# Applications of differential geometry to computer curves and surfaces

J. M. Duncan

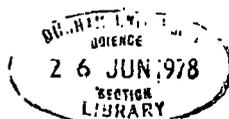# APPLICATIONS OF DIFFERENTIAL GEOMETRY

## TO COMPUTER CURVES AND SURFACES

by

### J. M. Duncan

A thesis presented for the degree of Doctor of Philosophy

of the University of Durham

December 1976

Department of Mathematics

University of Durham

# ACKNOWLEDGEMENTS

# C O N T E N T S

# ABSTRACT

This thesis realises the need for describing computer curves
and surfaces in terms of intrinsic quantities and certain
properties relative to the Euclidean space in which they are
embedded.

Chapter 1 introduces some of the ideas and problems involved
in what can be termed computational differential geometry.

Chapter 2 presents some analysis of the major types of
computer curves in terms of a number of shape control parameters.

Chapter 3 gives a similar analysis of computer surfaces.

Chapter 4 considers the calculus of variations in connection
with the minimal immersion and a particular invariantly
defined functional analogous to energy.

Chapter 5 applies the energy functional to a class of computer
curves.

Chapter 6 looks at a number of surfaces in relation to surface
mappings and distortion. Some mappings are also derived. This
generally involves the solution of non linear differential
equations the linearisation of which will almost certainly
remove the salient features of the theory.

A bibliography and a number of figures are provided following
chapter 6.

# Chapter 1

## Introduction

In recent years the introduction of the computer as an
additional tool in the industrial design process has
stimulated considerable interest in the development of
algorithms for curve and surface interpolation and
approximation {8-19}.   The purpose of this thesis has
been to apply some of the fundamental ideas of
differential geometry to this problem and to the problems
associated with deriving two dimensional patterns from
given three dimentional surface representations.

Traditionally the design process has involved a
tremendous amount of repetitive work which has been time
consuming, often irreversible and consequently expensive.
This process of producing sketches and drawings, making
models and applying numerous corrections and modifications
has served to emphasise the need for what is now
familiarly known as Computer Aided (Geometric) Design.

The basis of CAD could be stated briefly as the 'mathematical definition of shape'. More precisely one is interested in deriving for some shape - which may exist initially purely in the designers' mind - a mathematical representation as a surface in $E^3$ which will be of such form as can be held within a computer. The development of such computer hardware as 'real time' keyboards and graphics displays has meant that a designer can directly interact with the computer to produce this derivation by using the quick assimilation of computed results to make decisions and thereby efficiently guiding the computer until an acceptable representation has been obtained.

To achieve a surface representation it is clear that for all but the simplest of shapes geometrically an implicit equation would be unhandy if at all obtainable since we shall be mainly interested in local properties and having local control over the description. Consequently the technique generally adopted is that of considering the surface as being composed of a finite union of topological squares commonly known as 'patches' (11, 13, 14, 15) which are normally arranged to satisfy conditions which are analogous to a surface triangulation where:-

    (i)   no two patches have a common interior point,

    (ii)   to each side of a patch there correspond two and
           only two patches with this common side unless

the side of the patch lies on the boundary
of the surface in which case it belongs to only
one patch,

(iii) any two patches can be joined by a sequence of
patches where each patch in the sequence has one
and only one side in common with the next one in
the sequence,

(iv) all patches with a common vertex can be arranged
in a definite order so that consecutive patches
have a common side passing through the vertex.

An explicit form in which each point is expressed in terms
of two curvilinear coordinates also enables us to describe
a sense round the surface.  However, such a patch
representation is by no means unique and in addition to
the above we insist that there are side smoothness
conditions which will ensure that neighbouring patches
are glued together differentiably to form the complete
definition.

All functions are computed as truncated polynomials by
digital computers and so it seems reasonable to use
polynomials as the basis functions for both curves and
for the surface patches.   In practice for most
purposes cubics for curves and bicubics for surfaces are
optimal in that the degrees allow for sufficient shape
flexibility and the necessary smoothness criteria.

The problem of interpolation reduces to one of determining
suitable constraints to impose on properties such as arc
length, surface area and smoothness in order to uniquely
define the patch boundaries and interiors from the
vertices.    We insist however, that we do not impose
restrictions which are peculiar to the particular
parameterisation adopted and in particular we shall be
interested in intrinsic properties.    By using elements
of intrinsic geometry we can use the results and ideas
of Riemannian geometry which is mainly intrinsic and
for which the literative is vast {3}.    This would also
enable ideas to proceed along more general lines looking
at a particular field of tensors defined over
differentiable manifolds.


In placing our constraints we must pay attention to two
basic results from the theory of curves and surfaces in
$E^3$.    For curves, the fundamental existence and
uniqueness theorem states that for arbitrary continuous
functions $\kappa(s)$ and $\tau(s)$ defined on $a \leqslant s \leqslant b$ then there
exists, up to a rigid motion, a unique curve for which
$\kappa$ is the curvature, $\tau$ is the torsion and s is the arc
length.    Correspondingly, for surfaces if the functions
$E(u,v)$, $F(u,v)$, $G(u,v)$ are $C^2$ and $L(u,v)$, $M(u,v)$,
$N(u,v)$ $C^1$ on some open set containing $(u_o,v_o)$ and satisfy
the compatibility equations of Gauss and Codazzi and if
the quadratic differential form $Edu^2 + 2Fdudv + Gdv^2$ is

positive definite then there exists a unique surface, up to Euclidean motion, defined on a neighbourhood of $(u_o, v_o)$ having E, F, G and L, M, N as first and second fundamental coefficients. Length and area considerations force us to consider the theory of the calculus of variations to solve what amounts to a form of the general Plateau problem which can be stated as 'given an n-1 dimensional closed submanifold $\Gamma$ in $E^{n+k}$ then find a minimal submanifold $M^n$ with boundary $\partial M^n = \Gamma$ '.

Having achieved a suitable mathematical representation which will act as the geometric data base we can then move on to consider other phases of design which could be embodied in an overall interactive system. The mass production of fashion articles for instance usually requires specially shaped tools such as jigs, moulds and patterns and it would be beneficial if these can be acquired as soon as possible so that the manufacturer can catch the market at its peak. Thus if we could use the computer held definition to determine shapes for patterns, which are normally two dimensional, this would for instance facilitate automation of the drawing process with the aid of digital plotters, and the cutting process with the aid of laser beams or high pressure water jets.

The optimum mapping of a particular surface into $E^2$ will obviously depend upon constraints imposed by such things as the elasticity of the material, the available

machinery, the cost of the raw material and so on.
We will need to know whether the optimum mapping is
going to preserve some geometrical structure of the
surface.    This will not normally be possible because
of the restrictions this would involve and so we will
have to look at various norms for measuring the departure
from the considered ideal.    Such an ideal might be an
isometry for example or more generally conformal.    It is
well known that every point on a surface has a neighbourhood
which can be mapped conformally on some neighbourhood of
any other surface, and this result has been widely
exploited by cartographers.    The natural generalisation
of the conformal mapping is the quasiconformal mapping
(7) which now plays an active role in the theory of analytic
functions of a single complex variable.

For each chosen ideal we shall term the departure from
that ideal the 'stretch' or 'distortion' of the particular
mapping and the optimum solution will be the mapping having
minimum stretch.

Although this thesis will be concerned with applying
the fundamental ideas of differential geometry to computer
aided design an interesting and useful observation  is
that the computer could become a very attractive tool in
differential geometry.    A set which admits a patch
structure is a $C^r$ manifold with metric for some r

where r depends upon how the patches are joined
'together.     Conversely if a manifold can be cubulated to
give it a patch structure then the computer could be
used to calculate quantities on the manifold.    This would
overcome one significant problem which has existed in
the past in that for practical convenience many calculations
have been confined to a small number of manifolds often
only surfaces of revolution.

Considerable effort has gone into computer aided design
in recent years but there  is still a great need for
research in many areas before a fully integrated CAD
system could be considered as commercially viable.
Even  if CAD methods turn out to be no cheaper than
traditional methods they will certainly be quicker,
will allow more options to be considered and the
mathematical compatibility of such things as moulds and
patterns will mean  that the various components making
up the design should assemble together more accurately.

# Chapter 2

## Computer Curves in $E^3$

### Definition 2.1

A Cubic curve in $E^3$ is one for which there exists some parameter u in terms of which the equation of the curve can be written as

$$\underline{r}(u) = \underline{a}_0 + \underline{a}_1 u + \underline{a}_2 u^2/2! + \underline{a}_3 u^3/3! \tag{1}$$

where $\underline{a}_i \epsilon E^3$ for $i = 0,\ldots 3$

The parameter u is only indeterminate to within an affine transformation of the form $u \longrightarrow u' = \xi u + \eta$ where $\xi, \eta \epsilon R$ and so if we apply the restriction that the curve contains two particular points in $E^3$ with specified parameter values $u = \alpha$ and $u = \beta$ say, then the parameter is uniquely determined. For convenience we normally take $\alpha$ and $\beta$ to be 0 and 1 and let $0 \leqslant u \leqslant 1$. We will use primes to denote differentiation with respect to arc length s and a Natural representation of a curve is one for which s is the parameter.

Then $s(u) = \int_\alpha^U |\dot{\underline{r}}(u)| du = \int_\alpha^U f(u) du$ and so it is easy

to evaluate s using Gauss Quadrature with $s = \sum_{i=1}^{N} w_i f(u_i)$,

and verification that s is independent of the

parametrization follows immediately from the rule for

changing the variable of an integral.    We shall first

of all see that the cubic representation (1) places

restrictions on both the parameter u and the curve

itself as a geometric object.


## Proposition 2.2

The only cubic curve having a Natural representation is

the straight line.

To prove this, if

$\underline{r}(s) = \underline{a}_0 + \underline{a}_1 s + \underline{a}_2 s^2/2! + \underline{a}_3 s^3/3!$   then repeated

differentiation gives

$\underline{r}^{(iv)}(s) = \{\kappa''(s) - \kappa(s)\tau^2(s) - \kappa^3(s)\}\underline{n} + \{2\kappa'(s)\tau(s) + \kappa(s)\tau'(s)\}\underline{b}$

$- 3\kappa(s)\kappa'(s)\underline{t} = \underline{0}$   where we have used the Serret-Frenet

formulae :

$\underline{t}' = \kappa\underline{n}$ ,   $\underline{n}' = \tau\underline{b} - \kappa\underline{t}$ ,   $\underline{b}' = -\tau\underline{n}$

and where $\kappa$ and $\tau$ denote the curvature and torsion

respectively.


Hence $\kappa\kappa' = 2\kappa'\tau + \kappa\tau' = \kappa'' - \kappa\tau^2 - \kappa^3 = 0 \implies \kappa \equiv 0$

This means of course that we cannot take the arc length

as parameter if we restrict ourselves to cubics.

## Proposition 2.3

For given $\left\{ \underline{r}_0, \underline{r}_1, \underline{t}_0, \underline{t}_1 \right\} \subset E^3$ with $|\underline{t}_0| = |\underline{t}_1| = 1$
then there are $\infty^2$ cubic curves $\underline{r} : [0,1] \longrightarrow E^3$
such that

$$\underline{r}(i) = \underline{r}_i \text{ and } \underline{r}'(i) = \underline{t}_i \text{ for } i = 0,1$$

This follows since if $\underline{r}(u) = \underline{a}_0 + \underline{a}_1 u + \underline{a}_2 u^2/2! + \underline{a}_3 u^3/3!$
then $\dot{s}(u)\underline{t}(u) = \underline{a}_1 + \underline{a}_2 u + \underline{a}_3 u^2/2$ and the four coefficients
satisfy

$$
\begin{aligned}
\underline{a}_0 &= \underline{r}_0 \\
\underline{a}_1 &= \dot{s}(0)\underline{t}_0 \\
\underline{a}_2 &= 2! \left\{ 3(\underline{r}_1 - \underline{r}_0) - 2\dot{s}(0)\underline{t}_0 - \dot{s}(1)\underline{t}_1 \right\} \\
\underline{a}_3 &= 3! \left\{ 2(\underline{r}_0 - \underline{r}_1) + \dot{s}(0)\underline{t}_0 + \dot{s}(1)\underline{t}_1 \right\}
\end{aligned}
$$ (*)

which means we have a doubly infinite family of curves
parametrized by $\dot{s}(0)$ and $\dot{s}(1)$.

## Corollary

For given $\left\{ \underline{r}_0, \underline{r}_1, \underline{t}_0, \underline{t}_1 \right\} \subset E^3$ with $|\underline{t}_0| = |\underline{t}_1| = 1$
and given $\kappa_0, \kappa_1 \in \mathbb{R}$ then there is a unique cubic curve
$\underline{r}: [0,1] \longrightarrow E^3$ such that

$$\underline{r}(i) = \underline{r}_i, \ \underline{r}'(i) = \underline{t}_i, \ |\underline{r}''(i)| = \kappa_i \text{ for } i = 0,1$$

In fact since $\dot{\underline{r}}(u) \times \ddot{\underline{r}}(u) = \kappa(u)\dot{s}^3(u)\underline{b}$ where $\underline{b} = \underline{t} \times \underline{n}$
then we obtain $\dot{s}(0)$ and $\dot{s}(1)$ from :

$$2\{3\underline{t}_0 \times (\underline{r}_1 - \underline{r}_0) - \dot{s}(1)\underline{t}_0 \times \underline{t}_1\} = \kappa_0 \dot{s}^3(0)\underline{b}(0)$$

$$2\{3\underline{t}_1 \times (\underline{r}_0 - \underline{r}_1) - \dot{s}(0)\underline{t}_0 \times \underline{t}_1\} = \kappa_1 \dot{s}^3(1)\underline{b}(1)$$

For the general curve $\underline{r} = \underline{r}(u)$ it is normally convenient to have the curvature $\kappa$ and torsion $\tau$ expressed in terms of the derivatives of $\underline{r}$.

Since $\dot{\underline{r}}(u) = \dot{s}(u)\underline{t}(u)$ and

$$\ddot{\underline{r}}(u) = \ddot{s}(u)\underline{t}(u) + \dot{s}^2(u)\kappa(u)\underline{n}(u) \text{ then}$$

$\dot{\underline{r}}(u) \times \ddot{\underline{r}}(u) = \dot{s}^3(u)\kappa(u)\underline{b}(u)$ where $\underline{b}(u) = \underline{t}(u) \times \underline{n}(u)$ and differentiation gives

$$\dot{\underline{r}}(u) \times \dddot{\underline{r}}(u) = \frac{d}{du}\{\dot{s}^3(u)\kappa(u)\} \underline{b}(u) - \dot{s}^4(u)\kappa(u)\tau(u)\underline{n}(u).$$

Therefore $\quad \ddot{\underline{r}}(u).\{\dot{\underline{r}}(u) \times \dddot{\underline{r}}(u)\} = -\dot{s}^6(u)\kappa^2(u)\tau(u)$

But $\dot{r}^2(u) = \dot{s}^2(u)$ and $\{\dot{\underline{r}}(u) \times \ddot{\underline{r}}(u)\}^2 = \dot{s}^6(u)\kappa^2(u)$ and

so $\kappa^2(u) = \{\dot{\underline{r}}(u) \times \ddot{\underline{r}}(u)\}^2 / \{\dot{\underline{r}}^2(u)\}^3 \qquad\qquad (*)$

$$\tau(u) = \left[\dot{\underline{r}}(u), \ddot{\underline{r}}(u), \dddot{\underline{r}}(u)\right] / \{\dot{\underline{r}}(u) \times \ddot{\underline{r}}(u)\}^2 \qquad\qquad (**)$$

## Proposition 2.4

The curvature $\kappa$ and torsion $\tau$ of the cubic curve $\underline{r}:[0,1] \longrightarrow E^3$ take the boundary values :

$\kappa(0) = \{(\underline{a}_1 \times \underline{a}_2)^2/(\underline{a}_1^2)^3\}^{\frac{1}{2}}$

$\kappa(1) = \left[\{\underline{a}_1 \times \underline{a}_2 + \underline{a}_1 \times \underline{a}_3 + \underline{a}_2 \times \underline{a}_3/2\}/\{(\underline{a}_1 + \underline{a}_2 + \underline{a}_3/2)^2\}^3\right]^{\frac{1}{2}}$

$\tau(0) = \left[\underline{a}_1, \underline{a}_2, \underline{a}_3\right]/(\underline{a}_1 \times \underline{a}_2)^2$

$\tau(1) = \left[\underline{a}_1, \underline{a}_2, \underline{a}_3,\right]/\{\underline{a}_1 \times \underline{a}_2 + \underline{a}_1 \times \underline{a}_3 + \underline{a}_2 \times \underline{a}_3/2\}^2$

These results follow immediately by substitution of

$\dot{\underline{r}}(u) = \underline{a}_1 + \underline{a}_2 u + \underline{a}_3 u^2/2, \quad \ddot{\underline{r}}(u) = \underline{a}_2 + \underline{a}_3 u$ and $\dddot{\underline{r}}(u) = \underline{a}_3$

into equations (*) and (**).    Also as a consequence
of equations (*) and (**) we have

## Proposition 2.5

The general curve $\underline{r} = \underline{r}(u)$ is planar iff $[\dot{\underline{r}}, \ddot{\underline{r}}, \dddot{\underline{r}}] = 0$.

This follows because $[\dot{\underline{r}}, \ddot{\underline{r}}, \dddot{\underline{r}}] = \kappa^2 \tau \dot{s}^6$ and so $\tau = 0$
implies that $[\dot{\underline{r}}, \ddot{\underline{r}}, \dddot{\underline{r}}] = 0$.

Conversely if $[\dot{\underline{r}}, \ddot{\underline{r}}, \dddot{\underline{r}}] = 0$ and $\tau \neq 0$ at some point then
there is a neighbourhood of this point where $\tau \neq 0$ and so
$\kappa = 0$ in this neighbourhood.   But this means that the arc
is a straight line and so $\tau = 0$ which is contrary to the
hypothesis and so $\tau = 0$ at all points on $\underline{r}$.

Any discernible points along the curve will be of
particular interest from the point of view of design.

## Definition 2.6

A point of $\underline{r} = \underline{r}(u)$ at which $\underline{r}''(u) = \underline{0}$ is called a point
of inflexion.

## Proposition 2.7

If the cubic curve $\underline{r} : [0,1] \longrightarrow E^3$ has a point of
inflexion then :

$$\text{(i)} \quad [\underline{a}_1, \underline{a}_2, \underline{a}_3] = 0$$

and $\quad$ (ii) $\quad \dot{s}(0)\dot{s}(1)[\underline{t}_0, \underline{r}_1 - \underline{r}_0, \underline{t}_1] = 0$

where $\underline{r}_i = \underline{r}(i)$ and $\underline{t}_i = \underline{r}'(i)$ for $i = 0,1$.

The proof follows because from (*) and (**)

$$\kappa^2(u)\tau(u) = [\dot{\underline{r}}(u), \ddot{\underline{r}}(u), \dddot{\underline{r}}(u)]/\dot{s}^6(u)$$

and so $\kappa = 0$ means that

$$[\dot{\underline{r}}(u), \ddot{\underline{r}}(u), \dddot{\underline{r}}(u)] = [\underline{a}_1, \underline{a}_2, \underline{a}_3] = 0$$

or in terms of boundary values this means that

$\dot{s}(0)\dot{s}(1)[\underline{t}_0, \underline{r}_1-\underline{r}_0, \underline{t}_1] = 0$ by substitution of equations (+).

This means we have the following :

## Proposition 2.8

If the cubic curve $\underline{r} : [0,1] \longrightarrow E^3$ is such that either

(i) $[\underline{a}_1, \underline{a}_2, \underline{a}_3]$ is non zero

or (ii) $[\underline{t}_0, \underline{r}_1-\underline{r}_0, \underline{t}_1]$ is non zero

where $\underline{r}_i = \underline{r}(i)$, $\underline{t}_i = \underline{r}'(i)$, $i = 0,1$.

then $\underline{r}$ has no point of inflexion.

The proof follows from Proposition 2.7 if $\dot{s}(0)$ and $\dot{s}(1)$ are non zero.    And if $\dot{s}(0) = 0$ then

$$\dot{\underline{r}}(u) = 6u(1-u)(\underline{r}_1-\underline{r}_0) + (3u-2)\dot{s}(1)\underline{t}_1$$

$$\ddot{\underline{r}}(u) = 6(1-2u)(\underline{r}_1-\underline{r}_0) + 2(3u-1)\dot{s}(1)\underline{t}_1$$

$$\dddot{\underline{r}}(u) = -12(\underline{r}_1-\underline{r}_0) + 6\dot{s}(1)\underline{t}_1$$

and so $[\dot{\underline{r}}, \ddot{\underline{r}}, \dddot{\underline{r}}] = 0$ which means the curve is planar and
$[\underline{t}_0, \underline{r}_1-\underline{r}_0, \underline{t}_1] = 0$

And similarly if $\dot{s}(1) = 0$

If the curve is planar then the point of inflexion of

particular interest is the one where $\underline{n}$ changes sign,
and control over such points is desirable since one is
normally concerned in keeping the number to a minimum.

In the non planar case the analogy would be the point
where $\underline{b}' = \underline{0}$ and $\underline{b}$ changes sign, for at this point the
curve crosses its osculating plane.

To detect whether a curve $\underline{r} = \underline{r}(u)$ is to one side of
or on a particular plane $X$ then the obvious measure is
given by $(\underline{r}(u)-\underline{P}).\underline{N}$ where $\underline{P}$ is some point on $X$ and $\underline{N}$
is its normal.

In general we will only be able to define our curves
by cubics locally.    We would now like to consider the
problem of linking together several curve segments to
form an acceptable composite curve and to define in more
precise terms what we mean by an acceptable curve.

Let $\underline{r} : [a,b] \longrightarrow E^3$ be a given space curve C and let
us assume that C is smooth.   The arc length of C can be
used to provide a 'natural' cubic extension as follows :

We first assume that C is parametrized in what is the
only geometrically significant way that is by means of
arc length.
Since C is smooth we can write :

$$\underline{r}(s) = \underline{r}(c) + (s-c)\underline{r}'(c) + (s-c)^2/2! \ \underline{r}''(c) + \ldots\ldots$$

$$+ (s-c)^n/n! \underline{r}^{(n)}(c) + 0 (s-c)^{n+1}$$

where $a < c < b$.

If we use the Serret-Frenet formulae then

$$\underline{r}(s) = \underline{r}(c) + (s-c)\underline{t} + (s-c)^2 \kappa/2! \underline{n} + (s-c)^3/3!$$

$$\{\kappa'\underline{n} + \kappa\tau\underline{b} - \kappa^2\underline{t}\} + (s-c)^4/4! \{(\kappa'' - \kappa\tau^2 - \kappa^3)\underline{n} +$$

$$(2\kappa'\tau + \kappa\tau')\underline{b} - 3\kappa\kappa'\underline{t}\} + \ldots\ldots + 0(s-c)^{n+1}.$$

where $\underline{t}$, $\underline{n}$, $\underline{b}$, $\kappa$, $\tau$ are all evaluated at $s = c$.

Suppose now that we want to continue C be means of another

curve $\Gamma$, which is cubic, with $\Gamma$ given by $\hat{\underline{r}}$ : $[c,d] \longrightarrow E^3$.

We take as parameter for $\Gamma$ s the arc length of C, and we

let $\hat{s}$ be the arc length of $\Gamma$.   We write:

$$\hat{\underline{r}}(s) = \underline{r}(c) + (s-c)\underline{t} + (s-c)^2 \kappa/2! \underline{n} + (s-c)^3/3!$$

$$\{\kappa'\underline{n} + \kappa\tau\underline{b} - \kappa^2\underline{t}\}.$$

Then it is easy to show that at $s = c$ we have :

$$\hat{\underline{t}} = \underline{t}, \ \hat{\kappa} = \kappa, \ \hat{\underline{n}} = \underline{n}, \ \hat{\tau} = \tau, \ \hat{\underline{b}} = \underline{b}, \ \hat{\kappa}' = \kappa'$$

Thus we have found a smooth cubic continuation of the

curve C from the point $\underline{r}(c)$, $a < c < b$.   However we have

assumed that there are no other restrictions on $\Gamma$, and this

will not normally be the case.    In general we will be

looking to solve the following problem :

"Given $\left\{ \underline{b}_1, \ldots\ldots \underline{b}_{n+1} \right\}$ C $E^3$ we want to construct

a curve $\Gamma$ where :

(i)   $\Gamma$ passes through each $\underline{b}_i$ $i = 1, \ldots\ldots.n+1$

(ii)  $\Gamma$ is cubic between $\underline{b}_i$ and $\underline{b}_{i+1}$ $i = 1, \ldots\ldots n$

(iii) $\Gamma \in C^k$ , for a suitable $k$ . "

Such a curve is known as a Piecewise Parametric Cubic or Parametric Cubic Spline {8,10,23}. Thus the $i^{th}$ segment of $\Gamma$ from $\underline{b}_i$ to $\underline{b}_{i+1}$ is of the form

$$\underline{Y}_i(u) = \underline{a}_0^i + \underline{a}_1^i u + \underline{a}_2^i u^2/2! + \underline{a}_3^i u^3/3!, \underline{a}_j^i \in E^3, u \in [0,1]$$

To determine $\Gamma$ means we must determine for the n segments the 4n vector coefficients $\underline{a}_j^i$ for i = 1, .... n and j = 0, ...,3 or equivalently the 4n vector quantities $\underline{Y}_i(0)$, $\underline{Y}_i(1)$, $\underline{\dot{Y}}_i(0)$, $\underline{\dot{Y}}_i(1)$. Be requiring that $k \geqslant 0$ we are imposing 6n scalar conditions and to avoid corners we insist that $k \geqslant 1$, which places an additional 2n conditions. Jumps in the curvature vector are undesirable and so this means $k \geqslant 2$ and another 2n conditions. Now we could impose further continuity constraints to give us the final 2n conditions we now require. This would mean that in addition to continuity of $\underline{Y}_i$, $\underline{Y}_i'$, $\underline{Y}_i''$ we would also have continuity of torsion and rate of change of curvature. In fact if the points $\underline{b}_i$ i = 1,....n+1 are coplanar continuity of $\tau$ is assured and so one might instead ask for continuity of $\kappa''$. However in practice Figs{1-4} the curves resulting from excessive continuity restrictions at this discrete set of points indicate that we must relax some of the conditions in order to gain control over the behaviour of the curve away from this set. This inevitably means we must consider restrictions in some way related to arc length. We have already shown (Proposition 2.3) that for certain fixed end conditions then there are $\infty^2$ cubics to choose from for each segment depending upon the values of the end speeds $\dot{s}(0)$ and $\dot{s}(1)$. Also there

is no requirement for these speeds to be continuous

across segments since $C^1$ continuity only concerns the

tangent direction.    Therefore these could be used as

parameters for controlling the behaviour of the curve

away from the segment endpoints.    Manning {8} devised an

algorithm which determines the speeds based on an

approximation to a circular arc, and this has proved

to be effective in practice.    It is not clear however,

that any geometic significance can be attached to this form

of contraint in a non-planar situation or indeed when the

segment end conditions are not symmetrical.    Before

looking at this problem we consider the implications of

the continuity requirements.    ;

Suppose that at the point $\underline{b}_i \varepsilon E^3$ we are going to continue

the curve $\underline{\alpha}:[0,1] \longrightarrow E^3$ with $\underline{\beta}:[o,1] \longrightarrow E^3$ such that

$\underline{\alpha}^{(k)}(1) = \underline{\beta}^{(k)}(0)$ for some k.  Then if we express each

condition in terms of the parameter u we find :

(i) k = 0

$\quad \underline{\beta}(0) = \underline{\alpha}(1) = \underline{b}_i$

(ii) k = 1

$\quad$ Since, $\underline{\dot{\alpha}}(1) = \dot{s}(1)\underline{\alpha}'(1)$ and $\underline{\dot{\beta}}(0) = \dot{S}(0)\underline{\beta}'(0)$

$\quad$ then we require

$\quad \underline{\dot{\beta}}(0) = c\underline{\dot{\alpha}}(1)$ some scalar c > 0

(iii) k = 2

$\quad$ For $\alpha$, $\underline{\dot{\alpha}}(u) = \dot{s}(u)\underline{t}(u)$,

$\quad\quad \underline{\ddot{\alpha}}(u) = \dot{s}^2(u)\kappa(u)\underline{n}(u) + \ddot{s}(u)\underline{t}(u)$    and so

$\quad\quad \kappa(u)\underline{n}(u) = \{\underline{\ddot{\alpha}}(u) - \ddot{s}(u)\underline{t}(u)\}/\dot{s}^2(u).$

But $\ddot{s}(u) = \langle \dot{\underline{\alpha}}(u), \ddot{\underline{\alpha}}(u)\rangle / \dot{s}(u)$ , $\dot{s}^2(u) = \langle \dot{\underline{\alpha}}(u), \dot{\underline{\alpha}}(u)\rangle$ and

$\underline{t}(u) = \dot{\underline{\alpha}}(u)/\dot{s}(u)$  and so

$\kappa(u)\underline{n}(u) = \{\ddot{\underline{\alpha}}(u)/\langle \dot{\underline{\alpha}}(u),\dot{\underline{\alpha}}(u)\rangle$  $- \langle \dot{\underline{\alpha}}(u),\ddot{\underline{\alpha}}(u)\rangle \dot{\underline{\alpha}}(u)/\langle \dot{\underline{\alpha}}(u),\dot{\underline{\alpha}}(u)\rangle^2$

Thus we require :

$\{\ddot{\underline{\alpha}}(1)/\dot{\underline{\alpha}}^2(1)\} - \langle \dot{\underline{\alpha}}(1),\ddot{\underline{\alpha}}(1)\rangle \dot{\underline{\alpha}}(1)/\langle \dot{\underline{\alpha}}(1),\dot{\underline{\alpha}}(1)\rangle^2$

$\qquad = \{\ddot{\underline{\beta}}(0)/\dot{\underline{\beta}}^2(0)\} - \langle \dot{\underline{\beta}}(0),\ddot{\underline{\beta}}(0)\rangle \dot{\underline{\beta}}(0)/\langle \dot{\underline{\beta}}(0),\dot{\underline{\beta}}(0)\rangle^2$

But since $\dot{\underline{\beta}}(0) = c\dot{\underline{\alpha}}(1)$, (k=1), we find that on rearranging

we get

$c^2\ddot{\underline{\alpha}}(1) - \ddot{\underline{\beta}}(0) = \langle \dot{\underline{\alpha}}(1),c^2 \ddot{\underline{\alpha}}(1)-\ddot{\underline{\beta}}(0)\rangle \dot{\underline{\alpha}}(1)/\dot{\underline{\alpha}}^2(1)$ $\qquad\qquad$ (*)

Thus if we have $c^2\ddot{\underline{\alpha}}(1) - \ddot{\underline{\beta}}(0) = d\dot{\underline{\alpha}}(1)$ equation (*) holds

for all values of d.

