Mathematical Structure of Dual Amplitudes

by

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CONTENTS.

PREFACE

The work presented in this thesis was carried out in the Department of Mathematics, University of Durham, in the period October 1969 to September 1972 under the supervision of Dr. D.B. Fairlie, to whom the author would like to offer his sincere thanks for advice and continued encouragement throughout the above period.

The material in this thesis has not been submitted for any other degree in this or any other university. Part of Chapter II (4.3) and all of Chapter IV are claimed to be original, the latter based on a Durham preprint by Dr. Fairlie and the author.

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ABSTRACT

Firstly, the notion of duality is introduced and the generalized amplitude given in terms of the Koba - Nielsen variables. Furthermore, from the point of view of wave functions it is shown how the dynamical importance of spin implies an internal structure or extended description of hadrons which may be linked with duality (or internal symmetry).

Then, in the following chapter, we first of all describe how certain mathematical characteristics emerge as being desirable for a strong interaction theory and indicate how these appear in various models, the relationships between which we also explain. It then appears that the most suitable language in which to discuss these properties is that of two-dimensional Riemann surfaces, which are most prominent in the Analogue Model.

Chapter III indicates how the structure in dual models may be simply derived without an explicit physical interpretation (as in the functional integral formalism) - Ramond's Correspondence Principle.

In the final chapter we propose an approach which embodies the surface description and the mathematical attributes mentioned above (without a tachyon). Furthermore, it amounts to an extended description of interacting hadrons, which, when reconciled with Poincaré invariance, leads to the internal symmetry group of the original models.

CHAPTER 1

Introduction

There are two languages in terms of which we may discuss elementary interactions - field theoretic and the S - matrix approach. Each of these is involved in the development dual theory. The purpose of the introduction is to ex... the relationship between duality and each of these frameworks.

1. S - matrix

1.1 Duality

The idea of duality originated within the context of the S - matrix. Now, there are two representations of the S - matrix, one of which takes a simple form at low energies and the other at high energies.[2,3]

The former is directly related to the presence of resonance peaks in effective mass, corresponding to the formation of particles. These resonances, characterised by quantum numbers, form a spectrum with certain regular features.

a) particles belong to SU(3) multiplets of triality zero.

b) particles, when plotted on a Chew - Frautschi diagram, lie on straight lines (Regge trajectories)

ie. $J = a + bm^2, \quad \begin{cases} J = \text{spin} \\ m = \text{mass} \end{cases}$

(at least, to a good approximation.)

The other representation of the S - matrix is obtained by continuing the amplitude into the complex angular momentum plane and assuming that the singularities encountered are poles lying on Regge trajectories.[3] This approach (Regge pole theory), using analyticity and crossing properties of scattering amplitudes, exhibits a close connection between scattering at high energies and the mass spectrum[2] (Fig.1)

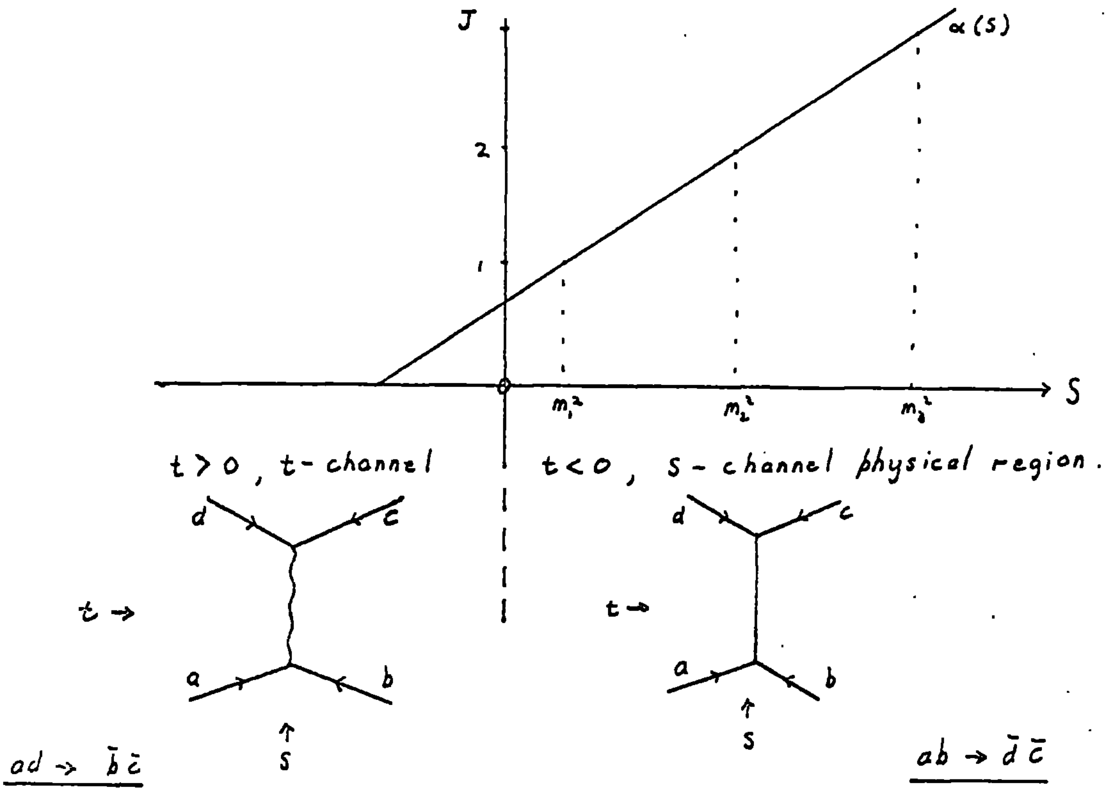


Fig.I (4 - point function)

Hence, we may find $\alpha(s)$ for

a) $S < 0$ from high energy data,

b) $S > 0$ from low energy data ie. masses, widths etc.,

with the following results,

i) All trajectories are approximately linear with a common slope $\alpha' \sim 1\ \mathrm{GeV}^{-2}$,

ii) trajectories are exchange degenerate,

iii) $\mathrm{Im}\,\alpha \ll \mathrm{Re}\,\alpha$, ie. narrow widths.

ie. we may approximate a trajectory by

$$\alpha(s) = \alpha(0) + \alpha' s .$$

(The pomeron trajectory merits special treatment, see later).

We therefore have for the S - channel physical region either a sum over resonance poles or a sum over Regge poles. These two expressions are inequivalent if the sums are stopped after a finite number of terms; ie. we must have an infinite number of resonances and Regge poles. In order to compare the two expressions we may use Finite Energy Sum Rules (FESR's)[3,4], eg. (for the four - point case)

$$\int_0^{\nu_1} d\nu \, \nu^N \, \mathrm{Im}\, T(\nu,t) \simeq \sum_n \frac{\beta_n \, \nu_1^{\alpha_n+N+1}}{\alpha_n+N+1} \qquad (1.1.1)$$

where $\nu = \dfrac{s-u}{4m^2}$, $(S = 1 + iT)$

and saturate ImT by resonance poles.

Hence, we have the scheme:

input is s - channel mass spectrum

output consists of trajectories and residues $\alpha_n(t)$, $\beta_n(t)$,

or vice versa.

ie. we have a FESR bootstrap[4] with s - channel resonances building up the cross - channel Regge poles.

ie. 

This is the origin of duality. In fact,[2] in an approximation of the amplitude by a meromorphic function ie. requiring only resonance poles in the reaction channels, then the amplitude may be written either as a sum over resonances in the s - and u channels or as a sum over Regge poles in the t - channel, ie. over resonances in the t - channel. If singularities other than poles are present then duality does not necessarily follow. (The interference model has no connection between s - & t - channel trajectories).

A special mention should be made concerning the Pomeron trajectory (invented to account for the asymptotic behaviour of cross sections). This trajectory is not normally included in the dual picture - ie. it is not dual to a sum of s - channel resonances. This is because

a) the pomeron (vacuum quantum numbers) is independent of s - channel quantum numbers ie. of s - channel resonance poles.

b) At high energies the pomeron is associated with the production of large numbers of particles ie. it is associated with cuts.

1.2 The Veneziano Model

Venziano[5] proposed an explicit form for A(s,t), (4 - point amplitude) possessing analyticity, crossing symmetry and duality (so satisfying FESR's), which took the form

$$A(s,t) = \frac{\beta}{\pi}\left\{V(\alpha_s,\alpha_t) + V(\alpha_u,\alpha_t) + V(\alpha_s,\alpha_u)\right\} \tag{1.2.1}$$

where,

$$V(x,y) = \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} = (1-x-y)\,B(1-x,1-y) \tag{1.2.2}$$

A(s,t) has simple poles at $\alpha_s = n$, for example, (no double poles), and is also Regge behaved. However, a fundamental defect was the violation of unitarity.

Duality follows from

$$V(x,y) = \sum_n \frac{C_n(y)}{x-n} = \sum_m \frac{C_m(x)}{y-m}$$

This amplitude has been compared with experimental data with not unfavourable results. For example in his letter[5], Lovelace points out that the Veneziano amplitude accounts reasonably well for the boson spectrum and some decay widths. He also draws certain correspondences between the predictions of this amplitude and those of current algebra. Indeed Koidalov[2] suggests that SU(6) and SU(3) may be of dynamical origin - ie. determined to a large extent by the requirement of duality. Of relevance here is the compatibility of duality and SU(3) zero triality multiplets, as demonstrated in duality diagrams.[6] This apparently close relation to quark ideas is brought out more clearly in a formulation of the generalised dual N - point function[7], in which quark lines are convenient mathematical constructs. Later, it will be shown that the most direct realization of this statement is the Analogue Model, which serves as the starting point for the author's contribution.

1.3 Generalization of the Veneziano Amplitude.

a) The extension to a five point process was given by Bordakci and Ruegg[7], after which Chan and Tsou Tsun[7] proposed an N - point formula. These amplitudes were analytic, crossing symmetric, possessed Regge asymptotic behaviour, and were "dual". In order to define what they meant by duality, Chan and Tsou Tsun introduced, for each Mandelstam channel, a conjugate variable: consider FIG. 2.

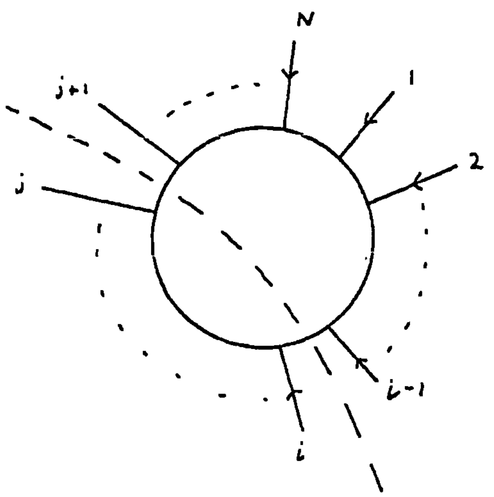


Fig.2.

Now, each partition of (1,----,N) defines a Mandelstam channel;

eg. P: (i, i + 1,---,j) (j + 1,-- N,1,--,i - 1) = (i, j)

We define $$x_{ij} = -1 - \alpha_{ij} = -1 - \alpha_{ij}(0) - \alpha'_{ij}(s_{ij})$$

where $$s_{ij} = \left(\sum_{k=i}^{j} p_k\right)^2$$

Then, two partitions P = (i j), and $\bar{P} = (\bar{i}, \bar{j})$ are dual to each other if

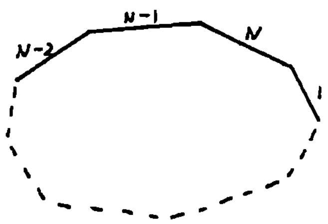
$$(i,\cdots,j) \cap (\bar{i},\cdots,\bar{j}) \neq \emptyset, \ (i,\cdots,j), \ (\bar{i},\cdots,\bar{j})$$

Now, the conjugate variables for two dual partitions are not allowed to vanish simultaneously (to prevent double counting). Hence, it may be shown that, if Up is the conjugate variable to P,

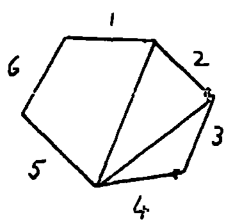
$$U_p = 1 - \prod_{\bar{P}} U_{\bar{P}} \quad , \quad U_p \in [0,1]$$

over all $\bar{P}$ dual to P.

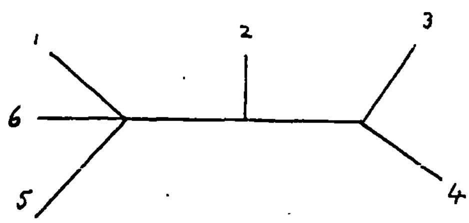
The number of independent variables is the number of variables which may vanish simultaneously (N - 3). In order to visualize dual and non-dual channels the following diagram (dual diagram) is useful.



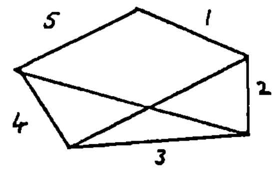
Each channel may be represented by a diagonal line eg.



corresponds to



Dual channels have intersecting diagonals

eg. 

does not correspond to a Feynman diagram.

b) Koba - Nielsen variables

Koba and Nielsen[7] found a mathematically pleasing solution to the conjugate variables, which opened the door to further progress in this field. To do so they considered the duality diagrams of Harari - Rosner[6] - ie. they built into the solution the notion of quarks (this is the formulation referred to in (1.2)). For, if each quark line in the duality diagram is denoted by a small letter, i, say, and if the meson containing this quark is represented by I and the meson containing the antiquark end of the quark line by $\bar{I}$, then a solution to the conjugate variable for the channel (i j) (ie. quark i and antiquark j form a resonance) is given by the anharmonic cross-ratio

$$U_{ij} = \frac{(z_I - z_{\bar{J}})(z_{\bar{I}} - z_J)}{(z_I - z_J)(z_{\bar{I}} - z_{\bar{J}})} , \qquad (1.3.1)$$

where z_A is a point on the unit circle in the complex plane, corresponding to the meson containing the quark a.

Then, the N - point amplitude (corresponding to this ordering) is defined as

$$\int_0^1 \prod_{|j-k|>1} \left[U_{jk} \right]^{x_{jk}} dV_N \qquad (1.3.2)$$

where

$$dV_N = \left\{ \prod_{k=1}^{N} | z_{k+2} - z_k |^{-1} \right\} |z_a - z_b||z_b - z_c||z_c - z_a| \prod_{\bar{a}\bar{b}\bar{c}} d\theta_i \qquad (1.3.3)$$

$\gamma_k = e^{i\theta_k}$, $\prod_{\bar{a}\,\bar{b}\,\bar{c}}$ means the product over all i, omitting a,b,c.

The choice of $\theta_{a,b,c}$ is quite arbitrary (because of the form of dV_N).

The reason for the omission of the triple integral is that a Mobius transformation

$$\gamma \to \gamma' = e^{i\varphi} \frac{(\gamma - re^{i\rho})}{(1 - re^{-i\rho}\gamma)} \quad , \quad (\varphi, \rho, r \text{ all real.}) \tag{1.3.4}$$

maps the unit circle onto itself while preserving the anharmonic cross-ratios. Therefore, we have a three-fold degeneracy. Hence the integration is over N - 3 parameters only.

The full amplitude is obtained by summing over non-cyclic permutations of the external momenta - this is quite different from Feynman diagram rules.

Now that duality has been described and an N - point function obtained via quark ideas, we now go on to consider some developments in wavefunctions relevant to dual models.

2. Field Theory.

2.1 Internal space

The success of Regge pole theory involved the combination of mass and spin into one picture - the Regge trajectory, thus giving spin a dynamical role as important as that of mass.

This latter point was suggested in 1964 by Lurçat[8] in connection with field theory. Therefore, just as in the case of mass, we should consider particles "off spin shell" ie. complex angular momentum. He argued that, to restrict oneself to wavefunctions defined only on Minkowski space is to force particles to have definite spin (for finite representatio... of the Poincaré group). For, the restriction to Minkowski space allows only point particles in the theory. Now, the physical idea of a point particle is that of a small body in the limit of its dimensions disappearing. However, in taking this limit we lose degrees of freedom - viz rotational degrees of freedom in the classical case. This is valid only if we may assume that the latter (ie. spin) play a minor role dynamically, compared with the translational degrees of freedom (i.e. mass). (Probably the success of Quantum Electrodynamics is due to the validity of neglecting higher spin ie. the electron may be treated as a point particle).

Therefore, if we are to assume spin dynamically important we should restore these degrees of freedom ie. the configuration space is the full Poincaré group manifold, in the relativistic case.

Even before Lurçat's paper, D. Finkelstein[8] (1955) considered the possibilities of adding internal degrees of freedom to particles, within the framework of quantum theory and special relativity. He characterizes a rigid body in classical physics in such a way as to lead naturally to a similar concept in relativistic physics, which is that the configuration space of such an entity is a homogeneous space of the Poincaré group. He then classifies all such possible spaces. (Proper subspaces of the full Poincaré group manifold correspond to symmetries in the internal configuration space).

More recently, Bacry and Kihlberg[9] (1969) also considered wavefunctions on homogeneous spaces, giving special attention to an 8 - dimensional space.

2.2 An example

In an attempt to combine mass and spin in one equation Bacry and Nuyts also considered this 8 - dimensional homogeneous space.[8] This space was formed from Minkowski space together with a two dimensional complex spinor space. Any point in this space may be written (x, ξ) where $x \in$ Minkowski space, and $\xi = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$. This point transforms as follows under the action of the Poincare group

$$(x, \xi) \xrightarrow[{[a, \Lambda]}]{} (\Lambda x \Lambda^{\dagger} + a, \Lambda \xi) \qquad (2.2.1)$$

where Λ is a 2 x 2 matrix of determinant one,

a, x are 2 x 2 hermitian matrices.

This induces the representation of $\overline{P}$ (the universal covering of P, the Poincaré group),

$$[T_{(a,\Lambda)} f](x,\xi) = f\left([a,\Lambda]^{-1}(x,\xi)\right) , \qquad (2.2.2)$$

from which are derived the Casimir equations

$$p^2 = p_\mu p^\mu = -\Box$$
$$W^2 = \Box D(D+1) , \quad where \quad D = \tfrac{1}{2}\xi^\alpha \frac{\partial}{\partial \xi^\alpha} . \qquad (2.2.3)$$

(This applies only to the case of analytic functions)

The authors then propose the Lagrangian,

$$\mathcal{L} = -a f^* f + f^*_\mu f^\mu + b (Df)^* (Df) \qquad (2.2.4)$$

where $f^*(x, \xi^*)$ is the conjugate of $f(x,\xi)$

This leads to the Euler equation

$$(-\Box - a - bD) f(x,\xi) = 0 \qquad (2.2.5)$$

ie. $m^2 = a + b\,s$, m = mass, s = spin.

ie. linear trajectories.

In the above, wavefunctions of particles with spin do not have indices, ie. they are scalars. The description of spin is accounted for by the presence of internal variables.

More recently, Brink and Kihlberg[8] applied the homogeneous space ideas to dual theory. Here, the Koba - Nielsen variables are treated as internal variables of a hadron wavefunction. The SL(2 $\mathbb{C}$) symmetry of dual models, (Chapter II), which accounts for duality, is interpreted as the Lorentz group acting on the internal part of the homogeneous space. Several other authors[9] have also made this identification in their respective treatments of dual models. In chapter IV this identification is realized in a structure similar to the above space of Bacry and Nuyts.

2.3 Extended Structure

a) On another line of approach in giving particles internal degrees of freedom, de Broglie et al[10] considered quantum numbers such as isospin, strangeness etc. as due to the extended structure of particles in space-time, ie. not, as is usual, due to symmetries in an abstract space, dissociated from Minkowski space. This idea was also put forward in Finkelstein's paper.

These considerations have received a lot of attention from Japanese physicists following on Yukawa's idea of elementary domains[10], invented in an effort to avoid the divergences of conventional field theory, which were thought to be the result of using point particles. In fact, in 1970, Hara and Goto proposed a "Deformable Sphere Model of Hadrons"[10] which accounted for SU(3) and SU(6) without invoking physical quarks, and also restricted hadrons (ie. excited states of the sphere) to zero triality. The authors point out the similarity with dual models - the sphere arising from three strings, while dual models are thought to be related to the dynamics of a string[11] - ie. SU(3) is a result of there being three space dimensions. However, the sphere is treated non-relativistically and the model has only been given for free hadrons.

Nevertheless, it is interesting to recall from I, that dual models have, in some way, SU(3) and zero triality built in, while giving some SU(6) results[5] (Lovelace).

b) A possible relationship between particle symmetries and space-time was also proposed by Takabayashi[12] in a multilocal theory of a relativistic oscillator model. Here, the free hadron, having an extended structure with relativistic internal motion, obeyed an infinite - component wave equation. By taking the limit of the number of oscillators to infinity Takabayashi made a direct link with the string model of Nambu and Susskind[11]. (The simplest way of handling the infinite number of oscillators is to order them in one dimension). In his papers[12] entitled "Detailed Wave Equation and Dual Amplitudes" he does not insert duality as a prerequisite, but it does appear as a characteristic feature of an extended structure. That the wave equation should hold at all points of the string is a stringent requirement and imposes a set of subsidiary conditions - the gauge relations of dual theory!

Remarks

In the introduction we have described how

a) duality arose from phenomenological considerations, containing some notion of quarks.

b) it is possible to visualize duality as a characteristic of extended structures, which may allow for internal symmetries.

Now, the thesis traces the mathematical formulation and development of dual theory, describing those points in its structure which are looked upon as being fundamental. Finally, an explanation is offered for the internal symmetry group, in terms of an extended structure in space-time (but quite different from the string model).

CHAPTER II

In order to introduce the main steps in the development of dual theory we employ the operatorial approach, by means of which the factorizability of the dual amplitudes is readily understood. In this way we see that it is possible to build a mathematical model which embodies, in a consistent manner, what are thought to be basic properties of a strong interaction theory.

These properties, and the structure in which they are realized, will become apparent in this chapter, thus laying the foundations on which the final chapter is based.

