Regge singularities in multiparticle processes

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REGGE SINGULARITIES IN MULTIPARTICLE PROCESSES

by

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Preface

The work presented in this thesis was carried out in the Department of Mathematics of the University of Durham between October 1974 and August 1977 under the supervision of Dr. W.J. Zakrzewski.

The material in this thesis has not been submitted previously for any degree in this or any other university. No claim of originality has been made for Chapter One and first section of Chapter Two; the remainder is claimed to be original except where otherwise indicated. Some parts of the chapters two, three and four are based on a paper by the author in collaboration with W.J. Zakrzewski and the remaining parts of the above chapters and the whole of Chapter Six are based on two papers by the author in collaboration with W.J. Zakrzewski and C. Barratt. Chapters Five and Seven contain some unpublished work by the author.

The author wishes to express his sincere thanks to Dr.W.J. Zakrzewski for his continuous help, guidance and patience throughout the stages of the present work, and also for critically reading the manuscript and for correcting the English. He furthermore wishes to extend his thanks and gratitude to Professor E.J. Squires and to the Lecturers of the Departments of Mathematics and Theoretical Physics from whom he benefited a great deal by attending their lectures during the last three years.

Content

Introduction

The concept of duality in strong interactions was first introduced in 1967 by Dolen, Horn and Schmid[1]. They studied the constraints imposed by analyticity and crossing symmetry using the techniques of finite energy sum rules and found that the direct channel resonances and crossed-channel Regge poles provided, in an average sense, equivalent descriptions of the same phenomena.

Soon after that Veneziano constructed a model[2] for the four-particle amplitude which satisfied most of the requirements of duality. This model exhibits an infinite number of poles in the s-channel or t-channel in such a way that duality is satisfied. (The sum over the poles in s-channel is equivalent to the sum over the poles in t-channel). In addition it was found that the model could be generalized to multiparticle amplitudes[3-5]. The resultant amplitudes exhibits factorization and possess similar properties.

Recently, Hoyer, Törnqvist and Webber[6], rekindled interest in the multi-Regge limits of scattering amplitudes with their observation that the conventional multi-Regge limit of the six-particle amplitude of the Veneziano model shows, for some values of momentum transfer, Regge behaviour corresponding to the exchange of a trajectory which does not couple to the two particle states and thus does not appear in two body processes. They were led to this discovery by the observation that the multi-Regge limit of the six-particle amplitude corresponding to the exchange of ordinary trajectories[7],

$$B_6^{\alpha} = D(\alpha_a, \tau_a) V(\alpha_a, \alpha_t; K_1) D(\alpha_t, \tau_t) V(\alpha_t, \alpha_b; K_2) D(\alpha_b, \tau_b)$$

has poles at nonsense wrong signature points resulting from two singular vertices and only one vanishing propagator. As the full amplitude has no singularities at these points eq.(1) cannot represent the asymptotic behaviour of the amplitude near these poles. Hoyer et al observed that (1) does not represent the correct asymptotic behaviour for $\alpha_t \leqslant -1$ and that there is an additional contribution corresponding to the exchange of a new trajectory $\beta_t = (\alpha_t - 1)/2$ which dominates for $\alpha_t \leqslant -1$. The contribution due to this new trajectory and its daughters has compensating poles at nonsense wrong signature points, making the full amplitude regular there. Recently Hoyer[8] analysed the behaviour of the eight point function in a helicity limit and found that this amplitude exhibited behaviour corresponding to the exchange of not only the ordinary trajectory and of the β trajectory but also of a further trajectory $\gamma = \frac{\alpha}{3} - 1$. This led him to speculate about a possible existence of a whole family of trajectories $\alpha_K = \frac{\alpha}{K} - \frac{K-1}{2}$, k = 1,2,3, whose k = 1,2,3 trajectories correspond to the α, β, γ trajectories mentioned above. He briefly discussed their properties of which probably the most important is the vanishing couplig of the α_K to states with not more than k particles.

The original discussion of Hoyer et al., which exhibited the β trajectory exchange, involved the double helicity limit of the six point function. Recently a discussion was given[9] of the behaviour of the six particle amplitude in the multi-Regge limit exhibiting the contribution of the β trajectory. In this paper it was also argued that the existence of new trajectories is not expected to be confined only to the Veneziano model and that most models are expected to exhibit unconventional behaviour for some values of the transferred momenta.

In the first chapter we review the Veneziano model, especially the parts which will be used in the next chapters. In chapter two, after a brief discussion of the six point amplitude, we determine the full contribution of the 2β exchange and one β exchange in the seven point function, and we obtain the $\beta-\alpha$-particle vertex. In addition we find that in spite of the equivalence of signaturization and twisting on the α trajectory level, they are different on the β trajectory level.

In chapter three we determine the full contribution of the linear 3β and $2\beta\gamma$ exchange in the eight point function. (In order to derive these results we use a mixed prescription for the analytical continuation; some variables are continued to their correct values straight away, some others are continued to different values first and are continued to their correct values after the asymptotic limit has been taken. We have not proved the validity of this prescription in general but we believe that it gives a correct final result and in the appendix we show this explicitly for the six point function where the correct analytical continuation is discussed in detail and shown to agree with the results of the mixed prescription).

We consider the factorization properties of the new trajectories by comparison of the expressions obtained for the seven point function and the eight point function. We find that due to the difference between signaturization and twisting the complete contribution of the 3β exchange to the eight point function does not factorize into a product of two complete 2β trajectory - one particle vertices, determined from the seven point function thus showing that simple factorization properties of the trajectory are not shared by its siblings. However, the factorization is violated only by a mismatch of some phase factors and the patterns of its breakdown suggests an underlying structure which we have so far been unable to unravel.

In chapter four we investigate the higher point functions and show that the mechanism which allows to exhibit the contributions of the β and γ trajectories to the six and eight point functions respectively when applied to higher point functions exhibits contributions coming from the exchange of further trajectories from the family suggested by Hoyer. Also we establish their behaviour under the operation of twisting.

In chapter five we derive the structure of 3β and of the $2\beta\alpha$- vertices, where we show that the 3β vertex vanishes whereas the structure of the $2\beta\alpha$-vertex is similar to the 3α vertex.

In chapter six we discuss possible phenomenological consequences of these new trajectories. We look at these trajectories in the conventional dual model (CDM) and the Neveu-Schwarz model (NSM), and show that in practice only the β_{π} of (NSM) can be of any phenomenological importance. We discuss the assumptions made in order to substantiate this claim and present proposals for the phenomenological search for the effects due to these trajectories.

In the last chapter we leave aside the Veneziano model and investigate the mutual relation between Regge poles and Regge cut and their contribution to the four point amplitude. It has been shown that if the Pomeron singularity is linear and has an intercept at one, the Regge cut corresponding to the exchange of two such poles gives a negative contribution to the total cross section (absorptive cut"). In Regge field theory language this fact corresponds to the imaginary triple Pomeron coupling vertex. In this thesis we investigate to what extent these results depend on the linear trajectory of the Pomeron. To study this problem we choose to consider the square root trajectory first suggested by Schwarz[9-a]. We find that the results are basically unchanged although on the way to this result we encounter some technical problems.

Chapter One - The Veneziano Model

In 1968 Veneziano constructed a model for the 2-2 scattering amplitude[2]. The model is based on a narrow resonance approximation and a linear form of Regge trajectories, and it possesses several desirable features, like crossing symmetry, resonance poles, and Regge behaviour.

For the simplest case of 4 neutral bosons with $J^P = 0^+$, the non-diffractive part of the amplitude is described by (see Fig.1)

$$A_4(s,t,u) = \beta\,[B_4(s,t) + B_4(u,t) + B_4(s,u)\,], \qquad (1\text{-}1)$$

where

$$B_4(s,t) \equiv B(-\alpha(s),-\alpha(t)) = \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \qquad (1\text{-}2)$$

with $\alpha(s) = \alpha' s + \alpha(0)$, etc. and has poles at $\alpha(s), \alpha(t) = n$, $n = \text{integer} \geqslant 0$. The gamma function in the denominator prevents double poles which occur at $\alpha(s)$ and $\alpha(t)$ being both integer. It can be shown that eq(1-2) and hence each of the three terms in (1-1) can be completely represented by a sum of narrow-resonance poles in either of two channels with residua that are polynomials of the appropriate order in the other channel variable.

$$B_4(s,t) = B(-\alpha(s),-\alpha(t)) = \sum_{n=0}^{\infty} \frac{(\alpha(t)+1)_n}{n!}\,\frac{1}{n-\alpha(s)}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha(s)+1)_n}{n!}\,\frac{1}{n-\alpha(t)}\,, \qquad (1\text{-}3)$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = (a+n-1)(a+n-2)\cdots(a+1)a.$$

In order to find the asymptotic behaviour of A(s,t,U) as $\alpha(S)\to\infty$ we write eq.(1-2) as

$$B_4(s,t)=\int_0^1 dx\, X^{-\alpha(t)-1}(1-x)^{-\alpha(s)-1}, \qquad (1\text{-}4)$$

and from now on for simplicity we assume unit slope for the trajectories i.e. $\alpha'=1$.

Since (1-4) is defined for negative values of α, we first take $S\to-\infty$, and keep t fixed, then continue in s. In this limit the region $X\approx0$ in the integral dominates. To exhibit this we make the following change of variable

$$X=y/(-s)$$

and use $\left(1+\frac{X}{S}\right)^{-S}\underset{S\to\infty}{\sim} e^{-x}$ to obtain

$$B_4(s,t)=(-s)^{\alpha(t)}\int_0^\infty dy\, y^{-\alpha(t)-1}e^{-y}, \qquad (1\text{-}5)$$

i.e. $B_4(s,t)=(-s)^{\alpha(t)}\Gamma(-\alpha(t))$.

Similarly, $B_4(u,t)$ gives

$$B_4(u,t)\underset{S\to\infty}{\sim}\Gamma(-\alpha(t))\,S^{\alpha(t)}, \qquad (1\text{-}6)$$

where to obtain (1-6) we have made use of $S+t+u=4m^2=$ fixed.

It is possible to show that the third term in (1-1) gives an exponentially vanishing contribution to the scattering amplitude provided the region in which the asymptotic analysis applies is the entire complex plane except for small wedges about the positive and negative real axis of an arbitrary small opening angle.

To find the total contribution to the scattering amplitude we add the two first terms in (1-1). In the full amplitude the contribution of the second term appears with a factor τ relative to the contribution of the first term (see Fig.2). (τ is the twisting operator which is equivalent to the signature factor on the α trajectory level, but as we shall see in the next chapters this is not the case for all siblings).

So

$$A_4(s,t) = \Gamma(-\alpha(t))\left[\tau + e^{-i\pi\alpha(t)}\right] s^{\alpha(t)} , \qquad (1\text{-}7)$$

where we have assumed $\beta = 1$ and have continued in s to its positive values.

Next we consider the five point amplitude[3] of the dual model. We write the Bardakci-Ruegg representation for this amplitude[3-5]:

$$B_5 = \int_0^1 dx_1 dx_2 \; x_1^{-\alpha_1-1} \; x_2^{-\alpha_2-1} \; (1-x_1)^{-\alpha(s_1)-1} \; (1-x_2)^{-\alpha(s_2)-1} \; (1-x_1x_2)^{-\alpha(s_{15})+\alpha(s_1)+\alpha(s_2)} \qquad (1\text{-}8)$$

with $\alpha(s_i) = s_i + \alpha(0)$, $\alpha_i \equiv \alpha(t_i) = t_i + \alpha(0)$.

The independent variables are (see Fig.3)

$t_1 = (P_1 - P_2)^2$ $\qquad$ $s_2 = s_{34} = (P_3 + P_4)^2$

$t_2 = (P_4 - P_5)^2$ $\qquad$ $s_{15} = s_1 s_2 K = (P_1 + P_5)^2$

$s_1 = s_{23} = (P_2 + P_3)^2$.

To obtain the double Regge limit of five point function, we first let $S_1, S_2, S_{15} \longrightarrow -\infty$ in such a way that $\frac{s_{15}}{s_1 s_2} = K =$ fixed and also keep t_1, t_2 fixed and then continue S's to their positive values. We make the usual change of variables

$$x_1 = \frac{y_1}{-s_1} \quad , \quad x_2 = \frac{y_2}{-s_2}$$

and use

$$\left(1+\frac{y}{s}\right)^{-s} \underset{s\to\infty}{\sim} e^{-y}$$

to obtain

$$B_5 \sim (-s_1)^{\alpha_1} (-s_2)^{\alpha_2} \int_0^\infty dy_1 \int_0^\infty dy_2 \; y_1^{-\alpha_1-1} \; y_2^{-\alpha_2-1}$$
$$\exp\{-y_1 - y_2 + y_1 y_2 K\} \,. \quad (1\text{-}9)$$

Next we apply the identity

$$e^{x} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda \, \Gamma(-\lambda) \, (-x)^{\lambda} \quad (1\text{-}10)$$

to the last term in the exponential in (1-9) and perform the y_1, y_2 and λ integrations to obtain [7,10]

$$B_5 \sim (-s_1)^{\alpha_1} (-s_2)^{\alpha_2} \Big\{ (-K)^{\alpha_1} \Gamma(-\alpha_2+\alpha_1) \Gamma(-\alpha_1) \, {}_1F_1\left(-\alpha_1, \alpha_2-\alpha_1+1; -\frac{1}{K_1}\right)$$
$$+ \alpha_1 \leftrightarrow \alpha_2 \Big\} \,. \quad (1\text{-}11)$$

To find the full contribution we add all relevant diagrams with relevant twisting factors and end up with (see Fig.4)

$$A_5(s_1, s_2; t_1, t_2; K_1) = \left\{ \Gamma(-\alpha_1)\left[\tau_1 + e^{-i\pi\alpha_1}\right] s_1^{\alpha_1} \right\} \left\{ \Gamma(-\alpha_2)\left[\tau_2 + e^{-i\pi\alpha_2}\right] s_2^{\alpha_2} \right\}$$

$$\left\{ \frac{\Gamma(\alpha_1-\alpha_2)\left[\tau_1\tau_2 + e^{-i\pi(\alpha_2-\alpha_1)}\right] K_1^{\alpha_1} \, {}_1F_1\left(-\alpha_1, \alpha_2-\alpha_1+1; -\frac{1}{K}\right.}{\Gamma(-\alpha_2)\left[\tau_2 + e^{-i\pi\alpha_2}\right]} \right.$$

$$\left. + \begin{matrix} \alpha_1 \longleftrightarrow \alpha_2 \\ \tau_1 \longleftrightarrow \tau_2 \end{matrix} \right\} \,. \quad (1\text{-}12)$$

Since Regge residues in Dual Resonance Model factorize[11], we can use (1-7) and (1-12) to define an α Reggeon Propagator and also a Reggeon-Reggeon-particle vertex (we assumed that the particle-particle-Reggeon vertex is one) as

$$D(\alpha_1,\tau_1)=\Gamma(-\alpha_1)\left[\tau_1+e^{-i\pi\alpha_1}\right]S^{\alpha_1} \qquad (1\text{-}13a)$$

$$V(\alpha_1,\alpha_2;K_1)=\left\{\frac{\Gamma(\alpha_1-\alpha_2)\left[\tau_1\tau_2+e^{-i\pi(\alpha_2-\alpha_1)}\right]K_1^{\alpha_1}\,{}_1F_1\left(-\alpha_1,\alpha_2-\alpha_1+1;-\frac{1}{K_1}\right)}{\Gamma(-\alpha_2)\left[\tau_2+e^{-i\pi\alpha_2}\right]}\right.$$

$$\left.+\begin{matrix}\alpha_1\longleftrightarrow\alpha_2\\ \tau_1\longleftrightarrow\tau_2\end{matrix}\right\}, \qquad (1\text{-}13)$$

respectively.

Eq. (1-13b) has nonsense wrong - signature zeros at

$$\alpha_1=-2n+\tfrac{1}{2}(1+\tau_1)\ ,\quad n=1,2,\cdots\quad ,\ \tau_1=+1,-1,\cdots \qquad (1\text{-}14)$$

and the vertex function in (1-13b) has poles at the same points. However in the double Regge limit of the five point amplitude

$$A_5=D(\alpha_1,\tau_1)V(\alpha_1,\alpha_2;K)\,D(\alpha_2,\tau_2) \qquad (1\text{-}15)$$

the poles of the vertex function are cancelled by the zeros of the propagator and the final result is finite. Thus we do not expect nonsense wrong - signature zeros in the five point amplitude. In the expression for the six point function in the multi-Regge limit (Fig.5)

$$A_6=D(\alpha_1,\tau_1)V(\alpha_1,\alpha_2;K_1)\,D(\alpha_2,\tau_2)\,V(\alpha_2,\alpha_3;K_2)\,D(\alpha_3,\tau_3) \qquad (1\text{-}16)$$

there are two singular vertices and only one vanishing propagator at $\alpha_2=-1,-2,\cdots$ with $\tau_2=+1,-1,\cdots$ respectively and we see that the amplitude is singular at these points. As we know that the exact amplitude is regular at these points, eq.(1-16) cannot be the correct asymptotic behaviour of A_6 at $\alpha_2 \approx -1$. However as it has been shown by Hoyer, Törnqvist and Webber[6] there is another contribution from the exchange of a new trajectory which is related to the α trajectory by $\beta=\frac{\alpha_2-1}{2}$, which develops a pole at $\alpha_2=-1$ in such a way that the residua of these poles cancel. It has also been shown[6,9] that in the region $\alpha_2+1<0$ this new contribution dominates, and that for $\alpha_2+1 \approx 0$ the contributions of both α and β are important. Hoyer et al. also have shown that the pole at $\alpha_2=-2$ which appears for $\tau=-1$ is cancelled by a daughter of the new trajectory which is parallel to β and spaced by $\frac{1}{2}$ unit. In the next chapter we discuss some properties of this new trajectory.

Chapter Two. The β Trajectory

2-1. The six point function and the β trajectory:

The contribution of the β trajectory exchange to the multi-Regge limit of the Veneziano model six point function has been discussed in detail in ref.(6,9,12). The two six point diagrams that contribute to the β exchange are shown in Fig.(6-a and b). Their contribution was given in ref.6, and it was also evaluated by a slightly different method in ref.9, and 12. Here we follow the method of ref.12 to derive the contribution of both these diagrams.

To determine the contribution of Fig.(6-a) we consider the Veneziano amplitude corresponding to diagram (7-a), and take the Regge limit corresponding to Fig.(6-a). The amplitude of Fig. (7-a) is defined for

$$\begin{array}{ll} S_{13} \sim -S_1 < 0 & S_{134} \sim -S_1 S_2 K_1 < 0 \\ S_{46} \sim -S_3 < 0 & S_{346} \sim -S_2 S_3 K_2 < 0 \\ S_{34} \sim +S_2 < 0 & S_{25} \sim +S_1 S_2 S_3 K_1 K_2 \phi < 0 \end{array} \qquad (2\text{-}1)$$

and so it requires continuation in S_2, K_1, K_2 *and* ϕ

$$\begin{array}{ll} S_2 = -S_2\, e^{-i\pi} & \\ K_1 = -K_1\, e^{i\pi} & \phi = e^{-2\pi i} \\ K_2 = -K_2\, e^{i\pi} & . \end{array} \qquad (2\text{-}2)$$

We choose to continue K_1, K_2, ϕ first, keeping S_2 at its negative values and to continue S_2 to positive values only after the asymptotic limit has been taken. In the appendix we justify this prescription. There we show how the rotation of the integration contours allows the continuation of S_2 to positive values before the asymptotic limit is taken and that both approaches lead to the same final result.

Thus we take the expression for the six point function shown in Fig. (7-a)

$$B_6 \cong \tau_1 \tau_3 \int_0^1 dx_1 \, dx_2 \, dx_3 \quad x_1^{-\alpha_1 - 1} \quad x_2^{-\alpha_2 - 1} \quad x_3^{-\alpha_3 - 1} \left(\frac{1 - x_1}{1 - x_1 x_2} \right)^{S_1}$$

$$\left(\frac{1 - x_3}{1 - x_2 x_3} \right)^{S_3} \left[\frac{(1 - x_2)(1 - x_1 x_2 x_3)}{(1 - x_1 x_2)(1 - x_2 x_3)} \right]^{-S_2} \cdot \left(\frac{1 - x_1 x_2}{1 - x_1 x_2 x_3} \right)^{S_1 S_2 K_1} \cdot$$

$$\left(\frac{1 - x_2 x_3}{1 - x_1 x_2 x_3} \right)^{S_2 S_3 K_2} \left(1 - x_1 x_2 x_3 \right)^{-S_1 S_2 S_3 K_1 K_2 \phi} \tag{2-3}$$

in which $\alpha_i = \alpha(t_i)$ and where for simplicity we have kept only the asymptotic parts of all trajectories in channels whose variables become large in the multi-Regge limit. The inclusion of non-asymptotic terms, especially in relation (2-1) complicates the discussion but does not alter the final result.

As the dominant asymptotic behaviour comes from $x_i \approx 0$, we expand the last six factors in (2-3) using

$$(1 - x)^a = \exp\left\{ \log(1 - x) \cdot a \right\} = \exp\left\{ -ax - \frac{ax^2}{2} \cdots \right\} .$$

We keep as few terms in these expansions as required by further cancellations. Thus in this case we keep only the two terms for the factor involving S_2, and only the leading terms for all other factors.

In this way we write

$$B_6 \sim \tau_1 \tau_3 \int_0^1 dx_1\, dx_2\, dx_3 \quad X_1^{-\alpha_1-1} \quad X_2^{-\alpha_2-1} \quad X_3^{-\alpha_3-1}$$

$$\exp\Big\{ -X_1 S_1 (1-X_2) - X_3 S_3 (1-X_2) + X_2 S_2 + X_2^2 \frac{S_2}{2} - X_1 X_2 S_1 S_2 K_1$$

$$- X_2 X_3 S_2 S_3 K_2 + X_1 X_2 X_3 S_1 S_2 S_3 K_1 K_2 \phi \Big\} \,.$$

(2-4)

We introduce a helicity-like integral

$$e^{+X_1 X_2 X_3 S_1 S_2 S_3 K_1 K_2 \phi} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} d\lambda\, \Gamma(-\lambda) \left(-X_1 X_2 X_3 S_1 S_2 S_3 K_1 K_2 \phi\right)^{\lambda} ,$$

change the variables $X_1 = \frac{y_1}{S_1}$, $X_3 = \frac{y_3}{S_3}$ and let $S_1, S_2 \to \infty$ obtaining

$$B_6 \sim \tau_1 \tau_3\, S_1^{\alpha_1} S_3^{\alpha_3} \frac{1}{2\pi i} \int_0^\infty dy_1 \int_0^\infty dy_3 \int_0^1 dX_2 \int_{\gamma - i\infty}^{\gamma + i\infty} d\lambda\, \Gamma(-\lambda)\, X_2^{-\alpha_2-1}\, y_1^{\lambda-\alpha_1-1}$$

$$y_3^{\lambda-\alpha_3-1} \exp\Big\{ -y_1 (1 - X_2 + X_2 S_2 K_1) - y_3 (1 - X_2 + X_2 S_2 K_2) + X_2 S_2 + X_2^2 \frac{S_2}{2} \Big\} (-X_2 S_2 K_1 K_2$$

We perform the y_1 and y_3 integrations obtaining

$$B_6 \sim S_1^{\alpha_1} S_3^{\alpha_3} \frac{\tau_1 \tau_3}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} d\lambda\, \Gamma(-\lambda) \int_0^1 dX_2\, X_2^{-\alpha_2-1} \exp\Big\{ X_2 S_2 + \frac{X_2^2 S_2}{2} \Big\}$$

$$\Gamma(\lambda - \alpha_1) \Gamma(\lambda - \alpha_3) (-X_2 S_2 K_1 K_2 \phi)^{\lambda} (1 - X_2 + X_2 S_2 K_1)^{\alpha_1 - \lambda} (1 - X_2 + X_2 S_2 K_2)^{\alpha_3 - \lambda} \,.$$

Next we perform the λ integration and obtain

$$B_6 \sim \tau_1 \tau_3\, S_1^{\alpha_1} S_3^{\alpha_3} \int_0^1 dX_2\, X_2^{-\alpha_2-1} \exp\Big\{ X_2 S_2 + \frac{X_2^2}{2} S_2 \Big\} (1 - X_2 + X_2 S_2 K_1)^{\alpha_1}$$

$$(1 - X_2 + X_2 S_2 K_2)^{\alpha_3}\, \Gamma(-\alpha_1) \Gamma(-\alpha_3)\, Z^{-\alpha_1}\, \Psi(-\alpha_1, \alpha_3 - \alpha_1 + 1; Z)$$

(2-5)

where $Z = \frac{-(1-x_2+x_2 s_2 K_1)(1-x_2+x_2 s_2 K_2)}{x_2 s_2 K_1 K_2 \phi}$ and $\psi(a,c;x)$ is the Tricomi function[13]. Using the relation of $\psi(a,c;x)$ to the confluent hyper-geometric function*

$$\psi(a,c;x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\, {}_1F_1(a,c;x) + \frac{\Gamma(c-1)}{\Gamma(a)} x^{1-c}\, {}_1F_1(a-c+1;2-c;x) \tag{2-6}$$

we can rewrite (2-5) as

$$B_6 \sim \tau_1 \tau_3 S_1^{\alpha_1} S_3^{\alpha_3} \int_0^1 dx_2\, x_2^{-\alpha_2-1} \exp\left\{x_2 s_2 + \frac{x_2^2 s_2}{2}\right\} (1-x_2+x_2 s_2 K_1)^{\alpha_1}$$

$$(1-x_2+x_2 s_2 K_2)^{\alpha_3} \left\{\Gamma(-\alpha_1)\Gamma(\alpha_1-\alpha_3)\, Z^{-\alpha_1}\, {}_1F_1(-\alpha_1;\alpha_3-\alpha_1+1;Z) + \alpha_1 \leftrightarrow \alpha_3\right\} \tag{2-7}$$

As the dominant asymptotic behaviour of the integral comes from $x_2 \gtrsim 0$ we change the variable of integration x_2 to $y_2 = x_2(-s_2)^P$, where $0 < P \leq 1$ and let $-s_2 \longrightarrow \infty$.

The expression (2-7) is not very suitable to find the contribution for P=1. In this case it is more convenient to start from eq.(2-4) and to introduce three helicity-like integrals for the last three factors in the exponential, then change the variables.

$$x_1 = \frac{y_1}{s_1}, \quad x_2 = \frac{y_2}{-s_2}, \quad x_3 = \frac{y_3}{s_3}$$

and let $s_1, (-s_2), s_3 \longrightarrow \infty$. After performing the y_1, y_2 and y_3 integrations we obtain[7,10]

$$B_6 \sim \tau_1 \tau_3\, S_1^{\alpha_1} (-S_2)^{\alpha_2}\, S_3^{\alpha_3} \left(\frac{1}{2\pi i}\right)^3 \int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda_1 d\lambda_2 d\lambda_3\, \Gamma(-\lambda_1)\Gamma(-\lambda_2)\Gamma(-\lambda_3)\, \phi^{\lambda_3}$$

$$\Gamma(\lambda_1+\lambda_3-\alpha_1)\Gamma(\lambda_2+\lambda_3-\alpha_3)\Gamma(\lambda_1+\lambda_2+\lambda_3-\alpha_2)(-K_1)^{\lambda_1+\lambda_3}(-K_2)^{\lambda_2+\lambda_3}.$$

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*Most of the special functions used in this thesis are in ref.13.

Next we change $\lambda_1 \to \lambda_1 - \lambda_3$, $\lambda_2 \to \lambda_2 - \lambda_3$ obtaining

$$B_6 \sim \frac{\tau_1 \tau_3}{(2\pi i)^3} s_1^{\alpha_1} s_3^{\alpha_3} (-s_2)^{\alpha_2} \int_{\gamma - i\infty}^{\gamma + i\infty} d\lambda_1 d\lambda_2 d\lambda_3 \, \Gamma(-\lambda_3) \Gamma(\lambda_1 - \alpha_1) \Gamma(\lambda_2 - \alpha_3)$$

$$\Gamma(\lambda_1 + \lambda_2 - \lambda_3 - \alpha_2) \Gamma(-\lambda_1 + \lambda_3) \Gamma(-\lambda_2 + \lambda_3) (-K_1)^{\lambda_1} (-K_2)^{\lambda_2} \phi^{\lambda_3} .$$

(2-8)

We perform the λ_3 integration and obtain

$$B_6 \sim \tau_1 \tau_3 s_1^{\alpha_1} (-s_2)^{\alpha_2} s_3^{\alpha_3} \left(\frac{1}{2\pi i}\right)^2 \frac{1}{\Gamma(-\alpha_2)} \int_{\gamma - i\infty}^{\gamma + i\infty} d\lambda_1 d\lambda_2 \, \Gamma(\lambda_1 - \alpha_1) \Gamma(\lambda_2 - \alpha_3)$$

$$\Gamma(-\lambda_1) \Gamma(-\lambda_2) \Gamma(\lambda_1 - \alpha_2) \Gamma(\lambda_2 - \alpha_2) (-K_1)^{\lambda_1} (-K_2)^{\lambda_2}$$

$${}_2F_1(-\lambda_1 , -\lambda_2 ; -\alpha_2 ; 1 - \phi) .$$

Next we use the formula for the analytical continuation of the hyper geometric function

$${}_2F_1(a , b ; c ; 1 - \phi) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} \phi^{-a} \, {}_2F_1\left(a , c-b ; a-b+1 ; \frac{1}{\phi}\right) + (a \leftrightarrow b)$$

(2-9a)

and also

$${}_2F_1(a , c-b ; a-b+1 ; 1) = \frac{\Gamma(a-b+1)\Gamma(-c+1)}{\Gamma(-b+1)\Gamma(-c+a+1)} \qquad (2\text{-}9b)$$

to obtain finally:

$$B_6 \sim \tau_1 \tau_3 s_1^{\alpha_1} (-s_2)^{\alpha_2} s_3^{\alpha_3} \frac{1}{\Gamma(-\alpha_2)} \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma - i\infty}^{\gamma + i\infty} d\lambda_1 d\lambda_2$$

$$\Gamma(-\lambda_1) \Gamma(\lambda_1 - \alpha_1) \Gamma(\lambda_1 - \alpha_2) (-K_1)^{\lambda_1}$$

$$\Gamma(-\lambda_2) \Gamma(\lambda_2 - \alpha_2) \Gamma(\lambda_2 - \alpha_3) (-K_2)^{\lambda_2}$$

$$\times\left[\frac{\sin\pi\lambda_2 \sin\pi(\lambda_1-\alpha_2)}{\sin\pi\alpha_2 \sin\pi(\lambda_1-\lambda_2)}\,\phi^{\lambda_1} + \frac{\sin\pi\lambda_1 \sin\pi(\lambda_2-\alpha_2)}{\sin\pi\alpha_2 \sin\pi(\lambda_2-\lambda_1)}\,\phi^{\lambda_2}\right] \qquad (2\text{-}10)$$

which is the most convenient form of exhibiting the contribution of the α trajectory exchange.

To investigate the case $0<p<1$ we start from eq. (2-7), and rewrite it as

$$B_6 \sim \tau_1\tau_3 s_1^{\alpha_1} s_3^{\alpha_3} (-s_2)^{p\alpha_2-1+p} \int_0^\infty dy_2\, y_2^{-\alpha_2-1)} \exp\left\{-y_2(-s_2)^{1-p} -\frac{y_2^2}{2}(-s_2)^{1-2p}\right\} A^{\alpha_1} B^{\alpha_3} \left\{\Gamma(-\alpha_1)\Gamma(\alpha_1-\alpha_3)\left(\frac{AB}{C\phi}\right)^{-\alpha_1} {}_1F_1\left(-\alpha_1, \alpha_3-\alpha_1+1; \frac{AB}{C\phi}\right) + \alpha_1 \longleftrightarrow \alpha_3\right\} \qquad (2\text{-}11)$$

where

$$A = 1 - y_2(-s_2)^{-p} - y_2(-s_2)^{1-p} K_1$$

$$B = 1 - y_2(-s_2)^{-p} - y_2(-s_2)^{1-p} K_2$$

$$C = y_2(-s_2)^{1-p} K_1 K_2 \quad .$$

To calculate the asymptotic behaviour of eq. (2-11) we first continue K_1, K_2, ϕ to their final values as given in (2-2) with $s_2 \longrightarrow -\infty$ and at the end continue s_2 to the positive values. Using ${}_1F_1(a,b;x) \xrightarrow[x\to\infty]{} \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b}$ in eq. (2-11) and keeping only the terms which are needed we are left with

$$B_6 \sim \tau_1\tau_3\, S_1^{\alpha_1} S_3^{\alpha_3} (-S_2)^{p\alpha_2-1+p} \int_0^\infty dy_2\, y_2^{-\alpha_2-2} \exp\left\{-\frac{y_2^2}{2}(-S_2)^{1-2p}\right\}$$

$$K_1^{\alpha_1} K_3^{\alpha_3} \left\{\phi^{\alpha_1}\Gamma(\alpha_1-\alpha_3)\Gamma(\alpha_3-\alpha_1+1) + \alpha_1 \longleftrightarrow \alpha_3\right\}$$

$$e^{-1/K_1-1/K_2}\; e^{+i\pi(\alpha_1+\alpha_3)} \; . \qquad (2\text{-}12)$$

For $\phi=1$ the term in the curly bracket vanishes. This is due to the cancellation between the two terms in (2-6). However for $\phi=e^{-2\pi i}$ which is the correct value for ϕ in this case, the two terms have different phases and the resultant expression is non zero

$$\left\{\phi^{\alpha_1}\Gamma(\alpha_1-\alpha_3)\Gamma(\alpha_3-\alpha_1+1) + \alpha_1 \longleftrightarrow \alpha_3\right\} = \begin{cases} 0 & \text{For } \phi=1 \\ -2\pi i\, e^{-i\pi(\alpha_1+\alpha_3)} & \text{For } \phi=e^{-2\pi i} . \end{cases}$$

The convergence of the y_2 integral requires $p=\frac{1}{2}$ showing that the effective power of S_2, which determines the form of the β trajectory is $\beta_2=\frac{\alpha_2-1}{2}$. Finally we perform the Y_2 integration in (2-12) and continue the result to positive values of S_2, obtaining:

$$B_6 \approx -\pi i\,\tau_1\tau_3 (S_1K_1)^{\alpha_1} \exp\left(-\frac{1}{K_1}\right) (S_3K_2)^{\alpha_3} \exp\left(-\frac{1}{K_2}\right) \Gamma(-1-\beta_2)$$

$$2^{-\beta_2-1}\; S_2^{\beta_2}\; e^{-i\pi\beta_2} . \qquad (2\text{-}13)$$

To derive the full contribution to the scattering amplitude we have to consider also the diagram shown in Fig.(6-b). Calculation of the expression corresponding to this diagram proceeds in the analogous way. We start with the expression corresponding to the diagram shown in Fig.(7b). Now the amplitude is defined for

$$\begin{array}{ll} S_{13} \sim -S_1 < 0 & S_{135} \sim -S_1S_2S_3K_1K_2\phi < 0 \\ S_{46} \sim -S_3 < 0 & S_{24} \sim +S_1S_2K_1 < 0 \\ S_{356} \sim -S_2 < 0 & S_{35} \sim S_2S_3K_2 < 0 \; . \end{array} \qquad (2\text{-}14)$$

and so requires continuation in K_1, K_2, ϕ

$$K_1 = - K_1 e^{-i\pi}$$
$$K_2 = - K_2 e^{-i\pi}$$
$$\phi = e^{2\pi i} \quad .$$

We start with

$$B_6' \sim \tau_1\tau_2\tau_3 \int_0^1 dx_1 dx_2 dx_3 \; x_1^{-\alpha_1-1} \; x_2^{-\alpha_2-1} \; x_3^{s_3} \left(\frac{1-x_1}{1-x_1x_2}\right)^{s_1}$$
$$\left(\frac{1-x_3}{1-x_2x_3}\right)^{-\alpha_3-1} \left(\frac{1-x_2x_3}{1-x_1x_2x_3}\right)^{s_2} \left(1-x_1x_2x_3\right)^{-s_1s_2K_1}$$
$$\left(\frac{(1-x_2)(1-x_1x_2x_3)}{(1-x_2x_3)(1-x_1x_2)}\right)^{-s_2s_3K_2} \left(\frac{1-x_1x_2}{1-x_1x_2x_3}\right)^{s_1s_2s_3K_1K_2\phi}$$

(2-15)

change the variable of integration x_3 to u_3 by $x_3 = \frac{1-u_3}{1-x_2u_3}$, and as before use $(1-x)^a = \exp\left\{-ax - \frac{ax^2}{2} \cdots\right\}$ for the last six brackets and keep only the terms which are necessary in further steps. In this way we obtain

$$B_6' \sim \tau_1\tau_2\tau_3 \int_0^1 dx_1 dx_2 du_3 \; x_1^{-\alpha_1-1} \; x_2^{-\alpha_2-1} \; u_3^{-\alpha_3-1}$$
$$\exp\left\{ -x_1 s_1 (1-x_2) - u_3 s_3 (1-x_2) - x_2 s_2 - \frac{x_2^2 s_2}{2} + x_1 x_2 s_1 s_2 K_1 \right.$$
$$\left. + x_2 u_3 s_2 s_3 K_2 - x_1 x_2 u_3 s_1 s_2 s_3 K_1 K_2 \phi (1-x_2) \right\} .$$

(2-16)

Next we write the last factor in the exponential as a helicity - like integral and proceed as before, obtaining

$$B_6' \sim \tau_1\tau_2\tau_3 \, s_1^{\alpha_1} s_3^{\alpha_3} \int_0^1 dx_2 \, x_2^{-\alpha_2-1} \exp\left\{-x_2 s_2 - \frac{x_2^2}{2} s_2\right\}$$

$$(1-x_2-x_2 s_2 K_1)^{\alpha_1} (1-x_2-x_2 s_2 K_2)^{\alpha_3} \left\{\Gamma(-\alpha_1)\Gamma(\alpha_1-\alpha_3) Z^{-\alpha_1}\right.$$

$$\left. {}_1F_1(-\alpha_1, \alpha_3-\alpha_1+1; Z) + \alpha_1 \longleftrightarrow \alpha_3\right\} . \tag{2-17}$$

Now

$$Z = \frac{(1-x_2-x_2 s_2 K_1)(1-x_2-x_2 s_2 K_2)}{x_2(1-x_2) s_2 K_1 K_2 \phi}$$

and we see that the difference between (2-17) and (2-7) lies only in the replacements:

$$\phi \longrightarrow \phi(1-x_2) \quad \text{and} \quad s_2 \longrightarrow -s_2 \, .$$

The α contribution comes from the change of variable $x_2 s_2 = y_2$. To exhibit this contribution we start from eq(2-16) and follow the same steps as in the previous case. The final result is exactly the same as (2-10) apart from the additional phase $e^{-i\pi\alpha_2}$ as expected.

