Vector cross product structures on manifolds

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VECTOR CROSS PRODUCT STRUCTURES

ON MANIFOLDS

by

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CONTENTS

INTRODUCTION

Vector cross products on vector spaces were first defined and studied from a topological standpoint by Eckmann [4]. Further, vector cross products have been studied from an algebraic standpoint in Brown and Grey [1].

Vector cross products are interesting for three reasons: first, they are a natural generalization of the concept of almost complex structure; secondly, a vector cross product on a manifold M generates unusual almost complex structures on certain submanifolds of M; thirdly, vector cross products provide an approach to the study of riemannian manifolds with holonomy group G_2 or Spin(7).

In Chapter (1) we give a description of vector cross products on vector spaces from a topological and then from an algebraic standpoint. We state the main results arrived at in [1] and [4].

In Chapter (2) we give an account of a work by Gray ([6]) dealing with vector cross products on manifolds and relating the geometry of certain submanifolds to the properties of a naturally induced almost complex structure. Some of the proofs presented in this Chapter are due to the author.

In Chapter (3) we determine completely all connected and complete nearly kahler hypersurfaces in R^7. We deduce that all nearly kahler 6-dimensional submanifolds in R^8 possess the nice property of pointwise constant type. Also for such submanifolds we obtain a formula relating the holomorphic sectional curvature to the type function. This, we believe, should make the study of the geometry of such submanifolds relatively easier. Further we generalize a result on the existence of almost complex structures on spheres by considering the class of all connected and compact hypersurfaces in R^{n+1} whose Gaussian curvature K_n vanishes nowhere.

Finally in Chapter (4) we make an attempt towards a generalization of the integrability notion to vector cross products. We discuss and evaluate possible ways of doing this.

CHAPTER 1

Vector Cross Products

In this Chapter we give an account of the development of the subject of vector cross product structures on vector spaces with an inner product.

1. Continuous vector cross products.

Let V be an n-dimensional real vector space and (,) the usual (positive definite) inner product. Eckmann [4] has defined a vector cross product on V to be a continuous map

$$P: V^r \longrightarrow V \quad (1 \leqslant r \leqslant n)$$

with the following properties

$$\Big(P(a_1, \dots, a_r),\ a_i \Big) = 0,\ 1 \leqslant i \leqslant r \qquad (1.1)$$

$$\Big(P(a_1, \dots, a_r),\ P(a_1, \dots, a_r) \Big) = \det \Big((a_i, a_j) \Big) \qquad (1.2)$$

The following theorem has been proved by Eckmann and Whitehead ([4] , [14]):

Theorem (1.1) A vector cross product exists in precisely the following cases:

n	is even,	r = 1	(1.3)
n	is arbitrary,	r = n-1	(1.4)
n = 3 or 7,		r = 2	(1.5)
n = 4 or 8,		r = 3	(1.6)

Eckmann and Whitehead proved theorem (1.1) using algebraic topology. Brown and Gray [1] considered multilinear vector cross products on vector spaces over arbitrary fields of characteristic not two with arbitrary non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$. Brown and Gray proved that vector cross products exist only in the cases listed in theorem (1.1), together with many more bilinear forms besides the positive definite one. In contrast to Eckmann's and Whitehead's method their technique is completely algebraic.

Two vector cross products P, P' with respect to the same bilinear form are said to be isomorphic if and only if there exists a linear map

$$\varphi : V \longrightarrow V \quad \text{s.t.}$$

$$\langle \varphi a, \varphi b \rangle = \langle a,b \rangle \text{ for all } a,b \in V \quad (1.7)$$

$$\varphi P(a_1,\ldots,a_r) = P'(\varphi a_1,\ldots,\varphi a_r) \quad (1.8)$$

The automorphism group of a given vector cross product P is the set of all linear maps

$$\varphi : V \longrightarrow V \quad \text{s.t.}$$

$$\langle \varphi a, \varphi b \rangle = \langle a,b \rangle \quad \text{for all } a,b \in V \quad (1.9)$$

$$\varphi P(a_1,\ldots,a_r) = P(\varphi a_1,\ldots,\varphi a_r) \quad (1.10)$$

We note that a vector cross product is a skew-symmetric tensor. This follows from linearization of (1.1) and the fact that $\langle\ ,\ \rangle$ is nondegenerate.

In the following sections we give an account of the classification of multilinear vector cross products. We omit most of the proofs since they are easily accessible in Brown and Gray [1].

2. Almost complex structures.

In this section we consider the case $r = 1$.

DEFINITION: An almost complex structure on V is a linear map

$$J : V \longrightarrow V \quad \text{s.t.} \quad J^2 = -1_V.$$

If moreover V is equipped with a bilinear symmetric form and $\langle Ja, Jb\rangle = \langle a,b\rangle$ for all $a,b \in V$ we say J is an almost hermitian structure.

PROPOSITION (2.1): A one-fold vector cross product on V is an almost hermitian structure. The converse is also true.

Proof: For a one-fold vector cross product P, linearization of (1.1) gives for all $a,b \in V$

$$\langle Pa,b\rangle + \langle a,Pb\rangle = 0 \qquad (2.1)$$

and $\langle Pa, Pb\rangle = \langle a,b\rangle$ follows from (1.2)

Hence for all $a,b \in V$, we have

$$\langle P^2a + a,b\rangle = \langle P^2a,b\rangle + \langle a,b\rangle$$

$$= -\langle Pa,Pb\rangle + \langle a,b\rangle = 0 \qquad (2.2)$$

Since $\langle\ ,\ \rangle$ is non-degenerate, (2.2) implies that for each $a \in V$,

$$P^2a + a = 0, \quad \text{or} \quad P^2 = -1_V.$$

Conversely let $J: V \to V$ be an almost hermitian structure. Then (1.2) is satisfied and furthermore,

$$\langle Ja,a\rangle = \langle J^2a,Ja\rangle = -\langle a,Ja\rangle \text{ i.e. } \langle Ja,a\rangle = 0$$

for all $a \in V$. Hence (1.1) is also satisfied.

Theorem (2.2): (i) A non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ admits a 1-fold vector cross product P, if and only if the quadratic form of $\langle\ ,\ \rangle$ has the form

$$\langle x,x\rangle = \alpha_1 (x_1^2 + x_2^2) + \dots + \alpha_m(x_{2m-1}^2 + x_{2m}^2) \qquad (2.3).$$

where $\alpha_1,\dots,\alpha_m \in F$, the ground field.

(ii) Any two one-fold vector cross products P,P' are isomorphic, when F is algebraically closed or F = R.

Proof: That V is of even dimension follows from the fact that V has an orthogonal basis of the form

$$\{e_1, Pe_1, e_2, Pe_2, \ldots, e_m, Pe_m\}$$

and with respect to such a basis the quadratic form of $\langle\ ,\ \rangle$ has the form (2.3) where $\alpha_j = \langle e_j, e_j\rangle$, $j = 1,..,m$. ($2m = n = \dim V$).

Conversely let $\{e_1, f_1, \ldots, e_m, f_m\}$ be an orthogonal basis of V, where $\langle e_i, e_i\rangle = \langle f_i, f_i\rangle = \alpha_i$, $i = 1,\ldots,m$.

Define P by $Pe_i = f_i$, $Pf_i = -e_i$. Then P is a one-fold vector cross product.

For (ii) we choose orthogonal bases of V of the form

$\{e_1, Pe_1, \ldots, e_m, Pe_m\}$; $\{f_1, P'f_1, \ldots, f_m, P'f_m\}$, where $\|e_i\|^2 = \|f_i\|^2 = \pm 1$, $1 \leq i \leq m$.

Define $\varphi: V \longrightarrow V$ by

$$\varphi e_i = f_i \quad ; \quad \varphi(Pe_i) = P'f_i \qquad i = 1,\ldots,m.$$

φ so defined satisfies (1.7) and (1.8).

3. Star operators.

Let V be a finite dimensional vector space with a non-degenerate symmetric bilinear form. Denote by ΛV the exterior algebra over V, $\Lambda V = \sum_{p=0}^{n} \Lambda^p V$, where $\Lambda^p V$ are the elements of order P. We extend $\langle , \rangle$ on ΛV by linearity and the formula

$$\langle e_1 \wedge \cdots \wedge e_p, f_1 \wedge \cdots \wedge f_q\rangle = \begin{cases} \det(\langle e_i, f_j\rangle) & \text{if } p = q. \\ 0 & \text{if } p \neq q. \end{cases}$$

$e_1, \ldots, e_p, f_1 \ldots, f_q \in V.$ (3.1)

This extension is symmetric and non-degenerate. $\bigwedge^n V$ is a 1-dimensional linear space for which we fix a basis element $w \in \bigwedge^n V$. For each $a \in \bigwedge^p V$, $1 \leq p < n$, we define a linear map $\mu_a : \bigwedge^{n-p} V \longrightarrow \bigwedge^n V$, by

$$\mu_a(b) = a \wedge b \quad \text{for each } b \in \bigwedge^{n-p} V.$$

Also we define a linear functional

$$\lambda : \bigwedge^n V \longrightarrow F, \text{ by}$$

$$\lambda(\xi) = \langle \xi, w \rangle \quad \text{for each } \xi \in \bigwedge^n V .$$

Since $\lambda \circ \mu_a : \bigwedge^{n-p} V \longrightarrow F$ is linear and $\langle \, , \, \rangle$ is non-degenerate on $\bigwedge^{n-p} V$, which we call $*a$ s.t.

$$\lambda \circ \mu_a(b) = \langle *a, b \rangle \text{ for all } b \in \bigwedge^{n-p} V \quad \text{or}$$

equivalently $\langle a \wedge b, w \rangle = \langle *a, b \rangle$ for all $b \in \bigwedge^{n-p} V$ (3.2).

The map $a \longrightarrow *a$ defines the star operator on $\bigwedge^p V \longrightarrow \bigwedge^{n-p} V$. Finally $*$ is defined on all of $\bigwedge V$ by linearity.

Next, let $\{e_1, \ldots, e_n\}$ be an orthogonal basis of V and write, $w = e_1 \wedge \cdots \wedge e_n$ then $\| w \|^2 = \prod_{i=1}^{n} \langle e_i, e_i \rangle = \alpha_1 \cdots \alpha_n = \mu$ where $0 \neq \alpha_i = \langle e_i, e_i \rangle$ is an element in the ground field F. Then for each basis element η of $\bigwedge^n V$, writing $\eta = \beta w$ for some $\beta \in F$ we get

$$\| \eta \|^2 = \beta^2 \| w \|^2 = \beta^2 \mu .$$

We say that $\langle \, , \, \rangle$ has discriminant 1, if there exists $w \in \bigwedge^n V$, s.t. $\| w \|^2 = 1$.

From the above it follows that $\langle\ ,\ \rangle$ has discriminant 1 if and only if for each basis element $\eta \in \Lambda^n V$, we have

$$\|\eta\|^2 = \lambda^2 \quad \text{for some } \lambda \in F.$$

PROPOSITION (3.1): Let $a \in \Lambda^p V$; then

$$**a = (-1)^{p(n-p)} \langle w, w\rangle a. \qquad (3.3)$$

$$\langle *a, *b\rangle = \langle w, w\rangle \langle a, b\rangle \qquad (3.4).$$

THEOREM (3.2): (i) A necessary and sufficient condition that $\langle\ ,\ \rangle$ possess an (n-1)-fold vector cross product is that the discriminant of $\langle\ ,\ \rangle$ be 1.

(ii) If the discriminant of $\langle\ ,\ \rangle$ is 1, then any (n-1)-fold vector cross product is given as follows: there exists $w \in \Lambda^n V$ with $\langle w, w\rangle = 1$ s.t.

$$P(a_1, \ldots, a_{n-1}) = *(a_1 \wedge \ldots \wedge a_{n-1}) \qquad (3.5)$$

for all $a_1, \ldots, a_{n-1} \in V$.

(iii) There are exactly two distinct (n-1)-fold vector cross products on V and they are isomorphic to each other.

4. Two and Three-fold Vector Cross products.

Two-fold and three-fold vector cross products are closely related to composition algebras, and so we present a few facts about them. Proofs and a detailed account are accessible in JACOBSON [9].

DEFINITION. A composition algebra W is an algebra equipped with a quadratic form N such that

(i) The bilinear form $\langle x, y\rangle = \frac{1}{2}\left[N(x+y)-N(x)-N(y)\right]$ is non-degenerate;

(ii) There is $e \in W$ such that $ex = xe = x$ for all $x \in W$.

(iii) $N(xy) = N(x)N(y)$ for all $x, y \in W$.

We recall the definition: $x \longrightarrow N(x)$ is a mapping of W into the base field F satisfying $N(\alpha x) = \alpha^2 N(x)$ for $\alpha \in F$ and having the property that the mapping defined by $N(x+y) - N(x) - N(y)$ is bilinear.

Since N is non-degenerate that is $N(x) \not\equiv 0$, (iii) implies that $N(e) = 1$. Hence the subspace Fe is not isotropic, that is, $Fe \cap (Fe)^{\perp} = 0$, and therefore we can write $W = Fe \oplus (Fe)^{\perp}$. Denoting $(Fe)^{\perp}$ by W_0, we have

$W = Fe \oplus W_0$. Next we define a mapping $a \longrightarrow \bar{a}$ from W into W as follows: for each $a \in W$ write $a = \alpha e + X$, $x \in W_0$ and set $\bar{a} = \alpha e - x$. One can easily check that $a \longrightarrow \bar{a}$ is an involution, that is, an antiautomorphism of period two. From the set of axioms defining a composition algebra we can easily deduce the following properties:

$$x^2 y = x(xy), \qquad yx^2 = (yx)x \tag{4.1}$$

(the alternative laws)

$$\bar{\bar{x}} = x, \quad \overline{yx} = \bar{x}\,\bar{y}, \quad x\bar{x} = \bar{x}x = N(x)e,$$

$$2\langle x, y\rangle e = x\bar{y} + y\bar{x} \tag{4.2}$$

$$\langle wy, zx\rangle + \langle wx, zy\rangle = 2\langle w, z\rangle\langle x, y\rangle \qquad (4.3)$$

$$\langle xy, xz\rangle = N(x)\langle y,z\rangle, \quad \langle xy, zy\rangle = \langle x,z\rangle N(y) \qquad (4.4)$$

$$\langle \bar{x},\bar{y}\rangle = \langle x,y\rangle, \quad \langle ax,y\rangle = \langle x,\bar{a}y\rangle, \quad \langle xa,y\rangle = \langle x,y\bar{a}\rangle \qquad (4.5)$$

$$x + \bar{x} = T(x)e, \quad T(x) \text{ in } F. \qquad (4.6)$$

We call $T(x)$ the trace of x.

Conversely if W is an alternative algebra with an identity and an involution $x \longrightarrow \bar{x}$ s.t. $x\bar{x} = N(x)e$ and $x + \bar{x} = T(x)e$, then $N(x)$ is a quadratic form permitting composition.

JACOBSON [9] proved that the only composition algebras are those of dimension; dim. $W = 1, 2, 4$ or 8, and W is not associative if dim. $W = 8$. Also the only quadratic forms admitting composition are those of signature $(o,8)$ or $(4,4)$ when dimension $W = 8$.

<u>THEOREM</u> (4.1): Two composition algebras are isomorphic if and only if their quadratic forms are equivalent. [9]

The following theorems are due to BROWN and GRAY [1].

<u>THEOREM</u> (4.2): (i) Let W be a composition algebra and let $V \subset W$ be the orthogonal complement of the identity e.

Define $P: V\text{x}V \longrightarrow V$ by

$$P(a, b) = ab + \langle a,b\rangle e \qquad (4.7)$$

Then P is a two-fold vector cross product on V.

(ii) Conversely if P is a two-fold vector cross product on V, let W be a vector space containing V as a subspace of co-dimension 1. Let $W = V \oplus \{e\}$ for some $e \in W \setminus V$ and extend $\langle\ ,\ \rangle$ to W by requiring $\langle e,e \rangle = 1$ and $\langle e,v \rangle = 0$. Define a multiplication in W by (4.7) and set $ex = xe = x$ for $x \in W$. Finally let $N(x) = \langle x, x \rangle$ for $x \in W$. Then W is a composition algebra and the dimension of the original space V is either 3 or 7.

(iii) Two two-fold vector cross products are isomorphic if and only if their corresponding composition algebras are isomorphic. The converse is also true.

<u>THEOREM</u> (4.3): Let V be a composition algebra with bilinear form $\langle\ ,\ \rangle_1$ and let $\alpha \neq 0$ be a field element. Then

$$P(a,b,c) = \alpha\left(-a(\bar{b}c) + \langle a,b \rangle_1 c + \langle b,c \rangle_1 a - \langle c,a \rangle_1 b\right) \quad (4.8)$$

and

$$P(a,b,c) = \alpha\left(-(a\bar{b}) c + \langle a,b \rangle_1 c + \langle b,c \rangle_1 a - \langle c,a \rangle_1 b\right) \quad (4.9)$$

are both three-fold vector cross products with respect to the bilinear form $\langle\ ,\ \rangle = \alpha \langle\ ,\ \rangle_1$. Also each three-fold vector cross product is one of these kinds.

In the case of 8-dimensional spaces the vector products defined by (4.8) and (4.9) are not isomorphic. Further BROWN and GRAY proved that there are no other vector cross products besides those described in the previous theorems.

To sum up we have the following table when $F = R$, the real numbers.

n and r	Number of isomorphism classes	Possible signatures
n even, r = 1	$n/2 + 1$	$n, n-4, \ldots, -n+4, -n$
r = n - 1	$[n/2] + 1$	$n, n-4, \ldots, n-2[n]$
n=7, r=2	2	7, -1
n=8, r=3	6	8, 0, -8

CHAPTER 2.

Vector Cross Products on Manifolds

In this Chapter we deal with multilinear vector products on differentiable manifolds, giving a detailed account of a paper by GRAY [6]. We give detailed proofs of those results for which a proof was not given or just outlined.

1. Existence of vector products on Manifolds.

The question arises as to which differentiable manifolds have on each tangent space a vector product which is continuous or differentiable as a tensor field on the manifold.

PROPOSITION (1.1) Let M be a pseudo-Riemannian manifold with signature (s,t). Then M admits a vector product structure P if and only if the structure group $O(s,t)$ of the frame bundle of M can be reduced to the automorphism group of P. The vector product is continuous or differentiable just as the reduction is continuous or differentiable.

Proof: First assume that the group $O(s,t)$ of the frame bundle of M admits a reduction to G, the automorphism group of P. Let Q be the reduced sub-bundle, $\pi : Q \longrightarrow M$ the natural projection.

Then each $u \in \pi^{-1}(x)$; $x \in M$, can be considered as a linear isometry $u: E_{s,t} \longrightarrow T_x(M)$, where $E_{s,t}$ is $R^{(s+t)}$, with a non-degenerate symmetric bilinear form $(,)$ of signature (s,t), $T_x(M)$ the tangent space to M at x. We denote the inverse of the above map by u^{-1}.

Given $X_1, \ldots, X_r \in T_x(M)$, we choose $u \in \pi^{-1}(x)$, and define $P(X_1, \ldots, X_r)$ as follows:

$$P(X_1, \ldots, X_r) = u\, P(u^{-1}X_1, \ldots, u^{-1}X_r) \qquad (1.1)$$

where P on the right hand side of (1.1) is the corresponding vector cross product on $E_{s,t}$. P, so defined, is independant of the choice of u, since if $u, \bar{u} \in \pi^{-1}(x)$, then there exists $a \in G$ s.t. $u = \bar{u}\, a$, and

$$u\, P(u^{-1}x_1, \ldots, u^{-1}x_r) = (\bar{u}a)\, P\left((\bar{u}a)^{-1} x_1, \ldots, (\bar{u}a)^{-1}x_r\right)$$

$$= \bar{u}\, a\, a^{-1}\, P(\bar{u}^{-1} X_1, \ldots, \bar{u}^{-1} X_r) = \bar{u}\, P(\bar{u}^{-1}X_1, \ldots, \bar{u}^{-1}X_r); \quad (1.2).$$

Since $P(a\xi_1, \ldots, a\xi_r) = a\, P(\xi_1, \ldots, \xi_r)$ for all $a \in G$, $\xi_1, \ldots, \xi_r \in R^{s+t}$.

Next we have,

$$\langle P(X_1, \ldots, X_r), X_j \rangle = \left(P(u^{-1}X_1, \ldots, u^{-1}X_r), u^{-1} X_j\right) = 0 \quad (1.3)$$

$$1 \leqslant j \leqslant r.$$

and

$$\langle P(X_1, \ldots, X_r), P(X_1, \ldots, X_r) \rangle = \left(P(u^{-1}x_1, \ldots, u^{-1}x_r), P(u^{-1}x_1, \ldots, u^{-1}x_r) \right)$$

$$= \det\left((u^{-1}x_i, u^{-1}x_i) \right) = \det\left(\langle x_i, x_j \rangle \right) \qquad (1.4)$$

Hence P is a vector cross product on M.