## (iv) k = 3

For $\alpha$, $\dddot{\underline{\alpha}}(u) = \{\dddot{\underline{\alpha}}(u) - 3\dot{s}(u)\ddot{s}(u)\kappa(u)\underline{n}(u) - \dddot{s}(u)\underline{t}(u)\}/\dot{s}^3(u)$

but $\dot{s}(u)\ddot{s}(u) = \langle \dot{\underline{\alpha}}(u),\ddot{\underline{\alpha}}(u)\rangle$ and so differentiating gives

$\qquad \dddot{s}(u) = \{\langle \dot{\underline{\alpha}}(u),\dddot{\underline{\alpha}}(u)\rangle + \ddot{\underline{\alpha}}^2(u) - \ddot{s}^2(u)\}/\dot{s}(u)$

or $\dddot{s}(u) = \{\langle \dot{\underline{\alpha}}(u),\dddot{\underline{\alpha}}(u)\rangle + \ddot{\underline{\alpha}}^2(u) - \langle \dot{\underline{\alpha}}(u),\ddot{\underline{\alpha}}(u)\rangle^2/$

$\qquad\qquad\qquad\qquad\qquad \langle \dot{\underline{\alpha}}(u),\dot{\underline{\alpha}}(u)\rangle\}/\dot{s}(u)$

Thus

$\dddot{\underline{\alpha}}(u) = \dot{s}^{-1}(u)\big[\dddot{\underline{\alpha}}(u)-3\langle \dot{\underline{\alpha}}(u),\ddot{\underline{\alpha}}(u)\rangle[(\ddot{\underline{\alpha}}(u)/\langle \dot{\underline{\alpha}}(u),\dot{\underline{\alpha}}(u)\rangle) -$

$\qquad \langle \dot{\underline{\alpha}}(u),\ddot{\underline{\alpha}}(u)\rangle\dot{\underline{\alpha}}(u)/\langle \dot{\underline{\alpha}}(u),\dot{\underline{\alpha}}(u)\rangle^2] - (\dot{\underline{\alpha}}(u)/\langle \dot{\underline{\alpha}}(u),\dot{\underline{\alpha}}(u)\rangle)$

$\qquad [\langle \dot{\underline{\alpha}}(u),\dddot{\underline{\alpha}}(u)\rangle + \ddot{\underline{\alpha}}^2(u) - \langle \dot{\underline{\alpha}}(u),\ddot{\underline{\alpha}}(u)\rangle^2/\dot{\underline{\alpha}}^2(u)]\big]/\dot{\underline{\alpha}}^2(u)$

and so we require

$$\dot{s}^{-1}(1)\left[\dddot{\underline{\alpha}}(1) - 3\langle\dot{\underline{\alpha}}(1),\ddot{\underline{\alpha}}(1)\rangle\left[(\ddot{\underline{\alpha}}(1)/\dot{\underline{\alpha}}^2(1)) - \langle\dot{\underline{\alpha}}(1),\ddot{\underline{\alpha}}(1)\rangle\,\dot{\underline{\alpha}}(1)/\right.\right.$$

$$\left.\langle\dot{\underline{\alpha}}(1),\dot{\underline{\alpha}}(1)\rangle^2\right] - (\dot{\underline{\alpha}}(1)/\dot{\underline{\alpha}}^2(1))\left[\langle\dot{\underline{\alpha}}(1),\dddot{\underline{\alpha}}(1)\rangle + \ddot{\underline{\alpha}}^2(1) - \right.$$

$$\left.\left.\langle\dot{\underline{\alpha}}(1),\ddot{\underline{\alpha}}(1)\rangle^2/\dot{\underline{\alpha}}^2(1)\right]\right]/\dot{\underline{\alpha}}^2(1) =$$

$$\dot{s}^{-1}(0)\left[\dddot{\underline{\beta}}(0) - 3\langle\dot{\underline{\beta}}(0),\ddot{\underline{\beta}}(0)\rangle\left[(\dddot{\underline{\beta}}(0)/\dot{\underline{\beta}}^2(0)) - \langle\dot{\underline{\beta}}(0),\ddot{\underline{\beta}}(0)\rangle\dot{\underline{\beta}}(0)/\right.\right.$$

$$\left.\langle\dot{\underline{\beta}}(0),\dot{\underline{\beta}}(0)\rangle^2\right] - (\dot{\underline{\beta}}(0)/\dot{\underline{\beta}}^2(0))\left[\langle\dot{\underline{\beta}}(0),\dddot{\underline{\beta}}(0)\rangle + \ddot{\underline{\beta}}^2(0) - \right.$$

$$\left.\left.\langle\dot{\underline{\beta}}(0),\ddot{\underline{\beta}}(0)\rangle^2/\dot{\underline{\beta}}^2(0)\right]\right]/\dot{\underline{\beta}}^2(0).$$

But since $\dot{\underline{\beta}}(0) = c\dot{\underline{\alpha}}(1)$           (k = 1)

     and   $\ddot{\underline{\beta}}(0) = c^2\ddot{\underline{\alpha}}(1) - d\dot{\underline{\alpha}}(1)$       (k = 2)

we find that this reduces to

$$\{\dddot{\underline{\beta}}(0) - c^3\dddot{\underline{\alpha}}(1) - 3cd\ddot{\underline{\alpha}}(1)\} \times \dot{\underline{\alpha}}(1) = \underline{0}$$

Collectively these results mean that :


## Proposition 2.9

If $\underline{\alpha} : [0,1] \longrightarrow E^3$ and $\underline{\beta} : [0,1] \longrightarrow E^3$ are two curves

satisfying :

    (i)    $\underline{\beta}(0) = \underline{\alpha}(1)$

   (ii)    $\dot{\underline{\beta}}(0) = c\dot{\underline{\alpha}}(1)$

 (iii)    $\ddot{\underline{\beta}}(0) = c^2\ddot{\underline{\alpha}}(1) - d\dot{\underline{\alpha}}(1)$

 (iv)    $\dddot{\underline{\beta}}(0) = c^3\dddot{\underline{\alpha}}(1) - 3cd\ddot{\underline{\alpha}}(1) + e\dot{\underline{\alpha}}(1)$     where $c, d, e \in \mathbb{R}$   $c > 0$

       are uniquely determined, then $\underline{\beta}$ is a $C^3$ extension of $\underline{\alpha}$.


For comparison (Figures 7, 8, 9) with results from

Chapter 5 we briefly describe two forms of curve

representation already widely used in computer aided design {12, 17, 18}.

Let I denote the unit interval, $[0,1]$ and let $\mu\{f\}$ denote the number of sign changes of the function $f : I \longrightarrow \mathbb{R}$.

## Definition 2.10

Let $A : C^r(I) \longrightarrow C^r(I)$ for some r such that

(i) $A(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 A(f_1) + \alpha_2 A(f_2)$ for $f_1, f_2 \in C^r(I)$, $\alpha_1, \alpha_2, \in \mathbb{R}$

(ii) $A(1) = 1$ where $1 : t \longmapsto 1 \ \forall \ t$

(iii) $A(\hat{I}) = \hat{I}$ where $\hat{I} : t \longmapsto t \ \forall \ t$

Then A is said to be Variation Diminishing if $\mu\{A(f)\} \leqslant \mu\{f\} \ \forall \ f \in C^r(I)$

This means that A(f) can only have as many points of inflexion as f.

## Definition 2.11

Let $B_m : C^r(I) \longrightarrow C^r(I)$ for each m be given by
$$B_m(f) = \sum_{i=0}^{m} f(\frac{i}{m})\lambda_i$$

where $\lambda_i : t \longmapsto \binom{m}{i} t^i (1-t)^{m-i}$ , $t \in I$

Then $B_m(f)$ is called the Bernstein approximation to f of degree m.

The Bernstein approximation has the following properties :

(i) $\quad \dot{\mu}\{B_m(f)\} \leqslant \dot{Z}\{B_m(f)\} \leqslant \dot{\mu}\{f\}$

where $\dot{Z}\{B_m(f)\}$ denotes the zeros of $B_m(f)$ on $I^0$.

(ii) If $f \in C^n(I)$ , $B_m^{(P)}(f) \longrightarrow f^{(P)}$ uniformly as

$m \longrightarrow \infty$ , $p = 0,1,.....n$

(iii) $B_m^{(P)}(f)\Big|_{t=0} = m!/(m-P)! \sum_{j=0}^{P} (-1)^{P-j} \binom{P}{j} f(\frac{j}{m})$

$B_m^{(P)}(f)\Big|_{t=1} = m!/(m-P)! \sum_{j=0}^{P} (-1)^{j} \binom{P}{j} f(\frac{m-j}{m})$

which means that these are simply functions of

$f(0), f(\frac{1}{m}).......f(\frac{P}{m})$ and $f(1), f(\frac{m-1}{m}),.....f(\frac{m-P}{m})$

(iv) If $P:I \longrightarrow R$ is a polynomial of degree m then

there is a unique set $\left\{ f(0), f(\frac{1}{m}),.....f(1) \right\}$

defined by

$f(\frac{j}{m}) = \sum_{r=0}^{j} \binom{j}{r} \{(m-r)!/m!\} P^{(r)} \Big|_{t=0}$

$= \sum_{r=0}^{m-j} (-1)^{r} \binom{m-j}{r} \{(m-r)!/m!\} P^{(r)} \Big|_{t=1}$

(v) If $\alpha_p \leqslant f^{(P)} \leqslant \beta_p$ then

$\alpha_p \leqslant m^P (m-P)!/m! \{B_m^{(P)}(f)\} \leqslant \beta_p$ for $p = 1,...m$

and $\alpha_0 \leqslant B_m(f) \leqslant \beta_0$

(vi) If $g:I \longrightarrow R$ is linear, then from (i)

$\dot{Z}\{B_m(f)-g\} = \dot{Z}\{B_m(f-g)\} \leqslant \dot{\mu}\{f-g\}$

which means that $B_m(f)$ does not oscillate about f.


Definition 2.12


Let $\underline{r}:I \longrightarrow E^3$ be given by

$\underline{r}(u) = \sum_{i=0}^{m} \lambda_i(u)\underline{d}_i$ , $\underline{d}_i \in E^3$ $i=0,....m$

Then $\underline{r}$ is the Bezier curve of degree m for $\left\{\underline{d}_i \epsilon E^3\right\}$.

Figure 5 shows typical Bezier curves for m = 3,5 and 6.

We see that $\underline{r}(0) = \underline{d}_0$ and $\underline{r}(1) = \underline{d}_m$, and since

$\lambda_i(u) > 0$, $u \epsilon I^0$ and

$$\sum_{i=0}^{m} \binom{m}{i} u^i (1-u)^{m-i} = \left[(1-u)+u\right]^m = 1 \text{ then if follows}$$

that $\underline{r}$ lies within the convex hull formed by the $\underline{d}_i$.


We confine our attention now to Bezier Cubic curves. For

any set of points $\left\{\underline{d}_i \epsilon E^3\right\}_{i=0}^{3}$ then the corresponding

Bezier cubic curve is given by

$$r(u) = \sum_{i=0}^{3} \lambda_i(u)\underline{d}_i$$

where $\quad \lambda_i(u) = \binom{3}{i} u^i (1-u)^{3-i}$, $u \epsilon I$.

We shall see that the points $\underline{d}_i$ control the geometry of

$\underline{r}$ in a nice way. Firstly, the curve contains $\underline{d}_0 = \underline{r}(0)$

and $\underline{d}_3 = \underline{r}(1)$ and the tangent directions at $\underline{d}_0$ and $\underline{d}_1$ are

$$\underline{t}(0) = 3(\underline{d}_1 - \underline{d}_0)/\dot{s}(0) \quad \text{and}$$
$$\underline{t}(1) = 3(\underline{d}_3 - \underline{d}_2)/\dot{s}(1)$$

Then since the equation for the osculating plane of $\underline{r}$

is given by

$$\left[ \underline{R}-\underline{r} , \dot{\underline{r}} , \ddot{\underline{r}} \right] = 0$$

at u = 0 we have

$$\left[ \underline{R}-\underline{d}_0, \; 3(\underline{d}_1-\underline{d}_0), \; 6(\underline{d}_0-2\underline{d}_1+\underline{d}_2) \right] = 0$$

This means that the osculating plane at $\underline{r}(0)$ contains

$\underline{d}_0, \underline{d}_1$ and $\underline{d}_2$.

Similarly the osculating place at $\underline{r}(1)$ contains $\underline{d}_1$, $\underline{d}_2$ and $\underline{d}_3$. Looking at this another way this means that $\underline{d}_2$ is the point of intersection of the osculating plane at $\underline{d}_0$ with the tangent line at $\underline{d}_3$. And similarly for $\underline{d}_1$. Unfortunately these are not unique if the curve is planar.

Further since $\kappa(u) = |\dot{\underline{r}}(u) \times \ddot{\underline{r}}(u)|/\dot{s}^3(u)$ we see that (Figure 5) :

$$\kappa(0) = 2|(\underline{d}_0-\underline{d}_1) + (\underline{d}_2-\underline{d}_1)|\sin\theta/3|\underline{d}_1-\underline{d}_0|^2$$

$$\kappa(1) = 2|(\underline{d}_3-\underline{d}_2) + (\underline{d}_2-\underline{d}_1)|\sin\phi/3|\underline{d}_3-\underline{d}_2|^2$$

and since $\tau(u) = [\dot{\underline{r}}(u), \ddot{\underline{r}}(u), \dddot{\underline{r}}(u)]/(\dot{\underline{r}}(u) \times \ddot{\underline{r}}(u))^2$

$$\tau(0) = 2|(\underline{d}_3-\underline{d}_0) + 3(\underline{d}_1-\underline{d}_2)|\cos\alpha / \varsigma_1 \quad \text{where}$$

$$\varsigma_1 = |\underline{d}_1-\underline{d}_0||(\underline{d}_0-\underline{d}_1) + (\underline{d}_2-\underline{d}_1)|\sin\theta$$

$$\tau(1) = 2|(\underline{d}_3-\underline{d}_0) + 3(\underline{d}_1-\underline{d}_2)|\cos\beta / \varsigma_2 \quad \text{where}$$

$$\varsigma_2 = |\underline{d}_3-\underline{d}_2||(\underline{d}_3-\underline{d}_2) + (\underline{d}_2-\underline{d}_1)|\sin\phi$$

One other curve form of particular interest {Figure 6} is a spline analog of the Bezier form {17} and this provides a type of moving average formulation for a series of points in $E^3$ where in general the resulting curve does not contain the points but is generated from them in a predictable manner as far as the shape is concerned.

### Definition 2.13

Let $t_0, t_1, \ldots, t_n$ be real numbers with $t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n$.

Then a Polynomial Spline of order k is a map $S : (t_0, t_n) \longrightarrow \mathbb{R}$ satisfying

(i) $S|(t_i, t_{i+1})$ is a polynomial of degree k-1

(ii) $S \in C^{k-2}$.

The real numbers $t_i$ are known as Knots and for fixed k and n such splines form a Vector Space over $\mathbb{R}$. A particular basis for this vector space is called the B-Spline basis and this was first introduced by Schoenberg {24}.


## Definition 2.14

For reals $t_0$, $t_1$, ......, $t_n$ the B-Spline basis of order k is given by the following recursive formulation.

$$B_{i,1}(t) = 1 \quad \text{for } t_i \leqslant t \leqslant t_{i+1}$$

$$B_{i,1}(t) = 0 \quad \text{for } t_i > t > t_{i+1}$$

$$B_{i,k}(t) = \{(t-t_i)/(t_{i+k-1}-t_i)\}B_{i,k-1}(t) +$$
$$\{(t_{i+k}-t)/(t_{i+k}-t_{i+1})\}B_{i+1,k-1}(t)$$

There are a number of alternative formulations {17}.
B-Splines have the following properties :

(i) $B_{i,k}(t) > 0 \quad$ for $t_i \leqslant t \leqslant t_{i+1}$

$\quad\quad B_{i,k}(t) = 0 \quad$ for $t_i > t > t_{i+1}$

(ii) $\sum_i B_{i,k}(t) = 1$

(iii) If r knots are coincident, say

$t_\alpha = t_{\alpha+1} = \cdots = t_{\alpha+r-1}$ then continuity is reduced and $B_{j,k} \in C^{k-r-1}$, $j=\alpha, \alpha+1, \ldots, \alpha+r-1$.

(iv) Calculations greatly simplify if $t_m = m \in N$, for each m.

(v) In the particular case where there are just two knots at m = 0 and m = 1 each with multiplicity k then the B-Spline basis reduces to the Bernstein basis with

$$B_{i,k}(t) = \binom{k-1}{i} t^i (1-t)^{k-1-i}, \quad i = 0, \ldots, k-1$$

(vi) The kth order B-Spline approximation to $f: [0,1] \longrightarrow \mathbb{R}$ is given by

$$B_k(f) = \sum_i f(\xi_i) B_{i,k} \quad \text{where the}$$
$$\xi_i = (t_{i+1} + t_{i+2} + \ldots + t_{i+k-1})/(k-1)$$

are termed the Nodes.

(vii) B-Spline approximation is variation diminishing.

(viii) In contrast to the Bernstein approximation, local changes in a function f produce only local changes in $B_k(f)$.

Definition 2.15

The B-spline curve form of order k for the set of points $\{\underline{P}_i\} \subset E^3$ , $i = 0, \ldots, n$ is given by

$$\underline{r}(u) = \sum_{i=0}^{n} B_{i-\frac{k}{2},k}(u)\underline{P}_i$$

The B-spline curve has the following basic properties :

(i) If k is even it can be shown that the knots of the spline are 'opposite' the $\underline{P}_i$ whereas if k is odd

then the knots are 'opposite' the points

$$\underline{Q}_i = (\underline{P}_i + \underline{P}_{i+1})/2$$

(ii) Since the curve can be regarded locally as a convex combination of $\underline{P}_\alpha, \ldots, \underline{P}_{\alpha+k-1}$ the curve lies within a series of local convex hulls.

(iii) For a given set $\{\underline{P}_i\}$, $i=0\ldots n$ then if $k=2$, then $\underline{r} \in C^0$ and interpolates the $\underline{P}_i$ linearly. As k tends to infinity then $\underline{r}$ tends to a single straight line.

(iv) If $\underline{r}$ is of order k and $\underline{P}_{\tau+1}, \underline{P}_{\tau+2} \ldots \underline{P}_{\tau+k}$ are collinear for some $\tau$ then $\underline{r}$ contains a linear segment on the line through $\underline{P}_\alpha$ and $\underline{P}_\beta$ where $\tau+1 \leqslant \beta < \alpha \leqslant \tau+k$.

(v) If $\underline{r}$ is of order k and $\underline{P}_{\tau+1}, \underline{P}_{\tau+2} \ldots \ldots, \underline{P}_{\tau+k-1}$ are collinear then $\underline{r}$ touches the line through $\underline{P}_\alpha$ and $\underline{P}_\beta$ where $\tau+1 \leqslant \beta < \alpha \leqslant \tau+k-1$.

The B-spline cubic curve for the sequence $\underline{P}_\alpha$ of points in $E^3$ is of the form

$$\underline{r}(u) = \sum_\alpha B(u-\alpha)\underline{P}_\alpha$$

where the B-spline scalar weighting function $B(u)$ is defined by

$$B(u) = \left[\xi_3(u-2) - 4.\xi_3(u-1) + 6.\xi_3(u) - 4.\xi_3(u+1) + \xi_3(u+2)\right]/6$$

where $\xi_k(a) = a^k$ for $a \geqslant 0$

and $\xi_k(a) = 0$ for $a < 0$

Explicitly $\underline{r}$ is given by

$$\underline{r}(u) = \{\xi_3(u-i-1)-4.\xi_3(u-i)+6.\xi_3(u-i+1)-4.\xi_3(u-i+2)+$$
$$\xi_3(u-i+3)\}\underline{P}_{i-1}/6$$
$$+\{\xi_3(u-i-2)-4.\xi_3(u-i-1)+6.\xi_3(u-i)-4.\xi_3(u-i+1)+$$
$$\xi_3(u-i+2)\}\underline{P}_i/6$$
$$+\{\xi_3(u-i-3)-4.\xi_3(u-i-2)+6.\xi_3(u-i-1)-4.\xi_3(u-i)+$$
$$\xi_3(u-i+1)\}\underline{P}_{i+1}/6$$
$$+\{\xi_3(u-i-4)-4.\xi_3(u-i-3)+6.\xi_3(u-i-2)-4.\xi_3(u-i-1)+$$
$$\xi_3(u-i)\}\underline{P}_{i+2}/6$$

$$\text{for } i \leqslant u \leqslant i+1$$

which means that $\underline{r}(i) = \{\underline{P}_{i-1}+4\underline{P}_i+\underline{P}_{i+1}\}/6$

Thus if $\underline{P}_{i-1} = \underline{P}_i = \underline{P}_{i+1}$ then $\underline{r}(i) = \underline{P}_i$

Differentiation with respect to $u$ gives

$$\underline{\dot{r}}(u) = \{\xi_2(u-i-1)-4.\xi_2(u-i)+6.\xi_2(u-i+1)-4.\xi_2(u-i+2)+$$
$$\xi_2(u-i+3)\}\underline{P}_{i-1}/2$$
$$+\{\xi_2(u-i-2)-4.\xi_2(u-i-1)+6.\xi_2(u-i)-4.\xi_2(u-i+1)+$$
$$\xi_2(u-i+2)\}\underline{P}_i/2$$
$$+\{\xi_2(u-i-3)-4.\xi_2(u-i-2)+6.\xi_2(u-i-1)-4.\xi_2(u-i)+$$
$$\xi_2(u-i+1)\}\underline{P}_{i+1}/2$$
$$+\{\xi_2(u-i-4)-4.\xi_2(u-i-3)+6.\xi_2(u-i-2)-4.\xi_2(u-i-1)+$$
$$\xi_2(u-i)\}\underline{P}_{i+2}/2$$

and so

$$\underline{\dot{r}}(i) = \{\underline{P}_{i+1}-\underline{P}_{i-1}\}/2$$

Thus the speed $\dot{s}(i) = 0$ if and only if $\underline{P}_{i+1} = \underline{P}_{i-1}$.
Also we can see that if $\underline{P}_{i-1}$, $\underline{P}_i$, $\underline{P}_{i+1}$, $\underline{P}_{i+2}$ are not
coplanar then $\dot{s}(u) \neq 0$ in $i \leqslant u \leqslant i+1$.

Differentiating again gives

$$\ddot{r}(u) = \{\xi_1(u-i-1)-4.\xi_1(u-i)+6.\xi_1(u-i+1)-4.\xi_1(u-i+2)+$$
$$\xi_1(u-i+3)\}P_{i-1}$$
$$+ \{\xi_1(u-i-2)-4.\xi_1(u-i-1)+6.\xi_1(u-i)-4.\xi_1(u-i+1)+$$
$$\xi_1(u-i+2)\}P_i$$
$$+ \{\xi_1(u-i-3)-4.\xi_1(u-i-2)+6.\xi_1(u-i-1)-4.\xi_1(u-i)+$$
$$\xi_1(u-i+1)\}P_{i+1}$$
$$+ \{\xi_1(u-i-4)-4.\xi_1(u-i-3)+6.\xi_1(u-i-2)-4.\xi_1(u-i-1)+$$
$$\xi_1(u-i)\}P_{i+2}$$

and so

$$\ddot{r}(i) = P_{i-1}-2P_i+P_{i+1}$$

Since

$$\dddot{r}(u) = \{\xi_0(u-i-1)-4.\xi_0(u-i)+6.\xi_0(u-i+1)-4.\xi_0(u-i+2)+$$
$$\xi_0(u-i+3)\}P_{i-1}$$
$$+ \{\xi_0(u-i-2)-4.\xi_0(u-i-1)+6.\xi_0(u-i)-4.\xi_0(u-i+1)+$$
$$\xi_0(u-i+2)\}P_i$$
$$+ \{\xi_0(u-i-3)-4.\xi_0(u-i-2)+6.\xi_0(u-i-1)-4.\xi_0(u-i)+$$
$$\xi_0(u-i+1)\}P_{i+1}$$
$$+ \{\xi_0(u-i-4)-4.\xi_0(u-i-3)+6.\xi_0(u-i-2)-4.\xi_0(u-i-1)+$$
$$\xi_0(u-i)\}P_{i+2}$$

then

$$\dddot{r}(i) = 3(P_{i-1}-P_i)+P_{i+1}$$ and so we find that the curvature is

$$\kappa(i) = 8\sqrt{\{(P_{i+1}\times P_{i-1})+(P_{i-1}\times P_i)+(P_i\times P_{i+1})\}^2/\{(P_{i+1}-P_{i-1})^2\}^3}$$

and the torsion,

$$\tau(i) = [P_{i-1}.P_i.P_{i+1}]/\{(P_{i+1}\times P_{i-1})+(P_{i-1}\times P_i)+(P_i\times P_{i+1})\}^2$$

The cubic weighting function B(u), which satisfies

   (i)    $B(u) \geqslant 0 \ \forall \ u$

  (ii)   $\sum_{\alpha} B(u-\alpha) = 1$

(iii)   there exists $\alpha \in R$ such that $B(u) = 0$ for all $|u| > \alpha$,

means that we can use the $\left\{ \underline{P}_\alpha \right\}$ as a means of shape prediction and modification of the B-spline cubic curve and so this curve form is particularly attractive for a design system from the designers' point of view.

## Chapter 3

## Computer Surfaces in $E^3$

### Definition 3.1

A Bicubic surface in $E^3$ is one for which there exist
parameters u and v in terms of which the equation of the
surface can be written as :

$$\underline{r}(u,v) = \sum_{i,j=0}^{3} \underline{a}_{ij} u^i v^j / i! j! \qquad \underline{a}_{ij} \in E^3$$

The parameters u and v are indeterminate to within
transformations of the form $u \longmapsto \alpha u + \beta$ and
$v \longmapsto \gamma v + \delta$ but if we wish the surface to pass
through four particular points in $E^3$ with specified
parameter values then the parameters are uniquely
determined.

For convenience we will normally assume that $\underline{r}$ is defined
on $[o,1] \times [o,1]$ and the four points chosen are the
four distinguishable boundary points $\underline{r}(i,j)$ for $i,j \in \{0,1\}$.
The four boundary curves $\underline{r}(u,i), \underline{r}(i,v)$ $i \in \{o,1\}$ are cubic
curves in the parameters u or v.

We will use $\underline{r}_1$ and $\underline{r}_2$ to denote the tangent vectors

$\partial \underline{r}(u,v)/\partial u$ and $\partial \underline{r}(u,v)/\partial v$ respectively and $\underline{N}$ for the unit surface normal where $\underline{N} = (\underline{r}_1 \times \underline{r}_2)/|\underline{r}_1 \times \underline{r}_2|$. We also use $x,y$ to denote the usual dot product or inner product of vectors $x,y$.