To derive the contribution of the β trajectory we change the variable $x_2(-s_2)^p = y_2$, $0 < p < 1$, (i.e. $x_2 s_2 \longrightarrow \infty$ as $s_2 \to \infty$) and proceed in the way as before. The justification for the analytical continuations is given in the appendix. Note that

$$Z \longrightarrow -\frac{1}{K_1\phi} - \frac{1}{K_2\phi} + \frac{x_2 s_2}{(1-x_2)\phi} \longrightarrow \infty \quad \text{as} \quad s_2 \to \infty \quad .$$

To determine the asymptotic behaviour of (2-17) we first continue K_1, K_2, ϕ, to their correct values, keeping S_2 large and positive. This allows us to take the asymptotic behaviour of ${}_1F_1$ functions:

$$ {}_1F_1(-\alpha_1, \alpha_3-\alpha_1+1; Z) \xrightarrow[Z\to\infty]{} \frac{\Gamma(\alpha_3-\alpha_1+1)}{\Gamma(-\alpha_1)} (X_2 S_2)^{-1-\alpha_3} \exp\left\{-\frac{1}{K_1}-\frac{1}{K_2}-\frac{X_2S_2}{1-X_2}\right\}. $$

Then we temporarily continue the resultant expression in (2-17) to $S_2<0$ find its asymptotic behaviour as $S_2 \longrightarrow -\infty$, and continue the result back to $S_2 \longrightarrow \infty$ obtaining:

$$ B_6' \sim \tau_1\tau_2\tau_3 S_1^{\alpha_1} S_3^{\alpha_3} K_1^{\alpha_1} K_2^{\alpha_3} \exp\{-1/K_1 - 1/K_2\} \int_0^1 dX_2 X_2^{-\alpha_2-1} \exp\left\{-X_2S_2 - \frac{X_2^2}{2}S_2 + \frac{X_2S_2}{1-X_2}\right\} \left\{\Gamma(\alpha_1-\alpha_3)\Gamma(\alpha_3-\alpha_1+1)\frac{\phi^{\alpha_1}}{S_2} e^{-i\pi(\alpha_1+\alpha_3)} + \alpha_1 \leftrightarrow \alpha_3\right\}. $$

(2-18)

or as $\phi = e^{+2\pi i}$,

$$ B_6' \approx \tau_1\tau_2\tau_3 S_1^{\alpha_1} S_3^{\alpha_3} K_1^{\alpha_1} K_2^{\alpha_3} \exp\left\{-\frac{1}{K_1}-\frac{1}{K_2}\right\} \cdot \frac{2\pi i}{S_2} \int_0^1 dX_2 X_2^{-\alpha_2-1} \exp\left\{\frac{X_2^2}{2}S_2\right\} $$

$$ = -\pi i \tau_1\tau_2\tau_3 (S_1K_1)^{\alpha_1} e^{-\frac{1}{K_1}} (S_3K_2)^{\alpha_3} e^{-\frac{1}{K_2}} \Gamma(-1-\beta_2) 2^{-\beta_2-1} S_2^{\beta_2} e^{-i\pi\beta_2}. $$

(2-19)

In addition to the above diagrams there are six other diagrams shown in Fig.(8), which contribute to the six point amplitude in the multi-Regge limit. They all correspond to $\phi=1$. Their contributions to the α_2 trajectory apart from the phase factors are exactly the same, and are such that the complete contribution corresponds to the fully signaturized expression

$$ {}^{\prime}A_6 = D(\alpha_1,\tau_1)V(\alpha_1,\alpha_2;K_1)\,D(\alpha_2,\tau_2)V(\alpha_2,\alpha_3;K_2)\,D(\alpha_3,\tau_3) $$

(2-20)

with

$$ D(\alpha,\tau) = \Gamma(-\alpha)\left(\tau+\bar{e}^{i\pi\alpha}\right)s^{\alpha}\,, $$

$$ V(\alpha_1,\alpha_2;K_1) = \left\{ \frac{\Gamma(\alpha_1-\alpha_2)\left[\tau_1\tau_2+e^{-i\pi(\alpha_2-\alpha_1)}\right]K_1^{\alpha_1}\,{}_1F_1\left(-\alpha_1,\alpha_2-\alpha_1+1;-\frac{1}{K_1}\right)}{\Gamma(-\alpha_2)\left[\tau_2+\bar{e}^{i\pi\alpha_2}\right]} + \begin{matrix}\alpha_1\leftrightarrow\alpha_2\\ \tau_1\leftrightarrow\tau_2\end{matrix} \right\} $$

and thus exhibits full Regge factorization.

However on the β trajectory level the expression in the curly brackets in (2-12) and in (2-18) vanish. Thus the new trajectory gives contribution to the full amplitude only when both α_1 and α_3 Reggeous are twisted (i.e. $\phi = e^{\pm 2\pi i}$). The complete contribution of the β trajectory is therefore given by

$$ A_6' = -\pi i\,\tau_1\tau_3\,(1+\tau_2)\,(s_1K_1)^{\alpha_1}\exp\left(-\frac{1}{K_1}\right)(s_3K_2)^{\alpha_3}\exp\left(-\frac{1}{K_2}\right) $$

$$ \bar{2}^{\beta_2-1}\,\Gamma(-1-\beta_2)\,s_2^{\beta_2}\,\bar{e}^{i\pi\beta_2}. \qquad (2\text{-}21) $$

As mentioned before for $\alpha_2 > -1$ the α_2 trajectory gives the dominant contribution to the scattering amplitude, whereas for $\alpha_2 < -1$ the β_2 trajectory is dominant; thus the dominant contribution is given by eq. (2.20) for $\alpha_2 > -1$, eq. (2.21) for $\alpha_2 < -1$ and for $\alpha_2 \approx -1$ both eq. (2.20) and (2.21) have to be taken into account (see Fig.9).

As it has been shown[6,9] each one of the eq. (2-20) and (2-21) develops a pole at $\alpha_2=-1$, but their residua cancel each other. From eq. (2-19) the structure of $\beta-\alpha-$ particle vertex can be read of as

$$V(\alpha,\beta,K)=\frac{1}{\Gamma(-\alpha)}K^{\alpha}e^{-\frac{1}{K}}$$

The phase factor of expression (2-21) shows that for the β trajectories, τ_2, does not have the meaning of signature and should presumably be thought of as representing some charge-like quantum number[9,12]. The S_2 dependence of the diagrams of Fig. (6a) and (6b), corresponding to the exchange of an α trajectorv is given by $(-S_2)^{\alpha_2}+\tau_2 S_2^{\alpha_2}=\left(\tau_2+e^{-i\pi\alpha_2}\right)S_2^{\alpha_2}$.

As signaturization involves symmetrization or antisymmetrization of the amplitude with respect to $S_2\longrightarrow -S_2$ we see that as

$$(-S_2)^{\alpha_2}+\tau_2 S_2^{\alpha_2}\longrightarrow S_2^{\alpha_2}+\tau_2(-S_2)^{\alpha_2}$$
$$=\tau_2\left((-S_2)^{\alpha_2}+\tau_2 S_2^{\alpha_2}\right),$$

corresponds also to the signature of the α_2 trajectory. Thus for the α trajectories in the Veneziano model signature and twisting - interpreted as a charge-like quantum number - are equivalent.

However expression (2-21) shows that this is not the case for the β trajectories as now the contribution of diagrams (6a) and (6b) gives

$$(-S_2)^{\beta_2}(1+\tau_2)\cdot$$

under

$$S_2\longrightarrow -S_2 \quad ; \quad (-S_2)^{\beta_2}(1+\tau_2)\longrightarrow S_2^{\beta_2}(1+\tau_2)$$

and we see that τ_2 (twisting) is inequivalent to the signaturization.

Looking at (2-21) we see that the β trajectory exchange contributes only to the amplitudes which are even under twisting $(\tau_2 = +1)$ and that the β trajectory is at least doubly degenerate as it gives contribution to amplitude of either signature.

2-2. The seven point function and the two β trajectories

In this section we determine the complete 2β contribution to the seven point function. To do this we determine the contribution corresponding to all four diagrams shown in Fig.(10).

First we consider the simplest diagram, (10a). To do this we consider the Veneziano amplitude corresponding to diagram (11a), and take the multi-Regge limit corresponding to Fig.(10a). The amplitude of Fig.(11a) is defined for[14]

$$S_{13} \sim -S_1 < 0$$

$$S_{57} \sim -S_4 < 0$$

$$S_{34} \sim S_2 < 0$$

$$S_{45} \sim S_3 < 0$$

$$S_{134} \sim -S_1 S_2 K_1 < 0$$

$$S_{345} \sim S_2 S_3 K_2 < 0$$

$$S_{457} \sim -S_3 S_4 K_3 < 0$$

$$S_{1345} \sim -S_1 S_2 S_3 K_1 K_2 \phi_1 < 0$$

$$S_{3457} \sim -S_2 S_3 S_4 K_2 K_3 \phi_2 < 0$$

$$S_{13457} \sim S_1 S_2 S_3 S_4 K_1 K_2 K_3 \phi_1 \phi_2 \eta < 0 \qquad (2\text{-}22)$$

Where in the multi-Regge limit K_1, K_2, K_3 are constant and ϕ_1, ϕ_2, η resemble ϕ in the six point function, and in the multi-Regge limit are all numerically +1, although in this case one of them (η) has to be continued to $\eta = e^{-2\pi i}$. Eq. (2-22) shows that the following continuations are required:

$$\begin{aligned} S_2 &= -S_2\, e^{-i\pi} & & \\ S_3 &= -S_3\, e^{-i\pi} & \phi_1 &= 1 \\ K_1 &= -K_1\, e^{i\pi} & \phi_2 &= 1 \\ K_2 &= -K_2\, e^{i\pi} & \eta &= e^{-2\pi i} \\ K_3 &= -K_3\, e^{i\pi} & & \end{aligned} \qquad (2\text{-}23)$$

We write the expression for the seven point function shown in Fig. (11a) as[5]

$$B_7 \sim \tau_1 \tau_4 \int^1 dx_1 dx_2 dx_3 dx_4 \; x_1^{-\alpha_1 - 1} \; x_2^{-\alpha_2 - 1} \; x_3^{-\alpha_3 - 1} \; x_4^{-\alpha_4 - 1}$$

$$\left(1 - x_1(1-x_2)\right)^{S_1} \left[1 - \frac{x_2(1-x_3)}{1 - x_2 x_3}\right]^{-S_2} \left[1 - \frac{x_3(1-x_2)}{1 - x_2 x_3}\right]^{-S_3} \left(1 - x_4(1-x_3)\right)^{S_4}$$

$$\left(1 - x_1 x_2 (1-x_3)\right)^{S_1 S_2 K_1} \left(1 - x_2 x_3\right)^{-S_2 S_3 K_2} \left(1 - x_3 x_4 (1-x_2)\right)^{S_3 S_4 K_3}$$

$$\left(1 - x_1 x_2 x_3\right)^{S_1 S_2 S_3 K_1 K_2 \phi_1} \left(1 - x_2 x_3 x_4\right)^{S_2 S_3 S_4 K_2 K_3 \phi_2} \left(1 - x_1 x_2 x_3 x_4\right)^{S_1 S_2 S_3 S_4 K_1 K_2 K_3 \phi_1 \phi_2}$$

(2-24)

and perform the relevant expansions obtaining

$$B_7 \sim \tau_1 \tau_4 \int_0^1 dx_1 dx_2 dx_3 dx_4 \; x_1^{-\alpha_1-1} \; x_2^{-\alpha_2-1} \; x_3^{-\alpha_3-1} \; x_4^{-\alpha_4-1}$$

$$\left(1-\frac{x_2(1-x_3)}{1-x_2x_3}\right)^{-s_2} \left(1-\frac{x_3(1-x_2)}{1-x_2x_3}\right)^{-s_3} \left(1-x_2x_3\right)^{-s_2s_3K_2}$$

$$\exp\Big\{-x_1s_1(1-x_2)-x_1x_2s_1s_2K_1(1-x_3)-x_1x_2x_3s_1s_2s_3K_1K_2\phi_1$$

$$+x_4s_4(1-x_3)-x_3s_3x_4s_4K_3(1-x_2)-x_2x_3x_4s_2s_3s_4K_2K_3\phi_2$$

$$+x_1x_2x_3x_4s_1s_2s_3s_4K_1K_2K_3\phi_1\phi_2\eta\Big\}.$$

To exhibit the α trajectory exchange in the two external channels we change the variables of integration

$$x_1 = \frac{y_1}{s_1}, \quad x_4 = \frac{y_4}{s_4}$$

and let $s_1, s_4 \rightarrow \infty$ obtaining

$$B_7 \sim \tau_1 \tau_4 \, s_1^{\alpha_1} s_4^{\alpha_4} \int_0^1 dx_2 dx_3 \; x_2^{-\alpha_2-1} \; x_3^{-\alpha_3-1} \left(1-\frac{x_2(1-x_3)}{1-x_2x_3}\right)^{-s_2} \left(1-\frac{x_3(1-x_2)}{1-x_2x_3}\right)^{-s_3}$$

$$\left(1-x_2x_3\right)^{-s_2s_3K_2} \int_0^\infty dy_1 dy_4 \; y_1^{-\alpha_1-1} \; y_4^{-\alpha_4-1}$$

$$\exp\Big\{-y_1\big[1-x_2+x_2s_2K_1(1-x_3)+x_2x_3s_2s_3K_1K_2\phi_1\big]$$

$$-y_4\big[1-x_3+x_3s_3K_3(1-x_2)+x_2x_3s_2s_3K_2K_3\phi_2\big]$$

$$+y_1y_4x_2x_3s_2s_3K_1K_2K_3\phi_1\phi_2\eta\Big\}.$$

Next we replace all $(1-x)^{-s}$ by the exponentials $\exp\{s(x+\frac{x^2}{2})\}$ as shortly we are going to let $-s_2, -s_3 \rightarrow \infty$ with $-x_i s_i \rightarrow \infty$ and $(-s_i)^{1/2} x_i$ held constant. Now we write the last term in the exponential as

$$e^{y_1 y_4 X_2 X_3 S_2 S_3 K_1 K_2 K_3 \phi_1 \phi_2 \eta} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} d\lambda \, \Gamma(-\lambda) \left(-y_1 y_4 X_2 X_3 S_2 S_3 K_1 K_2 K_3 \phi_1 \phi_2 \eta \right)^{\lambda},$$

perform y_1 and y_4 integrations and obtain

$$B_7 \sim \tau_1 \tau_4 S_1^{\alpha_1} S_4^{\alpha_4} \int_0^1 dX_2 dX_3 \, X_2^{-\alpha_2 - 1} X_3^{-\alpha_3 - 1} A^{\alpha_1} B^{\alpha_4} \exp\left\{ \frac{X_2 S_2 (1 - X_3)}{1 - X_2 X_3} \right.$$

$$+ \frac{X_2^2 S_2}{2} \left(\frac{1 - X_3}{1 - X_2 X_3} \right)^2 + \frac{X_3 S_3 (1 - X_2)}{1 - X_2 X_3} + \frac{X_3^2 S_3}{2} \left(\frac{1 - X_2}{1 - X_2 X_3} \right)^2 + X_2 X_3 S_2 S_3 K_2$$

$$\left. + \frac{X_2^2 X_3^2 S_2 S_3 K_2}{2} \right\} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} d\lambda \, \Gamma(-\lambda) \Gamma(\lambda - \alpha_1) \Gamma(\lambda - \alpha_4) \left(\frac{AB}{C\eta} \right)^{-\lambda} \qquad (2\text{-}25)$$

where

$$A = 1 - X_2 + X_2 S_2 K_1 (1 - X_3) + X_2 S_2 X_3 S_3 K_1 K_2 \phi_1$$

$$B = 1 - X_3 + X_3 S_3 K_3 (1 - X_2) + X_2 S_2 X_3 S_3 K_2 K_3 \phi_2$$

$$C = - X_2 S_2 X_3 S_3 K_1 K_2 K_3 \phi_1 \phi_2 \quad .$$

As before, we rewrite the helicity integral in (2-25) in terms of two ${}_1F_1$ functions, change the variables of integration

$$X_2 = \frac{y_2}{(-S_2)^{1/2}} \quad , \quad X_3 = \frac{y_3}{(-S_3)^{1/2}}$$

and let $-S_2, -S_3 \longrightarrow \infty$ obtaining

$$B_7 \sim \tau_1 \tau_4 S_1^{\alpha_1} (-S_2)^{\beta_2} (-S_3)^{\beta_3} S_4^{\alpha_4} \left(-\frac{1}{K_2} \right) (-K_1)^{\alpha_1} (-K_3)^{\alpha_4} \phi_1^{\alpha_1} \phi_2^{\alpha_4} \exp\left\{ -\frac{1}{K_1} - \frac{1}{K_2} - \frac{1}{K_3} \right.$$

$$\left\{ \eta^{\alpha_1} \Gamma(\alpha_1 - \alpha_4) \Gamma(\alpha_4 - \alpha_1 + 1) + \alpha_1 \longleftrightarrow \alpha_4 \right\} \int_0^\infty dy_2 dy_3 \, y_2^{-\alpha_2 - 2} y_3^{-\alpha_3 - 3}$$

$$\exp\left\{ -\frac{y_2^2}{2} - \frac{y_3^2}{2} + \frac{y_2^2 y_3^2}{2} K_2 \right\} , \qquad (2\text{-}26)$$

where $\beta_i = \frac{\alpha_i - 1}{2}$.

A simple change of variables allows us to rewrite the last term in (2-26) as

$$I = 2^{-\beta_2-\beta_3-4} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda \, \Gamma(-\lambda)\Gamma(\lambda-\beta_2-1)\Gamma(\lambda-\beta_3-1)\left(-\frac{1}{2K_2}\right)^{-\lambda}$$

$$= 2^{-\beta_2-\beta_3-4} \Big\{ \Gamma(-1-\beta_2)\Gamma(\beta_2-\beta_3)(-2K_2)^{\beta_2+1} {}_1F_1(-\beta_2-1, \beta_3-\beta_2+1; -\frac{1}{2K_2})$$

$$+ \Gamma(-1-\beta_3)\Gamma(\beta_3-\beta_2)(-2K_2)^{\beta_3+1} {}_1F_1(-\beta_3-1, \beta_2-\beta_3+1; -\frac{1}{2K_2})\Big\} .$$

thus showing that

$$B_7 \approx 2\pi i \, \tau_1\tau_4 \, e^{-i\pi\beta_2} e^{-i\pi\beta_3} S_1^{\alpha_1} S_2^{\beta_2} S_3^{\beta_3} S_4^{\alpha_4} \, 2^{-\beta_2-\beta_3-3} K_1^{\alpha_1} K_3^{\alpha_4}$$

$$\exp\Big\{-\frac{1}{K_1}-\frac{1}{K_2}-\frac{1}{K_3}\Big\} \Big\{ \Gamma(-1-\beta_2)\Gamma(\beta_2-\beta_3)(-2K_2)^{\beta_2} {}_1F_1(-\beta_2-1, \beta_3-\beta_2+1; -\frac{1}{2K_2}$$

$$+ \beta_2 \longleftrightarrow \beta_3 \Big\} . \qquad (2\text{-}27)$$

Comparing with the conventional two Reggeon-one particle vertex[7] we see that the two β-one particle vertex has essentially the same form.

Next we consider Fig.(10b). To find its 2β contribution we start with the amplitude corresponding to Fig.(11b), perform the required analytical continuations and take the asymptotic limit corresponding to Fig.(10b).

The Veneziano model amplitude for Fig.(11b) is given by[12]

$$B_7' \sim \tau_1\tau_3\tau_4 \int_0^1 dx_1 dx_2 dx_3 dx_4 \; x_1^{-\alpha_{12}-1} \; x_2^{-\alpha_{123}-1} \; x_3^{-\alpha_{1234}-1} \; x_4^{-\alpha_{57}-1}$$

$$(1-x_1)^{-\alpha_{13}-1} \; (1-x_2)^{-\alpha_{34}-1} \; (1-x_3)^{-\alpha_{46}-1} \; (1-x_4)^{-\alpha_{67}-1}$$

$$(1-x_1x_2)^{-\alpha_{134}+\alpha_{13}+\alpha_{34}} \; (1-x_2x_3)^{-\alpha_{346}+\alpha_{34}+\alpha_{46}} \; (1-x_3x_4)^{-\alpha_{467}+\alpha_{46}+\alpha_{6}}$$

$$(1-x_1x_2x_3)^{-\alpha_{1346}-\alpha_{34}+\alpha_{134}+\alpha_{346}} \; (1-x_2x_3x_4)^{-\alpha_{3467}-\alpha_{46}+\alpha_{346}+\alpha_{467}}$$

$$(1-x_1x_2x_3x_4)^{-\alpha_{13467}+\alpha_{1346}+\alpha_{3467}-\alpha_{346}} \quad . \tag{2-28}$$

In the multi-Regge limit

$$S_{57} \sim -s_4 \qquad S_{346} \sim s_2 s_3 s_4 K_2 K_3 \phi_2$$

$$S_{13} \sim -s_1 \qquad S_{13467} \sim s_1 s_2 s_3 K_1 K_2 \phi_1$$

$$S_{34} \sim s_2 \qquad S_{1346} \sim -s_1 s_2 s_3 s_4 K_1 K_2 K_3 \phi_1 \phi_2 \eta$$

$$S_{46} \sim s_3 s_4 K_3$$

$$S_{134} \sim -s_1 s_2 K_1$$

$$S_{467} \sim -s_3 \quad . \tag{2-29}$$

and as the amplitude is originally defined only when all these energies are negative it requires the continuations:

$$S_2 = -s_2 e^{-i\pi} \qquad \phi_2 = 1$$

$$K_1 = -K_1 e^{i\pi} \qquad \eta = e^{+2\pi i}$$

$$K_2 = -K_2 e^{i\pi}$$

$$K_3 = -K_3 e^{i\pi}$$

$$\phi_1 = e^{-2\pi i} \quad . \tag{2-30}$$

We change the variable of integration $u_4 = 1 - x_4$ and proceed as before obtaining:

$$B_7' = \tau_1\tau_3\tau_4 S_1^{\alpha_1} S_4^{\alpha_4} \int_0^1 dx_2 dx_3\, x_2^{-\alpha_2-1}\, x_3^{-\alpha_3-1} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda\, \Gamma(-\lambda)\,\Gamma(\lambda-\alpha_1)$$

$$\Gamma(\lambda-\alpha_4)\left(\frac{AB}{C\eta}\right)^{-\lambda} A^{\alpha_1} B^{\alpha_4} \exp\left\{ x_2 S_2 \frac{1-x_3}{1-x_2x_3} + x_2^2 S_2 \frac{(1-x_3)^2}{2(1-x_2x_3)} - x_3 S_3 \frac{1-x_2}{1-x_2x_3}\right.$$

$$\left. - x_3^2 S_3 \frac{(1-x_2)^2}{2(1-x_2x_3)^2} - x_2x_3S_2S_3K_2 - \frac{x_2^2x_3^2S_2S_3K_2}{2}\right\}, \qquad (2\text{-}31)$$

where

$$A = 1 - x_2 + x_2S_2K_1(1-x_3) - x_2x_3S_2S_3K_1K_2\phi_1$$

$$B = 1 - \frac{x_3S_3(1-x_2)K_3}{(1-x_3)(1-x_2x_3)} - \frac{x_2x_3S_2S_3K_2K_3\phi_2}{1-x_2x_3}$$

$$C = x_2x_3S_2S_3K_1K_2K_3\phi_1\phi_2 \;.$$

Next we perform the λ integration, rewrite the obtained ψ function in terms of ${}_1F_1$ functions and then letting $S_2S_3K_2 \to \infty$, replace each of them by its asymptotic form. Finally we continue ϕ_1, ϕ_2, η, K_1 and K_3 to their correct values as shown in (2-30) and at the same time continue S_3 to $S_3 = -S_3 e^{i\pi}$.

we obtain

$$\dot{B}_7' \sim \tau_1\tau_3\tau_4\, 2\pi i\, K_1^{\alpha_1} S_1^{\alpha_1} K_3^{\alpha_4} S_4^{\alpha_4} e^{-\frac{1}{K_1}-\frac{1}{K_2}-\frac{1}{K_3}} \int_0^1 dx_2dx_3\, x_2^{-\alpha_2-1}\, x_3^{\alpha_3-1}$$

$$\exp\left\{\frac{x_2^2S_2}{2} + \frac{x_3^2S_3}{2} + \frac{x_2^2x_3^2S_2S_3K_2}{2}\right\} \frac{1}{x_2x_3S_2S_3K_2}\,, \qquad (2\text{-}32)$$

where we have kept only the lowest non-vanishing powers of X_i in the coefficients of S_2 , S_3 and $S_2 S_3 K_2$ in the exponential.

As S_2, S_3 , K_2 are negative at this stage, we perform the X_2 and X_3 integrations and then perform the remaining continuations:

$$S_2 = -S_2 e^{-i\pi} \qquad K_2 = -K_2 e^{i\pi}$$
$$S_3 = -S_3 e^{-i\pi} \quad ,$$

obtaining

$$B_7' \sim \tau_1 \tau_4 \tau_3 \left(S_1^{\alpha_1} K_1^{\alpha_1} e^{-\frac{1}{K_1}} \right) \left(S_4^{\alpha_4} K_3^{\alpha_4} e^{-\frac{1}{K_3}} \right) \pi i \, 2^{-\beta_2 - \beta_3 - 2} \, S_2^{\beta_2} \, S_3^{\beta_3}$$
$$e^{-\frac{1}{K_1}} \, e^{-i\pi(\beta_2+\beta_3)} \left\{ \left(2K_2 e^{i\pi} \right)^{\beta_2} \Gamma(-1-\beta_2) \, \Gamma(\beta_2 - \beta_3) \right.$$
$$\left. {}_1F_1 \left(-\beta_2 - 1 ; \beta_3 - \beta_2 + 1 ; -\frac{1}{2K_2} \right) + \beta_2 \longleftrightarrow \beta_3 \right\} . \tag{2-33}$$

This result agrees (apart from the factor τ_3) with the expression (2-27) corresponding to the diagram (10a).

By symmetry the contribution of the diagram exhibited in Fig.(10c) is again the same.

Next we calculate the contribution of the diagram in Fig. (10d). The discussion along the lines as before, this time with the change of variables $u_4 = 1 - x_4$ and $u_1 = 1 - x_1$ gives

$$B_7'' \sim \tau_1 \tau_2 \tau_3 \tau_4 \, S_1^{\alpha_1} \, S_4^{\alpha_4} \int_0^1 dx_2 \, dx_3 \, x_2^{-\alpha_2 - 1} \, x_3^{-\alpha_3 - 1} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} d\lambda \, \Gamma(-\lambda) \Gamma(\lambda - \alpha_1)$$
$$\Gamma(\lambda - \alpha_4) \, A^{\alpha_1} B^{\alpha_4} \left(\frac{AB}{c\eta} \right)^{-\lambda} \exp \left\{ -x_2 S_2 \frac{1-x_3}{1-x_2 x_3} - x_2^2 S_2 \frac{(1-x_3)^2}{2(1-x_2 x_3)^2} \right.$$
$$\left. - x_3 S_3 \frac{1-x_2}{1-x_2 x_3} - x_3^2 S_3 \frac{(1-x_2)^2}{2(1-x_2 x_3)^2} + x_2 x_3 S_2 S_3 K_2 + \frac{x_2^2 x_3^2 S_2 S_3 K_2}{2} \right\}$$
$$\tag{2-34}$$

where

$$A = 1 - x_2 s_2 K_1 \frac{1-x_3}{(1-x_2)(1-x_2x_3)} + x_2 x_3 s_2 s_3 K_1 K_2 \phi_1 \frac{1}{1-x_2x_3}$$

$$B = 1 - x_3 s_3 K_3 \frac{1-x_2}{(1-x_3)(1-x_2x_3)} + x_2 x_3 s_2 s_3 K_2 K_3 \phi_2 \frac{1}{1-x_2x_3}$$

$$C = - \frac{x_2 x_3 s_2 s_3 K_1 K_2 K_3 \phi_1 \phi_2}{(1-x_2x_3)^2} \quad .$$

The obtained expression has to be continued to

$$K_1 = - K_1 e^{-i\pi}$$

$$K_2 = - K_2 e^{-i\pi} \qquad \phi_1 = e^{2\pi i}$$

$$K_3 = - K_3 e^{-i\pi} \qquad \phi_2 = e^{2\pi i}$$

$$\eta = e^{-2\pi i} \qquad .$$

We choose to perform this continuation in two steps: first we continue K_1, K_3, ϕ_1, ϕ_2 and η to their correct values and also S_2, S_3 to negative values $S_2 = -S_2 e^{+i\pi}$, $S_3 = -S_3 e^{i\pi}$. Not to let the phase of the argument of the ψ function, resultant from the λ integration go outside the range $\pm 3\pi/2$, we simultaneously continue K_2 to $K_2 = K_2 e^{-2\pi i}$ Thus the second step of continuations will involve $S_2 = -S_2 e^{-i\pi}$, $S_3 = -S_3 e^{-i\pi}$, $K_2 = -K_2 e^{i\pi}$ in agreement with the continuation required for all other contributions. Proceeding as before we obtain

$$B_7'' \sim \tau_1\tau_2\tau_3\tau_4 \, S_1^{\alpha_1} S_4^{\alpha_4} K_1^{\alpha_1} K_3^{\alpha_4} e^{-\frac{1}{R_1}-\frac{1}{K_2}-\frac{1}{K_3}} (2\pi i)\int_0^1 dx_2 dx_3 \, x_2^{-\alpha_2-1} x_3^{-\alpha_3-1}$$

$$\exp\left\{ \frac{x_2^2 S_2}{2} + \frac{x_3^2 S_3}{2} + \frac{x_2^2 x_3^2 S_2 S_3 K_2 e^{-2\pi i}}{2} \right\} \frac{1}{x_2 x_3 S_2 S_3 K_2} . \qquad (2\text{-}35)$$

This result agrees with the contributions of the other three diagrams except that now K_2 is evaluated on the other side of its cut. Thus when the final continuation is performed the contribution of this diagram is different as now the K_2 dependence is evaluated on the other side of its cut (i.e. in the final expression effectively $K_2 \longrightarrow -K_2 e^{-i\pi}$ and not $K_2 \longrightarrow -K_2 e^{i\pi}$ as in the expression for the other diagrams).

Notice that all four diagrams of Fig.(10) gave the same S_2,and S_3 dependence, apart from the above-mentioned difference in the continuation in K_2. This has come about as a result of the different exponential factors from the ${}_1F_1$ functions. They were just of such a form as to cancel the linear terms in x_2 and x_3 in the coefficients of S_2, S_3 and $S_2S_3K_2$ in the exponentials in (2-31) and (2-34) and also to change the sign of x_2^2 and x_3^2 terms so that they are all the same and ppsitive. This behaviour generalizes to higher point functions; the β contribution always comes in the same way. In the next chapter we shall show that this is the case for the triple-β exchange contribution to the eight point function. There we shall also derive the contribution of a two β-one γ exchange. Before these however, we exhibit the contribution of one β trajectory to the seven point function.

