rough estimate is now found for the probability that P reaches the n axis. Consider the region $m \geq 0,\ n \geq N_1$, where N_1 is an integer to be chosen later, and suppose that the boundary points of the region are temporarily made absorbing. Another process p_1, (similar to p in part (i)), is constructed on the same region with the boundary points still absorbing.

For $m > 0,\ n > N_1$, let

$$a_1 = (\frac{\alpha}{N_1})/(\gamma + \delta) \geq (\frac{\alpha}{n})/(\frac{\alpha}{n} + \frac{\beta}{m} + \gamma + \delta) = A(m,n),$$

$$b_1 = 0 \leq B(m,n),$$

$$c_1 = \gamma/(\alpha + \beta + \gamma + \delta) \leq \gamma/(\frac{\alpha}{n} + \frac{\beta}{m} + \gamma + \delta) = C(m,n),$$

$$d_1 = 1 - a_1 - b_1 - c_1,$$

where $d_1 \geq D(m,n)$ if

$$1 - (\frac{\alpha}{N_1})/(\gamma + \delta) - \gamma/(\alpha + \beta + \gamma + \delta) \geq \delta/(\frac{\alpha}{n} + \frac{\beta}{m} + \gamma + \delta),\ m > 0, n > N_1.$$

This holds if

$$1 - (\frac{\alpha}{N_1})/(\gamma+\delta) - \gamma/(\alpha+\beta+\gamma+\delta) \geq \delta/(\gamma+\delta),$$

and, since this holds for all large enough N_1, choose N_1 to be the least such integer. P and p_1 now satisfy conditions (10) of Lemma 3, applied to the region $m \geq 0$, $n \geq N_1$, and Lemma 4 may be applied to p_1. Hence there is a constant k_1 such that for all starting points (m,n) satisfying

(18) $$m \geq 0,\ n \geq N_1 + md_1/(c_1 - a_1) + k_1\{2m/\varepsilon\}^{\frac{1}{2}},$$

$$\begin{aligned} X(m,n) &= \Pr\{P \text{ reaches the n axis}\} \\ &\geq \Pr\{P \text{ reaches the line } m = 0,\ n > N_1 \\ &\qquad \text{before the line } m > 0,\ n = N_1\} \\ &\geq \Pr\{p_1 \text{ reaches the line } m = 0,\ n > N_1 \\ &\qquad \text{before the line } m > 0,\ n = N_1\} \\ &\geq 1 - \varepsilon/2. \end{aligned}$$

Now, using (16) and (18), choose N to be the least integer such that $N > \max\{N_0,\ M\alpha(\alpha+\beta+\delta)/(\beta\gamma),\ N_1 + Md_1/(c_1 - a_1) + k_1\sqrt{(2m/\varepsilon)}\}$. M is chosen in (16). Then, using (15), (16), (17) and (18), choose $k > 0$ such that

(19) $$m(\delta\gamma^{-1}+\eta) + k \geq N + (m-M)d/(c-a) + k_0\{2(m-M)/\varepsilon\}^{\frac{1}{2}},\quad m \geq M,$$

(20) $$m(\delta\gamma^{-1}+\eta) + k \geq N_1 + md_1/(c_1 - a_1) + k_1\{2m/\varepsilon\}^{\frac{1}{2}},\ 0 < m \leq M.$$

It follows immediately from (20), and the result of part (ii) that

$$X(m,n) \geq 1 - \varepsilon, \qquad 0 < m \leq M,\ n \geq m(\delta\gamma^{-1}+\eta) + k.$$

Also, noting the above definition of N, and using (19) and the results of (i) and (ii), it follows on applying the Strong Markov Theorem (see Chung [3], I, section 13) that when

$$m_0 \geq M, \quad n_0 \geq m_0(\delta\gamma^{-1}+\eta) + k,$$

$$\begin{aligned}
X(m_0,n_0) &= \Pr\{P \text{ reaches the } n \text{ axis starting from } (m_0,n_0)\} \\
&\geq \Pr\{P \text{ reaches the line } m = M,\ n > N \\
&\qquad \text{and then reaches the } n \text{ axis}\} \\
&= \sum_{r > N} \Pr\{P \text{ first reaches the line } m = M,\ n > N \text{ at} \\
&\qquad (M,r), \text{ and then reaches the } n \text{ axis}\} \\
&= \sum_{r > N} \Pr\{P \text{ first reaches the line } m = M,\ n > N \\
&\qquad \text{at } (M,r)\} \\
&\qquad \times \Pr\{P \text{ reaches the } n \text{ axis starting from } (M,r)\} \\
&\geq (1-\varepsilon/2) \sum_{r > N} \Pr\{P \text{ first reaches the line } m = M, \\
&\qquad n > N \text{ at } (M,r)\} \\
&\geq (1-\varepsilon/2)^2 \\
&\geq 1 - \varepsilon.
\end{aligned}$$

This completes the proof of Lemma 5. By employing a slightly more elaborate proof which allows M and N to depend on the value of the starting point (m,n), it is possible to replace the term $\eta m + k$ in the lemma by another proportional to $\sqrt{m}$.

The asymptotic behaviour of $X(m,n)$ is obtained by comparing $X(m,n)$ with $x(m,n)$, where, for the remainder of this section, $x(m,n)$ is the probability that a process p, as described just before Lemma 2, with $a = b = 0$, $c = \gamma/(\gamma+\delta)$, $d = \delta/(\gamma+\delta)$, reaches the n axis. The following lemmas are needed for the main proof. Lemma 6 is a generalisation of Lemma 1.

<u>Lemma 6.</u> Let $\{p_{ij}\}$ be the transition matrix of a Markov process on a countable state space S, and suppose the non empty set R of all absorbing points is reached almost certainly from any initial state. Let K be a subset of S, disjoint from R. Define the boundary of K to be the set L of points j not in K for which there exists i in K such that $p_{ij} > 0$. Let w_i be the probability that the process is absorbed in a subset C of R starting from i in S, and suppose there exist $\{v_i\}$ defined on $K \cup L$ such that

$$v_i \geq \sum_{j \in K \cup L} p_{ij} v_j, \qquad i \in K,$$

$$v_i \geq w_i, \qquad i \in L.$$

Then $v_i \geq w_i$ when $i \in K$.

<u>Proof:</u> Define $w_i^s = w_i$ when $s \geq 0$ and $i \in S - K$, $w_i^0 = 0$ when $i \in K$, and

$$w_i^{s+1} = \sum_j p_{ij} w_j^s, \qquad i \in K, \ s \geq 0.$$

Then, noting the definition of $K \cup L$, and using simple induction arguments based on the above equation and similar to those used in Lemma 1, it follows that

$$w_i^s \leq v_i, \qquad i \in K, \ s \geq 0,$$

$$w_i^s \leq w_i^{s+1} \leq w_i, \qquad i \in S, \ s \geq 0.$$

Hence there exists $\bar{w}_i = \lim_{s\to\infty} w_i^s$ such that $\bar{w}_i \leq w_i$ when $i \in S$, and $\bar{w}_i \leq v_i$ when $i \in K \cup L$. Also

$$\bar{w}_i = \sum_j p_{ij} \bar{w}_j, \qquad i \in K,$$

$$\bar{w}_i \geq \sum_j p_{ij} \bar{w}_j, \qquad i \in S - K.$$

Hence $1 + \bar{w}_i - w_i \geq 0$ since $w_i \leq 1$, and

$$1 + \bar{w}_i - w_i \geq \sum_j p_{ij}(1+\bar{w}_j - w_j), \qquad i \in S$$

Since absorption in R is certain, and $1 + \bar{w}_i - w_i = 1$ when

$i \in R$, it follows from Lemma 1 that $1 + \bar{w}_i - w_i \geq 1$ when $i \in S$, i.e. $\bar{w}_i \geq w_i$. Since $\bar{w}_i \leq w_i$ also, the required result now follows, i.e.

$$w_i = \bar{w}_i \leq v_i \qquad \text{when} \quad i \in K.$$

<u>Lemma 7.</u> Consider the process p defined before Lemma 6. For each positive ε and η there is a constant k such that

$$x(m,n) \geq 1 - \varepsilon, \qquad \text{when} \quad n \geq m(\delta\gamma^{-1}+\eta) + k,$$

$$x(m,n) \leq \varepsilon, \qquad \text{when} \quad n \leq m(\delta\gamma^{-1}-\eta) - k.$$

Also $x(m,n) = \Phi\left\{\dfrac{n\gamma - m\delta}{\sqrt{m\delta(\gamma+\delta)}}\right\} + O(m^{-\frac{1}{2}})$ as $m \to \infty$, where

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\, du, \qquad -\infty \leq t \leq \infty.$$

<u>Proof:</u> Consider the random variables Z and Z_m, as defined in the proof of Lemma 4, but applied to p in the present case. Clearly $x(m,n) = \Pr\{Z_m < n\}$, and it is easily found that $Z = Z_1$ has the generating function

$$G_1(\theta) = \sum_{n=0}^{\infty} \Pr\{Z_1 = n\}\theta^n = \sum_{n=0}^{\infty} d^n c\, \theta^n = \frac{c}{1 - d\theta}.$$

Then, since Z_m is the sum of m independent random variables each with the same distribution as Z_1,

$$G_m(\theta) = \sum_{n=0}^{\infty} \Pr\{Z_m = n\}\theta^n = \left(\frac{c}{1-d\theta}\right)^m.$$

Proceeding as in Lemma 4, it now follows that Z_m is finite with probability 1 since $G_m(1) = 1$, and Z_m has mean md/c and variance md/c^2. The first part of the lemma follows on applying the Tchebychev inequality, and noting that $c = \gamma/(\gamma+\delta)$, $d = \delta/(\gamma+\delta)$. Also Z_m is the sum of m independent random variables each with the same distribution as Z_1, and the distribution of Z_1 may easily be shown to have a finite third moment, so that it follows from a result on convergence to the normal law given by Gnedenko and Kolmogorov in [9], §40, **Theorem** 1, that

$$x(m,n) = \Pr\{Z_m < n\} = \Phi\left\{\frac{n - mdc^{-1}}{\sqrt{(mdc^{-2})}}\right\} + O\left(m^{-\frac{1}{2}}\right), \qquad \text{as} \quad m \to \infty.$$

This completes the proof of the second part of the lemma on noting the definitions of c and d.

The following lemma is a trivial consequence of Lemmas 5 and 7, and is needed in the proof of the main result of this section which follows it.

<u>Lemma 8.</u> Given positive ε and η, there is a constant $k > 0$ such that

$$X(m,n) \le \frac{\varepsilon}{2} \text{ and } x(m,n) \le \frac{\varepsilon}{2}, \text{ when } n \le m(\delta\gamma^{-1}-\eta) - k,$$

$$X(m,n) \ge 1 - \frac{\varepsilon}{2} \text{ and } x(m,n) \ge 1 - \frac{\varepsilon}{2}, \text{ when } n \ge m(\delta\gamma^{-1}+\eta) + k.$$

<u>Theorem 1.</u> $X(m,n) - \Phi\left\{\frac{n\gamma - m\delta}{\sqrt{m\delta(\gamma+\delta)}}\right\} \to 0$ uniformly as $m+n \to \infty$,

where $\Phi(t) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{t} e^{-u^2/2}du, \ -\infty \le t \le \infty.$

<u>Proof:</u> For any non negative function $v(m,n)$, let

$$\begin{aligned}\Delta v(m,n) &= (\alpha m+\beta n+\gamma mn+\delta mn)v(m,n) - \alpha m v(m+1,n) - \beta n v(m,n+1)\\ &\quad - \gamma mn\, v(m-1,n) - \delta mn\, v(m,n-1)\\ &= \alpha m\{v(m,n) - v(m+1,n)\} + \beta n\{v(m,n) - v(m,n+1)\}\\ &\quad +\gamma mn\{v(m,n) - v(m-1,n)\} + \delta mn\{v(m,n) - v(m,n-1)\}.\end{aligned}$$

Then $\Delta v(m,n) \ge 0$ corresponds to the condition $v_1 \ge \sum_j p_{ij}v_j$ applied to the discrete time process associated with the competition process, as described at the beginning of this section. Now consider $\Delta x(m,n)$. By using the generating function $G_m(\theta)$ in the proof of Lemma 7, and by interchanging the roles of m and n, it follows that

$$x(m,n) = \sum_{r=0}^{n-1}\binom{m+r-1}{m-1}c^m d^r = 1 - \sum_{r=0}^{m-1}\binom{n+r-1}{n-1}c^r d^n, \qquad m,n > 0.$$

Also $x(m,n) = 0$ and 1 on the positive parts of the m and n axes respectively.

Hence for $m,n > 0$,

$$\begin{aligned}\Delta x(m,n) &= \alpha m\{x(m,n) - x(m+1,n)\} + \beta n\{x(m,n) - x(m,n+1)\}\\ &\quad +(\gamma+\delta)mn\{x(m,n) - cx(m-1,n) - dx(m,n-1)\}\\ &= \alpha m\binom{m+n-1}{n-1}c^m d^n - \beta n\binom{m+n-1}{m-1}c^m d^n,\end{aligned}$$

(21) i.e. $\Delta\ x(m,n) = \frac{(\alpha-\beta)(m+n-1)!c^m d^n}{(m-1)!(n-1)!}$

Case (i): Suppose $\alpha = \beta$. Then $\Delta\ x(m,n) = 0$ when $m,n > 0$. Since $x(m,n)$ satisfies the appropiate boundary conditions on the axes, it follows from Lemma 1 that $x(m,n) \geq X(m,n)$. The same argument applies to $1 - x(m,n)$ and $1 - X(m,n)$ on interchanging the roles of m and n, and noting that absorption in the axes is certain for both processes. Hence $1 - x(m,n) \geq 1 - X(m,n)$. Thus in the case $\alpha = \beta$, the result is actually $X(m,n) = x(m,n)$.

Case (ii): Suppose $\alpha < \beta$. Then $\Delta\ x(m,n) < 0$ and $\Delta\{1-x(m,n)\} > 0$. The argument previously applied to $1 - x(m,n)$ and $1 - X(m,n)$ shows that

(22) $$X(m,n) \geq x(m,n), \qquad m,n \geq 0.$$

The next part of the proof shows that for every positive ε there is a function $v(m,n)$ and a constant M such that

$$X(m,n) \leq v(m,n) \leq x(m,n) + \varepsilon, \qquad m + n \geq M.$$

To construct $v(m,n)$, let

(23) $$v(m,n) = x(m,n) + \rho(m,n) + \varepsilon/2, \qquad m,n \geq 0,$$

where

(24) $$\rho(m,n) = \binom{m+n}{n} c^m d^n\ f(m+n).$$

$f(m+n)$ is chosen later, and the coefficient $\binom{m+n}{n} c^m d^n$ acts as a normalizing factor. Then

$$\Delta v(m,n) \geq 0 \text{ if and only if } \Delta\rho(m,n) \geq -\Delta x(m,n), \quad m,n > 0.$$

On substituting the expression (24) for $\rho(m,n)$, with $m,n > 0$,

$$\Delta\rho(m,n) = (\alpha m+\beta n+\gamma mn+\delta mn)\binom{m+n}{n} c^m d^n f(m+n)$$
$$-\alpha m\binom{m+n+1}{m+1} c^{m+1} d^n f(m+n+1) - \beta n\binom{m+n+1}{n+1} c^m d^{n+1} f(m+n+1)$$
$$-\gamma mn\binom{m+n-1}{m-1} c^{m-1} d^n f(m+n-1) - \delta mn\binom{m+n-1}{n-1} c^m d^{n-1} f(m+n-1)$$
$$= \frac{(m+n)!\,c^m d^n}{(m-1)!\,(n-1)!} \left\{(\gamma+\delta)[f(m+n) - f(m+n-1)] + \left(\frac{\alpha}{n} + \frac{\beta}{n}\right) f(m+n)\right.$$
$$\left. - \left(\frac{\alpha c}{(m+1)n} + \frac{\beta d}{m(n+1)}\right)(m+n-1)\; f(m+n+1)\right\}$$

(25) Let $$\tau(m,n) = \frac{(m+n)!\,c^m d^n}{(m-1)!\,(n-1)!},$$

(26) and $$f(m+n) = \lambda(m+n)^{\mu},$$

where μ is a constant with $0 < \mu < \frac{1}{2}$, and λ is chosen later.

Then for $m+n > 0$,

$$\lambda\mu(m+n)^{\mu-1} > f(m+n+1) - f(m+n) > \lambda\mu(m+n+1)^{\mu-1}.$$

Hence

$$\Delta\rho(m,n) \geq \tau(m,n) \left\{(\gamma+\delta)\lambda\mu(m+n)^{\mu-1} + \left(\frac{\alpha}{n} + \frac{\beta}{m}\right)\lambda(m+n)^{\mu}\right.$$

(27) $$\left. - (\alpha c+\beta d)\left(\frac{1}{m} + \frac{1}{n}\right)[\lambda(m+n)^{\mu} + \lambda\mu(m+n)^{\mu-1}]\right\}$$

$$= \tau(m,n)\; \lambda(m+n)^{\mu-1}\left\{\mu(\gamma+\delta) - \mu(\alpha c+\beta d)\left(\frac{1}{m} + \frac{1}{n}\right) - (\beta-\alpha)(m+n)\left(\frac{c}{m} - \frac{d}{n}\right)\right\}$$

Now choose

(28) $$\eta = \min\left\{\frac{d}{2c}\ ,\ \left(\frac{\mu(\gamma+\delta)}{3}\right) c^{-1}\left(1+\frac{2c}{d}\right)^{-1}\right\}$$

(29) Suppose $\left|n-mdc^{-1}\right| \le \eta m + k$, i.e. $\left|n-m\delta\gamma^{-1}\right| \le \eta m + k$,

where k is chosen according to Lemma 8. Then, for all large enough m and n,

$$\left|\frac{c}{m}-\frac{d}{n}\right|(m+n) \le \frac{c(\eta m+k)(m+n)}{mn}$$

$$= \eta c\left(1+\frac{m}{n}\right) + kc\left(\frac{1}{m}+\frac{1}{n}\right)$$

$$\le \eta c\left\{1+\frac{m}{m(d/c-\eta)-k}\right\} + kc\left(\frac{1}{m}+\frac{1}{n}\right)$$

$$\le \eta c\left(1+\frac{2c}{d}\right) + O(m^{-1}) + O(n^{-1})$$

By applying the definition (28) of η, and using (29) again, it follows that

$$\left|\frac{c}{m}-\frac{d}{n}\right|(m+n) \le \frac{\mu(\gamma+\delta)}{3} + O(m^{-1}) + O(n^{-1}),\quad \left|n-mdc^{-1}\right| \le \eta m + k.$$

Applying this last inequality to the inequality (27) for $\Delta\rho(m,n)$, it follows that there exists a constant M_0 such that

$$\Delta\rho(m,n) \ge \tau(m,n)\lambda(m,n)^{\mu-1}\mu(\gamma+\delta)/3$$

(30)

when $m + n \ge M_0$, $\left|n-m\delta\gamma^{-1}\right| \le \eta m + k$.

Now choose λ such that

(31) $$\lambda M_0^{\mu}\,\mu(\gamma+\delta)/3 \geq \beta - \alpha,$$

(32) $$\lambda \min_{m+n=M_0}\left\{\binom{m+n}{m}c^m d^n (m+n)^{\mu}\right\} \geq 1.$$

Then, using the expression (21) for $\Delta x(m,n)$, the inequalities (30), (31), and the definition (25) of $\tau(m,n)$, it follows that

(33) $\Delta\rho(m,n) \geq -\Delta x(m,n), \quad m + n \geq M_0, \ |n - m\delta\gamma^{-1}| \leq \eta m + k.$

Also, by the inequality (32) and equations (23), (24), (26),

(34) $$v(m,n) \geq \rho(m,n) \geq 1 \geq X(m,n), \qquad m + n = M_0.$$

Since k is chosen according to Lemma 8, it follows from (23) and Lemma 8 that

(35) $$X(m,n) \leq \frac{\varepsilon}{2} \leq v(m,n), \qquad n \leq m(\delta\gamma^{-1} - \eta) - k,$$

(36) $$X(m,n) \leq 1 \leq x(m,n) + \frac{\varepsilon}{2} \leq v(m,n), \qquad n \geq m(\delta\gamma^{-1} + \eta) + k.$$

Now consider the region $m + n > M_0$, $|n - m\delta\gamma^{-1}| < \eta m + k$. By the inequality (33) and equation (23), $\Delta v(m,n) \geq 0$ on this region. By the inequalities (34), (35) and (36), $v(m,n) \geq X(m,n)$ on the boundary of the region, where 'boundary' is used in the sense of Lemma 6. Hence by Lemma 6,

(37) $v(m,n) \geq X(m,n), \quad m + n > M_0, \ |n - m\delta\gamma^{-1}| < \eta m + k.$

From the standard proof of the de Moivre Laplace Limit Theorem, (e.g. see Feller [7], Chapter VII, Theorem 1),

$$\binom{m+n}{m} c^m d^n = O\left((m+n)^{-\frac{1}{2}}\right) \qquad \text{as} \quad m+n \to \infty.$$

Hence from (24) and (26), $\rho(m,n) = O\left((m+n)^{\mu-\frac{1}{2}}\right)$ as $m+n \to \infty$,

where $\mu - \frac{1}{2} < 0$. Thus there exists $M \geq M_0$ such that

$$0 < \rho(m,n) \leq \frac{\varepsilon}{2}, \qquad m+n \geq M.$$

Therefore, combining (22), (23), (37), (35) and (36), it now follows that

$$x(m,n) \leq X(m,n) \leq v(m,n) \leq x(m,n) + \varepsilon, \qquad m+n \geq M.$$

But $\varepsilon > 0$ is arbitrary. Hence $X(m,n) - x(m,n) \to 0$ uniformly as $m+n \to \infty$.

It is shown in case (i) that $X(m,n) = x(m,n)$ when $\alpha = \beta$, and in case (ii) that $X(m,n) - x(m,n) \to 0$ uniformly as $m+n \to \infty$ when $\alpha < \beta$. When $\alpha > \beta$ the proof of case (ii) may be applied with the roles of m and n interchanged, noting that both processes $\mathfrak{P}$ and $\mathfrak{p}$ are almost certain to be absorbed in the axes. Hence $X(m,n) - x(m,n) \to 0$ uniformly as $m+n \to \infty$ in all cases. Theorem 1 now follows from Lemma 7 if the additional condition $n < 2m\delta\gamma^{-1}$ holds. But if $n \geq 2m\delta\gamma^{-1}$, it follows from Lemma 8 that $X(m,n) \to 1$ uniformly as $m + n \to \infty$, and $\Phi\left\{\frac{n\gamma - m\delta}{\sqrt{m\delta(\gamma+\delta)}}\right\} \to 1$ uniformly as $m + n \to \infty$, also. Hence Theorem 1 follows.