If $\underline{\lambda} : [a,b] \longrightarrow S$ is a curve on a surface $S$ then $\underline{\lambda}(t) = \underline{r}(u(t),v(t))$ and since the arc length $s$ along $\underline{\lambda}$ is given by $s = \int_a^b \langle \underline{\dot{\lambda}}(t), \underline{\dot{\lambda}}(t) \rangle \, dt$ then along the parametric curves $v=v_c$ and $u=u_c$

$$s = \int_0^1 \sqrt{E(u,v_c)} \, du \quad \text{and} \quad s = \int_0^1 \sqrt{G(u_c,v)} \, dv.$$

where $E(u,v) = \langle \underline{r}_1, \underline{r}_1 \rangle$, $F(u,v) = \langle \underline{r}_1, \underline{r}_2 \rangle$,

$G(u,v) = \langle \underline{r}_2, \underline{r}_2 \rangle$ are the First Fundamental Coefficients.

Proposition 3.2

Given $\left\{ \underline{\alpha}_{ij}, \underline{\beta}_{ij}, \underline{\gamma}_{ij} \right\} \subset E^3$ where $i,j \in \{0,1\}$ then there are $\infty^{12}$ bicubic surfaces $\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$ such that :

(i)   $\underline{r}(i,j) = \underline{\alpha}_{ij}$

(ii)  $\underline{r}_1(i,j) = \underline{\beta}_{ij}$

(iii) $\underline{r}_2(i,j) = \underline{\gamma}_{ij}$    $i,j \in \{0,1\}$

This follows because there are twelve degrees of freedom remaining for specifying the sixteen coefficients of the surface equation.

## Corollary 1

Given $\{\underline{\alpha}_{ij} , \underline{\beta}_{ij} , \underline{\gamma}_{ij} , \underline{n}_{ij}\} \subset E^3$ , $i,j \in \{0,1\}$ then these

determine a unique bicubic surface $\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$

satisfying :

    (i)   $\underline{r}(i,j) = \underline{\alpha}_{ij}$ ,

   (ii)   $\underline{r}_1(i,j) = \underline{\beta}_{ij}$ ,

  (iii)   $\underline{r}_2(i,j) = \underline{\gamma}_{ij}$ ,

   (iv)   $\underline{r}_{12}(i,j) = \underline{n}_{ij}$

The vector $\underline{r}_{12}$ is often called the 'twist' vector which

is a misnomer because it is not an intrinsic property

of the surface.     To see this if we consider the sphere

$\underline{r} : [0,\pi] \times [0,2\pi] \longrightarrow E^3$    where

   $\underline{r}(u,v) = (a \sin u \cos v, a \sin u \sin v, a \cos v)$, then

     $\underline{r}_{12} = \underline{0}$ if and only if $u = \pi/2$.

Obviously the above Corollary would also hold if we

replaced condition (iv) by either $\underline{r}_{11}(i,j) = \underline{n}_{ij}$ or

$\underline{r}_{22}(i,j) = \underline{n}_{ij}$.

## Corollary 2

Given $\{\underline{\alpha}_{ij}, \underline{\beta}_{ij}, \underline{\gamma}_{ij}\} \subset E^3$ and reals $L_{ij}, M_{ij}, N_{ij}$ where

$i,j \in \{0,1\}$ then there is a unique bicubic surface

$\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$ such that

    (i)   $\underline{r}(i,j) = \underline{\alpha}_{ij}$,

   (ii)   $\underline{r}_1(i,j) = \underline{\beta}_{ij}$,

  (iii)   $\underline{r}_2(i,j) = \underline{\gamma}_{ij}$

(iv) $\langle \underline{r}_{11}(i,j), \underline{N}(i,j) \rangle = L_{ij}$

$\langle \underline{r}_{12}(i,j), \underline{N}(i,j) \rangle = M_{ij}$

$\langle \underline{r}_{22}(i,j), \underline{N}(i,j) \rangle = N_{ij}$

Then $L_{ij}$, $M_{ij}$, $N_{ij}$ are the values of the Second
Fundamental Coefficients $L(u,v), M(u,v), N(u,v)$ at the
four corners $\underline{r}(i,j)$.

## Corollary 3

Given $\{\underline{\alpha}_{ij}, \underline{\beta}_{ij}, \underline{\gamma}_{ij}\} \subset E^3$ and reals $H_{ij}$, $K_{ij}$ $i,j \in \{0,1\}$
then there are $\infty^4$ bicubic surfaces $\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$
such that :

(i) $\underline{r}(i,j) = \underline{\alpha}_{ij}$,

(ii) $\underline{r}_1(i,j) = \underline{\beta}_{ij}$,

(iii) $\underline{r}_2(i,j) = \underline{\gamma}_{ij}$

(iv) The mean $H(u,v)$ and Gaussian $K(u,v)$ curvatures
take the values $H(i,j) = H_{ij}$ and $K(i,j) = K_{ij}$
for $i,j \in \{0,1\}$.

## Corollary 4

Given $\{\underline{\alpha}_{ij}\} \subset E^3$ and reals $E_{ij}$, $F_{ij}$, $G_{ij}$, $L_{ij}$, $M_{ij}$, $N_{ij}$
for $i,j \in \{0,1\}$ then there are $\infty^{12}$ bicubic surfaces
$\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$ such that

(i) $\underline{r}(i,j) = \underline{\alpha}_{ij}$

(ii) $E(i,j) = E_{ij}$ , $F(i,j) = F_{ij}$ , $G(i,j) = G_{ij}$ ,
$L(i,j) = L_{ij}$ , $M(i,j) = M_{ij}$ , $N(i,j) = N_{ij}$ are

the values taken by the fundamental
coefficients at the corners.

Each of the coefficients $\underline{a}_{ij}$ in the surface equation
can be expressed in terms of $\underline{r}$ and its derivatives on the
boundary.    The boundary curves :

$$\underline{r}(u,o) = \underline{a}_{00} + \underline{a}_{10}u + \underline{a}_{20}u^2/2! + \underline{a}_{30}u^3/3! \qquad \text{and}$$

$$\underline{r}(o,v) = \underline{a}_{00} + \underline{a}_{01}v + \underline{a}_{02}v^2/2! + \underline{a}_{03}v^3/3!$$

are cubic curves and so we can write

(i)   $\underline{a}_{00} = \underline{r}(0,0)$

(ii)  $\underline{a}_{10} = \underline{r}_1(0,0)$

(iii) $\underline{a}_{01} = \underline{r}_2(0,0)$

(iv)  $\underline{a}_{20} = 2![3\{\underline{r}(1,0)-\underline{r}(0,0)\} - 2\underline{r}_1(0,0)-\underline{r}_1(1,0)]$

$$= \underline{r}_{11}(0,0)$

(v)   $\underline{a}_{02} = 2![3\{\underline{r}(0,1)-\underline{r}(0,0)\} - 2\underline{r}_2(0,0)-\underline{r}_2(0,1)]$

$$= \underline{r}_{22}(0,0)$

(vi)  $\underline{a}_{30} = 3![2\{\underline{r}(0,0)-\underline{r}(1,0)\}+ \underline{r}_1(0,0)+\underline{r}_1(1,0)]$

$$= \underline{r}_{111}(u,0)$

(vii) $\underline{a}_{03} = 3![2\{\underline{r}(0,0)-\underline{r}(0,1)\}+ \underline{r}_2(0,0)+\underline{r}_2(0,1)]$

$$= \underline{r}_{222}(0,v)$

Hence these coefficients have the significance of the
cubic curve coefficients as far as the boundary curves
$\underline{r}(u,0)$ and $\underline{r}(0,v)$ are concerned.

It is elementary but tedious to show that :

(viii) $\underline{a}_{11} = \underline{r}_{12}(0,0)$

(ix) $\quad \underline{a}_{21} = 2\left[3\{\underline{r}_2(1,0)-\underline{r}_2(0,0)\} - \underline{r}_{12}(1,0)-2\underline{r}_{12}(0,0)\right]$

$\qquad\qquad = \underline{r}_{121}(0,0)$

(x) $\quad \underline{a}_{12} = 2\left[3\{\underline{r}_1(0,1)-\underline{r}_1(0,0)\} - \underline{r}_{12}(0,1)-2\underline{r}_{12}(0,0)\right]$

$\qquad\qquad = \underline{r}_{212}(0,0)$

(xi) $\quad \underline{a}_{31} = 6\left[2\{\underline{r}_2(0,0)-\underline{r}_2(1,0)\} + \underline{r}_{12}(1,0)+\underline{r}_{12}(0,0)\right]$

$\qquad\qquad = \underline{r}_{1112}(u,0)$

(xii) $\quad \underline{a}_{13} = 6\left[2\{\underline{r}_1(0,0)-\underline{r}_1(0,1)\} + \underline{r}_{12}(0,1)+\underline{r}_{12}(0,0)\right]$

$\qquad\qquad = \underline{r}_{2221}(0,v)$

(xiii) $\quad \underline{a}_{22} = 4\left[9\{\underline{r}(0,0)+\underline{r}(1,1)-\underline{r}(1,0)-\underline{r}(0,1)\}+\right.$

$\qquad\qquad 3\{2(\underline{r}_1(0,0)-\underline{r}_1(0,1)) + \underline{r}_1(1,0)-\underline{r}_1(1,1)\} +$

$\qquad\qquad 3\{2(\underline{r}_2(0,0)-\underline{r}_2(1,0)) + \underline{r}_2(0,1)-\underline{r}_2(1,1)\} +$

$\qquad\qquad \left. 4\underline{r}_{12}(0,0)+\underline{r}_{12}(1,1)+2(\underline{r}_{12}(1,0)+\underline{r}_{12}(0,1))\right]$

$\qquad\qquad = \underline{r}_{1212}(0,0)$

(xiv) $\quad \underline{a}_{32} = -12\left[6\{\underline{r}(0,0)+\underline{r}(1,1)-\underline{r}(1,0)-\underline{r}(0,1)\} -\right.$

$\qquad\qquad 3\{\underline{r}_1(1,1)-\underline{r}_1(0,0)+\underline{r}_1(0,1)-\underline{r}_1(1,0)\} -$

$\qquad\qquad 2\{\underline{r}_2(1,1)-\underline{r}_2(0,1)+2(\underline{r}_2(1,0)-\underline{r}_2(0,0))\} +$

$\qquad\qquad \left. \{\underline{r}_{12}(1,1)+2(\underline{r}_{12}(0,0)+\underline{r}_{12}(1,0))+\underline{r}_{12}(0,1)\}\right]$

$\qquad\qquad = \underline{r}_{11122}(u,0)$

(xv) $\quad \underline{a}_{23} = -12\left[6\{\underline{r}(0,0)+\underline{r}(1,1)-\underline{r}(1,0)-\underline{r}(0,1)\} -\right.$

$\qquad\qquad 2\{\underline{r}_1(1,1)-\underline{r}_1(1,0)+2(\underline{r}_1(0,1)-\underline{r}_1(0,0))\} -$

$\qquad\qquad 3\{\underline{r}_2(1,1)-\underline{r}_2(0,0)+\underline{r}_2(1,0)-\underline{r}_2(0,1)\} +$

$\qquad\qquad \left. \{\underline{r}_{12}(1,1)+2(\underline{r}_{12}(0,0)+\underline{r}_{12}(0,1))+\underline{r}_{12}(0,1)\}\right]$

$\qquad\qquad = \underline{r}_{22211}(u,0)$

(xvi) $\quad \underline{a}_{33} = 36\left[4\{\underline{r}(0,0)+\underline{r}(1,1)-\underline{r}(1,0)-\underline{r}(0,1)\} -\right.$

$\qquad\qquad 2\{\underline{r}_1(1,1)-\underline{r}_1(0,0)+\underline{r}_1(0,1)-\underline{r}_1(1,0)\} -$

$\qquad\qquad 2\{\underline{r}_2(1,1)-\underline{r}_2(0,0)+\underline{r}_2(1,0)-\underline{r}_2(0,1)\} +$

$\qquad\qquad \left. \{\underline{r}_{12}(1,1)+\underline{r}_{12}(0,0)+\underline{r}_{12}(1,0)+\underline{r}_{12}(0,1)\}\right]$

$\qquad\qquad = \underline{r}_{222111}(u,v)$

The surface equation coefficients and the second fundamental coefficients are related in the following manner :

(i) at u=0, v=0

$$L = [\underline{a}_{10}, \underline{a}_{01}, \underline{a}_{20}]/|\underline{a}_{10} \times \underline{a}_{01}|$$

$$M = [\underline{a}_{10}, \underline{a}_{01}, \underline{a}_{11}]/|\underline{a}_{10} \times \underline{a}_{01}|$$

$$N = [\underline{a}_{10}, \underline{a}_{01}, \underline{a}_{02}]/|\underline{a}_{10} \times \underline{a}_{01}|$$

(ii) at u=1, v=0

$$L = [\underline{\alpha}_{10}, \underline{\beta}_{10}, \underline{a}]/|\underline{\alpha}_{10} \times \underline{\beta}_{10}|$$

$$M' = [\underline{\alpha}_{10}, \underline{\beta}_{10}, \underline{b}]/|\underline{\alpha}_{10} \times \underline{\beta}_{10}|$$

$$N = [\underline{\alpha}_{10}, \underline{\beta}_{10}, \underline{c}]/|\underline{\alpha}_{10} \times \underline{\beta}_{10}|$$

where $\underline{\alpha}_{10} = \underline{a}_{10} + \underline{a}_{20} + \underline{a}_{30}/2! = \underline{r}_1(1,0)$

$\underline{\beta}_{10} = \underline{a}_{01} + \underline{a}_{11} + \underline{a}_{21}/2! + \underline{a}_{31}/3! = \underline{r}_2(1,0)$

$\underline{a} = \underline{a}_{20} + \underline{a}_{30} = \underline{r}_{11}(1,0)$

$\underline{b} = \underline{a}_{11} + \underline{a}_{21} + \underline{a}_{31}/2! = \underline{r}_{12}(1,0)$

$\underline{c} = \underline{a}_{02} + \underline{a}_{12} + \underline{a}_{22}/2! + \underline{a}_{32}/3! = \underline{r}_{22}(1,0)$

(iii) at u=0, v=1

$$L = [\underline{\alpha}_{01}, \underline{\beta}_{01}, \underline{d}]/|\underline{\alpha}_{01} \times \underline{\beta}_{01}|$$

$$M = [\underline{\alpha}_{01}, \underline{\beta}_{01}, \underline{e}]/|\underline{\alpha}_{01} \times \underline{\beta}_{01}|$$

$$N = [\underline{\alpha}_{01}, \underline{\beta}_{01}, \underline{f}]/|\underline{\alpha}_{01} \times \underline{\beta}_{01}|$$

where $\underline{\alpha}_{01} = \underline{a}_{10} + \underline{a}_{11} + \underline{a}_{12}/2! + \underline{a}_{13}/3! = \underline{r}_1(0,1)$

$\underline{\beta}_{01} = \underline{a}_{01} + \underline{a}_{02} + \underline{a}_{03}/2! = \underline{r}_2(0,1)$

$\underline{d} = \underline{a}_{20} + \underline{a}_{21} + \underline{a}_{22}/2! + \underline{a}_{23}/3! = \underline{r}_{11}(0,1)$

$\underline{e} = \underline{a}_{11} + \underline{a}_{12} + \underline{a}_{13}/2! = \underline{r}_{12}(0,1)$

$\underline{f} = \underline{a}_{02} + \underline{a}_{03} = \underline{r}_{22}(0,1)$

(iv)  at u=1, v=1

$$L = [\underline{\alpha}_{11}, \underline{\beta}_{11}, \underline{g}] / |\underline{\alpha}_{11} \times \underline{\beta}_{11}|$$

$$M = [\underline{\alpha}_{11}, \underline{\beta}_{11}, \underline{h}] / |\underline{\alpha}_{11} \times \underline{\beta}_{11}|$$

$$N = [\underline{\alpha}_{11}, \underline{\beta}_{11}, \underline{l}] / |\underline{\alpha}_{11} \times \underline{\beta}_{11}|$$

where $\underline{\alpha}_{11} = \sum\limits_{i=1}^{3} \sum\limits_{j=0}^{3} \underline{a}_{ij} / (i-1)! j! \quad = \underline{r}_1(1,1)$

$\underline{\beta}_{11} = \sum\limits_{i=0}^{3} \sum\limits_{j=1}^{3} \underline{a}_{ij} / i! (j-1)! \quad = \underline{r}_2(1,1)$

$\underline{g} = \underline{a}_{20} + \underline{a}_{30} + \underline{a}_{21} + \underline{a}_{31} + (\underline{a}_{22} + \underline{a}_{32})/2!$

$$+ (\underline{a}_{23} + \underline{a}_{33})/3!$$

$$= \underline{r}_{11}(1,1)$$

$\underline{h} = \underline{a}_{11} + \underline{a}_{21} + \underline{a}_{12} + \underline{a}_{22} + (\underline{a}_{31} + \underline{a}_{13} + \underline{a}_{32} + \underline{a}_{23})/2!$

$$+ \underline{a}_{33}/4$$

$$= \underline{r}_{12}(1,1)$$

$\underline{l} = \underline{a}_{02} + \underline{a}_{03} + \underline{a}_{12} + \underline{a}_{13} + (\underline{a}_{22} + \underline{a}_{23})/2!$

$$+ (\underline{a}_{32} + \underline{a}_{33})/3!$$

$$= \underline{r}_{22}(1,1)$$

We can now derive the expressions for the Gaussian and mean curvatures evaluated at the corners in terms of the surface equation coefficients.

(i)  at u=0, v=0

$$K = (\lambda_{20}\lambda_{02} - \lambda_{11}^2) / (\xi_{00}|\underline{a}_{10} \times \underline{a}_{01}|^2)$$

$$H = (\lambda_{02} <\underline{a}_{10}, \underline{a}_{10}> + \lambda_{20} <\underline{a}_{01}, \underline{a}_{01}> - 2\lambda_{11} <\underline{a}_{10}, \underline{a}_{01}>) / (2\xi_{00}|\underline{a}_{10} \times \underline{a}_{01}|)$$

where $\lambda_{20} = [\underline{a}_{10}, \underline{a}_{01}, \underline{a}_{20}]$

$\lambda_{02} = [\underline{a}_{10}, \underline{a}_{01}, \underline{a}_{02}]$

$\lambda_{11} = [\underline{a}_{10}, \underline{a}_{01}, \underline{a}_{11}]$

$\xi_{00} = <\underline{a}_{10}, \underline{a}_{10}> <\underline{a}_{01}, \underline{a}_{01}> - <\underline{a}_{10}, \underline{a}_{01}>^2$

(ii)  at u=1, v=0

$$K = (\lambda_a\lambda_c - \lambda_b^2)/(\xi_{10}|\underline{\alpha}_{10} \times \underline{\beta}_{10}|)$$

$$H = (\lambda_c\langle\underline{\alpha}_{10}\cdot\underline{\alpha}_{10}\rangle + \lambda_a\langle\underline{\beta}_{10}\cdot\underline{\beta}_{10}\rangle - 2\lambda_b\langle\underline{\alpha}_{10}\cdot\underline{\beta}_{10}\rangle)/$$

$$(2\xi_{10}|\underline{\alpha}_{10} \times \underline{\beta}_{10}|)$$

where $\lambda_a = [\underline{\alpha}_{10}\cdot\underline{\beta}_{10}\cdot\underline{a}]$

$\lambda_b = [\underline{\alpha}_{10}\cdot\underline{\beta}_{10}\cdot\underline{b}]$

$\lambda_c = [\underline{\alpha}_{10}\cdot\underline{\beta}_{10}\cdot\underline{c}]$

$\xi_{10} = \langle\underline{\alpha}_{10}\cdot\underline{\alpha}_{10}\rangle\langle\underline{\beta}_{10}\cdot\underline{\beta}_{10}\rangle - \langle\underline{\alpha}_{10}\cdot\underline{\beta}_{10}\rangle^2$

(iii)  at u=0, v=1

$$K = (\lambda_d\lambda_f - \lambda_e^2)/(\xi_{01}|\underline{\alpha}_{01} \times \underline{\beta}_{01}|^2)$$

$$H = (\lambda_f\langle\underline{\alpha}_{01}\cdot\underline{\alpha}_{01}\rangle + \lambda_d\langle\underline{\beta}_{01}\cdot\underline{\beta}_{01}\rangle - 2\lambda_e\langle\underline{\alpha}_{01}\cdot\underline{\beta}_{01}\rangle)/$$

$$(2\xi_{01}|\underline{\alpha}_{01} \times \underline{\beta}_{01}|)$$

where $\lambda_d = [\underline{\alpha}_{01}\cdot\underline{\beta}_{01}\cdot\underline{d}]$

$\lambda_e = [\underline{\alpha}_{01}\cdot\underline{\beta}_{01}\cdot\underline{e}]$

$\lambda_f = [\underline{\alpha}_{01}\cdot\underline{\beta}_{01}\cdot\underline{f}]$

$\xi_{01} = \langle\underline{\alpha}_{01}\cdot\underline{\alpha}_{01}\rangle\langle\underline{\beta}_{01}\cdot\underline{\beta}_{01}\rangle - \langle\underline{\alpha}_{01}\cdot\underline{\beta}_{01}\rangle^2$

(iv)  at u=1, v=1

$$K = (\lambda_g\lambda_1 - \lambda_h^2)/(\xi_{11}|\underline{\alpha}_{11} \times \underline{\beta}_{11}|^2)$$

$$H = (\lambda_1\langle\underline{\alpha}_{11}\cdot\underline{\alpha}_{11}\rangle + \lambda_g\langle\underline{\beta}_{11}\cdot\underline{\beta}_{11}\rangle - 2\lambda_h\langle\underline{\alpha}_{11}\cdot\underline{\beta}_{11}\rangle)/$$

$$(2\xi_{11}|\underline{\alpha}_{11} \times \underline{\beta}_{11}|)$$

where $\lambda_g = [\underline{\alpha}_{11}\cdot\underline{\beta}_{11}\cdot\underline{g}]$

$\lambda_h = [\underline{\alpha}_{11}\cdot\underline{\beta}_{11}\cdot\underline{h}]$

$\lambda_1 = [\underline{\alpha}_{11}\cdot\underline{\beta}_{11}\cdot\underline{1}]$

$\xi_{11} = \langle\underline{\alpha}_{11}\cdot\underline{\alpha}_{11}\rangle\langle\underline{\beta}_{11}\cdot\underline{\beta}_{11}\rangle - \langle\underline{\alpha}_{11}\cdot\underline{\beta}_{11}\rangle^2$

In Chapter Two we used the Serret-Frenet equations to reduce conditions involving the derivatives of $\underline{t}$, $\underline{n}$ and $\underline{b}$ to conditions

involving $\underline{t}$, $\underline{n}$, $\underline{b}$ and the curvature and torsion scalars $\kappa$, $\tau$.    Analagously for surfaces, we can use the Gauss-Weingarten equations :

$$\underline{r}_{11}(u,v) = \Gamma_{11}^1(u,v)\underline{r}_1(u,v) + \Gamma_{11}^2(u,v)\underline{r}_2(u,v) + L(u,v)\underline{N}(u,v)$$

$$\underline{r}_{12}(u,v) = \Gamma_{12}^1(u,v)\underline{r}_1(u,v) + \Gamma_{12}^2(u,v)\underline{r}_2(u,v) + M(u,v)\underline{N}(u,v)$$

$$\underline{r}_{22}(u,v) = \Gamma_{22}^1(u,v)\underline{r}_1(u,v) + \Gamma_{22}^2(u,v)\underline{r}_2(u,v) + N(u,v)\underline{N}(u,v)$$

$$\underline{N}_1(u,v) = \beta_1^1(u,v)\underline{r}_1(u,v) + \beta_1^2(u,v)\underline{r}_2(u,v)$$

$$\underline{N}_2(u,v) = \beta_2^1(u,v)\underline{r}_1(u,v) + \beta_2^2(u,v)\underline{r}_2(u,v)$$

to express the derivatives of $\underline{r}_1$, $\underline{r}_2$ and $\underline{N}$ in terms of $\underline{r}_1$, $\underline{r}_2$, $\underline{N}$ and scalars $\Gamma_{ij}^k$, the Christoffel symbols of the second kind which depend upon the first fundamental coefficients and their derivatives, and $\beta_i^j$, the Weingarten coefficients which depend upon both the first and second fundamental coefficients.  · In fact

$$\Gamma_{11}^1 = (GE_1 - 2FF_1 + FE_2)/(2[EG-F^2])$$

$$\Gamma_{12}^1 = (GE_2 - FG_1)/(2[EG - F^2])$$

$$\Gamma_{22}^1 = (2GF_2 - GG_2 - FG_2)/(2[EG - F^2])$$

$$\Gamma_{11}^2 = (2EF_1 - EE_2 + FE_1)/(2[EG - F^2])$$

$$\Gamma_{12}^2 = (EG_1 - FE_2)/(2[EG - F^2])$$

$$\Gamma_{22}^2 = (EG_2 - 2FF_2 + FG_1)/(2[EG - F^2])$$

$$\beta_1^1 = (MF - LG)/(EG - F^2)$$

$$\beta_1^2 = (LF - ME)/(EG - F^2)$$

$$\beta_2^1 = (NF - MG)/(EG - F^2)$$

$$\beta_2^2 = (MF - NE)/(EG - F^2)$$

For given functions. E, F, G, L, M, N of u and v of sufficiently high class then in general there is no surface

$\underline{r} = \underline{r}(u,v)$ for which E, F, G, L, M, N are the first and second fundamental coefficients unless certain compatibility conditions are satisfied {1} . Using these conditions we can show in fact that K depends only upon the first fundamental coefficients and their derivatives and not upon the second fundamental coefficients.

The first fundamental coefficients have another important role to play and that is in calculating surface area.

## Definition 3.3

The surface area of the bicubic surface $\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$ is given by

$$A = \int_0^1 \int_0^1 \sqrt{[E(u,v)G(u,v) - F^2(u,v)]} \, du \, dv$$

We can obtain A using Gauss Quadrature where if we write

$$\mu(u,v) = E(u,v)G(u,v) - F^2(u,v)$$

then $\quad A = \sum_i \sum_j k_i \, l_j \, \mu(u_i, v_j)$

Hence we can evaluate A in terms of the coefficients $\underline{a}_{ij}$ or $\underline{r}$ and its derivatives at the corners.

We have now basically covered all of the geometric invariants which could conceivably be used in practice in defining our computer surfaces.

In general bicubics will not provide us with sufficient degrees of freedom to define each surface globally in terms of a single bicubic equation.

## Definition 3.4

A Piecewise bicubic surface is a mapping $\underline{R} : [0,m] \times [0,n] \longrightarrow E^3$

where $m, n \in N$ such that $\underline{R} \in C^0 [0,m] \times [0,n]$ and

$$\underline{R} \Big| [i-1,i] \times [j-1,j] = \underline{r}^{ij} \quad i,j \in N \quad 0 < i \leq m, \quad 0 < j \leq n$$

where $\underline{r}^{ij} : [0,1] \times [0,1] \longrightarrow E^3$ is a bicubic

surface and where $\underline{R}(U,V) = \underline{r}^{ij}(U-i+1, V-j+1)$ for all

$(U,V) \in [0,m] \times [0,n]$.

The general problem we are interested in solving is the

following :

'Given $\{ \underline{b}_{ij} \} \subset E^3$ where $i,j \in N \quad 0 \leq i \leq m, \quad 0 \leq j \leq n$ then we

would like to construct a unique piecewise bicubic surface

$\underline{R}$ such that

    (i)    $\underline{R}(i,j) = \underline{b}_{ij}$

    (ii)   $\underline{R} \in C^k [0,m] \times [0,n]$ for some k '

and this means we have to find mn bicubic surfaces each

satisfying certain boundary smoothness constraints.

Now any boundary curve of the bicubic surface $\underline{r}^{ij}$ is cubic

and uniquely determined by contraints at the two endpoints.

Hence adjacent bicubics sharing the same endpoint constraints

will share the complete boundary. Thus at each point p

of the common boundary U = const. of $\underline{r}^{ij}$ and $\underline{r}^{kl}$ we also

have $\underline{r}_2^{ij}\Big|_p = \underline{r}_2^{kl}\Big|_p$ . If in addition we constrain the

$\underline{a}_{ij}$ so that at this boundary $\underline{r}_1^{ij}\Big|_p = \underline{r}_1^{kl}\Big|_p$ then we will

have $C^1$ continuity across this boundary. We note that

this last condition can be achieved by placing constraints
at the boundary endpoints .. For example if $\underline{r}^{ij}(1,v) = \underline{r}^{kl}(0,v)$
is a common boundary of $\underline{r}^{ij}$ and $\underline{r}^{kl}$ then if

(i)    $\underline{r}_1^{ij}(0,0) = \underline{r}_1^{kl}(0,0)$

(ii)    $\underline{r}_{21}^{ij}(1,0) = \underline{r}_{21}^{kl}(0,0)$

(iii)   $\underline{r}_{221}^{ij}(1,0) = \underline{r}_{221}^{kl}(0,0)$

(iv)   $\underline{r}_{2221}^{ij}(1,0) = \underline{r}_{2221}^{kl}(0,0)$

then $\underline{r}_1^{ij}(1,v) = \underline{r}_1^{kl}(0,v)$ for all $v \in [0,1]$.