2-3. The seven point function and one β trajectory

In this section we discuss the contribution of one β exchange (in the second channel) to the seven point amplitude. We start with the simplest diagram which is shown in Fig.(12a). The corresponding expression is[14]:

$$B_7 \sim \tau_1\tau_3 \int_0^1 dx_1 dx_2 dx_3\, dx_4\, x_1^{-\alpha_1-1}\, x_2^{-\alpha_2-1}\, x_3^{-\alpha_3-1}\, x_4^{-\alpha_4-1}\, (1-x_2)^{-s_2-1}$$

$$\exp\{-x_1s_1(1-x_2) - x_1x_2s_1s_2K_1 - x_3s_3(1-x_2) - x_2x_3s_2s_3K_2$$

$$-x_3s_3x_4s_4K_3(1-x_2) + x_4s_4 - x_2x_3x_4\, s_2s_3s_4K_2K_3\phi_2$$

$$+ x_1x_2x_3\, s_2s_3\, K_1K_2\phi_1 + x_1x_2x_3x_4s_1s_2s_3s_4K_1K_2K_3\phi_1\phi_2\eta\}.$$

In the multi-Regge limit corresponding to Fig.(12a)

$$s_{13} \sim -s_1 < 0$$

$$s_{34} \sim +s_2 < 0$$

$$s_{476} \sim -s_3 < 0$$

$$s_{56} \sim +s_4 < 0$$

$$s_{134} \sim -s_1s_2K_1 < 0$$

$$s_{3476} \sim -s_2s_3K_2 < 0$$

$$s_{47} \sim -s_3s_4K_3 < 0$$

$$s_{13476} \sim s_1s_2s_3K_1K_2\phi_1 < 0$$

$$s_{347} \sim -s_2s_3s_4K_2K_3\phi_2 < 0$$

$$s_{1347} \sim +s_1s_2s_3s_4K_1K_2K_3\phi_1\phi_2\eta < 0$$

It requires the continuation

$$S_2 = -S_2\, e^{-i\pi}$$

$$S_4 = -S_4\, e^{-i\pi}$$

$$K_1 = -K_1\, e^{i\pi}$$

$$K_2 = -K_2\, e^{i\pi}$$

$$K_3 = -K_3\, e^{i\pi}$$

$$\phi_1 = e^{-2\pi i}$$

$$\phi_2 = 1$$

$$\eta = 1 .$$

Changing the variables of integration

$$X_1 = \frac{y_1}{S_1} \quad , \quad X_3 = \frac{y_3}{S_3} \quad , \quad X_4 = \frac{y_4}{-S_4}$$

and letting $S_1 , S_3 , (-S_4) \longrightarrow \infty$ we get

$$B_7 \sim \tau_1 \tau_3 S_1^{\alpha_1} S_3^{\alpha_3} (-S_4)^{\alpha_4} \int_0^1 dX_2 \, X_2^{-\alpha_2 - 1} (1-X_2)^{-S_2-1} \int_0^\infty dy_1 dy_3 dy_4$$

$$y_1^{-\alpha_1-1} \, y_3^{-\alpha_3-1} \, y_4^{-\alpha_4-1} \, e^{-y_4} \, \exp\{-y_1(1-X_2+X_2S_2K_1)$$

$$-y_3(1-X_2+X_2S_2K_1-y_4K_3+y_4X_2K_3-y_4X_2S_2K_2K_3\phi_2)$$

$$+y_1y_3X_2S_2K_1K_2\phi_1(1-y_4K_3\phi_2\eta)\} \, .$$

(2-36)

Next we introduce a helicity-like integration

$$\exp\{y_1y_3X_2S_2K_1K_2\phi_1(1-y_4K_3\phi_2\eta)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda \, \Gamma(-\lambda) (-y_1y_3X_2S_2K_1K_2\phi_1(1-y_4K_3\phi_2\eta$$

Perform the y_1 and y_3 integrations and obtain

$$B_7 \sim \tau_1\tau_3 S_1^{\alpha_1} S_3^{\alpha_3} (-S_4)^{\alpha_4} \int_0^1 dX_2 \, X_2^{-\alpha_2-1} (1-X_2)^{-S_2-1} \int_0^\infty dy_4 \, y_4^{-\alpha_4-1}$$

$$e^{-y_4} A^{\alpha_1} B^{\alpha_3} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda \, \Gamma(-\lambda)\Gamma(\lambda-\alpha_1)\Gamma(\lambda-\alpha_3)\left(\frac{AB}{C\phi_1}\right)^{-\lambda}$$

(2-37)

where

$$A = 1 - X_2 + X_2 S_2 K_1$$

$$B = (1-X_2)(1-y_4K_3) + X_2 S_2 K_2 (1-y_4 K_3 \phi_2)$$

$$C = -X_2 S_2 K_1 K_2 (1-y_4 K_3 \phi_2 \eta) \, .$$

Next we perform the λ integration, express the result in terms of two ${}_1F_1$ functions, change the variable of integration $X_2(-S_2)^{1/2} = y_2$ and letting $-S_2 \longrightarrow \infty$, obtain

$$B_7 \sim \tau_1 \tau_3 \, S_1^{\alpha_1} S_3^{\alpha_3} (-S_2)^{\beta_2} (-S_4)^{\alpha_4} \int_0^\infty dy_4 \int_0^\infty dy_2 \; y_2^{-\alpha_2-2} \, y_4^{-\alpha_4-1} \, e^{-\frac{y_2^2}{2}}$$
$$e^{-y_4} \, e^{-\frac{1}{K_1}-\frac{1}{K_2}} (-K_1)^{\alpha_1} (-K_3)^{\alpha_3} \Big\{ \phi_1^{\alpha_1} \Gamma(\alpha_1-\alpha_3)\Gamma(\alpha_3-\alpha_1+1)(1-y_4K_3\phi_2)^{\alpha_3-\alpha_1} \times$$
$$\times (1-y_4K_3\phi_2\eta)^{\alpha_1} + \phi_1^{\alpha_3} \Gamma(\alpha_3-\alpha_1)\Gamma(\alpha_1-\alpha_3+1)(1-y_4K_3\phi_2\eta)^{\alpha_3} \Big\} \, . \qquad (2\text{-}38)$$

After performing the y_2 integration we obtain (as $\phi_2 = \eta = 1$ in this case)

$$B_7 \sim \tau_1 \tau_3 \, S_1^{\alpha_1} (-S_2)^{\beta_2} S_3^{\alpha_3} (-S_4)^{\alpha_4} \, 2^{-\beta_2-2} \, \Gamma(-\beta_2-1) \, e^{-\frac{1}{K_1}-\frac{1}{K_2}} (-K_1)^{\alpha_1} (-K_2)^{\alpha_3}$$
$$\Big\{ \Gamma(\alpha_1-\alpha_3)\Gamma(\alpha_3-\alpha_1+1)\,\phi_1^{\alpha_1} + \alpha_1 \longleftrightarrow \alpha_3 \Big\} \int_0^\infty dy_4 \, y_4^{-\alpha_4-1}$$
$$(1-y_4K_3)^{\alpha_3} \, e^{-y_4} \, . \qquad (2\text{-}39)$$

Fdlowing Weis[15] we introduce

$$I_1 = \int_0^{1/K_3} dy_4 \, y_4^{-\alpha_4-1} \, e^{-y_4} (1-y_4K_3)^{\alpha_3} \, ,$$

and

$$I_2 = \int_{1/K_3}^{\infty} dy_4 \, y_4^{-\alpha_4-1} \, e^{-y_4} (y_4K_3-1)^{\alpha_3} \, ,$$

which allow us to rewrite (2-39) as

$$B_7 \sim (-2\pi i)\, e^{-i\pi\beta_2}\, e^{-i\pi\alpha_4}\, s_1^{\alpha_1}\, s_2^{\beta_2}\, s_3^{\alpha_3}\, s_4^{\alpha_4}\, (\tau_1\tau_3)\, 2^{-\beta_2-2}\, K_1^{\alpha_1}\, K_3^{\alpha_3}$$

$$e^{-\frac{1}{K_1}-\frac{1}{K_2}}\, \Gamma(-1-\beta_2)\left[I_1 + e^{i\pi\alpha_3} I_2\right] . \tag{2-40}$$

As $I_1 + e^{i\pi\alpha_3} I_2 = \Gamma^{-1}(-\alpha_3)\, V(\alpha_3, \alpha_4, K_3 - i\epsilon)$[15], we see that the appearnce of a β trajectory in the multi-Regge chain is consistent with the expectation based on Regge factorization. When compared with the chain of α trajectory exchanges, the appearance of a β trajectory affects only the two nearest vertices and propagator in a way that is already evident from the six point function; the vertices and propagators further away in the chain remain unchanged.

The complete contribution of one β exchange to the seven point function comes from addition of diagrams (12, a, b, c, d).

To find the contribution of (12b) we start with the amplitude corresponding to the diagram (13b). Perform the required analytical continuation and take the asymptotic limit corresponding to Fig. (12b). In the multi-Regge limit shown in Fig. (12b)

$$\begin{array}{ll}
s_{13} \sim -s_1 < 0 & s_{47} \sim -s_3 s_4 K_3 < 0 \\
s_{3567} \sim -s_2 < 0 & s_{135} \sim -s_1 s_2 s_3 K_1 K_2 \phi_1 < 0 \\
s_{2135} \sim -s_3 < 0 & s_{356} \sim s_2 s_3 s_4 K_2 K_3 \phi_2 < 0 \\
s_{56} \sim s_4 < 0 & s_{1356} \sim -s_1 s_2 s_3 s_4 K_1 K_2 K_3 \phi_1 \phi_2 \eta < 0 \\
s_{24} \sim s_1 s_2 K_1 < 0 & \\
s_{35} \sim s_2 s_3 K_2 < 0 & ,
\end{array} \tag{2-41}$$

and it requires the following continuations:

$$
\begin{aligned}
S_4 &= -s_4 e^{-i\pi} \\
K_1 &= -K_1 e^{-i\pi} \qquad \phi_2 = 1 \\
K_2 &= -K_2 e^{-i\pi} \\
&\qquad\qquad\qquad \eta = 1 \\
K_3 &= -K_3 e^{i\pi} \\
\phi_1 &= e^{2\pi i} \qquad\qquad .
\end{aligned}
$$

The corresponding expression is

$$
\begin{aligned}
B_7' \sim \tau_1\tau_2\tau_3 \int_0^1 & dx_1 dx_2 dx_3 dx_4 \quad x_1^{-\alpha_1-1} \; x_2^{-\alpha_2-1} \; x_3^{-\alpha_3-1} \; x_4^{-\alpha_4-1} \\
& (1-x_2)^{S_2} \; \exp\Big\{ -x_1 S_1 + \frac{x_1 x_2 S_1 S_2 K_1}{1-x_2} - x_1 x_2 x_3 S_1 S_2 S_3 K_1 K_2 \phi_1 \\
& - x_3 S_3 (1-x_2) + x_2 x_3 S_2 S_3 K_2 + x_4 S_4 - x_3 S_3 x_4 S_4 K_3 (1-x_2) + \\
& + x_2 x_3 x_4 S_2 S_3 S_4 K_2 K_3 \phi_2 - x_1 x_2 x_3 x_4 \; S_1 S_2 S_3 S_4 K_1 K_2 K_3 \phi_1 \phi_2 \eta \Big\} .
\end{aligned}
$$

(2-42)

Changing the variables of integration

$$
x_1 = \frac{y_1}{S_1} \qquad x_3 = \frac{y_3}{S_3} \qquad x_4 = \frac{y_4}{-S_4} ,
$$

letting $S_1, S_3, (-S_4) \to \infty$ and following the same steps as we did for Fig.(12a) we find a result which agrees with eq.(2-40) apart from a factor of τ_2.

The contributions of diagrams (12c) and (12d) can be evaluated in the same way, with the same final result (i.e. we obtain (2.40) with a factor τ_4 but without $e^{-i\pi\alpha_4}$). The addition of these four diagrams gives the full contribution of one β exchange (in the second channel) to the seven point function.

Chapter Three

The eight point function and new trajectories

3-1. The eight point function and three β trajectories

There are several contributions to the complete three β exchange in the multi-Regge limit of the eight point function. These contributions are indicated in Fig.(14). We start with the contribution of Fig.(14a), and introduce the multi-Regge variables $s_i, t_i\ ; i=1,\cdots 5\ ,\ K_j\ ; j=1,\dots 4$ and variables corresponding to ϕ_1, ϕ_2 and η of the seven point function, namely $\phi_1, \phi_2, \phi_3, \eta_1, \eta_2, \chi$ (see Fig.15a):

$$
\begin{aligned}
S_{13} &\sim -s_1 & S_{13456} &\sim -s_1 s_2 s_3 s_4 K_1 K_2 K_3 \phi_1 \phi_2 \eta_1 \\
S_{34} &\sim s_2 & S_{34568} &\sim -s_2 s_3 s_4 s_5 K_2 K_3 K_4 \phi_2 \phi_3 \eta_2 \\
S_{45} &\sim s_3 & S_{134568} &\sim s_1 s_2 s_3 s_4 s_5 K_1 K_2 K_3 K_4 \phi_1 \phi_2 \phi_3 \eta_1 \eta_2 \chi \\
S_{56} &\sim s_4 \\
S_{68} &\sim -s_5 \\
S_{134} &\sim -s_1 s_2 K_1 \\
S_{345} &\sim s_2 s_3 K_2 \\
S_{456} &\sim s_3 s_4 K_3 \\
S_{568} &\sim -s_4 s_5 K_4 \\
S_{1345} &\sim -s_1 s_2 s_3 K_1 K_2 \phi_1 \\
S_{3456} &\sim s_2 s_3 s_4 K_2 K_3 \phi_2 \\
S_{4568} &\sim -s_3 s_4 s_5 K_3 K_4 \phi_3
\end{aligned}
\qquad (3\text{-}1)
$$

As the amplitude is originally defined when all the above energies are negative it requires the continuation

$$\begin{array}{lll} S_2 = - S_2 e^{-i\pi} & K_4 = - K_4 e^{i\pi} & \\ S_3 = - S_3 e^{-i\pi} & \phi_i = 1 & i = 1, 2, 3 \\ S_4 = - S_4 e^{-i\pi} & \eta_i = 1 & i = 1, 2 \\ K_1 = - K_1 e^{i\pi} & \chi = e^{-2\pi i} & \\ K_2 = - K_2 e^{i\pi} & & \\ K_3 = - K_3 e^{i\pi} & & . \end{array}$$

The expression for the above eight point function is[5]

$$B_8 = \tau_1 \tau_5 \int_0^1 dx_1 dx_2 dx_3 dx_4 dx_5 \; x_1^{-\alpha_1 - 1} \; x_2^{-\alpha_2 - 1} \; x_3^{-\alpha_3 - 1} \; x_4^{-\alpha_4 - 1} \; x_5^{-\alpha_5 - 1}$$

$$\left(1 - \frac{x_2(1-x_3)}{1 - x_2 x_3}\right)^{-S_2} \left[1 - \frac{x_3(1-x_2)(1-x_4)}{(1-x_2x_3)(1-x_3x_4)}\right]^{-S_3} \left(1 - \frac{x_4(1-x_3)}{1 - x_3 x_4}\right)^{-S_4} \left(1 - \frac{x_2 x_3 (1-x_4)}{1 - x_2 x_3 x_4}\right)^{-S_2 S_3 K_2}$$

$$\left(1 - \frac{x_3 x_4 (1-x_2)}{1 - x_2 x_3 x_4}\right)^{-S_3 S_4 K_3} \left(1 - x_2 x_3 x_4\right)^{-S_2 S_3 S_4 K_2 K_3 \phi_2} \exp\Big\{ - x_1 S_1 (1-x_2)$$

$$- x_1 x_2 S_1 S_2 K_1 (1-x_3) - x_1 x_2 x_3 S_1 S_2 S_3 K_1 K_2 \phi_1 - x_1 x_2 x_3 x_4 S_1 S_2 S_3 S_4 K_1 K_2 K_3 \phi_1 \phi_2 \eta$$

$$- x_5 S_5 (1-x_4) - x_4 x_5 S_4 S_5 K_4 (1-x_3) - x_3 x_4 x_5 S_3 S_4 S_5 K_3 K_4 \phi_3$$

$$- x_2 x_3 x_4 x_5 S_2 S_3 S_4 S_5 K_2 K_3 K_4 \phi_2 \phi_3 \eta_2 + x_1 x_2 x_3 x_4 x_5 S_1 S_2 S_3 S_4 S_5 K_1 K_2 K_3 K_4 \phi_1 \phi_2 \phi_3$$

$$\eta_1 \eta_2 \chi \Big\} \qquad (3\text{-}2)$$

where we have already expanded some factors. Next we temporarily keep S_2, S_3 and S_4 fixed while letting $S_1, S_5 \to \infty$. The asymptotic behaviour corresponding to the α trajectory exchange in these channels is exhibited by the change of variables

$$x_1 = \frac{y_1}{S_1} \quad , \quad x_5 = \frac{y_5}{S_5} \quad .$$

we obtain

$$B_8 \sim \tau_1 \tau_5 \, S_1^{\alpha_1} \, S_5^{\alpha_5} \int_0^\infty dy_1 dy_5 \int_0^1 dx_2 dx_3 dx_4 \; y_1^{-\alpha_1-1} \; x_2^{-\alpha_2-1} \; x_3^{-\alpha_3-1} \; x_4^{-\alpha_4-1} \; y_5^{-\alpha_5-1}$$

$$\exp\Big\{ x_2 s_2 - x_2 s_2 x_3 + \frac{x_2^2}{2} s_2 + x_3 s_3 - x_3 s_3 x_2 x_4 + \frac{x_3^2}{2} s_3 + x_4 s_4 - x_4 s_4 x_3$$

$$+ \frac{x_4^2}{2} s_4 + x_2 x_3 s_2 s_3 K_2 + \frac{x_2^2 x_3^2 s_2 s_3 K_2}{2} - x_2 x_3 s_2 s_3 K_2 x_4 + x_3 x_4 s_3 s_4 K_3$$

$$+ \frac{x_3^2 x_4^2 \; s_3 s_4 K_3}{2} - x_3 x_4 s_3 s_4 K_3 x_2 + x_2 x_3 x_4 \, s_2 s_3 s_4 K_2 K_3 \phi_2 +$$

$$+ \frac{x_2^2 x_3^2 \; x_4^2 \; s_2 s_3 s_4 K_2 K_3 \phi_2}{2} \Big\}$$

$$\exp\Big\{ -y_1 (1 - x_2 + x_2 s_2 K_1 - x_2 s_2 x_3 K_1 + x_2 x_3 s_2 s_3 K_1 K_2 \phi_1 + x_2 x_3 x_4 s_2 s_3 s_4 K_1 K_2 K_3 \phi_1 \phi_2 \eta$$

$$- y_5 (1 - x_4 + x_4 s_4 K_4 - x_4 s_4 x_3 K_4 + x_3 x_4 s_3 s_4 K_3 K_4 \phi_3 + x_2 x_3 x_4 s_2 s_3 s_4 K_2 K_3 K_4 \phi_2 \phi_3 \eta_2)$$

$$+ y_1 y_5 \, x_2 x_3 x_4 s_2 s_3 s_4 K_1 K_2 K_3 K_4 \phi_1 \phi_2 \phi_3 \eta_1 \eta_2 \, x \Big\} , \qquad (3\text{-}3)$$

where we have used $(1-x)^{-s} \sim \exp\{xs + \frac{x^2 s}{2}\}$ as we shall shortly let $-s_2, -s_3, -s_4 \to \infty$ while keeping $x_2(-s_2)^{1/2}$, $x_4(-s_4)^{1/2}$ and $x_3(-s_3)^{1/2}$ fixed. We replace the last term in the second exponential in (3-3) by a helicity-like integral, perform the y_1 and y_5 integrations, let $-s_2, -s_3, -s_4 \to \infty$ and take the limit of the resultant ${}_1F_1$ functions, obtaining:

$$B_8 \sim \tau_1 \tau_5 \; S_1^{\alpha_1} \; S_5^{\alpha_5} \int_0^1 dx_2 dx_3 dx_4 \; \frac{x_2^{-\alpha_2-1}}{x_2 s_2} \; \frac{x_3^{-\alpha_3-1}}{x_3 s_3} \; \frac{x_4^{-\alpha_4-1}}{x_4 s_4} \left(- \frac{1}{K_2 K_3 \phi_2} \right)$$

$$\exp\Big\{ \frac{x_2^2 s_2}{2} + \frac{x_3^2 s_3}{2} + \frac{x_4^2 s_4}{2} + \frac{x_2^2 x_3^2 s_2 s_3 K_2}{2} + \frac{x_3^2 x_4^2 s_3 s_4 K_3}{2}$$

$$+ \frac{x_2^2 \, x_3^2 \, x_4^2 \, s_2 s_3 s_4 K_2 K_3 \, \phi_2}{2} \Big\} \left(-K_1 \phi_1 \eta_1\right)^{\alpha_1} \left(-K_4 \phi_3 \eta_2\right)^{\alpha_5} \times$$

$$\left\{\chi^{\alpha_1}\Gamma(\alpha_1-\alpha_5)\Gamma(\alpha_5-\alpha_1+1)+\alpha_1\longleftrightarrow\alpha_5\right\}e^{-\frac{1}{K_1}-\frac{1}{K_2}-\frac{1}{K_3}-\frac{1}{K_4}}\ .$$

(3-4)

Next we change the variables $X_i = \frac{y_i}{(-s_i)^{1/2}}$, i = 2,3,4, let $-s_i\rightarrow\infty$ drop the remaining unimportant factors and obtain

$$B_8\sim\tau_1\tau_5\ s_1^{\alpha_1}(-s_2)^{\beta_2}(-s_3)^{\beta_3}(-s_4)^{\beta_4}\ s_5^{\alpha_5}\left(-\frac{1}{K_2K_3\phi_2}\right)\left(-K_1\phi_1\eta_1\right)^{\alpha_1}\left(-K_4\phi_3\eta_2\right)^{\alpha_5}$$

$$\left\{\chi^{\alpha_1}\Gamma(\alpha_1-\alpha_5)\Gamma(\alpha_5-\alpha_1)+\alpha_1\longleftrightarrow\alpha_5\right\}2^{-\beta_2-\beta_3-\beta_4-5}\quad e^{-\frac{1}{K_1}-\frac{1}{K_2}-\frac{1}{K_3}-\frac{1}{K_4}}$$

$$\int_0^\infty du\,dw\,dz\quad u^{-\beta_2-2}\ w^{-\beta_3-2}\ z^{-\beta_4-2}\ \exp\{-u-w-z+uw(2K_2)$$

$$+wz(2K_3)-uwz(2K_2)(2K_3)\phi_2\}\ ,\qquad (3\text{-}5)$$

where we have also performed a further change of variables in the last integral to exhibit its similarity with a corresponding integral which arises in the study of a triple α exchange contribution to the multi-Regge limit of the six point function[16].

We perform the necessary continuation letting $\chi = e^{-2\pi i}$, comparing with the corresponding result for the triple α exchange we observe that the only difference is the additional exponential dependence on the Toller angles and the replacement $K_i\longrightarrow 2K_i$, $\alpha\rightarrow\beta+1$, in the final expression. The ϕ dependence is again given by the ${}_2F_1$ function.

Next we discuss the contribution of the diagram (14b). We start with the expression for the diagram in Fig.(15b)[12].

$$B'_8 = \tau_1\tau_2\tau_5 \int_0^1 dX_1 dX_2 dX_3 dX_4 dX_5 \; X_1^{-\alpha(S_{13})-1} X_2^{-\alpha_2-1} X_3^{-\alpha_3-1} X_4^{-\alpha_4-1} X_5^{-\alpha_5-1}$$

$$(1-X_1)^{-\alpha_1-1} (1-X_2)^{-\alpha(S_{24})-1} (1-X_3)^{-\alpha(S_3)-1} (1-X_4)^{-\alpha(S_4)-1} (1-X_5)^{-\alpha(S_{68})-1}$$

$$(1-X_1X_2)^{-\alpha(S_{124})+\alpha(S_{24})+\alpha_1} \quad (1-X_2X_3)^{-\alpha(S_{245})+\alpha(S_{24})+\alpha(S_3)}$$

$$(1-X_3X_4)^{-\alpha(S_{456})+\alpha(S_3)+\alpha(S_4)} \quad (1-X_4X_5)^{-\alpha(S_{568})+\alpha(S_4)+\alpha(S_{68})}$$

$$(1-X_1X_2X_3)^{-\alpha(S_{1245})+\alpha(S_{124})+\alpha(S_{245})-\alpha(S_{24})} \quad (1-X_2X_3X_4)^{-\alpha(S_{2456})+\alpha(S_{245})+\alpha(S_{456})-\alpha(}$$

$$(1-X_3X_4X_5)^{-\alpha(S_{4568})-\alpha(S_4)+\alpha(S_{456})+\alpha(S_{56})}$$

$$(1-X_1X_2X_3X_4)^{-\alpha(S_{578})-\alpha(S_{245})+\alpha(S_{1245})+\alpha(S_{2456})}$$

$$(1-X_2X_3X_4X_5)^{-\alpha(S_{137})-\alpha(S_{456})+\alpha(S_{2456})+\alpha(S_{4568})}$$

$$(1-X_1X_2X_3X_4X_5)^{-\alpha(S_{37})-\alpha(S_{2456})+\alpha(S_{378})+\alpha(S_{137})} \tag{3-6}$$

introduce U_1 by $X_1=1-U_1$ and proceed as in the previous case remembering that in the multi-Regge limit

$$S_{13} \sim -S_1 \rightarrow -\infty$$

$$S_{24} \sim S_1S_2K_1 \rightarrow +\infty$$

$$S_{68} \sim -S_5 \rightarrow -\infty$$

$$S_{124} \sim -S_2 \rightarrow -\infty$$

$$S_{245} \sim S_1S_2S_3K_1K_2\phi_1 \rightarrow +\infty$$

$$S_{568} \sim -S_4S_5K_4 \rightarrow -\infty$$

$$S_{456} \sim S_3S_4K_3 \rightarrow +\infty$$

$$S_{1245} \sim -S_2S_3K_2 \rightarrow -\infty$$

$$S_{2456} \sim S_1S_2S_3S_4K_1K_2K_3\phi_1\phi_2\eta_1 \rightarrow +\infty$$

$$S_{4568} \sim -S_3S_4S_5K_3K_4\phi_3 \rightarrow -\infty$$

$$S_{137} \sim -s_1 s_2 s_3 s_4 s_5 K_1 K_2 K_3 K_4 \phi_1 \phi_2 \phi_3 \eta_1 \eta_2 \chi \longrightarrow -\infty$$

$$S_{378} \sim -s_2 s_3 s_4 K_2 K_3 \phi_2 \longrightarrow -\infty$$

$$S_{37} \sim s_2 s_3 s_4 s_5 K_2 K_4 \phi_2 \phi_3 \eta_2 \longrightarrow +\infty \quad .$$

we obtain

$$B_8' \cong \tau_1 \tau_2 \tau_5 \, s_1^{\alpha_1} s_5^{\alpha_5} \int_0^1 dx_2 \, dx_3 \, dx_4 \, x_2^{-\alpha_2-1} x_3^{-\alpha_3-1} x_4^{-\alpha_4-1} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda$$

$$\Gamma(-\lambda)\Gamma(\lambda-\alpha_1)\Gamma(\lambda-\alpha_5)\left(\frac{AB}{c\chi}\right)^{-\lambda} A^{\alpha_1} B^{\alpha_5} \exp\Big\{ -x_2 s_2 \frac{1-x_3}{1-x_2x_3}$$

$$- x_2^2 s_2 \frac{(1-x_3)^2}{2(1-x_2x_3)^2} + x_3 s_3 \frac{(1-x_2)(1-x_4)}{(1-x_2x_3)(1-x_3x_4)} + x_3^2 s_3 \frac{(1-x_2)^2(1-x_4)^2}{2(1-x_2x_3)^2(1-x_3x_4)^2}$$

$$+ x_4 s_4 \frac{1-x_3}{1-x_3x_4} + x_4^2 s_4 \frac{(1-x_3)^2}{2(1-x_3x_4)^2} - x_2x_3 s_2 s_3 K_2 \frac{1-x_4}{1-x_2x_3x_4}$$

$$- x_2^2 x_3^2 s_2 s_3 K_2 \frac{(1-x_4)^2}{2(1-x_2x_3x_4)^2} + x_3 x_4 s_3 s_4 K_3 \frac{1-x_2}{1-x_2x_3x_4}$$

$$+ x_3^2 x_4^2 s_3 s_4 K_3 \frac{(1-x_2)^2}{2(1-x_2x_3x_4)^2} - x_2x_3x_4 s_2 s_3 s_4 K_2 K_3 \phi_2$$

$$- \frac{x_2^2 x_3^2 x_4^2 s_2 s_3 s_4 K_2 K_3 \phi_2}{2} \Big\} , \qquad (3\text{-}7)$$

where

$$A = 1 - x_2 s_2 K_1 \frac{(1-x_3)}{(1-x_2)(1-x_2x_3)} - x_2 x_3 s_2 s_3 K_1 K_2 \phi_1 \frac{1-x_4}{(1-x_2x_3)(1-x_2x_3x_4)}$$

$$- x_2 x_3 x_4 s_2 s_3 s_4 K_1 K_2 K_3 \phi_1 \phi_2 \eta_1 \frac{1}{1-x_2x_3x_4}$$

$$B = 1 - x_4 + x_4 s_4 K_4 (1-x_3) + x_3 x_4 s_3 s_4 K_3 K_4 \phi_3 (1-x_2)$$

$$- x_2 x_3 x_4 s_2 s_3 s_4 K_2 K_3 K_4 \phi_2 \phi_3 \eta_2$$

$$C = x_2 x_3 x_4 s_2 s_3 s_4 \kappa_1 \kappa_2 \kappa_3 \kappa_4 \phi_1 \phi_2 \phi_3 \eta_1 \eta_2 \ .$$

We proceed as before and perform the λ integration, rewrite the obtained Ψ function in terms of ${}_1F_1$ functions and then, letting $s_2 s_3 s_4 \kappa_2 \kappa_3 \phi_2 \longrightarrow \infty$, we replace each of them by its asymptotic form. The factor $\exp(\frac{AB}{c})$ in the asymptotic form of each ${}_1F_1$ function leads to the cancellation of some terms in the exponential in (3-7) in such a way that when $x, \eta_1, \eta_2, \kappa_1, \kappa_4$ are continued to their correct values, and s_2 to $s_2 = -s_2 e^{-i\pi}$ we obtain

$$B_8 \sim 2\pi i \tau_1 \tau_2 \tau_5 (\kappa_1 s_1)^{\alpha_1} e^{-\frac{1}{\kappa_1}} (\kappa_4 s_5)^{\alpha_5} e^{-\frac{1}{\kappa_4}} e^{-\frac{1}{\kappa_2}-\frac{1}{\kappa_3}} \int_0^1 dx_2 dx_3 dx_4$$

$$x_2^{-\alpha_2-1} x_3^{-\alpha_3-1} x_4^{-\alpha_4-1} \exp\left\{\frac{x_2^2 s_2}{2} + \frac{x_3^2 s_3}{2} + \frac{x_4^2 s_4}{2} + \frac{x_2^2 x_3^2 s_2 s_3 \kappa_2}{2}\right.$$

$$\left. + \frac{x_3^2 s_3 x_4^2 s_4 \kappa_3}{2} + \frac{x_2^2 x_3^2 x_4^2 s_2 s_3 s_4 \kappa_2 \kappa_3 \phi_2}{2}\right\} \frac{1}{x_2 x_3 x_4 s_2 s_3 s_4 \kappa_2 \kappa_3} ,$$

(3-8)

where again we have kept only the lowest non-vanishing powers of x_i in the coefficient of different terms in the exponential. Notice that although the coefficients of some terms in (3-7) were negative the cancellation with the terms from ${}_1F_1$ functions effectively changed the sign of all of them to being positive. The obtained result is the same, apart from the τ_2 factor, as the expression (3-5) corresponding to Fig.(14a).

A similar discussion can be given for other diagrams of Fig.(14). For the diagram of Fig.(14c) we start with the Veneziano amplitude for the diagram in Fig.(15c) and then change the variables $x_4 = 1 - u_4 u_5$, $x_5 = 1 - u_5$, The calculation along the lines as before, again gives the same result; i.e. we obtain (3-8) this time with $\tau_1\tau_3\tau_5$ factors. Similarly for all the other diagrams; for the diagram of Fig.(14e) corresponding to Fig.(15e) we change the variables $x_1=1-u_1$, $x_4=1-u_4u_5$, $x_5=1-u_5$ for that of Fig.(14f) corresponding to Fig.(15f) we take $x_1=1-u_1$, $x_5=1-u_5$ and for the one of Fig.(14h)(15h) we take $x_1=1-u_1$, $x_4=1-u_4$, $x_5=1-u_4u_5$ Straightforward although tedious calculations show that the result is again essentially of the same form.

Closer examination of all analytical continations required shows that although all eight diagrams lead to a similar expression the variables K_2, K_3 and ϕ_2 which appear in (3-8) have to be continued to different values. Thus

$\phi_2 = 1$ for diagrams 14-a, b, c, d, e and g

$\phi_2 = e^{-2\pi i}$ for 14-f and $\phi_2 = e^{+2\pi i}$ for 14-h

$K_2 = -K_2 e^{i\pi}$ for 14-a, b, c, d, f and g, and

$K_2 = -K_2 e^{-i\pi}$ for 14-e and 14-h

and similarly for K_3. To obtain the complete contribution we have to add all those eight terms continued to their correct values. Then we can compare the obtained result with the expression for the complete two β - Reggeon particle vertex in order to investigate the Regge factorization. We choose to follow the method employed by Weis in his study of the α trajectory factorization.[15]

We define

$$V(\beta_1,\beta_2;K\pm i\varepsilon)=\frac{e^{-\frac{1}{K}}}{K}\,2^{-\beta_1-\beta_2-4}\int_0^\infty dz_1dz_2\,z_1^{-\beta_1-2}\,z_2^{-\beta_2-2}\exp\{-z_1-z_2$$

$$+z_1z_2\,2\,(K\pm i\varepsilon)$$

$$=2^{-\beta_1-\beta_2-4}\,e^{-\frac{1}{K}}\,K^{-1}\,\Gamma(-\beta_1-1)\int_0^\infty dz_2\,z_2^{-\beta_2-2}\,e^{-z_2}\left[1-2z_2(K\pm i\varepsilon)\right]^{\beta_1+1}$$

$$=2^{-\beta_1-\beta_2-4}\,e^{-\frac{1}{K}}\,\frac{1}{K}\,\Gamma(-\beta_1-1)\left[I_1(\beta_1,\beta_2,K)+e^{\mp i\pi(\beta_1+1)}I_2(\beta_1,\beta_2,K)\right]$$

where I_1 and I_2 are given by

$$I_1=\int_0^{1/2K}dz_2\,z_2^{-\beta_2-2}\,e^{-z_2}\,(1-2z_2K)^{1+\beta_1}$$

$$I_2=\int_{1/2K}^{\infty}dz_2\,z_2^{-\beta_2-2}\,e^{-z_2}\,(2z_2K-1)^{1+\beta_1}$$

and are real for $K\geqslant 0$.

With these definitions we see that the full contribution of the seven point function is given by

$$B_7=2\pi i\,\tau_1K_1^{\alpha_1}S_1^{\alpha_1}e^{-\frac{1}{K_1}}\,\tau_4K_3^{\alpha_4}S_4^{\alpha_4}e^{-\frac{1}{K_3}}\,S_2^{\beta_2}S_3^{\beta_3}\,e^{-i\pi\beta_2-i\pi\beta_3}\,2^{-\beta_2-\beta_3-4}$$

$$e^{-\frac{1}{K_2}}\frac{1}{K_2}\,\Gamma(-1-\beta_2)\Big\{(1+\tau_2)(1+\tau_3)\left[I_1(\beta_2,\beta_3,K_2)-e^{i\pi\beta_2}I_2(\beta_2,\beta_3K_2)\right]$$

$$+2i\tau_2\tau_3\sin\pi\beta_2\,I_2(\beta_2,\beta_3,K_2)\Big\}. \tag{3-9}$$

A similar discussion of the complete 3β contribution to the eight point function gives

$$B_8=2\pi i\,\tau_1S_1^{\alpha_1}K_1^{\alpha_1}e^{-\frac{1}{K_1}}\,\tau_5K_4^{\alpha_5}S_5^{\alpha_5}e^{-\frac{1}{K_4}}\,S_2^{\beta_2}S_3^{\beta_3}S_4^{\beta_4}\,e^{-i\pi\beta_2-i\pi\beta_3-i\pi\beta_4}$$

$$2^{-\beta_2-\beta_3-\beta_4-6}\,\exp\left\{-\frac{1}{K_2}-\frac{1}{K_3}\right\}\frac{1}{K_2K_3}\,\Gamma(-1-\beta_2)\times$$

$$\Big\{\left[I_1(1)-e^{i\pi\beta_2}I_2(1)\right]\cdot\left[I_1(2)-e^{i\pi\beta_3}I_2(2)\right](1+\tau_2+\tau_3+\tau_4)$$

$$+\tau_2\tau_3\left[I_1(1)-e^{-i\pi\beta_2}I_2(1)\right]\cdot\left[I_1(2)-e^{-i\pi\beta_3}I_2(2)\right]$$

$$+\tau_3\tau_4\left[I_1(1)-e^{i\pi\beta_2}I_2(1)\right]\cdot\left[I_1(2)-e^{-i\pi\beta_3}I_2(2)\right]$$

$$+\tau_2\tau_4\left[I_1(1)I_1(2)-e^{i\pi\beta_3}I_1(1)I_2(2)-e^{i\pi\beta_2}I_2(1)I_1(2)+\right.$$

$$\left.+e^{-i\pi\beta_2+i\pi\beta_3}I_2(1)I_2(2)\right]+\tau_2\tau_3\tau_4\left[I_1(1)I_1(2)-e^{-i\pi\beta_3}I_1(1)I_2(2)\right.$$

$$\left.-e^{-i\pi\beta_2}I_2(1)I_1(2)+e^{i\pi\beta_2-i\pi\beta_3}I_2(1)I_2(2)\right]\Big\}, \qquad (3\text{-}10)$$

where $I_i(1)=I_i(\beta_2,\beta_3,K_2)$ and $I_i(2)=I_i(\beta_3,\beta_4,K_3)$.

