The weyl group and conjugacy classes

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THE WEYL GROUP AND CONJUGACY CLASSES

by

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Let G be a reductive group. Using the usual notation of the theory of algebraic groups : $B \supseteq T$ are respectively a fixed Borel Subgroup and a fixed maximal torus in G ; $\Phi = \Phi(G, T)$; $\Phi^+ = \Phi(B, T)$ etc. We let $\mathcal{C} = \{C(u_1), \ldots, C(u_\iota)\}$ be the set of unipotent conjugacy classes in G , where $C(u_i)$ is the class containing u_i , and $C(u_i) = C(u_j) \iff i = j$. Let β be the variety of Borel Subgroups, $u \in G$ unipotent and $\beta_u = \{\tilde{B} \in \beta \mid u \in \tilde{B}\}$. $A(u) = Z_G(u) / Z_G(u)^0$ acts on the irreducible components of β_u of maximal dimension. Let $c(u)_a$ be the number of such components fixed by $a \in A(u)$

<u>Basic assumption</u> If $C \in \mathcal{C}$ then $\exists$ $w \in W = W(G,T)$ such that $\overline{C \cap U_w^+} = U_w^+$

Results I) $|\{w \in W \mid \overline{C(u) \cap U_w^+} = U_w^+\}| = \frac{1}{|A(u)|} \sum_{a \in A(u)} c(u)_a^2$

II) (SPRINGER'S RESULT) $|W| = \sum_{i=1}^{\iota} \frac{1}{|A(u_i)|} \sum_{a \in A(u_i)} c(u_i)_a^2$

Now let $G = SL(n, K)$, $B \supseteq T$ be the groups of upper triangular matrices and diagonal matrices respectively. $W(G,T) \cong S_n$. Let $(k) = (k_1, \ldots, k_r)$ be an ordered partition of n, $d_{(k)}$ the dimension of the corresponding irreducible representation of S_n, N_{k_i} the $k_i \times k_i$ matrix with 1's on the superdiagonal and zeros elsewhere, $U_{(k)} = I + N_{k_1} \oplus N_{k_2} \oplus \cdots \oplus N_{k_r}$, and $\tilde{C}_w$, $w \in S_n$, the image of the Bruhat cell BwB under the canonical map $G \to \beta$.

<u>Result</u> $|\{w \in S_n \mid \tilde{C}_w \cap \beta_{u_{(k)}} \neq \phi , \overline{C(u_{(k)}) \cap U_w^+} = U_w^+\}| = d_{(k)}$

<u>Corollary</u> The number of irreducible components of $\beta_{u_{(k)}}$ of maximal dimension equals $d_{(k)}$.

PREFACE

The work presented in this thesis was carried out in the Department of Mathematics of the University of Durham between Jan. 1976 and Dec. 1977 under the supervision of Professor E.J. Squires.

The material in this thesis has not been submitted previously for any degree in this or any other university. No claim of originality is made for chapter one and most of chapter two. The remainder is claimed to be original. Chapters III, V and VI are based on two papers by the author in collaboration with Professor Squires. Chapter IV is based on some unpublished work by the author.

The author wishes to express his gratitude to Professor Squires for his help, guidance, continued encouragement throughout the course of this work and for critically reading the manuscript and correcting the English. He should also like to extend his thanks to the members of the Mathematics and Theoretical Physics department for numerous invaluable discussions.

ACKNOWLEDGEMENTS

I would like to thank Professor R. W. Richardson for his help in the work of this thesis, and also Professors R. Carter and T. A. Springer for the interest they have shown in this work.

I also acknowledge and thank the Science Research Council for their financial support whilst this work was being prepared.

TABLE OF CONTENTS

INTRODUCTION

Let G be a connected reductive algebraic group defined over an algebraically closed field K, T be a maximal torus in G, B a Borel Subgroup of G containing T, $W = W(G,T)$ the Weyl Group of G with respect to T, and $\mathcal{C}$ be the set of unipotent conjugacy classes in G. G. Lusztig has recently proved that the number of elements, $|\mathcal{C}|$, of $\mathcal{C}$ is finite. To each $w \in W$ we can associate a closed irreducible unipotent subgroup U_w^+, of B. Thus, we can associate to each $w \in W$ the unique element C of $\mathcal{C}$ which intersects U_w^+ in a dense open subset. In this way we obtain a map $\eta : W \to \mathcal{C}$. We make the following basic assumption: <u>if $C \in \mathcal{C}$ then there exists $w \in W$ such that $\overline{U_w^+ \cap C} = C$, or equivalently, η is surjective.</u>

This assumption holds in the following cases:

(i) G is a quasi-simple algebraic group for which the Carter Bala classification holds (see 3)

(ii) $G = SL(n,K)$

(iii) $G = SO(n,K)$ or $Sp(n,K)$, where K has infinite transcendence degree over its prime field and $char(K) \neq 2$.

Let β be the 'flag variety' of Borel Subgroups of G. If $u \in G$ is unipotent, then $\beta_u = \{\tilde{B} \in \beta \mid u \in \tilde{B}\}$ is a closed subvariety of β, and the finite group $A(u) = Z_G(u)/Z_G(u)^o$, where $Z_G(u)$ is the centralizer of u in G and $Z_G(u)^o$ is the identity component of $Z_G(u)$, acts on $\mathfrak{I}_u$, $\mathfrak{I}_u$ being the set of irreducible components of β_u of maximal dimension, as follows: if $F \in \mathcal{F}_u$ and $\gamma Z_G(u)^o \in A(u)$, then $\gamma Z_G(u)^o.F = \gamma F \gamma^{-1}$. If $c(u)_a$ denotes the number of elements of $\mathcal{F}_u$ fixed by $a \in A(u)$, then

$$|\eta^{-1}(C(u))| = \frac{1}{|A(u)|} \sum_{a \in A(u)} c(u)_a{}^2 \qquad \ldots \text{ I}$$

where $C(u)$ denotes the element of $\mathcal{L}$ containing u.

If $\mathcal{L} = \{C(u_1),\ldots,C(u_\ell)\}$, where $u_1,\ldots,u_\ell$ are unipotent elements of G, and $i \neq j$ implies that $C(u_i) \neq C(u_j)$, then it follows immediately that

$$|W| = \sum_{i=1}^{\ell} \frac{1}{|A(u_i)|} \sum_{a \in A(u_i)} c(u_i)_a{}^2 \qquad \ldots \text{ II}$$

In particular, if $G = SL(n,K)$, then $W \cong S_n$, the symmetric group on n elements, and $Z_G(u)$ is connected. Thus

$$|S_n| = \sum_{i=1}^{\ell} n_{u_i}{}^2, \text{ where } n_{u_i} = |\mathcal{F}_{u_i}|$$

This briefly covers the material of Chapter 2, the main result being II. Chapter 1 provides the necessary background material.

T. A. Springer presented the result we have labelled II in a seminar at Warwick University during Easter 1975. This suggested that I might be true. We later found that it was in fact an immediate consequence of the work we had completed prior to T. A. Springer's seminar. Springer's proof of II is algebraic in nature where as ours is geometric; ours is much easier. R. Steinberg has also obtained these results. We obtained our results just after Steinberg and quite independently.

In Chapter 3 we look specifically at the group $SL(n,K)$. We prove that our basic assumption is true for $SL(n,K)$, and then go on to look further at the fibres of the map η.

We assume that K has infinite transcendence degree over its prime field.

Let T be the maximal torus consisting of the diagonal matrices in SL(n,K), and B be the Borel Subgroup of upper traingular matrices in SL(n,K). Note that $W = W(SL(n,K),T)$ is isomorphic to S_n.

The unipotent conjugacy classes of SL(n,K) are in one to one correspondence with the ordered partitions $(k) = (k_1,\ldots,k_r)$ of n ($k_i \in \mathbb{Z}^+$ for $i = 1,\ldots,r$, $\sum_{i=1}^{r} k_i = n$, and $k_i \geq k_{i+1}$ for $i = 1,\ldots,r-1$). Also, there is a bijective correspondence between such partitions and the irreducible representations of S_n. We let $d_{(k)}$ denote the dimension of the representation corresponding to (k). It can be shown that

$$\sum_{\substack{\text{partitions} \\ (k)}} d_{(k)}^{2} = |S_n| \qquad \ldots \text{ III}$$

Let $\ell \in \mathbb{Z}^+$, then let N_ℓ be the $\ell \times \ell$ matrix with ones on the super-diagonal and zero's elsewhere. Also, if $(k) = (k_1,\ldots,k_r)$ is an ordered partition of n, then let $U_{(k)} = I + N_{k_1} \oplus \ldots \oplus N_{k_r}$. If $w \in W$, then let $\tilde{C}_w$ be the image of the Bruhat Cell BwB under the canonical morphism $G \to \beta$, $g \to {}^{g}B$.

Our interest lies in the set

$$N_{U_{(k)}} = \{w \in W \mid \tilde{C}_w \cap \beta_{U_{(k)}} \neq \emptyset, \quad \eta(w) = C(U_{(k)})\}$$

and we prove that

$$|N_{U_{(k)}}| = d_{(k)}.$$

This together with III and the results of Chapter 2 enables us to show that $n_u = d_{(k)}$, where u is any element of the unipotent conjugacy class of SL(n,K) corresponding to (k), and n_u is equal to the number of irreducible components of β_u of maximal dimension (cf page ii).

Finally, in Chapter 4 we show that our basic assumption holds for the groups SO(n,K) and Sp(n,K) (see page (i)). All we do in this chapter is to combine the work of Carter and Bala (3) and Gerstenhaber (5).

CHAPTER 1

BASIC CONCEPTS

Our aim in this chapter is to give a brief outline of the basic material which will be needed in later work. We begin by giving brief descriptions of Root Systems and Algebraic Varieties, and then go on to describe the structure of Linear Algebraic Groups. (Basic Reference, "Linear Algebraic Groups", by A. Borel).

1.1 ROOT SYSTEMS (SEE 2)

Let E be a finite dimensional, real Euclidean Space with inner product $(\, , \,)_E$.

1.1.1 Reflections:

If $\alpha \in E \sim \{0\}$ and $H_\alpha = \{v \in E \mid (v,\alpha) = 0\}$, then let τ_α denote the reflection of E in the hyperplane H_α. i.e.

$$\tau_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha \, , \quad \forall \; v \in E.$$

It is clear that $\tau_\alpha{}^2 = 1$.

1.1.2 Abstract Root Systems:

A subset, Φ, of E is called an abstract root system if it satisfies the following conditions:

(i) $0 \notin \Phi$, Φ is finite and spans E.

(ii) If $\alpha \in \Phi$, then $\tau_\alpha(\Phi) = \Phi$.

(iii) If $\alpha, \beta \in \Phi$, then $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.

(iv) If $\alpha \in \Phi$, $t \in \mathbb{R}$ and $t\alpha \in \Phi$, then $t = \pm 1$.

<u>Definition</u>: The dimension of E is called the <u>rank</u> of Φ.

1.1.3 The Equivalence of Root Systems

Suppose that E and E' are two, finite dimensional, real Euclidean Spaces with inner products $(,)_E$ and $(,)_{E'}$ respectively, and suppose that $\Phi \subset E$ and $\Phi' \subset E'$ are root systems, then Φ is said to be <u>equivalent</u> to Φ' if there exists a linear isomorphism $f: E \to E'$ such that:

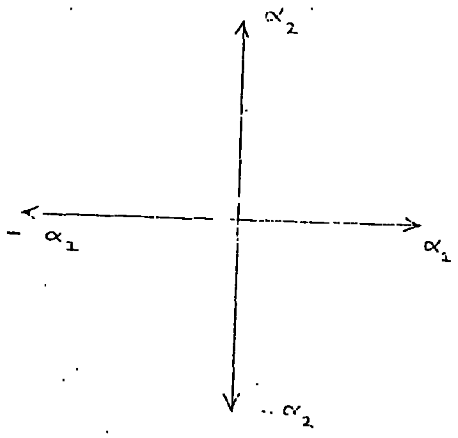
(i) $(v_1, v_2)_E = (f(v_1), f(v_2))_{E'}$ for all $v_1, v_2 \in E$.

(ii) f maps Φ bijectively onto Φ'.

1.1.4 Examples (see 8)

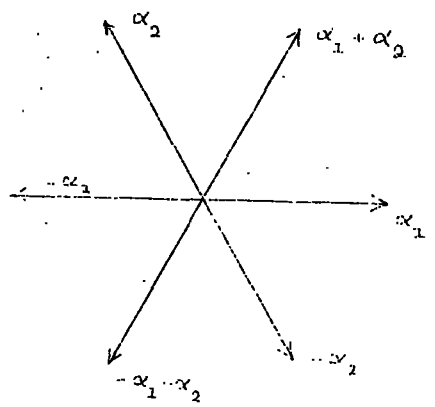
If $E = \mathbb{R}^2$ with the usual inner product, $((x_1,x_2), (y_1, y_2)) = x_1y_1 + x_2y_2$, then the subsets of E represented in the diagrams below are root systems. Moreover any other root system of rank 2 is equivalent to one of these:

1) $\underline{A_1 \oplus A_1}$



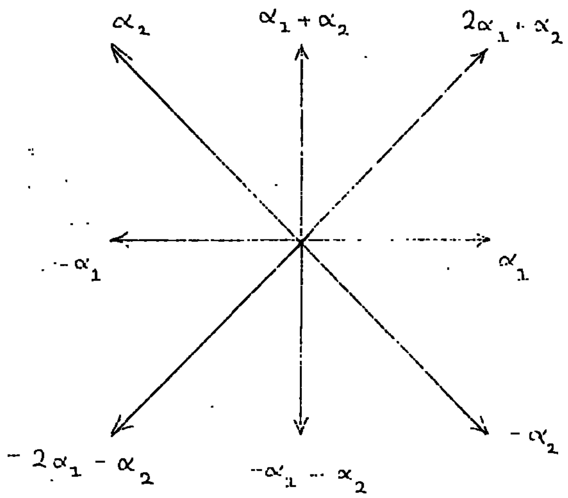
α_1 and α_2 are perpendicular and any ratio $|\alpha_1| : |\alpha_2|$ is permissible.

2) $\underline{A_2}$



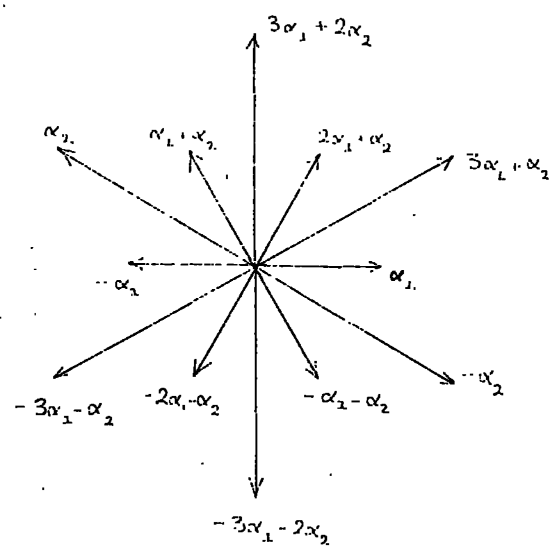
All the vectors have the same length and the angle between adjacent vectors is $\pi/3$.

3) B_2



$|\alpha_1| : |\alpha_2| = 1 : \sqrt{2}$. The angle between adjacent vectors is $\pi/4$.

4) G_2

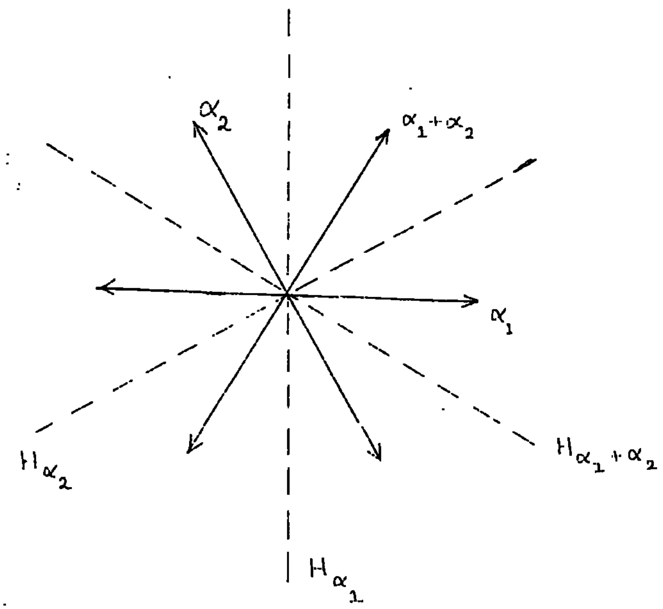


$|\alpha_1| : |\alpha_2| = 1 : \sqrt{3}$. The angle between adjacent vectors is $\pi/6$.

1.1.5 The Weyl Group:

Let $\Phi \subseteq E$ be a root system. Then the subgroup $W(\Phi)$ of $GL(E)$ generated by $\{\tau_\alpha \mid \alpha \in \Phi\}$ is a permutation group of Φ, and is thus finite. $W(\Phi)$ is called the Abstract Weyl Group of Φ.

Example. The Abstract Weyl Group of a root system Φ of type A_2 (see the above example) is isomorphic to S_3, the symmetric group on three elements. i.e.



$\tau_{\alpha_1}^{\;2} = \tau_{\alpha_2}^{\;2} = 1$ (see 1.1.1). From the above diagram it is clear that $\tau_{\alpha_1+\alpha_2} = \tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_1} = \tau_{\alpha_2}\tau_{\alpha_1}\tau_{\alpha_2}$. Hence $W(\Phi) = \{Id_E, \tau_{\alpha_1}, \tau_{\alpha_2}, \tau_{\alpha_1}\tau_{\alpha_2}, \tau_{\alpha_2}\tau_{\alpha_1}, \tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_1}\}$. We get the required isomorphism by mapping τ_{α_1} onto $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, and τ_{α_2} onto $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$.

1.1.6 Bases:

A subset π of Φ is called a basis of Φ if:

(i) π is a basis of E.

(ii) If $\alpha \in \Phi$, then $\alpha = \sum_{\beta \in \pi} m_\beta \beta$, where the m_β's are integers of like sign.

Bases exist; $W(\Phi)$ permutes the collection of bases simply transitively, and every root lies in at least one base. If a basis π of Φ is fixed, then its elements are called simple roots.

A subset ψ of Φ is said to be a set of positive roots if:

(i) $\alpha \in \psi$ if and only if $-\alpha \notin \psi$.

(ii) If $\alpha, \beta \in \psi$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \psi$.

Let $\mathcal{\Psi}$ denote the collection of all such sets of positive roots, and $\mathcal{T}$ the collection of bases. Then the map $\Gamma : \mathcal{T} \to \mathcal{\Psi}$, given by $\Gamma(\pi) = \{\alpha \in \Phi \mid \alpha = \sum_{\beta \in \pi} m_\beta \beta,\ m_\beta \geq 0\}$ for all $\pi \in \mathcal{T}$, is a bijection; i.e. if $\psi \in \mathcal{\Psi}$, then $\Gamma^{-1}(\psi) = \{\alpha \in \psi \mid \alpha - \beta \notin \psi, \forall \beta \in \psi\}$.

If π is a fixed basis of Φ, then we write Φ^+ for $\Gamma(\pi)$, and call $\Phi^- = \Phi \sim \Phi^+$ the set of negative roots with respect to π.

1.1.7 The Height Function

If $\alpha = \sum_{\beta \in \pi} m_\beta \beta \in \Phi$, then we put $h(\alpha) = \sum_{\beta \in \pi} m_\beta$. $h(\alpha)$ is called the height of the root α with respect to the basis π. The number $\max_{\alpha \in \Phi}(h(\alpha))$ is independent of the choice of π, and is called the height of the highest root.

1.1.8 Example

In Example 1.1.4, α_1 and α_2 are simple roots in the various root systems. In the root system of type B_2 α_1, α_2, $\alpha_1+\alpha_2$ and $2\alpha_1+\alpha_2$ are the positive roots, and $-\alpha_1$, $-\alpha_2$, $-\alpha_1-\alpha_2$ and $-2\alpha_1-\alpha_2$ are the negative roots corresponding to the basis $\{\alpha_1,\alpha_2\}$. Also, in this case, the height of the heighest root is 3.

We now fix an arbitary basis π of Φ.

1.1.9 The Length of the Elements of $W(\Phi)$

The elements of the set $\{\tau_\alpha \mid \alpha \in \pi\}$ are called fundamental reflections. $W(\Phi)$ is generated by the fundamental reflections (c.f. the example in section 1.1.5). The length $\ell(w)$ of $w \in W(\Phi)$ is the smallest non-negative integer q such that $w = \tau_{\alpha_1}\tau_{\alpha_2}\ldots\tau_{\alpha_q}$, where $\alpha_i \in \pi$

for $i = 1,\ldots,q$.

$\ell(w) = |\Phi^- \cap w\Phi^+|$

$= |\{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\}|$.

There exists a unique element w_o of $W(\Phi)$ for which $\ell(w_o)$ is a maximum. $w_o\Phi^+ = \Phi^-$ and $w_o = w_o^{-1}$.

1.1.10 Subsystems

Let S be a finite subset of E, and put $\mathbb{Z}S = \{v \in E \mid v = \sum_{u \in S} \lambda_u u; \lambda_u \in \mathbb{Z}\}$. Now if $J \subseteq \pi$, then $\Phi_J = \Phi \cap \mathbb{Z}J$ is a root system in the subspace of E spanned by J; Φ_J is called a subsystem of Φ. J is a basis of Φ_J. Also, $W(\Phi_J)$ can be identified, in the obvious way, with the subgroup W_J of $W(\Phi)$ generated by $\{\tau_\alpha \mid \alpha \in J\}$. If $\tau \in W_J$, then $\tau(\Phi \sim \Phi_J) = \Phi \sim \Phi_J$.

1.1.11 Dynkin Diagrams and Irreducible Root Systems

Suppose that $\pi = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$. The Dynkin Diagram, ∇, of Φ consists of ℓ nodes, each of which represents a distinct element of π. The nodes which represent the simple roots α_i and α_j are joined by $\dfrac{4(\alpha_i,\alpha_j)^2}{(\alpha_i,\alpha_i)(\alpha_j,\alpha_j)}$ bonds. Further, if $(\alpha_i,\alpha_j) \neq 0$ and $|\alpha_i| < |\alpha_j|$, then this is represented by an arrow which points from the node which represents α_i to the node which represents α_j.

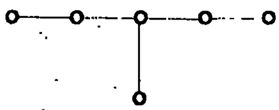
A root system is said to be irreducible if it cannot be written as the disjoint union of two, non-empty, mutually orthogonal subsets. Up to equivalence, the irreducible root systems are in one to one correspondence with the following Dynkin Diagrams:

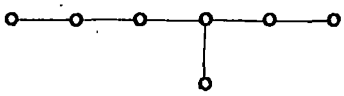
A_ℓ ($\ell \geq 1$)  (ℓ nodes)

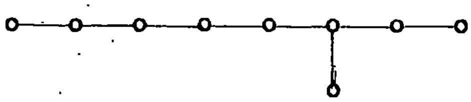
B_ℓ ($\ell \geq 2$)  (ℓ nodes)

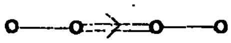
C_ℓ ($\ell \geq 3$)  (ℓ nodes)

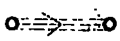
D_ℓ ($\ell \geq 4$)  (ℓ nodes)

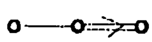
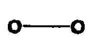
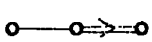
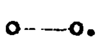
E_6 

E_7 

E_8 

F_4 

G_2 

If Φ is a root system, then $\Phi = \Phi_1 \cup \Phi_2 \cup \ldots \cup \Phi_p$, where each Φ_i is an irreducible subsystem of Φ, and Φ_i and Φ_j are mutually orthogonal whenever $i \neq j$. The subsystems Φ_i are called the <u>irreducible components</u> of Φ. The Dynkin Diagram of Φ is of the form $\nabla_1 \oplus \nabla_2 \oplus \ldots \oplus \nabla_p$, where ∇_i is the Dynkin Diagram of Φ_i, $i = 1,\ldots,p$. {If ∇_1 =  and ∇_2 = , then $\nabla_1 \oplus \nabla_2$ =  .}

1.2 ALGEBRAIC VARIETIES (SEE 9)

We will use K to denote an algebraically closed field.

1.2.1 Affine Algebraic Varieties:

An affine algebraic variety is a pair $(V, K[V])$, where:

(i) V is a set and $K[V]$ is a finitely generated, commutative K-algebra of K valued functions on V.

(ii) If $x, y \in V$ and $x \neq y$, then there exists $f \in K[V]$ such that $f(x) \neq f(y)$.

(iii) If $\psi : K[V] \to K$ is a K-algebra morphism, then $\exists\ x \in V$ such that $\psi(f) = f(x)$ for all $f \in K[V]$.

If $x \in V$, then we let e_x denote the evaluation at x (i.e. $e_x(f) = f(x)\ \forall\ f \in K[V]$). By (ii) and (iii) above it is clear that the map $V \to \mathrm{Hom}_{K\text{-alg}}(K[V],K)$, $x \to e_x$, is bijective. Also, we can identify the points of V with the maximal ideals of $K[V]$, i.e. $x \to \mathrm{Ker}(e_x)$.

In general we shall not distinguish between V and the pair $(V, K[V])$. $K[V]$ is called the co-ordinate ring of V.

1.2.2 The Zariski Topology

If $\mathfrak{a} \subseteq K[V]$, then we call $V(\mathfrak{a}) = \{x \in V \mid f(x) = 0\ \forall\ f \in \mathfrak{a}\}$ a closed subset of V. In this way we obtain a topology on V. This is called the Zariski Topology.

Note: If $\tilde{\mathfrak{a}}$ is the ideal in $K[V]$ generated by $\mathfrak{a}$, then $V(\mathfrak{a}) = V(\tilde{\mathfrak{a}})$. Thus, since the ideals of $K[V]$ are finitely generated, it is clear that there exists a finite subset $\mathfrak{b}$ of $\mathfrak{a}$ such that $V(\mathfrak{b}) = V(\mathfrak{a})$.

If $f \in K[V]$, then $V_f = \{x \in V \mid f(x) \neq 0\}$ is an open subset of V, and it is called a principal open set. The principal open sets form a basis for the topology on V; any open subset being the union of a finite number of principal open sets.

1.2.3 Morphisms of affine varieties

Let U and V be affine algebraic varieties. A function $\alpha : U \to V$ is called a morphism if, for all $f \in K[V]$, $f \circ \alpha \in K[U]$. If α is a morphism, then $\alpha^* : K[V] \to K[U]$, $\alpha^*(f) = f \circ \alpha$, is called the comorphism of α. If α is a morphism, then:

(i) For all $f \in K[V]$, $\alpha^{-1}(V_f) = U_{\alpha^* f}$. Thus α is continuous.

(ii) $e_{\alpha(u)} f = f(\alpha(u)) = e_u(\alpha^* f)$, and thus α is completely determined by α^*.

If $\alpha : U \to V$ and $\beta : V \to W$ are morphisms of affine varieties, then so is $\beta \circ \alpha$. Further $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$.

1.2.4 Subvarieties

If V is an affine variety and W is a closed subset of V, then W is an affine variety with co-ordinate ring $K[W] = K[V] / I[W]$, where $I[W] = \{f \in K[V] \mid f(x) = 0, \forall x \in W\}$. W is called a closed subvariety of V.

If $f \in K[V] - \{0\}$, then let $K[V]_f$ denote the localisation of $K[V]$ at f; i.e. $K[V]_f$ is the ring of fractions of the form $\frac{g}{f^r}$, where $g \in K[V]$ and r is a non-negative integer. V_f is an affine variety with co-ordinate ring $K[V]_f$ - note that if $x \in V_f$ and $\frac{g}{f^r} \in K[V]_f$, then $\frac{g}{f^r}(x) = \frac{g(x)}{f^r(x)}$.

1.2.5 Affine n-space

If we let $\{X_1,X_2,\ldots X_n\}$ be a set of n independent indeterminants, then $R = K[X_1,X_2,\ldots X_n]$ is, in the obvious way, a ring of K-valued functions on K^n. It is easy to see, by the Hilbert Nullstellensatz Theorem, that K^n is an affine variety with co-ordinate ring R.

If V is an affine variety, then there exists $u_1,u_2,\ldots,u_n \in K[V]$ such that $K[V] = K[u_1,u_2,\ldots u_n]$. Further, if $X_1,X_2\ldots X_n$ are n independent indeterminants, then we can define a K-algebra morphism $\xi^* : K[X_1,X_2,\ldots X_n] \to K[V]$ by putting $\xi^*(X_i) = u_i$ for $i = 1,\ldots,n$. Hence, we can obtain a morphism of algebraic varieties, $\xi: V \to K^n$, with comorphism ξ^*.

i.e. Identify V with the maximal ideals of $K[V]$, and K^n with the maximal ideals of $K[X_1,X_2,\ldots X_n]$. Now, if $\mathfrak{a}$ is a maximal ideal of $K[V]$, then put $\xi(\mathfrak{a}) = (\xi^*)^{-1}(\mathfrak{a})$.

It can be shown that:

(i) $\xi(V)$ is a closed subvariety of K^n.

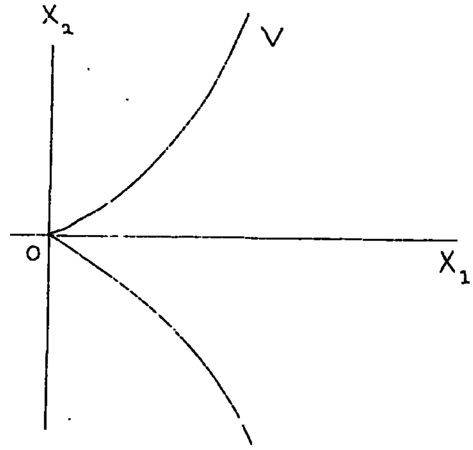
(ii) $\xi: V \to \xi(V)$ is an isomorphism of varieties.

Thus, there exists n such that V is isomorphic to a closed subvariety of K^n.

1.2.6 Examples

Note: if $\mathfrak{a}$ is an ideal in $K[V]$, then $I[V(\mathfrak{a})]$ = the radical of $\mathfrak{a}$. Also, if $f_1,f_2,\ldots,f_p \in K[X_1,X_2,\ldots X_n]$, then $(f_1,f_2,\ldots,f_p)$ denotes the ideal in $K[X_1,X_2,\ldots X_n]$ generated by $f_1,f_2,\ldots f_p$.

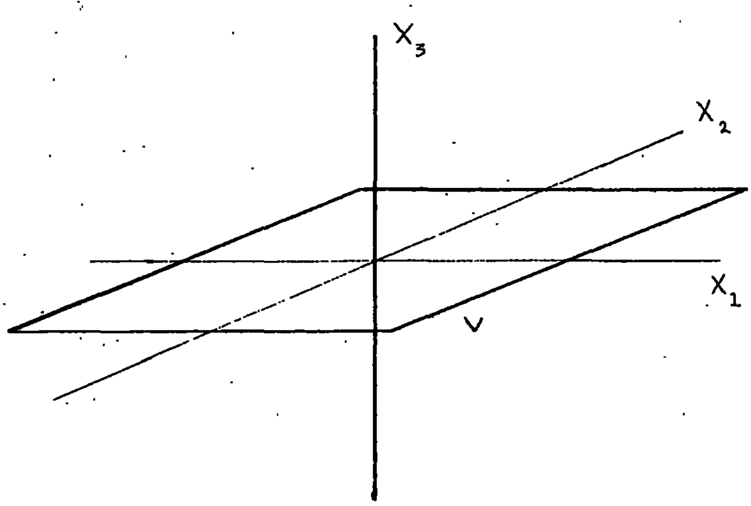
Example 1



$V = V((X_1^3 - X_2^2))$, and is a closed subvariety of K^2.

$I[V] = (X_1^3 - X_2^2)$, and thus $K[V] = K[X_1,X_2] / (X_1^3 - X_2^2)$.

Example 2



$V = V((X_1X_3, X_2X_3))$, and is a closed subvariety of K^3.

$I[V] = (X_1X_3, X_2X_3)$, and hence $K[V] = K[X_1, X_2, X_3] \,/\, (X_1X_3, X_2X_3)$.

Example 3

$M_n(K)$, the set of $n \times n$ matrices with coefficients in K, is an affine variety with co-ordinate ring $K[T_{11}, T_{12}, \ldots, T_{nn}]$, where $T_{ij}((a_{pq})) = a_{ij}$ for all $(a_{pq}) \in M_n(K)$; i.e. $M_n(K) \cong K^{n^2}$. $GL(n,K)$ is the principal open set $M_n(K)_D$, where $D \in K[T_{11}, T_{12}, \ldots, T_{nn}]$ is such that $D((a_{pq})) = \det((a_{pq}))$ for all $(a_{pq}) \in M_n(K)$. The co-ordinate ring of $GL(n,K)$ is $K[T_{11}, T_{12}, \ldots, T_{nn}, \frac{1}{D}]$.

1.2.7 Products of Affine Varieties

If U and V are affine varieties, then $K[U] \otimes K[V]$ is a finitely generated, commutative K-algebra of K-valued functions on $U \times V$. i.e. If $\sum_{i=1}^{n} f_i \otimes g_i \in K[U] \otimes K[V]$ and $(x,y) \in U \times V$, then $\sum_{i=1}^{n} f_i \otimes g_i \; ((x,y)) = \sum_{i=1}^{n} f_i(x)g_i(y)$. $U \times V$ is an affine variety with co-ordinate ring $K[U] \otimes K[V]$.

We note that the Zariski Topology on $U \times V$ is not the same as the product topology.

Example

The quadric $X_1^2 + X_2^2 = 1$ is a Zariski closed subset of $K^2 = K \times K$, but it is not a closed subset in the product topology on K^2.

1.2.8 Algebraic Varieties

We now extend our definition from affine varieties to varieties in general. An algebraic variety is a finite collection of affine varieties which have been suitably patched together; i.e. an algebraic variety is a topological space V for which there exists a finite open covering $\{U_i\}_{i=1}^n$ such that:

(i) Each U_i is an affine variety.