1.3. It was proved by Reuter in [20] that the expected time taken by the continuous time competition process to reach the absorbing states is finite for every starting point. In this section, the proof of this result is extended as follows.

__Theorem 2.__ For the competition process, the expected time to reach the set A of absorbing states from any non absorbing state is bounded.

__Proof:__ Let $\tau(m,n)$ now denote the expected time to reach A from (m,n)

The criterion (C) given by Reuter in [20] shows that if there exist finite $u(m,n) \geq 0$ such that

$$\Delta u(m,n) \geq 1, \qquad m,n > 0,$$

where

(38) $$\Delta u(m,n) = (\alpha m + \beta n + \gamma mn + \delta mn)u(m,n) - \alpha mu(m+1,n) - \beta nu(m,n+1) - \gamma mnu(m-1,n) - \delta mnu(m,n-1),$$

then the process almost certainly reaches A, and $\tau(m,n) \leq u(m,n) < \infty$. A bounded set of $u(m,n)$, which are a modification of those used by Reuter in [20], is now constructed.

Let

(39) $$u(m,n) = h\left\{2 - \frac{1}{m+1} - \frac{1}{n+1}\right\} + \frac{2k}{1-\rho}\left\{2 - \rho^m - \rho^n\right\}, \qquad m,n \geq 0,$$

where h, k and ρ, with $\rho < 1$, are constants to be determined later. The term $\left\{2 - \frac{1}{m+n} - \frac{1}{n+1}\right\}$ is suggested by the dominance of the coefficients γmn, δmn in (38) when m and n are large; the term $\{2 - \rho^m - \rho^n\}$ provides the correction needed near the axes $m = 0$ and $n = 0$. Clearly $u(m,n) \geq 0$ when $m,n \geq 0$. On substituting the expression (39) for $u(m,n)$ into that for $\Delta u(m,n)$ in (38),

$$\Delta u(m,n) = h\left\{-\frac{\alpha m}{(m+1)(m+2)} - \frac{\beta n}{(n+1)(n+2)} + \frac{\gamma n}{m+1} + \frac{\delta m}{n+1}\right\}$$
$$+ 2km\rho^{m-1}(\gamma n - \alpha\rho) + 2kn\rho^{n-1}(\delta m - \beta\rho).$$

Let $\rho = \min\left\{\frac{1}{2}, \frac{\gamma}{2\alpha}, \frac{\delta}{2\beta}\right\}$. Then

(40) $$\Delta u(m,n) \geq h\left\{-\frac{\alpha}{m} - \frac{\beta}{n} + \frac{\gamma n}{m+1} + \frac{\delta m}{n+1}\right\} + k\{\gamma\rho^m + \delta\rho^n\}, \qquad m,n > 0.$$

Now choose B and h such that

(41) $$\frac{\alpha+\beta}{B} = \frac{1}{4}\min(\gamma,\delta) \quad \text{and} \quad \frac{h}{4}\min(\gamma,\delta) = 1.$$

Then for m and $n > B$, it follows from (40) and (41) that

$$\Delta u(m,n) \geq h\left\{-\frac{\alpha+\beta}{B} + \frac{1}{2}\min(\gamma,\delta)\right\}$$
$$= \frac{h}{4}\min(\gamma,\delta)$$
$$= 1.$$

If m and $n > 0$, but m or $n \leq B$, then, using (40),

$$\Delta u(m,n) \geq - h(\alpha+\beta) + k \min(\gamma,\delta)\rho^{M}$$
$$= 1,$$

if k is suitably chosen. The two cases (i) m and $n > B$, and (ii) m and $n > 0$ but m or $n \leq B$, cover the region $m,n > 0$. Hence $\Delta u(m,n) \geq 1$ when $m,n > 0$, and it follows from the criterion that $\tau(m,n) \leq \sup_{m,n > 0} u(m,n) < \infty$.

1.4. It is shown by Chung ([3], Part II, § 19) that the spatial distribution of the successive steps of a continuous time process is the same as that for the corresponding 'jump process'. Thus, in the case of the competition process, the expected number of steps taken before absorption in the axes is the same as that for the associated discrete time process P, described at the beginning of section 1.2. and in Lemma 5. This last process is now compared with the process p described just before Lemma 6, using methods similar to those of section 1.2. The asymptotic behaviour of the expected number of steps taken before absorption is shown to be the same for P as for p, and may be calculated directly for the latter. The following lemmas are needed, the first being well known.

Lemma 9. Let $\{p_{ij}\}$ be the transition matrix of a discrete time Markov process on a countable set S which contains a subset A of absorbing states. Suppose the process is almost certain to reach A, and let e_i to be the expected time for this to happen if i is the initial state. Then $\{e_i\}$ is the least non negative solution of

$$y_i \geq 1 + \sum_j p_{ij} y_j, \qquad i \in S - A.$$

Proof: Without loss of generality, let the states of A be considered as one state, and let the expected number of steps taken before time r be

$$\begin{aligned} e_i^r &= \sum_{s=1}^{r} s\{p_{iA}^s - p_{iA}^{s-1}\} + r\{1 - p_{iA}^r\} \\ &= r - \sum_{s=0}^{r-1} p_{iA}^s, \end{aligned}$$

where p_{ij}^r is the r step transition probability, and $p_{ij}^0 = \delta_{ij}$.

If $e_i < \infty$, then

$$e_i = \sum_{s=1}^{\infty} s\{p_{iA}^s - p_{iA}^{s-1}\},$$

and as $r \to \infty$,

$$\begin{aligned} r\{1 - p_{iA}^r\} &= r \sum_{s=r+1}^{\infty} \{p_{iA}^s - p_{iA}^{s-1}\} \\ &\leq \sum_{s=r+1}^{\infty} s\{p_{iA}^s - p_{iA}^{s-1}\} \\ &\to 0. \end{aligned}$$

Hence $e_i^r \to e_i$ as $r \to \infty$. Clearly $e_A^r = 0$, and, when $i \in S - A$,

$$1 + \sum_j p_{ij} e_j^r = 1 + \sum_j p_{ij} \{ r - \sum_{s=0}^{r-1} p_{jA}^s \}$$

$$= (r+1) - \sum_{s=0}^{r} p_{iA}^s$$

$$= e_i^{r+1} .$$

(42) i.e.
$$e_i^{r+1} = 1 + \sum_j p_{ij} e_j^r$$

Suppose there exist $y_i \geq 0$ such that

$$y_i \geq 1 + \sum_j p_{ij} y_j , \qquad i \in S - A.$$

Clearly $y_i \geq e_i^0 = 0$. Suppose that for some $r \geq 0$, $y_i \geq e_i^r$ when $i \in S$. Then using the iterative construction (42),

$$y_i \geq 1 + \sum_j p_{ij} y_j \geq 1 + \sum_j p_{ij} e_j^r = e_i^{r+1}, \qquad i \in S - A.$$

Since $e_A^r = 0 \leq y_A$ for $r \geq 0$, it follows by induction that $y_i \geq e_i^r$ for all $r \geq 0$ and $i \in S$. Letting $r \to \infty$ in the inequality $y_i \geq e_i^r$ and in (42), it follows that

$$y_i \geq e_i, \ i \in S, \text{ and } e_i = 1 + \sum_j p_{ij} e_j, \qquad i \in S - A.$$

Now let $E(m,n)$ be the expected number of steps taken by the process P before absorption in the axes, and let $e(m,n)$ be the corresponding quantity for process p. Since, at each non absorbing state, P is less likely to move towards the axes than p, it would seem intuitively that $E(m,n) \geq e(m,n)$ when $m,n > 0$. This is now proved algebraically.

__Lemma 10.__ $E(m,n) \geq e(m,n)$ for all $m,n > 0$.

__Proof:__ It follows from Lemma 9 that, defining $e^r(m,n)$ in the same way as e_1^r,

$$e^{r+1}(m,n) = 1 + c\,e^r(m-1,n) + d\,e^r(m,n-1), \quad m,n > 0, \; r \geq 0.$$

By using methods similar to those of Lemma 2, or fairly obvious probabilistic considerations, it can be shown that

$$(43) \quad e^r(m-1,n) \leq e^r(m,n) \text{ and } e^r(m,n-1) \leq e^r(m,n), \quad m,n > 0, \; r \geq 0.$$

Clearly $E(m,n) \geq e^0(m,n)$. Suppose that, for some $r \geq 0$, $E(m,n) \geq e^r(m,n)$ for all $m,n \geq 0$. By applying an argument rather similar to that of Lemma 3, and observing that

$$C(m,n) = \gamma mn/(\alpha m + \beta n + \gamma mn + \delta mn) \leq \gamma/(\gamma + \delta) = c$$

and similarly $D(m,n) \leq d$, $A(m,n) \geq 0$, $B(m,n) \geq 0$, it follows, using (43), that

$$\begin{aligned} E(m,n) &= 1 + AE(m+1,n) + BE(m,n+1) + CE(m-1,n) + DE(m,n-1) \\ &\geq 1 + Ae^r(m+1,n) + Be^r(m,n+1) + Ce^r(m-1,n) + De^r(m,n-1) \\ &= 1 + ce^r(m-1,n) + de^r(m,n-1) + \{(C-c) + (D-d) + A+B\}\; e^r(m,n) \\ &\quad + (C-c)\{e^r(m-1,n) - e^r(m,n)\} + (D-d)\{e^r(m,n-1) - e^r(m,n)\} \\ &\quad + A\{e^r(m+1,n) - e^r(m,n)\} + B\{e^r(m,n+1) - e^r(m,n)\} \end{aligned}$$

$$\geq 1 + ce^{r}(m-1,n) + de^{r}(m,n-1)$$
$$= e^{r+1}(m,n).$$

Hence, by induction, $e^{r}(m,n) \leq E(m,n)$ for all $m,n > 0$ and $r \geq 0$, and the lemma follows on letting $r \to \infty$.

An estimate of the behaviour of $e(m,n)$ is now obtained.

<u>Lemma 11.</u> Given $\varepsilon > 0$, there is a constant k such that

$$\frac{m}{c} \geq e(m,n) \geq (1-\varepsilon)\frac{m}{c}, \qquad n \geq md/c + k\sqrt{m}.$$

<u>Proof:</u> Let $V(\theta,\phi) = \sum_{m,n} e(m,n)\,\theta^{m}\phi^{n}$ where $|\theta|, |\phi| < 1$. It follows from the equation

$$e(m,n) = 1 + ce(m-1,n) + de(m,n-1), \qquad m,n > 0,$$

that $V(\theta,\phi)$ is the solution of

$$V(\theta,\phi) = \frac{\theta\phi}{(1-\theta)(1-\phi)} + (c\theta + d\phi)V(\theta,\phi)$$

i.e. $$V(\theta,\phi) = \frac{\theta\phi}{(1-\theta)(1-\phi)(1-c\theta-d\phi)}.$$

On expanding this generating function, and identifying coefficients,

(44) $$e(m,n) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\binom{i+j}{i}c^{i}d^{j}$$

Suppose m is temporarily fixed, and let

$$a_{n} = \frac{c}{m}\{e(m,n+1) - e(m,n)\}, \qquad n \geq 0.$$

It follows from (44) that $a_n \geq 0$. Let

$$W(\phi) = \sum_{n=0}^{\infty} a_n \phi^n$$

$$= \frac{c}{m} \sum_{n=0}^{\infty} \sum_{i=0}^{m-1} c^i \binom{i+n}{i} d^n \phi^n, \quad \text{using (44)},$$

(45) $$\text{i.e. } W(\phi) = \frac{1}{m} \sum_{i=1}^{m} \left\{\frac{c}{1-d\phi}\right\}^i$$

Then as $n \to \infty$, $e(m,n)$ increases monotonically to

$$\frac{m}{c} \sum_{n=0}^{\infty} a_n = \frac{m}{c} W(1) = \frac{m}{c},$$

so that the first inequality in Lemma 11 holds. Now let S_i be the sum of i independent random variables, each having the generating function $\left\{\frac{c}{1-d\phi}\right\}$, with mean d/c and finite variance σ^2 say. Then S_i has mean id/c and variance $i\sigma^2$, so that by the Tchebychev inequality, if $n > id/c$,

$$\Pr\{S_i \geq n\} \leq \Pr\{|S_i - id/c| \geq n - id/c\} \leq i\sigma^2/(n-id/c)^2.$$

Hence, using (45),

$$1 - \frac{c}{m} e(m,n) = \sum_{r=n}^{\infty} a_n$$

$$= m^{-1} \sum_{i=1}^{m} P_r\{S_i \geq n\}$$

$$\leq \frac{1}{m} \sum_{i=1}^{m} \frac{i\sigma^2}{(n-id/c)^2}$$

$$\leq \frac{m\sigma^2}{(n-md/c)^2}$$

$$\leq \varepsilon,$$

provided that $n - md/c \geq \sqrt{m\sigma^2/\varepsilon}$. Taking $k = \sigma/\sqrt{\varepsilon}$ completes the proof.

An upper bound is now found for the difference $E(m,n) - e(m,n)$.

<u>Lemma 12.</u> $$E(m,n) = e(m,n) + O(\log \min(m,n)) .$$

<u>Proof:</u> $E(m,n)$ is the least non negative solution of

$$y_i \geq 1 + \sum_j p_{ij} y_j , \qquad i \notin A,$$

where $i = (m,n)$, and the p_{ij} represent the transition probabilities for the process P. Since $p_{ij} = (1-\delta_{ij}) q_{ij}/q_i$ when $i \notin A$, the above condition may be rewritten as

$$0 \geq q_i + \sum_j q_{ij} y_j , \qquad i \notin A,$$

i.e. $$\sum_{j \neq i} q_{ij}(y_i - y_j) \geq q_i , \qquad i \notin A.$$

For the competition process this condition becomes

$$\begin{aligned} \Delta y(m,n) &= \alpha m\{y(m,n) - y(m+1,n)\} + \beta n\{y(m,n) - y(m,n+1)\} \\ &\quad + \gamma mn\{y(m,n) - y(m-1,n)\} + \delta mn\{y(m,n) - y(m,n-1)\} \qquad (46) \\ &\geq \alpha m + \beta n + \gamma mn + \delta mn. \end{aligned}$$

Now let $y(m,n) = e(m,n) + \rho(m,n)$, where $\rho(m,n)$ is to be chosen later.

From (44), $e(m,n) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \binom{i+j}{i} c^i d^j$, where $c = \gamma/(\gamma+\delta)$, $d = \delta/(\gamma+\delta)$.

Then (46) holds if

$$- \alpha m\{\sum_{j=0}^{n-1} \binom{m+j}{m} c^m d^j\} - \beta n\{\sum_{i=0}^{m-1} \binom{n+i}{n} c^i d^n\}$$

(47) $$+ (\gamma+\delta)mn\{e(m,n) - ce(m-1,n) - de(m,n-1)\} + \Delta\rho(m,n)$$

$$\geq \alpha m + \beta n + (\gamma+\delta)m,n.$$

On observing that

$$\sum_{j=0}^{n-1} \binom{m+j}{m} c^m d^j \leq \sum_{j=0}^{\infty} \binom{m+j}{m} c^m d^j \leq c^{-1},$$

and similarly $\sum_{i=0}^{m-1} \binom{n+i}{n} c^i d^n \leq d^{-1}$,

and also $$e(m,n) = 1 + ce(m-1,n) + de(m,n-1),$$

it follows that condition (47) holds if

(48) $$\Delta\rho(m,n) \geq 2\alpha c^{-1}m + 2\beta d^{-1}n.$$

(49) Let $\rho(m,n) = \lambda \sum_{r=0}^{\min(m,n)} \frac{1}{r+1} + \frac{\mu}{1-\phi}\{2-\phi^m-\phi^n\}$

where λ, μ and $\phi < 1$, are positive constants to be chosen later.

Let $$\tau(m,n) = \sum_{r=0}^{\min(m,n)} \frac{1}{r+1}.$$

Then $\tau(m+1,n) - \tau(m,n) = \frac{1}{m+2}$ if $m < n$, and 0 otherwise, and similarly for $\tau(m,n+1) - \tau(m,n)$. Hence condition (48) hold if

$$\lambda\left\{ - \frac{\alpha m}{m+2} - \frac{\beta n}{n+2} + \frac{mn[\min(\gamma,\delta)]}{1+[\min(m,n)]}\right\}$$

$$+ \mu m\phi^{m-1}(\gamma n - \alpha\phi) + \mu n\phi^{n-1}(\delta m - \beta\phi) \geq 2\alpha c^{-1}m + 2\beta d^{-1}n.$$

Choose $\phi < 1$ such that $\alpha\phi < \gamma/2$, $\beta\phi < \delta/2$. Then the last condition holds if

$$\text{(50)}\quad \lambda\{\tfrac{1}{2}[\min(\gamma,\delta)][\max(m,n)] - \alpha - \beta\} + \mu\phi^{m-1}\alpha/2 + \mu\phi^{n-1}\delta/2 \geq 2\alpha c^{-1}m + 2\beta d^{-1}n$$

Choose λ and M such that this inequality holds when $\max(m,n) \geq M$. Choose μ such that

$$-\lambda(\alpha+\beta) + \mu[\min(\gamma,\delta)]\,\phi^{M}/2 \geq 2(\alpha c^{-1} + \beta d^{-1})M.$$

Then (50) holds when $\max(m,n) < M$. Hence $\Delta\{e(m,n) + \rho(m,n)\} \geq 0$ when $m,n > 0$, and it follows from Lemmas 9 and 10 that

$$e(m,n) \leq E(m,n) \leq e(m,n) + \rho(m,n)$$

It is clear from the definition (49) of $\rho(m,n)$ that $\rho(m,n) = O(\log\min(m,n))$, which completes the proof of the lemma.

Lemmas 11 and 12 may now be combined to prove :-

<u>Theorem 3.</u> $E(m,n) \sim (\gamma+\delta)\min(m/\gamma, n/\delta)$ uniformly as m and $n \to \infty$.

<u>Proof:</u> It follows from Lemma 11 in its original form, and in the form where the roles of m and n are interchanged that given $\varepsilon > 0$,

$$\frac{m}{c} \geq e(m,n) \geq \frac{m}{c}(1-\varepsilon) - O(m^{\frac{1}{2}}), \qquad \text{as } m \to \infty \text{ with } n \geq \frac{md}{c}.$$

Similarly $\frac{n}{d} \geq e(m,n) \geq \frac{n}{d}(1-\varepsilon) - O(n^{\frac{1}{2}})$, as $n \to \infty$ with $n \leq \frac{md}{c}$.

The theorem then follows on applying Lemma 12, and substituting for c and d.

Chapter 2.

An Epidemic Model.

2.1 The process considered in this chapter is a stochastic model of an epidemic in continuous time, used by Bartlett in [1], and is constructed in the following way. Let the state space consist of the elements (m,n), where $m,n = 0, 1, 2, 3, \ldots$, and, as in chapter 1, let the process be specified by its matrix of transition rates $Q = \{q_{ij}\}$. For $i = (m,n)$, where $m,n \geq 0$, let

$$q_{ij} = \lambda mn, \qquad j = (m-1,n+1),$$

$$q_{ij} = \mu n, \qquad j = (m,n-1),$$

$$q_{ij} = \nu\ , \qquad j = (m+1,n),$$

$$q_{ij} = \varepsilon\ , \qquad j = (m,n+1),$$

where λ, $\mu > 0$ and ν, $\varepsilon \geq 0$ are constants. Otherwise let $q_{ij} = 0$. Since $\lambda mn = 0$ on the n axis and $\mu n = 0$ on the m axis, such a process cannot escape from the region $m \geq 0$, $n \geq 0$. Although the process differs from the model considered by Reuter in [20] in having fewer transition rates, Theorem 1 in [20] shows that there is always a unique transition matrix $\{p_{ij}(t)\}$ corresponding to the given matrix Q.

If m and n denote the respective numbers of susceptible and infectious individuals in a population, then λmn represents the rate of infection, μn the rate of removal of infectives, ν the birth rate of the susceptibles, and ε the rate of immigration of new infectives. The main results obtained are (i) the expected time for the infectives first to die out behaves as $\mu^{-1} \log(m+n)$ when the starting point (m,n) goes to infinity, (ii) the stationary distribution of the irreducible process, with $\nu, \varepsilon > 0$, converges geometrically on the state space, so that the process tends to spend most of its time near the origin, and (iii) when ν is small, the stationary distribution for the irreducible process, with $\nu, \varepsilon > 0$, approximates to the stationary distribution for the reducible process with $\nu = 0, \varepsilon > 0$.

2.2 In this section, all the points of the m axis are made absorbing, so that the expected time for the infectives first to to die out now becomes the expected absorption time. Let $\tau(m,n)$ be the expected absorption time for the process starting at (m,n). Sufficiently close upper and lower bounds are now found for $\tau(m,n)$ to show that $\tau(m,n) \sim \mu^{-1} \log(m+n)$ as $m + n \to \infty$ with $n > 0$.