Conversely suppose that M has a vector cross product P. Let F be the bundle of orthonormal frames on M, and Q the set of all $u \in F$ such that

for all $\xi_1, \ldots, \xi_k \in R^n$, $(n = \dim M)$, we have

$$\bar{P}(\xi_1, \ldots, \xi_k) = u^{-1} P(u\xi_1, \ldots, u\xi_r) \qquad (1.5)$$

Then Q forms a sub-bundle with structure group G (the automorphism group of P).

<u>PROPOSITION</u> (1.2). A vector cross product on a differentiable manifold M gives rise to an orientation on M.

<u>Proof</u>: Since the automorphism group of a vector cross product is contained in the special orthogonal group of the associated bilinear form, then by proposition (1.1) the bundle of frames on M can be reduced to a bundle of oriented frames. This is equivalent to the orientability of M.

Thus the existence of an (n-1)-fold vector cross product globally on M is equivalent to the orientability of M. The existence of a two-fold vector cross product on a 7-dimensional manifold is equivalent to a reduction of O(7) or O(4,3) to G_2. Similarly the existence of a three-fold vector cross product on an 8-dimensional manifold is equivalent to a reduction of O(8) or O(4,4) to Spin (7) or Spin (4,3) principal bundle. Finally the existence of a one-fold vector cross product is equivalent to the reduction of O(2p, 2q) to a U(p,q) principal bundle. We note that on a paracompact almost complex manifold (M,J) we can construct a metric with respect to which J is a one-fold vector cross product. For, let g be a metric on M, and define $\bar{g}$ by

$$\bar{g}(X,Y) = g(X,Y) + g(JX,JY) \qquad (1.6)$$

for all $X, Y \in T(M)$.

Then J is a one-fold vector product with respect to $\bar{g}$.

Another structure that may exist on an 8-dimensional manifold is a cayley multiplication.

<u>DEFINITION</u>: Let M be an 8-dimensional pseudo-riemannian manifold whose metric $\langle \, , \rangle$ is positive definite or has signature $(4,4)$. A Cayley multiplication on M is a differentiable $(1,2)$ tensor field which for each $x \in M$ makes $T_x(M)$ an 8-dimensional composition algebra whose associated quadratic form is that induced by $\langle \, , \rangle$.

The identity vector field E, is that vector field which at each point $x \in M$ is the identity of the composition algebra at x. Finally conjugation is the map

$\sigma : \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ ($\mathcal{X}(M)$ denotes vector fields on M) ~~fields~~ defined by

$$\sigma(X) = 2\langle X, E\rangle E - X \qquad (1.7) \quad \text{for } X \in \mathcal{X}(M)$$

<u>THEOREM</u> (1.3). Let M be an 8-dimensional pseudo-riemannian manifold whose metric is either positive definite

or has signature (4,4). Consider the following conditions:

(A) M has a globally defined vector field E, with $\|E\|^2 = 1$ everywhere.

(B) M has one-kind of three-fold vector cross product.

(C) M has both kinds of three-fold vector cross products.

(D) M has a cayley multiplication.

Then $\big((A) \text{ and } (B)\big) \Longrightarrow \big((C) \text{ and } (D)\big)$,

$$(D) \Longrightarrow (C) \Longrightarrow (B) \quad \text{and} \quad (D) \Longrightarrow (A)$$

Proof: First assume (A) and (B), and let P_1 be a 3-fold vector cross product on M. Define $\sigma : \mathcal{H}(M) \longrightarrow \mathcal{H}(M)$ by $\sigma(X) = 2\langle X, E\rangle - X$ for $X \in \mathcal{H}(M)$.

We define P_2 by :

$$P_2(X,Y,Z) = \sigma P_1(\sigma X, \sigma Y, \sigma Z)$$

that P_2 is a vector cross product follows from the fact that:

$$\langle \sigma X, \sigma Y\rangle = \langle X, Y\rangle \quad \text{for all } X, Y \in \mathcal{H}(M)$$

and $\sigma^2 = 1$. Also P_1, P_2 are of different kind follows from equations (4.8) and (4.9) of Chapter 1.

For $A, B \in \mathcal{H}(M)$, we define

$$A \circ B = -P_1(A,E,B) + \langle A,E\rangle B + \langle B,E\rangle A - \langle A,B\rangle E.$$

Then the multiplication $\circ$, is bilinear, admits composition and $A.E = E.A = A$ for all $A \in \mathcal{H}(M)$. Thus M has a cayley multiplication. This proves the first implication.

It is obvious that (C) $\Longrightarrow$ (B), and (D) $\Longrightarrow$ (C) follows from equations (4.8) and (4.9) of Chapter 1.

To prove that (D) $\Longrightarrow$ (A) we take E to be the identity vector field of the cayley multiplication.

Next we consider induced vector cross products on a certain class of submanifolds.

<u>THEOREM</u> (1.4). Let $\bar{P}$ be an r-fold vector cross product on a manifold $\bar{M}$ with metric tensor $\langle\ ,\ \rangle$. Let M be an oriented submanifold of $\bar{M}$ such that the restriction of the metric tensor $\langle\ ,\ \rangle$ to M and to the normal bundle of M are non-degenerate and positive definite respectively. If $r > k$, where k is the co-dimension of M in $\bar{M}$, then $\bar{P}$ induces an $(r-k)$-fold vector cross product P on M in a natural way.

<u>Proof</u>: Define P by the following formula:

$$P(A_1, \ldots, A_{r-k}) = \bar{P}(N_1, \ldots, N_k, A_1, \ldots, A_{r-k})$$

for $A_1, \ldots, A_{r-k} \in \mathcal{H}(M)$.

Here $N_1, \ldots, N_k$ are orthonormal vector fields orthogonal to M, defined on an open subset of M and $N_1 \wedge \ldots \wedge N_k$ is compatible with the orientations of M and $\bar{M}$. P is independant of the choice of $N_1, \ldots, N_k$ having the properties mentioned above. For, if we choose $\bar{N}_1, \ldots, \bar{N}_k$, s.t.

$$\|\bar{N}_i\|^2 = 1 \;, \quad \langle \bar{N}_i, \bar{N}_j \rangle = \delta_{ij} \;, \quad \text{then}$$

$$\bar{N}_1 \wedge \cdots \wedge \bar{N}_k = \alpha \, N_1 \wedge \cdots \wedge N_k$$

for some real valued function α, on an open subset of M. Hence

$$1 = \|\bar{N}_1 \wedge \cdots \wedge \bar{N}_k\|^2 = \alpha^2 \|N_1 \wedge \cdots \wedge N_k\|^2 = \alpha^2 .$$

We deduce $\alpha = \pm 1$. Since $\bar{N}_1 \wedge \cdots \wedge \bar{N}_k$ is of the same orientation as $N_1 \wedge \cdots \wedge N_k$ then $\alpha = 1$, and we have

$$\bar{P}(\bar{N}_1, \ldots, \bar{N}_k, A_1, \ldots, A_{r-k}) = \bar{P}(\bar{N}_1 \wedge \cdots \wedge \bar{N}_k, A_1 \ldots, A_{r-k})$$

$$= \bar{P}(N_1 \wedge \cdots \wedge N_k, A_1, \ldots, A_{r-k}) = \bar{P}(N_1, \ldots, N_k, A_1, \ldots, A_{r-k})$$

Moreover,

$$\langle P(A_1, \ldots, A_{r-k}), A_j \rangle = \langle \bar{P}(N_1, \ldots, N_k, A_1 \ldots, A_{r-k}), A_j \rangle$$

$$= 0 \quad 1 \leqslant j \leqslant r-k ,$$

and

$$\langle P(A_1, \ldots, A_{r-k}), P(A_1, \ldots, A_{r-k}) \rangle$$

$$= \langle \bar{P}(N_1, \ldots, N_k, A_1, \ldots, A_{r-k}), \bar{P}(N_1, \ldots, N_k, A_1, \ldots, A_{r-k}) \rangle$$

$$= \det(\langle A_i, A_j \rangle) \;, \quad 1 \leqslant i, j \leqslant r-k.$$

Hence P is a vector cross product on M.

The question arises as to which spheres have v@ctor cross products. Since all spheres are orientable they have (n-1)-fold vector cross products. Also because S^7 is parallelizable it has a 2-fold vector cross product. For 1-fold and 3-fold vector cross products we state the following theorem.

<u>THEOREM</u> (1.5) Let S^n denote the unit sphere in R^{n+1} and let $\langle\ ,\ \rangle$ be the metric tensor on S^n induced from the usual positive definite one on R^{n+1}. IF S^n has a globally defined r-fold vector cross product then R^{n+1} has and (r+1)-fold continuous vector cross product in the vector space sense.

<u>Proof</u>: Let X_m denote the r-fold vector cross product on S^n at m. Define

$$P: (R^{n+1})^{r+1} \longrightarrow R^{n+1} \quad \text{as follows:}$$

For $a_1, \ldots, a_{r+1} \in R^{n+1}$, write

$$a_{r+1} = b + c \quad \text{where } b \text{ is the component of}$$

a_{r+1} orthogonal to $[a_1, \ldots, a_r]$. If $b = 0$, set

$$P(a_1, \ldots, a_{r+1}) = 0 .$$

If $b \neq o$, let $d = \| b \|^{-1} b$ and set

$$P(a_1, \ldots, a_{r+1}) = \| b \| X_d (a_1, \ldots, a_r)$$

P, so defined is linear in $a_1, \ldots, a_r$ but in general only continuous in a_{r+1}. P also satisfies the two axioms of a vector cross product.

COROLLARY (1.6) The only spheres with almost complex structures are S^2 and S^6.

Proof: If S^n has an almost complex structure, then R^{n+1} has a 2-fold vector cross product. Hence $n+1 = 3$ or 7, that is, $n = 2$ or 6.

COROLLARY (1.7) S^8 does not admit a 3-fold vector cross product.

Proof: If S^8 had a 3-fold vector cross product, then R^9 would have a continuous 4-fold vector cross product, which is impossible.

2. Differentiable Vector Cross Products.

Let M be a pseudo-riemannian manifold and let $\mathcal{H}(M)$ denote the lie algebra of differentiable vector fields on M. We denote by ∇ the riemannian connection of M. An r-fold differentiable (i.e. C^{∞}) vector cross product P on M is a tensor field on M of type $(1,r)$. With each such vector cross product P we associate a differential $(r+1)$-form φ defined by the following formula:

$$\varphi(X_1,\ldots,X_{r+1}) = \langle P(X_1,\ldots,X_r), X_{r+1}\rangle , \qquad (2.1)$$

for $X_1,\ldots,X_{r+1} \in \mathcal{H}(M)$.

<u>PROPOSITION</u> (2.1) Let $X, X_1,\ldots, X_{r+1} \in \mathcal{H}(M)$, then

$$\nabla_X(\varphi)(X_1,\ldots, X_{r+1}) = \langle \nabla_X(P)(X_1,\ldots,X_r), X_{r+1}\rangle \qquad (2.2)$$

<u>Proof</u>: $\nabla_X(\varphi)(X_1,\ldots,X_{r+1}) =$

$$X\,\varphi(X_1,\ldots,X_{r+1}) - \sum_{j=1}^{r+1} \varphi(X_1,\ldots, \nabla_X X_j,\ldots, X_{r+1})$$

$$= X\langle P(X_1,\ldots,X_r), X_{r+1}\rangle - \langle \sum_{j=1}^{r} P(X_1,\ldots, \nabla_X X_j,\ldots,X_r), X_{r+1}\rangle$$

$$- \langle P(X_1,\ldots,X_r), \nabla_X X_{r+1}\rangle .$$

For the first term on the R.H.S we have

$$X\langle P(X_1,\ldots,X_r),\ X_{r+1}\rangle = \langle \nabla_X P(X_1,\ldots,X_r),\ X_{r+1}\rangle$$

$$+ \langle P(X_1,\ldots,X_r),\ \nabla_X X_{r+1}\rangle$$

$$= \langle \nabla_X(P)\ (X_1,\ldots,X_r),\ X_{r+1}\rangle + \langle \sum_{j=1}^{r} P(X_1,\ldots,\nabla_X X_j,\ldots,X_r),\ X_{r+1}\rangle$$

$$+ \langle P(X_1,\ldots,X_r),\ \nabla_X X_{r+1}\rangle .$$

Hence we conclude that

$$\nabla_X(\varphi)\ (X_1,\ldots,X_{r+1}) = \langle \nabla_X(P)\ (X_1,\ldots,X_r),\ X_{r+1}\rangle .$$

A vector cross product P is said to be parallel if for each $X \in \mathcal{H}(M)$ we have $\nabla_X P = 0$. As to which manifolds admit parallel vector cross products we have:

<u>THEOREM</u> (2.2) A pseudo-riemannian manifold M has a parallel vector cross product P if and only if the holonomy group at each point $m \in M$ is a subgroup of the automorphism group of P.

<u>Proof</u>: The holonomy group H_m at $m \in M$ is the group of non-singular maps

$\tau_\gamma : M_m \longrightarrow M_m$ which are parallel translations

along piecewise differentiable closed curves γ starting at m.

The condition that $\nabla_x(P) = 0$ for all $X \in M_m$ and all $m \in M$ is equivalent to the condition that for each $m \in M$, γ a closed curve starting at m we have

$$\tau_\gamma P(X_1,\ldots,X_r) = P(\tau_\gamma X_1,\ldots,\tau_\gamma X_r), X_1,\ldots,X_r \in Mm .$$

Hence if P is parallel then τ_γ belongs to the automorphism group of P and conversely.

If a reimannian manifold has a parallel 2-fold or 3-fold vector cross product then the Holonomy group is contained in G_2 or Spin (7). BONAN, [2], has investigated compact riemannian manifolds with holonomy groups G_2 or Spin (7). He found that in the former case such manifolds have nonzero Betti numbers b_4 and b_3 and in the latter case such manifolds have non-zero Betti number b_4. Also it truns out , [2], that the Ricci curvature of a riemannian manifold with holonomy group G_2 or Spin (7) vanishes.

Now we look at a class of vector cross products which satisfy weaker conditions than that of being parallel.

<u>DEFINITIONS</u>: Let M be a pseudo-riemannian manifold with metric $\langle\, ,\rangle$, P an r-fold vector cross product on M with associated metric $\langle\, ,\rangle$. Let ∇ and δ denote the riemannian connection and coderivative, respectively, of M relative to $\langle\, ,\rangle$ and let φ denote the $(r+1)$-form determined by P.

(i) P is nearly parallel if $\nabla_{X_1}(P)(X_1,\ldots,X_r) = 0$

for all $X_1,\ldots,X_r \in \mathcal{X}(M)$.

(ii) P is almost parallel if $d\varphi = 0$

(iii) P is semi-parallel if $\delta\varphi = 0$

From now on we denote by $\mathcal{P}(s,t,r)$, $\mathcal{NP}(s,t,r)$, $\mathcal{AP}(s,t,r)$ and $\mathcal{SP}(s,t,r)$ the classes of r-fold vector cross products on pseudo-riemannian manifolds with signature (s,t) which are parallel, nearly parallel, almost parallel and semi-parallel respectively.

We have the following inclusion relations between the various classes.

<u>THEOREM</u> (2.3): We have the following inclusions:

(i) $\mathcal{P} \subseteq \mathcal{NP} \subseteq \mathcal{SP}$;

(ii) $\mathcal{P} \subseteq \mathcal{AP}$;

(iii) $\mathcal{P} = \mathcal{AP} \cap \mathcal{NP}$.

<u>Proof</u>: The inclusion $\mathcal{P} \subseteq \mathcal{NP}$ is obvious from the definition. On the other hand we have

$$\delta\varphi(X_1,\ldots,X_r) = -\sum_{i=1}^{n} \| E_i \|^{-2} \nabla_{E_i}(\varphi)(E_iX_1,\ldots,X_r) \tag{2.3}$$

$$d\varphi(X_1,\ldots,X_{r+2}) = \frac{1}{r+2}\sum_{i=1}^{r+2} (-1)^{i+1} \nabla_{X_i}(\varphi)(X_1,\ldots,\hat{X}_i,\ldots,X_{r+2}) \tag{2.4}$$

(for a torsion free connection)

where $X_1, \ldots, X_{r+2} \in \mathcal{H}(M)$, $\{E_1, \ldots, E_n\}$ is an orthogonal frame field with $\|E_i\|^2 = \pm 1$ and $\hat{X}_i$ means X_i removed.

It follows from (2.3) that if P is nearly parallel then $\delta\varphi = 0$ and hence P is semi-parallel. Also from (2.4) if P is parallel then $d\varphi = 0$ and hence P is almost parallel. Further if P is nearly parallel then

$$\nabla_{X_i}(\varphi)(X_1, \ldots, \hat{X}_i, \ldots, X_{r+2}) = (-1)^{i-1} \nabla_{X_1}(\varphi)(X_2, \ldots, X_{r+2}),$$

and hence in this case

$$d\varphi(X_1, \ldots, X_{r+2}) = \frac{(r+2)}{r+2} \nabla_{X_1}(\varphi)(X_2, \ldots, X_{r+2})$$

$$= \nabla_{X_1}(\varphi)(X_2, \ldots, X_{r+2})$$

If in addition P is almost parallel, then

$$\nabla_{X_1}(\varphi)(X_2, \ldots, X_{r+2}) = 0 \quad \text{for all}$$

$X_1, \ldots, X_{r+2} \in \mathcal{H}(M)$, and hence P is parallel.

In the case of 1-fold vector cross products, that is the class of all almost complex structures with $\langle JX, JY \rangle = \langle X, Y \rangle$ for $X, Y \in \mathcal{H}(M)$, there are two additional definitions which do not generalize to general vector cross products.

DEFINITIONS

(i) J is hermitian if $S = 0$, where S is the torsion tensor of J; a (1,2) tensor field given by the following formula:

$$S(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY],$$

for $X,Y \in \mathcal{X}(M)$.

(ii) J is quasi-kahler if $\nabla_X(J)(Y) + \nabla_{JX}(J)(JY) = 0$, for $X, Y \in \mathcal{X}(M)$.

Also for almost complex structures the term kahler is used instead of parallel.

Let $\mathcal{AH}$ denote the class of almost complex structures J with $\langle JX,JY \rangle = \langle X,Y \rangle$ for $X,Y \in \mathcal{X}(M)$.

For this class we have the following inclusion relations:

THEOREM (2.4)

(i) $\mathcal{NK} \subseteq \mathcal{QK} \subseteq \mathcal{SK}$;

(ii) $\mathcal{AK} \subseteq \mathcal{QK} \subseteq \mathcal{SK}$;

(iii) $\mathcal{K} \subseteq \mathcal{H}$ (hermitian);

(iv) $\mathcal{K} = \mathcal{H} \cap \mathcal{QK} = \mathcal{AK} \cap \mathcal{NK}$;

(v) $\mathcal{K}(0,2) = \mathcal{AH}(0,2)$;

(vi) If s,t are even and $s+t = 4$, then

$$\mathcal{K}(s,t) = \mathcal{NK}(s,t) \text{ and } \mathcal{AK}(s,t) = \mathcal{SK}(s,t).$$

We prove the following proposition first:

PROPOSITION (2.5): Let $X, X_1, \ldots, X_r \in \mathcal{H}(M)$. Then

$$\langle \nabla_X(P)(X_1,\ldots,X_r), \; P(X_1,\ldots,X_r) \rangle = 0$$

Proof: We have $\langle P(X_1,\ldots,X_r), P(X_1,\ldots,X_r) \rangle$

$= \langle X_1 \wedge \cdots \wedge X_r, \; X_1 \wedge \cdots \wedge X_r \rangle$ for all $X_1,\ldots,X_r \in \mathcal{H}(M)$.