1. N - scalar amplitude

1.1 The Operator Formalism[1]

This procedure allows dual amplitudes to have a Feynman - like interpretation - ie. the amplitude may be written as a sequence of vertices and propagators. These objects are described in terms of an infinite set of creation and annihilation operators, which have the commutation relations,

$$[a_\mu^n, a_\nu^m] = [a_\mu^{n\dagger}, a_\nu^{m\dagger}] = 0,$$
$$[a_\mu^n, a_\nu^{m\dagger}] = -g_{\mu\nu}\,\delta_{m,n} \qquad (2.1.1)$$

where μ, ν are Lorentz indices; n,m = 1,2------; $g_{\mu\nu}=(1,-1,-1,-1)$

Using these we define the more convenient operators α_μ^n, by

$$\left.\begin{aligned} \alpha_\mu^n &= -i\sqrt{n}\,a_\mu^n \\ \alpha_\mu^{-n} &= i\sqrt{n}\,a_\mu^{n\dagger} \\ \alpha_\mu^0 &= \sqrt{2}\,p_\mu \end{aligned}\right\} n = 1, 2, \cdots \qquad (2.1.2)$$

p_μ being the momentum operator;

the commutation relations now read

$$[\alpha_\mu^n , \alpha_\nu^m] = -n\, g_{\mu\nu}\, \delta_{m+n,0} \qquad (2.1.3)$$

The zeroth mode corresponds to a point-like particle, the higher modes to internal excitations. Following Ramond,[13] we assume that these internal excitations are periodic, with period 2 , say. The parameter describing the internal evolution we denote by , and introduce the generalized position and momentum operators,

$$Q_\mu = \sum_{n=0}^{\infty} q_\mu^n = q_\mu^0 + i \sum_{n=-\infty}^{\infty}{}' \frac{\alpha_\mu^n}{n}$$

$$P_\mu = \sum_{n=0}^{\infty} p_\mu^n = \sum_{n=-\infty}^{\infty} \alpha_\mu^n \; , \qquad (2.1.4)$$

where

$$q_\mu^n = \frac{1}{\sqrt{n}} (a_\mu^n + a_\mu^{n+}) = \frac{i}{n} (\alpha_\mu^n - \alpha_\mu^{-n})$$

$$q_\mu^0 = \frac{1}{\sqrt{\varepsilon}} (a_\mu^0 + a_\mu^{0+}) = x_\mu / \sqrt{2}$$

$$p_\mu^n = \frac{\sqrt{n}}{i} (a_\mu^n - a_\mu^{n+}) = (\alpha_\mu^n + \alpha_\mu^{-n})$$

$$p_\mu^0 = \frac{i\sqrt{\varepsilon}}{2} (a_\mu^0 - a_\mu^{0+}) = \sqrt{2}\, p_\mu = \alpha_\mu^0 \; ,$$

a_μ^{0+} being the creation operator for the zeroth mode.[14]

The presence of ε arises from a consideration of SU(1,1) represenations; here we are concerned with the $J = -\varepsilon/2$ representation[14], ε being equated to zero at the end of all calculations.

The internal motion is described by the Nambu Hamiltonian

$$H = -\sum_{n=0}^{\infty} (n + \tfrac{\varepsilon}{2})\, a^{n+} . a^n = -\sum_{1}^{\infty} \alpha^{-n} . \alpha^n - \tfrac{1}{2} \alpha^0 . \alpha^0 \qquad (2.1.5)$$

which gives rise to linear trajectories.

Therefore, the Heisenberg equations of motion

$$[H,F] = i \frac{dF}{d\tau} ,$$

imply

$$\left.\begin{aligned} Q_\mu(\tau) &= e^{i\tau H} Q_\mu e^{-i\tau H} \\ &= q^0_\mu + \tau p^0_\mu + i \sum_{-\infty}^{\infty}{}' \frac{\alpha^n_\mu}{n} e^{-in\tau} \end{aligned}\right\} \quad (2.1.6)$$

and, similarly,

$$P_\mu(\tau) = \sum_{-\infty}^{\infty} \alpha^n_\mu e^{-in\tau} = \frac{d}{d\tau} Q_\mu(\tau)$$

Hence,

$$[P_\mu(\tau), Q_\nu(\tau')] = 2\pi i g_{\mu\nu} [\delta(\tau'-\tau) + 1]$$

$$[P_\mu(\tau), P_\nu(\tau')] = -2\pi i g_{\mu\nu} \frac{d}{d\tau} \delta(\tau'-\tau) \quad (2.1.7)$$

$$[Q_\mu(\tau), Q_\nu(\tau')] = -\pi i g_{\mu\nu} \left[\int^{\tau} \delta(\tau'-\tau'')d\tau'' - \int^{\tau'} \delta(\tau-\tau'')d\tau'' \right]$$

$$= -\pi i g_{\mu\nu} \, \eta(\tau-\tau') \quad , \quad \eta(x) := \frac{x}{|x|} .$$

We may now define the <u>vertex</u> for the emission of an on-shell scalar of momentum k_μ,[15]

$$V_0(k, z) := \, : \exp[i\sqrt{2} k . Q(\tau)] :$$

$$= z^{-2k.p} e^{ik.x} \exp\left(\sqrt{2} k^\mu \sum_1^\infty \frac{\alpha^{-n}_\mu}{n} z^{-n}\right) \exp\left(-\sqrt{2} k^\mu \sum_1^\infty \frac{\alpha^n_\mu}{n} z^n\right) \quad (2.1.8)$$

$$= e^{i\tau H} V_0(k) e^{-i\tau H}$$

where

$$V_0(k) := V_0(k, 1) \quad , \text{i.e. } \tau = 0 \text{ in } z = e^{-i\tau} .$$

Then,

$$V_0(k, z) = z^H V_0(k) z^{-H}$$

The propagator of a Reggeon is (Beta function)

$$\begin{aligned} D[\alpha(s)] &= \int_0^1 dx \; x^{H-\alpha(s)-1} (1-x)^{a-1} \\ &= \sum_{\ell_0=0}^{\infty} \frac{(1-a)_{\ell_0}}{\ell_0 !} \left(H - \alpha(s) + \ell_0\right)^{-1}, \end{aligned} \qquad (2.1.9)$$

where $(Y)_r$ is Pochhammer's symbol

ie. $(Y)_r = Y(Y+1) \text{-----} (Y+r-1) = \dfrac{\Gamma(Y+r)}{\Gamma(Y)}$,

$\alpha(s) = a + bs$, and x is the Chan variable[7]
(ie. conjugate variable) for the Reggeon line.

Now we may write down the amplitude[1] for N scalars scattering off each other (Fig. 3.)

$$A_N(p_1 \cdots, p_N) = \langle 0; p_N | V_0(p_{N-1}) D \cdots\cdots D[\alpha(s_2)] V_0(p_2) | p_1; 0 \rangle \qquad (2.1.10)$$

where $S_i = \left(\sum_{j=1}^{i} p_j\right)^2$; $|0; k\rangle = e^{i2k.p}|0\rangle$, $|0\rangle$ being the vacuum defined by $\alpha_\mu^n |0\rangle = 0$, $n \geqslant 0$.

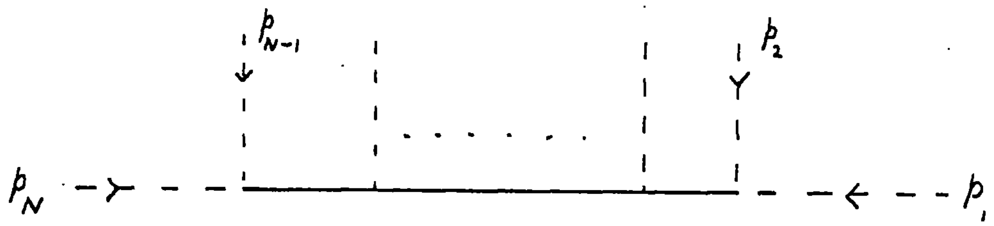


Fig. 3.

This expression is equivalent to that in (1.5).

1.2 Properties of A_N.

At this stage, in order to appreciate the ensuing development of the field, it is worthwhile looking at some of the properties of the N - point amplitude.

(i) First of all, it satisfies what are thought to be derirable requirements of an amplitude, since it is equivalent to the Bardakci - Ruegg formula.[1,7]

viz. a) Analyticity

b) Regge behaviour

c) Factorization

ie. consistency with the bootstrap hypothesis[17] in that all particles are bound states of others. By analyzing the intermediate states we may discover the spectrum of the model. It is found[1] that the degeneracy of states as a function of spin, d(J), increases exponentially as the mass

$$\text{ie.} \quad d(J) \propto \exp(b\, m)$$

This exponential dependence is also a feature of statistical models (Hagedorn, Frautschi)[16]

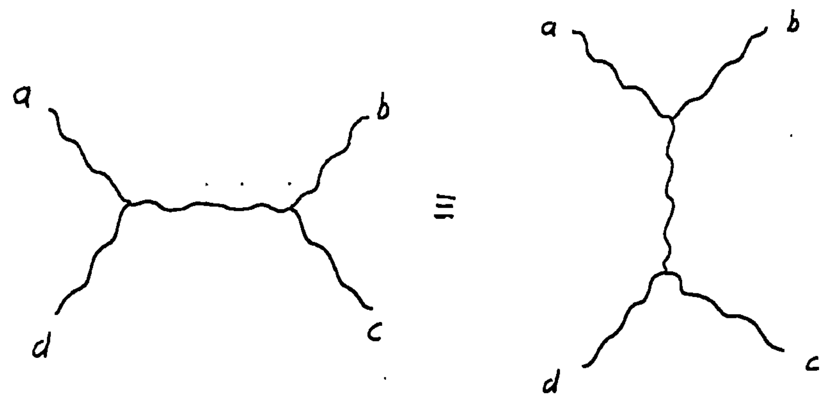
d) Duality

ie. A_N is invariant with respect to cyclic permutations of the external momenta. Neveu and Schwartz[15] show that this property depends on

i) Möbius invariance of A_N(eg $Z_1^{\,1} = Z_N$, $Z^1{}_2 = Z_1$, $Z_N^{\,1} = Z_{N-1}$)

ii) how $V_o(k_i, Z_i)$ commutes with $V_o(k_j, Z_i)$

We postpone, till after the introduction of the "twisting operator", the discussion of operatorial duality i.e.



where a,b,c,d, are arbitrary states.

e) Crossing symmetry

f) Resonance poles lie on linearly rising trajectories.

However, since all poles lie on the real axis (ie. zero widths), unitarity is violated. Later, in section 3, a procedure for implementing unitarity in a perturbative manner, will be outlined.

ii) If we now focus attention on those states obtained by a chain of propagators and vertices (definition of physical states) we find that they satisfy[1,15]

$$\left.\begin{aligned} L_0 |\varphi\rangle &= -m^2 |\varphi\rangle \\ L_+ |\varphi\rangle &= 0 \end{aligned}\right\} \qquad (2.1.11)$$

where

$$\begin{aligned} L_0 &:= -\tfrac{1}{2}\alpha_0\cdot\alpha_0 - \sum_1^\infty \alpha_{-n}\cdot\alpha_n \\ L_+ &= L_1 := -:\sum_0^\infty \alpha_{-n}\cdot\alpha_{n+1}: \\ L_- &= L_{-1} := L_+^\dagger \end{aligned} \qquad (2.1.12)$$

(: : denotes normal ordering)

The above operators give the following algebra,

$$[L_0, L_\pm] = \pm L_\pm \quad , [L_-, L_+] = 2L_0 \qquad (2.1.13)$$

ie. the algebra of SU(1,1) (or SL(2$\mathbb{R}$)) of projective transformations

$$z \rightarrow z' = \frac{\alpha z + \beta}{\gamma z + \delta}$$

This group appeared in connection with the Koba - Nielsen formulation of the N - point amplitude,[7] the latter being invariant with respect to the above transformations on the z's forming the conjugate variables. If we had chosen the z's to belong to the real line (which we can do, for all that is required of the z's is that the relevant cross-ratios are real and belong to the interval (0,1)), then the corresponding group to SU(1,1) is SL(2 $\mathbb{R}$) which maps the real axis onto itself.[15]

These considerations suggest that a study of the invariants of SU(1,1) or SL(2 $\mathbb{R}$) may be useful. Indeed, a number of papers have been written which do construct amplitudes from this point of view.[14] In fact some people[18] have gone further and suggest that SU(1,1) be regarded as a subgroup of SL(2 $\mathbb{C}$), the latter then being identified with the Lorentz group, which may allow a treatment of spin. Neverthless, at this stage there is little justification for this point of view.

However, the above group-theoretic investigations do not provide a natural framework for other operators which have proved to be of importance.

These operators, first found by Virasoro,[19] are defined as follows,[15]

$$L_n := -\frac{1}{2} : \sum_{m=-\infty}^{\infty} \alpha_{-m} \cdot \alpha_{m+n} : \qquad (n > 0)$$

$$= -\sum_{m=0}^{\infty} \alpha_{-m} \cdot \alpha_{m+n} - \frac{1}{2} \sum_{m=1}^{n-1} \alpha_m \cdot \alpha_{n-m} \qquad (2.1.14)$$

$$L_{-n} = L_n^{+} ,$$

with the algebra

$$[L_n , L_m] = (n-m) L_{n+m} + \frac{D}{12} n(n^2-1) \delta_{n+m,0} , \qquad (2.1.15)$$

(D = dimension of space-time).

The above operators, which include the SU(1,1) generators L_0, $L_{\pm}$, have the following effect on $V_0(k,z)$ [15]

$$[L_n , V_0(k,z)] = z^n \left(z \frac{d}{dz} - n k^2 \right) V_0(k,z) \qquad (2.1.16)$$

Therefore, if we restrict ourselves to the simple, but unphysical, case of $k^2 = m^2 = -1$, we obtain

$$[L_n , V_0(k,z)] = z \frac{d}{dz} [z^n V_0(k,z)] \qquad (2.1.17)$$

(Note, if $m^2 = 0$, then

$$[L_n , V_0(k,z)] = z^{n+1} \frac{d}{dz} V_0(k,z) = \frac{V_0(k, z + \epsilon z^{n+1}) - V_0(k,z)}{\epsilon} \qquad (2.1.18)$$

ie. an infinitesimal conformal transformation of $z \rightarrow f(z)$)

However, if we require all z's to lie on a circle (as in the Koba-Nielsen prescription) then $\frac{dz}{z}$ is a perfect differential and so, integration over the whole circle will yield zero,[14]

$$\text{ie.} \quad [L_n, V(k)] = 0 \quad (n \geq 1) \tag{2.1.19}$$

where
$$V(k) = \oint \frac{dz}{z} V_0(k,z)$$

These relations imply certain subsidiary conditions on physical states. Indeed, if $|\varphi\rangle$ is a physical state, then

$$L_n |\varphi\rangle = 0 \tag{2.1.20}$$

This follows from[15]

$$(L_0 - L_n - 1)\frac{1}{L_0 - 1} V_0(k) = \frac{1}{L_0 + n - 1} V_0(k)(L_0 - L_n - 1)$$

which implies

$$(L_0 - L_n - 1)\{D_0 V_0(k_2) \cdots D_0 V_0(k_{N-1})\} |0; k_N\rangle = 0$$

since

$$(L_0 - L_n - 1)|0; k_N\rangle = -(k_N^2 + 1)|0; k_N\rangle = 0$$

where $D_0 = \frac{1}{L_0 - 1}$ is the propagator for $m^2 = -1$.

Hence, we have an infinite set of conditions on the physical states. In fact, if $|s\rangle = L_{-n}|\psi\rangle$, $|\psi\rangle$ being <u>any</u> state in the Fock space, then $|s\rangle$ decouples from all physical states; for

$$\langle s|\varphi\rangle = \langle\psi|L_n|\varphi\rangle = 0$$

Such states $|s\rangle$ are termed "<u>spurious</u>"

This fact implies that there exist linear relations between residues of any state (pole). These relations are sufficient to remove all negative residues ie. ghost states arising from the indefinite Lorentz metric, which is the content of the following section.

1.3 Absence of Ghosts.

In their search for the physical spectrum in the Fock space, Del Giudice et al.[20] constructed a set of states which satisfied

$$L_n|\rangle = 0 \quad , \quad L_0|\rangle = |\rangle ;$$

they were orthonormal and had positive norm.

These states (DDF) are created from the vacuum by the operators A^i_{-n}, defined by

$$A^i_n = \frac{1}{2\pi i}\oint \frac{dz}{z}\,\frac{1}{2} : \left\{ \frac{p^i(z)}{\sqrt{2}} , V_0(nk,z)\right\}_+ :$$

$$A^i_{-n} = (A^i_n)^\dagger \quad , \quad n = 1,2,\cdots \quad , \quad i = 1,2,\cdots, D-2 ,$$

where D = space time dimension, and $k = \frac{1}{2}(1,0,0,-1)$.

Later, it was shown[21] that these states, for the special case D = 26, were the only physical states to couple. Indeed, introducing $K_n := k\cdot\alpha_n$, then all DDF states satisfy $K_n = L_n = 0$, n = 1,2---; the converse is also true. Now, if we let F^M = set of DDF states with R = M ($R = L_0 - p_0^2$, where $p_0 = p + Nk$, $p = (0,0,0,1)$), and G^M its orthogonal complement in the subset of <u>all</u> states with R = M, then neither F^M nor G^M contain null states, and, furthermore, F^M is positive definite.

Hence, for D = 26, there are no ghosts in the physical spectrum. In fact, any physical state $|\psi\rangle$ may be decomposed by

$$|\psi\rangle = |\varphi\rangle + |ns\rangle$$

where $|\varphi\rangle$ is a DDF state and $|ns\rangle$ is null spurious (a <u>null</u> state is one which is orthogonal to all physical states, including itself.)

The critical dimension D = 26 is precisely the one for which the Pomeron appears as a pole (and not a cut) (Lovelace[22]).

However, it is straightforward to show that, for $D < 26$ (in particular $D = 4$), there are still no ghosts.

2. N- Reggeon Amplitude

2.1 Introduction

We have seen the emergence of two important points in the last section, a) SU(1,1) algebra accounting for duality,
b) the gauge algebra which leads to the absence of ghosts.

The realization of the SU(1,1) group is that of conformal, one-to-one, mappings of the unit disk onto itself. In the case of the full complex plane the corresponding group is SL(2$\mathbb{C}$), which leads to the Virasoro amplitude.[9] Later we will show how the gauge group is also realized by transformations of the complex plane.

The prominence of the z - plane and projective transformations is emphasised by Lovelace's expression for the N - Reggeon vertex.[23] Indeed, in their introduction to the operator formalism, Alessandrini et al[1] found it convenient to introduce infinite - dimensional representations of projective transformations (canonical forms) to handle certain operators appearing in this approach. This correspondence was extended to the following:

If $\omega(\gamma) = \dfrac{\alpha\gamma+\beta}{\gamma\gamma+\delta}$, $\alpha\delta-\beta\gamma=1$, then the infinite dimensional matrix C_{nm} , corresponding to the transformation $\omega(\gamma)$ is defined by

$$\frac{\omega^n}{\sqrt{n}} = \sum_{m=0}^{\infty} C_{nm}\frac{\gamma^m}{\sqrt{m}} \quad , \quad n \geqslant 1 , \qquad \left(\text{ with the convention } \frac{1}{\sqrt{0}} = 1 \right)$$

$$C_{on} = (C^T)_{no} = \frac{1}{\sqrt{n}}\left(-\frac{\gamma}{\delta}\right)^n$$

$$C_{oo} = \log|\delta| \qquad (2.2.1)$$

The product of two such matrices is given by

$$(C_1 C_2)_{nm} = \sum_{k \neq 0} (C_1)_{nk} (C_2)_{km} + (C_1)_{n0} \delta_{0,m} + \delta_{n,0} (C_2)_{0m} \qquad (2.2.2)$$

This correspondence $\omega \rightarrow C$ is a representation of the group of projective mappings only if zero modes are absent.

Notation: if w takes f,g,h to f′, g′, h′, respectively, then we denote the corresponding infinite matrix by

$$\begin{pmatrix} f & g & h \\ f' & g' & h' \end{pmatrix}$$

2.2 The Reggeon Vertex.

When we consider the vertex for the emission of a scalar (ie. k_μ , the zero mode, only, is present), a possible generalization might be to replace k_μ^i by the generalized momentum $p_\mu^i(\tau)$ thus introducing higher modes into the emitted state. In this way $p_\mu^i(\tau)$ (considered as a c - number for the moment, with "Fourier" components α_n^i) characterizes the emitted Reggeon. However, since the description should be invariant under projective mappings (to ensure duality), we may compare two Reggeons only if their generalized momenta are each defined on the same interval. Therefore, we restrict z to some standard interval and introduce, for each Reggeon, a projective map g_i which performs the function of the Koba - Nielsen variable z_i in the scalar case. Then, following Kosterlitz and Saito,[24] the N - Reggeon vertex is

$$A_N = \int \oint_N^0 d\mu_N \, \langle 0| \exp\left\{ \frac{i}{2\pi} \sum_{\ell=1}^{N} \int_{-\pi}^{\pi} d\tau \, Q[g_\ell(z)] \cdot P_\ell(z) \right\} |0\rangle \qquad (2.2.3)$$

$$\times \exp\left\{ -\frac{1}{8\pi^2} \sum_{\ell=1}^{N} \iint d\tau \, d\tau' \, P_\ell(z) \, P_\ell(z') \log(z - z') \right\}$$

with , $z = e^{\lambda + i\tau}$, $(\lambda \to 0)$,

$$Q_\mu[g(z)] = \frac{x_\mu}{\sqrt{2}} - i\sqrt{2}\,p_\mu \log g(z) + \sum_1^\infty \frac{1}{n}\left[g(z)^{-n} a_\mu^n + g(z)^n a_\mu^{n+}\right] \quad (2.2.4)$$

$$P_\mu(z) = i\frac{\partial}{\partial\lambda} Q_\mu(z) = \sqrt{2}\,p_\mu - i\sum_1^\infty \sqrt{n}\left[a_\mu^n z^{-n} - a^{n+} z^n\right] \quad (2.2.5)$$

where $z = e^{i\tau}$, the standard interval being the unit circle ($-\pi \leqslant \tau \leqslant \pi$),

$$d\mu_N = \prod_{\ell=1}^{N} \frac{dz_\ell}{(z_{\ell+1} - z_\ell)} \left[\frac{dz_a\, dz_b\, dz_c}{(z_a - z_b)(z_b - z_c)(z_c - z_a)}\right]^{-1}$$

$$\phi_N^{\,o} = \prod_{\ell=1}^{N} \left(\frac{z_{\ell+1} - z_{\ell-1}}{z_{\ell+1} - z_\ell}\right) \quad , \text{ for } m^2 = -1 ;$$

z_ℓ being the usual Koba-Nielsen variable.

