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THE USE OF AUTONOMY IN THE DEVELOPMENT OF MATHEMATICAL
CONCEPTS IN PRIMARY AND MIDDLE SCHOOL CHILDREN

Joseph Elliot Zammarelli

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A Thesis presented for the Degree of Doctor of
Philosophy in the University of Durham

-1977-



"The activities of animal and man vary from machine-like automatisms to ingenious improvisations, according to the challenge they face. Other things being equal, a monotonous environment leads to the mechanization of habits, to stereotyped routines which, repeated under the same unvarying conditions, follow the same rigid, unvarying course "

-Arthur Koestler in
The Ghost in the Machine

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Abstract

The research reported in this thesis is divided up into three main sections.

The first section begins with a brief review of the literature on methods of conducting psychological research in the classroom. It is argued that an interest in the psychology of education must be followed by a methodology which takes into account the fact that schools are social institutions within our society, and that they have functions other than the teaching of cognitive skills. Learning which takes place in schools may therefore be seen as being embedded within a social milieu.

The classroom research therefore begins with an extended period of observations and interviews. Initially, the investigator operates as a non-participant, and gradually, as his presence becomes an unnoticed part of the school's routine, certain limited interactions are carried out with the children. Conversations and informal interviews are also held with the head and members of the teaching staff.

From all of these encounters, a simple model of mathematical learning in the classroom is put forward. This is supported by two scenarios taken from actual classroom situations. A hypothesis on mathematical learning in the classroom is then developed which states that children must experience a certain amount of autonomous activity if they are to formulate 'higher order' strategies for dealing with mathematical representations, structures, and problems.

Section two of the thesis contains a series of experiments which are designed to test aspects of this hypothesis in a controlled setting. This is accomplished by using three specially designed portable 'toys' which each contain rules or patterns relating to mathematical systems.

Positive evidence is found for the claim that an autonomous condition is more effective than a yoked, control or combined experience (hybrid) in promoting mathematical conceptualisation as scored by written and verbal measures.

In the third section of the thesis, the toys are brought into the classrooms in an attempt to integrate them with more formal instruction. Results of these efforts are assessed and documented via interviews, photographs, and samples of classroom work

Educational implications and suggestions for further research are then briefly discussed

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Part I Overall Perspective to the Research

Introduction

The central purpose of this thesis is to relate aspects of the current body of knowledge in cognitive psychology to some of the processes which go on in classrooms. More specifically, interest is centered on those time periods when children are being taught a specific subject or set of skills. Stated in this way, the question is initially a methodological one. Namely, how can we best utilize the various theories and models of man which deal with learning, problem solving, and creative invention to describe and prescribe methods of classroom teaching?

When viewed in this manner, the two goals of analysis of school learning and the development of so called 'learning environments' or methods of approach can be related under a common theoretical stance. However, what follows will not be an attempt to examine either the details or relative merits and weaknesses of any particular theory of learning or instruction. Rather it is primarily concerned with exploring and developing research techniques which are effective in dealing with the problems of educational design and evaluation. This is another way of saying that before good answers are found, the proper questions must be identified and formulated in ways which will yield potentially useful results. "Part of the problem is figuring out what the problem is", and perhaps this is best begun in our present area of concern, namely the psychology of education, once we identify the perspectives or paradigms which largely determine our basic assumptions and operating procedures. An important part of doing this involves an exploration of the issues and evidence which provide us with our basic assumptions in what follows. Once this is accomplished, I will review some of the implications which they hold for the overall design and carrying out of the research.



The first basic assumption is that from a psychological point of view, man can be seen as an active seeker and processor of information.¹ I believe that this notion must be dealt with in the classroom if one is to approach an understanding of how individuals process complex pieces of information every day in school. When successful, children formulate concepts that are able to deal with (i.e. help explain and predict) a whole range of observable occurrences. Therefore, one of the 'everpresent' concerns which governed the research was an attempt to identify those procedures in the classroom which were aimed at developing the student's conceptual framework in a given subject area (in this research maths was chosen as the subject area). Some of the more detailed reasons concerning the need to be aware of the process of learning as it occurs within school, and not merely as the outcomes of a series of evaluative tests or psychometric measurements, will be developed later. Suffice it to say for now that a belief in learning as a culturally dependent activity (see Bruner, esp. 1971) which transmits values and attitudes in concert with skills and information cannot be adequately explored by a series of measurements.

In particular, when numerical data which has been collected from a large sample is analysed, the form of the results may be seen as inadequate in that it cannot deal directly with questions concerning the development of the form and structure of any individual child's ability.² An important methodological implication of this belief is that a certain period of observation and/or intervention in the classroom is required if one is to deal with the context of the learning process

This leads to the second basic assumption in the research which is that school is a social institution which has functions other than the development of children's cognitive skills. In order to understand and evaluate the results of instruction, we must take into account the goals of the educational system as a force within our society. The belief here

is that one cannot ignore the context in which knowledge is transmitted. In order to develop a model which can help to explain how some of these contextual factors interact to affect learning in schools, we must first agree that there are certain values which come into play and which cannot be totally reduced to problems in the psychology of cognition. Further, we must be sensitive to the ideology and goals of the teachers and administrators within the educational system. Schooling, as well as learning, is context (culture) bound, and many of the important questions and issues (such as 'what kind of problems do we want our children to be concerned with?') are intimately wrapped up in the institutional functioning of education. Again, the implication is that we should, for at least part of the time during the research, make contact with the schools themselves in order to acquire data relating to the sociology of education.

Although it was not the purpose of this introduction to develop the specific methodology used in the research (that will follow next), the two basic assumptions concerning man as a processor of information and the sociological context of education and learning form the basis for what follows. In particular, they highlight the need to consider the process and not merely the products of the instructional system, and the need to establish an overall contextual perspective of the sociology of the classroom in which will be embedded particulars concerning the ways children learn. Both of these points also strongly imply that one should be required to enter into the particular school or classroom for at least part of the research, in order that direct observational evidence is acquired for both the overall context of the lessons, and its relationship to the progress of the individual student. Our statement of the overall problem can therefore be approximated by "How can we begin to develop and apply a methodology which will enable us to generate and test hypotheses about how children learn (maths) in a classroom setting?".

Chapter One The School as a Social Institution

Having said that my primary area of interest is the time periods in a classroom when children are being taught a specific subject, let me re-state my belief that one cannot ignore the context in which the knowledge is transmitted. Although it is not my intention to review the vast literature on social influences in school situations¹, it is important to realize that the criteria which are used to evaluate what and how a child has learned in school will be influenced by cultural factors within our society. Because of this, I would like to review some of the work on the sociology of knowledge, so that my theoretical orientation will be made clearer. Once this is done, I believe that the development of the research design which follows in part II will be better understood. Let me begin, then, with a brief historical outline and overview.

One of the problems with much of what is current in educational research is that it has not provided us with enough information to formulate any far-reaching program of educational change. Much of this can be traced to its heritage from the 1900's of stressing survey, measurement, testing and educational efficiency. Comparisons were made with the emerging industrialisation which itself in large part opened the need for more widespread education.

For example.

"Our schools are, in a sense, factories in which the raw products (children) are to be shaped and fashioned into products to meet the various demands of life. The specifications for manufacturing come from the demands of twentieth-century civilization, and it is the business of the school to build its pupils according to the specifications laid down. This demands good tools, specialised machinery, continuous measurement of production to see if it is according to specifications, the elimination of waste in manufacture, and a large variety in the output."²

This was written in 1916 when Terman, Ayres, and Thorndike were pioneering

the tests and measuring devices for this 'systems' approach. Administrators were urged to bring about 'scientific' school management.³ However, efficient utilisation of labour in manufacture is not the same as the acquisition, over a long period of time, of intellectual skills by the young. The post WWII period brought forth quite a few eloquent supporters of both the dangers of mere rote learning and the divorcing of education from human values. Max Wertheimer in 1945 wrote

"I now looked more thoroughly into customary methods, the ways of teaching arithmetic, the textbooks, the specific psychology books on which their methods were based. One reason for the difficulty became clearer and clearer: the emphasis on mechanical drill, on 'instantaneous response', on developing blind, piecemeal habits. Repetition is useful, but continuous use of mechanical repetition also has harmful effects."⁴

From his work within various classrooms he concludes the following concerning the ever-present use of the above mentioned surveys, statistics, etc

"The most important need in the experimental investigation of the problems seems to be not so much to get the quantitative answer 'How many children achieve a solution, how many fail, at what age?' etc., but to get at an understanding of what happens in good and bad processes."⁵

Yet most classroom structures of the 50's tended to conform to the more 'mechanical processes' of rote learning, fact acquisition, repetition, etc. Investigation of interpersonal exchange, an area which has been more recently found to be critical to learning within a human society, was almost forgotten. This can be traced in some ways to the adoption of a behaviouristic model of man which was developed from the traditions of learning and conditioning within psychology in the early to middle twentieth century.

Some of the work which provided an alternative forum came not from investigations of learning as a habit or routine, but when questions concerning thinking, problem solving and more open-ended or creative

situations were posed as legitimate problems for psychologists. Bartlett in 1958 provides one point of departure:

"The suggestion is perhaps that as thinking moves towards greater freedom one thing that happens is that the thinker is less and less concerned with the likelihood of items and more and more with that of packets, or groups of items. He is less detail-ridden, more 'schematic'-minded. If we should ask for the reasons why these lumping schematizing developments take place, our present answers can of course be no more than speculation. My guess is that there are two chief reasons - they are more efficient, and they are more fun."⁶

Bruner, in the Colorado Symposium concerns himself with the setting of code acquisition when he writes about the problem of instruction

"To sum up the manner and degree with which newly learned knowledge is coded generically can be influenced in a transient way by situational instruction and in a more permanent way by the regimen of one's past experience. One's 'attitude' toward learning, whether a transient or enduring thing, will then determine the degree to which one is equipped with coding systems that can be brought to bear on new situations and permit one to go beyond them."⁷

Later, in the same work, he writes

"General education does best to aim at being generic education, training men to be good guessers, stimulating the ability to go beyond the information given to probable reconstructions of other events "⁸

Thus we can see strong foundations for what was to become a virtual 'explosion' in the late 60's and early 70's of writers concerned with educational values, techniques of research, and classroom design. For example, Kimball (1963), from an anthropological point of view, writes that the mission of schools is to prepare us for 'membership' into the series of interconnected structures which serve our basic social needs in an industrial society. Jules Henry (1965), in a more impassioned expression of the purpose of schooling, leaves little doubt as to where he stands regarding the claims that schools are centres of enlightenment.

"The function of education has never been to free the mind and the spirit of man, but to bind them, and to the end that the mind and spirit of his children should never escape,

homo sapiens has employed praise, ridicule, admonition, accusation, mutilation and even torture to chain them to the culture pattern."⁹

Merton Kahne, who is a psychiatrist by training, points out (1969 and personal communication) that primitive societies do almost all of the teaching of practical skills and knowledge by a method of observation and apprenticeship. It is only the highly ritualised functions such as puberty rites which require any type of formalised instruction. Thus, secondary schools may be compared to an extended rite of adolescence, keeping the young out of the job market while providing them with an arena for instruction and peer contact which will go far in shaping their goals and attitudes into adulthood.

Other writers such as Jones (1968) decry the lack of genuine emotional involvement within our classrooms. He contends that the effectiveness of teaching materials is in proportion to their authenticity, and that even young children are very sensitive in responding to lies or false excitement in a teaching situation. However, dealing with the practical issue of integrating imagination into a curriculum becomes a much more difficult task than pointing out its need. Even still, some hint is given, (1968).

"A comprehensive theory of instruction should seek to prescribe not only optimal levels of intellectual uncertainty, risk and relevance, but also optimal levels of emotional involvement and personal curiosity."¹⁰

Jones also provides four conditions that seek to aid children to be 'creative in their schooling, the intention of which is to have them go beyond mere coping, to mastery and invention.' They are.

- (1) Stimulation
- (2) Play
- (3) Identification
- (4) Freedom from excessive drive

In a similar way, Smith and Geoffrey (1968), working in an Urban schoolroom in America, comment

". . . Anyone who attempts to change his classroom behaviour

must come to grips, theoretically and practically, with the informal social structure of the faculty, the cliques within the school, ...

. Innovations in curriculum, methods of teaching and pupil-teacher relationships often flounder, so it seems to us, because persons interested in educational change do not understand, or else they ignore, the implications of the variables in the model."¹¹

Similarly, from Leonard (1968):

"Learning eventually involves interaction between learner and environment, and its effectiveness relates to the frequency, variety and intensity of the interaction."¹²

Many of the research implications of this 'new breed' of writers concerned with problems of contemporary education can be summed up quite nicely by the expression of the following concern (from Jones, 1968)

"Traditionally, evaluation of new lessons, materials, and methods is conducted as a separate enterprise and kept at a purposeful distance from the heat of the classroom. This in the questionable interests of experimental rigor. Psychometrists are called in to find out what children know before and after the lesson or program at issue. The emphasis on achievement again, and the same cultivation of ignorance as regards process"¹³

Bruner is quoted in the same work as commenting that this is a bit like collecting military intelligence after the war is over! Patently, the present domain of educational psychology is 'under fire' for what has proven to be an embarrassingly sterile approach to the 'goings-on' in the classroom. Much of this has occurred as a result of the emphasis placed upon achievement rather than process, and on the practical rather than the possible.¹⁴

Although it would go beyond the scope of the present work to go into complete detail on the points made above, the research which follows was conducted under the assumption that there are many difficulties concerning the kinds of questions which 'traditional' educational research is organised to answer.

Widespread use of numerical and survey techniques in educational

research is probably as much a matter of available methods of gathering data as being the result of a particular theoretical orientation. Instead of coming to grips with what I have been calling the problems of sociological perspective and process vs. product modelling of man as a learner, researchers have often tended to rely upon correlations of various kinds to link-up whatever measuring device is employed with an external set of causations of future goals. But there are problems here also, for goals couched in terms of happiness or satisfaction also elude any meaningful 'static' evaluation or measurement. Further, more immediate and specific goal states which may be stated in behavioural terms are often too simple or primitive to deal with complex notions involving concept formation, insight, and long term attitude development.

One suggestion of an alternative perspective is present in some of the work which has been done recently and included in a book edited by Young (1971). G. Esland in a section entitled "The Sociology of Knowledge" relates the present trend of educational research to the 'objectivist' view of knowledge. Seen in this way, some of the inherent assumptions concerning the evaluation of pupils and schools are made clearer. For example.

"Objectivism has been firmly embedded in the norms and rituals of academic culture and its transmission. Through the procedures of psychological testing and school evaluation, the pupil and the curriculum have been reified. 'Bodies of knowledge' are presented for the child to learn and reproduce according to specified objective criteria. Educational psychology has been a powerful legitimating agency and rationalization for objectivism. As such, it has become an important form of social control."¹⁵

In the same volume, Bernstein relates the importance (and neglect) of the socialisation process within schools

"How a society selects, classifies, distributes, transmits and evaluates the educational knowledge it considers to be public, reflects both the distribution of power and the principles of social control. From this point of view, differences within and change in the organisation,

transmission and evaluation of educational knowledge should be a major area of sociological interest . . . Indeed, such a study is a part of the larger question of the structure and changes in the structure of cultural transmission. For various reasons, British sociologists have fought shy of this question. As a result, the sociology of education has been reduced to a series of input-output problems, the school has been transformed into a complex organisation or people-processing institution, the study of socialisation has been trivialized "16

Here the view is that the systems approach, an example of which was quoted earlier (footnote number 2), is reflective of the values and philosophical goals and presumptions which are embedded within any given society.

In summary then, although the work which follows will not deal primarily with the problem of the socialisation of knowledge or personal interaction in the classroom, the methodology which is developed will be sensitive to the implications which are generated from such a perspective. More specifically, the important points raised above can be summarised and listed with a general hint as to what will follow in the presentation of the methodology.

First, one cannot ignore the context in which knowledge is transmitted. Therefore, information must be available on at least the basic question of the student-teacher roles, student perception of the educational tasks, and social pressures and expectations.

Second, there is a great difficulty in examining 'cognitive processes' (which can be defined here as the children's models and strategies to deal with their perception of the academic work), totally outside of the classroom. Because of this, data from tests of I.Q., creativity, and achievement, can at the most provide a very incomplete notion of the environments where children learn best. Further, they are especially weak in telling us why children perform as they do under any particular set of conditions.

Third, it is important to take into account the philosophy of education of the administrators and teachers when dealing with the social determinants of the students' perceptions of knowledge. This need not necessarily be done by asking them for a written or verbal reply to a questioning of their individual philosophies. Rather, it is critical to be aware of those things which teachers consider important and unimportant for the children to be able to do. Just as an awareness of the context of the overall society in which the knowledge is being transmitted is significant, in like manner students' attitudes towards the classroom tasks are relevant when evaluating their skills and abilities.

Implications of these beliefs will be developed in Chapter Two where the overall design and specific methods used in the research will be outlined.

Chapter Two The Research Design

Before going into detail on the specific procedures that were used, I would first like to establish a contrast between the psychometric and an observationally based paradigm.¹ In the former, it is generally very important to specify those tests being used, the pre and post testing conditions, the method of data analysis, etc., before their actual application. This is generally cited as being the need for objectivity and scientific rigor. In the observationally based paradigm² (also called sociological/anthropological), the research methods are more oriented towards the establishment of a starting point or attitude set. Therefore, what I have listed below is not a blueprint for discovery, I do not believe that is possible. Rather, the details of the method must remain flexible as problems are encountered and new needs develop out of the flow of information. It is not a rigid format, but a backbone, so to speak, of my approach to the investigation of the learning process in the classroom.

Outline of Approach

- (1) OBSERVE
- (2) FOCUS
- (3) MODEL
- (4) DEVELOP HYPOTHESIS
- (5) EXPERIMENT (in the lab)
(in the classroom)
- (6) APPLY
- (7) THEORISE

The outline in numbered sequence is an abbreviation for the following. The work started with observations in a particular classroom in a specific school (1). The general area of concern was defined as those factors centering on the students' reaction to and handling of the instructional tasks in the classroom. As the research progressed (2) certain clusterings began to occur and patterns were found to exist. Here I focused in on

a certain set of variables and elaborated them. Relationships were postulated between these clustered factors (called classroom factors or simply "F's"), and a simple model (3) of the basic relationships during teaching was forwarded. From this model, a hypothesis (4) was developed that dealt with one area which I found to be relevant to my own interests, namely the problem of autonomy and feedback in the learning environment. A formalized version of this hypothesis (or simply "H"), was reviewed in the appropriate literature, and some support was found for its psychological foundations. The hypothesis was formalized into a series of experiments (5) in order that the factors could be put to an empirical test. It also permitted some other peripheral areas to be explored, both in the literature and by informal tests. These were generally areas that were not revealed in the observational stage by itself. The final two steps, bringing the change into the classroom (6), and making it part of a generalized theory of education (7) were not done in the experimental section of the research, but rather were dealt with in Part IV of the thesis.

Let me now go into a bit more detail concerning each of the first four steps of the research

Observation a base line was acquired through the method of non-participant observation. While this entailed the taking of detailed notes, no attempt was made to participate in or interfere with the classroom occurrences. The general areas of interest that helped determine the observational perspective were: 1) periods of instruction, 2) general student-teacher roles and relationships, including the language used, 3) the students' perceptions of the academic tasks, i.e. satisfactory levels of performance, teacher expectation, 'Passing' or 'Failing', etc.

Focusing. this occurred about 1/3 of the way through the total set of observations. It was in large part a result of my interest in the maths instruction and the strategies which I saw the students developing in response to their perception of the specific tasks at hand. In particular, there was a pattern of similar behaviour which took place in the groups of students who were having some difficulty and whose strategies "didn't work" or were not recalled long after the test or assignment. More specifically, why are some children able to 'go beyond the information given'³ and

extrapolate to solve previously unseen problems, while others must be provided with extremely detailed or rote procedures which they can apply, though not without some difficulty, to a set of problems which have the same basic structure?

Modelling is the procedure used to provide a 'dynamic' description of the learning processes. The statement of the model attempts to illuminate or highlight consistent aspects of the way the various children behave in the classroom: It also serves to incorporate their relationships and attitudes towards the teacher, the assignments, criteria of successful achievement, and so forth.

Observation, Focusing and Modelling thus form the first general stage of the research. As was mentioned earlier, the 'research attitude' developed at this stage is one of avoiding any severely restrictive or pre-determined methodological package. For a more detailed example of work carried out in this manner, the reader is referred to the following studies. Becker (1968), Perry (1968), Kahne (1969), Parlett and King (1971), and Henry (1971).

An examination of some of the difficulties which the methodology presents is discussed in an article entitled "Problems in Participant Observation" (Schwartz and Schwartz, 1955). The areas of data collection and analysis is dealt with in the same journal in a work by Vidich (1955) entitled "Participant Observation and the Collection and Interpretation of Data". While Becker deals with some of the problems encountered in his Medical School Study (cited above) in an article "Problems of Inference and Proof in Participant Observation" (1958).

These works, while by no means exhaustive or definitive, nevertheless give, I believe, a good introduction to the development and use of an observationally based technique in an educational (or institutional) setting.

Once the observations were completed, a hypothesis was presented concerning some particular area of interest contained within the model of

the basic teaching-learning process. Experimental conditions relating to the hypothesis were then devised to both 'test' predictions implied by the hypothesis, and, importantly, to probe and 'tease out' ideas which could lead to more generalized theories concerning the nature of human learning and thinking within the classrooms

The final step of constructing a critique of the current theoretical paradigms is reviewed in the concluding chapter, as a complete examination of them would be beyond the scope of the present work

Each of the two following chapters will deal with one main portion of the observational procedures. Chapter Three follows with the initial observations, while Chapter Four contains the focused observations and a presentation of the model and hypothesis.

