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DIFFERENTIAL EQUATIONS WITH SOLITON BEHAVIOUR

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A thesis presented for the degree of Doctor of Philosophy in the University of Durham.

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ABSTRACT.

Various non-linear wave equations are found to possess solitons - stable solitary waves which only undergo a change of position on collision with each other. It is shown in chapter 1, how the various soliton properties of the sine-Gordon equation, $u_{xy} = \sin u$, may be derived from its Backlund Transformation.

Most of the rest of the thesis consists of several attempts to find Backlund Transformations for other equations of the form $u_{xy} = F(u)$ by generalizing the usual form of the Backlund Transformation. The only exception to this is in chapter 2 where equations of the form $u_{xy} = A(x,y,u).u_x + B(x,y,u).u_y + C(x,y,u)$ are considered. The rest of chapter 2 considers the effect of allowing the Backlund Transformation to depend explicitly on the independent variables or on integrals of the dependent variables.

The rest of this thesis concentrates on allowing the Backlund Transformation to depend on derivatives only of the "old" and "new" variables, u and u'. It is found that if u and u' satisfy $u_{xy} = F(u)$ where $F'''(u) \nparallel K.F''(u)$ and $F''(u) \nparallel K.F(u)$ then there are no Backlund Transformations of the following form

Chapter 3.
$$u'_x = P(u,u';p_1,\ldots,p_N;q_1,\ldots,q_M)$$
$$u'_y = Q(u,u';p_1,\ldots,p_N;q_1,\ldots,q_M)$$

except possibly when $M = 1$, $N > 7$ and $F(u) = A_1.e^{cu} + A_2.e^{-2cu}$

Chapter 4.
$$u'_{xx} = P(u,u',u_x,u'_x,u_y,u_{xx},u_{yy})$$
$$u'_y = Q(u,u',u_x,u'_x,u_y,u_{xx},u_{yy})$$

Chapter 5.
$$\tfrac{1}{2}(p'_{N+1} - p_{N+1}) = P(p_0,p_1,\ldots,p_N;p'_0,\ldots,p'_N) \qquad N < 5$$
$$\tfrac{1}{2}(q' + q) = Q(p_0,p_1,\ldots,p_N;p'_0,\ldots,p'_N)$$

CONTENTS.

Chapter 4.

Chapter 5.

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CHAPTER 1.

§1. Introduction.

A soliton is a single wave pulse (solitary wave) which emerges from a collision with a similiar solitary wave having unchanged shape and speed. Non-linear wave equations which possess solitons are also found to possess an infinite number of conservation laws, Backlund Transformations and the initial value problem may be solved using linear methods only[1]. Half a dozen, or so, equations have been found which possess such soliton solutions.

The research for this thesis was started with the intention of finding suitable candidates for elementary particle field theories. So it is really only Lorentz invarient equations which are of interest. The only Lorentz invarient soliton equation known is the sine-Gordon equation. Consequently only equations related to the sine-Gordon equation will be considered in this thesis.

The sine-Gordon equation

$$u_{zz} - u_{tt} = \sin u \tag{1.1}$$

originally arose in connection with the theory of pseudopotential surfaces[2] but also occurs in many physical contexts[1]. For example, the self-induced transparency equations,[3] which incidently have soliton solutions, have the sine-Gordon equation as a limiting case. The sine-Gordon equation may be quantized as a field theory, where the solitons "survive quantization."[4] Note that (1.1) is Lorentz invarient in the sense that it is unchanged in form under the substitution

$$z' = (z-vt)(1-v^2)^{\frac{1}{2}} \quad , \qquad t' = (t-vz)(1-v^2)^{-\frac{1}{2}}$$

This thesis looks at the soliton concept from the viewpoint of Backlund Transformations. (Hereafter abreviated to B.T.). A B.T. for a given partial differential equation consists of two equations which enable one to construct new solutions to the given equation from a given solution. Sometimes the new solution satisfies a different equation but in most cases considered the above definition of an auto-Backlund transformation will apply. Note that the definition is (necessarily) vague because it is not clear what the best definition should be. It is part of the aim of this thesis to investigate this. The exact form of the B.T. will be given in each case considered.

It will be first shown that all the soliton properties of the sine-Gordon equation follow from its B.T. Hereafter sine-Gordon equation will be abreviated to S.G. The final section of this chapter is a summary of this thesis - the aims and the results.

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To conclude this section consider the following B.T.

$$u'_x = u_x - a.\exp \tfrac{1}{2}(u+u')$$
$$u'_y = -u_y - 2a^{-1}.\exp \tfrac{1}{2}(u'-u) \qquad (1.2)$$

If u is any solution of

$$u_{xy} = 0 \qquad (1.3)$$

and u' is any solution of (1.2) then u' must satisfy

$$u'_{xy} = \exp u' \qquad (1.4)$$

So by inserting the general solution of (1.3) into (1.2) one obtains the general solution of (1.4)

Note that here as throughout this thesis subscripts denote partial differentiation with respect to (w.r.t.) the variable displayed.

Because of the large number of equations involved the numbering of each chapter is independent of the others. This is not too confusing, I hope, since each chapter is almost self-contained. This seemed preferable to making the system of numbering of equations more complicated.

§2. Soliton Solutions.

The S.G. in characteristic coordinates is

$$u_{xy} = \sin u \qquad (2.1)$$

It has the B.T.

$$u'_x = u_x + 2a.\sin \tfrac{1}{2}(u'+u)$$
$$u'_y = -u_y + 2a^{-1}.\sin \tfrac{1}{2}(u'-u) \qquad (2.2)$$

where a is a constant.

If u is any solution of (2.1) then (2.2) have a solution. Further this solution u' of (2.2) must satisfy (2.1). One may alternatively replace u by u' in this statement.

Now $u = 0$ is a solution of (2.1). Inserting this into (2.2) and integrating gives

$$u' = 4.\tan^{-1}(\exp (ax + a^{-1}y + k)) \qquad (2.3)$$

where k is a constant. This is the single soliton solution. It is actually the derivative of u which is the solitary wave - the soliton. To obtain further solutions it is easiest to first derive the "theorem of permutability".

Let u_0 be any solution of (2.1). Then let $u' = u_1$ and $u' = u_2$ be two solutions of (2.2) with $u = u_0$ via constants a_1 and a_2 respectively. Define

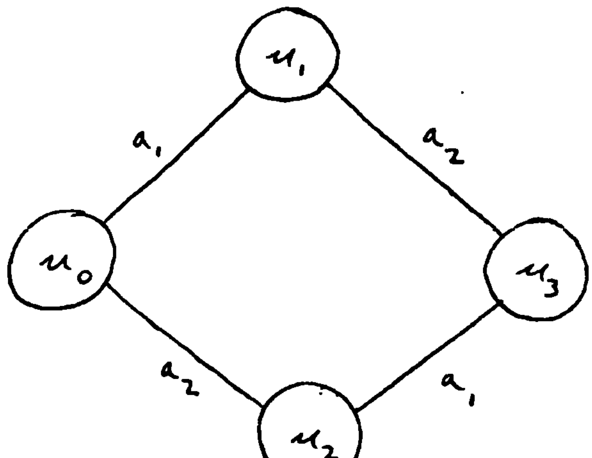
$$u_3 = u_0 + 4.\tan^{-1}((a_1+a_2)(a_1-a_2)^{-1}.\tan \tfrac{1}{4}(u_1-u_2)) \qquad (2.4)$$

It is then straight-foward, for example, to show that

$$\frac{\partial u_3}{\partial x} = \frac{\partial u_1}{\partial x} + 2a_2 \cdot \sin\left(\frac{u_1 + u_3}{2}\right)$$

$$\frac{\partial u_3}{\partial y} = -\frac{\partial u_1}{\partial y} + \frac{2}{a_2} \cdot \sin\left(\frac{u_3 - u_1}{2}\right)$$

So it has been shown that u_3 defined by (2.4) consists of a B.T. with parameter a_2 applied to u_1. It also consists of a B.T. with parameter a_1 applied to u_2. This permutability is illustrated in the Lamb diagram shown at right.



To obtain the two soliton solution to (2.1) one sets $u_0 = 0$ and u_1 and u_2 equal to the appropriate single soliton solution given by (2.3), in (2.4) :

$$u = 4 \cdot \tan^{-1}\left\{\left(\frac{a_1 + a_2}{a_1 - a_2}\right) \cdot \frac{\sinh\left[\frac{1}{2}(\zeta_1 - \zeta_2)\right]}{\cosh\left[\frac{1}{2}(\zeta_1 + \zeta_2)\right]}\right\} \tag{2.5}$$

where $\zeta_i = a_i \cdot x + a_i^{-1} \cdot y + k_i$, $i = 1,2$ (2.6)

The change of variable

$x = \frac{1}{2}(z + t)$, $y = \frac{1}{2}(z - t)$ (2.7)

takes one into real space-time, equation (1.1). Then

$\zeta_i = (z - v_i \cdot t)(1 - v_i^2)^{-\frac{1}{2}} + k_i$ $\quad v_i = (1 - a_i^2)(1 + a_i^2)^{-1}$ (2.8)

Note that if a_i is real then v_i is real with $|v_i| < 1$

Now make a Lorentz transformation with velocity

$$V = \frac{-1 - v_1 v_2 + \sqrt{(1 - v_1^2)(1 - v_2^2)}}{v_1 + v_2}$$

Also define

$$U = \frac{-1 + v_1 v_2 + \sqrt{(1 - v_1^2)(1 - v_2^2)}}{v_2 - v_1}$$

Then equation (2.5) is

$$\tan\left(\frac{u+2\pi}{4}\right) = \frac{1}{U}\cdot\frac{\cosh\left[\frac{z}{\sqrt{1-U^2}} + \frac{1}{2}(k_1+k_2)\right]}{\sinh\left[\frac{U.t}{\sqrt{1-U^2}} + \frac{1}{2}(k_1-k_2)\right]} \qquad (2.9)$$

$$\to \frac{1}{U}\left\{\exp\left[\frac{z-Ut}{\sqrt{1-U^2}} + k_2\right] + \exp\left[-\left(\frac{z+Ut}{\sqrt{1-U^2}} + k_1\right)\right]\right\}$$

$$\text{as } t \to +\infty$$

$$\to \frac{1}{U}\left\{-\exp\left[\frac{(z+Ut)}{\sqrt{1-U^2}} + k_1\right] - \exp\left[-\left(\frac{z-Ut}{\sqrt{1-U^2}}\right) - k_2\right]\right\}$$

$$\text{as } t \to -\infty .$$

Now the single soliton (2.3) is a kink of magnitude 2π from $u = 2n\pi$ at $z = -\infty$ to $u = 2(n+1)\pi$ at $z = +\infty$. From the above one sees that (2.9) at $t = -\infty$ represents a kink and an anti-kink moving in opposite directions. As $t \to +\infty$, it also represents a kink and an anti-kink but they have been displaced by an amount $2(1-U^2)^{-\frac{1}{2}}.\ln(U^{-1})$

Note that the constants a_i and k_i need not be real; all that is required is that u is real. So, for example, with $a_1 = a_2^* = \alpha + i\beta$, α and β real, in (2.5) one obtains the " breather " solution:

$$u = 4.\tan^{-1}\left\{\frac{\alpha}{\beta}\cdot\frac{\sin\left[\beta\left(x - \frac{y}{\alpha^2+\beta^2}\right) + \frac{1}{2}(k_1-k_2)\right]}{\cosh\left[\alpha\left(x + \frac{y}{\alpha^2+\beta^2}\right) + \frac{1}{2}(k_1+k_2)\right]}\right\}$$

One can go on, for example, to find solutions representing collisions of N solitons[1] and other breather-like solutions. This won't be pursued here though.

§3. Conservation Laws.

All soliton equations must conserve energy because the solitons retain their shape and do not " die out " over time. For example, the equation of conservation of energy for the sine-Gordon equation (2.1) in characteristic coordinates, is

$$\frac{\partial D}{\partial x} + \frac{\partial F}{\partial y} = 0 \qquad (3.1)$$

where $D = \frac{1}{2}u^2$, $F = \cos u$ (3.2)

The B.T. can now be used to generate an infinite number of conservation laws as follows. Let a in (2.2) be infinitesimal and take

$$u' = \sum_{i=0}^{\infty} u_i \cdot a^i \qquad (3.3)$$

Then (2.2b) gives on equating powers of a

$$\begin{aligned}
u_0 &= u \\
u_1 &= 2u_y \\
u_2 &= 2u_{yy} \\
u_3 &= 2u_{yyy} + \tfrac{1}{3}(u_y)^3 \\
u_4 &= 2u_{yyyy} + 2(u_y)^2.u_{yy} \\
u_5 &= 2u_{yyyyy} + 3(u_y)^2.u_{yyy} + 5u_y.(u_{yy})^2 + (3/20).(u_y)^5
\end{aligned} \qquad (3.4)$$

If one now inserts (3.3) into (3.1) and equates coefficients of powers of a then one an infinite set of conservation laws.

$$\frac{\partial D_i}{\partial x} + \frac{\partial F_i}{\partial y} = 0 \qquad i = 0, 1, \ldots \qquad (3.5)$$

With

$$D = \sum_{i=0}^{\infty} D_i \cdot a^i \quad , \quad F = \sum_{i=0}^{\infty} F_i \cdot a^i \qquad (3.6)$$

the first few are

$$\begin{aligned}
D_0 &= \tfrac{1}{2}(u_y)^2 \\
D_1 &= 2u_y.u_{yy} \\
D_2 &= 2u_y.u_{yyy} + 2(u_{yy})^2 \\
D_3 &= 2u_y.u_{yyyy} + 4u_{yy}.u_{yyy} + (u_y)^3.u_{yy}
\end{aligned} \qquad (3.7)$$

$$\begin{aligned}
F_0 &= \cos u \\
F_1 &= -2u_y.\sin u \\
F_2 &= -2(u_y)^2.\cos u - 2u_{yy}.\sin u \\
F_3 &= -2u_y.u_{yy}.\cos u - 2u_{yyy}.\sin u + (u_y)^3.\sin u
\end{aligned} \qquad (3.8)$$

Clearly the derivative of a conservation law is again a conservation law. From (3.7) one sees that D_1 and D_2 are just derivatives of D_0 but that D_3 is not. It will now be shown that the sequence of conservation laws given above contains an infinite number of laws, none of which is the derivative of the previous ones. Infact it will be shown that

D_0 and D_{2n+1}, $n > 0$, give rise to genuine conservation laws in that none is equal to a sum of derivatives of the others.[5]

Define the order of $(u_y)^{i_1} . (u_{yy})^{i_2} . \ . \ . \ . \ (u,_n)^{i_n}$ to be $i_1 + \ . \ . \ . + i_n$ where $u,_j$ is the jth derivative of u w.r.t. y. It is only necessary to look at second order terms in D to prove the result in the preceding paragraph since differentiation w.r.t. y leaves the order unchanged.

It is easy to show, by induction, that the first order term in $\frac{1}{n}u_n$ is $u,_n$. Then the second order term in D_n is

$$X_n = v_1 . v_{2n+2} + 2 \sum_{\ell=1}^{n} v_{\ell+1} . v_{2n+2-\ell} \quad , \ v_\ell = \frac{\partial^\ell u}{\partial y^\ell} \tag{3.9}$$

Then define

$$\begin{aligned} Y_n &= 4X_n - (X_{n-1})_{yy} = 3v_1 v_{2n+2} + 4v_2 v_{2n+1} + v_3 v_{2n} \ , \quad n > 1 \\ Y_1 &= X_1 - (X_0)_{yy} = -v_2 v_3 \quad , \quad Y_0 = X_0 = v_1 v_2 \end{aligned} \tag{3.10}$$

So it remains to prove that if

$$\sum_{k=0}^{n} a_k \left(\frac{\partial}{\partial y}\right)^{2(n-k)} Y_k = 0 \qquad n \geq 2. \tag{3.11}$$

then $a_0 = a_1 = \ . \ . \ . = a_n = 0$.

The coefficients of $v_1 v_6$, $v_2 v_5$ and $v_3 v_4$ in (3.11) for $n = 2$ give $a_0 + 3a_2 = 5a_0 + a_1 + 4a_2 = 10a_0 + 3a_1 + 3a_2 = 0$ which indeed imply $a_0 = a_1 = a_2 = 0$. Suppose the result (3.11) has been shown to be true for $n < N$ where $N > 1$. Clearly then the statement (3.11) is equivalent to the statement that if

$$\frac{\partial^2}{\partial y^2}\left[\sum_{i=0}^{N-1} b_i \, v_{i+1} v_{2N-i}\right] + b_N \left[3v_1 . v_{2N+2} + 4v_2 . v_{2N+1} + v_3 . v_{2N}\right] = 0 \tag{3.12}$$

then $b_0 = b_1 = \ . \ . \ . = b_N$.

The coefficients of $v_{i+1} v_{2N+2-i}$, $i = 1, \ . \ . \ . \ ,N$, in (3.12) give

$$\begin{aligned} &b_0 + 3b_N = 0 \\ &2b_0 + b_1 + 4b_N = 0 \\ &b_0 + 2b_1 + b_2 + b_N = 0 \\ &b_i + 2b_{i+1} + b_{i+2} = 0 \ , \ i = 1, \ . \ . \ . \ ,N-3 \quad \text{(not required for } N = 3 \text{)} \\ &b_{N-2} + 3b_{N-1} = 0 \end{aligned} \tag{3.13}$$

Take (a)+(c)-(b) in (3.13) to give $b_1 + b_2 = 0$. Then from (d) in (3.13) one has by induction that $b_i + b_{i+1} = 0$, $i = 1, \ . \ . \ . \ ,N-2$. In particular $b_{N-2} + b_{N-1} = 0$. Then (e) gives $b_{N-1} = 0$. Then one has $b_0 = b_1 = \ldots = b_N$ and the result is proved.

§4. Inverse Scattering.

The inverse scattering method allows one to solve the initial value problem for the given soliton equation, (1.1) or (2.1).[6] This method will be introduced via the B.T. In (2.2) replace u' by $\Gamma = \tan\frac{1}{2}(u'+u)$:

$$\begin{aligned} \Gamma_x &= \tfrac{1}{2}.u_x.(1+\Gamma^2) + a.\Gamma \\ \Gamma_y &= \Gamma.a^{-1}.\cos u + \tfrac{1}{2}(\Gamma^2-1).a^{-1}.\sin u \end{aligned} \tag{4.1}$$

The equation $\Gamma_x + 2P.\Gamma + Q.\Gamma^2 + R = 0$ is equivalent (though not uniquely) to

$$\begin{aligned} (w_1)_x + P.w_1 &= -R.w_2 \\ (w_2)_x - P.w_2 &= Q.w_1 \end{aligned} \qquad \text{where } \Gamma = (w_2)(w_1)^{-1}$$

So equation (4.1) with $a = 2i\zeta$ and after the change of variables (2.7) gives

$$\begin{aligned} (w_1)_z &= (-\tfrac{1}{2}i\zeta + i(8\zeta)^{-1}.\cos u).w_1 + (i(8\zeta)^{-1}.\sin u - \tfrac{1}{4}(u_z+u_t)).w_2 \\ (w_2)_z &= (i(8\zeta)^{-1}.\sin u + \tfrac{1}{4}(u_z+u_t)).w_1 + (\tfrac{1}{2}i\zeta - i(8\zeta)^{-1}.\cos u).w_2 \end{aligned} \tag{4.2}$$

$$\begin{aligned} (w_1)_t &= (-\tfrac{1}{2}i\zeta - i(8\zeta)^{-1}.\cos u).w_1 - (i(8\zeta)^{-1}.\sin u + \tfrac{1}{4}(u_z+u_t)).w_2 \\ (w_2)_t &= (-i(8\zeta)^{-1}.\sin u + \tfrac{1}{4}(u_z+u_t)).w_1 + (\tfrac{1}{2}i\zeta + i(8\zeta)^{-1}.\cos u).w_2 \end{aligned} \tag{4.3}$$

Note that if u satisfies (1.1) then (4.2) and (4.3) are consistent.

It is desired to solve (1.1) given u and u_t at $t = 0$. Only solutions of (1.1) which tend to zero (or a multiple of 2π) as $|z|$ tends to infinity will be considered. The method is then as follows.

Define functions v and w which are solutions of (4.2) at $t = 0$ and which have the asymptotic form

$$\begin{aligned} v &\to \begin{pmatrix}1\\0\end{pmatrix}.\exp\left\{-\frac{i}{2}.\left(\zeta - \frac{1}{4\zeta}\right).z\right\} \quad \text{as } z \to -\infty \\ w &\to \begin{pmatrix}0\\1\end{pmatrix}.\exp\left\{\frac{i}{2}\left(\zeta - \frac{1}{4\zeta}\right).z\right\} \quad \text{as } z \to +\infty \end{aligned} \tag{4.4}$$

Both v and w exist and are unique in the upper half ζ-plane.

If $w = (w_1, w_2)$ is a solution of (4.2) then $\bar{w} = (w_2{}^*, -w_1{}^*)$ is a linearly independent solution. (* denotes complex conjugate). So the pair of solutions w and $\bar{w}$ form a complete system of solutions and one may therefore write for any real ζ

$$v = a(\zeta).\bar{w} + b(\zeta).w \tag{4.5}$$

Now $a(\zeta)$ can be analytically continued to the upper half-plane and, in particular, the zeros ζ_j $(j = 1,\ldots,N)$ of $a(\zeta)$ in the upper half-plane are at the discrete eigenvalues of (4.2). At these values

$$v(z,\zeta_j) = c_j.w(z,\zeta_j) \tag{4.6}$$

Since (4.2) and (4.3) are consistent one may define $v(z,t)$, at fixed z, to be the solution of (4.3) which equals $v(z)$ defined by (4.4) at $t = 0$. Then v must satisfy (4.2) and (4.3) for all z and t. From (4.2) and (4.3) as $z \to +\infty$ since $u \to 0$ one has

$$v \to \begin{pmatrix} A_1 . \exp\left\{ -\frac{i}{2}\left(\zeta - \frac{1}{4\zeta}\right) z - \frac{i}{2}\left(\zeta + \frac{1}{4\zeta}\right) t \right\} \\ A_2 . \exp\left\{ \frac{i}{2}\left(\zeta - \frac{1}{4\zeta}\right) z + \frac{i}{2}\left(\zeta + \frac{1}{4\zeta}\right) t \right\} \end{pmatrix} \tag{4.7}$$

where A_1 and A_2 are constants. Also as $z \to +\infty$

$$w \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \exp\left\{ \frac{i}{2}\left(\zeta - \frac{1}{4\zeta}\right) z + \frac{i}{2}\left(\zeta + \frac{1}{4\zeta}\right) t \right\} \tag{4.8}$$

At fixed t, w and $\bar{w}$ are linearly independent solutions of (4.2) so one may define $a(\zeta,t)$ and $b(\zeta,t)$ by (4.5). Then as $z \to \infty$ one has from (4.5), (4.7) and (4.8) that $a(\zeta)$, $b(\zeta)$, ζ_j and c_j are all independent of t.

The solution to the initial value problem is then

$$u(z,t) = u(z,0) + \int_{\frac{1}{2}(z-t)}^{\frac{1}{2}(z+t)} K(y,y;t) . dy \tag{4.9}$$

where $K(x,y;t)$ is the solution of the integral equation

$$K(x,y) = B^*(x+y) - \int_x^\infty \int_x^\infty B^*(y+z) . B(\lambda+z) . K(x,\lambda)\, dz\, d\lambda$$

$$B(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b(\xi)}{a(\xi)} . \exp\left\{ i\xi x + \frac{it}{2\xi}\right\} . d\xi$$

$$- i \sum_{j=1}^{N} c_j . \exp\left\{ i\zeta_j x + \frac{it}{2\zeta_j}\right\}$$

§5. Summary.

This thesis is concerned with looking for B.T. for equations other than those for which B.T. are known. Here B.T. is used in the widest possible sense, as in section 1; the B.T. may involve integrals or derivatives of the two connected solutions, or it may involve additional auxilary dependent variables.

The original motivation for this was to look for suitable candidates for field theories. This was done from the viewpoint of B.T. because there is a well-defined procedure[7] for determining whether or not a given equation has a B.T. of the form under consideration. It is also known that all equations found to possess solitons also possess a B.T.[8] (By this I mean that the analytic as opposed to numerical solution is known). Finally, as has been shown in this chapter for S.G. , all the soliton properties follow once the B.T. is known.

It becomes immediately that only certain, very particular, equations possess B.T, One would like to understand why this is so. Pseudo-potentials seem to be a possible way of attacking this question.[9]

It is known from numerical studies[10] that certain equations, for example the double sine-Gordon equation

$$u_{xy} = \sin u + \tfrac{1}{2}\sin \tfrac{1}{2}u \tag{5.1}$$

possess solitary wave solutions which seem to pass through each other on collision - the soliton property. This result has taken over as the main motivation for this thesis and accounts for the final three chapters there-of.

The effect of explicit independent variable dependence is considered in chapter 2. In particular, one looks for B.T. of the form

$$\begin{aligned} u'_x &= P(x,y,u,u',u_x,u_y) \\ u'_y &= Q(x,y,u,u',u_x,u_y) \end{aligned} \tag{5.2}$$

when both u and u' satisfy

$$u_{xy} = A(x,y,u).u_x + B(x,y,u).u_y + C(x,y,u) \tag{5.3}$$

The case $A = B = 0$ has already appeared[11] and so the proof will not be repeated here. Infact only the case $P_{pp} \neq 0$ $(p = u_x)$ will be considered in detail. Solutions are found in this case in contrast to the situation when $A = B = 0$. The only B.T. that I have been able to find and which are not already known are contained in chapter 2.

Chapter 2 also contains some consideration of pseudopotentials

$$\begin{aligned} u'_x &= P(u,u',u_x,u_y) \\ u_y' &= Q(u,u',u_x,u_y) \end{aligned} \tag{5.4}$$

Here it is demanded that u satisfies

$$u_{xy} = F(u) \tag{5.5}$$

but u' is not made to satisfy any particular equation. Again this work will appear elsewhere[12] and so will not be repeated here. However some consideration is made of the case when u' does not satisfy a second order equation. Finally in chapter 2 all pseudopotentials of the form

$$\begin{aligned} u'_x &= P_0(u,u') + P_1(u,u').u_x \\ u'_y &= Q_0(u,u') + Q_1(u,u').u_y \end{aligned} \tag{5.6}$$

are found for equations of the form

$$u_{xy} = A(u).u_x + B(u).u_y + C(u) \tag{5.7}$$

If one wishes to find B.T. for (5.7) one still has quite a bit of work to do, but at least one can now say which equations of the form (5.7) cannot have B.T. of the form (5.4).

Neither of the above generalizations seems to leed far so for the rest of this thesis it will be assumed that there is no explicit independent variable dependence and also that u and u' satisfy the same equation. One possible generalization one could try is to consider that

$$(u_i)_{xy} = F_i(u_j) \qquad , i = 1,\ldots\ldots,N \tag{5.8}$$

has a B.T. of the form

$$\begin{aligned} (u_i')_x &= P_i(u_j,u_j',(u_j)_x,(u_j)_y) \\ (u_i')_y &= Q_i(u_j,u_j',(u_j)_x,(u_j)_y) \end{aligned} \qquad i = 1,\ldots,N \tag{5.9}$$

If $N = 2$ then one may replace (5.8) by

$$u_{xy} = F(u,u^*) \tag{5.9}$$

where u and F are complex. In this case I have shown that provided (5.9) cannot be written in the form

$$\begin{aligned} (u_1)_{xy} &= F(u_1) \\ (u_2)_{xy} &= G(u_1,u_2) \end{aligned} \tag{5.10}$$

then F must be of the form

$$F(u,u^*) = \sum_{i,j=1}^{4} A_{ij} . e^{\lambda_i . u + \lambda_j^* . u^*} \tag{5.11}$$

where A_{ij} and λ_j are constants. Because of the incomplete nature of the results and also because of (5.1)[10] the most interesting case (5.10) has been excluded I will not give the proof of the above result here. (The case (5.10) was excluded because if one wants to use (5.9) as a model field theory one would like, for example, to have rotational symmetry which excludes (5.10).)

If additional dependent variables are excluded then all the B.T. can depend on is derivatives and integrals of u and u'. Define

$$p_{nm} = \int dy_1 . dy_2 \dots dy_m . \frac{\partial^n u}{\partial x^n}$$

$$q_{nm} = \int dx_1 \dots dx_m . \frac{\partial^n u}{\partial y^n} \qquad (5.12)$$

$$u_{nm} = \int dx_1 \dots dx_n . dy_1 \dots dy_m . u$$

where n and m are non-negative integers. The most general B.T. would then involve some finite number of terms from u_{nm}, p_{nm}, q_{nm}, u'_{nm}, p'_{nm} and q'_{nm}. Not only are the equations that one must satisfy then very difficult to satisfy but the B.T. when obtained would appear to be of not much use in solving (5.5). My only attempt at including integrals in the B.T. is given at the end of chapter 2. The remaining chapters then only involve B.T. with derivatives of u and u'.

The easiest way to ensure that the B.T. is useful in solving the original equation is to allow only first derivatives of u'. That is take the B.T. to be

$$\begin{aligned} u'_x &= P(u,u';p_1,\dots,p_N;q_1,\dots,q_M) \\ u'_y &= Q(u,u';p_1,\dots,p_N;q_1,\dots,q_M) \end{aligned} \qquad (5.13)$$

where p_n is the nth derivative of u w.r.t. x and q_n is the nth derivative of u w.r.t. y. Note that as u satisfies (5.5) one not consider the mixed derivatives further.

There does not appear to be B.T. of the form (5.13) so one must try something more complicated. The next thing one can try is to allow x derivatives of u' greater than one but to restrict y derivatives of u' in the B.T. to one. This is done in chapters 4 and 5. In chapter 4, the B.T. is taken to be of the form

$$\begin{aligned} u'_{xx} &= P(u,u',u_x,u'_x,u_{xx},u_y) \\ u'_y &= Q(u,u',u_x,u'_x,u_{xx},u_y) \end{aligned} \qquad (5.14)$$

Without further simplification of the problem it rapidly becomes very difficult to solve the consistency conditions. Consequently in the final chapter u and u' are taken to satisfy

$$u_{xy} = \sin u + A.\sin \lambda.u \qquad (5.15)$$

and in analogy with the B.T. for S.G. , (2.2) , the B.T. is taken to be

$$\begin{aligned} p_n' + p_n &= P(p_0,p_1,\dots,p_{n-1};p_0',p_1',\dots,p_{n-1}') \\ u_y' - u_y &= Q(p_0,\dots,p_{n-1};p_0',\dots,p_{n-1}') \end{aligned} \tag{5.16}$$

Unfortunately no B.T. are found in these cases, although something seems to be going on when, for (5.13), $F(u) = A.\exp u + B.\exp -\frac{1}{2}u$. Further this seems to be a property associated with the functions

$$f_n(f_1, \dots, f_{n-1}) = e^{-\lambda\varphi}.\left(\frac{\partial}{\partial x}\right)^{n-1}.e^{\lambda\varphi}$$

rather than with the special form (5.13). That is, it seems to me that I have not chosen the right form for the u' dependence in the B.T. rather than there being no B.T. with higher derivatives for (5.5). One thing that does seem worth trying is to take the form

$$\begin{aligned} u_{xx}' + u_{xx} &= P(u,u',u_x,u_x',u_y,u_y') \\ u_{yy}' - u_{yy} &= Q(u,u',u_x,u_x',u_y,u_y') \end{aligned} \tag{5.17}$$

I considered the other forms first because it seemed that if they existed then they would be of more use in solving (5.5) than (5.17) would be. The only other possible candidate for a B.T. for (5.1) would seem to be (5.10).

CHAPTER 2.

§1. Introduction.

The effect of explicit independent variables, of not specifying the equation u' satisfies and of integrals in the B.T. is considered in this chapter. It also contains the only examples of B.T. which I have been able to find and which are not, I believe, known already.

Consider the equation

$$u_{xy} = F(x,y,u,u_x,u_y,u_{xx},u_{yy}) \tag{1.1}$$

Suppose this equation has a B.T. of the form

$$\begin{aligned} u'_x &= P(x,y,u,u',u_x,u_y) \\ u'_y &= Q(x,y,u,u',u_x,u_y) \end{aligned} \tag{1.2}$$

This last statement is to be interpreted to mean that for every solution u of (1.1), equation (1.2) has a solution u' which also satisfies (1.1).

It is assumed throughout this thesis that all functions are differentiable as many times as required.

Differentiate (1.2a) w.r.t. y and (1.2b) w.r.t. x and use (1.1) to obtain the equations which must be satisfied if the B.T. is to exist:

$$\begin{aligned} &F(x,y,u',P,Q,R,T) \\ &= \frac{\partial P}{\partial y} + \frac{\partial P}{\partial u}\cdot q + \frac{\partial P}{\partial u'}\cdot Q + \frac{\partial P}{\partial p}\cdot F(x,y,u,p,q,r,t) + \frac{\partial P}{\partial q}\cdot t \\ &= \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial u}\cdot p + \frac{\partial Q}{\partial u'}\cdot P + \frac{\partial Q}{\partial p}\cdot r + \frac{\partial Q}{\partial q}\cdot F(x,y,u,p,q,r,t) \end{aligned} \tag{1.3}$$

where $p = u_x$, $q = u_y$, $r = u_{xx}$, $t = u_{yy}$ and

$$R = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial u}\cdot p + \frac{\partial P}{\partial u'}\cdot P + \frac{\partial P}{\partial p}\cdot r + \frac{\partial P}{\partial q}\cdot F(x,y,u,p,q,r,t)$$

$$T = \frac{\partial Q}{\partial y} + \frac{\partial Q}{\partial u}\cdot q + \frac{\partial Q}{\partial u'}\cdot P + \frac{\partial Q}{\partial p}\cdot F(x,y,u,p,q,r,t) + \frac{\partial P}{\partial q}\cdot t$$

Now u is any solution of (1.1) so at any point (x,y) one may choose u, u_x, u_y, u_{xx}, u_{yy} arbitarily i.e. in (1.3) one may use x, y, u, p, q, r and t as independent variables. It is further assumed that, at (x,y), one may choose u' independently of u and its derivatives i.e. it is assumed that

$$u' \neq f(x,y,u,u_x,u_y,u_{xx},u_{yy}) \tag{1.4}$$

This means that (1.3) is now to be considered as two equations for the three unknowns $F(x,y,u,p,q,r,t)$, $P(x,y,u,u',p,q)$ and $Q(x,y,u,u',p,q)$ where x, y, u, u', p, q, r and t are all to be considered as independent variables. This sort of consideration may be carried out in all cases considered; the details however will not, in future, be spelt out.

It is equations like (1.3) which this thesis solves. It turns out that the number of B.T. is rather small. This makes it rather hard to see what is actually "going on."

From now on, equality will be used in the sense of "identically equal to." This means that if one has two functions A and B which are functions of some variable set Ω and satisfy $A.B = 0$ then one can deduce that $A = 0$ or $B = 0$ (or both). This is certainly true for all Ω near some Ω_0 and one could carry the analysis through allowing Ω near Ω_0 only. At the end one could then see if one could "patch together" the various pieces. For example, the equations $u_{xy} = \sin u$ and $u_{xy} = 0$ have B.T. but does the equation

$$u_{xy} = \begin{cases} 0 & u < 0 \\ \sin u & u > 0 \end{cases}$$

have a B.T. ? One can certainly satisfy (1.3) locally in this case but one runs into problems in trying to make (1.2) hold for all x and y. This problem will not be considered further. It will be assumed from now on that if an equation holds in some region then it holds everywhere. This is true, for example, if all functions under consideration are analytic.

The next section lists some trivial B.T. and explains why the case of F linear is always excluded from consideration. It has been shown elsewhere that if the F above depends on x , y and u only then, for a B.T. to exist, one must be able to convert (1.1) into

$$u_{xy} = H(u) \qquad , \qquad H''(u) = k.H(u) \tag{1.5}$$

by a change of scale and / or a displacement of the dependent variable. Sections 3 to 6 deal with the case when F is linear in both p and q (but not u) and when P is not linear in p. Apart from an obvious B.T. the result is that (1.1) must be reducible to

$$u_{xy} = A(u).u_x \qquad , \qquad A''(u) = K.A(u) \qquad , \quad K \text{ constant}$$

by a change of dependent and independent variables. So it appears that functions F satisfying $F''(u) = K.F(u)$, K constant, have some very special significance for the existence of B.T.

Sections 7 and 8 are then a consideration of pseudopotentials of the form (5.4) of the previous chapter, where u' does not satisfy a second order equation. Most of the discussion centers around the possibility of u' satisfying a Lorentz invarient equation. All pseudopotentials of the form (5.4) of chapter 1 are found elsewhere[12] and so the proof will not be repeated in this thesis. Also consideration of the case when u' satisfies a second order equation is not repeated here.

Sections 9 and 10 cover the problem given by (5.6) and (5.7) of the previous chapter. The final sections deals with my only attempt at including integrals in the B.T.

§2. Trivial Solutions.

First note that if F in (1.1) is linear in the dependent variable i.e. if

$$F = A_0 + A_1.u + A_2.u_x + A_3.u_y + A_4.u_{xx} + A_5.u_{yy} \tag{2.1}$$

where $A_0,\ldots,A_5$ are functions of x and y only, then the equation (1.1) trivially has a B.T. as follows. If v(x,y) is any solution of

$$v_{xy} = A_1.v + A_2.v_x + A_3.v_y + A_4.v_{xx} + A_5.v_{yy} \tag{2.2}$$

then, for all solutions u of (1.1) and all constants K, u' = u + K.v also satisfies (1.1). Dividing u' = u + K.v by v and differentiating gives the B.T. :

$$\begin{aligned} u'_x &= u_x + v^{-1}.v_x.(u'-u) \\ u'_y &= u_y + v^{-1}.v_y.(u'-u) \end{aligned} \tag{2.3}$$

So for the rest of this thesis it will be assumed that F is not linear in the dependent variable.

Now note that if (1.1) has the property that there exists a function f(x,y,u,K) such that for all solutions u of (1.1) and for some arbitrary constant K, u' = f(x,y,u,K) is also a solution of (1.1) then (1.1) has a B.T. exactly as in the linear case. As a not so transparent example consider the following. Let $A_0, \ldots, A_6$ and K be functions of x and y and let g be a function of y only. Further suppose that $A_0, \ldots, A_6$, K and g satisfy

$$\begin{aligned} &A_5.(g''(y) + (g')^2) + A_1 + A_3.g'(y) = 0 \\ &K_{xy} = A_1.K + g'(y).K_x + A_3.K_y + A_4.K_{xx} + A_5.K_{yy} \\ &K_{xx} \neq 0 \end{aligned} \tag{2.4}$$

Then the equation

$$u_{xy} = A_0 + A_1.u + g'(y).u_x + A_3.u_y + A_4.u_{xx} + A_5.u_{yy} + G(u_{xx}) \tag{2.5}$$

possesses a B.T. if G(r) is any function of period K_{xx} e.g. $G(r) = \sin(2\pi r(K_{xx})^{-1})$. The B.T. is infact

$$\frac{\partial u'}{\partial x} = e^{g(y)}\cdot v\left[e^{-g(y)}\cdot(u'-u-K), x\right] + \frac{\partial K}{\partial x} + \frac{\partial u}{\partial x}$$
$$\frac{\partial u'}{\partial y} = g'(y)\cdot(u'-u-K) + \frac{\partial K}{\partial y} + \frac{\partial u}{\partial y} \tag{2.6}$$

where $v(z,x)$ is defined implicitly by

$$z = x.v(z,x) + f(v(z,x)) \tag{2.7}$$

and f is an arbitary function.

Note that the general solution of (2.6) is

$$u' = u + K + (\,a.x + f(a)\,).\exp g(y)$$

where a is an arbitary constant and f is the function appearing in (2.7).

Infact if $F_{rr} \neq 0$ then it is not too difficult to show that the B.T. must be of this trivial type i.e. of the form

$$\frac{\partial}{\partial x}\left[A(x,y,u,u')\right] = 0$$
$$\frac{\partial}{\partial y}\left[A(x,y,u,u')\right] = 0$$

for some function A. This is most easily seen by looking for a pseudopotential where one has the freedom to replace u' by any function of x,y,u and u'.

§3.Quasilinear Case.

Consideration will now be restricted to equations of the form (1.1) but with F linear in p and q. That is consider

$$u_{xy} = A(x,y,u).u_x.u_y + B(x,y,u).u_x + C(x,y,u).u_y + D(x,y,u) \tag{3.1}$$

Make the change of dependent variable $v = g(x,y,u)$ where

$$g(x,y,z) = \int_0^z \exp\left\{-\int_0^\tau A(x,y,\varphi)\cdot d\varphi\right\}\cdot d\tau$$

One then sees that v satisfies an equation of the form (3.1) but with $A = 0$. (Note that $g_z > 0$ so one may always solve $v = g(x,y,u)$ for u.) Clearly if the original equation had a B.T. then the new equation would have one also. Hence, in place of (3.1) one need only consider

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial A}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial B}{\partial u}\cdot\frac{\partial u}{\partial y} + C = F(x,y,u,p,q) \tag{3.2}$$

where A , B and C are functions of x , y and u only.

Differentiate the first of (1.3) w.r.t. t to obtain $P_q = 0$. Similarly $Q_p = 0$. So one has that

$$P = P(x,y,u,u',p) \qquad Q = Q(x,y,u,u',q) \tag{3.3}$$

Only the case

$$P_{pp} \neq 0 \tag{3.4}$$

will be considered in the next few sections. So differentiate the second of (1.3) twice w.r.t. p and use (3.4) and F given by (3.2) .

$$A_{u'} = Q_{u'} \tag{3.5}$$

The coefficients of p^0 and p^1 in (1.3b) then give

$$0 = \frac{\partial Q}{\partial u} + \frac{\partial A}{\partial u} \cdot \frac{\partial Q}{\partial q} \tag{3.6}$$

$$\frac{\partial B}{\partial u'} \cdot Q + C(x,y,u') = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial q} \cdot \left[\frac{\partial B}{\partial q} \cdot q + C(x,y,u) \right] \tag{3.7}$$

The rest of this section is concerned with the case

$$B_{uu} \neq 0 \tag{3.8}$$

Differentiate (3.7) w.r.t. u' and use (3.5). Then diff. w.r.t. q

$$Q_q = 0$$

Then (3.6) gives $Q_u = 0$. Then from (3.5) one may without loss of generality (w.l.o.g.) take

$$Q = A(x,y,u') \tag{3.9}$$

Then (3.7) gives

$$B_u . A + C = A_{xu} \tag{3.10}$$

Substitute (3.9) into (1.3a) and equate coefficients of q^0 and q^1

$$0 = \frac{\partial P}{\partial u} + \frac{\partial B}{\partial u} \cdot \frac{\partial P}{\partial p} \tag{3.11}$$

$$\frac{\partial A}{\partial u'} \cdot P + \frac{\partial B}{\partial q'} \cdot A + C = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial u'} \cdot A(x,y,u') + \frac{\partial P}{\partial p} \cdot \left[\frac{\partial A}{\partial u} \cdot p + C(x,y,u) \right] \tag{3.12}$$

where A and C on the L.H.S. of (3.12) are functions of x , y and u'.

Add B_u times the p derivative of (3.12) to the u derivative of (3.12) and use $P_p \neq 0$ .The coefficient of p in this is

$$A_{uu} = 0 \tag{3.13}$$

Then one can see that the change of variable $v = a(x,y).u$ in (3.2) and (1.2) where $a_y + a.A_u = 0$ enables one w.l.o.g. to take

$$A_u = 0 \tag{3.14}$$

Further by the change of variable $v = u + a(x,y)$ one sees that w.l.o.g.

$$Q = 0 \tag{3.15}$$

Then (3.9) and (3.10) imply

$$A = C = 0 \tag{3.16}$$

Now (3.12) is

$$P_y = 0 \tag{3.17}$$

Then diff. (3.11) w.r.t. y

$$B_{yu} = 0 \tag{3.18}$$

Then because of the way B is defined in (3.2) and because of (3.17) one may w.l.o.g. choose $B = B(x,u)$.

Putting all this together one has that the equation

$$u_{xy} = A(x,u).u_y \tag{3.19}$$

has the B.T.

$$\begin{aligned} u'_x &= P(x,u',p-g) \\ u'_y &= \theta \end{aligned} \tag{3.20}$$

where g is a function of x and u only and satisfies $g_u = A(x,u)$.

The B.T. (3.20) is not interesting though because from (3.19) one has that p-g equals a function of x only. Also (3.20b) gives that u' is a function of x only. Equation (3.20a) is then only a relation between these functions of x.

§4. For $B_{uu} = 0$.

From the previous section one need only consider the case $B_{uu} = 0$. Then by a change of variable $v = a(x,y).u$ one sees that w.l.o.g. $B_u = 0$. So from the way B is defined in (3.2) one may take

$$B = 0 \tag{4.1}$$

Diff. (1.3a) twice w.r.t. q

$$P_{u'} \cdot Q_{qq} = 0 \tag{4.2}$$

If $P_{u'}$ is zero then the R.H.S. of (1.3a) is independent of u' which means since $P_{pp} \neq 0$ that both $A_{u'}$ and C are independent of u'. i.e. (3.2) is linear. So $P_{u'} \neq 0$ and (4.2) then gives

$$Q = Q_0(x,y,u,u') + Q_1(x,y,u,u').q \tag{4.3}$$

Equation (1.3b) then gives

$$Q_1 = Q_1(y) \tag{4.4}$$

$$Q_0 = A(x,y,u') - Q_1.A(x,y,u) + k(x,y) \tag{4.5}$$

Inserting these into (1.3b) gives w.l.o.g. since A is only defined up to a function of x and y that

$$\begin{aligned} C &= A_x \\ k &= k(y) \end{aligned} \tag{4.6}$$

Substitute Q given by the above into (1.3a) diff. w.r.t. p to obtain

$$A_{u'} - A_u = V_y + V_{u'}.(A(x,y,u')-Q_1.A(x,y,u)+k(y)) + V_p.(A_u.p+C(x,y,u)) \tag{4.7}$$

$$0 = V_u + Q_1.V_{u'} \tag{4.8}$$

where $\qquad V = \ln P_p \qquad$ (4.9)

Also (3.4) implies $\qquad V_p \neq 0 \qquad$ (4.10)

It is these equations which will be solved in the next three sections. The result is that by a change of variables equation (3.1) reduces to

$$u_{xy} = A(u).u_x \tag{4.11}$$

where $\qquad A''(u) = K.A(u) \qquad$ (4.12)

for some constant K.