(ii) $U_i \cap U_j$ is a principal open set in both U_i and U_j, and the identity map is an isomorphism of the two affine structures on $U_i \cap U_j$ (obtained from U_i and U_j).

(iii) $\{(x,x) \in U_i \times U_j \mid x \in U_i \cap U_j\}$ is a closed subset of $U_i \cap U_j$.

(iv) U is open in V if and only if $U \cap U_i$ is an open subset of U_i for each $i = 1,\ldots,n$.

If V is an algebraic variety, then we write $K[V]$ for the algebra of rational functions on V which are defined everywhere. A function f is said to be defined at $x \in V$ if for some affine open neighbourhood U_x of x, $f = g/h$ where $g, h \in K[U_x]$ (in the old sense) and $h(x) \neq 0$. Note that if V is an affine variety, then $K[V]$ as defined above coincides with $K[V]$ in the old sense.

A subset Z of V is said to be locally closed if it is the intersection of an open subset and a closed subset of V, or, equivalently, if it is open in its closure. If Z is a locally closed subset of V, then it has, in a natural way, the structure of an algebraic variety. It is called a subvariety of V.

1.2.9 Morphisms of varieties

Let V and W be algebraic varieties. A function $\alpha: V \to W$ is called a morphism of varieties if:

(i) α is continuous.

(ii) If S and T are open subsets of U and W respectively, and $\alpha(S) \subseteq T$, then there exists a K-algebra morphism, $\alpha_S{}^T: K[T] \to K[S]$, such that $\alpha_S{}^T(f) = f \circ \alpha$ for all $f \in K[T]$.

We denote the collection of maps $\alpha_S{}^T$ by α^*. α^* is called the comorphism of α.

1.2.10 Projective Varieties

Let $P_n(K)$ denote the set of lines, through the origin, in K^{n+1}. Also, if $(x_o,x_1,\ldots,x_n) \in K^{n+1} \sim \{0\}$, then let $[(x_o,x_1,\ldots,x_n)]$ denote the line in $P_n(K)$ on which $(x_o,x_1,\ldots,x_n)$ lies. It is clear that $[(x_o,x_1,\ldots,x_n)] = [(\bar{x}_o,\bar{x}_1,\ldots,\bar{x}_n)]$ if and only if there exists $k \in K^*$ such that $\bar{x}_i = kx_i$ for $i = 0,\ldots,n$.

Let $U_i = \{[(x_o,x_1,\ldots,x_n)] \mid x_i \neq o\}$ for $i = 0,1,\ldots,n$. Then each U_i is an affine algebraic variety with co-ordinate ring

$$K\left[\frac{X_o}{X_i}, \frac{X_1}{X_i}, \ldots, \frac{\hat{X}_i}{X_i}, \ldots, \frac{X_n}{X_i}\right], \text{ where}$$

$\frac{X_p}{X_i}([(x_o,x_1,\ldots,x_n)]) = \frac{x_p}{x_i}$ for $i = 0,1,\ldots,\hat{i},\ldots,n$. i.e. the map from U_i to K^n given by $[(x_o,x_1,\ldots,x_n)] \to \left(\frac{x_o}{x_i}, \frac{x_1}{x_i},\ldots,\frac{\hat{x}_i}{x_i},\ldots,\frac{x_n}{x_i}\right)$ is a bijection.

Now:

(i) $P_n(K) = \bigcup_{i=0}^{n} U_i$.

(ii) $U_i \cap U_j$ is a principal open set in both U_i and U_j (i.e. $U_i \cap U_j = (U_i)_{\frac{X_j}{X_i}} = (U_j)_{\frac{X_i}{X_j}}$), and the identity map is an isomorphism of the two affine structures on $U_i \cap U_j$.

(iii) $\{(x,x) \in U_i \times U_j \mid x \in U_i \cap U_j\} = V(\mathfrak{a})$, where

$$\mathcal{Q} = \{\frac{X_p}{X_i} \otimes \frac{X_i}{X_j} - 1 \otimes \frac{X_p}{X_j} \mid p = 0, 1, \ldots, \hat{i}, \ldots n\} \subseteq K[U_i] \otimes K[U_j].$$

So, if we define a topology on $P_n(K)$ by saying that $U \subseteq P_n(K)$ is open if and only if $U \cap U_i$ is an open subset of U_i for $i = 0, 1, \ldots n$, then it is clear that $P_n(K)$ has the structure of an algebraic variety. Any closed subvariety of $P_n(K)$ is called a projective variety.

1.2.11 Irreducible Components

A topological space is said to be irreducible if it is not the union of two proper closed subsets, or, equivalently, if every open subset is dense. Any topological space X has maximal irreducible subsets, these are closed and they cover X. They are called the irreducible components of X.

If Y is a subspace of X, then Y is irreducible if and only if $\overline{Y}$ is irreducible. Also, if X and Y are topological spaces, X is irreducible and α: $X \to Y$ is a continuous map, then $\alpha(X)$ is irreducible.

A variety has a finite number of irreducible components. If V and W are varieties with irreducible components $V_1, V_2, \ldots, V_r$ and $W_1, W_2, \ldots, W_s$ respectively, then $V \times W$ is a variety with irreducible components $V_i \times W_j$, $i = 1,2,\ldots,r$ and $j = 1,2,\ldots,s$.

Examples (i) In example 1 of section 1.2.6 V is an irreducible variety.
(ii) In example 2 of 1.2.6 the variety V has two irreducible components, namely the plane $X_3 = 0$, and the line $X_1 = X_2 = 0$.

1.2.12 Dimension

The dimension of a topological space X is defined to be the supremum of the lengths of chains, $F_o \subset F_1 \subset \ldots \subset F_n$, of distinct, irreducible closed subsets of X. Dim $X \in \mathbb{Z}^+$ or $\dim X = \pm\infty$ ($\text{Dim}\ \emptyset = -\infty$). If $x \in X$ then $\dim_x X = \text{Inf}\{\dim U \mid U$ is an open neighbourhood of $x\}$. If X_i $(1 \leq i \leq n)$ are closed subsets whose union is X, then $\dim X = \max_i \dim X_i$.

If V is a variety, then:

(i) $\dim V$ is finite.

(ii) if $V_1, V_2, \ldots, V_s$ are the irreducible components of V, then $\dim V = \max_i \dim V_i$.

(iii) if V is irreducible and U is an open subset of V, then $\dim U = \dim V$.

(iv) if W is a variety, then $\dim V \times W = \dim V + \dim W$.

<u>Examples</u> In example 1 of section 1.2.6 $\dim V = 1$, and for all $v \in V$ $\dim_v V = 1$. In example 2 of section 1.2.6 $\dim V = 2$; also, if $v = (x_1, x_2, 0)$ then $\dim_v V = 2$, and if $v = (0, 0, x_3)$, $x_3 \neq 0$, then $\dim_v V = 1$.

<u>1.2.13 Fibres of Morphisms</u>

If $\alpha: U \to V$ is a morphism of varieties, and $v \in \alpha(U)$, then the closed subvariety $\alpha^{-1}(v)$ of U is called the <u>fibre</u> of α over v.

Lemma:

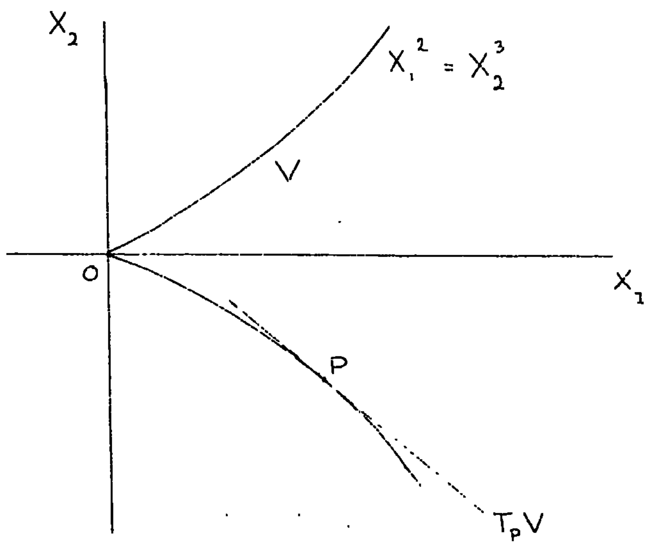
(i) If $\alpha: U \to V$ is a morphism of varieties with U irreducible and $\alpha(U)$ dense in V (such a morphism is said to be dominant), then $\dim \alpha^{-1}(v) \geq \dim U + \dim V$ for all $v \in \alpha(U)$. Further, equality holds for all v in some open-subset of $\alpha(U)$.

(ii) If $\alpha: U \to V$ is a morphism of varieties, then $\alpha(U)$ contains a dense open subset of $\overline{\alpha(U)}$.

1.2.14 Tangent Spaces

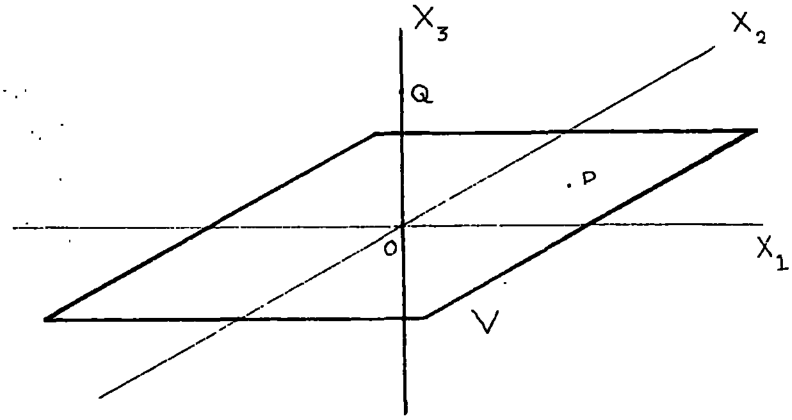
The definition of a tangent space T_vV to a variety V at a point $v \in V$ is quite involved, and although we give a definition below, an intuitive idea of what is meant by a tangent space is sufficient for our needs. Thus we begin with some examples:

(i)



As we saw in example 1 of section 1.2.6, V is an algebraic variety. The tangent space to V at the point P is the line indicated.
Dim $T_PV = 1$, and this is equal to $\dim_P V$ – when this occurs we say that P is a simple point. On the other hand $T_0V = K^2$ and $\dim_0 V \neq \dim T_0V$. 0 is called a singular point.

(ii)



V consists of the plane $X_3 = 0$, and the line $X_1 = X_2 = 0$. $T_PV = K^2$ and P is a simple point. $T_QV = K$ and Q is a simple point. $T_OV = K^3$ and O is a singular point.

We now give a formal definition of a tangent space. Let A be a commutative K-algebra, and let M be an A module (since A is commutative, we can regard M as a right and a left A module). A linear map $\delta: A \to M$ is called a derivation if $\delta(ab) = a\delta(b) + \delta(a)b$ for all a, b ϵ A. We let $Der_K(A,M)$ denote the K-vector space of derivations from A to M.

We note that if V is a variety, W is an open subvariety of V and $f \in K[V]$, then $f \mid W \in K[W]$.

Let V be an algebraic variety and $v \in V$. Also, let $\chi_v = \{U|U \text{ open in } V, v \in U\}$, and Γ_v be the disjoint union of the co-ordinate rings of the elements of χ_v. Suppose that $V_1, V_2 \in \chi_v$, $f \in K[V_1]$ and $g \in K[V_2]$, then we write $f \sim g$ if there exists $W \in \chi_v$ such that $W \subseteq V_1 \cap V_2$, and $f|W = g|W$.

$\sim$ is an equivalence relation on Γ_v, and the set, σ_v, of equivalence classes is called the stalk of V over v. If $f \in \Gamma_v$, then we let $[f]$ denote the corresponding element of σ_v. Also, if

$V_1, V_2 \in X_v$, $f \in K[V_1]$, $g \in K[V_2]$, $f' = f|V_1 \cap V_2$, $g' = g|V_1 \cap V_2$ and $k \in K$, then we write:

(i) $[f].[g] = [f'.g']$

(ii) $[f] + [g] = [f' + g']$

(iii) $k[f] = [kf]$

It is easy to see that these operations are well defined, and that σ_v has the structure of a commutative K-algebra. It can be shown that σ_v is a local ring. We let K_v denote the residue class field of σ_v (i.e. $K_v = \sigma_v/m_v$, where m_v is the maximal ideal of σ_v). K_v is isomorphic to K, and is, in the obvious way, a σ_v module. The <u>tangent space</u> to V at v is the K-vector space $Der_K(\sigma_v, K_v)$.

We have seen that a point $v \in V$ is said to be simple if $\dim_v V = \dim T_v V$. A variety is said to be <u>smooth</u> if all of its points are simple points.

If V is a variety, and Y is the set of the simple points of V, then:

(i) Y is an open dense subvariety of V.

(ii) The connected and the irreducible components of Y coincide.

If $\alpha: U \to V$ is a morphism of varieties, then we can differentiate α at $u \in U$ to get a linear map $d\alpha_u: T_u U \to T_{\alpha(u)} V$. i.e. If $X \in T_u U$ and $f \in \sigma_{\alpha(u)}$, then $d\alpha_u(X)f = X(\alpha^*(f))$, where $\alpha^*: \sigma_{\alpha(u)} \to \sigma_u$ is given by $\alpha^*([h]) = [h \circ \alpha]$ for all $[h] \in \sigma_{\alpha(u)}$.

1.3 LINEAR ALGEBRAIC GROUPS (SEE 1)

1.3.1 Algebraic Groups

A set G is called an algebraic group if:

(i) It is an algebraic variety

(ii) It is a group

(iii) The group operations μ: $G \times G \to G$, $\mu((x,y)) = x.y$, and i: $G \to G$, $i(x) = x^{-1}$, are morphisms of algebraic varieties.

G is called an affine algebraic group if it is an affine algebraic variety.

A map α: $G \to G'$ is called a morphism of algebraic groups if it is both a morphism of varieties, and a group homomorphism.

1.3.2 Linear Algebraic Groups

We have already seen that $GL(n,K)$ is an affine variety, and we will now show that it is an algebraic group.

i.e. If $G = GL(n,K)$, μ: $G \times G \to G$ is given by $\mu((X,Y)) = XY$ and i: $G \to G$ is given by $i(X) = X^{-1}$, then:

(i) The K-algebra morphism μ^*: $K[G] \to K[G] \otimes K[G]$, given by $\mu^*(T_{ij}) = \sum_{p=1}^{n} T_{ip} \otimes T_{pj}$ (see example 3 of section 1.2.6 for the notation), is such that $\mu^*(f)((X,Y)) = f(XY) = f \circ \mu((X,Y))$ for all $f \in K[G]$ and $X,Y \in G$. Thus μ is a morphism of affine varieties with comorphism μ^*.

(ii) The K-algebra morphism i^*: $K[G] \to K[G]$, given by $i^*(T_{ij}) = (-1)^{i+j}D^{-1}\det(T_{rs})_{r\neq j s\neq i}$, is such that $i^*(f)(X) = f(X^{-1}) = f \circ i(X)$ for all $f \in k[G]$ and $X \in G$. Thus i is a morphism of affine varieties with comorphism i^*.

A closed subgroup of $GL(n,K)$ is called a linear algebraic group.

Theorem (See 1) If G is an affine algebraic group, then G is isomorphic to a linear algebraic group.

1.3.3 The Identity Component

An algebraic group is smooth, and its irreducible and connected components coincide. We use G^o to denote the irreducible component of G which contains the identity element e. G^o is called the identity component of G. It is a closed normal subgroup of G, and G/G^o is a finite group.

Example The identity component of $O(n,K)$, the group of orthogonal matrices in $GL(n,K)$, consists of those matrices with determinant equal to one. Also, $O(n,K)/O(n,K)^o \cong \mathbb{Z}_2$.

1.3.4 Group Actions

An algebraic transformation space is a triple (G,V,α), where G is an algebraic group, V is an algebraic variety and $\alpha: G \times V \to V$, $(g,v) \to \alpha((g,v)) = g.v$, is a morphism of varieties such that:

(i) $e.v = v$ for all $v \in V$.

(ii) $g.(h.v) = (g.h).v$ for all $g,h \in G$ and $v \in V$.

We say that G acts on V. If $v \in V$, then $\alpha(G \times \{v\}) = G.v$ is called an orbit, and the closed subgroup $G_v = \{g \in G \mid g.v = v\}$ of G is called the isotropy group at v.

<u>Theorem</u> (See 1) If G acts on V, and $v \in V$, then:

(i) $G.v$ is locally closed

(ii) $G.v$ is smooth

(iii) $\dim G.v = \dim G - \dim G_v$

<u>Example</u> Int: $G \times G \to G$, $\mathrm{Int}((g,h)) = ghg^{-1}$, defines an action of G upon itself. The orbits of this action are called <u>conjugacy classes</u>, and the isotropy group $G_h = Z_G(h)$ is called the <u>centralizer</u> of h in G.

1.3.5 Homogeneous Spaces

If G is an affine algebraic group and H is a closed subgroup of G, then we can, in a natural way, give the coset space G/H the structure of an algebraic variety so that the canonical map $G \to G/H$ is a morphism of varieties. G/H is called a homogeneous space. If H is a normal subgroup of G, then G/H is an algebraic group. For further details see (1).

1.3.6 Semi Direct Products

Let G and H be closed subgroups of the algebraic group G' such that H is normalized by G (i.e. $gHg^{-1} = H$ for all $g \in G$). The cartesian product $H \times G$ can be given the structure of an algebraic group by defining a multiplication as follows:

$$(h_1,g_1).(h_2,g_2) = (h_1g_1h_2g_1^{-1},g_1g_2)$$

The map $H \times G \to G'$, $(h,g) \to hg$, is a morphism of algebraic groups. If it is an isomorphism, then G' is called the <u>semi direct product</u> of H and G; we write $G' = H.G$.

1.3.7 Lie Algebras

If G is an algebraic group, then the tangent space $\underline{g}$ to G at the identity element e can be given the structure of a Lie algebra (See 1). $\underline{g}$ is called the Lie algebra of G. If H is a closed subgroup of G, then the Lie algebra $\underline{h}$ of H can be identified with a subalgebra of $\underline{g}$. Also, if $\alpha: G \to G'$ is a morphism of algebraic groups, then we can differentiate α at e to get a Lie algebra morphism $d\alpha: \underline{g} \to \underline{g}'$.

Example The Lie algebra of $GL(n,K)$ is $\mathfrak{gl}(n,K)$, the set of $n \times n$ matrices with coefficients in K, and Lie bracket $[X,Y] = XY - YX$.

From now onwards we will assume that G is a connected affine algebraic group.

1.3.8 Unipotent and semi simple elements

If V is a vector space, then $X \in End(V)$ is said to be locally finite if $V = \sum_{\lambda \in \Lambda} V_\lambda$, where Λ is an indexing set, and each V_λ is a finite dimensional, X-invariant subspace of V.

Let $x \in G$, and consider the automorphism $\rho_x: G \to G$ given by $\rho_x(g) = gx \ \forall g \in G$. It is clear that $\rho_x^*: K[G] \to K[G]$ is a K-algebra automorphism, and that $\rho_{xy}^* = \rho_x^* \circ \rho_y^*$. Thus $\rho: G \to Aut_{K\text{-}alg}(K[G])$, $\rho(x) = \rho_x^* \ \forall x \in G$, is a group homomorphism. If $f \in K[G]$, then the subspace of $K[G]$ spanned by $\{\rho_x^*(f) \mid x \in G\}$ is a finite dimensional vector space. Thus if $x \in G$, then $\rho(x)$ is locally finite.

An element s of G is said to be semisimple if $\rho(s)$ is diagonalisable.

An element u of G is said to be unipotent if $\rho(u)$ is unipotent. i.e. locally, all the eigenvalues of $\rho(u)$ are equal to one.

<u>Theorem</u> (Jordan Decomposition) Each $g \in G$ may be written uniquely in the form $s.u$, where s is semi-simple, u is unipotent and s and u commute.

We let G_u denote the closed subgroup of G consisting of the unipotent elements of G, and if $G = G_u$, then we say that G is a <u>unipotent group</u>.

<u>Example</u> $u \in GL(n,K)$ is unipotent if and only if all of its eigenvalues are equal to 1, and $s \in GL(n,K)$ is semi-simple if and only if it is diagonalisable.

If $u \in G$ is unipotent, then for all $g \in G$, $\mathrm{Int}(g)u = gug^{-1}$ is also unipotent. We are thus able to talk about the unipotent conjugacy classes of G. We let $\mathcal{C}$ denote the set of unipotent conjugacy classes of G, and if $u \in G$ is unipotent, then we let $C(u)$ denote the element of $\mathcal{C}$ containing u.

<u>1.3.9 Character groups and one parameter subgroups</u>

Notation: we will use G_m to denote the multiplicative group K^*, and G_a to denote the additive group K.

An algebraic group morphism $\alpha: G \to G_m$ is called a <u>character</u> of G. We let $X(G)$ denote the set of characters of G. If $\chi_1, \chi_2 \in X(G)$, then we can obtain $\chi_1.\chi_2 \in X(G)$ by putting $\chi_1.\chi_2(g) = \chi_1(g)\chi_2(g)$ for all $g \in G$; this gives $X(G)$ the structure of an abelian group. By writing $g^\chi = \chi(g)$ we can adopt an additive notation for this group structure. $X(G)$ is called the character group of G.

A one parameter subgroup of G is an algebraic group morphism $\varepsilon: G_a \to G$.

Examples

(i) Let $D(n,K)$ denote the direct product of n copies of G_m, and if $1 \leq i \leq n$, then let $X_i \in X(D(n,K))$ be given by $X_i((k_1,k_2,\ldots,k_n)) = k_i$ for all $(k_1,k_2,\ldots,k_n) \in D(n,K)$. Now, if $X \in X(D(n,K))$, then $X = X_1^{p_1}.X_2^{p_2}\ldots X_n^{p_n}$, where $p_1,p_2,\ldots,p_n$ are integers. i.e. $X(D(n,K))$ is the free abelian group of rank n with basis $\{X_1,X_2,\ldots X_n\}$.

(ii) Let i and j be integers such that $1 \leq i, j \leq n$ and $i \neq j$. Also, let E_{ij} be the $n \times n$ matrix with 1 in the $(i,j)^{th}$ position and zeros elsewhere. Then the morphism $\varepsilon_{ij}: G_a \to GL(n,K)$, $\varepsilon_{ij}(k) = I + k\,E_{ij}$, is a one parameter subgroup of $GL(n,K)$.

1.3.10 The Adjoint Representation

Let V be a vector space over K, and $\dim V = n$ (n finite). Then, by choosing a basis of V, we can identify $GL(V)$ and $GL(n,K)$. Thus $GL(V)$ has the structure of an algebraic group (note that this structure is independent of the choice of the basis of V). A morphism $\alpha: G \to GL(V)$ is called a representation of G.

If we differentiate the isomophism Int (x): $G \to G$, $x \in G$, then we obtain a Lie algebra isomorphism Adx: $\underline{g} \to \underline{g}$, and hence a representation Ad: $G \to GL(\underline{g})$. This representation is called the Adjoint Representation.

Example If $g \in GL(n,K)$ and $X \in g\ell(n,K)$, then $Ad(g)X = gXg^{-1}$.

1.3.11 Tori and Roots

A torus is an algebraic group which is isomorphic to $D(n,K)$ for some n. Any connected, commutative algebraic group consisting entirely of semi simple elements is a torus, and if R is a torus, then $X(R)$ is a finitely generated, free abelian group (see example (i) of section 1.3.9).

We will use T to denote a maximal torus in an algebraic group G (maximal torii exist for reasons of dimension), and consider the representation $Ad: T \to GL(\underline{g})$. A non-zero element α of $X(T)$ (recall that we are using the additive notation for the group structure on $X(T)$) is called a root of G with respect to T if there exists $X \in \underline{g}$ such that $Adt.X = t^{\alpha}.X$ for all $t \in T$. We will use $\Phi(G,T)$, or more simply Φ, to denote the set of roots of G with respect to T.

If $\alpha \in \Phi(G,T)$, then

$$\underline{g}_{\alpha} = \{X \in \underline{g} \mid Adt.X = t^{\alpha}.X, \forall t \in T\}$$

is called the root space of $\underline{g}$ corresponding to the root α. If we let $\underline{g}^{o} = \{X \in \underline{g} \mid Adt.X = X\}$, then

$$\underline{g} = \underline{g}^{o} \oplus \coprod_{\alpha \in \Phi} \underline{g}_{\alpha}.$$

Example

The group T of diagonal matrices in $G = GL(n,K)$ is a maximal torus. Further:

(i) $\Phi(G,T) = \{\alpha_{ij} \mid i,j = 1,\ldots,n,\ i \neq j\}$, where $\alpha_{ij}(diag(k_1,k_2,\ldots,k_n)) = k_i/k_j$.

(ii) $\underline{g}_{\alpha_{ij}} = (E_{ij})$ and $\underline{g}^o$ is the set $\underline{t}$ of diagonal matrices in $g\ell(n,K)$

(iii) It is clear that

$$\underline{g} = \underline{t} \oplus \coprod_{\substack{i,j=1 \\ i\neq j}}^{n} (E_{ij})$$

Note that $\underline{t}$ is the Lie algebra of T.

1.3.12 Borel Subgroups

A maximal connected solvable subgroup of G is called a Borel Subgroup. All the Borel Subgroups of G are conjugate, i.e. if B_1 and B_2 are two such subgroups, then there exists $g \in G$ such that $B_1 = g\, B_2\, g^{-1} = {}^gB_2$.

All the maximal tori of a connected solvable group are conjugate. Thus, since any maximal torus of G is a maximal torus of some Borel Subgroup of G, we have that all the maximal tori of G are conjugate. The rank of G is the common dimension of the maximal torii of G.

If B is a Borel Subgroup of G, then:

(i) $G = \bigcup_{g\in G} {}^gB$

(ii) $N_G(B) = B$, $N_G(B)$ is the normalizer of B in G (see page 29)

(iii) If $T \subseteq B$ is a maximal torus of G, then B_u is normalised by T, and $B = T.B_u$.

Example The set, B, of upper triangular matrices in $GL(n,K)$ is a Borel Subgroup of $GL(n,K)$. $B_u = \{(a_{ij}) \in B \mid a_{ii} = 1\}$.

Let B be a fixed Borel Subgroup of G. Then G/B is a projective variety - see (1) for details. Also, if B' is another Borel Subgroup of G, then $B' = {}^hB$ for some h in G, and the map $G/B \to G/B'$,

$gB \to hgh^{-1}B'$, is an isomorphism of varieties. Thus G/B is, up to isomorphism, independent of the choice of B.

We can identify the set β, of Borel Subgroups of G with G/B, i.e. ${}^{g}B \to gB$. Thus β has the structure of a projective variety. If $S \subseteq G$, then $\beta^{S} = \{\tilde{B} \in \beta \mid S \subseteq \tilde{B}\}$ is a closed subvariety of β. The variety $\beta_u = \beta^{\{u\}}$, where u is a unipotent element of G, will play a considerable role in later work. In the above identification of β and G/B, β_u corresponds to $(G/B)_u = \{gB \in G/B \mid ugB = gB\}$; i.e. $ugB = gB \Longleftrightarrow g^{-1}ug \in B \Longleftrightarrow u \in {}^{g}B$.

1.3.13 <u>The Weyl Group</u>

If $S \subseteq G$, then $N_G(S) = \{g \in G \mid gSg^{-1} = S\}$ is called the <u>normalizer</u> of S in G, and $Z_G(S) = \{g \in G \mid gz = zg, \forall z \in S\}$ is called the <u>centralizer</u> of S in G.

$W(G,T) = N_G(T)/Z_G(T)$ is called the <u>Weyl Group</u> of G with respect to the maximal torus T.

Since all of the maximal tori of G are conjugate, $W(G,T)$ is, up to isomorphism, independent of the choice of T. (i.e. If T' is another maximal torus of G, then $T' = {}^{g}T$ for some $g \in G$. It is easy to see that $N_G(T') = {}^{g}N_G(T)$, $Z_G(T') = {}^{g}Z_G(T)$, and that the map $W(G,T) \to W(G,T')$, $nZ_G(T) \to gng^{-1}Z_G(T')$, is a group isomorphism).

We write W for $W(G,T)$. It can be shown that $N_G(T)^{o} = Z_G(T)^{o}$, and hence that W is finite.

If $w \in W$ and $\alpha \in X(T)$, then let $w.\alpha \in X(T)$ be defined as follows:

$w.\alpha(t) = \alpha(n_w^{-1} t\, n_w)$, where $n_w \in N_G(T)$ is mapped onto w by the canonical map $N_G(T) \to N_G(T)/Z_G(T)$. (Note that if $n_w' \in N_G(T)$ is also mapped onto w, then $n_w' = n_w z$, where $z \in Z_G(T)$. Therefore

$(n_w')^{-1}\, t\, n_w' = z^{-1} n_w^{-1}\, t\, n_w\, z = n_w^{-1} t\, n_w$ for all $t \in T$. Hence $w.\alpha$ is well defined.)

It is easy to see that W acts on $\phi(G,T)$.(i.e. if $\alpha \in \phi(G,T)$, then there exists $X \in \underline{g}$ such that $Adt.X = \alpha(t).X$ for all $t \in T$. Now $Adt(Adn_w.X) = Adn_w(Ad(n_w^{-1} t\, n_w).X) = Adn_w(\alpha(n_w^{-1} t\, n_w).X) = (w.\alpha(t)).Adn_w.X$. Thus $w.\alpha \in \phi(G,T)$.)

Example If $G = GL(n,K)$, and T is the group of diagonal matrices in G, then $(a_{ij}) \in N_G(T)$ if and only if there exists $\sigma \in S_n$, the symmetric group on n elements, such that $a_{\sigma(j)j} \neq 0$ and $a_{ij} = 0$ for all $i \neq \sigma(j)$. Also, $Z_G(T) = T$, and the map from W to S_n given by $(a_{ij})\, T \to \sigma$, where σ is as above, is a group isomorphism.

Now, if $w \in W$ corresponds to $\sigma \in S_n$, then $n_\sigma = (a_{ij})$, $a_{ij} = 0$ if $i \neq \sigma(j)$ and $a_{\sigma(j)j} = 1$, is an element of $N_G(T)$, and $w = n_\sigma T$. Also, $n_\sigma^{-1}\, \text{diag}\,(k_1,k_2,\ldots,k_n) n_\sigma = \text{diag}\,(k_{\sigma(1)},k_{\sigma(2)},\ldots,k_{\sigma(n)})$. Recall that $\phi(G,T) = \{\alpha_{ij} \mid i,j = 1,2,\ldots,n,\ i \neq j\}$ - see 1.3.11. It is clear that $w.\alpha_{ij}\, (\text{diag}\,(k_1,k_2,\ldots,k_n)) = k_{\sigma(i)}/k_{\sigma(j)}$, and hence that $w.\alpha_{ij} = \alpha_{\sigma(i)\sigma(j)}$.

1.3.14 Semi-simple and Reductive Groups

We say that an algebraic group G is quasi-simple if it contains no non-trivial connected closed normal subgroup.

If G is a connected affine algebraic group, then $R(G) = (\bigcap_{B \in \beta} B)^o$ is called the radical of G. It is a connected solvable normal subgroup of G, and contains all other such subgroups.

$R(G)_u$ is called the unipotent radical of G. It is a connected unipotent normal subgroup of G and contains all other such subgroups.

G is said to be semisimple if $R(G) = \{e\}$, and reductive if $R(G)_u = \{e\}$. It is clear that a quasi-simple group is semisimple .

If G is reductive, then $R(G)$ is central in G, and is thus a torus - See 1.3.11. Also, the commutator subgroup G' of G is semisimple, and $G = R(G).G'$.

Example $G = GL(n,K)$ is a reductive group. $R(G)$ is the group of scalar matrices in G, i.e. those matrices of the form λI, where $\lambda \in G_m$. Also, $G' = SL(n,K)$ and $G = R(G).SL(n,K)$.

Theorem (see 1). If G is a reductive group, then:

(i) $Z_G(T) = T$

(ii) $\Phi(G,T) = -\Phi(G,T)$

(iii) $\underline{g} = \underline{t} \oplus \coprod_{\alpha \in \Phi} \underline{g}_\alpha$, where $\underline{t}$ is the Lie algebra of T. Also, $\dim \underline{g}_\alpha = 1$ for all $\alpha \in \Phi$.

Further if $\alpha \in \Phi(G,T)$, then there exists a unique unipotent subgroup U_α of G having the following properties:

(i) The Lie algebra of U_α is $\underline{g}_\alpha$

(ii) If $w \in W$, then $n_w U_\alpha n_w^{-1} = U_{w.\alpha}$ - see 1.1.13 for the notation.

(iii) There exists an isomorphism $\varepsilon_\alpha : G_a \to U_\alpha$ such that for all $k \in G_a$ and $t \in T$, $t\,\varepsilon_\alpha(k)t^{-1} = \varepsilon_\alpha(\alpha(t).k)$

(iv) $G = \langle U_\alpha, T \mid \alpha \in \Phi(G,T) \rangle$.

Example If $G = GL(n,K)$, and T is the maximal torus in G consisting of the diagonal matrices, then:

(i) It is easy to see that $Z_G(T) = T$

(ii) $\Phi(G,T) = \{\alpha_{ij} \mid i,j = 1,\ldots,n,\ i \neq j\}$, and $-\alpha_{ij} = \alpha_{ji}$, i.e. $\Phi(G,T) = -\Phi(G,T)$.

(iii) We have already seen that $\underline{g}_{\alpha_{ij}} = (E_{ij})$, and that

$$\underline{g} = \underline{t} \oplus \coprod_{\substack{i,j=1 \\ i \neq j}} (E_{ij}), \text{ where } \underline{t} \text{ is the Lie algebra of } T.$$

Recall that $\varepsilon_{ij}: G_a \to G$ $(i \neq j)$ is given by $\varepsilon_{ij}(k) = I + kE_{ij}$.