Part II Research in the Classroom

Introduction

The specific procedural items which are suggested by the methodology developed in Chapter One can be enumerated as follows.

- (1) Step one is to locate a school where co-operation is found.
- (2) Next, observations are made, initially as a non-participant, in order to get a general outline of the day to day activities, and also to produce a baseline of data which can be referred to later in the study.
- (3) The philosophy of education, various goals and objectives of the teacher and the school are then obtained from observations, questions, comments, etc. From this, a list of desirable student traits are drawn up, from which an 'ideal student profile' is constructed.
- (4) The observations are then 'focused' with the central area of interest being the instructional and evaluative methods used in the maths curriculum. From this a basic model of student-teacher interaction is forwarded.
- (5) The behavioural implications of the model are contrasted with what was reported in the ideal student profile and the overall set of educational goals.
- (6) A hypothesis was then developed and put forward as a key concept valuable in understanding any discrepancy or inconsistency between the above goals and their 'cognitive' consequences.
- (7) Finally, an experimental test of this (initially to be conducted outside of the classroom) was suggested as a means of making the hypothesis more explicit, and also as a procedure which can yield experimental data.

As was stated earlier, the above is not intended to be a 'formula for discovery', but rather is a sequential format developed after the main portion of the research was carried out. Its presentation here is intended to be an introduction to the specific classroom work which follows.

Chapter Three Initial Observations

3.0 Introduction

Initial observations were conducted with the aim of acquiring a 'total' or 'world view' of the activities within the particular setting as they are perceived by the participants. Within this paradigm, one is expected to make a conscious effort to limit the number of pre-observational assumptions about both the overall purpose of the school system and the various classroom roles that are found to exist within it. From this point, one develops a set of relationships between various aspects of the structure of the situations which are observed.

The formulation of a specific structure of behavioural interaction provides a framework to help deal with the massive amounts of data that one tends to accumulate in any observational study.

In the present study the data was initially gathered by direct observation in the classroom from an unvarying location near one corner of the room. Early in the study, every attempt was made to avoid any interference or disruption of the conduct of the classroom lessons. This was accomplished in part by arriving and departing at a constant time each day. Also, movement in and out of the assigned 'observational' seat was avoided until the basic set of observations had been concluded.

In the later stages of the school visitations, observation was sometimes accomplished by moving from desk to desk, occasionally looking at what the students were working on, although verbal contact was still avoided. At this stage I also began to question the teacher in some detail about particular occurrences during the lessons, general issues of procedure, and her own educational philosophy.¹

Once the basic data was collected from the day to day observations,

analysis was begun to identify those factors which clustered around the stated concern, namely, the children's cognitive processes in the classroom.

A set of progressive focusings here helped in the selection of those factors found to be of particular interest. This concept of a "progressive focus" has been described elsewhere (Parlett, 1972) as a method of inquiry whereby one proceeds from a general set of observations to a more directed, systematic, and selective inquiry. It is in this series of steady progressions that certain patterns of behaviour emerged and became the focal points for a more specific search within the classroom. Following this, it was decided that a model could be developed under the general heading of cognitive processes, which itself would be naturally incorporated into the larger scheme of events. A further goal was to avoid an unnecessarily artificial view of the classroom processes based upon a set of pre-conceptions held by the observer or the teacher.

Still, the problem of bias cannot so easily be dismissed, (see Schwartz and Schwartz, 1955, referred to earlier). In analysing the data, every attempt was made to deal with any bias by recognizing and exploring its meaning and particular effect on the research. One cannot make a study or evaluation of a classroom without becoming concerned with questions of the progress of individual learners involved. Since personal values enter into consideration here, this issue reflects an overall concern with the expression of bias, especially realizing one's own stake in the research.

Even with the above problems, this general type of "open-ended" observation was found to be preferable to a set of tests or even a rigid and systematic checklist of observational events in the classroom. The former alternative would no doubt simplify the actual acquisition of data in addition to allowing one to quantify it for analysis. However, the

required sample size and need for rather strict experimental control could produce an artificial condition that would not be representative of the original processes that were being considered. In addition to this, when attempts are made to determine the scope of the study in advance, they can have the effect of limiting pursuit of unexpected or unexplained results in the data. Concentration on methods designed to handle and manipulate numerical data can prevent inclusion of other material which is often relegated to the role of being 'anecdotal' or 'subjective'. Data of this sort must often be used, however, if one is to explain some of the findings that may arise out of the study. The necessities of formulating a precise plan of research with specific tests usually make the possibilities of such explanation impractical. Thus they are often ignored.

As in the case of psychometric testing, checklists also dramatically restrict the possibility of noticing factors which were not at first assumed to be pertinent. In effect, the design of the observational or testing technique in both of the above cases will determine the general area of inquiry before contact is made with the actual classroom (the data base) ² Preference was therefore given to a method which allowed the final techniques to be developed out of the needs that presented themselves in the study. "The problem defines the methods used, and not vice versa." (Parlett, 1972)

Finally, in considering such factors as role playing, decision making of the teacher, group interaction, and so on, I have adopted an overall paradigm similar to the one expressed by Smith and Geoffrey (1968), in their examination of an urban classroom in America

"In considering participant-observation we have placed our emphasis on concept formation and model building in contrast to verification research "³

The adoption of this position summarizes the desire for a dynamic model which will permit one to incorporate within it a whole range of

factors that are pertinent to the overall goals of the educational research.

3 1 The Arrangement

Observations were conducted on a random basis and begun in the month of November. Both morning and afternoon⁴ sessions were watched, and every attempt was made to ensure that all of the particular kinds of activities that occurred were seen at least once.

All observations were recorded in a notebook for future reference. Initial attention was paid to the progression of the more structured events that occurred on the surface of the daily activities. These included notations concerning the curriculum, the students' movements from place to place in the room, the overall noise level, the teacher's instructions, and so on.

I was personally introduced to the teacher by the school's headmaster, through whom all of the administrative arrangements took place. Both the teacher and the headmaster were told that my general area of interest was in the psychology of the instructional process. However, in order to maintain as neutral a stance as possible relative to the students, I was introduced to them briefly at my first visit as "someone who is going to sit in on some of our classes". Since the school was occasionally used for both this type of observation and also practice teaching, I was assured that the students did have previous exposure to guests and visitors and that my presence in the room would not be that unusual. In fact, this became apparent in a short time, for the students' initial interest and occasional glance my way quickly faded and soon disappeared altogether. Although I did 'sense' an initial curiosity about what I was constantly writing down (no child ever approached me to ask about it though), I noticed that within a few days this too disappeared, perhaps after the

children came to realise that whatever it was I was watching and writing about, it wasn't a secret report on them - at least not individually!

3.2 The School

The school itself is a primary (but not infant) school located in a recently constructed building within a housing estate outside the centre of Durham City. It is a Church of England affiliate school, and the Church provides about 25% of the operating expenses. Religion is not provided as a separate subject although there is a short service each morning in the assembly hall. These are conducted by the headmaster and are attended by all 260 children. There are 8 classes, two sections each (arranged by birthday) for the 4 age groupings, 7-8, 8-9, 9-10, 10-11.

The children that I worked with were the older section of the 10-11 year olds. There were 23 students in the class and the absentee rate was usually between 5-10%. The physical arrangement of the classroom can be found in the appendix.

In talking with the headmaster, I was told that the school considered itself to be on the 'progressive' end of the spectrum, and that by and large they had adopted a stance which moved away from the notion of teaching 'subjects'. Except for reading, which was held at a specific time each day using the S.R.A. reading⁵ program, most (but not all) other subjects were conducted within an integrated curriculum arrangement. The school was particularly strong in its music program and utilised the concept of classroom project to teach writing, art, science, etc.

The teacher was personally most helpful and she willingly related her teaching methods and overall insights which she had gained through her experiences at various schools. Also, since she had been teaching for quite a few years, her teaching style was refined (when compared to new teachers) and her conduct within the classroom did not seem to be the least

bit affected by my presence. I have found from personal experience elsewhere that it is often more difficult to get permission from inexperienced teachers to observe them within the classroom, and when it is provided one's presence can often be seen to affect the classroom procedures. Thus, in these cases, it is difficult to surmise exactly what would have happened with no observer present.

3.3 General Observations

The children are given a large amount of freedom in their movements around the classroom as long as the level of noise does not rise to the point where it is objected to by the teacher. When this occurs, the teacher (referred to as Mrs. G), will object out loud with a comment such as "there is too much loud talking in here .. get on with your business quietly", whereupon the noise level drops dramatically for a while before rising again slowly.

The morning class begins after the assembly period, and in general it forms the more outwardly structured part of the students' (henceforth S's) day. In the afternoon S's are expected to choose what they want to do from a list of possibilities and suggestions which are written on the blackboard. S's with particular problems take this opportunity to bring them up to Mrs. G's desk. Thus, there is a certain amount of planning that the student must do independently if he is to finish the various project tasks on time.

Also, unlike the 'O' or 'A' level work, there is no set syllabus or curriculum which must be followed in order to reach a pre-stated goal by the end of the year. Instead, the more structured courses like maths, reading and spelling follow a formulated progression from one very general area to another in a sequence of gradually increasing difficulty. This is in direct contrast to the situation that will be facing the S's

in the next year, whether they attend either the large comprehensive school nearby or the smaller and more traditional school which had been a Grammar School until quite recently.

When I asked Mrs G about the educational transition from the 11 to the 12 year old period, she acknowledged that it is a marked contrast and takes some adjustment for most of the S's. However, it was seen by her as a kind of inevitability which stemmed from the formality which is an historic part of British higher education. Thus, secondary schools continue to be more like what they were in the past while many primary schools were either new or had changed to meet the different attitudes toward child rearing.

Mrs. G also pointed out that the system of exams given at the 'O' and 'A' levels require a syllabus which would define the basic abilities needed to pass the course. Moreover, Mrs. G noticed that the system of training people to teach at this level is conducted at the University which itself is often rigidly formal and impersonal in its method of approach

Interestingly, many of the points made above in the early parts of the pre-observational conversations with Mrs. G returned in a more subtle form as portions of daily classroom life clustered around this issue of the context and content of instructional tasks. More specifically, aspects of the 'hidden curriculum' (see Snyder, 1971) emerged which demonstrated a whole array of classroom cues that the students were found to be responding to in their own perception of the immediate classroom goals.

3 4 Ideal Student Profile

As a result of detailed discussions with Mrs. G concerning the process of education within the primary schools, the following list was made and presented as a profile of the kind of student attitudes and abilities the

school was attempting to foster. Aspects of this 'ideal' student will be referred to later when contrasts found between the focused classroom observations and the school's goals and objectives are compared in the development of the hypothesis on classroom learning.

Summary of information on the teacher's goals:

	In general, acceptable & desirable traits	In general, unacceptable, undesirable traits
<u>Academic</u>	thinker / insightful conceptualiser / genuine problem solver	replication / repetition rote performance / copying 'blind follower'
<u>Style</u>	original / creative clever / flexible novel approaches / critical thinking (healthy doubts)	stereotyped / slow dull / unimaginative rigid / inflexible

Ideal Student Profile. Works well in groups, but is independent also, has a value system but is not totally dominated by peers, has a strong self-image, but is not totally ego-centric, seeks approval, but does not need constant acknowledgement of small accomplishments, is original and creative, easily gains insight into the classroom tasks presented to him, seeks work outside the established project or curriculum and can extrapolate from information presented to uses in other situations, circumstances, and within different contexts, is a "self-starter".

Thus, the methodological problem discussed earlier of the way various values, goals, and legitimising agencies affect the psychological evaluation of learning tasks is dealt with in the present study by accepting as valid those statements from within 'the system' (i.e. the teachers and headmasters) as to the school's overall purpose. The ideal student profile serves as a reference point reflective of the goals of the day to day activities of the classroom. An important question is whether the processes occurring during periods of mathematical instruction provide the means for the development of the stated student traits.

3 5 Summary

It was one of the central arguments in the first part of this thesis that the development of a coherent "interactionist framework" from the observations must take into account the (often unstated) values and attitudes toward the entire process of schooling if they are to bear any generalizable truths. One cannot successfully adopt a reductionist position of simply referring to the means as teaching or instruction (with educational psychology as a 'legitimizing' advisor) through which the ends of an established set of skills, abilities, insights, etc are accomplished

One must therefore treat with great care any attempt at limiting such an inquiry to "fully controlled" conditions. It is not possible As most recently expressed by Kollos & Lundgren (1975):

"Pedagogically speaking, teaching has to be looked upon as an integral part of the educational system. The starting point is thus the system, and not the psychological processes within the individual learner or the teacher. From a theoretical point of view this will lead to a set of concepts not derivable from psychology, and a refutation of logical empiricism as the sole scientific basis for educational research."⁶

It must therefore be kept in mind, as the specific details of the data emerge, that educational problems may not be fully reduced to psychological ones. Rather, once one is provided with an overall social and educational framework, psychological methods and perspectives may be useful in identifying those factors which are most relevant to a model of the educational process. One can then go on to inquire how these factors interact to affect learning within the present school system. Only when such an internal relationship is realized can the methods of psychology be useful in dealing with some of the important areas of concern in the educational setting.

Having said this, I would like to proceed to the focused observations of the classroom activities.

Chapter Four A Model and Hypothesis of Mathematical Learning in the Classroom

4.1 The Classroom Factors

In the two examples of focused observation which will follow, the data was analysed by referring to what I have called Central Classroom Factors, (F's) which were developed from the overall observations and are presented here in the form of a list. Further, the structural assumptions made by the teacher in her conduct of the day to day lessons was also formalised and put into an outline which will be discussed later in the chapter.

This outline determined in large part the perspectives used to construct the model of classroom behaviour which follows directly from an examination of the focused observations.

Classroom Factors

- F (A): Discipline
- F (B). Mistakes and Failure
- F (C). Competition (and the development of acceptable standards of accomplishment)
- F (D) Assessment (assignments, exams, grades)
- F (E). Teacher - Student Roles

Explanation of the Classroom Categories.

Discipline (F-A)

This category consists of either the teacher singling out an individual student for behaving in an unacceptable way^a, or displaying dissatisfaction in general for the class's performance or actions (especially level of noise)^b. It can also include the establishment of disciplinary rules^c or guidelines designed to set limits on what will be tolerated.^d

(examples: - from notes taken on actual classroom occurrences)

- a) a student gets out of his place and is told

"go to your seat! and is given a gentle push in that direction.

- b) to the whole class: "I don't want to hear anything from any of you, we've a lot of work to do and we need your co-operation today."
- c) while regular teacher is away, another teacher assigns some work and says as he leaves the classroom: "Now that you've something to keep you quiet, shouldn't it do so?"
- d) teacher enters a noisy classroom and says sarcastically: "I can see a lot of work is being done - by Jove, those notebooks must be full!"

Mistakes and Failure (F-B)

This category appears most often in an actual assessment situation, as in the giving of a maths test. It supports the model of: /teaching (instruction)/ → /learning/ - which is subject to → /assessment/ by providing an undesirable set of behaviours, which can be labelled either as 'wrong' or more generally as 'failure'. Therefore, not only is being wrong a situation to be avoided, but certain attitudes are formed which make it very difficult to learn from mistakes. This may relate to some of the research on conceptualisation which shows the difficulty of incorporating negative information into an operation.¹ However, there are other implications besides this which could affect the ways that students develop their thought processes in school. For instance, a child may not know why an answer was wrong, and he may come to doubt the validity of the processes which lead to it. Continual reinforcement of such behaviour limits the range of strategy choices, and tends to reward conservative (high probability of

success per turn, low payoff per success) choices, and discourage adventurous (low probability of success per turn, highly successful payoff) thinking.²

Competition & Standards (F-C)

Again this is more prevalent in situations where the actual task to be done is overt and thrown open to the entire class. The notion of standards of accomplishment is necessarily a bit vague, but refers to the minimum the teacher will accept from any individual student in terms of effort expended.

The net effect of this factor is to isolate individuals during most task sessions, and all assessment periods, and clearly stresses the idea of individual vs. group responsibility for the accomplishment of work. Further, the particular teacher's attitude under this heading will delineate the areas considered to be 'cheating' (inappropriate), from those considered as co-operation (appropriate). To a certain extent, combined with different levels of student accomplishments, it helps to justify treating different students in various ways, even when their behaviour has been the same. (see the contrasts between notes No. 11 and No. 19 in scenario No. 1 which follows).

Assessment (F-D)

This forms the 'feedback' and 'categorising' function which follows the "I teach - you learn" model of classroom activity. An important point to note here, especially in the younger grades, is that students will sometimes question the specifics of the methods of assessment ("that was an unfair task" or "that exam was too hard" or "they had more time to do the project than we did") but rarely will they question the need for the procedure of assessment itself. This awareness on the part of the student of the form or context (the so-called "Hidden Curriculum referred

to earlier) has a great deal to do with the development of their strategies to deal with classroom problems. Having the right answer is more central than being able to get the right answer. As Holt³ has noted, knowledge can be externalised and children so often attempt to grab something "out there" rather than viewing their job as an internalisation of the processes and procedures of the subject area. They become estranged from the potential beauty of the structure of the material, or even its simplicity, and interest is often difficult to maintain in the subject material. In addition, classroom skills can come to depend upon a complex array of cues and hints which centre in on a highly specific and context-bound problem format. Children are thus able to do the examples which they are presented with, but are often unable to deal with unfamiliar, though only slightly altered versions of problems which embody the same level of conceptual difficulty.'

Teacher - Student Roles (F-E)

This is the largest category, both from the point of view of the number of examples which appear in the scenarios, and more fundamentally, the aspects of the basic teaching model which it encompasses. It follows from the teacher's role as a figure of authority, both socially and intellectually. From the student's point of view, the teacher has a large amount of control over the choice of material to be learned (input), the methods of instruction (technique), and the assessment of understanding or learning (output). She also has much to say about the relative merits of subject areas, and which knowledge is valuable and likely to be useful to the students in the future. This is especially true of the 11 year olds where reference to secondary school becomes more and more frequent as the year goes on.

The student's role is generally passive. By this I don't mean that they don't move about, question, resist, etc., but that they are limited as to any overt reactions which they might have in a classroom about what they really think about any particular subject area (see comment No. 2 in

scenario No. 2 for a good example of this). This reference to passivity also alludes to the student's dependence upon the teacher for prior approval to do many things. This is usually cited as being in the need for control - else anarchy could come to the classroom if all the students did as they saw fit when they saw fit!

Beyond this, however, Factor E is an important means of supporting the core assumption in the teacher's model of classroom learning; that is that learning follows from teaching. The amount of individual activity and autonomy which a teacher will permit follows, I believe, from the amount and the form of instruction which is to be taking place within the classroom.

It also forms the basis for identifying the appropriate forms for the knowledge to take. In most cases this can be seen as a demonstratable set of skills and abilities to carry out certain basic functions (see comment No. 3 in scenario No. 1). These skills are often demonstrated by what I have referred to as "fill in the blank" or closed-ended questions which are very common in the classroom (see comments No. 9, 18, 20 in scenario No. 1 and comments No. 2, 3 and 4 in scenario No. 2). Here the teacher has a fairly specific idea of what she wants to hear, as is often illustrated by the "yes, but ...", or the "yes, go on ..." type of response to a student's answer.

It is suggested that some of the behavioural consequences of this factor may be the unintentional and subtle pressures which are put on students causing them to avoid risky (and often creative) behaviour in the classroom. Also, it could be seen as fostering a kind of dependence (rather than the stated goal of independence), since students become more and not less reliant upon the teacher for the acquisition of further knowledge within an area, and importantly, further areas of knowledge. (the "Is this O.K.?" and "What do I do now?" syndromes.)

4.2 Two Scenarios

Under the general heading of the context and content of classroom knowledge and instruction, the above factors were found to be important in helping to formulate the ways children reacted to classroom tasks. More critically, how are the children's strategies viewed once we identify some of the methods used in the classroom to determine acceptable levels of performance and success? While many of these methods (for instance requiring neatness as a builder of 'character') do relate to the school's role as a socialising institution in our society, others (such as concentration on a final right answer as a rigid right or wrong rather than the thought process which lead to it) run counter to the educational goals stated in the student profile(see page 24).

Briefly, the children's General Classroom Strategy was to respond to the F's (Classroom Factors) by adopting an overall role of conformity (within acceptable levels of peer competition), orienting themselves toward performance on assessment tasks which were acceptable, and avoiding as far as possible failure, mistakes or any overly risky behaviour which could challenge the role of the teacher as authority or the image of the student in the eyes of the teacher. In short, children are very aware that certain things are expected of them and they often gear their work to the tasks which allegedly demonstrate their competence. Showing that you know something becomes a better investment, so to speak, than a deep understanding of a certain area of interest because it pays more dividends. Children respond to the standards of rewards and punishment and in general they are more successful when their strategies fully take into account the evaluation scheme. There are many instances of this in the literature, (Henry, 1965 & 1971, Leonard, 1968, and particularly Miller & Parlett, 1973) and I shall not dwell on it long, but rather continue to develop the model which follows. Herein lies the implications for a hypothesis on classroom tasks and the

teaching of effective learning methods.

Two separate situations, both involving maths, will be described below. Following the two texts will be an examination of the occurrences through the proposed model. Quotes given are those taken down at the time, and although they may vary slightly from what was actually said, every effort was made to maintain accuracy. The verbal activity is very important, particularly in the first scenario, as it highlights certain assumptions about the separation of the content and context during many of the classroom tasks.