Infact all equations satisfying (4.11) and (4.12) have a non-trivial B.T. In the case $A(u) = \cos u$ and $Q_1 = -1$ one has that

$$u_{xy} = u_x.\cos u \tag{4.13}$$

has a B.T.

$$\left.\begin{aligned} u'_x \,.\, u_x &= 1 - \cos(u'+u) \\ u'_y + u_y &= \sin u + \sin u' \end{aligned}\right\} \tag{4.14}$$

So for the problem given by (3.1) and (1.2) the inclusion of explicit x and y dependence, for $A = B = C = 0$ or $P_{pp} \neq 0$, gives no B.T. which are not obtainable by a change of variable from the problem without explicit x and y dependence. Consequently the problems $A = B = C = 0$ and $P_{pp} = 0$ in (3.1) and (1.2) are the only ones considered which involve explicit x and y dependence.

Now (4.7) to (4.10) will be solved.

For the rest of this section assume

$$A_{uuu} = c.A_{uu} \tag{4.15}$$

in (4.7) where c is a function of x and y only. It will then be shown that by a change of variable one may choose

$$A = \exp u \quad \text{or} \quad A = \tfrac{1}{2}u^2 \tag{4.16}$$

The result (4.11) and (4.12) is then proved in the case (4.15).

<u>Case 1.</u>

$$Q_1 = +1 \quad , \quad c \neq 0 \tag{4.17}$$

Define $\qquad v = u' - Q_1.u \qquad$ (4.18)

and use v instead of u'.

From (4.15) $\qquad A = a_0 + a_1.u + a_2.\exp c.u \qquad$ (4.19)

where a_0, a_1 and a_2 are functions of x and y only.

Then (4.7) and (4.8) give on using (4.6) and equating coefficients of $\exp cu$, $\exp Q_1cu'$, $u.\exp cu$, u and u^0

$$V_u = 0 \tag{4.20}$$

$$c_x = 0 \tag{4.21}$$

$$c = V_v \tag{4.22}$$

$$c = Q_1 \cdot V_v - V_p \cdot (c.p + (\ln a_2)_x) \tag{4.23}$$

$$0 = -Q_1'(v) \cdot V_v + (a_1)_x \cdot V_p \tag{4.24}$$

$$0 = V_y + V_v \cdot (a_0 \cdot (1-Q_1) + a_1 \cdot v + k) + V_p \cdot (a_1 \cdot p + (a_0)_x) \tag{4.25}$$

By replacing u by $c^{-1}.u$ (using (4.21)) in (3.2) and (1.2) one sees that w.l.o.g.

$$c = 1 \tag{4.26}$$

Then (4.22) and (4.23) give

$$V = v + (Q_1-1).\ln(p+(\ln a_2)_x) + d(x,y) \tag{4.27}$$

Substitute (4.27) into (4.25). The coefficient of $\ln(p + (\ln a_2)_x$ is

$$Q_1'(v) = 0 \tag{4.28}$$

Coeff. of v is $a_1 = 0$ (4.29)

Coeff. of $(p + (\ln a_2)_x)$ is $(\ln a_2)_{xy} + (a_0)_x = 0$ (4.30)

Coeff. of v^0 is $k + d_y = 0$ (4.31)

In (2.2) set $w = u + \ln a_2$ to obtain $w_{xy} = w_x . \exp w$. The result (4.16) is then proved in this case.

Case 2.

$$Q_1 = 1 \quad , \quad c \neq 0 \tag{4.32}$$

Set $v = u' - u$ (4.33)

and use v instead of u'. Then (4.8) gives

$$V_u = 0 \tag{4.34}$$

Then (4.7) using (4.6) and (4.34) gives

$$c_x = 0 \quad , \quad (a_1) = 0 \tag{4.35}$$

$$c.(e^{cv} - 1) = (e^{cv} - 1).V_v + V_p.(c.p + (\ln a_2)_x) \tag{4.36}$$

$$0 = V_y + (a_1.v + k).V_v + (a_1.p + (a_0)_x).V_p \tag{4.37}$$

From (4.35) one has w.l.o.g. that $c = 1$ (4.38)

The consistency condition on (4.36) and (4.37) is

$$0 = a_1 \cdot \frac{\partial V}{\partial v} + \left\{ \frac{\left[\frac{a_1}{a_2} \cdot \frac{\partial a_2}{\partial x} - \frac{\partial a_0}{\partial x} - \frac{\partial^2}{\partial x \partial y}(\ln a_2)\right]}{(e^v - 1)} - \frac{e^v(a_1 v + k)\left(p + \frac{1}{a_2} \cdot \frac{\partial a_2}{\partial x}\right)}{(e^v - 1)^2} \right\} \cdot \frac{\partial V}{\partial p} \tag{4.39}$$

Consistency condition on $(e^v-1)^{-1}.(4.36)$ and (4.39) is (p component)

$$\frac{\partial}{\partial v}\left\{-\frac{e^{v}(a_1 v + k)}{(e^{v}-1)^2}\right\} = a_1 \cdot \frac{\partial}{\partial v} \cdot \left\{\frac{1}{e^{v}-1}\right\} \tag{4.40}$$

Coefficient of v in (4.40) gives $a_1 = 0$ (4.40)

Then (4.40) gives $k = 0$ (4.41)

Then (4.39) gives $(a_0)_x + (\ln a_2)_{xy} = 0$ (4.42)

Then as in case 1 equation (4.16) is proved.

Case 3.

$$c = 0 \tag{4.43}$$

From (4.43) and (4.15) one may take

$$A = a_0 + a_1.u + a_2.u^2 \tag{4.44}$$

Define $v = u' - Q_1.u$ (4.45)

and use v instead of u'. Then (4.8) gives

$$V_u = 0 \tag{4.46}$$

Then (4.7) gives

$$0 = (Q_1^2 - Q_1).V_v + (\ln a_2)_x \cdot V_p \tag{4.47}$$

$$(1 - Q_1) = V_v.(Q_1.v - \tfrac{1}{2}.a_2^{-1}.Q_1'(y)) + V_p.(p + \tfrac{1}{2}.a_2^{-1}.(a_1)_x) \tag{4.48}$$

$$2.a_2.v = V_y + (a_0.(1-Q_1) + a_1.v + a_2.v^2 + k).V_v + (a_1.p + (a_0)_x).V_p \tag{4.49}$$

Note that if $a_2 = 0$ then (3.2) is linear. So

$$a_2 \neq 0 \tag{4.50}$$

Consistency condition on (4.47) and (4.48) after using (4.47) to eliminate derivatives of V gives by (4.10)

$$\begin{vmatrix} Q_1^2 - Q_1 & \frac{1}{a_2}\cdot\frac{\partial a_2}{\partial x} \\ Q_1(Q_1^2 - Q_1) & \frac{1}{a_2}\cdot\frac{\partial a_2}{\partial x} \end{vmatrix} = 0 \tag{4.51}$$

If $(a_2)_x \neq 0$ then (4.51) gives $Q_1^2 = Q_1$. But then (4.47) gives $(a_2)_x . V_p = 0$. So one must have

$$(a_2)_x = 0 \tag{4.52}$$

If $Q_1^2 \neq Q_1$ then (4.47) gives $V_v = 0$ and then the coefficient of v in (4.49) is $2a_2 = 0$ which contradicts (4.50). Hence

$$Q_1^2 = Q_1 \tag{4.53}$$

If $Q_1 = 0$ then one may integrate (4.48) to find the p dependence of V. If one then substitutes this expression for V into (4.49).$(p+\frac{1}{2}.a_2^{-1}.(a_1)_x)$ and equates coefficients of p and p^0 then one readily obtains that (4.16) must hold. So take

$$Q_1 = 1 \qquad (4.54)$$

Consistency on (4.47) and (4.48) gives

$$2a_2 \cdot v = (a_2 \cdot v^2 - k) \cdot \frac{\partial V}{\partial v} - \left\{ \frac{\partial a_0}{\partial x} - \frac{a_1}{2a_2} \cdot \frac{\partial a_1}{\partial x} + \frac{\partial}{\partial y}\left(\frac{1}{2a_2} \cdot \frac{\partial a_1}{\partial x} \right) \right\} \frac{\partial V}{\cancel{\partial k}} \qquad (4.55)$$

Consistency on (4.55) and (4.47) then gives $k = 0$ and

$$(a_0)_x - \tfrac{1}{2} \cdot a_1 \cdot a_2^{-1} \cdot (a_1)_x + (\tfrac{1}{2} \cdot a_2^{-1} \cdot (a_1)_x)_y = 0$$

By a change of variable one then sees that (4.16) must hold.

All cases have now been dealt with. It has then been found that if (4.15) holds then A must be given by (4.16).

§5. For $A_{uuu} \neq c.A_{uu}$.

Still considering (3.2) with (4.1) and (4.6) it has been shown in the previous section that one need only consider

$$A_{uuu} \neq c.A_{uu} \qquad (5.1)$$

where c is a function of x and y only. It is (4.7) and (4.8) which are being solved.

Take $\left(\frac{\partial}{\partial u} + Q_1 \cdot \frac{\partial}{\partial u'} \right)^n$, $n = 1,2,3$ of (4.7) to obtain three equations for the two unknowns $V_{u'}$ and V_p.

$$\begin{vmatrix} Q_1 \cdot \frac{\partial^2 A}{\partial u'^2} - \frac{\partial^2 A}{\partial u^2} & Q_1 \cdot \frac{\partial A}{\partial u'} - Q_1 \cdot \frac{\partial A}{\partial u} - Q_1'(y) & \frac{\partial^2 A}{\partial u^2} \cdot \cancel{k} + \frac{\partial C}{\partial u} \\ Q_1^2 \cdot \frac{\partial^3 A}{\partial u'^3} - \frac{\partial^3 A}{\partial u^3} & Q_1^2 \cdot \frac{\partial^2 A}{\partial u'^2} - Q_1 \cdot \frac{\partial^2 A}{\partial u^2} & \frac{\partial^3 A}{\partial u^3} \cdot \cancel{k} + \frac{\partial^2 C}{\partial u^2} \\ Q_1^3 \cdot \frac{\partial^4 A}{\partial u'^4} - \frac{\partial^4 A}{\partial u^4} & Q_1^3 \cdot \frac{\partial^3 A}{\partial u'^3} - Q_1 \cdot \frac{\partial^3 A}{\partial u^3} & \frac{\partial^4 A}{\partial u^4} \cdot \cancel{k} + \frac{\partial^3 C}{\partial u^3} \end{vmatrix} = 0 \qquad (5.2)$$

Note that if $Q_1 = 0$ then (4.5) gives $Q = A(x,y,u')$. The coefficient of q in (1.3a) is then $P_u = 0$. Then differentiate (1.3a) w.r.t. u

$$P_p \cdot (A_{uu} \cdot p + C_u) = 0$$

But then (3.4) implies that (3.2) is linear in u. Hence

$$Q_1 \neq 0 \qquad (5.3)$$

In (5.2) take the coefficient of p, add column 3 to column 1, and divide rows 1,2 and 3 by Q_1, Q_1^2, Q_1^3 respectively

$$\begin{vmatrix} \dfrac{\partial^2 A}{\partial u'^2} & \dfrac{\partial A}{\partial \varphi'} - k_0 & k_1 \\ \dfrac{\partial^3 A}{\partial u'^3} & \dfrac{\partial^2 A}{\partial \varphi'^2} - k_1 & k_2 \\ \dfrac{\partial^4 A}{\partial u'^4} & \dfrac{\partial^3 A}{\partial \varphi'^3} - k_2 & k_3 \end{vmatrix} = 0 \tag{5.4}$$

where k_0, k_1, k_2 and k_3 are functions of x, y and u only. Infact

$$k_0 = \frac{\partial A}{\partial u} + \frac{Q_1'(y)}{Q_1} \quad , \quad k_i = \frac{1}{Q_1^i} \cdot \left(\frac{\partial}{\partial u}\right)^{i+1} A \quad , \quad i = 1, 2, 3. \tag{5.5}$$

In (5.4) multiply columns 1 and 2 by k_1. Take $A_{u'u'}$ times col. 3 from col. 1 and $(A_{u'} - k_0)$ times col. 3 from col. 2. Then expand along row 1 using, from (5.5) that $k_1 \neq 0$.

$$\frac{\partial \Delta_1}{\partial u'} \cdot \Delta_2 = \frac{\partial \Delta_2}{\partial u'} \cdot \Delta_1 \tag{5.6}$$

where

$$\begin{aligned} \Delta_1 &= k_1 \cdot (A_{u'u'} - k_1) - k_2 \cdot (A_{u'} - k_0) \\ \Delta_2 &= k_1 \cdot (A_{u'u'u'} - k_1) - k_3 \cdot (A_{u'} - k_0) \end{aligned} \tag{5.7}$$

Now $\Delta_1 = 0$ contradicts (5.1) so integrating (5.6) gives $\Delta_2 = K(x,y) . \Delta_1$ i.e. there exist functions a_0, a_1 and a_2 of x and y only such that

$$A_{uuu} = a_0 + a_1 . A_u + a_2 . A_{uu} \tag{5.8}$$

Substitute (5.8) into (5.4) and take a_2.row 2 $-$ a_1.row 1 from row 3

$$\begin{vmatrix} \dfrac{\partial^2 A}{\partial u'^2} & \dfrac{\partial A}{\partial u'} - k_0 & k_1 \\ \dfrac{\partial^3 A}{\partial u'^3} & \dfrac{\partial^2 A}{\partial u'^2} - k_1 & k_2 \\ 0 & a_2 . k_1 + a_1 . k_0 - k_2 + a_0 & k_3 - a_2 . k_2 - a_1 . k_1 \end{vmatrix} = 0 . \tag{5.9}$$

If $k_3 \neq a_1 . k_1 + a_2 . k_2$ then define

$$\lambda = \frac{a_0 + a_1 k_0 + a_2 . k_1 - k_2}{k_3 - a_2 . k_2 - a_1 . k_1}$$

Take λ times col. 3 from col. 2 in (5.9) and expand along row 3. One obtains a similar equation to (5.6) which on integration w.r.t. u' gives

$$\rho_2 = K(x,y).\rho_1$$

where

$$\rho_1 = A_{u'} - k_0 - \lambda.k_1$$

$$\rho_2 = A_{u'u'} - k_1 - \lambda.k_2$$

But this contradicts (5.1). Hence it has been shown that

$$k_3 = a_1.k_1 + a_2.k_2 \tag{5.10}$$

Equation (5.10) using (5.5) and (5.8) to eliminate A_{uuuu} is

$$a_2.(1 - Q_1).A_{uuu} + a_1.(1 - Q_1^2).A_{uu} = 0 \tag{5.11}$$

By (5.1) the coefficients of A_{uu} and A_{uuu} in (5.11) must vanish. One then sees that $Q_1^2 = 1$ unless $a_1 = a_2 = 0$; in this case (5.9) and (5.1) imply $a_0 = k_2$ i.e. $A_{uuu} = Q_1^2.a_0$. Then from (5.8) one has in this case and so in all cases that

$$Q_1^2 = 1 \tag{5.12}$$

It will be shown in the rest of this section that

$$A_{uuuu} = K(x,y).A_{uu} \quad , \quad K \neq 0 \tag{5.13}$$

for some function K.

<u>Case 1.</u>

$$a_1 = a_2 = 0 \tag{5.14}$$

From (5.1), (5.8) and (5.14) one has

$$A = b_0 + b_1.u + b_2.u^2 + b_3.u^3 \quad , \quad b_3 \neq 0 \tag{5.15}$$

Define $\quad v = u' - Q_1.u \qquad$ (5.16)

and use v instead of u'. Equations (4.7) and (4.8) then give

$$V_u = 0 \tag{5.17}$$

$$(b_3)_x = 0 \tag{5.18}$$

$$0 = V_v.(b_2.(1-Q_1)+3b_3.v) + V_p.(3b_3.p + (b_2)_x) \tag{5.19}$$

$$2b_2.(1-Q_1) + 6b_3.v = V_v.(2b_2.v + 3b_3.v^2) + Q_1.V_p.(2b_2.p + (b_1)_x) \tag{5.20}$$

$$2b_2.v + 3b_3.v^2 = V_y + V_v.(b_0.(1-Q_1) + b_1.v + b_2.v^2 + b_3.v^3 + k) + V_p.(b_1.p + (b_0)_x) \tag{5.21}$$

Consistency on (5.19) and (5.20) gives another equation with (5.19) and (5.20) for the two unknowns V_v and V_p. Hence

$$= \begin{vmatrix} 0 & b_2(1-Q_1)+3b_3 v & 3b_3 p + \frac{\partial b_2}{\partial x} \\ 2b_2(1-Q_1)+6b_3 v & 2b_2 v + 3b_3 v^2 & Q_1\left[2b_2 p + \frac{\partial b_1}{\partial x}\right] \\ 6b_3\left[b_2(1-Q_1)+3b_3 v\right] & \begin{matrix}\left[2b_2+6b_3 v\right]\left[b_2(1-Q_1)+3b_3 v\right] \\ -3b_3\left[2b_2 v+3b_3 v^2\right]\end{matrix} & \begin{matrix} 2b_2 Q_1 \frac{\partial b_2}{\partial x} \\ -3b_3 Q_1 \frac{\partial b_1}{\partial x}\end{matrix} \end{vmatrix}$$

(5.22)

Multiply row 2 by $3b_3$. Take $2b_2.Q_1$ times row 1 from row 2. Then take row 2 from row 3. The only non-zero term in row 3 is then in the third column. Since $b_3 \neq 0$, the coefficient of v^2 then gives

$$2b_2.(b_2)_x = 3b_3.(b_1)_x \tag{5.23}$$

Take $3b_3.(5.20) - 2b_2.(5.19)$

$$V_v = 2(v + \tfrac{1}{3}b_2.b_3^{-1}.(1-Q_1))^{-1} \tag{5.24}$$

Then (5.19) gives

$$V_p = -2(p + \tfrac{1}{3}b_3^{-1}.(b_2)_x)^{-1} \tag{5.25}$$

Integrating (5.24) and (5.25) and substituting into (5.21) one obtains that $b_3 = 0$ contradicting (5.15). This completes this case.

Case 2.

$$a_2 \neq 0 \ , \ a_1 = 0 \tag{5.26}$$

Then from (5.11) and (5.8) one has

$$Q_1 = +1 \tag{5.27}$$

$$A = b_0 + b_1.v + b_2.v^2 + b_3.\exp c.v \quad , b_2 \neq 0 \ , b_3 \neq 0 \ , c \neq 0 \tag{5.28}$$

Then with $v = u' - u$ (5.29)

equations (4.7) and (4.8) give

$$V_u = 0 \tag{5.30}$$

$$(b_2)_x = c_x = 0 \tag{5.31}$$

$$c.(e^{cv}-1) = (e^{cv}-1).V_v + (cp + (\ln b_3)_x).V_p \tag{5.32}$$

$$0 = v.V_v + (p + \tfrac{1}{2}b_2^{-1}.(b_1)_x).V_p \tag{5.33}$$

$$2b_2.v = V_y + (b_1.v + b_2.v^2 + k).V_v + (b_1.p + (b_0)_x).V_p \tag{5.34}$$

The coefficient of p in the consistency equation between (5.32) and (5.33) then gives a contradiction.

Case 3.

$$a_2 \neq 0 \quad , \ a_2^2 + 4a_1 = 0 \tag{5.35}$$

Then from (5.8) and (5.11) one has

$$A = b_0 + b_1.u + (b_2 + b_3.u).\exp c.u \quad , b_3 \neq 0 \ , c \neq 0 \tag{5.36}$$

$$Q_1 = +1 \tag{5.37}$$

With $v = u' - u$ (5.38)

equation (4.8) gives $V_u = 0$. The coefficients of $u^1.e^{cu}$ in (4.7) give

$$c_x = 0 \tag{5.39}$$

$$c = V_v + V_p.(c.p + (\ln b_3)_x).(e^{cv}-1)^{-1} \tag{5.40}$$

$$(c.v + 1).e^{cv} - 1 = v.e^{cv}.V_v + V_p.(p + b_3^{-1}.(b_2)_x + b_2.(b_3^{-1})_x) \qquad (5.41)$$

Consistency on (5.40) and (5.41) gives three equations for the two unknowns V_v and V_p. The p component in the determinant then gives a contradiction.

Case 4.

From (5.8) and the previous cases it only remains to consider

$$a_2 \neq 0 \quad , \quad a_2^2 + 4a_1 \neq 0 \qquad (5.42)$$

in order to prove (5.13). Then one has

$$A = b_0 + b_3.u + b_1.\exp c_1.u + b_2.\exp c_2.u \qquad (5.43)$$

$$b_1 \neq 0 \,,\, b_2 \neq 0 \,,\, c_1 \neq 0 \,,\, c_2 \neq 0 \,,\, c_1 \neq c_2 \,,\, c_1 \neq -c_2 \qquad (5.44)$$

From (5.11), $Q_1 = +1$. So set $v = u' - u$ (5.45)

Then (4.8) is $V_u = 0$. So (4.7) gives

$$(c_1)_x = (c_2)_x = (b_3)_x = 0 \qquad (5.46)$$

$$c_i = V_v + V_p.(c_i.p + (\ln b_i)_x).(\exp(c_i.v) - 1)^{-1} \quad , i = 1,2 \qquad (5.47)$$

$$0 = V_y + V_v.(b_3.v + k) + V_p.(b_3.p + (b_0)_x) \qquad (5.48)$$

The p component of the consistency condition on (5.47) then readily gives a contradiction to (5.44).

§6. That $A_{uuu} = K.A_u$.

The proof of (4.11) and (4.12) will be completed in this section. From the previous section one has

$$A = b_0 + b_3.u + b_1.e^{cu} + b_2.e^{-cu} \quad , b_1 \neq 0 \,,\, b_2 \neq 0 \,,\, c \neq 0 \qquad (6.1)$$

It will now be shown that one may choose $b_0 = b_3 = 0$ and b_1 and b_2 constant. From (5.12) one must consider two cases.

Case 1.

$$Q_1 = +1 \qquad (6.2)$$

The relevant equations are (5.45) to (5.48) with $c_1 = -c_2 = c$. The consistency condition on (5.47) is

$$(\ln(b_1 + b_2))_x = 0 \qquad (6.3)$$

Then solving (5.47) for V_v and V_p and integrating gives

$$V = 2.\ln(\sinh \tfrac{1}{2}cv) - 2.\ln(p + c^{-1}.(\ln b_1)_x) + d(x,y) \qquad (6.4)$$

Substitute (6.4) into (5.48)

$$b_3 = k = d_y = 0 \qquad (6.5)$$

$$(b_0)_x + (c^{-1}.(\ln b_1)_x)_y = 0 \qquad (6.6)$$

By making a change of dependent variable and by rescaling y one sees that (4.11) and (4.12) is proved in this case.

Case 2.

$$Q_1 = -1 \tag{6.7}$$

With $v = u' - u$ (6.8)

equation (4.9) is $V_u = 0$. Then (4.7) gives

$$c_x = (b_3)_x = 0 \tag{6.9}$$

$$0 = V_y + V_v.(2.b_0 + b_3.v + k) + V_p.(b_3.p + (b_0)_x) \tag{6.10}$$

$$-c = V_v + V_p.(c.b_1.p + (b_1)_x).(b_1 + b_2.\exp -c.v)^{-1} \tag{6.11}$$

$$+c = V_v + V_p.(-c.b_2.p + (b_2)_x).(b_2 + b_1.\exp c.v)^{-1} \tag{6.12}$$

Consistency on (6.11) and (6.12) give

$$(\ln(b_1 + b_2))_x = 0 \tag{6.13}$$

Solving (6.11) and (6.12) for V_v and V_p and integrating

$$V = -2.\ln(p + c^{-1}.(\ln b_1)_x) + 2.\ln(b_1.e^{\frac{1}{2}cv} + b_2.e^{-\frac{1}{2}cv}) + d(x,y) \tag{6.14}$$

Substituting (6.14) into (6.10) gives that (6.5) and (6.6) must hold. So by a change of variable exactly as in the previous case one may show that (4.11) and (4.12) must hold.

The proof of (4.11) and (4.12) is now complete.

§7. Pseudopotentials.

Looking for B.T. makes it very difficult to see "what is going on." This is primarily because so few equations have B.T. One possible generalization that one may make to allow more equations to have transformations is not to specify the equations that the "new" dependent variable satisfies. In particular one considers the equations

$$\begin{aligned} u'_x &= P(u,u',u_x,u_y) \\ u'_y &= Q(u,u',u_x,u_y) \end{aligned} \tag{7.1}$$

where u satisfies $u_{xy} = F(u)$ (7.2)

but where u' is only specified by (7.1).

Unfortunately all such transformations, except one, require $F''(u) = K.F(u)$ for some constant K. All transformations of the form (7.1) and (7.2) are given in table 1 at the end of this chapter. The one exception is the potential

$$\begin{aligned} u'_x &= K_1.(u_x)^2 + 2.K_2.G(u) + K_3 \\ u'_y &= 2.K_1.G(u) + K_2.(u_y)^2 + K_4 \end{aligned} \tag{7.3}$$

where u satisfies $u_{xy} = G'(u)$ (7.4) and where $K_1,\ldots,K_4$ are constants. It is called a potential because the R.H.S. of (7.3) does not depend on u'.[9]

One possible use of these pseudopotentials is to demand that u' satisfy a second order equation. For example if (7.2) is

$$u_{xy} = \sin u \tag{7.5}$$

and the transformation cannot be converted to (7.3) by replacing u' by some function of u and u' then one has only two possible B.T. The first is just the auto-B.T. (2.2) of chapter 1. The second is

$$\begin{aligned} u'_x &= K^{-1}.(\sinh u' . \sin u + \cos u) \\ u'_y &= K + u_x.\cosh u' \end{aligned} \tag{7.6}$$

All B.T. via (7.3) are also given in table two at the end of this chapter.

This attack on the pseudopotentials in table one does not seem to have produced much useful information so one is tempted to ask if there is another possible use.

In general all solutions u' of (7.1) satisfy two third order equations, as in the following example. If u satisfies

$$u_{xy} = \tfrac{1}{4}.f(u).f'(u) \tag{7.7}$$

then the following equations have a solution i.e. are consistent.

$$\begin{aligned} (u'_x)^{\frac{1}{2}} &= u_x + \tfrac{1}{2}.a.f(u) \\ (u'_y)^{\frac{1}{2}} &= a.u_y + \tfrac{1}{2}.f(u) \end{aligned} \tag{7.8}$$

where a is a constant. If one differentiates (7.8a) w.r.t. y or (7.8b) w.r.t. x then one obtains

$$u'_{xy} = f'(u).(u'_x.u'_y)^{\frac{1}{2}} \tag{7.9}$$

One may solve (7.9) for u

$$u = g'\left(\frac{u'_{xy}}{\sqrt{u'_x . u'_y}}\right) \tag{7.10}$$

where g'(v) is the inverse of f'(u) such that

$$f(g'(v)) = v.g'(v) - g(v) \tag{7.11}$$

$$\left(g'(v) = \frac{dg}{dv} \quad \text{and} \quad f'(u) = \frac{df}{du} \right)$$

Differentiate (7.9) w.r.t. x and eliminate u , u_x , u_y to obtain that u' must satisfy

$$\left\{ u'_{xxy} - \tfrac{1}{2} u'_{xy} \cdot \left[\frac{u'_{xx}}{u'_x} + \frac{u'_{xy}}{u'_y} \right] \right\} \cdot g''\left(\frac{u'_{xy}}{\sqrt{u'_x \cdot u'_y}} \right)$$

$$= u'_x \cdot \sqrt{u'_y} - \tfrac{1}{2} \cdot a \cdot \sqrt{u'_x \cdot u'_y} \left\{ \frac{u'_{xy}}{\sqrt{u'_x \cdot u'_y}} \cdot g'\left(\frac{u'_{xy}}{\sqrt{u'_x \cdot u'_y}} \right) - g\left(\frac{u'_{xy}}{\sqrt{u'_x \cdot u'_y}} \right) \right\} \qquad (7.12)$$

Differentiating (7.9) w.r.t. y one obtains, in a similar way, a second equation which u' must satisfy.

Note that if u' is any solution of (7.12) (and the second equation) then u defined by (7.10) must satisfy (7.7). So (7.8) is really only "half a B.T." in that to go from (7.7) to (7.12) one uses the transformation (7.8) but to go from (7.12) to (7.7) one uses the substitution (7.10).

If one is to use (7.8) to solve (7.7) then one must be able to spot some special solutions or properties of (7.12). For example if the transformation (7.1) involves an arbitary constant which does not appear in the analogous equations to (7.12) then one could construct new solutions to (7.2) from a given solution as follows.

1). Let u_{old} be any solution of (7.2).

2) Integrate (7.1) to obtain u'_{old} which must satisfy two equations like (7.12) and which depends on some constant K_{old} which does not appear in the two third order equations.

3) Change the constant K_{old} to some new value K_{new} obtaining a new solution of the two third order equations, u'_{new} say.

4) Substitute u'_{new} into the analogous equation to (7.10) to obtain a new solution u_{new} to (7.2).

If one wants the new solution to (7.2) to correspond to the old "plus one soliton", in analogy with S.G., then the free parameter must correspond to the velocity of the soliton i.e. the two third order equations must be Lorentz invarient. It is this possibility which is considered in the next section.

§8. Lorentz Invariance.

Consider the equation

$$u_{xy} = G'(u) \qquad (8.1)$$

and the transformation

$$A_{u'} \cdot u'_x + A_u \cdot u_x = K_1 \cdot (u_x)^2 + 2 \cdot K_2 \cdot G(u) + K_3 \qquad (8.2a)$$

$$A_{u'}.u'_y + A_u.u_y = 2.K_1.G(u) + K_2.(u_y)^2 + K_4 \qquad (8.2b)$$

where A is a function of u and u' only.

Diff. (8.2a) w.r.t. y or (8.2b) w.r.t. x to obtain

$$A_{u'}.u'_{xy} = (2.K_1.p + 2.K_2.q - A_u).G'(u) - A_{uu}.p.q - A_{u'u}.(p.q' + p'.q) - A_{u'u'}.p'.q' \qquad (8.3)$$

where $p = u_x$, $q = u_y$, $p' = u'_x$, $q' = u'_y$.

Now differentiate (8.3) w.r.t. x and use the x derivative of (8.2a) to eliminate u_{xx} in favour of u'_{xx}.

$$u'_{xxy} = X_1.u'_{xx} + X_0 \qquad (8.4)$$

where X_0 and X_1 are functions of u , u' , p , p' , q and q' . Infact

$$X_1 = (2.K_1.G'(u) - A_{uu}.q - A_{u'u}.q').(2.K_1.p - A_u)^{-1} - (A_{u'})^{-1}.(A_{u'u}.q + A_{u'u'}.q') \qquad (8.5)$$

In principle, equations (8.2) and (8.3) can be solved for u , p and q. Equation (8.4) is then an equation for u' only.

It is now assumed that (8.4) considered as an equation which u' satisfies is Lorentz invarient. This means that if one lets $x \to a.x$ and $y \to a^{-1}.y$ in (8.4) (keeping u' unchanged) then the equation assumes the same form. Equations (8.2) and (8.3) may then be thought of as giving u , p and q as functions of u' , p' , q' and a.

What all this means is that one lets $u' \to u'$, $p' \to a^{-1}.p'$, $q' \to a.q'$, $u'_{xx} \to a^{-2}.u'_{xx}$, $u \to u(a)$, $p \to p(a)$ and $q \to q(a)$ where $u(a)$, $p(a)$ and $q(a)$ are given by (8.2) and (8.3). After this Lorentz transformation multiply (8.4) by a and then differentiate w.r.t. a and set $a = 1$. The resultant equation must then be identically zero in u', u'_{xy}, p' , q' and u'_{xx} . From (8.2) , (8.3) and (8.4) on differentiating w.r.t. a one obtains four equations for the three unknowns u_a , p_a and q_a .The determinant of coefficients in this must then vanish. The coefficient of u'_{xx} in this determinant is then

$$\begin{vmatrix} A_{uu'}.p' + A_{uu}.p - 2K_2.G'(u) & A_u - 2K_1 p & 0 & -A_{u'}.p' \\ A_{uu'}.q' + A_{uu}.q - 2K_1.G'(u) & 0 & A_u - 2K_2 q & A_{u'}.q' \\ X_2 & 2K_1.G'(u) - A_{uu}.q - A_{uu'}.q' & 2K_2.G'(u) - A_{uu}.p - A_{uu'}.p' & A_{uu'}.(p'q - pq') \\ \dfrac{\partial X_1}{\partial u} & \dfrac{\partial X_1}{\partial p} & \dfrac{\partial X_1}{\partial q} & -X_1 + \left(\dfrac{\partial}{\partial q'} - \dfrac{\partial}{\partial p'}\right)X_1 \end{vmatrix} \qquad (8.6)$$

where X_1 is given by (8.5) and is to be thought of as a function of u, u', p, p', q and q'. Also

$$\begin{aligned}X_2 = {} & (2.K_1.p + 2.K_2.q - A_u).G''(u) - A_{uu}.G'(u) - A_{uuu}.p.q - A_{u'u'u}.p'.q' \\ & - A_{u'uu}.(p.q' + p'.q) - (A_{u'})^{-1}.A_{u'u}.\Big\{(2.K_1.p + 2.K_2.q - A_u).G'(u) \\ & - A_{uu}.p.q - A_{u'u}.(p.q' + p'.q) - A_{u'u'}.p'.q'\Big\}\end{aligned} \tag{8.7}$$

Using (8.2) one may consider (8.6) to be an equation for u', u, p and q. It must be identically zero in these variables. If this were not the case then one could write u' equals some function of u and its derivatives. But one obtains u' from u by integrating (8.2) and not by a straight substitution.

The case $K_1 = K_2 = 0$ seems rather uninteresting so assume

$$K_1 \neq 0 \tag{8.8}$$

Multiply the last row of (8.6) by $(2.K_1.p - A_u)^2$ to obtain a polynomial in p (with ceofficients which are considered to be functions of u, u' and q only). The highest power of p which occurs is five. Infact the coefficient of p^5 is

$$\begin{vmatrix} A_{uu'}.q' + A_{uu}.q - 2K_1.G'(u) & A_u - 2K_2 q & A_{u'}.q' \\ -A_{u'u'u}.q' - A_{u'uu}.q + \dfrac{A_{u'u}}{A_{u'}}.\left[A_{u'u}.q + A_{u'u'}.q'\right] & -A_{uu'} & A_{uu'}.q \\ -4K_1.G''(u) + 2A_{uuu'}.q' + 2A_{uuu}.q & 2A_{uu} & A_{uu'}.q' - A_{uu}.q \\ \quad + \dfrac{A_{u'u}}{A_{u'}}.\left[2K_1.G'(u) - A_{uu}.q - A_{uu'}.q'\right] & & \quad + 2K_1.G'(u) \end{vmatrix} \tag{8.9}$$

It is (8.9) which will now be solved.

<u>Step 1.</u>

Suppose $\quad K_2 \neq 0 \qquad$ (8.10)

The highest power of q in (8.9) is then 5. Taking the coefficient of q^5 in (8.9) and integrating gives w.l.o.g.

$$A = a(u).u' + b(u) \tag{8.11}$$

Note that in (8.11) one is still free to replace u' by an arbitary function of u'.

Substituting (8.11) into (8.9) one sees that (8.9) is linear in u' with coefficients which are functions of u and q only. The highest power of q in the coefficient of u' in (8.9) is three. This term is infact

$$a'''(u) - a^{-1}.a''(u).a'(u) + a^{-2}.(a')^3 = 0 \tag{8.12}$$

The coefficient of u' in (8.9) is then

$$a'(u).(q.b'' - 2.K_1.G').(a^{-1}.(a')^2 - a'') + 2.a''.(2.K_1.G + K_4 - \tfrac{1}{2}.b'.q).(a^{-1}.(a')^2 - a'') = 0 \qquad (8.13)$$

If $a.a'' \neq (a')^2$ then the coefficient of q in (8.13) gives w.l.o.g. $b = 0$ because one can always add a constant to A or to u'. The coefficient of q^4 in the term independent of u' in (8.9) then gives on integrating that $a = (C_1.u + C_0)^2$ where C_0 and C_1 are constants. But with $b = 0$ equation (8.13) gives on integrating $G(u) + \tfrac{1}{4}.K_1^{-1}.K_4 = C_2.(a')^2 = 4.C_2.C_1^2.(C_0 + C_1.u)^2$ where C_2 is a constant. This means that the original equation (8.1) is linear. Hence

$$a.a'' = (a')^2 \qquad (8.14)$$

The coefficient of q^4 in (8.9) then gives that a is a constant, which one may take to be one because it is possible to multiply u' by a constant. That is

$$A = u' + b(u) \qquad (8.15)$$

Substitute (8.15) into (8.6). It is readily shown that

$$b''(u) \neq 0 \qquad (8.16)$$

The coefficient of q^3 in (8.6) is now

$$\begin{vmatrix} b''p - 2K_2G' & b' - 2K_1.p & \tfrac{1}{2}b'.p - 2K_2.G - K_3 \\ 2K_2G'' - b'''.p & -b'' & K_2.G' \\ b'.b''' - (b'')^2 - 4K_1K_2G'' & 4K_1.b'' & -\tfrac{1}{2}b'.b'' - 2K_1K_2G' \end{vmatrix} = 0 \qquad (8.17)$$

If $b''' = 0$ then the term independent of p in (8.17) gives on integrating that

$$(4.K_1.K_2.G' - b'.b'')^2 = C.(4.K_1K_2G + 2.K_1.K_3 - \tfrac{1}{2}(b')^2)$$

where C is a constant. But this means that the original equation is linear i.e. $G''' = 0$. Hence

$$b''' \neq 0 \qquad (8.18)$$

The coefficient of p^2 in (8.17) is now

$$b'.b'' = 4.K_1.K_2.G' \qquad (8.19)$$

The coefficient of p in (8.17) is then

$$(b')^2 = 4.K_1.(2.K_2.G + K_3)$$

But by symmetry one must also have

$$(b')^2 = 4.K_2.(2.K_1.G + K_4)$$

The transformation is then of the form (7.8) which is clearly not Lorentz invariant.

Step 2.

$$K_2 = 0 \tag{8.20}$$

The coefficient of q^2 in (8.9) gives on integrating

$$A = f(u' + a(u)) \tag{8.21}$$

The coefficient of q in (8.9) is then

$$(f'')^{-1}.f''' - (f')^{-1}.f'' = 2K_1G'.(a')^{-1}.(2K_1G + K_4)^{-1} - 2(a'.a'')^{-1}.a''' + 2.(a')^{-2}.a'' \tag{8.22}$$

which must equal a constant, k say. Integrating gives

$$\begin{aligned} f''.(f')^{-1} &= C_1.\exp k(u' + a) \\ 2.K_1.G + K_4 &= C_2.(a')^2.(a'')^{-2}.\exp -k.a \end{aligned} \tag{8.23}$$

It is now not too difficult to solve (8.9). For example if $k \neq 0$ then one obtains

$$\begin{aligned} a' &= (3/2).k^{-1}.(u + C_2)^{-1} \\ 2.K_1.G + K_4 &= C_3.(u + C_2)^{\frac{1}{2}} \end{aligned} \tag{8.24}$$

I have not checked whether the equations satisfied by u' are Lorentz invariant or not. (I suspect not). Note that it is only (8.9) which has been solved in this step. Still even if the equations are Lorentz invariant for (8.24) this is not very interesting and so the matter did not seem worth pursuing.

This completes this section. It has not proved possible to find Lorentz invariant equations which u' satisfies. With hindsight, perhaps this is to be expected: if one wants the transformation to involve a free parameter which does not appear in the equations satisfied by u' then it seems reasonable to suppose that the equations satisfied by u' must be "one order below the maximum." - in this case of second order. If this is the case then there should be a much easier method of proof than that given above.

§9. Quasilinear Equations.

In both this section and the next the equation

$$u_{xy} = A'(u).u_x + B'(u).u_y + C(u) \tag{9.1}$$

is considered. All B.T. of the form (1.2) have already been found, except for the case

$$\begin{aligned} u'_x &= P_0(u,u') + P_1(u,u').u_x \\ u'_y &= Q_0(u,u') + Q_1(u,u').u_y \end{aligned} \tag{9.2}$$

which will be considered now. (Explicit independent variable dependence will not be considered.) Rather than look for a B.T. it seems easier to look for pseudopotentials i.e. let the equation satisfied by u' be arbitary. This is easier because if one replaces u' by any function of u and u' then (9.2) retain the same form. The disadvantage of this approach is that one still has a lot of work to do to find all B.T.

By replacing u' by a suitable function of u and u' it is possible to choose

$$P_1 = 0 \tag{9.3}$$

The coefficient of pq in the x derivative of (9.2b) is then

$$(Q_1)_u = 0 \tag{9.4}$$

By replacing u' by a suitable function of u' only in (9.2) one may choose

$$Q_1 = 1 \tag{9.5}$$

since $Q_1 = 0$ implies that the transformation does not depend on u or its derivatives. The consistency condition on (9.2) implies

$$(Q_0)_u = - A'(u) \tag{9.6}$$

$$(P_0)_u + (P_0)_{u'} = B'(u) \tag{9.7}$$

$$(P_0)_{u'}.Q_0 = P_0.(Q_0)_{u'} + C(u) \tag{9.8}$$

Equations (9.6) and (9.7) give

$$\begin{aligned} P_0 &= W(u' - u) + B(u) \\ Q_0 &= Z(u') - A(u) \end{aligned} \tag{9.9}$$

Substituting (9.9) into (9.8)

$$A.W_v + B.Z_{u'} + C = Z.W_v - W.Z_{u'} \tag{9.10}$$

where $v = u' - u$ (9.11)

Suppose that

$$\text{and}\quad \begin{aligned} Z''' &\neq K_1.Z'' + K_2.Z' \\ W''' &\neq K_3.W'' + K_4.W' \end{aligned} \tag{9.12}$$

where $K_1,\dots,K_4$ are arbitary constants and primes denote differentiation w.r.t. u' or v as appropriate.

Differentiate (9.10) repeatedly w.r.t. u' keeping u fixed

$$\begin{aligned} &2(W''.Z''' - Z''.W''')(Z'.W''' - W'.Z''') \\ &\quad = (W''.Z'''' - Z''.W'''')(Z'.W'' - W'.Z'') \end{aligned} \tag{9.13}$$

Divide (9.13) by W''.W'.Z''.Z' and then diff. w.r.t. u' and v , treating u' and v as the independent variables.

$$\begin{aligned} &2\frac{d}{du'}\left(\frac{Z'''}{Z''}\right)\cdot\frac{d}{dv}\left(\frac{W'''}{W'}\right) + 2\frac{d}{du'}\left(\frac{Z'''}{Z'}\right)\frac{d}{dv}\left(\frac{W'''}{W''}\right) \\ &= \frac{d}{dv}\left(\frac{W''}{W'}\right)\frac{d}{du'}\left(\frac{Z''''}{Z''}\right) + \frac{d}{du'}\left(\frac{Z''}{Z'}\right)\cdot\frac{d}{dv}\left(\frac{W''''}{W''}\right) \end{aligned} \tag{9.14}$$

Divide by $\frac{d}{dv}\left(\frac{W''}{W'}\right)\cdot\frac{d}{du'}\left(\frac{Z''}{Z'}\right)$ and take $\frac{\partial^2}{\partial u'\partial v}$

Then separate variables and integrate

$$\frac{d}{du'}\left(\frac{Z'''}{Z''}\right) = A_1\cdot\frac{d}{du'}\left(\frac{Z'''}{Z'}\right) + A_2\,\frac{d}{du'}\left(\frac{Z''}{Z'}\right) \tag{9.15}$$

$$\frac{d}{dv}\left(\frac{W'''}{W''}\right) = -A_1\cdot\frac{d}{dv}\left(\frac{W'''}{W'}\right) + A_3\cdot\frac{d}{dv}\left(\frac{W''}{W'}\right) \tag{9.16}$$

where A_1, A_2 and A_3 are constants. Substitute (9.15) and (9.16) into (9.14) and separate variables toobtain , on integration ,

$$\frac{Z^{IV}}{Z''} = 2A_3\cdot\frac{Z'''}{Z'} + A_4\cdot\frac{Z''}{Z'} + A_7 \tag{9.17}$$

$$\frac{W^{IV}}{W''} = 2A_2\cdot\frac{W'''}{W'} - A_4\cdot\frac{W''}{W'} + A_8 \tag{9.18}$$

Integrate (9.15) and (9.16)

$$\left(\frac{Z'}{Z''} - A_1\right)\cdot Z''' = A_2\cdot Z'' + A_5\cdot Z' \tag{9.19}$$

$$\left(\frac{W'}{W''} + A_1\right)\cdot W''' = A_3\cdot W'' + A_6\cdot W' \tag{9.20}$$

Define $X = Z''.(Z')^{-1}$, $Y = W''.(W')^{-1}$ (9.21)

Diff. (9.19) w.r.t. u' and use (9.17) and (9.19) to eliminate Z'''' and Z''' . One then obtains a polynomial in X . But by (9.12) , X is not a constant. So all the coefficients of X^n must vanish.

$$0 = 2.A_1^2.A_2.A_3 + A_1^3.A_4 \tag{9.22}$$

$$0 = A_1.A_2.(1 + 4A_3 - A_2) + A_1^2.(A_5 - 3A_4 + A_1.A_7 - 2A_3.A_5) \tag{9.23}$$

$$2A_2^2 = A_2.(1 + 2A_3 + A_1.A_5) - 4A_1.A_3.A_5 + 3A_1(A_1.A_7 - A_4) + A_1.A_5 \tag{9.24}$$

$$3A_2.A_3 = 2A_3.A_5 - 3A_1.A_7 + A_4 \tag{9.25}$$

$$A_3^2 = A_7 \tag{9.26}$$

The coefficient of Y^4 in the similar equation from (9.18) and (9.20) is

$$0 = 2A_1^2.A_2.A_3 - A_1^3.A_4 \tag{9.27}$$

Equations (9.22) and (9.27) give

$$A_1^2.A_2.A_3 = A_1^3.A_4 \tag{9.28}$$

If $A_1 \neq 0$ then one may take $A_2 = 0$ using (9.28). (If $A_3 = 0$ then one obtains a contradiction exactly as in the case $A_2 = 0$ except that one uses the equations derived from (9.18) and (9.20) rather than those from (9.17) and (9.19).) With $A_2 = 0$ equations (9.23) to (9.26) readily give $A_5 = 0$. But then via (9.19) one has a contradiction to (9.12). Hence

$$A_1 = 0 \tag{9.29}$$

Then (9.24) is

$$2.A_2 = 1 + 2.A_3 \tag{9.30}$$

Note that if $A_2 = 0$ then (9.19) gives $Z''' = A_5.Z''$ which contradicts (9.12). In a similar manner from (9.18) and (9.20) one may derive $2.A_3 = 1 + 2.A_2$ which is incompatible with (9.30).