We can continue in this way to build up conditions involving
the surface equation coefficients which ensure that
properties like fundamental coefficients, curvatures, normals
and their derivatives are preserved across bicubic surface
boundaries as required. Although this will enable us
to select one from each of the mn 36-parameter families of
bicubic surfaces which combined interpolate the $\underline{b}_{ij}$ the
resulting piecewise bicubic will in general suffer a
similar disease to the 'over' continuous piecewise cubic
curve. Therefore we will have to relax some of these
boundary constraints in favour of some form of restriction
on the interior behaviour which inevitably means a
restriction on the surface area.

In fact for some applications we may just required
continuity of $\underline{N}$. At a common vertex p of four neighbouring
bicubic surfaces conditions forcing continuity of the tangent

directions of the boundary cubics are sufficient to ensure that all four surfaces have common normal at p. If $\underline{r}^{ij}$ and $\underline{r}^{kl}$ have common boundary $\underline{r}^{ij}(1,v) = \underline{r}^{kl}(0,v)$ then the normal will be continuous across the boundary if and only if $\underline{r}_1^{ij}(1,v)$, $\underline{r}_1^{kl}(0,v)$ and $\underline{r}_2^{kl}(0,v) \left[ = \underline{r}_2^{ij}(1,v) \right]$ are coplanar for each $v \in [0,1]$. Putting $\underline{r}_1^{ij}(1,v) = \underline{r}_1^{kl}(0,v)$ is sufficient to ensure this but it is not necessary since

$$\underline{r}_1^{kl}(0,v) \times \underline{r}_2^{kl}(0,v) = \mu \left[ \underline{r}_1^{ij}(1,v) \times \underline{r}_2^{ij}(1,v) \right]$$

is satisfied if

$$\underline{r}_1^{kl}(0,v) = \mu \underline{r}_1^{ij}(1,v) + \lambda \underline{r}_2^{ij}(1,v)$$

where $\mu, \lambda$ are scalars with $\mu > 0$.

And similarly for the V = const. boundaries.


We consider briefly some forms of the bicubic representation which are currently in use in CAD systems.


## Definition 3.5

Let $\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$ be given by

$$\underline{r}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} u^i (1-u)^{m-i} v^j (1-v)^{n-j} \underline{d}_{ij}$$

where $\{ \underline{d}_{ij} \} \subset E^3$, $m, n \in N$

Then $\underline{r}$ is called a Bezier surface {12}.


If $m = n = 3$ then $\underline{r}$ is the Bezier bicubic surface. We shall see that the coefficients $\underline{d}_{ij}$ bear nice relationships with

the surface boundary properties, {Figure 10} :

(i) $\underline{d}_{00}$ = $\underline{r}(0,0)$

(ii) $\underline{d}_{03}$ = $\underline{r}(0,1)$

(iii) $\underline{d}_{30}$ = $\underline{r}(1,0)$

(iv) $\underline{d}_{33}$ = $\underline{r}(1,1)$

Thus the points to be interpolated are coefficients in the surface equation.

(v) $3(\underline{d}_{10} - \underline{d}_{00})$ = $\underline{r}_1(0,0)$

(vi) $3(\underline{d}_{13} - \underline{d}_{03})$ = $\underline{r}_1(0,1)$

(vii) $3(\underline{d}_{30} - \underline{d}_{20})$ = $\underline{r}_1(1,0)$

(viii) $3(\underline{d}_{33} - \underline{d}_{23})$ = $\underline{r}_1(1,1)$

(ix) $3(\underline{d}_{01} - \underline{d}_{00})$ = $\underline{r}_2(0,0)$

(x) $3(\underline{d}_{03} - \underline{d}_{02})$ = $\underline{r}_2(0,1)$

(xi) $3(\underline{d}_{31} - \underline{d}_{30})$ = $\underline{r}_2(1,0)$

(xii) $3(\underline{d}_{33} - \underline{d}_{32})$ = $\underline{r}_2(1,1)$

Thus these coefficients of the surface equation lie on the tangent planes to the surface at the interpolation points:

(xiii) $6(\underline{d}_{00} - \underline{d}_{10} + \underline{d}_{20} - \underline{d}_{10})$ = $\underline{r}_{11}(0,0)$

(xiv) $6(\underline{d}_{03} - \underline{d}_{13} + \underline{d}_{23} - \underline{d}_{13})$ = $\underline{r}_{11}(0,1)$

(xv) $6(\underline{d}_{10} - \underline{d}_{20} + \underline{d}_{30} - \underline{d}_{20})$ = $\underline{r}_{11}(1,0)$

(xvi) $6(\underline{d}_{13} - \underline{d}_{23} + \underline{d}_{33} - \underline{d}_{23})$ = $\underline{r}_{11}(1,1)$

(xvii) $9(\underline{d}_{00} - \underline{d}_{01} + \underline{d}_{11} - \underline{d}_{10})$ = $\underline{r}_{12}(0,0)$

(xviii) $9(\underline{d}_{13} - \underline{d}_{12} + \underline{d}_{02} - \underline{d}_{03})$ = $\underline{r}_{12}(0,1)$

(xix) $9(\underline{d}_{31} - \underline{d}_{30} + \underline{d}_{20} - \underline{d}_{21})$ = $\underline{r}_{12}(1,0)$

(xx) $9(\underline{d}_{33} - \underline{d}_{32} + \underline{d}_{22} - \underline{d}_{23})$ = $\underline{r}_{12}(1,1)$

(xxi) $6(\underline{d}_{00} - \underline{d}_{01} + \underline{d}_{02} - \underline{d}_{01})$ = $\underline{r}_{22}(0,0)$

(xxii) $6(\underline{d}_{01} - \underline{d}_{02} + \underline{d}_{03} - \underline{d}_{02})$ = $\underline{r}_{22}(0,1)$

(xxiii)   $6(\underline{d}_{30} - \underline{d}_{31} + \underline{d}_{32} - \underline{d}_{31}) = \underline{r}_{22}(1,0)$

(xxiv)   $6(\underline{d}_{31} - \underline{d}_{32} + \underline{d}_{33} - \underline{d}_{32}) = \underline{r}_{22}(1,1)$

Therefore at the interpolation point u=0, v=0 ($= \underline{d}_{00}$) the second fundamental coefficients and the Gaussian and mean curvatures are :

$L(0,0) = 6|(\underline{d}_{00} - \underline{d}_{10}) + (\underline{d}_{20} - \underline{d}_{10})|\cos \alpha$

$M(0,0) = 9|(\underline{d}_{00} - \underline{d}_{01}) + (\underline{d}_{11} - \underline{d}_{10})|\cos \gamma$

$N(0,0) = 6|(\underline{d}_{00} - \underline{d}_{01}) + (\underline{d}_{02} - \underline{d}_{01})|\cos \beta$

$K(0,0) = (4n_1 n_2 - 9n_3^2)/9n_4$

$H(0,0) = (n_2|\underline{d}_{10} - \underline{d}_{00}|^2 + n_1|\underline{d}_{01} - \underline{d}_{00}|^2 -$

$$3n_3|\underline{d}_{10} - \underline{d}_{00}||\underline{d}_{01} - \underline{d}_{00}|\cos \delta)/3n_4$$

where

$n_1 = |(\underline{d}_{00} - \underline{d}_{10}) + (\underline{d}_{20} - \underline{d}_{10})|\cos \alpha$

$n_2 = |(\underline{d}_{00} - \underline{d}_{01}) + (\underline{d}_{02} - \underline{d}_{01})|\cos \beta$

$n_3 = |(\underline{d}_{00} - \underline{d}_{01}) + (\underline{d}_{11} - \underline{d}_{10})|\cos \gamma$

$n_4 = |\underline{d}_{10} - \underline{d}_{00}|^2 |\underline{d}_{01} - \underline{d}_{00}|^2 \sin^2 \delta$

with $\alpha, \beta, \gamma, \delta$ as shown in figure 10.

We can therefore see that the main advantage of this particular representation for CAD use is that properties of the surface are implied more directly by the designers' choice of the $\underline{d}_{ij}$.

In Chapter 2 we considered the Bezier cubic curve which we can write as:

$$\underline{r}(t) = t^3\underline{d}_3 + 3t^2\hat{t}\underline{d}_2 + 3t\hat{t}^2\underline{d}_1 + \hat{t}\underline{d}_0$$

where $t + \hat{t} = 1$.

The more obvious generalisation of the Bezier cubic curve is the triangular bicubic surface (Figure 11) :

$$\underline{r}(u,v) = u^3\underline{A} + 3u^2v\underline{B} + 3uv^2\underline{C} + v^3\underline{D} + 3u^2w\underline{E} +$$
$$3vw^2\underline{F} + 3uw^2\underline{G} + 3vw^2\underline{H} + w^3\underline{I} + 6uvw\underline{J}$$

where $u + v + w = 1$, $\left\{\underline{A},\underline{B},\underline{C}, ....\underline{J}\right\} \subset E^3$

For this particular representation

$$\underline{r}_1(u,v) = 3\left[u^2(\underline{A}-\underline{E}) + 2uv(\underline{B}-\underline{J}) + v^2(\underline{C}-\underline{F}) + 2uw(\underline{E}-\underline{G}) + 2vw(\underline{J}-\underline{H}) + w^2(\underline{G}-\underline{I})\right]$$

$$\underline{r}_2(u,v) = 3\left[u^2(\underline{B}-\underline{E}) + 2uv(\underline{C}-\underline{J}) + v^2(\underline{D}-\underline{F}) + 2uw(\underline{J}-\underline{G}) + 2vw(\underline{F}-\underline{H}) + w^2(\underline{H}-\underline{I})\right]$$

$$\underline{r}_{11}(u,v) = 6\left[u(\underline{A}-2\underline{E}+\underline{G}) + v(\underline{B}-2\underline{J}+\underline{H}) + w(\underline{E}-2\underline{G}+\underline{I})\right]$$

$$\underline{r}_{12}(u,v) = 6\left[u(\underline{B}-\underline{J}+\underline{G}-\underline{E}) + v(\underline{C}-\underline{F}+\underline{H}-\underline{J}) + w(\underline{J}-\underline{H}+\underline{I}-\underline{G})\right]$$

$$\underline{r}_{22}(u,v) = 6\left[u(\underline{C}-2\underline{J}+\underline{G}) + v(\underline{D}-2\underline{F}+\underline{H}) + w(\underline{F}-2\underline{H}+\underline{I})\right]$$

$$\underline{r}_{111}(u,v) = 6\left[\underline{A} - 3\underline{E} + 3\underline{G} - \underline{I}\right]$$

$$\underline{r}_{112}(u,v) = 6\left[\underline{B} - 2\underline{J} + \underline{H} - \underline{E} + 2\underline{G} - \underline{I}\right]$$

$$\underline{r}_{122}(u,v) = 6\left[\underline{C} - 2\underline{J} + \underline{G} - \underline{F} + 2\underline{H} - \underline{I}\right]$$

$$\underline{r}_{222}(u,v) = 6\left[\underline{D} - 3\underline{F} + 3\underline{H} - \underline{I}\right]$$

Along the boundary $u=0$,

$$\partial\underline{r}/\partial w = 3\left[v^2(\underline{F}-\underline{C}) + w^2(\underline{I}-\underline{G}) + 2vw(\underline{H}-\underline{J})\right]$$
$$\partial\underline{r}/\partial v = 3\left[v^2(\underline{D}-\underline{C}) + w^2(\underline{H}-\underline{G}) + 2vw(\underline{F}-\underline{J})\right]$$

Along $v=0$,

$$\partial\underline{r}/\partial u = 3\left|u^2(\underline{A}-\underline{B}) + w^2(\underline{G}-\underline{H}) + 2uw(\underline{E}-\underline{J})\right|$$
$$\partial\underline{r}/\partial w = 3\left|u^2(\underline{E}-\underline{B}) + w^2(\underline{I}-\underline{H}) + 2uw(\underline{G}-\underline{J})\right|$$

Along w=0,

$$\partial \underline{r}/\partial v = 3\left[u^2(\underline{B}-\underline{E}) + v^2(\underline{D}-\underline{F}) + 2uv(\underline{C}-\underline{J})\right]$$

$$\partial \underline{r}/\partial u = 3\left[u^2(\underline{A}-\underline{E}) + v^2(\underline{C}-\underline{F}) + 2uv(\underline{B}-\underline{J})\right]$$

Along the boundaries the normal is given by,

$$\underline{N}\Big|_{u=0} = \alpha(v)\left[v^4(\underline{F}-\underline{C}) \times (\underline{D}-\underline{C}) + w^4(\underline{I}-\underline{G}) \times (\underline{H}-\underline{G})\right.$$
$$+ v^2w^2\left[4(\underline{H}-\underline{J}) \times (\underline{F}-\underline{J}) + (\underline{I}-\underline{G}) \times (\underline{D}-\underline{C})+(\underline{F}-\underline{C}) \times (\underline{H}-\underline{G})\right]$$
$$+ 2v^3w\left[(\underline{H}-\underline{J}) \times (\underline{D}-\underline{C}) + (\underline{F}-\underline{C}) \times (\underline{F}-\underline{J})\right]$$
$$\left.+ 2vw^3\left[(\underline{I}-\underline{G}) \times (\underline{F}-\underline{J}) + (\underline{H}-\underline{J}) \times (\underline{H}-\underline{G})\right]\right]$$

where $\alpha(v) = 1/\left|\partial\underline{r}/\partial w \times \partial\underline{r}/\partial v\right|$

Similarly for $\underline{N}\Big|_{v=0}$ and $\underline{N}\Big|_{w=0}$

Thus continuity of normal $\underline{N}$ across u=0 puts severe
constraints on the coefficients of the surface equation
since we must already have $\underline{D}$, $\underline{F}$, $\underline{H}$, $\underline{I}$ as common coefficients
in order to have a common boundary.  For a  surface with
coefficients as above the continuity of $\underline{r}_1 \times \underline{r}_2$ across ·
u=0 requires that for the neighbouring surface (see
Figure 11) we position $\underline{P}$, $\underline{Q}$ and $\underline{R}$ such that $\left\{\underline{D}, \underline{C}, \underline{F}, \underline{Q}\right\}$,
$\left\{\underline{F}, \underline{J}, \underline{H}, \underline{P}\right\}$ and $\left\{\underline{H}, \underline{G}, \underline{I}, \underline{K}\right\}$ form three skew parallelograms.
This also means that exactly six triangular elements have
an internal vertex in the system as common vertex.   Thus
at each vertex we have a semi-regular plane hexagon.

For the above representation of a surface, the continuity
conditions are so restrictive that there are virtually
no degrees of freedom left.   This family of Bezier type
surfaces is therefore not rich enough and we can conclude

that bicubic triangles are unsuitable for surface fitting.
This is not altogether surprising since in constructing
smooth surfaces most of the difficulties arise from
crossing the boundaries  and in introducing the extra
boundary we can only be adding to our troubles.   We can
contrast this with finite element analysis where the
difficulties arise in jumping from one side of the element
to the other, and so the addition of the extra boundary
gives us an extra stepping stone.

The extension of B-spline curves to B-spline surfaces is
exactly analogous to the development of Bezier surfaces
from Bezier curves :

## Definition 3.6

If the equation of the surface can be expressed in terms
of B-spline functions as

$$\underline{r}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_{i-\frac{k}{2},k}(u) \; B_{j-\frac{k}{2},k}(v) \, \underline{P}_{ij} \; , \; \left\{ \underline{P}_{ij} \right\} \; C \; E^3$$

then $\underline{r}$ is called a B-spline surface {17}.

Surface representations of the types discussed above have
been derived to ensure that most of the engineering shapes
likely to be encountered can be  defined in sufficient
detail, albeit at a cost of smoothness in the sense of
differential geometry.   For other forms of analysis of
the resulting surface however smoothness becomes the critical

feature.    One way of achieving a compromise would be
to proceed from the bicubic definition to a smooth
approximation taking over properties of the bicubic.
This could obviously be done for instance by  using higher
order polynomials.    In so doing instead of considering
constraints on just one component of the tensor of second order
partial derivatives $\partial^2 r/\partial u^\alpha \partial u^\beta$ we could look at the tensor $x^1_{\alpha,\beta}$
where        $x^1_{\alpha,\beta} = \partial^2 x^1/\partial u^\alpha \partial u^\beta - \Gamma^\gamma_{\alpha\beta} x^1_j = \Omega_{\alpha\beta}N^1$
where $(N^1)$ is the unit normal and $(\Omega_{\alpha\beta})$ the second
fundamental form.

# Chapter 4

## Extremal Mappings of Riemannian Manifolds

We make use of the following notation:

(i) $TM^n = \bigcup_{p \in m} T_p M^n$ is the tangent bundle of the differentiable n-manifold M.

(ii) $H(TM \times TM, \mathbb{R})$ is the bundle of tensors of M of type (2,0). In general $H(\underbrace{TM \times TM \ldots \times TM}_{r} \times \underbrace{(TM)^* \ldots \times (TM)^*}_{s}, \mathbb{R})$ is the bundle of tensors on M of type (r,s).

(iii) A section of $H(TM \times TM, \mathbb{R})$ is a (2,0) tensor.

## Definition 4.1

A Riemannian manifold is a differentiable manifold together with a section g of $H(TM \times TM, \mathbb{R})$ satisfying

(i) $g(X,Y) = g(Y,X)$ , (ii) $g(X,X) = 0$ iff $X = 0$

where X,Y are vector fields.

We write $g(X,Y)$ as $<X,Y>$ and this is the first fundamental form on M. Thus for $M = E^n$ then $<X,Y> = \sum_{i=1}^{n} x^i y^i$.

## Definition 4.2

The second fundamental form on M is a section B of $H(TM \times TM, NM)$ given by $B(X,Y) = [\bar{\nabla}_X Y]^N$ the normal component of $\bar{\nabla}_X Y$ for the Riemannian connexion $\bar{\nabla}$.

Thus if $u^1,\ldots u^k$ are local coordinates at $x \in M \subset E^n$

$X = (x_1,\ldots,x_n) = (x_1(u^1,\ldots u^k),\ldots,x_n(u^1,\ldots u^k))$ and

$(g_{ij}) = (<\frac{\partial X}{\partial u}i , \frac{\partial X}{\partial u}j>)$. For a surface in $E^3$ then

$g_{ij} = <X_i,X_j>$ , $X_i = \partial X/\partial u^i$ i=1,2 and if

$N = X_1 \times X_2/|X_1 \times X_2|$ is unit normal then

$\bar{\nabla}_{X_j} X_i = \partial X_i/\partial u^j = \sum \Gamma^k_{ij} X_k + b_{ij}N$ in terms of tangential

and normal components. And for a curve $\gamma$ on M with tangent

$\sum \gamma^{r'} X_r$ then

$\partial(\sum \gamma^i X_i)/\partial t = [\gamma^{i''} + \gamma^{k'}\gamma^{r'}\Gamma^i_{kr}]X_i + \gamma^{k'}\gamma^{r'}b_{kr}N$

## Theorem 4.3

Given a Riemannian symmetric connexion $\nabla$ and a curve

$\gamma : [0,1] \longrightarrow M$ with $\gamma(0) = p$ and $Y_0 \in T_p M$ then there

is a unique $Y(t)$ along $\gamma$ such that $\nabla_{\gamma'} Y = 0$.

To see this if $X^1,\ldots X^n$ are coordinates at p,

$\nabla_{X_i} X_j = \sum_{k=1}^{n} \Gamma^k_{ij} X_k$ , $X_i = \partial/\partial X_i$

and $\gamma' = (X^{1'}(t),\ldots,X^{n'}(t))$ and

$\nabla_{\gamma'} Y = \nabla_{\gamma'} y^i X_i$

$= y^{i'} X_i + y^i \nabla_{\sum X'_j X_j} X_i$

$= y^{i'} X_i + y^i X'_j \Gamma^k_{ji} X_k$

Thus we have a linear system of differential equations
$y^{k'} + y^i x^{j'} \Gamma_{ij}^k = 0$ with initial condition $Y_0 = y^i(t)\big|_{t=0}$

Since $(|Y-\hat{Y}|^2)' = 2 <Y-\hat{Y}, \nabla_\gamma, (Y-\hat{Y})> = 0$

then $|Y-\hat{Y}|^2$ is constant and so since $Y(0) = \hat{Y}(0)$ we must

have uniqueness.


## Definition 4.4

A Geodesic is a curve $\gamma$ with acceleration vector field

$\nabla_\gamma, \gamma'$ zero.


Let $\Gamma$ be a closed $(n-1)$ manifold in $E^{n+k}$.   We say that

X spans $\Gamma$ if X is an n-submanifold and $\partial X = \Gamma$.

Let $X : D \longrightarrow E^{n+k}$, D closed ball in $E^n$ and $X(\partial D) = \Gamma$.


## Definition 4.5

A variation of X is a map $\bar{X} : D \times [-\epsilon, \epsilon] \longrightarrow E^{n+k}$
such that

   (i)  $\bar{X}_0(z) = X(z)$ for each $z \epsilon D$

   (ii) $\bar{X}_u(\partial D) = \bar{X}_0(\partial D) = X(\partial D)$ for each $u \epsilon [-\epsilon, \epsilon]$
where $\bar{X}_u(z) = \bar{X}(z,u)$

Thus for fixed $z \epsilon D$, $\bar{X}_u(z)$ is a curve with velocity vector

$V(z) = d(\bar{X}_u(z))/du\big|_{u=0}$ called the Variation  Vector Field
associated with the variation $\bar{X}$.   And $\bar{X}_u(D)$ is a surface

with first fundamental form $g_{ij}(u) = <\bar{X}_{u,i}, \bar{X}_{u,j}>$ and has area
$\int_D \sqrt{[\det g_{ij}(u)]} du^1 \dots du^n = \int_D dA_u$.

If $(u^1,\ldots,u^n)$ are coordinates for a domain D in $E^n$
and $X(u^1,\ldots,u^n)$ is a smooth map into $E^{n+k}$ which is
everywhere regular then X induces a Riemannian structure
on D and if we take $<\ ,\ >$ to be the usual inner product
on $E^{n+k}$ then $X^*(<\ ,\ >)$ is a metric tensor on D.

## Definition 4.6

$$g_{ij} = X^*(<\ ,\ >)(\partial/\partial u^i, \partial/\partial u^j) = <X_*(\partial/\partial u^i), X_*(\partial/\partial u^j)>$$
$$= \sum_1^{n+k} \partial X_k/\partial u^i \ \ \partial X_k/\partial u^j.$$

The volume element of $(g_{ij})$ is defined by
$dV = \sqrt{[\det(g_{ij})]}\,du^1 \ldots du^n$, and $\int_D dV$ is the 'volume' of X.

If $\bar{G} : D \times [-\epsilon.\epsilon] \longrightarrow E^{n+k}$ is a smooth variation of X
which leaves $\partial D$ fixed, that is $\bar{G}(z,0) = X(z)$ and
$\bar{G}(\partial D,s) = X(\partial D)$ then if $V(z) = d\bar{G}(z,s)/ds\Big|_{s=0}$ is the
variation vector field of $\bar{G}$ then for small enough $s, \bar{G}_s$
is regular and $(g_{s\,ij}) = X_s^*(<\ ,\ >)$ is a metric tensor with
volume element $dV_s = \sqrt{[\det(g_s)]}\,du^1 \ldots du^n$.

## Definition 4.7

If $M^n$ is a differentiable manifold and $X : M^n \longrightarrow E^{n+k}$
is regular then X induces a riemannian structure on $M^n$ as
above.   Furthermore $X_*$ is an isometry between each fibre
on $TM^n$ and its image in $TE^{n+k}$.   Such a map is called an
Isometric Immersion.

## Definition 4.8

An immersion $X : M^n \longrightarrow E^{n+k}$ is called Minimal if $H \equiv 0$ where H is its mean curvature vector field.

We shall see that an isometric immersion $X : M^n \longrightarrow E^{n+k}$ is minimal if and only if the volume of $M^n$ is stationary with respect to all variations with compact support in $M^n - \partial M^n$. We remark here that this carries over for immersions in $\bar{M}^{n+k}$ having $< , >$ as inner product, and in this case the induced metric from a regular map is

$$g_{ij} = \sum \partial X_\alpha / \partial u^i \, \partial X_\beta / \partial u^j \, \bar{g}_{\alpha\beta} \quad \text{where } \bar{g}_{\alpha\beta} \text{ is just } < , >$$

in local coordinates.

## Proposition 4.9

With X, D, $\bar{X}$ as previously defined then there is a normal vector field H along X (defined locally) such that for any variation $\bar{X}$ which leaves $\partial D$ fixed then

$$V'(0) = -\int_D <H,V> \, dV \quad \text{where V is the variation vector field}$$

of $\bar{X}$.

To see this one considers the varation $\bar{X}$ of X with variation vector field $V_{u_0}(z) = d(\bar{X}_u(z)/du \Big|_{u=u_0}$ and by taking the area $A(u) = \int_D \sqrt{[\det g_{ij}(u)]} \, du_1 \ldots du_n$ one simply evaluates $dA/du \Big|_{u=0}$, using the fact that V is zero on the boundary.

Suppose that $\gamma : [0,1] \longrightarrow M$ is a curve in M. The length of $\gamma$ is given by

$$L_0^1(\gamma) = \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle^{\frac{1}{2}} \, dt$$

If $\bar{\gamma} : [0,1] \times [-\epsilon, \epsilon] \longrightarrow M$ is a one parameter variation of $\gamma$ then $\bar{\gamma}_0(t) = \gamma(t) \; \forall \; t \in [0,1]$ and $\bar{\gamma}_u(0) = \gamma(0)$, $\bar{\gamma}_u(1) = \gamma(1) \; \forall \; u \in [-\epsilon, \epsilon]$. If we write $L_u = L_0^1(\bar{\gamma}_u)$ then $L_u = \int_0^1 \langle \bar{\gamma}_u'(t), \bar{\gamma}_u'(t) \rangle^{\frac{1}{2}} dt$ and using proposition 4.9 we have $d(L_u)/du \Big|_{u=0} = -\int_0^1 \langle V, H \rangle \langle \gamma'(t), \gamma'(t) \rangle^{\frac{1}{2}} \, dt$ where V is the variation vector field along $\gamma$, which depends upon the variation and $H = (\gamma'(t)/\langle \gamma'(t), \gamma'(t) \rangle^{\frac{1}{2}})'$ is the curvature vector which depends only on $\gamma$.

Thus $d(L_u)/du \Big|_{u=0} = 0$ if and only if $H \equiv 0$ and so $d(L_u)/du \Big|_{u=0} = 0$ for all variations fixing endpoints if and only if $\gamma$ is a geodesic. And if $M = E^n$ then $H = (\gamma'(t)/\langle \gamma'(t), \gamma'(t) \rangle^{\frac{1}{2}})' = 0$ means that $\gamma''(t) = 0$ which on integration just gives a linear equation.

If $X(s)$ is a curve in $E^n$ parameterized by arc length then $X''(s)$ is a vector normal to X and $X''(s) = 0$ if and only if X is locally the shortest curve between points on X. In this case $H = X''(s) = B(X'(s), X'(s))$ and if the curve is moved in the direction of $X''(s)$ then its length will be decreased.

Suppose that $\Gamma$ is a closed curve in $E^3$ and let $X : D \longrightarrow E^3$ be a surface which spans $\Gamma$. Then X has area $A_D(X) = \int_D \sqrt{[\det(g_{ij})]} \, du_1 du_2$ and if $\bar{X} : D \times [-\epsilon, \epsilon] \longrightarrow E^3$ is a smooth variation of X leaving $\partial D$ fixed with variation

vector field $V(z) = \partial \bar{X}(z,u)/\partial u \big|_{u=0}$ and $\bar{X}(z,u) \equiv \bar{X}_u(z)$

has metric $g_{ij}(u)$ then $d\ A(u)/du \big|_{u=0} = -\int <V,H>dA_0 = 0$

if and only if X is a minimal surface, where

$$A(u) = \int_D \sqrt{[\det g_{ij}(u)]}\ du_1 du_2.$$

If $X : M^2 \longrightarrow E^{2+k}$ is given locally in terms of conformal

coordinates then at each point $(g_{ij})$ is of the form

$\begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix}$ and the Laplacian $\Delta X = (\Delta X_1, \dots \Delta X_{2+k}) = H/\lambda^2$.

This means that X is minimal if and only if $\Delta X = 0$ if and

only if all the coordinate functions $X_1$ are harmonic.