Scenario No. 1 (November 11, 11.15 a.m.) The Maths Test*

<u>Actions</u>	<u>(coded Classroom F's)</u>	<u>Verbalizations</u>
(1) Mrs. G hands out the tests face down	<u>A</u> <u>C</u>	"Don't turn your paper over until you are told to do so"
(2)	<u>E</u>	"If you want anything, put your hand up - do not call out!"
(3) While giving out the test instructions, Mrs. G asks:	<u>E</u>	"What is the rule for rounding off to the nearest 1000th?"
(4) A student repeats a rule for doing it; this occurs again as she asks how one would find $\frac{1}{4}$ of a million if one didn't know offhand	<u>D</u>	
<u>Test begins at 11.25 a.m.</u>		
(5)	<u>C</u> <u>D</u>	"Keep your eyes off everybody's paper and just do your own"
(6) A student enters the classroom in order to pass into the room next door and many eyes turn to look, there is much eye movement in general some of the S's watch where Mrs. G goes, others begin to put up their hands	<u>B</u> <u>C</u>	(sounds of some 'grunts' of effort/disappointment)

* A copy of the questions given in this test may be found in the appendix.

- (7) Many hands are up now, children at the far end of the room must wait longer for Mrs. G to come over to them; 2 S's in the front of the room have yet to start (11.30 a.m.) and wait with hands raised; Mrs. G leaves an
- (8) S's desk saying:
- (9) At one S's desk she gives an interpretation to one of the questions on the exam saying:
- (10) A student who had attracted Mrs. G's attention and who proceeds to point to
- (11) question No. 2 on her paper is told:
- (12) It is now 11.40 and Mrs. G has not stopped going around for a moment
- (13) At the conclusion to each visit to one S, Mrs. G has said:
- (14) While walking from one S to another:
- (15) S asks a question and says Mrs. G answers it by explaining a bit and then asking 3 short-answer questions in a row, she then says:
- She returns in less than a minute to clarify further and he works on
- E
- B "You haven't got your thinking caps on!"
- E "What comes here ____?" (S answers)
"That's right!"
- C "Work it out for yourself, I don't mind telling you when your answer is wrong, but you have to work it out (eventually get it) for yourself"
- E "Remember, you must put No. 2 in words"
- C "Good girl"
- A "Keep your eyes on your own paper"
- C
- E "I don't understand this"
- E "That's all Gary"

- (16) Mrs. G stops at student's desk (hand was not raised) and asks: D "Are you stuck again?"
(S shakes head negatively)
- (17) Goes to S who has hand raised, says: E "That's right (in response) to a finger pointed at a question)
- (18) Time is now 11.55, 2-3 hands are still raised, Mrs. G visits another S E "How many ____?" asks Mrs. G; S responds, she replies, "That's right"
- (19) One S points to a question number, this S has not had many questions throughout the exam, therefore pointed finger is sufficient to C evoke response
- (20) Mrs. G breaks down one question: E "How many ____ in ____?" S responds, "That's right, now work the rest out for yourself"
- 12.05 p.m.
- (21) There is now more noise in the room and some students whisper 'Mrs G' as they raise their hands B
- (22) Mrs. G tells them to finish up and the test comes to an end. Some few students continue to work frantically on their papers, Mrs. G helps, but more briefly. Test ends

Scenario No. 2 (November 13, 11.00 a.m.) A Maths Lesson

<u>Actions</u>	<u>(coded Classroom F's)</u>	<u>Verbalizations</u>
<p>(S's are in 2 groups for maths, Mrs. G has the beta group, while some of the S's leave for the next room where the</p>		

- (1) alpha group meets Mrs. G A "Now, in your beta books - let's
enters (11.05) E have all this fussing stopped,
what have we been doing?"
"Graphs" one S replies
- (2) Mrs. G writes GRAPHS on
the board E "Why are they important"
- (3) Here Mrs. G asks a series of questions which require a single word
answer. I have termed them "fill in the blank" as they usually
take the form of "Graphs always contain ____?" or "Some graphs of
this kind are ____?", there being a known correct answer for each
question. When a student fails to correctly identify the missing
word or words in the sentence, Mrs. G will reply, "Well yes, but
what else ...?", meaning that the answer is not wrong, but that
there is a more specific or exact explanation that she is seeking.
- D & E
- (4) Mrs. G continues this procedure, introducing notions concerning the
types of graphs (pie, column, dot, line) and in general asking
questions which have a predetermined answer in mind.
- E
- (5) Her general method of
approach is to ask for the
general concept or rule
governing it and then ask for E
specific pieces of inform-
ation which result from it.
She continues for a while
sometimes calling on S's
who do not
- (6) have their hands up. To
one S. A "You were not listening because
you were playing with your hands!"
- (7) At 11.23, Mrs. G hands out
small sheets of lined graph
paper. Through the same
question-answer technique,
she makes sure that the
students know how to use D
the paper. The S's are
told to divide their paper
into four sections and
begin to work from their
workbooks (beta books).

- (8) Although a few students begin to work on their own, most require confirmation of points by Mrs. G, others rely on her for information D concerning aspects of the E work itself. Mrs. G is constantly on the move from desk to desk; the children appear to be concerned that they do not make any B mistakes C
- (9) Some children are confused about the fact that turning over the graph paper to put the written answers produces the paper's margin in an unfamiliar place. E
- (10) Mrs. G continues to walk around the room stressing the neatness of the work. E
- (11) "If you are not sure of anything, then ask" E
- (12) The S's continue to work until 12.00 when they put their papers away for next time, and the rulers and pencils that were handed out are collected. Session ends.

4.3 Analysis

Interpretations and comments below will correspond to the numbers that appear at the left of each episode in the scenarios. Here the five Classroom Factors will be referred to by their coded letters To review they are as follows

- A DISCIPLINE
B MISTAKES AND FAILURE

C COMPETITION
 D ASSESSMENT
 E TEACHER-STUDENT ROLES

INTERPRETATIONS

Scenario No. 1

- (1) A & C implication that action will be taken if the rule is violated
- (2) E direct example of S's dependence on the teacher
- (3) E problems consist of a) knowing, then b) applying correct rules to the questions on the test
- (4) D
- (5) C & D
- (6) B & C behaviour suggesting that there is an activity taking place which is distinct from the every day classroom activities, there is a greater tension and a feeling of getting the present material done, and done right
- (7) E S's rely on teacher for more information than is available from the problem as it is presented on the exam
- (8) B
- (9) E there is a correct answer which the teacher knows and the S must 'get', either by guessing, working it out by applying the formula, or asking the teacher for hints
- (10)
- (11) C minimum level of effort/accomplishment before the teacher's interaction is clearly spelled out
- (12) E teacher gives information related to the form of the required knowledge
- (13)
- (14) A & C
- (15) E (same as number 12 above)
- (16) D present classroom situation is clearer stated as a task
- (17) E
- (18) E (also the same as number 12 above)
- (19) C a change from No. 11 in that S who has not asked any questions before may be entitled to just point to a question in order to get more information
- (20) E (See No. 9)
- (21) B & C tension rises as 'deadline' for task approaches
- (22) FINISH

Scenario No. 2

- (1) A & E domain of interest for the present situation is clearly outlined
- (2) E continuation and statement of the importance of the task area; S's however are more concerned with 'completing the blanks' than any real statement of what they think is the importance of the area in question
- (3) D & E classic example of task area, see No. 2 above
- (4) E important point here is that the answers to the stated classroom questions are included in the presented material, in this case the workbooks, and the problem becomes one of location rather than extrapolation or derivation from what is already known
- (5) E more of the teacher's style, the important point here is that the rule in question is seen to be external from the information given and is therefore a matter of memorisation rather than having the S's come to see the relationship between the material and the information available from it; strict boundaries are presented around the content and context of the area of knowledge, in this case the embedding of information within a mathematical model, the graph
- (6) A S's must follow the course as prescribed by the teacher if there is to be advancement from one state of knowledge to the next. Failure to observe or exploring on one's own (i.e. not pay^{ing} attention) results in discipline and a chance to miss the transmission of a piece of knowledge (read: formula) or procedure
- (7) D assignment of task after insuring that students a) know how to use the paper b) follow the prescribed course of activity
- (8) D & E the important point here which is only implied by the model is that this series of activities demonstrates the fact that in general the procedures tend to make the S's more and not less reliant on the teacher for further work, this is in part true because of the teacher's almost total authority in deciding what areas are to be considered as proper for the enquiry into the subject matter and the assessment task at hand
- (9) B & C mistakes, as will be stated in the hypothesis, must be a part of the overall feedback in an exploratory encounter

- with the material. Here, they are part of a minimum standard required and sometimes they are used to activate punishment (mostly by embarrassment)
- (10) E the form (the paper's heading) is the primary area of concern
- (11) E another example of the above
- (12) E Mrs. G reinforces the S's need to rely upon her to do the work such that it turns out neat, is not wrong, and is what is wanted (as opposed to work which is conceptually correct, but incomplete because it does not address the specific task that is being tested)
- (13) COMPLETION of activity area for today (the clock is also a factor in structuring some of the S's encounters with the subject material)

4.4 The Model

The two examples presented above were by no means atypical of the conduct of the classroom during periods of instruction. The interpretation given and "Classroom Factors" are to be taken more as suggestions on my part than any kind of an exact analysis.

Presenting a high degree of descriptive detail provides a means for understanding exactly how I developed the following Model of the classroom interaction. Many such encounters as those described above were observed before the clustering of the categories were formulated into the general pattern of classroom interaction postulated to exist between the teacher and pupils.

The simple model presented below is not intended to explain all of the behaviour found to exist within the classroom, or even during the mathematical instruction. Rather, my hope is that it will highlight those factors which I found to be the most important and most consistent in determining both the ways children learn maths, and what it is to be accepted in the classroom as successful and as legitimate learning. It is here within the assumptions and rules⁵ which govern the roles in the

classroom that the contrast emerges between the desired behaviour set out in the ideal student profile and many of the more immediate constraints embedded in the day to day school regime. The hypothesis on maths learning which follows is an attempt to deal with the contrast

The Model Outlined

- * The teacher in her role of authority within the classroom serves as a legitimising agent for the value of the material to be taught
- * Within the framework of what she believes the learning of this particular material (maths) requires, the teacher makes an evaluation of the various levels of achievement of the students
- * Very importantly, the teacher believes that learning follows directly from teaching
- * Teaching is based upon the instruction of curricular material. This consists of introducing, demonstrating, explaining, questioning, correcting, providing assignments, evaluating (grading), and assessing the overall learning which occurred

The Curriculum is formed into a list of.

Subject Areas from which certain Topics are chosen for study. These are conducted through a series of Projects which can themselves be sub-divided into Tasks, Assignments, and Classroom work, these are assessed by demonstrating knowledge in classroom discussions, adequate performance on assignments, and passing grades on tests and exams

Therefore, the teacher decides

- * Value of material to be learned (includes the future usefulness, appropriateness, requirements of school, interest to the students)
- * Initial level of student abilities (includes where to set the 'standards', what to expect, and what level to pitch the classroom presentation)
- * Format which curricular functions take (tasks, assignments, projects, materials to be used)

Instruction proceeds under the MODEL that

Teaching ----- leads to ----- Learning

Learning is assessed as an end product, primarily, but not totally,

by performance in * Classroom discussions
 * Written work (assignments)
 * Tests and exams

4.5 The Model in Operation

Utilizing her role as a figure of authority, the teacher develops an initial evaluation⁶ of the general abilities, present state of knowledge, and particular skills of each student. In addition to this there is the student's concept of the importance of his overall performance in school as set by parents, brothers and sisters, peers, etc. This is characterised as the starting level. Through lectures, questions asked in the classroom, and written work, the teacher instructs the class in order to provide the students with an ordered sequence of information which deals with the area in question. This forms the basic method of transmitting knowledge and is sometimes referred to as 'chalk and talk'. As mentioned earlier, it proceeds under the assumption that the teacher knows something or has some skill which the children do not. The task at hand is for her to describe or illustrate the material in some way such that the students learn it. This is usually documented by a test, assignment, or in some cases a verbal demonstration of the ability in question. The students are thus transformed through temporary states of knowledge / skill until they reach the desired final level. By and large, the value of the required final state is an assumption backed up by the teacher who also occupies the role of assessor in the classroom.

4.6 The Hypothesis

Granted then that in some way the above factors interact to form the basis of the instructional programme. What implication does this hold for the educational psychologist? A key question here is are the processes shown above effective in satisfying the goals and assumptions which were taken to be part of the overall school system (i.e. the ideal student profile)? In one way I believe that they are not. This is in

the internal inconsistency which we find between the stated student profile on one hand, and the methods which are used to evaluate classroom work on the other.

Here I would like to introduce the concept of autonomy. In the hypothesis which follows, it is stated that a certain amount of autonomous and unstructured exploration must take place if an individual is to gain insight into problems that he is presented within the classroom. This is postulated as one of the important criteria in the process of concept formation. Without the autonomy to make decisions, gain feedback, and permit learning to occur from mistakes and initial failures, students are often forced into a pattern of working which is oriented towards accomplishment of evaluative tasks rather than exploration of all aspects of the material at hand. Speaking from the observations once again, the children were found to be responding to the form more than the content of the instructional material.

Following a statement of the hypothesis below, I will illustrate its potential usefulness when it is combined with the model by devising a set of experimental tests of it. In this way, I believe that the claimed advantages of the model's being a 'dynamic portrayal' of the classroom events can be more completely examined.

The Hypothesis on Mathematical Learning in the Classroom

"the basis of developing higher order conceptual strategies and gaining insights into the structure of the mathematical material within the classroom depends at least to a limited degree upon the amount of autonomy and self-structured activity which a student has to

- (a) observe, free from excessive pressure, the interactions of particular items operating under the rule or algorithm of the system. This would help to develop a set of internal representations of the groupings or patterns within the operation, i.e. notions of harmony, consistency, symmetry, equivalence, invariance, etc.
- (b) actively develop hypotheses of his own, i.e. to exhibit self-programming and self-starting activity, and to test these hypotheses

within a context which is free from the immediate evaluation of the results.

- (c) explore the 'boundaries' of the system, that is, to be able to discover, at least in part on his own, examples of where application would and would not be appropriate, to make mistakes and create hypotheses which are incorrect or incomplete and be able to learn from these instances by developing the ability to eventually distinguish them as negative instances. This relates to the 'boundary' point made above and further develops it as a kind of 'learning how to learn' by using feedback and a critical sense of doubt.
- (d) to reach a level of insight free from the imposition of a prematurely presented rote or 'overly-simplified' device by the teacher. This includes the importance attached to the formula used, especially when it is easily constrained by the context of the particular task, and further by the external necessity of having to complete the task at a particular point in time.
- (e) to be unconcerned with demonstrating this knowledge on an examination or under close scrutiny at least until the structure has been assimilated into existing forms of mathematical relationships. This point is concerned with the fact that a child may be able to perform certain operations without initially being able to explain how he has come to the correct answer. Certain evidence⁷ suggests that an attempt to have a child make explicit those procedures of conceptualisation which are successful at predicting accurate results could itself result in a regression, that is, loss of the ability.
- (f) have the ability to engage in periods of 'play' or 'reflection' when either no immediately discernable goal is at hand, or where there is no observable behaviour taking place at all.
- (g) to determine, at least in part, some of the form and order in which the information will be dealt with. This implies decisions about specific aspects at the level of the "Project" (see page 40) downward, in addition to thoughts concerning the value of the material and the appropriateness of its various applications. "

This is not intended to be a psychologically rigorous hypothesis on the development of mathematical concepts. Rather, it stems from the contrast between the ideal student profile with a combination of both the model of the structure of the classroom instruction, and information taken from experimental psychology. It is particularly intended to 'highlight' those parts of the day to day life in the classroom which present what I have found to be the most significant aspects of the contrast between the statement of the teacher's aims for the students, and the kinds of situations

which emerge out of her role as 'the teacher'.

There are three implications which the hypothesis on mathematical learning holds in relation to the kind of 'mathematical thought' which the students may come to develop from their encounters with the curriculum in schools.

The first concerns the selection of strategies which are chosen to deal with what the students themselves perceive to be the 'classroom tasks'. Here it is postulated that the structure of the classroom inhibits the development of higher order and more generalisable strategies, both because it often tends to reward 'the right answer' and further because it rarely presents opportunities or encouragement for autonomy to genuinely explore the mathematical system. This is no doubt in part due to the lack of suitable materials for such exploration, but it is tied up also, I believe, with the teacher's assumptions about a) what a classroom should 'look like', i.e. the "I teach, you learn" model, b) the control and authority of her role, which makes it quite difficult to permit individual activity, mistakes, and play within the set of expectations of both the pupils as well as the administration and parents, c) and, stemming from point b, there are pressures to ensure that a great deal of segmentisation takes place, that is that the children are adequately and properly prepared for secondary school, and that the 'requirements' and standards of the coursework and the school itself are being upheld. Since it is easier to demonstrate performances on standardised tests and examinations, and also because of their claim to being more uniform and objective than other forms of assessment, selection pressures are at work to reward and favour teaching methods which lead to these goals, and likewise instruction and grading which correspond to them.

The second of the three points follows directly. After the desired

strategies are selected, the conditions in the classroom greatly limit the attainment of any but a small set of highly predictable and uniform procedures for dealing with the classwork in this area. This simply means that a desire to foster true understanding or insight is not enough, and that the opportunities must also be present for a student to deal with the subject area in ways which will reflect the key items that are stated in the hypothesis on learning. In practical terms, this means an alteration of some aspects of the role relationships, as well as the development of certain 'internal' standards of accomplishment within the students which will ensure a healthy return from the "learning to learn from feedback" argument. This is in addition to the question of over-all logistics and scheduling which a situation where people were doing different things at different times would create. Clearly, aspects of the competitive and uniform standards (Factor C) would have to be altered toward greater tolerance and encouragement of natural differences between individuals.

The third point concerns the fact that the classroom often places proportionally much less value on the long term retention of specific concepts and also on the ability to generate them (as to their use or non-use) in unfamiliar situations, either within an area of study, or in situations which provide possible applications that are outside the immediate concerns of the area of study. Experience with students who have just taken an exam and are now 'free to forget it all', and Gestalt experimentation which showed a greater ability to remember partially completed rather than fully completed problems, both illustrate the kind of conflicts which are present between current knowledge on the acquisition and retention of concepts, strategies, and problem solving abilities, and the factors which determine the classroom's choice of a) the material to be learned, b) the methods which are used to teach it, and c) the means and value attached to 'objective' assessment of completion of the material, and acquisition of

the objectives of the subject area.

4.7 Summary

In this chapter I have presented two sequences of classroom behaviour as viewed through the context of the overall observations. The analysis which followed was derived from the Classroom Factors (F's) and the outline of the Curriculum (page 40). The consistent appearance of these F's in most of the instructional periods lead to the creation of the model of classroom behaviour.

It will be left to the reader to decide whether or not the examples presented here are contained within the general structure of the model. One must remember that no effort is being made to explain all of the occurrences in the classroom, but rather only those factors which most directly determine and affect the students' ability to react to the instructional material. It is here that the students' perspectives of the tasks within the classroom come together to formulate attitudes concerning the structure of classroom knowledge. In turn, these attitudes have been found to be most directly developed by events in the classroom which themselves are created by the need for evaluation, the teacher's role as authority over the domain of knowledge, mistakes and failures which are incorporated into a competitive and disciplined classroom function, and most importantly the lack of autonomy in the students' selection of those aspects of the material at hand.

In the following section I will present a possible direction for obtaining experimental evidence of the proposed hypothesis.

Part III An Experimental Investigation of the Hypothesis

Introduction

The following chapters which compose Part III of this thesis will report the results of a series of experiments designed to explore the hypothesis on autonomy and mathematical learning

Since the hypothesis put forward at the conclusion of Chapter Four was not in a rigorous enough format for direct experimental testing, but rather was much more speculative and descriptive, I would like to summarise its central points from a more experimental point of view. This distillation of the hypothesis will then be further sub-divided into three categories for the purpose of reviewing the appropriate psychological literature in each of these areas. This review of the experimental work reported in the literature will form the content of Chapter Five

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To summarise, the hypothesis has asserted that the conditions which are most likely to favour conceptualisation of mathematical material are those which permit autonomous as opposed to passive receipt of information, the opportunity to construct and test one's own hypothesis, opportunities to discover positive as well as negative examples of rules and to make unpunished mistakes, freedom from premature examination, the opportunity to have periods of play and reflection, and finally, to determine in part, the appropriate applications of the material in a variety of settings (i.e. as problems to solve)

I have chosen three areas of experimental psychology which relate to the above and which I will explore further. They are play, concept formation, and creative problem solving

First I will examine the cognitive (as opposed to the social or fantasy)

aspects of play, with particular reference to autonomous receipt of information

Second, I will review the work on concept formation, leaving out the large portions which deal with verbal associations and learning, and instead concentrating upon the mathematical and symbolic skills of classification, hypothesis testing, and the conditions affecting the selection of general strategies and reasoning.

Finally, a section will be used to discuss some of the implications of the large body of work done on problem solving. Here, particular attention will be paid to the effects of previous experience on hypothesis generation and the factors affecting the method of approach.

The present section (Part III) will be concluded with a summary of the four experiments (Chapters 6-9) which, when taken together, form the empirical evidence in support of the hypothesis. It will be argued that the experiments relate closely to the three areas of concern, namely, play, conceptualisation, and problem solving, and further, they hold several implications for the teaching of mathematics in the classroom

Chapter Five A Review of the Literature

Section One. Play

5.01 Overview

Interest in the area of play and exploration as it relates to problem solving and cognition can be traced from the classic work of Wolfgang Kohler (1926). Kohler believed that apes were capable of demonstrating insight when faced with a problem to solve. The "double-stick" problem¹, which has since become generally well known, was used to investigate this hypothesis. Kohler himself was not present when Sultan, a chimp, eventually did solve the problem. From the diary of the keeper, Kohler concluded that the two sticks were connected during a period of "free play" when Sultan was not considering the objective of acquiring the bananas outside of his cage.