So it has been shown that (9.12) cannot hold. Because of the symmetry in the problem between x and y one may w.l.o.g. assume

$$Z''' = K_1.Z'' + K_2.Z' \tag{9.31}$$

Suppose $Z'' \neq K.Z'$ (9.32)

for all constants K. Then substitute (9.31) into (9.13) to obtain a polynomial in X. Because of (9.32) the coefficients of X^n in this must be zero. Hence

$$2.W'.K_1.(K_1.W'' - W''') = W'.(W''.(K_1^2 + K_2) - W'''') \qquad (9.32)$$

$$K_1.K_2.W'.W'' + 2(K_1.W'' - W''')(K_2.W' - W''') = W''.(W'''' - W''.(K_1^2 + K_2)) \qquad (9.33)$$

$$W''' = 4.K_1.W'' + K_2.W' \tag{9.34}$$

Note that it has been assumed that

$$W'' \neq 0 \qquad \text{and} \qquad K_2 \neq 0 \tag{9.35}$$

Substituting (9.34) into (9.32) one sees that

$$K_1 = 0 \tag{9.36}$$

Equations (9.32) and (9.33) are now satisfied by virtue of (9.34).
If $K_2 = 0$ then one may show that $W''' = K_1.W''$.
So it has been shown that one of the following must hold

$Z'' = K.Z'$ (or $W'' = K.W'$)

or

$Z''' = K.Z'$ and $W''' = K.W'$ (9.37)

or

$Z''' = K.Z''$ and $W''' = K.W''$

The actual transformations corresponding to these cases will be dealt with in the next section.

§10. Quasilinear Solutions.

All pseudopotentials of the form (9.2) will now be found when u satisfies (9.1).

Case 1.

$$Z = Z_0 + Z_1.\exp ku' \quad , \quad Z_1 \neq 0 \quad , \quad k \neq 0$$
$$W' \neq W_0 + W_1.\exp kv \tag{10.1}$$

where Z_0, Z_1, W_0 and W_1 are constants.

Substitute (10.1) into (9.10) and diff. w.r.t. u treating u and v as independent variables. Then multiply by exp(-ku) and differentiate again w.r.t. u. Then because of (10.1) the coefficients of W', exp(kv) and terms independent of these must be zero. Integrating

$$A = A_0 + A_1.\exp ku$$
$$B = B_0 + B_1.\exp -ku \tag{10.2}$$
$$C = C_0 + C_1.\exp ku$$

Substitute (10.2) into (9.10) and equate coefficients of exp (ku)

$$A_1.W' + B_0.Z_1.k.\exp kv + C_1 = Z_1.(W' - k.W).\exp kv \tag{10.3}$$

$$A_0.W' + B_1.Z_1.k.\exp kv + C_0 = Z_0.W' \tag{10.4}$$

From (10.4) and (10.1) one has $A_0 = Z_0$, $B_1 = 0$ and $C_0 = 0$.

The equation (9.1) is then

$$u_{xy} = A_1.k.e^{ku}.u_x + C_1.e^{ku}$$

while the transformation is found from (10.3).

Case 2.

$$Z = Z_0 + Z_1.u' + Z_2.e^{ku'}$$
$$W = W_0 + W_1.v + W_2.e^{kv} \qquad Z_2 \neq 0 , k \neq 0 , W_2 \neq 0 \tag{10.5}$$

Coefficient of $u'.e^{ku'}$ in (9.10) is $W_2.k.Z_1.e^{-ku} = Z_2.k.W_1$. Hence

$$W_1 = Z_1 = 0 \tag{10.6}$$

Term independent of u' in (9.10) then implies

$$C = 0 \tag{10.7}$$

Then (9.10) is

$$(A - Z_0).W_2.e^{-ku} + (B + W_0).Z_2 = 0$$

Putting all this together one has that the equation

$$u_{xy} = Z_2.U'.u_x - W_2.(U.\exp -ku).u_y \tag{10.8}$$

has the transformation

$$u'_x = W_2.e^{k(u'-u)} - W_2.U.e^{-ku}$$
$$u'_y = u_y - Z_2.U + Z_2.e^{ku'} \qquad (10.9)$$

where U is any function of u only and Z_2 , W_2 and k are constants.

Case 3.

$$Z = Z_0 + Z_1.u' \quad , \qquad W'' \neq 0 \qquad (10.10)$$

where Z_0 and Z_1 are constants. Substitute (10.10) into (9.10) and diff. w.r.t. u keeping v fixed. Then $W'' \neq 0$ implies on integrating

$$A(u) = A_0 + Z_1.u$$
$$Z_1.B(u) + C(u) = C_0 \qquad (10.11)$$

where A_0 and C_0 are constants. Substitute (10.11) into (9.10) diff. w.r.t. v and use (10.10)

$$Z_1 = 0 \qquad (10.12)$$

So one has that the equation

$$u_{xy} = B'(u).u_y \qquad (10.13)$$

has the transformation

$$u'_x = W(u'-u) + B(u)$$
$$u'_y = u_y \qquad (10.14)$$

Case 4.

$$Z = Z_0 + Z_1.u' + Z_2.(u')^2$$
$$W = W_0 + W_1.v + W_2.v^2 \qquad (10.15)$$

The coefficients of u' in (9.10) with u fixed are

$$2.Z_2.W_2.u = Z_2.W_1 - W_2.Z_1 \qquad (10.16)$$

$$2.W_2.(Z_0-A) + Z_1.(W_1-2.W_2.u) = 2.Z_2.(W_0-W_1.u+W_2.u^2+B) + Z_1.(W_1-2.W_2.u) \qquad (10.17)$$

$$(Z_0 - A)(W_1 - 2.W_2.u) = C + Z_1.(W_0 - W_1.u + W_2.u^2 + B) \qquad (10.18)$$

If $W_2 \neq 0$ then (10.16) to (10.18) imply $Z_1 = 0$, $Z_0 = A$ and $C = 0$. But then this is just (10.13) and (10.14) . So take

$$Z_2 = W_2 = 0 \qquad (10.19)$$

One then has that the equation

$$u_{xy} = A'(u).u_x + B'(u).u_y - A(u) - B(u) + u \qquad (10.20)$$

has the transformation

$$u'_x = B(u) + u' - u$$
$$u'_y = u' - A(u) + u_y \qquad (10.21)$$

Case 5.

$$Z = Z_0 + Z_1.e^{ku'} + Z_2.e^{-ku'} \qquad Z_1 \neq 0 , Z_2 \neq 0 \qquad k \neq 0 \qquad (10.22)$$
$$W = W_0 + W_1.e^{kv} + W_2.e^{-kv} \qquad W_1 \neq 0 , W_2 \neq 0$$

The coefficients of $e^{nku'}$ in (9.10) then give

$$W_1.(Z_0 - A).e^{-ku} = Z_1.(W_0 + B) \qquad (10.23)$$

$$C = 0 \qquad (10.24)$$

$$W_2.(Z_0 - A).e^{ku} = Z_2.(W_0 + B) \qquad (10.25)$$

Cannot have both $Z_0 - A = 0$ and $W_0 + B = 0$ because then (9.1) is $u_{xy} = 0$. Hence equations (10.23) and (10.25) imply

$$Z_2.W_1.e^{-ku} = Z_1.W_2.e^{ku}$$

But this contradicts (10.22). Hence there are no transformations in this case.

§11. Integrals.

This section contains an attempt to generalize the usual definition (1.2) so as to find B.T. for

$$u_{xy} = F(u) \qquad (11.1)$$

In particular one allows P and Q to depend on integrals and derivatives of u. Note that if one allows integrals of functions of u and u' then (11.1) has the trivial B.T.

$$u'_x = a.u_x + \int(F(u') - a.F(u)).dy \qquad (11.2)$$
$$u'_y = b.u_y + \int(F(u') - b.F(u)).dx$$

where a and b are constants. Because of this and because u satisfies (11.1) one need only consider the B.T. to depend on the variables given in (5.12) of the previous chapter.

As a first attempt keep the form (1.2) but allow P and Q to depend on these variables in (5.12) of chapter 1 which involve at most one x integral and/or one y integral. One may then clear the integrals by defining

$$u = v_{xy} . \qquad (11.3)$$

Then one is looking for a B.T. for equations of the form

$$v_{xxyy} = F(v_{xy}) \qquad (11.4)$$

The B.T. is taken to be of the form

$$v'_{xxy} = P(v,v',p,p',q,q',r,r',s,s',t,t',\alpha,\beta,\gamma,\delta) \qquad (11.5)$$
$$v'_{xyy} = Q(v,v',p,p',q,q',r,r',s,s',t,t',\alpha,\beta,\gamma,\delta)$$

where $p = v_x$, $q = v_y$, $r = v_{xx}$, $s = v_{xy}$, $t = v_{yy}$, $\alpha = v_{xxx}$, $\beta = v_{xxy}$, $\gamma = v_{xyy}$ and $\delta = v_{yyy}$, and similar for the primed variables but with v' replacing v .

Differentiate (11.5a) w.r.t. y and (11.5b) w.r.t. x and use (11.4). It will be assumed that all derivatives of v up to and including fourth order and also all derivatives of v' up to and including third order can be taken as independent - except, of course v_{xxyy} , v'_{xxy} and v'_{xyy} . What this says is that v is any solution of (11.4), that (11.5) is the lowest order B.T. that one can write down and that two additional functions cannot be found such that the B.T. takes the form

$$
\begin{aligned}
v'_{xxx} &= A(v,v',p,p',q,q',r,r',s,s',t,t',\alpha,\beta,\gamma,\delta)\\
v'_{xxy} &= B(v,v',p,p',q,q',r,r',s,s',t,t',\alpha,\beta,\gamma,\delta)\\
v'_{xyy} &= C(v,v',p,p',q,q',r,r',s,s',t,t',\alpha,\beta,\gamma,\delta)\\
v'_{yyy} &= D(v,v',p,p',q,q',r,r',s,s',t,t',\alpha,\beta,\gamma,\delta)
\end{aligned}
\tag{11.6}
$$

The only serious assumption here is the one that the B.T. does not take the form (11.9) .

The coefficients of v_{xxxx} , v_{xxxy} , v_{xyyy} , v_{yyyy} , v'_{xxx} and v'_{yyy} in $(11.5a)_y$ and $(11.5b)_x$ give

$$
\begin{aligned}
&P_{t'} = P_\alpha = P_\gamma = P_\delta = 0\\
&Q_{r'} = Q_\alpha = Q_\beta = Q_\delta = 0
\end{aligned}
\tag{11.7}
$$

But then both P and Q are independent of α and δ . So the coefficients of both α and δ must be zero. Hence

$$P_t = Q_r = 0 \tag{11.8}$$

It will be assumed that F is not linear. It is then apparent that $P_{s'} = 0$ makes F linear (from $(11.5a)_y$) . Hence and similarly

$$P_{s'} \neq 0 \quad , \quad Q_{s'} \neq 0 \tag{11.9}$$

Then clearly from the x and y derivatives of (11.5) one has

$$
\begin{aligned}
P &= P_0 + P_1.r + P_2.r' + P_3.\beta\\
Q &= Q_0 + Q_1.t + Q_2.t' + Q_3.\gamma
\end{aligned}
\tag{11.10}
$$

where P_i and Q_i $(i= 0,\dots,3)$ are functions of v , v' , p , p' , q , q' , s and s' only.

The equations which then remain to be satisfied are

$$
\frac{\partial P_i}{\partial v}\cdot q + \frac{\partial P_i}{\partial v'}\cdot q' + \frac{\partial P_i}{\partial p}\cdot s + \frac{\partial P_i}{\partial p'}\cdot s' + \frac{\partial P_i}{\partial s'}\cdot Q_0 = \begin{cases} F(s') - P_2 . P_0 - P_3\, F(s) & i=0\\ -P_2 . P_i & i=1,2\\ -P_1 - P_2 P_3 & i=3 \end{cases}
\tag{11.11}
$$

$$0 = \frac{\partial P_i}{\partial q} + Q_1 \cdot \frac{\partial P_i}{\partial s'} \qquad i = 0, \dots, 3 \tag{11.12}$$

$$0 = \frac{\partial P_i}{\partial q'} + Q_2 \cdot \frac{\partial P_i}{\partial s'} \qquad i = 0, \dots, 3 \tag{11.13}$$

$$0 = \frac{\partial P_i}{\partial s} + Q_3 \cdot \frac{\partial P_i}{\partial s'} \qquad i = 0, \dots, 3 \tag{11.14}$$

$$\frac{\partial Q_i}{\partial r} \cdot p + \frac{\partial Q_i}{\partial r'} \cdot p' + \frac{\partial Q_i}{\partial q} \cdot s + \frac{\partial Q_i}{\partial q'} \cdot s' + \frac{\partial Q_i}{\partial s'} \cdot P_0 = \begin{cases} F(s') - Q_2 Q_0 - Q_3 \cdot F(s) & i = 0 \\ -Q_2 Q_i & i = 1, 2 \\ -Q_1 - Q_2 Q_3 & i = 3 \end{cases} \tag{11.15}$$

$$0 = \frac{\partial Q_i}{\partial p} + P_1 \cdot \frac{\partial Q_i}{\partial s'} \qquad i = 0, \dots, 3 \tag{11.16}$$

$$0 = \frac{\partial Q_i}{\partial p'} + P_2 \cdot \frac{\partial Q_i}{\partial s'} \qquad i = 0, \dots, 3 \tag{11.17}$$

$$0 = \frac{\partial Q_i}{\partial s} + P_3 \cdot \frac{\partial Q_i}{\partial s'} \qquad i = 0, \dots, 3 \tag{11.18}$$

It will be assumed throughout that

$$F'''(s) \neq K.F''(s) \tag{11.19}$$

for all constants K.

It will be shown in the rest of this section that

$$P_1 = P_2 = Q_1 = Q_2 = 0 \tag{11.20}$$

The proof proceeds by contradiction so assume that

$$Q_j \neq 0 \quad \text{where } j = 1 \text{ or } 2 \tag{11.21}$$

Note that if one can prove that (11.21) cannot hold then, by symmetry, one has that (11.20) must be true.

Take $\frac{\partial}{\partial q_j} + Q_j \cdot \frac{\partial}{\partial s'}$ of (11.11) and eliminate $\frac{\partial P_i}{\partial s'}$ using the $i = 0$ equation. ($q_1 = q$, $q_2 = q'$)

$$\frac{\partial P_i}{\partial s'}\cdot\left\{\frac{\partial P_0}{\partial r_j}+Q_j\cdot\frac{\partial P_0}{\partial f'}-Q_j\cdot F'(s)\right\}=\frac{\partial P_0}{\partial s'}\cdot\left\{\frac{\partial P_i}{\partial r_j}+Q_j\cdot\frac{\partial P_i}{\partial f'}\right\} \qquad (11.22)$$

Take $\left(\frac{\partial}{\partial q_j}+Q_j\cdot\frac{\partial}{\partial s'}\right)$ of (11.22), where $Z_j=\frac{\partial Q_j}{\partial q_j}+Q_j\cdot\frac{\partial Q_j}{\partial s'}$:

$$Z_j\left\{\frac{\partial P_i}{\partial s'}\cdot\left[\frac{\partial P_0}{\partial f'}-F'(s')\right]-\frac{\partial P_0}{\partial s'}\cdot\frac{\partial P_i}{\partial f'}\right\}=Q_j^2\cdot\frac{\partial P_i}{\partial s'}\cdot F''(s') \qquad (11.23)$$

Take $\left(\frac{\partial Z_j}{\partial q_j}+Q_j\frac{\partial Z_j}{\partial s'}\right)\left(\frac{\partial}{\partial q_j}+Q_j\frac{\partial}{\partial s'}\right)-Z_j\cdot\left(\frac{\partial}{\partial q_j}+Q_j\cdot\frac{\partial}{\partial s'}\right)^2$ of (11.23)

$$F''(s')\cdot\left\{3Z_j^2-Q_j\cdot\left[\frac{\partial Z_j}{\partial q_j}+Q_j\cdot\frac{\partial Z_j}{\partial s'}\right]\right\}+Z_j\cdot Q_j^2\cdot F'''(s')=0 \qquad (11.24)$$

If $\frac{\partial P_i}{\partial s'}\neq 0$ for i = 1,2 or 3 then one sees that (11.23) implies $Z_j\neq 0$. Further, by taking the operator occuring in (11.16), (11.17) or (11.18) acting on (11.24), depending on whether P_1, P_2 or P_3 is non-zero, one obtains two homogoneous linear equations for the two unknowns which are the coefficients of F''(s') and F'''(s') occuring in (11.24). But $Z_j\cdot Q_j^2\neq 0$ so the determinant of coefficients i.e. F''''(s').F''(s') - F'''(s').F'''(s') must be zero. But this contradicts (11.10). Hence

$$\frac{\partial P_i}{\partial s'}=0 \qquad i=1,2,3 \qquad (11.25)$$

Then it is easy to show, from (11.11) to (11.14) that

$P_1=P_2=0$ and P_3 is a constant . (11.26)

Take $\left(\frac{\partial}{\partial s}+P_3\cdot\frac{\partial}{\partial s'}\right)$ of (11.15) and use the i = 0 equation to eliminate P_0. Then for i = 1, 2 or 3 one has

$$\frac{\partial Q_i}{\partial s'}\left\{\frac{\partial Q_0}{\partial q}+P_3\cdot\frac{\partial Q_0}{\partial q'}-P_3\cdot F'(s')+Q_3\cdot F'(s)\right\}=\frac{\partial Q_0}{\partial s'}\cdot\left\{\frac{\partial Q_i}{\partial q}+P_3\frac{\partial Q_i}{\partial q'}\right\} \qquad (11.27)$$

Take $\left(\frac{\partial}{\partial s}+P_3\cdot\frac{\partial}{\partial s'}\right)$ of (11.27) and assume that

$$\frac{\partial Q_i}{\partial s'}\neq 0 \qquad i=1,2 \text{ or } 3 \qquad (11.28)$$

Then

$$Q_3.F''(s) = P_3^2.F''(s') \tag{11.29}$$

Take $\frac{\partial}{\partial s} + P_3.\frac{\partial}{\partial s'}$ of (11.29) and eliminate Q_3. Because of (11.19) one then sees that $P_3 = 0$. Then (11.29) gives $Q_3 = 0$. It is then easy to show that the B.T. (11.5) does not depend on the "old" variable i.e. does not depend on v or its derivatives. This is not very interesting and so will be excluded. Hence (11.28) cannot hold. So one has $\frac{\partial Q_i}{\partial s'} = 0$ for i = 1, 2 and 3. One may then show, from (11.15) to (11.18), that $Q_1 = Q_2 = 0$. This contradicts (11.21) and so the result is proved.

§12. For $Q_1 = Q_2 = 0$.

It is shown in this section that there are no B.T. of the form (11.5) for equations of the form (11.4).

From the previous section one has that (11.20) must hold.

$$P_1 = P_2 = Q_1 = Q_2 = 0 \tag{12.1}$$

Then (11.11) to (11.18) give, by inspection, that

$$\begin{aligned} P_0 &= A_0 + A_1.p + A_2.p' \quad , \qquad P_3 = A_3 \\ Q_0 &= B_0 + B_1.q + B_2.q' \quad , \qquad Q_3 = B_3 \end{aligned} \tag{12.2}$$

where A_i and B_i (i = 1,2,3,0) are functions of v, v', s and s' only. Substitute (12.2) into (11.11) to (11.18)

$$A_1 s + A_2 . s' + \frac{\partial A_0}{\partial s'} . B_0 = F(s') - A_3 . F(s) \tag{12.3}$$

$$\frac{\partial A_i}{\partial s'} . B_0 = 0 \qquad i = 1, 2, 3 \tag{12.4}$$

$$\frac{\partial A_i}{\partial v} + B_1 . \frac{\partial A_i}{\partial s'} = 0 \qquad i = 0, 1, 2, 3 \tag{12.5}$$

$$\frac{\partial A_i}{\partial v'} + B_2 . \frac{\partial A_i}{\partial s'} = 0 \qquad i = 0, 1, 2, 3 \tag{12.6}$$

$$\frac{\partial A_i}{\partial s} + B_3 . \frac{\partial A_i}{\partial s'} = 0 \qquad i = 0, 1, 2, 3. \tag{12.7}$$

$$B_1 . s + B_2 . s' + \frac{\partial B_0}{\partial s'} . A_0 = F(s') - B_3 . F(s) \tag{12.8}$$

$$\frac{\partial B_i}{\partial s'} . A_0 = 0 \qquad i = 1, 2, 3. \tag{12.9}$$

$$\frac{\partial B_i}{\partial v} + A_1 \cdot \frac{\partial B_i}{\partial s'} = 0 \qquad i = 0, 1, 2, 3 \tag{12.10}$$

$$\frac{\partial B_i}{\partial v'} + A_2 \cdot \frac{\partial B_i}{\partial s'} = 0 \qquad \text{"} \tag{12.11}$$

$$\frac{\partial B_i}{\partial s} + A_3 \cdot \frac{\partial B_i}{\partial s'} = 0 \qquad \text{"} \tag{12.12}$$

It will first be shown that $A_0 \neq 0$ and $B_0 \neq 0$. So suppose that

$$A_0 = 0 \tag{12.13}$$

It is then desired to prove a contradiction. Now if $B_0 \neq 0$ then (12.4) implies (with (12.5) to (12.7)) that A_1, A_2 and A_3 are constants. Yet (12.3) is $F(s') - A_2 . s' = A_3 . F(s) + A_1 . s$. This contradicts the hypothesis that F is not linear. Hence

$$B_0 = 0 \tag{12.14}$$

Consistency on (12.3) and (12.6) and also on (12.8) and (12.11) imply

$$A_2 . B_2 = A_2 . F'(s') = B_2 . F'(s') \tag{12.15}$$

That is, $A_2 = B_2$ is equal to 0 or $F'(s')$. If the latter is true then (12.11) is $F''(s').F'(s') = 0$ which contradicts (11.19). Hence

$$A_2 = B_2 = 0 \tag{12.16}$$

Then consistency on (12.5) and (12.3) implies $B_1 = 0$. Similarly (12.10) and (12.2) imply $A_1 = 0$. One then readily obtains a contradiction. Hence (12.3) cannot hold. Therefore

$$A_0 \neq 0 \quad , \quad B_0 \neq 0 \tag{12.17}$$

Then (12.4) to (12.12) imply that

A_i and B_i are constants for $i = 1, 2$ and 3 (12.18)

The consistency equations between (12.3) and (12.5), (12.6), (12.7) respectively are

$$A_2 \; B_i + (B_i - A_i) \frac{\partial A_0}{\partial s'} \cdot \frac{\partial B_0}{\partial s'} = B_i \cdot F'(s') \qquad i = 1, 2 \tag{12.19}$$

$$A_1 + A_2 B_3 + (B_3 - A_3) \cdot \frac{\partial A_0}{\partial s'} \cdot \frac{\partial B_0}{\partial s'} = B_3 \cdot F'(s') - A_3 \cdot F'(s) \tag{12.20}$$

Similarly from (12.8) to (12.12) one has

$$B_2 \cdot A_i + (A_i - B_i) \cdot \frac{\partial A_0}{\partial s'} \cdot \frac{\partial B_0}{\partial s'} = A_i \cdot F'(s') \qquad i = 1, 2. \tag{12.21}$$

$$B_1 + B_2 \cdot A_3 + (A_3 - B_3) \cdot \frac{\partial A_0}{\partial s'} \cdot \frac{\partial B_0}{\partial s'} = A_3 \cdot F'(s') - B_3 \cdot F'(s). \tag{12.22}$$

Adding the $i = 2$ equations and differentiating w.r.t. s' gives

$$A_2 = B_2 = 0 \tag{12.23}$$

Adding the $i = 1$ equations of (12.19) and (12.21) implies

$$A_1 + B_1 = 0 \tag{12.24}$$

Add (12.20) to (12.22)

$$A_3 + B_3 = 0 \tag{12.25}$$

If $A_1 \neq 0$ then adding $-A_3.A_1^{-1}$ times (12.19) to (12.20) and using (12.24) and (12.25) one sees that because F is not linear one has a contradiction. Therefore

$$A_1 = B_1 = 0$$

But one then sees that (11.4) and (11.5) have the form

$$s_{xy} = F(s)$$

$$s'_x = P(s,s',s_x,s_y)$$
$$s'_y = Q(s,s',s_x,s_y)$$

The result is known in this case and so there are no B.T. of the form (11.5) for equations of the form (11.4) except those which are already known .

Table 1 All Bäcklund Transformations from $\phi_{xy} = F(\phi)$

Case	$F(\phi)$	Bäcklund Transformation
1	$G'(\phi)$	$\phi_x' = K_1 . p^2 + 2K_2 . G(\phi) + K_3$ $\phi_y' = 2K_1 . G(\phi) + K_2 . q^2 + K_4$
2	$F(\phi)$ where $F''(\phi) = \mu^2 F(\phi)$ $\mu \neq 0$	$\phi_x' = \frac{\mu}{\lambda^2} . \cosh \lambda\phi' . F(\phi) + \frac{1}{\lambda^2} . F'(\phi)$ $\phi_y' = 1 + \frac{\mu}{\lambda} . q \sinh \lambda\phi'$ μ and λ are real or pure imaginary
3	Constant	$\phi_x' = \lambda_2 . (F\phi' - Kp) + A_1 . e^{\lambda_1 p} + K\lambda_2 \left[\frac{1 + \lambda_1 . p - e^{\lambda_1 p}}{\lambda_1}\right]$ $\phi_y' = \lambda_1 . F\phi' + K + A_2 e^{\lambda_2 q}$
4	0	$\phi_x' = \lambda\phi' . p + B(p)$ $\phi_y' = C(q) e^{\lambda\phi}$
5	$Ae^{\lambda\phi}$	$\phi_x' = Ae^{\lambda\phi}$ $\phi_y' = \frac{1}{2}\lambda q^2 + f(\phi' - q)$
6	0	$\phi_x' = 0$ $\phi_y' = Q(\phi', q)$
7	$K_0 + K_1 . \phi$	$\phi_x' = K_2(K_0 + K_1\phi) e^{\lambda\phi'} + \frac{K_1}{\lambda^2}$ $\phi_y' = 1 + K_2 \lambda q . e^{\lambda\phi'}$

Table 2 All Bäcklund Transformations to $\phi_{xy}' = f(\phi', \phi_x', \phi_y')$ via case 1 of table 1

Case	$F(\phi)$	$f(\phi', \phi_x', \phi_y')$	Transformation
1	$\frac{2a_2}{K_1K_2}.(2a_2\phi + a_1)$	$\pm 4a_2\sqrt{\frac{(\phi_x' - k_1)(\phi_y' - K_2)}{K_1K_2}}$	$\phi_x' = \frac{1}{2}K_2\left[p - \left(\frac{a_1 + 2a_2\phi}{K_2}\right)\right]^2 + k_1$ $\phi_y' = \frac{1}{2}K_1\left[q - \left(\frac{a + 2a_2\phi}{K_1}\right)\right]^2 + k_2$
2	$2\lambda(\phi + k)$	$-4\phi'\lambda - \frac{z_1z_2}{K_1K_2}.[z_1 + z_2 - 6\phi']$ where $z_1 = 2\phi' \pm \sqrt{4\phi'^2 - 2K_2(\lambda K_1 - p')}$ $z_2 = 2\phi' \pm \sqrt{4\phi'^2 - 2K_1(\lambda K_2 - q')}$	$\phi_x' = \frac{1}{2}K_2\left(\frac{p}{\phi + k_1}\right)^2 + \lambda K_1 - 2\phi'\left(\frac{p}{\phi + k_1}\right)$ $\phi_y' = \frac{1}{2}K_1\left(\frac{q}{\phi + k_1}\right)^2 + \lambda K_2 - 2\phi'\left(\frac{q}{\phi + k_1}\right)$
3	$G'(\phi)$	$\pm\sqrt{2K_1(q' - K_4)}$ $\times G'\left[G^{-1}\left(\frac{p' - K_3}{K_1}\right)\right]$	$\phi_x' = K_1.G(\phi) + K_3$ $\phi_y' = \frac{1}{2}K_1q^2 + K_4$
4	0	0	$\phi_x' = K_3 + \frac{1}{2}K_2p^2 - K_5p$ $\phi_y' = \frac{1}{2}K_1q^2 + K_4 - K_5q$
5	k_1	$\pm k_1\sqrt{2(\phi_y' - k_4)}$	$\phi_x' = k_2 + k_1\phi - k_3p$ $\phi_y' = \frac{1}{2}[q - k_3]^2 + k_4$
6	0	0	$\phi_x' = -p$ $\phi_y' = -q + f(\phi + \phi').(\frac{1}{2}q^2 + K)$.

CHAPTER 3.

§1. Introduction.

An attempt is made in this chapter to find B.T. for equations of the form

$$u_{xy} = F(u) \qquad (1.1)$$

other than the sine-Gordon. To do this the usual form of the B.T. considered previously needs to be extended. The first thing one might try is to allow higher derivatives than is usual in the B.T. The simplest case of this type is

$$\begin{aligned} u'_x &= P(u,u',p_1,p_2,\ldots,p_N;q_1,\ldots,q_M) \\ u'_y &= Q(u,u',p_1,p_2,\ldots,p_N;q_1,\ldots,q_M) \end{aligned} \qquad (1.2)$$

where M and N are positive integers and

$$p_i = \frac{\partial^i u}{\partial x^i} \quad , \qquad q_i = \frac{\partial^i u}{\partial y^i} \qquad (1.3)$$

The x and y subscripts denote partial differentiation. Note that one need not consider the mixed derivatives of u in the B.T. because u satisfies (1.1). The B.T. (1.2) is to be interpreted to mean that if u is any solution of (1.1) and if u' is any solution of (1.2) then u' satisfies (1.1) also i.e. $u'_{xy} = F(u')$

If one had a B.T. of the form (1.2) theh one could find u' given u by only integrating first order equations. If one allowed higher derivatives of u' then one would need to integrate equations of order greater than one which would make the B.T. of less use.

It will be assumed throughout that

$$\begin{aligned} &F''(u) \neq K.F(u) \\ \text{and} \quad &F'''(u) \neq K.F''(u) \end{aligned} \qquad (1.4)$$

for all constants K.

When looking for functions F which have the B.T. (1.2) one first proves that F satisfies a linear equation

$$F(u) = \sum_{i=1}^{n} \sum_{j=0}^{m} a_{ij}\, u^j \exp(\lambda_i u) \qquad (1.5)$$

where the λ_i are distinct but possibly complex. One also finds that $M=1$ or $N=1$.

It will be assumed throughout that there exists two constants λ in (1.5), say λ_1 and λ_2, such that for $i=1,2$ $a_{i0} \neq 0$, $a_{ij} = 0$ if $j>0$ If this is not the case then the proof should go through much the same. However, as the primary aim was to find B.T. for the double sine-Gordon equation, this has not been done. The problem, though, is set up in general in section 2.

It has been found that there are no equations of the form (1.1) with B.T. of the form (1.2) unless possibly $\lambda_1 + 2\lambda_2 = 0$. it is then shown that for $M=1$ one must have $N>7$ if a B.T. is to exist.

§2. Definitions.

Various properties of the functions

$$f_n(u, p_1, \ldots, p_{n-1}) = \left(\frac{\partial}{\partial x}\right)^{n-1} F(u) \qquad (2.1)$$

are discussed in this section. Some of these properties are known[13] but not, to my knowledge, equation (2.9) which is basic to this work.

The first few f_n are

$$\begin{aligned} f_1 &= F(u) \\ f_2 &= F'(u).p_1 \\ f_3 &= F''(u).p_1^2 + F'(u).p_2 \end{aligned} \qquad (2.2)$$

Further, if λ_1 and λ_2 are complex constants then define, for $k=1,2$ and for $n \geq 1$

$$f_n^{(k)} = \left\{ \left(\frac{\partial}{\partial x}\right)^{n-1} \exp(\lambda_k . u) \right\} . \exp(-\lambda_k . u) \qquad (2.3)$$

so $f_n^{(k)}$ is a function of $p_1, \ldots, p_{n-1}$ only and the first few are

$$f_1^{(k)} = 1$$
$$f_2^{(k)} = \lambda_k \cdot p_1 \qquad (2.4)$$
$$f_3^{(k)} = \lambda_k^2 \cdot p_1^2 + \lambda_k \cdot p_2$$
$$f_4^{(k)} = \lambda_k^3 \cdot p_1^3 + 3\lambda_k^2 \cdot p_1 \cdot p_2 + \lambda_k \cdot p_3$$
$$f_5^{(k)} = \lambda_k^4 \cdot p_1^4 + 6\lambda_k^3 \cdot p_1^2 \cdot p_2 + 4\lambda_k^2 \cdot p_1 \cdot p_3 + 3\lambda_k^2 \cdot p_2^2 + \lambda_k \cdot p_4$$
$$f_6^{(k)} = \lambda_k^5 \cdot p_1^5 + 10\lambda_k^4 \cdot p_1^3 \cdot p_2 + 15\lambda_k^3 \cdot p_1 \cdot p_2^2 + 10\lambda_k^3 \cdot p_1^2 \cdot p_3 + 10\lambda_k^2 \cdot p_2 \cdot p_3 + 5\lambda_k^2 \cdot p_1 \cdot p_4 + \lambda_k \cdot p_5$$

One could, instead of using (2.3), define $f_n^{(k)}$ by induction

$$f_{n+1}^{(k)} = \sum_{j=0}^{n-1} \binom{n-1}{j} \cdot p_{j+1} \cdot \lambda_k \cdot f_{n-j}^{(k)} \qquad n \geq 1$$
$$f_1^{(k)} = 1 \qquad (k = 1,2\ ) \qquad (2.5)$$

From (2.5) one may show by induction on n that for n >

$$f_n^{(k)} = \lambda_k \cdot p_{n-1} + (n-1) \cdot \lambda_k^2 \cdot p_1 \cdot p_{n-2} + \tfrac{1}{2}(n-1)(n-2) \cdot \lambda_k^2 \cdot (p_2 + \lambda_k \cdot p_1^2\ ) \cdot p_{n-3} + O(p_{n-4}) \qquad (2.6)$$

Define, for N>1

$$\mathcal{D}^{(N)} = \sum_{i=1}^{N-1} f_i \cdot \frac{\partial}{\partial p_i}$$
$$(2.7)$$
$$\mathcal{D}_k^{(N)} = \sum_{i=1}^{N-1} f_i^{(k)} \cdot \frac{\partial}{\partial p_i} \qquad k = 1,2$$

Define for N > 2

$$\mathcal{B}^{(N)} = \sum_{i=1}^{N-2} \kappa_{i+1} \cdot \frac{\partial}{\partial \kappa_i} \qquad (2.8)$$

From (2.5) for i < N , it may be shown, by induction on i, that

$$f_i^{(k)} = \left(\mathcal{B} + \lambda_k \cdot \kappa_1 \right)^{i-1} \cdot 1 \quad , \quad 1 \leq i \leq N \qquad (2.9)$$

In most cases the superscript is left off $\mathcal{D}$ and $\mathcal{B}$ and is assumed to be N .

Now if

$$\mathcal{a} = \sum_{i=1}^{M} a_i \cdot \frac{\partial}{\partial \kappa_i} \quad \text{and} \quad \mathcal{b} = \sum_{i=1}^{M} c_i \cdot \frac{\partial}{\partial \kappa_i} \qquad (2.10)$$

then

$$\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} = \sum_{i=1}^{M} \left[\mathcal{A}(c_i) - \mathcal{B}(a_i) \right] \cdot \frac{\partial}{\partial x_i} \tag{2.11}$$

This property will be used extensively throughout. From (2.7), (2.8), (2.9) and (2.11) for k = 1, 2 one has

$$\left(\mathcal{B}\mathcal{D}_k - \mathcal{D}_k\mathcal{B}\right)X = -\lambda_k \cdot \kappa_1 \cdot \mathcal{D}_k X \tag{2.12}$$

provided X is a function of $p_1, \dots, p_{N-2}$ only.

Define, for $n > 1$,

$$\mathcal{D}_{n+1} = \mathcal{D}_1 \mathcal{D}_n - \mathcal{D}_n \mathcal{D}_1 \tag{2.13}$$

Then

$$\mathcal{D}_n = \sum_{i=n-1}^{N-1} f_i^{(n)} \cdot \frac{\partial}{\partial x_i}, \qquad 2 \leq n \leq N \tag{2.14}$$

where

$$f_i^{(n+1)} = \mathcal{D}_1\left[f_i^{(n)}\right] - \mathcal{D}_n\left[f_i^{(1)}\right] \tag{2.15}$$

Define

$$\begin{aligned} \kappa_3 &= (\lambda_2 - \lambda_1) \\ \kappa_{n+2} &= \left(\tfrac{1}{2}n(n-1)\lambda_1 + n\lambda_2\right)\cdot\kappa_{n+1} \qquad n > 1 \end{aligned} \tag{2.16}$$

Also define $g_i^{(n)}$ by $f_i^{(n)} = \kappa_n \cdot g_i^{(n)}$ (2.17)

The first few $g_i^{(n)}$ are

$$\begin{aligned} g_2^{(3)} &= 1 \\ g_3^{(3)} &= 2(\lambda_1 + \lambda_2)\cdot p_1 \\ g_4^{(3)} &= 3(\lambda_1 + \lambda_2)\cdot p_2 + (3\lambda_1^2 + 5\lambda_1\cdot\lambda_2 + 3\lambda_2^2)\cdot p_1^2 \\ g_5^{(3)} &= 4(\lambda_1 + \lambda_2)\cdot p_3 + (12\lambda_1^2 + 19\lambda_1\cdot\lambda_2 + 12\lambda_2^2)\cdot p_1 p_2 + (4\lambda_1^3 + 9\lambda_1^2\cdot\lambda_2 + 9\lambda_1\cdot\lambda_2^2 + 4\lambda_2^3)\cdot p \\ g_6^{(3)} &= 5(\lambda_1 + \lambda_2)\cdot p_4 + (20\lambda_1^2 + 31\lambda_1\cdot\lambda_2 + 20\lambda_2^2)\cdot p_1 p_3 + (15\lambda_1^2 + 22\lambda_1\cdot\lambda_2 + 15\lambda_2^2)\cdot p_2^2 \\ &\quad + (30\lambda_1^3 + 64\lambda_1^2\cdot\lambda_2 + 64\lambda_1\cdot\lambda_2^2 + 30\lambda_2^3)\cdot p_1^2\cdot p_2 + (5\lambda_1^4 + 14\lambda_1^3\cdot\lambda_2 + 19\lambda_1^2\cdot\lambda_2^2 \\ &\qquad + 14\lambda_1\cdot\lambda_2^3 + 5\lambda_2^4)\cdot p_1^4 \end{aligned} \tag{2.18}$$

$$\left.\begin{aligned}
g_3^{(4)} &= 1\\
g_4^{(4)} &= (4\lambda_1 + 3\lambda_2).p_1\\
g_5^{(4)} &= (7\lambda_1 + 6\lambda_2).p_2 + (11\lambda_1^2 + 15\lambda_1.\lambda_2 + 6\lambda_2^2).p_1^2\\
g_6^{(4)} &= (11\lambda_1 + 10\lambda_2).p_3 + (48\lambda_1^2 + 68\lambda_1.\lambda_2 + 30\lambda_2^2).p_1.p_2\\
&\qquad + (26\lambda_1^3 + 50\lambda_1^2.\lambda_2 + 36\lambda_1.\lambda_2^2 + 10\lambda_2^3).p_1^3
\end{aligned}\right\} \tag{2.19}$$

$$\left.\begin{aligned}
g_4^{(5)} &= 1\\
g_5^{(5)} &= (7\lambda_1 + 4\lambda_2).p_1\\
g_6^{(5)} &= (14\lambda_1 + 10\lambda_2).p_2 + (32\lambda_1^2 + 34\lambda_1.\lambda_2 + 10\lambda_2^2).p_1^2
\end{aligned}\right\} \tag{2.20}$$

$$\left.\begin{aligned}
g_5^{(6)} &= 1\\
g_6^{(6)} &= (11\lambda_1 + 5\lambda_2).p_1\\
g_6^{(7)} &= 1
\end{aligned}\right\} \tag{2.21}$$

From (2.9) and (2.13) if X is a function of $p_1, \dots, p_{N-2}$ only

$$\mathcal{L}_3 \mathcal{B} X = \left[\mathcal{B}\mathcal{D}_3 + (\lambda_1 + \lambda_2) \varkappa_1 . \mathcal{D}_3 + \lambda_2 \mathcal{D}_2 - \lambda_1 \mathcal{D}_1\right] . X \tag{2.22}$$

Then one may prove, by induction, that for $n > 3$

$$\begin{aligned}
\mathcal{D}_n \mathcal{B} X = \Big\{ &\mathcal{B}\mathcal{D}_n + \left[(n-2)\lambda_1 + \lambda_2\right] \varkappa_1 . \mathcal{D}_n \\
&+ \left[\tfrac{1}{2}(n-2)(n-3)\lambda_1 + (n-2)\lambda_2\right] . \mathcal{D}_{n-1} \Big\} X
\end{aligned} \tag{2.23}$$

From (2.9), (2.12) and (2.15) one has for $0 < i < N$:

$$\begin{aligned}
f_{i+1}^{(3)} &= \mathcal{D}_1\left[f_{i+1}^{(2)}\right] - \mathcal{D}_2\left[f_{i+1}^{(1)}\right]\\
&= \mathcal{D}_1\left\{\left[\mathcal{B} + \lambda_2 \varkappa_1\right] f_i^{(2)}\right\} - \mathcal{D}_2\left\{\left[\mathcal{B} + \lambda_1 \varkappa_1\right] f_i^{(1)}\right\}\\
&= \left\{\left[\mathcal{B}\mathcal{D}_1 + \lambda_1 \varkappa_1 \mathcal{D}_1\right] + \lambda_2 + \lambda_2 \varkappa_1 \mathcal{D}_1\right\} f_i^{(2)}\\
&\quad - \left\{\left[\mathcal{B}\mathcal{D}_2 + \lambda_2 \varkappa_1 \mathcal{D}_2\right] + \lambda_1 + \lambda_1 \varkappa_1 \mathcal{D}_2\right\} f_i^{(1)}
\end{aligned}$$

$$f^{(3)}_{i+1} = \mathcal{B} f^{(3)}_{i} + (\lambda_1+\lambda_2)\chi_1 . f^{(3)}_{i} + \lambda_2 . f^{(2)}_{i} - \lambda_1 . f^{(1)}_{i} \qquad (2.24)$$

In a similar manner it is possible to prove that for $n > 3$

$$f^{(n)}_{i+1} = \mathcal{B} f^{(n)}_{i} + \left[(n-2)\lambda_1 + \lambda_2\right].\chi_1 . f^{(n)}_{i} + \left[\tfrac{1}{2}(n-2)(n-3)\lambda_1 + (n-2)\lambda_2\right]. f^{(n-1)}_{i} \qquad (2.25)$$

Since $f^{(n)}_{n-2} = 0$ one has from (2.25) that $f^{(n)}_{n-1}$ is a constant and

$f^{(n)}_{n-1} = (\frac{1}{2}(n-2)(n-3)\lambda_1 + (n-2)\lambda_2).f^{(n-1)}_{n-2}$, $n > 3$

Hence from (2.16) , (2.17) and (2.26) one has

$$g^{(n)}_{n-1} = 1 \quad , \quad n > 2 \qquad (2.26)$$

Also from (2.25) and (2.16) for $n > 3$ one has

$$g^{(n)}_{i+1} = \left\{ \mathcal{B} + \left[(n-2).\lambda_1 + \lambda_2\right].\chi_1 \right\} . g^{(n)}_{i} + g^{(n-1)}_{i} \qquad (2.27)$$

From (2.18) and (2.27) one may prove, by induction, that

$$g^{(n)}_{n} = (\tfrac{1}{2}(n^2-3n+4).\lambda_1 + (n-1).\lambda_2).p_1 \qquad (2.28)$$

$$g^{(n)}_{n+1} = \Big((1/6)(n^3-3n^2+8n-6).\lambda_1 + \tfrac{1}{2}n(n-1).\lambda_2\Big).p_2 + \Big((1/24)(3n^4-14n^3+33n^2-46n+48).\lambda_1^2 + \tfrac{1}{2}(n^3-3n^2+4n-2).\lambda_1.\lambda_2 + \tfrac{1}{2}(n^2-n).\lambda_2^2\Big).p_1^2 \qquad (2.29)$$

Provided $\lambda_2 + \lambda_1 \neq 0$ and $\lambda_1 + 2\lambda_2 \neq 0$ one may w.l.o.g. take $\kappa_n \neq 0$ for all n. This is because if $(n-2).\lambda_1 + 2\lambda_2 = 0$ for some $n > 2$ theninstead of the sequence defined by (2.13) one could use the sequence $\mathcal{D}_{n+1} = \mathcal{D}_2 \mathcal{D}_n - \mathcal{D}_n \mathcal{D}_2$, $n \geq 4$ and $\mathcal{D}_3 = \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1$
However the case $\lambda_1 + 2\lambda_2 = 0$ must be considered separately.