The expression in the curly brackets can be rewritten as

$$\left[I_1(1)-e^{i\pi\beta_2}I_2(1)\right]\left[I_1(2)-e^{i\pi\beta_3}I_2(2)\right](1+\tau_2)(1+\tau_3)(1+\tau_4)$$

$$+2i\sin\pi\beta_3\,\tau_3\tau_4(1+\tau_2)I_1(1)I_2(2)$$

$$+2i\sin\pi\beta_2\,\tau_2\tau_3(1+\tau_4)I_1(2)I_2(1)$$

$$-2i\sin\pi\beta_2\,e^{i\pi\beta_3}\,\tau_2(\tau_3+\tau_4)I_2(1)I_2(2)$$

$$-2i\sin\pi\beta_3\,e^{i\pi\beta_2}\,\tau_3\tau_4(1+\tau_2)I_2(1)I_2(2)\,. \qquad (3\text{-}11)$$

Comparison with the expression in the curly bracket in (3-9) shows that the expression in (3-11) does not factorise into a product of such terms. This lack of factorization stems from the appearance of additional terms proportional to $\sin\pi\beta_2\cos\pi\beta_3 I_2(1)I_2(2)$ in (3-11) and the mismatch of some coefficients.

Looking at (3-10) we see that when β_3 is continued to -2 the contribution of 3β exchange develops a pole whose residue is proportional to $1+\tau_3$. To see this we notice that when

$\beta_3 \sim -2$

$$I_1(2) \approx I_2(2) \approx \frac{1}{\beta_3+2}\, e^{-\frac{1}{2}K_3} (2K_3)^{1+\beta_4}$$

while

$$I_2(1) \sim e^{-\frac{1}{2}K_2} (2K_2)^{1+\beta_2}\, \Gamma(2+\beta_2).$$

Thus $I_1(2) - I_2(2) \sim$ Constant and the whole contribution of the pole at $\beta_3 = -2$ is contained in last two terms proportional to $\tau_2\tau_4$ and $\tau_2\tau_3\tau_4$ in (3-10). As $\sin\pi\beta_2\,\Gamma(2+\beta_2)\Gamma(-1-\beta_2) = \pi$ we see that the residue of this pole is proportional to $1-\tau_3$. It also has the expected symmetrical form and is not singular at the poles of the trajectories β_2 and β_4. As we mentioned before the structure of the 3β exchange contribution to the eight point function is very similar to the contribution of the 3α exchange to the six point function; in the next section we shall see that the poles of the 3β contribution are cancelled by the corresponding poles of the $\beta\gamma\beta$ exchange.

3-2. The eight point function and the γ trajectory

To exhibit the complete contribution of this exchange and to establish the behaviour of the γ trajectory under twisting we have to return to the unintegrated form of the 3β contribution given in eq. (3-8) and observe that as in the 3α case the appearance of the further trajectory stems from the necessity of continuation of ϕ_2 to $e^{\pm 2\pi i}$ which through the additional helicity integration and limits of ${}_1F_1$ functions leads to an additional factor $\sim \exp\{-\frac{x_3^2}{2} s_3\}$ which in turn allows for the scaling $x_3^3 s_3 =$ constant. As $\phi_2 \neq 1$ only for the diagrams of Figs.(14f and 14h) we have a complete parallel with the six point case.

To find the contribution of Fig.(14f) we start with the expression corresponding to the diagram (15f)^{11}:

$$
\begin{aligned}
B_8^{\gamma} = \tau_1\tau_2\tau_4\tau_5 \int_0^1 & dx_1\, dx_2\, dx_3\, dx_4\, dx_5\; x_1^{-\alpha(s_{13})-1}\; x_2^{-\alpha_2-1}\; x_3^{-\alpha_3-1}\; x_4^{-\alpha_4-1}\; x_5^{-\alpha(s_{68})-1} \\
& (1-x_1)^{-\alpha_1-1}\; (1-x_2)^{-\alpha(s_{24})-1}\; (1-x_3)^{-\alpha(s_{45})-1}\; (1-x_4)^{-\alpha(s_{57})-1}\; (1-x_5)^{-\alpha_5-1} \\
& (1-x_1x_2)^{-\alpha(s_{124})+\alpha(s_{12})+\alpha(s_{24})}\; (1-x_2x_3)^{-\alpha(s_{245})+\alpha(s_{24})+\alpha(s_{45})} \\
& (1-x_3x_4)^{-\alpha(s_{457})+\alpha(s_{45})+\alpha(s_{57})}\; (1-x_4x_5)^{-\alpha(s_{578})+\alpha(s_{57})+\alpha(s_{78})} \\
& (1-x_1x_2x_3)^{-\alpha(s_{1245})-\alpha(s_{24})+\alpha(s_{124})+\alpha(s_{245})} \\
& (1-x_2x_3x_4)^{-\alpha(s_{2457})-\alpha(s_{45})+\alpha(s_{245})+\alpha(s_{457})} \\
& (1-x_3x_4x_5)^{-\alpha(s_{4578})-\alpha(s_{57})+\alpha(s_{457})+\alpha(s_{578})} \\
& (1-x_1x_2x_3x_4)^{-\alpha(s_{12457})-\alpha(s_{245})+\alpha(s_{1245})+\alpha(s_{2457})} \\
& (1-x_2x_3x_4x_5)^{-\alpha(s_{24578})-\alpha(s_{457})+\alpha(s_{2457})+\alpha(s_{4578})} \\
& (1-x_1x_2x_3x_4x_5)^{-\alpha(s_{124578})-\alpha(s_{2457})+\alpha(s_{12457})+\alpha(s_{24578})} \; .
\end{aligned}
\tag{3-12}
$$

In the multi-Regge limit

$$S_{13} \sim -S_1 < 0$$

$$S_{124} \sim -S_2 < 0$$

$$S_{45} \sim +S_3 < 0$$

$$S_{875} \sim -S_4 < 0$$

$$S_{68} \sim -S_5 < 0$$

$$S_{24} \sim S_1 S_2 K_1 < 0$$

$$S_{1245} \sim -S_2 S_3 K_2 < 0$$

$$S_{8754} \sim -S_3 S_4 K_3 < 0$$

$$S_{57} \sim S_4 S_5 K_4 < 0$$

$$S_{245} \sim S_1 S_2 S_3 K_1 K_2 \phi_1 < 0$$

$$S_{36} \sim S_2 S_3 S_4 K_2 K_3 \phi_2 < 0$$

$$S_{457} \sim S_3 S_4 S_5 K_3 K_4 \phi_3 < 0$$

$$S_{124578} \sim -S_1 S_2 S_3 S_4 K_1 K_2 K_3 \phi_1 \phi_2 \eta_1 < 0$$

$$S_{368} \sim -S_2 S_3 S_4 S_5 K_2 K_3 K_4 \phi_2 \phi_3 \eta_2 < 0$$

$$S_{2457} \sim S_1 S_2 S_3 S_4 S_5 K_1 K_2 K_3 K_4 \phi_1 \phi_2 \phi_3 \eta_1 \eta_2 \chi < 0 \quad .$$

The change of variables of integration

$$X_1 = 1 - u_1 \quad , \quad X_5 = 1 - u_5$$

gives us

$$B_8^{\gamma} \approx \tau_1 \tau_2 \tau_4 \tau_5 \int_0^1 du_1 dX_2 dX_3 dX_4 du_5 \; u_1^{-\alpha_1 - 1} \; X_2^{-\alpha_2 - 1} \; X_3^{-\alpha_3 - 1} \; X_4^{-\alpha_4 - 1} \; u_5^{-\alpha_5 - 1}$$

$$(1 - u_1)^{S_1 - 1} \left(1 - \frac{X_2(1 - X_3)}{1 - X_2 X_3}\right)^{S_2} \left[1 - \frac{X_3(1 - X_2)(1 - X_4)}{(1 - X_2 X_3)(1 - X_3 X_4)}\right]^{-S_3} \left(1 - \frac{X_4(1 - X_3)}{1 - X_3 X_4}\right)^{S_4} \times$$

$$(1-u_5)^{S_5-1} \left[1-\frac{u_1x_2(1-x_3)}{(1-x_2)(1-x_2x_3)}\right]^{-S_1S_2K_1} \left[1-\frac{x_2x_3(1-x_4)}{1-x_2x_3x_4}\right]^{S_2S_3K_2}$$

$$\left[1-\frac{x_3x_4(1-x_2)}{1-x_2x_3x_4}\right]^{S_3S_4K_3} \left[1-\frac{x_4u_5(1-x_3)}{(1-x_4)(1-x_3x_4)}\right]^{-S_4S_5K_4}$$

$$\left[1-\frac{u_1x_2x_3(1-x_4)}{(1-x_2x_3)(1-x_2x_3x_4)}\right]^{-S_1S_2S_3K_1K_2\phi_1} (1-x_2x_3x_4)^{-S_2S_3S_4K_2K_3\phi_2}$$

$$\left[1-\frac{x_3x_4u_5(1-x_2)}{(1-x_2x_3x_4)(1-x_3x_4)}\right]^{-S_3S_4S_5K_3K_4\phi_3} \left(1-\frac{u_1x_2x_3x_4}{1-x_2x_3x_4}\right)^{S_1S_2S_3S_4K_1K_2K_3\phi_1\phi_2\eta_1}$$

$$\left[1-\frac{x_2x_3x_4u_5}{1-x_2x_3x_4}\right]^{S_2S_3S_4S_5K_2K_3K_4\phi_2\phi_3\eta_2}$$

$$\left[1-\frac{u_1x_2x_3x_4u_5}{(1-x_2x_3x_4)^2}\right]^{-S_1S_2S_3S_4S_5K_1K_2K_3K_4\phi_1\phi_2\phi_3\eta_1\eta_2\chi} . \qquad (3\text{-}13)$$

Next we temporarily keep S_2, S_3, S_4 fixed and let $S_1, S_5 \to \infty$. Then we change the variables of integration

$$u_1 = \frac{y_1}{S_1} \quad , \quad u_5 = \frac{y_5}{S_5}$$

and obtain

$$B_8^{\gamma} \approx \tau_1\tau_2\tau_4\tau_5 \, S_1^{\alpha_1} S_5^{\alpha_5} \int_0^{\infty} dy_1 \, dy_5 \int_0^1 dx_2 \, dx_3 \, dx_4$$

$$y_1^{-\alpha_1-1} \quad x_2^{-\alpha_2-1} \quad x_3^{-\alpha_3-1} \quad x_4^{-\alpha_4-1} \quad y_5^{-\alpha_5-1} \quad \times$$

$$\exp\Big\{-y_1\Big[1 - x_2 s_2 K_1 \frac{1-x_3}{(1-x_2)(1-x_2x_3)} - x_2x_3s_2s_3K_1K_2\phi_1 \frac{1-x_4}{(1-x_2x_3)(1-x_2x_3x_4)}$$

$$+ \frac{x_2x_3x_4s_2s_3s_4K_1K_2K_3\phi_1\phi_2\eta_1}{1-x_2x_3x_4}\Big]$$

$$-y_5\Big[1 - x_4s_4K_4\frac{1-x_3}{(1-x_4)(1-x_3x_4)} - x_3x_4s_3s_4K_3K_4\phi_3\frac{1-x_2}{(1-x_3x_4)(1-x_2x_3x_4}$$

$$+ \frac{x_2x_3x_4s_2s_3s_4K_2K_3K_4\phi_2\phi_3\eta_2}{1-x_2x_3x_4}\Big]\Big\}$$

$$\exp\Big\{-x_2s_2 - \frac{x_2^2s_2}{2} + x_2s_2x_3 + x_3s_3 + \frac{x_3^2}{2}s_3 + \frac{x_3^3}{3}s_3 - x_3s_3x_2 - x_3s_3x_4$$

$$-x_4s_4 - \frac{x_4^2s_4}{2} + x_4s_4x_3 - x_2x_3s_2s_3K_2 - \frac{x_2^2x_3^2s_2s_3K_2}{2} +$$

$$+x_2x_3x_4s_2s_3K_2 - x_3x_4s_3s_4K_3 - \frac{x_3^2x_4^2s_3s_4K_3}{2} + x_3x_4x_2s_3s_4K_3$$

$$+ x_2x_3x_4s_2s_3s_4K_2K_3\phi_2 + \frac{x_2^2x_3^2x_4^2s_2s_3s_4K_2K_3\phi_2}{2}\Big\}. \tag{3-14}$$

Next we replace the last term in the first exponential in (3-14) by a helicity-like integral, perform the y_1 and y_5 integrations and obtain

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda\,\Gamma(-\lambda)\Gamma(\lambda-\alpha_1)\Gamma(\lambda-\alpha_5)\left(\frac{AB}{Cx}\right)^{-\lambda} A^{\alpha_1} B^{\alpha_5}$$

where

$$A = 1 - x_2s_2K_1\frac{1-x_3}{(1-x_2)(1-x_2x_3)} - x_2x_3s_2s_3K_1K_2\phi_1\frac{1-x_4}{(1-x_2x_3)(1-x_2x_3x_4)}$$

$$+ \frac{x_2x_3x_4s_2s_3s_4K_1K_2K_3\phi_1\phi_2\eta_1}{1-x_2x_3x_4}$$

$$B = 1 - X_4 S_4 K_4 \frac{1 - X_3}{(1 - X_4)(1 - X_3 X_4)} - X_3 X_4 S_3 S_4 K_3 K_4 \phi_3 \frac{1 - X_2}{(1 - X_3 X_4)(1 - X_2 X_3 X_4)}$$

$$+ \frac{X_2 X_3 X_4 S_2 S_3 S_4 K_2 K_3 K_4 \phi_2 \phi_3 \eta_2}{1 - X_2 X_3 X_4}$$

$$C = - \frac{X_2 X_3 X_4 S_2 S_3 S_4 K_1 K_2 K_3 K_4 \phi_1 \phi_2 \phi_3 \eta_1 \eta_2}{(1 - X_2 X_3 X_4)^2} \, .$$

We perform the λ integration and take the limit of the resultant $_1F_1$ functions obtaining

$$B_g^{\gamma} \approx \tau_1 \tau_2 \tau_4 \tau_5 \, S_1^{\alpha_1} S_5^{\alpha_5} \int_0^1 dX_2 \, dX_3 \, dX_4 \; \frac{X_2^{-\alpha_2 - 1} \; X_3^{-\alpha_3 - 1} \; X_4^{-\alpha_4 - 1}}{- X_2 S_2 X_3 S_3 X_4 S_4 K_2 K_3 \phi_2}$$

$$\exp \Big\{ \frac{X_2^2 S_2}{2} + \frac{X_3^2 S_3}{2} + \frac{X_3^3 S_3}{3} + \frac{X_4^2 S_4}{2} +$$

$$\frac{X_2^2 X_3^2 S_2 S_3 K_2}{2} + \frac{X_3^2 X_4^2 S_3 S_4 K_3}{2} +$$

$$\frac{X_2^2 X_3^2 X_4^2 S_2 S_3 S_4 K_2 K_3 \phi_2}{2} \Big\} (- K_1 \phi_1 \eta_1)^{\alpha_1} (- K_4 \phi_3 \eta_2)^{\alpha_5}$$

$$\Big\{ \chi^{\alpha_1} \Gamma(\alpha_1 - \alpha_5) \Gamma(\alpha_5 - \alpha_1 + 1) + \chi^{\alpha_5} \Gamma(\alpha_5 - \alpha_1) \Gamma(\alpha_1 - \alpha_5 + 1) \Big\}$$

$$e^{- \frac{1}{K_1} - \frac{1}{K_2} - \frac{1}{K_3} - \frac{1}{K_4}} \, .$$

(3-15)

A further change of variables

$$X_2 = \frac{y_2}{(-s_2)^{1/2}} \quad , \quad X_4 = \frac{y_4}{(-s_4)^{1/2}} \quad , \qquad X_3 = \frac{y_3}{(-s_3)^{1/3}} \quad ,$$

and the introduction of a further helicity-like integral replacing the last term in the exponential in (3-15) lead in a similar way to

$$\begin{aligned} B_8^{\gamma} \sim \tau_1\tau_2\tau_4\tau_5 \; 2^{-\beta_2-\beta_4-2} \; 3^{-\gamma_3-3} \; \Gamma(-\gamma_3-2) \; s_1^{\alpha_1} \, (-s_2)^{\beta_2} \, (-s_3)^{\gamma_3} \, (-s_4)^{\beta_4} \, s_5^{\alpha_5} \\ \exp\left\{ -\frac{1}{K_1} - \frac{1}{K_4} - \frac{3}{2K_2} - \frac{3}{2K_3} \right\} (-K_1)^{\alpha_1} \, (-K_4)^{\alpha_5} \, (-2K_2)^{\beta_2} \, (-2K_3)^{\beta_4} \\ \phi_1^{\alpha_1} \, \phi_3^{\alpha_5} \, \eta_1^{\alpha_1} \, \eta_2^{\alpha_5} \; \left\{ \chi^{\alpha_1} \, \Gamma(\alpha_1-\alpha_5)\Gamma(\alpha_5-\alpha_1+1) + \alpha_1 \longleftrightarrow \alpha_5 \right\} \\ \left\{ \phi_2^{\beta_2} \, \Gamma(\beta_2-\beta_4)\Gamma(\beta_4-\beta_2+1) + \beta_2 \longleftrightarrow \beta_4 \right\} \; , \end{aligned}$$

(3-16)

Where we have used

$$\int_0^1 dx_3 \, \frac{x_3^{-\alpha_3-1}}{(x_3 s_3)(x_3^2 s_3)} \exp\left\{ \frac{x_3^3 \, s_3}{3} \right\} = 3^{-\frac{\alpha_3}{3}-2} \, (-s_3)^{\frac{\alpha_3}{3}-1} \, \Gamma\left(-1-\frac{\alpha_3}{3}\right)$$

$$= 3^{-\gamma_3-3} \, (-s_3)^{\gamma_3} \, \Gamma(-\gamma_3-2)$$

(3-17)

with $\quad \gamma_3 = \frac{1}{3}\alpha_3 - 1 \quad .$

Thus

$$B_8^{\gamma} \approx \tau_1 \tau_2 \tau_4 \tau_5 \; 2\pi^2 \; 3^{-\gamma_3-3} \; \Gamma(-\gamma_3-2) \; e^{-i\pi\beta_2} \; e^{-i\pi\beta_4} \; e^{-i\pi\gamma_3}$$

$$S_1^{\alpha_1} \; S_2^{\beta_2} \; S_3^{\gamma_3} \; S_4^{\beta_4} \; S_5^{\alpha_5} \; K_1^{\alpha_1} \; K_2^{\beta_2} \; K_3^{\beta_4} \; K_4^{\alpha_5}$$

$$\exp\left\{ -\frac{1}{K_1} - \frac{1}{K_4} - \frac{3}{2K_2} - \frac{3}{2K_3} \right\} .$$

(3-18)

The contribution of diagram (14h) can be calculated in the same way as the contribution of diagram (14f).

We start with the expression corresponding to Fig.(15h), change the variables of integration $X_1 = 1 - U_1$, $X_4 = 1 - U_4$, $X_5 = 1 - U_4 U_5$, and then $U_1 = y_1/S_1$, $U_5 = y_5/S_5$, replace the term in the exponential which contains y_1 , y_5 and all X_i $(i = 2, 3, 4)$ by a helicity-like integral (λ), perform the y_1, y_5 and then λ integration and take the limit of the resultant ${}_1F_1$ functions. We obtain an expression similar to eq.(3-15) with the following differences:

$$\exp\left\{ \frac{X_3^2 S_3}{2} \right\} \quad \text{and} \quad \exp\left\{ \frac{X_2^2 X_3^2 X_4^2 S_2 S_3 S_4 K_2 K_3 \phi_2}{2} \right\}$$ in (3-15) are replaced by:

$$\exp\left\{ -S_3 \left[X_3 + \frac{X_3^2}{2} + \frac{X_3^3}{3} - \frac{X_3}{1-X_3} \right] \right\} \quad \text{and} \quad \exp\left\{ \frac{X_2^2 X_3^2 X_4^2 S_2 S_3 S_4 K_2 K_3 \phi_2}{2(1-X_3)^2} \right\}$$

respectively. Then proceeding exactly as before we obtain the additional factor $\exp\left\{ -\frac{1}{2} \frac{X_3^2 S_3}{(1-X_3)^2} \right\}$ from the asymptotic form of the ${}_1F_1$ functions, ending up with the integral

$$\int_0^1 dx_3 \; x_3^{-\alpha_3-1} \frac{1}{x_3 s_3 x_3^2 s_3} \exp\left\{-\frac{x_3^3 s_3}{2}\right\} = s_3^{\gamma_3} \Gamma(-2-\gamma_3) \,,$$

with

$$\gamma_3 = \frac{\alpha_3}{3} - 1 \,.$$

Putting all factors together we see that the complete contribution of a $\beta\gamma\beta$ exchange to the eight point function is given by

$$B_8 = B_8^{\gamma} + B_8^{\prime\gamma} = 2\pi^2 \tau_1 \tau_2 \tau_4 \tau_5 \, 3^{-\gamma_3-3} \, \Gamma(-2-\gamma_3) \, e^{-i\pi\beta_2 - i\pi\beta_4}$$
$$\dot{s}_1^{\alpha_1} \, s_2^{\beta_2} \, s_4^{\beta_4} \, s_5^{\alpha_5} \, K_1^{\alpha_1} \, K_2^{\beta_2} \, K_3^{\beta_4} \, K_4^{\alpha_5}$$
$$\exp\left\{-\frac{1}{K_1} - \frac{3}{2K_2} - \frac{3}{2K_3} - \frac{1}{K_4}\right\} s_3^{\gamma_3} \left(e^{-i\pi\gamma_3} - \tau_3\right) .$$

(3-19)

Eq. (3-19) shows that on the γ trajectory level the signature factor and the twisting operator are again the same. Also it gives the structure of β-γ- particle vertex which is

$$V(\beta, \gamma; K) \sim K^{\beta} \, e^{-\frac{3}{2K}} \, \frac{1}{\Gamma(-\beta-1)} \,.$$

Chapter Four

Higher point functions and further trajectories

4-1. Higher point functions and further trajectories

The techniques used in the previous chapters can be applied to the higher point functions. Their use allows us to exhibit behaviour corresponding to the exchange of α, β and γ trajectories together with further trajectories from the family discussed by Hoyer[8]. The ten point function receives a contribution corresponding to the triple γ exchange, its full contribution exhibits a pole at the middle $\gamma=-3$ which is cancelled by a pole in a contribution corresponding to the exchange of a new trajectory δ with a slope $\frac{1}{4}$. Only two diagrams exhibit this exchange; they are: the one with twists on all Reggeon lines, and the one with twists on all lines apart from the δ trajectory line (see Figs.16a,16b). To derive the contribution of the δ exchange to the diagram (16a) we start with the Veneziano expression for the diagram (17a) and perform the change of variables

$$X_1 = 1 - u_1 u_2 \qquad X_6 = 1 - u_6$$

$$X_2 = 1 - u_2 \qquad X_7 = 1 - u_7 u_6$$

Next we proceed as in the γ case, except that this time we keep the terms up to X^4 in the exponential corresponding to (3-7) in the previous chapter. Next we introduce a helicity-like integration, perform the u_1 and u_7 integrations and continue $\xi_1 = \frac{S_{1,10}\cdot S_{3,8}}{S_{1,8}\cdot S_{3,10}}$ to $e^{-2\pi i}$.

Next we perform the helicity integral obtaining additional terms in the exponential from the asymptotic form of the resultant $_1F_1$ functions. This procedure is then repeated - this time with u_2^2, u_6^2 and $\rho_2 = \frac{s_{3,8} \cdot s_{4,7}}{s_{3,9} \cdot s_{2,8}}$ and then again with

x_3^3, x_5^3 and $\rho_3 = \frac{s_{47} \cdot s_{56}}{s_{26} \cdot s_{59}}$.

The resultant X_4 integration involves

$$\int_0^1 dx_4 \; x_4^{-\alpha_4-1} \frac{1}{-x_4^6 \, s_4^3} \exp\left\{ \frac{x_4^4 s_4}{4} \right\}$$

$$= (-s_4)^{\delta_4} \; 4^{-\delta_4-4} \; \Gamma(-3-\delta_4)$$

with

$$\delta_4 = \frac{\alpha_4}{4} - \frac{3}{2} .$$

To find the contribution of the diagram (16b), we start from the expression corresponding to the diagram (17b) and perform the change of variables

$x_1 = 1-u_1 u_2$ $\qquad$ $x_6 = 1-u_5 u_6 u_7$

$x_2 = 1-u_2$ $\qquad$ $x_7 = 1-u_5 u_6$

$x_5 = 1-u_5$

Then we proceed as before. The final X_4 integration this time is similar to, (4-1) with a relative negative sign, which shows that the phase factor associated with the complete δ trajectory contribution is given by $e^{-i\pi\delta_4}(1-\tau_4)$.

The next trajectory appears for the first time in the twelve point function and so on. The trajectory with slope $\frac{1}{n}$ appears for the first time in the $2n+2$ point function, Fig. (18a). It appears in the chain shown in Fig. (18) where α_i^K denotes trajectory with slope $\frac{1}{i}$ in the K^{th} channel (i.e. $\alpha_1^1 = \alpha_1$, $\alpha_2^2 = \beta_2$, $\alpha_1^{2n-1} = \alpha_{2n-1}$ in the previous notation).

To exhibit this exchange we proceed exactly as before. In the expression for the $2n+2$ point amplitude we introduce the multi-Regge variables S_i , t_i ; $i = 1, 2, \cdots 2n-1$; K_j , $j = 1, \cdots 2n-2$. and a score of variables corresponding to ϕ_1, ϕ_2 and η of the seven point function. The most important amongst them are those which symmetrically overlap the n^{Th} channel , (see Fig.19), i.e.

$$\rho_1 = \frac{S_{n,n+3} \cdot S_{n+1,n+2}}{S_{n,n+2} \cdot S_{n+1,n+3}} \quad , \quad \rho_2 = \frac{S_{n-1,n+4} \cdot S_{n,n+3}}{S_{n-1,n+3} \cdot S_{n,n+4}}$$

$$\rho_3 = \frac{S_{n-2,n+5} \cdot S_{n-1,n+4}}{S_{n-2,n+4} \cdot S_{n-1,n+5}} \quad , \ldots . \rho_{n-2} = \frac{S_{3,2n} \cdot S_{4,2n-1}}{S_{3,2n-1} \cdot S_{4,2n}}$$

$$\rho_{n-1} = \frac{S_{2,2n+1} \cdot S_{3,2n}}{S_{2,2n} \cdot S_{3,2n+1}} \quad , \qquad (4\text{-}2)$$

Where $S_{iK} = (P_i + P_{i+1} + \cdots P_K)^2$. The successive continuation in $\rho_{n-1}, \rho_{n-2}, \cdots \rho_2, \rho_1$ will, in an onion-peeling way, convert the exchange of $2n-1$, α trajectories into the exchange shown in Fig.(18). There are only two diagrams with ($2n+2$) external legs which contribute to the exchange of a trajectory with slope $\frac{1}{n}$, they are the one with a twist on all Reggeon lines, and the one with a twist on all lines but the n^{Th} Reggeon line. (Figs.18a, 18b).

We start with the simplest diagram namely (18a). In this case all ρ's will have to be continued to $e^{-2\pi i}$.

The order of external particles will be different depending on whether n is even or odd as shown in Figs.(19a, and 19b) respectively. We re-arrange the diagrams (19a) and (19b) as(20a) and (20b) respectively. The expression for (20a) will be as follows

$$B_{2n+2} = \tau_1\tau_2\cdots\tau_{n-1}\tau_{n+1}\cdots\tau_{2n-1}\int_0^1 dx_1\, dx_2 \cdots dx_{n-1}\, dx_n dx_{n+1} \cdots dx_{2n-1}$$

$$x_1^{-\alpha_{n,n-2}-1}\quad x_2^{-\alpha_{n,n-4}-1} \cdots\cdot x_{n-1}^{-\alpha_{n,n-1}-1}\quad x_n^{-\alpha_{n,n+1}-1}\quad x_{n+1}^{-\alpha_{n,n+2}-1} \cdots\cdots x_{2n-1}^{-\alpha_{n,n+7}-1}$$

$$(1-x_1)^{-\alpha_{n-2,n-4}-1} \cdots\cdot (1-x_n)^{-\alpha_{n+1,n+2}-1} \cdots\cdots (1-x_{2n-1})^{-\alpha_{n+7,n+5}-1}$$

$$,\quad (1-x_1x_2)^{-\alpha_{n-2,n-6}+\alpha_{n-2,n-4}+\alpha_{n-4,n-6}} \cdots\cdots$$

$$\left(1-x_1x_2\cdots x_{n-1}x_nx_{n+1}\cdots x_{2n-1}\right)^{-\alpha_{n-2,n+5}+\alpha_{n-2,n+7}+\alpha_{n-4,n+5}-\alpha_{n-4,n+7}} , \tag{4-3}$$

Where $\alpha_{n-i,n+j}$ is the trajectory whose argument $S_{n-i,n+j}$ involves all momenta between $n-i$ and $n-j$ as shown in Fig. (20a). We change the variables of integration by the following rule

$$\begin{aligned} x_{n-2} &= 1-u_{n-2} & x_{n+2} &= 1-u_{n+2} \\ x_1 &= 1-u_{n-2}u_{n-3} & x_{2n-1} &= 1-u_{n+2}u_{n+3} \\ x_{n-3} &= 1-u_{n-2}u_{n-3}u_{n-4} & x_{n+3} &= 1-u_{n+2}u_{n+3}u_{n+4} \\ x_2 &= 1-u_{n-2}u_{n-3}u_{n-4} & x_{2n-2} &= 1-u_{n+2}u_{n+3}u_{n+4}u_{n+5} \\ &\vdots & &\vdots \end{aligned}$$

With x_{n-1}, x_n, and x_{n+1} unchanged, although for convenience renamed u_{n-1}, u_n, and u_{n+1}.

Next, we temporarily keep $S_2, S_3, \ldots\ldots S_{2n-2}$ fixed but let $S_1, S_{2n-1} \rightarrow \infty$. The asymptotic behaviour corresponding to the exchange of α trajectories in these channels is exhibited by the change of variables

$$u_1 = \frac{y_1}{S_1} \quad , \quad u_{2n-1} = \frac{y_{2n-1}}{S_{2n-1}} \ .$$

We re-express all dependence on (u_i , $i = 2, \cdots\cdot 2n-2$) as a product of exponentials in which we perform an expansion in the power series of u_i keeping as many terms as will be required by further steps, and we replace the exponential which involves y_1 and y_5 and the product of all u_i by a helicity-like integral.

Next we perform the y_1 and y_{2n-1} integrations and obtain

$$S_1^{\alpha_1} S_{2n-1}^{\alpha_1^{2n-1}} \int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda\, \Gamma(-\lambda)\Gamma(\lambda-\alpha_1)\Gamma(\lambda-\alpha_1^{2n-1})\, A^{\alpha_1} B^{\alpha_1^{2n-1}} \left(\frac{AB}{C\rho_{n-1}}\right)^{-\lambda}$$

(4-5)

where

$$A = 1 - a_1 u_2 \frac{S_{234}}{S_2} + a_2 u_2 u_3 \frac{S_{2345}}{S_1} - a_3 u_2 u_3 u_4 \frac{S_{23456}}{S_1} +$$

$$\cdots\cdots + (-1)^n a_{n-2} u_2 u_3 \cdots u_{n-1} \frac{S_{23\cdots n-1}}{S_1} + (-1)^n a_{n-1} u_2 u_3 \cdots u_n . \frac{S_{23\cdots n}}{S_1}$$

$$+ (-1)^{n+1} a_n u_2 u_3 \cdots u_{n+1} \frac{S_{2\cdots n+1}}{S_1} + \cdots\cdots$$

$$+ (-1)^{2n-2} a_{2n-3} u_2 u_3 \cdots u_{2n-2} \frac{S_{23\cdots 2n}}{S_1} ,$$

$$B = 1 - b_1 u_{2n-2} \frac{S_{2n-1,2n+1}}{S_{2n-1}} + b_2 u_{2n-2} u_{2n-3} \frac{S_{2n+1,2n-2}}{S_{2n-1}} + \cdots$$

$$\cdots\cdots (-1)^n b_{n-2}\, u_{2n-2} u_{2n-3} \cdots u_{n+1} \frac{S_{2n+1,\, S_{n+2}}}{S_{2n-1}} + (-1) b_{n-1} u_{2n-2} \cdots u_n \frac{S_{2n+1,n+}}{S_{2n-1}}$$

$$\cdots\cdots\cdots\cdots + b_{2n-3}\, u_{2n-2} \cdots u_2 \frac{S_{2n+1,3}}{S_{2n-1}} ,$$

$$C = -c_1 u_2 u_3 \cdots u_{2n-2} \frac{S_{2n+1,2}}{S_1 S_{2n-1}} , \qquad (4-6)$$

where $S_{ij} = (P_i + P_{i+1} + \cdots P_j)^2$ involves all momenta between i, j, as shown in Fig. (19).

The coefficients of a_i , b_i , c_i are functions of u_i and are such that when $u_i \rightarrow 0$, $a_i , b_i , c_i \longrightarrow 1$. their explicit form is not very illuminating, and will not be given here. It can be easily found by the explicit calculation.

Next we perform the helicity-like integral and express the result in terms of two ${}_1F_1$ functions and then calculate the limit of each ${}_1F_1$ function for a large value of its argument (as we shall shortly let $(-s_i) \rightarrow \infty$ while keeping $u_i (-s_i)^P$ fixed with $P < 1$). We obtain an exponential factor $\exp\left(\frac{AB}{C}\right)$ from the asymptotic form of the ${}_1F_1$ functions. After cancellation with the other terms in the integrand we are left with

$$\exp\Big\{ \frac{u_2^2 s_2}{2} + \frac{u_3^2 s_3}{2} + \cdots\cdots + \frac{u_n^2 s_n}{2} + \cdots + \frac{u_{2n-2}^2 s_{2n-2}}{2} + \frac{u_2^2 u_3^2 s_2 s_3 K_2}{2} + \cdots\cdots + \frac{u_2^2 u_3^2 u_4^2 s_2 s_3 s_4 K_2 K_3 \phi_1}{2} + \cdots\cdots + \cdots\cdots + \text{higher orders in } u_i \Big\}$$

and a factor $\exp\left\{ -\frac{1}{K_1} - \frac{1}{K_2} - \frac{1}{K_3} \cdots\cdots - \frac{1}{K_{2n-2}} \right\}$.

The other factors which result from the y_1 and y_{2n-1} integrations and helicity integrations are $(-K_1)^{\alpha_1}$, $(-K_{2n-2})^{\alpha_1^{2n-1}}$, $\frac{K_1 K_{2n-1}}{-u_2 u_3 \ldots u_{2n-2} s_{3,2n}}$ and the phase factor necessary for the correct analytic continuation of the amplitude.

Next we repeat the previous procedure this time with u_i^2 replaced by u'_i; we change the variables of integration

$$u_2 = \frac{y_2}{(-s_2)^{1/2}} \quad , \quad u_{2n-2} = \frac{y_{2n-2}}{(-s_{2n-2})^{1/2}}$$

let $-s_2, -s_{2n-2} \longrightarrow \infty$ and write the term which contains $y_2, \; y_{2n-2}$ and all u_i as a helicity-like integral and then perform the y_2 and y_{2n-2} integrations to obtain

$$(-s_2)^{\alpha_2^2} \; (-s_{2n-2})^{\alpha_2^{2n-2}} \; 2^{-\alpha_2^2-\alpha_2^{2n-2}-2} \int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda \, \Gamma(-\lambda)$$

$$\Gamma(\lambda-\alpha_2^2)\Gamma(\lambda-\alpha_2^{2n-2}) \left(\frac{A'B'}{C' s_{n-2}}\right)^{-\lambda} \qquad (4\text{-}8)$$

where

$$A' = 1 + a_1' \, u_3^2 \, \frac{s_{3,5}}{s_2} + a_2' \, u_3^2 u_4^2 \, \frac{s_{3,6}}{s_2} + \cdots$$

$$B' = 1 + b_1' \, u_{2n-3}^2 \, \frac{s_{2n-2,2n}}{s_{2n-1}} + b_2' \, u_{2n-3}^2 \, u_{2n-4}^2 \, \frac{s_{2n-3,2n}}{s_{2n-1}} + \cdots$$

$$C' = -c_1' \, u_3^2 \, u_4^2 \cdots\cdots u_{2n-3}^2 \, \frac{s_{3,2n}}{s_2 s_{2n-2}} \; ,$$

where as before a_i', b_i', c_1' are functions of u_i and are such that when $u_i \longrightarrow 0$, $a_i', b_i', c_1' \longrightarrow 1$.