In a less well known study, Duncker (1945) used infants of 9 - 14 months in a task which required them to reach a toy placed some distance away from them. Duncker noted that most infants who were able to 'instrumentalise' the stick and solve the problem did so by some chance occurrence during play which enabled them to re-organise their perceptions.

However, in an influential piece of work, Birch (1945) demonstrated that the chimp solution could not be totally attributed to this Gestalt notion of a re-organised perception. He used (six) chimps, each of whom had detailed case histories of their behaviour on record. Only one chimp had ever been seen to manipulate sticks of any kind before, and when he was presented with the problem he immediately used the stick as a rake and solved it. One other chimp solved it 'accidentally' when he touched the banana with the stick and saw it move. The other four chimps could not solve the problem at all although their frustration after thirty minutes

revealed that they were interested in the prize! Birch then permitted them to manipulate the stick for three days in their cages. When next presented with the problem all chimps were able to solve it in less than one minute.

This raised the question of what it is in the experience which aided problem solving. In a later piece of work, Schiller (1952) showed that young chimps (under six years old) who had played with sticks were still unable to solve the problems. He claimed that stick use is not acquired through an experience such as play and manipulation, but rather is an innate motor pattern.²

Similarly, in an attempt to attribute a cognitive aspect to play, Sutton-Smith (1967) found that children in a kindergarten class gave more hypothetical uses for toys which they preferred during periods of free play. The difficulty with the study is that the total set of experiences with the toys might be interfering with the effects of play with the toys. Controlling for this in an elaborately designed experiment on four, five and six year olds, Dansky and Silverman (1973) found that children who had played with certain objects were able to name significantly more uses for the toys on a measure of associative fluency.

However, the question can still be raised as to whether other experiences which provide the same information would also be sufficient to give rise to the experimental results in these situations. Can an 'exploration', 'instruction', or 'yoked' demonstration of the information be as effective a pre-test experience as play? This form of the question provides a basis for much of the empirical work which is to follow.

5.02 Theoretical Considerations

From a theoretical point of view, Kohler saw play and exploration as two different kinds of activities. Working with children, Hutt (1966)

makes the same distinction. Others, notably Berlyne (1960) do not distinguish between the two.

Aspects of these arguments on the unique nature of play appear in the literature which has taken an interest in the evolutionary basis of play activity. See Lorenz (1972), Beach (1945), and particularly Eibl-Eibesfeldt (1970) in this regard. The common theme here is that the higher mammals take advantage of long periods of dependency in their youth to develop patterns of play and exploration. This in turn encourages manipulative curiosity which is helpful in an animal's gathering of knowledge in his environment. Thus, playful individuals within species may be favoured by natural selection. This in turn permits an elaborated form of behaviour which contains components that are found to be useful in the more serious (feeding, social development) aspects of later life.

Bruner (1972, 1974) comments on the flexible basis of play activity. He relates this to the psychological notion of 'rule-extraction' and calls this type of learning 'generic'³. Coining the term deixis, he argues that play and problem solving have a similar basis. The main difference can be seen in the nature of the goal. In play, the outcome is eventual and secondary to the activity which precedes it. However, problem solving works toward a particular goal which governs the specific patterns of the feature extracting activities.

There is support for this idea in the writings of Ellis (1973) as well as Herron and Sutton-Smith (1971). What appears to be of importance in the cognitive development of children is not the actual information extracted through play, but the ability to internalise methods of acquiring further information in addition to a flexible approach toward future problem solving and 'use of objects' tasks.

Relating this to intellectual development and writing in 1966,

Sutton-Smith contends that Piaget has attempted to reduce play to a function of thought and thereby deprives play of any genuinely constitutive role within thought. In his response, Piaget (1966) contends that the 'active mode' is presented throughout and that, in fact, " . all concepts are derived first from the action and then from the operation, which is another way of saying that concepts are the expression of an assimilation by schemes of transformation." (page 111)

However, it should be noted in the experimental work which follows that I will not be primarily concerned with the overall effects of play on the development of the child or with the theoretical issue of its exact role in assimilation/accommodation. Rather, play will be used as an activity which embodies the most important aspects of the notion of autonomy which were presented in the hypothesis on mathematical learning. Specifically, these are the freedom from tension and excessive drive, the ability to 'programme' one's own behaviour, and importantly, the development and use of certain 'constructs' which permit incoming information to be related to the patterns, structures and rules of mathematical systems.

Section Two. Concept Formation

5.11 Theoretical Overview

It has been the intention of this thesis to demonstrate the limitations involved in applying laboratory studies directly to classroom situations. Nevertheless, I believe that it will be useful to review some of the experimental work done on conceptualisation.

In reviewing this literature, I shall not be concerned with the large amount of work which has been carried out concerning the role of maturation, environment, and particularly, Piaget's stages of development on children's thinking. Rather, I will be primarily concerned with non-verbal concepts,

ie, those which are either mathematical or symbolic. Although there are many processes which take place while concepts are being formulated (motivation, the development of stages, the effects of various distractors, etc.), after Vinacke (1974), I shall concentrate on the idea that concepts "... are mediating processes rather than direct sensory data, and represent the elaboration, combination, and reorganisation of those data." (page 106)

For purposes of this thesis, the experimentals used will essentially involve the process of developing higher order mathematical concepts. In the tasks which will follow, the subject will be at an advantage if he can develop an organised system of relating the information which he has encountered into a conceptual framework. It will be argued that developing these patterns or relationships enable a subject (a) to have more flexibility in his approach, (b) to be able to generalise to other situations, and (c) to retain the information over long periods of time.

Most of the work done on this type of conceptualisation has focused on the attainment of the concept itself, i.e., "by observing errors, inquiring into reasons for the decisions made, comparing subjects with each other, and relating his response measures to variations in the task, to inferred cognitive characteristics, to motivational conditions and so on" (Vinacke, 1974 page 109)

A second area of research concerns itself with the strategies which subjects use in the development of concepts. This is best illustrated by the work of Bruner, Goodnow, and Austin (1956). In this work the authors propose that a distinction should be made between concept formation and attainment. In concept attainment, individuals are already in possession of the categories, and therefore they need only apply their definition to various instances to see which do and which do not meet them. In concept formation, which may be thought of as a more fundamental process,

it is the task of the subject to actually establish the categories themselves. While some authors would question the 'psychological reality' of such a distinction (Bolton, 1972) concept formation defined in this way closely resembles the processes which occur when subjects encounter the mathematical information presented to them in the chapters which follow

Although the work of Bruner and his colleagues at Harvard demonstrated that the processes involved in thinking should no longer be considered to be beyond the reach of empirical investigation, their central concern was still the strategies subjects developed to deal with the problems which they were presented.

In their 1956 work cited earlier, Bruner, et al pointed out that conjunctive, disjunctive, and relational concepts could each be associated with a particular rule of classification. Conjunctive rules are identified by the 'joint presence' of more than one defined attribute, disjunctive rules are identified by the presence of one or another defined attribute, while relational concepts are classified according to the rules which relate aspects of a particular instance, such as 'more or less than'. For example, those figures which have more stars than circles on them may be one such example

Variations on these rules may be found elsewhere in the literature, (Neisser and Weene, 1962, Laughlin, 1968). In general, this type of research deals with the kinds of factors which affect the subject's selection of strategies to deal with the problems presented to him. The questions to which I have addressed myself in the work on mathematical learning which follows are primarily concerned with the conditions under which conceptualisation takes place and as it affects the development of strategies and the results of assessment along a number of different parameters. This type of experiment also has a long history although it has been used primarily

in the learning of verbal concepts. Here the conditions varied are usually limited to the order of presentation, presentation interval, instructions given and the like.

5.12 Methods Used

Before reviewing the results of some of the experimental investigations which are found within this literature, I will first present a brief outline of some of the methods which have been used by investigators in the study of concept attainment. This will then be followed by a selection of some work, the results of which are of particular interest in a consideration of the conditions which affect conceptualisation.

The nature of the conceptual process, as one of organisation, has provided a basic methodological key used in most experiments. To quote Vinacke (1974) once again, "The subject must learn how to respond to stimuli in some fashion not determined by the properties of the stimuli considered as separate objects " (page 150)

Three techniques (as outlined by Vinacke) used to explore this process will be considered

(1) Introspection. Early investigators had subjects use self-report. Although this method is usually only of interest from an historical point of view, (i.e., Fisher's work, 1916), more recent work, (Rogers and Haygood, 1968) has suggested that direct questioning of subjects may yield information not otherwise obtained. In addition, work done by Stenild (1972) and Gagne and Smith (1962) provide somewhat contradictory results as to the role of verbalisations by subjects while conceptualising. It appears that in some tasks subjects benefit by their own verbalisations, while in others, the unconscious influences in the development of concepts may be disrupted by the level of awareness required by verbalisations. This may have ramifications on protocol procedures such as those used by Newell and

Simon (1972).

(2) Memory and Learning. The second type of task requires the subjects to respond (either verbally or by pressing a button) when the appropriate stimulus is presented. Most common are the experiments (Hull, 1920, Kuo, 1923, Trabasso and Bower, 1968) where the subject is first taught names for a series of stimuli and then is required to relate them to a second series which shares some common property with the first. These tasks usually resemble memory experiments in the first instance and require subjects to transfer learned material to a second set in order to test whether or not he can name them properly. With adults, non-sense syllables are often used.

(3) Problem Solving. This procedure is similar to the one above and most directly relates to the experiments presented in the following chapters. Subjects are presented with a collection of stimuli and are required to classify them in a way which satisfies the conditions established by the experimenter. This can involve sorting tasks (Grant and Berg, 1948), definition of the concept (Smoke, 1932) or a task where the subject must select those stimuli in an array which belong together (Heidbreder, 1949).

5.13 Experimental Work

Although I will not be working within this tradition in the experiments which follow, I believe that a brief review of the following work will be helpful in pointing out some of the factors which have been found to influence a subject's performance on a conceptual task. For example, there are various sorts of pre-training which will acquaint him with aspects of the stimuli. These can include practice, a variation in the interval between presentation of instances, the use of cues, attention directed to irrelevant dimensions, providing the subject with hints, and so on.

Bourne, Goldstein and Link (1964) found that there are beneficial

effects of allowing previously exposed instances to remain in view while the subject is solving a problem. Pishkin and Wolfgang (1965) found that negative feedback of information is of the least use to the subjects. Similarly, this difficulty of using negative information was found by Donaldson (1959), Wason (1968) and in the Bruner study cited earlier (Bruner, et al., 1956).

Laughlin (1969) found that feedback was important in the acquisition of conditional concepts and concludes "In summary, performance improved directly with increasing information specification, as specification of only one case resulted in less effective performance than specification of any two cases or of all four cases " (page 372)

The subject's perception of the task may also affect his performance. Thus, Archer, Bourne and Brown (1955) found that subjects who were told to be more analytical had less variability in their approaches and were more easily able to deal with complex tasks. More detailed information concerning the likelihood of one or another dimension sometimes aided subjects (Bornstein and Grier, 1968) while Wolfgang (1967) and Byers and Davidson (1967) both found evidence that verbalisations and a statement of current hypothesis increased performance. Byers and Davidson conclude from this that the role played by instructions is very important and that experimenters must ensure that they are followed and understood by the subjects.

A concept may be learned both by becoming familiar with positive examples which fulfill the criteria of the category, as well as by observing cases which are not included in the definition of the category.

Smoke's conclusion (1933) that concept attainment was not significantly greater when subjects worked with positive and negative instances than when they worked with positive instances alone was criticised by Hovland in 1952

Hovland argued that it was not obvious from the Smoke study whether or not the negative instances were actually low carriers of information or whether they presented difficulty to the subject who was attempting to assimilate the information which they do convey.

However, in 1953, using an experimental arrangement which was designed to control for the amount of information conveyed by positive and negative instances, Hovland and Weiss found that subjects did have more difficulty in using information concerning the attributes which are not in the concept to be learned. There are many other experiments which demonstrate that attainment is made easier as the subject has available to him a greater number of positive, as opposed to negative examples (see Mayzner, 1962, Schvaneveldt, 1966, Freiberger and Tulving, 1961). However, Johnson and White (1969) demonstrated that it was not the positive instances themselves which were critical, but the amount of usable information they contained. Thus, Fryatt and Tulving (1963) showed that continued practice by subjects reduces the differences in performance when using positive vs negative instances.

In a paper published in 1952, Underwood argued that one requirement for concept attainment is contiguity of stimuli. This is supported by the review of Dominowski (1965) with the warning that there have been too many problems explored in this area with too many different techniques used. This, he argues, has made data comparisons difficult and that some simple and direct measure is still required of what is and what is not retained in the memory after concept learning

Along these same lines, Haygood et al. (1969) found that by increasing the contiguity of positive instances, the attainment of both disjunctive as well as conjunctive concepts was aided.

Short term memory is very important for single-concept problems, but

the use of problems which embody more than one concept at a time require the subject to retain the correct hypothesis. This was more important than attention given to the stimulus itself (Restle and Emmerich, 1966)

Another dimension which has provided experimenters with an opportunity to vary the conditions of the task is the span from concrete to abstract. Evidence here indicates that performance increases as the stimuli presented becomes more concrete. Therefore, pictures are easier to deal with than words, and words dealing with physical objects (such as table or chair) are easier for the subject than words dealing with abstract concepts (such as beauty or truth). See Davidson (1952) for an elaboration of this theme.

Evidence of this kind, because of its consistency, has caused some authors (for instance, Harvey, Hunt and Schroder, 1961) to postulate a series of progressions or stages from more concrete systems to those which are more abstract and personal and do not necessarily depend upon externally verifiable rules for their existence

There is additional support for this idea from studies (Heidbreder, 1947, Wenzel and Flurry, 1948) which show an order within the conceptual process. Thus, objects themselves are the easiest to learn, their shape and colour more difficult, and the number of objects more difficult still. When subjects are given the opportunity to sort objects into their own categories, they will usually produce systems of classifications which are based upon the type of object or its use (such as with the class of objects containing athletic equipment) rather than on its shape, colour, or texture.

Finally, Mandler and Pearlstone (1966) varied the procedure of the typical experiment where the investigator imposes a rule or classification system on the data. Instead, they allowed subjects the freedom to sort cards (which had words on them) into their own categories. These rules

were then imposed on a second or 'yoked' group with the results being that the 'autonomous' group had fewer errors and took less time to attain the concepts.

This last example relates closely to the experimental designs which follow in that there will be an 'autonomous' or 'play' group which will be compared to a 'yoked' group. The difference in what follows is that the 'autonomous' group will not produce the concept which will be imposed upon the 'yoked' condition, but rather will be provided with an opportunity to select the order and rate of information input within a system which operates under a hidden or embedded mathematical rule. The 'yoked' group will have access to the same information (as their 'autonomous' partner) and both will be measured on their ability to attain an understanding of the system in question.

Before doing so, however, there remains the task of reviewing the conditions which affect solution in problem solving tasks

Section Three Problem Solving

5.21 The 'Classical' Background

Much of the early experimental work on problem solving had its origins in the work of the Gestalt psychologists. Many different kinds of tasks were used which ranged from puzzles to the solving of abstract mathematical problems in geometry. Although the experimental rigour in much of the early work was weak (by modern standards), nevertheless it provided a basis for more refined experimental techniques which followed.

Three aspects can be mentioned which underlie the contribution of most of the early work. First, they were concerned primarily with the reorganisation of perception* when human subjects are confronted with a

* the development of insight leading to 'productive thinking' - see Wertheimer, 1945.

problem to solve. This should be viewed against the background of most of the experimental work being carried out at the time which was either taking place with animals, or else in highly abstract situations.

This leads to the second point which is that the tasks used related to actual problems which one may encounter in day to day situations, albeit in a simplified form. They were 'whole' problems and attention was paid to the entire process of solving them rather than just the logic used, inferences made, or whatever. Third, most of the work hoped to generate interest in its potential application to educational situations. Thus, Wertheimer, concerned with the over-emphasis on repetition and past experience wrote.

"To study the use of what one has gained in past experience is of profound interest, but for our problem in a first approach, it is not decisive whether the material used stems from present or from past experience. What is important is its nature, whether a reasonable structure is grasped, and how it is brought in " (1945, page 69)

Wertheimer here was really expressing dismay at the emphasis on mechanical drill which he found in his role as school inspector. He felt very strongly that mechanical mastery should free the mind to do more important tasks rather than binding it slavishly to routines. In addition, problem solvers had to overcome the obstacles presented by an ego-centric point of reference. Here we find the expression of one of the essential points within the Gestalt tradition the fact that the solution lies clearly within the problem and that what is required in the first instance is a reorganisation of perception. Thus,

"The main cause of unreasonable, blind behaviour seems to be that a person sticks to an old view from sheer perseveration and habit, which make him ignore or even actively deny the more reasonable requirements that are clearly indicated in the situation." (Wertheimer, 1945, page 171)

Wertheimer's approach was extended by one of his pupils, Karl Duncker

(1926, 1945). In Duncker's view, the solution to a problem required a variation in certain of the relationships in component parts of the problem. Thus, potential 'solvers' were seen as those subjects who could look over a problem-centered situation in order to determine first, where the difficulty was, and second, how it could possibly be removed. In his classical "X-ray" problem⁴, Duncker states

"The final form of an individual solution is, in general not reached by a single step from the original setting of the problem, on the contrary, the principle, the functional value of the solution, typically arises first, and the final form of the solution in question develops only as this principle becomes successively more and more concrete. In other words, the general or 'essential' properties of a solution genetically precede the specific properties, the latter are developed out of the former." (1945, page 33)

The analysis of the problem has clear-cut implications for training experience. Katona (1940), reported a series of experiments in which he attempted to compare 'memorising' of a solution with methods designed to promote 'understanding' of its basic principles. Using both card tricks and match-stick puzzles, Katona found that the memorising group was at a slight advantage during the initial tasks, but as variations to the problems were introduced, the understanding group performed significantly better *

Maier (1930), noted that even if subjects were able to analyse the difficulty in attaining the goal within a problem, they may not be able to solve it without an awareness of one or more principles. Maier referred to these as sets or directions and he used a number of rather difficult problems which were designed to demonstrate the significance of these ideas. Using common materials such as chalk, wire, wood, clamps, etc., Maier had his subjects construct two pendulums which would swing over two different spots on the floor. The only group of subjects (of the five groups tested)

* refinements in this task and experiments carried out by Hilgard, et al (1953, 1954) will be mentioned near the end of this section.

which had any significant number of solvers (that is, in this case 36.3%) was the one which was given a demonstration of part of the solution with a directional hint supplied. Either part solution, or directional hints alone were not sufficient as the results showed. The total solvers in the other four groups were only 16% of the subjects.

In 1951, Reid gave one group of subjects several 'aids' designed to help them in their analysis of the final or goal state. This was to test Duncker's assertion that problem solving is greatly assisted by the introduction of such techniques. Reid found that aids which pointed to crucial requirements of the goal were much more helpful than other aids or no aids at all. Similarly, Burke, Maier and Hoffman (1966) devised a problem which used two sticks and a clamp as materials to construct a hatrack. Hints provided, such as, "The ceiling must be used", and "The clamp itself is the part the hat is hung on", were found to either inhibit an already adopted direction or provide an opportunity to find the correct direction toward an eventual solution of the problem. Thus, some hints can be effective both in re-directing fruitless approaches, and also in opening possibilities for alternative methods of approach.

Writing in 1951, Birch and Rabinowitz suggested that previous experience with objects might provide some basic information about the objects which can be applied by a subject who is attempting to vary the ways in which it can be used. To test this they used the problem first presented by Maier (1931) in which two strings hang from the ceiling out of reach of one another with the goal being to somehow tie the two together. The groups of subjects were given different experiences with the materials (an electrical switch or an electrical relay) which were potentially available as weights or plumb-bobs for the string. As predicted, the subjects who solved the problem more often used the electrical part which they did not previously observe in its proper function in the electrical circuit.

Along these same lines, Adamson (1952) showed that functional fixedness could be induced by requiring the subject to use an object (later needed as part of the solution) in a functional capacity completely unrelated to its later use.

Relating to the work on functional fixedness is Luchin's (1942) concept of Einstellung (or fixed tendency). In a series of experiments using water jars (of varying capacity) within a problem solving arrangement, Luchins found that subjects use the same patterns of relationships (between the jars) to solve problems even when more simple combinations were available. Further, Luchins demonstrated that subjects could be made to avoid some of these difficulties by writing "Don't be blind" on their papers prior to the presentation of the final problem.

Gardner and Runquist (1958) demonstrated experimentally that the strength of a particular set or fixed tendency is proportional to its use in pre-tasks and that it becomes progressively more difficult for 'experienced' subjects to shift to a new solution

To quote Vinacke (1974): " .. Duncker's principle of functional fixedness has received strong experimental support. Not only do directly perceived and habitual functions of objects interfere with their utilization in different or new ways, but factors that help to render the required functions more salient reduce fixedness of the prior or usual function (or the reverse) " (pages 260-261)

5.22 More Recent Experimental Work

The work reviewed here has demonstrated a number of basic considerations which must be taken into account and which may affect a subject's ability to solve a particular problem. Since this relates most directly to the structure of the problems presented in the next few chapters, I will cite the results of some of the other experimental work which has followed in

this 'classical' tradition.

At the conclusion of their 1956 experiment, Kurtz and Hovland made the following observation.