Case 2.

$$\lambda_1 + 2\lambda_2 = 0 \qquad (2.30)$$

Define

$$\mathcal{T}_1 = \mathcal{D}_1 \quad , \quad \mathcal{T}_2 = \mathcal{D}_2$$
$$\mathcal{T}_3 = \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1 \qquad (2.31)$$

Also define, for n a non-negative integer,

$$\left.\begin{aligned} \mathcal{J}_{8n+i} &= \mathcal{J}_2 \cdot \mathcal{J}_{8n+i-1} - \mathcal{J}_{8n+i-1} \mathcal{J}_2 && i = 4, 5 \\ \mathcal{J}_{8n+i} &= \mathcal{J}_1 \cdot \mathcal{J}_{8n+i-1} - \mathcal{J}_{8n+i-1} \cdot \mathcal{J}_1 && i = 6, 9, 11 \\ \mathcal{J}_{8n+i} &= \mathcal{J}_2 \cdot \mathcal{J}_{8n+i-2} - \mathcal{J}_{8n+i-2} \cdot \mathcal{J}_2 && i = 7, 8, 10 \end{aligned}\right\} \tag{2.32}$$

Then

$$\mathcal{J}_n = \sum_{i=1}^{N-1} t_i^{(n)} \cdot \frac{\partial}{\partial \chi_i} \tag{2.33}$$

It is then possible to show, by induction, that for $n \geq 0$

$$\left.\begin{aligned} 0 &= \mathcal{J}_1 \mathcal{J}_{8n+3} - \mathcal{J}_{8n+3} \cdot \mathcal{J}_1 \\ \mathcal{J}_{8n+4} \cdot \mathcal{B} &= \mathcal{B} \mathcal{J}_{8n+4} - 3\lambda_2 \cdot \mathcal{J}_{8n+3} \\ t_{i+1}^{(8n+4)} &= \mathcal{B}\, t_i^{(8n+4)} - 3\lambda_2 \cdot t_i^{(8n+3)} \end{aligned}\right\} \tag{2.34}$$

provided that the operators in the first and second lines act on functions of $p_1, \ldots, p_{N-2}$ only. In proving (2.34) one obtains

$$\left.\begin{aligned} 0 &= \mathcal{J}_1 \mathcal{J}_{8n+i} - \mathcal{J}_{8n+i} \cdot \mathcal{J}_1 && i = 4, 6, 9 \\ 0 &= \mathcal{J}_2 \cdot \mathcal{J}_{8n+i} - \mathcal{J}_{8n+i} \cdot \mathcal{J}_2 && i = 7, 10 \\ 2 \cdot \mathcal{J}_{8n+8} &= \mathcal{J}_1 \cdot \mathcal{J}_{8n+7} - \mathcal{J}_{8n+7} \cdot \mathcal{J}_1 \\ 2\, \mathcal{J}_{8n+11} &= \mathcal{J}_2 \cdot \mathcal{J}_{8n+9} - \mathcal{J}_{8n+9} \cdot \mathcal{J}_2 \end{aligned}\right\} \tag{2.35}$$

As an example of the proof of (2.35) define

$$a_i = \mathcal{T}_1\left[t_i^{(8n+4)}\right] - \mathcal{T}_{8n+4}\left[t_i^{(1)}\right]$$

Then, by induction, $a_i = 0$ since clearly $a_1 = 0$ and the $i = 4$ result of (2.35) is proved.

Also in the proof of (2.34) one obtains

$$\left.\begin{aligned}
t_{i+1}^{(8n+5)} &= \mathcal{B}\, t_i^{(8n+5)} + \lambda_2 \cdot \kappa_1 \cdot t_i^{(8n+5)} - 3\lambda_2 \cdot t_i^{(8n+4)} \\
t_{i+1}^{(8n+6)} &= \mathcal{B}\, t_i^{(8n+6)} - \lambda_2 \cdot \kappa_1 \cdot t_i^{(8n+6)} + \lambda_2 \cdot t_i^{(8n+5)} \\
t_{i+1}^{(8n+7)} &= \mathcal{B}\, t_i^{(8n+7)} + 2\lambda_2 \cdot \kappa_1 \cdot t_i^{(8n+7)} - 2\lambda_2 \cdot t_i^{(8n+5)} \\
t_{i+1}^{(8n+8)} &= \mathcal{B}\, t_i^{(8n+8)} - \lambda_2 \cdot t_i^{(8n+6)} + \lambda_2 \cdot t_i^{(8n+7)} \\
t_{i+1}^{(8n+9)} &= \mathcal{B}\, t_i^{(8n+9)} - 2\lambda_2 \cdot \kappa_1 \cdot t_i^{(8n+9)} + 2\lambda_2 \cdot t_i^{(8n+8)} \\
t_{i+1}^{(8n+10)} &= \mathcal{B}\, t_i^{(8n+10)} + \lambda_2 \cdot \kappa_1 \cdot t_i^{(8n+10)} - \lambda_2 \cdot t_i^{(8n+8)} \\
t_{i+1}^{(8n+11)} &= \mathcal{B}\, t_i^{(8n+11)} - \lambda_2 \cdot \kappa_1 \cdot t_i^{(8n+11)} + \lambda_2 \cdot t_i^{(8n+10)} - \lambda_2 \cdot t_i^{(8n+9)}
\end{aligned}\right\} \quad (2.36)$$

One may then prove, by induction, for $m \geq 0$ that for fixed m the first non-zero $t_i^{(m)}$ are as follows

$$\left.\begin{aligned}
&t_{6m+2}^{(8m+3)} = 3\lambda_2(81\lambda_2^6)^m, \quad t_{6m+3}^{(8m+4)} = -9\lambda_2^2(81\lambda_2^6)^m, \quad t_{6m+4}^{(8m+5)} = 27\lambda_2^3(81\lambda_2^6)^m \\
&t_{6m+5}^{(8m+6)} = 27\lambda_2^4(81\lambda_2^6)^m, \quad t_{6m+5}^{(8m+7)} = -54\lambda_2^4(81\lambda_2^6)^m, \quad t_{6m+6}^{(8m+8)} = -81\lambda_2^5(81\lambda_2^6)^m \\
&t_{6m+7}^{(8m+9)} = -162\lambda_2^6(81\lambda_2^6)^m, \quad t_{6m+8}^{(8m+10)} = 81\lambda_2^6(81\lambda_2^6)^m
\end{aligned}\right\} \quad (2.37)$$

<u>Case 3.</u>

Let $r_n^{(2)}$ be the coefficient of $k.u^{k-1}.e^{\lambda u}$ in $\left(\frac{\partial}{\partial x}\right)^{n-1}\left[u^k e^{\lambda u}\right]$

$$\therefore \left.\begin{aligned}
&r_1^{(2)} = 0 \\
&r_n^{(2)} = \sum_{i=1}^{n-1} \binom{n-1}{i} \cdot \kappa_i \cdot f_{n-i}^{(1)} \qquad n \geq 2
\end{aligned}\right\} \quad (2.38)$$

(with $\lambda = \lambda_1$)

Then from (2.9) and (2.38) one has

$$r^{(2)}_{n+1} = \mathcal{B} r^{(2)}_{n} + x_1 \cdot \left[f^{(1)}_{n} + \lambda_1 \cdot r^{(2)}_{n} \right] \tag{2.39}$$

Define

$$\mathcal{R}_1 = \mathcal{D}_1 \quad , \quad \mathcal{R}_2 = \sum_{i=1}^{N-1} r^{(1)}_i \cdot \frac{\partial}{\partial x_i} \tag{2.40}$$

$$\left.\begin{aligned} \mathcal{R}_n &= \mathcal{R}_1 \mathcal{R}_{n-1} - \mathcal{R}_{n-1} \cdot \mathcal{R}_1 \quad , \quad n \geqslant 3 \\ \mathcal{R}_n &= \sum_{i=1}^{N-1} r^{(n)}_i \cdot \frac{\partial}{\partial x_i} \end{aligned}\right\} \tag{2.41}$$

$$\therefore \mathcal{B} \mathcal{R}_2 - \mathcal{R}_2 \mathcal{B} = - x_1 \cdot \left[\mathcal{R}_1 + \lambda \mathcal{R}_2 \right] \tag{2.42}$$

Then from (2.9) , (2.12) , (2.39) , (2.41) and (2.42) :

$$r^{(3)}_{i+1} = \mathcal{B} r^{(3)}_i + 2 \lambda \cdot x_1 \cdot r^{(3)}_i + f^{(1)}_i + \lambda \cdot r^{(2)}_i \tag{2.43}$$

One may then prove, by induction on n, that for $n > 2$

$$r^{(n+1)}_{i+1} = \mathcal{B} r^{(n+1)}_i + n \lambda \cdot x_1 \cdot r^{(n+1)}_i + \tfrac{1}{2} n (n+1) \cdot r^{(n)}_i \tag{2.44}$$

and for $n > 1$

$$r^{(n+1)}_i = 0 \text{ if } i < n \quad , \quad r^{(n+1)}_n = 2^{1-n} . n . \left[(n-1)! \right]^2 . \lambda_1^{n-2} \tag{2.45}$$

This then completes the basic properties of f_n required.

§3.Lemma.

Let P be a function of $p_1 , \ldots , p_n$ i.e. in particular

$$\frac{\partial P}{\partial u} = 0 \quad , \quad \frac{\partial P}{\partial x_i} = 0 \text{ if } i > n \tag{3.1}$$

and suppose

$$\sum_{k=1}^{n} f_k \cdot \frac{\partial P}{\partial x_k} = 0 \tag{3.2}$$

where f_k are given by (2.1) . Further suppose that

$$F'(u) \neq K.F(u) \tag{3.3}$$

It will now be shown that P is a constant. So for the rest of this section assume

$$\frac{\partial \rho}{\partial f_n} \neq 0 \tag{3.4}$$

It is then desired to find a contradiction.

Diff. (3.2) w.r.t. u and use (3.4) to obtain that the determinant of coefficients must be zero. Note that from (2.1) f_n have the form

$$f_n = \sum_{k=1}^{n-1} \alpha_{nk} \cdot F^{(k)}(u) \quad , n \geq 2 \tag{3.5}$$

where $F^{(k)}(u) = \dfrac{d^k F}{du^k}$ and α_{nk} are functions of $p_1, \ldots, p_{n-1}$ only i.e. not of u. So one has that

$$\det(F^{(i+j-2)}(u)) = 0 \quad , \; i,j = 1,\ldots,n \tag{3.6}$$

Integrating (3.6) gives that F satisfies a linear equation, so

$$F(u) = \sum_{i=1}^{l} \sum_{j=0}^{l_i - 1} a_{ij} \cdot u^j \cdot e^{\lambda_i \cdot u} \tag{3.7}$$

where a_{ij} and λ_i are complex constants and l and l_i are positive integers.

Case 1.

Suppose $a_{ij} \neq 0$ for some $j > 0$ with $\lambda_i \neq 0$. In particular take $a_{1k} \neq 0$ for some $k > 0$, $a_{1j} = 0$ for $j > k$ and $\lambda = \lambda_1 \neq 0$ (3.8)

The coefficients of $u^k.\exp \lambda u$ and $k.u^{k-1}.\exp \lambda u$ in (3.2) are

$$\mathcal{R}_i \cdot \rho = 0 \tag{3.9}$$

for $i = 1, 2$. But then from the definition (2.41) one has that (3.9) holds for all positive integers i. In particular it holds for $i = n+1$. Then from (2.45) one has a contradiction. So it has been shown that (3.7) must be

$$F(u) = \sum_{i=0}^{L} A_i \cdot u^i + \sum_{i=L+1}^{n-1} A_i \cdot e^{\lambda_i \cdot u} \tag{3.10}$$

where L is a positive integer less than n, A_i are complex constants and λ_i are distinct, non-zero complex constants.

Case 2.

Suppose $A_L \neq 0$ for $L > 0$ (3.11)

The coefficients of u^L and u^{L-1} in (3.2) are then

$$\frac{\partial \rho}{\partial \kappa_1} = 0 \qquad \sum_{i=2}^{n} \kappa_{i-1} \cdot \frac{\partial \rho}{\partial \kappa_i} = 0 \tag{3.12}$$

But these contradict (3.4). So it has been shown that

$$L = 0 \tag{3.13}$$

Case 3.

Suppose there are two non-zero λ_i. That is take

$$\lambda_1 \neq 0 \quad \text{and} \quad \lambda_2 \neq 0 \tag{3.14}$$

The coefficients of $\exp \lambda_i u$ in (3.2) for $i = 1,2$ are

$$\mathscr{D}_i \, \rho = 0 \tag{3.15}$$

Then from the definition (2.13) one has that (3.15) is true for all positive integers i. Then $i = n+1$ in (3.15) gives, using (3.4), that $\kappa_{n+1} = 0$. From the discussion below (2.29) this implies

$$\lambda_1 + 2\lambda_2 = 0 \tag{3.16}$$

But then from (3.15) and (2.31) one must have

$$\mathscr{T}_i \, \rho = 0 \tag{3.17}$$

for $i = 1,2$. The definitions (2.31) and (2.32) then imply that (3.17) is true for all positive integers i. Then (2.37) gives a contradiction to (3.4). It now only remains to consider :

Case 4.

$$F(u) = A_0 + A_1 . \exp \lambda u \tag{3.18}$$

where λ, A_0 and A_1 are non-zero constants. Then (3.2) gives

$$\mathscr{D}_1 \rho = 0 \quad \text{with} \quad \lambda_1 = \lambda \quad ; \quad \frac{\partial \rho}{\partial \kappa_1} = 0 \tag{3.19}$$

The coefficient of p_1^{n-1} in the first of (3.19) then gives a contradiction to (3.4) and the lemma is proved.

§4. That M=1 or N=1.

Differentiate the first equation in (1.2) w.r.t. y and the second w.r.t. x and use (1.1) .

$$F(u') = \frac{\partial P}{\partial u}\cdot q_1 + \frac{\partial P}{\partial u'}\cdot Q + \sum_{i=1}^{N} f_i\cdot\frac{\partial P}{\partial p_i} + \sum_{i=1}^{M} q_{i+1}\cdot\frac{\partial P}{\partial q_i} \tag{4.1}$$

$$F(u') = \frac{\partial Q}{\partial u}\cdot p_1 + \frac{\partial Q}{\partial u'}\cdot P + \sum_{i=1}^{N} p_{i+1}\cdot\frac{\partial Q}{\partial p_i} + \sum_{i=1}^{M} h_i\cdot\frac{\partial Q}{\partial q_i} \tag{4.2}$$

where f_i are given by (2.1) and h_i are given by

$$h_i(u, q_1, \dots, q_{i-1}) = \left(\frac{\partial}{\partial y}\right)^{n-1} \cdot F(u) \tag{4.3}$$

The rest of this chapter is devoted to the solution of (4.1) and (4.2).

Diff. (4.1) w.r.t. q_{M+1} and (4.2) w.r.t. p_{N+1}

$$\frac{\partial P}{\partial q_M} = 0 \qquad \text{and} \qquad \frac{\partial Q}{\partial p_N} = 0 \tag{4.4}$$

W.l.o.g. both p_N and q_M appear in (1.2) . Because of (4.4) this means

$$\frac{\partial P}{\partial p_N} \neq 0 \qquad \text{and} \qquad \frac{\partial Q}{\partial q_M} \neq 0 \tag{4.5}$$

If P does not depend on u' then the R.H.S. of (4.1) is independent of u' which means that F is a constant. Therefore

$$\frac{\partial P}{\partial u'} \neq 0 \tag{4.6}$$

Diff. (4.1) w.r.t. q_M twice and use (4.4) and (4.6) . Then integrate to obtain (also similarly from (4.2))

$$\begin{aligned} P &= P_0 + P_1\cdot p_N \\ Q &= Q_0 + Q_1\cdot q_M \end{aligned} \tag{4.7}$$

where P_0 , P_1 , Q_1 , Q_0 are functions of $u, u'; p_1, \dots, p_{N-1}; q_1, \dots, q_{M-1}$ only.

Substitute (4.7) into (4.1) and (4.2) and equate coefficients of p_N and q_M. ($p_0 = q_0 = u$)

$$F(u') = \frac{\partial p_0}{\partial u'} \cdot Q_0 + \sum_{i=1}^{N-1} f_i \cdot \frac{\partial p_0}{\partial p_i} + f_N \cdot p_1 + \sum_{i=0}^{M-2} q_{i+1} \cdot \frac{\partial p_0}{\partial q_i} \tag{4.8}$$

$$0 = \frac{\partial p_1}{\partial u'} \cdot Q_0 + \sum_{i=1}^{N-1} f_i \cdot \frac{\partial p_1}{\partial p_i} + \sum_{i=0}^{M-2} q_{i+1} \cdot \frac{\partial p_1}{\partial q_i} \tag{4.9}$$

$$0 = \frac{\partial p_i}{\partial u'} \cdot Q_1 + \frac{\partial p_i}{\partial q_{M-1}} \quad , \quad i = 0, 1 . \tag{4.10}$$

$$F(u') = \frac{\partial Q_0}{\partial u'} \cdot p_0 + \sum_{i=0}^{N-2} p_{i+1} \cdot \frac{\partial Q_0}{\partial p_i} + \sum_{i=1}^{M-1} h_i \cdot \frac{\partial Q_0}{\partial q_i} + h_M \cdot Q_1 \tag{4.11}$$

$$0 = \frac{\partial Q_1}{\partial u'} \cdot p_0 + \sum_{i=0}^{N-2} p_{i+1} \cdot \frac{\partial Q_1}{\partial p_i} + \sum_{i=1}^{M-1} h_i \cdot \frac{\partial Q_1}{\partial q_i} \tag{4.12}$$

$$0 = \frac{\partial Q_i}{\partial u'} \cdot p_1 + \frac{\partial Q_i}{\partial p_{N-1}} \qquad i = 0, 1 . \tag{4.13}$$

The result is already known in the case M = 1 and N = 1 . So take

$$N > 1 \tag{4.14}$$

Take $\frac{\partial Q_1}{\partial u'} \cdot \left(p_1 \cdot \frac{\partial}{\partial u'} + \frac{\partial}{\partial p_{N-1}} \right)(4.11) - \frac{\partial Q_0}{\partial u'} \cdot \left(p_1 \cdot \frac{\partial}{\partial u'} + \frac{\partial}{\partial p_{N-1}} \right) \cdot (4.12)$:

$$\therefore \quad \frac{\partial Q_1}{\partial u'} \cdot \left[\frac{\partial Q_0}{\partial p_{N-2}} - p_1 . F'(u') \right] = \frac{\partial Q_0}{\partial u'} \cdot \frac{\partial Q_1}{\partial p_{N-2}} \tag{4.15}$$

Take $p_1 \cdot \frac{\partial}{\partial u'} + \frac{\partial}{\partial p_{N-1}}$ of (4.15)

$$\therefore \quad \frac{\partial Q_1}{\partial u'} \cdot \left\{ p_1^2 . F''(u') + F'(u') . \left[p_1 \cdot \frac{\partial p_1}{\partial u'} + \frac{\partial p_1}{\partial p_{N-1}} \right] \right\} = 0 \tag{4.16}$$

Note that (4.5) implies that $P_1 \neq 0$. If $\frac{\partial Q_1}{\partial u'} \neq 0$ then take

$Q_1 \frac{\partial}{\partial u'} + \frac{\partial}{\partial q_{M-1}}$ of (4.16) after dividing by $\frac{\partial Q_1}{\partial u'}$. One then obtains

a contradiction to either (1.4) or (4.5). Hence

$$\frac{\partial Q_1}{\partial u'} = 0 \tag{4.17}$$

Then (4.13) implies

$$\frac{\partial Q_1}{\partial p_{N-1}} = 0 \tag{4.18}$$

Equating coefficients of p_{N-1} down to p_1 in (4.12) then gives that Q_1 is a function of $q_1, \ldots, q_{M-1}$ only. The lemma of section 3 is then applicable and one sees that

Q_1 is a constant (4.19)

The rest of this section is devoted to proving that $M = 1$. So suppose

$$M > 1 \tag{4.20}$$

Then exactly as above for Q_1 one may show that

P_1 is a constant (4.21)

Take $\left(Q_1 \frac{\partial}{\partial u'} + \frac{\partial}{\partial q_{M-1}}\right)^2$ of (4.8)

$$Q_1^2 \cdot F''(u') = \frac{\partial P_0}{\partial u'} \cdot \left(Q_1 \frac{\partial}{\partial u'} + \frac{\partial}{\partial q_{M-1}}\right)^2 Q_0 \tag{4.22}$$

Take $\left[\left(Q_1 \frac{\partial}{\partial u'} + \frac{\partial}{\partial q_{M-1}}\right)^2 Q_0\right]\left(Q_1 \frac{\partial}{\partial u'} + \frac{\partial}{\partial q_{M-1}}\right)^3 - \left[\left(Q_1 \frac{\partial}{\partial u'} + \frac{\partial}{\partial q_{M-1}}\right)^3 Q_0\right]\left(Q_1 \frac{\partial}{\partial u'} + \frac{\partial}{\partial q_{M-1}}\right.$

of (4.8) and note that (4.5) implies $Q_1 \neq 0$

$$F''(u') \cdot \left(Q_1 \frac{\partial}{\partial u'} + \frac{\partial}{\partial q_{M-1}}\right)^3 Q_0 \doteq Q_1 \cdot F'''(u')\left(Q_1 \frac{\partial}{\partial u'} + \frac{\partial}{\partial q_{M-1}}\right)^2 Q_0 \tag{4.23}$$

Take

$F'''(u') - F''(u') \cdot \left(P_1 \frac{\partial}{\partial u'} + \frac{\partial}{\partial p_{N-1}}\right)$ of (4.23) to obtain a contradiction to (1.4) . Note that if $\left(Q_1 \cdot \frac{\partial}{\partial u'} + \frac{\partial}{\partial q_{M-1}} \right)^2 Q_0 = 0$ then (4.22) contradicts (1.4) or (4.5) . Hence (4.20) cannot hold and so it has been shown that

$$M = 1 \tag{4.24}$$

§5. That $(P_1)_{u'} \neq 0$.

It will be shown in this section that $\frac{\partial P_1}{\partial u'} \neq 0$. Since the proof proceeds by contradiction, for this section assume

$$\frac{\partial P_1}{\partial u'} = 0 \tag{5.1}$$

Then (4.9) , (4.10) and the lemma of section 3 imply

$$P_1 \text{ is a constant} \tag{5.2}$$

Take $\left(P_1 \cdot \frac{\partial}{\partial u'} + \frac{\partial}{\partial x_{N-1}} \right)^2$ of (4.11)

$$P_1^2 \cdot F''(u') = \frac{\partial Q_0}{\partial u'} \cdot \left(P_1 \cdot \frac{\partial}{\partial u'} + \frac{\partial}{\partial x_{N-1}} \right)^2 P_0 \tag{5.3}$$

Take $\left[\left(P_1 \frac{\partial}{\partial u'} + \frac{\partial}{\partial x_{N-1}} \right)^3 P_0 \right] \left(P_1 \frac{\partial}{\partial u'} + \frac{\partial}{\partial x_{N-1}} \right)^2 - \left[\left(P_1 \cdot \frac{\partial}{\partial u'} + \frac{\partial}{\partial x_{N-1}} \right)^2 P_0 \right] \left(P_1 \frac{\partial}{\partial u'} + \frac{\partial}{\partial x_{N-1}} \right.$

of (4.11) and use (4.5)

$$F''(u') \cdot \left(P_1 \cdot \frac{\partial}{\partial u'} + \frac{\partial}{\partial x_{N-1}} \right)^3 P_0 = P_1 \cdot F'''(u') \cdot \left(P_1 \cdot \frac{\partial}{\partial u'} + \frac{\partial}{\partial x_{N-1}} \right)^2 P_0 \tag{5.4}$$

Take $Q_0 \cdot \frac{\partial}{\partial u'} + \sum_{i=1}^{N-1} f_i \cdot \frac{\partial}{\partial x_i}$ of (5.4) and note that $Q_0 \neq 0$ by (4.5) and (4.24) . Also note that $\frac{\partial f_N}{\partial x_{N-1}} = F'(u)$.

$$F'''(u') \cdot \left(P_1 \cdot \frac{\partial}{\partial u'} + \frac{\partial}{\partial x_{N-1}} \right)^3 P_0 = P_1 \cdot F^{iv}(u') \cdot \left(P_1 \cdot \frac{\partial}{\partial u'} + \frac{\partial}{\partial x_{N-1}} \right)^2 P_0 \tag{5.5}$$

Now $\left(P_1 \dfrac{\partial}{\partial u'} + \dfrac{\partial}{\partial p_{N-1}}\right)^2 P_0 \neq 0$ by (5.3) and (4.5) so take $F'''(u').(5.4) - F''(u').(5.5)$ to obtain a contradiction to (1.4). Hence (5.1) cannot hold and it has been shown that

$$\frac{\partial P_1}{\partial u'} \neq 0 \tag{5.6}$$

§6. If N = 2.

The general method to be explained in the next section only works if $N > 2$. The case $N = 2$ needs to be done separately and so this section deals with it. For $N = 2$ the equations to be solved, namely (4.8) to (4.13), are, on using (4.19) and (4.24)

$$F(u') = \frac{\partial P_0}{\partial u'} . Q_0 + F(u) \cdot \frac{\partial P_0}{\partial p} + P_1 . F'(u) . p \tag{6.1}$$

$$0 = \frac{\partial P_1}{\partial u'} . Q_0 + F(u) \cdot \frac{\partial P_1}{\partial p} \tag{6.2}$$

$$0 = \frac{\partial P_i}{\partial u} + Q_1 . \frac{\partial P_i}{\partial u'} \qquad i = 0, 1 \tag{6.3}$$

$$F(u') = \frac{\partial Q_0}{\partial u'} . P_0 + Q_1 . F(u) + p \cdot \frac{\partial Q_0}{\partial u} \tag{6.4}$$

$$0 = \frac{\partial Q_0}{\partial u'} . P_1 + \frac{\partial Q_0}{\partial p} \tag{6.5}$$

where P_0, P_1 and Q_0 are functions of u, u' and p only while Q_1 is a constant.

Take $\left\{ F'(u) - F(u) \cdot \left(Q_1 \dfrac{\partial}{\partial u'} + \dfrac{\partial}{\partial u}\right)\right\} \left\{ \dfrac{\partial P_1}{\partial u'} . (6.1) - \dfrac{\partial P_0}{\partial u'} . (6.2)\right\}$:

$$P_1 . p . \left(F'(u).F'(u) - F(u).F''(u) \right) = F(u').F'(u) - Q_1 . F'(u').F(u) \tag{6.6}$$

So from (1.4) one has

$$P_1 = A_1(u,u') . p^{-1} \tag{6.7}$$

where A_1 is a function of u and u' only. From section 5

$$\frac{\partial A_1}{\partial u'} \neq 0 \tag{6.8}$$

So from (6.2) one may write

$$Q_0 = A_2(u,u').p^{-1} \tag{6.9}$$

If Q_0 does not depend on u' then R.H.S. of (6.4) is independent of u' and yet F is not a constant. Hence

$$\frac{\partial A_2}{\partial u'} \neq 0 \tag{6.10}$$

Then from (6.4) one has

$$P_0 = A_0(u,u').p \tag{6.11}$$

Substitute (6.7) , (6.9) and (6.11) into (6.1) to (6.5)

$$F(u') = \frac{\partial A_0}{\partial u'} . A_2 + A_0 . F(u) + A_1 . F'(u) \tag{6.12}$$

$$0 = \frac{\partial A_1}{\partial u'} . A_2 - A_1 . F(u) \tag{6.13}$$

$$0 = \frac{\partial A_i}{\partial u} + Q_1 . \frac{\partial A_i}{\partial u'} \qquad i = 0, 1. \tag{6.14}$$

$$F(u') = \frac{\partial A_2}{\partial u'} . A_0 + Q_1 . F(u) + \frac{\partial A_2}{\partial u} \tag{6.15}$$

$$0 = \frac{\partial A_2}{\partial u'} . A_1 - A_2 \tag{6.16}$$

Eliminate A_1 from (6.13) and (6.16) and integrate to obtain

$$\frac{\partial A_2}{\partial u'} = F(u) + F_1(u) . A_2 \tag{6.17}$$

where $F_1(u)$ is an arbitary function of u only. Then (6.16) is

$$A_2 . (1 - F_1(u).A_1) = F(u).A_1 \tag{6.18}$$

Then (6.13) is on using (6.8)

$$\frac{\partial A_1}{\partial u'} = 1 - F_1(u) . A_1 \tag{6.19}$$

Take $\quad \frac{\partial}{\partial u} + Q_1 . \frac{\partial}{\partial u'} \quad$ of (6.19) and use (6.14)

F_1 is a constant , k say (6.20)

First consider the case

$$k \neq 0 \tag{6.21}$$

Then (6.17) to (6.19) and (6.14) give on integrating

$$\begin{aligned} A_1 &= k^{-1} + K.\exp -k(u'-Q_1 u) \\ A_2 &= -k^{-1}.F(u).(1 + k^{-1}.K^{-1}.\exp k(u'-Q_1 u)) \end{aligned} \tag{6.22}$$

where K is a constant. Note that $K \neq 0$ by (6.8) .Substitute (6.22) into (6.15)

$$\begin{aligned} F(u') = F(u).(k.K)^{-1}.(Q_1 - A_0).e^{k(u'-Q_1 u)} + Q_1.F(u) \\ - k^{-1}.F'(u).(1 + k^{-1}.K^{-1}.\exp k(u'-Q_1 u)) \end{aligned} \tag{6.23}$$

Take $\quad F'(u) - F(u).\left(Q_1 . \frac{\partial}{\partial u'} + \frac{\partial}{\partial u} \right) \quad$ of (6.23)

$$\begin{aligned} F'(u).F(u') - Q_1.F(u).F'(u') = (F''(u).F(u) - F'(u).F'(u)).k^{-1} \\ \times (1 + k^{-1}.K^{-1}.\exp k(u'-Q_1 u)) \end{aligned} \tag{6.24}$$

Take $\quad \left(\alpha . \frac{\partial}{\partial u'} - \frac{\partial^2}{\partial u'^2} \right) \quad$ of (6.24)

$$Q_1.F(u).(F'''(u') - k.F''(u')) = F'(u).(F''(u') - k.F'(u')) \tag{6.25}$$

Since $F'(u) \neq K.F(u)$ for all constants K both sides of (6.25) must vanish. But R.H.S. is not zero by (1.4). Hence it has been shown that (6.21) cannot hold. Therefore

$$F_1 = 0 \tag{6.26}$$

Then (6.18) , (6.19) and (6.14) give

$$\begin{aligned} A_1 &= u' - Q_1.u + C \\ A_2 &= F(u).(u' - Q_1.u + C) \end{aligned} \tag{6.27}$$

where C is a constant. Substitute (6.27) into (6.15) and one obtains a contradiction exactly as above. This completes the case N = 2.

§7. F satisfies a Linear Equation.

In this section it will be shown that F satisfies a linear equation. First define

$$\mathcal{J} = Q_1 \cdot \frac{\partial}{\partial u'} + \frac{\partial}{\partial u} \tag{7.1}$$

Then take $\mathcal{J}^k$ of (4.9) . Then by section 5 the determinant of coefficients must be zero. Using (3.5) one has

$$\det\left[\mathcal{J}^k Q_0, F^{(k)}(u), \ldots, F^{(N-2+k)}(u)\right] = 0 \tag{7.2}$$

where the rows are as shown (k = 0,...,N-1).

Take $P_0 \cdot \frac{\partial}{\partial u'} + \sum_{i=0}^{N-2} k_{i+1} \cdot \frac{\partial}{\partial k_i}$ of (7.2) and use (4.11)

$$\therefore \det\left[Q_1^k \cdot F^{(k)}(u'), F^{(k)}(u), \ldots, F^{(N-2+k)}(u)\right]$$
$$+ k_1 \cdot \det\left[\mathcal{J}^k Q_0, F^k(u), \ldots, F^{(N-3+k)}(u), F^{(N-1+k)}(u)\right] = 0 \tag{7.3}$$

Take $P_1 \cdot \frac{\partial}{\partial u'} + \frac{\partial}{\partial k_{N-1}}$ of (7.3) and use N > 2 (from section 6)

$$\det\left[Q_1^k F^{(k+1)}(u'), F^{(k)}(u), \ldots, F^{(N-2+k)}(u)\right] = 0 \tag{7.4}$$

Equation (7.4) implies that F satisfies a linear equation and so

$$F(u) = \sum_{i=0}^{\ell} \sum_{j=0}^{\ell_i} a_{ij} \, u^j \, e^{\lambda_i u} \tag{7.5}$$

From now on however it will be assumed that there are no repeated roots. Exactly the same method as given below should work when there are repeated roots ; this will not be considered though. That is, from now on it will be assumed that F satisfies

$$F(u) = \sum_{i=1}^{N} A_i \, e^{\lambda_i u} \tag{7.6}$$

where the A_i and λ_i are complex constants with the λ_i distinct.

Define $\qquad v = u' - Q_1 . u \qquad$ (7.7)

and use v instead of u' as independent variable. Then (4.10) is

$$\frac{\partial P_i}{\partial u} = 0 \quad , \; i = 0 , 1 \tag{7.8}$$

Then from (4.9) and section 5 one has

$$Q_0 = \sum_{i=1}^{N} W_i . e^{\lambda_i u} \tag{7.9}$$

where W_i and P_i are functions of $p_1,\dots,p_{N-1}$ and v only.

From (1.4) at least two A_i must be non-zero. Take

$A_1 \neq 0$, $A_2 \neq 0$, $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_1 \neq \lambda_2 \qquad$ (7.10)

Now the coefficient of $\exp(\lambda_1 Q_1 u)$ on the L.H.S. of (4.8) is $A_1 . \exp(\lambda_1 v)$ which is not zero. Hence the R.H.S. must have a term in $\exp(\lambda_1 Q_1 u)$. But this means that there must be such a term in $F(u)$. So the coefficients of $\exp(\lambda_i Q_1 u)$ in (4.8) to (4.13) (i=1,2) are

$$A_i . e^{\mu_i v} = Z_i . \frac{\partial P_0}{\partial v} + \mathcal{D}_i . P_0 + f_N^{(i)} . P_1 \tag{7.11}$$

$$0 = Z_i . \frac{\partial P_1}{\partial v} + \mathcal{D}_i . P_1 \tag{7.12}$$

$$A_i . e^{\mu_i v} = (P_0 - Q_1 . p_1) . \frac{\partial Z_i}{\partial v} + \lambda_i Z_i p_1 + Q_1 + \mathcal{B} Z_i \tag{7.13}$$

$$0 = P_1 . \frac{\partial Z_i}{\partial v} + \frac{\partial Z_i}{\partial p_{N-1}} \tag{7.14}$$

These hold for $i = 1$, 2 where P_0 , P_1 , Z_1 and Z_2 are functions of $p_1,\dots,p_{N-1}$ and v only while Q_1 , λ_1 , λ_2 , μ_1 and μ_2 are constants. (Actually, $\lambda_i Q_1$ has been replaced by λ_i and μ_i is the old λ_i. Also the equations have been divided by some A_j and Z_j is the W_i of (7.9) divided by these A_j. The A_i of (7.11) to (7.14) are ratios of the A_i in (7.6).)

Also A_1 and A_2 are non-zero constants and $\mu_i = \lambda_i / Q_1$.

§8. A Condition on λ_i.

In this section an equation which involves λ_1 and λ_2 only will be derived. Define

$$\left.\begin{aligned} \mathcal{E}_i &= Z_i \cdot \frac{\partial}{\partial v} + \mathcal{D}_i \qquad , i = 1, 2 \\ \mathcal{C} &= (\rho_0 - Q_1 \cdot x) \cdot \frac{\partial}{\partial v} + \mathcal{B} \end{aligned}\right\} \tag{8.1}$$

$$\left.\begin{aligned} \mathcal{E}_{n+1} &= \mathcal{E}_1 \mathcal{E}_n - \mathcal{E}_n \mathcal{E}_1 \qquad n \geq 2 \\ \mathcal{E}_n &= Z_n \cdot \frac{\partial}{\partial v} + \mathcal{D}_n \end{aligned}\right\} \tag{8.2}$$

Then, by induction one has

$$\left.\begin{aligned} \mathcal{E}_n &= \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot (-1)^k \cdot \mathcal{E}_1^{n-2-k} \mathcal{E}_2 \mathcal{E}_1^k \quad , n \geq 2 \\ Z_n &= \mathcal{E}_1^{n-2} Z_2 + \sum_{k=0}^{n-3} \binom{n-2}{k} \cdot (-1)^{n+k} \mathcal{E}_1^k \mathcal{E}_2 \mathcal{E}_1^{n-3-k} Z_1 \\ &\qquad , n \geq 3 \end{aligned}\right\} \tag{8.3}$$

Then from (7.11) and (7.12) for n a positive integer

$$\mathcal{E}_n \rho_1 = 0 \tag{8.4}$$

$$\mathcal{E}_n \rho_0 = A_n \cdot e^{\mu_1 v} + B_n \cdot e^{\mu_2 v} - f_v^{(n)} \cdot \rho_1 \tag{8.5}$$

Also equations (7.13) and (7.14) are, for $i = 1, 2$:

$$\mathcal{C} Z_i = A_i \cdot e^{\mu_1 v} + B_i \cdot e^{\mu_2 v} - \lambda_i Z_i x_1 - Q_1 \tag{8.6}$$

$$0 = \rho_1 \cdot \frac{\partial Z_i}{\partial v} + \frac{\partial Z_i}{\partial x_{v-1}} \tag{8.7}$$

where A_i and B_i are constants for $i = 1, 2$. Infact

$$A_2 = B_1 = 0. \quad , \quad A_1 \neq 0 \quad , \quad B_2 \neq 0 \; , \mu_i = \lambda_i/Q_1 \tag{8.8}$$

$$\left.\begin{aligned} A_{n+1} &= \mathscr{E}_1 A_n - \mathscr{E}_n A_1 + Z_1 . \mu_1 . A_n - Z_n . \mu_1 . A_1 \\ B_{n+1} &= \mathscr{E}_1 B_n + Z_1 \mu_2 B_n \end{aligned} \quad n \geq 2 \right\} \tag{8.9}$$

Define

$$\left.\begin{aligned} \Delta_1 &= \det \left[f_1^{(k)}, \ldots, f_{N-1}^{(k)} \right] \\ \Delta_2 &= \det \left[f_1^{(k)}, \ldots, f_{N-2}^{(k)}, Z_k \right] \\ \Delta_3 &= \det \left[f_1^{(k)}, \ldots, f_{N-2}^{(k)}, A_k \right] \\ \Delta_4 &= \det \left[f_1^{(k)}, \ldots, f_{N-2}^{(k)}, B_k \right] \\ \Delta_5 &= \det \left[f_1^{(k)}, \ldots, f_{N-2}^{(k)}, f_N^{(k)} \right] \end{aligned} \right\} \tag{8.10}$$

The desired equation for λ_1 and λ_2 can now be written in terms of Δ_1 and Δ_5. It is shown in section 10 that this equation implies $\lambda_1 + 2\lambda_2 = 0$ if $N > 3$. The next section deals with the case $N = 3$. Now eliminate the $p_1, \ldots, p_{N-2}$ derivatives of P_0 and P_1 from (8.4) and (8.5) :

$$\Delta_1 \cdot \frac{\partial P_1}{\partial f_{N-1}} + \Delta_2 \cdot \frac{\partial P_1}{\partial v} = 0 \tag{8.11}$$

$$\Delta_1 \cdot \frac{\partial P_0}{\partial f_{N-1}} + \Delta_2 \cdot \frac{\partial P_0}{\partial v} = \Delta_3 \cdot e^{\mu_1 v} + \Delta_4 \cdot e^{\mu_2 v} - \Delta_5 \cdot P_1 \tag{8.12}$$

From (2.6) and (8.10) one has :

$$\Delta_5 = (-1)^N . (\lambda_1 - \lambda_2) . \kappa_{N-1} . p_{N-1} \tag{8.13}$$

Note that (8.13) holds even for $N = 3$ provided

$$\kappa_2 = 1 \tag{8.14}$$

Eliminate Δ_2 from (8.11) and (8.12)

$$\Delta_1 \cdot \left[\frac{\partial \rho_0}{\partial v} \cdot \frac{\partial \rho_1}{\partial x_{N-1}} - \frac{\partial \rho_0}{\partial x_{N-1}} \cdot \frac{\partial \rho_1}{\partial v} \right] + \frac{\partial \rho_1}{\partial v} \cdot \left[\Delta_3 \cdot e^{\mu_1 v} + \Delta_4 \cdot e^{\mu_2 v} - \Delta_5 \cdot \rho_1 \right] = 0 \tag{8.15}$$

Take $\mathcal{E}_k$ of (8.15) for k = 1 , 2 and use (8.15) and (8.11) to eliminate the derivatives of P_0 and P_1 .

$$\alpha_k . P_1 = \beta_k . \exp \mu_1 v + \gamma_k . \exp \mu_2 v \tag{8.16}$$

where

$$\left.\begin{aligned} \alpha_k &= (\mathcal{E}_k \Delta_1) . \Delta_5 + \lambda_k \Delta_1^2 - \Delta_1 . (\mathcal{E}_k \Delta_5) \\ \beta_k &= (\mathcal{E}_k \Delta_1) . \Delta_3 - \Delta_1 . (\mathcal{E}_k \Delta_3) - \mu_1 . \Delta_1 \Delta_3 Z_k + \mu_1 A_k \Delta_1 \Delta_2 \\ \gamma_k &= (\mathcal{E}_k \Delta_1) . \Delta_4 - \Delta_1 . (\mathcal{E}_k \Delta_4) - \mu_2 \Delta_1 \Delta_4 . Z_k + \mu_2 b_k . \Delta_1 \Delta_2 \end{aligned}\right\} \tag{8.17}$$

Eliminate P_1 from (8.16)

$$\alpha_1 . (\beta_2 . \exp \mu_1 v + \gamma_2 . \exp \mu_2 v) = \alpha_2 . (\beta_1 . \exp \mu_1 v + \gamma_1 . \exp \mu_2 v) \tag{8.18}$$

Take $\rho_1 \cdot \frac{\partial}{\partial v} + \frac{\partial}{\partial x_{N-1}}$ of (8.18) and use (8.16) to eliminate P_1 .

$$\begin{aligned} &(-1)^N . (\lambda_1 - \lambda_2) . K_{N-1} . \left\{ (\mathcal{E}_1 \Delta_1) . \left[\beta_2 . e^{\mu_1 v} + \gamma_2 . e^{\mu_2 v} \right] - (\mathcal{E}_2 \Delta_1) . \left[\beta_1 . e^{\mu_1 v} + \gamma_1 . e^{\mu_2 v} \right] \right\} \\ &\quad = (\mu_1 - \mu_2) . (\gamma_1 . \beta_2 - \beta_1 \gamma_2) . e^{(\mu_1 + \mu_2) v} \end{aligned} \tag{8.19}$$

Taking $\rho_1 \cdot \frac{\partial}{\partial v} + \frac{\partial}{\partial x_{N-1}}$ repeatedly of (8.19) one sees that the coefficients of $\exp \mu_1 v$, $\exp \mu_2 v$ and $\exp (\mu_1 + \mu_2) v$ must all vanish.

So if $\kappa_{N-1} \neq 0$ then

$$\left.\begin{aligned} (\mathcal{E}_1 \Delta_1) . \beta_2 &= (\mathcal{E}_2 \Delta_1) . \beta_1 \\ (\mathcal{E}_1 \Delta_1) . \gamma_2 &= (\mathcal{E}_2 \Delta_1) . \gamma_1 \end{aligned}\right\} \tag{8.20}$$

Use (8.20) to eliminate $_i$ and $_i$ from (8.16)

$$(\mathscr{C}_1\,\Delta_1)\,\alpha_2 = (\mathscr{C}_2\,\Delta_1)\,\alpha_1$$

i.e. $$(\mathscr{D}_1\,\Delta_1)\left[\mathscr{D}_2\,\Delta_5 - \lambda_2\,\Delta_1\right] = (\mathscr{D}_2\,\Delta_1)\left[\mathscr{D}_1\,\Delta_5 - \lambda_1\,\Delta_1\right] \tag{8.21}$$

This is the desired equation which involves λ_1 and λ_2 only.

§9. If $N = 3$.