<u>Lemma 1.</u>
$$\limsup_{\substack{m+n\to\infty \\ n>0}} \frac{\tau(m,n)}{\log(m.n)} \le \mu^{-1}.$$

<u>Proof:</u> An upper bound $u(m,n)$ is constructed using the criterion (C) given by Reuter in [20]. This criterion shows that absorption is certain and $\tau(m,n) \le u(m,n)$ when $u(m,n)$ is finite, nonnegative and satisfies

$$\Delta u(m,n) \ge 1, \qquad m \ge 0,\ n > 0,$$

where

$$\begin{aligned}\Delta u(m,n) = {} & \lambda mn\{u(m,n) - u(m-1,n+1)\} + \mu n\{u(m,n) - u(m,n-1)\} \\ (1) \qquad & + \nu\{u(m,n) - u(m+1,n)\} + \varepsilon\{u(m,n) - u(m,n+1)\}.\end{aligned}$$

Let $u(m,n)$ be defined for $m,n \ge 0$ by

$$(2) \qquad u(m,n) = \frac{1+c}{\mu}\sum_{r=0}^{m+n}\frac{1}{r+1} + A\sum_{r=0}^{m}\frac{1}{(r+1)(m+n+1-r)} + B\left\{\frac{1-\rho^{m+n}}{1-\rho}\right\},$$

where $c > 0$ is an arbitrary constant, and A, B, ρ, with $\rho < 1$, are positive constants to be chosen later. The first term in the expression (2) for $u(m,n)$ is suggested by the dominance of the term with coefficient μn in the expression (1) for $\Delta u(m,n)$ when $m = 0$ and n is large, and similarly the second term in the expression (2) for $u(m,n)$ is suggested by the dominance of the term in (1) with coefficient λmn when mn is large. The last term in (2) provides the correction needed near the origin. On substituting the expression (2) for $u(m,n)$ in (1),

$$\Delta u(m,n) = \left\{\frac{1+c}{\mu}\right\}\left\{\frac{\mu n}{m+n+1} - \frac{\nu+\varepsilon}{m+n+2}\right\} + B\rho^{m+n-1}\left\{\mu n - \rho(\nu+\varepsilon)\right\}$$

$$+A\left\{(\lambda mn+\mu n+\nu+\varepsilon)\sum_{r=0}^{m}\frac{1}{(r+1)(m+n+1-r)} - \lambda mn\sum_{r=0}^{m-1}\frac{1}{(r+1)(m+n+1-r)}\right.$$

$$\left.-\mu n\sum_{r=0}^{m}\frac{1}{(r+1)(m+n-r)} - \nu\sum_{r=0}^{m+1}\frac{1}{(r+1)(m+n+2-r)} - \varepsilon\sum_{r=0}^{m}\frac{1}{(r+1)(m+n+2-r)}\right\}$$

$$\geq \left\{\frac{1+c}{\mu}\right\}\left\{\frac{\mu n-\nu-\varepsilon}{m+n+1}\right\} + B\rho^{m+n}\{\mu n - \rho(\nu+\varepsilon)\}$$

$$+A\left\{\frac{\lambda mn}{(m+n)(n+1)} - \mu n\sum_{r=0}^{m}\frac{1}{(r+1)(m+n+1-r)(m+n-r)} - \frac{\nu}{(m+2)(n+1)}\right\}$$

Hence when $m \geq 0,\ n > 0$,

(3)
$$\Delta u(m,n) \geq \left\{\frac{1+c}{\mu}\right\}\left\{\frac{\mu n-\nu-\varepsilon}{m+n+1}\right\} + B\rho^{m+n}\{\mu n - \rho(\nu+\varepsilon)\}$$
$$+A\left\{\frac{\lambda m}{2(m+1)} - \mu\sum_{r=0}^{m}\frac{1}{(r+1)(m+n+1-r)} - \frac{\nu}{(m+2)(n+1)}\right\}.$$

The following inequalities are needed to deal with the above expression.

(4)
$$\sum_{r=0}^{m}\frac{1}{(r+1)(m+n+1-r)} \leq \frac{1}{m+n+2}\sum_{r=0}^{m+n}\left\{\frac{1}{r+1}+\frac{1}{m+n+1-r}\right\}$$
$$= O\left\{\frac{\log(m+n)}{m+n}\right\} \qquad \text{as } m+n \to \infty.$$

Hence there is a constant k such that

(5)
$$\sum_{r=0}^{m}\frac{1}{(r+1)(m+n+1-r)} \leq k, \qquad m \geq 0,\ n > 0.$$

Now choose $\rho < 1$ such that

(6) $$\rho(\nu+\varepsilon) \leq \frac{\mu}{2}.$$

Then using (3) and (4), it follows that when $m > 0$

$$\Delta u(m,n) \geq -\left\{\frac{1+c}{\mu}\right\}(\nu+\varepsilon) + A\left\{\frac{\lambda}{4} - O\left(\frac{\log(m+n)}{m+n}\right) - \frac{\nu}{(m+1)(n+1)}\right\}$$

Hence, by choosing suitably large M_0 and A,

(7) $$\Delta u(m,n) \geq 1, \qquad m > 0,\ n > 0,\ m+n \geq M_0.$$

Suppose $m = 0$. Then using (3),

$$\Delta u(m,n) \geq \left\{\frac{1+c}{\mu}\right\}\left\{\frac{\mu n-\nu-\varepsilon}{n+1}\right\} - A\left\{\frac{\mu}{n+1} + \frac{\nu}{2(n+1)}\right\}$$

Since A has already been chosen, there exists N_0 such that

(8) $$\Delta u(m,n) \geq 1, \qquad m = 0,\ n \geq N_0.$$

Let $M = \max(M_0, N_0)$. Then using (3), (5) and (6),

$$\Delta u(m,n) \geq -\left\{\frac{1+c}{\mu}\right\}(\nu+\varepsilon) - A(k\mu+\nu) + B\rho^M \mu/2,$$

$$n > 0,\ m+n \leq M.$$

Hence B may be chosen such that

(9) $$\Delta u(m,n) \geq 1, \qquad n > 0,\ m+n \leq M = \max(M_0, N_0).$$

On combining the inequalities (7), (8) and (9), it follows that $\Delta u(m,n) \geq 1$ when $m \geq 0$, $n > 0$. Hence the mean absorption time $\tau(m,n) \leq u(m,n) < \infty$. Then using (2) and (4)

$$u(m,n) \sim (1+c)\,\mu^{-1} \log(m+n) \qquad \text{as } m+n \to \infty.$$

Since c is arbitrary, the lemma follows.

A lower bound is now found for the mean absorption time $\tau(m,n)$. It is shown firstly that the process initially at (m,n), with $n > 0$, reaches a distance from the m axis of order $m+n$, with probability approaching 1 as $m \to \infty$. Secondly it is shown that $\tau(m,n) \geq \mu^{-1} \sum_{r=1} r^{-1}$. On combining these two results, a lower bound is obtained which gives

$$\liminf_{m+n \to \infty} \frac{\tau(m+n)}{\log(m+n)} \geq \mu^{-1}$$

Lemma 2. For each $\delta > 0$, the process initially at (m,n), with $n > 0$, reaches a distance at least $(1-\delta)(m+n)$ from the m axis with probability approaching 1 uniformly in n as $m \to \infty$.

Proof: Consider the region $m \geq M$, $n \geq 0$, where M is an integer to be chosen later, and all the boundary points are treated as absorbing. Let $X(m_0,n_0)$ be the probability that the epidemic process initially at (m_0,n_0) reaches the line $m = M$, $n \geq N$, where N is an integer to be chosen later. It follows from the discussion at the beginning of section 1.2 in chapter 1 that $X(m_0,n_0)$ is also the probability that the corresponding discrete time process reaches the line $m = M$, $n \geq N$ starting at (m_0,n_0). This discrete time process is now compared with a subsidiary process on the same region defined by the following transition probabilities: for $i = (m,n)$, with $m > M$, $n > 0$, let

$$p_{ij} = a = \frac{\nu}{\lambda M} \geq \frac{\nu}{\lambda mn + \mu n + \nu + \varepsilon}, \qquad j = (m+1,n-1),$$

(10) $$p_{ij} = d = \frac{\mu+\varepsilon}{\lambda M} \geq \frac{\mu n+\varepsilon}{\lambda mn+\mu n+\nu+\varepsilon}, \qquad j = (m,n-1),$$

$$p_{ij} = c = 1 - \frac{\mu+\varepsilon+\nu}{\lambda M} \leq \frac{\lambda mn}{\lambda mn+\mu n+\nu+\varepsilon}, \qquad j = (m-1,n+1),$$

where M is chosen to make $c > 0$. Let $x(m_0,n_0)$ be the probability that the subsidiary process reaches the line $m = M$, $n \geq N$, starting at (m_0,n_0). Using a proof similar to that of Lemma 3 in chapter 1, it is now shown that $X(m_0,n_0) \geq x(m_0,n_0)$ when $m_0 \geq M$, $n_0 \geq 0$.

In the iterative construction for $x(m,n)$, let the r step absorption probabilities be defined by

$$x^0(m,n) = 0, \qquad m > M,\ n > 0,$$

$$x^r(m,n) = 1, \qquad m = M,\ n \geq N,\ r \geq 0,$$

and $x^r(m,n) = 0$ elsewhere on the boundaries for $r \geq 0$; and for $m > M$, $n > 0$, $r \geq 0$,

(11) $$x^{r+1}(m,n) = cx^r(m-1,n+1) + dx^r(m,n-1) + ax^r(m+1,n).$$

As in Lemmas 1 and 2 of chapter 1, it is easily shown that

(12) $x^r(m-1,n+1) \geq x(m,n) \geq x^r(m,n-1), \quad m > M,\ n > 0,\ r \geq 0,$

$$x^r(m,n) \leq x^{r+1}(m,n) \to X(m,n), \quad \text{as} \quad r \to \infty;\ n \geq M,\ n \geq 0.$$

It follows from Lemma 1 in chapter 1 that $X(m,n)$ satisfies

(13)
$$X(m,n) = \left(\frac{\lambda mn}{q}\right)X(m-1,n+1) + \left(\frac{\mu n}{q}\right)X(m,n-1) + \left(\frac{\nu}{q}\right)X(m+1,n) + \left(\frac{\varepsilon}{q}\right)X(m,n+1),$$

for $m > M$, $n > 0$, where $q = q(m,n) = \lambda mn + \mu n + \nu + \varepsilon$.

Clearly $X(m,n) \geq x^0(m,n)$. Suppose that, for some $r \geq 0$, $X(m,n) \geq x^r(m,n)$. Then using (11), (12), (13), and the inequalities in (10), when $m \geq M$, $n \geq 0$,

$$\begin{aligned}
& X(m,n) \\
& \geq \left(\frac{\lambda mn}{q}\right)x^r(m-1,n+1) + \left(\frac{\mu n}{q}\right)x^r(m,n-1) + \left(\frac{\nu}{q}\right)x^r(m+1,n) + \left(\frac{\varepsilon}{q}\right)x^r(m,n+1) \\
& = \left\{cx^r(m-1,n+1) + \left(\frac{\lambda mn}{q} - c\right)x^r(m-1,n+1)\right\} \\
& \quad + \left\{dx^r(m,n-1) + \left[\left(\frac{\mu n}{q}\right)x^r(m,n-1) + \left(\frac{\varepsilon}{q}\right)x^r(m,n+1) - dx^r(m,n-1)\right]\right\} \\
& \quad + \left\{ax^r(m+1,n-1) + \left[\left(\frac{\nu}{q}\right)x^r(m+1,n) - ax^r(m+1,n-1)\right]\right\} \\
& \geq cx^r(m-1,n+1) + dx^r(m,n-1) + ax^r(m+1,n-1) \\
& \quad + \left(\frac{\lambda mn}{q} - c\right)x^r(m-1,n+1) + \left(\frac{\mu n+\varepsilon}{q} - d\right)x^r(m,n-1) + \left(\frac{\nu}{q} - a\right)x^r(m+1,n-1) \\
& \geq x^{r+1}(m,n) + \left(\frac{\lambda mn}{q} - c\right)x^r(m,n) + \left(\frac{\mu n+\varepsilon}{q} - d\right)x^r(m,n) + \left(\frac{\nu}{q} - a\right)x^r(m,n) \\
& = x^{r+1}(m,n) + \left\{\frac{\lambda mn+\mu n+\nu+\varepsilon}{q} - (c+d+a)\right\}x^r(m,n) \\
& = x^{r+1}(m,n).
\end{aligned}$$

Hence, by induction, $X(m,n) \geq x^r(m,n)$ for all $r \geq 0$, and, letting $r \to \infty$,

(14) $$X(m,n) \geq x(m,n), \qquad m \geq M,\ n \geq 0.$$

An estimate is now found for $x(m_0,n_0)$, with

(15) $$M = [m_0^{\frac{1}{2}}], \qquad N = [(1-\delta)(m_0+n_0)],$$

where the brackets $[\cdot]$ here denote the integer part of the contents. The subsidiary process is compared with processes having the same one step transition probabilities, but acting on various regions whose boundaries are assumed to be absorbing in each case. Then if (m_0,n_0) is the initial point,

$x(m_0,n_0) = P\{$Process on the region $m \geq M,\ n \geq 0$ reaches the line $m = M,\ n \geq N\}$

$\geq P\{$Process on the region $m \geq M$ reaches the line $m = M,\ n \geq N\}$

$- P\{$Process on the region $m \geq M,\ n \geq 0$ reaches the m axis$\}$

Hence

(16)

$x(m_0,n_0) \geq P\{$Process on the region $m \geq M,\ m+n \geq M+N$ reaches the line $m = M,\ n \geq N\}$

$- P\{$Process on the region $n \geq 0$ reaches the m axis$\}$, where the first term on the right of (16) follows from the structure of the transition probabilities. Although the process on the region $m \geq M$, $m+n \geq M+N$ takes place on a diamond shaped lattice, this may easily be reduced to the usual square lattice by the transformation

$$(17) \qquad m - M = m', \quad m + n - (M+N) = n',$$
$$\text{and} \quad m_0 - M = m_0', \quad m_0 + n_0 - (M+N) = n_0',$$

and the boundaries $m = M$, $m + n = M + N$ become the axes $m' = 0$, $n' = 0$.

For $i = (m', n')$ with $m', n' > 0$, the transition probabilities (10) become

$$(18) \qquad \begin{aligned} p_{ij} &= a = \frac{\nu}{\lambda M}, & j &= (m'+1, n'), \\ p_{ij} &= c = 1 - \frac{\mu+\varepsilon+\nu}{\lambda M}, & j &= (m'-1, n'), \\ p_{ij} &= d = \frac{\mu+\varepsilon}{\lambda M}, & j &= (m', n-1). \end{aligned}$$

Lemma 4 in chapter 1 may be applied when $c > a$, i.e. when M is large enough, to give

(19) $P\{\text{Process reaches the } n' \text{ axis}\} \geq 1 - \delta_0$,

if $n'_0 \geq m'_0 d/(c-a) + k(m'_0/\delta_0)^{\frac{1}{2}}$,

where $k = \{d(c+a-4ac)(c-a)^{-3}\}^{\frac{1}{2}}$.

Let $\delta_0 = m_0^{-\frac{1}{2}}$. Then from (15) and (18), $k = O\{m_0^{-1/4}\}$, and

$d/(c-a) = O\{m_0^{-\frac{1}{2}}\}$. Then (19) holds if

$$n_0' \geq m_0' \quad O\{m_0^{-\frac{1}{2}}\} + O\{(m_0')^{\frac{1}{2}}\}$$

i.e. $$m_0 + n_0 - (M+N) \geq (m_0 - M)O\{m_0^{-\frac{1}{2}}\} + O\{(m_0 - M)^{\frac{1}{2}}\},$$

using (17). It follows from the definitions (15) of M and N that the last inequality holds uniformly in n_0 for all large enough m_0. Thus (19) holds for all large enough m_0, say $m_0 \geq M_C$, amd may be rewritten as

(20) $P\{$Process on the region $m \geq M$, $m + n \geq M + N$ reaches the line $m = M$, $n > N$ starting from $(m_0, n_0)\}$

$$\geq 1 - m_0^{-\frac{1}{2}}, \qquad m_0 \geq M_0.$$

Also, by observing only the displacements in the n direction for the subsidiary process starting from (m_0, n_0), it follows that for all large enough m_0,

$P\{$Process on the region $n \geq 0$ reaches the m axis$\}$ $= \left(\frac{a+d}{c}\right)^{n_0}$

(21) $$\leq O(M^{-1})$$

$$= O\{m_0^{-\frac{1}{2}}\},$$

where $n_0 \geq 1$, $M = [m_0^{\frac{1}{2}}]$, and a, d, c are defined in (18). Hence by combining the inequalities (14), (16), (20) and (21),

$$X(m_0, n_0) \geq x(m_0, n_0) \geq 1 - O\{m_0^{-\frac{1}{2}}\}, \text{ uniformly in } n_0 \text{ as } m_0 \to \infty,$$

and the lemma now follows from the definition of $X(m_0,n_0)$.

A rough lower bound for the expected absorption time $\tau(m,n)$ is now obtained by comparing the epidemic process with a process which has only the transition rate μn,

i.e. $q_i = q_{ij} = \mu n$ when $i = (m,n)$, $j = (m,n-1)$ and $n \geq 0$.

<u>Lemma 3.</u> $$\tau(m,n) \geq \mu^{-1} \sum_{r=1}^{n} r^{-1}.$$

<u>Proof:</u> Using the construction method given by Feller, (see Feller [6] or Reuter [19]), let the r step transition probabilities be defined by

$$p_{ij}^{0}(t) = \delta_{ij} e^{-q_i t}, \qquad t \geq 0,$$

(22)

$$p_{ij}^{r+1}(t) = \delta_{ij} e^{-q_i t} + \int_0^t e^{-q_i(t-s)} \sum_{k \neq i} q_{ik} p_{kj}^{r}(s)ds, \quad r \geq 0, t \geq 0.$$

As mentioned in section 2.1, there is a unique transition matrix $\{p_{ij}(t)\}$ corresponding to the Q matrix for the epidemic process. Hence $p_{ij}^{r}(t) \to p_{ij}(t)$ as $r \to \infty$. Let the Laplace transforms of $p_{ij}^{r}(t)$ and $p_{ij}(t)$ with respect to t be $\phi_{ij}^{r}(\theta)$ and $\phi_{ij}(\theta)$ respectively. Then

$$\phi_{ij}^{r}(\theta) \to \phi_{ij}(\theta), \qquad \text{as } r \to \infty, \ \theta > 0.$$

On taking the Laplace transforms of equations (22),

$$\phi_{ij}^{0}(\theta) = \frac{\delta_{ij}}{q_i + \theta}, \qquad \theta > 0$$

(23)

$$\phi_{ij}^{r+1}(\theta) = \frac{\delta_{ij}}{q_i + \theta} + \sum_{k \neq i} \frac{q_{ik}\phi_{kj}^{r}(\theta)}{q_i + \theta}, \qquad r \geq 0,\ \theta > 0.$$

and on letting $r \to \infty$,

(24) $$\phi_{ij}(\theta) = \frac{\delta_{ij}}{q_i + \theta} + \sum_{k \neq i} \frac{q_{ik}\phi_{kj}(\theta)}{q_i + \theta}, \qquad \theta > 0.$$

Now suppose the absorbing states, i.e. the states of the m axis, are classed as one state, A say. For the epidemic process, let

$$\Phi_{m,n}^{r}(\theta) = \phi_{ij}^{r}(\theta),\ \Phi_{m,n}(\theta) = \phi_{ij}(\theta), \quad \text{when} \quad i = (m,n),\ j = A.$$

Then using equations (23),

(25) $$\Phi_{m,n}^{0}(\theta) = 0, \qquad n > 0,$$

(26) $$\Phi_{m,n}^{r}(\theta) = 1/\theta, \qquad n = 0,\ r \geq 0.$$

(27) $$\Phi_{m,n}^{r+1}(\theta) = \left\{\frac{\lambda mn}{q+\theta}\right\}\Phi_{m-1,n+1}(\theta) + \left\{\frac{\mu n}{q+\theta}\right\}\Phi_{m,n-1}^{r}(\theta)$$

$$+ \left\{\frac{\nu}{q+\theta}\right\}\Phi_{m+1,n}^{r}(\theta) + \left\{\frac{\varepsilon}{q+\theta}\right\}\Phi_{m,n+1}^{r}(\theta),$$

when $n > 0$, where $q = q(m,n) = \lambda mn + \mu n + \nu + \varepsilon$.

Now compare $\Phi_{m,n}^{r}(\theta)$ with $\Psi_n(\theta)$, the Laplace transform of the probability that the process, with the transition rates $q_i = q_{ij} = \mu n$ when $i = (m,n)$, $j = (m,n-1)$, is in the state O at time t. It follows from (24) that

$$(28) \qquad \Psi_n(\theta) = \frac{\mu n}{\mu n+\theta}\Psi_{n-1}(\theta)$$

$$= \frac{\mu n}{\mu n+\theta}\cdot\frac{\mu(n-1)}{\mu(n-1)+\theta}\cdots\frac{1}{\theta}, \qquad n \geq 0.$$

Clearly, using (25) and (26),

$$\Phi^{0}_{m,n}(\theta) = \delta_{n,0}/\theta \leq \Psi_n(\theta), \qquad n \geq 0,\ \theta > 0.$$

Suppose that for some $r \geq 0$, $\Phi_{m,n}(\theta) \leq \Psi_n(\theta)$ for $m,n \geq 0, \theta > 0$.

Then from (27) and (28)

$$\Phi^{r+1}_{m,n}(\theta) \leq \left\{\frac{\lambda mn}{q+\theta}\right\}\Psi_{n+1}(\theta) + \left\{\frac{\mu n}{q+\theta}\right\}\Psi_{n-1}(\theta) + \left\{\frac{\nu}{q+\theta}\right\}\Psi_n(\theta) + \left\{\frac{\varepsilon}{q+\theta}\right\}\Psi_{n+1}(\theta)$$

$$\leq \left\{\frac{\lambda mn+\nu+\varepsilon}{q+\theta}\right\}\Psi_n(\theta) + \left\{\frac{\mu n}{q+\theta}\right\}\left\{\frac{\mu n+\theta}{\mu n}\right\}\Psi_n(\theta)$$

$$= \Psi_n(\theta)$$

Hence by induction, $\Phi^{r}_{m,n}(\theta) \leq \Psi_n(\theta)$ for all m, n, r and θ, and on letting $r \to \infty$, $\Phi_{m,n}(\theta) \leq \Psi_n(\theta)$ for all m, n and θ. It was shown by Reuter in [20] that the expected absorption time

$$\tau(m,n) = \lim_{\theta \to 0+}\left\{\frac{1-\theta\,\Phi_{m,n}(\theta)}{\theta}\right\}.$$

Hence

$$\tau(m,n) \geq \lim_{\theta \to 0+}\left\{\frac{1-\theta\,\Psi_n(\theta)}{\theta}\right\} = \mu^{-1}\sum_{r=1} r^{-1},$$

which completes the proof of this lemma.

Lemma 4. $\liminf_{\substack{m+n\to\infty \\ n>0}} \frac{\tau(m,n)}{\log(m+n)} \geq \mu^{-1}$.