As before we perform the helicity-like integration, take the limit of the resultant ${}_1F_1$ functions and obtain an exponential factor $\exp\left(\frac{A'B'}{C'}\right)$ which again cancels the leading terms of the other exponential factors leaving us with

$$\exp\left\{-\frac{U_3^3 S_3}{3}-\frac{U_4^3 S_4}{3}-\cdots\cdots-\frac{U_{n-1}^3 S_{n-1}}{3}+\frac{U_n^3 S_n}{3}-\frac{U_{n+1}^3 S_{n+1}}{3}-\cdots\cdots\right.$$

$$-\frac{U_{2n-3}^3 S_{2n-3}}{3}+\frac{U_3^3 U_4^3}{3} S_3 S_4 K_3+\cdots\cdots+\frac{U_{n-2}^3 U_{n-1}^3 S_{n-1,n+1}}{3}$$

$$-\frac{U_{n-1}^3 U_n^3 S_{n,n+2}}{3}-\frac{U_n^3 U_{n+1}^3 S_{n+1,n+3}}{3}+\cdots\cdots$$

$$+\frac{U_{2n-4}^3 U_{2n-3}^3 S_{2n-3,2n-1}}{3}-\frac{U_3^3 U_4^3 U_5^3 S_3 S_4 S_5 K_3 K_4 \phi_2}{3}-\cdots\cdots$$

$$+\frac{U_{n-2}^3 U_{n-1}^3 U_n^3 S_{n-1,n+2}}{3}+\frac{U_{n-1}^3 U_n^3 U_{n+1}^3 S_{n,n+3}}{3}+$$

$$+\frac{U_n^3 U_{n+1}^3 U_{n+2}^3 S_{n+1,n+4}}{3}-\cdots\cdots$$

$$-\frac{U_{2n-5}^3 U_{2n-4}^3 U_{2n-3}^3 S_{2n-5,2n-1}}{3}+\cdots\cdots+\frac{U_3^3 U_4^3 \cdots U_{2n-3}^3 S_{4,2n-1}}{3}$$

$$\left.+\cdots+\text{higher orders in } U_i\right\}. \qquad (4\text{-}9)$$

We also obtain factors:

$$\exp\left\{-\frac{1}{2K_2}-\frac{1}{2K_3}\cdots-\frac{1}{2K_{2n-1}}\right\}, \quad (-K_2)^{\alpha_2^2+1} \quad (-K_{2n-3})^{\alpha_2^{2n-2}+1}$$

$$\cdot\left(-U_3^2 U_4^2 \cdots U_{2n-3}^2 S_{4,2n-1}\right)^{-1}, \text{ where } \alpha_2=\frac{\alpha_1}{2}-\frac{1}{2},$$

Phase factors necessary for the correct analytical continuation of the signaturized amplitude and an overall factor 2^{-2}.

To get the next trajectory we follow the same procedure as before, changing the variables of integrations.

$$u_3 = \frac{y_3}{s_3^{1/3}} \quad , \quad u_{2N-3} = \frac{y_{2N-3}}{(s_{2N-3})^{1/3}}$$

letting $s_3, s_{2N-3} \rightarrow \infty$ and writing the term in the exponential which contains y_3, y_{2N-3} and all u_i as a helicity-like integral and then perform the y_3 and y_{2N-3} integration and obtain

$$s_3^{\alpha_3^3} \quad s_{2N-3}^{\alpha_3^{2N-3}} \quad 3^{-\alpha_3^3 - \alpha_3^{2N-3}} \int_{\gamma - i\infty}^{\gamma + i\infty} d\lambda \, \Gamma(-\lambda)$$

$$\Gamma(\lambda - \alpha_3^3) \, \Gamma(\lambda - \alpha_3^{2N-3}) \quad \left(\frac{A'' B''}{C'' \rho_{N-3}} \right)^{-\lambda} , \qquad (4\text{-}10)$$

where

$$A'' = 1 + a_1'' \, u_4^3 \, \frac{s_{4,6}}{s_3} + \cdots$$

$$B'' = 1 + b_1'' \, \frac{u_{2N-4}^3 \, s_{2N-3,2N-1}}{s_{2N-3}} + \cdots$$

$$C'' = - c_1'' \, \frac{u_4^3 u_5^3 \cdots u_{2N-4}^3 \, s_{4,2N-1}}{s_3 s_{2N-3}}$$

and where again a_i'', b_i'', and c_1 are functions of u_i such that $u_i \rightarrow 0$ gives $a_i'', b_i'', c_1'' \rightarrow 1$.

Next we perform the helicity-like integration, take the limit of the resultant ${}_1F_1$ functions, obtaining an exponential factor $\exp\left\{\frac{A''B''}{c''}\right\}$ which again cancels the leading terms of the exponential and leaves us with

$$\exp\left\{\frac{u_4^4 S_4}{4} + \frac{u_5^4 S_5}{4} + \cdots\cdots + \frac{u_{2n-4}^4 S_{2n-4}}{4} + \frac{u_4^4 u_5^4 S_4 S_5 K_4}{4} + \right.$$

$$\cdots\cdots + \cdots\cdots \frac{u_4^4 u_5^4 \cdots u_{2n-4}^4 \; S_{5,2n-2}}{2} + \cdots$$

$$\left. + \text{higher orders in } u_i \right\} .$$

We proceed in the same way to obtain a contribution corresponding to the exchange of higher trajectories. The last integral will be of the form

$$\int_0^1 dU_n \, \frac{U_n^{-\alpha_n^n - 1}}{U_n^{n(n-1)/2} \, (-S_n)^{n-1}} \exp\left\{\frac{U_n^n S_n}{n}\right\}$$

It gives us

$$n^{-n-\alpha_n^n} \, (-S_n)^{\alpha_n^n} \, \Gamma\left(1-n-\alpha_n^n\right) ,$$

where $\alpha_n^n = \dfrac{\alpha_1^n}{n} - \dfrac{n-1}{2}$.

A quick glance at the phase factors which result from the helicity integration shows that we are left with the following product of terms,

$$\left\{ \rho_{n-1}^{\alpha_1^1} \Gamma(\alpha_1^1 - \alpha_1^{2n-1}) \Gamma(\alpha_1^{2n-1} - \alpha_1^1 + 1) + \alpha_1^1 \longleftrightarrow \alpha_1^{2n-1} \right\}$$

$$\left\{ \rho_{n-2}^{\alpha_2^2} \Gamma(\alpha_2^2 - \alpha_2^{2n-2}) \Gamma(\alpha_2^{2n-2} - \alpha_2^2 + 1) + \alpha_2^2 \longleftrightarrow \alpha_2^{2n-2} \right\}$$

$$\left\{ \rho_{n-3}^{\alpha_3^3} \Gamma(\alpha_3^3 - \alpha_3^{2n-3}) \Gamma(\alpha_3^{2n-3} - \alpha_3^3 + 1) + \alpha_3^3 \longleftrightarrow \alpha_3^{2n-3} \right\}$$

$$\vdots$$

$$\left\{ \rho_1^{\alpha_{n-1}^{n-1}} \Gamma(\alpha_{n-1}^{n-1} - \alpha_{n+1}^{n+1}) \Gamma(\alpha_{n+1}^{n+1} - \alpha_{n-1}^{n-1} + 1) + \alpha_{n+1}^{n+1} \longleftrightarrow \alpha_{n-1}^{n-1} \right\}. \qquad (4\text{-}11)$$

We perform the required continuations and observe that the final result can be written as

$$B_{2n+2} \sim \tau_1 \tau_2 \cdots \tau_{n-1} \tau_{n+1} \cdots \tau_{2n-1} \, S_1^{\alpha_1^1} (-S_2)^{\alpha_2^2} S_3^{\alpha_3^3} \cdots (-S_n)^{\alpha_n^n}$$

$$, \left((-1)^n S_{n+1}\right)^{\alpha_{n-1}^{n+1}} \cdots S_{2n+1}^{\alpha_1^{2n-1}} \, K_1^{\alpha_1^1} K_2^{\alpha_2^2} \cdots K_{2n-2}^{\alpha_1^{2n-1}}$$

$$\Gamma(+1-n-\alpha_n^n)\, n^{-n} \exp\left\{ \sum_{j=1}^{n-1} -\frac{1}{K_{2n-2-j}} \gamma_j \right\} \exp\left\{ \sum_{i=1}^{n-1} -\frac{1}{K_i} \gamma_i \right\}. \qquad (4\text{-}12)$$

where

$$\gamma_i = 1 + \frac{1}{2} + \cdots \frac{1}{i}$$

The contribution of the second diagram (Fig.18b) can be calculated in the same way. First we write the expression corresponding to the diagram (21), and change the variables of integration by the following rule

$$
\begin{aligned}
X_{n-2} &= 1-u_{n-2} \\
X_1 &= 1-u_{n-2}u_{n-3} \\
X_{n-3} &= 1-u_{n-2}u_{n-3}u_{n-4} \\
X_2 &= 1-u_{n-2}u_{n-3}u_{n-4}u_{n-5} \\
&\vdots \\
X_{n+1} &= 1-u_{n+1} \\
X_{2n-1} &= 1-u_{n+1}u_{n+2} \\
X_{n+2} &= 1-u_{n+1}u_{n+2}u_{n+3} \\
X_{2n-2} &= 1-u_{n+1}u_{n+2}u_{n+3}u_{n+4} \\
&\vdots
\end{aligned}
$$

where X_n and X_{n-1} remain unchanged. Then we continue following the steps used in the previous case. The final integral in this case is

$$\int_0^1 du_n \frac{u_n^{-\alpha_{n-1}^n}}{u_n^{n(n-1)/2}(-S_n)^{n-1}} \exp\left\{\frac{(-1)^n u_n^n S_n}{n}\right\} \tag{4-13}$$

which gives

$$\sim n^{-n-\alpha_n^n} \left((-1)^{n+1} S_n\right)^{\alpha_n^n} \Gamma(1-n-\alpha_n^n)$$

with $$\alpha_n^n = \frac{\alpha_1^n}{n} - \frac{n-1}{2}.$$

Eq.(4-13) shows that the properties of odd and even numbered sibling trajectories are different. All odd numbered sibling trajectories carry a phase factor of the form ($e^{-i\pi\alpha_n^n} \pm \tau_n$) and so for them twisting is equivalent to ± signature. The even-numbered trajectories have phase factor $e^{-i\pi\alpha_n^n}(1 \pm \tau_n)$ and for them the concepts of signature and twisting are not equivalent. The appearance of only one Γ function - for the α_n^n trajectory - demonstrates the decoupling of α_n^n trajectory from states with no more than $2n+2$ particles. Eq.(4-12) exhibits also the general form of the two Reggeon - one particle vertex function for the coupling of two adjacent Regge trajectories in the family, and shows that it is given by $V^{i,i+1}(\alpha_i^i, \alpha_{i+1}^{i+1}, K) = K^{\alpha_i^i} \exp\{-\frac{\gamma_i}{K}\} \frac{1}{\Gamma(-\alpha_i^i-(i+1)}$

Also, a further generalization of the discussion of chapter (2-2) shows that the two Reggeon - one particle vertex function for the coupling of two trajectories with the same slope, i.e. $V^{ll}(\alpha, \beta, K)$ has a similar form to the familiar two Reggeon - one particle vertex function (which corresponds to $l=1$); the only difference is the appearance of an additional exponential factor $\exp\{-\frac{\gamma_{i-1}}{K}\}$ and a replacement $K \rightarrow lK$. We have not been able to exhibit Regge exchanges which lead to the appearance of Reggeon vertices, coupling trajectories which are further apart in the familt (i.e. α^{l}, α^{l+K} with $K>1$). It appears that such couplings vanish, and this is borne out by simple expectations based on the factorization hypothesis.

The α_K trajectory intersects α_{K-1} at $\alpha_K = 1-K$, $(K=2,3,\cdots)$ For $\alpha_K < 1-K$ the α_K exchange dominates over the α_{K-1} exchange. However as $\alpha_K = \frac{\alpha}{K} - \frac{K}{2} + \frac{1}{2}$ the α_K exchange (in at least the 2K+2 point function) dominates for $\alpha < \frac{K(K-1)}{2}$ i.e. for large negative values of the transferred momentum. However we end up with a cautious remark.

The discussion above relied to some extent on the Regge factorization. However as we have shown in chapter 3 the new trajectories do not appear to satisfy any simple factorization, which makes this discussion a little suspect. However, the factorization is violated only by the mismatch of some phase factors and it appears that there is some underlying structure to the way the factorization is broken. We have not been able to understand more fully this structure but we believe that the discussion given in the last and in this chapter throws some light on this problem. At this stage however, it is not clear whether the determined twisting properties of the new trajectories should be associated with the trajectories themselves and not their couplings in the specific configurations that we have studied.

Chapter Five

Triple Reggeon Vertex

In this chapter we try to evaluate the structure of the following triple Reggeon vertices:

i) $\alpha - \beta - \beta$ vertex

ii) $\beta - \beta - \beta$ vertex.

We do this for several reasons. First of all it will allow us to understand better the factorization properties of the new trajectories. Also it will cast some light on the similarity and/or dissimilarity of the α and β trajectories. It will also provide us with some information on the structure of the coupling of two Reggeons to a particle with nonvanishing spin; we shall be able to see whether such a coupling can only be obtained by the continuation of the α trajectory or whether it can be given also by the continuation of the β trajectory. If a consistent picture emerges our results may be useful in a future construction of a field theory involving only Reggeons.

5-1. Two β - one α vertex

To obtain the structure of the $2\beta - \alpha$ vertex we consider the simplest diagram in which this vertex may appear, namely the one which is shown in Fig.(22a). We start with the diagram (23a), perform the required analytical continuations and take the asymptotic limit corresponding to Fig.(22a). We write the expression corresponding to diagram (23a), change the variable of integration X_3 to $X_3=1-U_3$ and obtain

$$
\begin{aligned}
B \approx \int_0^1 & dX_1 dX_2 dU_3 dX_4 dX_5 \; X_1^{-\alpha_1-1} X_2^{-\alpha_2-1} U_3^{-\alpha_3-1} X_4^{-\alpha_4-1} X_5^{-\alpha_5-1} \\
& \left(1-X_1(1-X_2)\right)^{S_1} \left[1-\frac{X_2(1-X_4)}{1-X_2X_4}\right]^{-S_2} (1-U_3)^{-S_3} \left[1-\frac{X_4(1-X_2)}{1-X_2X_4}\right]^{-S_4} \\
& \left(1-X_5(1-X_4)\right)^{S_5} \left(1-X_1X_2(1-X_4)\right)^{S_1S_2K_1} \left[1-\frac{X_2U_3}{1-X_2}\right]^{-S_2S_3K_2} \\
& \left[1-\frac{U_3X_4}{1-X_4}\right]^{-S_3S_4\cdot K_3} \left(1-X_2X_4\right)^{-S_2S_4K_5} \left(1-X_4X_5(1-X_2)\right)^{S_4S_5K_4} \\
& \left(1-X_1X_2U_3\right)^{S_1S_2S_3K_1K_2\phi_1} \left(1-X_1X_2X_4\right)^{S_1S_2S_4K_1K_5\phi_2} \\
& \left(1-U_3X_4X_5\right)^{S_3S_4S_5K_3K_4\phi_4} \left(1-X_2X_4X_5\right)^{S_2S_4S_5K_4K_5\phi_3} \\
& \left(1-X_1X_2X_4X_5\right)^{-S_1S_2S_4S_5K_1K_4K_5\phi_2\phi_3\eta} ,
\end{aligned}
$$

where we have introduced the multi-Regge variables $S_i, t_i, i=1,\cdots 5$; $K_j, j=1,\cdots 5$; ϕ_i $i=1,\cdots 4$, η as

$S_{13} \sim -S_1$

$S_{345} \sim S_2$

$S_{2!34} \sim S_3$

$S_{456} \sim S_4$

$S_{68} \sim -S_5$

$S_{1345} \sim -S_1S_2K_1$

$S_{34} \sim S_3S_2K_2$

$S_{56} \sim S_3S_4K_3$

$$S_{4568} \sim -S_4 S_5 K_4$$

$$S_{3456} \sim S_2 S_4 K_5$$

$$S_{134} \sim -S_1 S_2 S_3 K_1 K_2 \phi_1$$

$$S_{13456} \sim -S_1 S_2 S_4 K_1 K_5 \phi_2$$

$$S_{34568} \sim -S_2 S_4 S_5 K_4 K_5 \phi_3$$

$$S_{568} \sim -S_3 S_4 S_5 K_3 K_4 \phi_4$$

$$S_{134568} \sim S_1 S_2 S_4 S_5 K_1 K_4 K_5 \phi_2 \phi_3 \eta .$$

As the amplitude is originally defined when all these energies are negatove it requires the continuation

$$S_2 = -S_2 e^{-i\pi}$$

$$S_3 = -S_3 e^{-i\pi}$$

$$S_4 = -S_4 e^{-i\pi}$$

$$K_1 = -K_1 e^{i\pi}$$

$$K_2 = -K_2 e^{i\pi} \qquad \phi_1 = \phi_2 = \phi_3 = \phi_4 = 1$$

$$K_3 = -K_3 e^{i\pi}$$

$$K_4 = -K_4 e^{i\pi}$$

$$K_5 = -K_5 e^{i\pi}$$

$$\eta = e^{-2\pi i} .$$

As is well-known there is a nonlinear relation between K_2, K_3 and K_5 corresponding to the triple Reggeon vertex. We shall disregard it at this stage; it can be imposed on the result at the end of calculation.

Changing the variables of integration

$$x_1 = \frac{y_1}{s_1} \;, \quad x_3 = \frac{y_3}{-s_3} \;, \quad x_5 = \frac{y_5}{s_5}$$

we obtain

$$B \sim s_1^{\alpha_1} (-s_2)^{\alpha_3} s_5^{\alpha_5} \int_0^1 dx_2 dx_4 \int_0^\infty dy_3 \; x_2^{-\alpha_2-1} \; x_4^{-\alpha_4-1} \; y_3^{-\alpha_3-1}$$

$$\exp\Big\{ x_2 s_2 + \frac{x_2^2 s_2}{2} - x_2 s_2 x_4 - y_3 + x_4 s_4 + x_4^2 \frac{s_4}{2} - x_4 s_4 x_2$$

$$+ x_2 x_4 s_2 s_4 K_5 + \frac{x_2^2 x_4^2 s_2 s_4 K_5}{2} - x_2 s_2 y_3 K_2 - x_2^2 s_2 y_3 K_2$$

$$- x_4 s_4 y_3 K_3 - x_4^2 s_4 y_3 K_3 \Big\} \, I \;, \qquad (5\text{-}1)$$

where

$$I = \int_0^\infty dy_1 dy_5 \; y_1^{-\alpha_1-1} \; y_5^{-\alpha_5-1}$$

$$\exp\Big\{ -y_1 \big[1 - x_2 + x_2 s_2 (1 - x_4) K_1 + x_2 s_2 x_4 s_4 K_1 K_5 \phi_2 \big]$$

$$- y_5 \big[1 - x_4 + x_4 s_4 (1 - x_2) K_4 + x_2 x_4 s_2 s_4 K_4 K_5 \phi_3 \big]$$

$$+ y_1 y_3 x_2 s_2 K_1 K_2 \phi_1 + y_3 y_5 x_4 s_4 K_3 K_4 \phi_4$$

$$+ y_1 y_5 x_2 s_2 x_4 s_4 K_1 K_4 K_5 \phi_2 \phi_3 \eta \Big\} .$$

$$(5\text{-}2)$$

We replace the last three terms in the exponential in eq.(4-15) by three helicity-like integrals and perform the Y_1 and Y_5 integrations to obtain

$$I = \left(\frac{1}{2\pi i}\right)^3 \int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda\, d\rho\, dm\, \Gamma(-\lambda)\Gamma(-\rho)\Gamma(-m)\Gamma(\lambda+m-\alpha_1)$$
$$\Gamma(m+\rho-\alpha_5)\left(\frac{AB}{C\eta}\right)^{-m} A^{\alpha_1-\lambda} \quad B^{\alpha_5-\rho}$$
$$\left(-y_3X_2S_2K_1K_2\phi_1\right)^{\lambda} \left(-y_3X_4S_4K_3K_4\phi_4\right)^{\rho}$$

(5-3)

where

$$A = 1 - X_2 + X_2S_2K_1(1-X_4) + X_2X_4S_2S_4K_1K_5\phi_2$$

$$B = 1 - X_4 + X_4S_4(1-X_2)K_4 + X_2X_4S_2S_4K_4K_5\phi_3$$

$$C = -X_2X_4S_2S_4K_1K_4K_5\phi_2\phi_3 \; .$$

Next we perform the m integration obtaining:

$$I = \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda\, d\rho\, \Gamma(-\lambda)\Gamma(-\rho)\left(-y_3X_2S_2K_1K_2\right)^{\lambda} \left(-y_3X_4S_4K_3K_4\right)^{\rho}$$
$$(-K_1)^{\alpha_1-\lambda} \quad (-K_4)^{\alpha_5-\rho} \quad \frac{1}{-X_2X_4S_2S_4K_5}\left\{\eta^{\alpha_1-\lambda}\,\Gamma(\alpha_1-\lambda+\rho-\alpha_5)\right.$$
$$\left.\times\Gamma(\alpha_5-\rho+\lambda-\alpha_1) + \eta^{\alpha_5-\rho}\,\Gamma(\alpha_5-\rho-\alpha_1+\lambda)\Gamma(\alpha_1-\lambda+\alpha_5-\rho+1)\right\}$$

$$\exp\left\{-\frac{1}{K_1}-\frac{1}{K_4}-\frac{1}{K_5}-x_2s_2(1-x_4)-x_4s_4(1-x_2)-x_2s_2x_4s_4K_5\right\}. \tag{5-4}$$

Where we have set $\phi_1=\phi_2=\phi_3=\phi_4=1$.

Since $\left\{\eta^{\alpha}\Gamma(\alpha-\beta)\Gamma(\beta-\alpha+1)+\alpha\longleftrightarrow\beta\right\}=e^{-i\pi(\alpha+\beta)}(-2\pi i)$, when $\eta=e^{-2\pi i}$,

we obtain

$$I=-\frac{1}{2\pi i}(-K_1)^{\alpha_1}(-K_4)^{\alpha_5}e^{-\frac{1}{K_1}-\frac{1}{K_4}-\frac{1}{K_5}}e^{-i\pi(\alpha_1+\alpha_5)}\frac{1}{-x_2x_4s_2s_4K_5}$$

$$\exp\left\{-x_2s_2(1-x_4)-x_4s_4(1-x_2)-x_2s_2x_4s_4K_5\right\}\int_{\gamma-i\infty}^{\gamma+i\infty}d\lambda\, d\rho$$

$$\Gamma(-\lambda)\Gamma(-\rho)(y_3x_2s_2K_2)^{\lambda}(y_3x_4s_4K_3)^{\rho}e^{i\pi(\lambda+\rho)}. \tag{5-5}$$

We perform the λ and ρ integrations by using

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}d\lambda\,\Gamma(-\lambda)(-x)^{\lambda}=e^{x}$$

and obtain

$$I=-2\pi i(-K_1)^{\alpha_1}(-K_4)^{\alpha_5}e^{-\frac{1}{K_1}-\frac{1}{K_4}-\frac{1}{K_5}}e^{-i\pi(\alpha_1+\alpha_5)}\frac{1}{-x_2s_2x_4s_4K_5}$$

$$\exp\left\{-x_2s_2(1-x_4)-x_4s_4(1-x_2)+x_2s_2y_3K_2+x_4s_4y_3K_3-x_2x_4s_2s_4K_5\right\} \tag{5-6}$$

Substituting (4-19) into (4-14) we obtain

$$B \sim (-2\pi i)\, S_1^{\alpha_1} S_5^{\alpha_5} (-S_3)^{\alpha_3} (-K_1)^{\alpha_1} (-K_4)^{\alpha_5}\, e^{-\frac{1}{K_1}-\frac{1}{K_4}-\frac{1}{K_5}}\, e^{-i\pi(\alpha_1+\alpha_5)}$$

$$\int_0^\infty dy_3 \int_0^1 dX_2\, dX_4\, X_2^{-\alpha_2-1} X_4^{-\alpha_4-1} y_3^{-\alpha_3-1} \frac{1}{-X_2 X_4 S_2 S_4 K_5}$$

$$\exp\left\{ \frac{X_2^2 S_2}{2} + \frac{X_4^2 S_4}{2} - y_3 X_2^2 S_2 K_2 - y_3 X_4^2 S_4 K_3 + \frac{X_2^2 X_4^2 S_2 S_4 K_5}{2} \right\}. \qquad (5-7)$$

Next we change the variables $X_2 = y_2/(-S_2)^{1/2}$, $X_4 = y_4/(-S_4)^{1/2}$ and replace the last term in the exponential in (5-7) by a helicity-like integral, perform the Y_2 and Y_4 integrations and obtain:

$$B \sim (-2\pi i)\, S_1^{\alpha_1} S_5^{\alpha_5} (-S_3)^{\alpha_3} (-S_2)^{\beta_2} (-S_4)^{\beta_4} \left(\frac{1}{-K_5}\right) 2^{-\beta_2-\beta_4-2} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dn$$

$$\Gamma(-n) \int_0^\infty dy_3\, y_3^{-\alpha_3-1} e^{-y_3}\, \Gamma(n-\beta_2-1)\Gamma(n-\beta_4-1)(-2K_5)^n$$

$$(1-2y_3K_2)^{\beta_2+1-n} (1-2y_3K_3)^{\beta_4+1-n} (-K_1)^{\alpha_1} (-K_4)^{\alpha_5} e^{-i\pi(\alpha_1+\alpha_5)}$$

$$e^{-\frac{1}{K_1}-\frac{1}{K_4}-\frac{1}{K_5}}\,. \qquad (5-8)$$

We introduce two helicity-like integrations by applying the relation

$$(1+z)^a = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda \frac{\Gamma(-\lambda)\Gamma(\lambda-a)}{\Gamma(-a)} z^\lambda$$

to the terms in the last two brackets in (4-21), then perform the Y_3 integration and obtain:

$$B \sim -2\pi i\, S_1^{\alpha_1} (-S_2)^{\beta_2} (-S_3)^{\alpha_3} (-S_4)^{\beta_4} S_5^{\alpha_5} K_1^{\alpha_1} K_4^{\alpha_5} e^{-\frac{1}{K_1}-\frac{1}{K_4}-\frac{1}{K_5}}$$
$$2^{-\beta_2-\beta_4-2} \left(\frac{1}{-K_5}\right) \left(\frac{1}{2\pi i}\right)^3 \int_{\gamma-i\infty}^{\gamma+i\infty} dm\, dn\, dp\, \Gamma(-m)\Gamma(-n)\Gamma(-p)$$
$$\Gamma(m+n-\beta_2-1)\Gamma(p+n-\beta_4-1)\Gamma(m+p-\alpha_3)$$
$$(-2K_2)^m (-2K_3)^p (-2K_5)^m . \tag{5-9}$$

Eq.(5-9) shows that the structure of $\beta-\beta-\alpha$ vertex is similar to the $\alpha-\alpha-\alpha$ vertex[7], the only difference is the replacement

$$K_2, K_3, K_5 \longrightarrow 2K_2, 2K_3, 2K_5 \text{ and } \alpha_2, \alpha_4 \longrightarrow \beta_2+1, \beta_4+1$$

and the appearance of the additional factors $2^{-\beta_2-\beta_4-2}$ $e^{-\frac{1}{K_5}}$.

5-2. The triple β vertex

To obtain the structure of the triple β vertex we consider the diagram shown in Fig.(22b). We write the expression corresponding to Fig.(23b) and change the variables of integration

$$X_1 = 1-U_1$$
$$X_3 = 1-U_3$$
$$X_4 = 1-U_3 U_4$$

obtaining

$$B \sim \int_0^1 du_1\, dx_2\, du_3\, du_4\, dx_5\, dx_6\; u_1^{-\alpha_1-1}\; x_2^{-\alpha_2-1}\; u_3^{-\alpha_3-1}\; u_4^{-\alpha_4-1}\; x_5^{-\alpha_5-1}$$

$$x_6^{-\alpha_6-1}\; (1-u_1)^{s_1} \left[1-\frac{x_2(1-u_3)(1-x_5)}{1-x_2x_5(1-u_3)}\right]^{-s_2} (1-u_3)^{-s_3}$$

$$\left[1-\frac{u_4(1-x_2)(1-u_3)}{1-x_2(1-u_3)}\right]^{s_4} \left[1-\frac{x_5(1-u_3)(1-x_2)}{1-x_2x_5(1-u_3)}\right]^{-s_5} \left[1-x_6(1-x_5)\right]^{s_6}$$

$$\left[1-\frac{u_1x_2(1-u_3)(1-x_5)}{[1-x_2(1-u_3)][1-x_2x_5(1-u_3)]}\right]^{s_1s_2K_1} \left[1-\frac{x_2u_3}{1-x_2(1-u_3)}\right]^{-s_2s_3K_2}$$

$$(1-u_3u_4)^{s_3s_4K_3} \left[1-\frac{u_3x_5}{1-x_5(1-u_3)}\right]^{-s_3s_5K_4} \left[1-x_2x_5(1-u_3)\right]^{-s_2s_5K_6}$$

$$\left[1-x_5x_6(1-x_2)(1-u_3)\right]^{s_5s_6K_5} \left[1-\frac{u_1x_2u_3}{(1-x_2)[1-x_2(1-u_3)]}\right]^{s_1s_2s_3K_1K_2\phi_1}$$

$$\left[1-\frac{x_2u_3u_4(1-u_3)}{1-x_2(1-u_3)}\right]^{s_2s_3s_4K_2K_3\phi_2} \left[1-\frac{u_3u_4x_5}{1-x_5}\right]^{s_3s_4s_5K_3K_4\phi_3}$$

$$\left[1-x_2x_5x_6(1-u_3)\right]^{s_2s_5s_6K_5K_6\phi_5} \left(1-u_3x_5x_6\right)^{s_3s_5s_6K_4K_5\phi_4}$$

$$\left[1-\frac{u_1x_2x_5(1-u_3)}{1-x_2x_5(1-u_3)}\right]^{s_1s_2s_5K_1K_6\phi_6} \left[1-\frac{u_1x_2u_3u_4(1-u_3)}{[1-x_2(1-u_3)]^2}\right]^{-s_1s_2s_3s_4K_1K_2K_3\phi_1\phi_2\eta_1}$$

$$\left(1-u_1x_2x_5x_6\right)^{-s_1s_2s_5s_6K_1K_5K_6\phi_5\phi_6\eta_2}$$

$$\left(1-u_3u_4x_5x_6\right)^{-s_3s_4s_5s_6K_3K_4K_5\phi_3\phi_4\eta_3} \quad . \qquad (5\text{-}10)$$

Where we have already introduced the multi-Regge variables

$S_i \; (i=1,\ldots 6); K_j \quad j=1,\cdots 6 \; ; \; \phi_K \, , \; K=1,\ldots 6 \; , \; \eta_l \, , \; l=1,\ldots 3$

by observing that

$$S_{13} \sim - S_1$$
$$S_{32465} \sim S_2$$
$$S_{1324} \sim S_3$$
$$S_{46} \sim - S_4$$
$$S_{4657} \sim S_5$$
$$S_{79} \sim - S_6$$
$$S_{2465} \sim - S_1 S_2 K_1$$
$$S_{324} \sim S_2 S_3 K_2$$
$$S_{13246} \sim - S_3 S_4 K_3$$
$$S_{657} \sim S_3 S_5 K_4$$
$$S_{46579} \sim - S_5 S_6 K_5$$
$$S_{324657} \sim S_2 S_5 K_6$$
$$S_{24} \sim - S_1 S_2 S_3 K_1 K_2 \phi_1$$
$$S_{3246} \sim - S_2 S_3 S_4 K_2 K_3 \phi_2$$
$$S_{57} \sim - S_3 S_4 S_5 K_3 K_4 \phi_3$$
$$S_{6579} \sim - S_3 S_5 S_6 K_4 K_5 \phi_4$$
$$S_{3246579} \sim - S_2 S_5 S_6 K_5 K_6 \phi_5$$
$$S_{24657} \sim - S_1 S_2 S_5 K_1 K_6 \phi_6$$
$$S_{246} \sim S_1 S_2 S_3 S_4 K_1 K_2 K_3 \phi_1 \phi_2 \eta_1$$
$$S_{579} \sim S_3 S_4 S_5 S_6 K_3 K_4 K_5 \phi_3 \phi_4 \eta_3$$
$$S_{246579} \sim S_1 S_2 S_5 S_6 K_1 K_5 K_6 \phi_5 \phi_6 \eta_2 \, .$$

As the amplitude is originally defined when all these energies are negative it requires the following continuations:

$$+S_2 = -S_2 e^{-i\pi}$$

$$S_3 = -S_3 e^{-i\pi}$$

$$S_5 = -S_5 e^{-i\pi}$$

$$K_i = -K_i e^{i\pi} \qquad , \quad i=1,\ldots 6$$

$$\phi_i = 1 \qquad , \quad i=1,\ldots 6$$

$$\eta_i = e^{-2\pi i} \qquad , \quad i=1,2,3 \; .$$

From six K_i only five are independent. Again, at this stage we disregard this relation between them; it can be imposed at the end of calculations.