Now:

(i) $U_{\alpha_{ij}} = \varepsilon_{ij}(G_a)$, and the Lie algebra of $U_{\alpha_{ij}}$ is (E_{ij}).

(ii) If $\sigma \in S_n$, then $n_\sigma E_{ij} n_\sigma^{-1} = E_{\sigma(i)\sigma(j)}$, and hence $n_\sigma U_{\alpha_{ij}} n_\sigma^{-1} = U_{\alpha_{\sigma(i)\sigma(j)}}$.

(iii) If $t = \text{diag}(k_1,k_2,\ldots,k_n)$ and $k \in G_a$, then

$$\begin{aligned} t\,\varepsilon_{ij}(k)t^{-1} &= t(I + k\,E_{ij})t^{-1} \\ &= I + \frac{k_i}{k_j} k\,E_{ij} \\ &= \varepsilon_{ij}(\alpha_{ij}(t).k). \end{aligned}$$

(iv) $G = \langle T, U_{\alpha_{ij}} \mid i,j = 1,2,\ldots,n,\ i \neq j \rangle$.

1.3.15 The Roots of Semisimple Algebraic Groups

If G is a connected affine algebraic group of rank n, and T is a maximal torus of G, then $X(T)$ is a free abelian group of rank n (see 1.3.11). Hence the real vector space $E = \mathbb{R} \otimes_{\mathbb{Z}} X(T)$ has dimension n. We can identify $W(G,T)$ with a subgroup of $GL(E)$, i.e. if $w \in W$, then put $w(\sum_{i=1}^r a_i \otimes X_i) = \sum_{i=1}^r a_i \otimes w(X_i)$ (recall that W acts on $X(T)$) for all $\sum_{i=1}^r a_i \otimes X_i \in E$. Now, since W is finite, we can define a W-invariant, positive definite inner product on E.

Theorem If G is semisimple, then $\Phi(G,T)$ is an abstract root system in E, and the Abstract Weyl Group of $\Phi(G,T)$ is isomorphic to $W(G,T)$. (Note that the rank of $\Phi(G,T)$ will be equal to n.)

The isomorphism between $W(\Phi(G,T))$ and $W(G,T)$ is obtained as follows:

If $\alpha \in \Phi(G,T)$, then $Z_\alpha = \langle T, U_\alpha, U_{-\alpha} \rangle$ is a reductive group, $\Phi(Z_\alpha, T) = \{\alpha, -\alpha\}$, and $W(Z_\alpha, T) = \{1, \sigma_\alpha\}$, where $\sigma_\alpha^2 = 1$ and $\sigma_\alpha(\alpha) = -\alpha$. Also $N_{Z_\alpha}(T) \subseteq N_G(T)$, and thus $W(Z_\alpha, T) = N_{Z_\alpha}(T)/T \subseteq N_G(T)/T = W(G,T)$. Now τ_α, the reflection of E in the hyperplane perpendicular to α (see 1.1.1), is mapped onto σ_α.

Note: $\Phi(G,T)$ is, up to the equivalence of root systems, independent of the choice of T.

<u>Example</u> Let $G = SL(3,K)$, and T be the group of diagonal matrices in G. As far as the roots and the Weyl Group are concerned, there is no distinction between $SL(3,K)$ and $GL(3,K)$ (cf. 1.3.14 and 1.3.17). Therefore $\Phi(G,T) = \{\pm\alpha_{12}, \pm\alpha_{13}, \pm\alpha_{23}\}$ and $W(G,T) \cong S_3$. Now, $X(T)$ is the free abelian group of rank 2 generated by $\{\alpha_{12}, \alpha_{23}\}$, and thus $\{\alpha_{12}, \alpha_{23}\}$ is a basis of $E = \mathbb{R} \otimes_{\mathbb{Z}} X(T)$. We can define an S_3-invariant, positive definate inner product on E by putting $(\alpha_{12}, \alpha_{12}) = (\alpha_{23}, \alpha_{23}) = 1$, and $(\alpha_{12}, \alpha_{23}) = -\frac{1}{2}$ (note that if $\sigma \in S_3$ and $\alpha_{ij} \in \Phi(G,T)$, then $\sigma.\alpha_{ij} = \alpha_{\sigma(i)\sigma(j)}$). It is now easy to see that $\Phi(G,T)$ is the root system of type A_2 described in section 1.1.4. We note that

$$Z_{\alpha_{12}} = \left\langle T, \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ k' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; k,k' \in G_a \right\rangle ,$$

and $W(Z_{\alpha_{12}}, T) = \{Id, n_{12}T\}$, where

$$n_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

1.3.16 Quasi-simple Components

If G is a semisimple group, and $\{G_i \mid i \in I\}$ is the set of minimal closed connected normal subgroups of G of positive dimension, then:

(i) I is finite; i.e. $I = \{1,2,\ldots,n\}$

(ii) $(G_i,G_j) = \{e\}$ if $i \neq j$

(iii) The product morphism $G_1 \times G_2 \times \ldots \times G_n \to G$ is surjective, and has finite kernel

(iv) The decomposition $G = G_1.G_2.\ \ldots\ G_n$ corresponds precisely to the decomposition of $\Phi(G,T)$ into its irreducible components.

The groups G_i are called the quasi-simple components of G. It is clear that G is quasi-simple if and only if $\Phi(G,T)$ is irreducible. A quasi-simple group is said to be of type A_n if its root system is of type A_n, B_n if its root system is of type B_n, etc. (See 1.1.12).

Examples

(i) $SL(n,K)$ is of type A_n

(ii) $SO(2n+1,K)$, the group of $(2n+1) \times (2n+1)$ orthogonal matrices with determinant 1, is of type B_n

(iii) $SO(2n,K)$ is of type D_n

(iv) $Sp(n,K)$, the group of $2n \times 2n$ symplectic matrices, is of type C_n

These groups are called the classical groups.

1.3.17 The Structure of Reductive Groups

For the rest of this chapter we shall assume that G is reductive.

$G = R.G'$, where R is the radical of G, and G' is the commutator subgroup of G. Recall that R is a torus, and that G' is semisimple.

If T' is a maximal torus in G', then $T = R.T'$ is a maximal torus in G. Now:

(i) If $\alpha \in \Phi(G,T)$, then $\alpha \mid T' \in \Phi(G',T')$. Also, the map $\Phi(G,T) \to \Phi(G',T')$, $\alpha \to \alpha \mid T'$, is a bijection.

(ii) $W(G,T)$ is isomorphic to $W(G',T')$. (i.e. $g = rg_o$, $r \in R$ and $g_o \in G'$, is an element of $N_G(T)$ if and only if g_o is an element of $N_{G'}(T')$. Thus we can define an algebraic group morphism $\xi : N_G(T) \to N_{G'}(T')$ by putting $\xi(rg_o) = g_o$. It is easy to see that $\xi(T) = T'$, and that the induced map from $W(G,T)$ to $W(G',T')$ is a group isomorphism.)

From (i) and (ii) above it follows that $\Phi(G,T)$ has the structure of an abstract root system and that $W(\Phi(G,T))$ is isomorphic to $W(G,T)$.

Let T be a maximal torus in G. Then the choice of an element $B \in \beta^T$ is equivalent to the choice of a set of positive roots in $\Phi(G,T)$, and hence to the choice of a basis of $\Phi(G,T)$ (See 1.1.6). (i.e. if $B \in \beta^T$, then $\Phi(B,T)$ is a set of positive roots in Φ. Conversely, if Φ^+ is a set of positive roots of Φ, then $B = T.U$, where $U = \langle U_\alpha \mid \alpha \in \Phi^+ \rangle$, is the unique element of β^T such that $\Phi(B,T) = \Phi^+$.)

<u>Theorem</u> If $B \in \beta^T$, U is the unipotent radical of B, $\underline{b}$ is the Lie algebra of B, $\underline{u}$ is the Lie algebra of U, and $\Phi^+ = \Phi(B,T)$, then:

(i) $U = \langle U_\alpha \mid \alpha \in \Phi^+ \rangle$ and $\underline{u} = \coprod_{\alpha \in \Phi^+} \underline{g}_\alpha$

(ii) $\underline{b} = \underline{t} \oplus \coprod_{\alpha \in \Phi^+} \underline{g}_\alpha$

(iii) If $U^- = \langle U_\alpha \mid \alpha \in \Phi^- \rangle$, then $B^- = T.U$ is an element of β^T. B^- is called the Borel Subgroup opposite B.

(iv) If $\Phi^+ = \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$, the order being arbitary, then the product map $U_{\alpha_1} \times U_{\alpha_2} \times \ldots \times U_{\alpha_N} \to U$ is an isomorphism of varieties.

<u>Example</u> If $G = GL(n,K)$, and T is the maximal torus in G consisting of diagonal matrices, then B, the Borel Subgroup of upper triangular matrices in G, corresponds to the basis $\{\alpha_{12}, \alpha_{23}, \ldots, \alpha_{n-1\,n}\}$ of $\Phi(G,T)$. The Borel Subgroup opposite B is the group of lower triangular matrices in G.

1.3.18 The Bruhat Decomposition

Let T be a maximal torus in G, $B \varepsilon \beta^T$, U be the unipotent radical of B, and $\Phi^+ = \Phi(B,T)$.

If $S \subseteq G$ is normalized by T, and $w = n_wT \varepsilon W$, then we write ${}^wS = n_wSn_w^{-1}$; note that if $n_wT = n_w'T$, then $n_wSn_w^{-1} = n_w'S(n_w')^{-1}$. Also, if $w \varepsilon W$ then we put $A_w^+ = \{\alpha \varepsilon \Phi^+ \mid w^{-1}\alpha \varepsilon \Phi^+\}$ and $A_w^- = \{\alpha \varepsilon \Phi^+ | w^{-1}\alpha \varepsilon \Phi^-\}$.

We now consider two closed unipotent subgroups of G, namely $U_w^+ = U \cap {}^wU$, and $U_w^- = U \cap {}^wU^-$.

If $k \varepsilon G_a$, then $\varepsilon_\alpha(k) \varepsilon U_w^+ \Longleftrightarrow \alpha \varepsilon \Phi^+$ and $\varepsilon_\alpha(k) \varepsilon {}^wU$. Now, $\varepsilon_\alpha(k) \varepsilon {}^wU \Leftrightarrow n_w^{-1} \varepsilon_\alpha(k)n_w \varepsilon U$. But $n_w^{-1}\varepsilon_\alpha(k)n_w = \varepsilon_{w^{-1}(\alpha)}(k')$ for some $k' \varepsilon G_a$ (cf. 3.1.14). Thus $\varepsilon_\alpha(k) \varepsilon {}^wU \Leftrightarrow w^{-1}(\alpha) \varepsilon \Phi^+$.

$\therefore\ U_w^+ = \langle U_\alpha \mid \alpha \varepsilon A_w^+ \rangle$

Similarly

$$U_w^- = \langle U_\alpha \mid \alpha \varepsilon A_w^- \rangle$$

It is now clear that the product morphisms $U_w^+ \times U_w^- \to U$ and $U_w^- \times U_w^+ \to U$ are isomorphisms of varieties (see the theorem, part iii, in section 1.3.17 above).

If $w = n_wT \varepsilon W$, then we write $C_w = BwB = Bn_wB$; C_w is called the <u>Bruhat Cell</u> of G corresponding to w.

BRUHAT LEMMA

(a) G is the disjoint union of the double cosets BwB, and $Bw'B = BwB$ $\Longleftrightarrow w = w'$. Also, if $w \in W$, then the map $U_w^- \times B \to BwB$, $(u,b) \to u\,n_w b$, is an isomorphism of varieties.

(b) β is the disjoint union of the U orbits $\tilde{C}_w = U.{}^wB = \{{}^{uw}B \mid u \in U\}$. Also, if $w \in W$, then the map $U_w^- \to \tilde{C}_w$, $u \to {}^{uw}B$, is an isomorphism of varieties.

1.3.19 Parabolic Subgroups

A closed subgroup P of G is called a parabolic subgroup if it contains a Borel Subgroup, or, equivalently, if G/P is a projective variety.

Let π be the basis of Φ determined by B. If $J \subseteq \pi$, then we can define a map $h_J: \Phi \to \mathbb{Z}$ by putting $h_J(\alpha) = 0$ if $\alpha \in J$, $h_J(\alpha) = 2$ if $\alpha \in \pi - J$, and extending linearly. The closed subgroup $P_J = \langle U_\alpha, T \mid h_J(\alpha) \geqslant 0 \rangle$ contains B, and is thus a parabolic subgroup of G. Now:

(i) if $J, K \subseteq \pi$, and P_J is conjugate to P_K, then $J = K$.

(ii) if P is a parabolic subgroup of G containing B, then $P = P_J$ for some subset J of π

From (ii) it follows that if P is a parabolic subgroup of G, then P is conjugate to P_J, for some subset J of π. We say that P_J is the standard parabolic subgroup of G corresponding to J.

The Levi Decomposition $P_J = L_J.U_J$, where $L_J = \langle T, U_\alpha \mid h_J(\alpha) = 0 \rangle$, and $U_J = \langle U_\alpha \mid h_J(\alpha) > 0 \rangle$. U_J is the unipotent radical of P_J, and L_J is a reductive group with root system Φ_J (See 1.1.10).

A subgroup of G which is conjugate to the commutator subgroup R_J of L_J, for some $J \subset \pi$, is called a Regular Subgroup of Levi Type.

Note: If $\underline{p}_J$, $\underline{\ell}_J$ and $\underline{u}_J$ are the Lie algebras of P_J, L_J and U_J respectively, then $\underline{p}_J = \underline{\ell}_J \oplus \underline{u}_J$, i.e. $\underline{p}_J = \underline{t} \oplus \coprod_{h_J(\alpha) \geq 0} \underline{g}_\alpha$, $\underline{\ell}_J = \underline{t} \oplus \coprod_{h_J(\alpha) = 0} \underline{g}_\alpha$, and $\underline{u}_J = \coprod_{h_J(\alpha) > 0} \underline{g}_\alpha$.

Example Let $G = GL(6,K)$, T be the group of diagonal matrices in G, and B the group of upper triangular matrices in G, then $\Phi(G,T) = \{\alpha_{ij} \mid i,j = 1,\ldots,6,\ i \neq j\}$, and the basis of $\Phi(G,T)$ determined by B is $\{\alpha_{12},\alpha_{23},\alpha_{34},\alpha_{45},\alpha_{56}\}$. Let $J = \{\alpha_{12},\alpha_{34},\alpha_{45}\}$.

Then:

(i) P_J consists of matrices of the form:

$$\left(\begin{array}{cc|ccc|c} * & * & * & * & * & * \\ * & * & * & * & * & * \\ \hline 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ \hline 0 & 0 & 0 & 0 & 0 & * \end{array}\right)$$

where the $*$'s represent entries which need not be zero.

(ii) L_J consists of matrices of the form:

$$\left(\begin{array}{cc|ccc|c} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & * & * & * & 0 \\ 0 & 0 & * & * & * & 0 \\ 0 & 0 & * & * & * & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & * \end{array}\right)$$

(iii) U_J consists of matrices of the form:

$$\left(\begin{array}{cc|ccc|c} 1 & 0 & * & * & * & * \\ 0 & 1 & * & * & * & * \\ \hline 0 & 0 & 1 & 0 & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

i.e.

$\Phi_J = \{\alpha_{12}, \alpha_{34}, \alpha_{45}, \alpha_{35}, -\alpha_{12}, -\alpha_{34}, -\alpha_{45}, -\alpha_{35}\}$;

$P_J = \{(a_{ij}) \in G \mid a_{ij} = 0 \text{ if } i > j \text{ and } \alpha_{ij} \notin \Phi_J\}$;

$L_J = \{(a_{ij}) \in G \mid a_{ij} = 0 \text{ if } \alpha_{ij} \notin \Phi_J\}$, and

$U_J = \{(a_{ij}) \in G \mid a_{ii} = 1, \text{ and } a_{ij} = 0 \text{ if } i > j, \text{ or }$
$\alpha_{ij} \in \Phi_J\}$.

CHAPTER 2

SPRINGER'S RESULT

2.1 BACKGROUND

Recall that $\mathcal{C}$ denotes the set of unipotent conjugacy classes of an algebraic group G.

2.1.1 Lusztig's Result

George Lusztig has recently shown that if G is a reductive (connected) algebraic group, then $|\mathcal{C}|$ is finite. The proof of this is long and complicated and may be found in (6).

Note

The conjecture that $|\mathcal{C}|$ is finite for reductive groups has been an open question for some time. Prior to Lusztig's solution it was known that:

if G is a reductive group defined over K, and char(K) is a 'good' prime (see the definition below), then $|\mathcal{C}|$ is finite.

Definition A prime number p is said to be a 'good' prime for a reductive group G if:

(i) If G is quasi-simple, and of type:

A_n :	p	arbitary
B_n , C_n , D_n :	$p \neq$	2
G_2 , F_4 , E_6 , E_7 :	$p \neq$	2, 3
E_8 :	$p \neq$	2, 3, 5

(ii) If G is not quasi-simple, then P is 'good' with respect to each quasi-simple component of G.

2.1.2 The Carter Bala Classification of the Unipotent Conjugacy Classes of Quasi-simple Algebraic Groups

Let G be a reductive group, T a maximal torus in G, and $B \in \beta^T$. Let $\Phi = \Phi(G,T)$, and π be the basis of Φ determined by B. If $J \subseteq \pi$, then let ∇_J be the Dynkin Diagram (see 1.1.11) of Φ weighted with zero's and twos; a node being weighted with a zero if it represents an element of J, and with a two if it represents an element of $\pi \sim J$ (cf. the definition of h_J in 1.3.19).

We say that a diagram ∇_J is distinguished if $2N(0) + \ell - N(2) = 0$, where ℓ = rank G and $N(i) = |\{\alpha \in \Phi^+ \mid h_J(\alpha) = i\}|$ for $i = 0, 2$.

If P is a parabolic subgroup of G, then P is conjugate to the standard parabolic subgroup P_J for some subset J of π. We say that P is a distinguished parabolic subgroup of G if ∇_J is a distinguished diagram. It should be noted that this definition is completely independent of our initial choice of T and B.

Note that $\underline{p}_J = \coprod_{i=0}^{q} \underline{g}_{2i}$, where $\underline{g}_{2i} = \coprod_{h_J(\alpha)=2i} \underline{g}_\alpha$, and that P is a distinguished parabolic subgroup of G if and only if $\dim \underline{g}_0 + \ell = \dim \underline{g}_2$, i.e. $\dim \underline{\ell}_J = \dim \underline{g}_2$.

<u>Example</u> The only distinguished diagram of type A_n is $\overset{2}{\circ}\!-\!\overset{2}{\circ}\cdots\overset{2}{\circ}\!-\!\overset{2}{\circ}\!-\!\overset{2}{\circ}$, and thus the only distinguished parabolic subgroups of $SL(n,K)$ are the Borel Subgroups.

<u>Note</u>: we will look at distinguished diagrams in greater detail in Chapter 4.

Let $\mathcal{J}$ be the set of pairs (R,P_R), where R is a regular subgroup of G of Levi type, and P_R is a distinguished parabolic subgroup of R. We write $(R,P_R) \sim (\hat{R}\ \hat{P}_{\hat{R}})$ if there exists $g \in G$ such that $\hat{R} = {}^gR$, and $\hat{P}_{\hat{R}} = {}^gP_R$. $\sim$ is an equivalence relation on $\mathcal{J}$. Let $\mathcal{H}$ denote the corresponding set of equivalence classes, and $[(R,P_R)]$ the element of $\mathcal{H}$ containing (R,P_R).

<u>Lemma 1</u> The map $\xi: \mathcal{H} \to \mathcal{C}$, given by $\xi([(R,P_R)]) = C \Longleftrightarrow \overline{C \cap U_{P_R}} = U_{P_R}$, where U_{P_R} is the unipotent radical of P_R, is well defined.

<u>Proof</u> If $(R,P_R) \in \mathcal{J}$, then U_{P_R} is a closed irreducible subvariety of G. Also

$$U_{P_R} = \bigcup_{C \in \mathcal{C}} \overline{U_{P_R} \cap C}\,,$$ and hence

there exists $C \in \mathcal{C}$ such that $\overline{U_{P_R} \cap C} = U_{P_R}$.

If $C_1, C_2 \in \mathcal{C}$, and $\overline{C_1 \cap U_{P_R}} = \overline{C_2 \cap U_{P_R}} = U_{P_R}$, then $C_1 = C_2$. (i.e. C_1 and C_2 are locally closed (see 1.3.4), and hence $C_1 \cap U_{P_R}$ and $C_2 \cap U_{P_R}$ are open subsets of U_{P_R}. Therefore $C_1 \cap C_2 \cap U_{P_R} \neq \emptyset$, and thus $C_1 = C_2$.).

If $(R,P_R) \sim (\hat{R}, \hat{P}_{\hat{R}})$ then there exists $g \in G$ such that ${}^gR = \hat{R}$ and ${}^gP_R = \hat{P}_{\hat{R}}$. It is easy to see that ${}^gU_{P_R}$ is a maximal, connected, normal unipotent subgroup of $\hat{P}_{\hat{R}}$, and hence that ${}^{g}U_{P_R} = U_{\hat{P}_{\hat{R}}}$. Thus, if $\overline{C \cap U_{P_R}} = U_{P_R}$, then $\overline{C \cap U_{\hat{P}_{\hat{R}}}} = \overline{C \cap {}^{g}U_{P_R}} = {}^{g}\overline{C \cap U_{P_R}} = U_{\hat{P}_{\hat{R}}}$.

Let p be the characteristic of the base field K of G, and m be the height of the highest root of Φ (see 1.1.7). Then:

Theorem 2

If G is quasi-simple, and $p \geqslant 4m + 3$, then $\xi: \mathcal{H} \to \mathcal{L}$ is a bijection.

This is the Carter Bala classification theorem, and the proof can be found in (3). By classifying the elements of $\mathcal{H}$ Carter and Bala were able to classify the unipotent conjugacy classes of G.

2.1.3 The map η

Suppose that G is reductive, and that $W = N_G(T)/T$. If $w \in W$, then U_w^+ is a closed irreducible unipotent subgroup of G, and thus there exists a unique element $C \in \mathcal{L}$ such that $\overline{C \cap U_w^+} = U_w^+$. Hence we can define a map $\eta: W \to \mathcal{L}$ by putting $\eta(w) = C$ if and only if $\overline{U_w^+ \cap C} = U_w^+$.

In this chapter we shall be concerned with the map η, being motivated by the following theorem (this theorem is mentioned in the appendix of Carter and Bala's paper on unipotent conjugacy classes (3)):

<u>Theorem 3</u>

If G is quasi-simple and $p > 4m + 3$, then η is surjective.

<u>Proof</u> If $(R,P_R) \in \mathfrak{J}$, then there exists $g \in G$ such that ${}^gR = R_J$ (see 1.3.19), where J is some subset of π. Now, Φ_J is the root system of R_J, J is a basis of Φ_J, and ${}^gP_R$ is a parabolic subgroup of R_J. Hence, there exists $h \in R_J$ and a subset S of J such that ${}^{hg}P_R = P_{J,S}$, the standard parabolic subgroup of R_J determined by S. It is clear that $(R,P_R) \sim (R_J,P_{J,S})$; i.e. that we can write any element of $\mathcal{H}$ in the form $[(R_J,P_{J,S})]$.

Now, let C be an element of $\mathcal{L}$. By Theorem 2, there exists sets S and J, $S \subseteq J \subseteq \pi$, such that C intersects the unipotent radical $U_{J,S}$ of $P_{J,S}$ densely. Let Φ_S be the subsystem of Φ_J spanned by S. Now, R_J is the semi-simple part of the reductive group $L_J = \langle T,U_\alpha \mid \alpha \in \Phi_J\rangle$; i.e. $R_J = \langle T_o,U_\alpha \mid \alpha \in \Phi_J\rangle$, where T_o is the maximal torus in R_J such that $T = D.T_o$, D being the radical of L_J. Thus

$$P_{J,S} = \langle T_o, U_\alpha \mid \alpha \in \Phi_S \cup \Phi_J^+\rangle, \quad \text{and}$$
$$U_{J,S} = \langle U_\alpha \mid \alpha \in \Phi_J^+ - \Phi_S\rangle.$$

Let W_S and W_J be the abstract Weyl Groups of Φ_S and Φ_J respectively; we can assume that $W_S \subseteq W_J \subseteq W$. If w_S, w_J and w_o are the elements of greatest length (see 1.1.9) in W_S, W_J and W respectively, then:

(i) $w_S^{-1} = w_S$, $w_J^{-1} = w_J$ and $w_o^{-1} = w_o$.

(ii) w_J: $\Phi^+ - \Phi_J \to \Phi^+ - \Phi_J$, and $\Phi_J^+ \to \Phi_J^-$.

(iii) w_S: $\Phi^+ - \Phi_S \to \Phi^+ - \Phi_S$, $\Phi_J^+ - \Phi_S \to \Phi_J^+ - \Phi_S$, and hence $\Phi^+ - \Phi_J \to \Phi^+ - \Phi_J$. Also w_S: $\Phi_S^+ \to \Phi_S^-$.

(iv) w_o: $\Phi^+ \to \Phi^-$.

Let $\overline{w} = w_S w_J w_o$ and $\alpha \in \Phi^+$. If $\alpha \in \Phi^+ - \Phi_J$, then:

$$\overline{w}^{-1}(\alpha) = w_o w_J w_S(\alpha) \quad \varepsilon \quad w_o w_J(\Phi^+ - \Phi_J) \quad \text{from (iii)}$$

$$\varepsilon \quad w_o(\Phi^+ - \Phi_J) \quad \text{from (ii)}$$

$$\varepsilon \quad \Phi^- \quad \text{from (iv).}$$

If $\alpha \in \Phi_J^+ - \Phi_S$, then:

$$\overline{w}^{-1}(\alpha) \quad \varepsilon \quad w_o w_J(\Phi_J^+ - \Phi_S) \quad \text{from (iii)}$$

$$\varepsilon \quad w_o(\Phi_J^-) \quad \text{from (ii)}$$

$$\varepsilon \quad \Phi^+ \quad \text{from (iv).}$$

If $\alpha \in \Phi_S^+$, then:

$$\overline{w}^{-1}(\alpha) \quad \varepsilon \quad w_o w_J(\Phi_S^-) \quad \text{from (iii)}$$

$$\varepsilon \quad w_o(\Phi_J^+) \quad \text{from (ii)}$$

$$\varepsilon \quad \Phi^- \quad \text{from (iv)}$$

Hence $A_{\overline{w}}^+ = \Phi_J^+ - \Phi_S$.

$\therefore \quad U_{\overline{w}}^+ = U_{J,S}$, and $\eta(\overline{w}) = C$.

2.2 SPRINGER'S RESULT

2.2.1 Preliminary Results

Let G be a connected algebraic group acting on two algebraic varieties X and Y, with G transitive on Y. Let $\psi: X \to Y$ be a G-morphism, i.e. ψ is a morphism of algebraic varieties such that for all $g \in G$ $\psi(gx) = g\psi(x)$. It is clear that ψ is surjective, and that all of the fibres are isomorphic. In particular, all of the fibres have the same dimension. Let $y \in Y$, and put $F = \psi^{-1}(y)$.

Lemma 4 $\dim F = \dim X - \dim Y$.

Proof Y is the image of G under the morphism $G \to Y$, $g \to g.y$, and thus Y is irreducible. Let $\widetilde{X}$ be an irreducible component of X of maximal dimension. Then $G.\widetilde{X}$ is the image of the irreducible variety $G \times \widetilde{X}$ under the product morphism, and hence it is irreducible. But $\widetilde{X} \subseteq G.\widetilde{X}$, and thus $\widetilde{X} = G.\widetilde{X}$. It is now easy to see that the map $\widetilde{\psi} = \psi | \widetilde{X}$, from $\widetilde{X}$ to Y, is a surjective G-morphism. Hence $\dim \widetilde{\psi}^{-1}(y) = \dim \widetilde{X} - \dim Y$ (see 1.2.13). But $\dim \widetilde{X} = \dim X$ and $\widetilde{\psi}^{-1}(y) \subseteq F$. $\therefore$ $\dim F \geq \dim X - \dim Y$.

Let $\widetilde{F}$ be an irreducible component of F of maximum dimension, and $\widetilde{X} \supseteq \widetilde{F}$ be an irreducible component of X. Then the map $\psi_1 = \psi | \widetilde{X}$, from $\widetilde{X}$ to Y, is a surjective G-morphism. Hence $\dim \psi_1^{-1}(y) = \dim \widetilde{X} - \dim Y$. But $\dim \psi_1^{-1}(y) = \dim F$ and $\dim \widetilde{X} \leq \dim X$. $\therefore$ $\dim F \leq \dim X - \dim Y$.

Let $\dim F = m$ and $\dim X = n$. Let $\mathfrak{F} = \{F_1, F_2, \ldots, F_r\}$ be the set of irreducible components of F of dimension m, and $A(y) = G_y/G_y^o$, where G_y is the isotropy group of G at y. Note that $A(y)$ is a finite group.

<u>Lemma 5</u> $A(y)$ permutes the elements of $\mathfrak{F}$.

<u>Proof</u>

(i) $G_y^o.F_s = F_s$ for all $F_s \in \mathfrak{F}$ (cf. the proof of Lemma 4).

(ii) If $g \in G_y$, then gF_s is a closed, irreducible subvariety of F of dimension m. Hence $gF_s \in \mathfrak{F}$.

It is clear from (i) and (ii) above that $A(y)$ permutes the elements of $\mathfrak{F}$.

We now state the main result of this section.

<u>Lemma 6</u> (Counting Lemma) The number of irreducible components of X of maximal dimension is equal to

$$\frac{1}{|A(y)|} \sum_{a \in A(y)} c(y)_a ,$$

where $c(y)_a$ is the number of elements of $\mathfrak{F}$ fixed by a.

(Note that $\frac{1}{|A(y)|} \sum_{a \in A(y)} c(y)_a$ is independent of the choice of $y \in Y$).

The above counting lemma follows immediately from Lemmas 7 and 9 given below.

Let d be equal to the number of $A(y)$ orbits on $\mathfrak{F}$.

<u>Lemma 7</u> d = the number of irreducible components of X of dimension n.

<u>Proof</u> Let $\{X_1, X_2, \ldots, X_p\}$ be the set of irreducible components of X of dimension n.

(i) If $1 \leqslant i \leqslant p$, then $G.X_i = X_i$, and the map $\psi_i = \psi | X_i$, from X_i to Y, is a surjective G-morphism. Hence $\dim \psi_i^{-1}(y) = \dim X_i - \dim Y = \dim F$. But $\psi_i^{-1}(y) = F \cap X_i$, and hence $\dim F \cap X_i = \dim F$. Let $\widetilde{F}$ be an irreducible component of $F \cap X_i$ of maximal dimension. Then $\widetilde{F}$ is a closed, irreducible subset of F of dimension m.
$\therefore \widetilde{F} \varepsilon \mathfrak{F}$.

(ii) If $F_s \varepsilon \mathfrak{F}$, then $\overline{G.F_s}$ is invariant under G, and the map $\widetilde{\psi}_s = \psi | \overline{G.F_s}$, from $\overline{G.F_s}$ to Y, is a surjective G-morphism.
$\therefore \dim \widetilde{\psi}_s^{-1}(y) = \dim \overline{G.F_s} - \dim Y$. But $\dim \widetilde{\psi}_s^{-1}(y) = \dim F$, and hence $\dim \overline{G.F_s} = \dim X$. Now $\overline{G.F_s}$ is irreducible and hence it is equal to X_i for some i, $1 \leqslant i \leqslant p$.
$\therefore F_s \subseteq X_i \cap F$, i.e. F_s is an irreducible component of $X_i \cap F$ of dimension m.

(iii) Let $1 \leqslant s \leqslant r$, $1 \leqslant i, j \leqslant p$ and $g \varepsilon G_y$. If F_s and gF_s are irreducible components of $X_i \cap F$ and $X_j \cap F$ respectively, then $X_i = X_j$.
i.e. (a) $G.F_s \subseteq GX_i = X_i$; and
(b) $G.F_s \subseteq GX_j = X_j$. Thus, by the argument of (ii) above it is clear that $\overline{G.F_s}$ is equal to both X_i and X_j.

(iv) If F_{s_1} and F_{s_2} are two irreducible components of $F \cap X_i$, where $1 \leq s_1, s_2 \leq r$ and $1 \leq i \leq p$, then $F_{s_1} = gF_{s_2}$ for some $g \in G_y$.

i.e. $G.F_t$, $1 \leq t \leq r$, is the image of the variety $G \times F_t$ under the product map $G \times X \to X$. Hence, $G.F_t$ contains an open dense subset of $\overline{G.F_t}$ (see 1.2.13). Now, $\overline{G.F_{s_1}} = \overline{G.F_{s_2}} = X_i$ (cf (iii) above). Hence, $G.F_{s_j}$, $j = 1, 2$, contains an open dense subset of X_i. $\therefore$ $G.F_{s_1} \cap G.F_{s_2}$ contains an open dense subset U, of X_i. We can assume that U is invariant under G, since if its not, then we can replace it by G.U. The map $\tilde{\psi} = \psi \mid U$, from U to Y, is a surjective G-morphism, and hence:

$$\begin{aligned} \dim \tilde{\psi}^{-1}(y) &= \dim U - \dim Y \\ &= \dim X - \dim Y \\ &= \dim F. \end{aligned}$$

$$\begin{aligned} \text{But } \tilde{\psi}^{-1}(y) &\subseteq F \cap G.F_{s_1} \cap G.F_{s_2} \\ &= G_y.F_{s_1} \cap G_y.F_{s_2}. \end{aligned}$$

$\therefore$ $\dim G_y.F_{s_1} \cap G_y.F_{s_2} = \dim F.$

Let $G_y/G_y^{\,o} = \{g_1G_y^{\,o}, g_2G_y^{\,o}, \ldots, g_\ell G_y^{\,o}\}$. Then:

$$\begin{aligned} G_y.F_{s_1} \cap G_y.F_{s_2} &= \Big(\bigcup_{u=1}^{\ell} g_uF_{s_1}\Big) \cap \Big(\bigcup_{v=1}^{\ell} g_vF_{s_2}\Big) \\ &= \bigcup_{u,v=1}^{\ell} g_uF_{s_1} \cap g_vF_{s_2}. \end{aligned}$$

Hence there exists u and v, $1 \leq u, v \leq \ell$, such that $\dim g_uF_{s_1} \cap g_vF_{s_2} = \dim F$ (See 1.2.12). But $g_uF_{s_1}$ and $g_vF_{s_2}$ are irreducible components of F and thus $g_uF_{s_1} = g_vF_{s_2}$. The result now follows immediately.