Hence

$$X \langle P(X_1,\ldots,X_r), \; P(X_1,\ldots,X_r) \rangle$$

$$= X \langle X_1 \wedge \cdots \wedge X_r, \; X_1 \wedge \cdots \wedge X_r \rangle = 2 \sum_{j=1}^{r} \langle X_1 \wedge \cdots \wedge \nabla_X X_j \wedge \cdots \wedge X_r, \; X_1 \wedge \cdots \wedge X_r \rangle .$$

i.e. $\langle \nabla_X(P)(X_1,\ldots,X_r), \; P(X_1,\ldots,X_r) \rangle$

$$+ \sum_{j=1}^{r} \langle P(X_1,\ldots, \nabla_X X_j,\ldots,X_r), \; P(X_1,\ldots,X_r) \rangle$$

$$= \sum_{j=1}^{r} \langle X_1 \wedge \cdots \wedge \nabla_X X_j \wedge \cdots \wedge X_r, \; X_1 \wedge \cdots \wedge X_r \rangle ,$$

and since $\langle P(X_1,\ldots,\nabla_X X_j,\ldots,X_r), \; P(X_1,\ldots,X_r) \rangle$

$$= \langle X_1 \wedge \cdots \wedge \nabla_X X_j \wedge \cdots \wedge X_r, \; X_1 \wedge \cdots \wedge X_r \rangle ,$$

we have

$$\langle \nabla_X(P)(X_1,\ldots,X_r), \; P(X_1,\ldots,Xr) \rangle = 0.$$

Proof of theorem (2.4)

For an almost complex structure J- with $\langle JX, JY \rangle = \langle X, Y \rangle$, for $X, Y \in \mathcal{H}(M)$ - $\nabla_X(F)(JY,Z) = \nabla_X(F)(Y,JZ)$, (2.5)

and hence $\nabla_X(F)(JY,JZ) = -\nabla_X(F)(Y,Z)$ (2.6)

Hence using (2.5) and (2.6) we get

$$dF(X,Y,Z) + dF(JX,JY,Z) + dF(JX,Y,JZ) + dF(JY,X,JZ)$$

$$= 2\left(\nabla_X(F)(Y,Z) + \nabla_{JX}(F)(JY,Z)\right) \qquad (2.7)$$

Here F is the kahler 2-form defined by $F(X,Y) = \langle JX, Y \rangle$.

Also we have,

$$\nabla_X(J)(JY) = -J\nabla_X(J)(Y),$$

for all $X, Y \in \mathcal{H}(M)$. Hence if J is nearly kahler then

$$\nabla_{JX}(J)(JY) = -\nabla_X(J)(Y) \qquad (2.8)$$

Hence (i) follows from (2.8) and the fact that locally we can choose a frame field of the form $\{X_1, JX_1, X_2, \ldots, X_m, JX_m\}$.

(ii) follows from (2.7).

To prove (iii) and (iv) we make use of the formula: ([10])

$$4\langle \nabla_X(J)(Y), Z \rangle = 6\, dF(X, JY, JZ)$$

$$- 6\, dF(X,Y,Z) + \langle S(Y,Z), JX \rangle, \qquad (2.9)$$

and the fact that if $J \in \mathcal{QK}$ then

$$dF(JX,Y,Z) \quad = \quad dF(X,JY,Z) \quad = \quad dF(X,Y,JZ) \qquad (2.10)$$

t
To prove (v), let $\{X, JX\}$ be a local frame field. From (2.5) we see that $\nabla_X(F)\ (X, JX) = 0 = \nabla_{JX}(F)\ (X,JX)$,

and hence J is haklerian.

To prove (vi) we note that if $J \in \mathcal{Q}\mathcal{K}$; $\{X,JX,Y,JY\}$ a local frame field, then

$$dF(X,JX,Y) = dF(X,JX,JY) = dF(X,Y,JY) = dF(Y,JY,JX) = 0.$$

Hence $dF = 0$. i.e. $\mathcal{A}\mathcal{K}(s,t) = \mathcal{Q}\mathcal{K}(s,t)$.

Further, using proposition (2.5) we get

$$\delta F(X) = -\nabla_Y(F)\ (Y,X) - \nabla_{JY}(F)\ (JY,X) \quad ,$$

and similarly for JX, Y, JY. Hence if $J \in \mathcal{S}\mathcal{K}$ then $J \in \mathcal{Q}\mathcal{K}$.

Therefore $\mathcal{S}\mathcal{K}(s,t) = \mathcal{A}\mathcal{K}(s,t) = \mathcal{Q}\mathcal{K}(s,t)$. This fact, together with (i) and (iv), implies that $\mathcal{K}(s,t) = \mathcal{N}\mathcal{K}(s,t)$.

Next we investigate vector cross products induced on submanifolds.

Let M be a submanifold of a pseudo-riemannian manifold $\bar{M}$ such that the induced metric on M is non-degenerate. Let $\mathcal{X}(M) = \{ X|M: \quad X \in \mathcal{H}(\bar{M}) \}$; then we may write $\overline{\mathcal{H}}(M) = \mathcal{H}(M) \oplus \mathcal{H}(M)^{\perp}$, where $\mathcal{H}(M)^{\perp}$ is the set of vector fields orthogonal to M.

The configuration tensor $T: \mathcal{H}(M) \times \bar{\mathcal{H}}(M) \longrightarrow \bar{\mathcal{H}}(M)$ is defined by

$$T_X Y = \bar{\nabla}_X Y - \nabla_X Y \ , \quad \text{for} \quad X, Y \in \mathcal{H}(M) \ ;$$

and $$T_X Z = \pi \bar{\nabla}_X Z \ , \quad \text{for} \quad X \in \mathcal{H}(M), \ Z \in \mathcal{H}(M)^{\perp} .$$

Here $\bar{\nabla}$ and ∇ denote the riemannian connections of $\bar{M}$ and M, respectively, and π is the projection

$$\pi : \bar{\mathcal{H}}(M) \longrightarrow \mathcal{H}(M) .$$ Then, ([10]);

$$T_x Y = T_y X \ , \quad \text{for} \quad X, Y \in \mathcal{H}(M)$$

$$\langle T_x Z, Y \rangle = -\langle Z, T_x Y \rangle \ , \text{ for } \ X, Y \in \mathcal{H}(M), \ Z \in \mathcal{H}(M)^{\perp}$$

and $$\langle T_x Z, Y \rangle = \langle T_y Z, X \rangle \ , \text{ for } \ X, Y \in \mathcal{H}(M), \ Z \in \mathcal{H}(M)^{\perp}.$$

<u>PROPOSITION</u> (2.6) Let $\bar{M}$, M be psudo-riemannian manifolds with $\dim \bar{M} - \dim M = k$. Suppose $\bar{M}$ and M satisfy the hypotheses of theorem (1.4). Let the vector product $\bar{P}$ on $\bar{M}$ be r-fold $(r > k)$ so that the vector product P on M is $(r-k)$-fold. Then, for $X, X_1, \dots, X_{r-k} \in \mathcal{H}(M)$ we have:

$$\nabla_X(P)(X_1, \dots, X_{r-k}) = \pi \bar{\nabla}_x(\bar{P})(N_1, \dots, N_k; X_1, \dots, X_{r-k})$$

$$+ \sum_{j=1}^{k} \pi \bar{P}(N_1, \dots, T_x N_j, \dots, N_k \ ; X_1, \dots, X_{r-k}) \qquad (2.11)$$

Where $\{N_1, \dots, N_k\}$ is an orthonormal frame field of the normal bundle to M defined on an open subset of M.

Hence if $\bar{P}$ is parallel and M is totally geodesic in $\bar{M}$, then P is parallel.

<u>Proof</u>: We have $\nabla_x(P)\,(X_1,\ldots,X_{r-k}) = \nabla_x P(X_1,\ldots,X_{r-k})$

$$- \sum_{j=1}^{r-k} P(X_1,\ldots, \nabla_X Xj,\ldots,X_{r-k})$$

$$= \pi\,\bar{\nabla}_x\,\bar{P}(N_1,\ldots,N_k\,;\,X_1,\ldots,X_{r-k})$$

$$- \sum_{j=1}^{r-k} \bar{P}(N_1,\ldots,N_k\,;\,X_1,\ldots,\nabla_x X_j,\ldots,X_{r-k}) \quad .$$

Taking the first term on the R.H.S., we write

$$\pi\,\bar{\nabla}_x\bar{P}(N_1,\ldots,N_k, X_1,\ldots,X_{r-k}) = \pi\,\bar{\nabla}_x(\bar{P})(N_1,\ldots,N_k,X_1,\ldots,X_{r-k})$$

$$+ \sum_{j=1}^{k} \pi\,\bar{P}(N_1,\ldots,\bar{\nabla}_x N_j,\ldots,N_k\,;\,X_1,\ldots,X_{r-k})$$

$$+ \sum_{i=1}^{r-k} \pi\,\bar{P}\,(N_1,\ldots,N_k\,;\,X_1,\ldots,\nabla_x X_i,\ldots,X_{r-k}) \quad .$$

Hence we get

$$\nabla_x(P)(X_1,\ldots,X_{r-k}) = \pi\,\bar{\nabla}_x(\bar{P})\,(N_1,\ldots,N_k,X_1,\ldots,X_{r-k})$$

$$+ \sum_{j=1}^{k} \pi\,\bar{P}\,(N_1,\ldots,T_x N_j,\ldots,N_k,X_1,\ldots,X_{r-k}) \quad ,$$

where in the last term , $\bar{\nabla}_x N_j$ is replaced by $\pi\,\bar{\nabla}_x N_j$ since the normal component of $\bar{\nabla}_x N_j$ is a linear combination of

N_i's $(i \neq j)$.

We also note that if $\bar{P}$ is nearly parallel and M totally umbilic in $\bar{M}$ then P is nearly parallel.

<u>THEOREM</u> (2.7) Let $\bar{M}$ and M be riemannian manifolds (with dim $\bar{M}$-dim M = 1) which satisfy the hypotheses of Theorem (1.4). Assume $\bar{P} \in \mathcal{P}(0,8,3)$. If $P \in \mathcal{A}\mathcal{P}(0,7,2)$, then P is parallel and M is totally geodesic in $\bar{M}$.

<u>Proof</u>: We prove that T vanishes at each point $m \in M$. For that purpose choose vector fields X_1, X_2 such that at m $T_{X_i}N = K_i X_i$, i=1,2. Let $Z \in \mathcal{H}(M)$, then at m we have

$$0 = d\varphi(X_1, X_2, P(X_1,X_2), Z) = \nabla_{X_1}(\varphi)(X_2, P(X_1,X_2), Z)$$

$$- \nabla_{X_2}(\varphi)(X_1, P(X_1,X_2), Z) + \nabla_{P(X_1,X_2)}(\varphi)(X_1,X_2, Z)$$

$$- \nabla_Z(\varphi)(X_1, X_2, P(X_1,X_2)) .$$

Hence using (2.11) we get

$$0 = d\varphi(X_1,X_2, P(X_1,X_2), Z) = \langle \bar{P}(T_{X_1}N, X_2, P(X_1,X_2)), Z \rangle$$

$$- \langle \bar{P}(T_{X_2}N, X_1, P(X_1,X_2)), Z \rangle + \langle \bar{P}(T_{P(X_1,X_2)}N, X_1,X_2), Z \rangle$$

$$- \langle \bar{P}(T_Z N, X_1, X_2), P(X_1,X_2) \rangle .$$

For the last term we have,

$$\langle \bar{P}(X_1,X_2,T_Z N), \bar{P}(X_1,X_2,N) \rangle$$

$$= \det \begin{vmatrix} \|X_1\|^2 & 0 & 0 \\ 0 & \|X\|^2 & 0 \\ \langle T_Z N, X_1 \rangle & \langle T_Z N, X_2 \rangle & 0 \end{vmatrix} = 0$$

Hence we conclude that

$$0 = \langle \bar{P}(X_1, X_2, Z), K_2 P(X_1, X_2) \rangle$$

$$+ \langle \bar{P}(X_1, X_2, Z), T_{P(X_1, X_2)} N \rangle + \langle \bar{P}(X_1, X_2, Z), K_1 P(X_1, X_2) \rangle ,$$

or equivalently ,

$$\langle \bar{P}(X_1, X_2, Z), (K_1 + K_2) P(X_1, X_2) + T_{P(X_1, X_2)} N \rangle = 0 .$$

Since Z is arbitrary and

$\langle T_{P(X_1, X_2)} N, X_1 \rangle = \langle T_{P(X_1, X_2)} N, X_2 \rangle = 0$, we

conclude that $T_{P(X_1, X_2)} N = K_4 P(X_1, X_2)$, at m.

Next we choose $X_3 \in \mathcal{H}(M)$, orthogonal to X_1 , X_2 , $P(X_1, X_2)$ and such that $T_{X_3} N = K_3 X_3$, at m. Using the same method as above, we can see that ,

$$T_{P(X_1, X_3)} N = K_5 P(X_1, X_3) , \text{ and}$$

$$T_{P(X_2, X_3)} N = K_6 P(X_2, X_3) \text{ at m.}$$

Further using (2.11) we get

$$0 = d\varphi(X_1, X_2, X_3, Z) = (K_1+K_2+K_3) \langle \bar{P}(X_1, X_2, X_3) , Z \rangle$$

$$+ \langle T_{\bar{P}(X_1, X_2, X_3)}^{N} , Z \rangle \text{ , for all } Z \in \mathcal{H}(M).$$

Hence at m , $T_{\bar{P}(X_1, X_2, X_3)}^{N} = K_7 \bar{P}(X_1, X_2, X_3)$,

where $K_7 = -(K_1 + K_2 + K_3)$.

Set $X_4 = P(X_1, X_2)$, $X_5 = P(X_1, X_3)$, $X_6 = P(X_2, X_3)$

and $X_7 = \bar{P}(X_1, X_2, X_3)$. Then $\{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$

is a basis for the vector fields in a n b d of m such

that $T_{X_i}N = K_i X_i$, $i=1,\dots,7$, at m.

Also $d\varphi(X_i, X_j, X_k, Z) = 0$ implies

$$T_{\bar{P}(X_i, X_j, X_k)}^{N} = -(K_i + K_j + K_k) \bar{P}(X_i, X_j, X_k) \qquad (2.12)$$

Further making $\bar{M}_m$ into a composition algebra with identity N(m) , we may write ; $\bar{P}(X_i, X_j, X_k) = X_i.(X_j.X_k)$ (2.13)

$$P(X_i, X_j) = X_i . X_j \qquad (2.14)$$

and $X_i.(X_j.X_k) = -X_j(X_i.X_k)$ (2.15)

then (2.13) , (2.14) and (2.15) give the following system:

$$\bar{P}\,(X_i, X_j, X_k) = \pm\,\bar{P}\,(X_1, X_2, X_3) = \pm\, X_7 \,, \qquad (2.16)$$

for each $\{i, j, k\}$ one of the following combinations:

$\{1, 2, 3\}$; $\{1, 4, 5\}$; $\{1, 4, 6\}$; $\{3, 5, 6\}$

$$\bar{P}\,(X_i, X_j, X_k) = \pm\, X_6 \qquad (2.17)$$

for each $\{i, j, k\}$ one of the following combinations:

$\{1, 2, 5\}$; $\{1, 3, 4\}$.

and $$\bar{P}\,(X_2, X_3, X_4) = X_5 \qquad (2.18)$$

Using (2.12) , (2.16) , (2.17) and (2.18) we get the following system of equations:

$$\begin{aligned}
K_1 + K_2 + K_3 + K_7 &= 0 \\
K_1 + K_2 + K_5 + K_6 &= 0 \\
K_1 + K_3 + K_4 + K_6 &= 0 \qquad (2.19)\\
K_1 + K_4 + K_5 + K_7 &= 0 \\
K_2 + K_3 + K_4 + K_5 &= 0 \\
K_2 + K_4 + K_6 + K_7 &= 0 \\
K_3 + K_5 + K_6 + K_7 &= 0 \,.
\end{aligned}$$

The unique solution to this system is $K_1 = \ldots = K_7 = 0$.

Hence T vanishes at m. Since m is arbitrary, M is totally geodesic in $\bar{M}$. From (2.11) it follows that P is also parallel.

Let δ and $\bar{\delta}$ denote the coderivatives of M and $\bar{M}$ respectively; also $b_X (X \in \mathcal{H}(M))$ the contraction operator. If θ is a p-form then $b_X(\theta)(X_1, \ldots, X_{p-1}) = \theta(X, X_1, \ldots, X_{p-1})$,

for vector fields $X; X_1, \ldots, X_{p-1}$.

<u>THEOREM</u> (2.8) Let $\bar{M}$, M be pseudo-riemannian manifolds, with $\dim \bar{M} - \dim M = 1$, which satisfy the hypotheses of theorem (1.4). Then we have $\delta\varphi = -b_N(\bar{\delta}\bar{\varphi})$, where N is the unit normal of M in $\bar{M}$. Hence if $\bar{P} \in \mathcal{S}\mathcal{P}(s,t,r)$, then $P \in \mathcal{S}\mathcal{P}(s,t-1,r-1)$.

<u>Proof</u>: At an arbitrary point $m \in M$, we choose an orthogonal frame $\{E_1, \ldots, E_n\}$ such that $T_{E_j} N = k_j E_j \qquad j = 1, \ldots, n.$

At m we have

$$\delta\varphi(X_1, \ldots, X_{r-1}) = -\sum_{i=1}^{n} \| E_i \|^{-2} \nabla_{E_i}(\varphi)(E_i, X_1, \ldots, X_{r-1}) \tag{2.20}$$

Using (2.11) ; (2.20) becomes:

$$\delta\varphi(X_1, \ldots, X_{r-1}) = -\sum_{i=1}^{n} \| E_i \|^{-2} \langle \bar{\nabla}_{E_i}(\bar{P})(N, E_i, X_1 \ldots, X_{r-2}), X_{r-1} \rangle$$

$$-\sum_{i=1}^{n} \| E_i \|^{-2} \langle \bar{P}(T_{E_i}N, E_i, X_1, \ldots, X_{r-2}), X_{r-1} \rangle \tag{2.21}$$

$$= -\sum_{i=1}^{n} \| E_i \|^{-2} \langle \bar{\nabla}_{E_i}(\bar{P})(N, E_i, X_1, \ldots, X_{r-1}), X_{r-1} \rangle .$$

where the last equality is due to the choice of the frame $\{E_i\}$.

On the other hand at m, we have

$$b_N(\bar{\delta}\bar{\varphi})(X_1,\ldots,X_{r-1}) = \bar{\delta}(\bar{\varphi})(N,X_1,\ldots,X_{r-1})$$

$$= -\sum_{i=1}^{n} \| E_i \|^{-2} \bar{\nabla}_{E_i}(\bar{\varphi})(E_i,N,X_1,\ldots,X_{r-1})$$

$$- \bar{\nabla}_N(\bar{\varphi})(N,N,X_1,\ldots,X_{r-1})$$

$$= \sum_{i=1}^{n} \| E_i \|^{-2} \langle \bar{\nabla}_{E_i}(\bar{P})(N,E_i,X_1,\ldots,X_{r-2}), X_{r-1} \rangle \qquad (2.22)$$

comparing (2.21) and (2.22) we conclude that

$$\delta\varphi(X_1,\ldots,X_{r-1}) = -b_N(\bar{\delta}\bar{\varphi})(X_1,\ldots,X_{r-1}), \text{ at } m .$$

Since m is arbitrary we have

$$\delta\varphi = - b_N(\bar{\delta}\bar{\varphi}).$$

The above proof is different from that of GRAY's in which he treated the cases $\dim \bar{M} = 7$ and $\dim \bar{M} = 8$ separately.

Specializing the above results to the case where $\bar{M} = R^7$ or R^8 and $\bar{P}$ is the parallel translate of an ordinary 2- or 3-fold vector cross product we get;

<u>THEOREM</u> (2.9): Let M be an orientable hypersurface of $\bar{M} = R^7$ or R^8 with unit normal N satisfying $\| N \|^2 = 1$,

and let P denote the vector cross product on M determined by an ordinary vector cross product $\bar{P}$ on $\bar{M}$. Then

(i) P is semi-parallel.

(ii) If P is almost parallel and M is a 7-dimensional submanifold of R^8 with positive definite metric, then M is an open submanifold of a hyperplane and P is parallel.

Proof: (i) follows from theorem (2.8) and the fact that $\bar{P}$ is parallel. Also (ii) follows from theorem (2.7).

3. Relations between Curvature and Vector Cross Products.

As before let M be a pseudo-riemannian manifold with metric tensor $\langle\ ,\ \rangle$; P and r-fold vector cross product on M whose associated bilinear form is $\langle\ ,\ \rangle$. We also denote by φ the $(r + 1)$ differential form determined by P and $\langle\ ,\ \rangle$. We first define the following operations. Let θ be a p-form on M. Then for $X, Y, X_1, \ldots, X_p \in \mathcal{H}(M)$, we define:

$$\nabla\theta(X; X_1, \ldots, X_p) = \nabla_X (\theta)(X_1, \ldots, X_p)$$

$$= X\,\theta(X_1, \ldots, X_p) - \sum_{j=1}^{p} \theta(X_1, \ldots, \nabla_X X_j, \ldots, X_p) \quad . \qquad (3.1)$$

$$\nabla^2\theta(X; Y; X_1, \ldots, X_p) = \nabla_X (\nabla\theta)(Y; X_1, \ldots, X_p) ,$$

from which we easily see that

$$\nabla^2\theta\,(X;Y;) = (\nabla_X\nabla_Y)\,(\theta) - \nabla_{\nabla_X Y}\,(\theta) \quad . \qquad (3.2)$$

$$R_{XY}\,(\theta)\,(X_1,\dots,X_p) = -\sum_{i=1}^{p} \theta\,(X_1,\dots, R_{XY}X_i,\dots,X_p) \quad , \qquad (3.3)$$

or alternatively, $R_{XY}(\theta) = -\nabla^2\theta\,(X;Y;) + \nabla^2\theta\,(Y;X;)$.

$$\Delta\theta\,(X_1,\dots,X_p) = \sum_{i=1}^{p}\sum_{k=1}^{n} (-1)^{i+1}\,\|E_k\|^{-2}\, R_{X_iE_k}(\theta)\,(E_k,X_1,\dots\hat{X}_i,\dots X_p)$$

$$-\sum_{k=1}^{n}\|E_k\|^{-2}\,\nabla^2\theta\,(E_k;\,E_k;\,X_1,\dots,X_p) \quad , \qquad (3.4)$$

where $\{E_1,\dots,E_n\}$ is an orthogonal frame field on an open subset of M. Here $\nabla\theta$, $\nabla^2\theta$ are the first and second covariant derivatives of θ, $R_{XY} = \nabla_{[X,Y]} - [\nabla_X,\nabla_Y]$, and

$\Delta = d\,\delta + \delta\,d$, is the Laplacian.