Changing the variables of integration

$$u_1 = \frac{y_1}{s_1} \quad , \; u_4 = \frac{y_4}{s_4} \; , \; X_6 = \frac{y_6}{s_6}$$

we obtain

$$B \sim S_1^{\alpha_1} S_4^{\alpha_4} S_6^{\alpha_6} \int_0^1 dX_2\, du_3\, dX_5\; X_2^{-\alpha_2-1} X_5^{-\alpha_5-1} u_3^{-\alpha_3-1}$$

$$\left[1-\frac{X_2(1-u_3)(1-X_5)}{1-X_2X_5(1-u_3)}\right]^{-S_2} (1-u_3)^{-S_3} \left[1-\frac{X_5(1-u_3)(1-X_2)}{1-X_2X_5(1-u_3)}\right]^{-S_5}$$

$$\left[1-\frac{X_2u_3}{1-X_2(1-u_3)}\right]^{-S_2S_3K_2} \left[1-\frac{X_2X_5}{1-X_5(1-u_3)}\right]^{-S_3S_5K_4}$$

$$\left[1-X_2X_5(1-u_3)\right]^{-S_2S_5K_6} \qquad I \quad , \qquad (5\text{-}11)$$

where

$$
\begin{aligned}
I = \int_{0}^{\infty} & dy_1 dy_4 dy_6 \; y_1^{-\alpha_1-1} \; y_4^{-\alpha_4-1} \; y_6^{-\alpha_6-1} \\
& \exp\left\{ -y_1 \left[1 + x_2 s_2 K_1 \frac{(1-u_3)(1-x_5)}{[1-x_2(1-u_3)][1-x_2 x_5(1-u_3)]} \right] \right. \\
& \qquad \left. -y_4 \left[\frac{(1-x_2)(1-u_3)}{1-x_2(1-u_3)} + u_3 s_3 K_3 \right] - y_6 \left[1-x_5 + x_5 s_5 (1-x_2)(1-u_3) K_5 \right] \right. \\
& \exp\left\{ -y_1 x_2 s_2 u_3 s_3 K_1 K_2 \phi_1 \frac{1}{(1-x_2)[1-x_2(1-u_3)]} \right. \\
& \qquad -y_1 x_2 s_2 x_5 s_5 K_1 K_6 \phi_6 \frac{1-u_3}{1-x_2 x_5 (1-u_3)} \\
& \qquad -y_4 x_2 s_2 u_3 s_3 K_2 K_3 \phi_2 \frac{1-u_3}{1-x_2(1-u_3)} - y_6 x_2 s_2 x_5 s_5 K_5 K_6 \phi_5 (1-x_3) \\
& \qquad -y_4 u_3 s_3 x_5 s_5 K_3 K_4 \phi_3 \frac{1}{1-x_5} \\
& \qquad -y_6 u_3 s_3 x_5 s_5 K_4 K_5 \phi_4 \\
& \qquad + y_1 y_4 x_2 s_2 u_3 s_3 K_1 K_2 K_3 \phi_1 \phi_2 \eta_1 \frac{(1-u_3)}{[1-x_2(1-u_3)]^2} \\
& \qquad + y_1 y_6 x_2 s_2 x_5 s_5 K_1 K_5 K_6 \phi_5 \phi_6 \eta_2 (1-u_3) \\
& \qquad \left. + y_4 y_6 u_3 s_3 x_5 s_5 K_3 K_4 K_5 \phi_3 \phi_4 \eta_3 \right\} .
\end{aligned}
\tag{5-12}
$$

We replace the last nine terms in the exponential in (5-12) by nine complex helicity-like integrals, perform the Y_1, Y_4 and Y_6 integrations and obtain

$$I = \left(\frac{1}{2\pi i}\right)^9 \int_{\gamma - i\infty}^{\gamma + i\infty} d\lambda_1\, d\lambda_2\, d\lambda_3\, dm_1\, dm_2\, dm_3\, dp_1\, dp_2\, dp_3\, \Gamma(-\lambda_1)\Gamma(-\lambda_2)$$

$$\Gamma(-\lambda_3)\Gamma(-m_1)\Gamma(-m_2)\Gamma(-m_3)\Gamma(-p_1)\Gamma(-p_2)\Gamma(-p_3)$$

$$\Gamma(\lambda_1+\lambda_2+p_1+p_2-\alpha_1)\Gamma(m_1+m_3+p_2+p_3-\alpha_6)$$

$$\Gamma(p_1+p_3+\lambda_3+m_2-\alpha_4)\quad A^{\alpha_1-\lambda_1-\lambda_2-p_1-p_2}$$

$$B^{\alpha_4-\lambda_3-m_2-p_1-p_3} \qquad D^{\alpha_6-m_1-m_3-p_2-p_3}$$

$$\left(x_2 u_3 s_2 s_3 K_1 K_2 \phi_1 \frac{1}{(1-x_2)\left[1-x_2(1-u_3)\right]}\right)^{\lambda_1}$$

$$\left(x_2 s_2 x_5 s_5 K_1 K_6 \phi_6 \frac{1-u_3}{1-x_2x_5(1-u_3)}\right)^{\lambda_2} \left(\frac{x_2 u_3 s_2 s_3 K_2 K_3 \phi_2 (1-u_3)}{1-x_2(1-u_3)}\right)^{\lambda_3}$$

$$\left(x_2 x_5 s_2 s_5 K_5 K_6 \phi_5 (1-u_3)\right)^{m_1} \left(u_3 x_5 s_3 s_5 K_3 K_4 \phi_3 \frac{1}{1-x_5}\right)^{m_2}$$

$$\left(u_3 x_5 s_3 s_5 K_4 K_5 \phi_4\right)^{m_3} \left(-\frac{x_2 u_3 s_2 s_3 K_1 K_2 K_3 \phi_1 \phi_2 \eta_1 (1-u_3)}{\left[1-x_2(1-u_3)\right]^2}\right)^{p_1}$$

$$\left(-x_2 x_5 s_2 s_5 (1-u_3) K_1 K_5 K_6 \phi_5 \phi_6 \eta_2\right)^{p_2}$$

$$\left(-u_3 x_5 s_3 s_5 K_3 K_4 K_5 \phi_3 \phi_4 \eta_3\right)^{p_3}, \qquad (5\text{-}13)$$

where

$$A = 1 + X_2 S_2 K_1 \frac{(1-U_3)(1-X_5)}{\left[1-X_2(1-U_3)\right]\left[1-X_2X_5(1-U_3)\right]}$$

$$B = \frac{(1-X_2)(1-U_3)}{1-X_2(1-U_3)} + U_3 S_3 K_3$$

$$D = 1-X_5+X_5 S_5 (1-X_2)(1-U_3) K_5 \; .$$

Next we change the variables

$$\begin{aligned}
\lambda_1 &\longrightarrow \lambda_1 - P_1 \\
\lambda_2 &\longrightarrow \lambda_2 - P_2 \\
\lambda_3 &\longrightarrow \lambda_3 - P_1 \\
m_1 &\longrightarrow m_1 - P_2 \\
m_2 &\longrightarrow m_2 - P_3 \\
m_3 &\longrightarrow m_3 - P_3 \quad ,
\end{aligned}$$

perform the P_1, P_2 and P_3 integrations and taking the limit of ${}_1F_1$ functions we obtain

$$I = \left(\frac{1}{2\pi i}\right)^6 \int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda_1\, d\lambda_2\, d\lambda_3\, dm_1\, dm_2\, dm_3\, \Gamma(\lambda_1+\lambda_2-\alpha_1)\Gamma(m_1+m_3-\alpha_6)$$

$$\Gamma(\lambda_3+m_2-\alpha_4)\quad A^{\alpha_1-\lambda_1-\lambda_2}\quad B^{\alpha_4-\lambda_3-m_2}\quad D^{\alpha_6-m_1-m_3}$$

$$\left(X_2U_3S_2S_3K_1K_2\phi_1\frac{1}{(1-X_2)\left[1-X_2(1-X_3)\right]}\right)^{\lambda_1}\left(\frac{X_2X_5S_2S_5K_1K_6\phi_6(1-U_3)}{1-X_2X_5(1-U_3)}\right)^{\lambda_2}$$

$$\left(X_2U_3S_2S_3K_2K_3\phi_2\frac{1-U_3}{1-X_2(1-U_3)}\right)^{\lambda_3}\left(X_2X_5S_2S_5K_5K_6\phi_5(1-U_3)\right)^{m_1}$$

$$\left(U_3 X_5 S_3 S_5 K_3 K_4 \phi_3 \frac{1}{1-X_5} \right)^{m_2} \left(U_3 X_5 S_3 S_5 K_4 K_5 \phi_4 \right)^{m_3}$$

$$\left(\frac{-X_2 U_3 S_2 S_3 K_2}{1-X_2} \right)^{-\lambda_1-\lambda_3-1} \left(\frac{-X_5 U_3 S_5 S_3 K_4}{1-X_5} \right)^{-m_2-m_3-1}$$

$$\left(\frac{-X_2 X_5 S_2 S_5 K_6 (1-U_3)}{1-X_2 X_5 (1-U_3)} \right)^{-\lambda_2-m_1-1} \left\{ \eta_1^{\lambda_1} \Gamma(\lambda_1-\lambda_3) \Gamma(\lambda_3-\lambda_1+1) + \lambda_1 \longleftrightarrow \lambda_3 \right\}$$

$$\left\{ \eta_2^{m_1} \Gamma(m_1-\lambda_2) \Gamma(\lambda_2-m_1+1) + \lambda_2 \longleftrightarrow m_1 \right\} \left\{ \eta_3^{m_3} \Gamma(m_2-m_3) \Gamma(m_3-m_2+1) + m_2 \longleftrightarrow m_3 \right\}$$

$$\exp\left\{ - \frac{X_2 S_2 U_3 S_3 K_2}{1-X_2} - \frac{U_3 X_5 S_3 S_5 K_4}{1-X_5} - \frac{X_2 X_5 S_2 S_5 K_6 (1-U_3)}{1-X_2 X_5 (1-U_3)} \right\} .$$

Replacing the three curly brackets by $(-2\pi i)^3 \exp\{-i\pi(m_1+m_2+m_3+\lambda_1+\lambda_2+\lambda_3)\}$ and changing the variables

$$\lambda_1 \longrightarrow -\lambda_1-\lambda_2+\alpha_1$$

$$m_1 \longrightarrow -m_1-m_3+\alpha_6$$

$$m_2 \longrightarrow -m_2-\lambda_3+\alpha_4$$

allows us to perform the λ_1, m_1 and m_2 integrations obtaining

$$I = (-K_1)^{\alpha_1} (-K_3)^{\alpha_4} (-K_5)^{\alpha_6} \quad e^{-\frac{1}{K_1}-\frac{1}{K_3}-\frac{1}{K_5}} \quad e^{-i\pi(\alpha_1+\alpha_4+\alpha_6)}$$

$$\frac{(1-X_2)(1-X_5)\left[1-X_2 X_5 (1-U_3)\right]}{(-X_2 U_3 S_2 S_3 K_2)(-U_3 X_5 S_3 S_5 K_4)(-X_2 X_5 S_2 S_5 K_6} \cdot \frac{1}{1-U_3}$$

$$\exp\left\{ -x_2 s_2 \frac{(1-u_3)(1-x_5)}{1-x_2x_5(1-u_3)} - u_3 s_3 - x_5 s_5 \frac{(1-u_2)(1-u_3)}{1-x_2x_5(1-u_3)} \right.$$

$$\left. - x_2 u_3 s_2 s_3 K_2 \frac{1}{1-x_2} - u_3 x_5 s_3 s_5 K_4 \frac{1}{1-x_5} - \frac{x_2 x_5 s_2 s_5 K_6 (1-u_3)}{1-x_2x_5(1-u_3)} \right\}$$

$$\int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda_2 \, dm_3 \, d\lambda_3 \left(\frac{(1-x_2)(1-u_3)}{1-x_2(1-u_3)} \cdot \frac{\phi_2}{\phi_3} \right)^{\lambda_3}$$

$$\left(\frac{(1-x_5)}{1-x_2x_5(1-u_3)} \cdot \frac{\phi_4}{\phi_5} \right)^{m_3} \left([1-x_2(1-u_3)] \frac{\phi_6}{\phi_1} \right)^{\lambda_2} .$$

The last three brackets in the above expression are all one in the multi-Regge limit. Thus each of the λ_2, λ_3 and m_3 integrations gives zero and we see that the expression under consideration vanishes. This suggests that the triple β vertex vanishes or that it decouples from the amplitude under consideration. To understand this point further would require consideration of higher point amplitudes. Thus it appears that simple expectations based on assumed similarity between α and β trajectories are not borne out. This point deserves further study.

Chapter Six

Possible phenomenological implication of the new trajectories on the Pion trajectory[17]

6-1. Introduction

The new trajectories found in the previous chapters were determined in the conventional dual model (CDM). They provided contributions to the general scattering amplitude that dominate over the usual (α level) contribution at large enough values of transferred momentum. Thus the $\beta\,(= \frac{1}{2}\alpha - \frac{1}{2})$ trajectory dominates the α trajectory for $\alpha < -1$, the $\gamma\,(= \frac{1}{3}\,\cdot\cdot\, - 1)$ trajectory takes over for $\alpha < -3$ and so on. However, since, as is well known, the intercept α_0 of the Regge trajectory in this model has for reasons of consistency with unitarity of the derived dual field theory a value of unity we see that the β contribution only dominates for a momentum transfer (t) given by $t < -(1+\alpha_0)/\alpha' = -2/\alpha'$. Even if we assume 'real world' intercept $\alpha_0 = \frac{1}{2}$ for the ρ and $\alpha_0 = 0$ for the π the corresponding β trajectories will still be very difficult to observe experimentally since at such values of it their effects are likely to be masked by the contributions of Regge cuts. However, the CDM possesses no nontrivial internal symmetry and it is possible that the properties of new trajectories depend very strongly on the internal Quantum numbers. This is indeed supported by the consideration of the NSM[18,19] which contains a Quantum number resembling G-parity and thus distinguishes between the π and ρ trajectories. In this model the corresponding β trajectories β_π, β_ρ are shifted upwards by $3/4$ and $1/2$ respectively above their CDM counterparts. These shifts bring the β_π and β_ρ within the range of possible phenomenological interest, and so in this chapter we discuss their properties with this aim in mind. We take the positive view that the new trajectories of the NSM may appear in physical amplitudes and we consider how to exhibit their contributions, and their possible phenomenological consequences. Physical amplitudes involve baryons of course, and not knowing how to include their effective assume that their inclusion does not change the predictions of the Neveu-Schwarz scheme. We derive confidence for this assumption from the observation that the spectrum of states in the fermion-antifermion channel of the Ramond dual model agrees with that of the NSM, and also from the fact that the shift of the new trajectories in the NSM is associated with the absence of certain states in its spectrum.

In the next section we discuss the properties of the trajectory and discuss its phenomenological consequences. Then we briefly review the situation with other new trajectories in both the $G=+$ and $G=-$ sectors.

6-2. The β_π trajectory

The multi-Regge limit of the unsignaturized 6-point function in the NSM has been studied in reference 20, where it was shown that the relatively complicated form of the exchange in the pion channel in fact represented the exchange of 2 exchange-degenerate pairs of negative G parity trajectories; the two pairs $(\pi\text{-}H, \omega\text{-}A_2)$ being of opposite naturality and having the common trajectory function $\alpha = \frac{1}{2} + \alpha' t$ Thus the pion pole at $\alpha_\pi = 0$ is a tachyon. However, a good feature of the model, as shown in reference 20, is the explicit absence of a spin 1 state (H) from the pion-partner trajectory. Indeed, it is hoped that when and if the model is rendered more realistic by giving the π and ρ realistic intercepts, this good feature will be preserved. Thus, we use realistic intercepts while also assuming the absence of this state.

The fully signaturized form of the 6-point amplitude were obtained in ref.21., where it was demonstrated that the absence of this pole in the unsignaturized amplitude leads to the appearance of a wrong signature pole in the fully signaturized amplitude which thus needs to be cancelled by another, previously neglected, contribution to the amplitude. This other contribution is provided by the $\beta_\pi = \frac{1}{2}\alpha_\pi + \frac{1}{2}$ trajectory, which intersects the α_π trajectory at the pole position, is itself singular at $\beta_\pi = 1$ and provides the necessary cancellation.

From its form we see that $\beta_\pi > \alpha_\pi$ everywhere in the physical region so that the contribution to the amplitude of the β_π exchange dominates that of the pion exchange.

What are the phenomenological consequences and how do we look for the exchange experimentally?

The full contribution of α_π exchange to the multi-Regge limit of the 6-point amplitude of Fig.(24) was shown in ref.21 to have the form

$$A_6^{\pi-H} \sim \Gamma(1-\alpha_1)\, s_1^{\alpha_1}\, \xi_{\alpha_1}\, \Gamma(-\alpha_2)\, \frac{1-\alpha_2}{1-2\alpha_2}\, s_2^{\alpha_2}\, \xi_{\alpha_2}\, \Gamma(1-\alpha_3)\, s_3^{\alpha_3}\, \xi_{\alpha_3}$$

$$\left[\frac{\xi_{\alpha_2\alpha_1}}{\xi_{\alpha_2}} V_1(t_1,t_2;K_1) + \frac{\xi_{\alpha_1\alpha_2}}{\xi_{\alpha_1}} V_2(t_1,t_2;K_1) \right]$$

$$\left[\frac{\xi_{\alpha_2\alpha_3}}{\xi_{\alpha_2}} V_2(t_3,t_2;K_2) + \frac{\xi_{\alpha_3\alpha_2}}{\xi_{\alpha_3}} V_2(t_3,t_2;K_2) \right] \tag{6-1}$$

where

$$K_1 = \frac{s_{234}}{s_1 s_2}\ , \quad K_2 = \frac{s_{345}}{s_2\, s_3}\ ,$$

$$\xi_{\alpha_i} = e^{-i\pi\alpha_i} + \tau_i\ , \quad \xi_{\alpha_i\alpha_j} = e^{+i\pi(\alpha_j-\alpha_i)} + \tau_i\tau_j$$

The expression for the β_π trajectory contribution is easily shown to be

$$A_6^{\beta_\pi} \approx -i\pi\tau_1 s_1^{\alpha_1} K_1^{\alpha_1}\, e^{-\frac{1}{K_1}}\, s_2^{\beta_2}\, \Gamma(1-\beta_2)(1+\tau_2)\, 2^{-\beta_2+1}\, e^{-i\pi\beta_2}$$

$$K_2^{\alpha_3}\, e^{-\frac{1}{K_3}}\, \tau_3 s_3^{\alpha_3}\ . \tag{6-2}$$

The above expressions are valid in the multi-Regge limit; they also hold when the subenergies S_1, S_3 are not too large. From (6-2) we see that β exchange occurs only if the Reggeons α_1, α_3 are both twisted and this, along with the absence in (6-2) of the gamma functions of the α_1 and α_3 propagators shows that the β trajectory decouples from all quasi-2 body and quasi-3 body final stats. Thus the energy dependence of the diagram of Fig.(25) will, when neither pair of particles (23) or (45) resonates, exhibit an additional contribution corresponding to the exchange of the β_π trajectory. Since, in the physical region $\beta_\pi > \alpha_\pi$ this contribution dominates and its importance increases with the increase of rapidity gap between the clusters (23) and (45). This suggests the following possible phenomenological search for such effects:

Consider the process shown in Fig.(25) for which we again assume that the presence of nucleons does not alter the pattern. Isolate the "pion-exchange contribution - i.e. the contribution for which the exchange carries -meson quantum number. Study the behaviour of this process as a function of the incoming energy for different values of the invariant masses of the $\pi\pi$, πN final state clusters (we choose the π paired with the outgoing nucleon in such a way that the rapidity gap between this pair of particles and the other pair in the final state is as large as possible). The events in which neither pair of particles in the final state resonates should exhibit a different-enhanced-energy dependence, and a different dependence on the momentum transfer between the two pairs of particles. In practice the pion exchange can be isolated by choosing the momentum transfer as small as possible since at $t_2 \sim m_\pi^2 \sim 0$ the α_π has a pole whereas the β_π contribution is regular. For large values of $|t_2|$ the effects due to the β_π exchange should become more important, leading to an effective π trajectory which is almost flat (see Fig.(27), in which the intercepts have been shifted to their physical values). We emphasise that this prediction of an effectively flat pion trajectory applies only to genuine 6-body processes, the β_π decouples from any quasi-2 or 3 body. Final state system for which (excluding the effects of Regge cuts) $\alpha_{eff} = \alpha_\pi$.

Due to the practical difficulties involved with the above method of looking for β_π, we suggest the following alternative method, based on the possible breakdown of Regge factorization.

Consider the four processes shown in Fig.(27) in which the Regge line denotes exchange of a pion-like quantum number. For large incoming energy the amplitudes for the processes in Fig. (28a,b,c,) are approximately given by

$$A \sim V_{NN} S^{\alpha_\pi} V_{NN}$$

$$B \sim V_{NN\pi} S^{\alpha_\pi} V_{NN}$$

$$C \sim V_{\pi\pi\pi} S^{\alpha_\pi} V_{NN}$$

Where $V_{NN}, V_{NN\pi}, V_{\pi\pi\pi}$ denotes the (averaged) coupling of the trajectory to the systems of $2N$, $2N\pi$ and 3π respectively. If we denote the amplitude of the process (28d) as D, Regge factorization of the α_π trajectory shows that, in the absence of the β_π contribution, we expect

$$\frac{AD}{CB} \sim 1 \text{ asymptotically.}$$

However, the amplitude receives also a contribution from the β_π exchange. Thus

$$D = \frac{CB}{A} + E \quad , \quad \text{where } E \sim V'_{\pi\pi\pi} \, S^{\beta_\pi} \, V'_{\pi NN} \quad . \tag{6-3}$$

is this additional contribution. Thus we see that the presence of the β_π trajectory will show up through

$$\frac{AD}{CB} \neq 1 \quad \text{but} \quad \frac{AD}{CB} = 1 + \frac{EA}{CB} \longrightarrow \infty \tag{6-4}$$

$S \rightarrow \infty$ (for all fixed t_2), since $\beta_\pi > \alpha_\pi$.

Thus the detection of the breakdown of Regge factorization, together with the increase of this ratio with energy would be an indication of the additional β_π trajectory contribution.

The above is to some extent schematic since in practice there are several amplitudes needed to describe the above processes. Our problem rests in not knowing how to introduce the spin effects of fermions into the picture. We have assumed that such effects will not crucially affect the predictions. Further it is probable that the predictions will be modified by the effects of Regge cuts, which may shield the predicted effect completely. However, it is hard to find any compelling reason why they should mock the effect in 4. Thus a positive signal in 4 would lend some credence to the existence of the β_π trajectory and to its coupling pattern.

6-3. Other new trajectories

In the odd G-parity channel we have[21] in additional to the β_π a new trajectory corresponding to the α_ω ,namely $\beta_\omega = \frac{1}{2}\alpha_\omega$ This trajectory (extrapolating the discussion to physical intercepts) will only have any significant effect from around $|t| > \alpha_\omega(0) \sim 0.5$,Thus its effects may be difficult to observe experimentally.

To look for the other trajectories we need to consider processes with more particles in the final state. As is already well known the eight point function shown in Fig.(29) will receive contributions from the exchange of $\alpha_\rho, \beta_\rho, \gamma_\rho$ It has been shown[22] that the form of β_ρ and γ_ρ are

$$\beta_\rho(t) = \frac{1}{2}\alpha_\rho(t) \;, \quad \gamma_\rho(t) = \frac{1}{3}\alpha_\rho(t) + \frac{1}{3} \;,$$

respectively.

Of these , γ_ρ is probably more important phenomenologically since $\gamma_\rho(0) \sim \alpha_\rho(0)$ whereas $\beta_\rho(0) < \alpha_\rho(0)$. To find the form of further trajectories we need to consider higher and higher point functions. The analysis becomes more and more involved as the number of external particles increases, and general methods need to be developed. Nahm has recently, using partition function techniques,[23] derived the form of the leading trajectories in both G-parity channels considered together. His method gives

$$\beta_\pi = \frac{1}{2}\alpha_\pi + \frac{1}{2} \;, \quad \gamma_\rho = \frac{1}{3}\alpha_\rho + \frac{1}{3} \;, \quad \delta_\pi = \frac{1}{4}\alpha_\pi + \frac{1}{2}$$

$$\varepsilon_\rho = \frac{1}{5}\alpha_\rho , \frac{2}{5} \;, \quad \xi_\pi = \frac{1}{6}\alpha_\pi + \frac{1}{2} \;, \ldots$$

A summary of a precription which enables us to isolate the relevant term in NSM intergrand that corresponds to the trajectories found by Nahm is given in the appendix of ref.17. However, neither method yields the form of the remaining ρ, π trajectories, let alone the ω trajectories. To determine the former we take the above found trajectories and, taking a hint from the intersection pattern of the CDM, we look for a similar pattern in the NSM. The following scheme emerges:

$$\begin{array}{llllllll}
\beta_\rho = \alpha_\rho & \text{at} & \alpha_\rho = 0 & & \alpha_\pi = \beta_\pi & \text{at} & \alpha_\pi = 1 & \\
\gamma_\rho = \beta_\rho & '' & \beta_\rho = 1 & & \beta_\pi = \gamma_\pi & \cdot & \beta_\pi = -1 & \\
\delta_\rho = \gamma_\rho & '' & \gamma_\rho = -1 & & \gamma_\pi = \delta_\pi & , & \gamma_\pi = 2 & \\
\varepsilon_\rho = \delta_\rho & '' & \delta_\rho = 2 & & \delta_\pi = \varepsilon_\pi & \cdot & \delta_\pi = -2 & \\
\zeta_\rho = \varepsilon_\rho & '' & \varepsilon_\rho = -2 & \text{etc.} & \zeta_\pi = \varepsilon_\pi & , & \varepsilon_\pi = 3 & \text{etc.}
\end{array}$$

giving the trajectory functions of

$$\begin{array}{ll}
\alpha_\rho & \alpha_\pi \\
\beta_\rho = \frac{1}{2}\alpha_\rho & \beta_\pi = \frac{1}{2}\alpha_\pi + \frac{1}{2} \\
\gamma_\rho = \frac{1}{3}\alpha_\rho + \frac{1}{3} & \gamma_\pi = \frac{1}{3}\alpha_\pi \\
\delta_\rho = \frac{1}{4}\alpha_\rho & \delta_\pi = \frac{1}{4}\alpha_\pi + \frac{1}{2} \\
\varepsilon_\rho = \frac{1}{5}\alpha_\rho + \frac{2}{5} & \varepsilon_\pi = \frac{1}{5}\alpha_\pi \\
\zeta_\rho = \frac{1}{6}\alpha_\rho & \zeta_\pi = \frac{1}{6}\alpha_\pi + \frac{1}{2} .
\end{array}$$

This is the only consistent scheme that we can find. Further, we expect the ω trajectories to resemble those of the ρ channel (guided by the known $\beta_\omega = \frac{1}{2}\alpha_\omega$). Observe now that, if we use the physical intercepts $\alpha_\pi(0) = 0$,$\alpha_\rho(0) = \frac{1}{2}$, All trajectories with π-meson quantum numbers intersect at two points on the Chew-Frautchi plot, $J = 0, t = 0$ and $J = \frac{1}{2}, t = 0$ while trajectories in the ρ meson channel have intersections at $J = \frac{1}{2}, t = 0$ and $J = 0, t = -\frac{1}{2}$. Thus, in this scheme, we see that the highest lying "effective" trajectory in the π (ρ) channel, determined in processes with arbitrary numbers of produced particles will be as shown in Fig.(29a,b,); i.e. in both cases the trajectory flattens (approaching $\alpha = \frac{1}{2}$) for all negative values of momentum transfer. If however we restrict our considerations to a finite number of particles in the final state the effective trajectory acquires a slope of also for $t < 0$, as indicated by the dashed line in Fig.(29a(b)). The value of the slope depends on the number of final state particles decreasing as the latter increases.

Since the above is only true in high energy limits the trajectories in the rho channel may be difficult to establish experimentally so that at present energies the most hope must lie with the establishment of the β_π trajectory existence.

Chapter Seven

The Structure of triple Pomeron vertex in the case of square root trajectories

Introduction

As is well-known unitarity requires the existence of Regge cuts besides Regge poles. These two types of singularities can interact with each other and mutually influence each other. In the case of trajectories with intercept close to one the Regge cuts formed from such Regge poles are, for $t \approx 0$, also close to one and the interaction between these singularities can have very profound effects on the properties of each of them. To study the effects of such interaction, and the resultant effects on the asymptotic behaviour of the four point function, Gribov in 1968 introduced the so-called Reggeon field theory[24]. This theory, based on a Lagrangian, can reproduce all diagrams of the usual Regge expansion. To take account of the absorptive nature of the two Pomeron cut the basic coupling constant of the theory - the triple Reggeon vertex - is taken to be purely imaginary. This choice was shown to reproduce the signature rule for the Regge cut and seems to stand on very firm theoretical grounds. However, the original discussion was based on the linear form of the Pomeron Regge trajectory and one may wonder whether this result will have to be altered if the other form of the Pomeron trajectory is considered.

In this chapter we use the rules of Reggeon field theory to investigate the structure of the triple Pomeron vertex in the case of square root trajectories. The reason for introducing such trajectories was given in ref.(9a). Briefly, if the Pomeron is a single pole, $J=\alpha(t)$, then the branch point of a cut corresponding to the exchange of n-such poles is given by

$$J=\alpha_n(t) = n\alpha(t/n^2) - (n-1) . \tag{7-1}$$

Thus, if $\alpha'(o)$ is finite, $\alpha_n(o)=1$, $\alpha'_n(o) = \frac{1}{n}\alpha'(o)$ and all singularities $J=\alpha_n(t)$ intersect at the point (J = 1, t = o) on the Chew-Frautschi plot turning this point into an essential singularity. One way of avoiding this essential singularity as it has been pointed out in ref.(9a) is to require that the singularities arising from many Pomeron exchanges coincide with the Pomeron itself, i.e.

$$\alpha_n(t) \equiv n\alpha(t/n^2) - (n-1) = \alpha(t)$$

Thus $\alpha(t) = 2\alpha(t/4) - 1$ (7-2)

For the two Pomeron exchange. The solution is given by

$$\alpha(t) = 1+\gamma\sqrt{t}$$

Since an amplitude with vacuum quantum numbers should be regular at t = o, an additional singularity is required at

$$\tilde{\alpha}(t) = 1-\gamma\sqrt{t}$$

Thus the partial wave amplitude has the structure

$$a(J,t) = \frac{\beta(J,t)}{[J-\alpha(t)][J-\tilde{\alpha}(t)]} \tag{7-3}$$

with the requirement that $\beta(J,t)$ is analytic at t = o. For t = o the amplitude has a single pole or a double pole at J = 1 depending on whether $\beta(J, o)$ contains or does not contain a factor $(J - 1)$. We shall assume that in general $\beta(J,o)$ is finite at $J = 1$.

In the first section we evaluate the contribution of a single pole and of a two-Pomeron cut to the scattering amplitude and to the total cross section. In doing this we follow the technique used by De Tar[25] in the case of linear trajectories. We find that the contribution of the poles and of the cut are of opposite sign.

In the next section we use the rules of Reggeon field theory to determine the structure of the triple Pomeron vertex. We find that, as in the linear trajectory case, this vertex is pure imaginary.

7-1. The two-Pomeron cut

In order to obtain the contribution of the two Pomeron cut to the scattering amplitude, we start from a simple Pomeron pole. The contribution of a simple pole to the scattering amplitude can be evaluated using the Melin transform

$$A^{\tau}(s,t) = \frac{1}{2\pi i}\int_{0}^{\infty} \mathrm{disc}\, a^{\tau}(J,t)\, s^{J}\, dJ$$

or

$$A(s,t) = \frac{1}{2\pi i}\int_{c} dJ\; \Gamma(-J) \sum_{\tau} a^{\tau}(J,t)\,\frac{1}{2}\left[(-s)^{J} + \tau s^{J}\right],$$

where the contour c is shown in (Fig.31). (7-4)

Substituting (7-3) into (7-4) gives

$$A(s,t) = \frac{\Gamma(-\tilde{\alpha})\left[\tau + e^{-i\pi\tilde{\alpha}}\right] s^{\tilde{\alpha}}}{2(1-\tilde{\alpha})} + \frac{\Gamma(-\alpha)\left[\tau + e^{-i\pi\alpha}\right] s^{\alpha}}{2(1-\alpha)} \qquad (7\text{-}5)$$

where $\alpha, \tilde{\alpha} \equiv \alpha(t), \tilde{\alpha}(t)$.

Eq (7-5) can be written as

$$A(s,t) = \frac{s}{2i\gamma a}\left(A_1 - A_2\right)$$

where

$$a^{2} = -t$$

$$A_1 = \Gamma(-\tilde{\alpha})\, s^{-i\gamma a}\left[\tau + e^{-i\pi\tilde{\alpha}}\right]$$

$$A_2 = \Gamma(-\alpha)\, s^{i\gamma a}\left[\tau + e^{-i\pi\alpha}\right].$$

The contribution to the imaginary part of the amplitude for positive signature and at t = o is

$$\mathrm{Im}\, A(s,o) \approx \pi s \log s \,.$$

Using the optical theorem we see that the total cross-section is

$$\sigma_{tot} \approx \log s \,. \tag{7-6}$$

To obtain the contribution of a two Pomeron cut (shown in Fig.32) to the scattering amplitude we follow the calculations of ref.(24)and write

$$A(s,t) = -\frac{i}{(2\pi)^4} \int d^4 q_1\, B_a(M_a^2, t_1, t_2, t)\, B_b(M_b^2, t_1, t_2, t)$$

$$\left\{ \frac{\Gamma(-\alpha_1)}{2}\left[\frac{(-s)^{\alpha_1} + s^{\alpha_1}}{2(1-\alpha_1)}\right] + \frac{\Gamma(-\tilde{\alpha}_1)}{2}\left[\frac{(-s)^{\tilde{\alpha}_1} + s^{\tilde{\alpha}_1}}{2(1-\tilde{\alpha}_1)}\right]\right\}$$

$$\left\{ \frac{\Gamma(-\alpha_2)}{2}\left[\frac{(-s)^{\alpha_2} + s^{\alpha_2}}{2(1-\alpha_2)}\right] + \frac{\Gamma(-\tilde{\alpha}_2)}{2}\left[\frac{(-s)^{\tilde{\alpha}_2} + s^{\tilde{\alpha}_2}}{2(1-\tilde{\alpha}_2)}\right]\right\} \tag{7-7}$$

where $\alpha_i, \tilde{\alpha}_i \equiv \alpha_i(t_i), \tilde{\alpha}(t_i)$, are the leading trajectories in the t_i channel, and where we have defined the channel invariants as indicated in (Fig.32);

$$M_a^2 = (P_a - q_1)^2$$

$$M_b^2 = (P_b + q_1)^2$$

$$S = (P_a + P_b)^2$$

$$t_1 = q_1^2$$

$$t_2 = q_2^2$$

$$t_a = (P_a - P_{a'})^2$$

$$t_b = (P_b - P_{b'})^2 \; .$$

(7-8)

B_a and B_b are those parts of the two Pomeron - two particle scattering amplitude which have a discontinuity in M_a^2 and M_b^2 respectively. We have also set $\tau_i = +1$.

Next we introduce Sudakov variables[26]. To do this we define the basis vectors

$$P_\alpha = P_a - \frac{m^2}{s} P_b$$

$$P_\beta = P_b - \frac{m^2}{s} P_a$$

with $P_\alpha^2 \approx P_\beta^2 \approx O(1/s)$, $P_\alpha \cdot P_\beta = \frac{s}{2}$.

Any vector, in particular q_1 , may be written as

$$q_1 = \alpha P_\alpha - \beta P_\beta + \varphi_{1\perp}$$

Such that

$$\varphi_{1\perp} \cdot P_\alpha = \varphi_{1\perp} \cdot P_\beta = 0 \quad , \quad \varphi_{1\perp}^2 = - q_{1\perp}^2$$

and such that

$$q_1^2 = -\alpha\beta s - q_{1\perp}^2 \quad , \quad d^4 q_1 = \frac{1}{2} |s| \, d\alpha \, d\beta \, d^2 q_{1\perp} \, .$$

Next, we evaluate the channel invariants in terms of these variables. Eq (7-7) can be written as

$$A(s,t) = A_{\alpha_1\alpha_2} + A_{\alpha_1\tilde{\alpha}_2} + A_{\tilde{\alpha}_1\alpha_2} + A_{\tilde{\alpha}_1\tilde{\alpha}_2} \qquad (7\text{-}9)$$

where

$$A_{\alpha_1\alpha_2}^{(s,t)} = \int d^2 q_{1\perp} H \qquad (a)$$

with

$$H = \frac{\Gamma(-\alpha_1)\Gamma(-\alpha_2)}{16(1-\alpha_1)(1-\alpha_2)} \left[(-s)^{\alpha_1} + s^{\alpha_1}\right]\left[(-s)^{\alpha_2} + s^{\alpha_2}\right] I(s, M_a^2, M_b^2) \qquad (b)$$

$$I(s, M_a^2, M_b^2) = -\frac{i}{(2\pi)^4} \int \frac{1}{2}|s|\, d\alpha\, d\beta\, B_a(M_a^2)\, B_b(M_b^2) \qquad (c)$$

(7-10)

and with a similar expression for $A_{\alpha_1\tilde{\alpha}_2}$, etc.

To obtain the asymptotic behaviour of A(s,t), we evaluate its Mellin transform:

$$A^{\tau}(J,t) = \frac{\Gamma(J+1)}{2\pi i} \int_0^{\infty} ds\, s^{-J-1}\, \mathrm{disc}\, A^{\tau}(s,t) \qquad (7\text{-}11)$$

where

$$\mathrm{disc}\, A^{\tau}(s,t) = \mathrm{disc}_+ A(s,t) + \tau\, \mathrm{disc}_- A(-s,t),$$

and

$$A(s,t) = \frac{1}{2}\sum_{\tau} \left[A^{\tau}(s,t) + \tau A^{\tau}(-s,t)\right].$$

Then the inverse transform is asymptotically given by:

$$A(s,t) \underset{s\to\infty}{\sim} \frac{1}{2\pi i} \int_C dJ \, \Gamma(-J) \sum_{\tau} A^{\tau}(J,t) \, \frac{1}{2} \left[(-s)^J + \tau s^J \right] \cdot$$

(7-12)

In order to obtain the Mellin transform of (7-10a) it is convenient to evaluate first the Mellin transform of $I(s, M_a^2, M_b^2)$ in (7-10c). The Mellin transform of $I(s, M_a^2, M_b^2)$ has been explicitly calculated in ref.(25). It is given by:

$$I(J, q_{\parallel}, q_{\perp}) = \frac{1}{16\pi^2 \Gamma(J+1)} B_a^{\tau}(J, q_{\parallel}, q_{\perp}) \frac{\delta^{J+1}}{J+1} B_b^{\tau}(J, q_{\parallel}, q_{\perp})$$

where

$$B_a^{\tau}(J, q_{\parallel}, q_{\perp}) = \frac{\Gamma(J+1)}{2\pi i} \int^{\infty} dM_a^2 \, (M_a^2)^{-J-1} \, \mathrm{disc}\, B_a^{\tau}(M_a^2, q_{\perp}, q_{\parallel})$$

with a similar definition of $B^{\tau}{}_b$, δ is a low-energy cut off used in the definition of the multi-Regge limit (can be taken as $\delta = m_{\pi}^2$).