It will be shown in the next section that for $N > 3$ one must have $\lambda_2 + 2\lambda_1 = 0$. However the case $N = 3$ needs to be done separately. So in this section it will be shown that $\lambda_2 + 2\lambda_1 = 0$ in the case $N = 3$. For $N = 3$ the equations to be solved are (8.4) to (8.7). That is

$$\mathscr{C}_\alpha\,P_0 + (\lambda_\alpha\cdot x_2 + \lambda_\alpha^2\,x_1^2)\cdot P_1 = A_\alpha\cdot e^{\mu_1 r} + B_\alpha\cdot e^{\mu_2 r} \tag{9.1}$$

$$\mathscr{C}_\alpha\,P_1 = 0 \tag{9.2}$$

$$\mathscr{C}\,Z_\alpha + \lambda_\alpha\,x_1\cdot Z_\alpha + Q_1 = A_\alpha\cdot e^{\mu_1 r} + B_\alpha\cdot e^{\mu_2 r} \tag{9.3}$$

$$P_1\cdot\frac{\partial Z_\alpha}{\partial r} + \frac{\partial Z_\alpha}{\partial x_2} = 0 \tag{9.4}$$

where $$\left.\begin{aligned} \mathscr{C}_\alpha &= Z_\alpha\cdot\frac{\partial}{\partial r} + \frac{\partial}{\partial x_1} + \lambda_\alpha\cdot x_1\cdot\frac{\partial}{\partial x_2} \\ \mathscr{C} &= (P_0 - Q_1\cdot x_1)\cdot\frac{\partial}{\partial r} + x_2\cdot\frac{\partial}{\partial x_1} \end{aligned}\right\} \tag{9.5}$$

$$\left.\begin{aligned} \therefore\ \mathscr{C}_3 &= \mathscr{C}_1\,\mathscr{C}_2 - \mathscr{C}_2\,\mathscr{C}_1 = Z_3\cdot\frac{\partial}{\partial r} + (\lambda_2 - \lambda_1)\cdot\frac{\partial}{\partial x_2} \\ \text{where}\quad Z_3 &= \mathscr{C}_1\,Z_2 - \mathscr{C}_2\,Z_1 \end{aligned}\right\} \tag{9.6}$$

$$\left.\begin{aligned} \therefore\ \mathscr{C}_4 &= \mathscr{C}_1\,\mathscr{C}_3 - \mathscr{C}_3\,\mathscr{C}_1 = Z_4\,\frac{\partial}{\partial r} \\ \text{where}\quad Z_4 &= \mathscr{C}_1\,Z_3 - \mathscr{C}_3\,Z_1 \end{aligned}\right\} \tag{9.7}$$

Now (8.4) and section 5 give $\mathcal{E}_4 P_1 = 0$ and $\frac{\partial P_1}{\partial v} \neq 0$ so must have $Z_4 = 0$.

$$\therefore \quad \mathcal{E}_1 Z_3 = \mathcal{E}_3 Z_1 \tag{9.8}$$

$$\therefore \quad \mathcal{E}_4 = 0 \tag{9.9}$$

Similarly
$$\mathcal{E}_2 Z_3 = \mathcal{E}_3 Z_2 \tag{9.10}$$

Now

$$\mathcal{E}_3 P_0 + (\lambda_2^2 - \lambda_1^2) \cdot 2\chi_1 \cdot P_1 = -A_1 \cdot \mu_1 \cdot Z_2 \cdot e^{\mu_1 v} + B_2 \cdot \mu_2 \cdot Z_1 \cdot e^{\mu_2 v} \tag{9.11}$$

Let (9.9) act on P_0

$$(\lambda_2 - \lambda_1)(\lambda_1 + 2\lambda_2) P_1 = B_2 \cdot \mu_2 \cdot [\mathcal{E}_1 Z_1 + \mu_2 Z_1^2] \cdot e^{\mu_2 v} \tag{9.12}$$
$$- A_1 \cdot \mu_1 [2\mathcal{E}_1 Z_2 + \mu_1 Z_1 Z_2 - \mathcal{E}_2 Z_1] \cdot e^{\mu_1 v}$$

Similarly

$$(\lambda_2 - \lambda_1)(2\lambda_1 + \lambda_2) P_1 = B_2 \mu_2 [2\mathcal{E}_2 Z_1 - \mathcal{E}_1 Z_2 + \mu_2 Z_1 Z_2] \cdot e^{\mu_2 v}$$
$$- A_1 \mu_1 [\mathcal{E}_2 Z_2 + \mu_1 Z_2^2] \cdot e^{\mu_1 v} \tag{9.13}$$

Take $P_1 \cdot \frac{\partial}{\partial v} + \frac{\partial}{\partial \chi_2}$ acting on $(2\lambda_1 + \lambda_2)(9.12) - (\lambda_1 + 2\lambda_2)(9.13)$.

One sees that the coefficients of $\exp \mu_i v$ $(i = 1,2)$ must both vanish.

From (8.8) one then has

$$(2\lambda_1 + \lambda_2) \cdot [\mathcal{E}_1 Z_1 + \mu_2 \cdot Z_1^2] = (\lambda_1 + 2\lambda_2) \cdot [2\mathcal{E}_2 Z_1 - \mathcal{E}_1 Z_2 + \mu_2 Z_1 Z_2] \tag{9.14}$$

$$(2\lambda_1 + \lambda_2) \cdot [2\mathcal{E}_1 Z_2 + \mu_1 Z_1 Z_2 - \mathcal{E}_2 Z_1] = (\lambda_1 + 2\lambda_2) [\mathcal{E}_2 Z_2 + \mu_1 Z_2^2] \tag{9.15}$$

Now

$$\mathcal{E}\mathcal{E}_i \cdot Z_j = -(\lambda_i + \lambda_j) \cdot \chi_1 \cdot \mathcal{E}_i Z_j - \lambda_j Z_j + Z_i [A_j \cdot \mu_1 \cdot e^{\mu_1 v} + B_j \cdot \mu_2 \cdot e^{\mu_2 v}] \tag{9.16}$$

Take $\mathcal{E}$ of (9.14). Then by taking $P_1 \frac{\partial}{\partial v} + \frac{\partial}{\partial \chi_2}$ of this one sees that the coefficient of $\exp \mu_1 v$ must vanish. That is

$$(2\lambda_1 + \lambda_2)(\mu_1 + 2\mu_2).Z_1 = (\lambda_1 + 2\lambda_2)(2\mu_1 + \mu_2).Z_2 \tag{9.17}$$

But from (8.8) $\mu_i = \lambda_i/Q_1$,and if $Z_1 = Z_2$ then taking $\rho_1 \cdot \frac{\partial}{\partial v} + \frac{\partial}{\partial x_2}$ of $(9.3)_{k=1} - (9.3)_{k=2}$ gives a contradiction. Hence w.o.l.g. one may take $\lambda_1 + 2\lambda_2 = 0$ in the case $N = 3$.

§10. That $\lambda_2 + 2\lambda_1 = 0$.

This result has been proved in the previous section for $N = 3$ so for the rest of this section assume that

$$N > 3\ ,\ \lambda_2 + 2\lambda_1 \neq 0\ ,\ \lambda_1 + 2\lambda_2 \neq 0 \text{ and } \kappa_n \neq 0 \text{ for } n > 1 \tag{10.1}$$

One may also w.l.o.g. take

$$|\lambda_1/\lambda_2| > 1 \tag{10.2}$$

Define $z_i^{(n)}$ by

$$\begin{aligned} (\lambda_2 - \lambda_1).z_i^{(2)} &= f_i^{(2)} - f_i^{(1)} \\ z_i^{(n)} &= z_i^{(n-1)} - z_{n-1}^{(n-1)}.g_i^{(n)} \qquad n > 2 \end{aligned} \tag{10.3}$$

Then from (8.10) one sees that Δ_1 equals some constant (non-zero by (10.1)) times $z_{N-1}^{(N-1)}$ and that Δ_5 equals the same constant times $z_N^{(N-1)}$. Hence equation (8.21) is

$$\begin{aligned} &\left[\mathcal{D}_1 z_{N-1}^{(N-1)}\right]\cdot\left[\mathcal{D}_2 z_N^{(N-1)} - \lambda_2 \cdot z_{N-1}^{(N-1)}\right] \\ &= \left[\mathcal{D}_2 z_{N-1}^{(N-1)}\right]\cdot\left[\mathcal{D}_1 z_N^{(N-1)} - \lambda_1 \cdot z_{N-1}^{(N-1)}\right] \end{aligned} \tag{10.4}$$

Then from (2.4) , (2.9) and (10.3) one has

$$\left.\begin{aligned} &z_2^{(2)} = p_1 \\ &(\lambda_2 - \lambda_1)\cdot z_{i+1}^{(2)} = (\lambda_2 - \lambda_1)\cdot \mathcal{B} z_i^{(2)} + \lambda_2 \kappa_1 \cdot f_i^{(2)} - \lambda_1 \cdot \kappa_1 \cdot f_i^{(1)} \end{aligned}\right\} \tag{10.5}$$

Then from (2.24) and (2.25) one may prove, by induction, that

$$z^{(n)}_{i+1} = \mathcal{B}\, z^{(n)}_{i} + z^{(n)}_{n} \cdot g^{(n)}_{i} \tag{10.6}$$

where $$z^{(n)}_{n} = \mathcal{B}\, z^{(n-1)}_{n-1} - \lambda_1 \cdot \left[(n-2)\lambda_1 + \lambda_2\right] \cdot z^{(n-1)}_{n-1} \tag{10.7}$$

From (10.5) and (10.7) the first few $z^{(n)}_{n}$ are

$$\left.\begin{aligned}
z^{(3)}_{3} &= p_2 - (\lambda_1 + \lambda_2).p_1^2 \\
z^{(4)}_{4} &= p_3 - (4\lambda_1 + 3\lambda_2).p_1.p_2 + (2\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2).p_1^3 \\
z^{(5)}_{5} &= p_4 - (7\lambda_1 + 4\lambda_2).p_1.p_3 - (4\lambda_1 + 3\lambda_2).p_2^2 + (18\lambda_1^2 + 22\lambda_1\lambda_2 + 6\lambda_2^2).p_1^2.p_2 \\
&\qquad -(3\lambda_1 + \lambda_2)(2\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2).p_1^4
\end{aligned}\right\} \tag{10.8}$$

One may then prove, by induction, that

$$\mathcal{D}_1 z^{(n)}_{n} + \left[\tfrac{1}{2}(n-1)(n-2)\lambda_1 + (n-1)\lambda_2\right] z^{(n-1)}_{n-1} = 0 \tag{10.9}$$

$$\mathcal{D}_1 z^{(n)}_{n+1} = \lambda_1 \cdot z^{(n)}_{n} + g^{(n+1)}_{n+1} \cdot \mathcal{D}_1 \cdot z^{(n)}_{n} \tag{10.10}$$

From (10.8) and (10.9) one has, by induction, that for $n > 5$

$$\begin{aligned}
z^{(n)}_{n} = p_{n-1} &- \left\{ \tfrac{1}{2}(n^2-3n+4)\lambda_1 + (n-1)\lambda_2 \right\}.p_1.p_{n-2} \\
&- \left\{ (1/6)(n-1)(n^2-5n+12)\lambda_1 + \tfrac{1}{2}(n-1)(n-2)\lambda_2 \right\}.p_2.p_{n-3} \\
&+ \left\{ (1/24)(3n\ -22n^3+69n^2-98n+72)\lambda_1^2 + \tfrac{1}{2}(n-1)(n^2-4n+6)\lambda_1\lambda_2 \right. \\
&\qquad \left. + \tfrac{1}{2}(n-1)(n-2)\lambda_2^2 \right\}.p_1^2.p_{n-3} + 0(p_{n-4})
\end{aligned} \tag{10.11}$$

From (10.19) and (10.11) so dividing (10.4) by this gives on using (10.10) that

$$\mathcal{D}_2 z^{(N-1)}_{N} = \lambda_2 \cdot z^{(N-1)}_{N-1} + g^{(N)}_{N} \cdot \mathcal{D}_2 \cdot z^{(N-1)}_{N-1} \tag{10.12}$$

ie. $$\mathcal{D}_2 \left[z^{(N-1)}_{N} - g^{(N)}_{N} \cdot z^{(N-1)}_{N-1} \right] + \left[\tfrac{1}{2}(N^2-3N+4)\lambda_1 + (N-3)\lambda_2 \right] z^{(N}_{N} = 0$$

from (2.28) . Then from (10.3) one has

$$\mathcal{D}_2 \, z_N^{(N)} + \left[\tfrac{1}{2}(N^2 - 3N + 4)\lambda_1 + (N-2)\lambda_2 \right] z_{N-1}^{(N-1)} = 0 \qquad (10.13)$$

From (10.8) for $N = 4$ this is $(\lambda_1 - \lambda_2)(2\lambda_1 + \lambda_2).p_1^2 = 0$ which contradicts (10.1). So for the rest of this section one may take

$$N > 4 \qquad (10.14)$$

From (10.8) and (10.11) one may show that for $n > 4$

$$\mathcal{D}_2 \, z_n^{(n)} = - \left[\tfrac{1}{2}(n^2 - 3n + 4)\lambda_1 + (n-2)\lambda_2 \right] \kappa_{n-2}$$
$$+ \Big\{ \tfrac{1}{12}(3n^4 - 22n^3 + 69n^2 - 98n + 72)\lambda_1^2 \qquad (10.15)$$
$$+ \tfrac{1}{6}(5n^3 - 27n^2 + 52n - 36)\lambda_1\lambda_2 + \tfrac{1}{2}(n^2 - 3n)\lambda_2^2 \Big\} \kappa_1 \kappa_{n-3}$$
$$+ O(\kappa_{n-4})$$

From (10.8), (10.11) and (10.15) one sees that the coefficient of $p_1 p_{N-3}$ in (10.13) is $(\lambda_1 - \lambda_2)$ times

$(N^3 - 6N^2 + 17N - 12)\lambda_1 + 3(N^2 - 5N + 8)\lambda_2 = 0$

But this contradicts (10.2). So it has been shown that $\lambda_1 + 2\lambda_2 = 0$.

§11. For $N = 3$.

It has been shown that w.l.o.g.

$$\lambda_1 + 2\lambda_2 = 0 \qquad (11.1)$$

It has also been shown that $f_i^{(4)} = 0$ for all i. That is

$$\mathcal{D}_4 = \mathcal{D}_1 \mathcal{D}_3 - \mathcal{D}_3 \mathcal{D}_1 = 0 \quad , \quad \mathcal{D}_3 = \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1 . \qquad (11.2)$$

But $\mathcal{E}_4 = \mathcal{E}_1 \mathcal{E}_3 - \mathcal{E}_3 \mathcal{E}_1 = Z_4 \dfrac{\partial}{\partial v}$, $\mathcal{E}_4 \, \rho_1 = 0$ and

$\dfrac{\partial \rho_1}{\partial v} \neq 0$ then imply that

$$Z_4 = \mathcal{E}_1 Z_3 - \mathcal{E}_3 Z_1 = 0 \qquad (11.3)$$

$$\therefore \; \mathcal{E}_4 = 0 = \mathcal{E}_1 \mathcal{E}_3 - \mathcal{E}_3 \mathcal{E}_1 \qquad (11.4)$$

$$\therefore\ 0 = \mathscr{C}_4 \rho_0 = A_4 \cdot e^{\mu_1 v} + B_4 e^{\mu_2 v} - f_N^{(4)} \cdot \rho_1 \tag{11.4}$$

where
$$\begin{aligned} A_4 &= -A_1 \mu_1 \left[2\mathscr{C}_1 Z_2 - \mathscr{C}_2 Z_1 + \mu_1 Z_1 Z_2 \right] \\ B_4 &= B_2 \mu_2 \left[\mathscr{C}_1 Z_1 + \mu_2 Z_1^2 \right] \end{aligned} \tag{11.5}$$

By taking $\rho_1 \frac{\partial}{\partial v} + \frac{\partial}{\partial x_{N-1}}$ of (11.5) one sees that the coefficients of $\exp \mu_1 v$ and $\exp \mu_2 v$ must both vanish i.e. $A_4 = B_4 = 0$.

$$2\mathscr{C}_1 Z_2 - \mathscr{C}_2 Z_1 + \mu_1 Z_1 Z_2 = 0 \tag{11.6}$$

$$\mathscr{C}_1 Z_1 + \mu_2 Z_1^2 = 0 \tag{11.7}$$

Note that it is (8.4) to (8.7) which are being solved when (11.1) holds. From section 2 one would not expect the proof to repeat until one had done at least six cases. Hence only the lower values of N will be considered in this thesis. For the rest of this section take

$$N = 3 \tag{11.8}$$

From (2.14) one sees that

$$\mathscr{D}_2 - \mathscr{D}_1 = x_1 \cdot \mathscr{D}_3 = (\lambda_2 - \lambda_1) \cdot x_1 \cdot \frac{\partial}{\partial x_2} \tag{11.9}$$

Then (9.2) and section 5 imply

$$Z_2 - Z_1 = x_1 \cdot \left[\mathscr{C}_1 Z_2 - \mathscr{C}_2 Z_1 \right] = x_1 \cdot Z_3 \tag{11.10}$$

$$\therefore\ \mathscr{C}_2 - \mathscr{C}_1 = x_1 \cdot \mathscr{C}_3 \tag{11.11}$$

Now from (9.1) to (9.5)

$$\mathscr{L} \cdot \mathscr{C}_i X - \mathscr{C}_i \mathscr{L} X = -\lambda_i x_1 \cdot \mathscr{C}_i X \quad \text{if } \rho_1 \cdot \frac{\partial X}{\partial v} + \frac{\partial X}{\partial x_2} = 0 \tag{11.12}$$

Take $\mathscr{C}$ of (11.10) and use (9.3) and (11.12).

$$B_2.(1 - p_1.\mu_2.Z_1).\exp \mu_2 v \quad - \quad A_1.(1 - p_1.\mu_1.Z_2).\exp \mu_1 v$$
$$= p_1^{-1}.(p_2 - (\lambda_1 + \lambda_2).p_1^2).(Z_2 - Z_1) \tag{11.13}$$

From (11.6) and (11.10) one has

$$\mathscr{C}_1 Z_2 = -\left(\frac{Z_2 - Z_1}{\varkappa_1}\right) - \mu_1 . Z_1 . Z_2 \tag{11.14}$$

Take $\mathscr{C}_1$ of (11.13) and use (11.7) and (11.14):

$$2(Z_2 - Z_1) = \mu_2.p_1.Z_1.(Z_1 + 2.Z_2) \tag{11.15}$$

since $\mu_1 = -2\mu_2$. Then if one takes $\mathscr{C}_1$ acting on (11.15) then one readily obtains a contradiction. So it has been shown that there is no B.T. of the form (1.2) when $N = 3$.

§12. For $N = 4$.

It will be shown in this section that for $N = 4$ there are no B.T. of the form (1.2). Take, for this section,

$$N = 4 \quad \text{and} \quad \lambda_1 + 2\lambda_2 = 0 \tag{12.1}$$

Then from (2.4) and (2.7) one may show that

$$[\varkappa_2 - (\lambda_1 + \lambda_2)\varkappa_1^2].\mathscr{T} = (2\lambda_1 + \lambda_2)[\mathscr{D}_2 - \mathscr{D}_1 - \varkappa_1 . \mathscr{D}_3] \tag{12.2}$$

where $\mathscr{D}_3 = \mathscr{D}_1 \mathscr{D}_2 - \mathscr{D}_2 \mathscr{D}_1$ $\qquad \mathscr{T} = \mathscr{D}_2 \mathscr{D}_3 - \mathscr{D}_3 \mathscr{D}_2$. (12.3)

Then (8.4) and section 5 imply, from (12.2), that

$$(p_2 + \lambda_2.p_1^2).W = 3.\lambda_2(p_1.Z_3 + Z_1 - Z_2) \tag{12.4}$$

where $W = \mathscr{C}_2 Z_3 - \mathscr{C}_3 Z_2$, $\quad Z_3 = \mathscr{C}_1 Z_2 - \mathscr{C}_2 Z_1$ (12.5)

Take $\mathscr{C}$ of (12.4) and use (8.6) and (11.12)

$$[\kappa_3 + 2\lambda_2 \cdot \kappa_1 \cdot \kappa_2]\cdot W$$
$$= A_1\, e^{\mu_1 \nu} \cdot \{ [\kappa_2 + \lambda_2 \kappa_1^2][\mathcal{E}_2 Z_2 + \mu_1 Z_2^2]\cdot \mu_1 + 3\lambda_2 \cdot [1 - \kappa_1 \cdot \mu_1 \cdot Z_2$$
$$- B_2 \cdot e^{\mu_2 \nu} \cdot \{ \mu_2 \cdot [\kappa_2 + \lambda_2 \cdot \kappa_1^2]\cdot[2\mathcal{E}_2 Z_1 - \mathcal{E}_1 Z_2 + \mu_2 Z_1 Z_2] + 3\lambda_2 [1 - \kappa_1 \mu_2 \cdot Z_1$$
(12.6)

Now from (11.3)

$$\mathcal{E}_1 [\mathcal{E}_2 Z_2 - \mathcal{E}_2 Z_1] = [\mathcal{E}_1 \mathcal{E}_2 - \mathcal{E}_2 \mathcal{E}_1] Z_1$$

Then from (11.6) and (11.7)

$$\mathcal{E}_1^2 Z_2 = 2\mathcal{E}_1 [2\mathcal{E}_1 Z_2 + \mu_1 Z_1 Z_2] + \mathcal{E}_2 [\mu_2 Z_1^2]$$

Finally using (11.7) again gives

$$\mathcal{E}_1^2 Z_2 = 0 \qquad (12.7)$$

Then (11.3), (11.6) and (11.7) after a little algebra imply

$$2\mathcal{E}_2 Z_1 - \mathcal{E}_1 Z_2 + \mu_2 Z_1 Z_2 = -3Z_3 + 3\mu_2 Z_1 Z_2. \qquad (12.8)$$

$$\left.\begin{aligned}
\mathcal{E}_1 Z_1 &= -\mu_2 Z_1^2 \\
\mathcal{E}_1 Z_2 &= -Z_3 - \mu_1 Z_1 Z_2. \\
\mathcal{E}_1 Z_3 &= -\mu_2 Z_1 [2Z_3 + \mu_1 Z_1 Z_2] \\
\mathcal{E}_1 \mathcal{E}_2 Z_2 &= -W - \mu_1 Z_1 \mathcal{E}_2 Z_2 + \mu_1 Z_2 [2Z_3 + \mu_1 Z_1 Z_2]
\end{aligned}\right\} \qquad (12.9)$$

Take $\mathcal{E}_1$ of (12.6) and use (12.4)

$$\mathcal{E}_1 W = 0 \qquad (12.10)$$

Then take $\mathcal{E}_1$ of (12.4) and use (12.9) and (12.10)

$$2.Z_3.(1 - p_1.\mu_2.Z_1) = \mu_2.Z_1.(Z_1 + 2.Z_2 + \mu_1.p_1.Z_1.Z_2) \tag{12.11}$$

Take of (12.11) . Then taking of this one sees that the coefficient of $\exp \mu_1 v$ must vanish. That is

$$Z_2 - Z_1 = p_1.Z_3 \tag{12.12}$$

Then one obtains a contradiction exactly as in the case $N = 3$. That is use the proof in section 11 from equation (11.10) onwards. So it has been shown that for $N = 4$ there is no B.T. of the form (1.2).

13. For N = 5.

It will be shown that for $N = 5$ there are no B.T. of the form (1.2) . Take (11.1) to hold and use the definitions in (2.31) and (2.32) . One then sees that

$$[\varkappa_3 + 2\lambda_2\varkappa_1\varkappa_2]\cdot\mathcal{J}_5 = 9\lambda_2^2[\mathcal{J}_2 - \mathcal{J}_1 - \varkappa_1\mathcal{J}_3] + 3\lambda_2\cdot[\varkappa_2 + \lambda_2\varkappa_1^2]\;{}_4 \tag{13.1}$$

Given $\mathcal{E}_1$ and $\mathcal{E}_2$ define $\mathcal{E}_n$ for $n > 2$ by the same sequence as in (2.31) and (2.32) . Then define Z_n by

$$\mathcal{E}_n = Z_n\cdot\frac{\partial}{\partial v} + \mathcal{J}_n \tag{13.2}$$

Then from (8.4) , (13.1) , (13.2) and section 5 one has

$$(p_3 + 2.\lambda_2.p_1.p_2).Z_5 = 9.\lambda_2^2.(Z_2 - Z_1 - p_1.Z_3) + 3.\lambda_2.(p_2 + \lambda_2.p_1^2).Z_4 \tag{13.3}$$

Then (8.5) , (13.1) and (13.3) imply

$$\begin{aligned}
&27\lambda_2^3 p_1\cdot[\varkappa_4 + \lambda_2\varkappa_1\varkappa_3 + 2\lambda_2\cdot\varkappa_2^2 - 2\lambda_2^2\varkappa_1^2\varkappa_2] \\
&= \{9\lambda_2^2[A_2 - A_1 - \varkappa_1 A_3] + 3\lambda_2[\varkappa_2 + \lambda_2\varkappa_1^2]\cdot A_4 - [\varkappa_3 + 2\lambda_2\varkappa_1\varkappa_2]A_5\}e^{\mu_1} \\
&+ \{9\lambda_2^2[B_2 - B_1 - \varkappa_1 B_3] + 3\lambda_2[\varkappa_2 + \lambda_2\varkappa_1^2]B_4 - [\varkappa_3 + 2\lambda_2\varkappa_1\varkappa_2]B_5\}e^{\mu_2}
\end{aligned} \tag{13.4}$$

Where

$$\begin{aligned}
A_3 &= -A_1\mu_1 Z_2 & B_3 &= B_2\mu_2 Z_1 \\
A_4 &= \mathcal{E}_2 A_3 + A_3\mu_1 Z_2 & B_4 &= \mathcal{E}_2 B_3 + B_3\mu_2 Z_2 - B_2\mu_2 Z_3 \\
A_5 &= \mathcal{E}_2 A_4 + A_4\mu_1 Z_2 & B_5 &= \mathcal{E}_2 B_4 + B_4\mu_2 Z_2 - B_2\mu_2 Z_2
\end{aligned} \tag{13.5}$$

Take $\mathcal{C}$ of (13.3) and use (13.4)

$$Z_5 = 27.\lambda_2^3.P_1 \tag{13.6}$$

Now $\mathcal{T}_1 \mathcal{T}_5 = \mathcal{T}_5 \mathcal{T}_1$ so one must also have

$$\mathcal{C}_1 \mathcal{C}_5 = \mathcal{C}_5 \mathcal{C}_1 \tag{13.7}$$

Let the operator in (13.7) act on P_0. Hence

$$A_6.\exp \mu_1 v + B_6.\exp \mu_2.v + 81.\lambda_2^4.P_1 = 0 \tag{13.8}$$

Take $\rho_1 \frac{\partial}{\partial v} + \frac{\partial}{\partial \alpha_4}$ of this and use (13.6) to obtain that the coefficients of the exponentials must be zero. That is $A_6 = B_6 = P_1 = 0$. But this contradicts the result of section 5. So it has been shown that for $N = 5$ there are no B.T. of the form (1.2).

§14. For $N > 5$.

Define $s_i^{(n)}$ to be a constant times the $t_i^{(n)}$ of section 2 such that the first non-zero $s_i^{(n)}$ for fixed n is equal to 1. Then from (2.36) one may prove by induction that

$$\left.\begin{aligned}
&s_{6n+3}^{(8n+3)} = -2\lambda_2.p_1, \quad s_{6n+4}^{(8n+4)} = -2\lambda_2.p_1, \quad s_{6n+5}^{(8n+5)} = -\lambda_2.p_1 \\
&s_{6n+6}^{(8n+6)} = -2\lambda_2.p_1, \quad s_{6n+6}^{(8n+7)} = +\lambda_2.p_1, \quad s_{6n+7}^{(8n+8)} = 0 \\
&s_{6n+8}^{(8n+9)} = -2\lambda_2.p_1, \quad s_{6n+8}^{(8n+10)} = +\lambda_2.p_1
\end{aligned}\right\} \tag{14.1}$$

Define

$$\left.\begin{aligned}
(\lambda_2 - \lambda_1).z_i^{(2)} &= f_i^{(2)} - f_i^{(1)} \\
z_i^{(n)} &= z_i^{(n-1)} - z_{n-1}^{(n-1)}.s_i^{(n)} \qquad n = 3, 4, 5, 6.
\end{aligned}\right\} \tag{14.2}$$

Then

$$z_{i+1}^{(3)} = \mathcal{B} z_i^{(3)} + [\alpha_2 + \lambda_2.\alpha_1^2].s_i^{(3)} \tag{14.3a}$$

$$z_{i+1}^{(4)} = \mathcal{B} z_i^{(4)} + [\alpha_3 + 2\lambda_2 \alpha_1 \alpha_2].s_i^{(4)} \tag{14.3b}$$

$$z_{i+1}^{(5)} = \mathcal{D} z_i^{(5)} + \left[\kappa_4 + \lambda_2(\kappa_1\kappa_3 + 2\kappa_2^2) - 2\lambda_2^2\kappa_1^2\kappa_2\right] s_i^{(5)} \tag{14.3c}$$

$$z_{i+1}^{(6)} = \mathcal{D} z_i^{(6)} + \left[\kappa_5 + \lambda_2(2\kappa_1\kappa_4 + 5\kappa_2\kappa_3) - \lambda_2^2(\kappa_1^2\kappa_3 + 2\kappa_1\kappa_2^2) - 2\lambda_2^3\kappa_1^3\kappa_2\right] s \tag{14.3d}$$

$$\left.\begin{aligned}
z_3^{(3)} &= \kappa_2 + \lambda_2\kappa_1^2 \\
z_4^{(4)} &= \kappa_3 + 2\lambda_2\kappa_1\kappa_2 \\
z_5^{(5)} &= \kappa_4 + \lambda_2(\kappa_1\kappa_3 + 2\kappa_2^2) - 2\lambda_2^2\kappa_1^2\kappa_2 \\
z_6^{(6)} &= \kappa_5 + \lambda_2(2\kappa_1\kappa_4 + 5\kappa_2\kappa_3) - \lambda_2^2(\kappa_1^2\kappa_3 + 2\kappa_1\kappa_2^2) - 2\lambda_2^3\kappa_1^3\kappa
\end{aligned}\right\} \tag{14.4}$$

$$\left.\begin{aligned}
z_4^{(3)} &= \kappa_3 - 2\lambda_2^2\kappa_1^3 \\
z_5^{(4)} &= \kappa_4 + 2\lambda_2\kappa_2^2 - 4\lambda_2^2\kappa_1^2\kappa_2 \\
z_6^{(5)} &= \kappa_5 + 5\lambda_2\kappa_2\kappa_3 - \lambda_2^2(3\kappa_1^2\kappa_3 + 6\kappa_1\kappa_2^2) + 2\lambda_2^3\kappa_1^3\kappa_2 \\
z_7^{(6)} &= \kappa_6 + \lambda_2(7\kappa_2\kappa_4 + 5\kappa_3^2) - \lambda_2^2(5\kappa_1^2\kappa_4 + 16\kappa_1\kappa_2\kappa_3 + 2\kappa_2^3) \\
&\quad + 4\lambda_2^4\kappa_1^4\kappa_2
\end{aligned}\right\} \tag{14.5}$$

Now equation (10.4) must hold when the z's are defined by (14.2). One then easily sees that this is not true in the cases $N = 6$ and $N = 7$. (Use (14.4) and (14.5).) Hence it has been shown that there is no B.T. of the form (1.2) provided $N < 8$. I have not considered the situation when $N > 7$. It does seem that the method of this section is not suitable (and this is so because the analogous equations to (10.6) and (10.7) are very complicated.) The best hope of a method for general N seems to me to be that of section 13 , where hopefully an equation like (13.6) always holds. Still the method of this section does seem to suggest that there are no B.T. of the form (1.2) because equation (10.4) involves more and more terms the higher N is. (Note that in section 10, one only needed to consider the term $p_1 p_{N-3}$.)

The important thing to notice about this chapter is that the equation $\lambda_1 + 2\lambda_2 = 0$ comes about as a property of the operators involved. So if one chooses a more complicated u' dependence in the B.T. then one should still obtain this equation. The final two chapters are an attempt at such a more complicated u' dependence.

CHAPTER 4.

§1. Introduction.

Another attempt is made in this chapter to find B.T. for

$$u_{xy} = F(u) \tag{1.1}$$

Only derivatives of u and u' will be allowed in the B.T. There does not appear to be B.T. of the form (1.2) of the previous chapter so one must allow higher derivatives of u'. The simplest case of this type that one could have is

$$\begin{aligned} u'_{xx} &= P(u,u',u_x,u'_x,u_y,u_{xx},u_{yy}) \\ u'_y &= Q(u,u',u_x,u'_x,u_y,u_{xx},u_{yy}) \end{aligned} \tag{1.2}$$

One could make P and Q depend on higher derivatives of u. However this form has been chosen so as to make u and u' only occur to second order.

It will be shown in this chapter that there are no B.T. of the form (1.2). Both u and u' are assumed to satisfy (1.1).

It will be assumed throughout this chapter that

$$F'''(u) \neq K.F''(u) \qquad , \quad F''(u) \neq K.F(u) \tag{1.3}$$

Define $p = u_x$, $q = u_y$, $r = u_{xx}$, $t = u_{yy}$ and $p' = u'_x$ (1.4)

It will be assumed that u,u',p,p',q,r,t,u_{xxx} and u_{yyy} are independent.

Differentiate (1.2a) w.r.t. y and (1.2b) w.r.t. x. The coefficients of u_{xxx} and u_{yyy} are

$$P_t = Q_r = 0 \tag{1.5}$$

Then one must have

$$\begin{aligned} P &= P_0 + P_1.r \\ Q &= Q_0 + Q_1.t \end{aligned} \tag{1.6}$$

where P_i and Q_i (i = 1,2) are functions of u , u' , p , p' and q only.

Then one must have

$$F'(u')\cdot p' = \frac{\partial P_0}{\partial u}\cdot q + \frac{\partial P_0}{\partial u'}\cdot Q_0 + \frac{\partial P_0}{\partial p'}\cdot F(u') + \frac{\partial P_0}{\partial p}\cdot F(u) + P_1\cdot F'(u)\cdot p \tag{1.7}$$

$$0 = \frac{\partial P_1}{\partial u}\cdot q + \frac{\partial P_1}{\partial u'}\cdot Q_0 + \frac{\partial P_1}{\partial p'}\cdot F(u') + \frac{\partial P_1}{\partial p}\cdot F(u) \tag{1.8}$$

$$0 = \frac{\partial P_i}{\partial u'}\cdot Q_1 + \frac{\partial P_i}{\partial q} \qquad i = 0,1. \tag{1.9}$$

$$F(u') = \frac{\partial Q_0}{\partial u}\cdot p + \frac{\partial Q_0}{\partial u'}\cdot p' + \frac{\partial Q_0}{\partial p'}\cdot P_0 + \frac{\partial Q_0}{\partial q}\cdot F(u) + Q_1\cdot F'(u)\cdot q \tag{1.10}$$

$$0 = \frac{\partial Q_1}{\partial u}\cdot p + \frac{\partial Q_1}{\partial u'}\cdot p' + \frac{\partial Q_1}{\partial p'}\cdot P_0 + \frac{\partial Q_1}{\partial q}\cdot F(u). \tag{1.11}$$

$$0 = \frac{\partial Q_i}{\partial p'}\cdot P_1 + \frac{\partial Q_i}{\partial p} \qquad i = 0,1. \tag{1.12}$$

The rest of this chapter is concerned with the solution of (1.7) to (1.12) . It will be shown that these equations do not have a solution. (Subject to (1.3).)

§2. That $Q_1 = 0$.

It will be shown in this section that $Q_1 = 0$. Since the proof proceeds by contradiction assume that

$$Q_1 \neq 0 \tag{2.1}$$

Take $\frac{\partial P_1}{\partial u'}\cdot\left[\frac{\partial}{\partial q} + Q_1\cdot\frac{\partial}{\partial u'}\right]\cdot(1.7) - \frac{\partial P_0}{\partial u'}\cdot\left[\frac{\partial}{\partial q} + Q_1\cdot\frac{\partial}{\partial u'}\right]\cdot(1.8)$.

$$\frac{\partial P_1}{\partial u'}\cdot\left\{\frac{\partial P_0}{\partial u} + Q_1\cdot F'(u')\cdot\frac{\partial P_0}{\partial p'} - F''(u')\cdot p'\right\} = \frac{\partial P_0}{\partial u'}\cdot\left\{\frac{\partial P_1}{\partial u} + Q_1\cdot F'(u')\cdot\frac{\partial P_0}{\partial p'}\right\} \tag{2.2}$$

Taking $\left(Q_1\cdot\frac{\partial}{\partial u'} + \frac{\partial}{\partial q}\right)^n$, for $n = 1,2$, one sees , from (1.3), that $\frac{\partial P_1}{\partial u'} = 0$. But (1.8) and (1.9) then imply

$$P_1 \text{ is a constant} \tag{2.3}$$

Define $\qquad z = p' - P_1 \cdot p \qquad$ (2.4)

and use z instead of p'. Then (1.12) is

$$\frac{\partial Q_i}{\partial p} = 0 \qquad i = 0, 1 \tag{2.5}$$

If P_0 is not linear in p then (1.10) and (1.11) readily give that F is a constant. Hence

$$P_0 = P_2 + P_3 \cdot p \tag{2.6}$$

where P_2 and P_3 are functions of u , u' , z and q only. Then (1.7) to (1.12) give

$$F'(u')\, z = \frac{\partial P_2}{\partial u} \cdot q + \frac{\partial P_2}{\partial u'} \cdot Q_0 + \left[F(u') - P_1 \cdot F(u)\right] \cdot \frac{\partial P_2}{\partial z} + P_3 \cdot F(u) \tag{2.7}$$

$$\left[F'(u') - F'(u)\right] \cdot P_1 = \frac{\partial P_3}{\partial u} \cdot q + \frac{\partial P_3}{\partial u'} \cdot Q_0 + \left[F(u') - P_1 \cdot F(u)\right] \cdot \frac{\partial P_3}{\partial z} \tag{2.8}$$

$$0 = \frac{\partial P_i}{\partial u'} \cdot Q_1 + \frac{\partial P_i}{\partial q} \qquad i = 2, 3 \tag{2.9}$$

$$F(u') = \frac{\partial Q_0}{\partial u'} \cdot z + \frac{\partial Q_0}{\partial z} \cdot P_2 + \frac{\partial Q_0}{\partial q} \cdot F(u) + Q_1 \cdot F'(u) \cdot q \tag{2.10}$$

$$0 = \frac{\partial Q_1}{\partial u'} \cdot z + \frac{\partial Q_1}{\partial z} \cdot P_2 + \frac{\partial Q_1}{\partial q} \cdot F(u) \tag{2.11}$$

$$0 = \frac{\partial Q_i}{\partial u} + P_1 \cdot \frac{\partial Q_i}{\partial u'} + P_3 \cdot \frac{\partial Q_i}{\partial z} \qquad i = 0, 1. \tag{2.12}$$

Take $\quad \frac{\partial P_3}{\partial u'} \cdot \left(Q_1 \cdot \frac{\partial}{\partial u'} + \frac{\partial}{\partial q} \right)(2.7) - \frac{\partial P_2}{\partial u'} \cdot \left(Q \cdot \frac{\partial}{\partial u'} + \frac{\partial}{\partial q} \right) \cdot (2.8)$

$$\frac{\partial P_3}{\partial u'} \cdot \left\{ F'(u') \cdot \frac{\partial P_2}{\partial z} + \frac{1}{Q_1}\frac{\partial P_2}{\partial u} - F''(u') \cdot z \right\} = \frac{\partial P_2}{\partial u'} \cdot \left\{ F'(u') \cdot \frac{\partial P_3}{\partial z} + \frac{1}{Q_1}\frac{\partial P_3}{\partial u} - F''(u') \cdot P_1 \right\} \tag{2.13}$$

Taking $\left(Q_1 \cdot \frac{\partial}{\partial u'} + \frac{\partial}{\partial q} \right)^n$ for $n = 1, 2$ of (2.13) and using (1.3) one sees that

$$z \, \frac{\partial P_3}{\partial u'} = P_1 \cdot \frac{\partial P_2}{\partial u'} \tag{2.14}$$

$$\frac{\partial P_3}{\partial u'} \cdot \frac{\partial P_2}{\partial z} = \frac{\partial P_2}{\partial u'} \cdot \frac{\partial P_3}{\partial z} \tag{2.15}$$

$$\frac{\partial P_3}{\partial u'} \cdot \frac{\partial P_2}{\partial u} = \frac{\partial P_2}{\partial u'} \cdot \frac{\partial P_3}{\partial u} \tag{2.16}$$

Then (2.7) , (2.8) , (2.14) , (2.15) and (2.16) imply

$$P_3 . F(u) . \frac{\partial P_3}{\partial u'} = F'(u) . P_1 . \frac{\partial P_2}{\partial u'} \tag{2.17}$$

If $\frac{\partial P_3}{\partial u'} = 0$ and $P_1 \neq 0$ then diff. (2.8) w.r.t. u' to obtain a contradiction to (1.3) .

If $\frac{\partial P_3}{\partial u'} \neq 0$ then (2.14) and (2.17) imply $P_1 . F'(u) . z = P_1 . P_3 . F(u)$. Divide by P_1 and diff. w.r.t. u' to obtain a contradiction. Hence

$$P_1 = 0 \tag{2.18}$$

Then (2.8) , (2.9) and (2.14) give that

P_3 is a constant (2.19)

Define

$$w = z - P_3 . u \tag{2.20}$$

and use w instead of z . Then (2.12) is

$$\frac{\partial Q_i}{\partial u} = 0 \qquad i = 0, 1 . \tag{2.21}$$

Take $(Q_1)_w . (2.10) - (Q_0)_w . (2.11)$ and differentiate repeatedly w.r.t. u to obtain $(Q_1)_w = 0$, using (1.3) . Then (2.11) and (2.12) imply

Q_1 is a constant (2.22)

If $(Q_0)_w = 0$ then (2.10) implies that F is a constant. Hence $(Q_0)_w \neq 0$ and so (2.10) and (2.21) imply

$$P_2 = M_0 + M_1 . u + M_2 . F(u) + M_3 . F'(u) \quad , \quad \frac{\partial M_i}{\partial u} = 0, \ i = 0, \ldots, 3 \tag{2.23}$$

Equations (2.9) and (2.10) then imply

$$0 = Q_1 . \frac{\partial M_i}{\partial u'} + \frac{\partial M_i}{\partial q} \qquad i = 0, \ldots, 3 \tag{2.24}$$

$$F(u') = M_0 \cdot \frac{\partial Q_0}{\partial w} + w \cdot \frac{\partial Q_0}{\partial u'} \tag{2.25}$$

$$0 = \rho_3 \cdot \frac{\partial Q_0}{\partial u'} + M_1 \cdot \frac{\partial Q_0}{\partial w} \tag{2.26}$$

$$0 = M_2 \cdot \frac{\partial Q_0}{\partial w} + \frac{\partial Q_0}{\partial q} \tag{2.27}$$

$$0 = M_3 \cdot \frac{\partial Q_0}{\partial w} + Q_1 \cdot F'(u) \cdot q \tag{2.28}$$

If $M_3 = 0$ then (2.28) contradicts (1.3) or (2.1) .Hence

$$M_3 \neq 0 \tag{2.29}$$

Then (2.7) implies

$$F''(u) = b_0 + b_1 . u + b_2 . F(u) + b_3 . F'(u) \tag{2.30}$$

Then (2.7) implies

$$F'(u') \cdot w = (M_1 + b_0 \cdot M_3) q + \frac{\partial M_0}{\partial u'} \cdot Q_0 + [F(u') - \rho_3 \cdot q] \cdot \frac{\partial M_0}{\partial w} \tag{2.31}$$

$$F'(u') \cdot \rho_3 = b_1 \cdot M_3 \cdot q + \frac{\partial M_1}{\partial u'} \cdot Q_0 + [F(u') - \rho_3 \cdot q] \cdot \frac{\partial M_1}{\partial w} \tag{2.32}$$

$$0 = b_2 \cdot M_3 \cdot q + \rho_3 + \frac{\partial M_2}{\partial u'} \cdot Q_0 + [F(u') - \rho_3 \cdot q] \cdot \frac{\partial M_2}{\partial w} \tag{2.33}$$

$$0 = (M_2 + b_3 \cdot M_3) \cdot q + \frac{\partial M_3}{\partial u'} \cdot Q_0 + [F(u') - \rho_3 \cdot q] \cdot \frac{\partial M_3}{\partial w} . \tag{2.34}$$

Define $\rho = q - Q_1^{-1} . u'$ (2.35)

and use ρ instead of q . Then (2.24) implies

$$\frac{\partial M_i}{\partial u'} = 0 \qquad i = 0, \ldots, 3 \tag{2.36}$$

If $Q_0 = A_0 + A_1 . u' + A_2 . F(u')$ where A_0 , A_1 and A_2 are functions of w and ρ only, then one sees that the coefficients of $F(u')$ and $F'(u')$ in (2.25) cannot both vanish. Hence

$$Q_0 \neq A_0(w,\rho) + A_1(w,\rho).u' + A_2(w,\rho).F(u') \tag{2.37}$$

Then (2.35) and (2.34) must imply

$$\frac{\partial M_2}{\partial \rho} = \frac{\partial M_3}{\partial \rho} = 0 \tag{2.38}$$

Then (2.33) implies

$$\frac{\partial M_2}{\partial w} = b_2 M_3 = P_3 = 0 \tag{2.39}$$

Then (2.32) and $P_3 = 0$ imply

$$\frac{\partial M_1}{\partial \rho} = \frac{\partial M_1}{\partial w} = b_1 . M_3 = 0 \tag{2.40}$$

But (2.29) , (2.39) and (2.40) imply that $b_2 = b_1 = 0$. Then (2.30) contradicts (1.3) . So it has been shown that (2.1) cannot hold.

Hence

$$Q_1 = 0 \tag{2.41}$$

§3. For $Q_1 = 0$.

It has been shown in the previous section that $Q_1 = 0$. Then (1.9) gives that P_0 and P_1 are independent of q . Then (1.7) gives

$$Q_0 = Q_2 + Q_3.q \tag{3.1}$$

Then (1.7) to (1.12) imply

$$F'(u').p' = \frac{\partial P_0}{\partial u'}.Q_2 + \frac{\partial P_0}{\partial p'}.F(u') + \frac{\partial P_0}{\partial p}.F(u) + P_1.F'(u).p \tag{3.2}$$

$$0 = \frac{\partial P_1}{\partial u'}.Q_2 + \frac{\partial P_1}{\partial p'}.F(u') + \frac{\partial P_1}{\partial p}.F(u) \tag{3.3}$$

$$0 = \frac{\partial P_i}{\partial u} + \frac{\partial P_i}{\partial u'}.Q_3 \qquad i = 0, 1 . \tag{3.4}$$

$$F(u') = \frac{\partial Q_2}{\partial u}.p + \frac{\partial Q_2}{\partial u'}.p' + \frac{\partial Q_2}{\partial p'}.P_0 + Q_3 F(u) \tag{3.5}$$

$$0 = \frac{\partial Q_3}{\partial u}.p + \frac{\partial Q_3}{\partial u'}.p' + \frac{\partial Q_3}{\partial p'}.P_0 \tag{3.6}$$

$$0 = \frac{\partial Q_i}{\partial p} + P_1 \frac{\partial Q_i}{\partial p'} \qquad i = 2, 3 . \tag{3.7}$$

Take $\frac{\partial Q_3}{\partial p'} \cdot \left(\frac{\partial}{\partial p} + P_1 \cdot \frac{\partial}{\partial p'}\right)(3\cdot5) - \frac{\partial Q_2}{\partial p'} \cdot \left(\frac{\partial}{\partial p} + P_1 \cdot \frac{\partial}{\partial p'}\right)(3\cdot6)$.

$$\frac{\partial Q_3}{\partial p'} \cdot \left(\frac{\partial Q_2}{\partial u} + P_1 \cdot \frac{\partial Q_2}{\partial u'}\right) = \frac{\partial Q_2}{\partial p'} \cdot \left(\frac{\partial Q_3}{\partial u} + P_1 \cdot \frac{\partial Q_3}{\partial u'}\right) \qquad (3.8)$$

Taking $P_1 \cdot \frac{\partial}{\partial p'} + \frac{\partial}{\partial p}$ of (3.8) one sees that if this operator acting on P_1 is not zero then there is a function α such that

$$\frac{\partial Q_i}{\partial u'} + \alpha \cdot \frac{\partial Q_i}{\partial p'} = 0 \qquad i = 2,3$$

Take $\frac{\partial Q_3}{\partial p'} \cdot \left[\frac{\partial}{\partial u'} + \alpha \cdot \frac{\partial}{\partial p'}\right](3\cdot5) - \frac{\partial Q_2}{\partial p'} \cdot \left[\frac{\partial}{\partial u'} + \alpha \frac{\partial}{\partial p'}\right](3\cdot6)$ to obtain $\frac{\partial Q_3}{\partial p'} \cdot F'(u') = 0$. It will be assumed for the rest of this section that

$$\frac{\partial Q_3}{\partial p'} \neq 0 \qquad (3.9)$$

Hence

$$P_1 \cdot \frac{\partial P_1}{\partial p'} + \frac{\partial P_1}{\partial p} = 0 \qquad (3.10)$$

The rest of this section is devoted to proving that (3.9) cannot hold.