<u>PROPOSITION</u> (3.1): Let $X, Y, Z_1,\dots,Z_r \in \mathcal{X}(M)$; then

$$\nabla^2\varphi\,(X;\,Y;\,Z_1,\dots,Z_r,\,P\,(Z_1,\dots,Z_r))$$

$$= -\langle \nabla_X(P)\,(Z_1,\dots,Z_r)\ ,\ \nabla_Y\,(P)\,(Z_1,\dots,Z_r)\rangle . \qquad (3.5)$$

<u>Proof</u>: By (3.2) we have

$$\nabla^2 \varphi(X;\ Y;\ Z_1,\ldots,Z_r,\ P(Z_1,\ldots,Z_r))$$

$$= (\nabla_X \nabla_Y)\ (\varphi)\big(Z_1,\ldots Z_r, P(Z_1,\ldots,Z_r)\big)$$

$$- \nabla_{\nabla_X Y}(\varphi)\big(Z_1,\ldots,\ Z_r,\ P(Z_1,\ldots,Z_r)\big).$$

The term $\nabla_{\nabla_X Y}(\varphi)\big(Z_1,\ldots,Z_r,\ P(Z_1,\ldots,Z_r)\big)$ vanishes since $\langle \nabla_X(P)\ (Z_1,\ldots,Z_r),\ P(Z_1,\ldots,Z_r)\rangle = 0$ for all $X, Z_1,\ldots,Z_r \in \mathcal{H}(M)$. For the first term we have,

$$(\nabla_X \nabla_Y)\ (\varphi)\ (Z_1,\ldots,Z_r,\ P(Z_1,\ldots,Z_r))$$

$$= X\ \nabla_Y(\varphi)\ (Z_1,\ldots,Z_r,\ P(Z_1,\ldots,Z_r))$$

$$- \sum_{j=1}^{r} \nabla_Y(\varphi)\ (Z_1,\ldots,\nabla_X Z_j,\ldots Z_r,\ P(Z_1,\ldots Z_r))$$

$$- \nabla_Y(\varphi)\ (Z_1,\ldots,Z_r,\ \nabla_X P(Z_1,\ldots,Z_r))$$

$$= X \langle \nabla_Y(P)\ (Z_1,\ldots,Z_r),\ P(Z_1,\ldots,Z_r)\rangle$$

$$- \sum_{j=1}^{r} \langle \nabla_Y(P)\ (Z_1,\ldots,\nabla_X Z_j,\ldots,Z_r),\ P(Z_1,\ldots,Z_r)\rangle$$

$$- \langle \nabla_Y(P)\ (Z_1,\ldots,Z_r)\ ,\ \nabla_X P(Z_1,\ldots,Z_r)\rangle$$

$$= - \sum_{j=1}^{r} \langle \nabla_Y (P) (Z_1, \ldots, \nabla_X Z_j, \ldots, Z_r) , P(Z_1, \ldots, Z_r) \rangle$$

$$- \langle \nabla_Y (P) (Z_1, \ldots, Z_r) , \nabla_X (P) (Z_1, \ldots, Z_r) \rangle$$

$$- \sum_{j=1}^{r} \langle \nabla_Y (P) (Z_1, \ldots, Z_r) , P(Z_1, \ldots, \nabla_X Z_j, \ldots, Z_r) \rangle .$$

Moreover, $\sum_{j=1}^{r} \langle \nabla_Y (P)(Z_1, \ldots, Z_r), P (Z_1, \ldots, \nabla_X Z_j, \ldots Z_r) \rangle$

$$= - \sum_{j=1}^{r} \langle \nabla_Y (P) (Z_1, \ldots, \nabla_X Z_j, \ldots, Z_r), P(Z_1, \ldots, Z_r) \rangle ,$$

which follows from $\langle {}_Y(P)(Z_1, \ldots, Z_r), P(Z_1, \ldots, Z_r) \rangle = 0$.

Hence we conclude the result.

If R_{XY} denotes the curvature operator of $\langle \, , \, \rangle$, then the Ricci tensor K is defined by

$$K(X,Y) = \sum_{i=1}^{n} \| E_i \|^{-2} \langle R_{XE_i} Y , E_i \rangle ,$$

for $X, Y \in \mathcal{H} (M)$ and $\{E_1, \ldots, E_n\}$ is any orthogonal frame field on an open subset of M. Next we define an $(r+1)$-fold differential form γ .

DEFINITION: The Chern form γ of a vector cross product P is defined as follows:

$$(r+1)\,\gamma\,(X_1,\ldots,X_{r+1})$$

$$= \sum_{k=1}^{n} \sum_{i<j} (-1)^{r+i+j} \| E_k \|^{-2} \langle R_{X_iX_j}E_k,\ P(E_k,X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots X_{r+1}) \rangle ,$$

for $X_1,\ldots, X_{r+1} \in \mathcal{H}(M)$ and any orthogonal frame field $\{E_1,\ldots,E_n\}$. If $r=1$, i.e. P is an almost complex structure J, then

$$2\,\gamma\,(X,Y) = \sum_{k=1}^{n} \| E_k \|^{-2} \langle R_{XY}E_k,\ J\,E_k \rangle .$$

Moreover, if J is kahlerian (i.e. parallel) then

$$\gamma\,(X,Y) = K(JX,Y) \quad \text{and} \quad d\gamma = 0.$$

PROPOSITION (3.2) Let $X_1,\ldots,X_{r+1} \in \mathcal{H}(M)$, and let $\{E_1,\ldots,E_n\}$ be an orthogonal frame field on an open subset of M. Then

$$\Delta\varphi(X_1,\ldots,X_{r+1}) = -\sum_{k=1}^{n} \| E_k \|^{-2}\ \nabla^2\varphi\,(E_k;E_k;X_1,\ldots,X_{r+1})$$

$$- \sum_{i=1}^{r+1} (-1)^{i+r}\ K\,(X_i,P(X_1,\ldots,\hat{X}_i,\ldots,X_{r+1})) \tag{3.6}$$

$$- (r+1)\,\gamma\,(X_1,\ldots,X_{r+1})$$

Proof: By (3.4) we have:

$$\Delta\varphi(X_1,\ldots,X_{r+1}) = -\sum_{k=1}^{n} \| E_k \|^{-2} \nabla^2 \varphi(E_k, E_k, X_1,\ldots,X_{r+1})$$

$$+\sum_{i=1}^{r+1} \sum_{k=1}^{n} (-1)^{i+1} \| E_k \|^{-2} R_{X_iE_k}(\varphi)(E_k, X_1,\ldots,\hat{X}_i,\ldots,X_{r+1}) \cdot \quad (3.7)$$

For the second term of (3.7) we have,

$$\sum_{i=1}^{r+1} R_{X_iE_k}(\varphi)(E_k, X_1,\ldots,\hat{X}_i,\ldots,X_{r+1})$$

$$= -\sum_{i=1}^{r+1} \varphi(R_{X_iE_k}E_k, X_1,\ldots,\hat{X}_i\ldots,X_{r+1})$$

$$- \sum_{j<i} \varphi(E_k, X_1,\ldots, R_{X_iE_k}X_j,\ldots,\hat{X}_i,\ldots,X_{r+1})$$

$$- \sum_{j>i} \varphi(E_k, X_1,\ldots,\hat{X}_i,\ldots,R_{X_iE_k}X_j,\ldots,X_{r+1})$$

$$= - \sum_{i=1}^{r+1} (-1)^r \varphi(X_1,\ldots,\hat{X}_i,\ldots,X_{r+1}, R_{X_iE_k}E_k)$$

$$- \sum_{j<i} (-1)^{r-j} \varphi(E_k, X_1,\ldots,\hat{X}_j,\ldots,\hat{X}_i,\ldots,X_{r+1}, R_{X_iE_k}X_j)$$

$$+ \sum_{j<i} (-1)^{r-j} \varphi(E_k, X_1,\ldots,\hat{X}_j,\ldots,\hat{X}_i,\ldots,X_{r+1}, R_{X_jE_k}X_i)$$

Hence using the formula;

$$R_{X_jX_i}{}^{E_k} = R_{X_jE_k}{}^{X_i} - R_{X_iE_k}{}^{X_j} \quad , \quad \text{the above equation}$$

reduces to:

$$\sum_{i=1}^{r+1} R_{X_iE_k}(\varphi)(E_k, X_1, \ldots, \hat{X}_i, \ldots, X_{r+1})$$

$$= + \sum_{i=1}^{r+1} (-1)^r \langle P(X_1, \ldots, \hat{X}_i, \ldots, X_{r+1}) \, , \, R_{E_kX_i}{}^{E_k} \rangle$$

$$+ \sum_{j<i} (-1)^{r+j} \langle P(E_k, X_1, \ldots, \hat{X}_j, \ldots, \hat{X}_i, \ldots, X_{r+1}) \, , \, R_{X_jX_i}{}^{E_k} \rangle \, .$$

Hence $\sum_{k=1}^{n} \sum_{i=1}^{r+1} (-1)^{i+1} \| E_k \|^{-2} R_{X_iE_k}(\varphi)(E_k, X_1, \ldots, \hat{X}_i, \ldots X_{r+1})$

$$= -\sum_{i=1}^{r+1} (-1)^{r+i} K(X_i, P(X_1, \ldots, \hat{X}_i, \ldots, X_{r+1}))$$

$$- (r+1) \gamma (X_1, \ldots, X_{r+1}).$$ Hence we conclude the result.

<u>THEOREM</u> (3.3) Let $X_1, \ldots, X_r \in \mathcal{H}(M)$ and $\{E_1, \ldots, E_n\}$ be an orthogonal frame field on an open subset of M. Then

$$\Delta \varphi (X_1, \ldots, X_r, P(X_1, \ldots, X_r))$$

$$= \sum_{k=1}^{n} \| E_k \|^{-2} \| \nabla_{E_k}(P)(X_1, \ldots, X_r) \|^2 \tag{3.8}$$

$$= (r+1)\,\gamma(X_1,\dots,X_r,\ P(X_1,\dots,X_r)) + K(P(X_1,\dots,X_r),\ P(X_1,\dots,X_r))$$

$$+ \sum_{i=1}^{r} \sum_{j=1}^{r} (-1)^{i+j} \langle X_1\wedge\dots\wedge\hat{X}_i\wedge\dots\wedge X_r,\ X_1\wedge\dots\wedge\hat{X}_j\wedge\dots\wedge X_r\rangle K(X_i,X_j).$$

Proof: Theorem (3.3) follows from proposition (3.2), and the fact that

$$\sum_{i=1}^{r} (-1)^{i+r} \langle R_{E_k X_i}{}^{E_k},\ P(X_1,\dots,\hat{X}_i,\dots,X_r,\ P(X_1,\dots,X_r)) \rangle$$

$$= \sum_{i=1}^{r} (-1)^{i+r+1} \langle P(X_1,\dots,X_r),\ P(X_1,\dots,\hat{X}_i,\dots,X_r,\ R_{E_k X_i}{}^{E_k}) \rangle$$

$$= \sum_{i=1}^{r} - \langle P(X_1,\dots,X_r),\ P(X_1,\dots,R_{E_k X_i}{}^{E_k},\dots,X_r) \rangle ,$$

where $R_{E_k X_i}{}^{E_k}$ replaces X_i in $P(X_1,\dots,R_{E_k X_i}{}^{E_k},\dots X_r)$,

Further, $\sum_{i=1}^{r} \langle P(X_1,\dots,X_r),\ P(X_1,\dots R_{E_k X_i}{}^{E_k},\dots X_r) \rangle$

$$= \sum_{i=1}^{r} \sum_{j=1}^{r} (-1)^{i+j} K(X_i,X_j) \langle X_1\wedge\dots\wedge\hat{X}_i\wedge\dots\wedge X_r,\ X_1\wedge\dots\wedge\hat{X}_j\wedge\dots\wedge X_r\rangle ,$$

which follows from the formula

$$\| P(X_1,\dots,X_r)\|^2 = \| X_1\wedge\dots\wedge X_r\|^2 .$$

When a vector cross product P on a manifold M is parallel we have $\nabla_X \varphi = 0$ for $X \in \mathcal{H}(M)$, and hence $R_{XY}(\varphi) = 0$ $(X, Y \in \mathcal{H}(M))$. This places certain restrictions on the curvature operator. For example in (3.8) if P is parallel then $\Delta\varphi = 0$, and hence

$$(r+1)\gamma\Big(X_1,\dots,X_r,\ P(X_1,\dots,X_r)\Big) - K(P(X_1,\dots,X_r)\ P(X_1,\dots,X_r))$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{r}(-1)^{i+j}\langle X_1\wedge\dots\wedge\hat{X}_i\wedge\dots\wedge X_r, X_1\wedge\dots\wedge\hat{X}_j\wedge\dots\wedge X_r\rangle\, K(X_i,X_j)\ . \tag{3.9}$$

Conversely we have the following result for the class $\mathcal{A}\mathcal{P}\cap\mathcal{S}\mathcal{P}$.

<u>PROPOSITION</u> (3.4). Assume $P\in\mathcal{A}\mathcal{P}\cap\mathcal{S}\mathcal{P}$ and $\langle\ ,\ \rangle$ is positive definite. If the curvature operator R_{XY} has the same properties that would be satisfied if P were parallel, then P is in fact parallel.

<u>Proof</u>: If $P\in\mathcal{A}\mathcal{P}\cap\mathcal{S}\mathcal{P}$ then $\Delta\varphi = (d\delta+\delta d)\varphi = 0$.

Also by hypothesis, (3.9) is valid. Hence substituting in (3.8) we get

$$\sum_{k=1}^{n}\|\nabla_{E_k}(P)\ (X_1,\dots,X_r)\ \|^2 = 0$$

where $\{E_1,\dots,E_n\}$ is an orthonormal frame field on an open subset of M. Since $\langle\ ,\ \rangle$ is positive definite we deduce that P is parallel.

As an application of Proposition (3.4) we deduce a result of Tachibana [13] . The curvature operator of complex projective space (with usual metric and almost complex structure) is given by the formula

$$\langle R_{WX}Y, Z\rangle = K\Big(\langle W,Y\rangle\langle X,Z\rangle - \langle W,Z\rangle\langle X,Y\rangle$$

$$+ \langle JW,Y\rangle\langle JX,Z\rangle - \langle JW,Z\rangle\langle JX,Y\rangle$$

$$+ 2\langle JW,X\rangle\langle JY,Z\rangle\Big). \qquad (3.10)$$

Let M be a riemannian almost complex manifold whose curvature is given by (3.10). Proposition (3.4) shows that if the almost complex structure J of M is almost kahlerian then it is kahlerian. This is because if J is almost kahlerian than it is semi-kahlerian and (3.10) implies that (3.9) is valid and hence using (3.8) the result follows.

Further, for the case of positive definite metric we have two theorems.

<u>THEOREM</u> (3.5). Suppose P is a vector cross produce on a riemannian manifold M, P is in $\mathcal{AK}(0,n)$ $(n \geq 4)$, $\mathcal{AP}(0,7,2) \cap \mathcal{SP}(0,7,2)$ or $\mathcal{AP}(0,8,3) \cap \mathcal{SP}(0,8,3)$, and M is conformally flat. Denote by K and R the Ricci and Ricci scalar curvature of M.

(i) If K is positive semidefinite on M then it is identically zero and P is parallel.

(ii) If $R \geq 0$ on M, then R vanishes identically on M and P is parallel.

Proof: Suppose M is conformally flat. Then for $W, X, Y, Z \in \mathcal{H}(M)$,

$$\langle R_{WX}Y, Z \rangle = \frac{-R}{(n-1)(n-2)} \Big(\langle W,Y \rangle \langle X,Z \rangle - \langle W,Z \rangle \langle X,Y \rangle \Big)$$

$$+ \frac{1}{n-2} \Big(K(W,Y) \langle X,Z \rangle - K(W,Z) \langle X,Y \rangle$$

$$+ K(X,Z) \langle W,Y \rangle - K(X,Y) \langle W,Z \rangle \Big) \qquad (3.11)$$

Here R is the Ricci scalar curvature, K the Ricci curvature. (Ricci tensor);

$\left(R = \sum_{i=1}^{n} \| E_i \|^{-2} K(E_i, E_i) \right.$, for an orthogonal frame $\left. \{E_1, \ldots, E_n\} . \right)$

Using (3.11) and for $X_1, \ldots, X_r \in \mathcal{H}(M)$ mutually orthogonal we get,

$$(r+1)\, \gamma\big(X_1, \ldots, X_r, P(X_1, \ldots, X_r)\big) = \frac{-r(r+1)R}{(n-1)(n-2)} \| X_1 \wedge \cdots \wedge X_r \|^2$$

$$+ \frac{2r}{n-2} \sum_{i=1}^{r} K(X_i, X_i) \| X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_r \|^2$$

Hence if we choose E_i's so that $\| E_i \|^2 = 1$, (3.8) reduces to

$$\Delta \varphi \left(X_1, \ldots, X_r, P(X_1, \ldots, X_r) \right) = \sum_{k=1}^{n} \| \nabla_{E_k}(P)\ (X_1, \ldots, X_r) \|^2$$

$$+ \frac{r(r+1)R}{(n-1)(n-2)} \| X_1 \wedge \ldots \wedge X_r \|^2 + K \left(P(X_1, \ldots, X_r), P(X_1, \ldots, X_r) \right)$$

$$+ \left(1 - \frac{2r}{n-2}\right) \sum_{i=1}^{r} K(X_i, X_i) \| X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_r \|^2 \qquad (3.12)$$

According to the hypothesis the L.H.S. of (3.12(vanishes and each term on the R.H.S. ≥ 0. Hence each of these terms vanishes and (i) follows.

From (3.12) we also obtain

$$\sum_{j_1 \cdots j_r = 1}^{n} \Delta \varphi (E_{j_1}, \ldots, E_{j_r}, P(E_{j_1}, \ldots, E_{j_r}))$$

$$= \sum_{k, j_1, \ldots, j_r = 1}^{n} \| \nabla_{E_k}(P)(E_{j_1}, \ldots, E_{j_r}) \|^2$$

$$+ \frac{R\, n^{r-2}}{(n-1)(n-2)} \left[(r+1)n^3 - (r^2+3r+2)\, n^2 + (2r^2+5r-1)\, n - 2(r-1) \right].$$

(ii) follows from this equation.

<u>THEOREM</u> (3.6): Suppose P is a vector cross product on a riemannian manifold, and assume P is in $\mathcal{AK}(o,n)$ $(n \geq 4)$, $\mathcal{AP}(o,7,2) \cap \mathcal{SP}(o,7,2)$ or $\mathcal{AP}(o,8,3) \cap \mathcal{SP}(o,8,3)$. If M is a hypersurface of a flat riemannian manifold $\bar{M}$, then M cannot have positive Ricci scalar curvature.