If we put $f_\alpha = \partial X_\alpha/\partial u_1 - \partial X_\alpha/\partial u_2$ , $\alpha = 1,\dots.2+k$ then from

the Cauchy Riemann equations each $f_\alpha$ is analytic as a

complex function of $z = u_1 + iu_2$ and also

(i) $\quad \sum_\alpha f_\alpha^2 = 0$

(ii) $\quad \sum_\alpha |f_\alpha|^2 = 2\lambda^2$

## Proposition 4.10

Given that $f_1$, $f_2, \dots\dots, f_{2+k}$ are analytic functions of

$z = u_1 + iu_2$ on a simply connected domain $D \subset C$ such that they

satisfy (i) and (ii) above then there exists $X : D \longrightarrow E^{2+k}$

which is minimal and in conformal coordinates.

This follows by taking $X_\beta = Re\int f_\beta \qquad \beta = 1,\dots,2+k$

## Definition 4.11

Let $\gamma : [a,b] \longrightarrow M$ be a curve in M which is piecewise

smooth and continuous.    The energy of $\gamma$, (4.6) is given by

$$E_a^b(\gamma) = \int_a^b <\gamma',\gamma'>dt.$$

## Proposition 4.12

$$d \; E(\bar{\gamma})/du \Big|_{u=0} = - \; 2 \left[ \sum <U(t), \Delta_t V> + \int_a^b <U(t), \nabla_V V> dt \right]$$

where $\bar{\gamma} : [a,b] \times [-\epsilon, \epsilon] \longrightarrow M$ is a one parameter variation of $\gamma$, $\Delta_t V = V_{t+} - V_{t-}$ is the jump in the velocity vector $V(t) = \gamma'(t)$ at the corner $t$, $U(t) = \partial\bar{\gamma}/\partial u \Big|_{u=0}$ and $\nabla_V V$ is the acceleration vector of $\gamma$.

This result follows since

$$d \; E(\bar{\gamma})/du = d\left[\int_a^b <\bar{\gamma}', \bar{\gamma}'> dt\right]/du$$

$$= \int_a^b \left[<\nabla_u \bar{\gamma}', \bar{\gamma}'> + <\bar{\gamma}', \nabla_u \bar{\gamma}'>\right] dt$$

$$= 2 \int_a^b <\nabla_u \bar{\gamma}', \bar{\gamma}'> dt$$

$$= 2 \int_a^b <\nabla_{\bar{\gamma}'} U, \bar{\gamma}'> dt$$

If $\bar{\gamma} \in C^1 [t_{i-1}, t_i] \times [-\epsilon, \epsilon]$ $\quad i = 1, \ldots, m,$ $\quad t_0 = a,$ $\quad t_m = b$

then since

$$\partial <\partial\bar{\gamma}/\partial u, \gamma'>/\partial t = <\nabla_{\gamma'} U, \gamma'> + <\partial\bar{\gamma}/\partial u, \nabla_{\gamma'} \gamma'>$$

we have

$$\int_{t_{i-1}}^{t_i} <\nabla_{\gamma'} U, \gamma'> dt = <\partial\bar{\gamma}/\partial u, \gamma> \Big|_{t=t_{i-1}}^{t=t_i}$$

$$- \int_{t_{i-1}}^{t_i} <\partial\bar{\gamma}/\partial u, \nabla_{\gamma'} \gamma'> \; dt.$$

Now $\partial\bar{\gamma}/\partial u = 0$ at $t=0$ or $t=1$ and so

$$\int_a^b <\nabla_{\gamma'} U, \gamma'> dt = - \sum_{i=1}^{m-1} <\partial\bar{\gamma}/\partial u, \Delta_{t_i} \bar{\gamma}'>$$

$$- \int_a^b <\partial\bar{\gamma}/\partial u, \nabla_{\gamma'} \gamma'> dt$$

## Corollary

Stationary points of the energy function are geodesics.

This follows because if $\gamma$ is a geodesic then $\gamma$ is smooth

and $\nabla_V V = 0$ and so $d\,E(\bar{\gamma})/du\Big|_{u=0} = 0 \; \forall$ variations $\bar{\gamma}$.

Conversely if $d\,E(\bar{\gamma})/du\Big|_{u=0} = 0$ for all variations $\bar{\gamma}$

then $\nabla_V V = 0$ where defined and V is continuous so from

the uniqueness theorem for differential equations since

there is only one continuous solution then $\gamma$ is an unbroken

geodesic.

## Definition 4.13

Let $\gamma : [a,b] \longrightarrow M$ be a curve in M.   Then $\bar{\gamma}(t,u,w)$

is a two parameter variation of $\gamma$ if

$\bar{\gamma} : [a,b] \times [-\epsilon,\epsilon] \times [-\delta,\delta] \longrightarrow M$ with

$\bar{\gamma}(t,0,0) = \gamma(t)$

If we write $\gamma'(t) = V$, $U(t) = \partial\bar{\gamma}/\partial u$, $W(t) = \partial\bar{\gamma}/\partial w$ then

$U(t)$ is a vector field along the curve $\bar{\gamma}(t,u_0,w_0)$ for

fixed $u_0$, $w_0$   and $W(t) = \partial\bar{\gamma}(t,u_0,w)/\partial w\Big|_{w=w_0}$

## Proposition 4.14

For a smooth geodesic $\gamma$ defined on $[a,b]$ and a piecewise

smooth variation $\bar{\gamma}(t,u,w)$   then

$$\partial^2 E/\partial u \partial w\Big|_{u=w=0} = -2\Big[\ \sum_t <U_t, \Delta_t \nabla_V W>$$
$$+ \int_a^b <U, R(W,V)V + \partial^2 W/\partial t^2 > \; dt$$

for variation vector fields U,W of the variation $\bar{\gamma}$ of $\gamma$.

This follows from straightforward differentiation where

$R(W,V)V = \nabla_W \nabla_V V - \nabla_V \nabla_W V - \nabla_{[W,V]} V$   and where

$[W,V] = \partial/\partial w \; \partial/\partial t - \partial/\partial t \; \partial/\partial w = 0$.

## Corollary

If $\bar{\gamma}(t,u,w)$ is a smooth variation with $\bar{\gamma}(a,u,w) = p$ and $\bar{\gamma}(b,u,w) = q$ that is $U(a) = U(b) = W(a) = W(b) = 0$ then

$$\partial^2 E/\partial u \partial w = -2 \int_a^b <U, R(W,V)V + \partial^2 W/\partial t^2> \quad dt.$$

We shall use the following notation :

$$\Omega = \left\{ \gamma \,\middle|\, \gamma : [a,b] \longrightarrow M \right\}$$

$$\Omega_p^q = \left\{ \gamma \in \Omega \,\middle|\, \gamma(a) = p, \gamma(b) = q \right\}$$

$$T_\gamma \Omega = \left\{ U \,\middle|\, U \text{ is vector field along } \gamma \right\}$$

$$T_\gamma \Omega_p^q = \left\{ U \in T_\gamma \Omega \,\middle|\, U(p) = U(q) = 0 \right\}$$

We can therefore think of $E$, $E_*$, $E_{**}$ as mappings :

$$E : \Omega \longrightarrow \mathbb{R}, \quad E_* : T\Omega \longrightarrow \mathbb{R}, \quad E_{**} : T_\gamma \Omega \times T_\gamma \Omega \longrightarrow \mathbb{R}$$

such that if $W \in T_\gamma \Omega$ then $E_*(W) = \partial E/\partial w \big|_{w=0}$
and if $U, W \in T_\gamma \Omega$ then $E_{**}(U,W) = \partial^2 E/\partial u \partial w \big|_{u=w=0}$

## Propostion 4.15

If $\gamma \in \Omega_p^q$ is a piecewise smooth curve of shortest length on $M$ then $\gamma$ is a smooth geodesic and

$$E_*(U) = 0, \quad E_{**}(U,U) \geqslant 0 \quad \forall \; U \in T_\gamma \Omega_p^q.$$

To see this, since geodesics are stationary points of $E$ then $E_*(U) = 0$.

Also $\left[ L_a^b(\gamma) \right]^2 = \left[ \int_a^b <\gamma',\gamma'>^{\frac{1}{2}} dt \right]^2$

$$\leqslant \int_a^b <\gamma',\gamma'> dt \int_a^b 1 \, dt \quad \text{(Schwarz Inequality)}$$

$$= E_a^b(\gamma) \; (b-a)$$

But if $\gamma$ is a geodesic $L_a^b(\gamma) < L_a^b(\alpha)$ : $\forall \, \alpha \, \epsilon \, \Omega_p^q$ and so

$$[L_a^b(\gamma)]^2 \leq [L_a^b(\alpha)]^2 \leq E_a^b(\alpha)(b-a)$$

or $E_a^b(\gamma)(b-a) \leq E_a^b(\alpha)(b-a)$

Thus for any variation $\bar{\gamma}$ of $\gamma$, $E_a^b(\gamma) = E_a^b(\bar{\gamma}(0)) \leq E_a^b(\bar{\gamma})$

and $d^2E(\bar{\gamma})/du^2 \Big|_{u=0} \geq 0$

Physically we can interpret 'Energy' as follows. If a rubber band describes a curve $\Gamma$ when stretched between two points of a frictionless curved surface S then the potential energy arising from tension is proportional to the energy of $\Gamma$. In particular if we have an equilibrium state then the energy is minimised and $\Gamma$ is a geodesic on S.

Let $p$, $q \, \epsilon \, E^3$ and let $\lambda : [0,1] \longrightarrow E^3$ be any curve in $E^3$ with $\lambda(0) = p$ and $\lambda(1) = q$.

Then $\lambda \epsilon \Omega_p^q$ and the Energy of $\lambda$ is given by

$$E(\lambda) = \int_0^1 <\lambda'(t), \lambda'(t)> dt$$

We now consider $(\Omega_c^{12})_p^q \subset \Omega_p^q$ where

$(\Omega_c^{12})_p^q = \left\{ \lambda \epsilon \Omega_p^q \, \Big| \, \lambda \text{ is a cubic curve} \right\}$.

From this 12 parameter family of curves we can extract a 2 parameter subfamily $(\Omega_c^2)_p^q$ as follows :

Let $T_p$, $T_q \, \epsilon \, E^3$ with $|T_p| = |T_q| = 1$

then

$$(\Omega_c^2)_p^q = \left\{ \lambda \epsilon (\Omega_c^{12})_p^q \, \Big| \, \lambda'(t) \Big|_p = \alpha_\lambda T_p, \lambda'(t) \Big|_q = \beta_\lambda T_q \right\}$$

for some $\alpha_\lambda$, $\beta_\lambda \in \mathbb{R}$.

If $\bar{E}(\lambda)$ is $E(\lambda)$ restricted to the set $(\Omega_c^2)_p^q$ we will show in Chapter 5 that there is just one $\lambda \in (\Omega_c^2)_p^q$ which minimizes $\bar{E}$.

## Definition 4.16

Let $f : M^m \longrightarrow N^n$ be a smooth mapping of Riemannian manifolds M (compact) and N. If $(x^1,\ldots,x^m)$ and $(y^1,\ldots,y^n)$ are local coordinates at $p \in M$ and $f(p) \in N$ respectively then the energy of $f$ is given by

$$E(f) = \int_M (g^{ij}\hat{g}_{\alpha\beta}f_i^\alpha f_j^\beta) \cdot 1$$

where $ds^2 = g_{ij}dx^i dx^j$ and $d\hat{s}^2 = \hat{g}_{\alpha\beta}dy^\alpha dy^\beta$ are the metrics in local coordinates, and $f_i^\alpha = \partial f^\alpha/\partial x^i$, and $\cdot 1$ is the volume element of M.

The problem of deforming any given mapping into one for which $\delta E = 0$ is considered in {6}. For this the Euler equations turn out to be $\mathcal{T}^\alpha = 0$ where

$$T^\alpha = g^{ij}\left[\partial f_i^\alpha/\partial x^j - \Gamma_{ij}^h f_h^\alpha + \hat{\Gamma}_{\beta\gamma}^\alpha f_j^\gamma f_i^\beta\right]$$

where $\Gamma, \hat{\Gamma}$ are the Christoffel symbols on M,N and any map for which $T^\alpha = 0$ is a Harmonic map. If we take $M = S^1$ then $f$ is harmonic if and only if $f(S^1)$ is a closed geodesic on N.

# Chapter 5

## An Extremal curve in $E^3$

Using definition 2.1 for the cubic curve equation we can write any member of $(\Omega_c^2)_p^q$ in the form

$$\underline{r}(u) = \underline{a}_0 + \underline{a}_1 u + \underline{a}_2 u^2/2! + \underline{a}_3 u^3/3! \qquad \underline{a}_i \in E^3$$

where $u \in [0,1]$

## Definition 5.1

The energy of the cubic curve $\underline{r}$ is given by

$$\bar{E}(\underline{r}) = \int_0^1 \langle \underline{\dot{r}}(u), \underline{\dot{r}}(u) \rangle \, du$$

Then

$$\underline{r}(0) = \underline{a}_0 = p$$
$$\underline{\dot{r}}(0) = \dot{s}(0)\underline{t}(0)$$
$$\underline{\dot{r}}(1) = \dot{s}(1)\underline{t}(1)$$

and we recall from chapter 2 that each coefficient in the cubic equation can be written in terms of these as

$$\underline{a}_1 = \dot{s}(0)\underline{t}(0)$$
$$\underline{a}_2 = 6(\underline{r}(1)-\underline{r}(0)) - 4\dot{s}(0)\underline{t}(0) - 2\dot{s}(1)\underline{t}(1)$$
$$\underline{a}_3 = 12(\underline{r}(0)-\underline{r}(1)) + 6\dot{s}(0)\underline{t}(0) + 6\dot{s}(1)\underline{t}(1)$$

Writing

$$|\underline{r}(1)-\underline{r}(0)| = W$$

$$\langle\underline{r}(1)-\underline{r}(0),\underline{t}(0)\rangle = W \cos \theta$$

$$\langle\underline{r}(1)-\underline{r}(0),\underline{t}(1)\rangle = W \cos \phi$$

then

$$\langle\underline{\dot{r}}(u),\underline{\dot{r}}(u)\rangle = 9\alpha u^4 + 12\beta u^3 + 2\gamma u^2 + 4\delta u + \dot{s}^2(0)$$

where

$$\alpha = 4W(W-\dot{s}(0)\cos \theta - \dot{s}(1)\cos \phi) + \dot{s}^2(0) + \dot{s}^2(1)$$
$$+ 2\dot{s}(0)\dot{s}(1) \cos (\theta + \phi)$$

$$\beta = W(-6W+7\dot{s}(0)\cos \theta + 5\dot{s}(1)\cos \phi) - 2\dot{s}^2(0) - \dot{s}^2(1)$$
$$- 3\dot{s}(0)\dot{s}(1)\cos (\theta + \phi)$$

$$\gamma = 6W(3W-5\dot{s}(0)\cos \theta - 2\dot{s}(1)\cos \phi) + 11\dot{s}^2(0) - 2\dot{s}^2(1)$$
$$+ 11\dot{s}(0)\dot{s}(1) \cos (\theta + \phi)$$

$$\delta = 3\dot{s}(1)W \cos \theta - 2\dot{s}^2(0) - \dot{s}(0)\dot{s}(1) \cos (\theta + \phi)$$

And the energy of $\underline{r}$ is given by

$$\bar{E}(\underline{r}) = W(6W-\dot{s}(0)\cos \theta - \dot{s}(1) \cos \phi)/5$$
$$+ (2\dot{s}^2(0) + 2\dot{s}^2(1) - \dot{s}(0)\dot{s}(1) \cos (\theta + \phi))/15$$

from which it can be shown that $\bar{E}$ takes on its minimum when

$$\dot{s}(0) = 3W\left[4 \cos \theta + \cos \phi \cos (\theta + \phi)\right]/\left[16 - \cos^2(\theta + \phi)\right]$$
$$\dot{s}(1) = 3W\left[4 \cos \phi + \cos \theta \cos (\theta + \phi)\right]/\left[16 - \cos^2(\theta + \phi)\right]$$

(*)

In a symmetrical condition where $\theta = \phi$ then

$$\dot{s}(0) = \dot{s}(1) = 3W \cos \theta/(5-2 \cos^2\theta)$$

Figures 12-15 show some examples of minimum energy cubic curves. The boundary speeds (*) satisfy :

$$\dot{s}(i) > 0 \quad \text{iff} \quad \langle\underline{r}(1)-\underline{r}(0),\underline{t}(1)\rangle > 0 \quad i=0,1$$
$$\dot{s}(i) = 0 \quad \text{iff} \quad \langle\underline{r}(1)-\underline{r}(0),\underline{t}(1)\rangle = 0 \quad i=0,1$$

Using proposition 2.4 the curvature and torsion of

these cubics take the boundary values :

$\kappa(0) = (2\sqrt{f(\theta,\phi,\dot{s}(1))})/\dot{s}^2(0)$

$\kappa(1) = (2\sqrt{f(\phi,\theta,\dot{s}(0))})/\dot{s}^2(1)$

$\tau(0) = 3\dot{s}(1)[\underline{t}(0),\underline{r}(1)-\underline{r}(0),\underline{t}(1)]/\dot{s}(0)f(\theta,\phi,\dot{s}(1))$

$\tau(1) = 3\dot{s}(0)[\underline{t}(0),\underline{r}(1)-\underline{r}(0),\underline{t}(1)]/\dot{s}(1)f(\phi,\theta,\dot{s}(1))$

where $\dot{s}(0)$ and $\dot{s}(1)$ are given by equations (*) and where

$$f(a,b,\alpha) = 3W\left[3W \sin^2 a - 2\alpha(3 \cos b - \cos a \cos (a+b))\right]$$
$$+ \alpha \sin^2(a+b)$$

and for the minimum energy planar cubic the curvature

takes the boundary values

$\kappa(0) = 2\left[3W \sin \theta - \dot{s}(1) \sin (\theta + \phi)\right]/\dot{s}^2(0)$

$\kappa(1) = 2\left[3W \sin \phi - \dot{s}(0) \sin (\theta + \phi)\right]/\dot{s}2(1)$

where again $\dot{s}(0)$ and $\dot{s}(1)$ satisfy (*).

And if $\theta = \phi$ then we have

$\kappa(0) = \kappa(1) = 2 \sin \theta(5-4 \cos^2\theta)(5-2 \cos^2\theta)/3W \cos^2\theta$

Let $\xi = \left\{ p1,p2,\ldots\ldots p_n \right\} \subset E^3$ and let $\underline{r}^i \in (\Omega_c^2)_{p_i}^{p_{i+1}}$

Proposition 5.2

Given $\xi$ there is a unique piecewise cubic curve $\Gamma$ which

satisfies

(i)  $\Gamma$ contains $p_i$  i = 1,....n

(ii) $\Gamma$ is represented by $\underline{r}^i$ between $p_i$ and $p_{i+1}$ and

   $\underline{r}^1(0) = \underline{r}^n(1)$

(iii) $\Gamma \in C^2$

(iv) $\underline{r}^i$ minimizes $\bar{E}$ on $(\Omega_c^2)_{P_i}^{P_i+1}$ for $i=1,\ldots n$

This follows because at $p_i$ continuity of tangent direction and curvature vector means (proposition 2.9) that we must

have $\qquad \underline{\dot{r}}^i\Big|_{u=0} = c\,\underline{\dot{r}}^{i-1}\Big|_{u=1} \qquad\qquad$ for some $c > 0$

and $\qquad \underline{\ddot{r}}^i\Big|_{u=0} = c^2\underline{\ddot{r}}^{i-1}\Big|_{u=1} - d\,\underline{\dot{r}}^{i-1}\Big|_{u=1}$ for some $d$ .

By writing $\underline{\ddot{r}}^i\Big|_{u=0}$ and $\underline{\ddot{r}}^{i-1}\Big|_{u=1}$ in terms of the cubic curve coefficients and substituting in turn for these using the equations of proposition 2.3 then we find that the second condition for the continuity of curvature vector at $p_i$ reduces to (8) :

$$\mu_i\underline{t}_i = 3\Bigg[\langle\underline{\dot{r}}^i,\underline{\dot{r}}^i\rangle\Big|_{u=0}\ (\underline{r}^{i-1}(1)-\underline{r}^{i-1}(0))$$
$$+ \langle\underline{\dot{r}}^{i-1},\underline{\dot{r}}^{i-1}\rangle\Big|_{u=1}\ (\underline{r}^i(1)-\underline{r}^i(0))\Bigg]$$
$$- \langle\underline{\dot{r}}^{i-1},\underline{\dot{r}}^{i-1}\rangle^{\frac{1}{2}}\Big|_{u=1}\ \langle\underline{\dot{r}}^i,\underline{\dot{r}}^i\rangle\Big|_{u=0}\ \underline{t}_{i-1}$$
$$- \langle\underline{\dot{r}}^{i-1},\underline{\dot{r}}^{i-1}\rangle\Big|_{u=1}\ \langle\underline{\dot{r}}^i,\underline{\dot{r}}^i\rangle\Big|_{u=1}\ \underline{t}_{i+1}$$

where $\mu_i$ is an arbitrary scalar and where

$$\underline{t}_{i-1} = \underline{\dot{r}}^{i-1}/\langle\underline{\dot{r}}^{i-1},\underline{\dot{r}}^{i-1}\rangle^{\frac{1}{2}}\Big|_{u=0}$$

$$\underline{t}_{i+1} = \underline{\dot{r}}^i/\langle\underline{\dot{r}}^i,\underline{\dot{r}}^i\rangle^{\frac{1}{2}}\Big|_{u=1}$$

Hence these together with the equations (*) amount to

5n conditions involving

$$\langle\underline{\dot{r}}^i,\underline{\dot{r}}^i\rangle^{\frac{1}{2}}\Big|_{u=1}\ ,\ \langle\underline{\dot{r}}^i,\underline{\dot{r}}^i\rangle^{\frac{1}{2}}\Big|_{u=0}\ ,\ \mu_i,\ \underline{t}_i$$

which are sufficient in number to uniquely define the curve Γ.

## Definition 5.3

The energy $\bar{E}$ of the bicubic surface $\underline{r}$ : $[0,1] \times [0,1] \longrightarrow E^3$ is given by

$$\bar{E}(\underline{r}) = \int_0^1 \int_0^1 (E(u,v) + G(u,v)) \, du \, dv$$

where E and G are the two first fundamental coefficients.

The bicubic surface which minimizes $\bar{E}$ on some class of surfaces satisfying specified boundary conditions will be the surface which minimizes the Dirichlet integral {29} within the class.

## Chapter 6.

## Surface Mappings and Distortion

### Definition 6.1

A Ruled surface is a two dimensional submanifold M of $E^3$
with the property that each point $p \epsilon M$ lies on a straight
line, called the generator at p, which also lies on M.

It is easy to show that {1} on a ruled surface the
Gaussian curvature K is non positive.    Further :

### Definition 6.2

If M has the property that the tangent plane is constant
along generators then M is called Developable.

On a developable, the mean curvature is constant along
generators and a theorem due to Massey {2} states that
a closed connected surface is developable if and only
if the Gaussian curvature $K \equiv 0$.

If we consider the ruled surface $\underline{R} : [0,1] \times [0,1] \longrightarrow E^3$
defined by

$$\underline{R}(u,v) = \underline{R}(u,0) + v[\underline{R}(u,1) - \underline{R}(u,0)]$$

then this has v as the parameter along the generators.

And $\underline{R}$ is developable if

$$[\underline{R}(u,1) - \underline{R}(u,0), \ \underline{R}_1(u,0), \ \underline{R}_1(u,1)] = 0 \quad \forall \ u.$$

If we restrict ourselves to bicubics then the ruled bicubic $\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$ with parameter v along generators is given by

$$\underline{r}(u,v) = \underline{a}_{00} + \underline{a}_{10}\,u + \underline{a}_{20}u^2/2! + \underline{a}_{30}u^3/3!$$
$$+ v[\underline{a}_{01} + \underline{a}_{11}u + \underline{a}_{21}u^2/2! + \underline{a}_{31}u^3/3!]$$

And this is developable if $[\underline{a}(u), \underline{b}(u), \underline{c}(u)] = 0$ where

$$\underline{a}(u) = \underline{a}_{01} + \underline{a}_{11}u + \underline{a}_{21}u^2/2! + \underline{a}_{31}u^3/3!$$
$$\underline{b}(u) = \underline{a}_{10} + \underline{a}_{20}u + \underline{a}_{30}u^2/2!$$
$$\underline{c}(u) = \underline{a}_{11} + \underline{a}_{21}u + \underline{a}_{31}u^2/2!$$

## Proposition 6.3

Given $\{\underline{\alpha}_{ij}\} \subset E^3$ and real numbers $E_{ij}$, $F_{ij}$ then there are $\infty^4$ bicubic surfaces which are ruled, have v as parameter along generators and which satisfy

    (i) $\underline{r}(i,j) = \underline{\alpha}_{ij}$

    (ii) $E(i,j) = E_{ij}$, $F(i,j) = F_{ij}$   $i,j \in \{0,1\}$

        are the first fundamental coefficients E, F

        evaluated at the corners.

## Proposition 6.4

Given $\{\underline{\alpha}_{ij}, \underline{\beta}_{ij}\} \subset E^3$ then there is a unique ruled bicubic surface having v as parameter along generators such that

(i) $\underline{r}(i,j) = \underline{\alpha}_{ij}$

(ii) $\underline{r}_1(i,j) = \underline{\beta}_{ij}$  $i,j \in \{0,1\}$

And of course similar results hold for surfaces which are bicubics with u parameter generators.

## Definition 6.5

The Exponential map exp : $TM \longrightarrow M$ is defined as follows. If $(p,V) \in T_p M$ then $\exp(p,V) = \exp_p V$ is the value at $|V|$ of the unique geodesic $\gamma(t)$ in M such that

(i) $\gamma(0) = p$  (ii) $\gamma'(t)\Big|_{t=0} = V/|V|$

That is $\exp_p V = \gamma(|V|)$

## Theorem 6.6

Let $\lambda : [a,b] \longrightarrow M \subset E^3$ be a cubic curve on the developable surface M such that $\lambda(a) = \lambda(b)$ and suppose that $\lambda$ is contractible on M.   Then M is a plane.

The proof follows since if $\lambda$ is contractible on M then there is some closed disc inside $\lambda$ and we can find a generator G of M which cuts $\lambda$ twice.   Thus we construct the map

$\mu : [a,b] \longrightarrow [a,b]$  such that

$\lambda(\mu(t)) \in G$  if  $\lambda(t) \in G$.

Let $[c,d] \subset [a,b]$. Then we have smooth diffeomorphisms

$\alpha : [c,d] \longrightarrow [c,d] \times \mathbb{R}$ and $\beta = \lambda \times \exp : [c,d] \times \mathbb{R} \longrightarrow M$

where $\beta$ is given by

$\beta : (t,r) \longmapsto p \in M$

where p is the point of distance r along the generator

through $\lambda(t)$. Consider now just $\text{Im}\,\beta$ on $\lambda$. Let $V \subset M$ be

a neighbourhood containing this image. Since $\lambda$ is an

embedding there is a diffeomorphism $\gamma : V \longrightarrow U \subset E^2$.

Now take the projection $\delta : U \longrightarrow [e,f] \subset \mathbb{R}$.

Then $\mu = \delta\gamma\beta\alpha \quad : [c,d] \longrightarrow [e,f]$ is smooth.

Now since $\lambda$ is cubic we can write

$$\lambda(\mu(t)) = \lambda(t) + [\mu(t)-t]\lambda'(t) + [\mu(t)-t^2]/2! \; \lambda''(t)$$
$$+ [\mu(t)-t]^3/3! \; \lambda'''(t).$$

Since M is developable the tangent plane is constant along

generators and so

$$[\lambda(\mu(t)) - \lambda(t), \lambda'(t), \lambda'(\mu(t))] = 0 \qquad\qquad (+)$$

But

$$\lambda'(\mu(t))\mu'(t) = \lambda' + (\mu'-1)\lambda' + (\mu-t)\lambda'' + (\mu-t)(\mu'-1)\lambda''$$
$$+ (\mu-t)^2/2 \; \lambda''' + (\mu-t)^2(\mu'-1)/2 \; \lambda'''$$

Therefore (+) reduces to

$$(\mu(t)-t)[\lambda'', \lambda', \lambda''']/\alpha = 0 \qquad \alpha = \text{constant}$$

Thus $\lambda$ is a plane curve on each open subset of S where S

contains all members t of $[a,b]$ such that a generator

through $\lambda(t)$ cuts $\lambda$ again not tangentially.