The first exponential corresponds to the emission of N Reggeons P_μ^i , the second cancelling certain divergences occurring in the former (c.f. the Analogue Model where the divergences are seen as self-energies)

Now, in Lovelace's expression[23] the standard interval is the real axis. So the form of A_N becomes (Kosterlitz and Saito[24])

$$A_N = \int d\mu_N\, \phi_N^{\,o} \langle 0| \exp i\sum_{\ell=1}^{N} \int_{-\infty}^{1} d\xi\, Q[g_\ell(\xi)].K_\ell(\xi)|0\rangle \quad (2.2.6)$$

$$\times \exp\left\{-\frac{1}{2}\sum_{\ell=1}^{N} \iint K_\ell(\xi) K_\ell(\xi') \log|\xi - \xi'|\, d\xi\, d\xi'\right\}$$

with

$$\int_{-\infty}^{1} d\xi\, K_\ell(\xi)\,\xi^n = \alpha_n^\ell \quad , \quad \alpha_0^\ell = \sqrt{2}\,k_\ell$$

and $g_\ell(\xi)$ is defined by the associated infinite matrix

$$g_\ell = \begin{bmatrix} 0 & 1 & \infty \\ z_\ell & z_{\ell+1} & z_{\ell-1} \end{bmatrix}$$

which is the V^ℓ of Lovelace.[23]

Integrating we obtain (c.f. Kosterlitz and Saito[24])

$$A_N = \int d\mu_N \, \phi_N^0 \, \exp\left\{ -\frac{1}{2} \sum_{i \neq j} \langle \alpha^i | U^i V^j | \alpha^j \rangle \right\} \tag{2.2.7}$$

where each Reggeon is represented by an infinite vector $|\alpha^i\rangle$ with coherent state basis components α_n^i ,[23] and

$$U^i = \begin{bmatrix} \gamma_{i-1} & \gamma_i & \gamma_{i+1} \\ 0 & \infty & 1 \end{bmatrix} .$$

2.3 Canonical Forms of Operators

In addition to U^i, V^i defined above, there are certain other operators required in the discussion of loops (unitarity corrections). One of these, viz. the propagator, has already been mentioned. The part of this which depends on the harmonic oscillators, x^{L_0} , has the following associated infinite matrix[1,23]

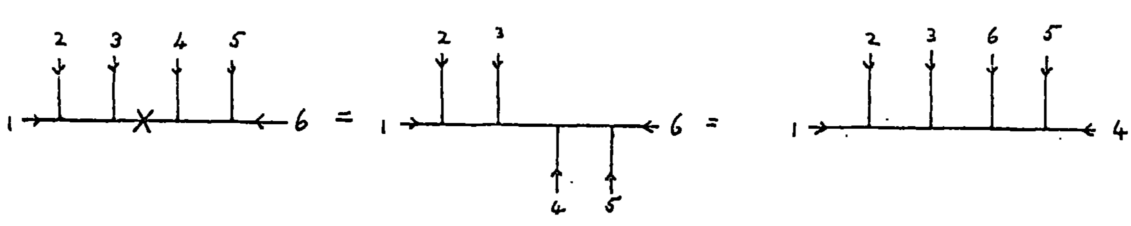
$$x^{L_0} = \begin{bmatrix} 0 & \infty & 1 \\ 0 & \infty & x \end{bmatrix}$$

Another construct is the twisting operator

$$\Omega = \begin{bmatrix} 0 & \infty & 1 \\ 1 & \infty & 0 \end{bmatrix}$$

which reverses the order of the external particles (think of the z_i s lying along the real axis; Ω reverses this line.)

ie. if X denotes the action of Ω to the right,



From this definition we have

$$\langle \varphi | x^{L_0} | \psi \rangle = \langle \varphi | \Omega^{+} x^{L_0} \Omega | \psi \rangle$$

and

$$\langle \varphi | \Omega^{+} \Omega | \psi \rangle = \langle \varphi | \psi \rangle$$

where $|\varphi\rangle, |\psi\rangle$ are physical states.

However, as operator equations, these relations do not hold. Nevertheless, by introducing a gauge transformation $(1 - x)^W$, $W = L_0 - L_-$ (which leaves a physical state unchanged), we may define another operator, θ , which performs the same functions as Ω' .

$$\theta(x) = \Omega (1-x)^W$$

We then have

$$\theta^{+}\theta = \mathbb{1}$$

$$[x^{L_0}\theta]^{+} = x^{L_0}\theta$$

The latter, called a twisted propagator, has an associated infinite matrix

$$x^{L_0}\theta = \begin{bmatrix} 0 & \infty & 1 \\ x & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/x & \infty & 1 \\ \infty & 1 & 0 \end{bmatrix}$$

2.4 Properties of A_N

a) Mobius invariance.

This is shown by the following argument due to Lovelace;[23] map each z_i onto ζ_i by the same Mobius transformation, thus implying

$$\begin{bmatrix} z_{j-1} & z_j & z_{j+1} \\ \zeta_{j-1} & \zeta_j & \zeta_{j+1} \end{bmatrix} \begin{bmatrix} \zeta_{i-1} & \zeta_i & \zeta_{i+1} \\ z_{i-1} & z_i & z_{i+1} \end{bmatrix} = 1$$

Hence, by sandwiching this between U^j and V^i, each $\gamma_i \rightarrow \zeta_i$, while leaving the expression unchanged (NB. $d\mu_N \phi_N^0$ is itself Mobius invariant). Therefore, just as in the case of the N - scalar amplitude we may arbitrarily fix any three z_i's.

b) Cyclic symmetry[23]

c) Factorization

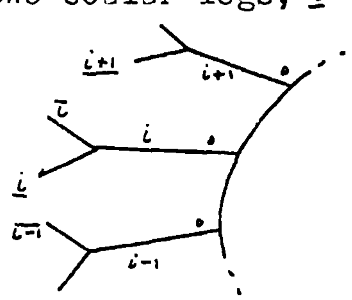
There are a number of forms of the N - Reggeon vertex which satisfy the above. These vertices differ because a Reggeon is invariant under a gauge transformation

ie. $\chi^W |a\rangle = |a\rangle$, $W = L_0 - L_-$.

For example, Alessandrini et al.[1] use Olive's[25] vertex to exhibit the bootstrap property. This vertex is given by the following

$$\int d\mu_{2N} \prod \left(\frac{1-x_i}{dx_i}\right) \left[G(N)\right]^2 \exp\left[\frac{1}{2}\sum_{i\neq j} a^{i+} X^i Y^j a^{j+}\right] |0,\ldots N\rangle$$

The x_i are Chan variables[7] for each external leg. The simplest way of treating these variables is by using Koba - Nielsen variables to express them, which is achieved by adding two scalar legs, $\underline{i}$, $\bar{i}$, to each external leg.

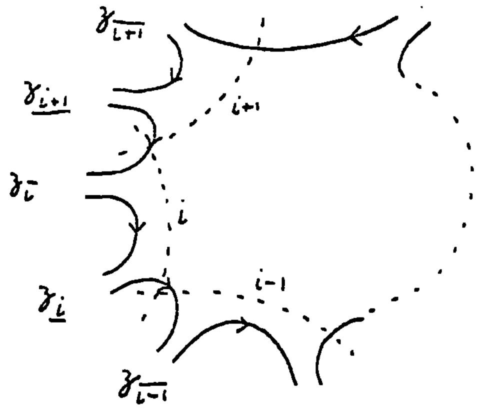


Then,

$$x_i = (\gamma_{\bar{i}}, \gamma_{\underline{i+1}}, \gamma_{\underline{i}}, \gamma_{\overline{i-1}})$$

$$= \left(\frac{\gamma_{\bar{i}} - \gamma_{\underline{i+1}}}{\gamma_{\bar{i}} - \gamma_{\overline{i-1}}}\right)\left(\frac{\gamma_{\underline{i}} - \gamma_{\overline{i-1}}}{\gamma_{\underline{i}} - \gamma_{\underline{i+1}}}\right)$$

as may be seen from



which is part of the duality diagram corresponding to the previous graph (c.f. equation (1.3.1))

The other terms are given by

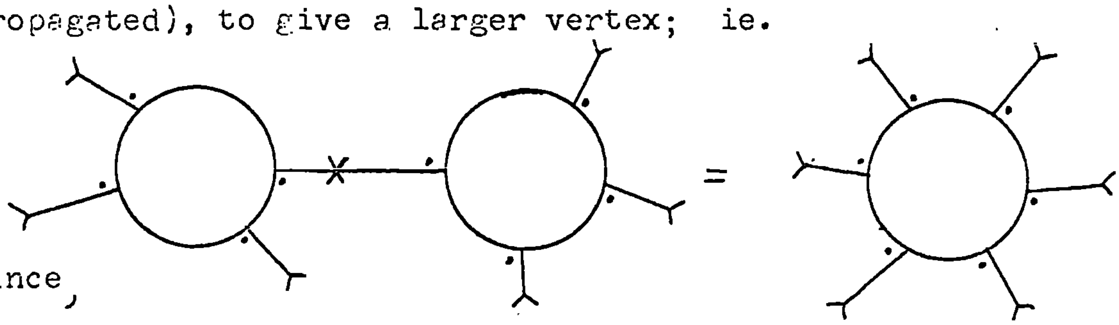
$$G(N) = - \prod_{1}^{N} \left(\frac{\gamma_{\bar{i}} - \gamma_{\underline{i+2}}}{\gamma_{\bar{i}} - \gamma_{\underline{i+1}}} \right)$$

$$X^{i} = \begin{bmatrix} \gamma_{\overline{i-1}} & \gamma_{\underline{i}} & \gamma_{\underline{i+1}} \\ 0 & \infty & 1 \end{bmatrix}$$

$$Y^{j} = \begin{bmatrix} \infty & 0 & 1 \\ \gamma_{\overline{j-1}} & \gamma_{\underline{j}} & \gamma_{\underline{j+1}} \end{bmatrix}$$

(The dot means that all external legs are on the same side as the dot.[1])

Now, Alessandrini et al.[1] showed that two vertices may be joined, using a twisted propagator θ (Ω for Lovelace's vertex,[23] so that in Olive's case no spurious states are propagated), to give a larger vertex; ie.



since,

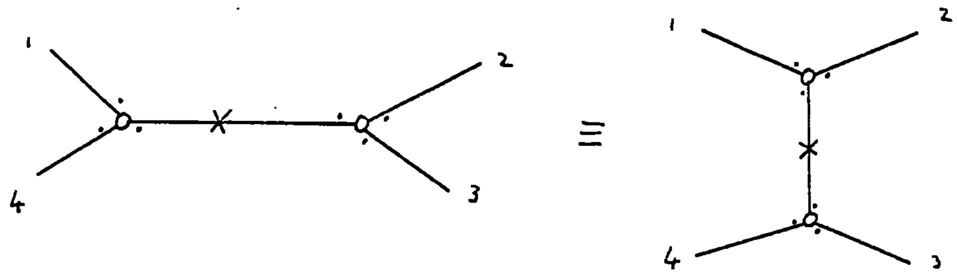
The relation between Olive's and Lovelace's vertices is given by [1]

$$\exp \sum_{i \neq j} \frac{1}{2} \left(a^{i+} X^i Y^j a^{j+}\right) |0_{1..N}\rangle =$$
$$\prod \left[1 - (\gamma_{\underline{r+1}}, \gamma_{\underline{r-1}}, \gamma_{\underline{r}}, \gamma_{\bar{r}})\right]^{W_r^+}$$
$$\times \exp \left[\frac{1}{2} \sum_{i \neq j} a^{i+} U^i V^j a^{j+}\right] |0_{1..N}\rangle$$

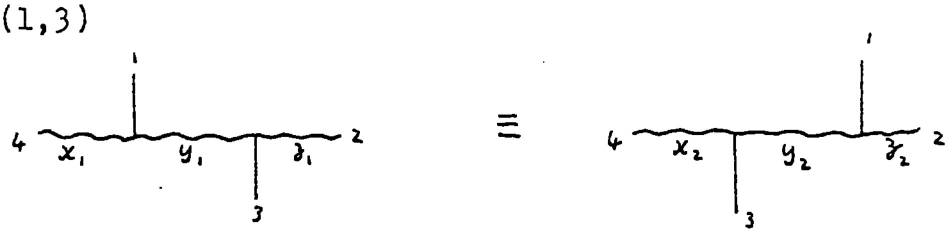
where $\gamma_{\underline{i}} = \gamma_{\bar{i}} = \gamma_i$, ($\Rightarrow X_i = 0$ ie. on mass-shell)

d) Operatorial Duality.

Using the above notation this property is the statement[1].



For example, in the case when two of the legs are scalars (1,3)



Let x_1, y_1, etc. be the Chan variables outlined. Then, in a tree amplitude containing either of the above graphs, we require that their contributions to the integrals be the same. ie. $R_1 = R_2$, where

$$R_1 = x_1^{L_0}\, \theta(x_1)\, V_0(p_1)\, y_1^{L_0}\, \theta(y_1)\, V_0(p_2)\, z_1^{L_0}\, d\mu(x_1)\, d\mu(y_1)\, d\mu(z_1)$$
$$R_2 = x_2^{L_0}\, V(p_2)\, y_2^{L_0}\, \theta(y_2)\, V(p_1)\, z_2^{L_0}\, \theta(z_2)\, d\mu(x_2)\, d\mu(y_2)\, d\mu(z_2)$$

with $d\mu(x) = dx \cdot x^{-2}$ $(m^2 = -1)$.

This is so if $x_1 = x_2$, $z_1 = z_2$

and $$y_2 = \frac{1 - y_1}{1 - y_1(x_1 + z_1 - x_1 z_1)}$$

which is the relation between Chan variables given by Chan in ref.7.

Remark

In the next section, on perturbative unitarity, the correct definition and manipulation of loops depends on two points[26].

a) Loop amplitude must be independent of the particular set of Reggeons chosen to be "sewn".

b) Internal lines may be dualized.

These two requirements follow from

a) Factorization

b) Operatorial duality.

3. Perturbative Unitarity - Multiloops.

3.1 The K.S.V. (Kikkawa, Sakita, Virasoro[27]) approach to the problem of rendering the dual N - point function unitary was to regard the latter as the Born term in a perturbative scheme. The elementary entity propagated is a tower of hadrons (required to satisfy duality and superconvergence). The higher order terms correspond to more than one of these towers being propagated. Each tower of hadrons is a Reggeon, and so multiloops are constructed by "sewing" Reggeons in a tree amplitude.

The following is merely an indication of the steps involved in calculating the M - loop amplitude. The point to notice is the emergence of an automorphic group and the corresponding Riemann surface, which plays a prominent role in the Analogue Model, thus enabling a simple statement of the unitarization programme to be made.

3.2 The M - loop Amplitude.[28,29]

We outline the "sewing" procedure using Lovelace's vertex, the one for Olive's being performed similarly.

a) We consider the N - Reggeon amplitude with the integrand

$$I = \prod_{1 \leq i < j \leq N} \exp \langle a^i | U^i V^j | a^j \rangle \qquad (2.3.1)$$

assuming that the Koba - Nielsen variables z_i are on the unit circle.

Introduce the infinite matrix

$$\Gamma = \begin{bmatrix} a & b & c \\ 1/a & 1/b & 1/c \end{bmatrix}$$

which corresponds to $z \longrightarrow 1/z$; this is not a Mobius matrix, but it acts consistently on Mobius matrices.[29]

If we map all the z_i's onto the real line by a projective transformation, then

$$U^i = (V^i)^{\dagger} \Gamma$$

and

$$I = \exp \left[\frac{1}{2} \sum_{i,j} \langle a^i | (V^i)^{\dagger} \Gamma V^j | a^j \rangle \right]$$

where the convention, $U^i V^i = 0$, is adopted.

b) Factorizing I by a real functional integral, we obtain[29]

$$I = [\det \Gamma]^2 \int \frac{\delta f}{\sqrt{\pi}} \exp \left\{ -\frac{1}{2} \langle f | \Gamma | f \rangle + \sum_{i=1}^{N} \langle f | V^i | a^i \rangle \right\}$$

where we have used

$$\int \frac{\delta f}{\sqrt{\pi}} \exp \left\{ - \langle f | A | f \rangle + \langle b | f \rangle \right\} =$$

$$= [\det A]^{-\frac{1}{2}} \exp\left\{ \frac{1}{4} \langle b | A^{-1} | b \rangle \right\},$$

$$\int \frac{\delta f}{\sqrt{\pi}} := \prod_n \int_{-\infty}^{\infty} \frac{df_n}{\sqrt{\pi}} \quad ;$$

A is any diagonalizable operator of finite non-zero Fredholm determinant det A.[28] Now, we wish to sew M pairs of Reggeons; let the μ^{th} pair be $\langle a^i | = \langle a_\mu |$, and $| a^j \rangle = | a_\mu \rangle$, and write

$$U_\mu = U^i = V_\mu^{+} \Gamma \; , \quad \tilde{U}_\mu = U^j = (\tilde{V}_\mu)^{+} \Gamma \; ,$$

NB. U_μ, $\tilde{U}_\mu$ differ because they depend on different Koba-Nielsen variables, viz. (z_{i-1}, z_μ , z_{i+1}) and (z_{j-1}, $\tilde{z}_\mu$, z_{j+1}).

Finally, let Δ_μ , the propagator joining the μ^{th} pair of Reggeons, have the Chan variable x_μ , and put all unsewn Reggeons in their ground state.

c) The trace of I over non-zero modes is given by functionally integrating over $| a_\mu \rangle$ (omitting the zero mode, which will be computed separately), with the measure[28,29]

$$\int \frac{1}{\pi} \, \delta(\mathrm{Re}\, a_\mu) \, \delta(\mathrm{Im}\, a_\mu) \, e^{-\langle a_\mu | a_\mu \rangle}$$

This yields

$$\text{Trace I} = [\det]^2 \int \frac{\delta f}{\sqrt{\pi}} \exp\left\{ -\frac{1}{2} \langle f | \left[1 - \sum_{\mu=1}^{M} (S_\mu + S_\mu^{+}) \right] \Gamma | f \rangle \right.$$

$$\left. + \sum_i (2\alpha')^{1/2} p_i \langle z_i | f \rangle + \sum_{\mu=1}^{M} \langle a_\mu^0 | X_\mu | f \rangle \right\} \qquad (2.3.2)$$

where ,

$$S_\mu = \tilde{V}_\mu \Delta_\mu U_\mu$$

$$X_\mu = \Delta_\mu V_\mu^{+} + \tilde{V}_\mu^{+} \qquad (2.3.3)$$

$$\langle z_i |_n = (\langle 0 | V^i)_n = z_i^{-n} / \sqrt{n}$$

(ref.29)

Then, integrating over δf, we obtain

$$\text{Trace } I = \frac{e^{-\frac{1}{2}F}}{\Delta^2}$$, where

$$F = \sum_{i,j} p_i p_j \langle \gamma_i | \frac{1}{1-\Sigma} | \gamma_j \rangle + \sum_{i,\mu} p_i \langle \gamma_i | \frac{1}{1-\Sigma} X_\mu^+ | a_\mu^\circ \rangle$$
$$+ \sum_{j,\nu} p_j \langle a_j^\circ | X_\nu \frac{1}{1-\Sigma} | \gamma_j \rangle + \sum_{\mu,\nu} \langle a_\nu^\circ | X_\nu \frac{1}{1-\Sigma} X_\mu^+ | a_\mu^\circ \rangle$$

after undoing that projective mapping which took the unit circle onto the real line:

$$\left.\begin{aligned} \Sigma &= \sum_{\mu=1}^{M} (S_\mu + S_\mu^+) \\ \Delta &= \det(1-\Sigma) \end{aligned}\right\} \qquad (2.3.4)$$

There are two points to note here[28,29]

i) S_μ, when restricted to non-zero modes, as it is here, is unitary (D_1^+ representation of SU(1,1))

Hence,

$$\Sigma = \sum_{\mu=1}^{M} (S_\mu + S_\mu^{-1}) \qquad (2.3.5)$$

ii) The absence of diagonal terms has the effect of forbidding S_μ, S_μ^{-1} to occur as neighbours in the expansion of $(1-\Sigma)^{-1}$

ie. $$(1-\Sigma)^{-1} = \sum_\alpha T_\alpha \qquad (2.3.6)$$

where each T_α occurs only once, and the set of all T_α is the group of projective transformations generated by the S_μ (infinite matrices of)

This identifies the <u>automorphic group</u> for the $\gamma_\mu, \tilde{\gamma}_\mu, x_\mu$ variables.

d) The integration over the zero modes (ie. the momenta) gives[29]

$$\Delta^{-2} (\det)^{-1} \exp\left\{ -\frac{1}{2} \sum_{i \neq j} p_i p_j N_B (\gamma_i, \gamma_j) \right\} \qquad (2.3.7)$$

where $N_B(z_i, z_j) =$

$$\sum_\alpha \langle z_i | T_\alpha | z_j \rangle + \sum_{\mu,\nu=1}^{M} \langle z_i | \frac{1}{1-\Sigma} (S_\mu^{-1} - 1) | \tilde{z}_\mu \rangle (A^{-1})_{\mu\nu} \langle \tilde{z}_\nu | (S_\nu - 1) \frac{1}{1-\Sigma} | z_j \rangle$$

& $$A_{\mu\nu} = \langle \tilde{z}_\nu | (S_\nu - 1) \frac{1}{1-\Sigma} (S_\mu^{+} - 1) | \tilde{z}_\mu \rangle .$$

Then, using Burnside's[30] work on Poincaré theta series and Abelian integrals, the following relations may be established

$$\sum_\alpha \langle z_i | T_\alpha | z_j \rangle \approx Re\ \omega_{z_i z_0}(z_j)$$

ω being the third Abelian integral with complex normalization;[28,29,30,31]

$$\sum_\alpha \langle z_i | T_\alpha (S_\mu^{-1} - 1) | \tilde{z}_\mu \rangle = Re\, \varphi_\mu(z)$$

φ_μ being the first Abelian integral (independent of $\tilde{z}_\mu$),

$$A_{\mu\nu} = Re\ B_{\mu\nu}$$

$B_{\mu\nu}$ being the period matrix.[31]

($\approx$ means equality up to factors cancelled by momentum conservation).