Before we go on to conclude the proof of this lemma, we note the following about (iv) above. Although $G.F_{s_j}$, $j = 1, 2$, is a constructable subset of X, it is not necessarily a subvariety. Thus, it was necessary to define ψ from U to Y, rather than from $GF_{s_1} \cap GF_{s_2}$ to Y.

(v) Consider the map

$$\Gamma: \{F_1, F_2, \ldots, F_r\} \to \{X_1, X_2, \ldots, X_p\}$$

defined by $\Gamma(F_s) = X_i$ if and only if F_s is an irreducible component of $F \cap X_i$.

By (ii) and (iii) above, Γ is well defined, i.e. if $F_s \in \mathcal{F}$, then there exists a unique element $X_i \in \{X_1, \ldots, X_p\}$ such that F_s is an irreducible component of $F \cap X_i$.

By (i) above, Γ is surjective.

By (iii) and (iv) above, each $A(y)$ orbit is mapped onto a single element of $\{X_1, \ldots, X_p\}$, and the images of any two such orbits are distinct.

The Lemma follows immediately.

Lemma 8 (Orbit Stabilizer Theorem) If Γ is a finite group acting transitively on a finite set E, then $|\Gamma| \div |\Gamma_v| = |E|$ for all $v \in E$.

Proof {Note that $\Gamma_v = \{\gamma \in \Gamma \mid \gamma.v = v\}$}. Let $\{\gamma_1\Gamma_v, \gamma_2\Gamma_v, \ldots, \gamma_r\Gamma_v\}$ be the set of left cosets of Γ with respect to Γ_v; we are of course assuming that $\gamma_1\Gamma_v, \ldots, \gamma_r\Gamma_v$ are all distinct. Then $E = \{\gamma_1 v, \ldots, \gamma_r v\}$, and $\gamma_i v \neq \gamma_j v$ for any $i, j, i \neq j$. The result follows immediately.

<u>Lemma 9</u> If Γ is a finite group acting on a finite set E, and if for each $\gamma \in \Gamma$, c_γ is equal to the number of elements of E fixed by γ, then:

$$\frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} c_\gamma = \text{number of } \Gamma \text{ orbits on } E.$$

<u>Proof</u> Let $\{O_1, O_2, \ldots, O_\ell\}$ be the set of Γ orbits on E. Also, if $1 \leq i \leq \ell$ and $\gamma \in \Gamma$, then put $c_\gamma^{\ i}$ equal to the number of elements of O_i fixed by γ. Let n_i be equal to the number of pairs $(\gamma, v) \in \Gamma \times O_i$ such that $\gamma.v = v$. It is clear that:

$$\sum_{\gamma\in\Gamma} c_\gamma^{\ i} = n_i = \sum_{v\in O_i} |\Gamma_v|$$

Now
$$\frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} c_\gamma = \sum_{i=1}^{\ell} \sum_{\gamma\in\Gamma} \frac{c_\gamma^{\ i}}{|\Gamma|}$$

$$= \sum_{i=1}^{\ell} \sum_{v\in O_i} \frac{|\Gamma_v|}{|\Gamma|}$$

$$= \sum_{i=1}^{\ell} \sum_{v\in O_i} \frac{1}{|O_i|} \qquad \text{(by Lemma 8)}$$

$$= \ell.$$

<u>2.2.3</u>

From now onwards G will denote a connected, reductive group, B a fixed Borel Subgroup of G, $T \subseteq B$ a maximal torus in G, and W the Weyl Group of G with respect to T. Also, we let ℓ = rank G, and U be the unipotent radical of B.

We now make the following basic assumption:

<u>Assumption I</u> <u>Given a unipotent conjugacy class C of G then there exists $w \in W$ such that $\overline{C \cap U_w^+} = U_w^+$.</u>

<u>Note</u>:

(i) Assumption I is equivalent to assuming that the map η (see page 45) is surjective.

(ii) If $\overline{C \cap U_w^+} = U_w^+$, then $U_w^+ \cap C$ is an open subset of U_w^+.

(iii) Assumption I implies that $|\mathcal{L}| \leqslant |W|$.

<u>Assumption I</u> holds in the following cases:

(i) $SL(n,K)$

(ii) $SO(n,K)$ and $Sp(n,K)$, given that $char(K) \neq 2$, and that K has infinite transcendence degree over its prime field.

(iii) For any algebraic group for which the Carter-Bala classification theorem holds.

We shall show that Assumption I holds in cases (i) and (ii) in chapters 3 and 4 respectively. See Theorem 3 for case (iii).

Recall that β is the variety consisting of the Borel Subgroups of G, and that if $u \in G$ is unipotent, then $\beta_u = \{\widetilde{B} \in \beta \mid u \in \widetilde{B}\}$ is a closed subvariety of β. Also let $C = C(u)$, $C(u)$ being the unipotent conjugacy class of G containing u.

We now give a result (Lemma 10) which is due to Steinberg (see 9).

Let $S = \{(v, B_1, B_2) \in C \times \beta \times \beta \mid v \in B_1 \cap B_2\}$ (it is clear that S is a closed subvariety of $C \times \beta \times \beta$), and let $\pi: S \to \beta \times \beta$ be given by $\pi((v, B_1, B_2)) = (B_1, B_2)$. G acts on $\beta \times \beta$ by conjugation, i.e. $g.(B_1, B_2) = ({}^gB_1, {}^gB_2)$. If $w \in W$, then let $X_w = (B, {}^wB)$, and put $S_w = \pi^{-1}(G.X_w)$. Note that:

(i) $S_w = \{(v, {}^gB, {}^{gw}B) \mid g^{-1}v \in B \cap {}^wB\}$.

(ii) $S_w \neq \emptyset \Longleftrightarrow C \cap (B \cap {}^wB) \neq \emptyset$

$\Longleftrightarrow C \cap U_w^+ \neq \emptyset$

(iii) $G.X_w$ is a subvariety of $\beta \times \beta$, and hence S_w is a subvariety of S, i.e. S_w is a locally closed subset of S.

<u>Lemma 10</u>

(i) S is the disjoint union of the S_w's.

(ii) If $S_w \neq \emptyset$; then $\dim S_w = \dim G - \ell + \dim U_w^+ \cap C - \dim U_w^+$.

(iii) $\dim S = \dim G - \ell$.

(iv) $\dim \beta_u = \dfrac{\dim Z_G(u) - \ell}{2}$

<u>Proof</u>

(i) Suppose that $(v, B_1, B_2) \in S$. Then there exists $g_1, g_2 \in G$ such that ${}^{g_i}B = B_i$, $i = 1, 2$. Also, there exist $b, b' \in B$, and $w \in W$ such that $g_1^{-1}.g_2 = bn_wb'$ (see the Bruhat Lemma in section 1.3.18). Put $g = g_1 b$. Now

${}^gB = {}^{g_1b}B = {}^{g_1}B = B_1$, and ${}^{gw}B = {}^{g_1bn_w}B = {}^{g_1bn_wb'}B = {}^{g_2}B = B_2$

$\therefore$ $(v, B_1, B_2) = (v, {}^gB\ {}^{gw}B)$

$\therefore$ $S = \bigcup_{w \in W} S_w$

Now suppose that $(v, {}^{g}B, {}^{gw}B) = (v_1, {}^{g_1}B, {}^{g_1w_1}B)$. Then $v = v_1$, ${}^{g}B = {}^{g_1}B$ and ${}^{gw}B = {}^{g_1w_1}B$. $\therefore$ $g = g_1.b$ for some $b \in B$, and $gn_w = g_1n_{w_1}b'$ for some $b' \in B$. Hence $g_1bn_w = g_1n_{w_1}b'$. $\therefore$ $bn_w = n_{w_1}b'$, and hence $Bn_wB = Bn_{w_1}B$. $\therefore$ $w = w_1$.

(ii) Suppose that $S_w \neq \emptyset$ or equivalently that $U_w^+ \cap C \neq \emptyset$. Let $S_w' = \{(v, g(B \cap {}^{w}B)) \in \mathbb{C} \times G/B \cap {}^{w}B \mid {}^{g^{-1}}v \in B \cap {}^{w}B\}$. It is easy to see that S_w is isomorphic to S_w', i.e. just consider the map $(v, {}^{g}B, {}^{gw}B) \to (v, g\, B \cap {}^{w}B)$. Let $\xi: S_w' \to G/B \cap {}^{w}B$ be the projection onto the second factor. G acts on S_w', i.e. $g_1.(v, g\, B \cap {}^{w}B) = ({}^{g_1}v, g_1g\, B \cap {}^{w}B)$, and also acts transitively on $G/B \cap {}^{w}B$ in the obvious way. It is clear that ξ is a surjective G morphism, and that $\xi^{-1}(B \cap {}^{w}B) \cong C \cap (B \cap {}^{w}B) = C \cap U_w^+$.

$$\begin{aligned}\therefore \dim S_w' &= \dim G/B \cap {}^{w}B + \dim C \cap U_w^+ \\ &= \dim G - \dim T.U_w^+ + \dim C \cap U_w^+ \\ &= \dim G - \ell - \dim U_w^+ + \dim C \cap U_w^+\end{aligned}$$

(iii) By (i) above, $\dim S \geq \dim S_w$ for all $w \in W$, and there exists $w \in W$ for which equality holds. Now, by Assumption I, $\exists\, w_o \in W$ such that $\dim U_{w_o}^+ = \dim U_{w_o}^+ \cap C$. $\therefore$ $\dim S_{w_o} = \dim G - \ell$. It is clear that $\dim S$ cannot be greater than $\dim G - \ell$, and thus that $\dim S = \operatorname{Dim} G - \ell$.

(iv) Let $\tilde{\xi}: S \to C$ be the projection onto the first factor. G acts on S, i.e. $g.(v, B_1, B_2) = ({}^{g}v, {}^{g}B_1, {}^{g}B_2)$, and acts transivitely on C by conjugation. It is clear that $\tilde{\xi}$ is a surjective G-morphism, and that $\tilde{\xi}^{-1}(u) \cong \beta_u \times \beta_u$.

$$\begin{aligned}\therefore \dim \beta_u \times \beta_u &= \dim S - \dim C \\ 2 \dim \beta_u &= \dim G - \ell - \dim C \\ &= \dim Z_G(u) - \ell \\ \therefore \dim \beta_u &= \frac{\dim Z_G(u) - \ell}{2}\end{aligned}$$

<u>Lemma 11</u> $\{\overline{S}_w \mid w \in n^{-1}(C)\}$ is the set of irreducible components of S of dimension $\dim G - \ell$. Further if $w_1, w_2 \in n^{-1}(C)$ and $w_1 \neq w_2$, then $\overline{S}_{w_1} \neq \overline{S}_{w_2}$.

<u>Proof</u>

(i) Suppose that $w \in n^{-1}(C)$. The variety $S_w'' = \{(v,g) \in C \times G \mid g^{-1} v \in B \cap {}^wB\}$ is isomorphic to $(U_w^+ \cap C) \times G$, the isomorphism being given by $(v,g) \to (g^{-1} v\, g, g)$. Now, $U_w^+ \cap C$ is irreducible (i.e. $\overline{U_w^+ \cap C} = U_w^+$ is irreducible), and hence S_w'' is irreducible. Let π: $S_w'' \to S_w$ be given by $\pi((v,g)) = (v, {}^gB, {}^{gw}B)$. It is clear that π is a surjective morphism, and hence that S_w is irreducible. Also, $\dim S_w = \dim S$ (cf. Lemma 10 part (ii)). Hence, $\overline{S}_w$ is an irreducible component of S of dimension $\dim G - \ell$.

(ii) Let Z be an irreducible component of S of dimension $\dim G - \ell$. $Z = \bigcup_{w \in W} \overline{Z \cap S_w}$, and hence there exists $w \in W$ such that $Z = \overline{Z \cap S_w} \subseteq \overline{S}_w$. Now, $\dim S_w = \dim \overline{S}_w$, and hence $\dim S_w = \dim G - \ell$.
$\therefore \dim U_w^+ = \dim C \cap U_w^+$ (see lemma 10 part (ii)),
i.e. $\overline{C \cap U_w^+} = U_w^+$. Hence $w \in n^{-1}(C)$ and by (i) above $\overline{S}_w$ is irreducible. $\therefore$ $\overline{S}_w = Z$.

(iii) If $\overline{S}_{w_1} = \overline{S}_{w_2}$, then $S_{w_1} \cap S_{w_2} \neq \emptyset$, and thus $w_1 = w_2$.

Let $A(u) = Z_G(u)/Z_G(u)^o$, and $\mathcal{L}$ be the set of irreducible components of β_u of maximal dimension. $Z_G(u)$ acts on β_u by conjugation, i.e. if $g \in Z_G(u)$ and $\widetilde{B} \in \beta_u$, then ${}^g\widetilde{B} \in \beta_u$. If $F \in \mathcal{L}$, then
(i) $gF \in \mathcal{L}$ for all $g \in Z_G(u)$; and (ii) $Z_G(u)^o.F = F$.
$\therefore$ $A(u)$ acts on $\mathcal{L}$. If $a \in A(u)$, then let $c(u)_a$ be the number of elements of $\mathcal{L}$ fixed by a.

Theorem 12

$$|\eta^{-1}(C)| = \frac{1}{|A(u)|} \sum_{a \in A(u)} c(u)_a^{\,2}.$$

Proof

Let $\mathcal{F}$ be the set of irreducible components of $\beta_u \times \beta_u$ of maximal dimension. $\mathcal{F} = \{F_1 \times F_2 \mid F_1, F_2 \in \mathcal{L}\}$, and $A(u)$ acts on $\mathcal{F}$ in the obvious way, i.e. $g\, Z_G(u)^o.(F_1, F_2) = (gF_1, gF_2)$. If $a \in A(u)$, then the number of elements fixed by a is $c(u)_a^{\,2}$.

Now consider the surjective G-morphism $\tilde{\xi}$: $S \to C$ (See the proof of lemma 10 pt. iv). Recall (i) that G acts on C by conjugation, and hence that the isotropy group of u is $Z_G(u)$; and (ii) that $\xi^{-1}(u) \cong \beta_u \times \beta_u$.

The action of $A(u)$ on $\mathcal{F}$ fits into the general framework described on page 46 . Hence by Lemma 6, we have that the number of irreducible components of S of maximal dimension is $\frac{1}{|A(u)|} \sum_{a \in A(u)} c(u)_a^{\,2}$.

The result now follows immediately from Lemma 11.

Let $\{u_1, u_2, \ldots, u_p\}$ be a set of unipotent elements of G such that (i) $\mathcal{L} = \{C(u_1), C(u_2) \ldots, C(u_p)\}$; and (ii) $C(u_i) = C(u_j) \Longleftrightarrow i = j$.

THEOREM (SPRINGER'S RESULT)

$$|W| = \sum_{i=1}^{p} \frac{1}{|A(u_i)|} \sum_{a_i \in A(u_i)} C(u_i)_{a_i}^{\,2}.$$

<u>Proof</u> The result follows immediately from theorem 12 and the obvious fact that $|W| = \sum_{i=1}^{p} | \eta^{-1}(C(u_i))|$.

Let $G = SL(n,K)$; T be the set of diagonal matrices in G, and u be a unipotent element of G. Then:

(i) G satisfies assumption I (we will prove this in the next chapter).

(ii) $W = W(G,T)$ is isomorphic to S_n, the symmetric group on n elements.

(iii) $Z_G(u)$ is connected.

Thus we obtain:

<u>Corollary 13</u>

$$n! = |S_n| = \sum_{i=1}^{p} n_{u_i}^{\ 2},$$

where $\{u_1, \dots, u_p\}$ are as described on page 56, and n_{u_i} is equal to the number of irreducible components of $\mathcal{B}_{u_i}$ of dimension $\frac{\dim Z_G(u_i) - \ell}{2}$.

2.3 BRUHAT CELLS

We finish this chapter by proving two lemmas which will prove useful later on.

Let G be reductive. Recall that $\mathcal{B} = \bigcup_{w \in W} \tilde{C}_w$, and that each element of $\tilde{C}_w$ can be written uniquely in the form ${}^{vw}B$, where $v \in U_w^-$ (see 1.3.18).

If $w \in W$, then let $\tau_w : U_w^- \times U_w^+ \to U$ be defined by $\tau_w((v,a)) = vav^{-1}$.

Lemma 14 Let u be a unipotent element of B, and $w \in W$. Then $\beta_u \cap \widetilde{C}_w \neq \emptyset \iff u \in \mathrm{Im}(\tau_w)$. In this case $\beta_u \cap \widetilde{C}_w$ is isomorphic to $\pi_1(\tau_w^{-1}(u))$, where $\pi_1 : U_w^- \times U_w^+ \to U_w^-$ is the projection onto the first factor.

Proof $\widetilde{C}_w \cap \beta_u = \{{}^{vw}B \mid v \in U_w^-,\ u \in {}^{vw}B\}$

(i) Suppose that $\widetilde{C}_w \cap \beta_u \neq \emptyset$. Then there exist $v \in U_w^-$ such that $u \in {}^{vw}B$, i.e. $v^{-1} u v \in B \cap {}^wB$.

$\therefore\ v^{-1} u v = a \in U_w^+$, and $\tau_w(v,a) = u$.

(ii) Suppose that $u \in \mathrm{Im}(\tau_w)$. Then $\exists\ (v,a) \in U_w^- \times U_w^+$ such that $v a v^{-1} = u$.

$\therefore\ {}^{v^{-1}}u = a \in B \cap {}^wB$, i.e. ${}^{vw}B \in \widetilde{C}_w \cap \beta_u$.

(iii) The last assertion follows easily.

Note that if $u \in U_w^+$, then $\beta_u \cap \widetilde{C}_w \neq \emptyset$.

Lemma 15 If $u \in U_w^+$ and $w \in \eta^{-1}(C(u))$, then $\dim \widetilde{C}_w \cap \beta_u = \dim \beta_u$.

Proof

$S_w = \{(v, {}^gB, {}^{gw}B) \in C(u) \times \beta \times \beta \mid v \in {}^g(B \cap {}^wB)\}$ is an irreducible variety of dimension $\dim G - \ell$ (see Lemma 11). Let $\pi : S_w \to C(u) \times \beta$ be given by $\pi(v, {}^gB, {}^{gw}B) = (v, {}^gB)$. Then $Y = \pi(S_w) = \{(v, {}^gB) \mid {}^{g^{-1}}v \in U_w^+\}$ is an irreducible subvariety of $C(u) \times \beta$. Now, $(u,B) \in Y$, and

$$\pi^{-1}((u, B)) = \{(u, B, {}^{bw}B) \mid u \in {}^{bw}B\}$$
$$\cong \widetilde{C}_w \cap \beta_u .$$

$\dim \beta_u \cap \tilde{C}_w \geq \dim S_w - \dim Y$. (see 1.2.13).

G acts on Y, i.e. $\bar{g}.(v, {}^gB) = ({}^{\bar{g}}v, {}^{\bar{g}g}B)$, and acts transitively on $C(u)$ by conjugation. The map $\tilde{\pi}: Y \to C(u)$ given by $\pi(v, {}^gB) = V$, is a surjective G-morphism. Hence $\dim \tilde{\pi}^{-1}(u) = \dim Y - \dim C(u)$. But

$$\tilde{\pi}^{-1}(u) \cong \{{}^gB \mid g^{-1}u\, g \in U_w^+\}$$
$$\subseteq \beta_u$$

$$\therefore \dim \beta_u \geq \dim Y - \dim C(u)$$

$$\therefore \dim Y \leq \dim \beta_u + \dim C(u).$$

$$\begin{aligned} \therefore \dim \beta_u \cap \tilde{C}_w &\geq \dim S_w - \dim \beta_u - \dim C(u) \\ &= \dim G - \ell - \dim \beta_u - \dim C(u) \\ &= \dim \beta_u \end{aligned}$$

(see the proof of part (iv) of Lemma 10).

$$\therefore \dim \beta_u \cap \tilde{C}_w = \dim \beta_u.$$

CHAPTER 3

SL(N,K)

In this chapter we look specifically at the group SL(n,K). Our main interest lies in the set $N_{U_{(k)}} = \{w \in W \mid \tilde{C}_w \cap \beta_{U_{(k)}} \neq \emptyset$ and $n(w) = C(U_{(k)})\}$ (see the introduction for the notation), where (k) is an ordered partition of n.

Before we look at $N_{U_{(k)}}$ we find it necessary:

(i) To describe SL(n,K) and to establish the relationship between the unipotent conjugacy classes of SL(n,K) and the ordered partitions of n (see sections 3.1.1 and 3.1.2). In the course of this we are able to show that our basic assumption (see 2.2.3) is true for SL(n,K) (see proposition 19).

(ii) To make a slight digression and look at the multiplication of matrices - see section 3.2.1. Our aim in this section is to prove Corollary 25.

Note: At the beginning of section 3.2 we impose the restriction that K has infinite transcendence degree over its prime field.

In sections 3.2.2 - 3.2.4 we look at the properties of $N_{U_{(k)}}$. We are able to find w_o, $w_1 \in N_{U_{(k)}}$ (see proposition 31) such that $w \in N_{U_{(k)}}$ if and only if $U_{w_o}^+ \subseteq U_w^+ \subseteq U_{w_1}^+$. This enables us to set up a bijective correspondence (proposition 33) between the elements of $N_{U_{(k)}}$ and the set of standard tableaux corresponding to (k) (see section 3.2.3 for the definition of a standard tableau). But we know (theorem 32) that the number of standard tableaux is equal to $d_{(k)}$. Thus we obtain Theorem 34 which states that:

$$|N_{U_{(k)}}| = d_{(k)}$$

Note that $d_{(k)}$ is the dimension of the irreducible representation of $S_n = W$ corresponding to (k).

Finally we show that the number of irreducible components of $\beta_{U_{(k)}}$ of maximal dimension is equal to $d_{(k)}$ (theorem 35).

It should be noted that in order to give complete proofs of the above results, it has been necessary to go into a great amount of somewhat tedious detail.

3.1 BACKGROUND

3.1.1 Description of SL(n,K)

$SL(n,K)$ is a quasi-simple algebraic group consisting of $n \times n$ matrices with coefficients in K, and determinant 1. The set, T, of diagonal matrices in $SL(n,K)$ is a maximal torus, and the set, B, of upper triangular matrices in $SL(n,K)$ is a Borel Subgroup containing T. U, the unipotent radical of B, consists of all those matrices in B with 1's on the diagonal.

$N_G(T)$ is the set of matrices (a_{ij}) for which there exists $\sigma \in S_n$, the symmetric group on n elements, such that $a_{\sigma(j)j} \neq 0$, and $a_{ij} = 0$ whenever $i \neq \sigma(j)$. $W(G,T) = N_G(T)/T$, and the map $W(G,T) \to S_n$, $(a_{ij})T \to \sigma$, where σ is as above, is a group isomorphism.

Let $<1,n>$ denote the set of integers $\{1, 2, \ldots, n\}$, $\Delta_n = \{(i,j) \in <1,n> \times <1,n> \mid i \neq j\}$, and $\Delta_n^+ = \{(i,j) \in \Delta_n \mid i < j\}$. If $(i,j) \in \Delta_n$, then let $\alpha_{ij}: T \to G_m$ be defined by $\alpha_{ij}(\mathrm{diag}\ (a_1, a_2, \ldots, a_n)) = a_i/a_j$. $\Phi(G,T) = \{\alpha_{ij} \mid (i,j) \in \Delta_n\}$, $\Phi^+ = \Phi(B,T) = \{\alpha_{ij} \mid (i,j) \in \Delta_n^+\}$, and the corresponding set of simple roots, π, is equal to $\{\alpha_{12}, \alpha_{23}, \ldots, \alpha_{n-1n}\}$. Recall that $t^{\alpha_{ij}+\alpha_{k\ell}} = \alpha_{ij}(t)\alpha_{k\ell}(t)$ for all $t \in T$. From this it is easy to see that $\alpha_{ij} + \alpha_{k\ell} \in \Phi(G,T)$ if and only if $j = k$ and $i \neq \ell$, and that in this case $\alpha_{ij} + \alpha_{k\ell} = \alpha_{i\ell}$.

If we identify Δ_n and $\Phi(G,T)$ in the obvious way, then Δ_n^+ corresponds to Φ^+, and the set $\{(1,2),(2,3)\ldots,(n-1,n)\}$ to π. Also, if $w \in W = S_n$, then $w.\alpha_{ij} = \alpha_{w(i)w(j)}$. Thus, the action of W on Δ_n^+ given by $w(i,j) = (w(i),w(j))$, corresponds to the action of W on $\Phi(G,T)$.

If $w \in W$, then $A_w^+ = \{(i,j) \in \Delta_n^+ \mid w^{-1}(i) < w^{-1}(j)\}$, $A_w^- = \{(i,j) \in \Delta_n^+ \mid w^{-1}(i) > w^{-1}(j)\}$, $U_w^+ = \{(a_{ij}) \in U \mid a_{ij} = 0$ if $(i,j) \in \Delta_n^+ - A_w^+\}$, and $U_w^- = \{(a_{ij}) \in U \mid a_{ij} = 0$ if $(i,j) \in \Delta_n^+ - A_w^-\}$.

The one parameter subgroup ε_{ij} corresponding to the root (i,j) is defined by $\varepsilon_{ij}(k) = I + kE_{ij}$ (see example (ii) of section 1.3.9).

We can make $s\ell(n,K)$, the set of $n \times n$ matrices with coefficients in K and trace zero, into a Lie algebra by putting $[X,Y] = XY - YX$ for all $X, Y \in s\ell(n,K)$. $s\ell(n,K)$ is the Lie algebra of $SL(n,K)$, $\underline{t}$, the Cartan Subalgebra consisting of the diagonal matrices in $s\ell(n,K)$, is the Lie algebra of T, and E_{ij} is a root vector corresponding to the root $(i,j) \in \Delta_n$. Also, if $w \in W$, then $\underline{u}_w^+ = \{X \in s\ell(n,K) \mid X + I \in U_w^+\}$, and $\underline{u}_w^- = \{X \in s\ell(n,K) \mid X + I \in U_w^-\}$.

3.1.2 Partitions and Conjugacy Classes

If N is the variety consisting of the nilpotent elements of $s\ell(n,K)$, and V the variety consisting of the unipotent elements of $SL(n,K)$, then the map $\Gamma: N \to V$, given by $\Gamma(X) = X + I$ is an isomorphism of varieties. Also, if $g \in SL(n,K)$, then $\Gamma(gXg^{-1}) = g\Gamma(X)g^{-1}$ for all $X \in N$. Thus, there is a bijective correspondence between the nilpotent conjugacy classes of $s\ell(n,K)$ and the unipotent conjugacy classes of $SL(n,K)$, i.e. the nilpotent conjugacy class, $C(X)$, containing X corresponds to the unipotent conjugacy class, $C(X + I)$, containing $X + I$.

__Lemma 16__ If X and Y are nilpotent elements of $s\ell(n,K)$, then:

(i) $C(X) \subseteq \overline{C(Y)}$ if and only if $\text{rank } X^i \leq \text{rank } Y^i$ for $i = 1,\dots,n$.

(ii) $C(X) = C(Y)$ if and only if $\text{rank } X^i = \text{rank } Y^i$ for $i = 1,2,\dots,n$.

Or, equivalently, if u and v are unipotent elements of $SL(n,K)$, then:

(i) $C(u) \subseteq \overline{C(v)}$ if and only if $\text{rank } (u - I)^i \leq \text{rank } (v - I)^i$ for $i = 1,2,\dots,n$.

(ii) $C(u) = C(v)$ if and only if $\text{rank } (u - I)^i = \text{rank } (v - I)^i$ for $i = 1,2,\dots,n$.

__Proof__ See 4.

__Definition__ An __ordered partition__ $(k) = (k_1, k_2,\dots, k_r)$ of n is a set $\{k_1,\dots, k_r\}$, of positive integers such that:

(i) $k_1 + k_2 + \ldots + k_r = n$.

(ii) $k_i \geqslant k_{i+1}$ for $i = 1,2,\ldots,r-1$.

The numbers k_i, $i = 1,2,\ldots,r$, are called the parts of the partition.

If k is a positive integer, then we use N_k to denote the $k \times k$ matrix with ones on the super diagonal, and zeros elsewhere.

Lemma 17 The nilpotent conjugacy classes of $s\ell(n,K)$ are in one to one correspondence with the ordered partitions of n; the nilpotent class corresponding to $(k) = (k_1, k_2,\ldots,k_r)$ being the one which contains $N_{(k)} = N_{k_1} \oplus N_{k_2} \oplus \ldots \oplus N_{k_r}$.

Proof See 4.

Note:

(i) The analogue in the unipotent case is obvious.

(ii) We write $U_{(k)} = I + N_{(k)}$.

Let $S \subseteq \Lambda_n^+$, $\underline{u}_S = \{(a_{ij}) \in s\ell(n,K) \mid a_{ij} = 0 \text{ if } (i,j) \notin S\}$, and $U_S = \{x \in SL(n,K) \mid x - I \in \underline{u}_S\}$. U_S is a closed, irreducible subvariety of $SL(n,K)$ consisting entirely of unipotent elements. Thus, there exists a unique unipotent conjugacy class, C_S, such that $\overline{C_S \cap U_S} = U_S$ (cf. the proof of Lemma 1).

<u>Definition</u> Let Φ_o be a root system, and Φ_o^+ a set of positive roots in Φ_o, then a subset Ψ of Φ_o^+ is said to be <u>closed</u> if $\alpha, \beta \in \Psi$, and $\alpha+\beta \in \Phi_o^+$ implies that $\alpha+\beta \in \Psi$.

<u>Lemma 18</u> If $\Psi \subseteq \Phi_o^+$, and Ψ and $\Phi_o^+ \sim \Psi$ are closed, then there exists $w \in W(\Phi_o)$ such that $A_w^+ = \Psi$.

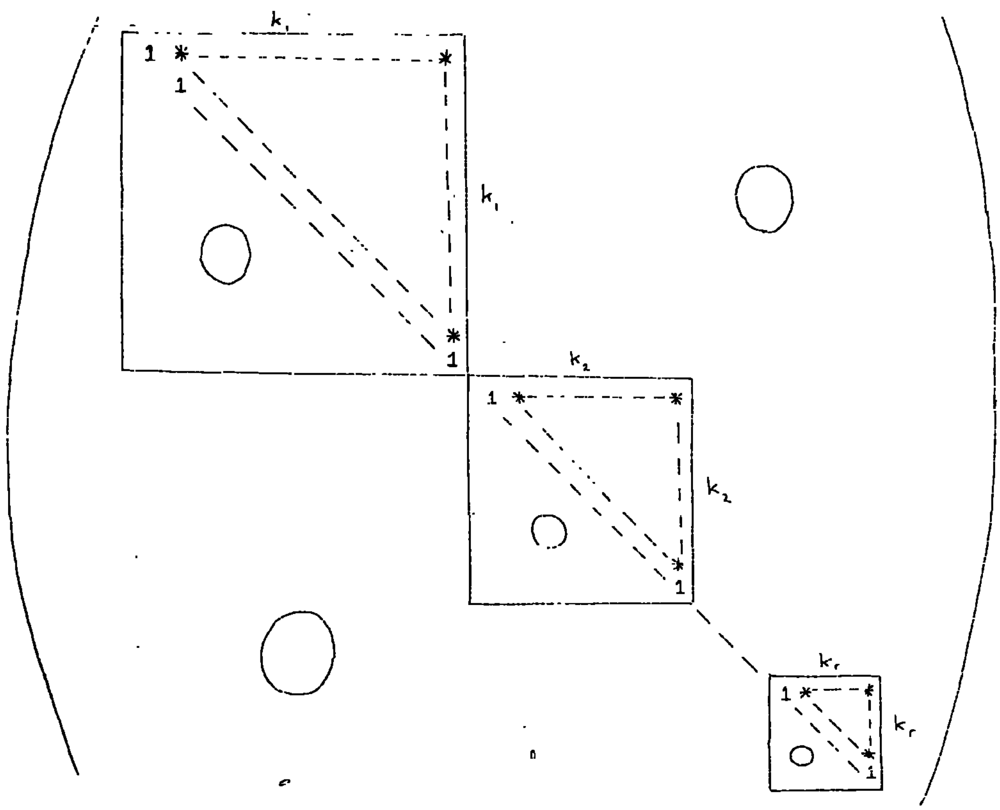
<u>Proof</u> See (10).

A subset S of Δ_n^+ is closed if $(i,j),(j,k) \in S$ implies that $(i,k) \in S$. We are now in a position to show that our basic assumptiom (see 2.2.3) is true for $SL(n,K)$.

<u>Proposition 19</u> If $(k) = (k_1, k_2, \ldots, k_r)$ is an ordered partition of n, then there exists $w_o \in S_n$ such that $\overline{C(U_{(k)})} \cap U_{w_o}^+ = U_{w_o}^+$

<u>Proof</u> Let $k_o = 0$, and put $S = \{(i,j) \in \Delta_n^+ \mid k_o + k_1 + \ldots + k_p < i,j < k_o + k_1 + \ldots + k_{p+1}$ for some $p = 0,1,\ldots,r-1\}$. It is easy to see that S and $\Delta_n^+ \sim S$ are closed (cf the diagram below). Thus, there exists $w_o \in S_n$ such that $U_S = U_{w_o}^+$. Now $U_{(k)} \in U_{w_o}^+$, and if $v \in U_{w_o}^+$, then $\mathrm{rank}(U_{(k)} - I)^i \geq \mathrm{rank}(v - I)^i$ for $i = 1,2,\ldots,n$. Hence $C(v) \subseteq \overline{C(U_{(k)})}$. It is now easy to see that $\overline{C(U_{(k)})} \cap U_{w_o}^+ = U_{w_o}^+$.