Consider two curves $\lambda : [a,b] \longrightarrow E^3$ and $\mu : [c,d] \longrightarrow E^3$
which are not coplanar. Then we can generate a developable
M which contains both $\lambda$ and $\mu$ as follows :

If $\lambda(r)\Big|_p$ and $\mu(t)\Big|_q$ are points of $\lambda$ and $\mu$ which lie on a
generator of M then

$$[\lambda-\mu,\lambda',\mu'] = 0 = f(r,t) \quad \text{say.}$$

And $\dfrac{\partial f}{\partial r} \delta r + \dfrac{\partial f}{\partial t} \delta t = 0$ and so

$$\delta r/\delta t = - [\lambda-\mu,\lambda',\mu'']/[\lambda-\mu,\lambda'',\mu']$$

$$= - \phi(r,t)$$

Hence we just need to solve

$$\frac{dr}{dt} = - \phi(r,t)$$

giving the solutions $r = \psi(t)$ where

$$f(r,\psi(t)) \equiv 0$$

Suppose now that we want to find some mapping f of a portion
of a curved surface M where $f : M \longrightarrow E^2$. We assume
that f is defined on $U \subset M$ where U contains at least two points
of M.

Let $x,y \in U$. Then we given the following quantitive
definition for the distortion of f.

Definition 6.7

The Scale of the map f for each pair $x,y \in U$ is the ratio

$$s(f) = d_E(f(x),f(y))/d_M(x,y)$$

where $d_E(f(x),f(y))$ is the Euclidean distance between $f(x),f(y) \in E^2$ and $d_M(x,y)$ is the geodesic distance between $x,y \in M$.

Ideally we would like $s(f)$ to be constant for all pairs of points in U, but this will not usually be possible.

## Definition 6.8

The minimum scale $s_1(f)$ is given by

$$s_1(f) = \inf_{x,y \in U} s(f)$$

and the maximum scale $s_2(f)$ is given by

$$s_2(f) = \sup_{x,y \in U} s(f)$$

Then $s_1(f)$ and $s_2(f)$ are the best possible constants such that

$$s_1 d_s(x,y) \leq d_E(f(x),f(y)) \leq s_2 d_M(x,y) \ \forall \ x,y \in U$$

This leads to the following definition for distortion which measures the extent to which $s(f)$ is non constant.

## Definition 6.9

The distortion $\delta_0$ of f is given by

$$\delta_0 = \log s_2/s_1$$

Then $0 \leq \delta_0 \leq \infty$ and $\delta_0$ is finite if and only if both $s_1, s_2 > 0$.

## Definition 6.10

The map $f_0$ on U has minimum distortion if $\delta_0(f_0) \leq \delta_0(f)$ for all maps f on U.

If we take $M = S^3$ and $U = D_\alpha$ is the closed disc of geodesic radius $r\alpha$. Thus $D_\alpha = \left\{ x \in S^3 : d_s(x, x_0) \leqslant r\alpha \right\}$ for some fixed point $x_0 \in S^3$. Then {25} there is one and only one minimum distortion map $f_0$ on $D_\alpha$. This map is $C^\infty$ and is the 'azimuthal equidistant projection', which preserves both distances and directions from $x_0$. And in this case $\delta_0(f_0) = \log(\alpha/\sin \alpha)$. This map is in fact the inverse of the exponential map.

If $M$ is smooth with Riemannian metric in terms of local coordinates given by $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ then we can consider $\Delta$, the associated Laplace - Beltrami operator, and a second Riemannian metric on $M$ of the form $\sigma^2 ds^2$ where $\sigma$ is positive and in $C^2$. The Gaussian curvature $\hat{K}$ associated with this new metric is given by {26} the formula $\hat{K} = (K - \Delta \log \sigma)/\sigma^2$. If $f : M \longrightarrow E^2$ is conformal and $\sigma$ is the infinitesimal scale function $\sigma(x) = \lim_{y \to x} s(f)$ then $\hat{K}\Big|_X$ is the Gaussian curvature of $E^2$ at $f\Big|_X$. But $\hat{K} \equiv 0$ and so the differential equation $\Delta \log \sigma = K$ must be satisfied.

Conversely given any solution $\sigma$ to the differential equation $\Delta \log \sigma = K$ the Riemannian metric $\sigma^2 ds^2$ has curvature $\hat{K} \equiv 0$. Thus {26} any sufficiently small connected open subset of $M$ with the metric $\sigma^2 ds^2$ can be mapped isometrically onto an open subset of the plane, which is unique up to rigid motions of the plane.

We consider now the construction of a particular mapping $f : M \longrightarrow E^2$ which arises from the inverse of the exponential map.    If M is a surface defined by $\underline{r} = \underline{r}(u^1, u^2)$ where $(u^1, u^2) \in [0,1] \times [0,1]$ then we first of all apply a linear transformation to $u^1, u^2$ so that we have $g_{ij}(1/2, 1/2) = \delta_{ij}$.

Now the differential equations for geodesics on M are

$$\frac{d^2 u^1}{ds^2} + \Gamma^1_{ij} \frac{du^i}{ds} \frac{du^j}{ds} = 0$$

$$\frac{d^2 u^2}{ds^2} + \Gamma^2_{ij} \frac{du^i}{ds} \frac{du^j}{ds} = 0$$

where the $\Gamma^k_{ij}$ are the Christoffel symbols.

By imposing the initial conditions that:

(i)   $u^1(0) = 1/2$  ,   $u^2(0) = 1/2$.

and by taking $a^1, a^2$ such that

(ii)  $du^1/ds \Big|_{s=0} = a^1$ and $du^2/ds \Big|_{s=0} = a^2$

this will give solutions of the form $u^1 = u^1(s, a^1, a^2)$ and $u^2 = u^2(s, a^1, a^2)$ and if we write

$$u^i = \sum A^i_{\lambda \mu \nu} (a^1)^\mu (a^2)^\nu s^\lambda$$

and solve for the A's then at $s = 1$ this gives

$$f^{-1} = u^i(1, a^1, a^2)$$

Let $M^m$ and $N^n$ be Riemannian manifolds with M compact and let $f : M^m \longrightarrow N^n$ be $C^1$.   At $p \in M$ take local coordinates $x^i$ and at $f(p)$ $y^i$ so that locally $y^\alpha = f^\alpha(x^1, \ldots, x^m)$ and

The differential $f_*$ has matrix representation $(\partial f^\alpha / \partial x^i)$ relative to these coordinates where $u^i \big|_p \longmapsto \left( \frac{\partial f^\alpha}{\partial x^i} u^i \right) \Big|_{f(p)}$

## Definition 6.11

(i) The distortion $\delta_1$ of $f$ at $p \in M$ is given by

$$\delta_1(f,p) = \sup_{u^i} \| f_*(u^i) \|^2 = \sup_{u^i} g'_{\alpha\beta} f^\alpha_i f^\beta_j u^i u^j$$

where $g$ and $g'$ are the metrics on $M$ and $N$

and $\| u^i \| = 1$

(ii) The distortion of $f$ is given by

$$\delta_1(f) = \sup_{p \in M} \delta_1(f,p)$$

Now if $f$ is an isometry then $g_{ij} = g'_{\alpha\beta} f^\alpha_i f^\beta_j$ and so as a measure of non-isometry we can take

$$\delta_1(p, u^i) = g'_{\alpha\beta} f^\alpha_i f^\beta_j u^i u^j - g_{ij} u^i u^j$$

Then the problem is to find the maximum of

$$g'_{\alpha\beta} f^\alpha_i f^\beta_j u^i u^j - 1 \text{ subject to } g_{ij} u^i u^j - 1 = 0$$

If we write

$$F(u) = g'_{\alpha\beta} f^\alpha_i f^\beta_j u^i u^j - 1 - \lambda(g_{ij} u^i u^j - 1)$$

then

$$\partial F / \partial u^i = g'_{\alpha\beta} f^\alpha_i f^\beta_j u^j - \lambda_{g_{ij}} u^j = 0$$

and so the distortion at $p$ will come from the characteristic roots of the equation

$$\det \left[ g'_{\alpha\beta} f^\alpha_i f^\beta_j - \lambda_{g_{ij}} \right] = 0$$

## Definition 6.12

If $p \in M$ and $f : M \longrightarrow N$ is such that for each $p \in M$ then

$\|f_*(u)\|/\|u\|$ is constant on $T_pM$, then $f$ is called conformal.

Every smooth oriented 2-manifold has a conformal structure. If $M$ and $N$ are oriented $C^2$ surfaces immersed in $E^3$ the Euclidean metric imposes a specific conformal structure upon $M$ and $N$. Conformal parameters may be introduced on $M$ for instance by means of isothermal coordinates.

At each point $p \in M^m$ let $A$ be the matrix representation of the differential $f_*$ relative to orthonormal bases of $T_pM^m$ and $T_{f(p)}N^n$ and let $A^t$ be the transpose of $A$. If rank $f_* =$ rank $A = r$ at each point then $r \leqslant \min(m,n)$ and rank $A^tA = r$.

If $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \cdots \cdots \geqslant \lambda_r > \lambda_{r+1} = \lambda_{r+2} \cdots = \lambda_m = 0$ are the characteristic roots of $A^tA$ then we consider the following ratio function $\|\Lambda^r f_*\|$ of volume elements of $M$ and $N$ {27}, where

$$\|\Lambda^r f_*\|^2 = \sum_{i_1 < \ldots < i_r}^{r} \lambda_{i_1} \cdots \lambda_{i_r}$$

Thus $\|\Lambda^1 f_*\| = \|f_*\|$ is the ratio of distances.

At each $p \in M$ let $S^{r-1}$ be a unit $(r-1)$ sphere in $T_pM$. If rank $f_* = r = \min(m,n)$ the image of $S^{r-1}$ under $f_*$ is an ellipsoid of dimension $r-1$.

Definition 6.13

$f : M \longrightarrow N$ is called K-quasiconformal if at each point

p∈M then a/b ⩽ K where a and b are the largest and smallest axes of the ellipsoid $f_*(S^{r-1})$ in $T_{f(p)}N$.

One may verify that f is K-quasiconformal if and only if $\lambda_1/\lambda_r \leq K^2$ at each point.

Let M and N be oriented $C^2$ surfaces immersed in $E^3$.

## Definition 6.14

f : M ⟶ N is called a Teichmüller mapping if isothermal coordinates u,v may be chosen in the neighbourhood of all but a discrete set of exceptional points on M so that the first fundamental forms at corresponding points of M and N are given by

$$ds^2 = \mu(u,v)(du^2 + dv^2)$$
$$d\hat{s}^2 = \hat{\mu}(u,v)(K^2du^2 + dv^2)$$

In fact a Teichmuller mapping is either a conformal mapping or else it is the K quasiconformal mapping characterised by a particular Beltrami differential {7}. It is known that {28} the mapping minimising K amongst all homeomorphisms between closed Riemann surfaces $R_1$ and $R_2$ of the same finite genus ⩾ 2, which are homotopic to some fixed homeomorphism between $R_1$ and $R_2$, always exists is unique and is Teichmüller. Thus a Teichmüller mapping always exists between any two closed oriented $C^2$ surfaces immersed in $E^3$ and each one will give the most nearly conformal mapping between the surfaces among homeomorphisms homotopic to it.

Let $\underline{r}$ : $D \longrightarrow S \subset E^3$ be a representation of some surface

S in $E^3$. We consider three mappings $f_1, f_2, f_3$ of S into

$E^2$.

(1) A mapping $f_1$ which has the property that $f_1\big|_{\partial S}$ has

a distortion equal to zero and is such that on S-$\partial$s the

distortion is a minimum.

Consider first of all the mapping of S onto its

parameter domain D. If the metrics on S and D are

$d\sigma^2$ and $ds^2$ then

$$d\sigma^2 = Edu^2 + 2Fdudv + Gdv^2 \text{ and } ds^2 = du^2 + dv^2$$

This mapping will produce stretching on $\partial$S and so

$d\sigma^2 \neq ds^2$ on the boundaries. Therefore we look

for a mapping h : $D' \longrightarrow D$ such that the metric $ds'^2$

on D' satisfies $ds'^2 = d\sigma^2$ on the boundaries. If

$z \epsilon D'$ let h(z) be given by $u = u(u',v')$, $v = v(u',v')$.

Then

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \partial u/\partial u' & \partial u/\partial v' \\ \partial v/\partial u' & \partial v/\partial v' \end{pmatrix} \begin{pmatrix} du' \\ dv' \end{pmatrix} = J_h \begin{pmatrix} du' \\ dv' \end{pmatrix}$$

And

$$d\sigma^2 = (du\ dv) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}$$

$$= (du'\ dv')\ J_h^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} J \begin{pmatrix} du' \\ dv' \end{pmatrix}$$

So the new first fundamental matrix is $J_h^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} J$

with parameters u', v'.

For a small displacement $\begin{pmatrix} du' \\ dv' \end{pmatrix}$ along $\partial D'$ we require

that $ds'^2 = d\sigma^2$ which means on $\partial D'$

$$(du')^2 + (dv')^2 = (du'\,dv')\, J_h^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} J \begin{pmatrix} du' \\ dv' \end{pmatrix}$$

We in fact find a conformal $h:D \longrightarrow D'$ which achieves this.  For a small displacement $dz$ on $\partial D$ we have displacement $h'(z)dz$ on $\partial D'$ and so to unstretch the boundary we require

$$|h'(z)|\,|dz| = d\sigma = \phi(z)|dz|$$

where $\phi(z) = d\sigma/|dz|$ is the stretch factor at the boundary point $z$.  We therefore require $h'(z)$ analytic in $D$ with $|h'(z)| = \phi(z)$ on $\partial D$.  If there is such an analytic $h'(z)$ then $\log|h'(z)| = u(z)$ is harmonic in $D$ and is therefore the solution of Dirichlets problem for the given boundary function $\log \phi(z)$.  Having obtained this unique $u(z)$ we can find the conjugate harmonic $v(z)$ from the Cauchy Riemann equations ($u_x = v_y$, $u_y = -v_x$) by integration. This will determine $v(z)$ up to an arbitrary additive constant, and

$$h'(z) = e^{u(z)+iv(z)}$$

and integrating gives us $h(z)$ which satisfies

$$|h'(z)| = \phi(z)$$

Then $h(z)$ certainly maps the boundary onto a curve with $ds'^2 = d\sigma^2$ but unfortunately we do not know that $\partial D'$ is a simple closed curve.

Now consider $\bar{h} : D' \longrightarrow D''$ such that $\bar{h}\Big|_{\partial D'}$ has

distortion zero and where $\bar{h}$ reduces the distortion

at the point of maximum distortion in $D'-\partial D'$.

Now $d\sigma^2 - ds^2 = (E-1)du^2 + 2Fdudv + (G-1)dv^2$.

At $z \in D-\partial D$ the distortion in the two principal

directions is given by the characteristic roots of $\left(g_{ij}\right)$.

Taking $\det\left(g_{ij} - \lambda\delta_{ij}\right) = 0$ we have

$$\lambda^2 - (E+G)\lambda + EG-F^2 = 0$$

and the solutions $\lambda_1, \lambda_2$ are the magnitudes of the

principal axes of the infinitesimal ellipse about

$\underline{r}\Big|_z$.

If we choose $\bar{h}$ such that

$$\delta = (\lambda_1-1)^2 + (\lambda_2-1)^2$$

is minimised this will give the required solution.

And in fact we can write

$$\delta = (\lambda_1+\lambda_2)^2 - 2\lambda_1\lambda_2 - 2(\lambda_1+\lambda_2) + 2$$

which is conveniently in terms of the trace and

determinant of the matrix.

(2)  A mapping $f_2 : S \longrightarrow \bar{S} \subset E^2$ having the property

that $f$ is conformal and is locally an isometry on $\partial S$.

We assume also that S is bicubic represented in terms

of parameters s and t by the equation

$$\underline{r}(s,t) = \sum_{j,k=0}^{3} \underline{a}_{jk}s^j t^k \qquad \underline{a}_{jk}\epsilon E^3$$

with $D = [0,1] \times [0,1]$.

On S let the metric $d\sigma^2$ be given by

$$d\sigma^2 = Eds^2 + 2Fdsdt + Gdt^2$$

and if $h : D' \longrightarrow D$ is given by $s=s(u,v), t=t(u,v)$

then

$$d\sigma^2 = E\left[s_u du + s_v dv\right]^2 + 2F\left[s_u du + s_v dv\right]\left[t_u du + t_v dv\right]$$
$$+ G\left[t_u du + t_v dv\right]^2$$

$$= \left[Es_u^2 + 2Fs_u t_u + Gt_u^2\right]du^2$$
$$+2\left[Es_u s_v + F(s_u t_v + s_v t_u) + Gt_u t_v\right]dudv$$
$$+\left[Es_v^2 + 2Fs_v t_v + Gt_v^2\right]dv^2$$

Then $f_2 : S \longrightarrow D'$ is conformal if and only if

$$Es_u s_v + F(s_u t_v + s_v t_u) + Gt_u t_v = 0 \quad \text{and}$$
$$E(s_u^2 - s_v^2) + 2F(s_u t_u - s_v t_v) + G(t_u^2 - t_v^2) = 0$$

Solving these non linear differential equations

will give a class of solutions from which we choose

the one giving the local isometry property on the

boundary.

Writing these equations in matrix form we have

$$(s_u \; t_u) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} s_v \\ t_v \end{pmatrix} = 0 \quad \text{and}$$

$$(s_u \; t_u) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} s_u \\ t_u \end{pmatrix} = (s_v \; t_v) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} s_v \\ t_v \end{pmatrix}$$

which combined give

$$\begin{pmatrix} s_u & t_u \\ s_v & t_v \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} s_u & s_v \\ t_u & t_v \end{pmatrix} = \begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix}$$

where $\phi$ is a scalar.

Writing $\underline{x} = \begin{pmatrix} s \\ t \end{pmatrix}$, $\underline{w} = \begin{pmatrix} u \\ v \end{pmatrix}$, $M^T M = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$

we have

$$(\partial \underline{x}/\partial \underline{w})^T M^T M (\partial \underline{x}/\partial \underline{w}) = \phi I$$

Then $\phi^{-1/2} M(\partial \underline{x}/\partial \underline{w})$ is orthogonal and so we can write

$$(\partial \underline{w}/\partial \underline{x}) = \begin{pmatrix} \theta & \psi \\ -\psi & \theta \end{pmatrix} M \qquad (+)$$

Now since

$$M = \begin{pmatrix} \sqrt{E} & F/\sqrt{E} \\ 0 & \sqrt{EG-F^2}/\sqrt{E} \end{pmatrix}$$

Then (+) becomes

$$\begin{pmatrix} u_s & u_t \\ v_s & v_t \end{pmatrix} \begin{pmatrix} 1/\sqrt{E} & -F/\sqrt{E(EG-F^2)} \\ 0 & \sqrt{E}/\sqrt{EG-F^2} \end{pmatrix} = \begin{pmatrix} \theta & \psi \\ -\psi & \theta \end{pmatrix}$$

Hence we have the two differential equations

$$E\partial v/\partial t = \sqrt{EG-F^2}\,\partial u/\partial s + F\partial v/\partial s$$

$$-E\partial u/\partial t = \sqrt{EG-F^2}\,\partial v/\partial s - F\partial u/\partial s$$

The uniqueness comes by insisting that f is locally an isometry on the boundary.

If the metric on D' is $d\sigma'^2$ with $d\sigma'^2 = du^2+dv^2$ then since along constant t, $d\sigma^2 = Eds^2$ and along constant s, $d\sigma^2 = Gdt^2$ then the required boundary conditions are

$$(\partial u/\partial s)^2 + (\partial v/\partial s)^2 = E \quad \text{for } t\epsilon\}0,1\{$$

$$(\partial u/\partial t)^2 + (\partial v/\partial t)^2 = G \quad \text{for } s\epsilon\}0,1\{$$

which will give uniqueness up to rotation and translation.


(3) A mapping $f_3 : S \longrightarrow \bar{S}\epsilon E^2$ which minimises a given norm for the departure from isometry.

Let the metric $ds^2$ on S be given by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

and let $h : D \longrightarrow D' = \bar{S}$ be given by

$x = x(u,v)$, $y = y(u,v)$ with metric $d\bar{s}^2$ on $\bar{S}$ given by

$$d\bar{s}^2 = \bar{E}du^2 + 2\bar{F}dudv + \bar{G}dv^2$$

where

$$\bar{E} = x_1^2 + y_1^2 \ , \ \bar{F} = x_1 x_2 + y_1 y_2 \ , \ \bar{G} = x_2^2 + y_2^2$$

Let $\mu = d\bar{s}^2/ds^2$.     Then

$$(E\mu - \bar{E})du^2 + 2(F\mu - \bar{F})dudv + (G\mu - \bar{G})dv^2 = 0$$

and the maximum and minimum values $\mu_1, \mu_2$ of $\mu$

arise from the condition

$$(F\mu - \bar{F})^2 = (E\mu - \bar{E})(G\mu - \bar{G})$$

Writing

$$H^2 = EG - F^2, \ \bar{H}^2 = \bar{E}\bar{G} - \bar{F}^2, \ J = (E\bar{G} + \bar{E}G - 2F\bar{F})/2$$

then these values are

$$\mu = \left[ J \pm \sqrt{(J^2 - H^2\bar{H}^2)} \right]/H^2$$

If $f_3$ is an isometry then $\mu_1 = \mu_2 = 1$ and

$$J = H^2 = \bar{H}^2, \ E = \bar{E} \ , \ F = \bar{F} \ , \ G = \bar{G}$$

Two obvious norms for measuring the amount by which $f_3$

distorts S are :

(a)   at each $p \varepsilon S$

$$\delta_1 (f_3, p) = \log(\mu_1/\mu_2)$$

Then $\delta_1 \geqslant 0$ and $\delta_1 = 0$ if and only if $\mu_1 = \mu_2$

Also $\delta_1 \longrightarrow \infty$ as $\mu_1 \longrightarrow \infty$ or $\mu_2 \longrightarrow 0$

In terms of the fundamental coefficients,

$$\delta_1 (f_3, p) = 2 \log (J + \sqrt{C})/H\bar{H}$$

where $C = J^2 - H^2\bar{H}^2$.

And a measure of the overall distortion of $f_3$

is $\delta_1(f_3) = 2\iint_D \log(J+\sqrt{C})/\bar{H}\ dudv$

(b) at each $p\epsilon S$

$$\delta_2(f_3,p) = (\mu_1+\mu_1^{-1}+\mu_2+\mu_2^{-1}-4)/2$$

Then $\delta_2 \geqslant 0$ and $\delta_2 = 0$ if and only if $\mu_1 = \mu_2 = 1$

Also $\delta_2 \to \infty$ as $\mu_1 \longrightarrow \infty$ or $\mu_2 \to 0$.

In terms of the fundamental coefficients,

$$\delta_2(f,p) = (J/\bar{H}^2 + J/H^2) - 2$$

The overall distortion of $f_3$ in this case is

$$\delta_2(f_3) = \iint_D \left[(J/\bar{H}^2 + J/H^2) - 2\right] H\ dudv$$

To obtain the mapping which minimizes the chosen norm we

can follow the normal methods of the calculus of variations.

If $x = \alpha(u,v)$ and $y = \beta(u,v)$ is the extremal mapping $f_0$

say then we consider $\hat{f}$ given by

$$\hat{\alpha}(u,v) = \alpha(u,v) + \delta\xi(u,v)$$

$$\hat{\beta}(u,v) = \beta(u,v) + \epsilon\eta(u,v)$$

Then

$\hat{E} = \bar{E} + 2\delta\alpha_1\xi_1 + 2\epsilon\beta_1\eta_1 + O(\delta^2) + O(\epsilon^2)$

$\hat{F} = \bar{F} + \delta(\alpha_1\xi_2 + \alpha_2\xi_1) + \epsilon(\beta_1\eta_2 + \beta_2\eta_1) + O(\delta^2) + O(\epsilon^2)$

$\hat{G} = \bar{G} + 2\delta\alpha_2\xi_2 + 2\epsilon\beta_2\eta_2 + O(\delta^2) + O(\epsilon^2)$

$\hat{H} = \bar{H}^2 + 2\epsilon\bar{H}(\alpha_1\eta_2 - \alpha_2\eta_1) + 2\delta\bar{H}(\beta_2\xi_1 - \beta_1\xi_2) + O(\delta^2) + O(\epsilon^2)$

Then

$$\delta_2(f) = \iint_D \left[(\hat{E}G + E\hat{G} - 2F\hat{F})(1/\hat{H}^2 + 1/H^2) - 2\right] H \, du \, dv$$

$$= \iint_D \left[(J/\bar{H}^2 + J/H^2) - 2\right] H \, du \, dv$$

$$+ 2\delta \iint_D (\gamma_a \xi_1 + \gamma_b \xi_2) \, du \, dv$$

$$+ 2\varepsilon \iint_D (\gamma_c \eta_1 + \gamma_d \eta_2) \, du \, dv + O(\delta^2) + O(\varepsilon^2)$$

where $\gamma_a, \gamma_b, \gamma_c, \gamma_d$ are functions of $E, F, G, H, \bar{E}, \bar{F}, \bar{G}, \bar{H}$ and $\partial\alpha/\partial u$, $\partial\alpha/\partial v$, $\partial\beta/\partial u$, $\partial\beta/\partial v$.

If $f_0$ is to minimise $\delta_2$ then we require

$$\iint_D (\gamma_a \partial\xi/\partial u + \gamma_b \partial\xi/\partial v) \, du \, dv \quad = 0$$

$$\iint_D (\gamma_c \partial\eta/\partial u + \gamma_d \partial\eta/\partial v) \, du \, dv \quad = 0$$

for arbitrary functions $\xi(u,v)$ and $\eta(u,v)$.

If $D = [0,1] \times [0,1]$ then we require

$$\partial\gamma_a/\partial u + \partial\gamma_b/\partial v \quad = 0$$

$$\partial\gamma_c/\partial u + \partial\gamma_d/\partial v \quad = 0$$

with the boundary conditons

$$\gamma_a\Big|_{u=0} = \gamma_c\Big|_{u=1} = \gamma_b\Big|_{v=0} = \gamma_d\Big|_{v=1} = 0$$

# Bibliography

{1}   Willmore T. J.

      'An Introduction to Differential Geometry'

Oxford University Press 1959

{2}   Hicks N. J.

      'Notes on Differential Geometry'

Van Nostrand Mathematical Studies 1965

{3}   Eisenhart L. P.

      'Riemannian Geometry'

Princeton University Press (1926, 2nd Edition 1949)

{4}   Milnor J.

      'Morse Theory'

Annals of Mathematics Studies No. 51, Princeton Un. Press

{5}   Simons J.

      'Minimal varieties in Riemannian manifolds'

Ann. of Math. 88 (1968)

{6}   Eells J, Sampson J. H.

      'Harmonic Mappings of Riemannian manifolds'

Amer. J. Math., 86 (1964)

{7}   Ahlfors L. V.

      'Lectures on Quasiconformal Mappings'

Van Nostrand Mathematical Studies 1966

{8}   Manning J. R.

      'A method of Constructing Smooth Curves'

Shoe and Allied Trades (Kettering) Research Report 251

{9}  Manning J. R.

    'Continuity conditions for spline curves'

    Computer Journal V.17. 1974

{10} Ferguson J.

    'Multivariable Curve Interpolation'

    J.Assoc. for Computing Machinery, II, 2, pp 221-228

{11} Coons S. A.

    'Surfaces for Computer Aided Design of Space Forms'

    Project MAC TR-41, M. I. T. 1967

{12} Bezier P. E.

    'Numerical Control - Mathematics and Applications'

    John Wiley & Sons Ltd., 1972

{13} Forrest A. R.

    'Curves and Surfaces for Computer Aided Design'

    Ph.d. Thesis, Cambridge University 1968

{14} Forrest A. R.

    'Computational Geometry'

    Proc. of Royal Society, London A321, pp 187-195

{15} Forrest A. R.

    'Computer Aided Design of Three Diml. Objects: A Survey'

    Proc. ACM/AICA Intern. Comp. Symp. Venice April 1972

{16} Thomas D. H.

    'Pseudospline Interpolation for Space Curves'

    Mathematics of Computation, Vol. 30, No. 133

{17} Riesenfeld R.

'Appl'ns of B-Spline approx'n to geometric problems in CAD'

Ph.d Thesis, Syracuse University 1972

{18} De Boor C.

'On calculating with B-Splines'

J. Approx. Theory Vol. 6 1977

{19} Cox M. G.

'The Numerical Evaluation of B-Splines'

N. P. L DNAC 4   1971

{20} Birkhoff G, Garabedian H. L.

'Smooth Surface Interpolation'

J. Math. Physics, 39,   1960 pp 258-268

{21} South N. E., Kelly J. P.

'Analytic Surface Methods'

Ford Motor Co., Dearborn N/C Development Unit. Dec. 1965

{22} Ahlberg J. H., Nilson E. N., Walsh J. L.

'The Theory of Splines and their applications'

Academic Press. New York   1967

{23} Greville T. N. E

'Intro. to Spline Fns., Theory & Appl'ns of Spline Fns'

Academic Press   1969

{24} Schoenberg I. J.

'On Variation Diminishing Approximation Methods'

(On Numerical Approximation, R. E. Langer)

University of Wisconsin Press   1959

{25} Milnor J.