Given the Mellin transform of $I(s, M_a^2, M_b^2)$ the Mellin transform of H in (7-10b) can be easily found. This was shown in the appendix of ref.(24).

Thus

$$H^{\tau}(J, q_{\parallel}, q_{\perp}) = C^{\tau}(J, \alpha_1, \alpha_2) \, I^{\tau}(J - \alpha_1 - \alpha_2)$$

$$C^{+}(J, \alpha_1, \alpha_2) = \frac{\Gamma(-J+\alpha_1+\alpha_2) \, \mathrm{Cos}\, \frac{1}{2}\pi (J - \alpha_1 - \alpha_2) \, \mathrm{Cos}\, \frac{1}{2}\pi\alpha_1 \, \mathrm{Cos}\, \frac{1}{2}\pi\alpha_2}{\Gamma(-J) \, \mathrm{Cos}\, \frac{1}{2}\pi J}$$

$$= \frac{\Gamma(J+1)\sin\frac{1}{2}\pi J\cos\frac{1}{2}\pi\alpha_1\cos\frac{1}{2}\pi\alpha_2}{\sin\frac{1}{2}\pi(J-\alpha_1-\alpha_2)\Gamma(J-\alpha_1-\alpha_2+1)}$$

$$\bar{C}(J,\alpha_1,\alpha_2) = \frac{\Gamma(-J+\alpha_1+\alpha_2)\sin\frac{1}{2}\pi(J-\alpha_1-\alpha_2)\cos\frac{1}{2}\pi\alpha_1\cos\frac{1}{2}\pi\alpha_2}{\Gamma(-J)\sin\frac{1}{2}\pi J}$$

$$= \frac{\Gamma(J+1)\cos\frac{1}{2}\pi J\cos\frac{1}{2}\pi\alpha_1\cos\frac{1}{2}\pi\alpha_2}{\cos\frac{1}{2}\pi(J-\alpha_1-\alpha_2)} .$$

(7-13)

Then

$$A^{\tau}_{\alpha_1\alpha_2}(J,q) = \int d^2q_{1\perp}\, C^{\tau}(J,\alpha_1,\alpha_2)\frac{1}{16\pi^3\Gamma(J-\alpha_1-\alpha_2+1)} B^{\tau}_a(J-\alpha_1-\alpha_2, q_{1\perp}, q_{\perp})$$

$$\frac{\delta^{J-\alpha_1-\alpha_2+1}}{J-\alpha_1-\alpha_2+1}\quad\frac{\Gamma(-\alpha_1)\Gamma(-\alpha_2)}{4(1-\alpha_1)(1-\alpha_2)}\; B^{\tau}_b(J-\alpha_1-\alpha_2, q_{1\perp}, q_{\perp}) .$$

(7-14)

Since the signature of a cut is the product of signatures of the exchanged trajectories, the cut appears only in the positive signatured amplitude if $\tau_1 = \tau_2 = +1$.

The expressions for $A_{\alpha_1\tilde{\alpha}_2}$, $A_{\tilde{\alpha}_1\alpha_2}$, and $A_{\tilde{\alpha}_1\tilde{\alpha}_2}$ are similar to eq. (7-14).

The factor C^{τ} does not, in fact, introduce singularities into the partial wave amplitude, since to avoid right-signature fixed poles, both B^{τ}_a and B^{τ}_b must vanish at the zeros of denominator of C^{τ} i.e.

$$N^{\tau}_{a,b} = \lim_{J \to -1} (J+1)\, B^{\tau}_{a,b}(J, q_{11}, q_{\perp}) \quad = \text{finite}$$

Using the Mellin transform (7-12) we obtain

$$A_1 = A_{\alpha_1\alpha_2}(s,t) = \frac{1}{32(2\pi)^3} \int d^2 q_{11} \frac{N_a^{+} N_b^{+}\, s^{\alpha_1+\alpha_2-1}}{\sin\frac{1}{2}\pi(\alpha_1+\alpha_2)}$$

$$\frac{\left[1+e^{-i\pi(\alpha_1+\alpha_2-1)}\right] \cos\frac{1}{2}\pi\alpha_1 \cos\frac{1}{2}\pi\alpha_2\, \Gamma(-\alpha_1)\Gamma(-\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}$$

Also:

$$A_2 = A_{\tilde{\alpha}_1\alpha_2} = A_{\alpha_1\alpha_2}(\alpha_1 \to \tilde{\alpha}_1)$$

$$A_3 = A_{\alpha_1\tilde{\alpha}_2} = A_{\alpha_1\alpha_2}(\alpha_2 \to \tilde{\alpha}_2)$$

$$A_4 = A_{\tilde{\alpha}_1\tilde{\alpha}_2} = A_{\alpha_1\alpha_2}(\alpha_1, \alpha_2 \to \tilde{\alpha}_1, \tilde{\alpha}_2)$$

(7-15)

When t=o and both trajectories are the same we see that

$$A_1 = \frac{1}{32(2\pi)^3} \int d^2 q_{11} N_a^{+} N_b^{+} \frac{s^{1+2i\gamma a}\left[1-e^{2\pi\gamma a}\right]\left[-\sin\frac{1}{2}\pi i\gamma a\, \Gamma(-1-i\gamma a)\right]^2}{(-i\gamma a)^2(-\sin\pi\gamma a)}$$

$$A_2 = A_3 = \frac{1}{32(2\pi)^3} \int d^2 q_{11} N_a^{+} N_b^{+} \frac{s\left[\sin\frac{1}{2}\pi i\gamma a\right]^2 \Gamma(-1-i\gamma a)\Gamma(-1+i\gamma a)}{(i\gamma a)^2}$$

7-2. The structure of triple Pomeron coupling of Reggeon field theory

In this section we try to use the techniques of Reggeon field theory[24,27] to evaluate the triple Pomeron vertex in the case of Schwarz trajectories. To do this we start with the diagram shown in (Fig.33). We define the propagator for a Reggeon of Momentum q as

$$G_0(q,E) = \frac{i}{\left(E+i\gamma\sqrt{q^2}\right)\left(E-i\gamma\sqrt{q^2}\right)}$$

where E = 1 - J and γ is real. Both E and q are conserved at each vertex. Next we calculate the contribution of this diagram. We find

$$G^{(1,1)}(E,q) = \left[\frac{i}{\left(E+i\gamma\sqrt{q^2}\right)\left(E-i\gamma\sqrt{q^2}\right)}\right]^2 \left(\frac{-i\lambda_0}{(2\pi)^{3/2}}\right)^2 \frac{1}{2} \int d^2q' dE'$$

$$\frac{(-1)}{\left(E'^2+\gamma^2 q'^2\right)\left[(E-E')^2+\gamma^2(q-q')^2\right]} + \frac{i}{\left(E+i\gamma\sqrt{q^2}\right)\left(E-i\gamma\sqrt{q^2}\right)}$$

$$= \left[\frac{i}{\left(E^2+\gamma^2 q^2\right)}\right]^2 \left[\frac{-i\lambda_0}{(2\pi)^{3/2}}\right]^2 I + \frac{i}{\left(E+i\gamma\sqrt{q^2}\right)\left(E-i\gamma\sqrt{q^2}\right)}$$

(7-18)

Where q, q', E and E$'$ are indicated in (Fig.33) and λ_0 is the triple Pomeron coupling constant. We perform the E$'$ integration in (7-18) and obtain

$$I = 2\pi i\,(I_1 - I_2) \tag{7-19}$$

where

$$I_1 = \int d^2q' \frac{1}{2i\gamma\sqrt{q'^2}} \; \frac{1}{E - i\gamma\sqrt{q'^2} - i\gamma\sqrt{(q-q')^2}} \; \frac{1}{E - i\gamma\sqrt{q'^2} + i\gamma\sqrt{(q-q')^2}}$$

$$I_2 = \int d^2q' \frac{1}{2i\gamma\sqrt{q'^2}} \; \frac{1}{E + i\gamma\sqrt{q'^2} - i\gamma\sqrt{(q-q')^2}} \; \frac{1}{E + i\gamma\sqrt{q'^2} + i\gamma\sqrt{(q-q')^2}}$$

or in terms of polar coordinates:

$$I_1 = \int dr\,d\theta \frac{1}{E^2 + \gamma^2 q^2 - 2E\gamma r + 2\gamma^2 |q| r \cos\theta} \frac{1}{2\gamma}$$

$$I_2 = \int dr\,d\theta \frac{1}{E^2 + \gamma^2 q^2 + 2E\gamma r + 2\gamma^2 |q| r \cos\theta} \frac{1}{2\gamma}$$

or

$$I_1 = \frac{\pi}{\gamma\sqrt{d}} \int_0^\infty dr \frac{1}{\sqrt{4\gamma^2 r^2 - 4E\gamma r + d}}$$

$$I_2 = \frac{\pi}{\gamma\sqrt{d}} \int_0^\infty dr \frac{1}{\sqrt{4\gamma^2 r^2 + 4E\gamma r + d}}$$

where $d = E^2+\gamma^2 q^2$.

Thus:

$$I = \frac{\pi^2 i}{\gamma^2\sqrt{E^2+\gamma^2 q^2}} \log \frac{4E\gamma + 4\gamma\sqrt{E^2+\gamma^2 q^2}}{4\gamma\sqrt{E^2+\gamma^2 q^2} - 4E\gamma}$$

$$= \frac{\pi^2 i}{\gamma^2\left(E^2+\gamma^2 q^2\right)^{1/2}} \log\left(\frac{2E^2 + 2E\sqrt{E^2+\gamma^2 q^2} + \gamma^2 q^2}{\gamma^2 q^2}\right)$$

and substituting these results into eq(7-18) we get:

$$G^{(1,1)} = \frac{+i\lambda_0^2}{2\pi\gamma^2\left(E^2+\gamma^2 q^2\right)} \left[\frac{1}{\sqrt{E^2+\gamma^2 q^2}} \log \frac{2E^2+2E\sqrt{E^2+\gamma^2 q^2}+\gamma^2 q^2}{\gamma^2 q^2}\right]$$

$$+ \frac{i}{\left(E - i\gamma\sqrt{q^2}\right)\left(E + i\gamma\sqrt{q^2}\right)}$$

(7-20)

or in terms of J and t , $(t=-q^2)$

$$A(J,t) = G^{(1,1)}(J,t) = \frac{i\lambda_0^2}{2\pi\gamma^2\left[(1-J)^2-\gamma^2 t\right]^{5/2}} \cdot$$

$$\log\left[\frac{2(1-J)^2 + 2(1-J)\sqrt{(1-J)^2-\gamma^2 t}}{-\gamma^2 t} + 1\right] + \frac{i}{(1-J)^2-\gamma^2 t}$$

(7-21)

Expression (7-20) has a branch cut as shown in (Fig.34) at $-i\gamma\sqrt{q^2} \leq E \leq +i\gamma\sqrt{q^2}$ together with two double poles which coincide with the branch points. The discontinuity across the cut is given by

$$\mathrm{disc}\, A(E,q) = \frac{2(-i\lambda_0)^2}{(-\gamma^2 q^2 - E^2)^{5/2}} .$$

The scattering amplitude consists of two terms; one corresponds to double poles and the cut, and the other corresponds to a single pole, i.e.

$$A(s,t) = A_1(s,t) + A_2(s,t)$$

To obtain the scattering amplitude we find the Mellin transform of (7-21). The Mellin transform of the second term (a simple pole) can be easily found and its contribution to the total cross section, by using the optical theorem is

$$\sigma_{tot}^{Pole} \sim \log s . \qquad (7\text{-}22).$$

Next we find the Mellin transform of the first term in (7-21) which corresponds to the contribution of double poles and the cut and it is given by

$$A_1(s,t) = \frac{i\lambda_0^2}{\pi} \int_{1-i\gamma\sqrt{-t}}^{1+i\gamma\sqrt{-t}} \frac{S^J}{\left[\gamma^2 t - (1-J)^2\right]^{5/2}} .$$

The above integral diverges. This results from the coincidence of the poles and the branch points of the cut. But since we are interested in the sign of the cut contribution we evaluate the controbution due to the cut only with no double pole. To do this we replace A_1 by A'_1 given by

$$A'_1 = \frac{i\lambda_0^2}{\pi} \int_{1-i\gamma\sqrt{-t}}^{1+i\gamma\sqrt{-t}} \frac{s^J \, dJ}{\left[\gamma^2 t - (1-J)^2\right]^{1/2}} .$$

This replacement corresponds to the replacement of the diagram in (Fig.33b) by the diagram shown in (Fig.35).

To perform this integral we change the variable of integration $1-J = iaX$ and then perform the X integration obtaining

$$A'_1(s,t) \approx i\lambda_0^2 \, s \, J_0(-a \log s)$$

(7-23)

where

$$-a^2 = \gamma^2 t < 0$$

The amplitude in (7-23) oscillates but since the total cross section is proportional to the imaginary part of the scattering amplitude at t = o, we see that the contribution of the cut to the total cross section is given by

$$\sigma_{tot}^{cut} \sim \lambda_0^2 J_0(0) \sim \lambda_0^2$$

(7-24)

and is constant.

In the first section of this chapter we pointed out that the contribution of a two Pomeron cut to the total cross section is negative as in the linear trajectory case. Comparing eq.(7-22) and eq.(7-24), according to the above statement λ_0^2 must be negative, i.e. $\lambda_0 = i r$, which shows that the triple Pomeron vertex is pure imaginary for both the square root trajectories and for the linear trajectories.

Conclusion

In the first section of this thesis we have studied the multi-Regge limit of the Veneziano amplitude and discussed those properties which are related to the appearance of new trajectories (siblings). The first member of these trajectories (β) appears for the first time in the six point function, it decouples from the states with two or three particle final states, it has positive signature and is related to the parent (α)trajectory by the relation $\beta = \frac{\alpha}{2} - \frac{1}{2}$. We found that the appearance of this trajectory alters the nearest Reggeon-Reggeon-particle vertices and Regge propagators and that the twisting operator is not equivalent to the signaturization , for this trajectory. We also showed that the coupling of 2β-particle has similar structure to the 2 α-particle vertex.

The next member of these trajectories (γ) appears for the first time in eight point function. Its slope is $\frac{1}{3}$ of α trajectory's slope and it is related to the α trajectory by $\gamma = \frac{\alpha}{3} - 1$. The twisting operator and signaturization are equivalent (up to the sign) on the γ trajectory level.

The next sibling trajectory which has slope $\frac{1}{4}$ of the α trajectory appears for the first time in ten point function, and so on. The trajectory with slope $\frac{1}{n}$ appears for the first time in $(2n+2)$ point function.

We have pointed out that the properties of odd and even numbered sibling trajectories are different. For all odd numbered sibling trajectories twisting operator is equivalent to ± signature, whereas for the even numbered sibling trajectories they are different.

We have also shown that the complete contribution of these trajectories to the multi particle amplitude do not satisfy conventional factorization requirement. We have also studied the triple Reggeon vertex and found out that the coupling of 3β trajectories vanishes and the $2\beta\alpha$ vertex is similar to the triple α vertex. Finally we discussed the phenomenological consequences of these trajectories, and showed that the β_π trajectory of the Neveu-Schwarz model should be easiest to detect experimentally.

In the last chapter we have studied the structure of the triple Pomeron vertex in Reggeon field theory in the case of square root trajectories and we have shown that barring some technical problems this vertex is expected to be imaginary like in the conventional theory (with linear trajectories).

Appendix

In this appendix we discuss the analytical continuations which we have to perform in order to determine the asymptotic behaviour of the amplitudes shown in Fig.(7) directly for positive s_2.

First we consider the amplitude of Fig.(7a). We start with the expression (2-3) and proceed as in section (2-1), except that this time we do not expand factors involving s_2 and x_2. (We shall perform the required analytical continuation to positive s_2, before we take the asymptotic limit of the amplitude). Proceeding as before we obtain

$$B_6 \sim s_1^{\alpha_1} \tau_1 s_3^{\alpha_3} \tau_3 \Gamma(-\alpha_1)\Gamma(-\alpha_3) \int_0^1 dx_2\, x_2^{-\alpha_2-1} (1-x_2)^{-s_2-1}$$

$$(1-x_2+x_2 s_2 K_2)^{\alpha_3-\alpha_1} \quad (-x_2 s_2 K_1 K_2 \phi)^{\alpha_1}$$

$$\psi\left(-\alpha_1, \alpha_3-\alpha_1+1; \frac{(1-x_2+x_2 s_2 K_1)(1-x_2+x_2 s_2 K_2)}{-x_2 s_2 K_1 K_2 \phi}\right) . \tag{A.1}$$

To perform a continuation of this expression to any s_2 we observe[28] that as

$$(1-x_2)^{-s_2-1} = -\frac{\mathrm{Disc}(x_2-1)^{-s_2-1}}{2i \sin \pi s_2}$$

the expression (A.1) can be rewritten as

$$B_6 \sim s_1^{\alpha_1} \tau_1 s_3^{\alpha_3} \tau_3 \Gamma(-\alpha_1)\Gamma(-\alpha_3) \frac{i}{2 \sin \pi s_2} \oint dx_2\, x_2^{-\alpha_2-1} (x_2-1)^{-s_2-1}$$

$$(1-x_2+x_2 s_2 K_2)^{\alpha_3-\alpha_1} \quad (-x_2 s_2 K_1 K_2 \phi)^{\alpha_1}$$

$$\psi\left(-\alpha_1, \alpha_3-\alpha_1+1; \frac{(1-x_2+x_2 s_2 K_1)(1-x_2+x_2 s_2 K_2)}{-x_2 s_2 K_1 K_2 \phi}\right) \tag{A.2}$$

where the contour of integration is shown in Fig. (3.6). Next the contour is distorted as in Fig. (3.7) and then the radius of the C_2 is increased to ∞, with corresponding increases in contours C_1 and C_3. In the limit of the contour C_2 lying at ∞ its contribution can be neglected, for as $|x_2| \to \infty$ the integrand behaves as

$$\sim x_2^{-\alpha_2+\alpha_1-1-s_2+\alpha_3-\alpha_1-1} \quad \Psi(\, , \, , -x_2 s_2) \sim x_2^{-\alpha_2+\alpha_1+\alpha_3-1-s_2}$$

and can be made to vanish fast enough by a suitable choice of $\alpha_1, \alpha_2, \alpha_3$ and s_2.

This allows us to keep only the contribution of contours C_1 and C_3.

Next we perform the required analytical continuation (2-2), and observe that during this continuation the phase of the argument z of the Ψ function changes but does not go outside the range $-\frac{3}{2}\pi < \arg z < \frac{3}{2}\pi$ guaranteeing no unexpected contribution from the previously neglected contour C_2.

Now we take the limit $s_2 \longrightarrow +\infty + i\epsilon$. Notice that for x_2 on C_1

$$(x_2-1)^{-s_2-1} = (1-x_2)^{-s_2-1} \, e^{-i\pi s_2} \quad \text{and} \quad \frac{e^{-i\pi s_2}}{-2i \sin \pi s_2} \to 1 \quad \text{as} \quad s_2 \to \infty + i\epsilon$$

while for x_2 on C_3 the corresponding factor vanishes exponentially

$$\frac{e^{i\pi s_2}}{-2i \sin \pi s_2} \longrightarrow 0$$

showing that the leading contribution comes entirely from C_1. The leading asymptotic behaviour comes from $x_2 \sim 0$. But

$$\psi(-\alpha_1, \alpha_3-\alpha_1+1; z) \underset{x_2 \to 0}{\sim} \psi\left(-\alpha_1, \alpha_3-\alpha_1+1; x_2 s_2 e^{i\pi} + \left(\frac{1}{K_1}+\frac{1}{K_2}\right)e^{i\pi}\right)$$

$$= \psi\left(-\alpha_1, \alpha_3-\alpha_1+1; \frac{(x_2 s_2 + 1/K_1 + 1/K_2)\, e^{-i\pi}}{e^{-2\pi i}}\right),$$

so, using the relation which describes the behaviour of the ψ function when its argument z encircles the origin[30], we obtain

$$\psi(-\alpha_1, \alpha_3-\alpha_1+1; z) = e^{-2\pi i(\alpha_3-\alpha_1)}$$

$$\psi\left(-\alpha_1, \alpha_3-\alpha_1+1; \left(x_2 s_2 + \frac{1}{K_1}+\frac{1}{K_2}\right)e^{-i\pi}\right)$$

$$+\left[1-e^{-2\pi i(\alpha_3-\alpha_1)}\right]\frac{\Gamma(\alpha_1-\alpha_3)}{\Gamma(-\alpha_3)}$$

$$ {}_1F_1\left(-\alpha_1, \alpha_3-\alpha_1+1; x_2 s_2 e^{-i\pi} - \frac{1}{K_1} - \frac{1}{K_2}\right). \tag{A.3}$$

The contribution due to the β exchange comes from the ${}_1F_1$ function in (A.3) and so we are left with the integral

$$B_6 \sim S_1^{\alpha_1} K_1^{\alpha_1} \tau_1 S_3^{\alpha_3} K_2^{\alpha_3} \tau_3 \Gamma(-\alpha_1)\Gamma(\alpha_1-\alpha_3)\left[1-e^{-2\pi i(\alpha_3-\alpha_1)}\right]$$

$$e^{-\pi i\alpha_1} s_2^{\alpha_3} \int_{C_1} dx_2 \; x_2^{-\alpha_2-1+\alpha_3}$$

$$ {}_1F_1\left(-\alpha_1, \alpha_3-\alpha_1+1; x_2 s_2 e^{-i\pi} - \frac{1}{K_1} - \frac{1}{K_2}\right).$$

The contour C_1 lies essentially along the imaginary axis; for small x_2 its phase is slightly over $\pi/2$, for larger x_2 its phase is just under $\pi/2$ (see Fig.38). As the β exchange comes from the small x_2 this way of drawing the contour justifies the approximation

$$
{}_1F_1\left(-\alpha_1, \alpha_3-\alpha_1+1; x_2 s_2 \bar{e}^{i\pi} - \frac{1}{K_1} - \frac{1}{K_2}\right)
$$

$$
\xrightarrow[s\to\infty]{} \exp\left\{-\frac{1}{K_1} - \frac{1}{K_2} - x_2 s_2\right\}\left(x_2 s_2 \bar{e}^{i\pi}\right)^{-\alpha_3-1} \frac{\Gamma(\alpha_3-\alpha_1+1)}{\Gamma(-\alpha_1)}
$$

which together with a further change of variable $x_2 = iz_2$ leads directly to the result obtained in section (2-1) exp.(2-13).

To discuss the amplitude corresponding to Fig.(7b) we proceed as in section (2-1) and obtain

$$
B_6' \sim s_1^{\alpha_1} s_3^{\alpha_3}\, \Gamma(-\alpha_1)\Gamma(-\alpha_3) \int_0^1 dx_2\, x_2^{-\alpha_2-1} \,(1-x_2)^{s_2-1}
$$

$$
\left(1-x_2-x_2 s_2 K_1\right)^{\alpha_3-\alpha_1} \left(x_2 s_2 (1-x_2) K_1 K_2 \phi\right)^{\alpha_1}
$$

$$
\psi\left(-\alpha_1, \alpha_3-\alpha_1+1; \frac{(1-x_2-x_2 s_2 K_1)(1-x_2-x_2 s_2 K_2)}{x_2(1-x_2) s_2 K_1 K_2 \phi}\right) .
$$

We perform a change of variables

$$\frac{x_2}{1-x_2} = u_2$$

and obtain

$$B'_6 \sim s_1^{\alpha_1} s_3^{\alpha_3} \Gamma(-\alpha_1)\Gamma(-\alpha_3)(K_1 K_2 s_2)^{\alpha_1} \int_0^{\infty} du_2 \; u_2^{-\alpha_2+\alpha_1-1}$$

$$(1+u_2)^{-s_2+\alpha_2-\alpha_1-\alpha_3} \qquad (1-u_2 s_2 K_2)^{\alpha_3-\alpha_1}$$

$$\psi\left(-\alpha_1, \alpha_3-\alpha_1+1; \frac{u_2 s_2}{\phi} - \frac{1}{K_1\phi} - \frac{1}{K_2\phi} + \frac{1}{u_2 s_2 K_1 K_2 \phi}\right).$$

Next we distort the u_2 contour until it runs essentially along the imaginary axis, like the x_2 contour in Fig.(38). This distortion is possible as the contribution due to the quarter circle at ∞ vanishes. Next we perform the required analytical continuation of K_1, K_2 and ϕ. As in the case of the untwisted Reggeon the phase of the argument of the ψ function does not go outside the range $-\frac{3\pi}{2} < \arg z < \frac{3\pi}{2}$ showing that the contribution of the quarter circle at ∞ does indeed vanish.

Next we proceed as before. The contribution corresponding to the β exchange comes from the ${}_1F_1$ function which expresses the behaviour of the ψ function when its argument encircles the origin. Again, the way the contour is drawn justifies the approximation of this ${}_1F_1$ function by its asymptotic behaviour; then the u_2 integration can be performed; it gives the previously obtained result (2-19).

Figure Captions

Fig.1. Diagrams for the Veneziano amplitude describing scattering of two spinless particles.

2. Diagrams which contribute to the Regge limit of four point amplitude.

3. Five point amplitude exhibiting notation used in the text.

4. Graphs for the Veneziano amplitude of the five point function.

5. The multi-Regge limit of the six point function.

6. The multi-Regge limit of the six point amplitude corresponding to the exchange of a β trajectory in the second channel.

7. The diagrams used for determining the Veneziano amplitude of six point functions.

8. The mutli-Regge limit of the six point amplitude corresponding to the exchange of $\beta\alpha$ trajectories.

9. A Chew-Frautschi plot of the α and β trajectories.

10. The multi-Regge limit of the seven point functions corresponding to the exchange of double β trajectories.

11. The diagrams used for determining the Veneziano amplitude of the seven point function.

12. The multi-Regge limit of the seven point amplitude corresponding to the exchange of one β trajectory.

13. The diagrams used for determining the Veneziano amplitude of the seven point function.

14. (a,b,c,d,e,g): The multi-Regge limit of the eight point function corresponding to the exchange of 3β trajectories.

(f,h): The multi-Regge limit of the eight point function corresponding to the exchange of 3β and one γ trajectories.

15. The diagrams used for determining the Veneziano amplitude of the eight point function.

16. The multi-Regge limit of the ten point functions corresponding to the exchange of δ trajectory.

Fig.17. The diagrams used for determining the Veneziano amplitude of ten point function.

18. The multi-Regge limit of $(2n+2)$ point function corresponding to the exchange of a trajectory with slope $\frac{1}{n}$ in the $n\pi$ channel.

19. The diagrams corresponding to the Veneziano amplitude of the $(2n+2)$ point function.

20. The diagrams corresponding to the Veneziano amplitude of the $(2n+2)$ point function.

21. The diagrams corresponding to the Veneziano amplitude of the $(2n+2)$ point function.

22.a. The simplest diagram which gives contribution to the $2\beta\alpha$ vertex.

b. The simplest diagram which gives contribution to the 3β vertex.

23.a. The diagram used for determining the Veneziano amplitude of the eight point function.

b. The diagram used for determining the Veneziano amplitude of the nine point function.

24. The multi-Regge limit of the six point function that exhibits contribution of α_π and β_π trajectories in the second channel.

25. A single Regge limit of the six point function.

26. A single Regge limit of the six point function corresponding to the exchange of β_π trajectory.

27. The effective π trajectory after taking the β_π contribution into account.

28.a. A single Regge limit of the four point amplitude.

b, c, d: A single Regge limit of the five point amplitude.

29. The multi-Regge limit of the eight point function corresponding to the exchange of α_ρ, β_ρ and γ_ρ trajectories.

30. The effective trajectories in π and ρ channels.

31. The integration contour for the Mellin transform.

32. The two-Pomeron - cut diagram.

33. The lowest order contribution of the $G^{(1,1)}$

Fig.34. The Regge structure in the E plane cut.

35. The modified Regge cut diagram (with external lines amputated).

36. The integration contour in the X_2 plane.

37. The modified integration contour in the X_2 plane.

38. The part of the contour integral which gives non zero contribution in the Regge limit.

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30. Ref.13, p.264, exp.17.

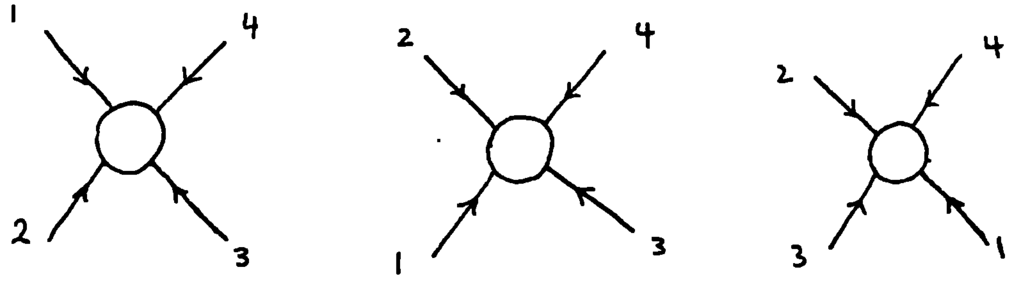
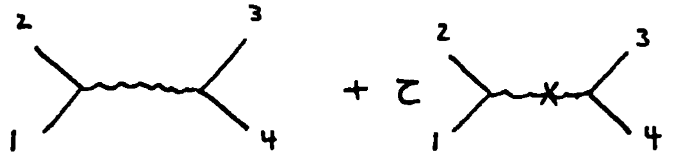


Fig 1.



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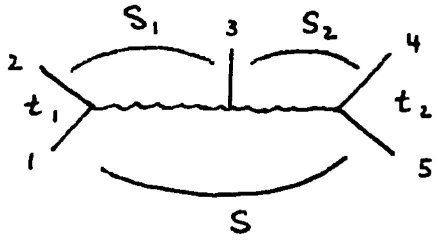

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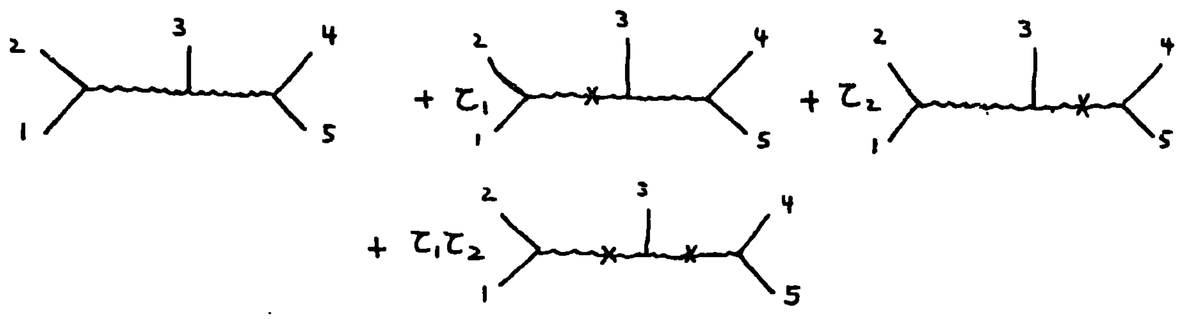

Fig 4.