<u>Case 1.</u>

$$\frac{\partial P_1}{\partial p'} \neq 0 \qquad (3.11)$$

Then from (3.10) one sees that u , u' , p and z may be used as independent variables where $z = z(u,u',p,p')$ is given by

$$\begin{aligned} p' &= f(u,u',z).p + z \\ P_1 &= f(u,u',z) \end{aligned} \qquad (3.12)$$

Equations (3.6) and (3.7) imply

$$P_0 = A_0(u,u',z) + A_1(u,u',z).p + A_2(u,u',z).p^2 \qquad (3.13)$$

since both Q_2 and Q_3 independent of p' gives a contradiction to (1.3). After some re-arranging and defining A_3 by (3.23) one has from (3.2) to (3.7) that

$$\left.\begin{aligned}
\mathcal{D}_1 &= \frac{\partial}{\partial z} + \frac{Q_2}{[F(u') - f.F(u)]} \cdot \frac{\partial}{\partial u'} \\
\mathcal{D}_2 &= \frac{\partial}{\partial u} + Q_3 \cdot \frac{\partial}{\partial u'} \\
\mathcal{D}_3 &= \frac{\partial}{\partial u} + f \cdot \frac{\partial}{\partial u'} + A_3 \cdot \frac{\partial}{\partial z} \\
\mathcal{D}_4 &= \frac{\partial}{\partial u'} + \frac{A_0}{z} \cdot \frac{\partial}{\partial z}
\end{aligned}\right\} \tag{3.14}$$

$$\mathcal{D}_1 f = 0 \tag{3.15}$$

$$\mathcal{D}_1 A_0 = \frac{[F'(u') \cdot z - A_1 . F(u)]}{[F(u') - f . F(u)]} \tag{3.16}$$

$$\mathcal{D}_1 A_1 = \frac{[F'(u') \cdot f - 2 A_2 . F(u) - f . F'(u)]}{[F(u') - f . F(u)]} \tag{3.17}$$

$$\mathcal{D}_1 A_2 = 0 \tag{3.18}$$

$$\mathcal{D}_1 A_3 = \frac{[F'(u') . f - A_2 . F(u) - f . F'(u)]}{[F(u') - f . F(u)]} \tag{3.19}$$

$$\mathcal{D}_2 f = 0 \tag{3.20}$$

$$\mathcal{D}_2 A_i = 0 \qquad i = 0, 1, 2, 3. \tag{3.21}$$

$$\mathcal{D}_3 Q_i = 0 \qquad i = 2, 3. \tag{3.22}$$

$$\mathcal{D}_3 f = A_2 \tag{3.23}$$

$$\mathcal{D}_4 Q_3 = 0 \tag{3.24}$$

$$\mathcal{D}_4 Q_2 = [F(u') - Q_3 . F(u)] . z^{-1} \tag{3.25}$$

Consistency on (3.15) and (3.20) gives, using (3.11), that

$$\mathcal{D}_1 Q_3 = \mathcal{D}_2 \left[\frac{Q_2}{F(u') - f.F(u)} \right] \tag{3.26}$$

$$\mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1 = 0 . \tag{3.27}$$

$$\therefore \quad 0 = \left[\mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1 \right] (A_3 - A_1) = \mathcal{D}_2 \left[\frac{A_2 . F(u)}{F(u') - f.F(u)} \right] \tag{3.28}$$

If $A_2 \neq 0$ then divide by it and differentiate w.r.t. z to obtain a contradiction to (3.9). Hence

$$A_2 = 0 \tag{3.29}$$

Now (3.17), (3.21) and (3.27) one has

$$\left[\mathcal{D}_2 \mathcal{D}_1 - \mathcal{D}_1 \mathcal{D}_2 \right] A_1 = 0$$

$$\begin{aligned} \therefore \quad & Q_1 . \left\{ F''(u') . \left[F(u') - f.F(u) \right] - F'(u') . \left[F'(u') - F'(u) \right] \right\} \\ & = F''(u) . \left[F(u') - f.F(u) \right] - f.F'(u) . \left[F'(u') - F'(u) \right] \end{aligned} \tag{3.30}$$

Take of (3.30) and eliminate Q_1. One obtains a polynomial in f with coefficients which are functions of u and u' only. By (3.11), then, one has that these coefficients must be zero. The coefficient of f^3 is

$$F''(u').F(u).F'(u).F''(u') = (F''(u).F(u) + F'(u).F'(u')).F'''(u').F(u) \tag{3.31}$$

Without too much difficulty one then obtains that this contradicts (1.3). So it has been shown that (3.11) cannot hold.

Case 2.

$$\frac{\partial P_1}{\partial p'} = 0 \tag{3.32}$$

Then (3.3), (3.4) and (3.10) imply

$$P_1 \text{ is a constant} \tag{3.33}$$

Define

$$z = p' - P_1 . p \tag{3.34}$$

and use z instead of p'. Then (3.5) to (3.7) imply

$$\frac{\partial Q_i}{\partial z} = 0 \qquad , i = 2, 3 . \tag{3.35}$$

$$P_0 = P_3(u,u',z) + P_4(u,u',z).p \tag{3.36}$$

Substitute this into (3.2) to (3.7)

$$F'(u').z = \frac{\partial P_3}{\partial u'}.Q_2 + \frac{\partial P_3}{\partial z}.\left[F(u') - P_1.F(u)\right] + P_4.F(u) \tag{3.37}$$

$$F'(u').P_1 = \frac{\partial P_4}{\partial u'}.Q_2 + \frac{\partial P_4}{\partial z}.\left[F(u') - P_1.F(u)\right] + P_1.F'(u). \tag{3.38}$$

$$0 = \frac{\partial P_i}{\partial u} + Q_3.\frac{\partial P_i}{\partial u'} \qquad i = 3, 4 . \tag{3.39}$$

$$F(u') = \frac{\partial Q_2}{\partial u'}.z + \frac{\partial Q_2}{\partial z}.P_3 + Q_3.F(u) \tag{3.40}$$

$$0 = \frac{\partial Q_3}{\partial u'}.z + \frac{\partial Q_3}{\partial z}.P_3 . \tag{3.41}$$

$$0 = \frac{\partial Q_i}{\partial u} + P_1.\frac{\partial Q_i}{\partial u'} + P_4.\frac{\partial Q_i}{\partial z} \qquad i = 2, 3. \tag{3.42}$$

Consistency on (3.41) and (3.42) using (3.9) is

$$\left[\frac{\partial}{\partial u} + P_1.\frac{\partial}{\partial u'} + P_4.\frac{\partial}{\partial z}\right]\left(\frac{P_3}{z}\right) = \left(\frac{\partial}{\partial u'} + \frac{P_3}{z}.\frac{\partial}{\partial z}\right)P_4 \tag{3.43}$$

Consistency on (3.40) and (3.42) is then

$$(P_1.F'(u') - Q_3.F'(u)).z = P_4.(F(u') - Q_3.F(u)) \tag{3.44}$$

Take $\frac{\partial}{\partial u} + Q_3.\frac{\partial}{\partial u'}$ of (3.44) and eliminate P_4 . Take $\frac{\partial}{\partial u} + P_1.\frac{\partial}{\partial u'} + P_4\frac{\partial}{\partial z}$ of the result and eliminate $\frac{\partial Q_i}{\partial u} + Q_1.\frac{\partial Q_i}{\partial u'}$ If $P_1 = 0$ then one obviously has a contradiction to (1.3) . If $P_1 \neq 0$ then one has an expression, linear in Q_3 and having no explicit z dependence. Because of (3.9) the coefficient of Q_3 in this as well as the term independent of Q_3 must vanish. Take the coefficient of Q_3, multiply by $F(u')$ and subtract from $F(u)$ times the term independent of Q_3. One then has a contradiction to (3.9).

Hence (3.9) cannot be true.

§4. P_1 is a constant.

It has been shown in the previous section that (3.9) cannot hold. From this, (3.6) and (3.7) one has that

$$Q_3 \text{ is a constant} \tag{4.1}$$

Note that it is equations (3.2) to (3.7) which are being solved. It is desired to prove that P_1 is a constant, so for the rest of this section assume

$$\frac{\partial P_1}{\partial u'} \neq 0 . \tag{4.2}$$

It is then desired to find a contradiction. Eliminate Q_2 from (3.2) and (3.3). Take $\frac{\partial}{\partial u} + Q_3 . \frac{\partial}{\partial u'}$ repeatedly of this to obtain

$$\begin{vmatrix} F(u') & F(u) & F'(u').p' - P_1.F'(u).p \\ Q_3.F'(u') & F'(u) & Q_3.F''(u').p' - P_1.F''(u).p \\ Q_3^2.F''(u') & F''(u) & Q_3^2.F'''(u').p' - P_1.F'''(u).p \end{vmatrix} = 0 \tag{4.3}$$

Note that this implies that

$$Q_3 \neq 0 \tag{4.4}$$

Case 1.

$$P_1 \neq A(u,u').p'.p^{-1} \tag{4.5}$$

Then the coefficients of p' and $P_1.p$ in (4.3) must both vanish. Looking at the u dependence of the coefficient of p' in (4.3) one sees that F must satisfy

$$F''(u) = a_0.F(u) + a_1.F'(u) \tag{4.6}$$

where a_0 and a_1 are constants. Substituting (4.6) into the p' component of (4.3) and using (1.3) one obtains that

$$Q_3 = +1 \tag{4.7}$$

From (4.6) and (1.3) one has that $F(u) = A_1.\exp(c_1u) + A_2.\exp(c_2u)$ or $F(u) = (A_1 + A_2u).\exp(cu)$. Only the former case will be considered here, although the latter case is similar. That is take

$$F(u) = A_1.\exp(c_1u) + A_2.\exp(c_2u) \tag{4.8}$$

where A_1 , A_2 , c_1 and c_2 are constants, and where

$$A_1 \neq 0 \quad , \quad A_2 \neq 0 \quad , \quad c_1 \neq 0 \quad , \quad c_2 \neq 0 \quad , \quad c_1 \neq c_2 \text{ and } c_1 \neq -c_2 \tag{4.9}$$

Define

$$w = \tfrac{1}{2}(u'+u) \qquad , \qquad v = \tfrac{1}{2}(u'-u) \tag{4.10}$$

Then (3.3) and (3.4) imply

$$\frac{\partial \rho_i}{\partial w} = 0 \quad , \; i = 0, 1. \tag{4.11}$$

$$\rho_0 = 2A_2.W_2(v,x,x').e^{c_2 \cdot w} + 2A_1.W_1(v,x,x').e^{c_1 \cdot w} \tag{4.12}$$

Equations (3.2) to (3.7) then are, for $i = 1,2$

$$c_i.x'.e^{c_i v} = W_i.\frac{\partial \rho_0}{\partial v} + e^{c_i v}.\frac{\partial \rho_0}{\partial x'} + e^{-c_i v}.\frac{\partial \rho_0}{\partial x} + \rho_1.x.c_i.e^{-c_i v} \tag{4.13}$$

$$0 = W_i.\frac{\partial \rho_1}{\partial v} + e^{c_i v}.\frac{\partial \rho_1}{\partial x'} + e^{-c_i v}.\frac{\partial \rho_1}{\partial x} \tag{4.14}$$

$$e^{c_i v} - e^{-c_i v} = (x'-x).\frac{\partial W_i}{\partial v} + c_i(x'+x).W_i + 2\rho_0.\frac{\partial W_i}{\partial x'} \tag{4.15}$$

$$0 = \frac{\partial W_i}{\partial x} + \rho_1.\frac{\partial W_i}{\partial x'} \tag{4.16}$$

Note that (4.15) and (4.16) imply

$$\frac{\partial W_i}{\partial x'} \neq 0 . \tag{4.17}$$

Eliminate P_0 from (4.15) and take $\frac{\partial}{\partial x} + \rho_1 \cdot \frac{\partial}{\partial x'}$

$$(\rho_1 - 1)\left[\frac{\partial W_2}{\partial x'}.\frac{\partial W_1}{\partial v} - \frac{\partial W_1}{\partial x'}.\frac{\partial W_2}{\partial v}\right] = (\rho_1+1)\left[c_2 W_2.\frac{\partial W_1}{\partial x'} - c_1.W_1.\frac{\partial W_2}{\partial x'}\right] \tag{4.18}$$

If $\rho_1 \frac{\partial \rho_1}{\partial p'} + \frac{\partial \rho_1}{\partial p} \neq 0$ then take $\rho_1 \frac{\partial}{\partial p'} + \frac{\partial}{\partial p}$ of (4.18) to see that both sides of (4.18) must vanish. This , (4.15) , (4.17) and (4.2) imply that

$$c_1 . W_1 . \left[e^{c_2 v} - e^{-c_2 v} \right] = c_2 . W_2 . \left[e^{c_1 v} - e^{-c_1 v} \right]$$

Diff. w.r.t. p' and use the fact that the R.H.S. of (4.18) is zero to eliminate W_2. One then obtains a contradiction. Hence

$$\rho_1 . \frac{\partial \rho_1}{\partial p'} + \frac{\partial \rho_1}{\partial p} = 0 \qquad (4.19)$$

Eliminate derivatives of P_1 from (4.14) and (4.19) using (4.2)

$$W_1 . \left[e^{c_2 v} - \rho_1 . e^{-c_2 v} \right] = W_2 . \left[e^{c_1 v} - \rho_1 . e^{-c_1 v} \right] \qquad (4.20)$$

Consistency on (4.15) and (4.16) is

$$\mathcal{D}_3 W_i + c_i . (\rho_1 + 1) W_i = 0 \qquad (4.21)$$

where

$$\left. \begin{aligned} \mathcal{D}_3 &= (\rho_1 - 1) \frac{\partial}{\partial v} + X \frac{\partial}{\partial p'} \\ X &= 2 \left[\frac{\partial \rho_0}{\partial p} + \rho_1 . \frac{\partial \rho_0}{\partial p'} \right] - \left[(p' - p) . \frac{\partial \rho_1}{\partial v} + 2 \rho_0 . \frac{\partial \rho_1}{\partial p'} \right] \end{aligned} \right\} \qquad (4.22)$$

Take $\mathcal{D}_3$ of (4.20) and use (4.20) to eliminate W_1 . Take $W_i \frac{\partial}{\partial v} + e^{c_i v} \frac{\partial}{\partial p'} + e^{-c_i v} \frac{\partial}{\partial p}$ repeatedly of the result to see that the coefficients of $\exp(c_1 - c_2)v$, $\exp(c_2 - c_1)v$, $\exp(c_1 + c_2)v$ and $\exp(-c_1 - c_2)v$ must vanish. The coefficient of $\exp(c_1 + c_2)v$ is $(c_1 - c_2) . P_1 = 0$ which is a contradiction. Hence (4.5) cannot hold.

<u>Case 2.</u>

$$P_1 = A(u,u') . p' . p^{-1} \qquad (4.23)$$

Then (3.3) gives

$$Q_0 = B(u,u') . (F(u) . p^{-1} - F(u') . (p')^{-1}) \qquad (4.24)$$

Take $P_1 \cdot \frac{\partial}{\partial p'} + \frac{\partial}{\partial p}$ of this to obtain a contradiction. Hence (4.2) cannot hold. That is

$$\frac{\partial P_1}{\partial u'} = 0$$

But then (3.3) and (3.4) imply that

P_1 is a constant (4.25)

§5. P_0 is linear.

The equations to be solved are (3.2) to (3.7) where P_1 and Q_3 are constants. Define

$$z = p' - P_1 . p \tag{5.1}$$

and use z instead of p'. Then (3.5) and (3.7) give

$$\frac{\partial Q_2}{\partial p} = 0 \tag{5.2}$$

$$P_0 = Z_0(u,u',z) + Z_1(u,u',z).p \tag{5.3}$$

Substitute (5.3) into (3.2) to (3.6)

$$F'(u').z = \frac{\partial Z_0}{\partial u'}.Q_2 + [F(u') - P_1.F(u)].\frac{\partial Z_0}{\partial z} + Z_1.F(u) \tag{5.4}$$

$$F'(u').P_1 = \frac{\partial Z_1}{\partial u'}.Q_2 + [F(u') - P_1.F(u)].\frac{\partial Z_1}{\partial z} + P_1.F'(u) \tag{5.5}$$

$$0 = \frac{\partial Z_i}{\partial u} + Q_3.\frac{\partial Z_i}{\partial u'} \qquad i = 0, 1. \tag{5.6}$$

$$F(u') = \frac{\partial Q_2}{\partial u'}.z + \frac{\partial Q_2}{\partial z}.Z_0 + Q_3.F(u) \tag{5.7}$$

$$0 = \frac{\partial Q_2}{\partial u} + P_1.\frac{\partial Q_2}{\partial u'} + Z_1.\frac{\partial Q_2}{\partial z} \tag{5.8}$$

As in the previous section, the proof will not be given in the case $F(u) = (A_1+A_2u).\exp(cu)$ because it is very similar to the case under consideration. I have carried through the proof in this case and found, as in section 4, that (4.2) cannot hold and in the present section one finds that (5.9) cannot hold.

For this section assume that

$$Z_1 \neq A(u,u').z \tag{5.9}$$

It will be shown in this section that (5.9) cannot hold.

If Z_0 is independent of u' then consistency on (5.4) and (5.6) gives that $\frac{\partial Z_0}{\partial z}$ equals a constant times z since, by (1.3), one has $\left(\frac{\partial}{\partial u} + Q_3 \cdot \frac{\partial}{\partial u'}\right)\left[\frac{F'(u')}{F(u)}\right] \neq 0$. But then (5.4) gives a contradiction to (5.9). Hence one has that

$$\frac{\partial Z_0}{\partial u'} \neq 0 \tag{5.10}$$

If $\frac{\partial Z_1}{\partial u'} = 0$ then (5.5) and (5.6) give that Z_1 is a constant and that $P_1 = 0$. Define $w = z - Z_1.u$ and use w instead of z. Then (5.7) and (5.8) imply that

$Z_0 = W_0(u',w) + W_1(u',w).u + W_2(u',w).F(u)$

Substitute this into (5.6) and use (1.3) to obtain $W_2 = 0$. One then sees that the B.T. does not depend on u or its derivatives. Hence

$$\frac{\partial Z_1}{\partial u'} \neq 0 \tag{5.11}$$

Eliminate Q_2 from (5.4) and (5.5). Divide by $(Z_0)_{u'}$ and diff. w.r.t. z

$$F'(u')\cdot\frac{\partial}{\partial z}\cdot\left\{\left(z\cdot\frac{\partial Z_1}{\partial u'}\right)\cdot\left(\frac{\partial Z_0}{\partial u'}\right)^{-1}\right\} - F(u)\cdot\frac{\partial}{\partial z}\cdot\left\{\left(Z_1\cdot\frac{\partial Z_1}{\partial u'}\right)\left(\frac{\partial Z_0}{\partial u'}\right)^{-1}\right\}$$
$$= \left[F(u') - P_1\cdot F(u)\right]\frac{\partial}{\partial z}\cdot\left\{\left[\frac{\partial Z_1}{\partial u'}\cdot\frac{\partial Z_0}{\partial z} - \frac{\partial Z_0}{\partial u'}\cdot\frac{\partial Z_1}{\partial z}\right]\cdot\left(\frac{\partial Z_0}{\partial u'}\right)^{-1}\right\} \tag{5.12}$$

If both terms on the L.H.S. are zero then integrating up gives a contradiction to (5.9) or (5.10). So take $\frac{\partial}{\partial u} + Q_3\cdot\frac{\partial}{\partial u'}$ repeatedly of (5.12) to obtain

$$\begin{vmatrix} F'(u') & F(u) & F(u') \\ Q_3 . F''(u') & F'(u) & Q_3 . F(u') \\ Q_3^2 . F'''(u') & F''(u) & Q_3^2 F''(u') \end{vmatrix} = 0 \tag{5.13}$$

The u dependence of (5.13) implies

$$F''(u) = a_0 . F(u) + a_1 . F''(u) \tag{5.14}$$

So, by the discussion below (5.8) take

$$F(u) = A_1 . \exp(c_1 u) + A_2 . \exp(c_2 u) \tag{5.15}$$

Substituting (5.14) into (5.13) gives

$$Q_3 = +1 \tag{5.16}$$

Define $$v = u' - u \tag{5.17}$$

and use v instead of u' . Then (5.6) and (5.4) or (5.5) imply

$$\frac{\partial Z_i}{\partial u} = 0 \quad , \quad i = 0, 1. \tag{5.18}$$

$$P_0 = A_1 . W_1 . (v, z) . e^{c_1 . u} + A_2 . W_2 (v, z) . e^{c_2 u} \tag{5.19}$$

Substitute (5.19) into (5.4) to (5.8) to obtain, for i = 1,2

$$c_i . e^{c_i . v} . z = \frac{\partial Z_0}{\partial v} . W_i + (e^{c_i . v} - P_1) . \frac{\partial Z_0}{\partial z} + Z_1 \tag{5.20}$$

$$c_i . (e^{c_i v} - 1) . P_1 = \frac{\partial Z_1}{\partial v} . W_i + (e^{c_i v} - P_1) . \frac{\partial Z_1}{\partial z} \tag{5.21}$$

$$e^{c_i v} - 1 = \frac{\partial W_i}{\partial v} . z + \frac{\partial W_i}{\partial z} . Z_0 . \tag{5.22}$$

$$0 = c_i . W_i + (P_1 - 1) . \frac{\partial W_i}{\partial v} + Z_1 . \frac{\partial W_i}{\partial z} \tag{5.23}$$

Note that if $P_1 = 0$ then (5.21) implies $W_2 . \exp(c_1 v) = W_1 . \exp(c_2 v)$. Take the operator in (5.23) acting on this to obtain $W_1 = A(u) . z^{-1}$. But then (5.23) gives a contradiction to (5.9) . Therefore

$$P_1 \neq 0 \tag{5.24}$$

Define

$$W_3 = W_2 \cdot \frac{\partial W_1}{\partial v} + \left(e^{c_2 v} - P_1\right) \cdot \frac{\partial W_1}{\partial z} - W_1 \cdot \frac{\partial W_2}{\partial v} - \left(e^{c_1 v} - P_1\right) \cdot \frac{\partial W_2}{\partial z} \tag{5.25}$$

$$\therefore \; (P_1 - 1) \cdot \frac{\partial W_3}{\partial v} + Z_1 \cdot \frac{\partial W_3}{\partial z} = -(c_1 + c_2) \cdot W_3 \tag{5.26}$$

Use the consistency condition on (5.21) to eliminate derivatives of Z_1 from (5.21) :

$$W_3 \left\{ (c_1 - c_2) \cdot \left[e^{(c_1+c_2)v} - P_1 \right] + (c_2 - P_1 . c_1) . e^{c_1 v} - (c_1 - P_1 c_2) e^{c_2 v} \right\}$$

$$- W_1 W_2 \cdot \left\{ (c_1 - c_2)^2 . e^{(c_1+c_2)v} + c_1 \left(c_2 - P_1 . c_1\right) . e^{c_1 v} + c_2 \left(c_1 - P_1 c_2\right) . e^{c_2 v} \right\}$$

$$- (P_1 - 1) \left\{ c_1^2 . W_2^2 . e^{c_1 v} + c_2^2 . W_1^2 . e^{c_2 v} \right\} = 0 \tag{5.27}$$

Take $\left(P_1 - 1\right) \frac{\partial}{\partial v} + Z_1 \cdot \frac{\partial}{\partial z}$ repeatedly of (5.27) to eliminate W_3, $W_1 W_2$, W_1^2 and W_2^2. One sees that the coefficient of $\exp(2c_1+c_2)v$ or of $\exp(c_1+2c_2)v$ in the result must vanish. In either case one has

$$(c_1 - P_1 . c_2)(P_1 . c_1 - c_2)(P_1 - 1) = 0 \tag{5.28}$$

If $P_1 \neq 0$ then one may w.l.o.g. take $c_1 = P_1 . c_2$. Then from the above mentioned sequence of equations from (5.27) one may prove that $W_3 = c_1 . W_1 . W_2$ and $W_1 = P_1^2 . W_2 . \exp(-c_2 v)$. Take $z \frac{\partial}{\partial v} + Z_0 \cdot \frac{\partial}{\partial z}$ of this last equation to obtain $W_1 = A(v) . z^{-1}$, which, from (5.23) gives a contradiction to (5.9). Hence

$$P_1 = 1 \tag{5.29}$$

Consistency on (5.21) is then

$$\left[\frac{\partial}{\partial z} + \frac{W_1}{(e^{c_1 v} - 1)} \cdot \frac{\partial}{\partial v} \right] \left[\frac{W_2}{(e^{c_2 v} - 1)} \right] = \left[\frac{\partial}{\partial z} + \frac{W_2}{(e^{c_2 v} - 1)} \cdot \frac{\partial}{\partial v} \right] \left[\frac{W_1}{(e^{c_1 v} - 1)} \right] \tag{5.30}$$

Take $z\cdot\frac{\partial}{\partial v} + Z_0\cdot\frac{\partial}{\partial z}$ of (5.30)

$$z = A(v).W_1^{-1} + B(v).W_2^{-1} \tag{5.31}$$

where

$$\left.\begin{aligned} A(v)\cdot\{c_1^2\cdot e^{c_1v}\cdot(e^{c_1v}-1)^{-2} - c_2^2\cdot e^{c_2v}\cdot(e^{c_2v}-1)^{-2}\} \\ = -c_2\cdot(e^{c_1v}-1)\cdot(e^{c_2v}-1)^{-1}\cdot(2\cdot e^{c_2v}-1) \\ B(v)\cdot\{c_1^2\cdot e^{c_1v}\cdot(e^{c_1v}-1)^{-2} - c_2^2\cdot e^{c_2v}\cdot(e^{c_2v}-1)^{-2}\} \\ = c_1\cdot(e^{c_2v}-1)\cdot(e^{c_1v}-1)^{-1}\cdot(2\,e^{c_1v}-1) \end{aligned}\right\} \tag{5.32}$$

Diff. (5.31) w.r.t. z, multiply by Z_1 and use (5.23). Solve this and (5.31) for W_1 and W_2.

$$\begin{aligned} W_1 &= A.(c_1-c_2)(Z_1-c_2z)^{-1} \\ W_2 &= B.(c_2-c_1)(Z_1-c_1z)^{-1} \end{aligned} \tag{5.33}$$

Substitute (5.33) into (5.30) and use (5.21), to obtain for a constant K :

$$A(v).(\exp c_1v - 1)^{-1} = B(v).(\exp c_2v - 1)^{-1}.K$$

From (5.32) one sees that this cannot be true.

So it has been shown that (5.9) cannot hold. This completes this section.

6. Conclusion.

The proof that there are no B.T. of the form (1.2) will be completed in this section. From the previous section one has that (5.9) cannot hold. Hence

$$Z_1 = A(u,u').z \tag{6.1}$$

Then (5.5) and (5.11) imply

$$Q_2 = B(u,u').z^{-1} \tag{6.2}$$

Then (5.7) implies

$$Z_0 = C(u,u').z^2 \tag{6.3}$$

Substitute (6.1) to (6.3) into (5.4) to (5.8)

$$F'(u') = \frac{\partial C}{\partial u'}.B + [F(u') - P_1.F(u)].2C + A.F(u) \qquad (6.4)$$

$$F'(u').P_1 = \frac{\partial C}{\partial u'}.B + [F(u') - P_1.F(u)].A + P_1.F'(u) \qquad (6.5)$$

$$0 = \frac{\partial A}{\partial u} + Q_3.\frac{\partial A}{\partial u'} \qquad (6.6)$$

$$0 = \frac{\partial C}{\partial u} + Q_3.\frac{\partial C}{\partial u'} \qquad (6.7)$$

$$F(u') = \frac{\partial B}{\partial u'} - BC + Q_3.F(u) \qquad (6.8)$$

$$0 = \frac{\partial B}{\partial u} + P_1.\frac{\partial B}{\partial u'} - A.B \qquad (6.9)$$

Note that (5.11) is

$$\frac{\partial A}{\partial u'} \neq 0 \qquad (6.10)$$

Eliminate B from (6.4) and (6.5) .Repeatedly take the operator in (6.6) and (6.7) acting on this and use (6.10) .

$$\begin{vmatrix} F'(u') & F(u') & F'(u) & F(u) \\ Q_3 F''(u') & Q_3.F'(u') & F''(u) & F'(u) \\ Q_3^2.F'''(u') & Q_3^2.F''(u') & F'''(u) & F''(u) \\ Q_3^3.F^{iv}(u') & Q_3^3.F'''(u') & F^{iv}(u) & F'''(u) \end{vmatrix} = 0 \qquad (6.11)$$

Suppressing the u' dependence one may replace the first two columns in (6.11) by $(0,1,a_1,a_2)$ and $(1,a_3,a_4,a_5)$ respectively. Take F(u).col 2 from col 4 and F'(u).col 2 from col 3 . Expand along row 1 .

In the resultant 3X3 determinant again use the constant column to expand along row 1 . In the resultant 2X2 determinant one sees that the first column is the derivative of the second. Integrating this one obtains that

$$F'''(u) = b_0.F(u) + b_1.F'(u) + b_2.F''(u) \tag{6.12}$$

where b_0 , b_1 and b_2 are constants.

Case 1.

$$F''(u) \neq c_0.F(u) + c_1.F'(u) \tag{6.13}$$

From (6.11) , (6.12) and (6.13) it is not too difficult to see that

$$\begin{array}{ll} & Q_3 = +1 \\ \text{or} & \\ & Q_3 = -1 \quad \text{and} \quad F'''(u) = K.F'(u) \end{array} \tag{6.14}$$

If $P_1 = Q_3$ then (6.6) and (6.7) give that A and C are independent of u when u and $v = u' - P_1.u$ are used as independent variables. Take $\frac{\partial^3}{\partial u^3} - b_0 - b_1 \frac{\partial}{\partial u} - b_2 . \frac{\partial^2}{\partial u^2}$ of (6.5) where u and v are used as independent variables and use (6.12) and (6.14) :

$$B_{uuu} = b_0.B + b_1.B_u + b_2.B_{uu} \tag{*}$$

With u and v as independent variables, equation (6.9) gives, on integrating, that $B = D(v).\exp A.u$

Substituting this into (*) one obtains that A(v) is a constant. This contradicts (6.10). So one must have

$$P_1 \neq Q_3 \tag{6.15}$$

Consistency on (6.8) and (6.9) imply

$$F'(u) + F'(u') = 2.A.F(u) + 2.C.(\, F(u') - P_1.F(u)\,) \tag{6.16}$$

Define
$$v = u' - Q_3.u \tag{6.17}$$

and use u and v as independent variables. Then (6.6) and (6.7) imply

$$A_u = C_u = 0 \tag{6.18}$$

If $Q_3 = -1$ then (6.14) gives $F(u) = F_0 + F_1.e^{cu} + F_2.e^{-cu}$ where F_0 , F_1 and F_2 are constants. substituting this into (6.16) one readily obtains a contradiction. Hence one must have

$$Q_3 = +1 \tag{6.19}$$

If $F(u) = (F_0 + F_1.u + F_2.u^2).e^{cu}$ or $F(u) = (F_0 + F_1.u).\exp c_1u + F_2.\exp c_2u$ then (6.16) readily gives a contradiction. So take

$$F(u) = F_1.\exp c_1u + F_2.\exp c_2u + F_3.\exp c_3u \tag{6.20}$$

where F_i and c_i are constants.

Substitute (6.20) into (6.16), equate coefficients of $\exp c_i u$ for $i = 1,2,3$. Then eliminate A and C from these equations

$$(c_2-c_1)(e^{c_2 v}+1)(e^{c_3 v}-e^{c_1 v}) = (c_3-c_1)\cdot(e^{c_3 v}+1)(e^{c_2 v}-e^{c_1 v}) \tag{6.21}$$

Since c_1, c_2 and c_3 are distinct, one sees that (6.21) can only be satisfied if

$$c_3 = c_1 + c_2 = 0 \tag{6.22}$$

or equivalent e.g. $c_2 = c_3 + c_1 = 0$. So one may take

$$F(u) = K_0 + K_1.e^{cu} + K_2.e^{-cu} \tag{6.23}$$

Substitute (6.23) into (6.16) and solve for A and C.

$$\begin{aligned} C &= \tfrac{1}{2}c.\coth\tfrac{1}{2}cv \\ A &= C.(P_1-1) \end{aligned} \tag{6.24}$$

Substitute (6.23) into (6.4) or (6.5) to obtain

$$B = K_0.W_0(v) + K_1.W_1(v).e^{cu} + K_2.W_2(v).e^{-cu} \tag{6.25}$$

The terms independent of u in (6.5) and (6.9) are then

$$0 = \frac{\partial A}{\partial v}.W_0 + (1-P_1).A$$

$$0 = (P_1-1)\frac{\partial W_0}{\partial v} - A.W_0$$

From this and (6.24) one easily obtains a contradiction. So (6.13) cannot hold.

<u>Case 2</u>.

$$F''(u) = c_0.F(u) + c_1.F'(u) \tag{6.26}$$

Substitute (6.26) into (6.11) to obtain a polynomial in $(F'(u'))/(F(u'))$ with constant constant coefficients.From (1.3) these coefficients must vanish. It is then not too difficult to see that this can only happen if

$$Q_3 = +1 \tag{6.27}$$

Define $v = u' - u$ (6.28)

and use u and v as independent variables. Then (6.6) and (6.7) are

$$A_u = C_u = 0 \tag{6.29}$$

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Only the case

$$F(u) = K_1.\exp k_1 u + K_2.\exp k_2 u \tag{6.30}$$

will be considered here. The case $F(u) = (K_0 + K_1 u).\exp ku$ is very similar and like the present case does not give rise to any B.T.

From (1.3) one may take

$$K_1 \neq 0 \ , \ K_2 \neq 0 \ , \ k_1 \neq 0 \ , \ k_2 \neq 0 \ , \ k_1 \neq k_2 \text{ and } k_1 \neq -k_2 \tag{6.31}$$

Now equation (6.4) or (6.5) gives, on using (6.29) and (6.30), that

$$B = K_1.W_1(v).\exp k_1 u + K_2.W_2.\exp k_2 u \tag{6.32}$$

Then (6.4) to (6.9), for $i = 1,2$, are

$$k_i . e^{k_i v} = \frac{dC}{dv} . W_i + (e^{k_i v} - P_1).2C + A \tag{6.33}$$

$$k_i . e^{k_i v} . P_1 = \frac{dA}{dv} . W_i + (e^{k_i v} - P_1). A + P_1 . k_i \tag{6.34}$$

$$e^{k_i v} = \frac{dW_i}{dv} - C . W_i + 1 \tag{6.35}$$

$$0 = k_i . W_i + (P_1 - 1) \frac{dW_i}{dv} - A . W_i \tag{6.36}$$

Eliminate derivatives of W_i from (6.35) and (6.36) . Differentiate the result w.r.t. v and use (6.33) to (6.35) to eliminate derivatives of A , C and W_i . One then has that

$$2A + 2C . (e^{k_i v} - P_1) = k_i . (e^{k_i v} + 1) \tag{6.37}$$

Eliminate A from (6.37)

$$2C . (e^{k_1 v} - e^{k_2 v}) = k_1 (e^{k_1 v} + 1) - k_2 (e^{k_2 v} + 1) \tag{6.38}$$

Note that if $P_1 = 1$ then (6.36) implies $A = k_1 = k_2$.

From (6.37) and (6.38)

$$2A \ (e^{k_1 v} - e^{k_2 v}) = k_1 (e^{k_1 v} + 1)(P_1 - e^{k_2 v}) - k_2 (e^{k_2 v} + 1)(P_1 - e^{k_1 v} \tag{6.39}$$

Also from (6.35) and (6.36) :

$$\left.\begin{aligned} W_1 \cdot (k_2 - k_1) \cdot (e^{k_2 v} + 1) &= 2(\rho_1 - 1)(e^{k_1 v} - e^{k_2 v}) \\ W_2 \cdot (k_2 - k_1) \cdot (e^{k_1 v} + 1) &= 2(\rho_1 - 1) \cdot (e^{k_1 v} - e^{k_2 v}) \end{aligned}\right\} \quad (6.40)$$

Substitute (6.39) and (6.40) into (6.36)

$$e^{(k_1+k_2) \cdot v} \cdot (5k_1 - 3k_2) - 2k_1 \cdot e^{2k_2 v} + \rho_1 (k_2 - k_1)$$
$$+ \left[(4 - \rho_1) k_1 - k_2\right] \cdot e^{k_1 v} + \left[(\rho_1 - 2) k_2 - k_1\right] \cdot e^{k_2 v} = 0$$

One then sees that, because of (6.31), this cannot be true.

So one sees that (1.1) has no B.T. of the form (1.2). The next chapter is another attempt to find B.T. for (1.1).

CHAPTER 5.

§1. Introduction.

A final attempt will be made in this chapter to find B.T. for equations of the form

$$u_{xy} = F(u) \tag{1.1}$$

by allowing the B.T. to depend on derivatives of u and u' only. One is only interested in B.T. which are useful in solving (1.1). So one would like the B.T. to be as simple as possible. The easiest way that this can be done and which has not been considered previously is if one equation in the B.T. is an ordinary differential equation i.e. take

$$p'_{N+1} = P(u,p_1,\ldots,p_M;u',p'_1,\ldots,p'_N) \tag{1.2}$$

for one of the equations of the B.T. pair. Here p_i and p'_i are the ith derivatives of u and u' respectively.

If the B.T. does not reduce to first order equations for u' (considered in chapter 3) and if neither of the equations in the B.T. reduce to an ordinary differential equation then it is hard to see how the B.T. could be useful in solving (1.1). Unfortunately from the result of this chapter there does not seem to be any B.T. of this type for (1.1).

If one differentiates (1.2) w.r.t. y and assumes that both u and u' satisfy (1.1) then if u' actually appears in (1.2) then one may take the second equation in the B.T. to be

$$u'_y = Q(u,p_1,\ldots,p_M;u',p'_1,\ldots,p'_N;u'_y) \tag{1.3}$$

Actually, from the lemma of section 3 of chapter 3, one sees that the second equation must be of this form provided only that the simplest form of the first equation is (1.2).

To look for all B.T. of the form (1.2) and (1.3) is very difficult. The main motivation for this chapter was the numerical results for the double sine-Gordon equation. Also, in all examples considered so far one has found that F(u) of (1.1) must satisfy a linear equation. So, for the rest of this chapter, instead of (1.1) take the equation to be

$$u_{xy} = \sin u + K.\sin cu \tag{1.4}$$

where K and c are constants with

$$K \neq 0 \quad \text{and} \quad c.(c^2 - 1) \neq 0 \tag{1.5}$$

Even with this simplification it is very difficult to work with the forms (1.2) and (1.3) .A reasonable assumption seems to be that one cannot tell whether u or u' is the "new" variable . Infact, in analogy with the S.G. it will be assumed that the B.T. has the form

$$\begin{aligned} \tfrac{1}{2}(p'_{N+1} - p_{N+1}) &= P(p_0,p_1,\dots,p_N;p'_0,p'_1,\dots,p'_N) \\ \tfrac{1}{2}(q' + q) &= Q(p_0,p_1,\dots,p_N;p'_0,p'_1,\dots,p'_N) \end{aligned} \tag{1.6}$$

Note that if there is a B.T. of this form then since $-u$ is a solution of (1.4) whenever u is there is also a B.T. of the form $p'_{N+1} + p_{N+1} = P$; $q' - q = Q$.

It is proved in the rest of this chapter that for $N < 5$ there are no B.T. of the form (1.6) for (1.4).

For $n > 0$,
$$\begin{aligned} \frac{\partial p_n}{\partial y} &= A_n.\sin u + B_n.\cos u + K.C_n.\sin cu + K.D_n.\cos cu \\ &= \frac{\partial^{n+1}}{\partial x^n \partial y}.u = \left(\frac{\partial}{\partial x}\right)^{n-1}\left[\sin u + K.\sin cu\right] \end{aligned} \tag{1.7}$$

Now this is the definition of A_n , B_n , C_n and D_n which are functions of $p_1,\dots,p_{n-1}$ only. The first few are

$$\left.\begin{array}{llll} A_1 = 1 & B_1 = 0 & C_1 = 1 & D_1 = 0 \\ A_2 = 0 & B_2 = p_1 & C_2 = 0 & D_2 = c.p_1 \\ A_3 = -p_1^2 & B_3 = p_2 & C_3 = -c^2.p_1^2 & D_3 = c.p_2 \\ A_4 = -3.p_1.p_2 & B_4 = p_3 - p_1^3 & C_4 = -3.c^2.p_1.p_2 & D_4 = c.p_3 - c^3.p_1^3 \end{array}\right\} \tag{1.8}$$

Define

$$\begin{aligned} p'_n &= \alpha_n + \beta_n \\ p_n &= \alpha_n - \beta_n \end{aligned} \tag{1.9}$$

Differentiate the first of (1.6) w.r.t. y and the second w.r.t. x. The coefficient of q' - q in the first must vanish i.e.

$$\frac{\partial P}{\partial \beta_0} = 0 \tag{1.10}$$

Also the coefficient of α_{N+1} in the second must vanish i.e.

$$\frac{\partial Q}{\partial \alpha_N} = 0 \tag{1.11}$$

If P does not depend on u or u' then the coefficients of sin u' and cos u' in the y derivative of the first of (1.6) must vanish. The lemma of section 3 of chapter 3 is then applicable and one sees that $P - \frac{1}{2}p'_{N+1}$ cannot depend on p'_n for any n. But P does not depend on p'_{N+1}. Therefore

$$\frac{\partial P}{\partial \alpha_0} \neq 0 \tag{1.12}$$

Then from the y derivative of the first of (1.6) one has

$$Q = Q_1.\sin\beta_0 + Q_2.\cos\beta_0 + K.Q_3.\sin c\beta_0 + K.Q_4.\cos c\beta_0 \tag{1.13}$$

Now consider the x derivative of the second equation in (1.6). If Q does not depend on β_N then one sees that it cannot depend on β_{N-1} or . . . or β_0. But then the R.H.S. is independent of β_0 while the L.H.S. is not. Hence

$$\frac{\partial Q}{\partial \beta_N} \neq 0 \tag{1.14}$$

Also $$P = P_0 + P_1.\alpha_N \tag{1.15}$$

Substitute (1.13) and (1.15) into the y derivative of (1.6a) and into the x derivative of (1.6b). Define

$$\left.\begin{aligned} \mathcal{D}_1 &= P_0 \frac{\partial}{\partial \beta_N} + \sum_{i=0}^{N-2} \alpha_{i+1} \cdot \frac{\partial}{\partial \alpha_i} + \sum_{i=1}^{N-1} \beta_{i+1} \cdot \frac{\partial}{\partial \beta_i} \\ \mathcal{D}_2 &= \frac{\partial}{\partial \alpha_{N-1}} + P_1 \cdot \frac{\partial}{\partial \beta_N} \end{aligned}\right\} \tag{1.16}$$

$$\left(\mathcal{D}_1 = P_0 \cdot \frac{\partial}{\partial \beta_1} \quad \text{if} \quad N=1\right)$$

$$\mathcal{J}_1 = Q_1 \cdot \frac{\partial}{\partial \alpha_0} + \sum_{n=1}^{N-1} \left[\tfrac{1}{2}(A_n' - A_n) \cdot \cos \alpha_0 - \tfrac{1}{2}(B_n' - B_n) \sin \alpha_0 \right] \cdot \frac{\partial}{\partial \alpha_n}$$
$$+ \sum_{n=1}^{N} \left[\tfrac{1}{2}(A_n' + A_n) \cdot \cos \alpha_0 - \tfrac{1}{2}(B_n' + B_n) \cdot \sin \alpha_0 \right] \cdot \frac{\partial}{\partial \beta_n} \tag{1.17a}$$

$$\mathcal{J}_2 = Q_2 \cdot \frac{\partial}{\partial \alpha_0} + \sum_{n=1}^{N-1} \left[\tfrac{1}{2}(A_n' + A_n) \cdot \sin \alpha_0 + \tfrac{1}{2}(B_n' + B_n) \cdot \cos \alpha_0 \right] \cdot \frac{\partial}{\partial \alpha_n}$$
$$+ \sum_{n=1}^{N} \left[\tfrac{1}{2}(A_n' - A_n) \cdot \sin \alpha_0 + \tfrac{1}{2}(B_n' - B_n) \cdot \cos \alpha_0 \right] \cdot \frac{\partial}{\partial \beta_n} \tag{1.17b}$$

$$\mathcal{J}_3 = Q_3 \cdot \frac{\partial}{\partial \alpha_0} + \sum_{n=1}^{N-1} \left[\tfrac{1}{2}(C_n' - C_n) \cdot \cos c\alpha_0 - \tfrac{1}{2}(D_n' - D_n) \sin c\alpha_0 \right] \cdot \frac{\partial}{\partial \alpha_n}$$
$$+ \sum_{n=1}^{N} \left[\tfrac{1}{2}(C_n' + C_n) \cdot \cos c\alpha_0 - \tfrac{1}{2}(D_n' + D_n) \cdot \sin c\alpha_0 \right] \cdot \frac{\partial}{\partial \beta_n} \tag{1.17c}$$

$$\mathcal{J}_4 = Q_4 \cdot \frac{\partial}{\partial \alpha_0} + \sum_{n=1}^{N-1} \left[\tfrac{1}{2}(C_n' + C_n) \cdot \sin c\alpha_0 + \tfrac{1}{2}(D_n' + D_n) \cdot \cos c\alpha_0 \right] \cdot \frac{\partial}{\partial \alpha_n}$$
$$+ \sum_{n=1}^{N} \left[\tfrac{1}{2}(C_n' - C_n) \cdot \sin c\alpha_0 + \tfrac{1}{2}(D_n' - D_n) \cdot \cos c\alpha_0 \right] \cdot \frac{\partial}{\partial \beta_n} \tag{1.17d}$$

From (1.7) one sees that, for $n > 1$

$$B_n = E_n + p_{n-1}$$
$$D_n = F_n + c.p_{n-1} \tag{1.18}$$

where E_n and F_n are functions of $p_1, \ldots, p_{n-2}$ only.