'A problem in Cartography'.

{26} Struik D. J.

'Lectures on Classical Differential Geometry'

Addison-Wesley, Reading, Mass.

{27} Lu Y. G.

'Holomorphic mappings of complex manifolds'

J. Differential Geometry 2  (1968)

{28} Bers L.

'Quasiconformal Mappings and Teichmüllers Theorem'

Conf. on Analytic Functions, Princeton 1960

{29} Courant R

'Dirichlets principle, conformal mapping and

minimal surfaces'

Interscience, New York  1953.

$\tau = 0$

$\tau \equiv 0$
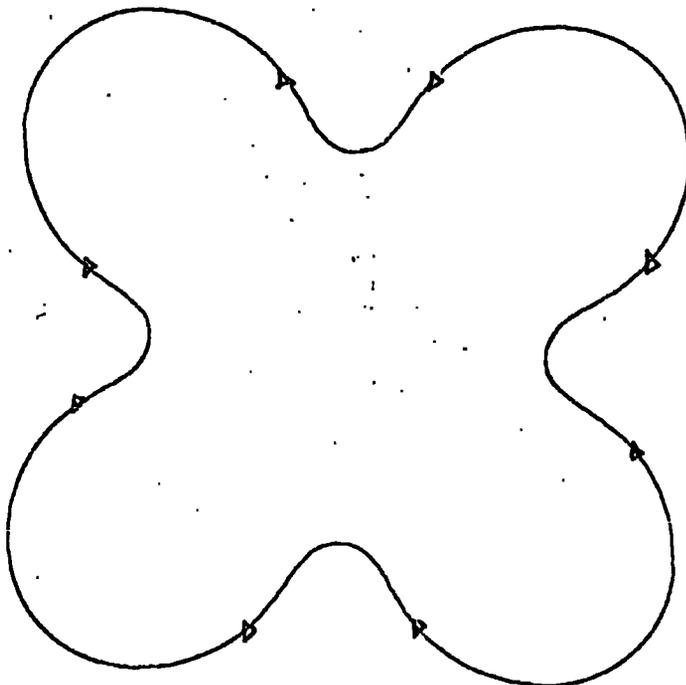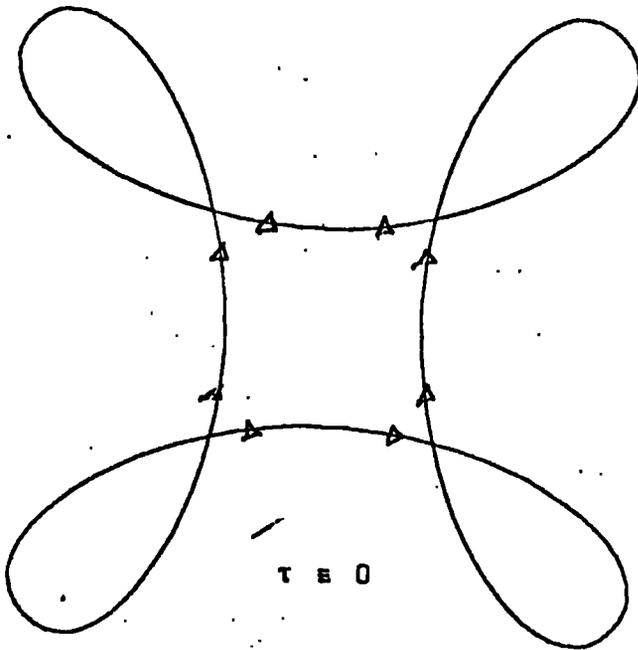
$C^3$ Piecewise Cubics

Figure 1

$\tau = 0$

$\tau = 0$

$\tau = 0$
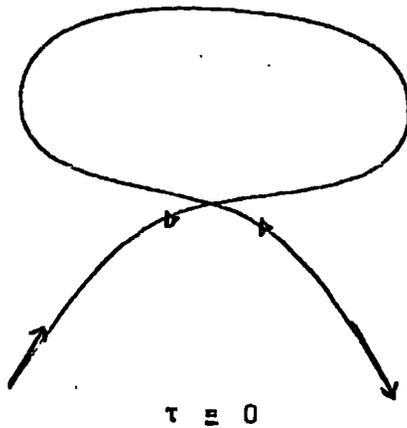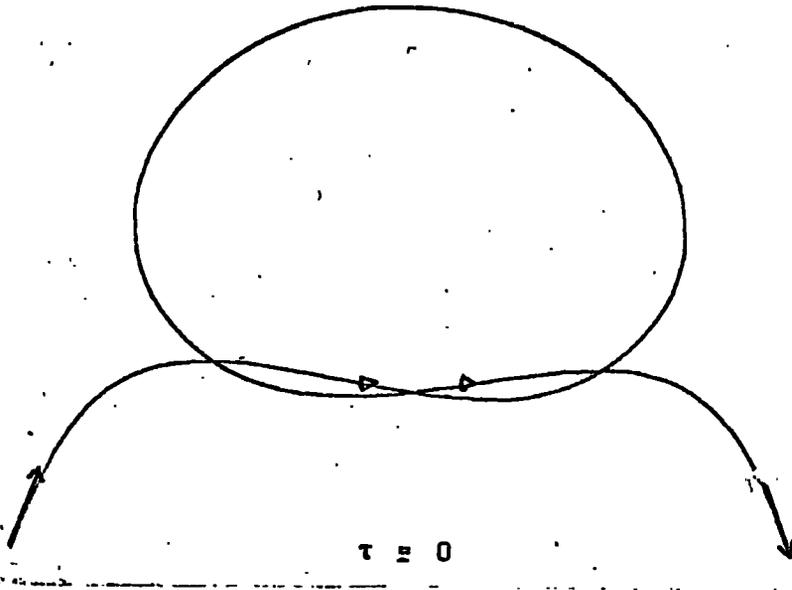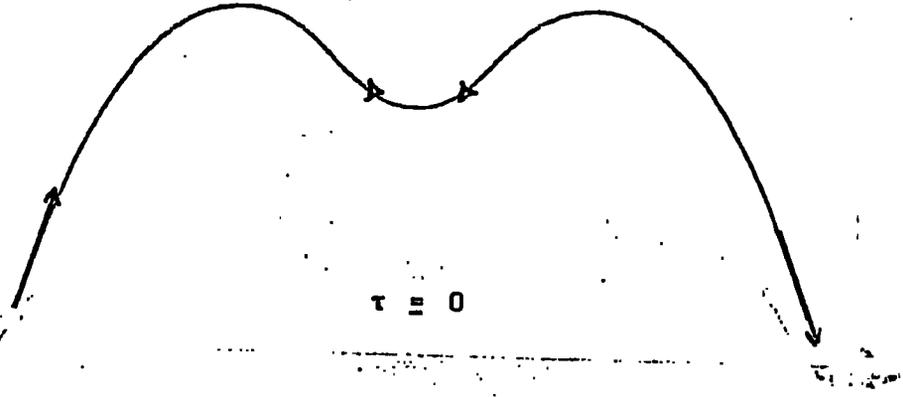
$C^3$ Piecewise Cubics

Figure 2

$\tau \neq 0$

$\tau \neq 0$

$\tau \neq 0$
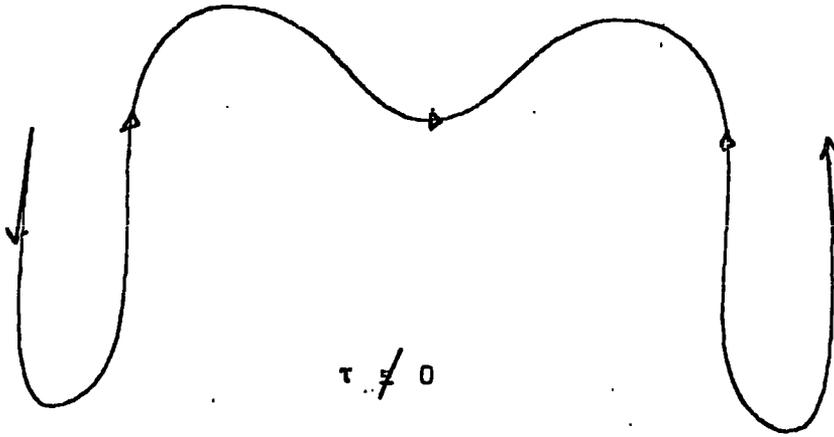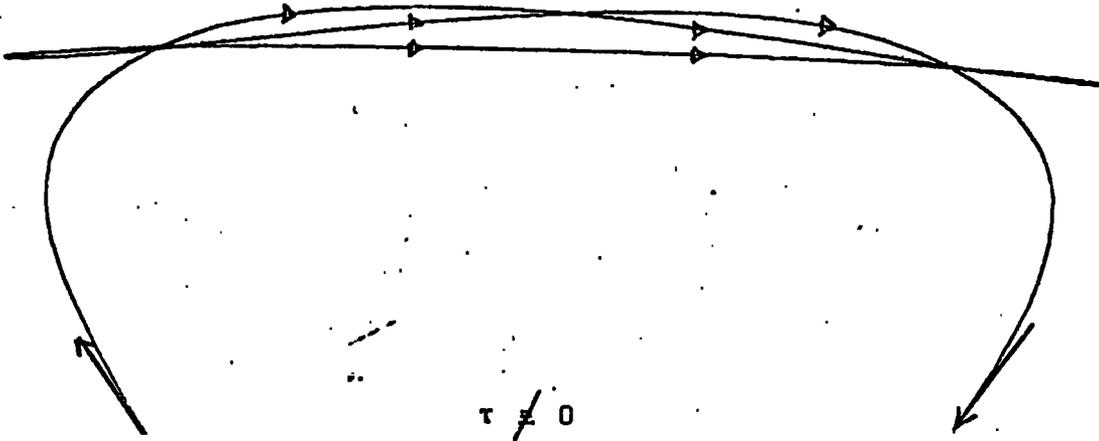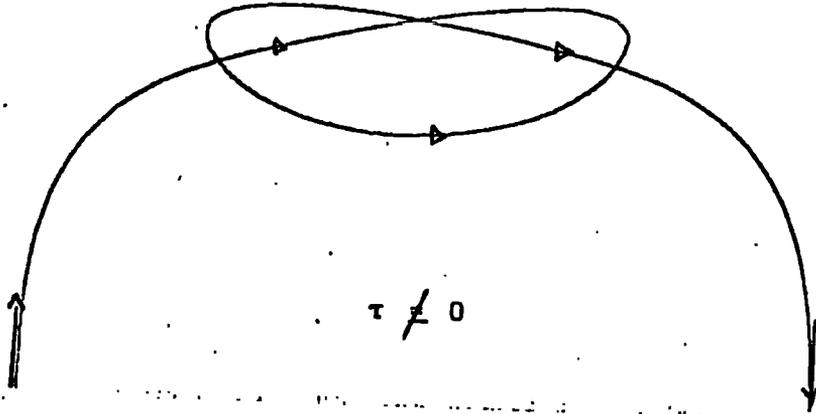
$C^3$ Piecewise Cubics

Figure 3

$\tau = 0$

$C^3$ Piecewise Cubics

Figure 4
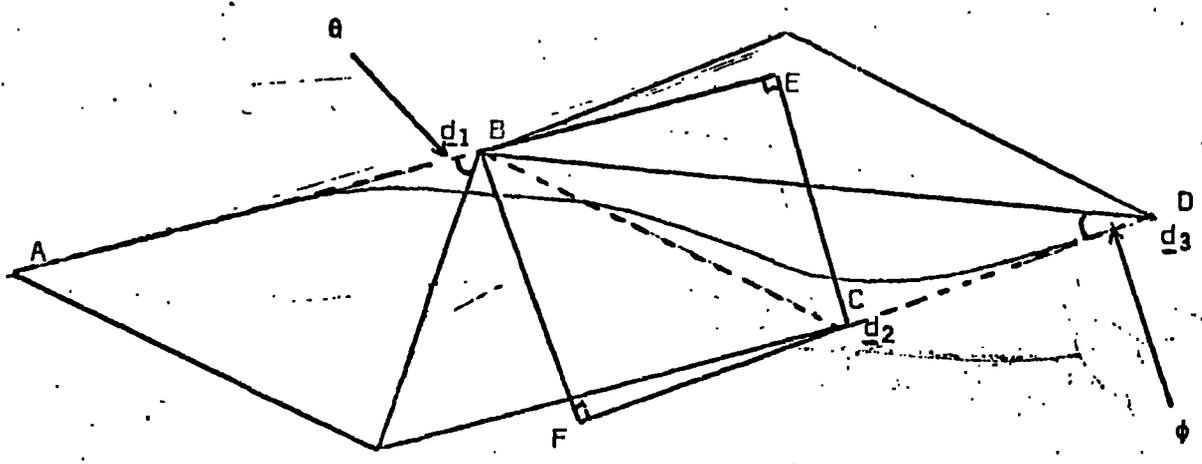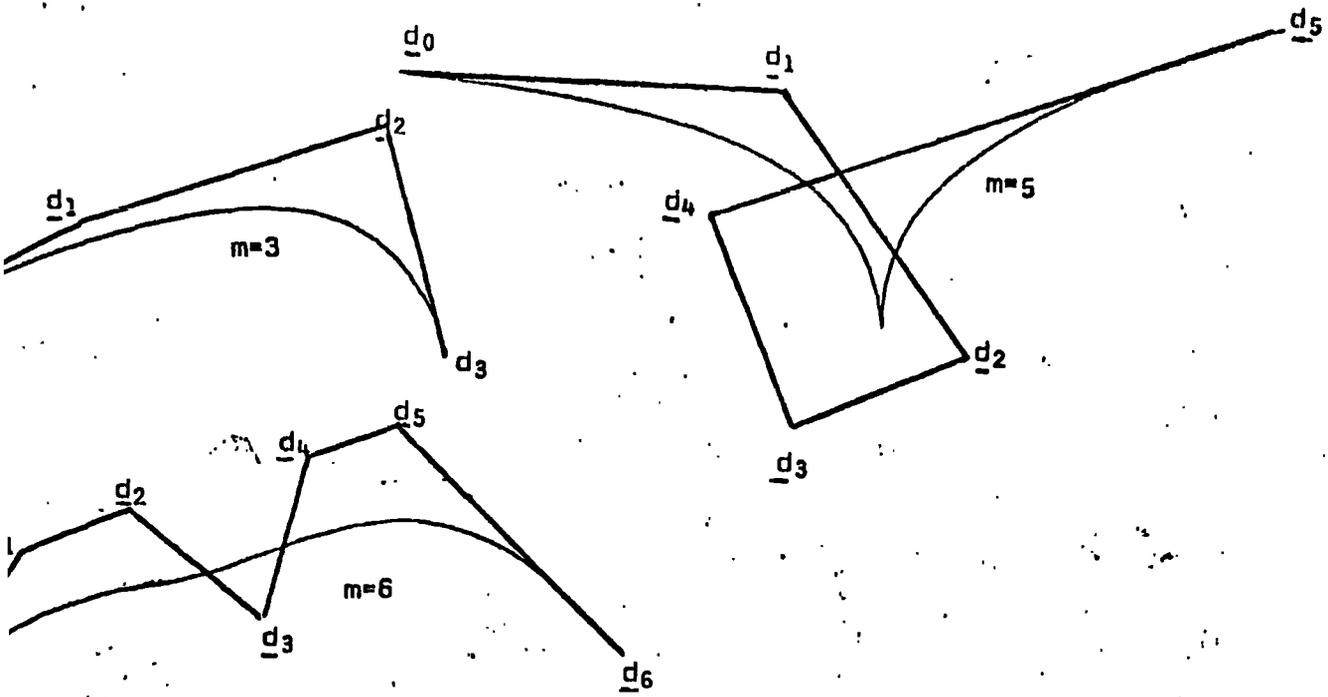
$$\kappa(0) = 2|\overline{CE}| / 3|\overline{AB}|^2 \qquad \kappa(1) = 2|\overline{BF}| / 3|\overline{CD}|^2$$

$$\tau(0) = 2|\overline{CH}|\cos\alpha / |\overline{AB}||\overline{CE}|$$

$$\tau(1) = 2|\overline{BK}|\cos\beta / |\overline{CD}||\overline{BF}|$$

where $\alpha$ is angle between $\overline{CH}$ & $\overline{AB} \times \overline{BG}$

and $\beta$ is angle between $\overline{BK}$ & $\overline{CD} \times \overline{CJ}$

$$(\overline{AB} = \dot{\underline{r}}(0) \qquad \overline{BG} = \ddot{\underline{r}}(0))$$

$$(\overline{CD} = \dot{\underline{r}}(1) \qquad \overline{CJ} = \ddot{\underline{r}}(1))$$
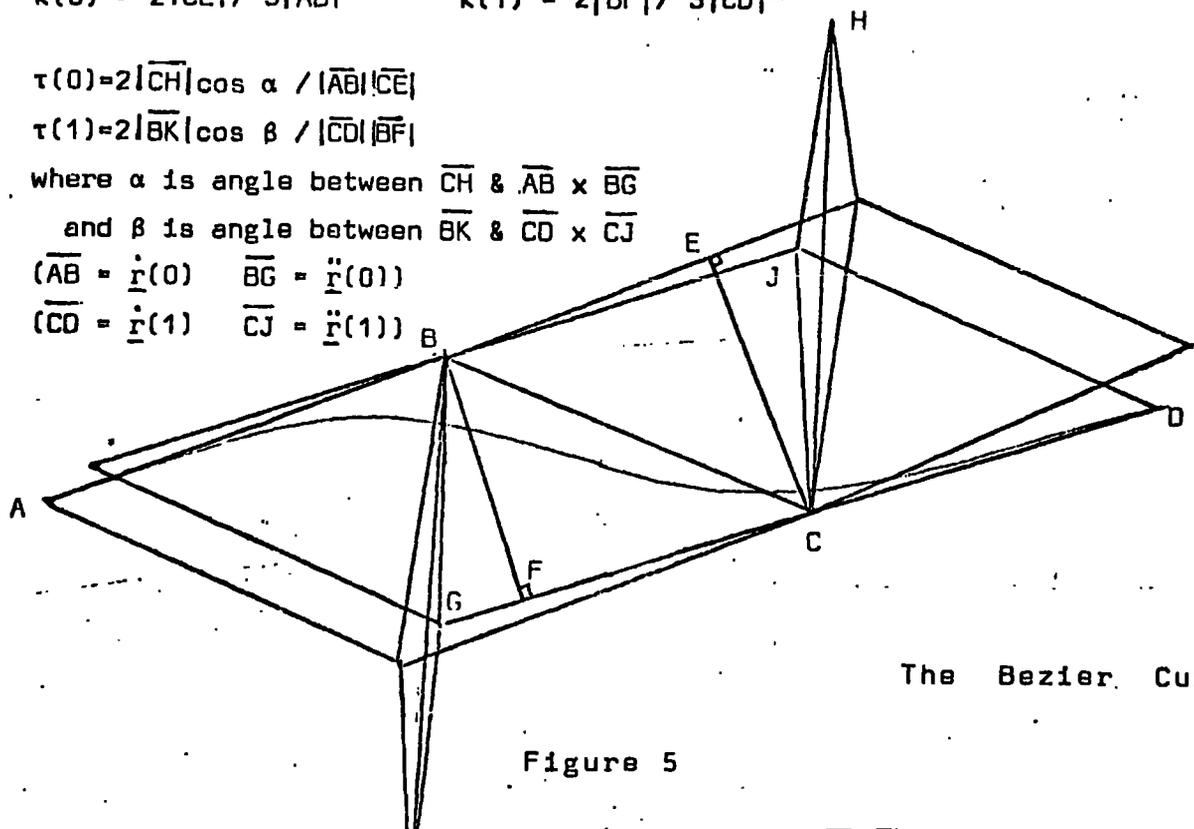
The Bezier Curve

Figure 5
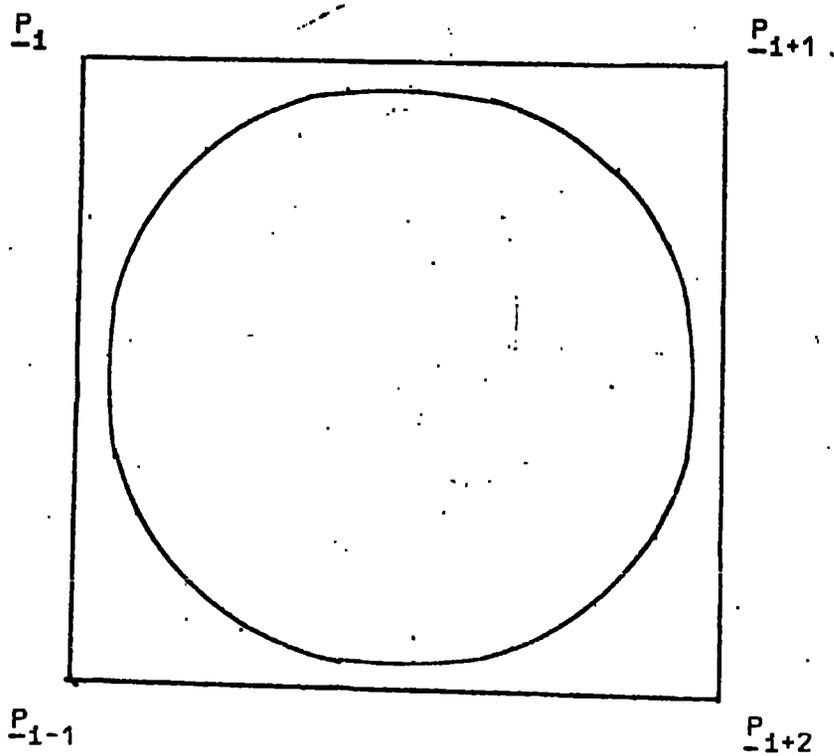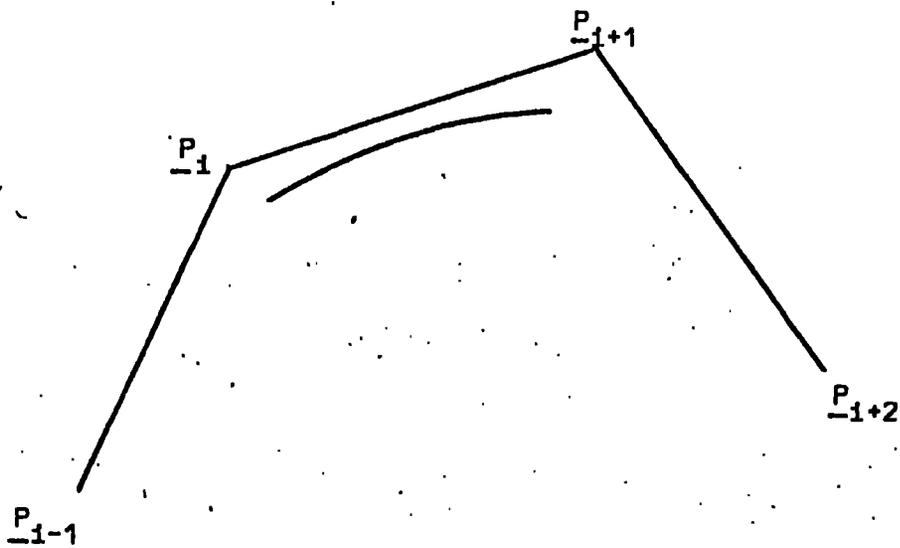
Typical B-Spline Cubic Curves

Figure 6

Bezier Curves

Figure 7

Extremal Cubics

$d_1$
$d_2$

$d_0$
$d_3$
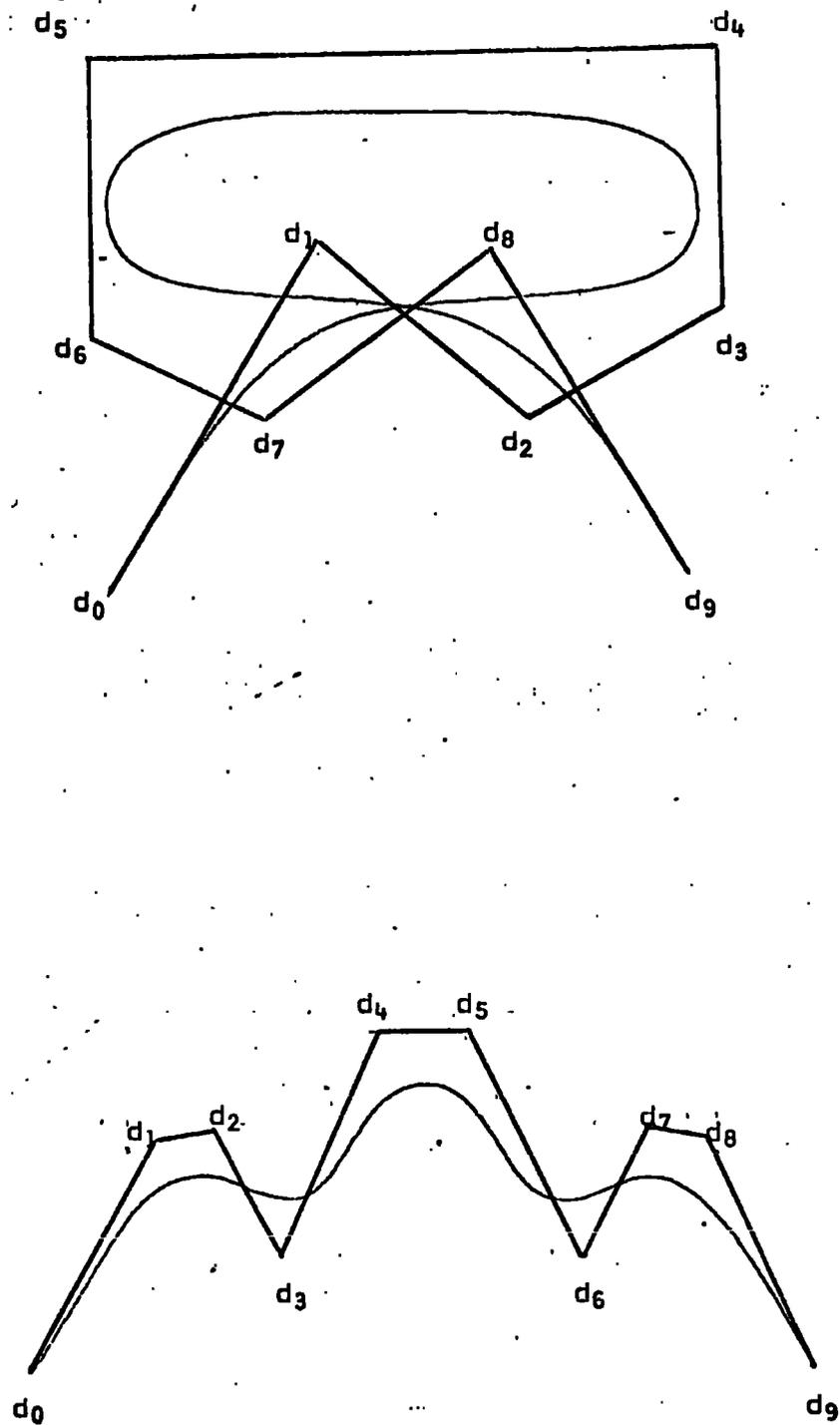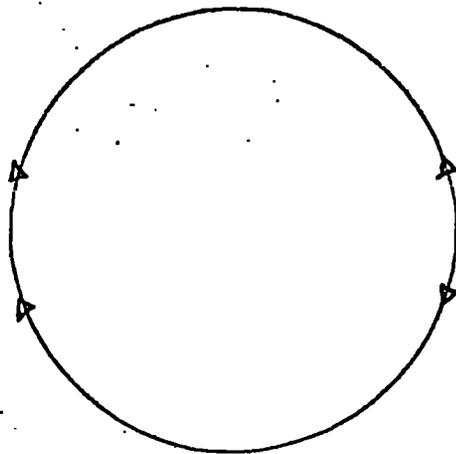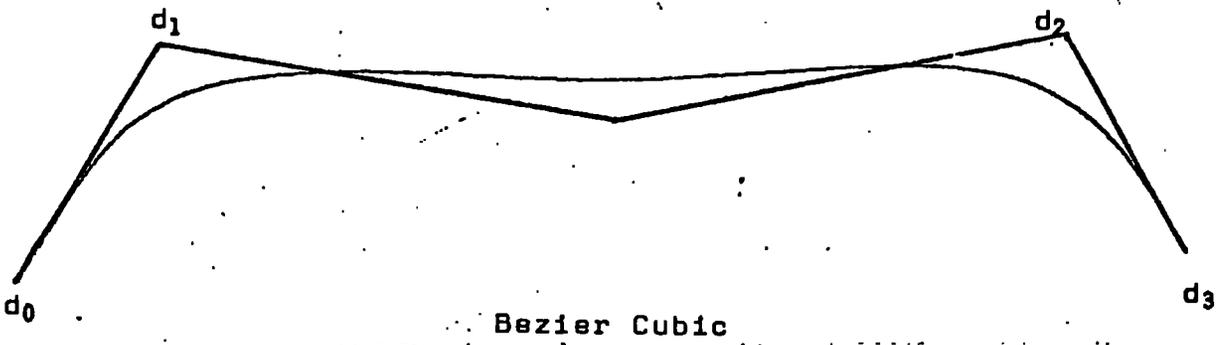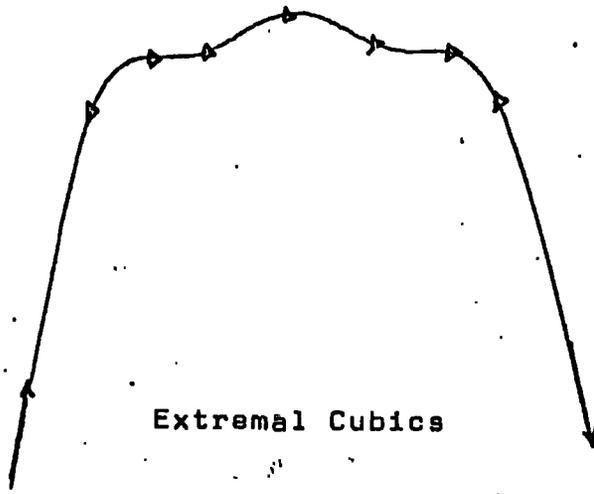
Bezier Cubic