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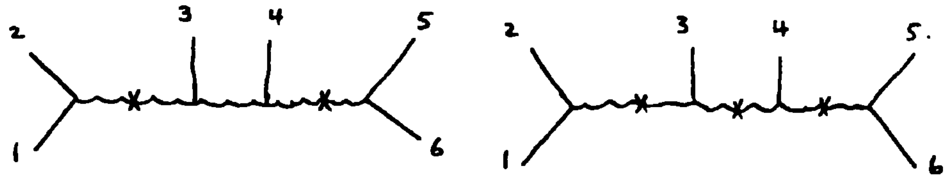

a) b)

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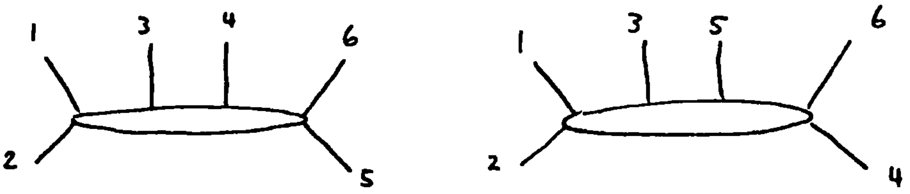

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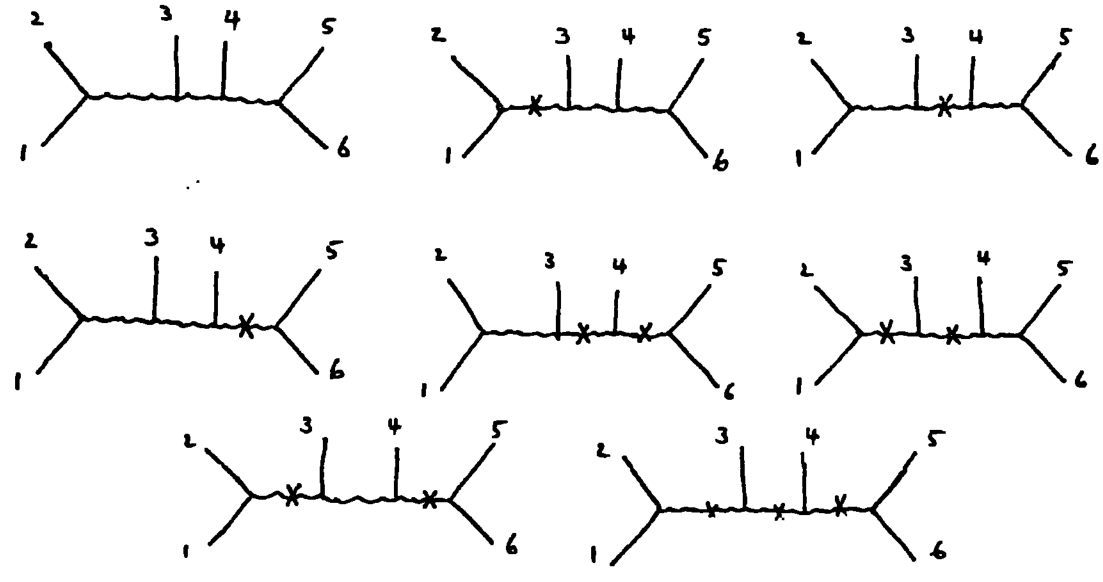

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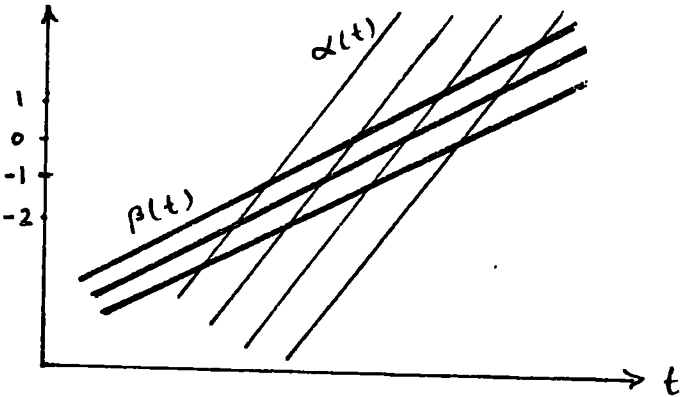

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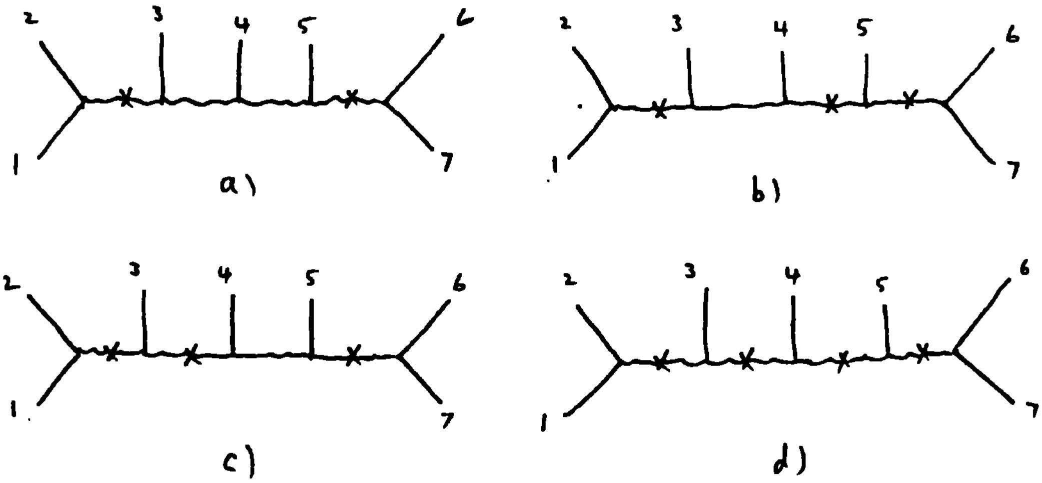

a) b) c) d)

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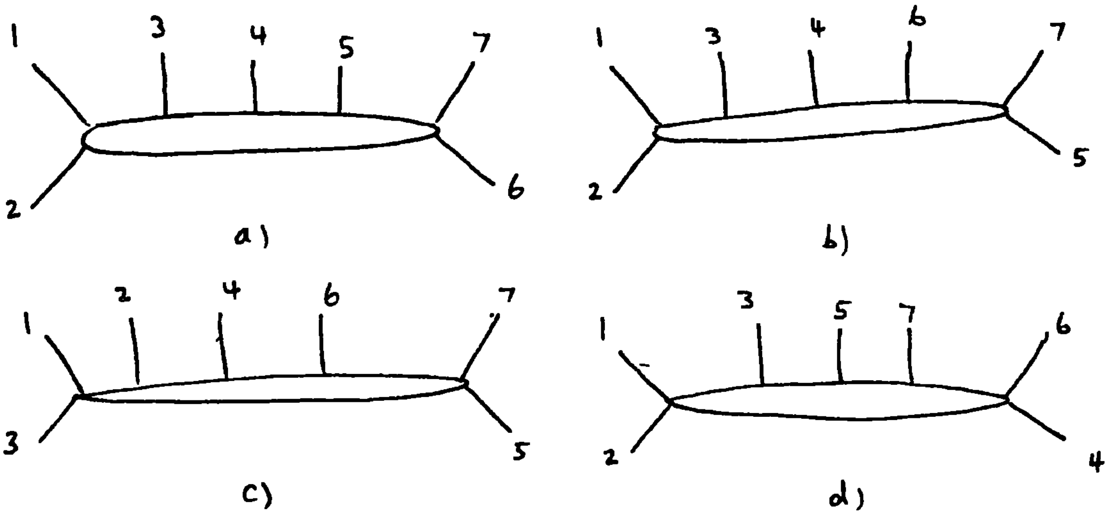

a) b) c) d)

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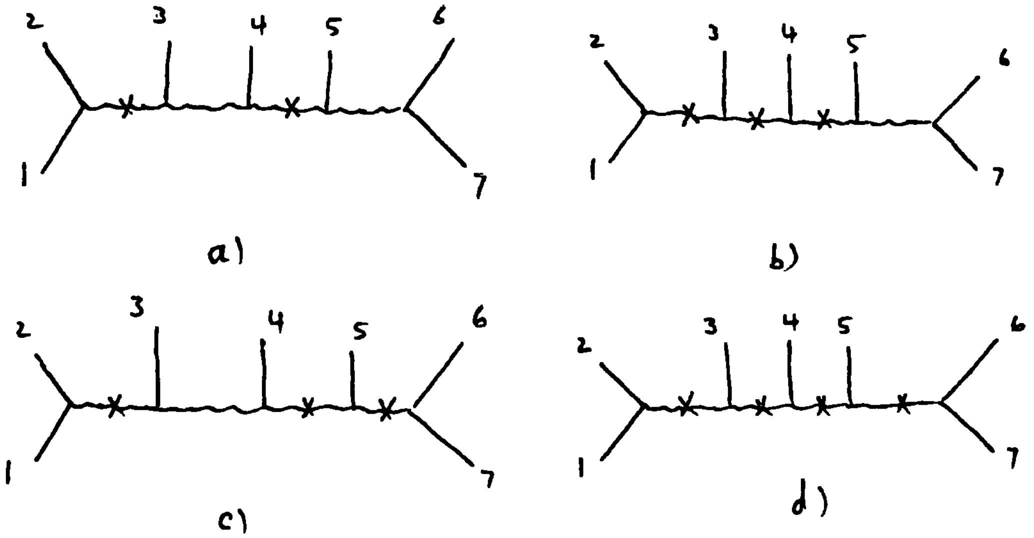

a) b) c) d)

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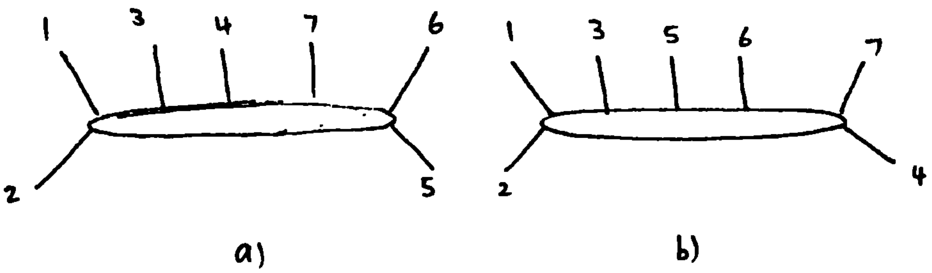

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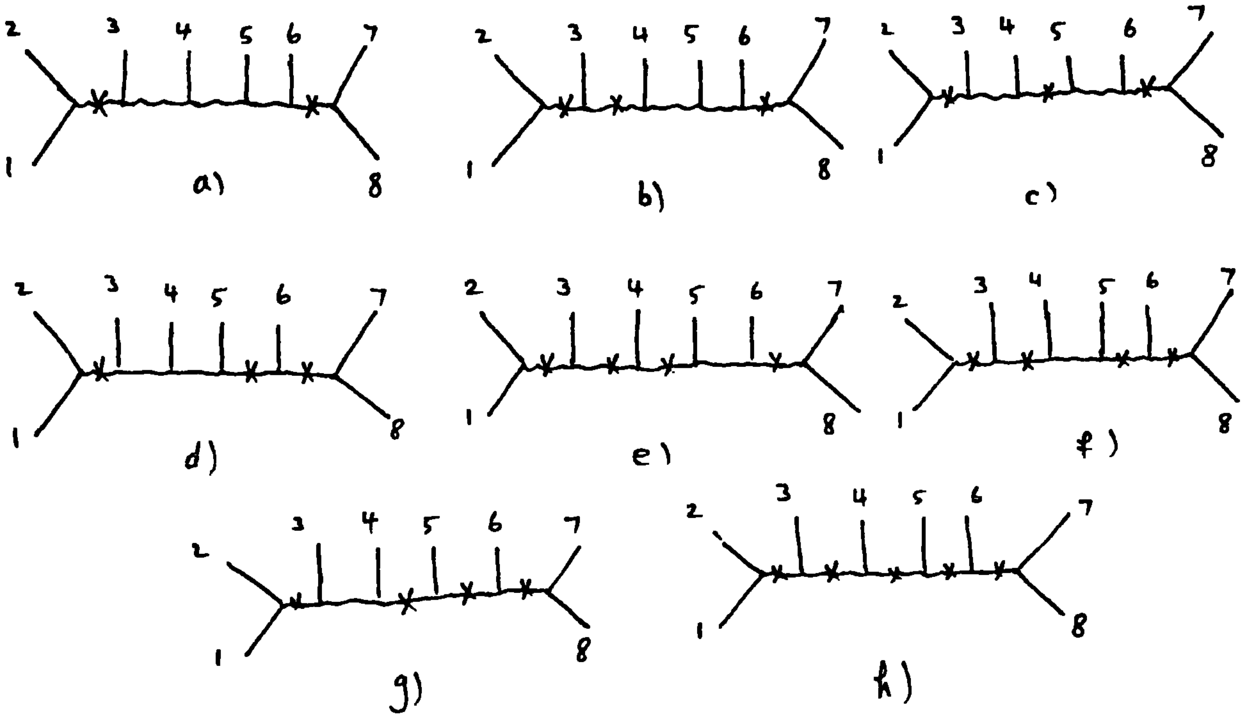

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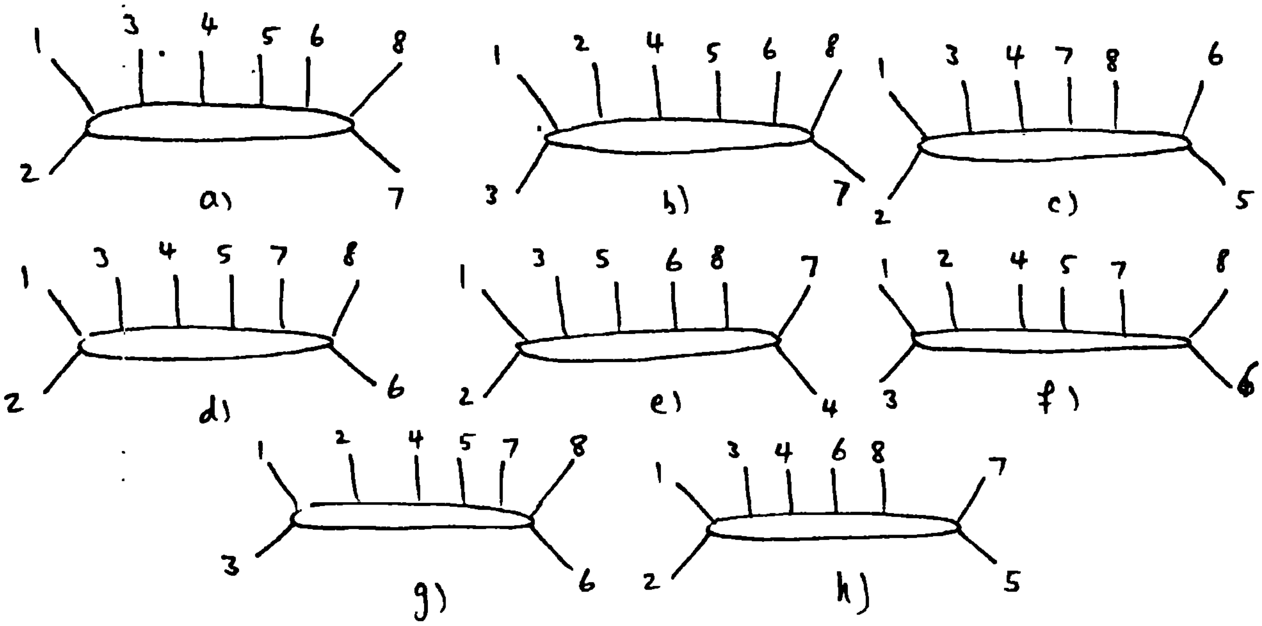

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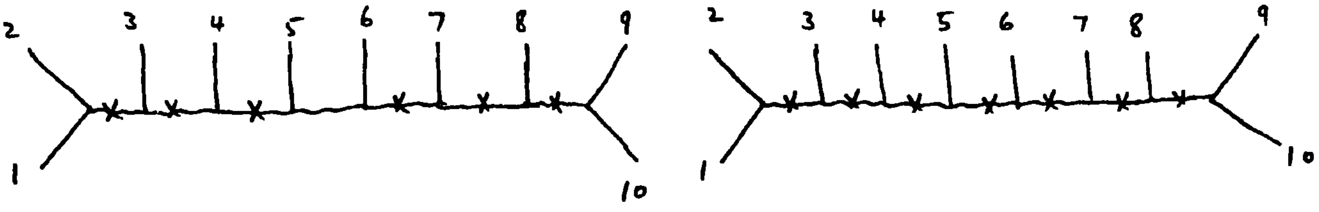

a) b)

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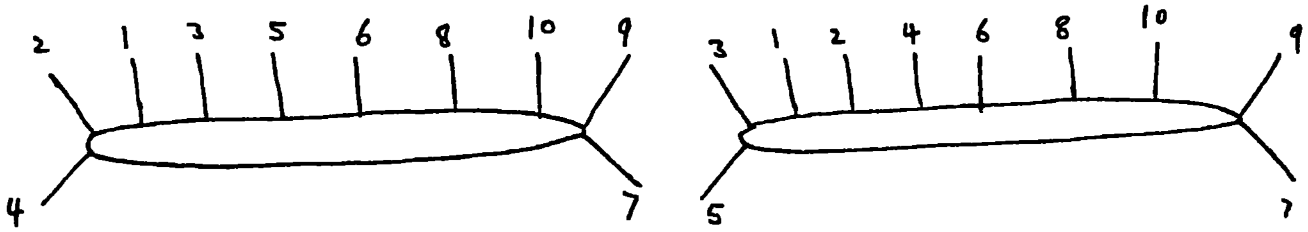

a) b)

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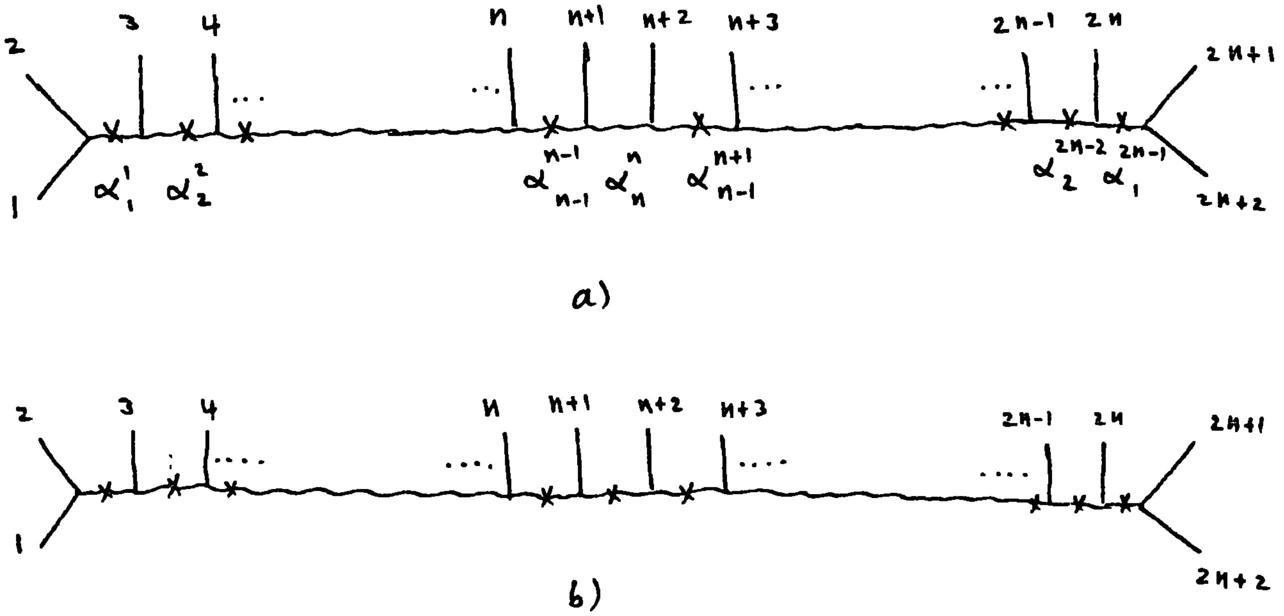
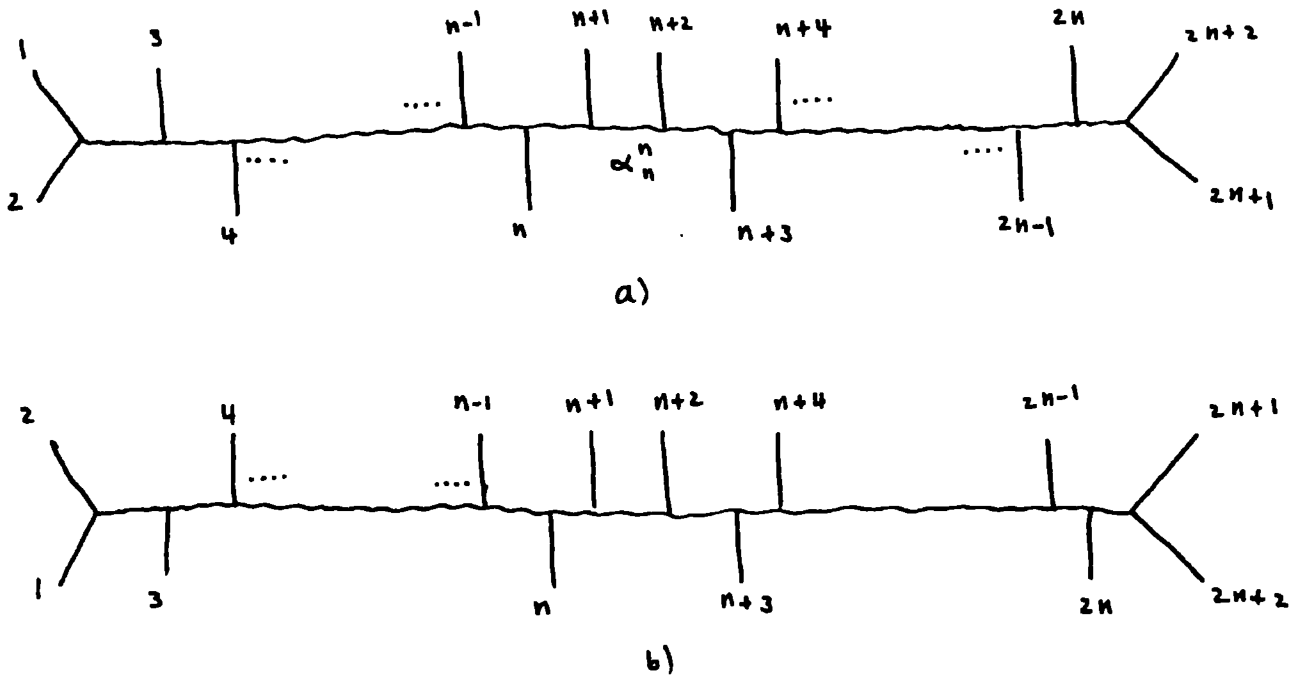


a)

b)

Fig 18.



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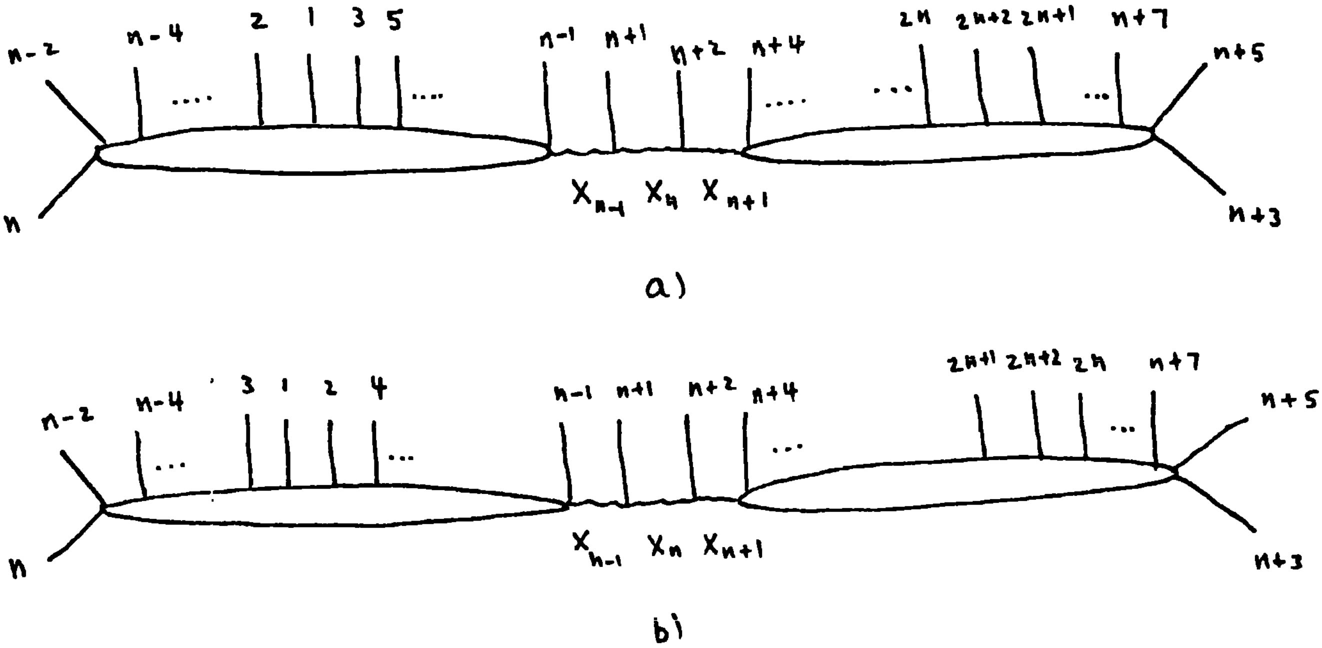

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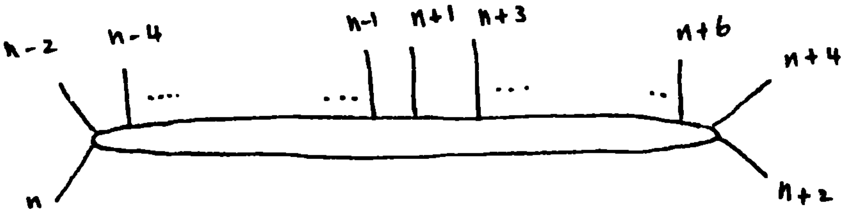

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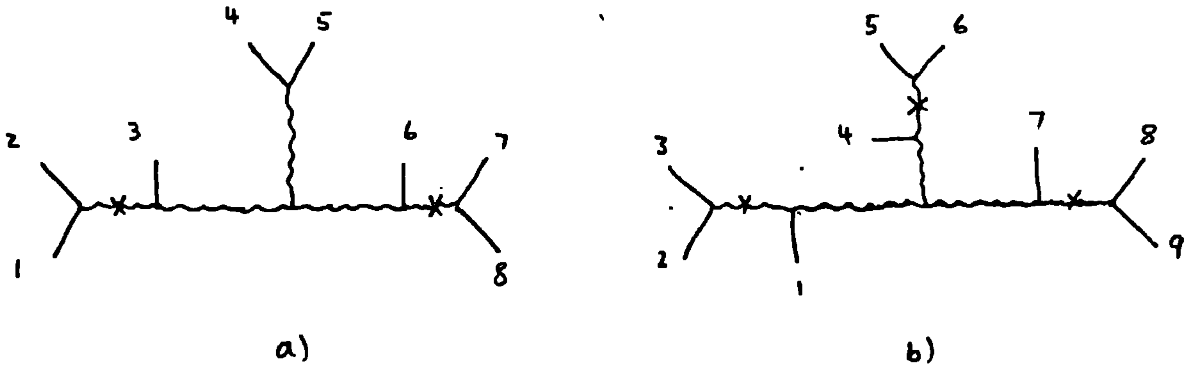
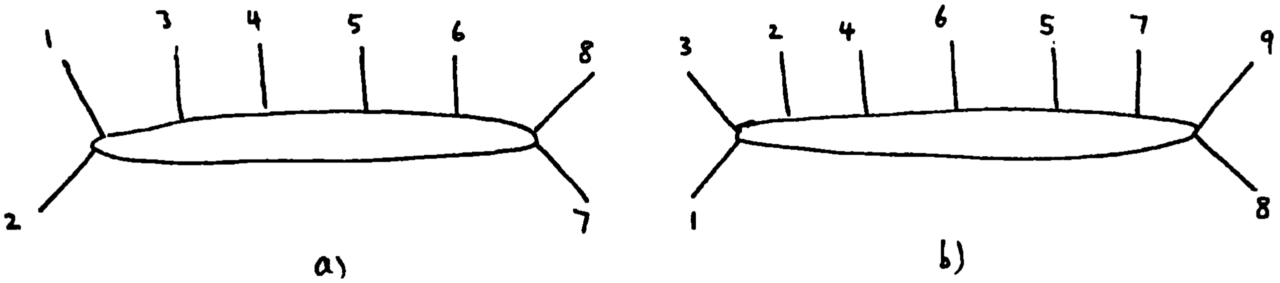


a) b)

Fig 22



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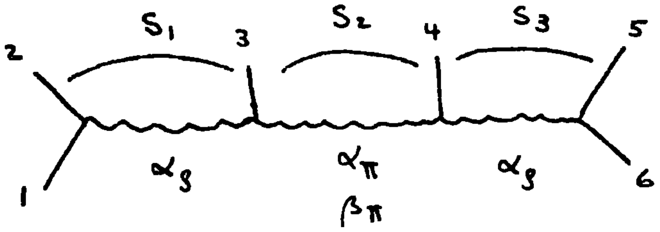

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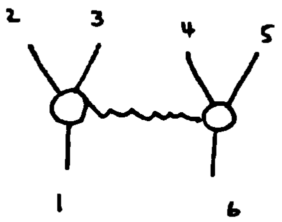

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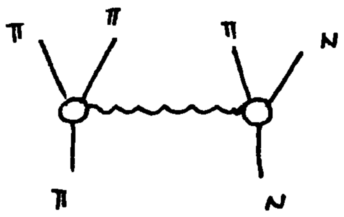

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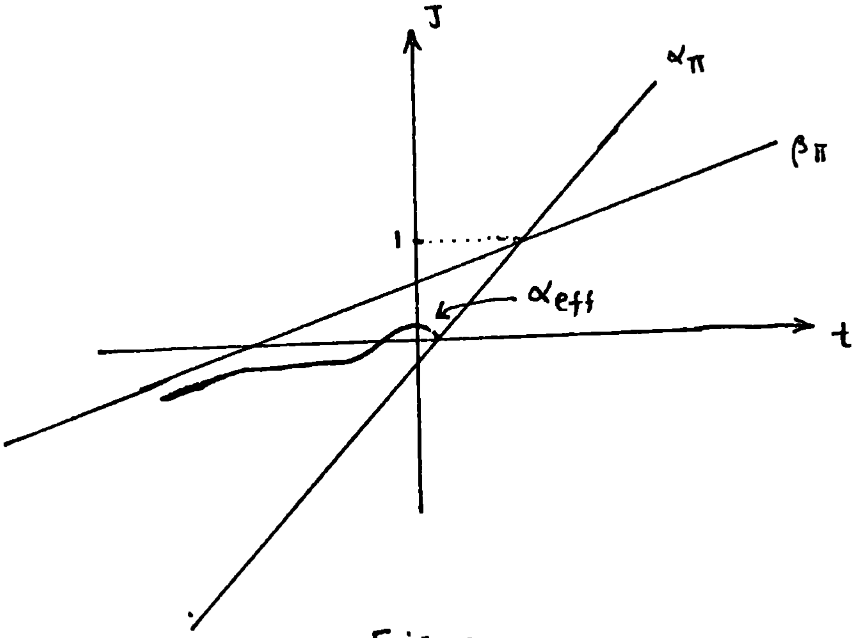

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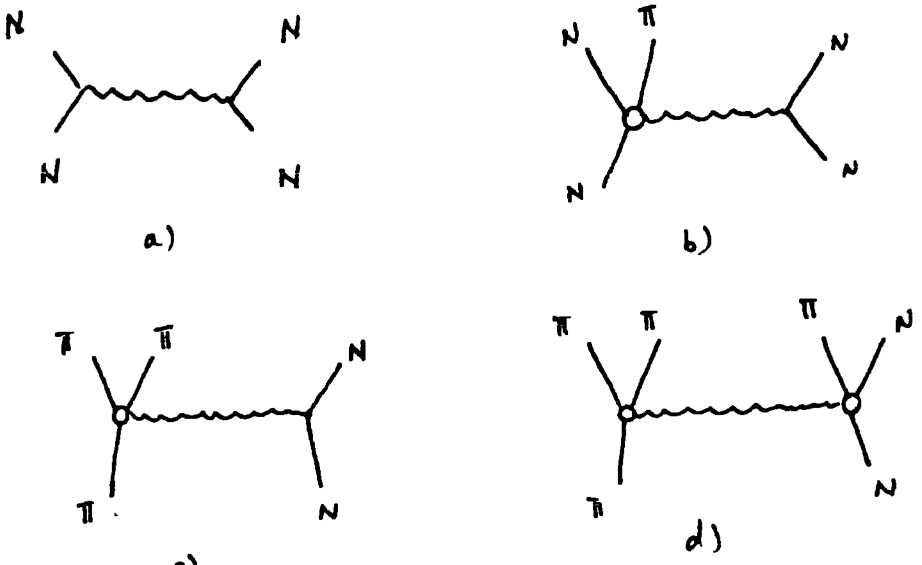

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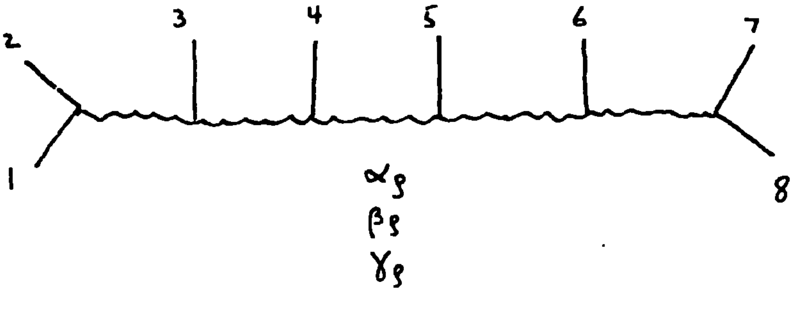

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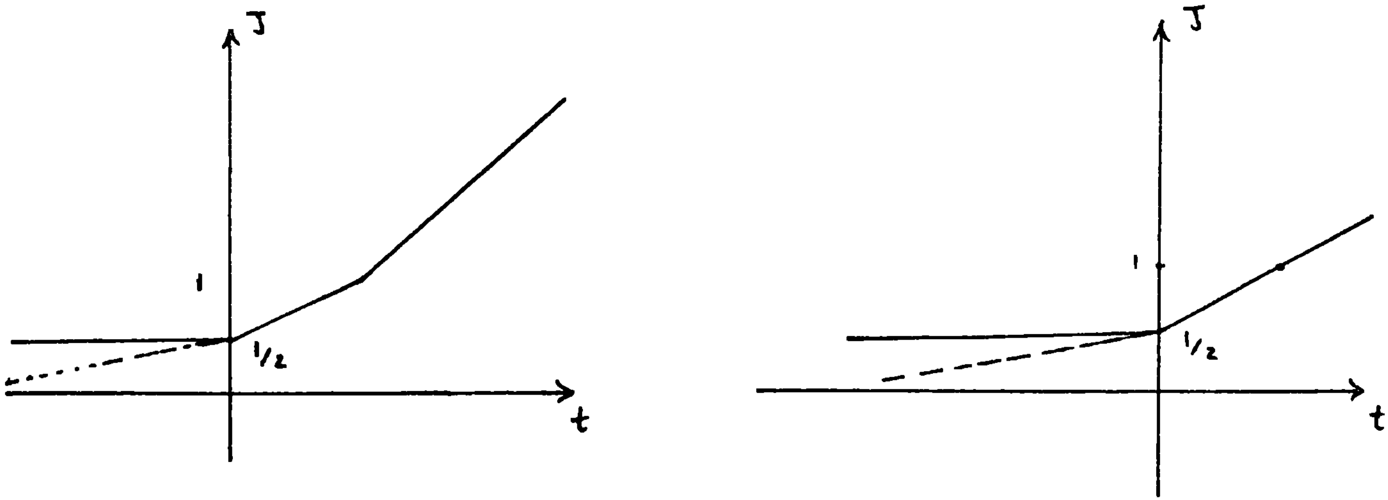
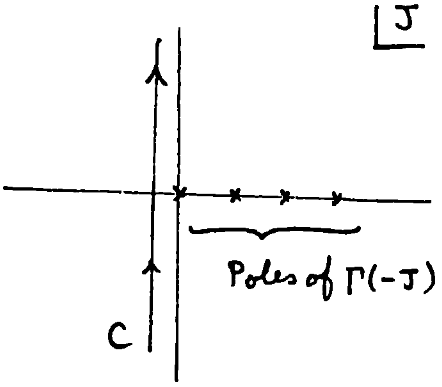


π channel

ρ channel

Fig 30.



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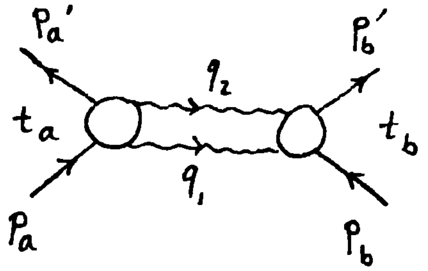

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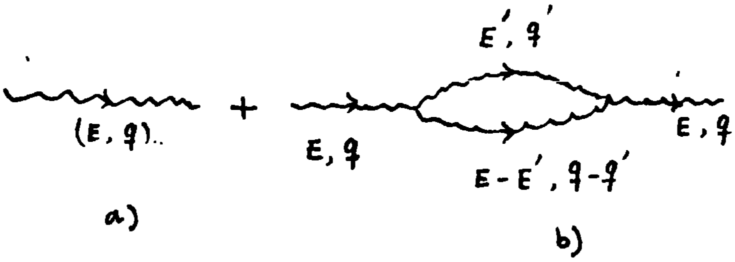

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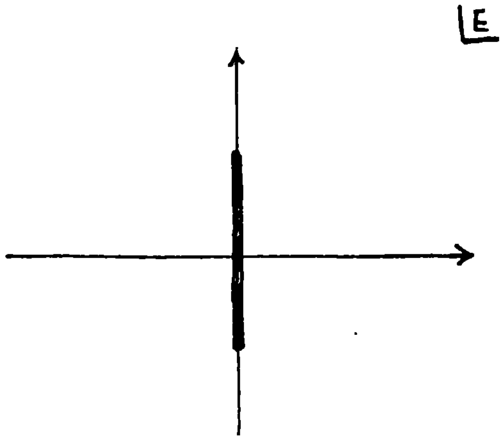

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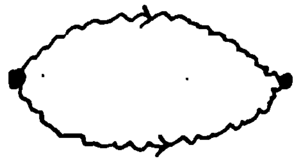
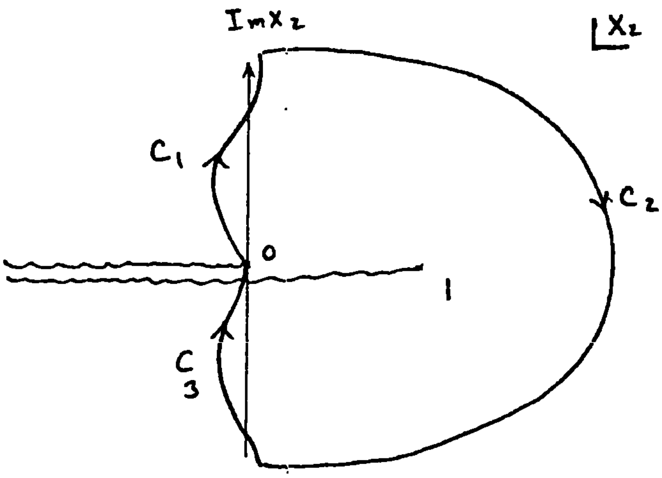


Fig 35

Fig 36



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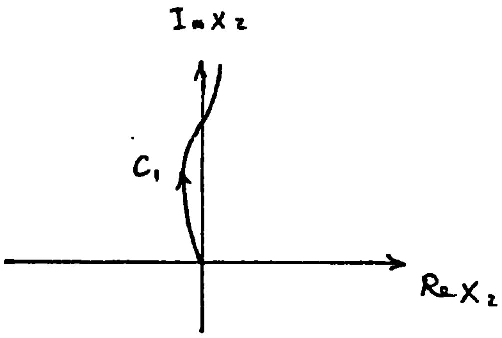

Fig 38