The equations to be solved are

$$\mathcal{D}_1 Q_1 = \beta_1 \cdot Q_2 . \tag{1.19}$$

$$\mathcal{D}_1 Q_2 = \sin \alpha_0 - Q_1 \cdot \beta_1 \tag{1.20}$$

$$\mathcal{D}_1 Q_3 = c \beta_1 Q_4 \tag{1.21}$$

$$\mathcal{D}_1 Q_4 = \sin c\alpha_0 - c\beta_1 \cdot Q_3 . \tag{1.22}$$

$$\mathcal{D}_2 Q_i = 0 \quad , \quad i = 1, \ldots, 4 \tag{1.23}$$

$$\mathcal{J}_1 P_1 = -\sin \alpha_0 . \tag{1.24}$$

$$\mathcal{J}_2 P_1 = 0 \tag{1.25}$$

$$\mathcal{J}_3 P_1 = -c . \sin c\alpha_0 . \tag{1.26}$$

$$\mathcal{J}_4 P_1 = 0 . \tag{1.27}$$

$$\begin{aligned} \mathcal{J}_1 P_0 = {} & \tfrac{1}{2}\left(A_{N+1}' + A_{N+1}\right) . \cos\alpha_0 - \tfrac{1}{2}\left(E_{N+1}' + E_{N+1}\right) . \sin\alpha_0 \\ & - \left[\tfrac{1}{2}(A_N' - A_N) \cos\alpha_0 - \tfrac{1}{2}(B_N' - B_N) . \sin\alpha_0\right] . P_1 \end{aligned} \tag{1.28}$$

$$\begin{aligned} \mathcal{J}_2 P_0 = {} & \tfrac{1}{2}\left(A_{N+1}' - A_{N+1}\right) . \sin\alpha_0 + \tfrac{1}{2}\left(B_{N+1}' - B_{N+1}\right) . \cos\alpha_0 \\ & - \left[\tfrac{1}{2}(A_N' - A_N) . \sin\alpha_0 + \tfrac{1}{2}(B_N' + B_N) . \cos\alpha_0\right] . P_1 \end{aligned} \tag{1.29}$$

$$\begin{aligned} \mathcal{J}_3 P_0 = {} & \tfrac{1}{2}\left(C_{N+1}' + C_{N+1}\right) . \cos c\alpha_0 - \tfrac{1}{2}\left(F_{N+1}' + F_{N+1}\right) . \sin c\alpha_0 \\ & - \left[\tfrac{1}{2}(C_N' - C_N) . \cos c\alpha_0 - \tfrac{1}{2}(D_N' - D_N) . \sin c\alpha_0\right] . P_1 \end{aligned} \tag{1.30}$$

$$\begin{aligned} \mathcal{J}_4 P_0 = {} & \tfrac{1}{2}\left(C_{N+1}' - C_{N+1}\right) . \sin c\alpha_0 + \tfrac{1}{2}\left(D_{N+1}' - D_{N+1}\right) . \cos c\alpha_0 \\ & - \left[\tfrac{1}{2}(C_N' + C_N) . \sin c\alpha_0 + \tfrac{1}{2}(D_N' + D_N) \cos c\alpha_0\right] . P_1 \end{aligned} \tag{1.31}$$

It is equations (1.19) to (1.31) which one wishes to solve. The rest of this chapter is devoted to solving these when $N < 5$. It is found that in this case they have no solution.

§2. Basic Results.

Three small results are found in this section. The first is the case $N = 1$. The second is to find how the operators in (1.16) and (1.17) commute. The third is that P_1 must depend on β_N.

For $N = 1$, if Q_3 is independent of β_1 then (1.21) implies $Q_4 = 0$. But then (1.22) cannot hold. Hence

$$\frac{\partial Q_3}{\partial \beta_1} \neq 0 \tag{2.1}$$

Consistency on (1.21) and (1.23) is then

$$\mathcal{D}_2\left(\frac{\rho_0}{\beta_1}\right) = \frac{\rho_0}{\beta_1} \cdot \frac{\partial \rho_1}{\partial \beta_1} \tag{2.2}$$

Consistency on (1.22) and (1.23) is then

$$\mathcal{D}_2\left(\frac{\sin c.\alpha_0}{\beta_1}\right) = 0$$

i.e. $\quad P_1 = c.\beta_1.\cot c\alpha_0 \quad$ (2.3)

Similarly from (1.19), (1.20) and (1.23) one has

$P_1 = \beta_1.\cot \alpha_0 \quad$ (2.4)

But (2.3) and (2.4) cannot both hold. This completes the case $N = 1$. So from now on one may take

$$N > 1 \tag{2.5}$$

From section 2 of chapter 3 and, in particular, equation (2.9) of that section one may prove, in the notation of this chapter that

$$\mathcal{D}_1\mathcal{F}_1 - \mathcal{F}_1\mathcal{D}_1 = \beta_1\mathcal{F}_2 + \left[\tfrac{1}{2}(A_N' - A_N).\cos\alpha_0 - \tfrac{1}{2}(B_N' - B_N)\sin\alpha_0\right]\mathcal{D}_2 \tag{2.6a}$$

$$\mathcal{D}_1\mathcal{F}_2 - \mathcal{F}_2\mathcal{D}_1 = -\beta_1\mathcal{F}_1 + \left[\tfrac{1}{2}(A_N' + A_N).\sin\alpha_0 + \tfrac{1}{2}(B_N' + B_N)\cos\alpha_0\right].\mathcal{D}_2 \tag{2.6b}$$

$$\mathcal{D}_1\mathcal{F}_3 - \mathcal{F}_3\mathcal{D}_1 = c.\beta_1\mathcal{F}_4 + \left[\tfrac{1}{2}(C_N' - C_N).\cos c.\alpha_0 - \tfrac{1}{2}(D_N' - D_N).\sin c\alpha_0\right].\mathcal{D}_2 \tag{2.6c}$$

$$\mathcal{D}_1\mathcal{F}_4 - \mathcal{F}_4\mathcal{D}_1 = -c.\beta_1.\mathcal{F}_3 + \left[\tfrac{1}{2}(C_N' + C_N).\cos c\alpha_0 + \tfrac{1}{2}(D_N' + D_N).\cos c\alpha_0\right].\mathcal{D}_2 \tag{2.6d}$$

$$\mathcal{D}_2\mathcal{F}_i = \mathcal{F}_i\mathcal{D}_2 \qquad i = 1, \ldots, 4. \tag{2.7}$$

Finally it will be shown that P_1 must depend on β_N. So suppose

$$\frac{\partial P_1}{\partial \beta_N} = 0 \tag{2.8}$$

It is then desired to find a contradiction. Differentiate (1.24) to (1.27) w.r.t. β_N and use (1.14)

$$\frac{\partial P_1}{\partial \alpha_0} = 0 \tag{2.9}$$

But then the coefficients of $\sin \alpha_0$ and $\cos \alpha_0$ in (1.24) and (1.25) must vanish i.e.

$$\sum_{n=1}^{N-1} A_n \cdot \frac{\partial P_1}{\partial \kappa_n} = 0 \tag{2.10}$$

$$\sum_{n=1}^{N-1} B_n \cdot \frac{\partial P_1}{\partial \kappa_n} = -\frac{1}{2} \tag{2.11}$$

Since $N < 5$ only is being considered one may replace N in (2.10) and (2.11) by 4. I suspect that the result that (2.8) cannot hold for any N but it is not necessary to prove it here. So (2.10) and (2.11) may be replaced by

$$\frac{\partial P_1}{\partial \kappa_1} - \kappa_1^2 \cdot \frac{\partial P_1}{\partial \kappa_3} = 0 \tag{2.12}$$

$$\kappa_1 \cdot \frac{\partial P_1}{\partial \kappa_2} + \kappa_2 \cdot \frac{\partial P_1}{\partial \kappa_3} = -\frac{1}{2}. \tag{2.13}$$

Consistency on these is

$$\frac{\partial P_1}{\partial \kappa_2} = 0 \tag{2.14}$$

But then, from (2.12) and (2.13) one sees that (2.8) cannot hold. Therefore

$$\frac{\partial P_1}{\partial \beta_N} \neq 0 \quad , \quad N \leq 4 \tag{2.15}$$

§3. For $\mathcal{D}_2 \rho_1 \equiv 0$.

It will be shown, at the end of this section that

$$\mathcal{D}_2 \rho_1 = 0 \tag{3.1}$$

For the present assume that (3.1) is true. Define

$$\mathcal{D}_3 = \mathcal{D}_2 \mathcal{D}_1 - \mathcal{D}_1 \mathcal{D}_2 \tag{3.2}$$

$$\therefore \mathcal{D}_3 = (\mathcal{D}_2 \rho_0 - \mathcal{D}_1 \rho_1) \cdot \frac{\partial}{\partial \beta_N} + \frac{\partial}{\partial \alpha_{N-2}} + \rho_1 \cdot \frac{\partial}{\partial \beta_{N-1}} \tag{3.3}$$

Then (1.19) to (1.23) imply

$$\mathcal{D}_3 Q_i = 0 \quad , i = 1, \ldots, 4 \tag{3.4}$$

From (1.23), (3.4) and (3.3) one has

$$0 = (\mathcal{D}_2 \mathcal{D}_3 - \mathcal{D}_3 \mathcal{D}_2) Q_i = \{\mathcal{D}_2 [\mathcal{D}_2 \rho_0 - \mathcal{D}_1 \rho_1] - \mathcal{D}_3 \rho_1\} \cdot \frac{\partial Q_i}{\partial \beta_N}$$

From this and (1.14) one has

$$\mathcal{D}_2 [\mathcal{D}_2 \rho_0 - \mathcal{D}_1 \rho_1] = \mathcal{D}_3 \rho_1 . \tag{3.5}$$

$$\therefore \mathcal{D}_2 \mathcal{D}_3 = \mathcal{D}_3 \mathcal{D}_2 \tag{3.6}$$

Now $\qquad \mathcal{D}_2 \mathcal{D}_1 \rho_1 = (\mathcal{D}_3 + \mathcal{D}_1 \mathcal{D}_2) \rho_1 \qquad$ from (3.2)

$\qquad\qquad = \mathcal{D}_3 \rho_1 \qquad$ from (3.1)

Take $\mathcal{D}_2$ of (3.5) to obtain

$$\mathcal{D}_2^3 \rho_0 = 0 \tag{3.7}$$

So one may write

$$\rho_0 = R_0 + R_1 \cdot \alpha_{N-1} + R_2 \cdot \alpha_{N-1}^2 \tag{3.8}$$

where

$$\mathcal{D}_2 R_i = 0 \quad , i = 0, 1, 2 . \tag{3.9}$$

Then (3.5) is

$$\mathcal{D}_2 \mathcal{D}_1 P_1 = \mathcal{D}_3 P_1 = \tfrac{1}{2} \mathcal{D}_2^2 P_0 = R_2 \tag{3.10}$$

So one may write

$$\mathcal{D}_1 P_1 = R_3 + R_2 . \alpha_{N-1} \tag{3.11}$$

where (3.9) holds for $i = 3$. From (1.26) , (1.27) and (2.6) one has

$$\begin{aligned} \mathcal{J}_3 [\mathcal{D}_1 P_1] &= - c^2 . \alpha_1 . \cos c.\alpha_0 \\ \mathcal{J}_4 [\mathcal{D}_1 P_1] &= - c^2 . \beta_1 . \sin c\alpha_0 \end{aligned} \tag{3.12}$$

Note that one may read off $\mathcal{J}_1 [\mathcal{D}_1 P_1]$ and $\mathcal{J}_2 [\mathcal{D}_1 P_1]$ from (3.12) merely by taking $c = 1$. So here and in all future cases only the equations for $i = 3,4$ will be written down explicitly.

The case $N = 2$ can now be dealt with quite easily.

<u>Case 1.</u>

$$N = 2 \tag{3.13}$$

Take $\mathcal{D}_2$ of (3.12)

$$\left.\begin{aligned} \mathcal{J}_3 R_2 &= - c^2 . \cos c\alpha_0 \\ \mathcal{J}_4 R_2 &= 0 \end{aligned}\right\} \tag{3.14}$$

$$\left.\begin{aligned} \mathcal{J}_3 R_3 &= 0 \\ \mathcal{J}_4 R_3 &= - R_2 . \sin c\alpha_0 - c^2 . \beta_1 . \sin c\alpha_0 \end{aligned}\right\} \tag{3.15}$$

Eliminate derivatives of P_1 from (1.25) , (1.27) and (3.1)

$$Q_2 .(c.\beta_1 .\cos c\alpha_0 - P_1 .\sin c\alpha_0) = Q_4 .(\beta_1 .\cos \alpha_0 - P_1 .\sin \alpha_0) \tag{3.16}$$

Then (3.9) , (3.15) and (3.16) imply

$$\begin{aligned} (R_2 + \beta_1).\sin \alpha_0 .(c.\beta_1 .\cos c\alpha_0 - P_1 .\sin c\alpha_0) \\ = (R_2 + c^2 .\beta_1).\sin c\alpha_0 .(\beta_1 .\cos \alpha_0 - P_1 .\sin \alpha_0) \end{aligned} \tag{3.17}$$

Take of (3.17) after dividing by $\sin \ _0 . \sin c \ _0$:

$$(R_2 + \beta_1).(-c^2.\beta_1.\mathrm{cosec}^2\, c\alpha_0) = (R_2 + c^2.\beta_1).(-\beta_1.\mathrm{cosec}^2\, \alpha_0) \tag{3.18}$$

Note that if $Q_2 = 0$ then (1.25) with (3.1) contradict (2.15).

Differentiate (3.18) w.r.t. β_2 to obtain $\dfrac{\partial R_2}{\partial \beta_2} = 0$.

Then differentiate (3.17) w.r.t. β_2 and use (2.15) to obtain a contradiction.

Case 2.

It has now been shown that

$$N > 2 \tag{3.19}$$

From (1.7) one may take, for $n > 2$,

$$\begin{aligned} C_n &= -(n-1).c^2.p_1.p_{n-2} + H_n(p_1,\ldots,p_{n-3}) \\ D_n &= c.p_{n-1} + F_n(p_1,\ldots,p_{n-3}) \end{aligned} \tag{3.20}$$

Then from (1.28) to (1.31) , (2.7) ,(3.8) , (3.11) and (3.12) one has

$$\mathcal{J}_3 R_3 = -c^2.\alpha_1.\cos c\alpha_0 - R_2.\left[\tfrac{1}{2}(C'_{N-1} - C_{N-1})\cos c\alpha_0 - \tfrac{1}{2}(D'_{N-1} - D_{N-1})\sin c \right. \tag{3.21a}$$

$$\mathcal{J}_4 R_3 = -c^2.\beta_1.\sin c\alpha_0 - R_2.\left[\tfrac{1}{2}(C'_{N-1} + C_{N-1}).\sin c\alpha_0 + \tfrac{1}{2}(D'_{N-1} + D_{N-1}).\cos c \right. \tag{3.21b}$$

$$\mathcal{J}_i R_2 = 0 \quad , i = 1,2,3,4 . \tag{3.22}$$

$$\mathcal{J}_3 R_1 = -N.c^2.\alpha_1.\cos c\alpha_0 - 2R_2\left[\tfrac{1}{2}(C'_{N-1} - C_{N-1}).\cos c\alpha_0 - \tfrac{1}{2}(D'_{N-1} - D_{N-1})\sin c \right. \tag{3.23a}$$

$$\mathcal{J}_4 R_1 = -N.c^2.\beta_1.\sin c\alpha_0 - 2R_2\left[\tfrac{1}{2}(C'_{N-1} + C_{N-1}).\sin c\alpha_0 + \tfrac{1}{2}(D'_{N-1} + D_{N-1}).\cos c \right. \tag{3.23b}$$

$$\begin{aligned} \mathcal{J}_3 R_0 = {} & \left[\tfrac{1}{2}(H'_{N+1} + H_{N+1}) - Nc^2\beta_1.\beta_{N-1} - \tfrac{1}{2}(C'_N - C_N).R_1 - \tfrac{1}{2}(C'_{N-1} - C_{N-1}).R \right. \\ & \cdot \cos c \\ & + \left[\tfrac{1}{2}(D'_N - D_N).R_1 - \tfrac{1}{2}(F'_{N+1} + F_{N+1}) + \tfrac{1}{2}(D'_{N-1} - D_{N-1}).R_1\right].\sin c\alpha_0 \end{aligned} \tag{3.24a}$$

$$\begin{aligned} \mathcal{J}_4 R_0 = {} & \left[\tfrac{1}{2}(H'_{N+1} - H_{N+1}) - Nc^2\alpha_1.\beta_{N-1} - \tfrac{1}{2}(C'_N + C_N).R_1 - \tfrac{1}{2}(C'_{N-1} + C_{N-1})R_1\right].\sin \\ & + \left[\tfrac{1}{2}(F'_{N+1} - F_{N+1}) - \tfrac{1}{2}(F'_N + F_N).R_1 + c(\beta_N - R_1.\alpha_{N-1}) - \tfrac{1}{2}(D'_{N-1} + D_{N-1})R_1\right] \\ & . \cos c \end{aligned} \tag{3.24b}$$

The rest of this section is devoted to proving that (3.1) is true. So, for the rest of this section assume

$$\mathcal{D}_2 \rho_1 \neq 0 \tag{3.25}$$

From (3.3) and (3.4) one has

$$\left[\frac{\partial Q_i}{\partial \alpha_{N-2}} \cdot \frac{\partial Q_j}{\partial \beta_N} - \frac{\partial Q_j}{\partial \alpha_{N-2}} \cdot \frac{\partial Q_i}{\partial \beta_N}\right] + \rho_1 \cdot \left[\frac{\partial Q_i}{\partial \beta_{N-1}} \cdot \frac{\partial Q_j}{\partial \beta_N} - \frac{\partial Q_j}{\partial \beta_{N-1}} \cdot \frac{\partial Q_i}{\partial \beta_N}\right] = 0 \tag{3.26}$$

Take $\mathcal{D}_2$ of (3.26) and use (3.25)

$$\frac{\partial Q_i}{\partial \beta_{N-1}} \cdot \frac{\partial Q_j}{\partial \beta_N} = \frac{\partial Q_j}{\partial \beta_{N-1}} \cdot \frac{\partial Q_i}{\partial \beta_N} \tag{3.27}$$

Then, from (1.14) one may write

$$\frac{\partial Q_i}{\partial \beta_{N-1}} + Z_1 \cdot \frac{\partial Q_i}{\partial \beta_N} = 0 \qquad i = 1, \ldots, 4 \tag{3.28}$$

for some function Z_1. Then from (3.26) and (3.27) one may write

$$\frac{\partial Q_i}{\partial \alpha_{N-2}} + Z_2 \cdot \frac{\partial Q_i}{\partial \beta_N} = 0 \qquad i = 1, \ldots, 4 \tag{3.29}$$

Take $\dfrac{\partial}{\partial \alpha_{N-2}} + Z_2 \cdot \dfrac{\partial}{\partial \beta_N}$ of (1.19) to (1.22)

$$\frac{\partial Q_i}{\partial \alpha_{N-3}} + Z_3 \cdot \frac{\partial Q_i}{\partial \beta_N} = 0 \qquad i = 1, \ldots, 4 \tag{3.30}$$

Continuing in this fashion one obtains

$$\frac{\partial Q_i}{\partial \alpha_0} + Z_N \cdot \frac{\partial Q_i}{\partial \beta_N} = 0 \qquad i = 1, \ldots, 4 \tag{3.31}$$

for some function Z_N.

If $\dfrac{\partial Q_1}{\partial \beta_N} = \dfrac{\partial Q_3}{\partial \beta_N} = 0$ then (3.28) is $\dfrac{\partial Q_1}{\partial \beta_{N-1}} = \dfrac{\partial Q_3}{\partial \beta_{N-1}} = 0$

Then differentiate (1.19) and (1.21) w.r.t. β_N to obtain a contradiction to (1.14). So one has

$$\frac{\partial Q_1}{\partial \beta_N} \neq 0 \quad \text{or} \quad \frac{\partial Q_3}{\partial \beta_N} \neq 0 \tag{3.32}$$

Take $\frac{\partial}{\partial \alpha_0} + Z_N \cdot \frac{\partial}{\partial \beta_N}$ of (1.19) and (1.21) and use (3.28) and (3.32)

$$\frac{\partial P_0}{\partial \alpha_0} + Z_N \cdot \frac{\partial P_0}{\partial \beta_N} = \mathcal{D}_1 Z_N + Z_1 Z_N \tag{3.33}$$

Take $\frac{\partial}{\partial \alpha_0} + Z_N \cdot \frac{\partial}{\partial \beta_N}$ of (1.20) and (1.22) and use (3.28) and (3.33) to obtain a contradiction. So it has been shown that (3.25) cannot hold.

§4. A Change of Variable.

Use (2.15) to define

$$z = P_1(\alpha_0,\dots,\alpha_{N-1};\beta_1,\dots,\beta_N) \tag{4.1}$$

and use z instead of β_N. Also define $X = X(\alpha_0,\dots,\alpha_{N-1};\beta_1,\dots,\beta_{N-1};z)$ such that $\beta_N = X$ i.e.

$$z = P_1(\alpha_0,\dots,\alpha_{N-1};\beta_1,\dots,\beta_{N-1};X) \tag{4.2}$$

Equations (1.23) and (3.9) are

$$\left.\begin{aligned} \frac{\partial Q_i}{\partial \alpha_{N-1}} &= 0 \quad , \; i = 1, \dots, 4 \\ \frac{\partial R_i}{\partial \alpha_{N-1}} &= 0 \quad , \; i = 0, \dots, 3 \,. \end{aligned}\right\} \tag{4.3}$$

Also (3.1) gives on integrating

$$X = z.\alpha_{N-1} + R_4 \tag{4.4}$$

where R_4 does not depend on α_{N-1}.

$$\mathcal{D}_1 \to \sum_{i=0}^{N-2} \alpha_{i+1} \cdot \frac{\partial}{\partial \alpha_i} + \sum_{i=1}^{N-2} \beta_{i+1} \cdot \frac{\partial}{\partial \beta_i} + (z \cdot \alpha_{N-1} + R_4)\frac{\partial}{\partial \beta_{N-1}} + (R_3 + R_2 \cdot \alpha_{N-1})\frac{\partial}{\partial z}$$

$$\left.\begin{aligned}
&\mathcal{D}_3 \to \frac{\partial}{\partial\alpha_{N-2}} + z\cdot\frac{\partial}{\partial\beta_{N-1}} + R_2\cdot\frac{\partial}{\partial z}\\
&\text{Define}\\
&\mathcal{D}_4 = \sum_{i=0}^{N-3}\alpha_{i+1}\frac{\partial}{\partial\alpha_i} + \sum_{i=1}^{N-2}\beta_{i+1}\cdot\frac{\partial}{\partial\beta_i} + R_4\cdot\frac{\partial}{\partial\beta_{N-1}} + R_3\cdot\frac{\partial}{\partial z}
\end{aligned}\right\}\quad(4.5)$$

$$\therefore\ \mathcal{D}_1 \Rightarrow \mathcal{D}_4 + \alpha_{N-1}\cdot\mathcal{D}_3$$

Now (3.11) is

$$\left.\begin{aligned}
R_1 &= R_3 + \mathcal{D}_3\, R_4\\
R_0 &= \mathcal{D}_4\, R_4
\end{aligned}\right\}\qquad(4.6)$$

If one defines R_0 and R_1 by (4.6) then one finds that (3.23) and (3.24) are satisfied provided (1.25), (1.26), (3.21) and (3.22) are true. (Use (2.6)). So then (1.19) to (1.27), (3.21) and (3.22) are

$$\left.\begin{aligned}
\mathcal{D}_3 &= \frac{\partial}{\partial\alpha_{N-2}} + z\cdot\frac{\partial}{\partial\beta_{N-1}} + R_2\cdot\frac{\partial}{\partial z}\\
\mathcal{D}_4 &= \sum_{i=0}^{N-3}\alpha_{i+1}\cdot\frac{\partial}{\partial\alpha_i} + \sum_{i=1}^{N-2}\beta_{i+1}\cdot\frac{\partial}{\partial\beta_i} + R_4\cdot\frac{\partial}{\partial\beta_{N-1}} + R_3\cdot\frac{\partial}{\partial z}
\end{aligned}\right\}\quad(4.7)$$

$$\begin{aligned}
\mathcal{J}_3 = {}& Q_3\cdot\frac{\partial}{\partial\alpha_0} + \sum_{n=1}^{N-2}\left[\tfrac{1}{2}(C_n'-C_n).\cos c\alpha_0 - \tfrac{1}{2}(D_n'-D_n).\sin c\alpha_0\right]\cdot\frac{\partial}{\partial\alpha_n}\\
&- c.\sin c\alpha_0.\frac{\partial}{\partial z} + \sum_{n=1}^{N-1}\left[\tfrac{1}{2}(C_n'+C_n).\cos c\alpha_0 - \tfrac{1}{2}(D_n'+D_n)\sin c\alpha_0\right]\cdot\frac{\partial}{\partial\beta_n}
\end{aligned}\quad(4.8a)$$

$$\begin{aligned}
\mathcal{J}_4 = {}& Q_4\cdot\frac{\partial}{\partial\alpha_0} + \sum_{n=1}^{N-2}\left[\tfrac{1}{2}(C_n'+C_n)\sin c\alpha_0 + \tfrac{1}{2}(D_n'+D_n)\cos c\alpha_0\right]\cdot\frac{\partial}{\partial\alpha_n}\\
&+ \sum_{n=1}^{N-1}\left[\tfrac{1}{2}(C_n'-C_n).\sin c\alpha_0 + \tfrac{1}{2}(D_n'-D_n).\cos c\alpha_0\right]\cdot\frac{\partial}{\partial\beta_n}
\end{aligned}\quad(4.8b)$$

$$\mathcal{D}_4 Q_1 = \beta_1 \cdot Q_2 \qquad\qquad \mathcal{D}_4 Q_3 = c \cdot \beta_1 \cdot Q_4 \tag{4.9}$$

$$\mathcal{D}_4 Q_2 = \sin \alpha_0 - Q_1 \cdot \beta_1 \qquad\qquad \mathcal{D}_4 Q_4 = \sin c\alpha_0 - c \cdot \beta_1 \cdot Q_3 \tag{4.10}$$

$$\mathcal{D}_3 Q_i = 0 \quad , \quad i = 1, \ldots, 4 \tag{4.11}$$

$$\begin{aligned} \mathcal{J}_3 R_4 = {} & \left[\tfrac{1}{2} (C_N' + C_N) \cdot \cos c\alpha_0 - \tfrac{1}{2} (F_N' + F_N) \cdot \sin c\alpha_0 \right] \\ & - z \cdot \left[\tfrac{1}{2} (C_{N-1}' - C_{N-1}) \cdot \cos c\alpha_0 - \tfrac{1}{2} (D_{N-1}' - D_{N-1}) \cdot \sin c\alpha_0 \right] \end{aligned} \tag{4.12a}$$

$$\begin{aligned} \mathcal{J}_4 R_4 = {} & \left[\tfrac{1}{2} (C_N' - C_N) \sin c\alpha_0 + \tfrac{1}{2} (D_N' - D_N) \cdot \cos c\alpha_0 \right] \\ & - z \left[\tfrac{1}{2} (C_{N-1}' + C_{N-1}) \cdot \sin c\alpha_0 + \tfrac{1}{2} (D_{N-1}' + D_{N-1}) \cdot \cos c\alpha_0 \right] \end{aligned} \tag{4.12b}$$

$$\mathcal{J}_3 R_3 = - c^2 \cdot \alpha_1 \cdot \cos c\alpha_0 - R_2 \cdot \left[\tfrac{1}{2} (C_{N-1}' - C_{N-1}) \cdot \cos c\alpha_0 - \tfrac{1}{2} (D_{N-1}' - D_{N-1}) \sin c\alpha_0 \right. \tag{4.13a}$$

$$\mathcal{J}_4 R_3 = - c^2 \cdot \beta_1 \cdot \sin c\alpha_0 - R_2 \left[\tfrac{1}{2} (C_{N-1}' + C_{N-1}) \cdot \sin c\alpha_0 + \tfrac{1}{2} (D_{N-1}' + D_{N-1}) \cos c\alpha_0 \right. \tag{4.13b}$$

$$\mathcal{J}_i R_2 = 0 \quad , \quad i = 1, \ldots, 4 \tag{4.14}$$

Also equations (2.6) give

$$\mathcal{D}_3 \mathcal{J}_i = \mathcal{J}_i \mathcal{D}_3 \qquad i = 1, \ldots, 4 \tag{4.15}$$

$$\begin{aligned} \mathcal{D}_4 \mathcal{J}_3 - \mathcal{J}_3 \mathcal{D}_4 = {} & c \cdot \beta_1 \cdot \mathcal{J}_4 + \left[\tfrac{1}{2} (C_{N-1}' - C_{N-1}) \cos c\alpha_0 - \tfrac{1}{2} (D_{N-1}' - D_{N-1}) \sin c \right. \\ & \times \mathcal{D}_3 \end{aligned} \tag{4.16a}$$

$$\mathcal{D}_4 \mathcal{J}_4 - \mathcal{J}_4 \mathcal{D}_4 = - c \cdot \beta_1 \cdot \mathcal{J}_3 + \left[\tfrac{1}{2} (C_{N-1}' + C_{N-1}) \sin c\alpha_0 + \tfrac{1}{2} (D_{N-1}' + D_{N-1}) \cos c\alpha_0 \right] \cdot \mathcal{D} \tag{4.16b}$$

The rest of this section is devoted to proving that

$$\left| \frac{\partial Q_i}{\partial \beta_{N-2}} \quad \frac{\partial Q_i}{\partial \beta_{N-1}} \quad \frac{\partial Q_i}{\partial z} \right| \neq 0 \tag{4.17}$$

where this means that all 3×3 determinants with rows as shown cannot vanish (i takes on any 3 distinct values from 1 , 2 , 3 or 4)

If $\frac{\partial Q_1}{\partial z} = 0$ then $\mathcal{D}_3 Q_1 = 0$ gives $\frac{\partial Q_1}{\partial \alpha_{N-2}} = \frac{\partial Q_1}{\partial \beta_{N-1}} = 0$.

Take $\mathcal{D}_3$ of (4.9) to obtain $\frac{\partial Q_1}{\partial \alpha_{N-3}} = \frac{\partial Q_1}{\partial \beta_{N-2}} = 0$

Continuing in this fashion gives that Q_1 is a constant. Then (4.9) is $Q_2 = 0$. But then (4.10) cannot hold. Hence

$$\frac{\partial Q_1}{\partial z} \neq 0 \tag{4.18}$$

If $\frac{\partial Q_i}{\partial \beta_{N-1}} + X \cdot \frac{\partial Q_i}{\partial z} = 0$, $i = 1,2,3,4$, then (4.11) implies

$\frac{\partial Q_i}{\partial \alpha_{N-2}} + X_{N-2} \cdot \frac{\partial Q_i}{\partial z} = 0, \ i = 1, \dots, 4$. Take $\frac{\partial}{\partial \alpha_n} + X_n \cdot \frac{\partial}{\partial z}$ of (4.9) and (4.10) (working down from $n = N-2$ to $n = 1$.) to obtain

$$\frac{\partial Q_i}{\partial \alpha_n} + X_n \cdot \frac{\partial Q_i}{\partial z} = 0 \ ; \ i = 1, \dots, 4 \ ; \ n = 0, \dots, N-2 .$$

Take $\frac{\partial}{\partial \alpha_0} + X_0 \cdot \frac{\partial}{\partial z}$ of (4.9) and use (4.18) and

$$- X \cdot \left(\frac{\partial R_4}{\partial \alpha_0} + X_0 \cdot \frac{\partial R_4}{\partial z} \right) + \left(\frac{\partial R_3}{\partial \alpha_0} + X_0 \cdot \frac{\partial R_3}{\partial z} \right) - \mathcal{D}_4 X_0 = 0$$

Take $\frac{\partial}{\partial \alpha_0} + X_0 \frac{\partial}{\partial z}$ of (4.10) to obtain a contradiction.

$$\therefore \ \frac{\partial Q_i}{\partial \beta_{N-1}} + X \cdot \frac{\partial Q_i}{\partial z} \neq 0 \quad , \ i = 1, 2, 3 \text{ or } 4 \tag{4.19}$$

To prove (4.17) it is then only necessary to prove that (4.20) cannot hold (because of (4.18) and (4.19)

$$\frac{\partial Q_i}{\partial \beta_{N-2}} + X^{(1)}_{N-2} \cdot \frac{\partial Q_i}{\partial \beta_{N-1}} + X^{(2)}_{N-2} \cdot \frac{\partial Q_i}{\partial z} = 0 \ , \ i = 1, \dots, 4 \tag{4.20}$$

So for the rest of this section assume that (4.20) is true. It is then desired to find a contradiction.

Take $\frac{\partial}{\partial\beta_n} + X_n^{(1)} \cdot \frac{\partial}{\partial\beta_{N-1}} + X_n^{(2)} \cdot \frac{\partial}{\partial z}$ of (4.9) and (4.10) , working down from n = N-2 to n = 2 ,defining $X_n^{(1)}$ and $X_n^{(2)}$ as one goes :

$$\frac{\partial Q_i}{\partial\beta_n} + X_n^{(1)} \cdot \frac{\partial Q_i}{\partial\beta_{N-1}} + X_n^{(2)} \cdot \frac{\partial Q_i}{\partial z} = 0 \qquad \begin{array}{l} i = 1,\dots,4 \\ n = 1,\dots,N-2 \end{array} \tag{4.21}$$

Similarly from (4.9) , (4.10) and (4.11) :

$$\frac{\partial Q_i}{\partial\alpha_n} + X_n^{(3)} \cdot \frac{\partial Q_i}{\partial\beta_{N-1}} + X_n^{(4)} \cdot \frac{\partial Q_i}{\partial z} = 0 \qquad \begin{array}{l} i = 1,\dots,4 \\ n = 0,\dots,N-2 \end{array} \tag{4.22}$$

Take $\frac{\partial}{\partial\alpha_0} + X_0^{(3)} \frac{\partial}{\partial\beta_{N-1}} + X_n^{(4)} \cdot \frac{\partial}{\partial z}$ of (4.9) and (4.10)

$$\frac{\partial Q_i}{\partial\beta_{N-1}} + X_{N-1}^{(6)} \cdot \frac{\partial Q_i}{\partial z} = \begin{cases} 0 & i = 1 \text{ or } 3 \\ X_{N-1}^{(5)} \cdot \cos\alpha_0 & i = 2 \\ c \cdot X_{N-1}^{(5)} \cdot \cos c\alpha_0 & i = 4 \end{cases} \tag{4.23}$$

where

$$X_{N-1}^{(5)} \neq 0 \tag{4.24}$$

Then (4.9) , (4.10) , (4.21) , (4.22) and (4.23) give

$$X_1^{(7)} \cdot \frac{\partial Q_1}{\partial z} = \beta_1 \cdot Q_2 \tag{4.25}$$

$$X_1^{(7)} \cdot \frac{\partial Q_3}{\partial z} = c \cdot \beta_1 \cdot Q_4 \tag{4.26}$$

Take $\frac{\partial Q_3}{\partial z} \cdot \left[\frac{\partial}{\partial\beta_{N-1}} + X_{N-1}^{(6)} \cdot \frac{\partial}{\partial z}\right] \cdot (4.25) - \frac{\partial Q_1}{\partial z} \cdot \left[\frac{\partial}{\partial\beta_{N-1}} + X_{N-1}^{(6)} \frac{\partial}{\partial z}\right] \cdot (4.$ and use (4.23) and (4.24) :

$$\frac{\partial Q_3}{\partial z} \cdot \cos\alpha_0 = c^2 \cdot \cos c\alpha_0 \cdot \frac{\partial Q_1}{\partial z} \tag{4.27}$$

Multiply by $X_1^{(7)}$ and use (4.25) and (4.26)

Take $\mathcal{D}_4$ and then $\frac{\partial}{\partial\beta_{N-1}} + X_{N-1}^{(6)} \cdot \frac{\partial}{\partial z}$ of the result to obtain a contradiction. So it has been shown that (4.17) is true

§5. That $N > 3$.

Define
$$\mathcal{D}_5 = \mathcal{D}_3\,\mathcal{D}_4 - \mathcal{D}_4\,\mathcal{D}_3 \tag{5.1}$$

$$\therefore\ \mathcal{D}_5 = \frac{\partial}{\partial\alpha_{N-3}} + z\cdot\frac{\partial}{\partial\beta_{N-2}} + (\mathcal{D}_3 R_4 - R_3)\cdot\frac{\partial}{\partial\beta_{N-1}} + (\mathcal{D}_3 R_3 - \mathcal{D}_4 R_2)\cdot\frac{\partial}{\partial z} \tag{5.2}$$

Then from (4.9) to (4.11)
$$\mathcal{D}_5\,\varphi_i = 0, \quad i = 1, \dots, 4 \tag{5.3}$$

Now
$$\mathcal{D}_3\mathcal{D}_5 - \mathcal{D}_5\mathcal{D}_3 = R_2\cdot\frac{\partial}{\partial\beta_{N-2}} + \left[\mathcal{D}_3(\mathcal{D}_3 R_4 - R_3) - (\mathcal{D}_3 R_3 - \mathcal{D}_4 R_2)\right]\frac{\partial}{\partial\beta_{N-1}}$$
$$+ \left[\mathcal{D}_3(\mathcal{D}_3 R_3 - \mathcal{D}_4 R_2) - \mathcal{D}_5 R_2\right]\cdot\frac{\partial}{\partial z} \tag{5.4}$$

But $(\mathcal{D}_3\mathcal{D}_5 - \mathcal{D}_5\mathcal{D}_3)\,\varphi_i = 0$ and (4.17) then imply

$$R_2 = 0. \tag{5.5}$$

$$\mathcal{D}_3^2 R_4 = 2\,\mathcal{D}_3 . R_3. \tag{5.6}$$

$$\mathcal{D}_3^2 R_3 = 0 \tag{5.7}$$

From (5.7) and (5.6) one may write

$$R_3 = T_2 + T_3.\alpha_{N-2} \tag{5.8}$$
$$R_4 = T_0 + (T_1 + T_2).\alpha_{N-2} + T_3.\alpha_{N-2}^2 \tag{5.9}$$

where
$$\mathcal{D}_3\,T_i = 0 \quad , i = 0, \dots, 3 \tag{5.10}$$

The rest of this section deals with the case

$N = 3$

In this case the equations to be solved are (4.9) to (4.14) where (4.7) and (4.8) are

$$\left.\begin{aligned}\mathcal{D}_3 &= \frac{\partial}{\partial\alpha_1} + z\cdot\frac{\partial}{\partial\beta_2}\\ \mathcal{D}_4 &= \alpha_1\cdot\frac{\partial}{\partial\alpha_0} + \beta_2\cdot\frac{\partial}{\partial\beta_1} + R_4\cdot\frac{\partial}{\partial\beta_2} + R_3\cdot\frac{\partial}{\partial z}\end{aligned}\right\} \quad (5.11)$$

$$\begin{aligned}\mathcal{J}_3 &= Q_3\cdot\frac{\partial}{\partial\alpha_0} + \cos c\alpha_0\cdot\frac{\partial}{\partial\beta_1} - c\cdot\alpha_1\cdot\sin c\alpha_0\cdot\frac{\partial}{\partial\beta_2} - c\cdot\sin c\alpha_0\cdot\frac{\partial}{\partial z}\\ \mathcal{J}_4 &= Q_4\cdot\frac{\partial}{\partial\alpha_0} + \sin c\alpha_0\cdot\frac{\partial}{\partial\alpha_1} + c\cdot\beta_1\cdot\cos c\alpha_0\cdot\frac{\partial}{\partial\beta_2}\end{aligned} \quad (5.12)$$

Take $\mathcal{D}_3$ of (4.13)

$$\left.\begin{aligned}\mathcal{J}_3 T_3 &= -c^2\cdot\cos c\alpha_0\\ \mathcal{J}_4 T_3 &= 0\end{aligned}\right\} \quad (5.13)$$

$$\left.\begin{aligned}\mathcal{J}_3 T_2 &= 0\\ \mathcal{J}_4 T_2 &= -(c^2\cdot\beta_1 + T_3)\cdot\sin c\alpha_0\end{aligned}\right\} \quad (5.14)$$

Eliminate derivatives of T_3 from (5.10), $\mathcal{J}_2 T_3 = 0$ and $\mathcal{J}_4 T_3 = 0$

$$Q_4\cdot[\beta_1\cdot\cos\alpha_0 - z\cdot\sin\alpha_0] = Q_2\cdot[c\cdot\beta_1\cdot\cos c\alpha_0 - z\cdot\sin c\alpha_0] \quad (5.15)$$

Then (5.10) and (5.14) imply

$$\begin{aligned}&(c^2\beta_1 + T_3)\cdot\sin c\alpha_0\cdot[\beta_1\cdot\cos\alpha_0 - z\cdot\sin\alpha_0]\\ &\quad = (\beta_1 + T_3)\cdot\sin\alpha_0\cdot[c\cdot\beta_1\cdot\cos c\alpha_0 - z\cdot\sin c\alpha_0]\end{aligned} \quad (5.16)$$

Take $\mathcal{J}_2$ of (5.16) after multiplying by $(\sin\alpha_0\cdot\sin c\alpha_0)^{-1}$ to obtain a contradiction. So it has been shown that $N > 3$. Infact the rest of this chapter deals only with the case $N = 4$.

§6. Lemma.