Hence, the momentum integration yields the first Abelian integrals required to change the third Abelian integrals from complex to real normalization.[31] Thus, $N_B(z_i, z_j)$ is Neumann's function for the Riemann surface B corresponding to the automorphic group above.[30,31]

e) <u>The Riemann Surface B.</u>

If $$S_\mu(z) = \frac{\alpha_\mu z + \beta_\mu}{\gamma_\mu z + \delta_\mu}, \quad \alpha_\mu \delta_\mu - \beta_\mu \gamma_\mu = 1,$$

then S_μ takes the isometric circle[32]

$$C_\mu : \ |\gamma_\mu z + \delta_\mu| = 1$$

into the isometric circle

$$\tilde{C}_\mu : \ |\gamma_\mu z - \alpha_\mu| = 1 .$$

The plane outside these $2M$ circles ($\mu = 1, \cdots, M$) is the fundamental region[30,32] for the automorphic group. The Riemann surface B is obtained by identifying corresponding points on $C_\mu, \tilde{C}_\mu$ to give a sphere with M handles

If

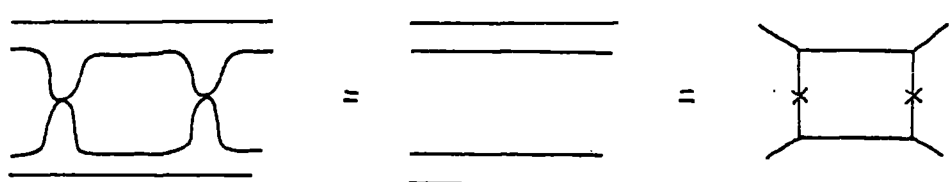
$$S_\mu = \begin{bmatrix} \infty & 0 & 1 \\ \gamma_{j-1} & \gamma_j & \gamma_{j+1} \end{bmatrix} \begin{bmatrix} 0 & \infty & 1 \\ x_\mu & 1 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{i-1} & \gamma_i & \gamma_{i+1} \\ 0 & \infty & 1 \end{bmatrix} ,$$

for $\Delta(x_\mu)$ a twisted propagator, then, following Lovelace[28] and reading right to left; unit circle anticlockwise $\longrightarrow$ real axis leftwards $\longrightarrow$ real axis rightwards $\longrightarrow$ unit circle anticlockwise, ie. the interior of the unit circle is transformed into itself. Alternatively, if $\Delta(x_\mu)$ is an untwisted propagator, then the interior of the disk is mapped into its exterior. Thus, using twisted propagators to sew Reggeons results in orientable surfaces, while using untwisted propagators yields non-orientable surfaces. In the former the automorphic group is Fuchsian[32], the latter Kleinian[32]. (Convergence of the Poincaré series is discussed by Lovelace[28])

Now, the generators of the automorphic group, S_μ, each depend on three parameters $\gamma_\mu, \tilde{\gamma}_\mu, x_\mu$, corresponding to the three parts of S_μ. Hence, subtracting three of these parameters for similarity transformations of the S_μ[26,32] (which merely gives a conformally equivalent Riemann surface), we have the automorphic group depending on $3M - 3$ parameters. Normally, topologically equivalent surfaces of genus M will be related by $3M-3$ complex parameters.[31] However, all the operators, S_μ, used above, leave the unit circle invariant, thus showing that we are dealing with a symmetric Riemann surface[31] (ie. each is the double of some open surface). Therefore, the surface depends on $3M - 3$ real parameters only. Variation of these parameters corresponds to varying the surface B while retaining the same topology. Now, Neumann's function is singular at those points of this $3M - 3$ dimensional domain corresponding to a change in topology ie. when a hole shrinks to zero or two holes touch. Domain variational theory[31] may be used to analyse these singularities (Alessandrini and Amati[26]).

In the one - loop case the singularity occuring when the radius of the inner circle (of the annulus whose double is a torus) vanishes, has been investigated[33] and a renormalization suggested which is unique (maintaining duality and Regge behaviour), and renders a finite contribution. Physically, the divergence is due to the exponential dependence of the number of intermediate states.

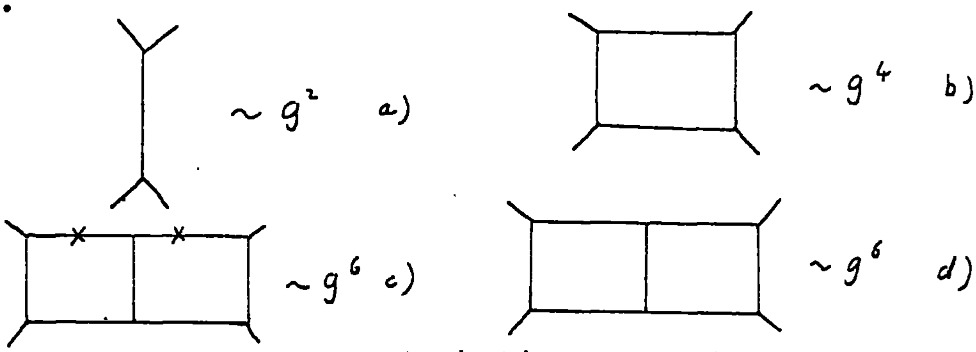
In the analysis of the one-loop case by D. Gross et al.[33], there appeared a new singularity in the orientable non-planar graphs. This singularity, a cut, corresponded to vacuum quantum numbers and was therefore associated with the Pomeron.

ie. 

However, because it was a cut its presence was incompatible with perturbative unitarity. In the case of $D = 26$ (N - scalar amplitude, section (1.3)) the Pomeron appears as a pole[22], thus, hopefully, complying with unitarity requirements.

According to the above procedure of unitarizing the dual amplitude the order of a diagram in the perturbative series is determined by the number of vertices appearing in it.

eg.



g = perturbation parameter.

Remarks: From the above account we see that the Koba-Nielsen plane appears as the complex plane on which the automorphic group of a particular graph acts to give the corresponding Riemann surface. The latter two structures (group and surface), together with the associated Neumann's function were given a more primary role in the Analogue Model. In this model, factorization is not evident but it does provide a natural framework for the notion of duality.

This approach to dual amplitudes is now introduced.

4. The Analogue Model.

4.1 Introduction

This approach originated in a paper by Nielsen[34] (also Susskind[35]), in which the integrand of the generalized Veneziano amplitude was written as the exponential of the extremal value of a certain integral.

To show this he took the Koba - Nielsen variables on the unit circle and considered the unit disk to be a homogeneous conducting medium which carried various flows of quantum numbers. For example in the case of momentum there were four fields each corresponding to one component of momentum. He then worked out the energy dissipated in the disk by the N momenta p_i^{μ} (for the N - point amplitude) entering at the point z_i on the boundary of the disk. This energy is the integral referred to in the previous paragraph; we have

$$E = \sigma \sum_{i \neq j} p_i \cdot p_j \log |z_i - z_j| ,$$

omitting the self-energies. Hence, the integrand is $\exp - E$, (σ = resistivity of the medium).

ie. $\prod_{i \neq j} |z_i - z_j|^{-\sigma p_i \cdot p_j}$

Therefore, assuming $\alpha(s) = 1 + \sigma s$, the above assertion may be proved.

Nielsen[34] also gave a physical interpretation in which hadronic material was viewed as a one dimensional continuum (or infinitely long chain of "molecules"). Thus, the hadron's evolution traced out a world sheet. He viewed interactions between hadrons as "tunnel effects", the amplitude for crossing a barrier being given by $\int \mathcal{D}(\text{paths}) \exp\left[-\int_{\text{path}} \Delta E(t)\, dt\right]$

ΔE being a measure of the violation of energy conservation for a given path; ie. the greater the violation the less probable was the penetration. However, ΔE of a string is an integral over the length of the string. Hence, the exponent is a surface integral of a quantity which can be suitably chosen because of the freedom of choice in defining the parameter of length.

In the same paper[34] he suggests why the hadronic material is one-dimensional. For, if we approximate the world-sheet by a dense network of Feynman vertices and propagators, then the only non-trivial case is when the world volume is two dimensional. The particles propagated are called partons and the Feynman formula for such a network uses an electrical analogy not dissimilar from the analogue model, viz.[36,37]

$$\text{Amplitude} = \int_0^\infty \cdots \int_0^\infty \frac{1}{C^2} \exp i\Big[\sum_{i,j} R_{ij}\, p_i \cdot p_j - \sum_k m_k \alpha_k\Big] \prod d\alpha_k$$

R_{ij} is the "resistance" of the diagram considered as an electrical network of lines with resistance α_k (Feynman papameter) between the two positions of entry of "currents" p_i^μ, p_j^μ.

4.2 Formulation of Model

Later (Fairlie and Nielsen[37]) the Analogue model was regarded as a non-particle approach to strong interactions. (Prior to the emergence of dual amplitudes it was assumed that an amplitude was dominated by nearby singularities.) The new terminology was to be based on the geometry of duality diagrams.[6] To each diagram was associated a two-dimensional open surface[37]. "Currents" of momenta were allowed to flow on this surface, the sources being the momenta of the external particles which entered at the boundary of the surface. Reggeons are represented by a distribution of momenta; in fact the coherent state parameters of a Reggeon may be regarded as the multipole moments of this distribution.[38]

The amplitude is given by a functional integral over all possible surfaces, each surface being weighted by exp(-E), E being the energy dissipated on the surface (as explained in § (4.1).)

$$A = \frac{\int (\exp -E)\, d(\text{configurations})}{\int d(\text{configurations})}$$

We may classify the surfaces according to their topology; so that, in one class we sum over all topologically equivalent surfaces. However, we require a measure for the integration over conformally inequivalent surfaces (since, by the nature of the problem, the energy is the same for conformally equivalent surfaces); this depends on the quantum-mechanical nature of the problem and is not given by the model. However, Lovelace[28,29] has proved the equivalence of this approach to that of the operator formalism, as was indicated in the previous section; the amplitude for a given topology being the sum of

$$\exp\left[-\frac{1}{2}\sum_{i\neq j} p^i \cdot p^j \; N_B(\gamma_i,\gamma_j)\right]$$

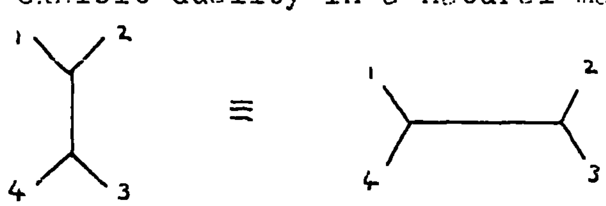
ie. $\exp(-E)$

with the measure (equation (2.3.7)) $\left[\Delta^2 \det A\right]^{-1}$

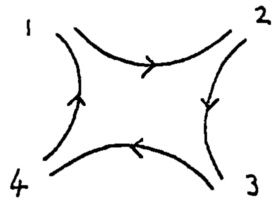
The sum over topologically inequivalent surfaces, to ensure equivalence with the operator approach, is done using the factor g^β, g being the perturbative parameter, depending on the number of external particles and the genus of the surface involved.

Note on Duality Diagrams.[6,39]

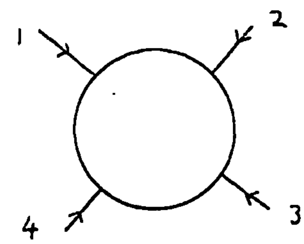
These diagrams are well-known for their demonstration of duality; so it is not surprising that the corresponding surfaces exhibit duality in a natural manner:

eg. a) 

has the duality diagram (each line traces the motion of a quark)

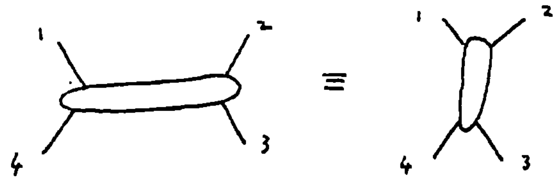


and the corresponding analogue surface



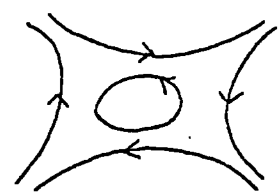
However, all simply-connected surfaces are conformally equivalent (Riemann Mapping Theorem[31]);

hence

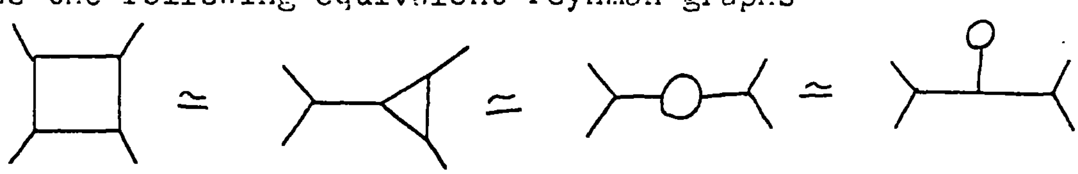


ie. each of these diagrams gives the same result (NB. summing over all surfaces in this case means summing over all positions of the external momenta)

b) The one duality diagram

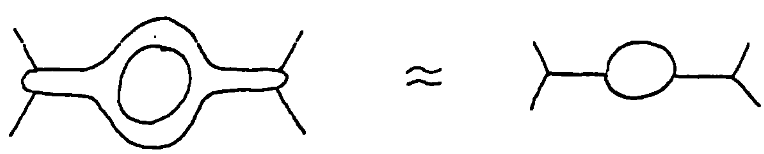


has the following equivalent Feynman graphs[39]



any two of which are related by duality transformations; the latter may be realized on the above duality diagram by deformations of the analogue surface (annulus) while maintaining the same topology;

e.g.

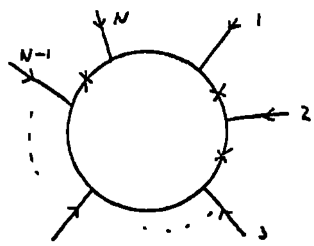


However, annuli of different radii are conformally equivalent only if the ratio of the radii is the same in each case. Therefore, the measure on non-conformal annuli is a function of this ratio.[37]

As an illustration we give the calculations for the orientable and non-orientable surfaces of genus one, and also the measure for each, which may be derived from the dense Feynman network corresponding to each surface.

4.3 Surfaces of genus = 1

In the operator formalism a typical one-loop diagram is



These diagrams may be divided into two groups;

viz. those with

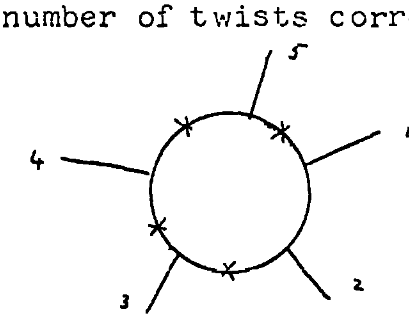
a) even number

b) odd number, of twists.

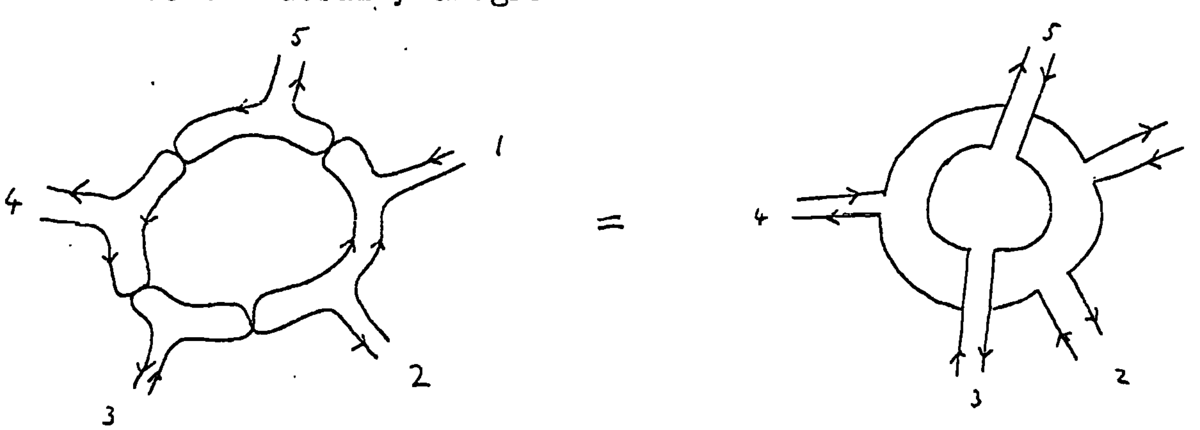
Consider the first class;

a) Orientable surfaces

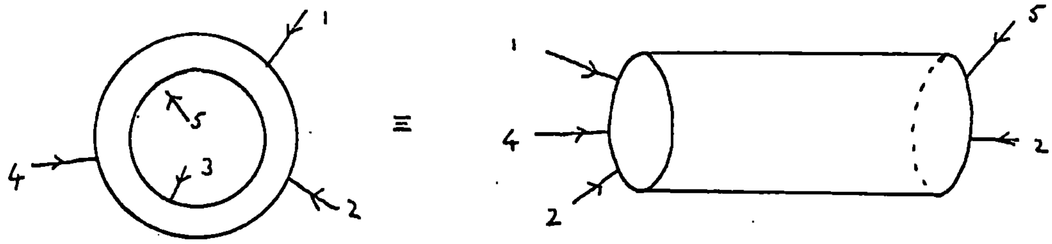
A diagram with an even number of twists corresponds to an orientable surface eg.



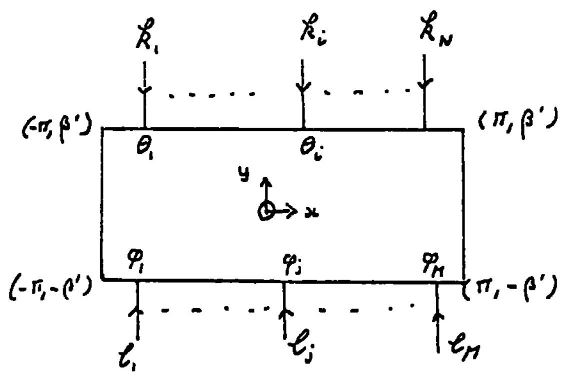
has the duality diagram



and the corresponding analogue surface is an annulus (or cylinder),



We represent the general case by a cylinder with N momenta entering at one circumference and M at the other. The cylinder is characterized, in the (x,y) plane, by the rectangle



with the proviso that functions in the (x,y) plane are functions on the cylinder only if they are periodic in x with period 2

The most general harmonic function on the cylinder is therefore

$$\varphi^\mu(x,y) = \sum_{n=1}^{\infty} \Big\{ \sin nx \left[a_n^\mu \sinh ny + b_n^\mu \cosh ny \right] + \cos nx \left[c_n^\mu \sinh ny + d_n^\mu \cosh ny \right] \Big\}$$

This must satisfy the boundary conditions

$$\left.\frac{\partial\varphi^\mu}{\partial y}\right|_{y=\beta'} = -\sum_{1}^{N} k_i^\mu \, \delta(x-\theta_i)$$

$$\left.\frac{\partial\varphi^\mu}{\partial y}\right|_{y=-\beta'} = \sum_{1}^{M} l_i^\mu \, \delta(x-\varphi_i) ,$$

thus determining the coefficients a,b etc.

Therefore (Appendix A1.) we find that the energy dissipated (σ = resistivity)

$$\mathcal{E}_1 = \frac{\sigma}{\pi^2}\left\{ \sum_{i<j} 2k_i\cdot k_j \log\left[\frac{\vartheta_1(\nu_{ij}|\tau)}{q_0\,\rho^{1/4}}\right] + \sum_{i<j} 2\ell_i\cdot\ell_j \log\left[\frac{\vartheta_1(\nu'_{ij}|\tau)}{q_0\,\rho^{1/4}}\right] + \sum_{i,j} 2k_i\cdot\ell_j \log\left[\frac{\vartheta_0(u_{ij}|\tau)}{q_0}\right]\right\}$$

with

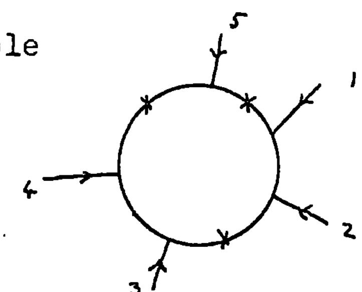
$$\nu_{ij} = \frac{\theta_j - \theta_i}{2\pi}, \quad \nu'_{ij} = \frac{\varphi_j - \varphi_i}{2\pi}, \quad u_{ij} = \frac{\varphi_j - \theta_i}{2\pi}$$

$$\tau = 2i\beta'/\pi = i\beta/\pi, \quad \rho = e^{\pi i\tau}.$$

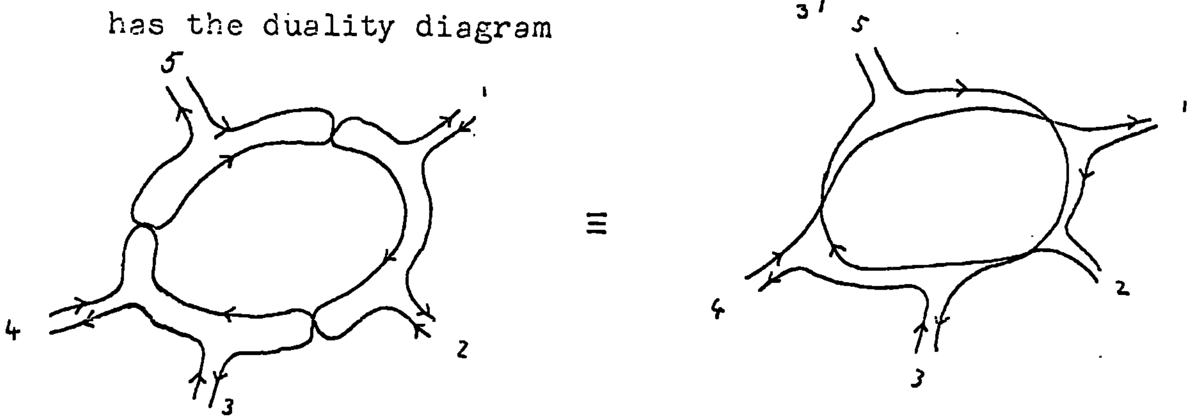
b) <u>Non-orientable surfaces</u>.

A diagram with an odd number of twists has an analogue surface which is non-orientable

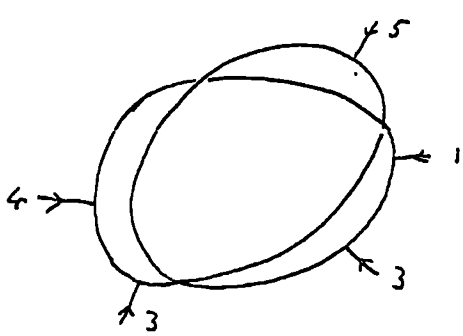
eg.



has the duality diagram

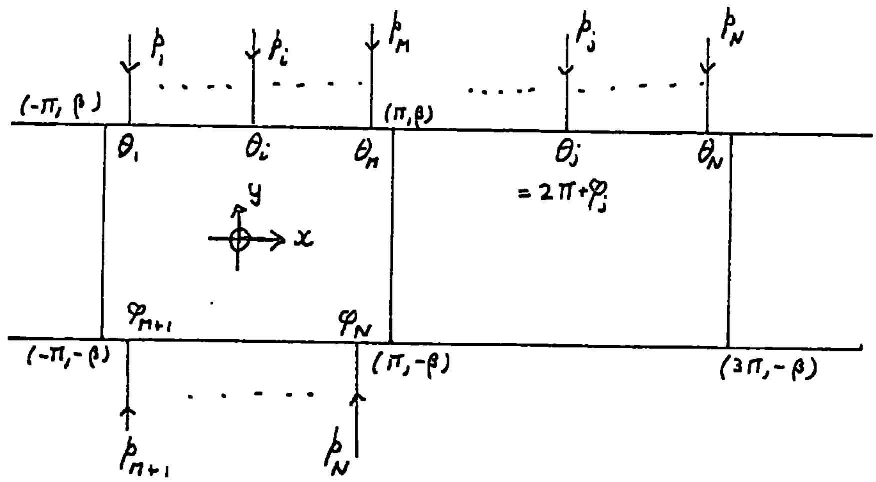


and the corresponding analogue surface is



ie. a Mobius band.

We may represent a Mobius band in the (x,y) plane by a rectangle with a suitable identification of the boundaries. Consider the figure



The point (x, β) is identical to $(x+2\pi, -\beta)$,

and (π, y) is identical to $(-\pi, -y)$.