Proof: By Lemma 2, for each positive δ and η there exists M such that the epidemic process initially at (m_0,n_0), with $m_0 \geq M$, reaches a distance at least $(1-\delta)(m_0+n_0)$ from the m axis with probability at least $1-\eta$. Let the event $E(m,n)$ be defined by

$E(m,n) = \{$The epidemic process initially at (m_0,n_0) reaches a distance at least $(1-\delta)(m_0+n_0)$ from the m axis for the first time at $(m,n)\}$

Then, using Lemma 3, and the Strong Markov Theorem for continuous time (see Chung, [3], II, section 9), it follows that, if $m_0 \geq M$,

$\tau(m_0,n_0) =$

$\mathcal{E}\{$Absorption time of the process initially at $(m_0,n_0)\}$

$\geq \sum_{m,n} \mathcal{E}\{$Absorption time $\mid E(m,n)\}\ P\ \{E(m,n)\}$

$\geq \sum_{m,n} \mathcal{E}\{$Time to reach m axis after first reaching $(m,n) \mid E(m,n)\}\ P\ \{E(m,n)\}$

$= \sum_{m,n} \mathcal{E}\{$Absorption time of the process initially at $(m,n)\}\ P\ \{E(m,n)\}$

$$= \sum_{m,n} \tau(m,n) \quad P\{E(m,n)\}$$

$$\geq \{ \inf_{n \geq (1-\delta)(m_0+n_0)} \tau(m,n)\} \sum_{m,n} P\{E(m,n)\}$$

$$\geq \mu^{-1} \sum_{1 \leq r \leq (1-\delta)(m_0+n_0)} r^{-1} (1-\eta)$$

$$\sim (1-\eta)\mu^{-1} \log(m_0+n_0) \qquad \text{as} \quad m_0 + n_0 \to \infty.$$

If however $m_0 < M$, it follows directly from Lemma 3 that

$$\tau(m_0,n_0) \geq \mu^{-1} \sum_{r=1}^{n_0} r^{-1} \sim \mu^{-1}\log(m_0+n_0) \qquad \text{as } m_0 + n_0 \to \infty.$$

Hence
$$\liminf_{\substack{m_0+n_0 \to \infty \\ n_0 > 0}} \frac{\tau(m_0,n_0)}{\log(m_0+n_0)} \geq \frac{1-\eta}{\mu}.$$

But η may be chosen arbitrarily small, and therefore the lemma follows.

On combining the results of Lemmas 1 and 4, the following is obtained.

<u>Theorem 1.</u> $\tau(m,n) =$

$\mathcal{E}\{$Absorption time of the epidemic process initially at $(m,n)\}$

$\sim \mu^{-1} \log(m+n)$, as $m+n \to \infty$, $n > 0$.

2.3 In this section, the stationary behaviour of the irreducible epidemic process, with ν and $\varepsilon > 0$, is examined.

Since the process is irreducible, the transition probabilities $p_{ij}(t)$ have limits π_j independent of i as $t \to \infty$. It is shown firstly that the process is non-dissipative, i.e. $\sum_j \pi_j = 1$, before the rate of convergence of $\sum_j \pi_j$ is considered. Let $\pi_j = \pi(m,n)$ when $j = (m,n)$ with $m,n \geq 0$.

Theorem 2. $$\sum_{m,n} \pi(m,n) = 1$$

Proof: The criterion (D) given by Reuter in [20] may be generalised, with only a slight alteration in the proof, by replacing the single state I, (in the notation of [20]), by a finite set. The generalized criterion shows that, since the epidemic process is irreducible, and is uniquely specified by its transition rates as mentioned earlier, a sufficient condition for the process to be non dissipative is the existence of a function $u(m,n) \geq 0$ such that

$\Delta u(m,n) \geq 1$ for all but a finite set of (m,n), with $m,n \geq 0$, where

$$(29) \quad \Delta u(m,n) = \lambda mn\{u(m,n) - u(m-1,n+1)\} + \mu n\{u(m,n) - u(m,n-1)\} + \nu\{u(m,n) - u(m+1,n)\} + \varepsilon\{u(m,n) - u(m,n+1)\}.$$

Let $u(m,n) = m + n + C\rho^n/(1-\rho)$, where C and ρ are positive constants, with $\rho < 1$. Then

(30) $\Delta u(m,n) = \mu n - \nu - \varepsilon + C\rho^{n-1}(\lambda mn\rho - \mu n + \varepsilon\rho), \qquad m,n \geq 0.$

Let $C = (1+\nu+\varepsilon)/\varepsilon$, and $\rho = \frac{1}{2}$. Then

(31) $\Delta u(m,n) = 1, \qquad m \geq 0,\ n = 0.$

Using (30), choose N such that

(32) $\Delta u(m,n) \geq 1, \qquad m \geq 0,\ n \geq N,$

and then choose M such that

(33) $\Delta u(m,n) \geq 1 \qquad m \geq M,\ 0 < n < N.$

On combining (31), (32) and (33) it follows that $\Delta u(m,n) \geq 1$ on all but a finite set of (m,n), with $m,n \geq 0$, and the criterion then shows that the process is non-dissipative.

It is now shown that the stationary distribution converges geometrically on the state space, with a uniformity in ν which is used later. The following simple lemma is needed to obtain bounds on the $\pi(m,n)$'s.

Lemma 5. For any finite set A,

$$\sum_{i \in A} \sum_{j \notin A} \pi_i q_{ij} = \sum_{i \notin A} \sum_{j \in A} \pi_i q_{ij} .$$

Proof: It is known (see Kendall and Reuter, [13], Theorem 8) that $\sum_i \pi_i q_{ij} = 0$ for each j, i.e.

$$\sum_{i \neq j} \pi_i q_{ij} = \pi_j q_j = \sum_{\alpha \neq j} \pi_j q_{j\alpha} .$$

On summing the loft and right hand sides of the above equation over $j \in A$,

$$\sum_{j \in A} \left\{ \sum_{i \neq j} \pi_i q_{ij} \right\} = \sum_{j \in A} \left\{ \sum_{\alpha \neq j} \pi_j q_{j\alpha} \right\}$$

i.e. $$\sum_{j \in A} \left\{ \sum_{\substack{i \neq j \\ i \in A}} + \sum_{\substack{i \neq j \\ i \notin A}} \right\} \pi_i q_{ij} = \sum_{j \in A} \left\{ \sum_{\substack{\alpha \neq j \\ \alpha \in A}} + \sum_{\substack{\alpha \neq j \\ \alpha \notin A}} \right\} \pi_j q_{j\alpha}$$

i.e. $$\sum_{j \in A} \sum_{i \notin A} \pi_i q_{ij} = \sum_{j \in A} \sum_{\alpha \notin A} \pi_j q_{j\alpha}$$

which proves the lemma, since the series considered above are of positive terms and are convergent.

<u>Theorem 3.</u> For each δ and $h > 0$, with $\delta + h/(h+\varepsilon) < 1$, there exists a constant k such that

$$\sum_{m+n \geq R} \pi(m,n) \leq k\left(\frac{\nu}{\nu+\varepsilon} + \delta\right)^R, \qquad R \geq 0,\ 0 < \nu < h,$$

and hence $$\pi(m,n) = O\left\{\left(\frac{\nu}{\nu+\varepsilon} + \delta\right)^{m+n}\right\} .$$

<u>Proof:</u> Suppose $0 < \nu < h$ and let

$$A = \{(m,n) : m \leq M,\ m+n \leq M+N\}, \qquad \text{where } M \geq 0,\ N \geq 1.$$

On applying the lemma to the competition process, with this definition of A,

$$(34)\qquad \sum_{\substack{m = M+1 \\ 1 \le n \le N-1}} \pi(m,n)\lambda mn + \sum_{\substack{0 \le m \le M \\ m+n = M+N+1}} \pi(m,n)\mu n$$

$$= \sum_{\substack{m = M \\ 0 \le n \le N-1}} \pi(m,n)\nu + \sum_{\substack{0 \le m \le M \\ m+n = M+N}} \pi(m,n)(\nu+\varepsilon).$$

Hence

$$(35)\qquad \sum_{\substack{m = M+1 \\ 1 \le n \le N-1}} \pi(m,n) + \sum_{\substack{0 \le m \le M \\ m+n = M+N+1}} \pi(m,n)$$

$$\le \left\{\frac{\nu+\varepsilon}{\min[\lambda(M+1),\mu N]}\right\} \left\{ \sum_{\substack{m = M \\ 0 \le n \le N-1}} \pi(m,n) + \sum_{\substack{0 \le m \le M \\ m+n = M+N}} \pi(m,n)\right\}.$$

If $\delta > 0$ is constant, there exist integers M_0, N_0 such that

$$(36)\qquad \frac{\nu+\varepsilon}{\min[\lambda(M+1),\mu N]} \le \delta, \qquad M \ge M_0,\ N \ge N_0,\ 0 < \nu < h.$$

Now let $B(M,N) = \sum_{\substack{m = M \\ 0 \le n \le N-1}} \pi(m,n)$ and $C(M,N) = \sum_{\substack{0 \le m \le M \\ m+n = M+N}} \pi(m,n)$.

Then it follows from (35) and (36) that

$$(37)\qquad \{B(M+1,N) - \pi(M+1,0)\} + \{C(M+1,N) - \pi(M+1,N)\}$$

$$\le \delta\{B(M,N) + C(M,N)\}, \qquad M \ge M_0,\ N \ge N_0,\ 0 < \nu < h.$$

Since $\pi(M+1,N-1) \le$ the left hand side of (35), it follows from this, on replacing N by $N+1$ and using (36) and (37), that

(38) $\pi(M+1,N)$

$$\leq \left\{\frac{\nu+\varepsilon}{\min[\lambda(M+1),\mu(N+1)]}\right\}\left\{\sum_{\substack{m=M\\0\leq n\leq N}}\pi(m,n) + \sum_{\substack{0\leq m\leq M\\m+n=M+N+1}}\pi(m,n)\right\}$$

$$\leq \delta\{[B(M,N)+C(M,N)] + [C(M+1,N)-\pi(M+1,N)]\}$$

$$\leq \delta(1+\delta)\{B(M,N)+C(M,N)\}, \qquad M\geq M_0,\ N\geq N_0,\ 0<\nu<h .$$

Also, by applying the equation $\sum_i \pi_i q_{ij} = 0$ with $j=(M+1,0)$,

(39) $\pi(M+1,0)(\nu+\varepsilon) = \pi(M,0)\,\nu + \pi(M+1,1)\,\mu$,

so that, on using (37),

(40) $$\pi(M+1,0) = \left(\frac{\nu}{\nu+\varepsilon}\right)\pi(M,0) + \left(\frac{\mu}{\nu+\varepsilon}\right)\pi(M+1,1)$$

$$\leq \left(\frac{\nu}{\nu+\varepsilon}\right)B(M,N) + \left(\frac{\mu}{\varepsilon}\right)\{B(M+1,N)-\pi(M+1,0)\}$$

$$\leq \left(\frac{\nu}{\nu+\varepsilon}+\frac{\delta\mu}{\varepsilon}\right)\{B(M,N)+C(M,N)\},$$

$$M\geq M_0,\ N\geq N_0,\ 0<\nu<h .$$

Hence, on adding the inequalities obtained in (37), (38) and (40),

(41) $B(M+1,N) + C(M+1,N)$

$$\leq \left\{\delta+\delta(1+\delta)+\frac{\nu}{\nu+\varepsilon}+\frac{\delta\mu}{\varepsilon}\right\}\{B(M,N)+C(M,N)\},$$

$$M\geq M_0,\ N\geq N_0,\ 0<\nu<h .$$

Since δ is arbitrary, it follows from (41) that, given any $\delta > 0$, with $\delta + h/(h+\varepsilon) < 1$, there exist M_1 and N_1 such that

$$B(M+1,N) + C(M+1,N) \leq \left\{ \frac{\bar{\nu}}{\nu+\varepsilon} + \delta \right\} \{ B(M,N) + C(M,N) \} ,$$

$$M \geq M_1, \; N \geq N_1, \; 0 < \nu < h .$$

Clearly for N fixed, say $N = N_1$, and for $R \geq M_1 + N_1$, it follows that

$$\sum_{m+n \geq R} \pi(m,n) \leq \sum_{M \geq R-N_1} \{ B(M,N_1) + C(M,N_1) \}$$

$$\leq \sum_{M \geq R-N_1} \left\{ \frac{\bar{\nu}}{\nu+\varepsilon} + \delta \right\}^{M-M_1} \{ B(M_1,N_1) + C(M_1,N_1) \}$$

$$= O\left\{ \left(\frac{\bar{\nu}}{\nu+\varepsilon} + \delta \right)^R \right\} .$$

Hence the theorem follows when k is suitably chosen.

It is now possible to deduce from Theorem 3 that, as $\nu \to 0+$, the stationary distribution $\pi(m,n)$ of the epidemic process tends to that for the process given by the limiting case $\nu = 0$.

<u>Theorem 4.</u> For all small enough values of ν,

$$\sum_{m \geq M, n \geq 0} \pi(m,n) = \{ O(\nu) \}^M ,$$

where $O(\nu)$ is independent of M, and, as $\nu \to 0+$,

$$\pi(0,n) \to \{ (\tfrac{\bar{\varepsilon}}{\mu})^n/n! \} / \{ \sum_{r=0}^{\infty} (\tfrac{\bar{\varepsilon}}{\mu})^r/r! \}$$

and $\pi(m,n) \to 0$, when $m > 0$.

<u>Proof</u>: The proof of the first part follows easily from (34) and (39) which show that, for all $M \geq 0$, $N \geq 1$,

$$(42) \quad \lambda(M+1) \sum_{n=1}^{N-1} \pi(M+1,n)\, n \leq \nu \sum_{n=0}^{N-1} \pi(M,n) + (\nu+\varepsilon) \sum_{\substack{0 \leq m \leq M \\ m+n=M+N}} \pi(m,n) ,$$

$$(43) \quad \pi(M+1,0) \leq (\tfrac{\bar{\nu}}{\varepsilon})\, \pi(M,0) + (\tfrac{\bar{\mu}}{\varepsilon})\, \pi(M+1,1) .$$

Letting $N \to \infty$ in (42), it follows that

$$(44) \quad \sum_{n=1}^{\infty} \pi(M+1,n)\, n \leq (\tfrac{\nu}{\lambda}) \sum_{n=0}^{\infty} \pi(M,n), \qquad M \geq 0.$$

Hence, using (43) and (44),

$$\begin{aligned} \sum_{n=0}^{\infty} \pi(M+1,n) &= \pi(M+1,0) + \sum_{n=1}^{\infty} \pi(M+1,n) \\ &\leq (\tfrac{\nu}{\varepsilon})\, \pi(M,0) + (1 + \tfrac{\mu}{\varepsilon}) \sum_{n=1}^{\infty} \pi(M+1,n) \\ &\leq \{(\tfrac{\nu}{\varepsilon}) + (1 + \tfrac{\mu}{\varepsilon})(\tfrac{\nu}{\lambda})\} \sum_{n=0}^{\infty} \pi(M,n) \\ &= O(\nu) \sum_{n=0}^{\infty} \pi(M,n), \end{aligned}$$

where $O(\nu)$ is independent of M. The first part of the lemma follows on iterating the above inequality , and the last part of the lemma is a trivial consequence.

To obtain the behaviour of $\pi(0,n)$ as $\nu \to 0+$, apply Lemma 5, now with $A = \{(m,n) : m = 0,\ n < N\}$, for $N \geq 1$. then

(45) $$\sum_{n=0}^{N-1} \pi(0,n)\nu + \pi(0,N-1)\varepsilon = \sum_{n=1}^{N-2} \pi(1,n)\lambda n + \mu N\pi(0,N),$$

where the sum $\sum_{n=1}^{N-2} \pi(1,n)$ is null when $N \leq 2$. It then follows from (42) with $M = 0$ that

$$\sum_{n=1}^{N-2} \pi(1,n)\ \lambda n \leq \nu \sum_{n=0}^{\infty} \pi(0,n) = O(\nu)$$

so that from (45), $\pi(0,N) = (\frac{\varepsilon}{\mu N})\pi(0,N-1) + O(\nu)$, and thus

(46) $$\pi(0,n) = (0,0)\ (\tfrac{\varepsilon}{\mu})^n / n! + O(\nu)$$

where in this case $O(\nu)$ may depend on n. Given arbitrary constants η and $h > 0$, it follows, using Theorem 3, that there exists a fixed integer N such that

(47) $$\sum_{m+n \leq N} \pi(m,n) \geq 1 - \eta, \qquad 0 < \nu < h,$$

(48) $$\sum_{n=N+1}^{\infty} (\tfrac{\varepsilon}{\mu})^n / n! \leq \eta.$$

By the first part of the theorem, it follows from (47) that

$$1 - \sum_{n \leq N} \pi(0,n) \leq \eta + O(\nu), \qquad 0 < \nu < h.$$

Hence from (46),

$$|1-\pi(0,0)\sum_{n=0}^{N}(\tfrac{\varepsilon}{\mu})^n/n!| \leq \eta + O(\nu). \qquad 0 < \nu < h,$$

and from (48).

$$|1-\pi(0,0)\sum_{n=0}^{\infty}(\tfrac{\varepsilon}{\mu})^n/n!|$$

$$\leq |1-\pi(0,0)\sum_{n=0}^{N}(\tfrac{\varepsilon}{\mu})^n/n!| + \pi(0,0)\sum_{n=N+1}^{\infty}(\tfrac{\varepsilon}{\mu})^n/n!$$

$$\leq 2\eta + O(\nu)$$

$$\to 2\eta, \qquad \text{as } \nu \to 0+.$$

But η is arbitrary, so that $\lim_{\nu\to 0+}\pi(0,0) = \{\sum_{n=0}^{\infty}(\frac{\varepsilon}{\mu})^n/n!\}^{-1}$.

Therefore, using (46), $\lim_{\nu\to 0+}\pi(0,n) = \{(\frac{\varepsilon}{\mu})^n/n!\}/\{\sum_{r=0}^{\infty}(\frac{\varepsilon}{\mu})^r/r!\}$,

which completes the proof of Theorem 4.

The actual limiting values of $\pi(m,n)$ as $\nu \to 0+$ may also be found by using a result of J. Lamperti ([14], §4, Example) on the convergence of stationary solutions. Let the stationary distribution of the epidemic process be denoted by $\{\pi_i\}$ when $\nu \geq 0$, where for convenience the states are now labelled $i = 0, 1, 2, \ldots$. Suppose the matrix of jump transition probabilities is defined by $p_{ij} = (1-\delta_{ij})q_{ij}/q_i$. Then, for the given epidemic process, $\sum_j p_{ij} = 1$ when $\nu \geq 0$, and q_{ij}, q_i and p_{ij} are all continuous to the right at $\nu = 0$. In the following arguments, the values of the quantities considered when $\nu = 0$ are distinguished by a tilde if necessary. The problem is to show $\pi_i \to \tilde{\pi}_i$ as $\nu \to 0+$.

A criterion given by Foster ([8], Theorem 2) may be generalised to show that the jump process contains an ergodic set of states if there exists a non negative solution $\{u(m,n)\}$ of

$$\Delta u(m,n) \geq \lambda mn + \mu n + \nu + \varepsilon$$

for all but a finite number of (m,n) with m, $n \geq 0$, where $\Delta u(m,n)$ is defined as in (29). A solution is given by

$$u(m,n) = Am + Bn + C\rho^n/(1-\rho),$$

where A, B, C and ρ are constants, with $A > B + 1 > 2, \rho = \frac{1}{2}$, and C chosen such that $C\varepsilon - A\nu - B\varepsilon \geq \nu + \varepsilon$. It follows that the jump process is ergodic when $\nu > 0$, and, when $\nu = 0$, the jump process possesses on ergodic subset of states (the n axis), which is reached with probability 1 from any other state. It is well known (e.g. see Kendall and Reuter, [13], Theorem 1) that there exists a unique non negative solution $\{z_i\}$ of

(49) $$\sum_i z_i p_{ij} = z_i \qquad \text{for each } j, \text{ with } \sum_i z_i = 1.$$

It is also known (see Kendall and Reuter, [13], Theorem 8) that

$$\sum_i \pi_i q_{ij} = 0 \qquad \text{for each } j, \text{ with } \sum_i \pi_i = 1,$$

(50) i.e. $\sum_i \pi_i q_i p_{ij} = \pi_j q_j$ for each j, with $\sum_i \pi_i = 1$,

since $p_{ij} = (1-\delta_{ij})q_{ij}/q_i$. It follows from Theorem 3 when $\nu > 0$, and from calculating π_i explicitly when $\nu = 0$ that $\sum_j \pi_j q_j < \infty$. Therefore, since the non negative solution of (49) is unique, it follows from (50) that

(51) $z_i = \pi_i q_i / \sum_j \pi_j q_j$ for each j, when $\nu \geq 0$.

Hence

(52) $\pi_i = \left(\frac{z_i}{q_i}\right) / \sum_j \left(\frac{z_i}{q_i}\right)$ for each j, when $\nu \geq 0$.

Now let 0 be a state which is ergodic for all $\nu \geq 0$ (e.g the origin $(0,0)$ in the $m\ n$ plane). For $\nu > 0$, define $x_i = z_i/z_0$ so that

$$\sum_i \pi_i p_{ij} = x_j, \qquad \text{for all } j \text{ when } \nu > 0, \text{ with } x_0 = 1.$$

Lamperti's result in [14] is applicable to irreducible recurrent processes, but may easily be extended to cover the present situation for the jump process corresponding to the epidemic process. From this it follows that there exist $\{X_i\}$ such that $x_i \to X_i$ as $\nu \to 0+$, and

(53) $$\sum_i X_i \bar{p}_{ij} = X_j, \qquad \text{with } X_0 = 1.$$

Clearly $z_i = x_i / \sum_j x_j$ when $\nu > 0$, since $x_i = z_i/z_0$ and $\sum_i z_i = 1$. It is now shown that $\sum_i x_i \to \sum_i X_i < \infty$ as $\nu \to 0+$, so that $z_i = x_i / \sum_j x_j \to X_i / \sum_j X_j$. Then, from (53) and the uniqueness of the solution of (49), it follows that

$\bar{z}_i = X_i / \sum_j X_j$ and $z_i \to \bar{z}_i$ as $\nu \to 0+$.

For the given process, Theorem 3 shows that for each $\delta > 0$, there exists N such that

$$\sum_{i > N} \pi_i q_i < \delta\varepsilon \qquad \text{when } 0 < \nu < \nu_0 \text{ say.}$$

Hence it follows from (51) that since $q_i \geq \varepsilon$ and $\sum_i \pi_i = 1$,

$$\sum_{i > N} z_i = \sum_{i > N} \pi_i q_i / \sum_j \pi_j q_j < \delta\varepsilon / \sum_j \pi_j \varepsilon = \delta.$$

when $0 < \nu < \nu_0$. Then, since $x_i = z_i / z_0$,

(54) $$\sum_{i > N} x_i < \frac{\delta}{z_0}, \qquad 0 < \nu < \nu_0,$$

and therefore

$$\frac{1}{z_0} = \sum_{i \leq N} x_i + \sum_{i > N} x_i < \sum_{i \leq N} x_i + \frac{\delta}{x_0},$$

i.e. $$\frac{1-\delta}{z_0} < \sum_{i \leq N} x_i, \qquad 0 < \nu < \nu_0.$$

Therefore $\limsup_{\nu \to 0+} \left(\frac{1}{z_0}\right) < \infty.$

Thus, since $\frac{1}{z_0} = \sum_i x_i$, the series $\sum_i x_i$ is bounded as $\nu \to 0+$, and from (54) converges uniformly for all small enough $\nu > 0$. It is then easily shown that $\sum_i X_i < \infty$ and $\sum_i x_i \to \sum_i X_i$ as $\nu \to 0+$.