<u>Proof</u>: Let $\{E_1, \ldots, E_n\}$ be an orthonormal frame field on an open subset of M such that at a point m, $T_{E_i} N = \lambda_i E_i$, $i=1, \ldots, n$. Then by the Ganss equation we have

$$\langle R_{E_i E_j} E_k, E_l \rangle = \lambda_i \lambda_j (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}),$$

and hence

$$(r+1)\, \gamma(E_1, \ldots, E_{r+1}) = \sum_{k=1}^{n} \sum_{i<j} (-1)^{r+i+j} \langle R_{E_i E_j} E_k, P(E_k, \ldots, \hat{E}_i, \ldots, \hat{E}_j, \ldots, E_{r+1}) \rangle$$

Since $R_{E_i E_j} E_k = \lambda_i \lambda_j \delta_{i_k} E_j - \lambda_i \lambda_j \delta_{jk} E_i$, then

$$\sum_{i<j} \sum_{k=1}^{n} (-1)^{r+i+j} \langle R_{E_i E_j} E_k, P(E_k, \ldots, \hat{E}_i, \ldots, \hat{E}_j, \ldots E_{r+1}) \rangle$$

$$= \sum_{i<j} \lambda_i \lambda_j \langle E_{r+1}, P(E_1, \ldots, E_r) \rangle$$

$$+ \sum_{i<j} \lambda_i \lambda_j \langle E_{r+1}, P(E_1, \ldots, E_r) \rangle$$

$$= \left(\sum_{i \neq j} \lambda_i \lambda_j \right) \varphi(E_1, \ldots, E_{r+1}) .$$

Hence $(r+1)\, \gamma(E_1, \ldots, E_{r+1}) = \left(\sum_{i \neq j} \lambda_i \lambda_j \right) \varphi(E_1, \ldots, E_{r+1})$, and therefore for each $X_1,\dots,X_{r+1} \in \mathcal{H}(M)$, we have at m,

$$(r+1)\,\gamma(X_1,\dots,X_{r+1}) = \Big(\sum_{i\neq j}\lambda_i\lambda_j\Big)\,\varphi(X_1,\dots,X_{r+1}).$$

Substitution for γ in (3.8) gives:

$$\Delta\varphi(E_1,\dots,E_r,\ P(E_1,\dots,E_r))$$

$$= \sum_{k=1}^{n}\|\nabla_{E_k}(P)(E_1,\dots,E_r)\|^2 + R$$

$$+ K\big(P(E_1,\dots,E_r),\ P(E_1,\dots,E_r)\big)$$

$$+ \sum_{i=1}^{r} K(E_i,E_i)\ ;$$

Noting that for an orthonormal set

$$\{E_1,\dots,E_r\},\ \langle E_1\wedge\dots\wedge\hat{E}_i\wedge\dots\wedge E_r,\ E_1\wedge\dots\wedge\hat{E}_j\wedge\dots\wedge E_r\rangle$$

$$= \delta_{ij} \quad \text{and}$$

$$\sum_{i\neq j}\lambda_i\lambda_j\,\varphi(E_1,\dots,E_r,\ P(E_1,\dots,E_r))$$

$$= \Big(\sum_{i\neq j}\lambda_i\lambda_j\Big)\langle P(E_1,\dots,E_r),\ P(E_1,\dots,E_r)\rangle$$

$$= \sum_{i\neq j}\lambda_i\lambda_j = R\ .$$

From the above equation we obtain;

$$\sum_{j_1,\dots j_r=1}^{n} \Delta\varphi\left(E_{j_1},\dots,E_{j_r}, P(E_{j_1},\dots,E_{j_r})\right)$$

$$= \mathbb{R}^{r-2}\left(2n^2 + (r+1)n - r+1\right)R$$

$$+ \sum_{k,j_1,\dots,j_{r}=1}^{n} \left\| \nabla_{E_k}(P)\,(E_{j_1},\dots,E_{j_r}) \right\|^2 .$$

The result follows from this equation.

Next we investigate the class $\mathcal{N}\mathcal{K}$ (nearly kahler).

<u>THEOREM</u> (3.7). Assume P is nearly parallel. Then for $X, Z_1,\dots,Z_r \in \mathcal{H}(M)$, we have

$$\| \nabla_X(P)\,(Z_1,\dots,Z_r) \|^2 = \langle R_{XZ_1} X \wedge Z_2 \wedge\dots\wedge Z_r,\ Z_1 \wedge\dots\wedge Z_r \rangle$$

$$+ \sum_{i=2}^{r} \langle P(X,Z_2,\dots,R_{XZ_1}Z_i,\dots,Z_r)\ ,\ P(Z_1,\dots,Z_r) \rangle$$

$$- \langle R_{XZ_1}\ P(X,Z_2,\dots,Z_r)\ ,\ P(Z_1,\dots,Z_r) \rangle \qquad (3.13)$$

<u>Proof</u>: For $X, Z_1,\dots,Z_{r+1} \in \mathcal{H}(M)$, we have

$$R_{XZ_1}(\varphi)\,(X,Z_2,\dots,Z_{r+1})$$

$$= (\nabla^2\varphi)\,(Z_1;X;X, Z_2,\dots,Z_{r+1})$$

$$- (\nabla^2\varphi)\,(X;Z_1;X, Z_2,\dots,Z_{r+1})$$

Taking the first term we have;

$$\nabla^2 \varphi\,(z_1;\ x;\ x, z_2, \ldots, z_{r+1})$$

$$= (\nabla_{z_1} \nabla_x \varphi)\ (x, z_2, \ldots, z_{r+1}) - (\nabla_{\nabla_{z_1} x} \varphi)\ (x, z_2, \ldots, z_{r+1})$$

$$= z_1\ (\nabla_x \varphi)(x, z_2, \ldots, z_{r+1}) \quad - (\nabla_x \varphi)\ (\nabla_{z_1} x, z_2, \ldots, z_{r+1})$$

$$- \sum_{i=2}^{r+1} (\nabla_x \varphi)\ (x,\ z_2, \ldots, \nabla_{z_1} z_i, \ldots, z_{r+1})$$

$$- (\nabla_{\nabla_{z_1} x} \varphi)\ (x,\ z_2, \ldots, z_{r+1}) \quad .$$

Since P is nearly parallel, the first and third terms in the R.H.S. of the last equation both vanish. Also the second and fourth terms cancel each other. Hence if P is nearly parallel then

$$R_{xz_1}(\varphi)\ (x, z_2, \ldots, z_{r+1})$$

$$= \ -\nabla^2 \varphi\ (x;\ z_1;\ x,\ z_2, \ldots, z_{r+1}) \quad . \qquad (3.14)$$

On the other hand, if P is nearly parallel then

$$\nabla^2 \varphi\,(x;\ z_1;\ x,\ z_2, \ldots,\ z_{r+1})$$

$$= (\nabla_x \nabla_{z_1} \varphi)\ (x, z_2, \ldots, z_{r+1}) - (\nabla_{\nabla_x z_1} \varphi)\ (x, z_2, \ldots, z_{r+1})$$

$$= X\ (\nabla_{Z_1}\varphi)\ (X, Z_2, \ldots, Z_{r+1}) \quad - (\nabla_{Z_1}\varphi)\ (\nabla_X X, Z_2, \ldots, Z_{r+1})$$

$$- \sum_{i=2}^{r+1} (\nabla_{Z_1}\varphi)\ (X, Z_2, \ldots, \nabla_X Z_i, \ldots, Z_{r+1})$$

$$- (\nabla_{\nabla_X Z_1}\varphi)\ (X, Z_2, \ldots, Z_{r+1})$$

$$= - X\ (\nabla_X\varphi)\ (Z_1, \ldots, Z_{r+1}) \quad + (\nabla_{\nabla_X X}\varphi)\ (Z_1, \ldots, Z_{r+1})$$

$$+ \sum_{i=2}^{r+1} (\nabla_X\varphi)\ (Z_1, Z_2, \ldots, \nabla_X Z_i, \ldots, Z_{r+1})$$

$$+ (\nabla_X\varphi)\ (\nabla_X Z_1, Z_2, \ldots, Z_{r+1}) \ .$$

The last step in the above equation follows from the fact that, for the class nearly parallel we have,

$$\nabla_X(\varphi)\ (Z_1, \ldots, Z_{r+1}) = - \nabla_{Z_1}(\varphi)\ (X, Z_2, \ldots, Z_{r+1}) \ .$$

Also in the R.H.S. of the last equation the first, third and fourth terms together reduce to

$- (\nabla_X\nabla_X\varphi)\ (Z_1, \ldots, Z_{r+1})$, and hence

$$\nabla^2\varphi(X; Z_1;\ X,\ Z_2, \ldots, Z_{r+1}) = - \nabla^2\varphi\ (X;\ X;\ Z_1, \ldots, Z_{r+1}) \ .$$

~~and~~ Therefore if P is nearly parallel then

$$R_{XZ_1}(\varphi)\ (X, Z_2, \ldots, Z_{r+1}) = \nabla^2\varphi\ (X;\ X;\ Z_1, \ldots, Z_{r+1}) \ .$$

Putting $Z_{r+1} = P(Z_1,\dots,Z_r)$, and noting that

$$\nabla^2\varphi(X,X,Z_1,\dots,Z_r, P(Z_1,\dots Z_r))$$

$$= -\langle \nabla_X(P)(Z_1,\dots,Z_r), \nabla_X(P)(Z_1,\dots,Z_r) \rangle ,$$

(Proposition (3.1)), we conclude that

$$\| \nabla_X(P)(Z_1,\dots,Z_r) \|^2 = - R_{XZ_1}(\varphi)(X,Z_2,\dots, P(Z_1,\dots,Z_r)) . \quad (3.15)$$

Expansion of the R.H.S. of (3.15) gives the result.

<u>COROLLARY</u> (3.8). The curvature operator of an almost Hermitian manifold whose almost complex structure is nearly parallel satisfies the identities:

$$\langle \nabla_W(J)(X), \nabla_Y(J)(Z) \rangle = \langle R_{WX}Y, Z \rangle - \langle R_{WX}JY, JZ \rangle \quad (3.15')$$

$$\langle R_{JWJX}JY, JZ \rangle = \langle R_{WX}Y, Z \rangle \quad (3.16)$$

$$K(JW,JX) = K(W,X) \quad (3.17)$$

$$\gamma(JW,JX) = \gamma(W,X) \quad (3.18)$$

<u>Proof</u>: As a special case of (3.15') we get

$$\| \nabla_X(J)(Y) \|^2 = \langle R_{XY}X, Y \rangle - \langle R_{XY}JX, JY \rangle \quad (3.19)$$

Linearization of (3.19) together with the Bianchi first identity and the fact that

$$\nabla_U(J)(V) + \nabla_{JU}(J)(JV) = 0 \quad \text{for all } U, V \in \mathcal{H}(M),$$

gives (3.15).

From (3.15) we get

$$\langle R_{WX}JY, JZ \rangle = \langle R_{WX}Y, Z \rangle - \langle \nabla_W(J)(X), \nabla_Y(J)(Z) \rangle$$

and $\langle R_{JWJX}Y, Z \rangle = \langle R_{YZ}JW, JX \rangle$

$$= \langle R_{YZ}W, X \rangle - \langle \nabla_Y(J)(Z), \nabla_W(J)(X) \rangle$$

and hence $\langle R_{WX}JY, JZ \rangle = \langle R_{JWJX}Y, Z \rangle$,

from which it follows that

$$\langle R_{JWJX}JY, JZ \rangle = \langle R_{WX}Y, Z \rangle .$$

This proves (3.16).

To prove (3.17) we take an orthogonal frame field $\{E_1, \ldots, E_n\}$ and write

$$K(JW,JX) = \sum_{i=1}^{n} \|E_i\|^{-2} \langle R_{E_i JW} E_i, JX \rangle$$

$$= \sum_{i=1}^{n} \|E_i\|^{-2} \langle R_{JE_i W} JE_i, X \rangle, \text{ (using (3.16))}.$$

Since $\{JE_1, \ldots, JE_n\}$ is an orthogonal frame field such that $\| JE_i \|^2 = \| E_i \|^2$, it follows that

$$\sum_{i=1}^{n} \| E_i \|^{-2} \langle R_{JE_iW} JE_i , X \rangle = K(W,X) .$$

This proves (3.17). For (3.18) we have

$$2\gamma(JW,JX) = \sum_{k=1}^{n} \| E_k \|^{-2} \langle R_{JWJX} E_k , JE_k \rangle$$

$$= - \sum_{k=1}^{n} \| E_k \|^{-2} \langle R_{WX} JE_k , E_k \rangle \qquad \text{(using (3.16))}$$

$$= \sum_{k=1}^{n} \| E_k \|^{-2} \langle R_{WX} E_k \; JE_k \rangle = 2\gamma(W,X).$$

Hence $\gamma(JW,JX) = \gamma(W,X)$.

4. 4. <u>6-Dimensional almost Complex Manifolds defined by means of a 3-fold vector cross product</u>.

In this section we use theorem (1.4) to generate a class of six-dimensional almost complex manifolds. We study these manifolds using the configuration tensor T (i.e. the second fundamental form).

Let $\bar{M}$ be a non-degenerate pseudo-riemannian manifold, of dimension 8, with a 3-fold vector cross product defined on it. It follows that the metric tensor $\langle\ ,\ \rangle$ on $\bar{M}$ is positive definite, negative definite or has signature (4,4). Since the case of negative definite metric is essentially the same as that of positive definite metric, we exclude it from our discussion.

Now let M be a non-degenerate 6-dimensional submanifold of $\bar{M}$ such that the restriction of $\langle\ ,\ \rangle$ to the normal bundle of M is positive definite. It follows that the metric tensor of M is positive definite or has signature (4,2).

If further, we assume that M is orientable, then the normal bundle of M is orientable since $\bar{M}$ is orientable. It follows that the normal bundle has an almost complex structure J. Choosing a unit normal N on an open subset of M, we have $N \wedge JN$ is independant of N and may be extended to a global vector 2-field on M and $\| N \wedge JN \|^2 = 1$.

<u>DEFINITION</u>: Let P denote the vector cross product of $\bar{M}$. Then $J: \mathcal{H}(M) \to \mathcal{H}(M)$ is defined by $JA = P(N,JN,A)$ for $A \in \mathcal{H}(M)$. That J is an almost complex structure follows from theorem (1.4).

<u>PROPOSITION</u> (4.1). Let $A, B \in \mathcal{H}(M)$ and let N be a unit normal on an open subset of M. Then

$$P(JA,B,N) + J\,P(A,B,N) = -\langle JA,B\rangle N - \langle A,B\rangle JN \tag{4.1}$$

$$P(JA,JB,N) + P(A,B,N) = 2 \langle JA,B \rangle JN \qquad (4.2)$$

Furthermore exactly one of the following equations holds (depending on the kind of 3-fold vector cross product on $\bar{M}$) for a given vector cross product P.

$$JP(N,A,B) = P(JN,A,B) \qquad (4.3)$$

$$JP(N,A,B) = -P(JN,A,B) - 2\langle JA,B \rangle N \qquad (4.4)$$

Proof: For all $A, B \in \mathcal{H}(M)$ we have

$$\langle P(A,JA,N), B \rangle = \langle P(N,B,A), P(N,JN,A) \rangle$$

$$= \tfrac{1}{2} \left\{ \|P(N,B+JN,A)\|^2 - \|P(N,JN,A)\|^2 - \|P(N,B,A)\|^2 \right\}$$

$$= \tfrac{1}{2} \left\{ \|A\|^2 (\|B\|^2 + 1) - \langle A,B \rangle^2 - \|A\|^2 - \|A\|^2 \|B\|^2 + \langle A,B \rangle^2 \right\} = 0.$$

Hence $P(A, JA, N)$, for $A \in \mathcal{H}(M)$, is always orthogonal to M. Hence for $C \in \mathcal{H}(M)$;

$$\langle P(JA, B, N) + JP(A, B, N), C \rangle$$

$$= \langle P(C, JA, N) + P(A, JC, N), B \rangle$$

The above equation follows from the alternating character of the form $\varphi(A,B,C,D) = \langle P(A,B,C),D \rangle$ and the fact that $\langle JA, B \rangle = -\langle A, JB \rangle$ for $A, B \in \mathcal{H}(M)$.

Also $\langle P(C, JA, N) + P(A, JC, N), B \rangle$

$$= \langle P(A+C, J(A+C), N), B \rangle$$

$$-\langle P(A, JA, N), B\rangle - \langle P(C, JC, N), B\rangle = 0 .$$

Hence $P(JA, B, N) + J\,P(A, B, N)$ is orthogonal to M.

Furthermore $\langle P(JA, B, N) + J\,P(A, B, N), N\rangle$

$$= -\langle P(A, B, N), JN\rangle = -\langle P(N, JN, A), B\rangle = +\langle JA, B\rangle ,$$

and $\langle P(JA, B, N) + J\,P(A, B, N), JN\rangle$

$$= -\langle P(N, JN, B), JA\rangle = -\langle A, B\rangle .$$

Hence (4.1) follows.

In (4.1) writing JB for B we get,

$$P(JA, JB, N) + J\,P(A, JB, N) = -\langle A,B\rangle N + \langle JA, B\rangle JN.$$

Also from (4.1) we get,

$$P(A, B, N) - JP(A, JB, N) = \langle JA, B\rangle JN + \langle A, B\rangle N .$$

Adding the above two equations gives (4.2).

To prove that one of (4.3) and (4.4) holds we may, because of (4.1), take A, JA, B, JB and $P(JN, A, B)$ to be linearly independant. Moreover both $JP(JN, A, B)$ and $P(N, A, B)$ are orthogonal to these five vector fields. Their components in the normal bundle are given by

$$\langle J\,P(JN, A, B), JN\rangle = -\langle JA, B\rangle$$

and

$$\langle P(N, A, B), JN\rangle = \langle JA, B\rangle .$$

Hence there exist functions α and β on M such that

$$\alpha \left(JP(JN, A, B) + \langle JA, B \rangle JN \right)$$

$$= \beta \left(P(N, A, B) - \langle JA, B \rangle JN \right)$$

Taking norms of both sides of the above equation gives $\alpha = \pm \beta$.

If $\alpha = +\beta$ then (4.4) holds and if $\alpha = -\beta$ then (4.3) holds.

However, if both equations hold then it follows that,

$$P(N, A, B) = -\langle JA, B \rangle JN$$

Taking norms of both sides of the above equation gives

$$\|A\|^2 \|B\|^2 - \langle A,B \rangle^2 = \langle JA, B \rangle^2$$

However this equation is false whenever A, B, JA, JB are non-isotropic and orthogonal.

For manifolds with a cayley multiplication there are two kinds of 3-fold vector cross products, each inducing an almost complex structure. For these almost complex structures we have the following result.

<u>PROPOSITION</u> (4.2) Let $\bar{M}$ be an 8-dimensional pseudo-riemannian manifold with a differentiable cayley multiplication and let M be a non-degenerate orientable 6-dimensional submanifold of $\bar{M}$ such that the restriction of the metric tensor of $\bar{M}$ to the normal bundle of M is positive definite. Let P_1, P_2 be the 3-fold vector cross products defined by cayley multiplication and are given explicitly by (4.8) and (4.9) of Chapter one with $\alpha = 1$, respectively. Then,

(i) P_1 defines an almost complex structure on M which satisfies (4.3).

(ii) P_2 defines an almost complex structure on M which satisfies (4.4).

(iii) The almost complex structures on M defined by P_1 and P_2 coincide if and only if M is always normal to the identity vector field of the cayley multiplication.

<u>Proof</u>: To prove (i) we have

$$P_1 (X,Y,Z) = - X(\bar{Y}Z) + \langle X,Y \rangle Z + \langle Y,Z \rangle X - \langle X,Z \rangle Y \quad \text{for all } X, Y, Z \in \mathcal{H}(\bar{M}).$$

Let $A, B \in \mathcal{H}(M)$ and let N be a unit normal on an open subset of M, then

$$JP_1(N,A,B) + \langle JA,B \rangle N = J\Big(P_1(N,A,B) - \langle JA,B \rangle JN\Big) .$$

Since $P_1(N,A,B) - \langle JA,B \rangle JN$ is tangent to M it follows that

$$JP_1(N,A,B) + \langle JA,B \rangle N = P_1(N,JN, P_1 (N,A,B))$$

$$= - P_1\Big(JN, N, P_1 (N, A, B)\Big) =$$

$$= JN\cdot\Big(\bar{N}\cdot P_1 (N, A, B)\Big) + \langle P_1 (N, A, B), JN \rangle N$$

$$= JN\cdot\Big(\bar{N}\cdot(-N(\bar{A}.B) + \langle A, B \rangle N)\Big) - \langle JB, A \rangle N$$

$$= -JN\cdot (\bar{A}\cdot B) + \langle A, B \rangle JN + \langle JA, B \rangle N$$

$$= P_1 (JN, A, B) + \langle JA, B \rangle N .$$

Hence

$$JP_1(N, A, B) = P_1 (JN, A, B) .$$

This proves (i).

To prove (ii) we have

$$P_2(X, Y, A) = -(X\bar{Y})\, Z + \langle X,Y \rangle Z + \langle Y, Z \rangle X$$

$$- \langle X, Z \rangle Y ,$$

for all $X, Y, Z \in \mathcal{H}(\bar{M})$.