The double of the Mobius band is the rectangle of length 4π and width 2β. Now, any function on the band must be of period 4π in x, and satisfy the above identifications; so, any harmonic function on the band has the form

$$\varphi^\mu(x,y) = \sum_{n=1}^{\infty}\Big\{(a_n^\mu \sin nx + b_n^\mu \cos nx)\cosh ny + \left(c_n^\mu \sin \overline{n-\tfrac{1}{2}}\,x + d_n^\mu \cos \overline{n-\tfrac{1}{2}}\,x\right)\sinh \overline{n-\tfrac{1}{2}}\,y\Big\}$$

NB. $$-\frac{\partial\varphi^\mu}{\partial y}(x+2\pi,-\beta) = \frac{\partial\varphi^\mu}{\partial y}(x,\beta)$$

The boundary conditions are

$$\left.\frac{\partial\varphi^\mu}{\partial y}\right|_{y=\beta} = -\sum_{i=1}^{N} p_i^\mu\,\delta(x-\theta_i)$$

Then, from the Appendix we find that

$$\mathcal{E}_2 = \frac{2\sigma}{\pi} \sum_{i \neq j} p_i \cdot p_j \log \left[\frac{\vartheta_1\left(\frac{\nu_{ij}}{2} \middle| \frac{\tau}{2}\right) \vartheta_3\left(\frac{\nu_{ij}}{2} \middle| \frac{\tau}{2}\right)}{\rho^{1/4} \quad q_0} \right]$$

is the energy dissipated in a Mobius band of resistivity σ .

c) <u>Relation with ψ functions of Gross et al.</u>[33]

i) <u>Orientable diagrams</u>

In the notation of Gross et al.[33] we may show that

$$\psi(C_{ji}) = \frac{\log \omega}{2\pi} \quad \frac{\vartheta_1(\nu_{ij}|\tau)}{q_0^3 \quad q^{1/4}}$$

$$\psi_T(C_{ji}) = \frac{\log \omega}{2\pi} \quad \frac{\vartheta_0(\nu_{ij}|\tau)}{q_0^3 \quad q^{1/4}}$$

where $\quad C_{ji} = x_{i+1}\, x_{i+2} \cdots x_j$

$C_{ii} = \omega = \prod x_i \;,$ the product being over all

Chan variables x_i .

Further

$$\nu_{ij} = \frac{\theta_j - \theta_i}{2\pi} = \frac{\log C_{ji}}{\log \omega}$$

and

$$\tau = \frac{2i\beta}{\pi} = \frac{-2\pi i}{\log \omega}$$

In their formula for the one loop case (even number of twists) $\psi(C_{ji})$ is chosen if there are an even number of twists between p_i and p_j , ie. p_i, p_j enter on the same circumference of the cylinder.

$\psi_T(C_{ji})$ is used for an odd number of twists between p_i and p_j ,

ie. p_i, p_j are on different circumferences.

Therefore, there is agreement between the two approaches.

ii) Non-orientable diagrams.

Again we may show that[33]

$$\psi_N^{\frac{1}{2}}(c_{ji}) = \frac{\log \omega}{\pi} \frac{\tilde{\vartheta}_1\left[\frac{\nu_{ij}}{2} \mid \frac{\tau}{2}\right] \tilde{\vartheta}_3\left[\frac{\nu_{ij}}{2} \mid \frac{\tau}{2}\right]}{q_0^4 \, q_2^2 \, q'^{1/4}}$$

$$\psi_{NT}^{\frac{1}{2}}(c_{ji}) = \frac{\log \omega}{\pi} \frac{\tilde{\vartheta}_1\left[\frac{\nu_{ij}+1}{2} \mid \frac{\tau}{2}\right] \tilde{\vartheta}_3\left[\frac{\nu_{ij}+1}{2} \mid \frac{\tau}{2}\right]}{q_0^4 \, q_2^2 \, q'^{1/4}}$$

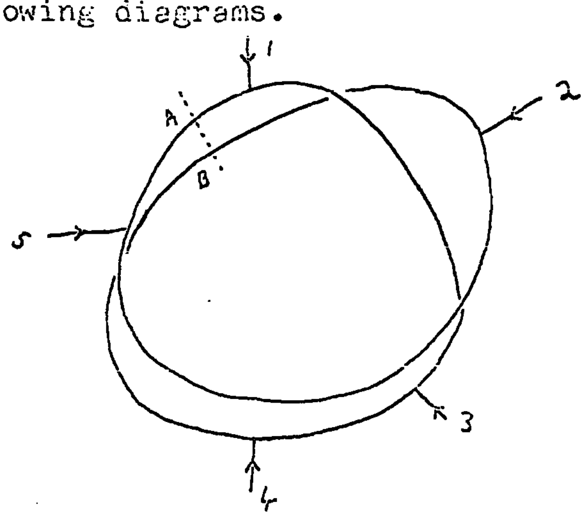
where $\quad q_2 = \prod_1^\infty (1 + q^{2n-1})$

(c_{ji} , ν_{ij} , τ as before).

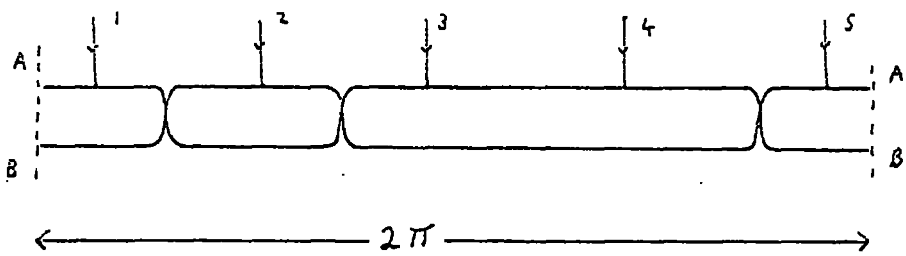
As in the case of orientable diagrams, use $\psi_N(c_{ji})$ if there are an even number of twists between p_i and p_j , $\psi_{NT}(c_{ji})$ if there are an odd number of twists.

(N = non-orientable , T = twisted)

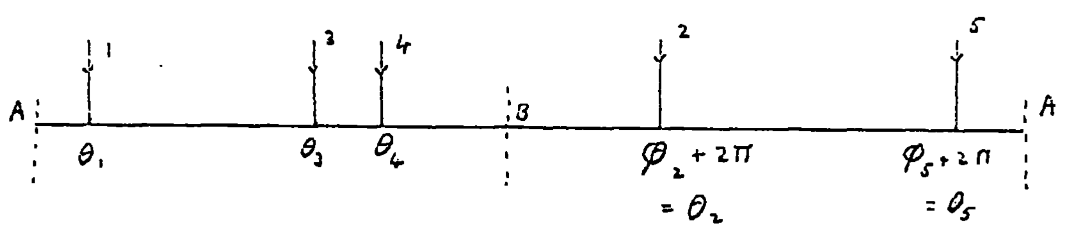
To understand this in terms of the Mobius band consider the following diagrams.



Cut the above diagram across AB and open out to give:-



If we call the length of this band 2π, then the Mobius strip has an edge of length 4π; ie.

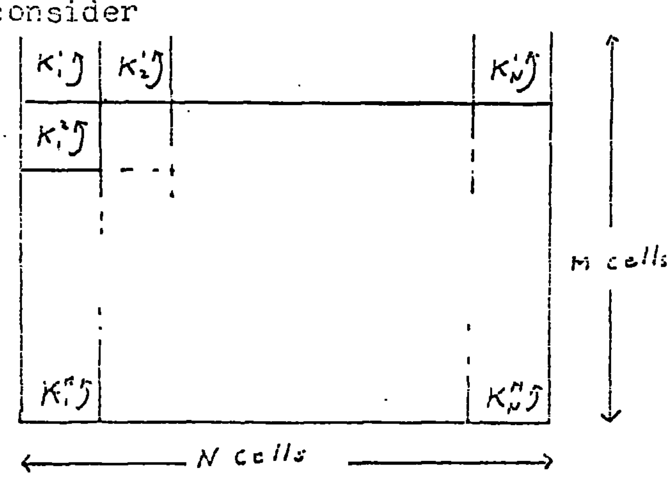


Hence, if p_i, p_j, are separated by an odd number of twists we must augment the difference in their positions by 2π to be in accord with the analogue formula; for

$$\nu_{ij}' = \frac{\varphi_j - \theta_i + 2\pi}{2\pi} = \nu_{ij} + 1$$

d) <u>The Measure</u>

We show how to obtain the measure for the cylinder by approximating the conducting surface by a dense Feynman net. For, consider



in which the top end joins the bottom to form a cylinder. The circulating momenta are labelled K_j^i as shown. If the resistance of one edge of a cell is σ , and

$$X := (K_1^1, \cdots, K_N^1, K_1^2 \cdots, K_N^2, \cdots\; K_N^M)$$

then the energy dissipated for the above distribution, is

$$\mathcal{E} = \sigma X \mathcal{A} X',$$

with

$$\mathcal{A} = \begin{bmatrix} A & -I & & & 0 & -I \\ -I & A & -I & & & 0 \\ 0 & -I & A & \ddots & 0 & \\ & & \ddots & \ddots & \ddots & 0 \\ & 0 & & \ddots & \ddots & -I \\ -I & & & 0 & -I & A \end{bmatrix} \qquad (MN \times MN)$$

with

$$A = \begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & 0 \\ & -1 & 4 & & \\ & & & \ddots & -1 \\ & 0 & & -1 & 4 \end{bmatrix}$$

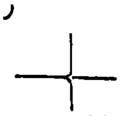
Hence, from the appendix, we obtain

$$(\det \mathcal{A}) = \prod_{p=1}^{\infty} q^{-2p}(1-q^{2p})^2$$

for $N \to \infty$, $\frac{M}{N} = k$ = a constant, $q = e^{-k\pi/2}$

Now, when a Gaussian functional integral is taken over all distributions on the surface, the final answer[41] is the product of the classical solution and

$$Q = g^V \iint \exp(-\sigma X \mathcal{A} X')\, dX,$$

where g is the strength of the coupling ,

V is the number of vertices (= NM + M = kN(N + 1)),

and the integration is over all X,

ie
$$\prod_{i,j} \int_{-\infty}^{\infty} dK_j^i = \iint dX.$$

In the Appendix it is shown that

$$Q \underset{N\to\infty}{\longrightarrow} \left[\sigma^{kN^2} |A| \right]^{-2}$$

Hence, the measure required is

$$\left[f(\omega)\right]^{-4} \lim_{N\to\infty} \left[\sigma^{-2kN^2} q^{2N(N+1)} g^{kN(N+1)} \right]$$

where

$$f(\omega) = \prod_1^{\infty} (1-\omega^n) \quad , \quad \omega = q^2$$

The limit converges if

$$\sigma = 1 \; , \quad g^k = q^{-2}$$
$$\text{i.e. } g = e^{\pi} \; (\sim 23).$$

If we also join the sides of the above network then we have a torus. We may calculate the weight attached to this torus by replacing C_N by

$$A_N = \begin{bmatrix} 4 & -1 & & & -1 \\ -1 & 4 & -1 & & 0 \\ & -1 & 4 & \ddots & \\ & 0 & & \ddots & -1 \\ -1 & & & -1 & 4 \end{bmatrix}$$

in the Appendix calculation. We then obtain

$$|A|^{-\frac{1}{2}} = \prod_{\ell} \rho^{\ell} (1-\rho^{2\ell})^{-1}$$

$$\rho = e^{-k\pi} = e^{-\beta} \; , \quad k = \frac{M}{N} = \beta/\pi \; ;$$

this agrees with the cylinder calculation when one remembers that the double[31] of the cylinder has length $2\beta = 4\beta'$.

We may obtain the measure for the Mobius band, now that the torus measure is known. For, the τ' of the Mobius strip is related to the τ of the cylinder by

$$\tau' = \tau + 1/2$$

(ie. the ends of the cylinder are rotated relatively to one another by π.)

Hence, q is replaced by iq

ie. w is replaced by - w

ie. f(w) is replaced by f(- w).

5. The Functional Integral Formulation

5.1 Introduction

The calculations in section 4 have been performed in momentum space. The reformulation of the problem in configuration space allows an identification to be made with the functional integral approach of Hsue, Sakita and Virasoro[41]. Gervais and Sakita[42] extended these considerations to multiloop amplitudes (but not including non-orientable diagrams), providing the framework to bridge the gap between the operator language and that of the analogue model. In this approach the degeneracy of conformally equivalent diagrams is directly linked with the Virasoro algebra.[19]

Their amplitudes are defined by "functional integrals which correspond to transition amplitudes of quantum - mechanical systems of strings with imaginary time."[42] The latter aspect arises because the Lagrangian for a free string

$$\mathcal{L}(\varphi) = -\tfrac{1}{2}\left\{\left(\frac{\partial\varphi}{\partial x}\right)^2 - \left(\frac{\partial\varphi}{\partial t}\right)^2\right\} \tag{2.5.1}$$

becomes the Lagrangian used in the functional integral approach on identifying $y^2 = -t^2$

ie. t is pure imaginary, since y is real.

5.2 Analogue Model.

To see the connection with the analogue model we consider the N - Reggeon vertex. Now, each amplitude is built up from "rudimental" amplitudes. The rudimental amplitude in this case is the functional average of

$$\exp\left[i\sqrt{2\pi} \sum_{\ell=1}^{N} \int_0^{\pi} p_\ell(\xi) .\ \Phi[g_\ell(\xi)]\, d\xi \right] \tag{2.5.2}$$

over $\Phi_\mu(x,y)$ on the unit disk (or, in fact, any simply - connected region D), where $\xi \in$ boundary of the disk. As in Kosterlitz and Saito's form of the N - Reggeon vertex.[24]

$g_\ell : \xi \longrightarrow$ part of curve in (x,y) plane where the momentum $p_\ell(\xi)$ is emitted.

$p_\ell(\xi)$ are N momentum distributions corresponding to the N Reggeons.

The functional average of a functional of Φ_μ , $A(\Phi_\mu)$ say, is given by

$$\langle A(\Phi)\rangle_D := \frac{1}{n_0} \int \mathcal{D}^4(\Phi(x,y)) A \exp \iint_D \mathcal{L}(\Phi)\, dx\, dy \tag{2.5.3}$$

where,

$$\mathcal{L}(\Phi) = -\frac{1}{2}\left\{ \left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 \right\}$$

$$n_0 = \left[(2\pi)^4 \delta^4(0)\right]^{-1} \int \mathcal{D}^4(\Phi) \exp\left[\iint \mathcal{L}(\Phi)\, dx\, dy \right].$$

The method of evaluation of these functional averages exhibits the link with the analogue model. For, let $N(\gamma,\gamma')$ be Neumann's function for the domain D[31] (not necessarily simply - connected) of definition of the functions we are considering, and make the change of integration variables $\Phi \longrightarrow \Phi'$, where

$$\Phi = \Phi' - \frac{i}{\sqrt{2\pi}} \sum_{k=1}^{N} \int_0^{\pi} d\xi\, p_k(\xi)\, N[\gamma, g_k(\xi)]$$

(g_k allows the p_k to be defined on the standard interval, $0 \leq \xi \leq \pi$, ie half of the unit circle)

Now, translational invariance implies

$$\mathcal{D}^4(\Phi) = \mathcal{D}^4(\Phi')$$

(Φ , Φ' differ by a constant function)

The exponent in the integrand of the functional average is

$$\iint \mathcal{L}(\Phi)\,dx\,dy + i\sqrt{2\pi}\sum_{\ell}\int_0^{\pi} p_\ell(\xi)\,\Phi[q_\ell(\xi)]\,d\xi$$

$$= \iint \mathcal{L}(\Phi')\,dx\,dy + \frac{1}{2}\sum_{i,j}\iint p_i(\xi)\,p_j(\xi')\,N[q_i(\xi), q_j(\xi')]\,d\xi\,d\xi'.$$

Hence,

$$\left\langle \exp\left\{ i\sqrt{2\pi}\sum_{\ell}\int_0^{\pi} p_\ell(\xi)\,\Phi[q_\ell(\xi)]\,d\xi \right\}\right\rangle$$

$$= (2\pi)^4\,\delta^4\left(\sum_{\ell=1}^{N}\int_0^{\pi} p_\ell(\xi)\,d\xi\right)\exp\left[\frac{1}{2}\sum_{i,j}\iint p_i(\xi)\,p_j(\xi')\,N[q_i(\xi), q_j(\xi')]\,d\xi\,d\xi'\right]$$

The first factor ensures the conservation of momentum, while the exponent of the second is the energy dissipated in the relevant domain of resistivity $1/2$, with "currents" of momenta $p_i(\xi)$ entering on the boundary.

The self-energies are subtracted in the definition of the amplitude. For example the N - scalar amplitude is defined by[42] (intercept, $\alpha(0) = 1$)

$$(2\pi)^4\,\delta^{(4)}\left[\sum_{i=1}^{N} k_i\right] V_N(k_1, \cdots, k_N) := \frac{1}{C}\int\cdots\int \lim_{\varepsilon\to 0}\prod_{i=1}^{N} d\theta_i \frac{\left\langle \exp i\sqrt{2\pi}\sum_{1}^{N} k_i\cdot\Phi(z_i)\right\rangle}{E(z_i, k_i^2)}, \qquad (2.5.4)$$

where, $E(z_i, k_i^2) = \exp[\text{self-energy of } i^{th} \text{ source}]$

$$= \exp\left[k_i^2 \iint dx\,dy\,dx'\,dy'\,\rho_i(x,y)\,\rho_i(x',y')\,\log|z-z'|\right]$$

$$\rho_i(x,y) = \delta(x - \overline{1-\varepsilon}\cos\theta_i)\,\delta(y - \overline{1-\varepsilon}\sin\theta_i)$$

$$z_i = e^{i\theta_i}, \quad z = x+iy,$$

$$C = \int_0^{2\pi} d\theta_1 \int_{\theta_1}^{\theta_1+2\pi} d\theta_2 \int_{\theta_2}^{\theta_2+2\pi} d\theta_3\, |(z_1-z_2)(z_2-z_3)(z_3-z_1)|^{-1}.$$

This approach, which is equivalent to the operator formalism,[41,42] also furnishes the measure associated with a diagram. For, if D is the domain of definition of Φ_μ, then the corresponding measure is given by

$$\exp \iint_D \mathcal{L}(\Phi)\,dx\,dy$$

For the one-loop case, for example,

$$\int \mathcal{D}^4[\Phi] \exp \iint \mathcal{L}(\Phi)\,dx\,dy$$

over all Φ_μ defined on the annulus (or torus) gives the partition function as calculated previously.

5.3 Operator Formalism

When factorization in the functional integral approach is considered, it is found that the relationship of $\Phi_\mu(\xi)$ to $p_\mu(\xi)$ is that of a generalized position to a generalised momentum. Indeed, Φ_μ may be quantized as follows

$$\Phi_\mu(\xi) = \frac{1}{\sqrt{2\pi}}\left\{x_\mu + \sum_{n=1}^{\infty}\left(\frac{2}{n}\right)^{1/2}\cos n\xi\,(a_\mu^n + a_\mu^{n\dagger})\right\} \tag{2.5.5}$$

the canonical momentum being

$$\Pi_\mu(\xi) = \frac{1}{\sqrt{2\pi}}\left\{2p_\mu + \sum_{n=1}^{\infty}\frac{1}{i}(2n)^{1/2}\cos n\xi\,(a_\mu^n - a_\mu^{n\dagger})\right\} \tag{2.5.6}$$

where $[x_\mu, p_\nu] = i g_{\mu\nu}$,

$0 \leq \xi \leq \pi$,

and a_μ^n, $a_\mu^{n\dagger}$ are the creation and annihilation operators introduced in section 1.

We may associate this field with a free string of length π, if we use imaginary time so that the Lagrangian is

$$\mathcal{L}(\Phi) = -\frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right] \tag{2.5.7}$$

(It is assumed, when such an identification is made, that the Wick rotation, to the Lorentz metric, can be made after all calculations have been completed).

Furthermore, the Hamiltonian for this system may be derived from the given Lagrangian, and is found to be the Nambu Hamiltonian,

$$H = p^2 + \sum_1^\infty n\, a^{n\dagger} . a^n$$

as in the operator formalism.

Now, the relation between Φ_μ, above, and Q_μ, of the operator model, is

$$\Phi_\mu (z, \bar{z}) = \frac{1}{2} \left[Q_\mu (z) + Q_\mu (\bar{z}) \right]$$

where $|z| = 1$, ref.24.

The explanation for the above is to be found in the definition of the standard interval for each case. For, Gervais and Sakita[42] use the half-circle, $0 \leq \xi \leq \pi$, $(z = e^{i\xi})$, whereas in Ramond's work[13] the full circle is used.

We may carry this two-dimensional field model further by deriving those operators which generate the transformations leaving the Lagrangian invariant (Noether's theorem). In this way we find,

a) SL(2 $\mathbb{R}$) ie. Mobius invariance (Duality)

b) Conformal invariance; the generators of the conformal transformations are the Ln's:

$$L_n : z \rightarrow z'$$

where

$$(z')^n = z^n + n\varepsilon$$

(L_0, $L_{\pm 1}$ generate SL(2 $\mathbb{R}$))

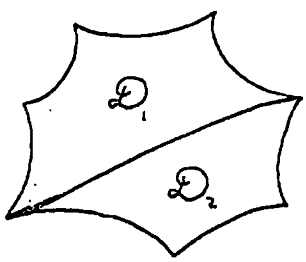
The operators are obtained by exponentiating the L_n's to give $\mathcal{O}(g)$, say, where g is a conformal mapping. These act on Φ_μ as follows,

$$\mathcal{O}(g)\,\Phi_\mu(z,\bar{z})\,\mathcal{O}^{-1}(g) = \Phi_\mu(z',\bar{z}')$$

with

$$z' = g(z).$$

Hence, duality and the gauge algebra are seen to be properties of the Lagrangian, Indeed, factorization itself (going back to the beginning of this section) follows from the local nature of the Lagrangian. For, from the diagram



$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$$

we have

$$\int_{\mathcal{D}} \mathcal{L}\,d\omega = \int_{\mathcal{D}_1} \mathcal{L}\,d\omega + \int_{\mathcal{D}_2} \mathcal{L}\,d\omega$$

ie. $$S = S_1 + S_2$$

Therefore, $e^{S} = e^{S_1} \cdot e^{S_2}$

This, together with the fact that the measure factorizes similarly, accounts for the bootstrap property.

The three requirements, duality, gauge algebra, factorization (all guaranteed by the properties of the Lagrangian) restrict the Lagrangians possible. Indeed, Gervais and Sakita[43] introduce new Lagrangians which yield other dual amplitudes already arrived at by other means. However, functional intergration does unify the descriptions of dual amplitudes given in this chapter.