It has now been shown that $z_i \to \bar{z}_i$ as $\nu \to 0+$, where, from (52),

$$\pi_i = \left(\frac{\bar{z}_i}{\bar{q}_i}\right) / \sum_j \left(\frac{\bar{z}_j}{\bar{q}_j}\right) \quad \text{when } \nu > 0, \text{ and}$$

(55)
$$\bar{\pi}_i = \left(\frac{\bar{z}_i}{\bar{q}_i}\right) / \sum_j \left(\frac{\bar{z}_i}{\bar{q}_i}\right).$$

It remains to show that $\pi_i \to \bar{\pi}_i$. But

$$\left| \sum_i \frac{z_i}{q_i} - \sum_i \frac{\bar{z}_i}{\bar{q}_i} \right| \le \sum_i |z_i - \bar{z}_i|/q_i + \sum_i \bar{z}_i \left| \frac{1}{q_i} - \frac{1}{\bar{q}_i} \right|$$

Since $q_i \ge \varepsilon$ when $\nu \ge 0$, $z_i \to \bar{z}_i$ and $\sum_i z_i = \sum_i \bar{z}_i = 1$, it follows that

$$\sum_i |z_i - \bar{z}_i|/q_i \le \varepsilon^{-1} \sum_i |z_i - \bar{z}_i| \to 0 \qquad \text{as } \nu \to 0+.$$

Also by dominated convergence, since $q_i \to \bar{q}_i$,

$$\sum_i z_i \left| \frac{1}{q_i} - \frac{1}{\bar{q}_i} \right| \to 0 \qquad \text{as } \nu \to 0+.$$

Hence $\sum_i z_i/q_i \to \sum_i \overline{z}_i/\overline{q}_i$ as $\nu \to 0+$, and the result now follows from (55) .

Chapter 3.

The limiting behaviour of first hitting times for a general recurrent random walk in two dimensions

3.1. Firstly consider the problem of first hitting times in one dimension. For a simple random walk on the integers $0, \pm 1, \pm 2, \ldots$, with transition probabilities $p_{ij} = \frac{1}{2}$ when $j = i \pm 1$, Feller ([7], Chapter III) has shown that the probability that, starting at 0, the first passage through y occurs before time ty^2 tends as $y \to \infty$ to

$$(1) \qquad 1 - (2/\pi)^{\frac{1}{2}} \int_0^{t^{-\frac{1}{2}}} e^{-s^2/2}\, ds ,$$

which is the positive stable distribution of order $\frac{1}{2}$. In the case of one-dimensional Brownian motion, it may be shown (see Lévy, [15], Chapter III) that the corresponding first passage probability has exactly the value (1) for all y. It is shown in the following sections that this analogy carries over into two dimensions, and the limiting behaviour of the distribution of first hitting times is found for a random walk on a lattice, in both discrete and continuous time, and for Brownian motion. As a corollary, the limiting behaviour of the distribution of first hits on an axis is found for a restricted class of three-dimensional 'recurrent' random walks.

3.2. The first hitting time problem is now investigated for two-dimensional Brownian motion. Let $T(r)$ be the time at which the first passage through a disc of radius a occurs, when the starting point is at a distance r from the centre of the disc. F. Spitzer [21] has shown that the Laplace transform of the distribution of $T(r)$ is given by

$$(2) \qquad \int_{t=0}^{\infty} e^{-\lambda t} P\{T(r) \leq t\} \, dt = \frac{K_0(r\sqrt{2\lambda})}{\lambda K_0(a\sqrt{2\lambda})} ,$$

where the real part of λ is positive, and K_0 is the modified Bessel function of the second kind and zero order. By inverting this transform, the following limit may be obtained.

<u>Theorem 1.</u> For each $\alpha \geq 2$, $P\{T(r) \leq r^{\alpha}\} \to 1 - 2\alpha^{-1}$, as $r \to \infty$.

It is interesting to compare this result, and the analogous result of Theorem 2 later on, with the following result of Doney [4] for the symmetric random walk in discrete time and on a three-dimensional lattice. If such a process starts at $(x,y,0)$, with $r = (x^2+y^2)^{\frac{1}{2}}$, the probability that, when the first passage through the z axis occurs, the magnitude of the displacement in the z direction is less than r^{α}, where $\alpha \geq 1$, tends to $1-\alpha^{-1}$ as $r \to \infty$.

<u>Proof of Theorem 1:</u> On applying the complex inversion formula for the Laplace transform to (2), (see Widder [25]),

$$
\begin{aligned}
(3) \quad P\{T(r) \leq t\} &= \lim_{\tau \to \infty} \frac{1}{2\pi i} \int_{c-i\tau}^{c+i\tau} \frac{e^{\lambda t} K_0(r\sqrt{2\lambda})}{\lambda K_0(a\sqrt{2\lambda})} d\lambda \\
&= \lim_{\tau \to \infty} \frac{1}{2\pi i} \int_{c-i\tau}^{c+i\tau} \frac{e^{z} K_0(rt^{-\frac{1}{2}}\sqrt{2z})}{z K_0(at^{-\frac{1}{2}}\sqrt{2z})} dz
\end{aligned}
$$

where the second integral follows on substituting $z = \lambda t$, and noting that $c > 0$ is arbitrary in this case. Suppose $t = r^{\alpha}$ where $\alpha \geq 2$ is fixed. Then the limiting behaviour of $P\{T(r) \leq r^{\alpha}\}$ may be obtained on integrating by parts in (3), thereby introducing suitable dominating factors. Firstly let

$$
(4) \quad g_r(u) = \frac{K_0(rt^{-\frac{1}{2}}\sqrt{2z})}{K_0(at^{-\frac{1}{2}}\sqrt{2z})} = \frac{K_0[r^{1-\alpha/2}\sqrt{2(c+iu)}]}{K_0[ar^{-\alpha/2}\sqrt{2(c+iu)}]}, \qquad z = c+iu.
$$

Then $$P\{T(r) \leq r^{\alpha}\} = \lim_{\tau \to \infty} \frac{1}{2\pi} \int_{-\tau}^{\tau} \frac{e^{c+iu}}{c+iu} g_r(u)\, du,$$

and, on integrating by parts,

$$
\begin{aligned}
(5) \quad P\{T(r) \leq r^{\alpha}\} = \lim_{\tau \to \infty} \frac{1}{2\pi} \Bigg\{ &\left[\frac{e^{c+iu}}{i(c+iu)} g_r(u)\right]_{-\tau}^{\tau} - \\
&\int_{-\tau}^{\tau} \left[\frac{e^{c+iu}}{i(c+iu)} \frac{d}{du} g_r(u) - \frac{e^{c+iu}}{(c+iu)^2} g_r(u)\right] du \Bigg\}
\end{aligned}
$$

The following properties of $K_0(z)$ are now needed, (see Watson [23]) :- In the half plane with the real part of z positive, $K_0(z)$ and $K_0'(z)$ are analytic, and $K_0(z)$ has no zeros;

(6) $K_0(z) \sim -\log z$, and $K_0'(z) \sim -1/z$, as $z \to 0$;

(7) and $K_0(z) \sim e^{-z}(2\pi/z)^{\frac{1}{2}}$, and $K_0'(z) \sim -e^{-z}(2\pi/z)^{\frac{1}{2}}$, as $z \to \infty$.

Using (6) and (7) to construct bounds for K_0 , it follows from (4) that, when $\alpha \geq 2$, $g_r(u)$ is bounded for all u , and $g_r(u) \to 1 - 2\alpha^{-1}$ as $r \to \infty$ for all fixed u . Hence

(8) $$\left[\frac{e^{c+iu}}{i(c+iu)} g_r(u)\right]_{-\tau}^{\tau} \to 0 \qquad \text{as } \tau \to \infty .$$

Also, by applying dominated convergence, with $|g_r(u)|$ bounded for all r and u ,

(9)
$$\begin{aligned}\lim_{r\to\infty} \lim_{\tau\to\infty} \frac{1}{2\pi} \int_{-\tau}^{\tau} \frac{e^{c+iu}}{(c+iu)^2} g_r(u)\,du &= \lim_{r\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{c+iu}}{(c+iu)^2} g_r(u)\,du \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{c+iu}}{(c+iu)^2} \{\lim_{r\to\infty} g_r(u)\}\,du \\ &= (1-2\alpha^{-1}) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{c+iu}}{(c+iu)^2}\,du \\ &= 1 - 2\alpha^{-1} ,\end{aligned}$$

since $$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{x(c+iu)}}{(c+iu)^2}\,du = \lim_{\tau\to\infty} \frac{1}{2\pi i} \int_{c-i\tau}^{c+i\tau} \frac{e^{x\lambda}}{\lambda^2}\,d\lambda ,$$

which is the inversion formula for the function $h(x)$ with

Laplace transform λ^{-2}, i.e. $h(x) = x$. Now consider

$$\frac{d}{du} g_r(u) = \frac{d}{du} \frac{K_0[r^{1-\alpha/2}\sqrt{2(c+iu)}]}{K_0[a r^{-\alpha/2}\sqrt{2(c+iu)}]}$$

$$= \frac{i\, r^{1-\alpha/2}}{\sqrt{2(c+iu)}} \frac{K_0'[r^{1-\alpha/2}\sqrt{2(c+iu)}]}{K_0[a r^{-\alpha/2}\sqrt{2(c+iu)}]}$$

$$- \frac{i\, a\, r^{-\alpha/2}}{\sqrt{2(c+iu)}} \frac{K_0[r^{1-\alpha/2}\sqrt{2(c+iu)}]\, K_0'[a r^{-\alpha/2}\sqrt{2(c+iu)}]}{\{K_0[a r^{-\alpha/2}\sqrt{2(c+iu)}]\}^2}$$

Using (6) and (7) to construct bounds for K_0 and K_0', it follows that when $\alpha \geq 2$

$$\frac{d}{du} g_r(u) = O(|c+iu|^{-\frac{1}{2}}),$$

where the right hand side is independent of r, and $\frac{d}{du} g_r(u) \to 0$ as $r \to \infty$ when u is fixed. Hence by applying dominated convergence

$$\lim_{r\to\infty} \lim_{\tau\to\infty} \int_{-\tau}^{\tau} \frac{e^{c+iu}}{c+iu} \left\{\frac{d}{du} g_r(u)\right\} du = \lim_{r\to\infty} \int_{-\infty}^{\infty} \frac{e^{c+iu}}{c+iu} \left\{\frac{d}{du} g_r(u)\right\} du$$

$$= \int_{-\infty}^{\infty} \frac{e^{c+iu}}{c+iu} \left\{\lim_{r\to\infty} \frac{d}{du} g_r(u)\right\} du \qquad (10)$$

Therefore, on combining (8), (9) and (10) with (5), the proof of Theorem 1 is completed.

Theorem 1 may easily be extended to a more general region which is bounded and contains an open subset, by comparing the first hitting times with those for discs contained in and contained by the region.

3.3. Suppose that a general recurrent random walk in discrete time and on a two-dimensional lattice of points (x_1, x_2) where $x_1, x_2 = 0, \pm 1, \pm 2, \ldots$, is defined as follows. Let $\{X_n\}$, $n = 1, 2, \ldots$, be a sequence of independent identically distributed random variables, taking two-dimensional values with integer components. Consider a particle moving at random, whose position at time n is given by $S_n = S_o + X_1 + \ldots + X_n$, $n = 0, 1, 2, \ldots$, where S_o is the position at time 0. Let the characteristic function of the distribution of X_n be

$$\varphi(\theta) = \mathcal{E}\{e^{i\theta . X_n}\} = \sum_x e^{i\theta . x} P\{X_n = x\} ,$$

where $\theta = (\theta_1, \theta_2)$, $x = (x_1, x_2)$ and $\theta . x = \theta_1 x_1 + \theta_2 x_2$. Suppose the process defined by the random variable S_n is irreducible, and can in fact reach any point of the lattice with positive probability. As pointed out by F.Spitzer [22], a number theoretic argument shows that this last condition is equivalent to the condition that $\varphi(\theta) = 1$ only if both components of θ are multiples of 2π. Suppose also that the random variables X_n have zero mean and a finite moment of order greater than 2, i.e. $\mathcal{E}\{X_n = 0\}$, and there exists $\delta > 0$ such that $\mathcal{E}\{|X_n|^{2+\delta}\} < \infty$, where $|X_n|$ denotes the length of the vector X_n. Under these assumptions, the following result, which is analogous to that for two-dimensional Brownian motion, may be obtained.

Theorem 2. Let $T(x)$ be the time taken for the process to reach the origin starting at position x. Then for each $\alpha \geq 2$,

$$P\{T(x) \leq |x|^{\alpha}\} \to 1 - 2\alpha^{-1} \qquad \text{as } x \to \infty,$$

where $|x|$ denotes the length of the vector x.

The following two lemmas are needed.

Lemma 1. Let $a_n(x) = P\{T(x) > n\}$, and let $A(x,z) = \sum_{n=0}^{\infty} a_n(x) z^n$, where $0 < z < 1$. Then

$$A(x,z) = \frac{\iint \frac{1-e^{i\theta . x}}{1-z\varphi(\theta)}\, d\theta}{(1-z) \iint \frac{d\theta}{1-z\varphi(\theta)}},$$

where the double integration is with respect to the components of θ, and is to be taken over the square $-\pi \leq \theta_1 \leq \pi$, $-\pi \leq \theta_2 \leq \pi$, unless otherwise stated.

Proof: In the customary notation let p^n_{xy} be the probability that the process reaches y at the n th. step, starting from x , with $p_{xy}^{0} = \delta_{xy}$. Let f_{xy}^{n} be the probability that the process reaches y for the first time at the n th. step, starting from x . The generating function for f_{x0}^{n} , with $x \neq 0$, may be obtained by using the formula

$$p_{x0}^{n} = f_{x0}^{n}\, p_{00}^{0} + f_{x0}^{n-1}\, p_{00}^{1} + \dots + f_{x0}^{1}\, p_{00}^{n-1} , \qquad n \geq 1.$$

On multiplying this by z^n, with $0 < z < 1$, and summing over $n \geq 1$, with $x \neq 0$,

$$\sum_{n=1}^{\infty} p_{x0}^{n}\, z^{n} = \sum_{n=1}^{\infty} \sum_{r=0}^{n-1} f_{x0}^{n-r}\, p_{00}^{r}\, z^{n} = \sum_{n=1}^{\infty} f_{x0}^{n}\, z^{n} \sum_{r=0}^{\infty} p_{00}^{r}\, z^{r} ,$$

where all the above sums are finite. Hence

$$\text{(11)} \qquad \sum_{n=0}^{\infty} f_{x0}^{n}\, z^{n} = \sum_{n=0}^{\infty} p_{x0}^{n}\, z^{n} \Big/ \sum_{r=0}^{\infty} p_{00}^{r}\, z^{r} , \qquad x \neq 0.$$

Since

$$a_n(x) = P\{T(x) > n\} = \sum_{r=n+1}^{\infty} f_{x0}^{r} ,$$

It follows that when $x \neq 0$ and $0 < z < 1$,

$$\text{(12)} \qquad \begin{aligned} A(x,z) &= \sum_{n=0}^{\infty} a_n(x)\, z^n \\ &= \sum_{n=0}^{\infty} \sum_{r=n+1}^{\infty} f_{x0}^{r}\, z^{n} \\ &= \sum_{r=1}^{\infty} \sum_{n=0}^{r-1} f_{x0}^{r}\, z^{n} \\ &= \sum_{r=0}^{\infty} f_{x0}^{r} \left(\frac{1 - z^{r}}{1 - z} \right) \end{aligned}$$

It is shown independently, in Lemma 2 (21), that $\sum_{n=0}^{\infty} p_{00}^{n} = \infty$. Hence, since the process is irreducible, it follows from the general theory (see Feller [7]) that the process is recurrent and $\sum_{n=0}^{\infty} f_{x0}^{n} = 1$. Now (11) and (12) may be combined to give

$$A(x,z) = \frac{1 - \sum_{n=0}^{\infty} f_{x0}^{n} z^{n}}{1-z} \tag{13}$$
$$= \frac{\sum_{r=0}^{\infty} p_{00}^{r} z^{r} - \sum_{n=0}^{\infty} p_{x0}^{n} z^{n}}{(1-z)\sum_{r=0}^{\infty} p_{00}^{r} z^{r}} .$$

If $\varphi(\theta) = \mathcal{E}\{e^{i\theta.X_n}\}$, then $[\varphi(\theta)]^n = \mathcal{E}\{e^{i\theta.S_n} | S_0 = 0\}$, and

$$p_{xy}^{n} = P\{S_n = y | S_0 = x\}$$
$$= P\{S_n = y-x | S_0 = 0\}$$
$$= \frac{1}{4\pi^2} \iint [\varphi(\theta)]^n e^{-i\theta.(y-x)} d\theta .$$

Hence $$\sum_{n=0}^{\infty} p_{xy}^{n} z^{n} = \sum_{n=0}^{\infty} \frac{z^n}{4\pi^2} \iint [\varphi(\theta)]^n e^{-i\theta.(y-x)} d\theta$$

$$= \frac{1}{4\pi^2} \iint \sum_{n=0}^{\infty} [z\varphi(\theta)]^n e^{-i\theta.(y-x)} d\theta \tag{14}$$

$$= \frac{1}{4\pi^2} \iint \frac{e^{-i\theta.(y-x)}}{1 - z\varphi(\theta)} d\theta ,$$

where the summation and integration may be interchanged since the function considered has modulus $\leq z^n$, and is therefore absolutely convergent under summation and integration over the given limits. The lemma is now completed by using (13) and (14) to give

$$A(x,z) = \frac{\iint \frac{d\theta}{1 - z\varphi(\theta)} - \iint \frac{e^{i\theta.x}}{1 - z\varphi(\theta)}\, d\theta}{(1-z)\iint \frac{d\theta}{1 - z\varphi(\theta)}}$$

$$= \frac{\iint \frac{1 - e^{i\theta.x}}{1 - z\varphi(\theta)}\, d\theta}{(1-z)\iint \frac{d\theta}{1 - z\varphi(\theta)}} .$$

The result for the asymptotic behaviour of the first hitting time $T(r)$ for two-dimensional Brownian motion depends on the behaviour of the Laplace transform $\int_0^\infty e^{-\lambda t}\, P\{T(r) \leq t\}\, dt$ when λ is near 0 and r is large. The behaviour of the generating function $A(x,z) = \sum_{n=0}^{\infty} P\{T(r) > n\}\, z^n$ is now examined when z is near 1 and x is large.

<u>Lemma 2.</u> If $z = 1 - \eta$, with $\eta > 0$ small, then

$$A(x,z) = \frac{2 \log|x| - f(x,\eta) + O(1)}{-\eta\{\log \eta + O(1)\}} ,$$

with
$$f(x,\eta) = 2 \int_0^{k|x|} \frac{1 - J_0(u)}{u(1 + u^2/c^2\eta|x|^2)}\, du ,$$

where $O(1)$ and the constants $c, k > 0$ are independent of x and η, and J_0 is the Bessel function of first kind and zero order.

Proof: Consider the behaviour of $\varphi(\theta)$ near $\theta = 0$. Following a method due to F.Spitzer [22], let

$$h(u) = e^{iu} - (1 + iu - u^2/2) .$$

Suppose $\mathcal{E}\{|X_n|^{2+\delta}\} < \infty$. Then a constant d may be chosen such that

$$|h(u)| \leq d\,|u|^{2+\delta} \qquad \text{for all real } u ,$$

and, by applying Schwarz's inequality,

$$\begin{aligned} |\mathcal{E}\{h(\theta.X_n)\}| &\leq d\,\mathcal{E}\{|\theta.X_n|^{2+\delta}\} \\ &\leq d\,\mathcal{E}\{(\theta.\theta)^{1+\delta/2}\,(X_n.X_n)^{1+\delta/2}\} \\ &\leq d\,|\theta|^{2+\delta}\,\mathcal{E}\{|X_n|^{2+\delta}\} , \end{aligned}$$

i.e. $$\mathcal{E}\{h(\theta.X_n)\} = O(|\theta|^{2+\delta}) .$$

Now let $\mu(\theta) = 2\,\mathcal{E}\{(\theta.X_n)^2\}$, which is a positive definite quadratic form in the components θ_1, θ_2 of θ. Then

$$\begin{aligned} \varphi(\theta) &= \mathcal{E}\{e^{i\theta.X_n}\} \\ &= \mathcal{E}\{1 + i(\theta.X_n) - (\theta.X_n)^2/2 + h(\theta.X_n)\} \\ &= 1 - \tfrac{1}{2}\mathcal{E}\{(\theta.X_n)^2\} + \mathcal{E}\{h(\theta.X_n)\} , \end{aligned}$$

i.e. $$\varphi(\theta) = 1 - \mu(\theta) + O(|\theta|^{2+\delta}) .$$

Hence if η and θ are both small,

$$1 - z\varphi(\theta) = 1 - (1-\eta)\{1 - \mu(\theta) + O(|\theta|^{2+\delta})\}$$
$$= \{\eta + \mu(\theta)\}\{1 + O(|\theta|^{\delta})\} ,$$

so that

(15) $$\frac{1}{1 - z\varphi(\theta)} = \frac{1 + O(r^{\delta})}{\eta + \mu(\theta)}$$ for all small enough η and θ.

The quadratic form $\mu(\theta)$ may be reduced to r^2 by making a transformation involving a rotation and a linear change of scale in the θ plane, say

(16)
$$\theta_1 = r\{\sigma_1 \cos(\psi-\psi_0) + \sigma_2 \sin(\psi-\psi_0)\} ,$$
$$\theta_2 = r\{-\sigma_1 \sin(\psi-\psi_0) + \sigma_2 \cos(\psi-\psi_0)\} ,$$

where r and ψ are polar coordinates, and σ_1, σ_2 and ψ_0 are constants. Hence (15) may now be rewritten as

(17) $$\frac{1}{1 - z\varphi(\theta)} = \frac{1 + O(r^{\delta})}{\eta + r^2} , \qquad r \leq r_0 , \quad \eta \leq \eta_0 ,$$

where r_0 and η_0 are suitable small constants.