The rest of this chapter deals with the case

$$N = 4 \tag{6.1}$$

Define

$$w = \beta_{N-1} - z.\alpha_{N-2} \tag{6.2}$$

and use w instead of β_{N-1}. Then equations (4.9) to (4.14) are

$$\left.\begin{aligned} \mathcal{D}_5 &= \frac{\partial}{\partial\alpha_1} + z\cdot\frac{\partial}{\partial\beta_2} + T_1\cdot\frac{\partial}{\partial w} + T_3\cdot\frac{\partial}{\partial z} \\ \mathcal{D}_6 &= \alpha_1\cdot\frac{\partial}{\partial\alpha_0} + \beta_2\cdot\frac{\partial}{\partial\beta_1} + w\cdot\frac{\partial}{\partial\beta_2} + T_0\cdot\frac{\partial}{\partial w} + T_2\cdot\frac{\partial}{\partial z} \end{aligned}\right\} \tag{6.3}$$

$$\mathcal{F}_3 = Q_3\cdot\frac{\partial}{\partial\alpha_0} + \cos c\alpha_0\cdot\frac{\partial}{\partial\beta_1} - c.\alpha_1.\sin c\alpha_0\cdot\frac{\partial}{\partial\beta_2} - c.\sin c\alpha_0\cdot\frac{\partial}{\partial z}$$
$$+ \left[c\beta_1\cdot z\cdot\sin c\alpha_0 - c^2(\alpha_1^2+\beta_1^2)\cos c\alpha_0\right]\cdot\frac{\partial}{\partial w} \tag{6.4a}$$

$$\mathcal{F}_4 = Q_4\cdot\frac{\partial}{\partial\alpha_0} + \sin c\alpha_0\cdot\frac{\partial}{\partial\alpha_1} + c\beta_1\cdot\cos c\alpha_0\cdot\frac{\partial}{\partial\beta_2} \tag{6.4b}$$
$$+ \left[c.\beta_2\cdot\cos c\alpha_0 - 2c^2.\alpha_1.\beta_1.\sin c\alpha_0 - c.\alpha_1.z.\cos c\alpha_0\right]\cdot\frac{\partial}{\partial w}$$

$$\mathcal{D}_6 Q_1 = \beta_1\cdot Q_2. \qquad \mathcal{D}_6 Q_3 = c.\beta_1\cdot Q_4 \tag{6.5}$$

$$\mathcal{D}_6 Q_2 = \sin\alpha_0 - \beta_1\cdot Q_1 \qquad \mathcal{D}_6 Q_4 = \sin c\alpha_0 - c.\beta_1.Q_3 \tag{6.6}$$

$$\mathcal{D}_5 Q_i = 0 \qquad i = 1, \ldots, 4 \tag{6.7}$$

$$\mathcal{F}_i\, T_3 = 0 \qquad i = 1, \ldots, 4 \tag{6.8}$$

$$\left.\begin{aligned} \mathcal{F}_3 T_2 &= -c^2.\alpha_1.\cos c\alpha_0 + T_3.c\beta_1.\sin c\alpha_0 \\ \mathcal{F}_4 T_2 &= -c^2.\beta_1.\sin c\alpha_0 - T_3.c.\alpha_1.\cos c\alpha_0 \end{aligned}\right\} \tag{6.9}$$

$$\left.\begin{aligned} \mathcal{Y}_3 T_1 &= -2c^2.\alpha_1.\cos c\alpha_0 + T_3.c.\beta_1.\sin c\alpha_0 \\ \mathcal{Y}_4 T_1 &= -2c^2.\beta_1.\sin c\alpha_0 - T_3.c.\alpha_1.\cos c\alpha_0 \end{aligned}\right\} \tag{6.10}$$

$$\begin{aligned} \mathcal{Y}_3 T_0 = {} & \left[c.\beta_2.z + c^3(\alpha_1^3 + 3\alpha_1.\beta_1^2) + (T_1+T_2).c.\beta_1\right].\sin c\alpha_0 \\ & + \left[2c^2.\alpha_1.\beta_1.z - 3c^2.\beta_1.\beta_2.\right]\cos c\alpha_0 \end{aligned} \tag{6.11a}$$

$$\begin{aligned} \mathcal{Y}_4 T_0 = {} & \left[c^2.z(\alpha_1^2 + \beta_1^2) - 3c^2.\alpha_1.\beta_2\right].\sin c\alpha_0 \\ & + \left[c\mu - c^3(3\alpha_1^2.\beta_1 + \beta_1^3) - c.\alpha_1.(T_1+T_2)\right].\cos c\alpha_0 \end{aligned} \tag{6.11b}$$

Further equation (4.16) gives

$$\mathcal{Y}_i \mathcal{D}_5 = \mathcal{D}_5 \mathcal{Y}_i \qquad i = 1, \dots, 4. \tag{6.12}$$

$$\left.\begin{aligned} \mathcal{D}_6 \mathcal{Y}_3 &= \mathcal{Y}_3 \mathcal{D}_6 + c.\beta_1.\mathcal{Y}_4 - c.\beta_1.\sin c\alpha_0.\mathcal{D}_5 \\ \mathcal{D}_6 \mathcal{Y}_4 &= \mathcal{Y}_4 \mathcal{D}_6 - c.\beta_1.\mathcal{Y}_3 + c.\alpha_1.\cos c\alpha_0.\mathcal{D}_5 \end{aligned}\right\} \tag{6.13}$$

Finally (4.17) is

$$\left| \frac{\partial Q_i}{\partial \beta_2} \quad \frac{\partial Q_i}{\partial \mu} \quad \frac{\partial Q_i}{\partial z} \right| \neq 0 \tag{6.14}$$

In the rest of this section it is shown that if $\mathcal{Y}_i X = 0$, $i = 1, \dots, 4$ then X is a constant, provided $T_3 \neq 0$. So assume

$$T_3 = 0 \tag{6.15}$$

and

$$\left.\begin{aligned} & \mathcal{Y}_i X = 0, \quad i = 1, \dots, 4 \\ & X \text{ is not a constant} \end{aligned}\right\} \tag{6.16}$$

From (6.9) and (6.10)

$$\left.\begin{aligned} \mathcal{Y}_3 (2T_2 - T_1) &= T_3.c.\beta_1.\sin c\alpha_0 \\ \mathcal{Y}_4 (2T_2 - T_1) &= -T_3.c.\alpha_1.\cos c\alpha_0 \end{aligned}\right\} \tag{6.17}$$

Define

$$\left.\begin{aligned} \mathcal{J}_5 &= \mathcal{J}_1\mathcal{J}_2 - \mathcal{J}_2\mathcal{J}_1 \\ \mathcal{J}_6 &= \mathcal{J}_3\mathcal{J}_4 - \mathcal{J}_4\mathcal{J}_3 \end{aligned}\right\} \qquad (6.18)$$

Then from (6.8) , (6.16) and (6.17) one has

$$\mathcal{J}_i . X = 0 \qquad i = 1, \ldots, 6 \qquad (6.19)$$

$$\mathcal{J}_6\,(2T_2 - T_1) = c^2.T_3.\left[-\beta_1.Q_4.\cos c\alpha_0 + \alpha_1.Q_3.\sin c\alpha_0\right] \qquad (6.20)$$

From (6.17) it is not too difficult to show that

$$\frac{\partial}{\partial\alpha_0}\,(2T_2 - T_1) \neq 0 \qquad (6.21)$$

Now (6.19) are 6 equations for the derivatives of X w.r.t. the 6 independent variables in the problem. So the determinant of coefficients must vanish. But, from (6.17) and (6.20) one may replace the column corresponding to $\frac{\partial X}{\partial\alpha_0}$ by R.H.S. of (6.17) and (6.20). Then since $T_3 \neq 0$ one has, after some re-arranging (add β_1.col 5 to col 1 ; divide col 1 by α_1 ; add β_1.z.col 5 + $(\beta_2 - \alpha_1.z)$.col 1 to col 6 ; add β_1.col 1 - α_1.col 5 to col 4 ; divide row 6 by c.) that

$$\begin{vmatrix} 0 & 0 & \cos\alpha_0 & 0 & -\sin\alpha_0 & -(\alpha_1^2 + \beta_1^2).\cos\alpha_0 \\ -\cos\alpha_0 & \sin\alpha_0 & 0 & 0 & 0 & -2.\alpha_1.\beta_1.\sin\alpha_0 \\ 0 & 0 & \cos c\alpha_0 & 0 & -c.\sin c\alpha_0 & -c^2.(\beta_1^2 + \alpha_1^2).\cos c\alpha_0 \\ -c.\cos c\alpha_0 & \sin c\alpha_0 & 0 & 0 & 0 & -2.c^2.\alpha_1.\beta_1.\sin c\alpha_0 \\ Q_1.\sin\alpha_0 & Q_1.\cos\alpha_0 & Q_2.\sin\alpha_0 & 1 & Q_2.\cos\alpha_0 & Y_1 \\ c.Q_3.\sin c\alpha_0 & Q_3.\cos c\alpha_0 & Q_4.\sin c\alpha_0 & 1 & c.Q_4.\cos c\alpha_0 & Y_2 \end{vmatrix} = \qquad (6.22)$$

where

$$Y_1 = -2.Q_1.\alpha_1.\beta_1.\cos\alpha_0 - Q_2.(\alpha_1^2 + \beta_1^2).\sin\alpha_0$$

$$Y_2 = -2.c^2.\alpha_1.\beta_1.Q_3.\cos c\alpha_0 - c^2.Q_4.(\alpha_1^2 + \beta_1^2).\sin c\alpha_0$$

Take $\mathcal{D}_5$ of (6.22). Because of (6.7) it only acts on α_1. So one sees that the coefficient of α_1^2 in (6.22) must vanish. That is

$$c.Q_4.\cos\alpha_0 = Q_2.\cos c\alpha_0 \tag{6.23}$$

Take $\mathcal{D}_6$ and then $\mathcal{D}_5$ of (6.23)

$$Q_4.\sin\alpha_0 = Q_2.\sin c\alpha_0 \tag{6.24}$$

Using (6.5) and (6.6) one then has a contradiction. So it has been proved that (6.16) cannot hold. That is

If $T_3 \neq 0$ and $\mathcal{J}_1 X = 0$ then X is a constant (6.25)

§7. That $T_3 \neq 0$.

For this section assume that

$$T_3 = 0 \tag{7.1}$$

One then wishes to find a contradiction. Note that the equations to be solved are (6.5) to (6.11). Define

$$\mathcal{D}_7 = \mathcal{D}_5\mathcal{D}_6 - \mathcal{D}_6\mathcal{D}_5 \tag{7.2}$$

$$\therefore \mathcal{D}_7 = \frac{\partial}{\partial\alpha_0} + \gamma\cdot\frac{\partial}{\partial\beta_1} + (T_1 - T_2)\cdot\frac{\partial}{\partial\beta_2} + (\mathcal{D}_5 T_0 - \mathcal{D}_6 T_1)\cdot\frac{\partial}{\partial\mu} + (\mathcal{D}_5 T_2)\cdot\frac{\partial}{\partial\gamma} \tag{7.3}$$

Then (6.5) to (6.7) give

$$\mathcal{D}_7 Q_i = 0 \quad , \quad i = 1,\dots,4 \tag{7.4}$$

Now $(\mathcal{D}_5\mathcal{D}_7 - \mathcal{D}_7\mathcal{D}_5)Q_i = 0$, $i = 1,\dots,4$ and (6.14) imply

$$\mathcal{D}_5(T_1 - 2T_2) = 0 \tag{7.5}$$

$$\mathcal{D}_5^2 T_2 = 0 \tag{7.6}$$

$$\mathcal{D}_5(\mathcal{D}_5 T_0 - \mathcal{D}_6 T_1) = \mathcal{D}_7 T_1 \tag{7.7}$$

$$\therefore \mathcal{D}_5\mathcal{D}_7 = \mathcal{D}_7\mathcal{D}_5 \tag{7.8}$$

From (7.2), (7.5), (7.6) and (7.8) one may write

$$\left.\begin{aligned} T_2 &= V_1 + V_2.\alpha_1 \\ T_1 &= V_0 + 2.V_2.\alpha_1 \\ \mathcal{D}_6 V_2 &= V_3 + V_4.\alpha_1 \end{aligned}\right\} \qquad (7.9)$$

where

$$\mathcal{D}_5 V_i = 0 \quad , \quad i = 0, \ldots, 4 \qquad (7.10)$$

Equations (6.9), (6.10) and (6.13) then imply

$$\left.\begin{aligned} \mathcal{J}_3 V_0 &= 0 \\ \mathcal{J}_4 V_0 &= -2c^2.\beta_1.\sin c\alpha_0 - 2V_2.\sin c\alpha_0 \end{aligned}\right\} \qquad (7.11)$$

$$\left.\begin{aligned} \mathcal{J}_4 V_1 &= -c^2.\beta_1.\sin c\alpha_0 - V_2.\sin c\alpha_0. \\ \mathcal{J}_3 V_1 &= 0 \end{aligned}\right\} \qquad (7.12)$$

$$\left.\begin{aligned} \mathcal{J}_3 V_2 &= -c^2.\cos c.\alpha_0 \\ \mathcal{J}_4 V_2 &= 0 \end{aligned}\right\} \qquad (7.13)$$

$$\left.\begin{aligned} \mathcal{J}_3 V_3 &= 0 \\ \mathcal{J}_4 V_3 &= -c^3.\beta_1.\cos c\alpha_0 - V_4.\sin c\alpha_0 \end{aligned}\right\} \qquad (7.14)$$

$$\left.\begin{aligned} \mathcal{J}_3 V_4 &= c^3.\sin c\alpha_0. \\ \mathcal{J}_4 V_4 &= 0 \end{aligned}\right\} \qquad (7.15)$$

From (7.13) it is not too difficult to show that

$$\frac{\partial V_2}{\partial w} \neq 0 \tag{7.16}$$

Define $\eta = \beta_2 - \alpha_1 . z$ (7.17)

and use α_0, α_1, β_1, η, w and z as independent variables.

Define $\xi = V_2(\alpha_0,\alpha_1,\beta_1,z,\eta,w)$ (7.18)

and use ξ instead of w (which is allowed because of (7.16).)

Then (6.7) and (7.10) are

$$\left.\begin{aligned} \frac{\partial Q_i}{\partial \alpha_1} &= 0 \qquad i=1,\ldots,4 \\ \frac{\partial V_i}{\partial \alpha_1} &= 0 \qquad i=0,\ldots,4 \end{aligned}\right\} \tag{7.19}$$

Then (7.12), (7.13) and (7.15) imply

$$\left.\begin{aligned} \mathcal{J}_3 &= Q_3 \cdot \frac{\partial}{\partial \alpha_0} + \cos c\alpha_0 \cdot \frac{\partial}{\partial \beta_1} - c \cdot \sin c\alpha_0 \cdot \frac{\partial}{\partial z} - c^2 \cos c\alpha_0 \cdot \frac{\partial}{\partial \xi} \\ \mathcal{J}_4 &= Q_4 \cdot \frac{\partial}{\partial \alpha_0} + [c \cdot \beta_1 \cdot \cos c\alpha_0 - z \cdot \sin c\alpha_0] \cdot \frac{\partial}{\partial \eta} \end{aligned}\right\} \tag{7.20}$$

$$\left.\begin{aligned} \mathcal{J}_3 V_1 &= 0 \\ \mathcal{J}_4 V_1 &= -c^2 \cdot \beta_1 \cdot \sin c\alpha_0 - \xi \cdot \sin c\alpha_0 \end{aligned}\right\} \tag{7.21}$$

$$\left.\begin{aligned} \mathcal{J}_3 V_4 &= c^3 \cdot \sin c\alpha_0 \\ \mathcal{J}_4 V_4 &= 0 \end{aligned}\right\} \tag{7.22}$$

From (7.22) one may prove that

$$\frac{\partial V_4}{\partial \alpha_0} \neq 0 \tag{7.23}$$

Eliminate derivatives of V_4 from and

$$Q_2.(c.\beta_1.\cos c\alpha_0 - z.\sin c\alpha_0) = Q_4.(\beta_1.\cos\alpha_0 - z.\sin\alpha_0) \qquad (7.24)$$

But then (7.21) and (7.24) imply

$$(\xi + \beta_1).\sin\alpha_0.(c.\beta_1.\cos c\alpha_0 - z.\sin c\alpha_0) = (c^2.\beta_1 + \xi).\sin c\alpha_0.(\beta_1.\cos\alpha_0 - z.\sin\alpha_0)$$

which is clearly false. So it has been shown that (7.1) must be false. Therefore

$$T_3 \not\equiv 0 \qquad (7.25)$$

§8. For $T_3 \not\equiv 0$.

From (6.8), (6.25) and (7.25) one has that

T_3 is a non-zero constant (8.1)

Define

$$\mathcal{D}_7 = \frac{\partial}{\partial\alpha_0} + z\cdot\frac{\partial}{\partial\beta} + (T_1 - T_2)\cdot\frac{\partial}{\partial\beta_2} + (\mathcal{D}_5 T_0 - \mathcal{D}_6 T_1)\frac{\partial}{\partial\mu} + \mathcal{D}_5 T_2\cdot\frac{\partial}{\partial z} \qquad (8.2)$$

$$= \mathcal{D}_5\mathcal{D}_6 - \mathcal{D}_6\mathcal{D}_5 \qquad (8.3)$$

Then (6.6) to (6.7) imply

$$\mathcal{D}_7 Q_i = 0, \quad i = 1, \ldots, 4 \qquad (8.4)$$

Define

$$\mathcal{D}_8 = \mathcal{D}_5\mathcal{D}_7 - \mathcal{D}_7\mathcal{D}_5 \qquad (8.5)$$

$$= T_3\frac{\partial}{\partial\beta_1} + \mathcal{D}_5(T_1 - 2T_2)\cdot\frac{\partial}{\partial\beta_2} + [\mathcal{D}_5(\mathcal{D}_5 T_0 - \mathcal{D}_6 T_1) - \mathcal{D}_7 T_1]\cdot\frac{\partial}{\partial\mu} + \mathcal{D}_5^2\cdot T_2\cdot\frac{\partial}{\partial z} \qquad (8.6)$$

Then (6.7) and (8.4) imply

$$\mathcal{D}_8 Q_i = 0 \qquad i = 1, \ldots, 4 \qquad (8.7)$$

Now $(\mathcal{D}_5\mathcal{D}_8 - \mathcal{D}_8\mathcal{D}_5)Q_i = 0, \ i = 1, \ldots, 4$ and (6.14) imply

$$\mathcal{D}_5^2\,(T_1 - 3T_2) = 0 \tag{8.8}$$

$$\mathcal{D}_5^3\,T_2 = 0 \tag{8.9}$$

$$\mathcal{D}_5\left[\mathcal{D}_5(\mathcal{D}_5 T_0 - \mathcal{D}_6 T_1) - \mathcal{D}_7 T_1\right] - \mathcal{D}_8 T_1 = 0 \tag{8.10}$$

$$\therefore \quad \mathcal{D}_5\,\mathcal{D}_8 = \mathcal{D}_8\,\mathcal{D}_5 \tag{8.11}$$

From (8.8) , (8.9) , (8.3) , (8.5) and (8.11) one may write

$$\left.\begin{aligned} T_1 &= U_0 + U_1.\alpha_1 - 3.U_2.\alpha_1^2 \\ 2.T_2 - T_1 &= U_3 + U_4.\alpha_1 + U_2.\alpha_1^2 \\ \mathcal{D}_6\,U_4 &= U_5 + U_6.\alpha_1 + U_7.\alpha_1^2 \end{aligned}\right\} \tag{8.12}$$

where

$$\mathcal{D}_5\,U_i = 0 \qquad , i = 0, \ldots, 7 \tag{8.13}$$

Then (6.9) , (6.10) , (6.12) , (6.13) and (8.12) imply

$$\left.\begin{aligned} \mathcal{J}_3\,U_0 &= T_3.c.\beta_1.\sin c\alpha_0 \\ \mathcal{J}_4\,U_0 &= -(2c^2.\beta_1 + U_1).\sin c\alpha_0 \end{aligned}\right\} \tag{8.14}$$

$$\left.\begin{aligned} \mathcal{J}_3\,U_1 &= -2c^2.\cos c\alpha_0. \\ \mathcal{J}_4\,U_1 &= -T_3.c.\cos c\alpha_0 + 6\,U_2.\sin c\alpha_0 \end{aligned}\right\} \tag{8.15}$$

$$\mathcal{J}_i\,U_2 = 0 \qquad , i = 1, \ldots, 4 \tag{8.16}$$

$$\left.\begin{aligned} \mathcal{J}_3\,U_3 &= T_3.c.\beta_1.\sin c\alpha_0 \\ \mathcal{J}_4\,U_3 &= -U_4.\sin c\alpha_0 \end{aligned}\right\} \tag{8.17}$$

$$\left.\begin{aligned} \mathcal{J}_3 U_4 &= -T_3.c.\cos c\alpha_0 - 2U_2.\sin c\alpha_0 \\ \mathcal{J}_4 U_4 &= T_3.c.\beta_1.\sin c\alpha_0 \end{aligned}\right\} \tag{8.18}$$

$$\left.\begin{aligned} \mathcal{J}_3 U_5 &= T_3.c^2.\beta_1.\cos c\alpha_0 + 2c.\beta_1.U_2.\sin c\alpha_0 . \\ \mathcal{J}_4 U_5 &= -U_6.\sin c\alpha_0 \end{aligned}\right\} \tag{8.19}$$

$$\left.\begin{aligned} \mathcal{J}_3 U_6 &= 0 \\ \mathcal{J}_4 U_6 &= (T_3.c^2 - 2U_7).\sin c\alpha_0 \;\; -2c.U_2.\cos c\alpha_0 \end{aligned}\right\} \tag{8.20}$$

$$\mathcal{J}_i U_7 = 0 \qquad i = 1,..,4 \tag{8.21}$$

Then (6.25) , (7.25) , (8.16) and (8.21) imply

U_2 and U_7 are constants (8.22)

If $\frac{\partial U_4}{\partial w} = 0$ then diff. (8.18) w.r.t. w to obtain $\frac{\partial U_4}{\partial \alpha_0} = 0$.
One then readily obtains a contradiction, from (8.18) . Therefore

$$\frac{\partial U_4}{\partial w} \neq 0 \tag{8.23}$$

Define
$$\begin{aligned} \eta_1 &= z - T_3.\alpha_1 \\ \eta_2 &= \beta_2 - \alpha_1.z + \tfrac{1}{2}.T_3.\alpha_1^2 \end{aligned} \tag{8.24}$$

and use α_0 , α_1 , β_1 , η_1 , η_2 and w as independent variables.

Define
$$\xi = U_4(\alpha_0,\alpha_1,\beta_1,\eta_1,\eta_2,w) \tag{8.25}$$

and use ξ instead of w . (allowed because of (8.23).)

Then (6.7) and (8.13) are

$$\left.\begin{aligned} \frac{\partial Q_i}{\partial \alpha_1} &= 0 \qquad i = 1,2,3,4 \\ \frac{\partial U_i}{\partial \alpha_1} &= 0 \qquad i = 0,1,2,3,5,6,7 \end{aligned}\right\} \tag{8.26}$$

Let $w = Z(\alpha_0, \alpha_1, \beta_1, \eta_1, \eta_2, \xi)$ be the inverse of (8.25). Then, from (8.13) for $i = 4$ one may write (after integrating)

$$Z = U_8 + \tfrac{1}{2}\cdot U_1 \cdot \alpha_1^2 - U_2 \cdot \alpha_1^3 + U_0 \cdot \alpha_1 \tag{8.27}$$

Then from (6.5), (6.6), (8.14) to (8.20) one has

$$\left.\begin{aligned} \mathcal{D}_9 &= \eta_2 \cdot \frac{\partial}{\partial \beta_1} + \tfrac{1}{2}(U_3 + U_0)\frac{\partial}{\partial \eta_1} + U_8 \cdot \frac{\partial}{\partial \eta_2} + U_5 \cdot \frac{\partial}{\partial \xi} \\ \mathcal{D}_8 &= T_3 \cdot \frac{\partial}{\partial \beta_1} - 2U_2 \cdot \frac{\partial}{\partial \eta_1} - \xi \cdot \frac{\partial}{\partial \eta_2} + 2U_7 \cdot \frac{\partial}{\partial \xi} \\ \mathcal{D}_7 &= \frac{\partial}{\partial \alpha_0} + \eta_1 \cdot \frac{\partial}{\partial \beta_1} + \tfrac{1}{2}(U_1 + \xi)\frac{\partial}{\partial \eta_1} + \tfrac{1}{2}(U_0 - U_3)\frac{\partial}{\partial \eta_2} + U_6 \frac{\partial}{\partial \xi} \end{aligned}\right\} \tag{8.28}$$

$$\mathcal{J}_3 = Q_3 \cdot \frac{\partial}{\partial \alpha_0} + \cos c\alpha_0 \cdot \frac{\partial}{\partial \beta_1} - c \cdot \sin c\alpha_0 \cdot \frac{\partial}{\partial \eta_1} \tag{8.29a}$$

$$\mathcal{J}_4 = Q_4 \cdot \frac{\partial}{\partial \alpha_0} - T_3 \sin c\alpha_0 \cdot \frac{\partial}{\partial \eta_1} + \left[c \cdot \beta_1 \cdot \cos c\alpha_0 - \eta_1 \sin c\alpha_0\right]\frac{\partial}{\partial \eta_2}$$
$$- \left[T_3 \cdot c \cdot \cos c\alpha_0 + 2U_2 \cdot \sin c\alpha_0\right] \cdot \frac{\partial}{\partial \xi} \tag{8.29b}$$

$$\mathcal{D}_9 Q_1 = \beta_1 Q_2 \qquad \mathcal{D}_9 \cdot Q_3 = c \cdot \beta_1 \cdot Q_4 \tag{8.30}$$

$$\mathcal{D}_9 Q_2 = \sin \alpha_0 - \beta_1 \cdot Q_1 \qquad \mathcal{D}_9 Q_4 = \sin c\alpha_0 - c \cdot \beta_1 \cdot Q_3 \tag{8.31}$$

$$\mathcal{D}_7 Q_i = 0, \quad i = 1, \ldots, 4 \tag{8.32}$$

$$\mathcal{D}_8 Q_i = 0 , \quad i = 1, \ldots, 4 \tag{8.33}$$

$$\left.\begin{aligned} \mathcal{J}_3 U_0 &= T_3 . c . \beta_1 . \sin c\alpha_0 \\ \mathcal{J}_4 U_0 &= -(2c^2 . \beta_1 + U_1) . \sin c\alpha_0 \end{aligned}\right\} \tag{8.34}$$

$$\left.\begin{aligned} \mathcal{J}_3 U_1 &= -2c^2 . \cos c\alpha_0 \\ \mathcal{J}_4 U_1 &= -T_3 . c . \cos c\alpha_0 + 6 U_2 . \sin c\alpha_0 \end{aligned}\right\} \tag{8.35}$$

$$\left.\begin{aligned} \mathcal{J}_3 U_3 &= T_3 . c . \beta_1 . \sin c\alpha_0 \\ \mathcal{J}_4 U_3 &= -\xi . \sin c\alpha_0 \end{aligned}\right\} \tag{8.36}$$

$$\left.\begin{aligned} \mathcal{J}_3 U_5 &= T_3 . c^2 . \cos c\alpha_0 + 2c . \beta_1 . U_2 . \sin c\alpha_0 \\ \mathcal{J}_4 U_5 &= -U_6 . \sin c\alpha_0 \end{aligned}\right\} \tag{8.37}$$

$$\left.\begin{aligned} \mathcal{J}_3 U_6 &= 0 . \\ \mathcal{J}_4 U_6 &= (T_3 . c^2 - 2U_7) . \sin c\alpha_0 - 2c . U_2 . \cos c\alpha_0 \end{aligned}\right\} \tag{8.38}$$

$$\left.\begin{aligned} \mathcal{J}_3 U_8 &= c . \beta_1 . \eta_1 . \sin c\alpha_0 - c^2 . \beta_1^2 . \cos c\alpha_0 \\ \mathcal{J}_4 U_8 &= c . \eta_2 . \cos c\alpha_0 - U_0 . \sin c\alpha_0 \end{aligned}\right\} \tag{8.39}$$

Also (6.14) is

$$\left| \frac{\partial Q_i}{\partial \eta_1} \quad \frac{\partial Q_i}{\partial \eta_2} \quad \frac{\partial Q_i}{\partial \xi} \right| \neq 0 \tag{8.40}$$

$$\left.\begin{aligned}
\mathcal{D}_8 \mathcal{T}_i &= \mathcal{T}_i \mathcal{D}_8 \qquad i = 1, \cdots, 4 \\
\mathcal{D}_7 \mathcal{T}_3 &= \mathcal{T}_3 \mathcal{D}_7 \\
\mathcal{D}_7 \mathcal{T}_4 &= \mathcal{T}_4 \mathcal{D}_7 + \sin c\alpha_0 \cdot \mathcal{D}_8 \\
\mathcal{D}_9 \mathcal{T}_3 &= \mathcal{T}_3 \mathcal{D}_9 + c \cdot \beta_1 \cdot \mathcal{T}_4 \\
\mathcal{D}_9 \mathcal{T}_4 &= \mathcal{T}_4 \mathcal{D}_9 - c \cdot \beta_1 \cdot \mathcal{T}_3 + \sin c\alpha_0 \cdot \mathcal{D}_7
\end{aligned}\right\} \qquad (8.41)$$

It will be shown in the next two sections that (8.30) to (8.39) have no solutions.

§9. That $U_7 \neq 0$.

For this section assume

$$U_7 = 0 \qquad (9.1)$$

A contradiction is then found as follows.

Define

$$\begin{aligned} \upsilon_1 &= U_2 . \beta_1 + \tfrac{1}{2} . T_3 . \eta_1 \\ \upsilon_2 &= T_3 . \eta_2 + \xi . \beta_1 \end{aligned} \qquad (9.2)$$

and use α_0, β_1, υ_1, υ_2 and ξ as independent variables.

Now (8.33) is

$$\frac{\partial Q_i}{\partial \beta_1} = 0 \qquad i = 1, \ldots, 4 \qquad (9.3)$$

Also (8.40) is

$$\left| \frac{\partial Q_i}{\partial \upsilon_1} \quad \frac{\partial Q_i}{\partial \upsilon_2} \quad \frac{\partial Q_i}{\partial \xi} \right| \neq 0 \qquad (9.4)$$

Diff. (8.32) w.r.t. β_1 and (8.30) and (8.31) twice w.r.t. β_1 and use (9.4). Integrate the resultant equations to obtain

$$Z_1 = 2 . U_2 . T_3^{-1} . (\upsilon_1 - U_2 . \beta_1) + \tfrac{1}{4} . T_3 . (U_1 + \xi) \qquad (9.5a)$$

$$Z_2 = 2 . \xi . T_3^{-1} . (\upsilon_1 - U_2 . \beta_1) + \tfrac{1}{2} . T_3 . (U_0 - U_3) + \beta_1 . U_6 \qquad (9.5b)$$

$$Z_3 = U_6 \qquad (9.5c)$$

$$Z_4 + Z_5 . \beta_1 = U_2 . T_3^{-1} . (\upsilon_2 - \xi . \beta_1) + \tfrac{1}{4} . T_3 . (U_0 + U_3) \tag{9.5d}$$

$$Z_6 + Z_7 . \beta_1 = \xi . T_3^{-1} . (\upsilon_2 - \xi . \beta_1) + U_8 . T_3 + U_5 . \beta_1 \tag{9.5e}$$

$$Z_8 + Z_9 . \beta_1 = U_5 \tag{9.5f}$$

where

$$\frac{\partial Z_i}{\partial \beta_1} = 0 \qquad , i = 1, \ldots, 9 .$$

Then (8.28) to (8.39) imply

$$\left.\begin{aligned} \mathcal{D}_7 &= \frac{\partial}{\partial \alpha_0} + Z_1 . \frac{\partial}{\partial \nu_1} + Z_2 . \frac{\partial}{\partial \nu_2} + Z_3 . \frac{\partial}{\partial \xi} \\ \mathcal{D}_{10} &= Z_4 . \frac{\partial}{\partial \nu_1} + Z_6 . \frac{\partial}{\partial \nu_2} + Z_8 . \frac{\partial}{\partial \xi} \\ \mathcal{D}_{11} &= Z_5 . \frac{\partial}{\partial \nu_1} + Z_7 . \frac{\partial}{\partial \nu_2} + Z_9 . \frac{\partial}{\partial \xi} \end{aligned}\right\} \tag{9.6}$$

$$\mathcal{Y}_3 = Q_3 . \frac{\partial}{\partial \alpha_0} + \left[U_2 . \cos c\alpha_0 - \tfrac{1}{2} . c . T_3 . \sin c\alpha_0 \right] . \frac{\partial}{\partial \nu_1} + \xi . \cos c\alpha_0 . \frac{\partial}{\partial \nu_2} \tag{9.7}$$

$$\mathcal{D}_7 Q_i = 0 \qquad , i = 1, \ldots, 4 \tag{9.8}$$

$$\left.\begin{aligned} \mathcal{D}_{10} Q_1 &= 0 & \qquad \mathcal{D}_{10} Q_3 &= 0 \\ \mathcal{D}_{10} Q_2 &= \sin \alpha_0 & \qquad \mathcal{D}_{10} Q_4 &= \sin c\alpha_0 \end{aligned}\right\} \tag{9.9}$$

$$\left.\begin{aligned} \mathcal{D}_{11} Q_1 &= Q_2 & \qquad \mathcal{D}_{11} Q_3 &= c . Q_4 \\ \mathcal{D}_{11} Q_2 &= -Q_1 & \qquad \mathcal{D}_{11} Q_4 &= -c . Q_3 \end{aligned}\right\} \tag{9.10}$$

$$\mathcal{Y}_3 Z_1 = -c . U_2 . \sin c\alpha_0 - \tfrac{1}{2} . c^2 . T_3 . \cos c\alpha_0 \tag{9.11}$$

$$\mathcal{Y}_3 Z_2 = Z_3 . \cos c\alpha_0 - c . \omega . \sin c\alpha_0 \tag{9.12}$$

$$\mathcal{Y}_3 Z_3 = 0 \tag{9.13}$$

$$\mathcal{J}_3 Z_4 = -Z_5 \cdot \cos c\alpha_0 \tag{9.14}$$

$$\mathcal{J}_3 Z_5 = \tfrac{1}{2} \cdot T_3^2 \cdot c \cdot \sin c\alpha_0 \tag{9.15}$$

$$\mathcal{J}_3 Z_6 = (Z_8 - Z_7) \cdot \cos c\alpha_0 \tag{9.16}$$

$$\mathcal{J}_3 Z_7 = 2c \cdot \mathcal{J}_1 \cdot \sin c\alpha_0 + Z_9 \cdot \cos c\alpha_0 . \tag{9.17}$$

$$\mathcal{J}_3 Z_8 = -Z_9 \cdot \cos c\alpha_0 \tag{9.18}$$

$$\mathcal{J}_3 Z_9 = T_3 \cdot c^2 \cdot \cos c\alpha_0 + 2c \cdot U_2 \cdot \sin c\alpha_0 \tag{9.19}$$

Also (8.41) implies

$$\left.\begin{aligned} \mathcal{D}_7 \mathcal{J}_3 &= \mathcal{J}_3 \mathcal{D}_7 \\ \mathcal{D}_{10} \mathcal{J}_3 &= \mathcal{J}_3 \mathcal{D}_{10} + \cos c\alpha_0 \cdot \mathcal{D}_{11} \\ \mathcal{D}_{11} \mathcal{J}_3 &= \mathcal{J}_3 \mathcal{D}_{11} + c \cdot \mathcal{J}_4 \end{aligned}\right\} \tag{9.20}$$

Now $(\mathcal{D}_{11}\mathcal{D}_7 - \mathcal{D}_7\mathcal{D}_{11}) Q_i = 0, \; i = 1, .., 4$ and (8.40) imply

$$\left.\begin{aligned} \mathcal{D}_7 Z_5 &= \mathcal{D}_{11} Z_1 \\ \mathcal{D}_7 Z_7 &= \mathcal{D}_{11} Z_2 \\ \mathcal{D}_7 Z_9 &= \mathcal{D}_{11} Z_3 \end{aligned}\right\} \tag{9.21}$$

$$\therefore \; \mathcal{D}_{16} \mathcal{D}_7 = \mathcal{D}_7 \mathcal{D}_{16} \tag{9.22}$$

Define

$$\left.\begin{aligned} \mathcal{D}_{12} &= \mathcal{D}_7 \mathcal{D}_{10} - \mathcal{D}_{10} \mathcal{D}_7 \\ \mathcal{D}_{13} &= \mathcal{D}_7 \mathcal{D}_{12} - \mathcal{D}_{12} \mathcal{D}_7 \\ \mathcal{D}_{14} &= \mathcal{D}_7 \mathcal{D}_{13} - \mathcal{D}_{13} \mathcal{D}_7 \end{aligned}\right\} \tag{9.23}$$

$$\left.\begin{aligned} \mathcal{D}_{12} &= Z_{10} \cdot \frac{\partial}{\partial \nu_1} + Z_{11} \cdot \frac{\partial}{\partial \nu_2} + Z_{12} \cdot \frac{\partial}{\partial \xi} \\ \mathcal{D}_{13} &= Z_{13} \cdot \frac{\partial}{\partial \nu_1} + Z_{14} \cdot \frac{\partial}{\partial \nu_2} + Z_{15} \cdot \frac{\partial}{\partial \xi} \\ \mathcal{D}_{14} &= Z_{16} \cdot \frac{\partial}{\partial \nu_1} + Z_{17} \cdot \frac{\partial}{\partial \nu_2} + Z_{18} \cdot \frac{\partial}{\partial \xi} \end{aligned}\right\} \tag{9.24}$$

Then (9.20) and (9.23) give

$$\left.\begin{aligned} \mathcal{Y}_3 \mathcal{D}_{10} &= \mathcal{D}_{10} \mathcal{Y}_3 - \cos c\alpha_0 \cdot \mathcal{D}_{11} \\ \mathcal{Y}_3 \mathcal{D}_{12} &= \mathcal{D}_{12} \mathcal{Y}_3 + c \cdot \sin c\alpha_0 \cdot \mathcal{D}_{11} \\ \mathcal{Y}_3 \mathcal{D}_{13} &= \mathcal{D}_{13} \mathcal{Y}_3 + c^2 \cdot \cos c\alpha_0 \cdot \mathcal{D}_{11} \end{aligned}\right\} \tag{9.25}$$

From (9.11) to (9.19), (9.21), (9.24) and (9.25) one has

$$\mathcal{Y}_3 Z_{10} = c \cdot Z_5 \cdot \sin c\alpha_0 . \tag{9.26}$$

$$\mathcal{Y}_3 Z_{11} = Z_{12} \cdot \cos c\alpha_0 + c \cdot Z_7 \cdot \sin c\alpha_0 . \tag{9.27}$$

$$\mathcal{Y}_3 Z_{12} = c \cdot Z_9 \cdot \sin c\alpha_0 \tag{9.28}$$

$$\mathcal{Y}_3 Z_{13} = c^2 \cdot Z_5 \cdot \cos c\alpha_0 \tag{9.29}$$

$$\mathcal{Y}_3 Z_{14} = Z_{15} \cdot \cos c\alpha_0 + c^2 \cdot Z_7 \cdot \cos c\alpha_0 \tag{9.30}$$

$$\mathcal{Y}_3 Z_{15} = c^2 \cdot Z_9 \cdot \cos c\alpha_0 \tag{9.31}$$

$$\mathcal{J}_3\, Z_{16} = -c^3 . Z_5 . \sin c\alpha_0 \tag{9.32}$$

$$\mathcal{J}_3\, Z_{17} = Z_{18} . \cos c\alpha_0 - c^3 Z_7 . \sin c\alpha_0 \tag{9.33}$$

$$\mathcal{J}_3\, Z_{18} = -c^3 . Z_9 . \sin c\alpha_0 \tag{9.34}$$

Also (9.8) , (9.9) , (9.10) and (9.23) imply

$$\left.\begin{aligned}
\mathcal{D}_{10}(Q_1, Q_2, Q_3, Q_4) &= (0, \sin\alpha_0, 0, \sin c\alpha_0) \\
\mathcal{D}_{12}(Q_1, Q_2, Q_3, Q_4) &= (0, \cos\alpha_0, 0, c.\cos c\alpha_0) \\
\mathcal{D}_{13}(Q_1, Q_2, Q_3, Q_4) &= (0, -\sin\alpha_0, 0, -c^2 \sin c\alpha_0) \\
\mathcal{D}_{14}(Q_1, Q_2, Q_3, Q_4) &= (0, -\cos\alpha_0, 0, -c^3 \cos c\alpha_0)
\end{aligned}\right\} \tag{9.35}$$

From (9.35) :

$$\left[\cos\alpha_0 . (c^2 . \mathcal{D}_{10} + \mathcal{D}_{13}) - \sin\alpha_0 (c^2 . \mathcal{D}_{12} + \mathcal{D}_{14})\right] Q_i = 0, \quad i = 1, \ldots, 4$$

This , (9.4) and (9.24) imply

$$\cos\alpha_0 . (c^2 . Z_4 + Z_{13}) = \sin\alpha_0 . (c^2 . Z_{10} + Z_{16}) \tag{9.36}$$

$$\cos\alpha_0 (c^2 . Z_6 + Z_{14}) = \sin\alpha_0 (c^2 . Z_{11} + Z_{17}) \tag{9.37}$$

$$\cos\alpha_0 (c^2 . Z_8 + Z_{15}) = \sin\alpha_0 (c^2 Z_{12} + Z_{18}) \tag{9.38}$$

Take $\mathcal{J}_3$ of (9.36) and (9.38) to obtain

$$\begin{aligned} c^2 . Z_4 + Z_{13} &= c^2 . Z_{10} + Z_{16} = 0 \\ c^2 . Z_8 + Z_{15} &= c^2 . Z_{12} + Z_{18} = 0 \end{aligned} \tag{9.39}$$

Then $\mathcal{J}_3$ of (9.37) gives

$$c^2 . Z_6 + Z_{14} = c^2 . Z_{11} + Z_{17} \tag{9.40}$$

Then (9.6) , (9.24) , (9.39) and (9.40) imply

$$c^2 \mathcal{D}_{10} + \mathcal{D}_{13} = 0 \tag{9.41}$$

But from (9.35)

$$\left(c^2 \mathcal{D}_{10} + \mathcal{D}_{13}\right) Q_2 = c^2 \sin \alpha_0 - \sin \alpha_0 \neq 0$$

This is a contradiction. Hence (9.1) cannot hold. Therefore

$$U_7 \neq 0 \tag{9.42}$$

§10. Conclusion.

The proof that there can be no B.T. of the form (1.6) for (1.4) will be concluded in this section. From (9.42):

$$U_7 \neq 0 \tag{10.1}$$

Define
$$\left.\begin{aligned} \zeta_1 &= U_2 \cdot \beta_1 + \tfrac{1}{2} \cdot T_3 \cdot \eta_1 \\ \zeta_2 &= U_7 \cdot \beta_1 - \tfrac{1}{2} \cdot T_3 \cdot \xi \\ \zeta_3 &= U_7 \cdot \eta_2 + \tfrac{1}{4} \cdot \xi^2 \end{aligned}\right\} \tag{10.2}$$

and use α_0, ζ_1, ζ_2, ζ_3 and ξ as independent variables. Then (8.33) and (8.40) are

$$\frac{\partial Q_i}{\partial \xi} = 0 \qquad i = 1, \ldots, 4 \tag{10.3}$$

$$\left| \frac{\partial Q_i}{\partial \zeta_1} \quad \frac{\partial Q_i}{\partial \zeta_2} \quad \frac{\partial Q_i}{\partial \zeta_3} \right| \neq 0 \tag{10.4}$$

Exactly as in the previous section (diff. (8.30) and (8.31) twice w.r.t. ξ and (8.32) w.r.t. ξ and use (10.4).) one may take (8.28) to (8.39) to give

$$\left.\begin{aligned} \mathcal{D}_7 &= \frac{\partial}{\partial \alpha_0} + w_1 \cdot \frac{\partial}{\partial \zeta_1} + w_2 \cdot \frac{\partial}{\partial \zeta_2} + w_3 \cdot \frac{\partial}{\partial \zeta_3} \\ \mathcal{D}_{15} &= w_4 \cdot \frac{\partial}{\partial \zeta_1} + w_5 \cdot \frac{\partial}{\partial \zeta_2} + w_6 \cdot \frac{\partial}{\partial \zeta_3} \\ \mathcal{D}_{16} &= w_7 \cdot \frac{\partial}{\partial \zeta_1} + w_8 \cdot \frac{\partial}{\partial \zeta_2} + w_9 \cdot \frac{\partial}{\partial \zeta_3} \end{aligned}\right\} \tag{10.5}$$

$$\mathcal{Y}_3 = Q_3 \cdot \frac{\partial}{\partial \alpha_0} + \left[U_2 \cdot \cos c\alpha_0 - \tfrac{1}{2} c \, T_3 \cdot \sin c\alpha_0 \right] \frac{\partial}{\partial \xi_1} + U_7 \cos c\alpha_0 \cdot \frac{\partial}{\partial \xi} \quad (10.6)$$

$$\mathcal{D}_7 \, Q_i = 0 \qquad i = 1, \ldots, 4 \quad (10.7)$$

$$\left.\begin{array}{ll} \mathcal{D}_1 \, Q_1 = Q_2 & \mathcal{D}_1 \, Q_3 = c . Q_4 \\ \mathcal{D}_1 \, Q_2 = -Q_1 & \mathcal{D}_1 \, Q_4 = -c . Q_3 \end{array}\right\} \quad (10.8)$$

$$\left.\begin{array}{ll} \mathcal{D}_{16} \, Q_1 = 0 & \mathcal{D}_{16} \, Q_3 = 0 \\ \mathcal{D}_{16} \, Q_2 = \sin \alpha_0 & \mathcal{D}_{16} \, Q_4 = \sin c\alpha_0 \end{array}\right\} \quad (10.9)$$

$$\mathcal{Y}_3 W_1 = -\tfrac{1}{2} \cdot c . T_3^2 \cdot \cos c\alpha_0 \quad (10.10)$$

$$\mathcal{Y}_3 W_2 = - c . U_7 . \sin c\alpha_0 \quad (10.11)$$

$$\mathcal{Y}_3 W_3 = 0 \quad (10.12)$$

$$\mathcal{Y}_3 W_4 = \tfrac{1}{2} T_3^2 . c . \sin c\alpha_0 \quad (10.13)$$

$$\mathcal{Y}_3 W_5 = -\tfrac{1}{2} (T_3 . c)^2 \cdot \cos c\alpha_0 - T_3 . U_2 . c . \sin c\alpha_0 \quad (10.14)$$

$$\mathcal{Y}_3 W_6 = 2c . T_3^{-1} (U_7 . \xi_1 - U_2 . \xi_2) \sin c\alpha_0 - c^2 . \xi_2 . \cos c\alpha_0 \quad (10.15)$$

$$\mathcal{Y}_3 W_7 = - W_4 \cdot \cos c\alpha_0 \quad (10.16)$$

$$\mathcal{Y}_3 W_8 = - W_5 . \cos c\alpha_0 \quad (10.17)$$

$$\mathcal{Y}_3 W_9 = - W_6 . \cos c\alpha_0 \quad (10.18)$$

Define

$$\left.\begin{aligned}
\mathcal{D}_{17} &= \mathcal{D}_7 \mathcal{D}_{16} - \mathcal{D}_{16} \mathcal{D}_7 \\
\mathcal{D}_{18} &= \mathcal{D}_7 \mathcal{D}_{17} - \mathcal{D}_{17} \mathcal{D}_7 \\
\mathcal{D}_{19} &= \mathcal{D}_7 \mathcal{D}_{18} - \mathcal{D}_{18} \mathcal{D}_7
\end{aligned}\right\} \qquad (10.19)$$

$$\left.\begin{aligned}
\mathcal{D}_{17} &= w_{10} \cdot \frac{\partial}{\partial \zeta_1} + w_{11} \frac{\partial}{\partial \zeta_2} + w_{12} \frac{\partial}{\partial \zeta_3} \\
\mathcal{D}_{18} &= w_{13} \cdot \frac{\partial}{\partial \zeta_1} + w_{14} \cdot \frac{\partial}{\partial \zeta_2} + w_{15} \cdot \frac{\partial}{\partial \zeta_3} \\
\mathcal{D}_{19} &= w_{16} \cdot \frac{\partial}{\partial \zeta_1} + w_{17} \cdot \frac{\partial}{\partial \zeta_2} + w_{18} \frac{\partial}{\partial \zeta_3}
\end{aligned}\right\} \qquad (10.20)$$

Exactly as in the previous section one may show that

$$c^2 \cdot \mathcal{D}_{16} + \mathcal{D}_{18} = 0$$

and obtain a contradiction. (This operator acting on Q_2 is not zero.)

So it has been shown that there is no B.T. of the form (1.6) for (1.4), for $N < 5$. It seems, to me, very strange that one must work so hard to prove this. It seems to suggest that there are B.T. but that thecorrect u' dependence has not been chosen. Or perhaps one really does need to include integrals or explicit independent variable dependence or extra dependent variables. I still find it most amazing that it is so difficult to find B.T. It seems to me that the easiest way to proceed is to use the results of section 3 of chapter 3 in an expression which does not specify what u' satisfies.

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