Remarks

Now that we have presented three approaches to dual amplitudes, we may discuss the merits and demerits of each in an attempt to ascertain those characteristics which might remain in future modifications.

a) The Operator Formalism.

At the beginning of this chapter was presented a mathematical model for constructing amplitudes which, starting from the assumption of infinitely rising linear trajectories and narrow widths, incorporated in an apparently consistent manner the following attributes. a) analyticity b) crossing - symmetry c) duality d) factorization (explicitly exhibited in this approach — Feymman - like diagrams) e) Regge asymptotic behaviour. Unitarity was to be enforced perturbatively and there were no ghosts in the model.

However, quite apart from the fact that it did not produce a realistic spectrum, there were some drawbacks. The first of these was that the ground state was a tachyon. Furthermore, the operator formalism makes use of the Koba - Nielsen variables which span two-dimensional surfaces. The emergence of these surfaces in terms of space-time is not accounted for in this framework. Indeed, the surfaces play a role secondary to the operators in this technique, although they (ie. the surfaces) give the most convenient terminology for discussing the singularities of the amplitude[26].

b) The Analogue Model.

In this approach the two-dimensional surfaces (akin to duality diagrams and so incorporating something of the notion of quarks - c.f Chapter I) are naturally accomodated within it. However, the language of this model makes no mention of space-time, nor of its relation to the, perhaps basic, surfaces (apart from the momentum flows which are constructed on the surfaces). Indeed, there is no attempt to offer a physical interpretation, as for example in the functional integration formalism (ie. no physical, space-time, meaning for the complex plane is given). The geometry and related constructs depend critically on the Euclidean metric and Laplace's equation, compared with the functional approach where a hyperbolic equation is more appropriate. Thus, it is quite abstract, and, as such, may be a fruitful area in which to explore possible modifications and, later, interpretations, once the relation with external space-time has been elucidated.

It is interesting to note that if we regard the integrations over the γ_i's as part of the sum over all possible configurations, then the classical contribution is the extremal value of

$$\prod_{i \neq j} |\gamma_i - \gamma_j|^{-p_i \cdot p_j}$$

with respect to variations of the γ_i. This expression, without the differentials, is Mobius invariant if $p_i^2 = m^2 = 0$. This is to be compared with the ideas of Chapter IV. (In fact the classical contribution is the same in each case).

However, the bootstrap property (factorization) is not clear in the Analogue prescription and the gauge algebra (resulting from the conformal invariance properties of the surface functions - c.f. the functional integration approach) has not been realized. Furthermore, there is no means, within the model, of determining the measure on the space of conformally inequivalent surfaces. The measure is a characteristic of the quantum mechanical nature of the problem and (section on sewing Reggeons) unitarity. As there is no space-time interpretation for this model the question of the measure must be left unanswered.

c) <u>The Functional Integration Formalism</u>

At first sight this approach seems to be the most preferable of the three, since it may be linked with either of the first two in a direct manner. It supercedes the operator formalism since it incorporates the Riemann surfaces of the Analogue Model (but only applies to orientable surfaces - ie. spurious free). However, although factorization is not manifest it may be proved. Furthermore, it provides the measure and yields the gauge algebra, both missing from the Analogue Model. The gauge algebra arises from the exploitation of the conformal degeneracy of the surfaces. These latter points are possible because of the physical interpretation given to this approach - ie. the dynamics of a string. However, the motion of the string evolves in imaginary time (the Wick rotation has to be assumed possible). A proper treatment (ie. real time) would, of course, destroy the nature of the Analogue geometry; the only reason for doing so seems to be to force a physical interpretation on the dual models. Since the latter have not been fully developed it would appear prudent to consider the structure in these models, when expressed in an abstract, mathematical manner, before attempting to impose a physical interpretation. For, the Koba - Nielsen variable[7] (and so the surfaces) emerged as convenient mathematical constructs with no definite preconceived physical meaning (indeed duality diagrams,[6] first introduced as helpful mnemonics, do not have an explicit space - time significance).

CHAPTER III

1. Introduction

There were a number of approaches to dual amplitudes which were important to the development of the subject as a whole. In addition to those already mentioned was the group-theoretic method; SU(1,1), $SL(2\,\mathbb{R})$[14,18] invariants, or $SL(2\,\mathbb{C})$ which led to the Virasoro amplitude - Bars and Gursey.[18] However, perhaps one of the most appealing, in its introduction of structure in dual models, was Ramond's attempt[13] at a field theoretic interpretation of the latter. It certainly proved useful in guiding researchers to an extension of the original model.

In Ramond's work, the structure - SU(1,1), gauge algebra, spectrum - was exposed more readily than in other models by considering free wave equations of hadrons[13,44] instead of the field in interaction ie. the actual amplitude.

As in the previous chapter, section one, it is assumed that hadrons have a structure with periodic internal excitations. The position and momentum operators are generalized to the Q_μ, P_μ of equation (2.1.6), with the Hamiltonian H (generating the spectrum) given in equation (2.1.5). Now, Ramond[13] assumes that all physical quantities are averages, over the internal period, of their generalized counterparts; ie. the average of an operator $A(\tau)$ is

$$\langle A(\tau)\rangle := \frac{1}{2\pi}\int_0^{2\pi} d\tau\, A(\tau) \qquad (3.1.1)$$

e.g.

$$\langle Q_\mu(\tau)\rangle = q_\mu$$
$$\langle P_\mu(\tau)\rangle = p_\mu ,$$

the physical position and momentum (in appropriate units)
N.b. When applied to products of operators normal ordering must be enforced.

2. The Free Boson Equation[13]

We now restrict ourselves to bosons and consider the Klein - Gordon equation for a particle of mass m,

ie. $$m^2 = p_\mu p^\mu = \langle P_\mu(\tau)\rangle \langle P_\mu(\tau)\rangle \tag{3.2.1}$$

At this point Ramond introduced what he called a "Correspondence Principle", by means of which products of averages were replaced by the average of normal ordered products,

ie. (3.2.1) becomes

$$m^2 = :\langle P_\mu(\tau)\, P^\mu(\tau)\rangle: \tag{3.2.2}$$

ie. $$H - m^2 = 0$$

However, if we go further and require equation (3.2.2) to hold for all modes and not just the average (ie. zero mode), then we have

$$:P_\mu(\tau)\, P^\mu(\tau): - m^2 = 0 \tag{3.2.3}$$

which becomes, on expanding,

$$\sum_{n=-\infty}^{\infty} L_n e^{-in\tau} - m^2 = 0 \tag{3.2.4}$$

where

$$\begin{aligned} L_n &= \langle :P_\mu(\tau)\, P^\mu(\tau): e^{in\tau}\rangle \\ L_{-n} &= L_n^{\dagger} \\ L_0 &= H, \end{aligned} \tag{3.2.5}$$

are the Virasoro gauge operators.[19]

(c.f. Takabayashi,[12] who obtains the gauge operators by requiring his wave equation to hold at all points of the string - "Detailed Wave Equation and Dual Amplitudes".)

Then, for physical states $|\varphi\rangle$, $|\psi\rangle$ we require

$$\sum_{n=-\infty}^{\infty} \langle\varphi| L_n |\psi\rangle e^{-in\tau} - m^2 = 0$$

ie. it is sufficient to have

$$L_n|\varphi\rangle = 0 \qquad n = 1, 2, \cdots$$

and

$$L_0|\varphi\rangle = m^2|\varphi\rangle$$

It is possible to extend the above and introduce generalized Lorentz generators by the Correspondence Principle;

$$\varepsilon_{\mu\nu\rho\sigma}\, q_\rho p_\sigma = \varepsilon_{\mu\nu\rho\sigma} \langle Q_\rho(\tau)\rangle \langle P_\sigma(\tau)\rangle$$

$$\rightarrow \; : \varepsilon_{\mu\nu\rho\sigma} \langle Q_\rho(\tau) P_\sigma(\tau)\rangle :$$

$$= \varepsilon_{\mu\nu\rho\sigma} \left\{ q_\rho p_\sigma - i \sum_{m=1}^{\infty} a_\rho^{m+} a_\sigma^{m} \right\} =: M_{\mu\nu}{}^{B}$$

These operators satisfy the Lorentz algebra.

By introducing an interaction vertex $V(\tau)$, for the emission of a meson, we may obtain an N - particle amplitude. However, for consistency, $V(\tau)$ must be compatible with the gauge conditions. If we use

$$V(\tau) = \; : \exp[i\, P(\tau) . Q(\tau)] :$$

as in the operator formalism, then, for compatibility with the above requirements, we must impose the restriction,

$$m^2 = -1,$$

giving the generalized Veneziano amplitude.

Having considered a free boson equation in this way, which yielded the gauge algebra with ease, Ramond then suggested a generalization of the Dirac equation,[44] leading to an enlargement of this algebra.

3. The Free Fermion Equation[44]

(i) The form of this equation is, by the Correspondence Principle,

$$: \langle \Gamma_\mu(\tau)\, p^\mu(\tau) \rangle : \; - \; m = 0 \tag{3.3.1}$$

where $p_\mu(\tau)$ is as before and $\Gamma_\mu(\tau)$ is a generalization of the Dirac matrices γ_μ.

To determine $\Gamma_\mu(\tau)$ Ramond required

a) that they reduce to the Dirac matrices when there are no internal excitations

ie. $$\langle \Gamma_\mu(\tau) \rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} d\tau\, \Gamma_\mu(\tau) = \gamma_\mu$$

b) that they satisfy the relations

$$\{\Gamma_\mu(\tau), \Gamma_\nu(\tau')\} = 2 g_{\mu\nu}\, \delta(\tau-\tau') \quad (\mathrm{mod}\ 2\pi) \tag{3.3.2}$$

which is the simplest requirement consistent with the anticommutation relations of the γ_μ.

c) that they also obey

$$\Gamma_\mu^\dagger(\tau) = \gamma_0\, \Gamma_\mu(\tau)\, \gamma_0 \tag{3.3.3}$$

- for simplicity

These three conditions dictate a unique Fourier expansion of $\Gamma_\mu(\tau)$ viz.

$$\Gamma_\mu(\tau) = \gamma_\mu + i\sqrt{2}\,\gamma_5 \sum_{-\infty}^{\infty}{}' d_\mu^{\,n}\, e^{-in\tau} \tag{3.3.4}$$

where, $$\{d_\mu^{\,n}, d_\nu^{\,m}\} = -g_{\mu\nu}\,\delta_{m+n,0} \qquad n, m = 1, 2, \ldots$$

$$d_\mu^{-n} = d_\mu^{\,n\dagger}$$

$$[d_\mu^{\,n}, \alpha_\nu^{\,m}] = 0.$$

(ii) The spectrum is given by considering

$$\langle \Gamma . P \rangle \langle \Gamma . P \rangle - m^2 = 0$$

ie. (3.3.5)

$$\langle P^2 \rangle - \frac{i}{4} \langle \Gamma_\mu(\tau) \frac{d}{d\tau} \Gamma_\mu(\tau) \rangle - m^2 = 0$$

ie.

$$p^2 - m^2 + \sum_1^\infty \left\{ \alpha^n . \alpha^{-n} + n d^n . d^{-n} \right\} = 0$$

thus giving linear trajectories.

That part of the Hamiltonian depending on the d's is

$$H_F = -\frac{i}{4} \langle \Gamma . \frac{d}{d\tau} \Gamma \rangle ,$$

and that part on the α's

$$H_B = \langle P^2(\tau) \rangle .$$

We then find

$$[H_F , \Gamma_\mu(\tau)] = i \frac{d}{d\tau} \Gamma_\mu(\tau) \qquad (3.3.6)$$

Further,

$$M^F_{\mu\nu} := \frac{i}{2} \langle \Gamma_\mu \Gamma_\nu \rangle$$

satisfy the Lorentz algebra; so

$$M_{\mu\nu} = M^B_{\mu\nu} + M^F_{\mu\nu}$$

give the total generators of the Lorentz group.

(iii) As in the boson case a gauge algebra emerges. For, if we define

$$L^B_{\pm n} = \langle e^{\pm in\tau} : P^2(\tau) : \rangle$$

as before, and (3.3.7)

$$L^F_{\pm n} = -\frac{i}{4} \langle e^{\pm in\tau} : \Gamma_\mu(\tau) \frac{d}{d\tau} \Gamma_\mu(\tau) : \rangle$$

and require equation (3.3.5) to hold for all internal modes, then we have

$$\langle \varphi | \sum_{-\infty}^{\infty} L_n e^{-in\tau} - m^2 | \psi \rangle = 0 \qquad (3.3.8)$$

where

$$L_n = L_n^B + L_n^F$$

Now, the L_n^F obey the same algebra as the L_n^B, and commute with the latter. Hence the L_n also generate the Virasoro algebra.

Therefore,

$$L_n | \varphi \rangle = 0 \qquad , n = 1, 2, \cdots$$

where $|\varphi\rangle$ satisfies equation (3.3.1)

We may go further and require (3.3.1) to hold for all modes,

$$\langle \varphi | : \Gamma(\tau) . P(\tau) : - m | \psi \rangle = 0$$

ie.

$$\langle \varphi | \sum_{-\infty}^{\infty} F_n e^{-in\tau} - m | \psi \rangle = 0$$

ie.

where $$F_n = \langle e^{in\tau} : \Gamma(\tau) . P(\tau) : \rangle \qquad (3.3.9)$$

Hence, we obtain further gauge conditions; viz.

$$F_n | \varphi \rangle = 0 \quad , n = 1, \cdots \qquad (3.3.10)$$

and equation (3.3.1) becomes

$$F_0 - m = 0$$

The F_n have the algebra[44]

$$[L_n , F_m] = \frac{1}{2} (2m - n) F_{n+m}$$

$$\{ F_n , F_m \} = 2 L_{n+m} \qquad (3.3.11)$$

(iv) Neveu and Schwartz[45] introduced an interaction into Ramond's fermion equation, by means of a vertex for the emission of a meson. This vertex was compatible with the gauge conditions (3.3.8) and (3.3.10) only if $m^2 = -{}^1/2$, m being the mass of the ground state of the emitted meson.

There is, in fact, a close connection between the above structure and that existing in the pion model of Neveu and Schwartz[15] - the amplitude of the latter being obtained by factorizing the amplitude for a fermion emitting mesons, in the quark - antiquark channel near the first pole ($m^2 = -{}^1/2$).[45] Furthermore, the new gauge operators are similar to those of Ramond, ie. the F_n, when the pion model is formulated in the second Fock space introduced by Neveu, Schwartz and Thorn.[15]

Now, people have proposed other extensions of the original model which add to the existing structure — Bardakci - Halpern[46] and Clavelli[47] models, described in an SU(1,1) context by Ramond.[14] However, the Neveu - Schwartz approach has more gauge conditions leading to a no-ghost theorem. (Goddard and Thorn[21] demonstrate this point, showing in particular that for a space - time dimension of ten, the Pomeron singularity appears as a pole). Moreover, the spectrum of the model shows considerable improvement over the original. We refer the reader to the papers quoted for a discussion of the merits and demerits of this model.

The purpose of this chapter was to present an alternative means of deriving the gauge algebra in a straightforward manner, but which did not require a physical interpretation such as that in the functional integral approach. Furthermore, we described an extension of this which augmented the existing gauge algebra. This new algebra is isomorphic to that of the Neveu - Schwartz model.

In the next chapter we outline an alternative framework within which may be accomodated Ramond's Correspondence Principle. Any future modifications to this new approach may develope along lines similar to those given in this chapter.

CHAPTER IV

Introduction

From the remarks made at the end of Chapter II it would appear that the most appropriate mathematical structure is based on the language of two-dimensional surfaces (which originated from quark duality diagrams). Indeed, using the functional integration formalism the Mobius group and gauge conditions may be attributed to the properties of the Lagrangian defined on such surfaces. However, this approach to dual models depends on the, improper, treatment of a string evolving in imaginary time. Hence, a possible modification of present considerations would be to use real time in a string model of hadrons,several authors have done this, taking for the most natural Lagrangian, the area of the world-sheet traced out by the string (a direct extension of the Lagrangian for a point particle, viz. the invariant length of the world-line). Nevertheless, this does involve a basic change in the character of the surfaces from the original formulation, the hyperbolic wave equation now being appropriate,whereas Laplace's equation was formerly.

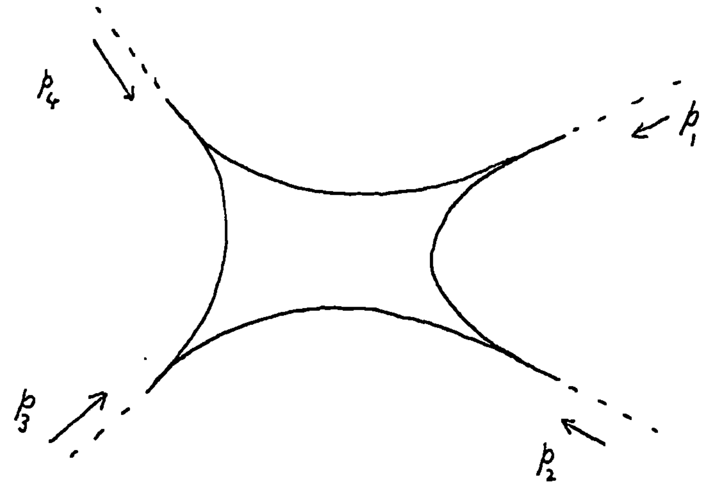
However, there is another line of attack which remains closer to the original model, proceeding, as it does, by a direct extension of the mathematical treatment of the Analogue approach. It involves interpreting the complex plane (or regions thereof) as a parameter space for the description of surfaces in a Minkowski space. The equations expressing the extremum of the energy dissipated in an Analogue surface are read as the conditions for the represented surface in

Minkowski space to be of minimal area. Thus, this method is also based on the geometry of surfaces, although here the character of the surfaces remains unchanged (definite metric). Furthermore, again in contrast to the string model, the surface is not intrinsic to any one hadron, but is primarily an expression of the extension of the interaction between hadrons rather than of the extension of the hadrons themselves ie. the dynamics of the interaction is determined by the geometry of the system. For, in the case of scalar particles scattering for example, the surface is bounded by the momenta of these particles. One consequence of this is that we may only discuss the scattering of at least three particles (to give a finite surface) - quite an acceptable situation.

(It is interesting to note that, via bootstrap ideas, the compositeness of a particle is the result of all possible interactions which, from the above, are governed by geometrical considerations. Hence, the description of one particle is in some way determined by the geometry of surfaces (Newman[59]) ie. extension of interaction of hadrons implies internal structure for hadrons.)

As a result of this approach we find that we are constrained to have a ground state of zero rest mass - ie. no tachyon. Furthermore, we are able to incorporate Ramond's Correspondence Principle, in a modified form, which allows a derivation of the Virasoro operators and the subsidiary conditions (missing in the Analogue approach). Moreover, by the very nature of the model there is a direct correspondence between the internal and the external coordinates, which is such that "rotations" of the latter (Lorentz group) appear as linear fractional transformations of the complex plane. In fact, the Lorentz group, acting on the momenta, manifests itself on the Koba - Nielsen variables as Mobius transformations. This fact adds to the relevance of the work of Domokos et al, Bars and Gursey, Bacry and Brink, who assumed the above identification. Indeed, the internal group, SL(2 $\mathbb{R}$ or $\mathbb{C}$), had assumed such a paramount role in dual models that various authors (Ramond and Clavelli) developed a group - theoretical prescription for constructing dual amplitudes based on the representations of SL(2 $\mathbb{R}$). (SL(2$\mathbb{C}$) applicable in the context of the full plane, leading to Virasoro-type amplitudes eg. Bars & Gursey; Shapiro - Z_i's allowed to roam over the full sphere instead of only a circle.)

However, there remains the problem of interpreting the surfaces in a space-time context. In the string model this is apparently straightforward (viz. world-sheet), although there is the problem of introducing an interaction between strings in a geometrical manner. Recalling the origin of the Analogue surfaces (duality diagrams) one might expect the represented surfaces to retain some of the characteristics of a quark-like structure, although at present it is not at all clear how this is realized. Furthermore, at first sight the boundary appears unusual - the momenta being tangential rather than normal to the "interaction region". However, (see section 1.4) by taking the real part instead of the imaginary part of the "complex potential" we may obtain a surface with a typical boundary



This point may be resolved once the surface itself has been given an explicit meaning.

On a note of speculation, the author would like to draw the reader's attention to some work of Newman and others in which internal structure is given to particles. The dynamics (velocity and acceleration of the particle - non-quantum treatment) is determined by a two-dimensional geometry. In fact, the distortions of the surface of a sphere in three-space yield the internal degrees of freedom of the particle. Similarly, in the paper by Goto and Hara, deformations of a sphere (also in space - non-relativistic treatment) produce a hadronic spectrum incorporating SU(3) without quarks. These indications may provide an interesting line for future research.

1. Minimal Surfaces in Minkowski Space

A. Classical Description

1.1 Representation of Surfaces.

We first indicate how surfaces in a Euclidean space are described, given a boundary (J. Douglas,[49] R. Courant,[50] for example.)

Suppose we are given a set of k contours (Γ) in this Euclidean space, each of definite sense; then we may construct a surface $\mathcal{S}$ of a prescribed topological type - viz. genus p , and either orientable or non-orientable - bounded by (Γ) as follows.

Let R be any symmetric Riemann surface (ie. there exists an involutory transformation T of R onto itself[49]), whose semi-surface,[49] R$'$, has genus p and k boundaries (C) which are topologically equivalent to (Γ). Further, if we allow x_μ(u,v) to be any single-valued, continuous vector function which takes the same value at conjugate points (ie. points equivalent under T) of R, and which also maps (C) into (Γ), then the surface traced out by x_μ, of genus p and bounded by (Γ), is orientable or not orientable according as R is of the first or second kind in the Klein sense[49] (ie. whether the locus of points invariant under T separates R or not) ie. x_μ transforms R$'$ topologically into $\mathcal{S}$. This is a particular representation of $\mathcal{S}$. The most general representation is obtained by topologically transforming R into itself or a homeomorphic surface while preserving conjugate points. We denote a particular representation by (g, R) if g maps the boundaries (C) of R into (Γ).

NB. in the language of references 30,31, R is the Schottky double of R′, the locus of those points invariant under T being the boundaries of the open surface.

Now, any Riemann surface of genus p and assigned orientability may be represented by the fundamental region of a group of linear fractional transformations (eg. Burnside[30], Lehner[32]), ie. the automorphic group of the surface. So, any surface in Euclidean space, or Minkowski space if the metric is space-like (for if this is not so then the metric is indefinite and so clearly the surface cannot be conformal to a region of the Non-Euclidean plane), may be represented by a normal polygon[30,32] (fundamental region) in the complex plane together with a mapping of the boundaries of the polygon into the boundaries of the surface.

1.2 Plateau's Problem:- briefly stated is

To find g and $X_\mu(u,v)$ such that the area of the surface $\mathcal{S}$ (NB. topological character already prescribed) is a minimum.