Now consider the numerator and denominator of the expression for $A(x,z)$ in Lemma 1. The numerator may be written as

$$\iint \frac{1-e^{i\theta.x}}{1-z\varphi(\theta)}\,d\theta = \iint\limits_{\substack{|\theta_1|,|\theta_2|\le\pi \\ r>r_0}} \frac{1-e^{i\theta.x}}{1-z\varphi(\theta)}\,d\theta + \iint\limits_{\substack{|\theta_1|,|\theta_2|\le\pi \\ r\le r_0}} \frac{1-e^{i\theta.x}}{1-z\varphi(\theta)}(\sigma_1^2+\sigma_2^2)r\,dr\,d\psi,$$

where $r(\sigma_1^2+\sigma_2^2)$ is the Jacobian of the transformation (16). Since $\varphi(\theta)$ is continuous, and $\varphi(\theta)=1$ only if both θ_1 and θ_2 are multiples of 2π, $\varphi(\theta)$ is bounded away from 1 if θ is bounded away from 0 in the region $|\theta_1|, |\theta_2| \le \pi$. It follows that the above integral over $r>r_0$ is bounded independently of x and z when $0<z<1$. To tackle the integral over $r\le r_0$, suppose firstly that $x = (|x| \cos\psi_1, |x| \sin\psi_1)$. Then using (16),

$$\begin{aligned}\theta.x &= r|x|\{\sigma_1[\cos\psi_1 \cos(\psi-\psi_0) - \sin\psi_1 \sin(\psi-\psi_0)] \\ &\qquad + \sigma_2[\cos\psi_1 \sin(\psi-\psi_0) + \sin\psi_1 \cos(\psi-\psi_0)]\} \\ &= r|x|\{\sigma_1 \cos(\psi-\psi_0+\psi_1) + \sigma_2 \sin(\psi-\psi_0+\psi_1)\} \\ &= c\,r\,|x| \cos(\psi-\psi_2),\end{aligned} \tag{18}$$

where c and ψ_2 are constants, and c is independent of x. Then, if $\eta\le\eta_0$, it follows from (17) that

$$\iint_{\substack{|\theta_1|,|\theta_2|\le\pi \\ r\le r_0}} \frac{1-e^{i\theta.x}}{1-z\varphi(\theta)}\; r\,dr\,d\psi$$

$$= \iint_{r\le r_0} (1-e^{i\theta.x})\;\frac{1+O(r^{\delta})}{\eta+r^2}\; r\,dr\,d\psi$$

$$= \iint_{r\le r_0} (1-e^{i\theta.x})\left\{\frac{1}{r^2}-\frac{\eta}{r^2(\eta+r^2)}+O(r^{\delta-2})\right\} r\,dr\,d\psi .$$

Clearly the contribution to the integral from the term $O(r^{\delta-2})$ is bounded, independently of x and η. Using (18),

$$\begin{aligned}\int_{\psi=0}^{2\pi}(1-e^{i\theta.x})\,d\psi &= 2\pi-\int_0^{2\pi} e^{i\,c\,r\,|x|\,\cos(\psi-\psi_2)}\,d\psi\\ &= 2\pi-\int_0^{2\pi} e^{i\,c\,r\,|x|\,\cos\psi}\,d\psi\\ &= 2\pi-4\int_0^{\pi/2}\cos(c\,r\,|x|\cos\psi)\,d\psi\\ &= 2\pi\{1-J_0(c\,r\,|x|)\},\end{aligned}$$

since J_0 has the representation (see Watson [23])

$$J_0(u)=\frac{2}{\pi}\int_0^{\pi/2}\cos(u\cos\psi)\,d\psi .$$

Since $J_0(u)=1+O(u^2)$ as $u\to 0$, and $J_0(u)=O(u^{-\frac{1}{2}})$ as $u\to\infty$, (see Watson [23]), it follows that when x is large,

$$\begin{aligned}\iint_{r\le r_0}(1-e^{i\theta.x})\left(\frac{1}{r^2}\right) r\,dr\,d\psi &= 2\pi\int_{r=0}^{r_0}\{1-J_0(c\,r\,|x|)\}\,\frac{dr}{r}\\ &= 2\pi\int_{u=0}^{k|x|}\{1-J_0(u)\}\,\frac{du}{u}\\ &= 2\pi\log|x|+O(1),\end{aligned}$$

where $k = cr_o$ and $O(1)$ are independent of x and η when $\eta \leq \eta_o$. Also

$$\iint_{r \leq r_o} (1 - e^{i\theta.x}) \frac{\eta}{r^2(\eta + r^2)} \, r \, dr \, d\psi = 2\pi \int_{r=0}^{r_o} \frac{\{1 - J_o(c\, r\, |x|)\}\eta r}{r^2(\eta + r^2)} \, dr$$

$$= 2\pi \int_0^{k|x|} \frac{1 - J_o(u)}{u\{1 + u^2/c^2\eta|x|^2\}} \, du$$

$$= \pi \, f(x, \eta) \, .$$

Hence

$$(19) \qquad \iint \frac{1 - e^{i\theta.x}}{1 - z\varphi(\theta)} \, d\theta = \pi(\sigma_1^2 + \sigma_2^2)\{2 \log|x| + f(x,\eta) + O(1)\}$$

where $O(1)$ is independent of x and η when $\eta \leq \eta_o$.

The integral in the denominator of the expression for $A(x,z)$ in Lemma 1 may be written as

$$\iint \frac{d\theta}{1 - z\varphi(\theta)} = \iint_{\substack{|\theta_1|, |\theta_2| \leq \pi \\ r > r_o}} \frac{d\theta}{1 - z\varphi(\theta)} + (\sigma_1^2 + \sigma_2^2) \iint_{\substack{|\theta_1|, |\theta_2| \leq \pi \\ r \leq r_o}} \frac{r \, dr \, d\psi}{1 - z\varphi(\theta)}$$

The integral over $r > r_o$ is bounded independently of x and z when $0 < z < 1$, since $\varphi(\theta)$ is then bounded away from 1 in the range of integration. Using (17) the integral over $r \leq r_o$, with $\eta \leq \eta_o$, may be written as

$$\iint_{r \le r_0} \frac{r\,dr\,d\psi}{1 - z\varphi(\theta)} = \iint_{r \le r_0} \frac{1 + O(r^\delta)}{\eta + r^2}\, r\,dr\,d\psi$$

$$= 2\pi \int_{r=0}^{r_0} \frac{r\,dr}{\eta + r^2} + O(1)$$

$$= -\pi \log \eta + O(1) \,,$$

where $O(1)$ is independent of $\eta \le \eta_0$ and x. Hence

$$(20) \qquad \iint \frac{d\theta}{1 - z\varphi(\theta)} = \pi(\sigma_1^2 + \sigma_2^2)\{-\log \eta + O(1)\} \,,$$

and the lemma follows on combining (19) and (20). Also from (20),

$$(21) \qquad \sum_{n=0}^{\infty} p_{00}^n = \lim_{z \to 1-} \frac{1}{4\pi^2} \iint \frac{d\theta}{1 - z\varphi(\theta)} = \infty \,,$$

which is the condition required in Lemma 1.

<u>Proof of Theorem 2:</u> Let $B(x,t) = \sum_{n \le t} a_n(x)$. Then, on substituting $z = e^{-\lambda}$, with λ small, say $0 < \lambda < \lambda_0$, in Lemma 2,

$$(22) \qquad \int_0^\infty e^{-\lambda t}\, d_t B(x,t) = A(x, e^{-\lambda}) = \frac{2 \log|x| - h(x,\lambda) + O(1)}{-\lambda\{\log \lambda + O(1)\}} \,,$$

where $O(1)$ is independent of x and λ, and

$$(23) \quad h(x,\lambda) = f(x, 1 - e^{-\lambda}) = 2 \int_0^{k|x|} \frac{1 - J_0(u)}{u\{1 + u^2/c^2(1-e^{-\lambda})\,|x|^2\}}\, du.$$

The Tauberian Theorems 98 and 108 in [10] use a method due to Karamata to obtain the asymptotic behaviour of a function from the behaviour of its Laplace Stieltjes transform, with respect to λ say, near $\lambda = 0$. Since $A(x, e^{-\lambda})$ is the Laplace transform of a series of positive terms, this real variable method may conveniently be modified here, by considering $A(x, e^{-\lambda})$ for large values of x as well as for small values of λ, to obtain the behaviour of $B(x, |x|^{\alpha})$ when $\alpha \geq 2$.

By Theorem 99 in [10], for each real function g, Riemann integrable on $(0,1)$, and for each $\varepsilon > 0$, there exist polynomials

$$p(u) = \sum_{s=0}^{i} p_s u^s \quad \text{and} \quad q(u) = \sum_{s=0}^{j} q_s u^s \qquad \text{such that}$$

$$(24) \qquad p < g < q \quad \text{and} \quad \int_0^{\infty} e^{-t} \{ q(e^{-t}) - p(e^{-t}) \} \, dt < \varepsilon .$$

Then, since $B(x,t)$ increases with t,

$$(25) \qquad \begin{aligned} \int_0^{\infty} e^{-\lambda t} g(e^{-\lambda t}) \, d_t B(x,t) &\geq \int_0^{\infty} e^{-\lambda t} p(e^{-\lambda t}) \, d_t B(x,t) \\ &= \int_0^{\infty} e^{-\lambda t} \sum_{s=0}^{i} p_s e^{-s\lambda t} \, d_t B(x,t) \\ &= \sum_{s=0}^{i} p_s \int_0^{\infty} e^{-(s+1)\lambda t} \, d_t B(x,t) \\ &= \sum_{s=0}^{i} p_s A\{x, e^{-(s+1)\lambda}\} \end{aligned}$$

Using (22), with λ small,

$$A\{x, e^{-(s+1)\lambda}\} = \frac{2\log|x| - h\{x,(s+1)\lambda\} + O(1)}{-(s+1)\lambda\{\log\lambda + O(1)\}}$$

(26)

$$= -\frac{2\log|x|}{(s+1)\lambda\log\lambda} + \Delta_s(x,\lambda) \quad ,$$

where

$$(27) \qquad \Delta_s(x,\lambda) = \frac{O\{h[x,(s+1)\lambda]\} + O\left(\frac{\log|x|}{\log\lambda}\right)}{(s+1)\lambda\log\lambda} \quad ,$$

where $O(.)$ is now independent of x and λ, as $\lambda \to 0+$, but may depend on s. Let $g(u) = u^{-1}$ when $e^{-1} \le u \le 1$, and 0 otherwise, i.e. $g(e^{-t}) = e^t$ when $0 \le t \le 1$, and 0 otherwise. Then, since $B(x,0) = 0$, it follows from (25) and (26) that

$$B(x,\lambda^{-1}) = \int_0^\infty e^{-\lambda t} g(e^{-\lambda t})\, d_t B(x,t)$$

$$\ge \sum_{s=0}^{i} p_s \left\{-\frac{2\log|x|}{(s+1)\lambda\log\lambda} + \Delta_s(x,\lambda)\right\}$$

$$= -\left(\frac{2\log|x|}{\lambda\log\lambda}\right) \sum_{s=0}^{i} p_s (s+1)^{-1} + \sum_{s=0}^{i} p_s \Delta_s(x,\lambda).$$

But

$$\sum_{s=0}^{i} p_s (s+1)^{-1} = \int_0^\infty e^{-t} \sum_{s=0}^{i} p_s e^{-st}\, dt$$

$$= \int_0^\infty e^{-t} p(e^{-t})\, dt$$

$$\ge 1 - \varepsilon ,$$

using (24) and the above definition of g. Hence

$$(28) \qquad B(x,\lambda^{-1}) \ge -\frac{(1-\varepsilon)2\log|x|}{\lambda\log\lambda} + \sum_{s=0}^{i} p_s \Delta_s(x,\lambda) \quad .$$

So far λ has been small but independent of x. Now let $\lambda = |x|^{-\alpha}$, where $\alpha \geq 2$ is fixed. Then, since $J_o(u) = 1 - O(u^2)$ as $u \to 0$, $J_o(u) = O(u^{-\frac{1}{2}})$ as $u \to \infty$, and

$$\{1 - e^{-(s+1)\lambda}\} |x|^2 \sim (s+1) |x|^{-\alpha+1}, \qquad \text{as} \quad x \to \infty,$$

it follows from (23) that $h\{x,(s+1)\lambda\} = O(1)$ as $x \to \infty$. Therefore, using (27) with $\lambda = |x|^{-\alpha}$, and $\alpha \geq 2$ fixed,

$$\Delta_s(x,\lambda) = |x|^{\alpha} O\left(\frac{1}{\log|x|}\right) \qquad \text{as} \quad x \to \infty,$$

where $O(.)$ may depend on s, so that from (28)

$$B(x,|x|^{\alpha}) \geq |x|^{\alpha} \left\{ \frac{2}{\alpha}(1-\varepsilon) + O\left(\frac{1}{\log|x|}\right)\right\} \qquad \text{as} \quad x \to \infty.$$

Since $\varepsilon > 0$ is arbitrary, it now follows that

$$\liminf_{x \to \infty} \frac{B(x,|x|^{\alpha})}{|x|^{\alpha}} \geq \frac{2}{\alpha}, \qquad \alpha \geq 2.$$

The previous argument may be repeated similarly beginning with the inequality $g < q$ to obtain

$$\limsup_{x \to \infty} \frac{B(x,|x|^{\alpha})}{|x|^{\alpha}} \leq \frac{2}{\alpha}, \qquad \alpha \geq 2.$$

Hence for $\alpha \geq 2$, $B(x,|x|^{\alpha}) \sim 2|x|^{\alpha}/\alpha$, as $x \to \infty$.

The behaviour of $P\{T(x) > |x|^{\alpha}\}$ as $x \to \infty$ is now easily obtained since

$$B(x,t) = \sum_{n \leq t} P\{T(x) > n\},$$

where $P\{T(x) > n\}$ is monotonically decreasing as n increases. Thus, if $\alpha \geq 2$ and $\delta > 0$,

$$B(x, |x|^{\alpha+\delta}) - B(x, |x|^{\alpha}) \leq |x|^{\alpha+\delta} P\{T(x) > |x|^{\alpha}\} ,$$

and on dividing both sides by $|x|^{\alpha+\delta}$ and letting $x \to \infty$,

$$\liminf_{x \to \infty} P\{T(x) > |x|^{\alpha}\} \geq 2(\alpha+\delta)^{-1} .$$

Hence, since $\delta > 0$ is arbitrary,

$$\liminf_{x \to \infty} P\{T(x) > |x|^{\alpha}\} \geq 2\alpha^{-1} , \qquad \alpha \geq 2 .$$

Similarly $$B(x, |x|^{\alpha}) \geq |x|^{\alpha} P\{T(x) > |x|^{\alpha}\} ,$$

so that $$\limsup_{x \to \infty} P\{T(x) > |x|^{\alpha}\} \leq 2\alpha^{-1} , \qquad \alpha \geq 2 .$$

Hence $$\lim_{x \to \infty} P\{T(x) > |x|^{\alpha}\} = 2\alpha^{-1} \qquad \alpha \geq 2 ,$$

which completes the proof of Theorem 2 .

The following deduction may be made easily now.

<u>Corollary.</u> Theorem 2 holds when the origin is replaced by any finite set of points S , and $|x|$ by the distance r of the starting point from S .

<u>Proof:</u> Suppose y is a point in S. Then for each $\alpha \geq 2$

$$\begin{aligned} & P\{\text{ Process reaches } S \text{ by time } r^{\alpha}\} \\ (29) \qquad & \geq P\{\text{ Process reaches } y \text{ by time } r^{\alpha}\} \\ & \to 1 - 2\alpha^{-1} \qquad \text{as } r \to \infty . \end{aligned}$$

Suppose $\alpha \geq 2$ and $\delta > 0$. Then by a simple application of the Strong Markov Theorem (see Chung [3], I, section 13) it follows that

$$P\{\text{Process reaches the origin by time } r^{\alpha+\delta}\}$$

$$\geq P\{\text{Process reaches } S \text{ by time } r^{\alpha}, \text{ and then reaches the origin within a time } r^{\delta} \text{ of first hitting } S\}$$

$$= \sum_{y \in S} P\{\text{Process first reaches } S \text{ by time } r^{\alpha} \text{ starting at } y\}$$

$$\times P\{\text{Process reaches the origin by time } r^{\delta} \text{ starting at } y\}.$$

The second term in each of the last summands tends to 1 as $r \to \infty$. Since S is finite, it then follows that

$$\limsup_{r \to \infty} P\{\text{Process first reaches } S \text{ by time } r^{\alpha}\} \leq 1 - 2(\alpha+\delta)^{-1}.$$

The corollary follows on combining this inequality with (29) and noting that δ may be chosen arbitrarily small.

3.4. The first hitting time problem is now investigated for a random walk on the same two-dimensional lattice, but in continuous time. Let $\{p_{xy}(t)\}$ be the matrix of transition probabilities, and suppose that the process is specified by the transition rates $p'_{xy}(0) = q_{0,y-x}$, where

$$0 < \sum_{y \neq 0} q_{0y} = -q_{00} < \infty .$$

Since the transition rates q_{xy} are bounded, a short functional analysis argument shows that if $P(t) = \{p_{xy}(t)\}$ and $Q = \{q_{xy}\}$, then $P(t)$ is uniquely determined and $P(t) = e^{tQ}$. Hence if

$$\varphi(\theta) = \sum_x q_{0x}\, \epsilon^{i\theta.x} ,$$

then
$$p_{xy}(t) = \frac{1}{4\pi^2} \iint e^{t\varphi(\theta)}\, e^{-i\theta.(y-x)}\, d\theta ,$$

where the double integration is to be taken over the range $-\pi \le \theta_1 \le \pi$, $-\pi \le \theta_2 \le \pi$ unless otherwise stated. Suppose that

$$\varphi(\theta) = -\mu(\theta) + O(|\theta|^{2+\delta})$$

where $\delta > 0$, and $\mu(\theta)$ is a positive definite quadratic form in the components θ_1 and θ_2 of θ. This is equivalent to the condition that the first jump positions have zero mean and a finite moment of order greater than 2. Since the real part of $\varphi(\theta)$ is not positive,

$$\int_0^\infty e^{-\lambda t}\, p_{xy}(t)\, dt = \int_0^\infty e^{-\lambda t} \left\{ \frac{1}{4\pi^2} \iint e^{t\varphi(\theta)}\, e^{-i\theta.(y-x)}\, d\theta \right\} dt$$

(30)
$$= \frac{1}{4\pi^2} \iint \left\{ \int_0^\infty e^{-(\lambda-\varphi(\theta))t}\, dt \right\} e^{-i\theta.(y-x)}\, d\theta$$

$$= \frac{1}{4\pi^2} \iint \frac{e^{-i\theta.(y-x)}}{\lambda - \varphi(\theta)}\, d\theta ,$$

when $\lambda > 0$, where the order of integration may be changed since the integral is absolutely convergent under the given limits of integration.

To obtain the Laplace transform of the first hitting time probabilities, firstly let $g_{x0}(t)$ be the probability that the process first reaches 0 later than time t, starting from $x \neq 0$. Then

$$p_{x0}(t) = -\int_0^t p_{00}(t-s)\, d_s g_{x0}(s) .$$

Now let $$\hat{p}_{xy}(\lambda) = \int_0^\infty e^{-\lambda t}\, p_{xy}(t)\, dt ,$$

and $$B(x,t) = \int_0^t g_{x0}(s)\, ds .$$

Then since $g_{x0}(s)$ decreases as s increases,

$$\begin{aligned}
\hat{p}_{x0}(\lambda) &= \int_{t=0}^\infty e^{-\lambda t}\, p_{x0}(t)\, dt \\
&= \int_{t=0}^\infty e^{-\lambda t} \{ -\int_{s=0}^t p_{00}(t-s)\, d_s g_{x0}(s) \}\, dt \\
&= -\int_{s=0}^\infty e^{-\lambda s}\, d_s g_{x0}(s) \int_{t=0}^\infty e^{-\lambda t}\, p_{00}(t)\, dt \\
&= -\hat{p}_{00}(\lambda) \{ [e^{-\lambda s}\, g_{x0}(s)]_{s=0}^\infty + \lambda \int_{s=0}^\infty e^{-\lambda s} g_{x0}(s)\, ds \} \\
&= \hat{p}_{00}(\lambda) \{ 1 - \lambda \int_{s=0}^\infty e^{-\lambda s}\, d_s B(x,s) \} .
\end{aligned}$$

Hence using the above and (30),

$$\int_0^\infty e^{-\lambda t}\, d_t B(x,t) = \frac{\hat{p}_{00}(\lambda) - \hat{p}_{x0}(\lambda)}{\lambda\, \hat{p}_{00}(\lambda)} = \frac{\iint \frac{1 - e^{i\theta . x}}{\lambda - \varphi(\theta)}\, d\theta}{\lambda \iint \frac{d\theta}{\lambda - \varphi(\theta)}} ,$$

The methods used in the discrete time case may now be applied to show that Theorem 2 and its corollary also hold in the continuous time case, except that , in the proof of the corollary, the continuous time version of the Strong Markov Theorem is now needed (see Chung, [3], II, section 9).

3.5. The following result is now obtained from Theorem 2. Suppose a random walk, in discrete time, and on a three-dimensional lattice of points (x,y,z) where $x, y, z = 0, \pm 1, \pm 2, \ldots$, is defined as follows. Let X_n , $n = 1, 2, \ldots$, be a sequence of independent identically distributed random variables, with values on the three-dimensional lattice, whose components in the $x\,y$ plane have the properties of the two-dimensional random variables defined at the beginning of section 3.3 . Suppose also that X_n cannot simultaneously possess non zero components in the z direction and the $x\,y$ plane, and that the z direction component has zero mean and finite variance. Suppose the position of the random walk at time n is given by $S_n = S_o + X_1 + \ldots + X_n$, $n = 0, 1, 2, \ldots$, where S_o is the initial position at time 0. Let Z be the point of the z axis which is reached first starting from $(x,y,0)$, with $r = (x^2 + y^2)^{\frac{1}{2}}$.

<u>Corollary.</u> If $\alpha \geq 1$, $P\{|Z| \leq r^{\alpha}\} \to 1 - \alpha^{-1}$ as $r \to \infty$.

As mentioned at the beginning of section 3.2, this result was proved directly in the important case of the simple symmetric random walk by Doney [4].