Manning Cubics
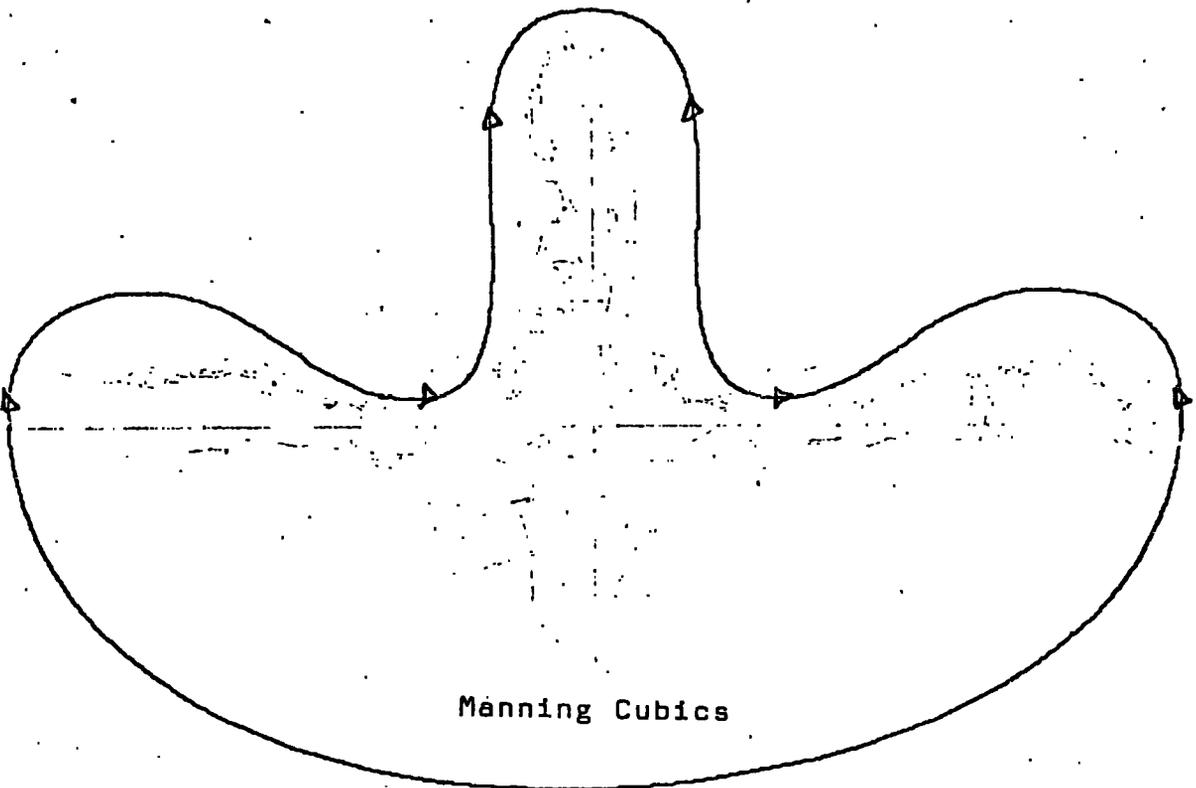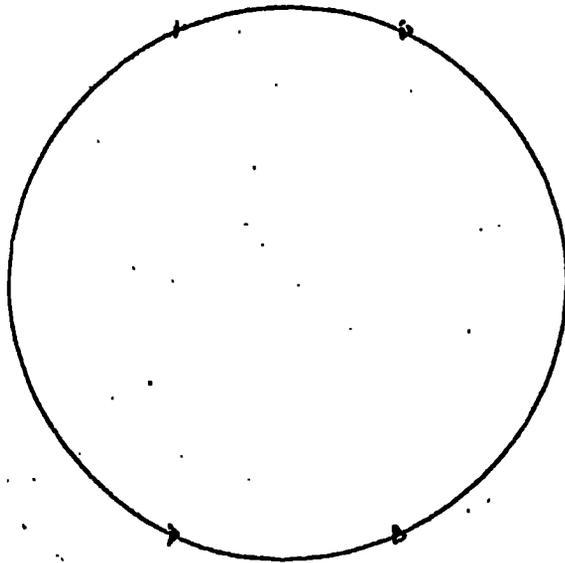
Figure 8

Manning Cubics

Figure 9
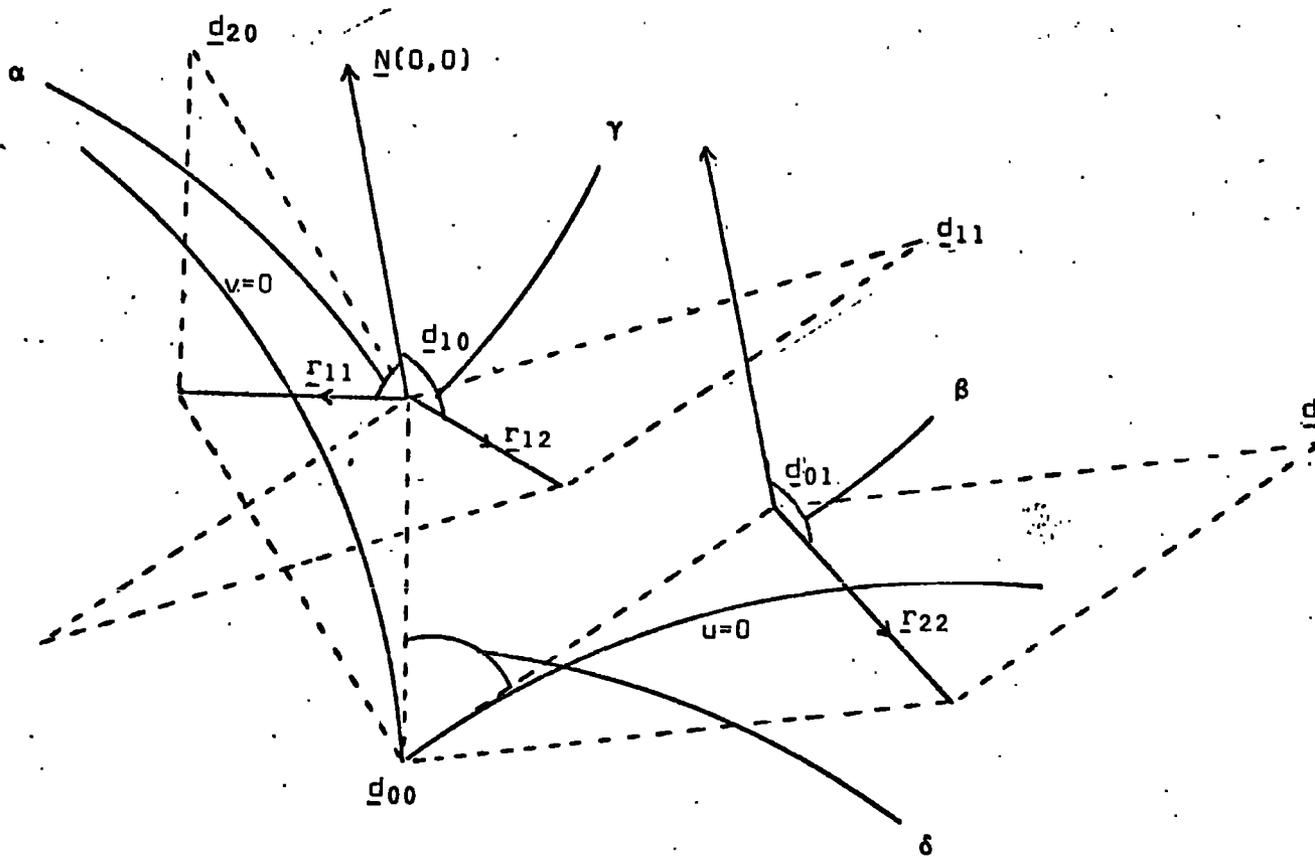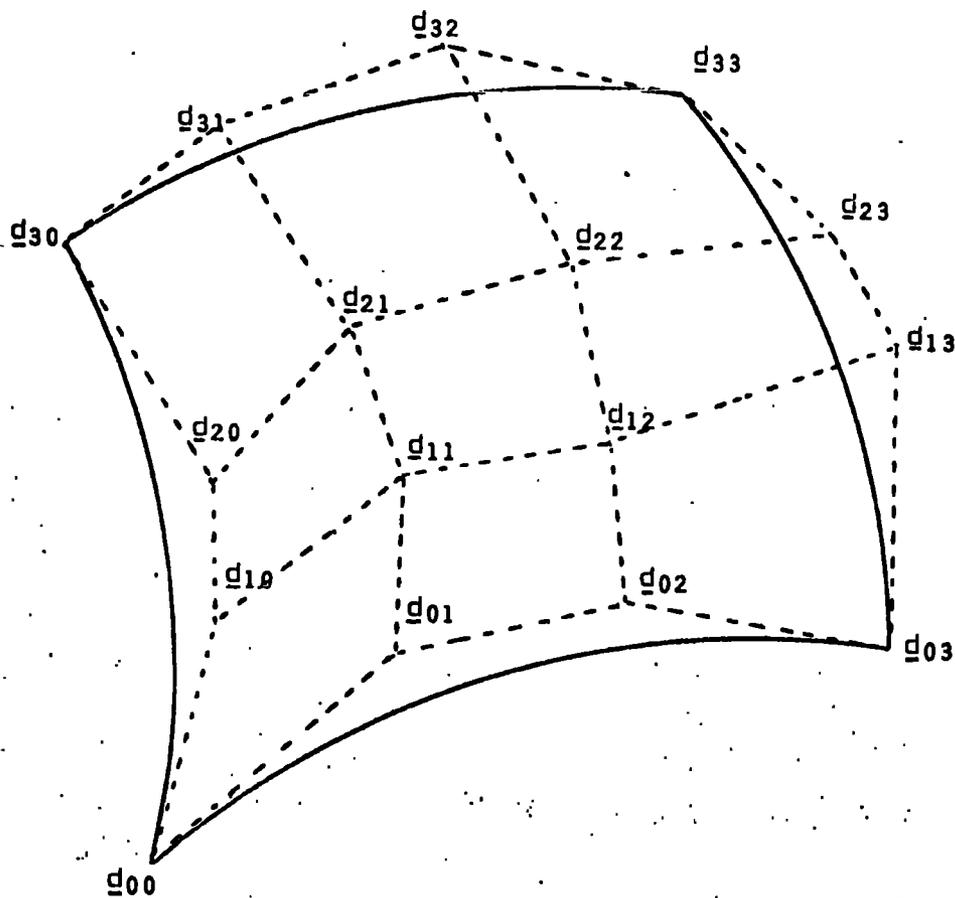
The Bezier Bicubic Surface

Figure 10
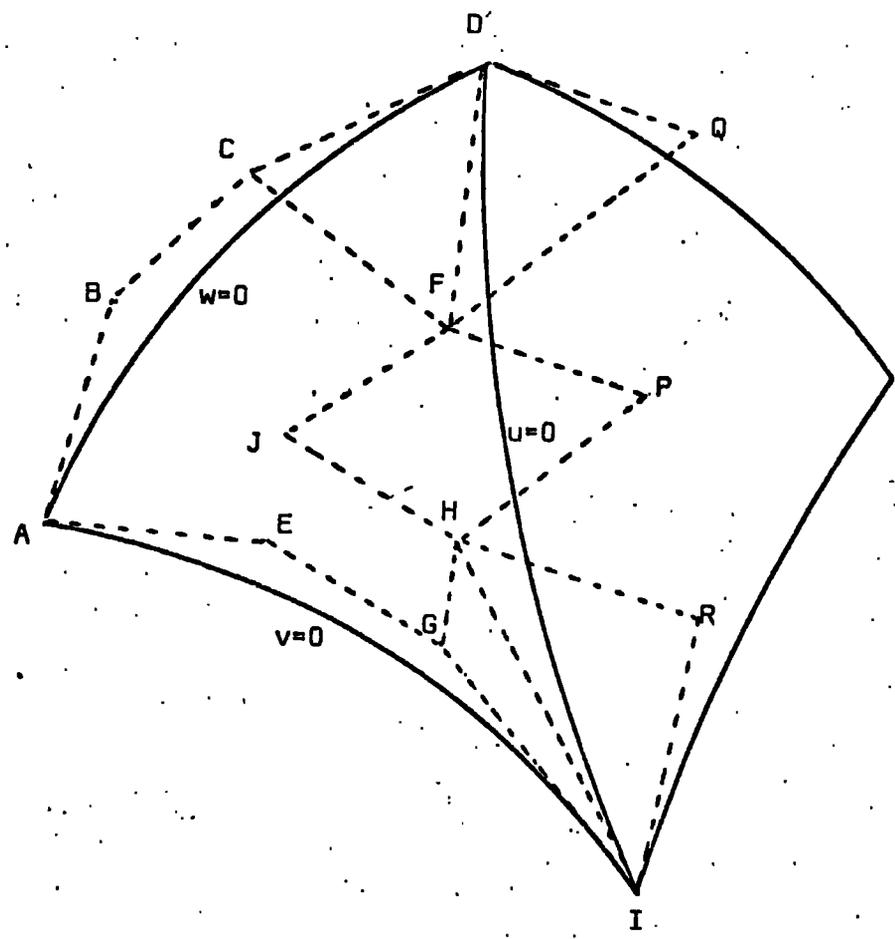
Triangular Surface Boundary Continuity

Figure 11
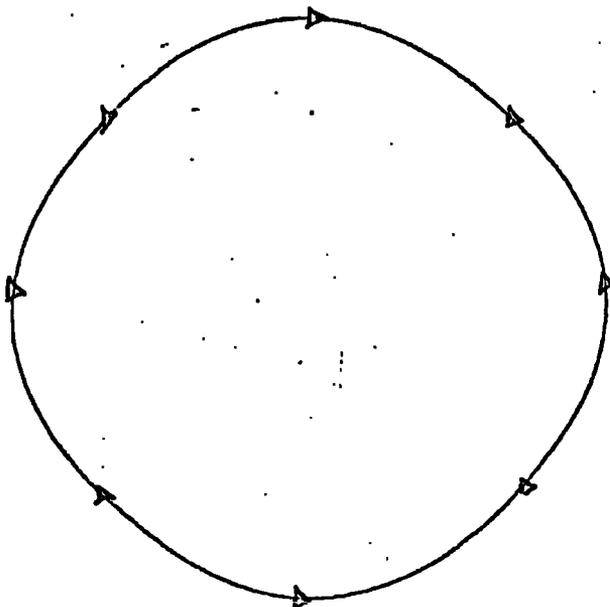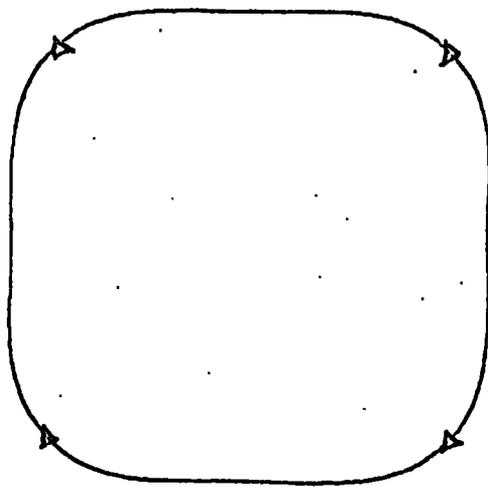
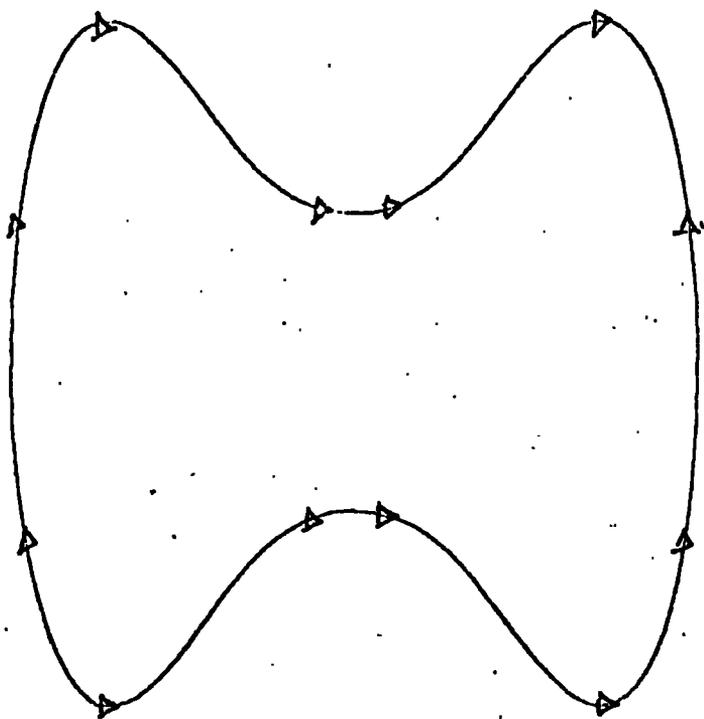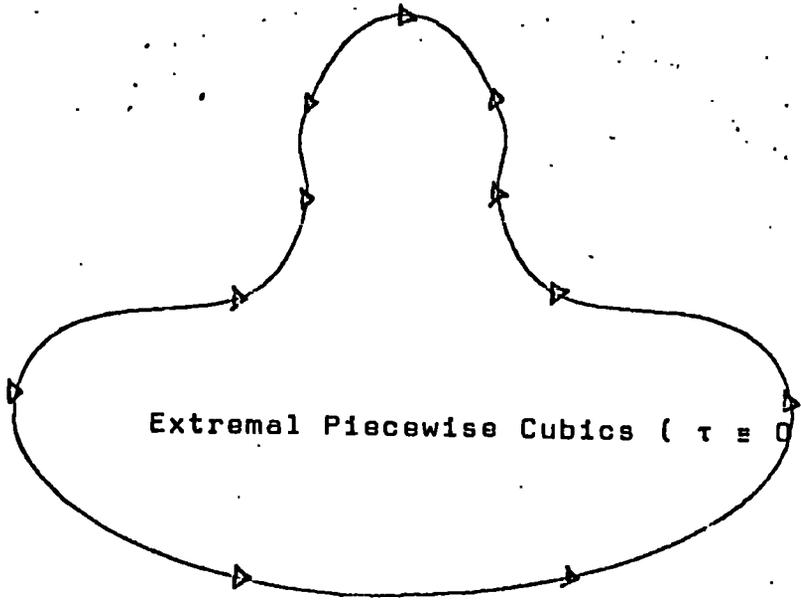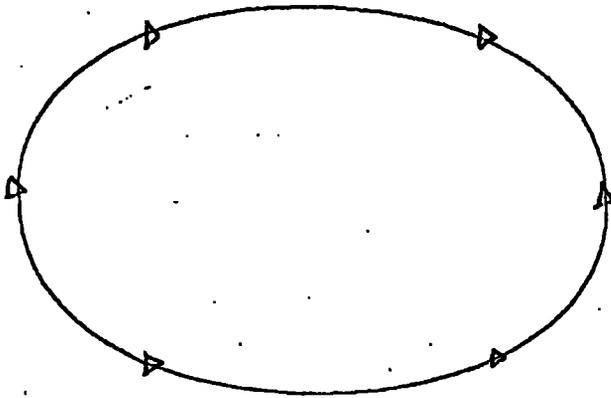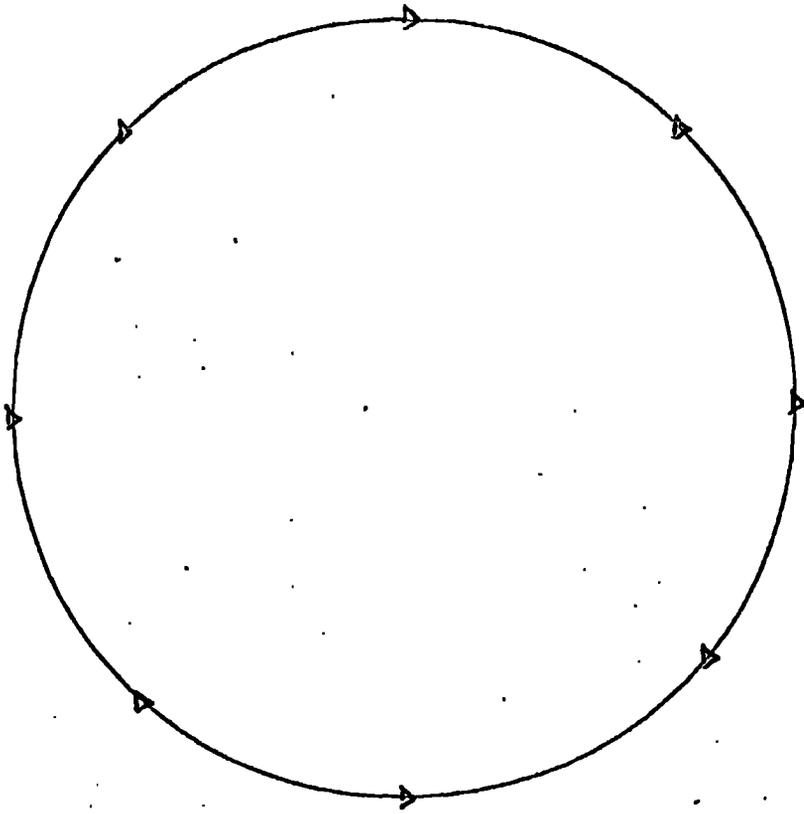Extremal Piecewise Cubics ( $\tau = 0$ )

Figure 12
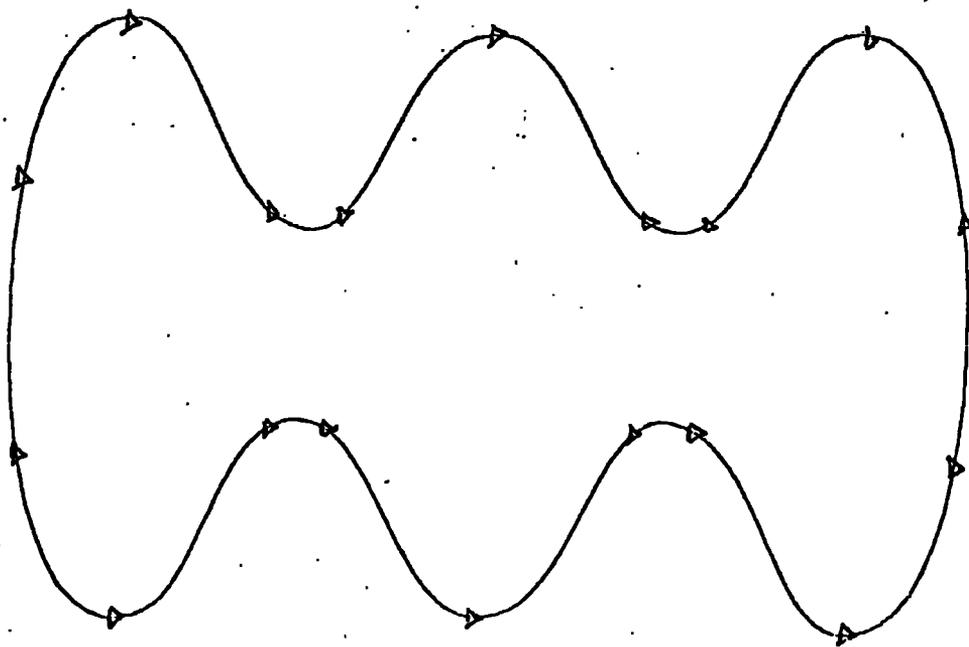
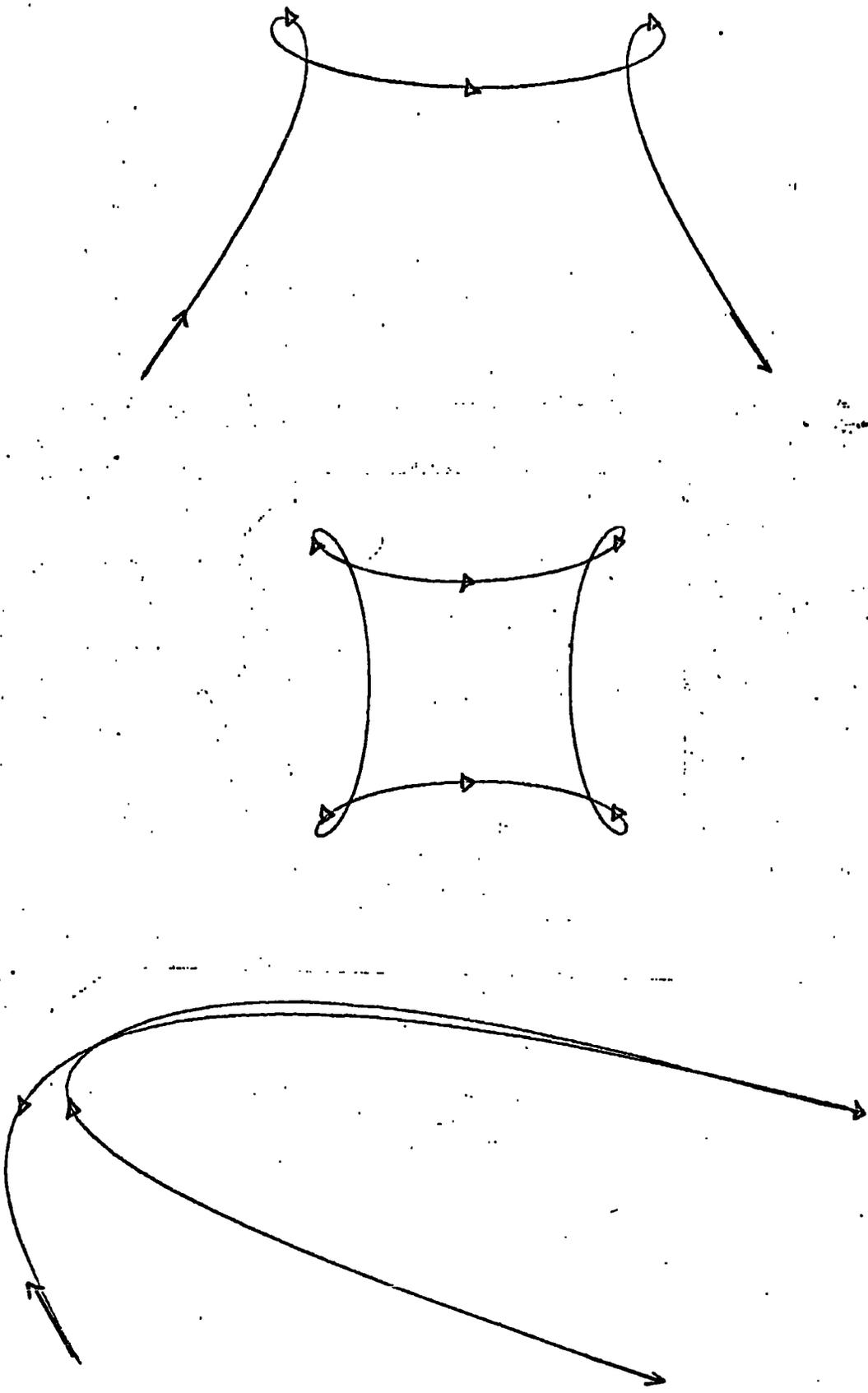Extremal Piecewise Cubics ( τ ≡ 0 )

Figure 13

Extremal Piecewise Cubics ( $\tau \equiv 0$ )

Figure 14

Extremal Piecewise Cubics ( $\tau \neq 0$ )

Figure 15