For a fuller statement and discussion of the problem we refer the reader to J. Douglas[49] and R. Courant[50]. Suffice it to note here that firstly, if isometric parameters are used, then Euler's equations for the extremum of the area of $\mathcal{S}$ reduce to the requirement that each $X_\mu(u,v)$ be a harmonic function of (u,v). Secondly, the corresponding Dirichlet integral, the energy integral of the Analogue Model, equals the area integral (up to a factor of 2) when isometric coordinates are employed. Hence, a minimisation of the former, to give the classical potential, implies the minimisation of the latter. Therefore, by varying the boundary conditions of the Analogue

Model (while maintaining the same contours of the surface boundary) to reach an extremum of the energy integral (corresponding to the variation of g), we may solve Plateau's Problem; for we know the solution to the Analogue problem (given by 28,29,30,31 and also by J. Douglas 49).

1.3 Characteristic Equation for Minimal Surfaces.

Since, using isometric parameters, Euler's equations for the area functional imply that $X_\mu(u,v)$ are harmonic, we may regard them as the imaginary parts of holomorphic functions $f_\mu(z)$, with $z = u + iv$.

ie. $$f_\mu(z) = y_\mu(u,v) + i x_\mu(u,v)$$

where y_μ is the conjugate harmonic function (ie. the potential if x_μ is the streamline function in the Analogue language).

Then, using the Cauchy - Riemann equations, we obtain

$$\begin{aligned} \phi(z) &:= \left[\frac{df_\mu}{dz}\right]\left[\frac{df^\mu}{dz}\right] \\ &= \left(\frac{\partial x_\mu}{\partial v} + i\frac{\partial x_\mu}{\partial u}\right)^2 \\ &= -(E - G - 2iF) \end{aligned} \qquad (4.1.1)$$

where,

$$E = \left(\frac{\partial x_\mu}{\partial u}\right)^2, \quad F = \left(\frac{\partial x_\mu}{\partial u}\cdot\frac{\partial x^\mu}{\partial v}\right), \quad G = \left(\frac{\partial x_\mu}{\partial v}\right)^2$$

ie. the metric is

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

Hence,

$$\phi(z) \equiv 0 \qquad (4.1.2)$$

for isometric coordinates (u,v).

Therefore, if we have a harmonic vector function, then it describes a minimal surface if the corresponding characteristic equation (4.1.2) is satisfied. For if this is so, then the parameters (u,v) are isometric, thus implying that Euler's equations (now Laplace's equation) have the above vector function as a solution.

<u>Null Coordinates</u> (required later)

These are defined by

$$ds^2 = 0 \iff dp\,dq = 0$$

where (p, q) are null parameters. If we have a pair of coordinates (u,v), we may put

$$p = z = u + iv$$

and $q = \bar{z} = u - iv$, giving

$$ds^2 = E'dp^2 + 2F'dp\,dq + G'dq^2$$

where

$$4E' = E - G - 2iF$$
$$4F' = E + G$$
$$4G' = E - G + 2iF.$$

When (u,v) are isometric, then (4.1.2) implies

$$ds^2 = E\,dp\,dq = E\,dz\,d\bar{z} \qquad (4.1.3)$$

NB. The element of area is

$$dA = \sqrt{EG - F^2}\,du\,dv = E\,dp\,dq ,$$

for null coordinates.

B. Quantisation of Characteristic Equation

1.4 Subsidiary Conditions and Absence of Tachyon

Since there is no direct physical model we cannot quantize in the manner of the functional integration formalism. However, we may expand the surface vector $f_\mu(z)$ in powers of z, with operator-valued coefficients, similar to Ramond.[13,14]

ie. $$F_\mu(z) = \frac{i x_\mu}{\sqrt{2}} + \sqrt{2}\, p_\mu \log z + i \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(a_\mu^n z^n + a_\mu^{n\dagger} z^{-n} \right) \qquad (4.1.4)$$

so that
$$X_\mu(u,v) = \mathcal{I}m\, F_\mu(z),$$

is a harmonic operator-valued function. (c.f. equation (2.2.4)) (X, F are the quantized forms of x, f respectively).

We then require (u,v) to be isometric parameters ie. (4.1.2) must hold; since

$$\frac{d}{dz} F_\mu(z) = -\frac{1}{z} \sum_{-\infty}^{\infty} \alpha_\mu^n z^n$$

$\left(a_\mu^n ,\ \alpha_\mu^n ,\right.$ etc. as given in chapter II$\left.\right)$, then

$$:\phi(z): \;=\; :\left[F_\mu'(z) \right]^2 :$$

$$= \frac{1}{z^2} \sum_{-\infty}^{\infty} L_n z^n \qquad (4.1.5)$$

where,
$$L_n := \;: \sum_{m=-\infty}^{\infty} \alpha_{m+n} \cdot \alpha_{-m} :$$

ie. the usual Virasoro operators with the gauge algebra

$$\left[L_n, L_m \right] = (n-m) L_{n+m} + \frac{1}{3} n (n^2-1)\, \delta_{n+m,0} .$$

In this way we incorporate Ramond's method of deriving the gauge algebra.

Hence, our "equations of motion" reduce to

$$\sum_{-\infty}^{\infty} L_n z^n = 0 \tag{4.1.6}$$

for all z. However, this is too stringent a condition. The situation is similar to that in Q.E.D. where we do not stipulate that

$$\partial_\mu A^\mu |\text{physical}\rangle = 0$$

but merely that

$$\partial_\mu A^{\mu(+)} |\text{physical}\rangle = 0$$

(Schweber[52] p 246), where $A^{\mu\,(+)}$ is the destruction part of A^μ.

In our case, we require (4.1.6) to hold only between physical states. Therefore, we have the gauge conditions

$$L_n |\text{physical}\rangle = 0 \tag{4.1.7}$$

for n = 0,1,2,---

NB. for no internal excitations

$$L_0 = p^2 = 0$$

ie. the ground state has zero rest-mass.

(Gauge conditions also obtained in this way by Minami[51].)

1.5 Q.E.D. Analogy

The comparison with Q.E.D. may be extended in a formal manner as follows. If we write

$$\frac{d}{dz} X_\mu = P_\mu + i Q_\mu$$

and put

$$F_{\mu\nu} = P_\mu Q_\nu - P_\nu Q_\mu$$

then, the area Lagrangian $\mathcal{L}$ may be expressed as

$$\mathcal{L} = \sqrt{EG-F^2} = (F_{\mu\nu}F^{\mu\nu})^{1/2} \approx \mathcal{L}_{Q.E.D.}^{1/2}$$

since $E = P^2$, $F = P.Q$, $G = Q^2$

Now, in Q.E.D.[52] variation of the Lagrangian yields

$$\Box A_\mu = 0$$

only provided $\partial_\mu A^\mu = 0$

ie. $p. A = 0$

Therefore, by identifying $P_\mu \Leftrightarrow p_\mu$ and $i\, Q_\mu \leftrightarrow -e A_\mu$ the gauge condition reads

$$P.Q. = F = 0$$

which is part of the condition for isometric coordinates (leading to Laplace's equation analogous to $\Box A_\mu = 0$.)

Indeed, if we consider the equation for a scalar particle of mass m containing the effects of the electromagnetic field

$$(p_\mu - e A_\mu)^2 = m^2$$

(Feynman[53], Principle of Minimal Electromagnetic Coupling) and write down the analogous equation for zero mass,

$$(P_\mu + i\, Q_\mu)^2 = 0$$

then we obtain the characteristic equation

ie. $E - G - 2iF = 0$

The "square root" of this equation may give the analogue to Ramond's generalisation of the Dirac equation, although in our case, zero mass, the neutrino equation would be more appropriate (c.f. section 3).

NB. in the usual formulations of the original dual model P_μ was treated as a momentum variable also.

Remark It has been suggested[54] that the compatibility of Poincaré invariance with the dynamical realization of duality might lead to a situation of no ghosts. The authors of the relevant paper cite the case of quantum electrodynamics in which the compatibility between Poincaré invariance and long-range interaction gives constraints, the gauge identities, which destroy the ghosts. They then proceed to construct a model, "relativistic string", in which the equation of motion (of the string) appears as a geometric constraint in Minkowski space; the gauge conditions are a consequence of the geometric nature of the model.

In the approach advocated in this chapter a similar state of affairs exists from a geometric point of view - the Lagrangian in each case is a geometric invariant viz. an area. The subsidiary conditions follow from the minimisation of the respective areas.

However, in our approach, the reconciliation of the extended nature of the interaction of hadrons with Poincaré invariance leads to the internal symmetry group SL(2 $\mathbb{C}$) (section 3).

2. Discussion of the Minimal Surface.

2.1 The Scattering of N Scalars.[55]

If we consider the case of N scalars with momentum p_i^{μ}, i = 1,---,N, then in the Analogue Model the Born term corresponds to the current distribution on a simply - connected surface - usually taken to be the unit disk. The complex potential for N currents p_i^{μ} entering the disk at the points z_i is

$$f_{\mu}(z) = \sum_{j=1}^{N} p_{j\mu} \log(z - z_i) \qquad (4.2.1)$$

Now, according to the characteristic equation (4.1.2), $\mathrm{Im}\, f_{\mu}(z)$ describes a minimal surface if and only if

$$\left[\frac{d}{dz} f_{\mu}\right]^2 = \sum_{i,j} \frac{p_i \cdot p_j}{(z-z_i)(z-z_j)} = 0 \qquad (4.2.2)$$

For $z \sim z_i$, in this equation, the dominant term is proportional to $p_i^2 = m^2$. Hence, we must have all ground masses zero. The above expression is equivalent to a $(N - 2)^{th}$ order polynomial in z. However, the coefficient of z^{N-2} is $\sum_{i,j} p_i \cdot p_j = (\sum p_i)^2 = 0$, (conservation of momentum). Hence, we have (N - 3) conditions to be met by the z_i, which characterize the parametrization of the boundary. These (N - 3) constraints determine the (N - 3) Chan variables in terms of Lorentz invariants (formed from the p_i^{μ})

Now, equation (4.2.2) is equivalent to

$$2 \sum_{i,j} \frac{p_i \cdot p_j}{(z_i - z_j)(z - z_i)} = 2 \sum_i \frac{a_i}{z - z_i} \qquad (4.2.3)$$

where $$a_i = \sum_{j=1}^{N} \frac{p_i \cdot p_j}{z_i - z_j}$$

However, (4.2.3.) must hold for all z, in particular for $z \sim z_i$. Hence, $a_i = 0$, is equivalent to (4.2.2)

(NB. We may obtain this result by minimizing the energy, $\sum p_i \cdot p_j \log |z_i - z_j|$, with respect to each z_i.)

i.e. $$\sum_{j=1}^{N} \frac{p_i \cdot p_j}{(z_i - z_j)} = 0 . \qquad (4.2.4)$$

This equation is invariant under SL(2 $\mathbb{C}$) transformation of the z's (ie. $z \rightarrow \frac{\alpha z + \beta}{\gamma z + \delta}$). For,

$$\sum_{i=1}^{N} \frac{p_i \cdot p_j (z_i - u)(z_j - v)}{(z_i - z_j)(u - v)} = \left[\sum_{i=1}^{N} \frac{p_i \cdot p_j}{(z_i - z_j)} (z_i - u) \right] \frac{(z_j - v)}{(u - v)}$$

$$= \left\{ \sum_{i=1}^{N} p_i \cdot p_j \frac{(z_i - z_j) + (z_j - u)}{(z_i - z_j)} \right\} \frac{(z_j - v)}{(u - v)}$$

$$= \left\{ \sum_{i=1}^{N} \frac{p_i \cdot p_j}{(z_i - z_j)} \right\} \frac{(z_j - v)(z_j - u)}{(u - v)}$$

= 0, if and only if $a_i = 0$, for u,v are quite arbitrary. The invariance under the given transformations follows since the coefficient of $p_i \cdot p_j$ in the first expression is a cross-ratio. Therefore, we should be able to express our solutions in the form of cross-ratios (SL(2$\mathbb{C}$) invariants) or equivalently in terms of Lorentz invariants. This point is clarified in section 3.

2.2 Explicit Example

For the case of N = 5, the conditions that the surface is minimal are

$$\sum_{i,j} p_i \cdot p_j \, \gamma_i \gamma_j = 0$$

and

$$\sum_{i,j} p_i \cdot p_j / \gamma_i \gamma_j = 0 .$$

If we define

$$x = \frac{(\gamma_1 - \gamma_2)(\gamma_3 - \gamma_4)}{(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_4)} , \qquad x' = 1 - x$$

$$X = \frac{(p_1 \cdot p_2)(p_3 \cdot p_4)}{(p_1 \cdot p_3)(p_2 \cdot p_4)}$$

and

$$X' = \frac{(p_1 \cdot p_4)\,(p_2 \cdot p_3)}{(p_1 \cdot p_3)\,(p_2 \cdot p_4)} ,$$ corresponding to x',

then x is the solution to (Appendix A5)

$$\frac{X}{x} + \frac{X'}{x'} = 1 \tag{4.2.5}$$

- the other cross-ratios obeying analogous equations.

For the four-point case the solution simplifies to

$$x = -\frac{p_1 \cdot p_2}{p_1 \cdot p_3} = -\frac{s}{t} \tag{4.2.6}$$

where $s = (p_1 + p_2)^2 , \; t = (p_1 + p_3)^2 , \; u = (p_1 + p_4)^2$

This leads to the four-point classical amplitude

$$A(s,t,u) = (-s)^{-\alpha' s} (-t)^{-\alpha' t} (-u)^{-\alpha' u} \tag{4.2.7}$$

using $s + t + u = 0$. This is Regge behaved as $s \to \infty, t$ fixed, for

$$A \to e^{\alpha' t} t^{-\alpha' t} s^{-\alpha' t}$$

($s \to -\infty$, is similar)

In order to treat the general case it proves to be convenient to introduce spinors allied with some modification of a work of Eisenhart on surfaces in four dimensions. This is presented after a discussion of the boundary of the minimal surface.

2.3 The Boundary of the Born Term Surface

The minimal surface is given by

$$\begin{aligned} x_\mu(z) &= \mathrm{Im} \sum_{i=1}^{N} p_{i\mu} \log(z - z_i) \\ &= \sum_{i=1}^{N} p_{i\mu} \varphi_i \end{aligned} \tag{4.2.8}$$

for z_i satisfying the characteristic equation (4.1.2); where

$$\varphi_i = \arg(z - z_i)$$

Let $z_j = e^{i\theta_j}$, $z = re^{i\theta}$, so that the boundary is given by $r = 1$.

Then we have,

$$\varphi_i = \alpha_i + \beta_i$$

where

$$\alpha_i = \tfrac{1}{2}(\theta + \theta_i)$$

and

$$\tan \beta_i = \frac{r+1}{r-1} \tan \frac{\theta - \theta_i}{2}$$

with $r = 1 - \varepsilon$, $\varepsilon \geqslant 0$, (for $r = 1$ (4.2.8) describes four points as z varies over the unit circle, ie. x_μ jumps from point to point; we obtain the boundary by z travelling just inside $r = 1$)

For

$$\theta = \theta_j - \delta \quad , \quad \beta_j = \pi/2 \quad , \quad \delta \geqslant 0$$

$$\theta = \theta_j \quad , \quad \beta_j = 0$$

$$\theta = \theta_j + \delta \quad , \quad \beta_j = -\frac{\pi}{2} \; .$$

We now put

$$\chi_j = \frac{1}{2} \sum_{i=1}^{N} p_i \theta_i - \pi \sum_{i=1}^{j} p_i + \frac{\pi}{2} p_j$$

so that

$$\chi_{j+1} = \chi_j - \frac{\pi}{2} (p_j + p_{j+1})$$

Then

$$\chi(z) = \begin{cases} \chi_j + \frac{\pi}{2} p_j & , \quad \theta = \theta_j - \delta \\ \chi_j & , \quad \theta = \theta_j \\ \chi_j - \frac{\pi}{2} p_j & , \quad \theta = \theta_j + \delta . \end{cases}$$

So that as θ crosses θ_j, χ goes from $\chi + \frac{\pi}{2} p_j$ through χ_j, at $\theta = \theta_j$, to $\chi_j - \frac{\pi}{2} p_j$.

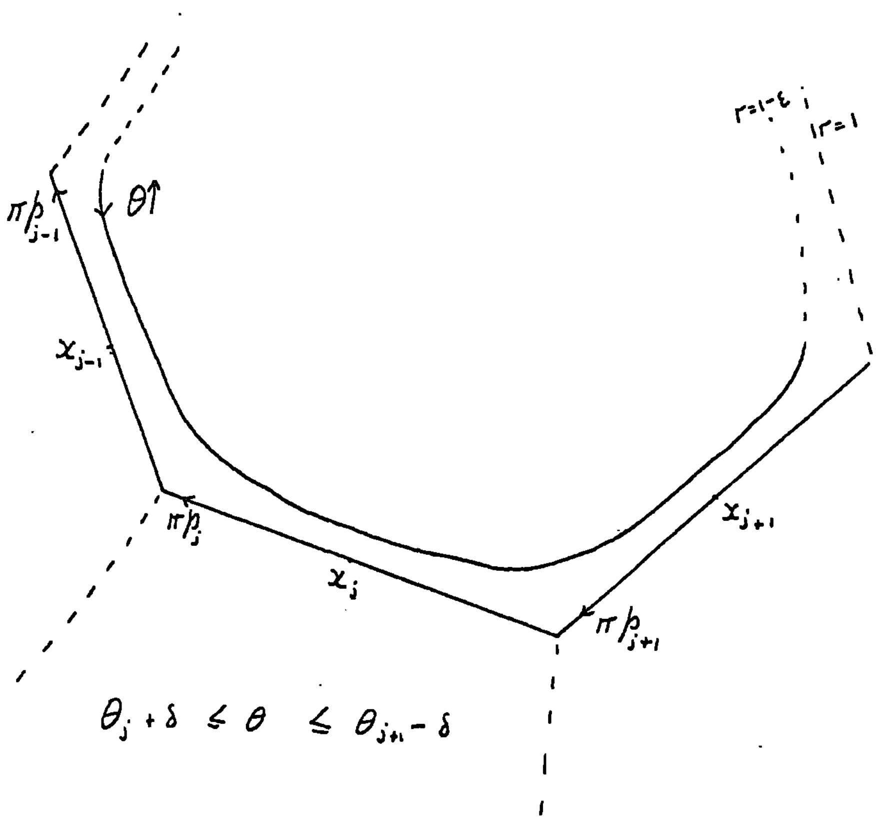


Figure of Boundary

As $r \to 1^-$ then χ remains at each vertex for a longer period, until r reaches 1 when

$$\chi_\mu(\gamma) = \frac{1}{2}\sum_{i=1}^{N} p_{i\mu}\,\theta_i - \pi\sum_{i=1}^{N} p_{i\mu}\,\Theta(\theta-\theta_i)$$

- stream-function for the Analogue Problem.

2.4 The General Solution to the Characteristic Equation

From the results of Appendix A4 we have a general expression for a surface in Minkowski space. However, from the Analogue approach we also have an expression for the stream function ie. for certain values of the z_i, the minimal vector. Therefore, by imposing the general form on the Analogue function we may determine those z_i which satisfy the characteristic equation.

For example in the case of N scalars scattering we have

$$X_\mu(z) = \mathrm{Im}\sum_{j=1}^{N} p_{j\mu}\log(z-z_j)$$

$$= \mathrm{Re}\sum_{j=1}^{N} \frac{p_{j\mu}}{i}\log(z-z_j)$$

Then, using the general form for X_μ we find that

$$4g = i\sum_{j=1}^{N}(p_j^1 - ip_j^2)\{(z-z_j)\log(z-z_j) - z\}$$

$$4f = -i\sum_{j=1}^{N}(p_j^0 - p_j^3)\{(z-z_j)\log(z-z_j) - z\}$$

(g,f and their relation to X_μ are given in Appendix A4).

Using these expressions we may calculate X_μ and equate it to the stream - function above. We find that, for consistency, we require

$$z_j = \frac{p_j^1 + ip_j^2}{p_j^0 - p_j^3} \qquad (4.2.9)$$

ie. each four - momentum determines a point in the complex plane.

We may understand this relation by noting that for z close to z_j , $dX_\mu \sim \frac{1}{\varepsilon} p_{j\mu}$, from the stream - function ($\varepsilon = z - z_j \sim 0$). More generally, for a distribution of momenta $P_{j\mu}(\xi_j)$ (ie. Reggeon) we have

$$X_\mu(z) = \mathrm{Im}\sum_{j=1}^{N}\int_{C_j} d\xi_j\, P_{j\mu}(\xi_j)\log(z-\xi_j)$$

where $\rho_{j\mu}(\xi_j)$ is spread over C_j . Hence, at $z = \xi_j$ we obtain

$$\xi_j = \frac{dx_1(\xi_j) + i\,dx_2(\xi_j)}{dx_0(\xi_j) - dx_3(\xi_j)}$$

$$= \frac{\rho_{j1}(\xi_j) + i\,\rho_{j2}(\xi_j)}{\rho_{j0}(\xi_j) - \rho_{j3}(\xi_j)}$$

since $\quad dx_\mu(z) \simeq \frac{1}{z-\xi_j}\,\rho_{j\mu}(\xi_j)$, for $z \sim \xi_j$

3. Relation Between Internal and External Symmetries

3.1 Spinors and SL(2 $\mathbb{C}$)

In order to see the convenience of introducing spinors, first consider the neutrino - type equation (NB. the ground state is massless),

$$(\sigma . p) \xi = 0 \qquad (4.3.1)$$

where $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$, $\sigma_\mu = (I, \underset{\sim}{\sigma})$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and

σ_i are the Pauli matrices

ie

$$\begin{pmatrix} p_0 - p_3 & -p_1 + i p_2 \\ -p_1 - i p_2 & p_0 + p_3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \qquad (4.3.2)$$

This implies that the solution satisfies

$$z := \xi_2 / \xi_1 = \frac{p_1 + i p_2}{p_0 + p_3} \qquad (4.3.3)$$

Now, the Lorentz group has the action

$$\xi \rightarrow \xi' = \begin{pmatrix} c\,\xi_2 + d\,\xi_1 \\ a\,\xi_2 + b\,\xi_1 \end{pmatrix} , \quad ad - bc = 1 .$$

ie.

$$z \rightarrow z' = \frac{a z + b}{c z + d}$$

ie. a fractional linear transformation.

This suggests that in Eisenhart's[56] modified equations we should choose the parameter z as

$$z = \frac{dx_1 + i\, dx_2}{dx_0 + dx_3} \qquad (4.3.4)$$

which would lead to equation (4.3.3)

It is in this manner that the internal group SL(2 $\mathbb{C}$) is directly related to the Lorentz group (indeed the Poincaré group since dx_μ is the relevant quantity) acting on Minkowski space (x_μ).