In this case we have

$$JP_2(N, A, B) + \langle JA, B \rangle N = P_2\big(P_2(A, B, N), N, JN\big)$$

$$= -\big(P_2(A, B, N).\bar{N}\big).JN - \langle P_2(A, B, N), JN \rangle N$$

$$= \Big(\big((A.\bar{B}).N - \langle A, B \rangle N\big).\bar{N}\Big).JN - \langle JA, B \rangle N$$

$$= (A.\bar{B}).JN - \langle A, B \rangle JN - \langle JA, B \rangle N$$

$$= -P_2(A, B, JN) - \langle JA, B \rangle N .$$

Hence

$$JP_2(N, A, B) = -P_2(JN, A, B) - 2\langle JA, B \rangle N,$$

which proves (ii).

To prove (iii) we note that the almost complex structures J_1 and J_2 are given by

$$J_1A = P_1(N, JN, A) = -N.(\overline{JN}.A) ,$$

and $$J_2A = P_2(N, JN, A) = -(N.\overline{JN}).A ,$$

for all $A \in \mathcal{H}(M)$. Hence $J_1 = J_2$ if and only if

$N.(\overline{JN}.A) = (N.\overline{JN}).A$ (4.5) for all $A \in \mathcal{H}(M)$.

In a composition algebra the associator A is defined by $A(a,b,c) = (ab)c - a(bc)$. Also $A(a,a,b) = A(a,b,b) = 0$ implies that A is alternating. Hence if N, JN are both orthogonal to E (the identity) and we choose A such that

$\langle A,E \rangle = 0 = \langle JN.N,A \rangle$, then

$$(N.JN).A - N.(JN.A) = (JN.A).N - JN.(A.N)$$
$$= - \left[N.(JN.A) - JN.(N.A)\right] . \qquad (4.6)$$

In the second step we use the formula

$$X\bar{Y} + Y\bar{X} = 2\langle X,Y \rangle E.$$

If $J_1 = J_2$ it follows from (4.5) and (4.6) that

$$N.(JN.A) - JN(N.A) = 0 \qquad (4.5')$$

Also using the formula $a(\bar{b}x) + b(\bar{a}x) = 2\langle a,b\rangle x$, which is a linearization of $a(\bar{a}x) = (a\bar{a})x = N(a)x$, we get

$$N.(JN.A) + JN(N.A) = 0 \qquad (4.5'')$$

From (4.5') and (4.5") we get

$N.(JN.A) = 0$. However we can choose A such that $N.(JN.A) \neq 0$. Hence we conclude that E cannot be orthogonal to both N and JN.

Writing $E = aN + bJN + cC$ where b, say, $\neq 0$, and $C \in \mathcal{H}(M)$, we have $\overline{JN} = 2bE - JN$. Substituting the last two equations in

$(\overline{JN}.A).N - \overline{JN}(A.N) = 0$, gives

$$c\Big((N.C).A - N.(C.A)\Big) = 0 \quad \text{for all} \quad A \in \mathcal{H}(M).$$

Since the algebra is non-associative, we can find $A \in \mathcal{H}(M)$ such that

$(N.C).A - N.(C.A) \neq 0$, and hence we must have $c = 0$.

On the other hand if E is orthogonal to M everywhere, we set $N=E$. With this choice of N (4.5) is valid and hence $J_1=J_2$.

Let M be an orientable 6-dimensional submanifold of R^8, with the cayley multiplication at the origin of R^8 translated over all of R^8 in a natural way.

If M is contained in the hyperplane of pure cayley numbers (i.e. those orthogonal to the identity) then the almost complex structures on M defined by means of the two 3-fold vector cross products coincide with Calabi's almost complex structure [3] . On the other hand if M is not contained in the hyperplane of pure cayley numbers, then by proposition (4.2) the induced almost complex structures are different. For example, on S^6 we obtain some new almost complex structures which are hermitian with respect to the natural metric on S^6.

Next we investigate conditions, in terms of the second fundamental form, that the almost complex structure J defined by $JA = P(N,JN,A)$ be in one of the classes; $\mathcal{K}$, $\mathcal{NK}$, $\mathcal{AK}$, $\mathcal{QK}$, $\mathcal{SK}$ or $\mathcal{H}$.

First we establish the following lemma.

LEMMA (4.3) Let $L: \mathcal{H}(M)^{\perp} \longrightarrow \mathcal{H}(M)$ be a pointwise linear map. Then for $A, B \in \mathcal{H}(M)$ and $Z \in \mathcal{H}(M)^{\perp}$ we have

$$\langle P(LZ,JZ,A) + P(Z,LJZ,A), B \rangle$$
$$= \langle P(Z,LJZ \pm JLZ,A), B \rangle \qquad (4.7)$$

Proof. From (4.3), (4.4) it follows that

$$\langle P(LZ,JZ,A), B \rangle = \langle \pm JP(LZ,Z,A), B \rangle, \qquad (4.8)$$

Where if (4.3) holds then "±" is to be taken " + " and if (4.4) holds then "±" is to be taken "-" .

Also from (4.1) we get

$$\langle \pm JP(LZ,Z,A), B \rangle = \langle \mp P(JLZ,Z,A), B \rangle$$
$$= \langle \pm P(Z, JLZ, A), B \rangle . \qquad (4.9)$$

Hence combining (4.8) and (4.9) we get

$$\langle P(LZ, JZ, A) + P(Z, LJZ, A), B \rangle$$
$$= \langle \pm P(Z, JLZ, A) + P(Z, LJZ, A), B \rangle$$
$$= \langle P(Z, LJZ \pm JLZ, A), B \rangle .$$

We denote by F and φ the differential forms (on M and $\bar{M}$) associated with J and P.

THEOREM (4.4). Suppose $P \in \mathcal{P}$. Then $J \in \mathcal{K}$ if and only if

$$T_A JB \pm JT_A B = 0 \qquad (4.10)$$

for all $A, B \in \mathcal{H}(M)$.

Proof: For $A, B, C \in \mathcal{H}(M)$; N a unit normal on an open subset of M we have

$$\nabla_A(F)(B,C) = \bar{\bar{\nabla}}_A(\varphi)(N, JN, B, C)$$

$$+ \langle P(T_A N, JN, B) + P(N, T_A JN, B), C \rangle .$$

The above equation is a special case of (2.11), noting that

$$\nabla_A(F)(B,C) = \langle \nabla_A(J)(B), C \rangle ,$$

and $\bar{\bar{\nabla}}_A(\varphi)(N, JN, B, C) = \langle \bar{\bar{\nabla}}_A(P)(N, JN, B), C \rangle$.

Let $LZ = T_A Z$ for $Z \in \mathcal{H}(M)^1$. By Lemma (4.3) we have

$$\nabla_A(F)(B, C) = \bar{\bar{\nabla}}_A(\varphi)(N, JN, B, C)$$

$$+ \langle P(N, T_A JN \pm JT_A N, B), C \rangle \qquad (4.11)$$

If $P \in \mathcal{P}$, then (4.11) reduces to

$$\nabla_A(F)(B,C) = \langle P(N, T_A JN \pm JT_A N, B), C \rangle . \qquad (4.12)$$

$J \in \mathcal{K}$ if and only if $\nabla_A(F)(B,C) = 0$ for all $A, B, C \in \mathcal{H}(M)$. By (4.12) this is equivalent to the condition that

$T_A JN \pm JT_A N = 0$ for all $A \in \mathcal{H}(M)$ and unit normals N .

Hence for $B \in \mathcal{H}(M)$,

$$< T_A JN, B > \pm < JT_A N, B > = 0 ,$$

and therefore $- < T_A B, JN > \pm < T_A JB, N > = 0,$

which reduces to

$$< JT_A B, N > \pm < T_A JB, N > = 0 ,$$

for all $A, B \in \mathcal{H}(M)$, and unit normals N. The last equation gives the result.

<u>THEOREM</u> (4.5). Suppose $P \in \mathcal{P}$. Then $J \in \mathcal{H}$ if and only if

$$T_A A + T_{JA} JA = 0 , \text{ for } A \in \mathcal{H}(M) .$$

<u>Proof</u>. By (4.12) we have

$$\nabla_{JA}(F)(JB, C) = < P(N, T_{JA}JN \pm JT_{JA}N, JB), C > .$$

Also by (4.2);

$$< P(N, \pm JT_{JA}N, JB), C > = - < P(N, \pm T_{JA}N, B), C > ,$$

and $< P(N, T_{JA}JN, JB), C > = < P(N, JT_{JA}JN, B), C > .$

Combining the above two equations we get

$$\nabla_{JA}(F)(JB,C) = < P(N, JT_{JA}JN \mp T_{JA}N , B) , C > .$$

Hence

$$\nabla_A(F)(B, C) - \nabla_{JA}(F)(JB, C) =$$

$$= \langle P(N, T_A JN - J T_{JA} JN \pm (JT_A N + T_{JA} N), B), C \rangle .$$

Now $J \in \mathcal{H}$ if and only if

$$\nabla_A(F)(B,C) - \nabla_{JA}(F)(JB,C) = 0 \quad ([7]) .$$

Hence $J \in \mathcal{H}$ if and only if

$$(T_A JN - JT_{JA} JN) \pm (JT_A N + T_{JA} N) = 0 .$$

The last equation is equivalent to

$$J(T_A B + T_{JA} JB) \pm (T_A JB - T_{JA} B) = 0 , \qquad (4.14)$$

for all $A, B \in \mathcal{H}(M)$.

Now (4.14) implies (4.13). Moreover linearization of (4.13) gives

$$T_A B + T_{JA} JB = 0 , \qquad (4.15)$$

And hence putting JB instead of B in (4.15) gives:

$$T_A JB + T_{JA} J^2 B = 0 ,$$

and therefore

$$T_A JB - T_{JA} B = 0 \qquad (4.16).$$

(4.15) and (4.16) imply (4.14).

<u>COROLLARY</u> (4.6). Suppose $P \in \mathcal{P}$ and $J \in \mathcal{H}$. Then

(i) M is a minimal variety of $\bar{M}$

(ii) If R_{AB} and $\bar{R}_{AB}$ denote the curvature operators of M

and $\bar{M}$ respectively, then for $A, B, C, D \in \mathcal{H}(M)$,

$$\langle R_{AB}C, D \rangle - \langle R_{JAJB}{}^{JC}, JD \rangle = \langle \bar{R}_{AB}C, D \rangle - \langle \bar{R}_{JAJB}{}^{JC}, JD \rangle \quad (4.17)$$

(iii) If $\bar{M}$ is flat, then the Ricci curvature K and the chern form γ of M satisfy the relations

$$K(JA, JB) = K(A, B) = \gamma(JA, B) = -\gamma(A, B) ,$$

for $A, B \in \mathcal{H}(M)$.

(iv) On holomorphic fields of 2-planes tangent to M the holomorphic curvature is less than or equal to the corresponding sectional curvature of $\bar{M}$.

Proof. (i) follows from (4.13) and the fact that we can choose a frame field, on an open subset of M, of the form $\{E_1, JE_1, \ldots, E_m, JE_m\}$ where $n = 2m = \dim M$. Relative to such a frame the mean curvature normal H is given by

$$H = \sum_{i=1}^{m} (T_{E_i}E_i + T_{JE_i}JE_i) = 0,$$

on account of (4.13).

(ii) follows from the Gauss equation.

(iii) If $\bar{M}$ is flat then (4.17) reduces to $\langle R_{AB}^{\ C}, D \rangle =$ $\langle R_{JAJB}^{\ JC}, JD \rangle$. An application of this equation gives the result.

(iv) We have by the Gauss equation that for $A \in \mathcal{H}(M)$

$$K_{AJA} = \langle T_A A, T_{JA} JA \rangle - \| T_A JA \|^2 + \bar{K}_{AJA}$$

$$= - \| T_A A \|^2 - \| T_A JA \|^2 + \bar{K}_{AJA} .$$

<u>COROLLARY</u> (4.7) Suppose $P \in \mathcal{P}$, $J \in \mathcal{H}$, and the metric of $\bar{M}$ is positive definite. If the sectional curvature of $\bar{M}$ is non-positive, then M is non-compact.

<u>Proof</u>. By Corollary (4.6) M is a minimal submanifold. This implies, [12], that M cannot be compact.

Next we investigate the class $\mathfrak{QK}$.

<u>THEOREM</u> (4.8). Suppose $P \in \mathcal{P}$. Then $J \in \mathfrak{QK}$ if and only if for all $A \in \mathcal{H}(M)$,

$$J(T_A A - T_{JA} JA) \pm 2 T_A JA = 0 . \qquad (4.18)$$

<u>Proof</u>. As in theorem (4.5) we have,

$$\nabla_A(F)(B,C) + \nabla_{JA}(F)(JB, C) =$$

$$\langle P(N, T_A JN + J T_{JA} JN \pm (J T_A N - T_{JA} N)', B), C \rangle \cdot \quad (4.19)$$

$J \in \mathfrak{QK}$ if and only if for all $A, B, C \in \mathcal{H}(M)$,

$\nabla_A(F)(B,C) + \nabla_{JA}(F)(JB,C) = 0$. From (4.19)

this is equivalent to

$$T_A JN + J T_{JA} JN \pm (J T_A N - T_{JA} N) = 0 \qquad (4.20)$$

for all A $\mathcal{H}(M)$. Taking the scalar product of (4.20) with $B \in \mathcal{H}(M)$ gives:

$$J(T_A B - T_{JA} JB) \pm (T_A JB + T_{JA} B) = 0 \qquad (4.21)$$

for all $A, B \in \mathcal{H}(M)$.

Now, (4.21) implies (4.18) (take B=A). On the other hand, linearization of (4.18) gives (4.21).

PROPOSITION (4.9) Suppose $P \in \mathcal{P}$ and $J \in \mathcal{DK}$. Then the holomorphic curvature of M is given by the formula

$$2\, K_{AJA} = \| T_A A \|^2 + \| T_{JA} JA \|^2 - 6 \| T_A JA \|^2 + 2\, \bar{K}_{AJA}, \qquad (4.22)$$

for $A \in \mathcal{H}(M)$, $\| A \|^2 = \pm 1$.

Proof: By the Gauss equation we have

$$K_{AJA} = \langle T_A A, T_{JA} JA \rangle - \| T_A JA \|^2 + \bar{K}_{AJA}. \qquad (4.23)$$

Also (4.18) gives $\| T_A A - T_{JA} JA \|^2 = 4 \| T_A JA \|^2$.

Hence $\| T_A A \|^2 + \| T_{JA} JA \|^2 - 2 \langle T_A A , T_{JA} JA \rangle$

$$= 4 \| T_A JA \|^2.$$

And therefore, $\langle T_A A, T_{JA} JA \rangle = \frac{1}{2} \| T_A A \|^2 + \frac{1}{2} \| T_{JA} JA \|^2 - 2 \| T_A JA \|^2$.

Substitution in (4.23) gives,

$$2\, K_{AJA} = \| T_A A \|^2 + \| T_{JA} JA \|^2 - 6 \| T_A JA \|^2 + 2\, \bar{K}_{AJA}.$$

THEOREM (4.10). Suppose $P \in \mathcal{P}$. Then $J \in \mathcal{N}\mathcal{K}$ if and only if there exist a 1-form β on the normal bundle of M such that

$$T_A JN \pm J\,T_A N = \beta(JN)A \pm \beta(N)\,JA$$

for all $A \in \mathcal{H}(M)$.

Proof: We have $\nabla_A(F)(A,B) = \langle P(N, T_A JN \pm JT_A N, A), B \rangle$,

for $A, B \in \mathcal{H}(M)$. If $J \in \mathcal{N}\mathcal{K}$ then

$$T_A JN \pm JT_A N = \beta(A,JN)A + \alpha(A,JN)JA \,. \qquad (4.24)$$

Since $J \in \mathcal{Q}\mathcal{K}$ it follows by (4.20) and (4.24) that

$\alpha(A,JN) = \pm\, \beta(A,N)$ and β is independant of A.

From (4.24) it also follows that β is linear on the normal bundle.

On the other hand if (4.24) holds then

$\nabla_A(F)(A,B) = 0$ for all $A, B \in \mathcal{H}(M)$ and hence J is nearly parallel.

THEOREM (4.11). Suppose $P \in \mathcal{P}$ and $J \in \mathcal{A}\mathcal{K}$. Then M is a minimal variety of $\bar{M}$.

Proof: Let N be a unit normal on an open subset of M. We choose an orthogonal frame field on this open subset of the form $\{E_1, JE_1, E_2, JE_2, E_3, JE_3\}$ such that

$\|E_i\|^2 = \pm 1$ and $P(N, E_i, E_j) = \|E_k\|^2 E_k$, where

(i, j, k) is a cyclic permutation of $(1, 2, 3)$. Then

$$< H, JN > = \sum_{i=1}^{3} \| E_i \|^2 < T_{E_i}E_i + T_{JE_i}JE_i, JN >$$

$$= \sum_{i=1}^{3} \| E_i \|^2 < JT_{JE_i}JN - T_{E_i}JN, E_i >$$

$$= \sum_{G} < P(N,E_j,E_k) , J T_{JE_i}JN - T_{E_i}JN >$$

$$= \sum_{G} < P(N,J T_{JE_i}JN - T_{E_i}JN, E_j) , E_k > .$$

Hence

$$< H, JN > = \sum_{G} < P(N,JT_{JE_i}JN - T_{E_i}JN, E_j), E_k > . \qquad (4.25)$$

Here H is the mean curvature normal and the above equation is obtained using the definition of H, the symmetry properties of the second fundamental form and the special choice of the frame field.

Using an equation derived in theorem (4.5) namely;

$$\nabla_A(F)(B,C) - \nabla_{JA}(F)(JB,C)$$

$$= < P(N,T_A JN - JT_{JA}JN \pm (JT_A N + T_{JA}N), B), C > ,$$

(4.25) reduces to

$$< H,JN > = - \sum_{G} \left\{ \nabla_{E_i}(F)(E_j,E_k) - \nabla_{JE_i}(F)(JE_j,E_k) \right\}$$

$$\pm \sum_{G} < P(N,JT_{E_i}N + T_{JE_i}N , E_j) , E_k > \qquad (4.26).$$

Since J is quasi-kahler it follows that

$$- \sum_{\mathfrak{S}} \left\{ \nabla_{E_i}(F)\,(E_j, E_k) - \nabla_{JE_i}(F)\,(JE_j, E_k) \right\}$$

$$= -2 \sum_{\mathfrak{S}} \nabla_{E_i}(F)\,(E_j, E_k) = -2dF(E_1, E_2, E_3) = 0 .$$

Also, $\sum_{\mathfrak{S}} \langle P(N, JT_{E_i}N + T_{JE_i}N, E_j), E_k \rangle$

$$= \sum_{\mathfrak{S}} \langle P(N, E_j, E_k), JT_{E_i}N + T_{JE_i}N \rangle$$

$$= \sum_{i=1}^{3} \langle E_i, JT_{E_i}N + T_{JE_i}N \rangle \, \| E_i \|^2 = 0$$

The last equality follows from the symmetry properties of the second fundamental form.

Since N is an arbitrary unit normal it follows that $H \mp 0$.

CHAPTER 3.

Nearly Kahler Manifolds.

In this chapter we investigate 6-dimensional nearly kahler manifolds defined by means of a 2- or 3-fold vector cross product.

1. Nearly Kahler Manifolds defined by a 2- or 3-fold vector cross product.

First we state the following notions introduced by GRAY [8].

DEFINITION. Let M be an almost hermitian manifold. Then M is said to be of constant type at $m \in M$ provided that for all $x \in M_m$ we have

$$\| \nabla_X(J)\ (Y) \| = \| \nabla_X(J)\ (Z) \| \quad , \text{ whenever}$$

$$\langle X, Y \rangle = \langle JX, Y \rangle = \langle X, Z \rangle = \langle JX, Z \rangle = 0 \,,$$

and $\| Y \| = \| Z \|$. If this holds for all $m \in M$ we say that M has (pointwise) constant type. Finally if M has pointwise constant type and for $X, Y \in \mathcal{X}(M)$, with $\langle X, Y \rangle = \langle JX, Y \rangle = 0$, the function

$$\| \nabla_X(J)\ (Y) \|^2 \quad \text{is constant whenever} \quad \| X \| = \| Y \| = 1,$$

then we say that M has global constant type.

PROPOSITION (1.1) Let M be a nearly kahler manifold. Then M has (pointwise) constant type if and only if there exists $\alpha \in \mathcal{F}(M)$ such that

$$\| \nabla_W(J)\ (X) \|^2 = \alpha \left\{ \|W\|^2 \|X\|^2 - \langle W,X\rangle^2 - \langle W,JX\rangle^2 \right\}, \quad (1.1)$$

for all $W, X \in \mathcal{H}(M)$. Furthermore, M has global constant type if and only if (1.1) holds with a constant α.

__Proof__: Assume that M has pointwise constant type. Then by definition and the assumption that M is nearly kahler, we have, that for $X, Y \in M_m$, $m \in M$;

$$\| \nabla_X(J)\ (Y) \|^2 = \alpha_m, \text{ whenever } \langle X,Y\rangle = \langle JX,Y\rangle = 0,$$

$\|X\| = \|Y\| = 1$ and α_m a constant.