"Finally, the effect of grouping instances may depend upon the general manner in which S's set about to solve the problem. Although not much is known about the conditions determining choice of approach, the present authors have observed that S's differ in the extent to which they make use of information conveyed by concept instances in formulating verbal hypotheses about the nature of a particular concept." (page 242)

Here we find an attempt to link subject performance to the kinds of strategies which they employ during a form of problem solving. What other evidence can be cited which bears on this issue of the conditions which affect successful solution?

Anthony (1966) used a simple mapping problem to see if subjects worked backward or forward in finding the solution to a problem. Sixteen of the twenty subjects made their first attempt by working forward while only four initially worked backward, even though the particular problem was much easier to solve if one began at the conclusion and worked backward. However, with successive trials, subjects showed a willingness and a tendency to switch directions (within each trial) suggesting that the direction of problem solving ability depends both upon the kind of information given initially, (in the form of instructions, etc.) in addition to the information which arises out of the total set of problems to be solved.

In a series of experiments using the Katona card tricks (mentioned earlier), Hilgard, et al. (1953, 1954), found that even the 'understanding' group made some errors which they 'shouldn't' have made. Although some of the errors were careless and depended upon the mechanical features of the methods used to solve the problems, others showed that "The tendency to revert to a method of rote memorisation appears when the 'understanding'

methods fail " (page 461)

Finally, there are the experiments which require the subjects to adopt a specific orientation to the problem in order to solve it. Sometimes this may require trial and error behaviour, it may require memory or the use of a rule or set of heuristics, while still others require recentering or understanding of the desired goal state.

Durkin (1937) stated that there are three patterns or methods which are used in general problem solving. They are:

- (1) Trial and error
- (2) Insight (that is, perceptual reorganisation)
- (3) Gradual analysis

In Chapter Nine, while analysing one of the experiments, I will be making further use of (3), gradual analysis, in order to inquire into methods and strategies used by the subjects as they search a visual domain. Specifically, can the patterns found in the data of those individuals who acquire insight into a set of relationships be identified with regard to their level of performance on related tasks and the hypotheses they make while surveying the incoming data in an active/passive search mode?

Bartlett (1958) in his work on thinking emphasised the 'adventurous' and 'gap-filling' nature of this process. This pattern is useful in what he termed 'closed systems'⁵, that is, where there are a limited amount of components to be used in a situation.

Johnson (1960) attempts to differentiate various problem solving phases from the solution itself. The first phase, which he calls preparation, includes the occurrence of a large number of errors. In a simple experiment (Johnson, Lincoln and Hall, 1961) using a lighted two-part panel, certain component features of items (i.e., descriptives) were

put in groups down the left side of a display and objects (i.e., exemplars) were put down the right side. Only one side of the panel could be lit at once and subjects themselves controlled the timing. The results showed that as the number of descriptives per group increased, preparation time (left side of panel lighted) increased, but solution time (right side of panel) did not. This provides some evidence that preliminary aspects of problem solving are in some ways separable from the solutions required.

Goodnow and Postman (1955) showed that subjects had a tendency to search for principles behind a solution even when this was not warranted. Using a card trick task where the subject has to choose the correct one from a number of possibilities, they report:

"This experiment demonstrates the occurrence of probability learning in a problem-solving situation. Confronted with a two-choice task, S's learned to respond in accordance with the probabilities of alternative outcomes even though they did not recognize the task as a probability situation and attempted to find a lawful solution to the problem " (page 21)

Later, Goodnow and Pettigrew (1956) used a 'one armed bandit' where the payoff was secretly determined by the experimenter (although the subjects believed that it was controlled by the machine, that is, operating under some rule of 'chance'). Their findings relate to the overall issue of hypothesis formulation.

"The S's use their choices as direct tests of specific hypothesis rather than as tools for data gathering with hypothesis testing held in abeyance. As a result, the information gathered is related to a specific hypothesis, and if the latter should prove to be wrong, it is only with difficulty that the information can be transformed and made relevant to another hypothesis. As a rule, S's do not transform information but start from scratch again with their next hypothesis. Furthermore, S's frequently become so preoccupied with their level of success and failure that they can pay attention to little else. Finally, and most important, in the course of directly testing one hypothesis after another, the variability of their own responses obscures any event pattern. It is as if one tried to make observations in a scientific enquiry without a standard set of operations." (page 385)

They continue

"When the initial and final patterns are the same, the initial experience may lend a useful distinctiveness to the initial pattern ." (page 391)

Markova (1969) has noted that the complexity of the problem often influences the kind of hypothesis which is formed. Therefore, simple versions of problems yield general hypotheses while complex versions yield hypotheses which are differentiated with respect to a number of sub-goals. Simon and Simon (1962) have coined the term 'selective heuristics' in their study of problem solving based on the game of chess. By applying these strategies successfully, one can discover how difficult combinations of moves are developed.

This evidence has been summarised as follows:

"People typically approach problems by formulating and testing hypotheses, tactics not readily identified from merely the objective characteristics of the task. Nor is hypothesis forming necessarily appropriate, and, if not, the subject finds the task more difficult than it really is. Success in solving problems thus depends more on experience with relevant situations, on the complexities of the problem, and on the accessibility of information, than on failure to develop and test hypotheses." (page 299, Vinacke, 1974)

5.3 Overall Summary of Chapter Five

There is evidence in the literature that autonomy (or 'play') has potentially beneficial effects on later problem solving and conceptualisation tasks. In addition, the review of the literature can be distilled into the following items which should be taken into account in any experimental situation involving concept formation or problem solving. They are

For Concept Formation

- (A) The feedback given to the subject
- (B) The presence of negative information
- (C) The subject's perception of the task (instructions)
- (D) The order and rate of presentation of items of data
- (E) The stages from concrete to abstract (of the ease of formulating items into concepts)

For Problem Solving.

- (F) The Gestalt notion of a 'reorganised perception'
- (G) Functional fixedness (and Einstellung)
- (H) The spectrum of memorisation to insight (in assessing successful performance on a concept task)
- (I) The use of hints or directional sets (relating to 'C' above)
- (J) Subjects' previous experience with components (both in a manipulative and symbolic context)
- (K) The tendency of subjects to form hypotheses (even when it is not an efficient method of dealing with data)

For the Assessment of Subjects.

- (L) What are the strategies used by the subjects, how did they develop, how are they used, how effective, flexible and generalisable are they? (effective refers to the performance in the task, flexible is how easily they are able to be changed when they are found to be ineffective, and generalisable is their use in other related tasks.)
- (M) The insight or understanding gained, i.e., the difficulties of subjective measurement
- (N) The ability to predict events or to solve problems, the development of performance criteria; objective measurement

Overall, this presents us with the following outline and structure for the experimental design

The Task: It should provide the subject with a 'dynamic' situation which relates to a mathematical concept that has not been previously seen and which can be explored or otherwise received passively in terms of 'bits' of data. There should be the allowance for several different modes or levels of perceiving and ordering the data.

The Conditions of the Experimental Groups The access to the 'data' should be varied from an autonomous (or play) condition where subjects choose the form and order of the information to a 'yoked' condition where they receive a pre-specified set of data. There can also be 'instructional' (the same set of data prepared for the entire group), and 'hybrid', (some time for exploration and some of the time passive). Of course, a control group should also be used to provide a base line and guard against a higher score in one of the two groups above, but an overall depression when compared with the base-line.

To review our progress thus far Classroom observation/participation yielded data, which lead to the model of mathematics teaching in the classroom. The results of this model in terms of student learning and development (as illustrated by a 'close up' look and analysis of two separate situations) was found to be in direct contrast to the goals of the teachers and the school as formulated in the Ideal Student Profile.

As a result, it was postulated that a certain amount of autonomy was required if an individual was to gain insight into problems encountered in the classroom This lead to an elaboration of the hypothesis which states some of the conditions required to promote insightful learning of mathematical concepts.

A review of the related literature was then carried out and some support was found for the general notion that individuals who played or otherwise were granted a 'free' or 'autonomous' condition to organise incoming data were better able to generalise from it and to maintain it over periods of time (i.e., Katona, 1940) They also produced fewer errors and took less time to attain the concepts (i.e., Mandler and Pearlstone, 1966)

What is therefore suggested is a series of experiments which employ both objective and subjective means of assessment in exploring the conditions which affect a subject's ability to formulate a 'higher-order' mathematical concept. In addition, armed with certain limited implications from the experimental evidence, we shall return to the classroom to investigate possible applications of such principles.

Chapter Six The Effects of Autonomy on Mathematical Concept Formation
Part I, University Students

6.0 Introduction

Following the conditions set out at the conclusion of the previous chapter, an experiment was designed to test the hypothesis on play and autonomy.

As was mentioned in the review of the literature, there has been much theoretical speculation about the cognitive implications of play but direct, experimental support for the thesis is limited. The idea that play experience can lead to the solution of problems is implicit in the Gestalt tradition, for example, in Kohler's (1926) work on insight in apes or Wertheimer's (1945) studies of novel solutions in productive thinking mentioned earlier. The importance of the subject's own activity in intellectual development is also stressed in the theories of Piaget (1950), Bruner (1966) and Dienes (1963). However, there are only a small number of studies which have attempted to demonstrate the importance of play for cognitive advance.

Among these is the study on young children by Silva (1976) in which a play group was 'yoked' to a passive group which received the same information about a set of sticks and clamps as their active 'yoke-mates'. The results demonstrated the effectiveness of play in aiding the solution of a Kohler-type insight task. In a similar vein, the present study is designed to explore the hypothesis that play can be a useful preliminary experience for the more advanced problem-solving involved in mathematical concept formation.

Since the concept of play appears to overlap with the concept of "creativity", a related area of research is that which has attempted to link tests of divergent thinking with mathematical ability. Bennett (1974),

reviewing this literature, points to the contradictory evidence available - Hasan and Butcher (1966), for example, finding a significant relationship between divergent thinking and mathematical ability, whereas Richards and Bolton (1971) found that divergent thinking ability contributed only minimally to performance on a wide range of mathematical tests. However, these latter authors do suggest that further studies should concentrate on exploring the implications of activity methods of teaching mathematics for productive thinking problems of the type described by Wertheimer (1945). The present experimental study pursues this suggestion through an assessment of the effects of active play on the structural thinking involved in mathematical concept formation.

The learning of two mathematical groups was chosen as a basis for the conceptual tasks. This was taken from the work done by Dienes and Jeeves (1965) in their monograph Thinking in Structures. The use of 'mod-4' and 'Klein' groups was modelled after the two groups presented on page 19 of their work. There they are referred to as 'Cyclic' or ' M_4 ' and 'Klein' (see diagram 6a).

In the overall introduction to their experimental work, Dienes and Jeeves posed their central question in the following way

"How do we sort out the apparent chaos of our environment into anything like order?" (page 15)

From there, they go on to outline the conditions required in an experimental investigation of the above:

"To study the process it is necessary to establish some experimental chaos, that is a situation which is almost bound to appear chaotic to a subject upon a first encounter. Ways must be provided for the subject to sort out this chaos, enabling him to work out a model which has predictive value. When the subject has worked out a model which has 100 per cent predictive value he has sorted out the chaos, and has fabricated the order, according to which he evaluates and predicts events. In order to approximate our situation to a real one, we should give the subject a reasonable choice of strategies in the sorting-out process, yet the choices must

be provided in an experimentally controlled way, so as to make the results of the sorting by different subjects comparable." (page 15)

In the present experiment, we will be more concerned with the differences found between groups (that is, 'autonomy', 'yoked' and 'instructional') rather than between individuals within the same group.

Dienes and Jeeves continue

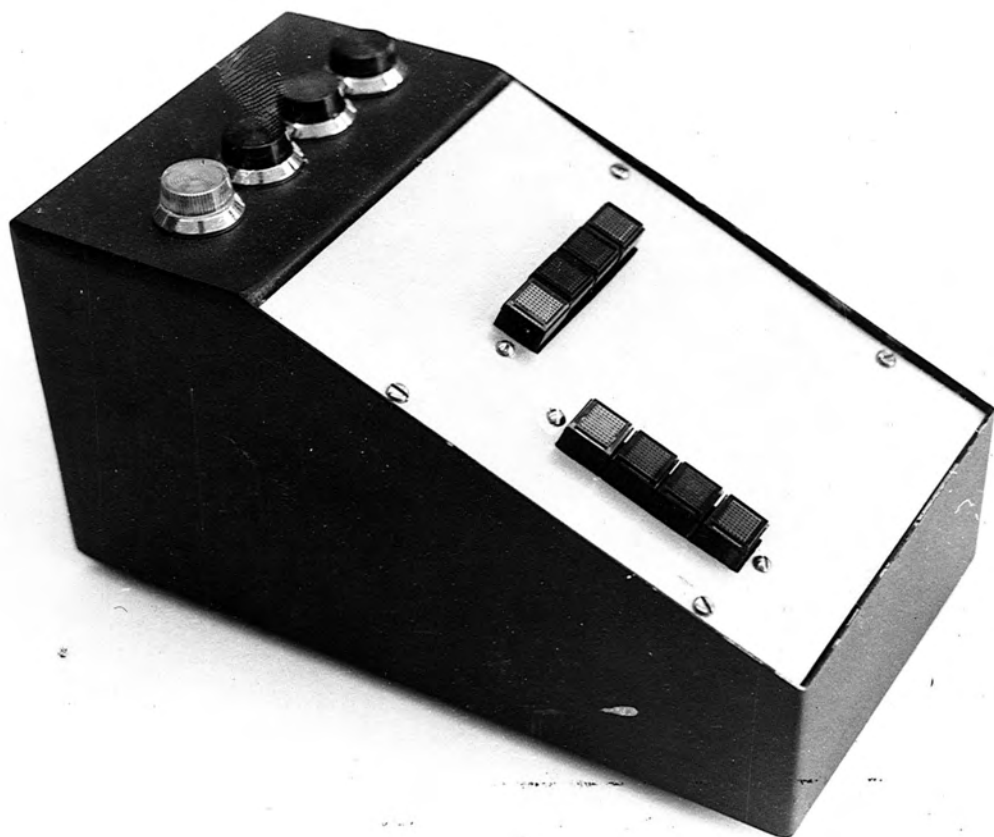
"The only sure way to satisfy this condition is to construct a task which no subject is likely to have come across, and one involving a chaos that cannot be sorted out by well-worn strategies. In order to study developmental differences, it is also desirable to construct tasks which children can sort out and which are at the same time not trivial for adults.

Situations involving mathematical groups are the most likely to satisfy all the above conditions. These also have the advantage of providing mathematical learning situations, the results of which might be used to predict how learning would take place in other similar mathematical learning situations." (page 15)

6.1 The Toy

In addition to the conditions above, there was also the practical difficulty of the method of presenting information which could be both autonomous and 'playful'. For our purposes the mathematical abstractions had to be structured such that they could be represented on a piece of machinery to be used as our pre-task 'toy'. In order to do this, the information contained in the concept task must permit a sequential presentation that can be both controlled by a subject, and further, recorded and replicated by a different subject at a later time. By the very nature of mathematical groups, this sort of presentation can readily be accomplished on a 'toy' which related the pushing of buttons and the lighting of lights to the rules which govern members in a group.

The toy was designed to be both portable and easily manipulated with the subjects being able to push the buttons at will (see photo). The toy has two rows of four buttons each, one horizontal and one vertical. In



BASE FOUR TOY

each row the coloured buttons were arranged in the order yellow, green, red, blue, and the face of the buttons lit when they were pushed to indicate that they were 'on'. Buttons were turned off by pressing another button in that row, and each of the two rows, in this sense, were independent of the other. In addition, on the top of the machine there were four lights, also in the order Y, G, R, B. Pressing one button on each of the horizontal and vertical columns caused one of the lights to light according to the rules of the mathematical modulo-4 group. The task given to the S's in group one was simply to push the buttons and watch the lights. S's in group two and group three did this according to a tape recorded set of instructions which they could control as far as the pacing was concerned.

Thus, one group could be given a period of free play which would be video-taped for later duplication by a 'yoke-mate'*. In addition, a standard audio tape recording was made which would provide a third experimental group ('instructional') with a program designed to reveal the operation of the modulo-4 group by showing the patterns of the relationships in a systematic way.

As was mentioned earlier, the design of the toy parallels the function of the modulo-4 task that was presented to the S's. Therefore, it can be seen as a device for presenting information equivalent with that needed to solve the initial problem, and directly related to a potential approach to the second problem. Because of the autonomy presented to group one, it was expected that they would gain greater insight into the workings of the toy, and importantly, that they would both be able to apply this insight to the first task (modulo-4), and transfer this insight to a changed but related situation (Klein). It should be noted that this control over the

* once it had been recorded on audio tape as a set of instructions as to which buttons to press on the toy

specific order that the information was acquired is the only difference between each S in the play group and his corresponding yoke-mate

Thus, using the methodology developed by Dienes and Jeeves, the aim of this first experiment is to confirm the prediction that a group which has the benefit of autonomous activity with the specially designed toy will gain greater insight into a related concept formation task than either a group which simply replicates the actions of the play group or ^{a group} which goes through a standard instrumental programme using the toy.

6.20 Method

6.21 Selection of Subjects

Twenty-four volunteers, (thirteen male and eleven female), reading various courses at Durham University were used as subjects. They were divided up randomly into three groups labelled play, yoked and instructional. This was done before the experimenter first met them. Maths majors were excluded after a pilot study found that their performances on the task, due to its very nature, were uniformly high.

6.22 Concept Tasks

Three separate but related concept tasks were used in the experiment. They are called two groups, modulo-4, and Klein, and were taken directly from the work cited earlier (Dienes and Jeeves, 1965). The two group* task was used first in order to assure that the S's thoroughly understood the nature of the procedure. It was not, therefore, used in the scoring. The modulo-4 task was then given and scored according to the criterion outlined below. Wooden blocks of four different colours (yellow, green, red and blue, as on the machine), were used as opposed to cards used by

* the two group is shown in diagram 6a

Dienes and Jeeves The procedure can be summarised as follows The experimenter placed one of his coloured blocks on an area of the table labelled as the centre. The S then selected one of his four blocks to place down in his area. As a result of these two blocks, and according to the pre-existing pattern of the numerical group theory, the experimenter then placed a third block on the area of the table designated as resultant, while verbalising the combination, i.e., "red plus red makes yellow" The S was permitted to view the combination for a few seconds before both the central block and his block were removed, and the resultant block moved over toward the centre to become the new centre colour. The S then chose another, or the same, coloured block to place in his area The sequence was continued until the S felt as though he could accurately predict the resultant block for any of the sixteen colour combinations When the S desired a check on this, a standard verbal test was administered During the test the S's were permitted to manipulate their own blocks but were not permitted to look at the instructor's set which were placed out of sight If the S succeeded in the test, he was then interviewed in order to confirm or contradict the available information as to the method he used to accomplish the task * If the S failed, he was reassured that there was no penalty, and the sequence with the blocks continued until he felt ready for another test. There was no limit placed on the number of tests given, and all S's eventually completed the task When time permitted, the Klein group was presented to the S's and scored in order to see whether or not they had any difficulty shifting their strategy to a new arrangement.

6.3 Procedure

Each S was taken to a small room, seated at a desk, and told to play

* a step-by-step explanation of this procedure, illustrated with pictures, can be found in the appendix, Table C

with some toys (wooden blocks, a Newton's cradle, a 'slinky', etc). These were not related to the scoring procedure but rather were intended to help subjects in all the groups to relax, develop a good rapport with the experimenter, and avoid a conscious feeling of being assessed. Post-experimental debriefing sessions found that almost all subjects enjoyed the tasks and that there was a minimum of anxiety as to individual performance. It should be noted however that this was an informal impression and that more objective measurements were not made due to severe time limitations. All S's were video-taped during the sessions with their knowledge and approval. There were no objections to this procedure. Besides verifying that all S's had indeed 'played' initially, this video was used in recording the exact sequence of buttons pressed by the autonomous group when they were later limited to 'playing' with the specially designed toy. This was ^{used} later when the experimenter made an audio tape for each of the yoked S's.

After five minutes in the room alone playing with the toys on the desk, the subjects were told to 'Stop', the toys were set aside, and a set of instructions was given about the mechanical 'toy' which they were then shown for the first time. These instructions provided them with the basic information about how to turn the buttons on the toy 'on' and 'off' In addition, all groups were told that the pressing of the button lit all the lights on the top of the toy in an undisclosed way.

The autonomous group was then given another five minute period of 'free play' (although, again, they were not told in advance how long they had to play).

The yoked and instructional groups* were given verbal instructions

* these are referred to as groups throughout all of the experimental write-ups although it should be noted that the subjects were always seen individually

from a tape recording as to which buttons they were to press. They did have available to them a switch which they could use to control the pacing of the tape. It should be stressed once again that members of the instructional group had a standard tape which they all listened to (separately), while each member of the yoked group was instructed to press buttons on the toy according to the way one of the members of the autonomous group (and hereafter described as his/her 'yoke-mate'), played with the toy in an earlier session.

After completing the session with the toys, subjects were explained the concept formation (with the coloured wooden blocks - described earlier) in terms of a problem to be solved. It was emphasised that the experimenter was interested not so much in the eventual solution, (which they were assured was within their power to accomplish), but rather in the methods used while getting to that solution. Thus, S's were encouraged to think out loud, and were told to do the task in an efficient way without any need to rush or conform to any time limit.

Once the subjects successfully passed the test, they were given the post experimental interview to determine how they ^{had} proceeded. Based upon this and evidence accumulated during their work, a score of between one and four (1.0, 1.5, 2.0, 4.0) was assigned to them.

6.4 Scoring of the Task

Three scores were recorded for each subject. First, a record was kept of the number of blocks which each subject required before attaining the concept. The lower the number, the faster the concept was attained. The second and third scores taken were subjective ratings based upon an interview with each subject after they had completed the modulo-4 task and the Klein group (when time permitted). The score from the modulo-4 was designed to assess the level of understanding attained after the

experiences with the toy, which is programmed according to the modulo-4 system. The scores from the Klein task were therefore intended to test the effects of transfer to another mathematical group which had a completely different set of operational properties

The ratings of 'insight' or 'level of conceptualisation' attained was done on the scale of 1-4 mentioned earlier and was based upon the following criteria.