<u>Proof of corollary:</u> If the initial position is $(x,y,0)$, with $r = (x^2 + y^2)^{\frac{1}{2}}$, let T be the total number of steps taken before reaching the z axis for the first time, H the total number of non zero horizontal steps taken parallel to the xy plane, and V the total number of zero steps and steps taken in the vertical z directions. (Thus $T = H + V$.)

By observing only the motion parallel to the xy plane, it follows from Theorem 2 that, if $\alpha \geq 2$,

(31)
$$P\{T \leq r^{\alpha}\} \to 1 - 2\alpha^{-1} \quad \text{as } r \to \infty .$$

Now suppose $\alpha > 1$, δ is small, with $0 < \delta < \alpha - 1$, and $\lambda > 0$ is large. Then for all large enough r,

(32)
$$\begin{aligned} P\{|Z| \leq r^{\alpha}\} &\geq P\{|Z| \leq \lambda r^{\alpha-\delta}\} \\ &\geq P\{|Z| \leq \lambda r^{\alpha-\delta},\ T \leq r^{2(\alpha-\delta)}\} \\ &\geq P\{|Z| \leq \lambda\sqrt{T},\ T \leq r^{2(\alpha-\delta)}\} \\ &\geq P\{T \leq r^{2(\alpha-\delta)}\} - P\{|Z| > \lambda\sqrt{T}\} . \end{aligned}$$

Clearly from (31)

(33) $$P\{T \le r^{2(\alpha-\delta)}\} \to 1 - (\alpha-\delta)^{-1}, \qquad \text{as } r \to \infty .$$

and it remains to show that the second term $P\{|Z| > \lambda\sqrt{T}\}$ is small.

(34) $$\text{Let} \quad \eta = \tfrac{1}{2}\min(\rho, 1-\rho) ,$$

where ρ is the probability that X_n has zero x and y components. Then

(35)
$$\begin{aligned} &P\{|Z| > \lambda\sqrt{T}\} \\ &= P\{|Z| > \lambda\sqrt{T},\ \eta T < V < (1-\eta)T\} \\ &\qquad + P\{|Z| > \lambda\sqrt{T},\ \eta T \ge V \text{ or } V \ge (1-\eta)T\} \\ &\le P\{|Z| > \lambda\sqrt{V/(1-\eta)},\ V > \eta r\} + P\{V \le \eta T\} + P\{V \ge (1-\eta)T\}, \end{aligned}$$

since $\eta T < V$ implies $V > \eta H \ge \eta r$. Now put

(36)
$$\begin{aligned} &P\{|Z| > \lambda\sqrt{V/(1-\eta)},\ V > \eta r\} \\ &\qquad = \sum_{v > \eta r} P|Z| > \lambda\sqrt{v/(1-\eta)} \mid V=v\}\ P\{V=v\} , \end{aligned}$$

and, using $T = H + V$,

(37)
$$\begin{aligned} P\{V \le \eta T\} &= P\{V \le \eta H/(1-\eta)\} \\ &= \sum_h P\{V \le \eta h/(1-\eta) \mid H=h\}\ P\{H=h\} \end{aligned}$$

Then let the one step transition probabilities of the random walk be denoted by (i) p_{ab} for non zero steps parallel to the xy plane, where the suffices of p_{ab} are two dimensional vectors, and (ii) $q_{\alpha\beta}$ for zero steps or steps parallel to the z axis, where the suffices of $q_{\alpha\beta}$ are scalars. Then for all a and α,

$$\sum_b p_{ab} + \sum_\beta q_{\alpha\beta} = 1 \; ; \quad \text{and} \quad \sum_\beta q_{\alpha\beta} = \rho \; .$$

If the random walk is initially at vector position a in the xy plane,

$$(38) \qquad P\{H=h,\ V=v,\ Z=z\} = \sum_{\substack{a_i \neq 0 \\ \alpha_j}} \binom{h+v-1}{v} p_{a a_1} p_{a_1 a_2} \cdots p_{a_{h-1} 0}\, q_{0 \alpha_1} q_{\alpha_1 \alpha_2} \cdots q_{\alpha_{v-1} z} \; .$$

Hence $P\{V=v,\ Z=z\} = \sum_h P\{H=h,\ V=v,\ Z=z\}$

$$= \Big\{ \sum_h \Big[\sum_{a_i \neq 0} \binom{h+v-1}{v} p_{a a_1} \cdots p_{a_{h-1} 0} \Big] \Big\} \sum_{\alpha_j} q_{0 \alpha_1} \cdots q_{\alpha_{v-1} z} \; ,$$

and

$$P\{V=v\} = \sum_z P\{V=v,\ Z=z\}$$

$$= \Big\{ \sum_h \Big[\sum_{a_i \neq 0} \binom{h+v-1}{v} p_{a a_1} \cdots p_{a_{h-1} 0} \Big] \Big\} \rho^v \; .$$

Thus

$$P\{Z=z \mid V=v\} = \frac{P\{Z=z,\ V=v\}}{P\{V=v\}} = \sum_{\alpha_j} (q_{0 \alpha_1}/\rho) \cdots (q_{\alpha_{v-1} z}/\rho),$$

and it follows that the conditional distribution of steps in the vertical direction has zero mean and finite variance.

The central limit theorem (see Gnedenko and Kolmogorov [9], § 35, Theorem 4) then shows that the conditional probabilities in (36) have a limit as $v \to \infty$ which is $O(\lambda^{-1})$. Hence there exists a constant c such that

$$(39) \qquad \limsup_{r \to \infty} P\{ |Z| > \lambda\sqrt{V/(1-\eta)},\ V > \eta r \} \leq \frac{c}{\lambda} .$$

Similarly from (38)

$$P\{H=h,\ V=v\} = \sum_z P\{H=h,\ V=v,\ Z=z\}$$

$$= \sum_{a_i \neq 0} p_{a a_1} \cdots p_{a_{h-1} 0} \binom{h+v-1}{v} \rho^v ,$$

and

$$P\{H=h\} = \sum_v P\{H=h,\ V=v\} = \sum_{a_i \neq 0} p_{a a_1} \cdots p_{a_{h-1} 0} (1-\rho)^{-h}.$$

Thus

$$P\{V=v \mid H=h\} = \frac{P\{H=h,\ V=v\}}{P\{H=h\}} = \binom{h+v-1}{v} (1-\rho)^h \rho^v .$$

Considering the conditional probabilities in (37) and using the definition of η in (34), it follows that

$$P\{V \leq \eta h/(1-\eta) \mid H=h\} \leq \sum_{v \leq \frac{\rho h}{2(1-\rho)}} \binom{h+v-1}{v} (1-\rho)^h \rho^v .$$

The last expression is the tail of a negative binomial distribution, which has mean $h\rho/(1-\rho)$ and variance $h\rho/(1-\rho)^2$. It is easily shown, using Tchebychev's inequality (e.g. see

Feller [7], Chapter IX), that this tail tends to **zero** as $h \to \infty$. Hence, since $H \geq r$ in (37),

(40) $$P\{V \leq \eta T\} \to 0 , \qquad \text{as } r \to \infty.$$

Similarly it may be shown that

(41) $$P\{V \geq (1-\eta) T\} \to 0 , \qquad \text{as } r \to \infty .$$

Therefore, on combining (39), (40) and (41) with (35),

$$\limsup_{r \to \infty} P\{|Z| > \lambda\sqrt{T}\} \leq \frac{c}{\lambda} ,$$

and thus, on combining (32) and (33),

$$\liminf_{r \to \infty} P\{|Z| \leq r^{\alpha}\} \geq 1 - \frac{1}{\alpha-\delta} - \frac{c}{\lambda} .$$

But δ may be chosen arbitrarily small, and λ arbitrarily large, so that

(42) $$\liminf_{r \to \infty} P\{|Z| \leq r^{\alpha}\} \geq 1 - \frac{1}{\alpha} .$$

Now suppose $\alpha \geq 1$, $\delta > 0$ and $\varepsilon > 0$, where δ and ε are both small. Then, for all large enough r ,

(43) $$\begin{aligned} P\{|Z| > r^{\alpha}\} &\geq P\{|Z| > \varepsilon r^{\alpha+\delta}\} \\ &\geq P\{|Z| > \varepsilon r^{\alpha+\delta} , \ T > r^{2(\alpha+\delta)}\} \\ &\geq P\{|Z| > \varepsilon\sqrt{T} , \ T > r^{2(\alpha+\delta)}\} \\ &\geq P\{T > r^{2(\alpha+\delta)}\} - P\{|Z| \leq \varepsilon\sqrt{T}\} . \end{aligned}$$

From (31) it follows that

(44) $$P\{T > r^{2(\alpha+\delta)}\} \to \frac{1}{\alpha+\delta} \qquad \text{as } r \to \infty .$$

Using methods similar to those used earlier to consider $P\{|Z| > \lambda\sqrt{T}\}$, it may be shown that there exists a constant c such that

$$\limsup_{r \to \infty} P\{|Z| \leq \varepsilon\sqrt{T}\} \leq c\,\varepsilon .$$

It now follows from (43) and (44) that

$$\liminf_{r \to \infty} P\{|Z| > r^{\alpha}\} \geq \frac{1}{\alpha+\delta} - c\,\varepsilon .$$

But ε and δ are arbitrary, so that

$$\liminf_{r \to \infty} P\{|Z| > r^{\alpha}\} \geq \frac{1}{\alpha} , \qquad \alpha \geq 1 .$$

The corollary now follows on combining the last result with (42).

Chapter 4.

A Simple Unsymmetric Random Walk in the Half and Quarter Planes

4.1. Suppose a particle performs a simple random walk on a two-dimensional lattice of points (x,y), where $x,y = 0, \pm 1, \pm 2, \ldots$, and let X_n be the position of the particle at time n. Let the one step transition probabilities be defined by

$$p^1(x,y;x',y') = P\{X_{n+1}=(x',y')|X_n=(x,y)\}, \qquad n = 1,2,\ldots ,$$

which takes the values p_1, q_1 when (x',y') is immediately to the right or left respectively of (x,y), and takes the values p_2, q_2 when (x',y') is immediately above or below respectively (x,y), where p_1, q_1, p_2, q_2 are positive constants and $p_1+q_1+p_2+q_2 = 1$. Henze [11] has shown that the n step transition probabilities

$$p^n(x,y;x',y') = P\{X_{n+m}=(x',y')|X_m=(x,y)\}, \qquad n,m \geq 0,$$

are given by

$$\frac{2^n}{4\pi^2}\left(\frac{p_1}{q_1}\right)^{\frac{x'-x}{2}}\left(\frac{p_2}{q_2}\right)^{\frac{y'-y}{2}} \times$$

$$\iint \left(\sqrt{p_1q_1}\cos\alpha + \sqrt{p_2q_2}\cos\beta\right)^n \cos\alpha(x'-x)\ \cos\beta(y'-y)\ d\alpha\, d\beta ,$$

where the double integration is to be taken over the square $-\pi \leq \alpha \leq \pi$, $-\pi \leq \beta \leq \pi$, unless otherwise stated. By using a reflection argument, Henze in [11] showed that, if each point of the x axis is made absorbing, the n step transition probabilities in the upper half plane become

$$(1) \qquad p^n(x,y;x',y') - \left(\frac{q_2}{p_2}\right)^y p^n(x,-y;x',y')$$

$$= \frac{2^{n+1}}{4\pi^2}\left(\frac{p_1}{q_1}\right)^{\frac{x'-x}{2}}\left(\frac{p_2}{q_2}\right)^{\frac{y'-y}{2}} \times$$

$$(2) \qquad \iint \left(\sqrt{p_1q_1}\cos\alpha + \sqrt{p_2q_2}\cos\beta\right)^n \cos\alpha(x'-x)\,\sin\beta y'\,\sin\beta y\,d\alpha\,d\beta,$$

where multiplying by $(q_2/p_2)^y$ replaces $p_2^{\,y}$ by $q_2^{\,y}$ in $p^n(x,-y;x',y')$, and thus the negative term in (1) accounts for those paths in the whole plane which reach the x axis before (x',y'). A similar argument shows that the n step transition probabilities in the positive quarter plane with absorbing axes are given by

$$\frac{2^{n+2}}{4\pi^2}\left(\frac{p_1}{q_1}\right)^{\frac{x'-x}{2}}\left(\frac{p_2}{q_2}\right)^{\frac{y'-y}{2}} \times$$

$$\iint \left(\sqrt{p_1q_1}\cos\alpha + \sqrt{p_2q_2}\cos\beta\right)^n \sin\alpha x'\,\sin\alpha x\,\sin\beta y'\,\sin\beta y\,d\alpha\,d\beta.$$

Various questions are now investigated on the distribution of first hits on the axes in the half and quarter planes. For the upper half plane case with $p_2 \neq q_2$, it is shown that, starting at $(0,y)$, the distribution of first hits on the x axis, given that the axis is reached at all, obeys the central limit law as $y \to \infty$. But when $p_2 = q_2$ and $p_1 \neq q_1$, the distribution of first hits on the x axis, when suitably normalised, tends to the stable law of order $\frac{1}{2}$ as $y \to \infty$. In the case of the quarter plane, it is found that similar results hold for each axis considered, except when the expected step of the random walk is perpendicular to the axis considered.

4.2. In the case of the random walk in the upper half plane, let $S(y)$ be the point of the x axis which is reached first starting from $(0,y)$, if the x axis is reached at all, and let $S(y)$ be undefined otherwise. Using (2) and the transformation

$$(3) \qquad 2\sqrt{p_2 q_2}\cosh\mu = 1 - 2\sqrt{p_1 q_1}\cos\alpha, \qquad \text{with } \mu \geq 0.$$

Henze [11] showed that

$$(4) \quad P\{S(y)=m\} = \frac{1}{2\pi}\left(\frac{p_1}{q_1}\right)^{\frac{m}{2}}\left(\frac{p_2}{q_2}\right)^{-\frac{y}{2}} \int_{\alpha=-\pi}^{\pi} e^{-yu} \cos\alpha m \, d\alpha.$$

By changing the path of integration in the complex α plane, it is now possible to obtain the characteristic function $\varphi_y(u)$ of the distribution of $S(y)$, where

$$\varphi_y(u) = \mathcal{E}\{e^{iuS(y)}\} = \sum_{m=-\infty}^{\infty} P\{S(y)=m\}\ e^{ium}.$$

The transformation (3) may be generalised for complex α by putting

$$(5) \qquad \zeta = \frac{1 - 2\sqrt{p_1 q_1}\cos\alpha}{2\sqrt{p_1 q_1}} ,$$

and then letting

$$(6) \qquad \psi = \zeta - (\zeta^2 - 1)^{\frac{1}{2}}.$$

To make ψ analytic, the ζ plane is cut along $-1 \leq \zeta \leq 1$. Then ψ is an analytic function of α, with $\alpha = u+iv$, if the α plane is cut along

$$(7) \qquad u = 2r\pi, \qquad r = 0, \pm 1, \pm 2, \ldots ,$$
$$\frac{1 - 2\sqrt{p_2 q_2}}{2\sqrt{p_1 q_1}} \leq \cosh v \leq \frac{1 + 2\sqrt{p_2 q_2}}{2\sqrt{p_1 q_1}} .$$

For real α, $\psi(\alpha) = e^{-\mu}$ and is even, so that (4) becomes

$$(8) \qquad P\{S(y)=m\} = \frac{1}{2\pi}\left(\frac{p_1}{q_1}\right)^{\frac{m}{2}} \left(\frac{p_2}{q_2}\right)^{-\frac{y}{2}} \int_{-\pi}^{\pi} \{\psi(\alpha)\}^y\ e^{-i\alpha m}\, d\alpha .$$

Now let $a = \log\sqrt{p_1/q_1}$. Then

$$\cosh a = \frac{p_1 + q_1}{2\sqrt{p_1q_1}} \leq \frac{1 - 2\sqrt{p_2q_2}}{2\sqrt{p_1q_1}} ,$$

with equality only when $p_2 = q_2$. Hence from (7), $\psi(\alpha)$ is analytic in the region $-\pi \leq u \leq \pi$, $0 \leq |v| \leq |a|$, where $\alpha = u+iv$, except for singularities at $\alpha = \pm ia$ when $p_2=q_2$, at which $\psi(\alpha)$ is continuous. Therefore, using Cauchy's theorem for contour integrals, it follows from (8) that

$$P\{S(y)=m\} =$$

$$\left\{ \int_{-\pi}^{-\pi-ia} + \int_{-\pi-ia}^{\pi-ia} + \int_{\pi-ia}^{\pi} \right\} \frac{1}{2\pi}\left(\frac{p_1}{q_1}\right)^{\frac{m}{2}} \left(\frac{p_2}{q_2}\right)^{-\frac{y}{2}} \{\psi(\alpha)\}^y e^{-i\alpha m}\, d\alpha$$

(9)

$$= \frac{1}{2\pi} \int_{-\pi-ia}^{\pi-ia} \left\{\left(\frac{p_2}{q_2}\right)^{-\frac{1}{2}} \psi(\alpha)\right\}^y \left(\frac{p_1}{q_1}\right)^{\frac{m}{2}} e^{-i\alpha m}\, d\alpha$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \{\varphi(u)\}^y e^{-i\alpha m}\, d\alpha ,$$

where

(10) $$\varphi(u) = \left(\frac{p_2}{q_2}\right)^{-\frac{1}{2}} \psi(u-ia), \qquad a = \log\sqrt{p_1/q_1} .$$

It is easily verified that $\frac{d}{du}\{\varphi(u)\}^y$ is Riemann integrable over $-\pi \leq u \leq \pi$, so that $\{\varphi(u)\}^y$ is of bounded variation over $-\pi \leq u \leq \pi$. Therefore by Jordan's test $\{\varphi(u)\}^y$ is equal to the sum of its Fourier series, and thus, from (9) and (10), the characteristic function of $S(y)$ is

(11) $$\varphi_y(u) = \sum_{m=-\infty}^{\infty} P\{S(y)=m\}\ e^{ium} = \{\varphi(u)\}^y.$$

By considering only the behaviour of the random walk in the y direction, or by using the above formula, it follows that

(12) $$\sum_{m=-\infty}^{\infty} P\{S(y)=m\} = \{\varphi(0)\}^y = \{\min(1, q_2/p_2\}^y.$$

Suppose W_r is the displacement in the x direction between the time when the line $y = r$ is first reached, supposing that this event does occur, and the time when the line $y = r-1$ is first reached, if this event occurs at all. Then the random variables W_r, $r=1,2,\ldots$, are independent, and each has the same distribution as $S(1)$ and, from (11), the characteristic function $\varphi(u)$. Thus $S(y) = W_1+\ldots+W_y$, where this sum must be undefined if one of the summands does not occur. By applying an improved version of the central limit theorem when $p_2 \neq q_2$, the following result may be obtained, with an error term which is used when the quarter plane case is considered later.

<u>Theorem 1.</u> If $p_2 \neq q_2$, then as $y \to \infty$,

$$\frac{P\{S(y)\leq m\}}{\{\min(1, q_2/p_2)\}^y} = \Phi\left\{\frac{m-\mu y}{\sigma\sqrt{y}}\right\} + O(y^{-\frac{1}{2}})$$

where the last term is independent of m , and $$\mu = \frac{p_1 - q_1}{|p_2 - q_2|} ,$$

$$\sigma^2 = \frac{p_1 + q_1}{|p_2 - q_2|} + \frac{p_2 + q_2}{|p_2 - q_2|} \left(\frac{p_1 - q_1}{p_2 - q_2} \right)^2 ,$$

and $$\Phi(t) = \text{erf } t = (1/\sqrt{2\pi}) \int_{-\infty}^{t} e^{-s^2/2} \, ds.$$

Proof: It is easily found that, when $p_2 \neq q_2$,

$$\varphi'(0) = i\mu \min(1, q_2/p_2) ,$$

$$\varphi''(0) = -(\sigma^2 + \mu^2) \min(1, q_2/p_2) ,$$

and $\varphi(u)$ has a finite fourth derivative at $u = 0$. Hence, using (12), $\varphi(u)/\min(1, q_2/p_2)$ is the characteristic function of a random variable which is finite with probability 1 , has mean μ , variance σ^2, and a finite third moment. Since $S(y)$ is the sum of y independent random variables, each with the characteristic function $\varphi(u)$, the theorem now follows from a result on convergence to the normal law given by Gnedenko and Kolmogorov in [9], §40, Theorem 1.

The connection between the limiting behaviours in the cases $p_2 > q_2$ and $p_2 < q_2$ is in fact a special case of a general result obtained by O'N. Waugh in 'Conditioned Markov Processes' [24].

When $p_2 = q_2$, it follows from (12) that the x axis is reached with probability 1. However the analysis used when $p_2 \neq q_2$ clearly breaks down when $p_2 = q_2$ since $\varphi(u)$ is then not analytic at $u = 0$. A method given by Gnedenko and Kolmogorov in [9], §39 and §40, is now adapted to cover this case. Theorem 1 in [9], §39, may be modified as follows by replacing the second condition and adding an extra condition.