Now for $W, X \in \mathcal{H}(M)$, write

$$\bar{X} = X - \langle X,W\rangle \frac{W}{\|W\|^2} - \langle X,JW\rangle \frac{JW}{\|W\|^2}.$$

Then $\langle \bar{X}, W\rangle = 0 = \langle \bar{X},JW\rangle$

and
$$\|\bar{X}\|^2 = \|X\|^2 + \frac{\langle X,W\rangle^2}{\|W\|^2} + \frac{\langle X,JW\rangle^2}{\|W\|^2}$$

$$- 2\frac{\langle X,W\rangle^2}{\|W\|^2} - 2\frac{\langle X,JW\rangle^2}{\|W\|^2}$$

$$= \|X\|^2 - \frac{\langle X,W\rangle^2}{\|W\|^2} - \frac{\langle X,JW\rangle^2}{\|W\|^2}.$$

Further,

$$\left\| \nabla_{\frac{W}{\|W\|}}(J)\ \left(\frac{\bar{X}}{\|\bar{X}\|}\right) \right\|^2 = \frac{1}{\|W\|^2}\ \frac{1}{\|\bar{X}\|^2}\ \| \nabla_W(J)\ (X) \|^2$$

In the above equality we use the fact that

$$\| \nabla_W(J)\ (\bar{X}) \|^2 = \| \nabla_W(J)\ (X) \|^2 ,$$

which is valid since J is nearly kahler.

Hence $\| \nabla_W(J)\ (X) \|^2 = \alpha \| W \|^2 \ \| \bar{X} \|^2$

$$= \alpha \left\{ \| W \|^2 \ \| X \|^2 - < W,X >^2 - < W,JX >^2 \right\} .$$

The converse is obviously true.

PROPOSITION (1.2). Let M be a connected nearly kahler manifold of pointwise constant type α.

If $(\nabla^2 J)\ (X_j X_j) = 0$, for $X \in \mathcal{H}\ (M)$, then M has global constant type.

Proof. Linearization of (1.1) gives:

$$< \nabla_W(J)\ (X),\ \nabla_W(J)\ (Y) > =$$

$$= \alpha \left\{ \| W \|^2 < X,Y > - < W,X > < W,Y > - < W,JX > < W,JY > \right\} . \quad (1.2)$$

On an open subset of M let W,X be vector fields such that $\| W \| = \| X \| = 1$, $< W,X > = < W,JX > = 0.$ Then

$\| \nabla_W(J)\ (X) \|^2 = \alpha$, and hence

$W \| \nabla_W(J)\ (X) \|^2 = W\alpha$, and therefore

$$2 \langle (\nabla_W^2 J)(X), \nabla_W(J)(X) \rangle + 2 \langle (\nabla_W J)(\nabla_W X), \nabla_W(J)(X) \rangle = W\alpha \quad (1.3)$$

Using (1.2), we see that the second term in the L.H.S. of (1.3) vanishes.

Hence for vector fields $W, X,$ such that $\|W\| = \|X\| = 1$, $\langle W, X \rangle = \langle W, JX \rangle = 0$, we have

$$2 \langle (\nabla_W^2 J)(X), \nabla_W(J)(X) \rangle = W\alpha \quad . \qquad (1.4)$$

Now let $m \in M$ be an arbitrary point in M, $U(m)$ a normal nbd of m and γ_t any minimal geodesic in $U(m)$ starting at m, and parametrized by arc length.

We choose $X \in M_m$, such that $\|X\| = 1$,

$\langle X, \dot{\gamma}_0 \rangle = 0 = \langle X, J\dot{\gamma}_0 \rangle$, and define X_t along γ_t by parallel translating X along γ_t. Since M is nearly kahler it follows that $J\dot{\gamma}_t$ is parallel along γ_t.

Hence

$\langle X_t, \dot{\gamma}_t \rangle = 0 = \langle X_t, J\dot{\gamma}_t \rangle$ and

$$\|X_t\| = \|\dot{\gamma}_t\| = 1 \quad .$$

Substituting in (1.4) we get

$$2 \langle (\nabla_{\dot{\gamma}_t}^2 J)(X_t), (\nabla_{\dot{\gamma}_t} J)(X_t) \rangle = \dot{\gamma}_t \alpha \quad . \qquad (1.5)$$

If $(\nabla^2 J)(X;X;) = 0$ then $\nabla^2_X J = \nabla_{\nabla_X X} J$

Hence $\nabla^2_{\dot{\gamma}_t} J = \nabla_{\nabla_{\dot{\gamma}_t}\dot{\gamma}_t} J = 0$ and (1.5)

reduces to $\dot{\gamma}_t \alpha = 0$. Hence α is constant along geodesics emanating from m. Since such geodesics cover a nbd of m it follows that α is locally constant. If M is connected then α is a global constant.

Let $\bar{M}$ be a 7-dimensional riemannian manifold with a 2-fold vector cross product P ; M an orientable 6-dimensional submanifold of $\bar{M}$. Then, as in Chapter 2, we can define an almost complex structure on M as follows; for each $X \in \mathcal{X}(M)$, we set

$J(X) = P(N,X)$, (1.6) where N is a unit normal vector field on M choosen in such a way that it is compatible with the orientations of M and $\bar{M}$.

In the special case when $\bar{M} = R^7$ with the ordinary 2-fold vector cross product obtained by parallel translation, we have the following result;

THEOREM (1.3). Let M be a connected orientable hypersurface of R^7, with the almost complex structure defined in (1.6). If M is nearly kahler then the following are true:

(i) M has constant non-negative sectional curvature.

(ii) M is Einstein.

(iii) M has global constant type, the type of M being the sectional curvature.

(iv) M has non-negative holomorphic bisectional curvature.

We first prove the following Lemma:

LEMMA (1.4). A nearly kahler hypersurface (M, J) of R^7 is totally umbilic.

Proof: For each X, Y $\in \mathcal{H}(M)$, we have

$$\nabla_X(J)(Y) = \nabla_X JY - J\nabla_X Y$$

$$= \nabla_X P(N,Y) - P(N,\nabla_X Y)$$

$$= \pi\,\bar{\nabla}_X P(N,Y) - P(N,\bar{\nabla}_X Y)$$

where $\pi : \bar{\mathcal{H}}(M) \longrightarrow \mathcal{H}(M)$ is the natural projection, $\bar{\nabla}$, ∇ are the riemannian connections on R^7, M, respectively, and in the second term $\nabla_X Y$ is replaced by $\bar{\nabla}_X Y$ since the normal component of $\bar{\nabla}_X Y$ is parallel to N.

Hence

$$\nabla_X(J)(Y) = \pi\,\bar{\nabla}_X(P)(N,Y) + \pi\,P(\bar{\nabla}_X N, Y)$$

$$+\ \pi\,P(N, \bar{\nabla}_X Y) - P(N, \bar{\nabla}_X Y)\ . \qquad (1.7)$$

Since $\pi\,P(N,\bar{\nabla}_X Y) = P(N,\bar{\nabla}_X Y)$ and P is parallel, (1.7) reduces to:

$$\nabla_X(J)(Y) = \pi\,P(\bar{\nabla}_X N, Y)\ , \quad X, Y \in \mathcal{H}(M)\ .$$

Hence $\nabla_X(J)(X) = \pi\,P(\bar{\nabla}_X N, X)$ (1.8)

If J is nearly kahler than (1.8) gives:

$$\langle P(\bar{\nabla}_X N, X), Y \rangle = 0 \quad (1.9) \quad \text{for all } X, Y \in \mathcal{H}(M).$$

Also for each $X, Y \in \mathcal{H}(M)$ we have

$$\nabla_X(J)(JY) = \nabla_X(-Y) - J\nabla_X JY$$

$$= -\nabla_X Y - J\left(\nabla_X(J)(Y) + J\nabla_X Y\right)$$

$$= -J\nabla_X(J)(Y).$$

Now if J is nearly kahler, then

$$\nabla_{JX}(J)(JY) = -J\nabla_{JX}(J)(Y) = J\nabla_Y(J)(JX)$$

$$= -J^2\nabla_Y(J)(X) = -\nabla_X(J)(Y).$$

Hence $\nabla_X(J)(Y) + \nabla_{JX}(J)(JY) = 0$, and therefore

for all $X, Y, Z \in \mathcal{H}(M)$, we have

$$\langle \nabla_X(J)(Y), Z \rangle + \langle \nabla_{JX}(J)(JY), Z \rangle = 0 \quad . (1.10)$$

Hence (1.7) and (1.10) give

$$\langle P(\bar{\nabla}_X N, Y), Z \rangle + \langle P(\bar{\nabla}_{JX} N, JY), Z \rangle = 0 \quad (1.11)$$

Using the alternating character of the trilinear vector valued function;

$$[A, B, C] = (A \times B) \times C - \langle A, C \rangle B + \langle B, C \rangle A$$

where $A \times B = P(A, B)$, and noting that $JY = N \times Y = P(N, Y)$ we deduce that

$$P(\bar{\bar{\nabla}}_{JX}N, JY) = P(J\bar{\bar{\nabla}}_{JX}N, Y) + 2 \langle Y, \bar{\bar{\nabla}}_{JX}N \rangle N .$$

Hence (1.11) reduces to

$$\langle P(\bar{\bar{\nabla}}_X N + J\bar{\bar{\nabla}}_{JX}N, Y), Z \rangle = 0 \quad (1.12),$$ for all $X, Y, Z \in \mathcal{H}(M)$.

Now for each $X \in \mathcal{H}(M)$, $\bar{\bar{\nabla}}_X N$ is tangent to M. Writing $T_X N$ for $\bar{\bar{\nabla}}_X N$, then it is well known that

$X \longrightarrow T_X N$, $X \in M_m$ is a symmetric endomorphism on M_m for each $m \in M$. Then for a nearly kahler hypersurface M we have,

$$\langle P(T_X N, X), Y \rangle = 0 = \langle P(Y,X), T_X N \rangle, \quad (1.9')$$

for all $X, Y \in \mathcal{H}(M)$, and

$$\langle P(Y, Z), T_X N + J T_{JX} N \rangle = 0, \quad (1.12')$$

for all $X, Y, Z \in \mathcal{H}(M)$.

To complete the proof, we first establish the following proposition:

<u>PROPOSITION</u> (1.5). Let $m \in M$, $0 \neq X \in M_m$. Then for each $Z \in M_m$, such that $Z \perp X$, $Z \perp JX$, there exists $Y \in M_m$ and $P(Y, X) = Z$.

Proof: Consider the endomorphism

$$P(\cdot\ , X) : [X]^{\perp} \text{ (in } R^7_m) \longrightarrow [X]^{\perp} \text{ (in } R^7_m)$$

$$: Y \longmapsto P(Y, X) .$$

If $Y \in [X]^{\perp}$, $Y \neq 0$, then

$$\| P(Y, X) \|^2 = \| X \|^2 \| Y \|^2 \neq 0$$

Hence $Y \longmapsto P(Y, X)$ is an automorphism, and therefore if $Z \in [X]^{\perp}$ i.e. $Z \perp X$, then there exists $Y \in [X]^{\perp}$ such that $P(Y, X) = Z$. Moreover since $Z \perp JX$ it follows that

$0 = \langle P(Y,X) , P(N,X) \rangle = \| X \|^2 \langle Y,N \rangle$, and hence $Y \perp N$.

Now using proposition (1.5), (1.9) becomes

$$T_X N = \alpha X + \beta J X \quad , \qquad (1.13)$$

for each $X \in \mathcal{H}(M)$, where α , β are functions on M . Also (1.12) becomes $T_X N + J T_{JX} N = 0$, for each $X \in \mathcal{H}(M)$ or equivalently $J T_X N = T_{JX} N$ (1.14)

The fact that α and β in (1.13) are independant of X follows from the linearity of $T_X N$ in X .

Now $\langle T_X N , JX \rangle = - \langle J T_X N , X \rangle$

$$= - \langle T_{JX} N , X \rangle = - \langle T_X N , JX \rangle \quad .$$

where in the second step we use (1.14) and in the last step we use the symmetry of $X \longrightarrow T_X N$.

Hence $< T_X N , JX > = 0$ which implies that $\beta = 0$,

and therefore $T_X N = \alpha X$, (1.15)

for all $X \in \mathcal{H}(M)$, and α is a real valued function on M . Since $X \longrightarrow T_X N$ is a differentiable (1,1) tensor field on M , we have trace $X \longrightarrow T_X N = n\alpha$ is differentiable. Here $n = \dim M$. This completes the proof of lemma (1.4).

We proceed to show that α is a constant function.

Writing $T_X N = T(X)$, we have $T = \alpha I$, I is the identity transformation. Using the Codazzi equation for a hypersurface in E^{n+1} , i.e. $\nabla_X(T)(Y) = \nabla_Y(T)(X)$ we get $(X\alpha)Y - (Y\alpha)X = 0$. Locally in a nbd U we may choose vector fields X, Y on U such that X_x and Y_x are linearly independant for each $x \in U$ and get $X_x(\alpha) = Y_x(\alpha) = 0$. This implies that α is a constant in U.

<u>Proof of Theorem</u> (1.3) It follows from Lemma (1.4) that, for each $X \in \mathcal{H}(M)$, $T(X) = \alpha X$. (1.16)

Also since M is connected then α is a constant function on M. For a nearly kahler manifold it is known that

$$< R_{XY}X, Y > - < R_{XY}JX, JY > = \| \nabla_X(J)(Y) \|^2 . \qquad (1.17)$$

Also for a hypersurface in E^{n+1} we have,

$$R_{XY}Z = < T(X), Z > T(Y) - < T(Y), Z > T(X) \qquad (1.18)$$

where $R_{XY}Z = \nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y] Z$ is the curvature operator on M, T the symmetric endomorphism defined earlier.

Substituting $T = \alpha I$ in (1.18) gives

$$R_{XY}Z = \alpha^2 \left[< X,Z > Y - < Y,Z > X \right] . \qquad (1.19)$$

To prove (i) we take a 2-plane $p = X \wedge Y$. Then the sectional curvature is given by

$$K(P) = \frac{< R_{XY}X , Y >}{\| X \wedge Y \|^2}$$

$$= \frac{\alpha^2 \left[\|X\|^2 \|Y\|^2 - < X,Y >^2 \right]}{\| X \wedge Y \|^2} = \alpha^2 \geq 0.$$

For (ii) we have

$$Ric(X,Y) = \sum_{i=1}^{n} < R_{E_i X}^{E_i} , Y > , \text{ where } \{E_i\}$$

is a local orthonormal frame field on M. Hence from (1.19) we get

$$\text{Ric}\,(X,Y) = \alpha^2 \left[\sum_i \langle E_i, E_i \rangle \langle X,Y \rangle - \sum_i \langle E_i, X \rangle \langle E_i, Y \rangle \right]$$

$$= n\,\alpha^2 \langle X,Y \rangle - \alpha^2 \langle X,Y \rangle$$

$$= \alpha^2 (n-1) \langle X, Y \rangle .$$

Again using (1.19) we get

$$\langle R_{XY}X, Y \rangle - \langle R_{XY}JX, JY \rangle$$

$$= \alpha^2 \left[\| X \|^2 \| Y \|^2 - \langle X,Y \rangle^2 - \langle X,JY \rangle^2 \right] ,$$

and therefore

$$\| \nabla_X(J)\,(Y) \|^2 = \alpha^2 \left[\| X \|^2 \| Y \|^2 - \langle X,Y \rangle^2 - \langle X,JY \rangle^2 \right] .$$

It then follows from proposition (1.1) that M has global constant type, the type being $\alpha^2 = K$.

To prove (iv), the holomorphic bisectional curvature B is given by

$$B_{XY} \| X \|^2 \| Y \|^2 = \langle R_{XJX}Y , JY \rangle$$

for $X, Y \in \mathcal{H}(M)$, $X \neq 0 \neq Y$. Using (1.19) we get

$$\langle R_{XJX}Y , JY \rangle = \frac{\alpha^2 \left[\langle Y,X \rangle \langle JX,JY \rangle - \langle Y,JX \rangle \langle X,JY \rangle \right]}{\| X \|^2 \| Y \|^2}$$

$$= \frac{\alpha^2 \left[\langle Y,X \rangle^2 + \langle X,JY \rangle^2 \right]}{\| X \|^2 \| Y \|^2} \geq 0 .$$

<u>THEOREM</u> (1.6). Let (M,f) be a connected orientable hypersurface in R^7 ($f:M \longrightarrow R^7$ is an isometric imersion). Further assume M is complete as a riemannian manifold. If $f(M)$ is nearly kahlerian with respect to the induced almost complex structure then either M is isometrically imbedded as a hyperplane or M is isometrically imbedded as a hypersphere of a certain radius.

<u>Proof</u>: Theorem (1.6) follows from Lemma (1.4) and the following theorem [10].

<u>THEOREM</u> (1.7) Let (M^n,f) be a connected hypersurface in E^{n+1}, where M is complete as a riemannian manifold. If $f(M)$ is umbilic at each point then either $f(M)$ is a hyperplane or $f(M)$ is a hypersphere of a certain radius. In both cases $f:M^n \longrightarrow E^{n+1}$ is an isometric imbedding.

Next we give a generalization of theorem (1.5) of Chapter 2.

<u>THEOREM</u> (1.8). Let M be a connected compact hypersurface in R^{n+1} $(n \geq 2)$. If the Gaussian curvature K_n of M never vanishes on M, then the following are equivalent:

(i) M admits an almost complex structure J;

(ii) The dimension of M is either 2 or 6.

<u>Proof</u>: If the Gaussian curvature K_n of M never vanishes on M then, ([10]), M is orientable and the spherical map of Gaus $\varphi : M \longrightarrow S^n$ is a diffeomorphism.

(ii) $\Longrightarrow$ (i). Assume that the dimension of M is either 2, or 6. Then in this case M is either an orientable hypersurface in R^3 or an orientable hypersurface in R^7. In both cases M admits an almost complex structure, namely that induced by the ordinary 2-vector cross product in R^3 or R^7.

(i) $\Longrightarrow$ (ii). Let J_m denote the almost complex structure on M at $m \in M$. We define $P:(R^{n+1})^2 \longrightarrow R^{n+1}$ as follows:

for $a_1, a_2 \in R^{n+1}$ we write $a_2 = b + c$ where b is the component of a_2 orthogonal to a_1. If b=o we set $P(a_1, a_2) = 0$. If $b \neq o$ let $d = b\|b\|^{-1}$, Then d is a point on S^n and we write $m = \varphi^{-1}(d)$ and set

$$P(a_1, a_2) = \|b\| \; J_m(a_1). \qquad (1.20)$$

P, so defined, is continuous since both φ and J are continuous. Also we have using (1.20):

$$\langle P(a_1, a_2), a_1 \rangle = \langle \|b\| J_m(a_1), a_1 \rangle = 0,$$

$$\langle P(a_1, a_2), a_2 \rangle = \|b\| \langle J_m(a_1), b + c \rangle$$

$$= \|b\| \langle J_m(a_1), b \rangle + \|b\| \langle J_m(a_1), c \rangle .$$

Since C is parallel to a_1 the second term in the R.H.S. of the above equation vanishes. Also since b is orthogonal to M at m , and $J_m(a_1)$ is tangent to M at m , it follows that the first term also vanishes. Hence

$$< P(a_1, a_2) , a_2 > = 0$$

Finally,

$$\| P(a_1, a_2) \|^2 = \| b \|^2 \| a_1 \|^2 = \| a_1 \wedge b \|^2 = \| a_1 \wedge a_2 \|^2 .$$

Hence P is a continuous 2-fold vector cross product, and therefore n + 1 = 3 or 7 which implies that the dimension of M is either 2 or 6.

Note: We may note that hypersurfaces of dimension 8 satisfying the hypotheses of theorem (1.8) have no three-fold vector cross products. If they were to admit a 3-fold vector cross product then, using a similar technique to that of theorem (1.8), we would be able to define a 4-fold vector cross product on R^9 which is impossible.

Finally, for nearly kahler 6-dimensional submanifolds of an 8-dimensional manifold we have the following result.

THEOREM (1.9) Let $\bar{M}$ be a pseudo-riemannian 8-dimensional manifold with a parallel 3-fold vector cross product P. Let M be a 6-dimensional orientable submanifold in $\bar{M}$ such that the restriction of the metric tensor of $\bar{M}$ to M is non-degenerate; and positive definite on the normal bundle of M. If M is nearly kahler with respect to the induced almost complex structure J , then

(1) M has pointwise constant type.

(2) The holomorphic sectional curvature of M is given by

$K_{AJA} = \mu^2 - 2 \| T_A JA \|^2 + , \bar{K}_{AJA}$, (1.21) where

A is such that $\| A \|^2 = \pm 1$, and μ^2 is the type of M.