Total memorisation of the blocks task without any elaboration or awareness of a pattern or relationships of any kind was given a score of one. A two was awarded when the S used a method which relied very heavily upon memory, but which included some basic relationships as aids in acquiring mastery of the system. Such aids which commonly occurred were noticing that yellow plus any colour always yielded yellow, or in the Klein arrangement noticing that any two of the same colours combined together to always give yellow. (see diagram 6a) Category three was attained when the S was able to describe the workings of the blocks in terms of a total set of heuristics, that is, several rules and strategies, which when used together could form a systematic approach able to predict any combination of the blocks. In order to attain this category, the criterion of category two must also have been reached, and, in addition, evidence must be available that the blocks were being manipulated in a systematic way that the S could express in each and every example presented to him. Usually this was accomplished by placing the blocks in a special order or arrangement and pointing to them as the procedure was explained. Finally, a top score of four was given only if the S was able to explain the workings of the blocks in terms of one rule which could be totally mechanised, that is, demonstrated to work in all cases presented. The complexity of the statement of the rule was not considered crucial, as long as the workings of the blocks were seen as being universally governed by it. As it was

mentioned before, half scores (such as 1.5) were permitted for a small number of S's who were clearly between stages or in transition. It was found in a few cases that S's were able to elevate their score by one-half step during the questioning period, presumably based upon further thought and information acquired there. In these cases the score was given which represented the level attained during the actual experimental task.

6.5 Results

The results of the experiment are presented in Table 6b. Individual scores are given and the profiles of each yoked subject can be found directly to the right of the play subject to which he or she corresponds. Play and instructional subjects are presented in the order that they were tested.

Table 6c gives the results of the statistical tests performed on the data. Mann-Whitney U tests were used throughout. Averages of each group are also presented. As can be seen, the averages of the scores on the modulo-4 task are separated by almost exactly one unit, starting from the play group, then the yoked group, and finally the instructional group. Following the prediction that the play group, due to the autonomy that they were presented over the situation, would perform better, one-tailed tests were used. However, two-tailed tests were used in the comparisons between the yoked and instructional groups.

Significance was found in the play vs. yoked and instructional conditions, both when they were compared individually, and when they were combined. This holds true for both the scores on the modulo-4 task and the number of blocks seen. Tests were not made on the Klein task due to a lack of data.

6.6 Discussion

Significant differences^{were} found between the play and non-play groups on the two main criteria of the task, namely the level of abstraction or generalisability the S's were able to attain, and the number of blocks seen before solution. With only one exception, equal or better abstract scores were gained by S's in the play group over their individual yoke-mates. Also, with only minor exception, a lower number of seen blocks were required by each play S when compared to his yoke-mate. This implies the possible existence of a gradient of 'good' or more productive play sequences vs. 'less good' or non-productive play sequences.

The almost uniform drop of the average score by one unit from the play group (score 3.19) to the yoked group (score: 2.13) to the instructional group (score: 1.67) corresponds to a similar rise in the number of blocks needed to be seen before a solution was reached. From this we can conclude that the autonomous or play condition was a more productive means of encountering the toy for a five minute period than either of the two non-play conditions. Although the yoked group scores are consistently higher than the instructional group, despite the intentional design of providing "ideally sequenced" information in the standard tape, there is an apparent benefit for the instructional group, who obtain a higher average score on the Klein task (1.67) than the yoked group (0.9). The play group's average score here again was about one unit higher, (2.57), although it is to be noted that because not all of the subjects were given the Klein task due to lack of time, these suggestions must be considered only as tentative. Interestingly, the only two S's who fully operationalised the procedure outside of the play group were both in the yoked group and both were the only two S's (again of those tested) to fail completely on the Klein task. This evidence suggests a greater flexibility on the part of the S's in the play group when they are required to shift their strategy

on a similar problem. Both yoked S's drew a blank and demonstrated behaviour similar to the rigid adherence to a set noted by Luchins, (1942).

Thus the general implication of these results is that it is the autonomy of play that differentiates it from the other forms of pre-task activity. A subject given play experience is enabled to "programme" his own activities and this may help him formulate and change his own hypotheses about the situation (in this case the toy) more effectively than a passive recipient of information can. Autonomous activity leads, then, more rapidly to greater insight and a higher level of understanding.

On this interpretation the results reported here support Bruner's (1957) contention that instruction should enable a student to go beyond the information given and is in accord with the suggestion made by Dienes and Jeeves (1965) that an important feature of this process may be permitting the student the opportunity to correct his own false hypotheses.

6.70 Critique

The first experiment done on University students was useful as it provided answers to a number of very basic questions about the task. First of all, it was learned that students were willing and able to treat the 'machine' as a 'toy', and that they did usually find playing with it interesting and enjoyable.

Secondly, the categories within the scoring system were found to be a potentially effective way of rating the level of mathematical conceptualisation of each subject. As in the Dienes and Jeeves work (which suggested a more polar choice of 'operational' or 'memory' system - see page 126 of their work), I did get a spectrum of scores with each category represented by more than one subject.

However, there were also some serious shortcomings of the experimental

design. These are discussed under the three main headings below. The critical points in the discussion which follow will therefore form the basis of the changes and improvements in the experiment reported in the next chapter.

6.71 The Scoring System

There are three questions which can be raised with regard to the 1-4 scoring system. The first is simply this: Why have four categories? Would three be sufficient? Or, on the other hand, should five or even six categories be attempted?

The second point relates to the first. Is the scoring system consistent, and can others be taught to use it? Will two or more different scorers give similar ratings on the same subject? In other words, even if there is a theoretical consistency within the four groupings, will other scorers use the criteria in a way which provides for the same, or nearly the same cut-off points?

Finally, in the improved experimental design, the interviews must be taped recorded and made available to a 'blind' scorer who has no knowledge of the subject's previous experimental condition.

6.72 The Groups

Again there are three main points to be made here. First, we should include a control group which will provide us with an indicator of baseline performance. Without such a group, we cannot be sure that play is actually a more efficient way to encounter the information imbedded within the toy, or whether it is just less 'inhibitory' than the other two conditions.

Next, in order to relate the experiments more closely to the original hypothesis and also to possible latter implications for teaching in the

classroom, children of primary and middle schools (aged between 9-12 years) should be used as subjects. In addition, the experimental sessions should, if possible, take place within or near the actual school environment.

We should also have more pre-task information available on each individual subject. These might include ratings of mathematical ability in the classroom, evidence of an ability to deal with symbolic mathematical systems, memory ability, and so on. This will enable us to ensure that the groups are indeed 'mixed' with regard to these factors, and will permit us to be more confident in attributing differences in scores to the effects of the play and non-play experiences with the toy.

6.73 The Yoking

Yoking of the subjects in the present experiment relied upon audio instructions about which buttons to press. Although the subjects were therefore getting the same information from the lit buttons and lights, and even though they could control the pacing of the tape and stop it completely, there is still the difficulty of the rate of information which is eventually presented to each subject. Stated simply, it too often came too slowly. For example, let us suppose that an autonomous subject rapidly pushed YELLOW, GREEN, RED, and BLUE and then went in reverse and pressed BLUE (which in fact is already down), RED, GREEN, and YELLOW to see what effect this sequencing had on the lights lighting. These actions were difficult to describe quickly on the audio tape and the yoked group was therefore in danger of 'missing out' on a visual pattern in the toy. Verbal instructions were limited in their speed by the yoked subject's ability to duplicate the actions of pressing the buttons.

In addition, there is the potential interference of a mixed mode of operation here. That is, yoked and instructional subjects had to attend to their own audio system to listen for instructions and this could have

interfered with their process of conceptualising.

Clearly then, a method had to be devised which provided visual information at the same rate on the toy to the yoked subjects without their being required to listen for instructions about the pushing of buttons.

6.8 Summary

The aim of this experiment was to assess the effects of a previous play experience on a test of concept formation. Twenty-four University students were randomly divided into three groups of eight students each. Each group was given a different set of instructions on a pre-test procedure of exploring the relationships between pushing buttons and lighting coloured lights on a specially designed 'toy'. Members of group one were permitted total autonomy as to which buttons they pressed for the five minute period, members of group two were given an instructional tape which had them press the buttons in the same sequence as a member of group one, members of group three followed a standard taped instruction designed to provide access to the patterns in the toy with maximum efficiency. Each S was then administered a concept formation task which related to the relationships existing in the 'toy'. The results showed that the play group gained a greater insight into the task and required fewer trials prior to solution. These results were then discussed briefly within the theoretical framework provided. A critique was made which posed some important questions in addition to giving specific suggestions about improving and adding to the existing experimental design.

Diagram 6a

	Y	G	R	B
Y	Y	G	R	B
G	G	R	B	Y
R	R	B	Y	G
B	B	Y	G	R

'Cyclic' or 'Modulo 4'
Groups

	Y	G	R	B
Y	Y	G	R	B
G	G	Y	B	R
R	R	B	Y	G
B	B	R	G	Y

Klein Group

	Y	G
Y	Y	G
G	G	Y

Two Group

Table 6b

Group 1 PLAY

Sex	Modulo-4 score	Blocks seen	Klein score
M	4	40	3
M	4	10	3
F	4	42	nt*
M	2	113	2
M	3	48	3
F	2	75	2
M	3.5	35	3
F	3	74	2
average	3 19	54 6	2 57

(yoked to →)

" " " " " " " "

Group 2 YOKED

Sex	Modulo-4 score	Blocks seen	Klein score
M	2	64	2
F	1	80	nt*
M	4	111	0
F	1	147	1 5
F	4	66	0
F	1	86	1
M	2	59	nt*
M	2	70	nt*
average	2 13	85 4	0 9

Group 3 INSTRUCTIONAL

Sex	Modulo-4 score	Blocks seen	Klein score
F	1 5	79	2
F	1 5	137	nt*
M	1	121	nt*
M	1	155	1
M	1	98	nt*
F	1	34	2
F	1	110	nt*
M	1 5	63	nt*
average	1 19	99 6	1 67

* refers to no time

LEVELS OF STATISTICAL SIGNIFICANCE¹

<u>Scores on the Modulo-4 Task</u>			
Play vs. Non-Play*	Play vs. Yoked p < .05	Play vs Instructional TOTAL (p<.001)	Yoked vs Instructional N.S.
<u>Number of Blocks Seen Before Attaining Solution</u>			
Play vs Non-Play p < .025	Play vs. Yoked p .05	Play vs. Instructional p < .025	Yoked vs. Instructional N.S.

¹ Mann-Whitney U test used throughout, all play conditions are t = one tailed, yoked vs. instructional condition is t = two tailed

* Non-Play refers to the yoked and instructional conditions combined.

Table 6c

Chapter Seven The Effects of Autonomy on Mathematical Concept Formation Part II, Middle School Children

7.0 Introduction

The present chapter reports the results of an experiment which employed an improved method of yoking subjects in addition to certain other alterations as described below. It was carried out in a Middle School within facilities provided by the teaching staff with approval of the Headmaster. The space made available consisted of two adjacent rooms in one corner of the building. They were sometimes used during the week for individual music practice and teaching.

Before beginning the experiment, I spent over a month in the school observing maths lessons, talking to students and generally taking part in all of the school-time activities. These included eating lunch with the children, playing football during the break, lecturing one geography lesson, and even helping out on an overnight weekend at an open-air camp. Because of these activities, the children saw me as part of the school routine, perhaps somewhere between a practice teacher, mathematics specialist, and all-round assistant to the teaching staff. This meant that the experimental sessions were perceived by the children as a special part of school-time activities rather than as something totally separated from them, i.e., an experiment within a room in the psychology department. Thus, even though subjects were selected by the teachers, there were always volunteers who were eager to take part!

In addition, before work was begun within the school, a new set of pilot studies was carried out with the help of an undergraduate project group within the Durham Department of Psychology. Because of practicalities, we again used University students as subjects in an attempt to solve the problems raised in the earlier critique. These efforts resulted in the

improved version of the toy described below, and an alteration of the experimental design which was then used with the Middle School children

7.1 The Yoking

In order to eliminate the need for a set of tape-recorded instructions for the yoked group, a second toy was built. It was identical to the first and was connected to it by means of a long, detachable, electrical cable. Both toys remained battery powered, so that they could be used singularly without any cables or wires.

When they were used with the cable connecting them, one toy was designated as the 'autonomous toy' and the other as the 'yoked toy'. They were set up and used in two separate but adjacent rooms on tables placed there. We now had a situation which enabled the yoked subject to watch his toy without any interference from audio instructions. In addition, the subject saw the buttons and lights lighting at the same time and at the same rate as the autonomous subject who was in the other room exploring the toy. However, for purposes of the experiment, the subjects were not told about the cable yoking and it was deduced from their general response (or more accurately lack of it) that they either took little notice of the cable or else thought that it was a mains lead.

7.2 The Scoring

In the pilot study mentioned earlier, several different scoring systems were attempted. None of the alternatives to the 1-4 scale proved to be satisfactory. In particular, it was found that a more elaborate set of categories which went from 1-5 caused confusion when scores of 2, 3 or 4 were given. In fact, these scores could not be adequately described either in theoretical terms or for purposes of scoring the subjects.

Tape recordings were also made of what had become a pretty standardised interviewing procedure. This is not to say that a specific series of

questions was read out loud or that the structure of the interview itself was made more formal. Rather, the interviews were still kept as informal 'chats' following the task, but the experimenters (two second year undergraduates from the project group and myself) became much more consistent in their choice of questions asked. This caused the format of the interviews to become more uniform making it easier to apply the 1-4 categories.

Finally, there is the question of 'double-blind' scoring. In a pilot study (which used 24 subjects studying at the University), two of the second year students were used to score the data. Their inter-scorer correlation was significant at the .001 level*. Because of this, we were more confident of what had been our earlier speculation, that is, the categories did indeed represent 'stages' in the development of insight. We now had a way of storing information (from the tape recorded interviews) about a subject's level of mathematical attainment on the conceptualisation task. This could then be compared with the same subject's performance after an experience with the toy and was accomplished with the use of the newly trained 'blind' scorers who would not be told anything (i.e, sex, year, ability, experimental group or session, etc.) about the subjects. These improvements were incorporated into the experimental design discussed below.

7.3 The Experimental Design

The first change made in the design of the experiment was mentioned earlier. It involves the use of children as subjects instead of University students. In addition, the experiments were conducted in the school in an environment which was conducive to both play and concentration.

* correlation $r_s = 0.984$, See Potts (1976)

The most important design change (besides the use of the two connected toys giving perfect 'yoking'), involved the amount of information acquired about each subject before they were exposed to the toy. The total set of scores used in the experiment can be divided into two groups consisting of Initial and Final measurements (see Tables 7a-c). The initial measurements were taken before the children played with the toy, while the final scores were taken after the experience with the toy. In addition, the instructional group was replaced by a control group which had no experience with the toy. This enabled a base-line to be established which was required for comparing changes that occurred in the play and yoked groups after their experiences with the toy.

The initial or pre-toy scores can be described as follows (from Tables 7a-c)

Year; Half the children were from the second year class in the Middle School (about 10½ years old upon entry), and half were from the third year class (about 11½ years old upon entry) They were evenly matched in the three groups.

Sex; Males and females were also evenly divided within the groups. In addition, all combinations of M/F were arranged in the yoking of subjects.

Teacher's Rating, This is a rating of 1-12 which compares a student with others in the same year group. The lower the number given in the rating, the better their maths ability

Blocks, This represents the number of blocks seen by the subject before solving the task. Note that unlike the previous experiment, here the Dienes and Jeeves task was administered before contact with the toy in order to provide information about the child's ability to deal with mathematical groups.

Conceptual Level, This is the 1-4 score on the interview taken immediately after the blocks task was completed. It was scored from a tape-recording by an experienced 'blind' scorer.

Memory, This represents the subject's score on remembering the 16 combinations from the blocks task. It was taken about four days later and immediately preceded contact with the toy. See the appendix, Table D, for an example of the question sheet used.

The Final or post-toy ratings were

Conceptual Level, This is taken from a second tape recorded interview following experience with the toy (or 7½ minutes of sitting quietly for members of the control group).

Memory Score, This was obtained by requiring each subject to fill in a 4 x 4 chart with 16 coloured blocks (4 each of Yellow, Green, Red, and Blue). The subjects did this immediately after exposure to the toy (or sitting quietly) and they were permitted to refer to any patterns or designs made with the colours of the blocks on the 4 x 4 grid while they were explaining the task to the experimenter in the interview session.

7.40 Method

7.41 Selection of Subjects

Twenty-four children, aged from 10 years 2 months to 12 years 7 months, were selected from the second and third years of a Middle School. For the purpose of later ensuring that no group had an initial advantage or disadvantage with regard to mathematical ability, each child was ranked by the teacher on a 1-12 scale in comparison with others of the same year who took part in the experiment, ^{after the children were placed in one of the three year groups.} Table 7d shows that there were no significant differences between the play, yoked and control groups in this respect.

7.42 The Concept Task

The concept task used was the same as in the previous experiment. Because of the additional time required by the use of two sessions, and the large number of blocks required prior to solution*, there was no time for testing on the Klein group.

7.43 Scoring of Concept Task

The number of blocks were recorded that were required to be put forward

* this was due to a very large extent to the fact that the children, unlike the University students, did the task before using the toy.

by the S before he could achieve an 80% success at predicting the 16 combinations of blocks.* In addition, two scores were derived from this concept task, both before and after the intervening condition. The first score represents the subject's ability to remember the results of combining any two blocks and was assessed by a questionnaire prior to the intervening condition and by asking a child to complete a four-by-four matrix after the intervening condition. Thus, in both cases, the score is out of a total of 16. The second score was a qualitative evaluation of the child's level of understanding of the concept task. Following completion of the task, each S was interviewed and asked how he knew which colour combinations produced the resultant colours. The child's responses were tape-recorded and scored by an independent, experienced rater who had no knowledge of the experimental condition to which the child was assigned. It is these scores which are reported in the text. It should be noted that there is substantial agreement between this and the experimenter's ranking. Of the 48 ratings, (i.e. 24 Ss ranked twice) 32 were identical and the other 16 differed by the average of .7. Each child was scored on a 1-4 scale both before and after the intervening experience based upon the same criteria as in the first experiment. To reiterate Total memorisation of the block task without any elaboration or awareness of pattern of relationships of any kind was given a score of one. A two was awarded when the S used a method which relied very heavily upon memory, but which included some basic relationships as aids in acquiring mastery of the system. Such aids which commonly occurred were noticing that yellow plus any colour

* this was lowered from the 100% required in the previous experiment, also primarily because of time considerations - requiring a perfect score could have taken well over an hour, especially for some of the second year subjects. Clearly, this would have had adverse effects on the rest of the procedure, especially when the children were told that they would have no difficulty with the task.

always yielded yellow. Category three was attained when the S was able to describe the workings of the blocks in terms of a total set of heuristics, that is, several rules and strategies, which when used together could form a systematic approach able to predict any combination of the blocks. In order to attain this category, the criterion of category two must also have been reached, and, in addition, evidence must be available that the blocks were being manipulated in a systematic way that the S could express in each and every example presented to him. Usually this was accomplished by placing the blocks in a special order or arrangement and pointing to them as the procedure was explained. Finally, a top score of four was given only if the S was able to explain the workings of the blocks in terms of one rule which could be totally mechanised, that is, demonstrated to work in all cases presented. The complexity of the statement of the rule was not considered crucial, as long as the workings of the blocks were seen as being universally governed by it. As was mentioned before, half scores (such as 1.5) were permitted for a small number of S's who were clearly between stages or in transition. It was found in a few cases that S's were able to elevate their score by one-half step during the questioning period, presumably based upon further thought and information acquired there. In these cases the score was given which represented the level attained during the actual experimental task.

7.44 Procedure

Each S was first given the modulo-4 task. Three groups of S's were then formed, taking into account year, sex and teacher's ranking of mathematical ability. S's were called back to a second session 4 days later and each was given a sheet of paper which listed the 16 colour combinations and was instructed to remember as many resultants as possible guessing when unsure. The four original colours of the concept task

were present as a set of four blocks during this phase.* After completion a child experienced one of the three possible intervening experiences. S's in the play condition were given $7\frac{1}{2}$ minutes to play with and explore the toy, S's in the yoked condition observed the combinations of lights and their resultants, whilst S's in the control condition had no further experience but returned for testing after $7\frac{1}{2}$ minutes. Finally, each S was presented with a chart which contained a list of the four colours vertically and horizontally, thus forming a four-by-four matrix. They were given 16 blocks, four of each colour, and told to complete the matrix. They were asked to explain how they had accomplished this and their explanation was tape-recorded for independent scoring.

7.45 Results

The results are presented in Tables 7a, 7b and 7c, and Table 7d records the results of applying Mann-Whitney U tests to the data. Averages are given for all of the scores. It may be seen that there are no significant differences between the groups on the four initial measures. On the other hand, the final concept scores show that there is a high degree of statistical significance between the play group and the two other groups. There are no significant differences between the control and the yoked groups.

7.46 Discussion

These results provide a striking confirmation of the hypothesis that play with the specially designed toy can lead to a greater understanding of the rules embodied in a mathematical concept and a better memory for

* see Appendix Table E for photos of the 16 blocks as presented, and when correctly arranged

such rules than can be provided by observation of the same stimuli but without manipulation. Indeed, it is only subjects in the play condition who advanced to higher levels of abstraction, as assessed by the scale of conceptual level. Thus the general implication of these results is that it is the autonomy allowed to the subjects in the play condition that enables them to profit from the information embodied in the toy and potentially available also to the yoked subjects. A subject given play experience can "programme" his own activities and this may help him to formulate and change his own hypotheses more effectively than a passive recipient of information can. Autonomous activity leads, then, more rapidly to greater insight and a higher level of understanding.