<u>Note</u>: An element v of $U_{w_o}^{+}$ has the form



Before we go on to look at the subset $N_{U_{(k)}}$ of W (see the introduction to this chapter) we need to establish some preliminary results. We do this in the following section.

3.2 THE WEYL GROUP AND CONJUGACY CLASSES

From now onwards we assume that K has infinite transcendence degree over its prime field.

3.2.1 Some Preliminary Results

Definition If $S \subseteq \Delta_n^+$, then $(a_{ij}) \in \underline{u}_S$ is said to be generic if $\{a_{ij} \mid (i,j) \in S\}$ is a set of algebraically independent transcendentals over the prime field. $u \in U_S$ is said to be generic if $u - I$ is a generic element of $\underline{u}_S$.

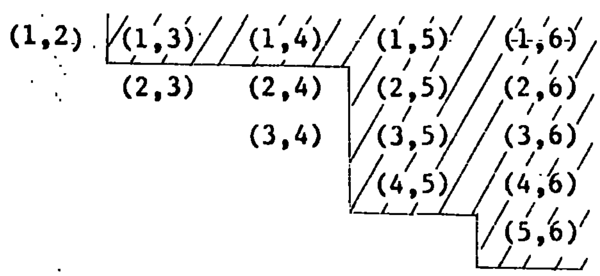
Lemma 20 If $u \in U_S$ is generic, then $C(u) = C_S$.

Proof It is clear that if $v \in U_S$, then $\text{rank } (v - I)^i \leq \text{rank } (u - I)^i$ for $i = 1, 2, \ldots, n$. Thus $C(v) \subseteq \overline{C(u)}$. It now follows immediately that $\overline{C(u) \cap U_S} = U_S$.

Definition A subset S of Δ_n^+ is said to be triangular if $(i,j) \in S$ implies that $(i-1,j),(i,j+1) \in S$.

Note: The notion of triangular subsets can be found in M. Gerstenhaber's paper (5) on Classical Groups. We shall be using his ideas in the next chapter.

<u>Example</u> Let $n = 6$. We can display the elements of Δ_6^+ in a triangular array.



The elements of Δ_6^+ which lie in the shaded region form a triangular subset S, of Δ_6^+. Note that the elements of Δ_6^+ which lie above, and to the right of an element (i,j) of S, are elements of S.

<u>Definition</u> Let (k_1, k_2) be an ordered partition of n. Then we say that a subset S of Δ_n^+ is triangular with respect to (k_1, k_2) if:

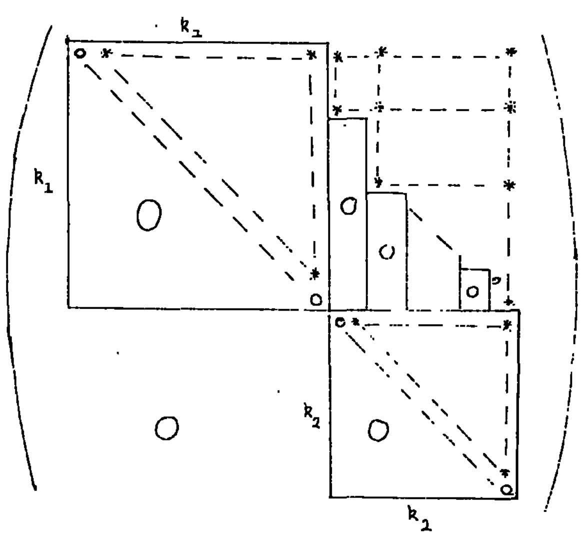
(i) S is triangular.

(ii) $S \subseteq \{(i,j) \in \Delta_n^+ \mid 1 \leq i \leq k_1, \text{ and } k_1 < j \leq n\}$.

For the rest of this chapter we shall assume that S is triangular with respect to (k_1, k_2).

Let $\xi(S) = \{(i,j) \in \Delta_n^+ \mid 1 \leq i \leq k_1 \text{ and } k_1 < j \leq n\} \sim S$.

Then a generic element, X_o, of $\underline{u}_{\Delta_n^+} \sim \xi(S)$ has the form:



where the *'s denote non-zero entries.

Our aim in this section is to describe X_o^t, where t is a positive integer. We achieve this in Proposition 24. In particular, we obtain Corollary 25 which we will need in the proof of Proposition 31.

Before we can prove Proposition 24, however, we need to consider some of the properties of the triangular subset S.

The triangular subset S

We define a sequence, $(I_o, J_o), (I_1, J_1), \ldots, (I_{p+1}, J_{p+1})$, of elements of Δ_n^+ as follows:

(i) Put $(I_o, J_o) = (1, k_1)$.

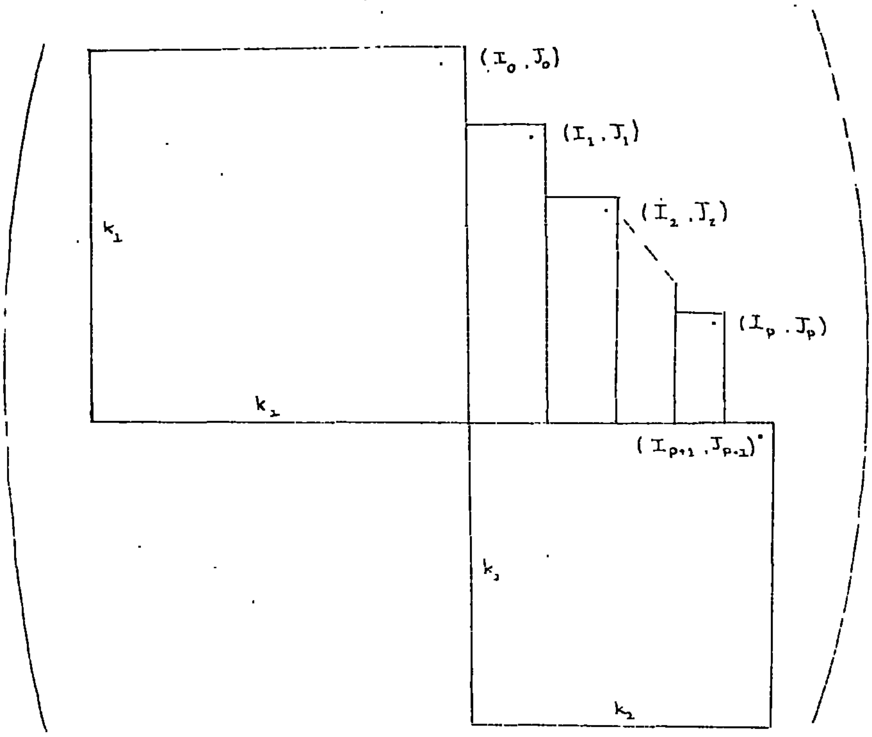
(ii) Suppose that $(I_o, J_o), (I_1, J_1), \ldots, (I_r, J_r)$ have been defined. Then:

(a) If there exists $(i,j) \in \xi(S)$ such that $j > J_r$, then put $I_{r+1} = \min\{i \in \langle 1, n\rangle \mid \exists\ j > J_r$ such that $(i,j) \in \xi(S)\}$, and $J_{r+1} = \max\{j \in \langle 1,n\rangle \mid (I_{r+1},j) \in \xi(S)\}$.

(b) If there does not exist $(i,j) \in \xi(S)$ with $j > J_r$, and $I_r \neq k_1 + 1$, then put $(I_{r+1},J_{r+1}) = (k_1 + 1, n)$.

(c) If $I_r = k_1 + 1$, then the sequence finishes with (I_r,J_r).

In the diagram below we indicate the positions of $(I_o,J_o),(I_1,J_1)\ldots,(I_p,J_p)$ and (I_{p+1},J_{p+1}), and we also note that this sequence gives us a complete discription of S.



We now go on to define the sequence, $S^1 \supset S^2 \supset \ldots \supset S^m = \emptyset$, of <u>characteristic subsets</u> of S.

(i) Put $S^1 = S$.

(ii) Suppose that $S^1, S^2, \ldots, S^d$ have been defined. Then:

(a) If $S^d \neq \emptyset$, then put $L^d = \{(i,j) \in S^d \mid (i+1,j), (i,j-1) \notin S^d\}$, and let $S^{d+1} = S^d - L^d$.

(b) If $S^d = \emptyset$, then the sequence ends with S^d.

<u>Note</u>:

(i) If $S^d \neq \emptyset$, then $L^d \neq \emptyset$, and thus the above inclusions are strict.

(ii) Each S^d, $d = 1,\ldots,m$, is a triangular subset of Δ_n^+ with respect to (k_1, k_2).

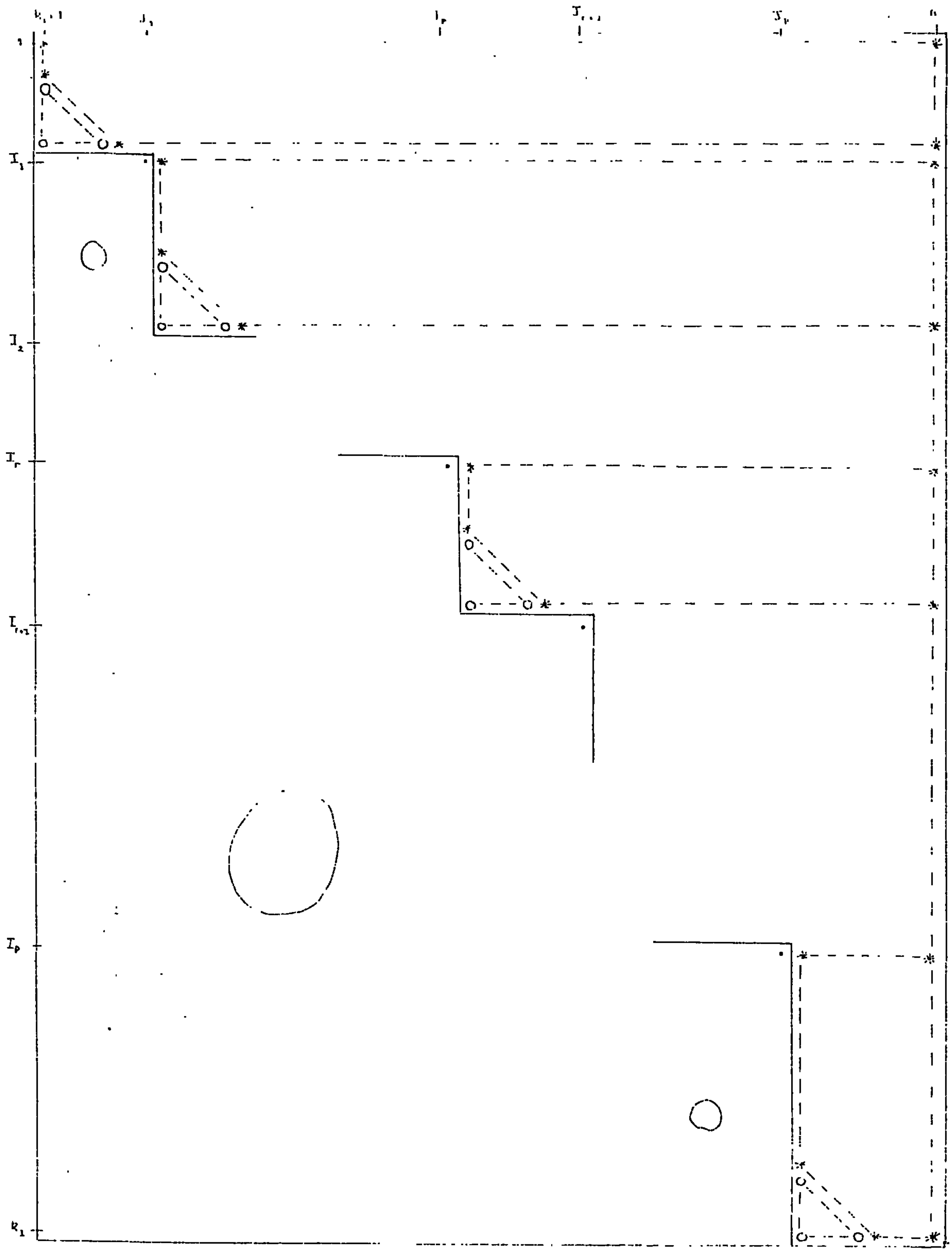
(iii) S^d is called the <u>d^{th} characteristic subset of S</u>

<u>Example</u> Let $n = 10$, and $(k_1, k_2) = (6, 4)$.

	7	8	9	10
1	α	β	β	β
2	0	0	β	β
3	0	0	α	β
4	0	0	0	β
5	0	0	0	β
6	0	0	0	α

If the positions with non-zero entries represent the elements of S^d, then the positions with entries α represent the elements of L^d, and the positions with entries β represent the elements of S^{d+1}.

It will prove helpful, when we come to look at Lemmas 21 and 22, to represent S^d diagramatically. So we let the set $\{(i,j) \mid 1 < i < k_1, k_1 < j \leqslant n\}$ be represented by a rectangular array of $k_1 \times k_2$ nodes, and the 'diagram' below represent S^d.



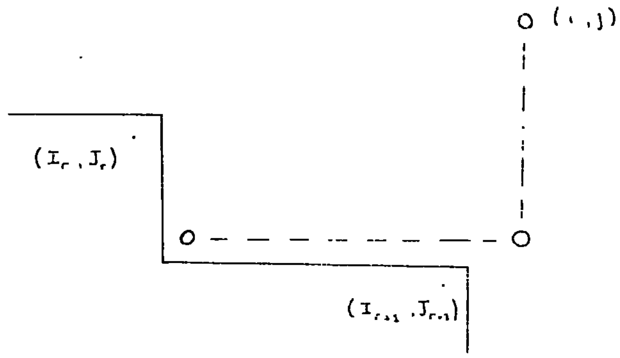
i.e. The positions with entries 0 represent the elements of $\xi(S^d)$, those with entries * represent the elements of S^d, and those to the right, and above the bold lines represent the elements of S.

The scheme below indicates how the 'diagram' for S^{d+1}, is obtained from the 'diagram' for S^d.

Row i:	* . . . . *	Row i:	* . . . . *
i+1:	* . . . . *		
Row i:	* . . * * . . . . . . *	Row i:	0 * . . . . *
i+1:	0 . . 0 * . . . . . . *		
	j		j
Row i:	0 . . 0 * . . . . . . *	Row i:	0 . . 0 * . . . . . . *
i+1:	0 . . 0 * . . . . . . *		
			j+1
Row i:	0 . . 0 * . . * * . . *	Row i:	0 . . . 0 * . . . . . *
i+1:	0 . . 0 0 . . 0 * . . *		
Row i:	0 0	Row i:	0 0
Row k_1:	* . . . . *	Row k_1:	0 * . . . . *
	j		j+1
Row k_1:	0 . . 0 * . . . . . . *	Row k_1:	0 . . . 0 * . . . . . *

<u>Lemma 21</u> If $i < I_{r+1}$, $j > J_r$ and $(i,j) \in S - S^d$, then $I_{r+1} - i + j - J_r \leq d$.

<u>Proof</u> The relevant part of the 'diagram' for S^d has the form:



To prove the lemma it is sufficient to notice that the number of zeros indicated must be less than d.

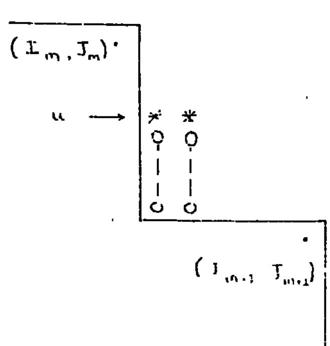
<u>Lemma 22</u> If $I_r \leqslant i < I_{r+1}$, $J_r < j < n$, $(i,j) \in S^d$ and $(i+1,j+1) \notin S^d$, then:

(i) $i = I_{r+1} - 1$.

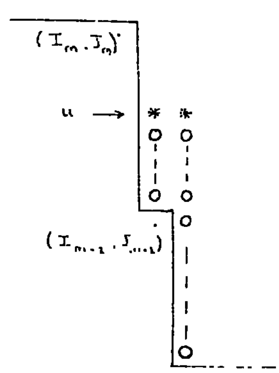
(ii) $j - J_r \geqslant d$.

<u>Proof</u> Case (i) $r \neq p$.

(A) If $I_m \leqslant u < I_{m+1}$, $(u, J_m + 1) \in S^\ell$ and $(u + 1, J_m + 2) \in \Delta_n^+ - S^\ell$, then $u = I_{m+1} - 1$, for otherwise we would get a 'diagram'



or



for S^{ℓ}, and this is not possible. It follows that $(I_{m+1} - 1, J_m + 1) \varepsilon S^{\ell}$, and hence that $\ell = 1$.

(B) We now return to the situation described in the Lemma. Rows i and $i+1$ of the 'diagram' for S^d have the form:

$$\begin{array}{lccccccccccccccccccc} & & & & J_r & & & & v & & & j & j+1 & & & q & & & \\ \text{Row } i & 0 & . & . & 0 & 0 & . & . & 0 & * & . & . & * & * & . & . & * & * & . & . & * \\ \text{Row } i+1 & 0 & . & . & 0 & 0 & . & . & 0 & 0 & . & . & 0 & 0 & . & . & 0 & * & . & . & * \end{array}$$

{we include the cases where $v=j$, $v=J_{r+1}$, $q=n$ etc.}.

It is clear that $v - J_r < d + 1$ (see Lemma 21). By looking at the diagrams on page 73, it can be seen that rows i and $i+1$ of $X_o^{d-(v-J_r-1)}$ have the form:

$$\begin{array}{lcccccccccccccccc} & & & & J_r & & & & & & & & & & & & \\ \text{Row } i{:} & 0 & . & . & 0 & * & . & . & . & . & * & * & . & . & . & . & * \\ \text{Row } i+1 & 0 & . & . & 0 & 0 & . & . & . & . & 0 & * & . & . & . & . & * \end{array}$$

i.e. $(i, J_r + 1) \varepsilon S^{d - (v-J_r-1)}$ and $(i + 1, J_r + 2) \notin S^{d - (v-J_r-1)}$.

Hence by part A, we have that $i = I_{r+1} - 1$. Also we have that $v - J_r = d$, and hence that $j - J_r \geqslant d$.

Case (ii) $r = p$

It is trivial to show that the lemma is true in this case.

<u>Description of X_o^t</u>

Recall that $X_o = (a_{ij})$ is a generic element of $\underline{u}_{\Delta_n^+} - \xi(S)$.

Lemma 23 If $d \in \mathbb{Z}$, $X_o^d = (b_{ij})$, $X_o^{d+1} = (c_{ij})$, and $a_{rp}, b_{pq} \neq 0$ for some $p \in \langle 1,n \rangle$, then $c_{rq} \neq 0$.

Proof If $\ell \in \mathbb{Z}$, and $(i,j) \in \langle 1,n \rangle \times \langle 1,n \rangle$ then put

$$f_{ij}^{\ell} = \sum_{m_1,m_2,\dots m_{\ell-1}=1}^{n} \mu^{\ell}_{im_1m_2\dots m_{\ell-1}j} X_{im_1} X_{m_1m_2} \dots X_{m_{\ell-1}j} ,$$

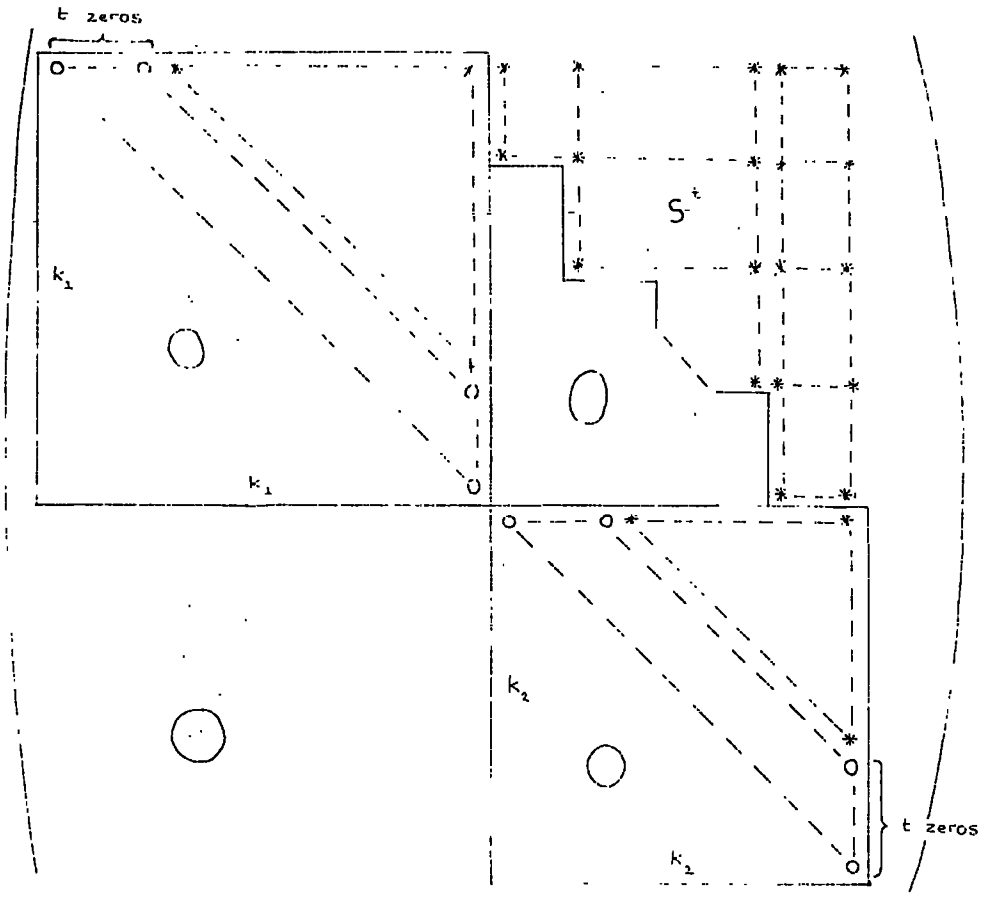
where $\{X_{11}, X_{12}, \dots, X_{nn}\}$ is a set of n^2 independent indeterminants, and

$$\mu^{\ell}_{im_1m_2\dots m_{\ell-1}j} = \begin{cases} 0 & \text{if } 0 \in \{a_{im_1}, a_{m_1m_2}, \dots, a_{m_{\ell-1}j}\} . \\ 1 & \text{otherwise} \end{cases}$$

f_{ij}^{ℓ} is a polynomial in the variables X_{vw}, $(v,w) \in \Delta_n^+ - \xi(S)$, with coefficients in the prime field of K. Also $f_{ij}^{\ell}(X_o) = (X_o^{\ell})_{ij}$. Hence, since the non-zero coefficients of X_o are algebraically independent over the prime field of K, it is clear that $(X_o^{\ell})_{ij} = 0$ if and only if $f_{ij}^{\ell} = 0$.

Now, $b_{pq} \neq 0$ and thus $f_{pq}^{d} \neq 0$, i.e. $\exists\ m_1,m_2,\dots,m_{d-1} \in \langle 1,n \rangle$ such that $0 \notin \{a_{pm_1}, a_{m_1m_2},\dots,a_{m_{d-1}q}\}$. Further, $f_{rq}^{d+1} = \mu^{d+1}_{rpm_1m_2\dots m_{d-1}q} X_{rp}X_{pm_1} \dots X_{m_{d-1}q}$ + other terms. But $\mu^{d+1}_{rpm_1m_2\dots m_{d-1}q} \neq 0$ since $0 \notin \{a_{rp}, a_{pm_1},\dots,a_{m_{d-1}q}\}$. Hence $f_{rq}^{d+1} \neq 0$, and $c_{rq} \neq 0$.

Proposition 24 $X_o^t = (h_{ij}) \in \underline{u}_{\Delta_n^+ \smile (\xi(S^t) \cup R^t)}$, where $R^t = \{(i,j) \in \Delta_n^+ \mid 1 \leq i,j \leq k_1$ or $k_1 < i,j \leq n$, and $j - i \leq t - 1\}$. Further, if $(i,j) \in \Delta_n^+ - (\xi(S^t) \cup R^t)$, then $h_{ij} \neq 0$. i.e. X_o^t has the form



where the $*$'s represent non-zero elements.

<u>Proof</u> It is clear that the proposition is true when $t = 1$. We shall assume that it is true for $X_o^d = (b_{ij})$, and prove that it is true for $X_o^{d+1} = (c_{ij})$.

We need only look at the situation in the top right hand $k_1 \times k_2$ block of X_o^{d+1}, since, using Lemma 23, the situation in the two central blocks is easily taken care of.

(I) We begin by looking at columns $k_1 + 1$ of X_o^d and X_o^{d+1}. Either column $k_1 + 1$ of X_o^d is zero, or there exists $i \in \mathbb{Z}$, $1 \leqslant i \leqslant k_1$, such that $b_{u\, k_1+1} \neq 0$ for $1 \leqslant u \leqslant i$, and $b_{u\, k_1+1} = 0$ for $i < u \leqslant n$. If column $k_1 + 1$ of X_o^d is zero, then column $k_1 + 1$ of X_o^{d+1} is zero. On the other hand, if i is as above, then:

(i) If $0 < u \leqslant i - 1$, then $a_{ui} \neq 0$. Also $b_{ik_1+1} \neq 0$, and thus $c_{uk_1+1} \neq 0$ (see Lemma 23).

(ii) If $u \geqslant i$, then $a_{uv} = 0$ for all $v < i$. Also $b_{vk_1+1} = 0$ for all $v > i$, and thus $c_{uk_1+1} = 0$.

(II) We now look at columns q and $q+1$ of X_o^d, $k_1 < q < n$, and derive the form of the first k_1 rows of column $q+1$ of X_o^{d+1}. Three possibilities arise for columns q and $q+1$ of X_o^d, i.e. they are of the form:

Case (i)		Col q	Col q+1	
		*	*	
		.	.	
		.	.	
		.	.	
	p	*	*	
		0	*	
$p \neq i$				
		.	.	
		.	.	
	i	0	*	
		0	0	
		.	.	
		.	.	
	k_1	0	0	
		*	*	If $q-k_1-d+1 \geqslant 0$ there are thus many non-zero entries. Otherwise all the entries are zero.
		.	.	
		.	.	
		*	*	
		0	*	
		0	0	
		.	.	
		.	.	
		0	0	

Note: we include the cases where $p = 0$ or $i = k_1$.

Case (ii)	Col q	Col q+1
	*	*
	.	.
	.	.
i	*	*
	0	0
	.	.
	.	.
k_1	0	0
	*	*
	.	.
	.	.
	.	.
	*	*
	0	*
	0	0
	.	.
	.	.
	.	.
	0	0

Same number of non-zero's as above

Note: We include the case where $i = k_1$, but not where $i = 0$.

Case (iii)	Col q	Col q+1
	0	0
	.	.
	.	.
	.	.
k_1	0	0
	*	*
	.	.
	.	.
	.	.
	*	*
	0	*
	0	0
	.	.
	.	.
	0	0

Same number of non-zero entries as above

We note that if $I_r \leq i < I_{r+1}$, then row i of X_o has the form

$$\begin{matrix} 0 & 0 & . & . & 0 & * & . & . & * & 0 & . & . & 0 & * & . & . & * \\ & & & & i & & & & k_1 & & & & J_r & & & & \end{matrix}$$

Hence, if $i \leq u \leq k_1$ and $J_r - k_1 \geq q - k_1 - d + 1$, i.e. $q - d \leq J_r - 1$, then $c_{uq+1} = 0$. Further, if $q - d \geq J_r$ then $c_{iq+1} \neq 0$.

Case (i)

(A) If $1 \leq u < i$, then $a_{ui} \neq 0$. Also $b_{i\ q+1} \neq 0$, and hence $c_{uq+1} \neq 0$ (see Lemma 23).

(B) Suppose that $I_r \leq i < I_{r+1}$. Then either $q = J_r$ or $(i,q) \varepsilon S - S^d$. If $q = J_r$, then it follows immediately that $q - d \leq J_r - 1$. On the other hand, if $(i,q) \varepsilon S - S^d$, then:

$$\begin{aligned} I_{r+1} - i + q - J_r &\leq d \quad \text{(see lemma 21)} \\ q - d &\leq J_r - (I_{r+1} - i) \\ &\leq J_r - 1 \end{aligned}$$

Thus we have that $c_{u\ q+1} = 0$ for all $i \leq u \leq k_1$.

Case (ii)

(A) If $1 \leq u < i$, then $c_{u\ q+1} \neq 0$ (see part A of case (i)).

Now suppose that $I_r \leq i < I_{r+1}$. $(i,q) \varepsilon S^d$ and $(i+1,q+1) \notin S^d$, and thus (see lemma 22) $i = I_{r+1} - 1$ and $q - d \geq J_r$. Hence:

(B) $c_{i\ q+1} \neq 0$

(C) Either $q \leq J_{r+1}$, or $(I_{r+1}, q) \varepsilon S - S^d$. If $q \leq J_{r+1}$, then it follows trivially that $q - d \leq J_{r+1} - 1$. On the other hand, if

$(I_{r+1}, q) \in S - S^d$, then:

$$I_{r+2} - I_{r+1} + q - J_{r+1} \leq d$$
$$q - d \leq J_{r+1} - (I_{r+2} - I_{r+1})$$
$$\leq J_{r+1} - 1$$

(note that if $(I_{r+1}, q) \in S$, then $r < p$).

Thus, if $i + 1 = I_{r+1} \leq u \leq k_1$, then $c_{u\ q+1} = 0$.

Case (iii)

(A) If $S = \emptyset$, then it is clear that $c_{u\ q+1} = 0$ for $1 \leq u \leq k_1$.

(B) Suppose that $S \neq \emptyset$, and let r be such that $I_r \leq 1 < I_{r+1}$ (note that $r = 0$ or 1). Either $q \leq J_r$ or $(1,q) \in S - S^d$. If $q < J_r$, then it follows trivially that $q - d \leq J_r - 1$. On the other hand, if $(1,q) \in S - S^d$, then:

$$I_{r+1} - 1 + q - J_r \leq d$$
$$q - d \leq J_r - (I_{r+1} - 1)$$
$$\leq J_r - 1.$$

Hence $c_{u\ q+1} = 0$ for all u, $1 \leq u \leq k_1$.

(III) We can summarise the results obtained in (I) and (II) above as follows: if $(i,j) \in S^{d+1}$, then $c_{ij} \neq 0$, and if $(i,j) \in \xi(S^{d+1})$, then $c_{ij} = 0$.

Thus the theorem is true for X_o^{d+1}. The proof is completed by induction.

Corollary 25

If $S \not\subseteq \tilde{S} = \{(i,j) \mid 1 \leq i \leq k_1,\ k_1 < j \leq n,\ \text{and}\ j - i > k_2\}$, then $X_o^{k_1} \neq 0$.

Proof

If $S \neq \tilde{S}$, then there exists $(i,j) \in S$ such that $j - i = k_2$. Suppose that $I_r < i < I_{r+1}$. If $X_o^{k_1} = 0$, then (by Proposition 24) $(1,n) \in S - S^{k_1}$, and thus (by Lemma 21) $I_{r+1} - 1 + n - J_r \leq k_1$.

But $J_r < j$ and $I_{r+1} - 1 \geq i$.

$\therefore I_{r+1} - 1 + n - J_r > i + n - j = k_1$.

Thus, we get a contradiction, and conclude that $X_o^{k_1} \neq 0$.

Having obtained the main results of this section, we go on to prove Lemma 26 and Corollary 27. We will need the latter later on.

Lemma 26 Let $X = (a_{ij})$ be an $m \times n$ matrix such that $a_{ij} = 0$ whenever $i - j \geq m - n$, and let $Y = (b_{ij})$ be an $n \times p$ matrix such that $b_{ij} = 0$ whenever $i - j \geq n - p - t$, where t is a non- negative integer. Then $XY = (c_{ij})$, where $c_{ij} = 0$ whenever $i - j \geq m - p - t - 1$.

Proof If $i \geq j + m - p - t - 1$, then $a_{ir} = 0$ whenever

$r \leq j + m - p - t - 1 - (m - n)$

$= j + n - p - t - 1$

and $b_{rj} = 0$ whenver $r \geq j + n - p - t$.

$$\therefore c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj} = 0.$$

Corollary 27

If $p \in \langle 1, \ell \rangle$, then let $X_p = (a_{ij}^{p})$ be an $m_p \times n_p$ matrix, and suppose that:

(i) $m_{p+1} = n_p$ for $p = 1 \ldots, \ell-1$.

(ii) $a_{ij}^{\ p} = 0$ whenever $i - j \geq m_p - n_p$.

Then $X_1 X_2 \ldots X_\ell = (c_{ij})$ where $c_{ij} = 0$ whenever $i - j \geq m_1 - m_\ell - \ell + 1$.

Proof Use induction.

3.2.2 The subset N_u of S_n

Let $(k) = (k_1, k_2, \ldots, k_r)$ be an ordered partition of n, and put $U_{(k)} = I + N_{k_1} \oplus N_{k_2} \oplus \ldots \oplus N_{k_r}$. We will use u to denote $U_{(k)}$ when there is no possibility of confusion. Our aim in this section is to calculate $|N_u|$, where $N_u = \{w \in S_n \mid \eta(w) = C(u)$ and $\tilde{C}_w \cap \beta_u \neq \emptyset\}$.

Lemma 28

(i) If $w \in S_n$, then $\beta_u \cap \tilde{C}_w \neq \emptyset \iff u \in U_w^{\ +}$.

(ii) There exists a unique element w_o in N_u such that $U_{w_o}^{\ +} \subseteq U_w^{\ +}$ for all $w \in N_u$.