3.2 Cross-ratios

We now return to the problem of the solutions to the characteristic equation which, we said, could be expressed either as cross-ratios or in terms of Lorentz invariants. In the following we make the relation between these forms explicit, using the spinors introduced in (3.1).

For, given two spinors ξ, η we may form the Lorentz scalar[52]

$$\xi^T C \eta = \xi_1 \eta_2 - \xi_2 \eta_1$$

with
$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Hence, if we associate a spinor ξ^j with each momentum p^j_μ such that

$$z_j = \xi^j_2 / \xi^j_1 ,$$

we may write,

$$\xi^j = N_j \begin{pmatrix} 1 \\ z_j \end{pmatrix},$$ N_j being a normalisation factor depending only on p^j_μ . Then

$$(\xi^j)^T C \xi^i = N_i N_j (z_i - z_j)$$

so that, for example,

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} = \frac{\left[(\xi^1)^T C\, \xi^2\right]\left[(\xi^3)^T C\, \xi^4\right]}{\left[(\xi^1)^T C\, \xi^3\right]\left[(\xi^2)^T C\, \xi^4\right]} ,$$

which is Lorentz invariant.

Conclusion

So far we have only discussed the Classical Born term. For higher order terms the procedure would be similar to that employed by Lovelace and Alexandrini in the perturbative approach to the dual model, although, as in the Analogue Model, there is no prescription for a measure, since as yet, time is not incorporated. In fact, Douglas[49] used the theory of Abelian Integrals to construct solutions to the Plateau Problem in terms of multi-dimensional theta functions.

However, there remains the difficulty of calculating the Quantum Born term, which involves the functional integration of $EG - F^2$ over all x_μ and boundary parametrisations. We cannot assume that the Quantum Born term is proportional to to the Classical expression, as in the case of a quadratic Lagrangian (Feynman & Hibbs[57]). The string-model also faces this problem, which is similar to the quantisation problem in General Relativity.[58]

Nevertheless, it is the view of the author that the importance of dual models lies in the emergence of a set of mathematical characteristics for a strong interaction theory. These attributes (duality or $SL(2\,\mathbb{C})$ internal symmetry, conformal invariance leading to no ghosts) have been combined in an approach which has no tachyon and also relates the internal symmetry to the homogeneous Lorentz group. In fact, there is a relation between the internal coordinates (z) and the external Minkowski space. Furthermore, the surface is extended (c.f. Takabayashi,[12] duality a consequence of hadrons being extended) in Minkowski space, but it is not yet clear how to regard this point. The world-sheet of the relativistic string model also exists in Minkowski space (here it is real space-time), but the former cannot be interpreted in the same manner. However, what does seem important is the geometry, ie. the metric, for it is the latter which appears in the equations governing the model - ie. a local condition.

The structure obtained using the approach of the present chapter is similar in some respects to that advocated by some authors on homogeneous space techniques,[8] by means of which internal degrees of freedom are given to a system. These degrees of freedom span a homogeneous space of the Lorentz group. In fact, as explained in the introduction, Bacry and Nuyts[8] suggest an internal space composed of two-dimensional complex spinors (replace z by $\xi = (\xi_1, \xi_2)$).

Furthermore, now that the relevance of the Lorentz group to the internal symmetries is understood, it may be possible to include half-integer spin in the framework; perhaps by attaching spin coordinates at each point of the surface. The preceding sentences suggest that, if this approach is to prove fruitful, one should consider ways of introducing the remaining two coordinates. The Koba-Nielsen variable z may be regarded as a stereographic projection from a sphere in three-space. This may lead to a connection with some of the ideas of Newman et al,[59] in which a two-dimensional geometry determines the dynamics of the particle.

However, the present formulation does allow the retention of the desirable characteristics of dual models, while evading the tachyon, and also provides a possible framework on which to build extensions to achieve hopefully, more realistic conseouences.

A1 Cylinder Calculation

$$\varphi_\mu = \sum_{n=1}^{\infty}\Big\{\sin nx\,(a_n^\mu \sinh ny + b_n^\mu \cosh ny) + \cos nx\,(c_n^\mu \sinh ny + d_n^\mu \cosh ny)\Big\}$$

$$\left.\frac{\partial\varphi_\mu}{\partial y}\right|_{y=\beta'} = -\sum_{i=1}^{N} k_i^\mu\,\delta(x-\theta_i) = -\sum_{i=1}^{N} k_i \sum_{n=-\infty}^{\infty} \frac{e^{in(x-\theta_i)}}{2\pi}$$

$$= -\frac{1}{\pi}\sum_{i=1}^{N} k_i^\mu \sum_{n=1}^{\infty}(\sin nx \sin n\theta_i + \cos nx \cos n\theta_i)$$

$$\left.\frac{\partial\varphi_\mu}{\partial y}\right|_{y=-\beta'} = \frac{1}{\pi}\sum_{i=1}^{M}\sum_{n=1}^{\infty} \ell_i^\mu(\sin nx \sin n\varphi_i + \cos nx \cos n\varphi_i)$$

Equating coefficients,

$$a_n^\mu = \frac{\sum_1^M \ell_i^\mu \sin n\varphi_i - \sum_1^N k_j^\mu \sin n\theta_j}{\pi n \cosh n\beta'}, \quad b_n^\mu = -\frac{\sum_1^M \ell_i^\mu \sin n\varphi_i + \sum_1^N k_j^\mu \sin n\theta_j}{\pi n \sinh n\beta'},$$

$$c_n^\mu = \frac{\sum_1^M \ell_i^\mu \cos n\varphi_i - \sum_1^N k_i^\mu \cos n\theta_i}{\pi n \cosh n\beta'}, \quad d_n^\mu = -\frac{\sum_1^M \ell_i^\mu \cos n\varphi_i + \sum_1^N k_i^\mu \cos n\theta_i}{\pi n \sinh n\beta'},$$

$$\text{Energy} = \mathcal{E}_1 = \sigma\int_{\text{bdry.}} \varphi . \frac{\partial\varphi}{\partial n}\, ds$$

$$= \sigma\sum_{i=1}^{N} k_i . \varphi(\theta_i, \beta') + \sigma\sum_{j=1}^{M} \ell_j . \varphi(\varphi_j, -\beta')$$

$$= -\frac{2\sigma}{\pi}\Big\{\sum_{i\neq j}\frac{k_i . k_j \cos n(\theta_i-\theta_j)\cosh 2n\beta'}{n \sinh 2n\beta'} + \sum_{i\neq j}\frac{\ell_i . \ell_j \cos n(\varphi_i-\varphi_j)\cosh 2n\beta'}{n\sinh 2n\beta'} + 2\sum_{i=1}^{N}\sum_{j=1}^{M} k_i . \ell_j \frac{\cos n(\theta_i-\varphi_j)}{n\sinh 2n\beta'}\Big\}$$

$$= \frac{2\sigma}{\pi}\Big\{\sum_{i\neq j} k_i . k_j \log\Big[\frac{\vartheta_1[\nu_{ij}|\rho]}{q_0\ \rho'^{1/4}}\Big] + \sum_{i\neq j}\ell_i . \ell_j \log\Big[\frac{\vartheta_1[\nu'_{ij}|\rho]}{q_0\ \rho'^{1/4}}\Big] + \sum_{i,j} k_i . \ell_j \log\Big[\frac{\vartheta_0[u_{ij}|\rho]}{q_0}\Big]\Big\} \quad \text{(Ref. 60)}$$

$$\nu_{ij} = \frac{\theta_i-\theta_j}{2\pi}, \quad \nu'_{ij} = \frac{\varphi_i-\varphi_j}{2\pi}, \quad u_{ij} = \frac{\theta_i-\varphi_j}{2\pi}, \quad \rho = e^{-2\beta'} = e^{\pi i\tau},$$

$$q_0 = \prod_1^{\infty}(1-\rho^{2n}).$$

A2 Mobius Calculation

$$\varphi^\mu = \sum_{n=1}^{\infty}\Big\{\cosh ny\,[a_n^\mu \sin nx + b_n^\mu \cos nx] + \sinh \overline{n-\tfrac{1}{2}}\,y\,[c_n^\mu \sin\overline{n-\tfrac{1}{2}}\,x + d_n^\mu \cos\overline{n-\tfrac{1}{2}}\,x]\Big\}$$

$$\left.\frac{\partial\varphi^\mu}{\partial y}\right|_{y=\beta} = -\sum_i p_i^\mu\,\delta(x-\theta_i)$$

$$= -\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_i p_i^\mu\left[\sin\frac{mx}{2}\sin\frac{m\theta_i}{2} + \cos\frac{mx}{2}\cos\frac{m\theta_i}{2}\right].$$

Hence,

m even $=2n \Rightarrow$ $$a_n^\mu = \frac{-1}{\pi n\sinh n\beta}\sum_1^N p_i^\mu \sin n\theta_i$$

$$\& \quad b_n^\mu = \frac{-1}{\pi n\sinh n\beta}\sum_1^N p_i^\mu \cos n\theta_i$$

m odd $=2n-1 \Rightarrow$ $$c_n^\mu = \frac{-1}{\pi(n-\frac{1}{2})\cosh(n-\frac{1}{2})\beta}\sum_{i=1}^N p_i^\mu \sin(n-\tfrac{1}{2})\theta_i$$

$$\& \quad d_n^\mu = \frac{-1}{\pi(n-\frac{1}{2})\cosh(n-\frac{1}{2})\beta}\sum_1^N p_i^\mu \cos(n-\tfrac{1}{2})\theta_i$$

$$\text{Energy} = \mathcal{E}_2 = \sigma\int_{\text{bdry}}\varphi\cdot\frac{\partial\varphi}{\partial n}\,ds = \sigma\sum_1^N p_i\cdot\varphi(\theta_i,\beta)$$

$$= -\frac{\sigma}{\pi}\sum_{i,j,n} p_i\cdot p_j\left\{\frac{\cos n(\theta_i-\theta_j)\cosh n\beta}{n\sinh n\beta} + \frac{\cos(n-\frac{1}{2})(\theta_i-\theta_j)\sinh(n-\frac{1}{2})\beta}{(n-\frac{1}{2})\cosh(n-\frac{1}{2})\beta}\right\}$$

$$= \frac{2\sigma}{\pi}\sum_{i\neq j} p_i\cdot p_j \log\left\{\frac{\vartheta_1\left[\frac{\nu_{ij}}{2}\,\middle|\,\frac{\tau}{2}\right]\vartheta_3\left[\frac{\nu_{ij}}{2}\,\middle|\,\frac{\tau}{2}\right]}{q_0\;\rho^{1/4}}\right\} \qquad \text{(Ref. 60)}$$

$$\rho = e^{-\beta}, \quad \tau = i\beta/\pi, \quad \nu_{ij} = \frac{\theta_i-\theta_j}{2\pi}$$

A3 Measure Calculation

a) Consider the matrix A, and diagonalize it by means of a matrix B such that

$$B' A B = diag(\lambda_1, \cdots, \lambda_N) = D$$

and

$$B' B = I$$

Writing $\mathcal{B} = \text{diag.}(B, \cdots, B)$

we find also

$$\mathcal{B}' \mathcal{A} \mathcal{B} = \begin{bmatrix} D & -I & & & -I \\ -I & D & -I & O & \\ & -I & D & & \\ & O & & & -I \\ -I & & & -I & D \end{bmatrix}$$

We may now interchange rows and columns to give

$$|\mathcal{B}' \mathcal{A} \mathcal{B}| = |diag.(A_1, \cdots, A_N)|$$

where

$$A_p := \begin{bmatrix} \lambda_p & -1 & & & -1 \\ -1 & \lambda_p & & O & \\ & & & & \\ & O & & & -1 \\ -1 & & & -1 & \lambda_p \end{bmatrix}_{M \times M}$$

This reordering leaves the determinant unchanged; for, to finish up with all λ's diagonal requires that the parity of row changes equal the parity of column changes.

Therefore,

$$|\mathcal{A}| = |\mathcal{B}'\mathcal{A}\mathcal{B}| = \prod_{p} |A_p| .$$

b) <u>Determination of λ_p</u>

We wish to solve

$$|A - \lambda I| = 0$$

Let

$$C_N = \begin{bmatrix} a & -1 & & & 0 \\ -1 & a & -1 & & \\ & -1 & a & & \\ & & & \ddots & \\ 0 & & & & -1 \\ & & & -1 & a \end{bmatrix}_{N \times N} , \quad a = 4 - \lambda ,$$

$$= a\,C_{N-1} - C_{N-2}$$

$$= \frac{\sin (N+1)\alpha}{\sin \alpha}$$

where $a = 2\cos\alpha$

Hence,

$$C_N = 0 , \text{ for } \alpha = \alpha_p = \frac{p\pi}{N+1} , \quad p = 1, \dots, N$$

ie. $\lambda_p = 4 - 2\cos\frac{p\pi}{N+1}$

c) <u>Determination of $|A_p|$</u>

We note that

$$|A_p| = \lambda_p C_{M-1} - 2C_{M-2} - 2$$

with $a = \lambda_p$ in C_M .

Hence,

$$|A_p| = -4\sin^2\left(\frac{M}{2}\beta_p\right)$$

where,

$$2\cos\beta_p = \lambda_p = 4 - 2\cos\frac{p\pi}{N+1}$$

ie.

$$\sin\tfrac{1}{2}\beta_p = i\sin\frac{p\pi}{2(N+1)}$$

Therefore, for $p \ll N$,

$$\beta_p = i\alpha_p = \frac{i p\pi}{N+1}.$$

If we denote the ratio of the circumference to the length of the cylinder network by k = M/N, then

$$|A_p| = 4\sinh^2\left[\frac{kNp\pi}{2(N+1)}\right] \longrightarrow 4\sinh^2\left(p\frac{k\pi}{2}\right)$$

as $N \to \infty$ (dense Feynman net.)

Hence,

$$|\mathcal{A}| = \prod_{p=1}^{\infty}|A_p| = \prod_p e^{pk\pi}(1-e^{-pk\pi})^2$$

By putting

$$q = e^{-k\pi/2}$$

we obtain

$$|\mathcal{A}| = \prod_{p=1}^{\infty} q^{-2p}(1-q^{2p})^2.$$

d) <u>The Functional Integration over K^i_j</u>

Consider,

$$I = \iint \exp\left[-\alpha\pi Y'LY\right]dY,$$

where $Y' = (y_1, \ldots, y_n)$, L symmetric. Then, if $L = B'DB$, D = diag $(d_1, \cdots \cdots, d_n)$, we may write $\tilde{Y}$ = BY to give

$$I = \iint exp\,[-\alpha\pi\tilde{Y}'D\tilde{Y}]\,d\tilde{Y} \qquad (if\ B'B = I)$$

$$= \int\cdots\int exp\,\Big[-\sum_{i=1}^{n}(\tilde{y}_i^2\pi\alpha d_i)\Big]\prod_{j=1}^{n} d\tilde{y}_j$$

$$= \prod_{i=1}^{n}(\alpha d_i)^{-\frac{1}{2}} = [\alpha^n|L|]^{-1/2} .$$

Similarly,

$$\iint exp\,[-\sigma\pi X'\mathcal{A}X]\,dX = [\sigma^{kN^2}|\mathcal{A}|]^{-2},$$

X being a MN = kN^2 row matrix; the Lorentz index makes each integral four-fold, giving the power - 2.

A4 Modification of Eisenhart's Equations.

Let $x_\mu(p, q)$ describe a surface imbedded in Minkowski space. Then

$$ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2 \qquad \text{(A4.1)}$$

where

$$E = \left(\frac{\partial x_\mu}{\partial p}\right)^2 , \quad F = \frac{\partial x_\mu}{\partial p}\frac{\partial x^\mu}{\partial q} , \quad G = \left(\frac{\partial x_\mu}{\partial q}\right)^2$$

and the element of area is

$$V\,dp\,dq = \sqrt{EG - F^2}\,dp\,dq . \qquad \text{(A4.2)}$$

We wish to minimise $\iint V\,dp\,dq$; the Euler equations are (Forsyth p.313)

$$\frac{\partial}{\partial p}\left[\frac{1}{V}\left(G\frac{\partial x^\mu}{\partial p} - F\frac{\partial x^\mu}{\partial q}\right)\right] + \frac{\partial}{\partial q}\left[\frac{1}{V}\left(E\frac{\partial x^\mu}{\partial q} - F\frac{\partial x^\mu}{\partial p}\right)\right] = 0 \qquad \text{(A4.3)}$$

If we now use null parametric coordinates, ie. $E = G = 0$, these equations reduce to

$$\frac{\partial^2}{\partial p\,\partial q} x^\mu = 0 , \quad \text{since} \quad V = iF , \qquad \text{(A4.4)}$$

Hence x^μ = (function of p) + (function of q)

These functions are only constrained by the conditions

$E = 0 = G$

Now, $E = 0$ implies

$$\frac{\dfrac{\partial x^1}{\partial p} + i\dfrac{\partial x^2}{\partial p}}{\dfrac{\partial x^0}{\partial p} - \dfrac{\partial x^3}{\partial p}} = - \frac{\dfrac{\partial x^0}{\partial p} + \dfrac{\partial x^3}{\partial p}}{\dfrac{\partial x^1}{\partial p} - i\dfrac{\partial x^2}{\partial p}} = \text{some function of } p. \qquad \text{(A4.5)}$$

We may define p by setting it equal to these fractions

ie.
$$p := \frac{\frac{\partial x^1}{\partial p} + i\frac{\partial x^2}{\partial p}}{\frac{\partial x^0}{\partial p} - \frac{\partial x^3}{\partial p}} . \qquad \text{(A4.6)}$$

Hence, $$\frac{\partial x^1}{\partial p} + i\frac{\partial x^2}{\partial p} = p\left(\frac{\partial x^0}{\partial p} - \frac{\partial x^3}{\partial p}\right) = 2pf''(p)$$

and $$\frac{\partial x^0}{\partial p} + \frac{\partial x^3}{\partial p} = -p\left(\frac{\partial x^1}{\partial p} - i\frac{\partial x^2}{\partial p}\right) = 2pg''(p),$$ say;

since each side is a function of p only.

Therefore, solving for $\partial x^\mu/\partial p$ we find that

$$\left.\begin{aligned}
\frac{\partial x^1}{\partial p} &= pf''(p) - g''(p)\\
i\frac{\partial x^2}{\partial p} &= pf''(p) + g''(p)\\
\frac{\partial x^3}{\partial p} &= pg''(p) - f''(p)\\
\frac{\partial x^0}{\partial p} &= pg''(p) + f''(p)
\end{aligned}\right\} \qquad \text{(A4.7)}$$

We apply similar considerations to $G = 0$, with $\phi(q)$ replacing $f(p)$, $\psi(q)$ replacing $g(p)$. Now, because $x^\mu(p,q)$ describes a real surface in Minkowski space we must have p conjugate to q, and similarly $f = \bar{\phi}$, $g = \bar{\psi}$. Then, integrating (A4.7), we obtain.

$$\begin{aligned}
x^1 &= 2\,Re\,[pf' - f - g']\\
x^2 &= 2\,Re\,\tfrac{1}{i}[pf' - f + g']\\
x^3 &= 2\,Re\,[pg' - g - f']\\
x^0 &= 2\,Re\,[pg' - g + f'] ,
\end{aligned} \qquad \text{(A4.8)}$$

In (A4.8) the functions $f(p)$, $g(p)$, are completely arbitrary.

If we put $p = z = u + iv$, $q = \bar{z} = u - iv$, then (§(1.3) of Chapter IV) we see that u,v are isometric parameters.

A5 <u>The four & five point functions.</u>

a) <u>4-point</u> We require the solution to

$$\sum_{i=1}^{4} \frac{p_i \cdot p_1}{z_i - z_1} = 0 , \qquad \text{(taking } j = 1 \text{)}$$

ie.

$$\frac{p_1 p_2}{z_1 - z_2} + \frac{p_1 p_3}{z_1 - z_3} + \frac{p_1 p_4}{z_1 - z_4} = 0 \tag{A5.1}$$

We use the notation $(ij) = p_i \cdot p_j$, $[ij] = z_i - z_j$.
Multiplying equation (A5.1) by $[41]$ we obtain

$$(12)\frac{[41]}{[21]} + (13)\frac{[41]}{[31]} + (14) = 0 ,$$

ie.

$$(12)\left[1 + \frac{[42]}{[21]}\right] + (13)\frac{[41]}{[31]} + (14) = 0 ,$$

ie.

$$(13)\frac{[43]}{[31]} + (12)\frac{[42]}{[21]} = 0 ,$$

ie.

$$\frac{[12][34]}{[13][24]} = - \frac{(12)}{(13)}$$

ie.

$$\omega = - \frac{s}{u} , \qquad \begin{aligned} s &= (p_1 + p_2)^2 = 2 p_1 \cdot p_2 \\ t &= (p_2 + p_3)^2 = 2 p_2 \cdot p_3 \\ u &= (p_1 + p_3)^2 = 2 p_1 \cdot p_3 \end{aligned} \tag{A5.2}$$

$$\omega = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$

b) <u>5-point</u> In this case we have two conditions. We may take

$$\sum_{i=2}^{5} \frac{p_i \cdot p_1}{\gamma_i - \gamma_1} = 0 \quad \& \quad \sum_{i=1}^{5} \frac{p_i \cdot p_2}{\gamma_i - \gamma_2} = 0 , \qquad \text{(A5.3)}$$

These may be written

$$(13) + (14)x + (15)y = 0$$

$$(23) + (24)\frac{1}{x} + (25)\frac{1}{y} = 0 , \qquad \text{(A5.4)}$$

using the previous notation, and where

$$x = \frac{[24][13]}{[23][14]} , \quad y = \frac{[25][13]}{[23][15]}$$

Eliminating y we obtain

$$x^2 (23)(14) + [(13)(23) + (14)(24) - (15)(25)]\, x + (13)(24) = 0$$

If we put

$$X = \frac{(24)(13)}{(23)(14)} \quad \& \quad X' = \frac{(21)(34)}{(23)(14)} ,$$

corresponding to x & (1 - x) respectively, then we may show that the above equation becomes

$$x^2 + (X' - X - 1)x + X = 0$$

ie.
$$\frac{X}{x} + \frac{X'}{x'} = 1 , \quad \text{where } x' = (1-x) , \qquad \text{(A5.5)}$$

$$[\text{c.f. } x + x' = 1]$$

A similar equation holds for y.

<u>Note</u>. For the 4-point case $X = \left(\frac{u}{t}\right)^2$, $X' = \left(\frac{s}{t}\right)^2$

Hence,

$$x^2 + t^{-2}(s^2 - u^2 - t^2)x + u^2 t^{-2} = 0,$$

ie.

$$x^2 + t^{-2}(2ut)x + u^2 t^{-2} = 0,$$

ie. $\left(x + \frac{u}{t}\right)^2 = 0, \rightarrow$

$x = -\frac{u}{t}$, equivalent to previous solution.

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