Another interesting finding is that there is a significant difference in the initial conceptual scores between the second and third year children. Even some of the better second year subjects only scored a "1" on the pre-task ratings. Overall 25% of the second years scored a "2"*, while only 25% of the third years failed to score a "2" on the initial blocks task. This provides interesting implications for a 'progression' of conceptual attainment determined more by school year or age than by individual ability. These suggestions must be tentative, however, as the difference was lost after the subjects were given an experience with the toy. This finding may relate in some way to the difficulties which some children (especially in the 8-10 age group) seem to have with modern mathematics courses which include an introduction to the properties of formal groups. One suggestion which follows from this work and extensive classroom observations in this area is that younger children must have a more immediate and perhaps even manipulative experience with examples of materials

* "2" was the highest score earned in this pre-toy task, that is, all subjects earned either a "1" (memory) or a "2" (noticing yellow or similar operation.)

which represent these concepts*. Highly abstracted or symbolic instruction may be more likely to produce confusion or rote performance in which the child lacks any genuine understanding of the mathematical significance of the operation.

7.5 Post Script

Since it was not clear from the results gained from this experiment whether the conclusions would hold for a rating of the task taken over longer periods of time, I returned to the school during the following academic year (after a full six months had passed since the experiment was completed) and I administered the second memory task to the subjects once again. This was done with the 4 x 4 grid as in the final results. The children were also given an informal interview, but this was not tape recorded. Some of the children were either not available while I was at school (a two day period) or they had left it entirely. There were five such (NA - for Not Available) subjects out of the total 24 tested. However, the remaining 19 scores shown (see Table 7e) strongly support the section of the hypothesis which states that autonomy is required in order to 'integrate' the mathematical structures as part of an individual's method of perceiving relationships and patterns. If this is not done, they can too soon be forgotten. This is what has happened to the majority of yoked and control subjects, although there were a few exceptions. In particular this occurred with one of the yoked and two of the control subjects who were the highest scorers (each earning a '2 5') outside of the play group. Also note that the one subject in the play group who received the lowest score on the long-term task (earning a score of 10 out of a possible 16)

* the suggestion made here is in accord with similar findings and recommendations put forward by Z. P. Dienes (1963) which relate to the development of the 'Dienes Blocks'

had previously earned the lowest conceptual score in the play group

It should be mentioned that these scores which were collected six months later were obtained with only the following instructions given to each individual child:

"Do you remember the game we played with the four different coloured blocks?" (blocks are shown and the child nods)

"Well, I want you to try to remember, in any way that you can, how all of the blocks combined and fill in this chart again. I realise that it was a long time ago and that it may be difficult, so don't worry if you've forgotten one or two, do the best you can and fill in all of the blank spaces."

The child was then given the 16 coloured blocks and the chart but was not permitted to touch or play with the toy. It was kept within sight for the members of the play and the yoked group however.

The statistical analysis shown in Table 7f indicates that significant differences remain between the play and the yoked and control groups. In addition, the drop in scores in the non-play group is statistically significant, but is not for members of the play group. Clearly then those subjects who played with the toy and earned high scores on the conceptual interview (1-4 scale) were able to retain the rules which governed the patterns on the chart. Information from the interviews demonstrated that they were not merely relying upon memory.* This was also strikingly apparent by the methods which the play group members used to put the blocks down on the chart. It was not at all random or in simple sequence, but was carefully built up, altered, checked and re-checked. In contrast, the non-play subjects finished much more quickly and had little idea about which of their choices were correct and which were not.

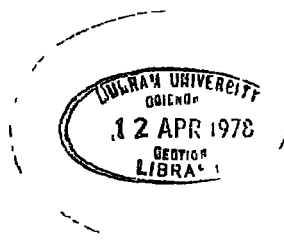
From this, it may be suggested that the methods developed by the

*See appendix J

autonomous individuals to bring 'order' to a 'chaotic' situation were able to be retained and used (with understanding) over long periods of time even though no periods of demonstration were given in order to show how the mathematical system operated. What would happen if actual demonstration of a mathematical arrangement was included as part of the procedure forms the basis for the next chapter's experiment.

7.6 Summary

The aim of this experiment was to assess the effects of a previous play experience on a test of concept formation involving base four arithmetic. Twenty-four children from 10 to 12 years of age were divided into three groups of eight subjects each, matched on teachers' ratings of mathematical ability. A concept formation task was given as an initial measure of ability. Members of group one were permitted total autonomy as to which buttons they pressed on a specially designed toy whose operation embodied base four rules, members of group two were yoked to members of group one so that they observed the same sequence of information, members of group three had no experience with the toy. Each S was then administered a further concept formation task which involved the same rules existing in the "toy". The results showed that group one gained a greater insight into the task and remembered the information more effectively in both the short and long term.



Individual and Average Scores on the Concept Task - Initial and Final

Group 1 Play

Year (2 or 3)	Sex	Teacher's rating of mathematical ability	(a) Blocks required prior to solution	Initial Concept Scores			Final Concept Scores		
				(b) Memory	(c) Conceptual level	(c) Conceptual level	(b) Memory	(c) Conceptual level	(c) Conceptual level
2	M	10	295	4	1	16	3.5	3.5	
2	M	9	148	8	2	16	3.5	3.5	
2	F	1	300	12	1	16	4	4	
2	F	7	257	10	1	16	3.5	3.5	
3	M	11	342	9	1-	16	3.5	3.5	
3	M	4	196	14	2	16	3.5	3.5	
3	F	12	236	8	1	14	3.5	3.5	
3	F	9	158	11	2	16	2.5	2.5	
Average				242	9.50	1.37	15 75	3.43	

Table 7a

Group 2 Yoked

Year	Sex	Teacher's rating	(a) Blocks	Initial Concept Scores			Final Concept Scores		
				(b) Memory	(c) Conceptual level	(b) Memory	(c) Conceptual level	(b) Memory	(c) Conceptual level
2	M	6	280	12	1	16	2		
2	M	5	232	7	1	12	2		
2	F	4	180	12	2	12	2		
2	F	3	213	15	1	9	1		
3	M	7	196	8	1	12	2	5	
3	M	10	144	12	2	14	2		
3	F	6	194	7	2	11	1.5		
3	F	2	210	12	2	12	2		
Average•			206	10.62	1.5	12.25	1.87		

Table 7b

Group 3 Control

Year	Sex	Teacher's rating	<u>Initial Concept Scores</u>			<u>Final Concept Scores</u>		
			(a) Blocks	(b) Memory	(c) Conceptual level	(b) Memory	(c) Conceptual level	
2	M	8	342	12	1	12	12	2
2	M	2	180	11	2	11	11	2
2	F	11	368	3	1	6	6	1
2	F	12	318	8	1	9	9	1
3	M	8	232	8	1	6	6	1
3	M	1	274	12	2	16	16	2.5
3	F	3	126	15	2	16	16	2.5
3	F	5	254	12	2	9	9	2.5
Average			262	10 12	1.5	10.62	10.62	1 81

Table 7c

Table 7d

Summary of Mann-Witney U tests

	Play v yoked	Play v control	Play v non play	Control v yoked
1 Blocks seen	U = 20 5 NS	U = 27 5 NS	U = 57 NS	U = 18 NS
2 Memory	U = 26 NS	U = 27 NS	U = 75 NS	U = 30 5 NS
3 Toy 1-4 scale	U = 28 NS	U = 28 NS	U = 56 NS	U = 32 NS
4 Teachers rating	U = 17 5 NS	U = 24 5 NS	U = 42 NS	U = 28,5 NS
1 Memory	U = 5 002	U = 9 02	U = 14 .002	U = 2.5 NS
2 Conceptual level 1-4	U = 0 5 001	U = 1 5 001	U = 2 001	U = 31 NS

PRE CONDITIONS

POST CONDITIONS

LONG-TERM DATAGroup 1 PLAY

Sex	Year	Final memory score	Long-term memory score
M	2	16	16
M	2	16	10
F	2	16	16
F	2	16	16
M	3	16	NA*
M	3	16	16
F	3	14	7
F	3	16	16

(yoked to →)

Group 2 YOKED

Sex	Year	Final memory score	Long-term memory score
M	2	16	4
M	2	12	NA
F	2	12	6
F	2	9	NA
M	3	12	9
M	3	14	9
F	3	11	NA
F	3	12	14

Group 3 CONTROL

Sex	Year	Final memory score	Long-term memory score
M	2	12	4
M	2	11	6
F	2	6	4
F	2	9	5
M	3	6	3
M	3	16	16
F	3	16	16
F	3	9	NA

Table 7e

Table 7f

Change in Scores from Final Memory Score to Long-term Memory Score		
Within the play group N.S	Within the yoked and control groups combined $p < .05^*$	
Comparisons of Long-term Memory Scores		
Play vs. Yoked $p < .01$	Play vs Control $p < .01$	Yoked vs Control N S

* Mann-Whitney U test used throughout, all conditions are $t =$ two tailed

Chapter Eight The Relative Effectiveness of Play and Demonstration
On a Binary-Digital "Toy"

8.0 Introduction

In the previous experiment on base four learning it was found that a group of children, aged from 10 to 12 years, who were able to manipulate freely a specially designed "toy" embodying base four mathematical operations, showed greater insight into a related test of concept formation than either a group yoked to them and hence being exposed to the same information but without autonomous activity or a group which had no experience of the toy. Play with the toy, it was argued, allowed the S to check his hypotheses and change them when necessary. However, although such a finding supports the hypothesis that conceptual development may be promoted by allowing S's freedom to discover solutions for themselves, there remains the significant educational problem of integrating such activity into an ongoing programme of instruction. A more complete answer to this question would involve a detailed programme of collaboration with practicing teachers. However, aspects of the problem may be explored initially in a more restricted experimental fashion by introducing demonstration of the relationships embodied in a toy along with free play activity.

The question to which this experiment is addressed may thus be stated which of the following three approaches provides the most effective way of permitting a child to develop the mathematical concept of base two autonomous activity with the toy, demonstration of the toy, or a combination of the two?

In the previous chapters also, for purposes of assessment of the degree of concept learning attained a fourfold classification of subjects' responses to questions in a structured interview was developed. The aim

of this classification was to give S's a score according to their level of understanding of the concept, ranging from rote memorisation of specific instances through stages in which certain rules only are recognised to a level at which the S possesses an idea of the system of rules as a whole. A subsidiary aim of the present study is to ascertain the relationship between such an assessment of the level of thinking attained by the subject and a traditional question and answer assessment in which the S's task is to give correct answers to specific questions. We would predict an overall positive correlation between the two forms of assessment, but a detailed comparison of them might be instructive in revealing the ways in which a subject's level of understanding is reflected in paper-and-pencil tests.

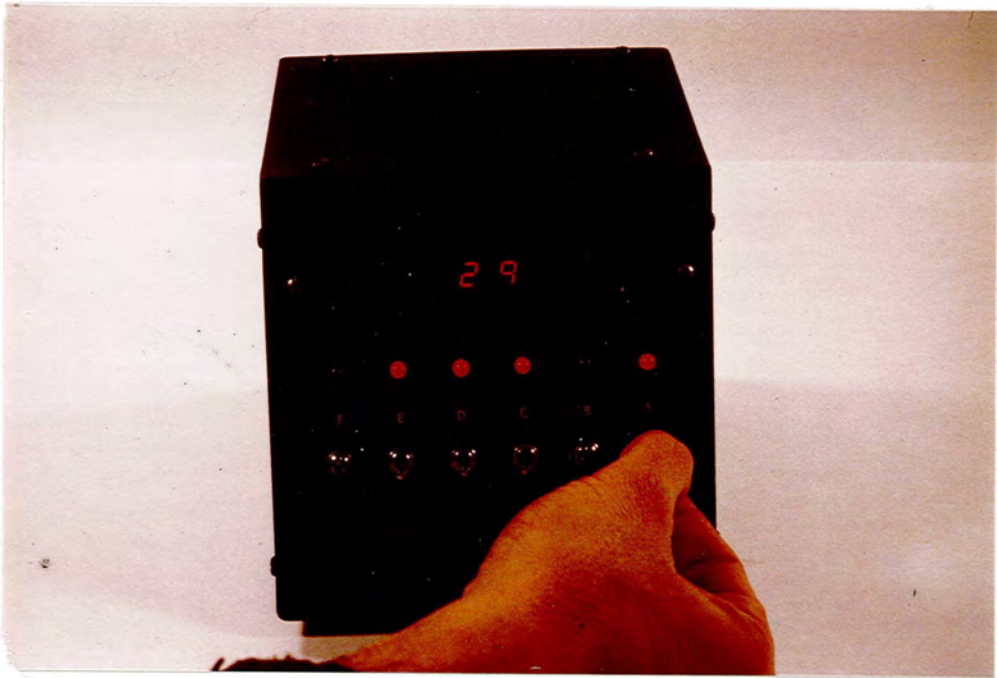
8.10 Method

8.11 Selection of Subjects

Twenty-four children, 9 female and 15 male between the ages of 9 and 12, were selected from the summer session of a day care centre in an inner-city area of a large East Coast city in the U S. Volunteer co-operation was secured from the centre which ran a full-day, five day a week programme of activities for the children. The experimental sessions were integrated into the daily schedule and every effort was made to ensure that they were perceived by the children to be part of the centre's activities. Subsequent discussions with the director indicated that this had been the case.

8.12 The Toy

A toy was specially designed to embody base two operations. As can be seen from the photograph, the toy was designed to be both portable and easily manipulated by children. The small round lights above the letters across the front of the toy indicate when each switch is in the "ON" (down) position. The position of the letters themselves corresponds to the



BINARY (Base 2) TOY

place values of base 2 arithmetic (see Table 8a). The digital read out on the top of the toy corresponds to the value in base 10 of the numbers represented by the lights on the front of the toy. By flicking the individual switches on and off, various combinations of lighting the lights appear, which can represent the numbers in base 10 from 0 to 63. Therefore, when each light is "ON" or "OFF" as operated by its switch, it represents the numerals 1 or 0 respectively, in base 2. See Rosenthal (1965, p 57-60) for more details about base 2 operations

8.2 Procedure

The 24 children were placed into 3 groups of 8, matched according to age, sex and ability. The groups were labelled demonstration, hybrid and autonomy. Children in all 3 of the groups were seen separately in one area of the day care centre. They were each given a demonstration about how to turn the toy on. This was followed by an explanation by the experimenter of how the operation of the switches corresponded to the lights. This was done with the digital number panel covered. The number panel was then uncovered and switch 'A' was shown to equal 01. The child was told to see if he could work out how the toy worked, and to notice the relationship between switches and lights, and the numbers that appeared.

Children in the autonomous group were given 20 minutes to play with the toy

Children in the demonstration group were told to watch while the experimenter operated the switches. They saw each number that came up but were not told what it was. A list of the 63 possible numbers arranged in a random order was demonstrated twice. This also took about 20 minutes.

Children in the hybrid group were told to watch while the experimenter went through half the list of 63 numbers. This took approximately 5 minutes. It was followed by a 10 minute play period, exactly half the time of the

autonomy group. Finally, the hybrid group again watched while the experimenter went through the second half of the list for another 5 minutes. Thus the overall time was also 20 minutes, about half of which was spent on play, and half of which was spent going through the numerical list once

At the end of the 20 minutes each child was given a written test consisting of 20 items.* This was followed by a tape recorded interview lasting about five minutes. They were then asked for any personal comments about the toy, the test, or the interview

8.3 Scoring

The written test was scored on a scale of 0 to 20. The children were then ranked in order (1 to 24) on the basis of their scores on this test, tied scores being treated by averaging the rankings of subjects with the same score.

The interview was conducted in the following manner. The first set of questions were open-ended (e.g. "What did you notice about the toy?"), thus giving the child the opportunity to explain without any guidance from the interviewer. General questions were then asked about the operation of the toy (e.g. "Did you notice any relationship between the lights and the numbers?") and, finally, more specific questions were included, such as "Did you notice which number came up when light E was lit?" or, "Did you notice which letters made the larger numbers?" The interview was tape-recorded for scoring purposes. The scorer of the taped interview was familiar with the toy and had previous experience of other children, playing with it. However, he was not present at any of the experimental sessions and of course had no information about the experimental group to which a child belonged. The following criteria were used to rank the

* a copy of this can be found in the appendix, Tables F and G

children on the basis of the interview. The first group of children were those who conceptualised the operation of the toy, understood the correspondence of lights to numbers individually, understood that they added together when more than one was lit, and finally that there was a relationship between the lights expressible by the notion of 'doubling'. It is important to note here that those children in the uppermost section explained the toy spontaneously during the open-ended section of the interview.

The second group of children only noticed a few specific things about the toy, e.g. different lights made different numbers, which letters made larger numbers, and some lights made certain numbers. In general they responded only to specific questions and did not seem to be aware of the overall algorithm which governed the toy. The third group of children, by the very nature of their responses, got very little out of their experiences with the toy. When asked either general or specific questions they tended to use the same words and phrases and to merely repeat what had been explained to them in the first place. They usually could do little more than relate back how they noticed lights lighting and numbers appearing. They were not able to accurately respond to any of the specific questions.

After the children had been placed in the three categories the taped transcript of each interview was analysed a second time in order to rank the children in each category. It should be noted at this point that for purposes of comparison the S's were equally distributed among three categories (i.e. 8 in each), and, whilst these categories also correspond to the criteria listed above there were borderline cases who were assigned to one category or another.

8.4 Results

Table 8a shows the distribution of S's ranks according to the interview

and the written test assessment. Mann-Whitney U tests (two-tailed) were carried out on these data (see Table 8b). For interview rankings, the comparison between autonomy and demonstration groups was significant ($U = 8, p < .02$), the comparison between hybrid group and demonstration group approached significance ($U = 15, p < .1$), while the comparison between autonomy and hybrid groups was not significant. All of the comparisons based on the written test rankings failed to attain significance. There was, however, a significant correlation between interview and test rankings ($r_s = .69, p < .01$).

8.5 Discussion

The results of this experiment must be considered in relation to two forms of assessment. With respect to the interview assessment, the present study fails to confirm the hypothesis that demonstration of relationships is as effective a way of teaching a mathematical concept as allowing the child to discover^{it} for himself or that it constitutes an effective supplement to discovery learning. With respect to the rankings obtained from the interview, although not all of the comparisons reach statistical significance, there is an overall tendency for insight into the task to be inversely related to the amount of demonstration. We can only suppose that the beneficial effects of play derived from the child's freedom to test his own hypotheses and that the lack of impact of the demonstration was a result of the child's inability or lack of opportunity to relate the incoming information to his existing ideas. If formal instruction is to build upon the achievements of informal learning, it would seem that it should begin from the child's own cognitive structures. This is in agreement with the suggestion put forward by Dienes (1963).

Although the test assessment correlates significantly with the interview assessment, it emerges as a less sensitive measure of competence. Whilst those children who did well in the interview also scored highly on

the test, children in the second category, as assessed by the interview, that is, those who did not understand the toy fully, were not distinguished from the lowest group of children by the written test. This lack of differentiation may be due to the written test encouraging children to guess or to the influence of test anxiety which might especially affect the performance of those who are beginning to master a task and thus lower their performance. If this latter hypothesis were to be substantiated, the traditional reliance upon paper-and-pencil tests of assessment would be open to criticism for communicating to certain students a too negative assessment of their ability. This possibility deserves further investigation.

In addition, it is possible that the children who are permitted to actively seek information from the toy are in a more favourable position to formulate a representation* of its operation. This results in part from the child's direct control over the material in a way which allows him to structure his own approach to a new source of knowledge.

One implication which this holds for the classroom teacher is to reconsider both the setting and techniques used in the evaluation of mathematical learning. These methods should take into account the process of how children develop and structure concepts, rather than attempting to fit 'traditional' (paper and pencil) tests to notions of what results from mathematical learning.

Therefore, particular attention should be taken by those primary schools which employ open classrooms, discovery learning and the use of special equipment in their educational programmes. Teachers should be made aware that a fundamental aspect of learning in schools is the development of the child's perception of legitimate mathematical knowledge

* a term used by Piaget, see Copeland (1974), p. 240

Since the teacher is a powerful influence in this regard, care must be taken in order that too great a separation does not occur between what is encouraged as productive learning, and acceptable methods of measuring performance on assessment tasks.

- - - - -

In the experimental investigations of the hypothesis carried out thus far, two general areas of interest have emerged. The first is the process of autonomous exploration of a mathematical system which is represented dynamically on a small electronic toy. It has been argued that subjects who explore such a toy actively make and check hypothesis about the nature of its operation. It is this structuring and restructuring of the patterns and relationships which enables them to eventually integrate the information into their own conceptual framework. With the use of a rule or a set of rules, they can explain the operation of elements within the mathematical systems (either with coloured blocks and lights or numerals), they can extrapolate to other, related tasks (as with the Klein group), and they can retain a basic 'insight' or understanding of the concept over a long period of time (the six month study)

The second area of interest is more directly related to the integration of these experiments within an educational setting. It can be posed in the form of the following question "How can periods of individual play or informal exploration be combined with more formal, structured, or group activities to foster the development of children's mathematical learning in schools?"

More specifically, it has been shown that certain limited periods of autonomy are useful in the attainment of specific mathematical concepts. However, can play with these toys aid more general understanding of numerical systems when subjects are required to solve specific problems?

That is, will children who have played with the toys benefit when they are faced with a problem to solve which indirectly relates to some aspect of other number bases?