(iii) If $w \in N_u$, then $A_w^{\ -} \cap \pi = \{(i, i+1) \mid i \in \{k_1, k_1 + k_2, \ldots, k_1 + \ldots + k_{r-1}\}\}$

Proof

(i) Recall that if $w \in S_n$, then $\tau_w\colon U_w^{\ -} \times U_w^{\ +} \to U$ is defined as follows:

$\tau_w((v,a)) = vav^{-1}$ for all $(v,a) \in U_w^{\ -} \times U_w^{\ +}$.

Also, recall that $\widetilde{C}_w \cap \beta_u \neq \emptyset$ if and only if $u \in \mathrm{Im}(\tau_w)$ (see lemma 14).

(a) Suppose that $u = \tau_w((v,a))$, where $v = (v_{ij}) \in U_w^-$ and $a = (a_{ij}) \in U_w^+$. Put $v^{-1} = (c_{ij})$.

$$(v.v^{-1})_{i\,i+1} = \sum_{p=1}^{n} v_{ip}\, c_{p\,i+1}$$

$$= v_{ii}\, c_{ii+1} + v_{ii+1}\, c_{i+1i+1}$$

$$= c_{ii+1} + v_{ii+1}$$

(i.e. $v_{ip} = 0$ whenever $i > p$, $c_{pi+1} = 0$ whenever $p > i+1$, $v_{ii} = 1$ and $c_{i+1i+1} = 1$).

Hence $c_{ii+1} + v_{ii+1} = 0$ for $i = 1,\dots,n-1$.

Now:

$$(vav^{-1})_{ii+1} = \sum_{p,q=1}^{n} v_{ip}\, a_{pq}\, c_{qi+1}$$

$$= \sum_{i\leq p\leq q\leq i+1} v_{ip}\, a_{pq}\, c_{q\,i+1}$$

$$= v_{ii}\, a_{ii}\, c_{i\,i+1} + v_{ii}\, a_{i\,i+1}\, c_{i+1i+1} + v_{ii+1}\, a_{i+1i+1}\, c_{i+1i+1}$$

$$= c_{i\,i+1} + a_{i\,i+1} + v_{i\,i+1}$$

$$= a_{i\,i+1}.$$

Hence, since $u = vav^{-1}$, we have that $a_{i\,i+1} \neq 0$ if and only if $i \in R = \langle 1,n\rangle - \{k_1, k_1 + k_2, \dots, k_1 + \dots + k_r\}$. Thus $D = \{(i,i+1) \mid i \in R\} \subseteq A_w^+$.

$\therefore\ u \in U_w^+$.

(b) If $u \in U_w^+$ then it is clear that $u \in \mathrm{Im}(\tau_w)$.

(ii) Let w_o be the element of S_n described in Proposition 19. If $w \in N_u$, then $D \subseteq A_w^+$ (see the proof of part (i) above). But $A_{w_o}^+ = \mathbb{Z}D \cap \phi^+$ (see page 7 for the definition of $\mathbb{Z}D$), and thus $A_{w_o}^+ \subseteq A_w^+$ (A_w^+ being a closed subset of ϕ^+). $\therefore$ $U_{w_o}^+ \subseteq U_w^+$.

It is clear that w_o is unique.

(iii) If $w \in N_u$, then $D \subsetneq A_w^+ \cap \pi$ (see i above). Also, if X_o is a generic element of U_w^+, then $C(X_o) = C(u)$, and thus rank $(X_o - I)^i$ = rank $(u - I)^i$ for $i = 1, 2, \ldots, n$. This is obviously false if $\pi - D \not\subseteq A_w^- \cap \pi$. The result follows immediately.

Lemma 29

If $w \in N_u$, $k_o = 0$, and $(i,j) \in A_w^-$, where $k_o + k_1 + \ldots + k_{s-1} < i \leq k_o + k_1 + \ldots + k_s$ and $k_o + \ldots + k_{t-1} < j \leq k_o + \ldots + k_t$ for some $(s,t) \in \Delta_r^+$, then $(\ell,m) \in A_w^-$ whenever $i \leq \ell \leq k_o + \ldots + k_s$ and $k_o + \ldots + k_{t-1} < m \leq j$.

Proof

We will let α_i denote $\alpha_{i\,i+1}$.

(i,j) corresponds to $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{k_o+\ldots+k_s} + \ldots + \alpha_{k_o+\ldots+k_{t-1}} + \ldots + \alpha_{j-1}$.

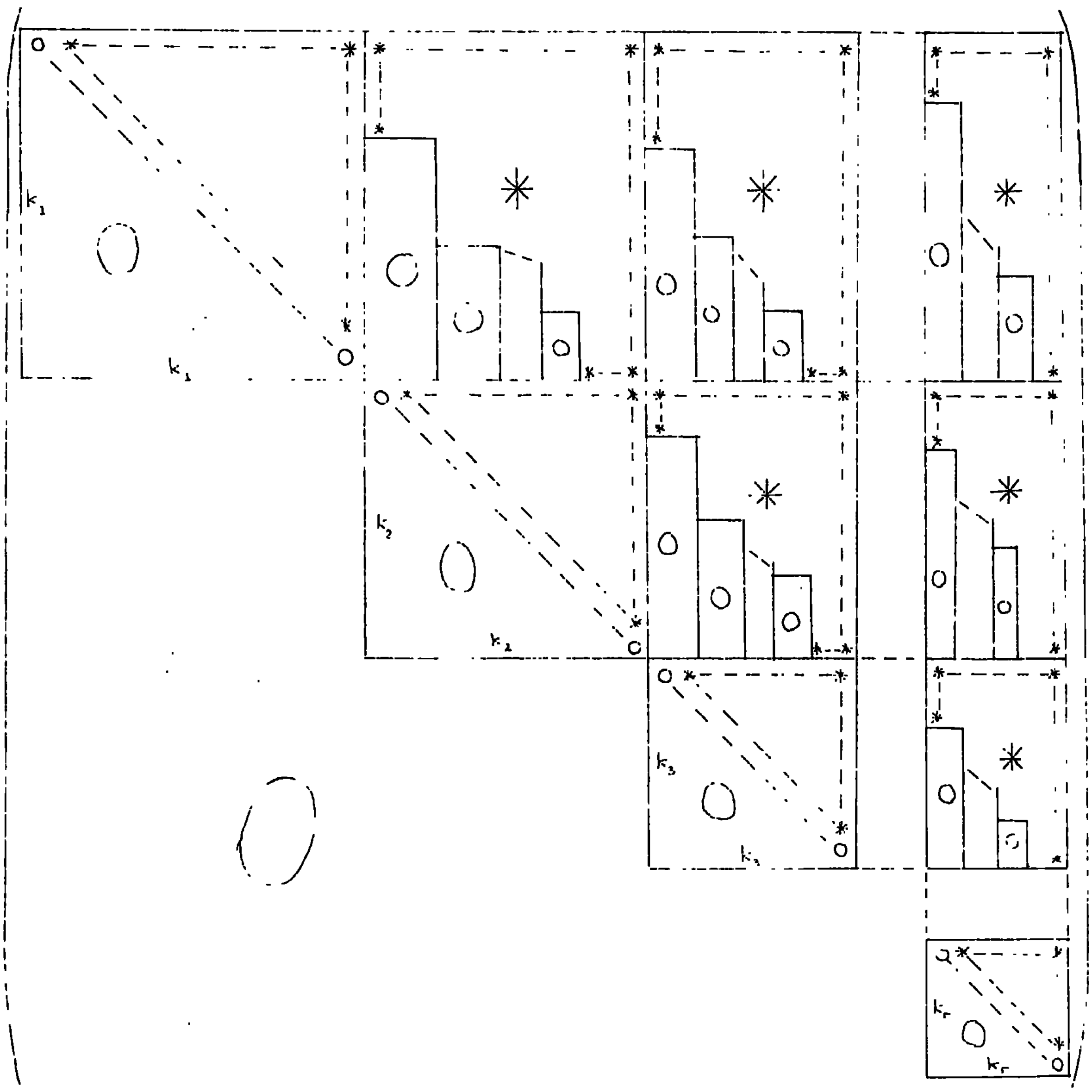
If ℓ and m are as above then

$$\alpha_i + \ldots + \alpha_{\ell-1},\ \alpha_m + \ldots + \alpha_{j-1} \in A_w^+$$

(see Lemma 28(iii)). Thus, since A_w^+ is closed, it follows that if $\alpha_\ell + \ldots + \alpha_{k_o+\ldots+k_s} + \ldots + \alpha_{k_o+\ldots+k_{t-1}} + \ldots + \alpha_{m-1} \in A_w^+$, then $\alpha_{ij} \in A_w^+$. $\therefore$ $(\ell,m) \in A_w^-$.

We now need to extend our definition of triangular subsets. If s and t, $s \geq t$, are positive integers, then a subset P, of $\langle 1,s\rangle \times \langle 1,t\rangle$ is said to be <u>triangular</u>, if $(i,j) \in P$ implies that $(i-1,j)$, $(i,j+1) \in P$. Also, we say that an $s \times t$ matrix $A = (a_{ij})$, is <u>triangular</u> if there exists a triangular subset P of $\langle 1,s\rangle \times \langle 1,t\rangle$ such that $a_{ij} \neq 0 \Longleftrightarrow (i,j) \in P$.

Let $w \in N_u$, and X_o be a generic element of U_w^+. Then, by Lemmas 28 and 29, it is clear that $(X_o - 1)$ has the form:



where the *'s represent non-zero entries i.e. If we divide $(X_o - 1)$ up into blocks A_{IJ}, $(I,J) \in \langle 1,r\rangle \times \langle 1,r\rangle$, in the obvious way, then:

(i) $A_{IJ} = 0$ if $I > J$

(ii) $A_{II} = (a_{ij}(I))$, where $a_{ij}(I) = 0$ if and only if $i \geq j$.

(iii) If $I < J$, then $A_{IJ} = (a_{ij}(I,J))$ is a triangular matrix.

We now define a subset D of Δ_n^+ as follows:

If $(I,J) \in \Delta_r^+$, then put

$D^{IJ} = \{(i,j) \mid k_o + \ldots + k_{I-1} < i \leq k_o + \ldots + k_I,$

$k_o + \ldots + k_{J-1} < j \leq k_o + \ldots + k_J,$ and $j - i \leq k_{I+1} + \ldots + k_J\}$.

$$D = \bigcup_{(I,J) \in \Delta_r^+} D^{IJ}.$$

Lemma 30

There exists a unique element $w_1 \in S_n$ such that $U_{w_1}^+ = U_{\Delta_n^+} - D$.

Proof We need to show that D and $\Delta_n^+ - D$ are closed subsets of Δ_n^+ (see Lemma 18).

(i) Suppose that (i,p) and (p,j) are elements of D. Then:

(a) $\exists$ $(I,P) \in \Delta_r^+$ such that $k_o + \ldots + k_{I-1} < i \leq k_o + \ldots + k_I$,

$k_o + \ldots + k_{P-1} < p \leq k_o + \ldots + k_P$, and

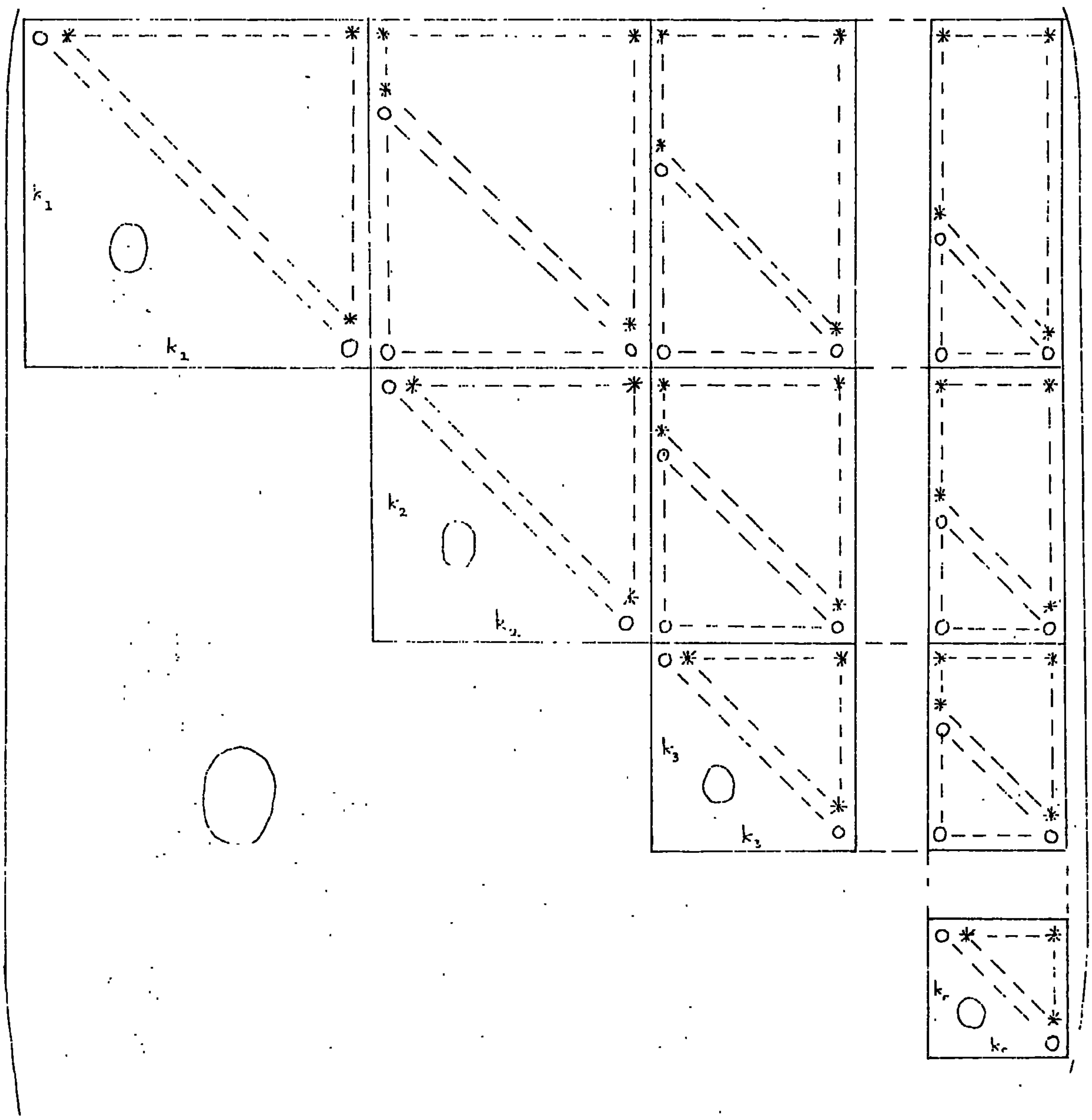
$p - i \leq k_{I+1} + \ldots + k_P$.

(b) $\exists$ J, $P < J \leq r$, such that $k_o + \ldots + k_{J-1} < j \leq k_o + \ldots + k_J$, and $j - p \leq k_{P+1} + \ldots + k_J$.

It is clear (see a and b above) that $j - i \leq k_{I+1} + \ldots + k_J$. Hence, it is easy to see that $(i,j) \in D$. Thus D is closed.

(ii) Similarly, $\Delta_n^+ \sim D$ is closed.

If X_o is a generic element of $U_{w_1}^+$, then $X_o - 1$ has the form:



where the *'s represent non-zero entries.

Proposition 31

$$w \in N_u \Longleftrightarrow U_{w_o}^+ \subseteq U_w^+ \subseteq U_{w_1}^+$$

Proof

(I) Let $w \in N_u$. Then, by Lemma 28, $U_{w_o}^+ \subseteq U_w^+$.

To show that $U_w^+ \subseteq U_{w_1}^+$, we need to show that $A_{w_1}^- \subseteq A_w^-$.

If $(I,J) \in \Delta_r^+$, then put $E^{IJ} = \{(i,j) \in A_w^- \mid k_o + \ldots + k_{I-1} < i \leqslant k_o + \ldots + k_I$, and $k_o + \ldots + k_{J-1} < j \leqslant k_o + \ldots + k_J\}$. It is clear that A_w^- is the disjoint union of the sets E^{IJ}. We need to show that $D^{IJ} \subseteq E^{IJ}$, for all $(I,J) \in \Delta_r^+$.

(A) If (I,P), $(P,J) \in \Delta_r^+$, $D^{IP} \subseteq E^{IP}$ and $D^{PJ} \subseteq E^{PJ}$, then $D^{IJ} \subseteq E^{IJ}$.

i.e. If $(i,j) \in D^{IJ}$, then

$k_o + k_1 + \ldots + k_{I-1} < i \leqslant k_o + \ldots + k_I$,

$k_o + k_1 + \ldots + k_{J-1} < j \leqslant k_o + \ldots + k_J$,

and $j - i \leqslant k_{I+1} + \ldots + k_J$.

$\therefore \quad k_o + \ldots + k_{J-1} - i < j - i \leqslant k_{I+1} + \ldots + k_J$

$\therefore \quad k_o + \ldots + k_{J-1} - (k_{I+1} + \ldots + k_J) < i$

$\therefore \quad k_o + \ldots + k_I - k_J < i$

But $k_J \leqslant k_P$, and thus

$k_o + \ldots + k_I - k_P < i \leqslant k_o + \ldots + k_I$.

$\therefore k_o + \ldots + k_I - k_P + k_{I+1} + \ldots + K_P$

$< i + k_{I+1} + \ldots + k_P \leqslant k_o + \ldots + k_P$

$\therefore k_o + \ldots + k_{P-1} < i + k_{I+1} + \ldots + k_P \leqslant k_o + \ldots + k_P$

$\therefore (i, i + k_{I+1} + \ldots + k_P) \in D^{IP} \subseteq A_w^- \ldots$ (a).

Also, since $j - i \leqslant k_{I+1} + \dots + k_J$, it is clear that $j - (i + k_{I+1} + \dots + k_P) < k_{P+1} + \dots + k_J$. Hence

$(i + k_{I+1} + \dots + k_P, j) \in D^{PJ} \subseteq A_w^-$ . . . (b).

Now A_w^- is closed, and hence by a and b above, we have that $(i,j) \in A_w^-$.

$\therefore (i,j) \in E^{IJ}$.

(B) $D^{I\ I+1} \subseteq E^{I\ I+1}$ for $I = 1, \dots, r - 1$.

i.e. Suppose that $D^{I\ I+1} \not\subseteq E^{I\ I+1}$ for some $I \in \langle 1, r - 1\rangle$. Let X_o be a generic element of U_w^+, and divide $(X_o - 1)^{k_I}$ into blocks B_{PQ}, $(P,Q) \in \langle 1,r\rangle \times \langle 1,r\rangle$, in the obvious way. By Corollary 25, it is clear that $B_{I\ I+1} \neq 0$, and hence that rank $(X_o - 1)^{k_I} >$ rank $(u - 1)^{k_I}$. This is not possible.

(C) From A and B above, it is clear that $D^{IJ} \subseteq E^{IJ}$ for all $(I,J) \in \Delta_r^+$.

(II) Suppose that $U_{w_o}^+ \subseteq U_w^+ \subseteq U_{w_1}^+$.

$u \in U_{w_o}^+ \subseteq U_w^+$ and hence, by Lemma 28, $\mathcal{B}_u \cap \mathcal{C}_w \neq \emptyset$... (1)

Let $p \in \mathbb{Z}^+$, and X_o be a generic element of U_w^+. Partition $X_o - 1$ into blocks A_{IJ}, $(I,J) \in \langle 1,r\rangle \times \langle 1,r\rangle$, and $(X_o - 1)^p$ into blocks $B_{IJ} = (b_{ij}(I,J))$ -

$$B_{IJ} = \sum_{T_1, T_2, \dots, T_{p-1} = 1}^{n} A_{IT_1} A_{T_1 T_2} \dots A_{T_{p-1} J}.$$

Now: (a) If $I > J$, then $B_{IJ} = 0$

(b) $b_{ij}(I,I) = 0 \iff j - i < p - 1$ (cf. Proposition 24)

(c) If $I < J$, then

$$B_{IJ} = \sum_{I \leq T_1 \leq \ldots \leq T_{p-1} \leq J} A_{IT_1} A_{T_1 T_2} \cdots A_{T_{p-1} J}.$$

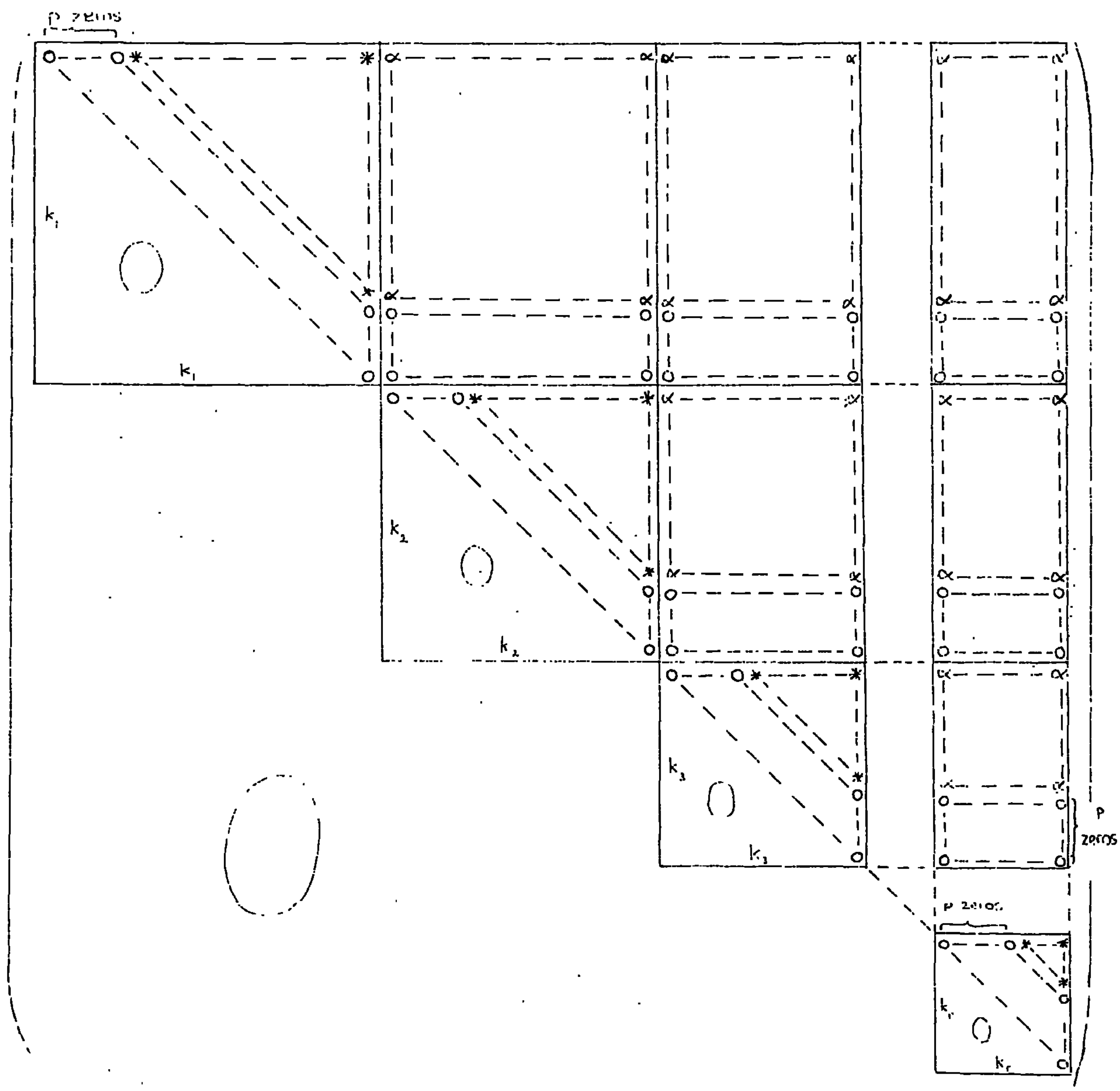
But $U_w^+ \subseteq U_{w_1}^+$, and hence: if $(P,Q) \in \Delta_r^+$, then

$A_{PQ} = (a_{ij}(P,Q))$, where $a_{ij}(P,Q) = 0$ whenever $i-j \geq k_P - k_Q$.

Thus $b_{ij}(I,J) = 0$ whenever $i-j \geq k_I - k_J - p+1$ (see Corollary 27).

In particular, we have that the last p rows of B_{IJ} are zero.

By (a), (b) and (c) above, it is clear that $(X_o - 1)^p$ has the form:



where the *'s represent non-zero elements, and the α's represent elements which may, or may not be zero.

Thus, it is easy to see that $\text{rank}(X_o - 1)^p = \text{rank}(u - 1)^p$. Hence $\eta(w) = C(u)\ldots$ (2)

$\therefore$ $w \in N_u$ (see (1) and (2) above).

3.2.3 Young's Diagrams

Let $(\ell) = (\ell_1, \ell_2, \ldots, \ell_s)$ be an ordered partition of n. Then we can associate with (ℓ) a Young's Diagram:

```
* * * . . . . . . . * . . . . . . . . . * . . . . . . . . .*   (ℓ1 nodes)
* * * . . . . . . . * . . . . . . . . . *                      (ℓ2 nodes)
. . .               .
. . .               .
. . .               .
. . .               .
* * * . . . . . . .*                                           (ℓs nodes)
```

Example The partition (4, 2, 1) has a Young's Diagram:

```
* * * *
* *
*
```

We say that a node is in the position (i,j) if it is in row i and column j.

If we place the numbers $1 \ldots n$ at the nodes, in any order, then we obtain a tableau.

Example

$$\begin{matrix} 4 & 1 & 3 & 2 \\ 5 & 6 & & \\ 7 & & & \end{matrix}$$

is a tableau for the partition (4, 2, 1).

Let ∇ be a tableau for the partition (ℓ), and let n_{ij} be the entry in the $(i,j)^{th}$ position. Then we say that ∇ is a standard tableau if $n_{ij} > n_{ij+1}, n_{i+1\,j}$.

Example

$$\begin{matrix} 7 & 6 & 3 & 2 \\ 5 & 1 & & \\ 4 & & & \end{matrix}$$

is a standard tableau for the partition (4, 2, 1).

Theorem 32

(i) There is a bijective correspondence between the ordered partitions of n and the irreducible representations of the group S_n. Further, the dimension, $d_{(\ell)}$, of the irreducible representation of S_n corresponding to the partition (ℓ) is equal to the number of standard tableaux for (ℓ).

(ii.) $$\sum_{\substack{\text{Ordered partitions} \\ (\ell) \text{ of } n}} d_{(\ell)}^{2} = |S_n|$$

<u>Proof</u> See (7).

3.2.4 The number of elements in N_u

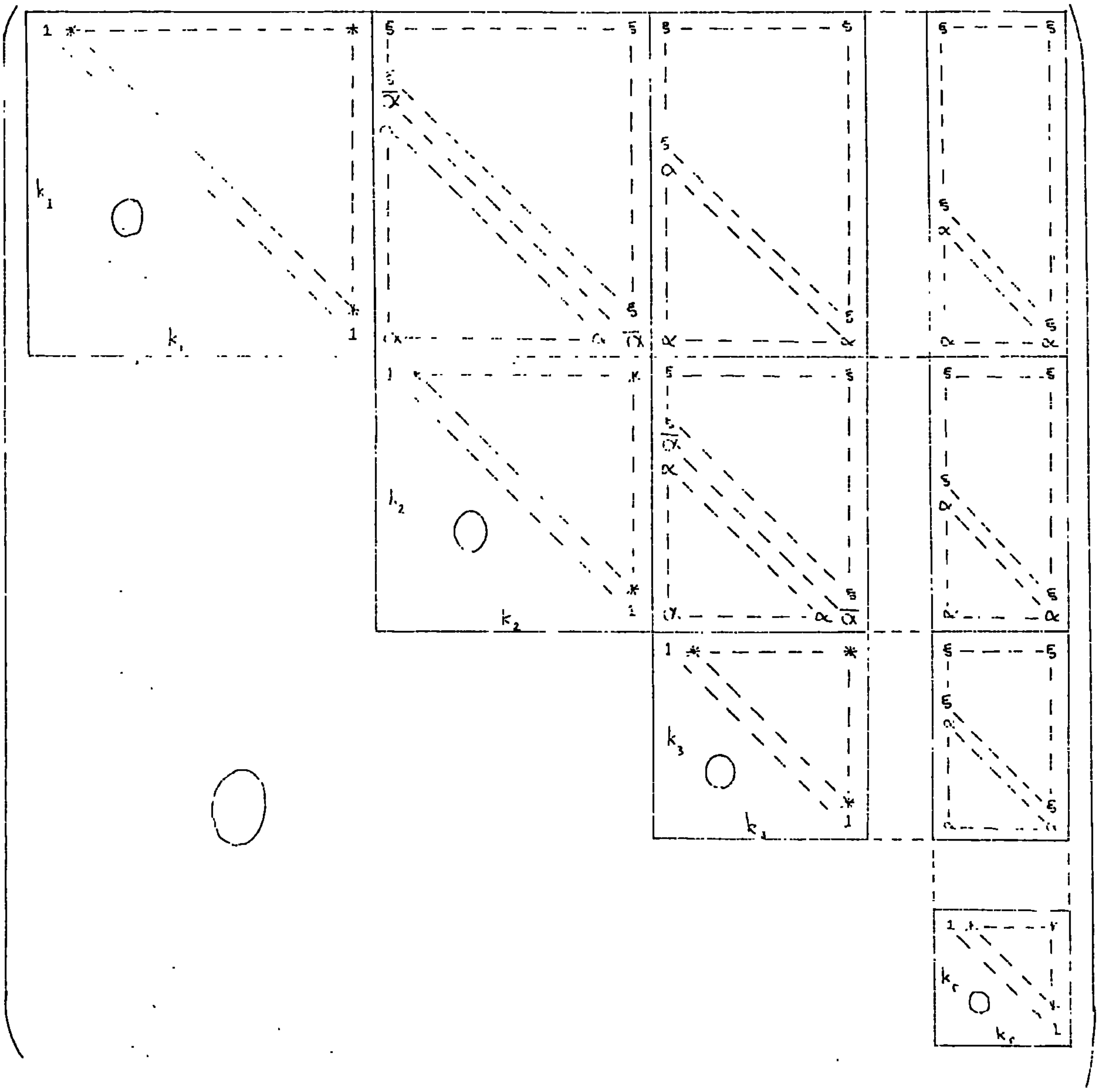
<u>Proposition 33</u>

$w \in N_u$ if and only if

$$\begin{array}{cccccccccc}
w^{-1}(k_1) & w^{-1}(k_1-1) & w^{-1}(k_1-2) & \cdots & w^{-1}(k_1+1-k_r) & \cdots & w^{-1}(k_1+1-k_2) & \cdots & w^{-1}(1) \\
w^{-1}(k_1+k_2) & w^{-1}(k_1+k_2-1) & w^{-1}(k_1+k_2-2) & \cdots & w^{-1}(k_1+k_2+1-k_r) & \cdots & w^{-1}(k_1+1) & & \\
\vdots & \vdots & \vdots & & \vdots & \swarrow & & & \\
w^{-1}(n) & w^{-1}(n-1) & w^{-1}(n-2) & \cdots & w^{-1}(k_2+\cdots+k_{r-1}+1) & & & &
\end{array}$$

is a standard tableau for (k).

<u>Proof</u> $w \in N_u \iff U_{w_0}^{+} \subseteq U_{w}^{+} \subseteq U_{w_1}^{+}$, i.e. a generic element of U_w^{+} has the form:



where the $*$'s represent non-zero entries, and thus elements of A_w^+; the α's and $\bar{\alpha}$'s represent zero entries, and thus elements of A_w^-; and the ξ's represent entries which may, or may not be, zero.

Note that by Lemma 29 and the fact that A_w^- is closed, a necessary and sufficient condition for the α's to represent elements of A_w^- is that the $\bar{\alpha}$'s represent elements of A_w^-.

So, we have that $w \in N_u \iff$

(i) $(i,i+1) \in A_w^+$ whenever $i \in \langle 1,n\rangle \sim \{k_1, k_1 + k_2, \ldots, k_1 + \ldots + k_r\}$;

and

(ii) $\forall I = 1,\ldots, r-1$, $(i, k_{I+1} + i) \in A_w^-$ whenever

$k_1 + \ldots + k_I - k_{I+1} < i \leqslant k_1 + \ldots + k_I$.

$\therefore w \in N_u \iff$

(i) $w^{-1}(i) < w^{-1}(i+1)$ whenever $i \in \langle 1\ n\rangle \sim \{k_1, k_1 + k_2, \ldots, k_1 + \ldots + k_r\}$

and

(ii) $\forall I = 1,\ldots, r-1$, $w^{-1}(i) > w^{-1}(k_{I+1} + i)$ whenever

$k_1 + \ldots + k_I - k_{I+1} < i \leqslant k_1 + \ldots + k_I$.

The result now follows easily.

<u>Theorem 34</u>

$|N_{U_{(k)}}| = d_{(k)}$ (recall that in the above work, we simplified our notation by writing u for $U_{(k)}$).

<u>Proof</u> The result follows immediately from theorem 32 and proposition 33.

<u>Theorem 35</u>

Let $G = SL(n,K)$, and (k) be an ordered partition of n. Then the number of irreducible components of $\beta_{U_{(k)}}$ of maximal dimension (see pages 29 and 64 for the notation) is equal to $d_{(k)}$.

Proof

If $w \in N_{U_{(k)}}$, then $\dim \tilde{C}_w \cap \beta_{U_{(k)}} = \dim \beta_{U_{(k)}}$ (see lemma 15).