<u>Proof</u>: If M is nearly kahler then as in Chapter 2, it follows that there exists a 1-form β on the normal bundle such that

$$T_A JN \pm J T_A N = \beta(JN) A \pm \beta(N) JA \quad , \quad (1.22)$$

for all $A \in \mathcal{H}(M)$. Further, we also have

$$\nabla_X(J)(Y) = \pi P(N, T_X JN \pm JT_X N, Y) \quad , \quad (1.23)$$

for all $X, Y \in \mathcal{H}(M)$. Since

$$\pi P(N, T_X JN \pm JT_X N, Y) = P(N, T_X JN \pm JT_X N, Y)$$
$$+ < JY, T_X JN \pm JT_X N > JN ,$$

it follows that

$$\| \nabla_X(J)(Y) \|^2 = \| P(N, T_X JN \pm JT_X N, Y) + < JY, T_X JN \pm JT_X N > JN \|^2$$

$$= \| Y \|^2 \| T_X JN \pm J T_X N \|^2 - < T_X JN \pm J T_X N, Y >^2$$

$$- < T_X JN \pm J T_X N, J Y >^2 \quad (1.24)$$

Writing $\beta(N) = \beta_1$, $\beta(JN) = \beta_2$ and using (1.22) we see that

$$\| T_X JN \pm J T_X N \|^2 = (\beta_1^2 + \beta_2^2) \| X \|^2 ;$$

$$< T_X JN \pm JT_X N, Y >^2 = \beta_2^2 < X,Y >^2 + \beta_1^2 < JX,Y >^2$$
$$\pm 2 \beta_1 \beta_2 < X,Y > < JX,Y > ,$$

$$< T_X JN \pm JT_X N, JY >^2 = \beta_2^2 < X,JY >^2 + \beta_1^2 < JX,JY >^2 \pm 2 \beta_1 \beta_2 < X,JY > \times$$
$$\times < JX,JY >$$

substituting the last three equations in (1.24) we get

$$\| \nabla_X (J)(Y) \|^2 = (\beta_1^2 + \beta_2^2) \left\{ \| X \|^2 \| Y \|^2 - < X,Y >^2 - < X,JY >^2 \right\} \quad (1,25).$$

Next we prove that $(\beta_1^2 + \beta_2^2)$ is independant of the choice of (N,JN). Let $(\bar{N}, J\bar{N})$ be orthonormal and compatible with the orientations of M and $\bar{M}$, then we have

$$\begin{pmatrix} \bar{N} \\ J\bar{N} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_1 \end{pmatrix} \begin{pmatrix} N \\ JN \end{pmatrix} ; \quad \alpha_1^2 + \alpha_2^2 = 1 ;$$

$\bar{\beta}_1 = \beta(\bar{N})$ and $\bar{\beta}_2 = \beta(J\bar{N})$. We have

$$\bar{\beta}_1^2 + \bar{\beta}_2^2 = \| T_A J\bar{N} \pm JT_A \bar{N} \|^2$$

$$= \| \alpha_1 (T_A JN \pm JT_A N) + \alpha_2 J (JT_A N \pm T_A JN) \|^2$$

$$= \alpha_1^2 \| T_A JN \pm JT_A N \|^2 + \alpha_2^2 \| JT_A N \pm T_A JN \|^2$$

$$= \alpha_1^2 \left(\beta_1^2 + \beta_2^2 \right) + \alpha_2^2 \left(\beta_1^2 + \beta_2^2 \right) = \beta_1^2 + \beta_2^2 .$$

Hence $(\beta_1^2 + \beta_2^2) = \mu^2$ (say) is a differentiable function on M and (1) follows from (1.25).

Taking the scalar product of (1.22) with A gives; $(\|A\|^2 = \pm 1)$

$$-\langle T_A A, JN \rangle \pm \langle T_A JA, N \rangle = \pm \beta_2 . \qquad (1.26)$$

Also scalarly multiplying (1.22) with JA gives

$$-\langle T_A JA, JN \rangle \mp \langle T_A A, N \rangle = \pm \beta_1 . \qquad (1.27)$$

Squaring (1.26) and (1.27) and adding we get

$$\| T_A A \|^2 + \| T_A JA \|^2 \pm 2 \langle JT_A A, T_A JA \rangle = \mu^2 . \qquad (1.28)$$

Since M is quasi-kahler we have by theorem (4.8) of Chapter 2:

$$\pm 2 T_A JA = - J(T_A A - T_{JA} JA) \qquad (1.29)$$

and therefore

$$\pm 2 \langle T_A JA, JT_A A \rangle = - \| T_A A \|^2 + \langle T_A A, T_{JA} JA \rangle \qquad (1.30)$$

Hence (1.28) and (1.30) give

$$\| T_A JA \|^2 + \langle T_A A, T_{JA} JA \rangle = \mu^2 \qquad (1.31)$$

By the Gauss equation we have

$$K_{AJA} = \langle T_A A, T_{JA} JA \rangle - \| T_A JA \|^2 + \bar{K}_{AJA}$$

Hence using (1.31) we get

$$K_{AJA} = \mu^2 - 2 \| T_A JA \|^2 + \bar{K}_{AJA} .$$

CHAPTER 4

Integrability of a Vector Cross Product Structure

For an almost complex structure on a differentiable manifold a certain concept of integrability is defined. In what follows we make an attempt towards a generalization of that concept to vector cross products.

1. Integrability of an almost complex structure

DEFINITION. Given A and B two tensor fields of type (1,1) on a differentiable manifold M, the torsion tensor S, of type (1,2), of A and B is defined as

$$S(X,Y) = [AX, BY] + [BX, AY] + AB[X,Y] + BA[X,Y] - A[X, BY] - A[BX,Y] - B[X,AY] - B[AX, Y], \quad X,Y \in \mathcal{H}(M).$$

The construction of S was discovered by NIJENHUIS [11]. In the special case when A=J, B=J (J an almost complex structure on M) the torsion of J is given by

$$S(X,Y) = 2\left\{[JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\right\},$$

for $X, Y \in \mathcal{H}(M)$.

A simple calculation shows that every almost complex structure J on a 2-dimensional orientable manifold has vanishing torsion. For an arbitrary almost complex structure we state the following well known result [10];

<u>THEOREM</u> (1.1). A C^∞ - almost complex structure is a complex structure if and only if it has no torsion.

Next we give an alternative formulation of the integrability of an almost complex structure.

<u>DEFINITION</u>. An affine connection ∇ on an almost complex manifold (M,J) is called an almost complex affine connection if $\nabla J = 0$. For the existence of such connections we state the following([10]);

<u>THEOREM</u> (1.2). Every almost complex manifold M admits an almost complex affine connection such that its torsion T is given by

$$S = 8T ,$$

where S is the torsion of the almost complex structure J of M.

<u>Proof</u>: Take an arbitrary torsion-free affine connection ∇ on M, and let Q be the tensor field of type (1,2) defined by

$$4Q(X,Y) = (\nabla_{JY} J)X + J(\nabla_Y J)X + 2J(\nabla_X J)Y , \quad (1.1)$$

$X, Y \in \mathfrak{X}(M)$.

Consider the affine connection $\tilde{\nabla}$ defined by

$$\tilde{\nabla}_X Y = \nabla_X Y - Q(X,Y).$$

A calculation shows that $\tilde{\nabla}$ is the desired connection.

<u>COROLLARY</u> (1.3). An almost complex manifold M admits a torsion-free almost complex affine connection if and only if

the almost complex structure has no torsion.

Hence it follows that an almost complex structure is integrable if and only if there exists a torsion-free connection ∇ , and $\nabla J=0$.

Although the concept of integrability is defined for an almost complex structure which need not be almost hermitian - (that is an almost complex structure J satisfying $\langle JX,JY \rangle = \langle X,Y \rangle$, $X, Y \in \mathcal{H}(M)$) - and for almost complex manifolds which need not carry a riemannian structure, it seems natural to attempt a generalization of the concept of integrability to arbitrary vector cross products. This is so because vector cross products are generalized almost hermitian structures.

2. Integrability of a vector cross product

A natural generalization of the concept of integrability to an arbitrary vector cross product would be to construct a generalized torsion tensor of an arbitrary vector r-from (that is, an alternating tensor field of type $(1,r)$). For this purpose we give a brief description of a method by A. FROLICHER and A. NIJENHUIS ([5]).

On a differentiable manifold M let S_p denote the module of scalar p-forms on M , and V_p the module of vector p-forms on M.

Let $L \in V_l$, $W \in S_q$. Then $W \wedge L \in V_{q+1}$ is defined by

$$W \wedge L\ (U_1,\ldots,U_{q+1}) = \frac{1}{(q+1)!} \sum_z \operatorname{sgn}(z)\ W\ (U_{z_1},\ldots,U_{z_q})\ L\ (U_{z_{q+1}},\ldots, U_{z_{q+l}}) \qquad (2.1)$$

Also for $W \in S_q$, $L \in V_l$, $W \,\bar{\wedge}\, L \in S_{q+l-1}$ is defined by

$$W \,\bar{\wedge}\, L(U_1,\ldots,U_{q+l-1}) = \frac{1}{(q+l-1)!} \sum_z \operatorname{sgn}(z) W\Big(L(U_{z_1},\ldots,U_{z_l}), U_{z_{l+1}},\ldots,U_{z_{l+q-1}}\Big) \qquad (2.2)$$

For $q = o$, $W \,\bar{\wedge}\, L = 0$

Further for $L \in V_l$, $M \in V_m$, $M \,\bar{\wedge}\, L \in V_{l+m-1}$ is defined by

$$M \,\bar{\wedge}\, L(U_1,\ldots,U_{1+m-1}) = \frac{1}{(1+m-1)!} \sum_z \operatorname{sgn}(z) M\Big(L(U_{z_1},\ldots,U_{z_l}), U_{z_{l+1}},\ldots,U_{z_{l+m-1}}\Big) \qquad (2.3)$$

Here z runs over all permutations of $(1,\ldots,m+1-1)$ and the U's are vector fields on M.

Locally on an open subset of M we may choose a frame field $\{e_1,\ldots,e_n\}$ with dual frame $(e_1^*,\ldots,e_n^*)$. Relative to such a frame, and for $L \in V_l$ we set

$$L^j = e_j^* \circ L \quad ; \quad L = \sum_{j=1}^{n} L^j \otimes e_j \,.$$ Then the

operations defined above assume the following simple forms:

$$W \wedge L = \sum_{j=1}^{n} (W \wedge L^j) \otimes e_j \qquad (2.1)$$

$$W \bar{\wedge} L = \sum_{j=1}^{n} L^j \wedge C_{e_j} W \qquad (2.2)$$

$$M \bar{\wedge} L = \sum_{j,i=1}^{n} (L^j \wedge C_{e_j} M^i) \otimes e_i \qquad (2.3)$$

Here C_{e_j} denotes contraction with respect to e_j and

$(C_{e_j} W)(U_1, \ldots, U_{q-1}) = q\, W(e_j, U_1, \ldots, U_{q-1})$ for vector

fields $U_1, \ldots, U_{q-1}$.

DEFINITIONS. The mapping $D: S \longrightarrow S$ $(S = \sum_{q=0}^{n} S_q)$

is called a derivation of degree r on S if

a) $Dk = 0$, $k \in R$ (the reals)

b) $DS_p \subset S_{p+r}$,

c) $D(\varphi + \psi) = D\varphi + D\psi$,

d) $D(\pi \wedge W) = D\pi \wedge W + (-1)^{pr} \pi \wedge DW$; $\pi \in S_p$, $W \in S_q$.

The commutator $[D_1, D_2]$ of two derivations D_1 of degree r_1 and D_2 of degree r_2, is a derivation of degree $r_1 + r_2$ defined by

$$[D_1, D_2] = D_1 D_2 - (-1)^{r_1 r_2} D_2 D_1 .$$

A derivation D of degree r on S is of type d_* if

$Dd = (-1)^r d D$.

We note that the commutator of two derivations of type d_* is a derivation of type d_* .

PROPOSITION (2.1) Every derivation D on S of type d_* is determined uniquely by its action on S_0. With every derivation D of degree r of type d_* there is associated a vector form L of degree r, and

$$DW = [L, W] = dw \barwedge L + (-1)^r d (W \barwedge L). \qquad (2.4)$$

Conversely, every mapping $D: S \longrightarrow S$ of the form (2.4) is a derivation of type d_* denoted by d_L .

PROPOSITION (2.2) Given any two vector forms $L \in V_l$, $M \in V_m$, there exists a vector form $[L, M] \in V_{l+m}$ which is uniquely determined by the condition

$$[d_L, d_M] = d_{[L,M]} .$$

The vector form $[L,M]$ depends differentiably on L and M , and satisfies the following rules:

a) $[M,L] = (-1)^{lm+1} [L, M]$,

b) $[L, [M, W]] - (-1)^{lm} [M, [L, W]] = [[L,M], W]$,

c) $\left[1, M\right] = 0$.

In terms of a local frame field $(e_1, \ldots, e_n)$,

$L^j = e_j^* \circ L$ we have for $L \in V_\ell$

$$\left[L, L\right] = 2 \sum_{i,j=1}^{n} \left(L^j \wedge \mathcal{L}_{e_j} L^i - d L^j \wedge C_{e_j} L^i \right) \otimes e_i \, .$$

Here $\mathcal{L}$ denotes lie differentiation and C denotes contraction.

We observe that the torsion tensor S of an almost complex structure J, as defined earlier, agrees with the vector 2-form given by proposition (2.2) when $M = J$ and $L = J$. This suggests that we may define the torsion of an arbitrary r-fold vector cross product P, to be the vector $2r$-form $\left[P, P\right]$. Then by proposition (2.2) we have

$$\left[P, P\right] = (-1)^{r^2+1} \left[P, P\right].$$

Hence if r is even, then $\left[P, P\right] \equiv 0$. This implies that any 2-fold vector cross product has vanishing torsion. This makes such a generalization inadequate.

In section 1 we have seen that an almost complex structure is integrable if and only if there exists a torsion-free connection ∇ such that $\nabla J = 0$.

Now let ∇ be a torsion-free connection on M and set,

$\tilde{\nabla}_X Y = \nabla_X Y - Q(X,Y)$, where Q is a tensor field of type $(1,2)$. We easily see that $\tilde{\nabla}$ is an affine connection whose torsion $\tilde{T}$ is given by

$$\tilde{T}(X,Y) = -Q(X,Y) + Q(Y,X) \qquad (2.5)$$

Further, for an r-fold vector cross product P we have,

$$\tilde{\nabla}_X(P)(X_1,\dots,X_r) = \tilde{\nabla}_X P(X_1,\dots,X_r) - \sum_{j=1}^{r} P(X_1,\dots,\tilde{\nabla}_X X_j,\dots,X_r)$$

$$= \nabla_X(P)(X_1,\dots,X_r) - Q(X,P(X_1,\dots,X_r)) + \sum_{j=1}^{r} P(X_1,\dots,Q(X,X_j),\dots,X_r). \qquad (2.6)$$

From (2.5) and (2.6) we deduce the following;

PROPOSITION (2.3)

For an r-fold vector cross product P on a riemannian manifold M with riemannian connection ∇ the following are equivalent;

a) There exists on M a ~~symmetric tensor field Q of type (1,2)~~ torsion free connection $\tilde{\nabla}$ such that $\tilde{\nabla} P = 0$.

(b) There exists on M a symmetric tensor field Q of type (1,2) such that

$$\nabla_X(P)(X_1,\dots,X_r) = Q(X, P(X_1,\dots,X_r)) - \sum_{j=1}^{r} P(X_1,\dots,Q(X,X_j),\dots,X_r), \qquad (2.7)$$

for all $X, X_1,\dots,X_r \in \mathcal{H}(M)$.

We note that when $r=1$, Q is given in terms of J and its derivatives by (1.1). It is not clear whether a Q satisfying (2.7) exists. However, if P is parallel then $Q = 0$ gives the desired connection.

Now let M be an n-dimensional differentiable manifold. We denote by F the principal frame bundle over M with structure group $GL(n, R)$. If $G \subset GL(n,R)$ is a lie subgroup, then a G-structure on M is a reduction of F to a principal subbundle with group G. R^n considered as a differentiable manifold carries a special G-structure whose fibre at $x \in R^n$ is the set of all linear mappings

$$i_x \circ A: \quad R^n \longrightarrow R^n_x \quad ; \quad A \in G ,$$

where $i_x : R^n \longrightarrow R^n_x$ is the cannonical isomorphism of R^n onto its tangent space at x. Any G-structure which is locally equivalent to this G-structure is said to be integrable or flat. (Equivalently, this means that one can choose coordinates $(x_1, \ldots, x_n)$ locally such that $(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ is a local cross-section of the G-structure.)

Next, let M and $\bar{M}$ be two n-dimensional manifolds and let $f: M \longrightarrow \bar{M}$ be a diffeomorphism. f induces a map, $f_* : F \longrightarrow \bar{F}$, of the corresponding principal frame bundles. Suppose that both M and $\bar{M}$ possess G-structures. If f_* maps the frames of one into the frames of the other then f is called structure preserving.

Now let M and $\bar{M}$ be two n-dimensional manifolds; let

$E \longrightarrow M$ be a G-structure on M ; let $\bar{E} \longrightarrow \bar{M}$ be a G-structure on $\bar{M}$. Let $p \in M$ and f a diffeomorphism of a nbd of p into $\bar{M}$. We say that f preserves the G-structure to Kth order at p if some frame belonging to E at p gets mapped by f_* onto a frame, A, belonging to $\bar{E}$ and if $F_* (E)^{\not{}}$ and $\bar{E}$ have contact of order K at A as submanifolds of $\bar{F}$.

GUILLEMIN [15], defined a G-structure on an n-dimensional manifold M to be uniformly k-flat at $p \in M$, if there exists a nbd $U(p)$ and a diffeomorphism of $U(p)$ onto a nbd $U(0) \subset R^n$ which is Kth order structure preserving at p. GUILLEMIN [15], found that the obstructions to constructing such mappings, which are structure preserving to order K for arbitrary large K , are tensors of type $H^{k,2}(g)$ defined on E , where $H^{k,i}(g)$ are the bigraded homology groups of a certain chain complex associated with g, the lie algebra of G.

Let $gl(V)$ be the lie algebra of linear endomorphisms of an n-dimensional real vector space V. Let g be a lie subalgebra of $gl(V)$. The Kth prolongation $g^{(k)}$ of g is by definition, the vector space of all symmetric $(k+1)$-linear mappings:

$$a: (X_1,\ldots,X_{k+1}) \in V \times V \times \ldots \times V \longrightarrow a(X_1,\ldots,X_{k+1}) \in V$$

such that, for arbitrary fixed $X_1,\ldots,X_k \in V$, the linear endomorphism of V which sends X_{k+1} into $a(X_1,\ldots,X_k,X_{k+1})$ is an element of g. The definition of $g^{(k)}$ can be written as

$$g^{(k)} = g \otimes S^k(V^*) \cap V \otimes S^{k+1}(V^*) ,$$

Where $S^k(V^*)$ is the space of all k-linear symmetric mappings $a: V^k \longrightarrow R$. We say that g is of finite type if $g^{(k)} = 0$ for some k. If $g^{(k-1)} \neq 0$ and $g^{(k)} = 0$, g is said to be of type k. We note that $g^{(o)} = g$.

According to [15], if g is of type k then $H^{r,2}(g) = 0$ for $r > k$. Hence for G-structures of type k, obstructions to integrability lie in $H^{r,2}(g)$, for $r \leq k$.

We have seen in Chapter 2 that a vector cross product on a riemannian manifold M is a G-structure on M, where G is the automorphism group of the vector cross product. Further, since G is contained in the special orthogonal group of the riemannian structure it is of type 1. It follows that obstruction to integrability in this case may lie in $H^{o,2}(g)$ and $H^{1,2}(g)$. We note that a necessary condition for integrability of a vector cross product in this sense is that the riemannian structure is flat. For an almost complex structure the corresponding G-structure is $GL(n,c)$ considered as a lie subgroup of $GL(2n, R)$. In this case one can show that g is of infinite type and $H^{i,j} = 0$ for $i > o$ and all j. The only obstruction to integrability lies in $H^{o,2}(g)$ and is the torsion tensor of the almost complex structure.

Finally let P be a vector cross product on a pseudo-riemannian manifold M ; $E \longrightarrow M$ the associated G-structure and Γ an arbitrary linear connection in E. Then:

PROPOSITION (2.4) ([15]) Let Q be an arbitrary point of E. The lowest order structure tensor of $E \longrightarrow M$ at Q is the cohomology class in $H^{0,2}(g)$ of the torsion tensor of Γ at Q.

Hence if this structure tensor vanishes then there exist torsion free connections with respect to which P is parallel.

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