<u>Theorem.</u> Let A, T, and $\varepsilon > 0$ be constants, $F(x)$ a non decreasing function, and $G(x)$ a function of bounded variation. Suppose $f(t) = \int_{-\infty}^{\infty} e^{itx}\,dF(x)$ and $g(t) = \int_{-\infty}^{\infty} e^{itx}\,dG(x)$. If

1. $F(-\infty) = G(-\infty)$ and $F(\infty) = G(\infty)$;

2. There exists $\eta_0 > 0$ such that $F(x)$ and $G(x) = O(e^{\eta_0 x})$ as $x \to \infty$;

3. $G'(x)$ exists for all x and $|G'(x)| \leq A$;

4. $\int_{-T}^{T} \left|\frac{f(t)-g(t)}{t}\right| dt = \varepsilon$;

5. For t real and $\eta > 0$

$$f(t+i\eta)-g(t+i\eta) = O(|t+i\eta|) \qquad \text{as } t \to 0 \text{ and } \eta \to 0+,$$

then to every number $k > 1$ there corresponds a finite positive number $c(k)$ depending only on k such that

$$|F(x)-G(x)| \leq k\frac{\varepsilon}{2\pi} + c(k)\frac{A}{T} .$$

Proof: As in the proof of the original theorem, it is sufficient to consider the case $A = T = 1$, and then introduce the functions

$$H(x) = \frac{3}{8\pi}\left(\frac{\sin x/4}{x/4}\right)^4 ,$$

and

$$h(t) = \begin{cases} 0 , & |t| \geq 1, \\ 2(1-|t|)^3 , & \frac{1}{2} \leq |t| \leq 1, \\ 1-6t^2+6|t|^3, & 0 \leq |t| \leq \frac{1}{2}, \end{cases}$$

where

$$h(t) = \int_{-\infty}^{\infty} e^{itx} H(x)\, dx .$$

It is easily verified, using condition 2 and partial integration when $x < 0$, that the integrals for $f(t+i\eta)$ and $g(t+i\eta)$ converge locally uniformly for real t and uniformly for η when $0 \leq \eta \leq$ constant $< \eta_0$. It may then be shown that $f(t+i\eta)$ and $g(t+i\eta)$ are continuous in η when $0 \leq \eta \leq$ constant $< \eta_0$, locally uniformly for real t . On integrating by parts,

$$\frac{f(t+i\eta)-g(t+i\eta)}{-i(t+i\eta)} = \int_{-\infty}^{\infty} e^{(it-\eta)}\{F(x)-G(x)\}\, dx, \qquad 0 < \eta < \eta_0.$$

Let $$v_\eta(x) = \int_{-\infty}^{\infty} H(x-y)\, e^{-\eta y}\{F(y)-G(y)\}\, dy, \qquad 0 \leq \eta < \eta_0.$$

Then if $0 < \eta < \eta_0$,

$$\frac{f(t+i\eta)-g(t+i\eta)}{-i(t+i\eta)}\, h(t) = \int_{-\infty}^{\infty} e^{itx}\, v_\eta(x)\, dx ,$$

and, since $v_\eta(x)$ is integrable over $(-\infty,\infty)$ and is of bounded variation over any finite interval,

$$v_\eta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\, \frac{f(t+i\eta)-g(t+i\eta)}{-i(t+i\eta)}\, h(t)\, dt .$$

Hence, using the definition of $v_\eta(x)$ and noting that $h(t) = 0$ for $|t| \geq 1$,

$$\int_{-\infty}^{\infty} H(x-y)\, e^{-\eta y} \{F(y)-G(y)\}\, dy = \frac{1}{2\pi}\int_{-1}^{1} e^{-itx}\, \frac{f(t+i\eta)-g(t+i\eta)}{-i(t+i\eta)}\, h(t)dt .$$

The proof of the modified theorem now follows without any change from the methods used after equation (3) in the proof of the original theorem.

The following result may be obtained using this last theorem, with an error term which is needed later when the quarter plane case is considered. The actual limiting behaviour of $S(y)$ might be anticipated by firstly observing only the vertical motion of the random walk and using a result for the first passage time through the x axis analogous to that given by Feller in [7] (Chapter III, section 8(c)) for the limiting behaviour of the first passage time of the simple random walk in one dimension.

The limiting behaviour of $S(y)$ is then obtained by an argument similar to that used in the last section of chapter 3 .

<u>Theorem 2.</u> If $p_1 > q_1$ and $p_2 = q_2$, then as $y \to \infty$,

$$P\{S(y) \leq m\} = G(m/cy^2) + O(y^{-1})$$

where the last term is independent of m, $c=(p_1-q_1)/2p_2$, and $G(x)$ is now the distrbution function

$$G(x) = \begin{cases} 0 , & x \leq 0, \\ (2/\pi)^{\frac{1}{2}} \int_{x^{-\frac{1}{2}}}^{\infty} e^{-s^2/2} ds , & x > 0, \end{cases}$$

which is the positive stable distribution of order $\frac{1}{2}$.

<u>Proof:</u> The proof consists of satisfying the conditions of the previous theorem with suitable values of A, T, η_0, and ε , taking

$$F(x) = P\{S(y) \leq cy^2 x\} .$$

Condition 1 is satisfied since the x axis is reached with probability 1 . If the motion in the x direction only is considered, without the x axis being absorbing, it is easily shown that the random walk reaches a distance $m > 0$ to the left of its starting point with probability $(q_1/p_1)^m$.

It follows that

$$P\{S(y)\le x\} \le (p_1/q_1)^x \qquad \text{when } x \le 0,$$

so that condition 2 is satisfied with $\eta_o = cy^2 \log(p_1/q_1)$.

By using the same type of contour change as in (9), which establishes that $\varphi(u)$ is the characteristic function of the distribution function $P\{S(y)\le x\}$, it is easily verified that, if u is real and $0 \le v < \log(p_1/q_1)$,

$$\sum_{m=-\infty}^{\infty} P\{S(y)=m\}\, e^{(iu-v)} = \{\varphi(u+iv)\}^y.$$

Hence, if $0 \le \eta < cy^2 \log(p_1/q_1)$,

$$\begin{aligned} f(t+i\eta) &= \int_{-\infty}^{\infty} e^{i(t+i\eta)x}\, dF(x) \\ &= \int_{-\infty}^{\infty} e^{i(t+i\eta)x}\, d_x P\{S(y)\le cy^2 x\} \\ &= \int_{-\infty}^{\infty} e^{i(t+i\eta)c^{-1}y^{-2}x}\, d_x P\{S(y)\le x\} \\ &= \left\{\varphi\left(\frac{t+i\eta}{cy^2}\right)\right\}^y . \end{aligned}$$

But from (5), (6), (7) and (10), if $\alpha = -i \log\sqrt{p_1/q_1} + u$, where u is small and $-\frac{\pi}{2} < \arg u \le \frac{3\pi}{2}$, then as $|u| \to 0$,

$$\zeta = 1 - icu + O(|u|^2) ,$$

and

$$\begin{aligned} \varphi(u) &= \psi\{u - i \log\sqrt{p_1/q_1}\} \\ &= 1 - (-2icu)^{\frac{1}{2}} + O(|u|) , \end{aligned}$$

taking $\arg\{(-2icu)^{\frac{1}{2}}\} = \frac{1}{2}(-\frac{\pi}{2} + \arg u)$ with $-\frac{\pi}{2} < \arg u \le \frac{3\pi}{2}$.

$$\text{Hence} \quad f(t+i\eta) = \left\{\varphi\left(\frac{t+i\eta}{cy^2}\right)\right\}^y$$

$$= e^{y \log\left\{1 - \left[-2ic\left(\frac{t+i\eta}{cy^2}\right)\right]^{\frac{1}{2}} + O\left(\left|\frac{t+i\eta}{cy^2}\right|\right)\right\}}$$

$$(13) \qquad = e^{y\left\{-\left[-2ic\left(\frac{t+i\eta}{cy^2}\right)\right]^{\frac{1}{2}} + O\left(\left|\frac{t+i\eta}{cy^2}\right|\right)\right\}}$$

$$= e^{-[-2i(t+i\eta)]^{\frac{1}{2}}}\left\{1 + O\left(\left|\frac{t+i\eta}{cy^2}\right|\right)\right\}$$

as $\left|\frac{t+i\eta}{cy^2}\right| \to 0$. Also, using [5], it follows that

$$(14) \qquad g(t+i\eta) = \int_{-\infty}^{\infty} e^{(it-\eta)x}\, dG(x) = e^{-[-2i(t+i\eta)]^{\frac{1}{2}}}$$

Therefore, from (13) and (14) with y fixed, condition 5 is satisfied. Also since

$$\left|e^{-(2it)^{\frac{1}{2}}}\right| = e^{-t^{\frac{1}{2}}},$$

it follows from (13) and (14) with $\eta = 0$ that

$$\int_{-y}^{y} \left|\frac{f(t)-g(t)}{t}\right| dt = \int_{-y}^{y} e^{-t^{\frac{1}{2}}} |t|^{-1}\, O\left(\left|\frac{t}{y}\right|\right) dt$$

$$= O(y^{-1}), \qquad \text{as } y \to \infty.$$

Therefore condition 4 holds with $T = y$ and $\varepsilon = O(y^{-1})$. Condition 3 is satisfied with **A** a suitably chosen constant and independent of y, so that all the conditions of the modified theorem are now satisfied, and theorem 2 follows immediately.

4.3. The results of theorems 1 and 2 are now applied to find the limiting distribution of first hits on the x and y axes of the quarter plane. If the random walk in the upper half plane starts at (x,y), let $S(x,y)$ be the point of the x axis which is reached first, if the x axis is reached at all. Also, if the random walk in the positive quarter plane starts at (x,y), let $T(x,y)$ be the point of the x axis which is reached first, if the x axis is reached at all and the y axis is not reached earlier. By applying the reflection argument mentioned in section 4.1, it follows that, when x, y, and $m > 0$,

$$(15) \qquad P\{T(x,y)=m\} = P\{S(x,y)=m\} - \left(\frac{q_1}{p_1}\right)^x P\{S(-x,y)=m\},$$

$$(16) \text{ and } \qquad P\{T(x,y)=m\} = P\{S(x,y)=m\} - \left(\frac{p_1}{q_1}\right)^m P\{S(x,y)=-m\}.$$

$$(17) \text{ Also let } \qquad P\{T(x,y) > m\} = P\{S(x,y) > m\} - R(x,y;m) .$$

The problem is now considered in various cases, in each of which the expected step of the random walk is non zero. These cases are determined by the direction of the drift vector, that is the direction of the expected step of the random walk.

<u>Case (i):</u> $p_1 > q_1$, $p_2 < q_2$. It is known that, if such a random walk in the whole plane starts at (x,y) with x, $y > 0$,

the x axis is reached with probability 1, and the y axis with probability $(q_1/p_1)^x$. Therefore

$P\{T(x,y)>0\}$

= P{The random walk in the upper half plane hits the x axis}

- P{The random walk in the positive quarter plane hits the y axis, and does not hit the x axis earlier}

$\geq$ 1 - P{The random walk in the right hand half plane hits the y axis}

$= 1 - (q_1/p_1)^x$.

Hence, if the random walk starts at (x,y) in the positive quarter plane, the probability that the x axis is reached first tends to 1 as $x \to \infty$. Also, if θ is finite and $x+\mu y+\theta\sigma\sqrt{y} \geq 0$ as x and $y \to \infty$, it follows from (15) and Theorem 1 that

$$P\left\{\frac{T(x,y)-(x+\mu y)}{y} > \theta\right\} = P\{T(x,y) > x+\mu y+\theta\sigma\sqrt{y}\}$$

$$= P\{S(x,y) > x+\mu y+\theta\sigma\sqrt{y}\} + O((q_1/p_1)^x)$$

$$\to 1 - \Phi(\theta) .$$

Thus, when the random walk starts at (x,y), the distribution of first hits on the y axis tends to the normal distribution with the expected hit at $x+\mu y$ and variance $\sigma^2 y$ as x and $y \to \infty$.

Case (ii): $p_1 < q_1$, $p_2 < q_2$.

Put $$P\{T(x,y) > 0\} = P\{S(x,y) > 0\} - R(x,y;0) ,$$

where, using (16), (17) and Theorem 1,

$$\begin{aligned} R(x,y;0) &= \sum_{m=1}^{\infty} (p_1/q_1)^m \, P\{S(x,y)=-m\} \\ &\leq P\{-y^{\varepsilon} \leq S(x,y) < 0\} + \sum_{m>y^{\varepsilon}} (p_1/q_1)^m \\ &= O(y^{\varepsilon-\frac{1}{2}}) \qquad \text{as } y \to \infty , \end{aligned}$$

where $0 < \varepsilon < \frac{1}{2}$. It then follows from Theorem 1 that as $y \to \infty$

$$\begin{aligned} P\{T(x,y) > 0\} &= P\{S(x,y) > 0\} - o(1) \\ &= 1 - \Phi\left(-\frac{x+\mu y}{y}\right) + o(1) . \end{aligned}$$

In this case $\mu = (p_1-q_1)/|p_2-q_2| < 0$, so that if θ is a finite constant and $x+\mu y \sim \theta\sigma\sqrt{y}$, i.e.

$$\frac{x}{|p_1-q_1|} - \frac{y}{|p_2-q_2|} \sim \theta \left(\frac{x}{|p_1-q_1|}\right)^{\frac{1}{2}} \left(\frac{p_1+q_1}{(p_1-q_1)^2} + \frac{p_2+q_2}{(p_2-q_2)^2}\right)^{\frac{1}{2}} ,$$

as x and $y \to \infty$, then the probability that the random walk starting at (x,y) reaches the x axis first tends to $1 - \Phi(-\theta) = \Phi(\theta)$. Similarly, if $x+\mu y+\theta\sigma\sqrt{y} \geq 0$ as x and y $\to \infty$, then by applying (17) and Theorem 1,

$$\begin{aligned} P\left\{\frac{T(x,y)-(x+\mu y)}{y} > \theta\right\} &= P\{T(x,y) > x+\mu y+\theta\sigma\sqrt{y}\} \\ &= P\{S(x,y) > x+\mu y+\theta\sigma\sqrt{y}\} - R(x,y;x+\mu y+\theta\sigma\sqrt{y}) \\ &\to 1 - \Phi(\theta) , \end{aligned}$$

since $0 \leq R(x,y\; x+\mu y+\theta\sigma\sqrt{y}) \leq R(x,y;0) \to 0$ as $y \to \infty$. Thus, as in case (i), when the random walk starts at (x,y) the limiting distribution of first hits on the positive x axis as x and $y \to \infty$ is the same as for the normal distribution with variance $\sigma^2 y$, and mean $x+\mu y$, where the mean is the point at which the drift vector meets the x axis. By symmetry, equivalent results hold for the first hitting probabilities on the y axis.

<u>Case (iii):</u> $p_1 > q_1$, $p_2 > q_2$. It follows from Theorem 1 that the same limiting distributions as in case (ii) may be obtained in this case for the random walk starting at (x,y), if the hitting probabilities for the x and y axes are divided by $(q_2/p_2)^y$ and $(q_1/p_1)^x$ respectively. Therefore the probability of hitting either axis, starting from (x,y), tends to 0 as x and $y \to \infty$

<u>Case (iv):</u> $p_1 < q_1$, $p_2 = q_2$.

Put $$P\{T(x,y) > 0\} = P\{S(x,y) > 0\} - R(x,y;0),$$

where, using (16), (17) and Theorem 2,

$$\begin{aligned} R(x,y;0) &= \sum_{m=1}^{\infty} (p_1/q_1)^m \, P\{S(x,y) = -m\} \\ &\leq P\{-y \leq S(x,y) < 0\} + \sum_{m>y} (p_1/q_1)^m \\ &= O(y^{-\frac{1}{2}}) \qquad \text{as } y \to \infty . \end{aligned}$$

It then follows from Theorem 2, with $c = (q_1-p_1)/2p_2$ now, that as $y \to \infty$

$$P\{T(x,y) > 0\} = P\{S(x,y) > 0\} - o(1)$$
$$= G\left(\frac{x}{cy^2}\right) + o(1) .$$

Therefore, if θ is a non negative constant and $x \sim \theta cy^2$ as x and $y \to \infty$, the probability that the x axis is reached first, starting from (x,y), tends to $G(\theta)$. Similarly, if $x - \theta cy^2 \geq 0$ as x and $y \to \infty$, then by applying (17) and Theorem 2 ,

$$P\left\{\frac{T(x,y)-x}{cy^2} > -\theta\right\} = P\{T(x,y) > x-\theta cy^2\}$$
$$= P\{S(x,y) > x-\theta cy^2\} - R(x,y;x-\theta cy^2)$$
$$\to G(\theta) ,$$

since $0 \leq R(x,y;x-\theta cy^2) \leq R(x,y;0) \to 0$ as $y \to \infty$. Thus the distribution of first hits on the positive x axis has the same limiting behaviour as in the corresponding half plane case.

Now consider the first hitting probabilities on the positive y axis, and let $T_2(x,y)$, $S_2(x,y)$, μ_2 and σ_2 be the quantities for the y axis corresponding to $T(x,y)$, $S(x,y)$, μ and σ respectively for the x axis. Clearly $\mu_2 = 0$ and $(\sigma_2)^2 = 2p_2/|p_1-q_1|$ in the present case. It then follows from (15) and Theorem 1, with the roles of x and y interchanged, that as x and $y \to \infty$

$$P\{T_2(x,y) > 0\} = P\{S_2(x,y) > 0\} - P\{S_2(x,-y) > 0\}$$

$$= 1 - \Phi(-y/\sigma_2\sqrt{x}) - \{1 - \Phi(y/\sigma_2\sqrt{x})\} + O(x^{-\frac{1}{2}}) \ .$$

Therefore, if θ is a non negative constant and $y \sim \theta\sigma_2\sqrt{x}$ as x and $y \to \infty$, the probability that the y axis is reached first, starting from (x,y), tends to $\Phi(\theta) - \Phi(-\theta)$. Similarly, if $y + \theta\sigma_2\sqrt{x} \geq 0$ as $x \to \infty$,

$$P\left\{\frac{T_2(x,y)-y}{\sigma_2\sqrt{x}} > \theta\right\} = P\{T_2(x,y) > y + \theta\sigma_2\sqrt{x}\}$$

$$= P\{S_2(x,y) > y + \theta\sigma_2\sqrt{x}\} - P\{S_2(x,-y) > y + \theta\sigma_2\sqrt{x}\}$$

$$= 1 - \Phi(\theta) - \left\{1 - \Phi\left(\theta + \frac{2y}{\sigma_2\sqrt{x}}\right)\right\} + O(x^{-\frac{1}{2}}) \ .$$

Therefore if $y \sim \theta_o\sigma_2\sqrt{x}$ as x and $y \to \infty$,

$$P\left\{\frac{T_2(x,y)-y}{\sigma_2\sqrt{x}} > \theta\right\} \to 1 - \Phi(\theta) - \{1 - \Phi(\theta+2\theta_o)\}$$

$$= \Phi(\theta+2\theta_o) - \Phi(\theta) \ .$$

Thus the limiting distribution of first hits on the positive y axis, for the random walk starting at (x,y), is the same as for the normal distribution with variance $(\sigma_2)^2$ and mean x , (where the mean is the point at which the drift vector meets the y axis), except that a non negligible term must be subtracted, which corresponds to the probability of hits from the reflection of the starting point in the x axis. The limiting results for the probabilities of hitting the x or y axes first are clearly complementary, although they are written in different forms for convenience.

Case (v): $p_1 > q_1$, $p_2 = q_2$. An argument, similar to that used in case (i), shows that the probability that the x axis is reached first, starting from (x,y), tends to 1 as $x \to \infty$. Also, applying (15) and Theorem 2, with $c = (p_1-q_1)/2p_2$ now, it follows that, if θ is finite and $x+\theta cy^2 \geq 0$ as x and $y \to \infty$,

$$
\begin{aligned}
P\left\{\frac{T(x,y)-x}{cy^2} > \theta\right\} &= P\{T(x,y) > x+\theta cy^2\} \\
&= P\{S(x,y) > x+\theta cy^2\} - O((q_1/p_1)^x) \\
&= 1 - G(\theta) + o(1) .
\end{aligned}
$$

Thus when x and $y \to \infty$, the distribution of first hits on the positive x axis for the random walk starting at (x,y) behaves just as in the half plane case.

The results for the behaviour of the first hitting probabilities in the remaining cases for the orientation of the drift vector now follow from the cases already considered by interchanging the roles of x and y. To the first order of approximation, the limiting distribution of first hitting probabilities on an axis of the quarter plane is therefore the same as for the half plane case, except when the drift vector is directed perpendicular to and towards the axis considered.

References

[1] M.S. Bartlett, Some stochastic models in ecology and epidemiology, Hotelling Festschrift, Stanford, Stanford University Press, 1960, pp. 89-96.

[2] M.S. Bartlett, Monte Carlo studies in ecology and epidemiology, Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, 1961, Vol. 4, pp. 39-55.

[3] K.L. Chung, Markov Chains with Stationary Transition Probabilities, Springer, Berlin, 1960.

[4] R.A. Doney, Some Problems on Random Walks, Thesis, Durham University, 1964.

[5] A. Erdélyi et al., Tables of Integral Transforms, Bateman Manuscript Project, McGraw Hill, New York, 1954, Vol. 1.

[6] W. Feller, On the integro-differential equations of purely discontinuous Markoff processes, Trans. Am. Math. Soc., Vol. 48 (1940), pp. 488-515; ibid. Vol. 58 (1945), p.474.

[7] W.Feller, An introduction to Probability Theory and its Applications, John Wiley and Sons, New York, 1957, (Second edition), Vol. 1.

[8] F.G. Foster, On the stochastic matrices associated with certain queuing processes, Ann. Math. Stat., Vol. 24 (1953), pp. 355-360.

[9] B.V. Gnedenko and A.N. Kolmogorov, Limit distributions for sums of independent random variables, English translation by K.L. Chung, Addison-Wesley, Cambridge, Mass., 1954.

[10] G.H. Hardy, Divergent Series, Clarendon Press, Oxford, 1949.

[11] E. Henze, Zur Theorie der diskreten unsymmetrischen Irrfahrt, Z. Angew. Math Mech., Band 41 (1961), pp.1-9.

[12] D.G. Kendall, Deterministic and stochastic epidemics in closed populations, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, 1956, Vol. 4, pp. 149-165.

[13] D.G. Kendall and G.E.H. Reuter, The calculation of the ergodic projection for Markov chains and processes with a countable infinity of states, Acta. Math., Vol. 97 (1957), pp.103-144.

[14] J. Lamperti, Monotonic solutions of certain integral equations, Arch. Rational Mech. Anal., Vol. 13 (1963), pp. 309-320.

[15] P. Lévy, Processus stochastiques et mouvement brownien, Gauthier-Villars, Paris, 1948.

[16] E. Lukacs, Characteristic Functions, Griffin, London, 1960.

[17] W.H. McCrea and F.J.W. Whipple, Random paths in two and three dimensions, Proc. Roy. Soc. Edinburgh, Sect. A, Vol. 60 (1939/40), pp. 281-298.

[18] J. Newman, T. Park and E.L. Scott, Struggle for existence. The Triboleum model: biological and statistical aspects, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, 1956, Vol. 4, pp. 41-79.

[19] G.E.H. Reuter, Denumerable Markov processes and the associated contraction semigroups on 1, Acta. Math., Vol. 97 (1957), pp. 1-48.

[20] G.E.H. Reuter, Competition Processes, Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, 1961, Vol 2, pp. 421-430.

[21] F. Spitzer, Some theorems concerning 2-dimensional Brownian motion, Trans. Amer. Math. Soc., Vol. 87 (1958), pp. 187-197.

[22] F. Spitzer, Recurrent random walk and logarithmic potential, Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, 1961, pp. 515-534.

[23] G.N. Watson, Theory of Bessel Functions, Cambridge University Press, 1922.

[24] W.A. O'N. Waugh, Conditioned Markov processes, Biometrika, Vol. 45 (1958), pp. 241-249.

[25] D.V. Widder, The Laplace Transform, Princeton University Press, 1941.