Thus, if X is an irreducible component of $\tilde{C}_w \cap \beta_{U_{(k)}}$ of maximal dimension, then $\overline{X}$ is an irreducible component of $\beta_{U_{(k)}}$ of maximal dimension. Also, if $w' \in N_{U_{(k)}}$, $w \neq w'$, and X' is an irreducible component of $\tilde{C}_{w'} \cap \beta_{U_{(k)}}$ of maximal dimension, then $X \neq X'$. Thus

$$|N_{U_{(k)}}| \leq n_{U_{(k)}}, \quad \text{i.e.} \quad d_{(k)} \leq n_{U_{(k)}}.$$

But $\sum_{\text{partitions }(k)} d_{(k)}^2 = |S_n|$ (see theorem 32), and

$\sum_{\text{partitions }(k)} n_{U_{(k)}}^2 = |S_n|$ (see corollary 13) . $\therefore$ $n_{U_{(k)}} = d_{(k)}$.

CHAPTER 4

SO(N,K) AND SP(N,K)

In this chapter we will show that our basic assumption (see 2.2.3) is true for the classical groups $SO(n,K)$ and $Sp(n,K)$, where K has infinite transcendence degree over its prime field, and $char(K) \neq 2$. To achieve this we just combine the work of Carter and Bala, and M. Gerstenhaber.

<u>4.1 BACKGROUND</u>

<u>4.1.1 Distinguished diagrams of type B_ℓ, C_ℓ and D_ℓ</u> (see 3)

In the following we shall assume that all the partitions are ordered (see 3.1.2).

Let $(k) = (k_1, k_2, \ldots, k_r)$ be a partition of n, and put $(k)^* = (\lambda_1, \lambda_2, \ldots, \lambda_s)$, where $\lambda_i = |\{k_j \mid k_j \geq i\}|$. $(k)^*$ is a partition of n, and it is called the <u>duol</u> of (k).

We note that $(k)^{**} = (k)$.

<u>Example</u> The partition $(4, 3, 2, 2)$ has Young's Diagram

```
*  *  *  *
*  *  *
*  *
*  *
```

Reading off the columns we see that $(k)^* = (4, 4, 2, 1)$

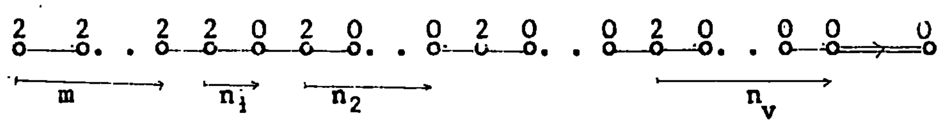
If $p \in \mathbb{Z}^+$, and $z_1, z_2, \ldots, z_p$ are non-negative integers, then we will use $(p^{z_p}, (p-1)^{z_{p-1}}, \ldots, 1^{z_1})$ to denote the partition

$$(\underbrace{p,\ldots,p}_{z_p}, \underbrace{(p-1),\ldots,(p-1)}_{z_{p-1}},\ldots, \underbrace{1,\ldots,1}_{z_1}), \quad \text{of} \quad \sum_{i=1}^{p} i\, z_i .$$

If $(k) = (k_1, k_2, \ldots, k_r)$, then $(k)^* = (r^{k_r}, (r-1)^{k_{r-1}-k_r}, (r-2)^{k_{r-2}-k_{r-1}}, \ldots, 1^{k_1-k_2})$.

(A) <u>The distinguished diagrams of type B_ℓ have the form:</u>

(I) 

(II) 

,

where m and the n_i are obtained as follows:

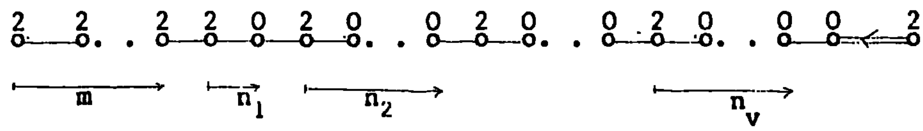
Let $(k) = (k_1, k_2, \ldots, k_r)$, $r > 1$, be a partition of $2\ell + 1$ into distinct odd parts, and put $\lambda_i = \dfrac{k_i - 1}{2}$ for $i = 1,\ldots, r$. If $\lambda_r \neq 0$, then put $(\lambda) = (\lambda_1,\ldots,\lambda_r)$, and if $\lambda_r = 0$, then put $(\lambda) = (\lambda_1,\ldots,\lambda_{r-1})$. $(\lambda)^* = (n_{v-1}, n_{v-2},\ldots,n_1, \underbrace{1,\ldots,1}_{m+1})$, and

$$n_v = \begin{cases} \dfrac{n_{v-1}}{2} & \text{if } n_{v-1} \text{ is even (i.e. if } \lambda_r = 0) \\ \dfrac{n_{v-1} - 1}{2} & \text{if } n_{v-1} \text{ is odd (i.e. if } \lambda_r \neq 0) \end{cases}$$

(Note that $n_i \geqslant n_{i-1}$ for $i = 2,\ldots, v - 1$).

Distinct partitions, $(k) = (k_1,\ldots,k_r)$, $r > 1$, give rise to distinct diagrams of type II, and we associate the diagram (I) with the partition $(2\ell + 1)$.

(B) <u>The distinguished diagrams of type C_ℓ are of the form:</u>

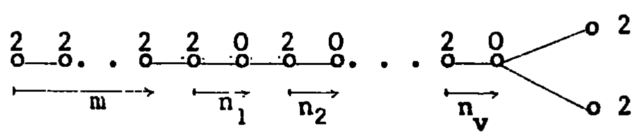


where m and the n_i are obtained as follows:

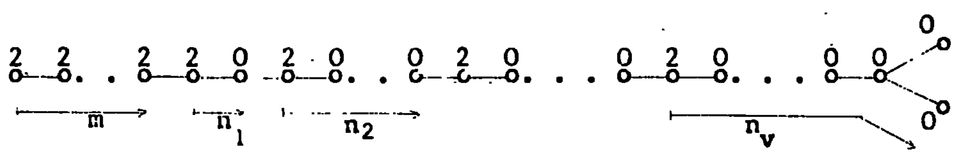
Let $(k) = (k_1, k_2, \dots, k_r)$ be a partition of ℓ into distinct parts, then $(k)^* = (n_v, n_{v-1}, \dots, n_1, \underbrace{1, \dots, 1}_{m+1})$. Each partition, (k), gives rise to a distinct diagram. Note that

$n_i \geqslant n_{i-1}$ for $i = 2, 3, \dots, v$.

(C) <u>The distinguished diagrams of type D_ℓ are of the form:</u>

I) 

where all the n_i's are equal to 2, m is odd if ℓ is odd, and m is even if ℓ is even.

II) 

where m and the n_i are obtained as follows:

Let $(k) = (k_1, k_2, \dots, k_r)$, $r \geqslant 4$, be a partition of 2ℓ into distinct odd parts, and put $\lambda_i = \frac{k_i - 1}{2}$. If $\lambda_r \neq 0$, then put

$(\lambda) = (\lambda_1, \dots, \lambda_r)$, and if $\lambda_r = 0$, then put $(\lambda) = (\lambda_1, \dots, \lambda_{r-1})$.

$(\lambda)^* = (n_{v-1}, n_{v-2}, \dots, n_1, \underbrace{1, \dots, 1}_{m+1})$, and

$$n_v = \begin{cases} \dfrac{n_{v-1} + 1}{2} & \text{if } n_{v-1} \text{ is odd (i.e. if } \lambda_r = 0) \\ \dfrac{n_{v-1}}{2} & \text{if } n_{v-1} \text{ is even (i.e. if } \lambda_r \neq 0) \end{cases}$$

(Note that $n_i \geqslant n_{i-1}$ for $i = 2, \dots, v-1$)

Each partition gives rise to a distinct diagram.

We can associate to each partition (k_1, k_2) of 2ℓ into distinct odd parts a diagram of type I, i.e. if $\lambda_i = \frac{k_i - 1}{2}$, $i = 1, 2$, $(\lambda) = (\lambda_1, \lambda_2)$ if $\lambda_2 \neq 0$, and $(\lambda) = (\lambda_1)$ if $\lambda_2 = 0$, then $(\lambda)^* = (n_v, n_{v-1}, \ldots, n_1, \underbrace{1, \ldots, 1}_{m+1})$. The correspondence thus obtained is bijective.

4.1.2 Description of the groups SO(n,K) and Sp(n,K) See (5)

Let E be a vector space of dimension n over K, and let $\{e_1, e_2, \ldots, e_n\}$ be a basis of E. We can define a non-degenerate, symmetric bilinear form on E by putting $(e_i, e_j) = 1$ if $i + j = n + 1$, and $(e_i, e_j) = 0$ otherwise. $M_n(K)$, the set of $n \times n$ matrices with coefficients in K, acts on E with respect to the basis $\{e_1, e_2, \ldots, e_n\}$ in the usual way.

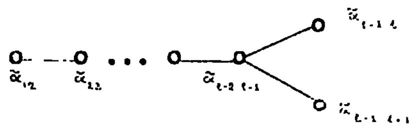
$SO(n,K) = \{X \in M_n(K) \mid (Xv, Xw) = (v,w)$ for all $v, w \in E$, and $\det X = 1\}$ is a quasi-simple algebraic group of type $\begin{cases} B_\ell & \text{if } n = 2\ell + 1 \\ D_\ell & \text{if } n = 2\ell \end{cases}$

If $X = (a_{ij})$, and $\sigma = (i,j) \in \langle 1,n \rangle \times \langle 1,n \rangle$, then we write $X_\sigma = a_{ij}$. Let $\Gamma: \langle 1,n \rangle \times \langle 1,n \rangle \to \langle 1,n \rangle \times \langle 1,n \rangle$ be given by $\Gamma((i,j)) = (n+1-j, n+1-i)$, and if $X \in M_n(K)$, then let X^Γ be the element of $M_n(K)$ such that $(X^\Gamma)_\sigma = X_{\Gamma(\sigma)}$. $SO(n,K) = \{X \in M_n(K) \mid X.X^\Gamma = I$ and $\det X = 1\}$.

Let T_o be the maximal torus consisting of the diagonal matrices in $G = SO(n,K)$, B_o be the Borel Subgroup of upper triangular matrices in G, and if $(i,j) \in \Delta_n^+$ and $i+j < n+1$, then let $\tilde{\alpha}_{ij}: T_o \to G_m$ be defined by $\tilde{\alpha}_{ij}\{\mathrm{diag}(a_1, \ldots, a_n)\} = a_i/a_j$. Then $\Phi_o = \Phi(G, T_o) = \{\tilde{\alpha}_{ij} \mid (i,j) \in \Delta_n$ and $i+j < n+1\}$, $\Phi_o^+ = \Phi(B_o, T_o) = \{\tilde{\alpha}_{ij} \mid (i,j) \in \Delta_n^+$ and $i+j < n+1\}$, and the corresponding basis is

$$\pi_o = \begin{cases} \{\tilde\alpha_{12}, \tilde\alpha_{23}, \ldots, \tilde\alpha_{\ell\ \ell+1}\} & \text{if } n = 2\ell + 1 \\ \{\tilde\alpha_{12}, \tilde\alpha_{23}, \ldots, \tilde\alpha_{\ell-1\ \ell}, \tilde\alpha_{\ell-1\,\ell+1}\} & \text{if } n = 2\ell. \end{cases}$$

The Dynkin Diagram for $SO(2\ell + 1, K)$ is , and the Dynkin Diagram for $SO(2\ell, K)$ is



$so(n,K) = \{X \in M_n(K) \mid (Xv,w) + (v,Xw) = 0$ for all $v, w \in E\}$ $= \{X \in M_n(K) \mid X + X^\Gamma = 0\}$ is the Lie algebra of $SO(n,K)$.

<u>Example</u> $so(5,K)$ is the set of matrices of the form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ a_{21} & a_{22} & a_{23} & 0 & -a_{14} \\ a_{31} & a_{32} & 0 & -a_{23} & -a_{13} \\ a_{41} & 0 & -a_{32} & -a_{22} & -a_{21} \\ 0 & -a_{41} & -a_{31} & -a_{21} & -a_{11} \end{pmatrix}$$

Recall that if $(i,j) \in \Delta_n$, then E_{ij} is the $n \times n$ matrix with a 1 in the $(i,j)^{th}$ position and zeros elsewhere. $R_{ij} = E_{ij} - E_{ij}^{\ \Gamma}$, $(i,j) \in \Delta_n$ and $i + j < n + 1$, is a root vector corresponding to the root $\tilde\alpha_{ij}$, and $\tilde\varepsilon_{ij}: G_a \to SO(n,K)$, $\tilde\varepsilon_{ij}(t) = I + tR_{ij} + \frac{t^2}{2} R_{ij}^{\ 2}$, is the one parameter subgroup corresponding to $\tilde\alpha_{ij}$.

<u>Note</u>: If $n = 2\ell$ then $R_{ij}^{\ 2} = 0$ (we are of course assuming that $(i,j) \in \Delta_n$ and that $i + j < n + 1$).

On the other hand if $n = 2\ell + 1$ then:

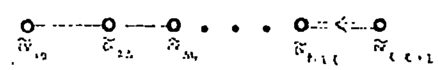
(i) $R_{i\,\ell+1}^{\;2} = -E_{i\,n+1-i}$ and $R_{\ell+1\,j}^{\;2} = -E_{n+1-j\,j}$.

(ii) $R_{ij}^{\;2} = 0$ if $i \neq \ell + 1$ and $j \neq \ell + 1$.

Now suppose that $n = 2\ell$, and define a skew symmetric bilinear form on E by putting $(e_i, e_j) = 1$ if $i + j = 2\ell + 1$ and $1 \leqslant i \leqslant \ell$, $(e_i, e_j) = -1$ if $i + j = 2\ell + 1$ and $\ell < i \leqslant 2\ell$, and $(e_i, e_j) = 0$ otherwise. $Sp(n,K) = \{X \varepsilon M_n(K) \mid (Xv, Xw) = (v,w) \quad v, w \varepsilon E\}$ is a quasi-simple group of type C_ℓ, it is called the <u>symplectic group</u>.

Let I_ℓ be the $\ell \times \ell$ unit matrix and put $\Lambda = I_\ell \oplus (-I_\ell)$. If $X \varepsilon M_n(K)$, then put $X^{\Gamma'} = \Lambda X^{\Gamma} \Lambda$. $Sp(n,K) = \{X \varepsilon M_n(K) \mid X^{\Gamma'}.X = I\}$.

Let T_o be the maximal torus consisting of the diagonal matrices in $G = Sp(n,K)$, B_o be the Borel Subgroup of upper triangular matrices in G, and if $(i,j) \varepsilon \Delta_n^+$ and $i + j \leqslant n + 1$, then let $\tilde{\alpha}_{ij} : T_o \to G_m$ be defined by $\tilde{\alpha}_{ij}(\mathrm{diag}\,(a_1,\ldots,a_n)) = a_i/a_j$. Then $\Phi_o = \Phi(G,T_o) = \{\tilde{\alpha}_{ij} \mid (i,j) \varepsilon \Delta_n,\ i + j \leqslant n + 1\}$, $\Phi_o^+ = \Phi(B_o, T_o) = \{\tilde{\alpha}_{ij} \mid (i,j) \varepsilon \Delta_n^+,\ i + j \leqslant n + 1\}$, and the corresponding basis π_o is the set $\{\tilde{\alpha}_{12}, \tilde{\alpha}_{23}, \ldots, \tilde{\alpha}_{\ell\,\ell+1}\}$. $Sp(n,K)$ has a Dynkin Diagram



$sp(n,K) = \{X \varepsilon M_n(K) \mid (Xv,w) + (v,Xw) = 0$ for all $v,w \varepsilon E\} = \{X \varepsilon M_n(K) \mid X + X^{\Gamma'} = 0\}$ is the Lie algebra of $Sp(n,K)$.

Example $sp(6,K)$ is the set of matrices of the form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{15} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{24} & a_{14} \\ a_{41} & a_{42} & a_{43} & -a_{33} & -a_{23} & -a_{13} \\ a_{51} & a_{52} & a_{42} & -a_{32} & -a_{22} & -a_{12} \\ a_{61} & a_{51} & a_{41} & -a_{31} & -a_{21} & -a_{11} \end{pmatrix}$$

$R_{ij}' = E_{ij} - E_{ij}^{\Gamma'}$, $(i,j) \in \Delta_n$ and $i + j \leqslant n + 1$, is a root vector corresponding to the root $\tilde{\alpha}_{ij}$, and $\tilde{\varepsilon}_{ij}: G_a \to Sp(n,K)$, $\tilde{\varepsilon}_{ij}(t) = I + t R_{ij}'$ is the one parameter subgroup corresponding to $\tilde{\alpha}_{ij}$.

4.1.3 Partitions and Conjugacy Classes

First of all we note that if $A, B \in M_n(K)$, then:

$$\begin{aligned} (A\,B)^{\Gamma}{}_{ij} &= \sum_{p=1}^{n} A_{n+1-j\;p} \; B_{p\;n+1-i} \\ &= \sum_{q=1}^{n} A_{n+1-j\;n+1-q} \; B_{n+1-q\;n+1-i} \\ &= \sum_{q=1}^{n} (B^{\Gamma})_{iq} \; (A^{\Gamma})_{qj} \\ &= (B^{\Gamma} A^{\Gamma})_{ij} \end{aligned}$$

$$\therefore \; (A\,B)^{\Gamma} = B^{\Gamma} A^{\Gamma}.$$

Also, $$\begin{aligned} (A\,B)^{\Gamma'} &= \Lambda (A\,B)^{\Gamma} \Lambda = \Lambda B^{\Gamma} A^{\Gamma} \Lambda \\ &= \Lambda B^{\Gamma} \Lambda\Lambda \; A^{\Gamma} \Lambda \\ &= B^{\Gamma'} A^{\Gamma'}. \end{aligned}$$

It is now easy to see that Γ and Γ' are anti-automorphisms of $M_n(K)$.

Lemma 35 If s is an anti-automorphism of $M_n(K)$, and A and B are similar matrices in $M_n(K)$ such that $AA^s = BB^s = I$, or $A + A^s = B + B^s = 0$, then there exists $C \in M_n(K)$ such that $C\,C^s = I$ and $C\,A\,C^s = B$.

Proof See (5)

If X is a nilpotent {unipotent} element of $M_n(K)$, then X $\{X - I\}$ is similar to $N_{(k)}$ (see page 64 for the notation) for some partition (k), of n. We write $t(X) = (k)$.

Let $s = \Gamma$ or Γ^*, $G = \{X \in M_n(K) \mid XX^s = 1$ and $\det X = 1\}$, and $\underline{g} = \{X \in M_n(K) \mid X + X^s = 0\}$.

If X and Y are two nilpotent {unipotent} elements of $\underline{g}$ $\{G\}$ and $C(X) = C(Y)$, $C(X)$ and $C(Y)$ being the nilpotent {unipotent} conjugacy classes in $\underline{g}$ $\{G\}$ which contain X and Y respectively, then $t(X) = t(Y)$. If C is a nilpotent {unipotent} conjugacy class in $\underline{g}$ $\{G\}$, then we write $t(C) = t(X)$, where X is an arbitrary element of C. It is clear that if C' is another nilpotent {unipotent} conjugacy class of $\underline{g}$ $\{G\}$, then $t(C) = t(C') \Rightarrow C = C'$ (see Lemma 35 above).

Lemma 36 Let $\mathcal{N}$ be the set of nilpotent conjugacy classes in $\underline{g}$, and $\mathcal{V}$ be the set of unipotent conjugacy classes in G, then we can define a bijective map $\rho: \mathcal{N} \to \mathcal{V}$ such that:

(i) if $C_1, C_2 \in \mathcal{N}$ and $C_1 \subseteq \overline{C_2}$, then $\rho(C_1) \subseteq \overline{\rho(C_2)}$

(ii) If $C \in \mathcal{N}$, then $t(C) = t((\rho(C))$.

<u>Proof</u> Let N be the variety of nilpotent elements of $\underline{g}$, and V be the variety of unipotent elements of G. If $A \varepsilon N$, then

$$B = \frac{I + A}{I - A} = I + 2A + 2A^2 + \ldots + 2A^n \text{ is a}$$

unipotent element of $M_n(K)$. Further, since s is an anti automorphism, it is clear that $\left(\frac{I + A}{I - A}\right)^s = \frac{I + A^s}{I - A^s}$, and hence that $BB^s = 1$. So we can define a map $\xi : N \to V$ by putting $\xi(A) = \frac{I + A}{I - A}$.

ξ is in fact an isomorphism of varieties, i.e. $\xi^{-1}(B) = \frac{B - I}{B + I}$ for all $B \varepsilon V$. Now, if $g \varepsilon G$ then $\xi(g A g^{-1}) = g\xi(A)g^{-1}$ for all $A \varepsilon N$, and hence we are able to define ρ. It is clear that if $C_1, C_2 \varepsilon \mathcal{N}$ and $C_1 \subseteq \overline{C_2}$ then $\rho(C_1) \subseteq \overline{\rho(C_2)}$.

If X and Y are elements of $M_n(K)$, X is nilpotent, Y is non-singular, and X and Y commute, then XY is nilpotent. Recall that $M_n(K)$ acts on the vector space E. If $1 \leq i \leq n$ then $X^iY^i : E \to E$, and thus rank X^iY^i = dim E - dim Ker X^iY^i. But Y^i is non-singular, and thus rank X^iY^i = dim E - dim Ker X^i = rank X^i. Hence $t(X) = t(XY)$ (see Lemma 16). As above let $B \varepsilon V$ and put $A = \frac{B - I}{B + I}$. Applying the above result to $(B - I)$ and $(B + I)^{-1}$, we get that $t(A) = t(B - I) = t(B)$. Hence if $C \varepsilon \mathcal{N}$ then $t(C) = t(\rho(C))$.

<u>Lemma 37</u> If $(k) = (k_1, k_2, \ldots, k_r)$ is a partition of n, then:

(i) A necessary and sufficient condition for there to exist a nilpotent {unipotent} element X, of $so(n,K)$ $\{SO(n,K)\}$, such that $t(X) = (k)$, is that each even part of (k) appears an even number of times.

(ii) A necessary and sufficient condition for there to exist a nilpotent {unipotent} element X, of $sp(n,K)$ $\{Sp(n,K)\}$, such that $t(X) = (k)$, is that each odd part of (k) appears an even number of times.

Proof See (5) for the nilpotent case. The unipotent case is obtained by applying Lemma 36.

Definition If each even part of a partition (k) appears an even number of times, then we say that (k) is orthogonal. If each odd part of a partition (k) appears an even number of times then we say that (k) is symplectic.

It is clear (see Lemma 35) that the nilpotent {unipotent} conjugacy classes of so(n,K) {SO(n,K)} are in one-to-one correspondence with the orthogonal partitions of n, and that the nilpotent {unipotent} conjugacy classes of sp(n,K) {Sp(n,K)} are in one-to-one correspondence with the symplectic partitions of n.

If (k) and (k') are partitions of m and n respectively, then we use $(k) \oplus (k')$ to denote the partition of $m + n$ obtained by taking the parts of (k) and (k') together and rearranging them in descending order.

If $(k) = (k_1, k_2, \ldots, k_r)$ is a partition of n, then we can obtain an orthogonal partition $(k)_o$ of n, as follows:

If $r = 1$ and k_1 is odd, then put $(k)_o = (k_1)$; and if k_1 is even, then put $(k)_o = (k_1 - 1, 1)$. If $r > 1$ and k_1 is odd, then put $(k)_o = (k_1) \oplus (k_2, \ldots, k_r)_o$; if k_1 is even and $k_1 \neq k_2$, then put $(k)_o = (k_1 - 1) \oplus (k_2 + 1, k_3, \ldots, k_r)_o$; and if k_1 is even and $k_1 = k_2$, then put $(k)_o = (k_1, k_2) \oplus (k_3, \ldots, k_r)_o$.

Similarly we can obtain a symplectic partition $(k)_s$, of n (we are of course assuming that n is even in this case), i.e:

If $r = 1$, then put $(k)_{\rho} = (k_1)$. If $r > 1$ and k_1 is even, then put $(k)_{\rho} = (k_1) \oplus (k_2,\ldots,k_r)_{\rho}$; if k_1 is odd and $k_1 \neq k_2$, then put $(k)_{\rho} = (k_1 - 1) \oplus (k_2 + 1, k_3,\ldots, k_r)_{\rho}$; and if k_1 is odd and $k_1 = k_2$, then put $(k)_{\rho} = (k_1, k_2) \oplus (k_3,\ldots, k_r)_{\rho}$.

Example

$(7, 6, 6, 4, 2, 2, 1)_o = (7, 6, 6, 3, 3, 1, 1, 1)$

$(7, 6, 6, 4, 2, 2, 1)_{\rho} = (6, 6, 6, 4, 2, 2, 2)$

We now leave partitions for the time being, and go on to look at triangular subsets.

Recall that a subset S of Δ_n^+ is said to be triangular if $(i,j) \in S$ implies that $(i-1,j), (i,j+1) \in S$. We say that a triangular subset S of Δ_n^+ is symmetric if $(i,j) \in S$ implies that $\Gamma((i,j)) \in S$.

If S is a triangular subset of Δ_n^+, then we define a subset I of $\langle 1, n \rangle$, called the first characteristic sequence of S, as follows: the first element of I is 1, and if $i \in I$, then its successor is the least j such that $(i,j) \in S$, if such a j does not exist then the sequence ends with i. Now suppose that the characteristic sequences $I_1 = I, I_2,\ldots,I_{p-1}$ have been defined for S, and let $J_{p-1} = J$ be their union. We define the pth characteristic sequence, I_p, of S as follows: the first element of I_p is the first element of $\langle 1,n \rangle - J$, and if $i \in I_p$, then its successor is the least $j \in \langle 1,n \rangle - J$ such that $(i,j) \in S$; if such a j does not exist then the sequence ends with i. This process of defining characteristic sequences can continue until for some r $I_1 \cup I_2 \cup \ldots \cup I_r = \langle 1,n \rangle$; $I_1,\ldots, I_r$ constitutes a complete set of characteristic sequences for S. If we let k_i equal the number of elements in the sequence I_i, then $(k) = (k_1, k_2, \ldots, k_r)$ is a partition of n. We write $t(S) = (k)$.

Example

(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, 7)	(1, 8)
	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, 7)	(2, 8)
		(3, 4)	(3, 5)	(3, 6)	(3, 7)	(3, 8)
			(4, 5)	(4, 6)	(4, 7)	(4, 8)
				(5, 6)	(5, 7)	(5, 8)
					(6, 7)	(6, 8)
						(7, 8)

As before (cf. page 68) we display the elements of Δ_8^+ in a triangular array, and let S be the triangular subset of Δ_8^+ consisting of those elements in the shaded region.

$I_1 = \{1, 4, 7\}$, $I_2 = \{2, 5, 8\}$

$I_3 = \{3, 6\}$, and $t(S) = (3, 2, 2)$

Notation If G_o is a classical group with Lie algebra $\underline{g}_o$, and $\underline{v}$ is a closed irreducible subvariety of $\underline{g}_o$ consisting entirely of nilpotent elements, then there exists a unique nilpotent conjugacy class C of $\underline{g}_o$ such that $\overline{C \cap \underline{v}} = \underline{v}$; we write $t(\underline{v}) = t(C)$.

Now, if S is a triangular subset of Δ_n^+, then $\underline{u}_S$ (see page 64 for the notation) is a closed irreducible subvariety of $s\ell(n,K)$ consisting entirely of nilpotent elements. Further, if S is symmetric, then $so\underline{u}_S = so(n,K) \cap \underline{u}_S$ and $sp\underline{u}_S = sp(n,K) \cap \underline{u}_S$ (in the latter case we are assuming that n is even) are closed irreducible subvarieties of $so(n,K)$ and $sp(n,K)$, respectively, consisting entirely of nilpotent elements.

<u>Lemma 38</u> If S is a triangular subset of Δ_n^+, then $t(\underline{u}_S) = t(S)$. Further, if S is symmetric then $t(so\underline{u}_S) = t(S)_o$ and $t(sp\underline{u}_S) = t(S)_{\sigma}$.

<u>Proof</u> See (5).

Recall that $\underline{g} = so(n,K)$ $\{sp(n,K)\}$, $G = SO(n,K)$ $\{Sp(n,K)\}$, and that if X is a nilpotent element of $\underline{g}$, then $C(X)$ denotes the nilpotent conjugacy class in $\underline{g}$ containing X.

<u>Lemma 39</u> If X and Y are nilpotent elements of $\underline{g}$, then:

(i) $C(X) = C(Y) \Longleftrightarrow \text{rank } X^i = \text{rank } Y^i$ for $i = 1,\ldots,n$.

(ii) $C(X) \subseteq \overline{C(Y)} \Longleftrightarrow \text{rank } X^i \leq \text{rank } Y^i$ for $i = 1,\ldots,n$.

<u>Proof</u>

(i) follows immediately from Lemmas 16 and 35.

(ii) see 5.

The analogue in the unipotent case is obvious and follows immediately from Lemma 36.

4.2 THE BASIC ASSUMPTION FOR SO(n,K) AND Sp(n,K)

Let $\underline{g}$ and G be as above.

If $D_o \subseteq E_o \subseteq \pi_o$, π_o being the basis of the root system of G described on pages 102 and 103, then let $\tilde{R}_{E_o}$ be the regular subgroup of G of Levi type corresponding to E_o (i.e. $\tilde{R}_{E_o}$ is the semi-simple part of $\tilde{L}_{E_o} = \langle T_o, \tilde{U}_{\tilde{\alpha}_{ij}} \mid \tilde{\alpha}_{ij} \in (\Phi_o)_{E_o} \rangle$, where $\tilde{U}_{\tilde{\alpha}_{ij}} = \tilde{\varepsilon}_{ij}(G_a)$ (see pages 102 + 104)), $\tilde{P}_{E_oD_o}$ be the standard parabolic subgroup of $\tilde{R}_{E_o}$ corresponding to the subset D_o of E_o, and $\tilde{\underline{u}}_{E_oD_o}$ be the Lie algebra of the unipotent radical $\tilde{U}_{E_oD_o}$, of $\tilde{P}_{E_oD_o}$.

If $D \subseteq E \subseteq \pi$, π being the basis of the root system of SL(n,K) described on page 63, then let R_E be the regular subgroup of SL(n,K) of Levi type corresponding to the subset E of π, P_{ED} be the standard parabolic subgroup of R_E corresponding to the subset D of E, and $\underline{u}_{ED}$ be the Lie algebra of the unipotent radical U_{ED}, of P_{ED}.

Lemma 40 If C_o is a nilpotent conjugacy class of $\underline{g}$, then there exist sets D_o and E_o, $D_o \subseteq E_o \subseteq \pi_o$, such that:

(i) $\tilde{P}_{E_oD_o}$ is a distinguished parabolic subgroup of $\tilde{R}_{E_o}$, and C_o intersects $\tilde{\underline{u}}_{E_oD_o}$ in a dense open subset.

(ii) There exists two sets D and E, $D \subseteq E \subseteq \pi$, such that

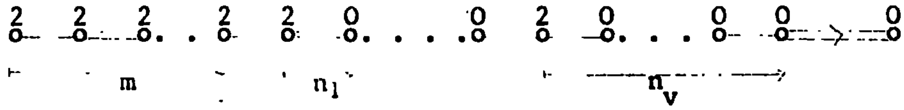
$$\tilde{\underline{u}}_{E_oD_o} = \underline{g} \cap \underline{u}_{ED} \quad \text{and} \quad \tilde{U}_{E_oD_o} = G \cap U_{ED}.$$

Proof

(I) The proof of part (i) for $so(2\ell + 1, K)$.

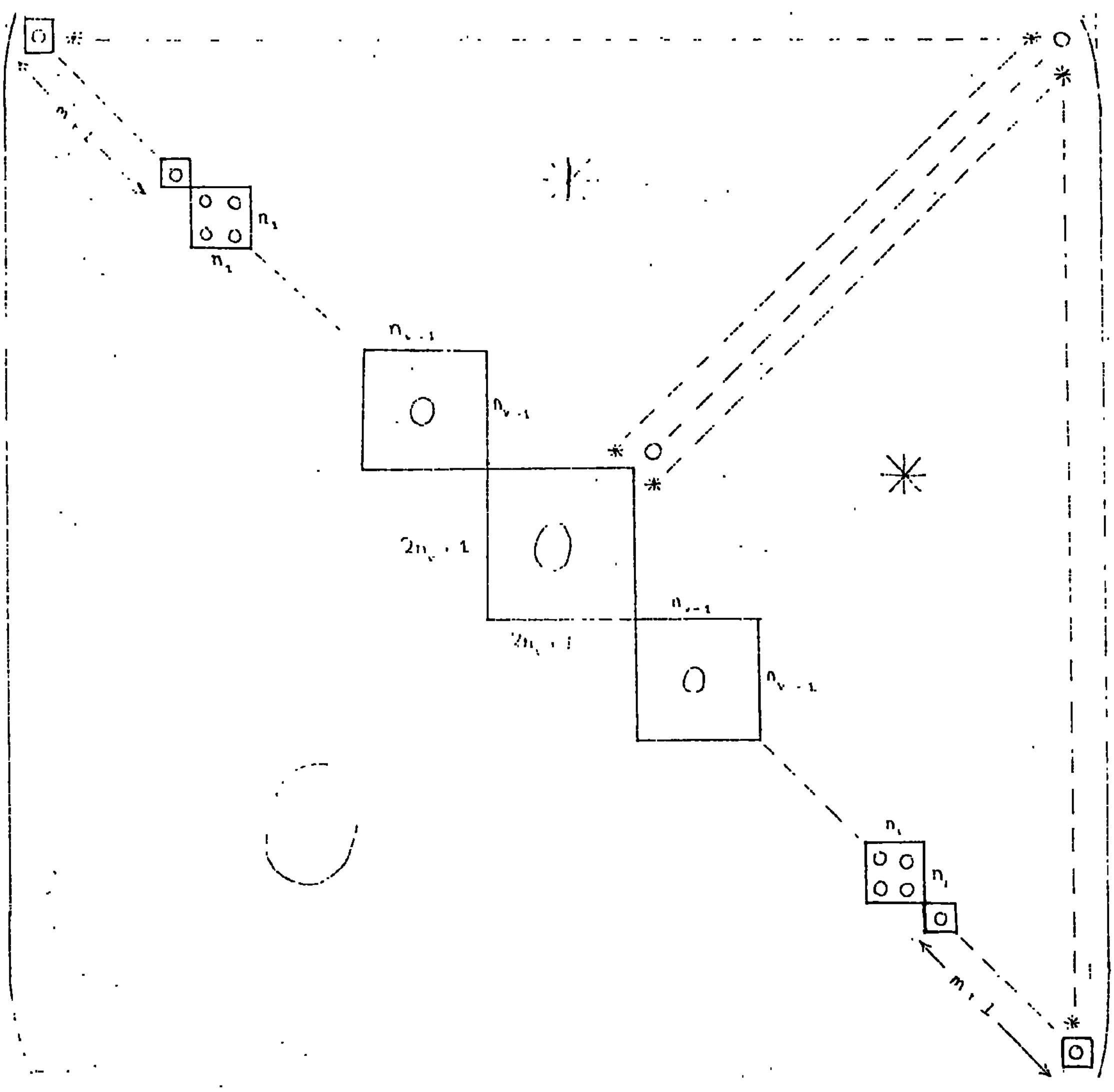
1) Let $(k) = (k_1, k_2, \ldots, k_r)$, $r > 1$, be a partition of $2\ell + 1$

into distinct odd parts, and let



be the distinguished Dynkin Diagram of type B_ℓ obtained from (k) (see page 99) - note that the nodes in the above diagram represent elements of π_o. Let D_o be the set of those simple roots which have weight zero in the above diagram, $F = \{(i,j) \in \Delta_{2\ell+1}^+ \mid i + j < 2\ell + 2$ and $h_{D_o}(\tilde{\alpha}_{ij}) \geq 2\}$ - see 1.3.19 for the definition of h_{D_o}, $\delta = \{(i,j) \in \Delta_{2\ell+1}^+ \mid i + j = 2\ell + 2$ and $i \leq \ell - n_v\}$, and $S = F \cup \delta \cup \Gamma(F)$. S is a symmetric triangular subset of $\Delta_{2\ell+1}^+$ and $so\underline{u}_S = \tilde{\underline{u}}_{D_o}$, the Lie algebra of the unipotent radical of the standard parabolic subgroup of $SO(2\ell + 1,K)$ corresponding to the subset D_o of π_o.

i.e. An element X, of $so\underline{u}_S$ has the form



where the $*$'s represent entries which may or may not be zero. Recall that X is antisymmetric about the antidiagonal.

Let $I_1, \ldots, I_t$ be a complete set of characteristic sequences of S. It is easy to see that $|I_i|$ is equal to the number of blocks in the above diagram with sides greater than or equal to i. Thus $t(S) = (2n_v + 1, n_{v-1}, n_{v-1}, \ldots, n_1, n_1, \underbrace{1, \ldots, 1}_{2m+2})^*$. Let

$\lambda_i = \dfrac{k_i - 1}{2}$ for $i = 1, \ldots, r$.

(a) If $\lambda_r \neq 0$, then put $(\lambda) = (\lambda_1, \ldots, \lambda_r)$. In this case $2n_v + 1 = n_{v-1} = r$ (see page 99). Now:

$$(k)^* = (r^{k_r}, (r-1)^{k_{r-1} - k_r}, \ldots, 1^{k_1 - k_2})$$

$$= (r^{2\lambda_r + 1}, (r-1)^{2(\lambda_{r-1} - \lambda_r)}, \ldots, 1^{2(\lambda_1 - \lambda_2)})$$

$$= (r) \uplus (\lambda)^* \uplus (\lambda)^*$$

$$= t(S)^*.$$

$\therefore$ $t(S) = (k)$ and hence $t(\tilde{\underline{u}}_{D_o}) = (k)_o = (k)$ (see Lemma 38).

(b) If $\lambda_r = 0$, then put $(\lambda) = (\lambda_1, \ldots, \lambda_{r-1})$. In this case $2n_v + 1 = n_{v-1} + 1 = r$. Now

$$(k)^* = (r, (r-1)^{2\lambda_{r-1}}, (r-2)^{2(\lambda_{r-2} - \lambda_{r-1})}, \ldots, 1^{2(\lambda_1 - \lambda_2)})$$

$$= (r) \uplus (\lambda)^* \uplus (\lambda)^*$$

$$= t(S)^*.$$

Thus $t(S) = (k)$ and $t(\tilde{\underline{u}}_{D_o}) = (k)_o = (k)$.

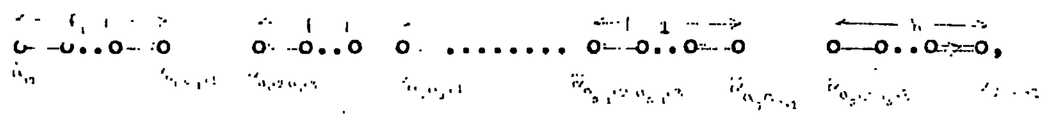
2) The partition $(k) = (2\ell + 1)$ corresponds to the distinguished diagram $\overset{2}{\circ}\!-\!\overset{2}{\circ}\!-\!\overset{2}{\circ} \cdots \overset{2}{\circ}\!\Rightarrow\!\overset{2}{\circ}$. D_o, the set of those simple roots which are weighted with zero in the above diagram, is the empty set, and $\tilde{\underline{u}}_{D_o}$ (see 1 for the notation) is the Lie algebra of the unipotent

radical of B_o, the Borel Subgroup of upper triangular matrices in $SO(2\ell + 1,K)$. It is easy to see that $t(\tilde{\underline{u}}_{D_o}) = (k)$.

3) Let C_o be a nilpotent conjugacy class of $so(2\ell + 1,K)$ and (k) be the corresponding orthogonal partition of $2\ell + 1$ (See page 107). Three possibilities arise:

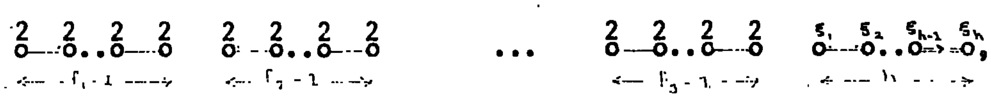
(a) (k) is a partition of $2\ell + 1$ into distinct odd parts. In this case the result follows immediately from (1) and (2) above, i.e. if $E_o = \pi_o$, and D_o is obtained as in (1) and (2), then C_o intersects $\tilde{\underline{u}}_{E_oD_o} = \tilde{\underline{u}}_{D_o}$ in a dense open subset.

(b) $(k) = (f_1, f_1, \ldots, f_s, f_s) \oplus (h_1, \ldots, h_p)$, where $h_1, h_2, \ldots, h_p$ are distinct odd integers and $h_1 + h_2 + \ldots + h_p = 2h + 1 > 1$. Let $E_o = \pi_o - \{\tilde{\alpha}_{i\ i+1} \mid i = f_1 + \ldots + f_q$ for some $q = 1, 2, \ldots, s\}$. $\tilde{R}_{E_o}$ has Dynkin Diagram



where $a_i = f_1 + \ldots + f_i - 1$ for $i = 1, 2, \ldots, s$.

Let V be the distinguished diagram



where the distinguished diagram $V_1 = \overset{\xi_1}{o}\!-\!\overset{\xi_2}{o} \ldots \overset{\xi_{h-1}}{o}\!\Rightarrow\!\overset{\xi_h}{o}$ is obtained from the partition $(h_1, \ldots, h_p)$ (see 1 and 2 above) - note that if $h = 1$, then $V_1 = \overset{2}{O}_{\alpha_{\ell\ \ell+1}}$.

Let D_o be the subset of E_o consisting of those simple roots which have weight zero in the above diagram.

An element X of $\tilde{u}_{E_oD_o}$ has the form

where the *'s represent entries which may or may not be zero. Recall that X is antisymmetric about the antidiagonal.

Let $\tilde{P}_o$ be the distinguished parabolic subgroup of $so(2h + 1,K)$ corresponding to the weighted Dynkin Diagram ∇_1, $\tilde{u}_o$ be the Lie algebra of the unipotent radical of P_o, C_1 be the nilpotent conjugacy class of $so(2h + 1,K)$ which intersects $\tilde{\underline{u}}_o$ densely, and $X \in C_1 \cap \tilde{\underline{u}}_o$. $X_o = N_{f_1} + N_{f_2} + \ldots + N_{f_s} + \tilde{X} + (-N_{f_s}) + \ldots + (-N_{f_2}) + (-N_{f_1})$ is an element of $u_{E_oD_o}$, and if $Y \in u_{E_oD_o}$, then rank $Y^i \leq$ rank X^i for $i = 1, 2, \ldots, 2\ell + 1$. Thus (see Lemma 39) $C(X_o) \cap u_{E_oD_o} = \underline{u}_{E_oD_o}$, $C(X_o)$ being the nilpotent conjugacy class of $so(2\ell + 1,K)$ containing X_o. Also (see (1) and (2)) it is easy to see that $t(X_o) = (k)$. Hence $C(X_o) = C_o$.

(c) $(k) = (f_1, f_1, \ldots, f_s, f_s) + (1)$. Let $E_o = \pi_o - \{\breve{\alpha}_{i\,i+1} \mid i = f_1, \ldots, f_q$ some $q = 1,\ldots,s\}$, and D_o be the empty set. $\tilde{R}_{E_o}$ has Dynkin Diagram

$$\overset{}{\underset{\breve{\alpha}_{12}}{o}}\ \underbrace{o..o}_{f_1 - 1}\ o \quad o\ \underbrace{o..o}_{f_2 - 1}\ o \quad \ldots \quad o\ \underbrace{o\,..\,o}_{f_s - 1}\ \underset{\breve{\alpha}_{\ell-1\,\ell}}{o}$$

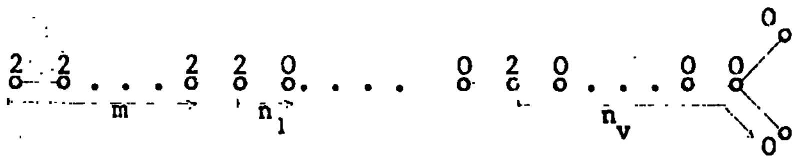
and an element X of $\underline{u}_{E_oD_o}$ has the form

where the *'s represent entries which may or may not be zero. Recall that X is anti symmetric about the antidiagonal.

It is easy to see that C_o will intersect $\tilde{\underline{u}}_{E_o D_o}$ in a dense open subset.

(II) The proof of (i) for $so(2\ell,K)$.

1) Let $(k_1, k_2,\ldots, k_r)$, $r > 2$, be a partition of 2ℓ into distinct odd parts, and let



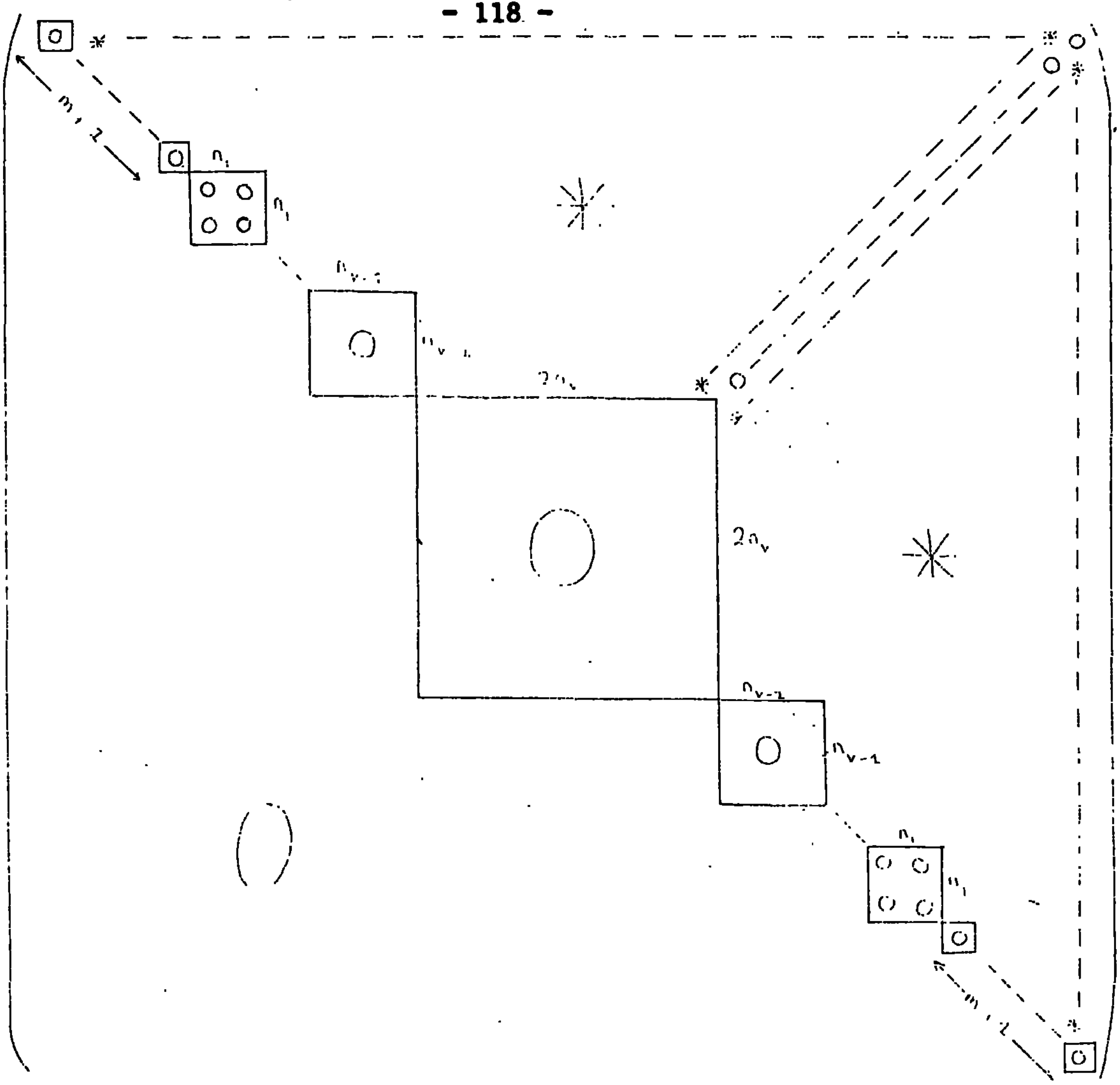
be the corresponding distinguished Dynkin Diagram of type D_ℓ (note that the nodes represent elements of π_o). Let D_o be the set of those simple roots which are weighted with a zero in the above diagram,

$F = \{(i,j) \in \Delta^+_{2\ell} \mid i + j < 2\ell + 1,\ h_{D_o}(\tilde{\alpha}_{ij}) \geq 2\}$,

$\delta = \{(i,j) \in \Delta^+_{2\ell} \mid i + j = 2\ell + 1 \text{ and } i \leq \ell - n_v\}$, and

$S = F \cup \delta \cup \Gamma(F)$. S is a symmetric triangular subset of $\Delta^+_{2\ell}$ and $so\underline{u}_S = \tilde{\underline{u}}_{D_o}$, the Lie algebra of the unipotent radical of the standard parabolic subgroup, of $SO(2\ell,K)$ corresponding to the subset D_o of π_o.

i.e. an element X of $so\underline{u}_S$ has the form



where the *'s represent entries which may or may not be zero. Recall that X is anti symmetric about the antidiagonal.

It is easy to see that $t(S) = (2n_v, n_{v-1}, n_{v-1}, \ldots, n_1, n_1, \underbrace{1,\ldots,1}_{2m+2})^*$

Let $\lambda_i = \dfrac{k_i - 1}{2}$, $i = 1, \ldots, r$.

(a) If $\lambda_r \neq 0$, then put $(\lambda) = (\lambda_1, \ldots, \lambda_r)$. In this case $2n_v = n_{v-1} = r$ (see page 100). Now

$$(k)^* = (r^{2\lambda_r+1}, (r-1)^{2(\lambda_{r-1}-\lambda_r)}, \ldots, 1^{2(\lambda_1-\lambda_2)})$$

$$= (r) \oplus (\lambda)^* \oplus (\lambda)^*$$

$$= t(S)^*.$$

$\therefore$ $t(S) = (k)$ and hence $t(\underline{u}_{D_o}) = (k)_o = (k)$ (see Lemma 38).

(b) If $\lambda_r = 0$, then put $(\lambda) = (\lambda_1,\dots,\lambda_{r-1})$. In this case

$2n_v = n_{v-1} + 1 = r$ (see page 100). Now

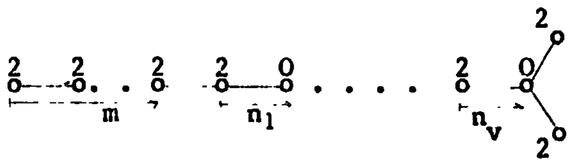
$$(k)^* = (r,\ (r-1)^{2\lambda_{r-1}},\ (r-2)^{2(\lambda_{r-2}-\lambda_{r-1})},\dots,1^{2(\lambda_1-\lambda_2)})$$

$$= (r) \oplus (\lambda)^* \oplus (\lambda)^*$$

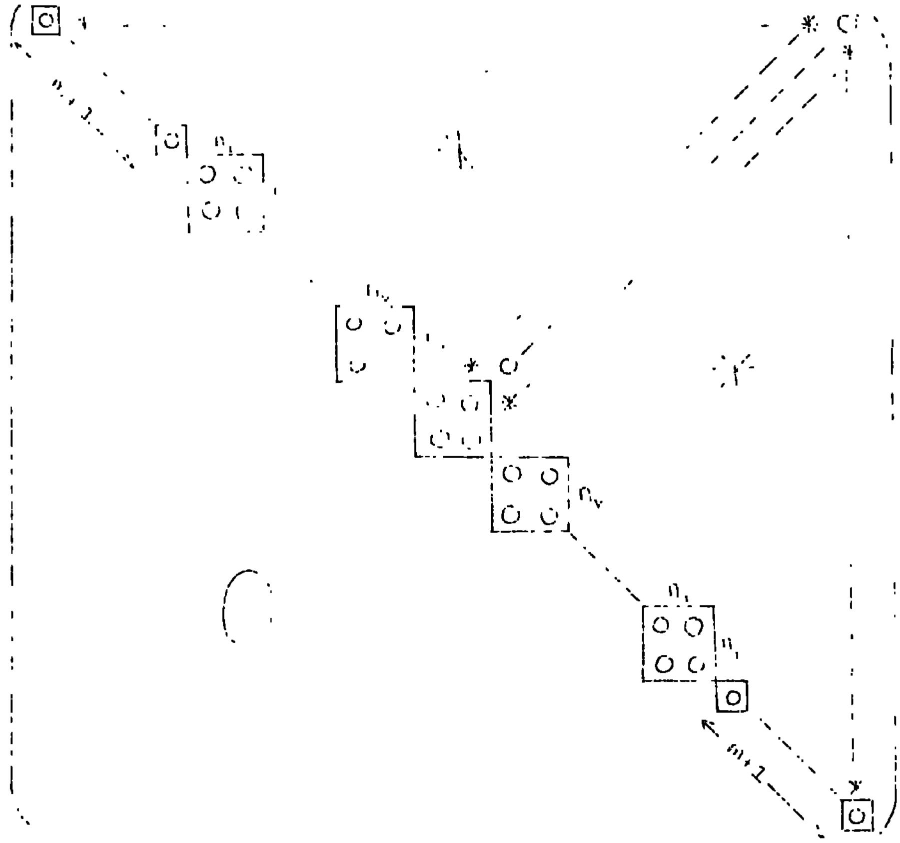
$$= t(S)^*$$

$\therefore$ $t(S) = (k)$ and $t(\underline{u}_{D_o}) = (k)_o = (k)$.

2) If (k_1, k_2) is a partition of 2ℓ into distinct odd parts, then let



be the corresponding Dynkin Diagram of type D_ℓ. Let D_o be the set of those simple roots which are weighted with a zero in the above diagram, $F = \{(i,j) \in \Delta_{2\ell}^+ \mid i + j < 2\ell + 1,\ h_{D_o}(\tilde{\alpha}_{ij}) \geq 2\}$,
$\delta = \{(i,j) \in \Delta_{2\ell}^+ \mid i + j = 2\ell + 1$ and $i < \ell - 1\}$, and
$S = F \cup \delta \cup \Gamma(F)$. S is a symmetric triangular subset of $\Delta_{2\ell}^+$, and
$so\underline{u}_S = \underline{\hat{u}}_{D_o}$. i.e. an element of X of $so\underline{u}_S$ has the form



where the *'s represent entries which may or may not be zero; X is anti symmetric about the antidiagonal. It is easy to see that

$$t(S) = (\underbrace{2, 2, \ldots, 2}_{2v+1}, \underbrace{1, \ldots, 1}_{2m+2})^*$$

$$= (2v + 2m + 3,\ 2v + 1)$$

$$= (k_1, k_2) \qquad \text{(see page 101)}$$

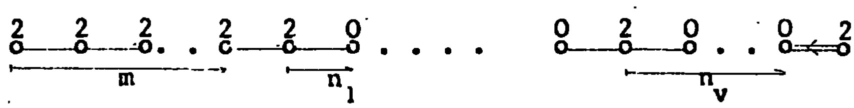
$$\therefore\ t(\tilde{\underline{u}}_{D_o}) = (k_1, k_2)_o = (k_1, k_2).$$

Note: if $(k_1, k_2) = (2\ell - 1, 1)$, then D_o is equal to the empty set and $\tilde{\underline{u}}_{D_o}$ is the Lie algebra of the unipotent radical of the Borel Subgroup, B_o, of upper triangular matrices in $SO(2\ell,K)$.

3) Let C_o be a nilpotent conjugacy class of $so(2\ell,K)$. To find two sets D_o and E_o, $D_o \subseteq E_o \subseteq \Pi_o$, such that $\tilde{P}_{E_oD_o}$ is a distinguished parabolic subgroup of $\tilde{R}_{E_o}$ and $\overline{C_o \cap \tilde{\underline{u}}_{E_oD_o}} = \tilde{\underline{u}}_{E_oD_o}$, follow the same procedure as in I part 3.

(III) The proof of part (i) for $sp(2\ell,K)$.

1) If $(k) = (k_1, k_2, \ldots, k_r)$ is a partition of 2ℓ into distinct even parts, then let $\lambda_i = k_i/2$ for $i = 1, 2, \ldots, r$, and

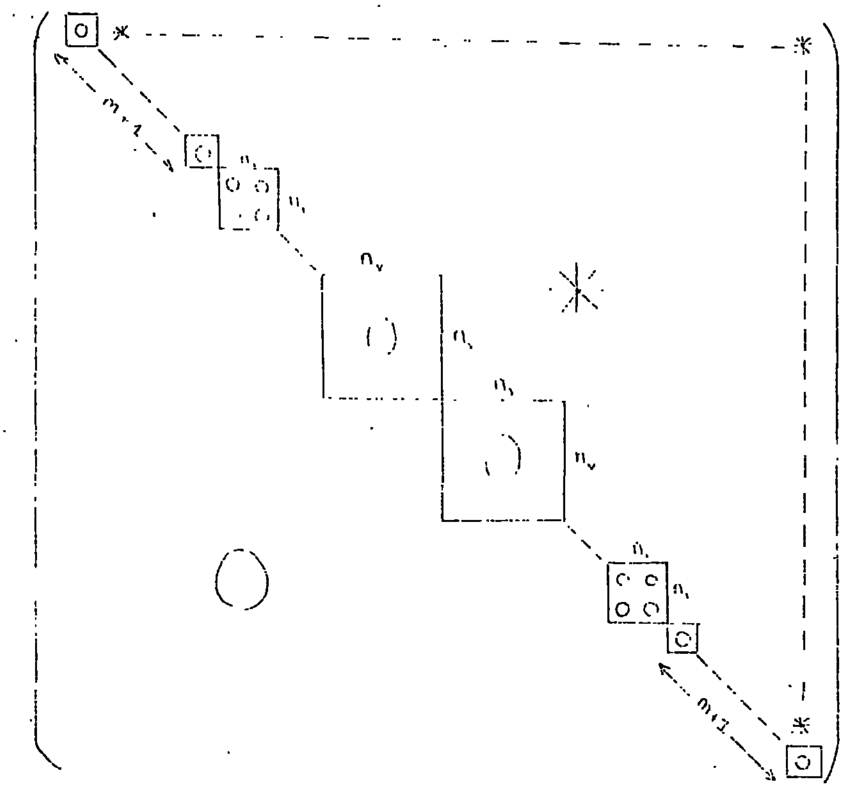


be the distinguished Dynkin Diagram of type C_ℓ corresponding to the partition $(\lambda) = (\lambda_1, \ldots, \lambda_r)$. Let D_o be the set of those simple roots weighted with zero in the above diagram,

$F = \{(i,j) \in \Delta_{2\ell}^+ \mid i + j \leqslant 2\ell + 1$ and $h_{D_o}(\tilde{\alpha}_{ij}) \geqslant 2\}$, and

$S = F \cup \Gamma(F)$. S is a symmetric triangular subset of $\Delta_{2\ell}^{+}$ and $sp\underline{u}_S = \ddot{u}_{D_o}$, the Lie algebra of the unipotent radical of the standard parabolic subgroup of $Sp(2\ell,K)$ corresponding to the subset D_o of π_o.

i.e. an element X of $sp\underline{u}_S$ has the form



where the $*$'s represent entries which may or may not be zero. Recall that $X + X^{\Gamma'} = 0$ (see page 103). It is easy to see that $t(S) = (n_v, n_v, \ldots, n_1, n_1, \underbrace{1,\ldots,1}_{2m+2})^* = (k)$, and hence that $t(\ddot{u}_{D_o}) = (k)_\rho = (k)$ (see Lemma 38).

2) Let C_o be a nilpotent conjugacy class of $sp(2\ell,K)$. To obtain two sets D_o and E_o, $D_o \subseteq E_o \subseteq \pi_o$, such that $\tilde{P}_{E_oD_o}$ is a distinguished parabolic subgroup of $\tilde{R}_{E_o}$ and $\overline{C_o \cap \underline{\tilde{u}}_{E_oD_o}} = \underline{\tilde{u}}_{E_oD_o}$ proceed as in I part (3).

Proof of part (ii)

Recall that B is the Borel Subgroup of SL(n,K) consisting of uppertriangular matrices, that U is the unipotent radical of B, and that $\underline{u}$ is the Lie algebra of U.

It is easy to see that we can choose two sets D and E, $D \subseteq E \subseteq \pi$, such that $\underline{u} \cap \underline{u}_{ED} = \tilde{\underline{u}}_{E_oD_o}$, where D_o and E_o are obtained as in the proof of part (i), i.e. all we need to note is that in the case where $\underline{g} = so(2\ell,K)$ if $\tilde{\alpha}_{\ell-1\,\ell+1} \in D_o$ {or E_o}, then $\tilde{\alpha}_{\ell-1\,\ell} \in D_o$ {or E_o}. It is now fairly easy to see that $U \cap U_{ED} = \tilde{U}_{E_oD_o}$, and hence we obtain the desired result.

Proposition 41

If C is a unipotent conjugacy class of G, then there exists two sets D_o and E_o, $D_o \subseteq E_o \subseteq \pi_o$, such that $\tilde{P}_{E_oD_o}$ is a distinguished parabolic subgroup of $\tilde{R}_{E_o}$, and $\overline{\tilde{U}_{E_oD_o} \cap C} = \tilde{U}_{E_oD_o}$.

Proof

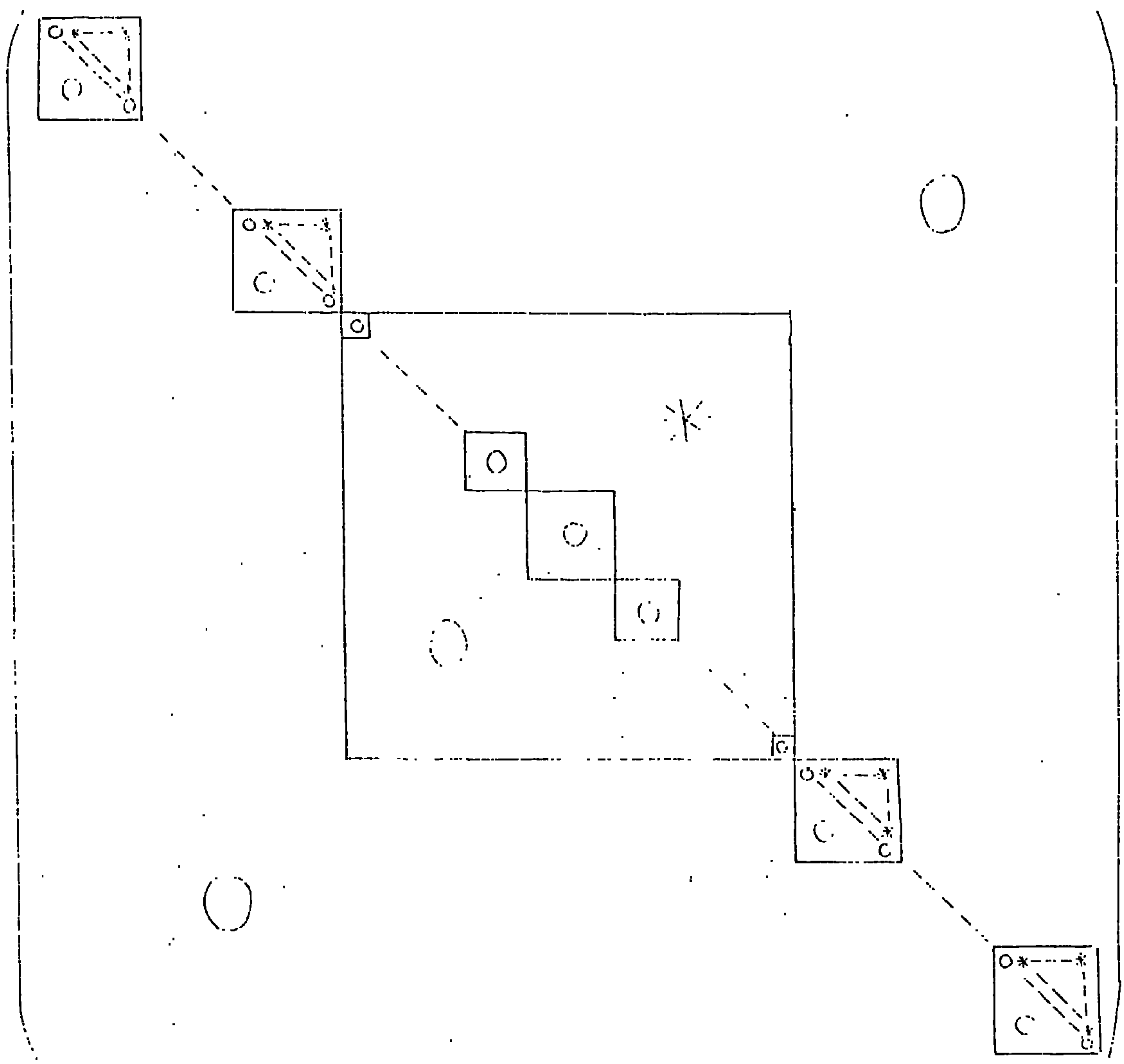
Let C_o be the nilpotent conjugacy class of $\underline{g}$ such that $\rho(C_o) = C$ (see Lemma 36). By Lemma 40, there exist four sets D_o, E_o, D and E, $D_o \subseteq E_o \subseteq \pi_o$ and $D \subseteq E \subseteq \pi$, such that:

(i) $\tilde{P}_{E_oD_o}$ is a distinguished parabolic subgroup of $\tilde{R}_{E_o}$ and

$\overline{\tilde{\underline{u}}_{E_oD_o} \cap C_o} = \tilde{\underline{u}}_{E_oD_o}$.

(ii) $\tilde{\underline{u}}_{E_oD_o} = \underline{g} \cap \underline{u}_{ED}$ and $\tilde{U}_{E_oD_o} = G \cap U_{ED}$.

Let N be the variety consisting of the nilpotent elements of $\underline{g}$, and V be the variety consisting of the unipotent elements of G. We have already seen (see the proof of Lemma 36) that the map $\xi: N \to V$, $\xi(X) = \frac{I + X}{I - X}$, is an isomorphism of varieties. If $X \in \tilde{\underline{u}}_{E_o D_o}$, then $X \in \underline{u}_{ED}$ and thus $I + X, I - X \in U_{ED}$ (recall that $U_{ED} = \{u \in SL(n,K) \mid u - I \in \underline{u}_{ED}\}$ (see page 64)). Hence $\xi(X) \in U_{ED} \cap G = \tilde{U}_{E_o D_o}$. On the other hand, if $Y \in \tilde{U}_{E_o D_o}$, then $Y - I \in \underline{u}_{ED}$ and $\frac{I}{I+Y} = \frac{1}{2} v$, where $v \in U_{ED}$. If we note that $Y - I$ and $v - I$ have the form



then it is easy to see that $\xi^{-1}(Y) = \frac{Y - I}{Y + I} \in \underline{u}_{ED} \cap \underline{g} = \tilde{\underline{u}}_{E_o D_o}$.

Thus ξ maps $u_{E_o D_o}$ bijectively onto $\grave{U}_{E_o D_o}$; i.e. $\xi: \underline{\tilde{u}}_{E_o D_o} \to \grave{U}_{E_o D_o}$ is an isomorphism of varieties. Therefore $\tilde{U}_{E_o D_o} = \xi(\overline{\ddot{u}_{E_o D_o} \cap C_o})$ $= \overline{\xi(\underline{\tilde{u}}_{E_o D_o})} \cap \overline{\xi(C_o)} = \overline{\tilde{U}_{E_o D_o} \cap C}$.

Theorem 42

Our basic assumption that η is surjective (see 2.2.3) is true for $SO(n,K)$ and $Sp(n,K)$.

Proof This follows immediately from proposition 41. (cf. Theorem 3).

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