This second area of interest will be explored in the final section (Part IV) of this thesis. The next chapter will conclude this section, (Part Three), with a set of experiments on a third toy. This work was designed in order to take a more detailed look at the relative effectiveness of different kinds of search strategies. Are there different 'gradients' of 'better' play and can they be identified by the search strategies employed by subjects? A record of these strategies will be made and then related to the subject's organisation of visual information and performance when playing the tactical board game of 'Battleships'

8.6 Summary

Twenty-four children between the ages of 9 and 12 were divided into three groups - labelled autonomy, hybrid, and demonstration. They were each given a 20 minute exposure to a binary-digital "toy" which differed as follows the autonomy group was permitted a full 20 minutes of time for "free play", the demonstration group watched passively as number sequences were presented to them on the toy, the hybrid group had 10 minutes of each kind of experience. All children were given a tape-recorded interview and a written test, each of which was scored without knowledge of the experimental condition to which the child had been exposed and ranked in order (1-24). Scores from the taped interview showed significantly better results from those groups with at least some period of autonomy. Similar trends were found in the written test, but did not attain statistical significance. It was suggested that a higher level of mathematical conceptualisation was, at least to an extent, the result of an ability to actively make and check hypotheses during the task.

TABLE 8A

	Autonomy	Hybrid	Demonstration
1st group (top 8 children)	6	2	0
2nd group (middle 8 children)	1	4	3
3rd group (lowest 8 children)	1	2	5

Distribution of children's ranks according to taped interviews

	Autonomy	Hybrid	Demonstration
1st group (top 9 children)	4	2	3
2nd group (middle 6 children)	1	4	1
3rd group (lowest 9 children)	3	2	4

Distribution of children's ranks according to written test

●	●	●	●	●	●
F	E	D	C	B	A
32	16	8	4	2	1

Values on the Binary Toy

STATISTICAL SUMMARY

Scores on the Interview Rankings		
Autonomy vs Demonstration p < .02	Hybrid vs Demonstration p < 1	Autonomy vs Hybrid N S.
Interview and Written Test Results. Correlation on a Spearman Rank Correlation		
r _s = .69 Significant at p < .01		

Table 8b

Chapter Nine Autonomy and the Development of Strategies
A Detailed Look

9.0 Introduction

The present chapter contains three studies which were carried out on a third 'toy'. As was stated in the discussion of Chapter Eight, these experiments attempt to closely examine whether or not a subject's performance on a particular task can be directly related to his/her experience with the toy.

The first study will therefore be another test of the hypothesis on autonomy. However, this time it will be related to an exploration of visual patterns on a 5 x 5 display board. As with the mathematical toys, subjects will be scored with respect to the number of items remembered both initially and in the long term. In addition, they will be grouped and ranked on the basis of a tape recorded interview.

The second study will use a much smaller set of subjects (5 in each of two groups) in order to pursue the possibility that the relationships which were apparent between each play subject and his yoke-mate in the first study was due to a gradient of more or less organised play and exploration.

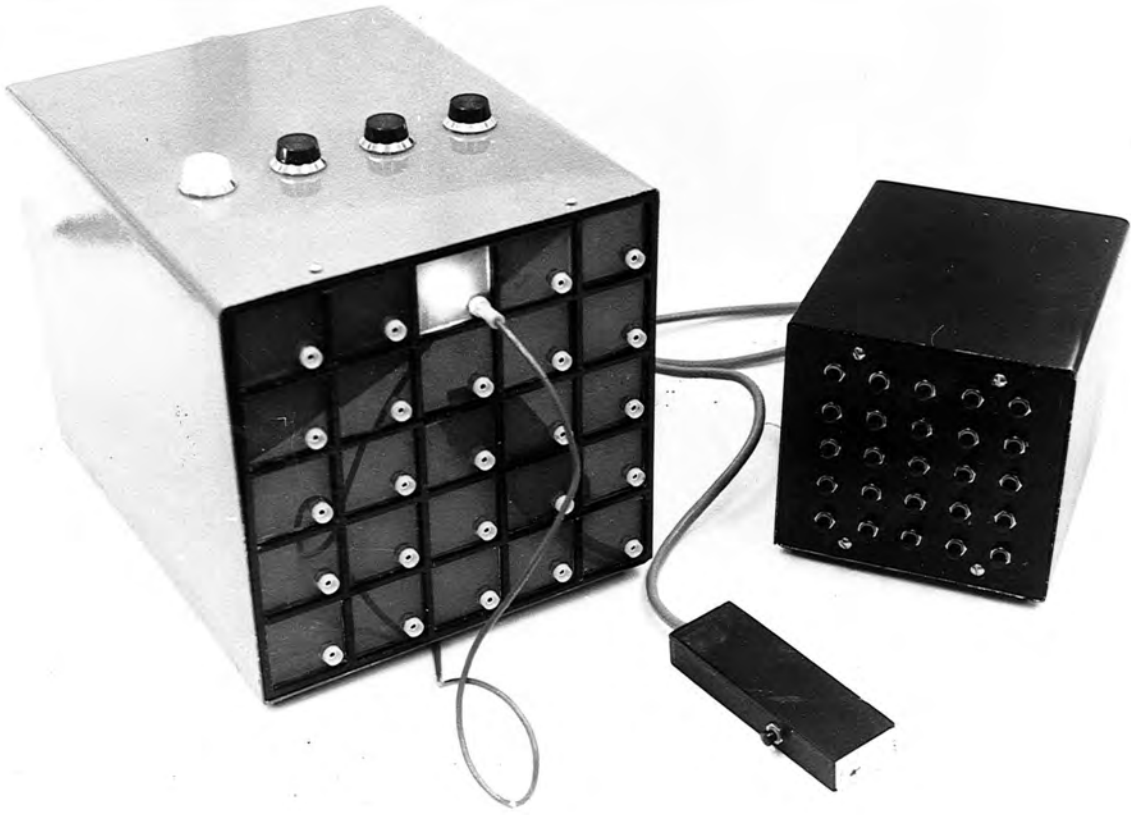
Finally, in a third task, the same ten children who took part in the second study will be given an opportunity to apply some of the strategies they may have developed to a problem solving situation. This will take place in a game of 'Battleships' played with an experimenter in a relaxed and friendly, but competitive, atmosphere. In this game the children are required to make their own initial 'Battleships' arrangement, in addition to developing a strategy which will be effective in 'defeating' the opposition's arrangement. The game, which is described in more detail later, provides an opportunity to compare the subject's perception

of patterns in an initial exploration session with their own placement of ships and use of strategy. Certain interesting correlations emerge in the study, and although any general conclusion must be tentative because of the small number of subjects used, nevertheless I believe that the insights gained justifies the use of this kind of detailed analysis on a small number of subjects.

9.1 The Toy

The appearance and operation of the toy pictured in the photo can be described in the following way. Although slightly larger than the two previous toys, it is still self-contained and portable. On the front of the toy there are 25 opaque panels, each of which has a light bulb behind it. In the lower right hand corner of each panel there is a small socket. A probe attached to a wire coming out of the toy can be used to explore the panels by placing it into any one of the 25 small yellow sockets. When this is done, the panel will light up along with one of the four coloured lights (YELLOW, GREEN, RED, BLUE) found on the top of the toy. In this way the toy can be explored in order to discover what colour is represented by each panel in the rows and columns. It should be noted, however, that because there is only one probe wire, that a maximum of one panel can be lighted at a time. Thus, the information which a subject obtains from each probing of the toy is usually soon forgotten unless it can be related to previous pieces of information and built up into a series of patterns or even a single overall arrangement.

There are two additional pieces of equipment pictured with the toy. These relate to the recording and playing back of information to and from the play and yoked subjects. The play subjects, as always, are left on their own within a room during their sessions. Their pattern of exploring the toy is recorded by observing the small black box pictured below the toy. This is attached to the toy by a removable cable and watched from



THE 5 x 5 'TOY'



THE BLACK BOX RECORDING DEVICE

another room. The 25 small red lights on its face correspond to the 25 panels which a play subject may explore in any order.

The yoking takes place by using the other black box pictured to the right of the toy. When required, this can also be attached to the toy by a long cable. With a yoked subject sitting in front of the toy, the 25 buttons are pressed in the same order as recorded from the play subject. The sequence for the yoke-mate is thus the following. sitting in front of the toy, he or she sees one panel light for a brief moment. This is caused by the experimenter and does not light any of the coloured lights on the top of the toy. The yoked subject must do this himself by placing the probe into the socket of whichever panel is lit. This action will re-light the panel (since the action of the experimenter indicating which panel to choose lit the panel for only a second or so), and also light the appropriate coloured light on the top of the toy. The yoked subject is therefore performing an action identical to the play subject with the critical difference once again being that he or she does not have any autonomy over which panel to explore.

9.20 The First Task

9.21 Selection of Subjects

Sixteen subjects, 9 male and 7 female between the ages of 10 and 12 were selected from the summer session of the same day care centre which was used in the previous experiment. All subjects were volunteers and had not participated in the other experiment, although by now some of them had begun to know the experimenter personally.

9.22 Procedure

The 16 children were randomly divided up into two groups of eight. However, there was an effort made to keep the male/female ratio and pairings from becoming too one-sided. Children in both groups were seen

separately in one area of the day care centre where they were given informal instructions on how to play with the toy. They were told that when they put the probe into one of the sockets, the panel would light up. In addition, it was pointed out that this action would also light one of the four coloured lights on the top of the toy. The panel and coloured light would stay lit for as long as the probe was kept in the socket. As an example of this, each child was shown panel '4B' (the panel on the fourth row down and the second column across) which lit up GREEN. The child was told to notice which panel lit up which light and to try and remember it. Children were also told to notice and remember any pattern or shapes which the coloured lights might make on the toy.

Children in the autonomous group were given 15 minutes to play with the toy and each move they made was recorded.

Children in the yoked group were given a programme to follow in the manner described earlier. Each one of these programmes corresponded to the exact sequence of play of the subject's 'yoke-mate'. These sessions sometimes took slightly longer or shorter than the 15 minutes given to the play subjects. Their exact time depended upon whether or not the autonomous child had played quickly, or more slowly and deliberately. However, yoked children were able to spend as much or as little time with each panel lit as they required, so that they were able to have a large amount of control over the pacing of the exploration.

At the conclusion to each session, the child was given a 5 x 5 chart and told to fill in the 25 spaces with either yellow, green, red or blue as remembered from the toy. This was followed by a tape recorded interview which was intended to assess the extent to which the child noticed and used the pattern of colours embedded within the panels.

In addition, each child was given the chart task 10 days after their

session with the toy. This was recorded and scored on a scale of 0-25 in the same manner as the initial chart task.

9.23 Programme of the Toy

The arrangement of the pattern of the toy can be seen in diagram 9a. For purposes of reference, the rows have been numbered from 1 to 5, and the columns have been lettered from A to E. In the 25 panels shown, 1 lit up the yellow light, while 8 lit up green, 8 red, and 8 blue.

If these were arranged in a random order and only seen one at a time, it would be quite difficult for 10-12 year old children to remember very many more colours than one would expect from chance. However, as can be seen from the chart, there is a definite pattern in the way the colours are distributed. It can be described as follows: there is only one yellow light and it is in the centre, (panel 3C). It is completely surrounded by 8 green lights. This leaves only the outer ring which consists of alternating red and blue lights. The red lights appear in each of the corners and therefore can be thought to 'begin' the alternating series. It should be pointed out here that since the operation of the toy only permitted one coloured light to be lit at a time, the pattern found in this programme was never seen to be as obvious as if all of the colours could be revealed at once.

This programme was chosen after another arrangement (see diagram 9a) was found to be too difficult for this age group. Information from the same pilot study also indicated that 15 minutes was enough time for a complete investigation of the toy without allowing too much time for the child to become bored or disinterested.

9.24 Scoring

As was mentioned earlier, there were three scoring sessions. The first occurred immediately after the experience with the toy and was an

attempt to measure the number of colours which the child noticed. This was scored on a scale of 0-25. Because there were only four possible colours for each square, a score of 8-9 would be attained by a random placement of colours.

Each child was then interviewed and a 'blind' scorer* placed the 16 subjects into four groups depending upon the following criteria: group one noticed no patterns, and attempted to reproduce the colours totally from memory. Group two noticed that there was a yellow in or near the centre and that the other colours seemed to 'radiate' out from it. Group three noticed the yellow in the centre surrounded by green and then red and blue in some sequence. Finally, group four noticed the entire pattern, that is, they could accurately describe the ring of alternating red and blue, with red being present in each corner. It was interesting that the five subjects who attained the perfect score of 25 (4 play and 1 yoked), all were able to describe this pattern. A perfect description, it was found, almost guaranteed a perfect score by the very nature of its accuracy.

After each subject was put into one of the four groups, the tapes were heard again and subjects were compared within groups. On the basis of this, each child was ranked within the group. Since there were many borderline cases between the groups, the scores are shown in rank order from 1-16, with a score of 1 being the highest level of conceptualisation attained. Even though it was difficult to distinguish between groups, a second listening to the tapes found that there were usually clear indicators distinguishing one child from another.

The third score given was based upon the same chart task as the first score, but this one was administered ten full days after the subject's

* in this case, the author

experience with the toy. For this score, each child was again seen individually and the toy was present, but the child was not permitted to touch it. Subjects were first asked to review what they remembered about the patterns in the toy and then they were given the blank piece of paper to fill in. In this review, the interviewer made a conscious attempt to not tell the child if any of the patterns they were describing were correct or not. The slight overall rise in scores in both the play and yoked groups is most probably due to the fact that it came after both the formal interview and this brief review of the patterns. The explanations given by the children probably made them slightly more aware of the exact positions of some of the blocks, although the change in scores is not statistically significant.

9.25 Results

Table 9b shows the distribution of subjects' rankings according to the interview, as well as their initial and final chart scores. For the interview rankings, Mann-Whitney U tests (two-tailed) revealed a significant difference between the play and yoked groups ($U = 12, p < .05$). In addition, the correlation between the interview rank and the final chart score for both groups of subjects combined is very high ($r_s = .96$). This is significant at $p < .001$. Comparisons based solely upon written test information showed a consistently larger average for the play group (17.9 compared to 13.3 for the initial task, and 19.4 compared with 14.0 for the final task). The play group also had twice as large an increase in average score from the initial to the final chart task (1.5 compared with .7). Chart task scores by themselves did not attain statistical significance on a rank order test (Mann-Whitney). However, there was a high correlation, $r_s = .86$, between individuals in the play and yoked groups on their final chart task. This is statistically significant at the .05 level.

9.26 Discussion

The results of the interview rankings once again fail to confirm the hypothesis that a kind of demonstration of the patterns contained within this particular toy is as effective a method as autonomous exploration in the development of a child's perception and understanding of the arrangement. This result is particularly interesting because the method of yoking which was used permitted the passive subjects to go through the same physical motion as the play subjects in addition to allowing them control over the pacing of the probes

As in the previous experiment, written test scores on their own, while showing a strong trend in the direction of play being a more useful experience in remembering the 25 colours, do not attain statistical significance at the .05 level, even though they correlate very highly with the interview rankings.

What is also of great interest is the final result reported. Namely, the final chart scores for the play and yoked groups show that there is a relationship between the score which was attained by the play subject and the score attained by the play subject's yoke-mate. This provides indirect evidence that there may be different levels of 'play', and that some strategies of exploring the toy are more effective in revealing the patterns than others

This hypothesis can be given a more direct test on the toy by recording the order of exploration of a group of play subjects and then having a blind scorer rank the subjects on the basis of this play. This ranking would be based upon the organisation of the strategy used by the subject, specifically with regard to the way that it revealed the patterns found within the toy. It would be interesting to see if this ranking could be related to the subject's performance on a task based upon the exploration, and in addition, whether or not this relationship would hold for subjects

who were yoked to the same strategy. These questions form the basis for the next two tasks which follow.

9.30 The Second Task

9.31 Introduction

The aim of the second task is to take a closer look at the kind of strategies which individual subjects employ. If, as suspected, some play subjects use a procedure which is more effective than others in revealing the patterns and relationships within a toy, are there some circumstances where the differences will show up on the scores of their yoke-mates?

In order to begin to answer this question, the programme of the toy was changed to that shown in diagram 9c. Children were exposed to this pattern individually in order to familiarise them with the operation of the toy. In addition, for purposes of the games which were to follow, the children were shown three configurations, each of which represented a different war ship. Thus, in the figure, the single Green panel represents a destroyer, the two Yellow panels together represent submarines, and the three Red panels are air-craft carriers. The remaining Blue spaces are the background or sea. It was pointed out to the children that these colours made up the boats and that submarines and air-craft carriers could be found either up and down or across, but never diagonally.

The children in the play group probed the toy and told the experimenter every time they found a complete boat. In this way the children came to know all of the boats by name. Similarly, the yoked group was passively shown the same exploratory pattern as their play-mate. They too told the experimenter when they noticed a complete boat and therefore also learned all of the boats by name. It took each child about five minutes to find all of the different kinds of boats.

When it was felt that each child was comfortable playing with the toy, in addition to being able to recognise all of the boats, the experimental session began.

The Experimental Session

9.32 Selection of Subjects

Ten subjects, six male and four female were divided up randomly into two groups of play and yoked. The children were all aged from 10 to 12 years, and they came from the same day care centre used previously.

9.33 Procedure

First, the pattern of the toy was changed from the introductory pattern to that shown in diagram 9c. As can be seen, there are three destroyers (Green) two submarines (Yellow), and one air-craft carrier (Red) arranged on the panels of the toy. Because of the wiring of the toy, no colour could be lit up by more than eight panels, so Blue was no longer used as the sea. Instead, the children were told that when no colour came on, that represented the sea surrounding the ships. Since 'no colour' is perceived as more different from Yellow, Green, and Red, i.e. the ships, than Blue was, this actually simplified things for the children.

Before exploring the toy, children were asked if they could remember which colours made up the various different kinds of boats. The experimenter reviewed any that were not familiar, making sure that each child knew them all well before beginning the task.

The autonomous group was given ten minutes to explore the toy. They were instructed to play with it while looking for all of the different kinds of boats, remembering where it was that they saw each one of them. Their play sequence was recorded by the experimenter.

The yoked group went through the same procedure, exploring in the

sequence which was determined by their play-mate

After the exploration, each child was given a 5 x 5 grid and 6 Green, 6 Red and 6 Yellow blocks. They were then asked to fill in the grid with the blocks as they remembered them from the toy. It was pointed out to them that there were more blocks than boats so that there would be some left over. It was thought that the task would be too easy if the exact number of blocks was provided. The number of correct 'hits' each child made was recorded.

9.34 Scoring

There are two scores for each child. The first is the number of correct 'hits' which they made on the 5 x 5 grid. This is a number from 0-10, with the maximum score being earned if they could correctly recall the position of all of the ships. No child was able to do this.

The second score was a ranking of the five play sequences recorded from the play subjects. This was first viewed subjectively by a blind scorer and then analysed on paper. Rankings were made on the basis of how much information was revealed about the relationships which exist between the colours making up the boats. More specifically, an algorithm was used which operated on the following simple criteria: every time a 'hit' was made on either a submarine (where 2 consecutive Yellow lights made up a boat) or an air-craft carrier (where the boat was made by three consecutive Red lights), the next panel which the child explored was noted to see whether or not it took place in one of the adjacent squares where the other part of the boat may have been found. Thus the "X" squares should be a more probable area of search than the "0" areas if the child was

0	0	X	0	0	
0	X	"HIT"	X	0	"HIT" ON A YELLOW OR RED SQUARE
0	0	X	0	0	

A PORTION OF THE 6 x 6 BOARD

relating the lights to boats. In the five play sequences ranked in this way, a large division was found between those which were ranked 1, 2 and 3, and the two sequences which were ranked 4 and 5. Clearly then, some of the children appeared to be actively looking for boats, while others were seemingly just exploring coloured lights without any apparent active perception of the pattern of relationships between them.

9.35 Results and Discussion

The results may be found in Table 9d. A Spearman rank order test found that there was a correlation ($r_s = .90$) between the yoked group and the 'level' of play information received. This is significant at $p < .05$. On the number of hits made, the play group did average a slightly higher score than the yoked group, but the difference was not as large as one may have expected. Looking into the possible reason for this, it was found that members of the yoked group averaged a higher score on a written test of basic maths ability than the play group. The yoked group averaged 19.0 (raw score) on the maths placement test, while the play group averaged only 15.2. These figures represent percentage scores of 68% and 54% respectively. While they only approach statistical significance ($p < .05$), nevertheless, they could be part of the reason why the play group did not attain scores which were greater in their difference to the yoked group. Because of this, any comparisons between the two groups must be made with great care. Although the scores on the maths ability are not as evenly distributed overall as one may have wished, we can still compare members of each group to the ranking of the information given to the sequence which they experienced.

When this is carried out in the yoked group, although the number of subjects is small, the relationship between performance on the task and the organisation of the play which they were shown does seem to hold. It is especially satisfying to see the large difference in scores between

subjects 1, 2, and 3 in both groups and subjects 4 and 5. This division was anticipated from the scoring of the five play sequences.

Another difficulty with making certain conclusions from these results is that there is a lack of control information about individual subjects. As was stated above, since a 'random' distribution of the 10 subjects produced a yoked group which was more able according to the written maths test given, we cannot really compare each pair of play-yoked subjects. One way to solve this problem while still using the same 10 subjects is to have each child act as a control for himself. This is what is done in the third task reported below. The design of the experiment reflects my interest in the use of strategies within problem solving situations which more closely approximate 'real world' concerns.

9.40 The Third Task

9.41 Procedure

In this experiment, the same ten children who participated in the previous task were used as subjects. The procedure used here was much more playful than any of the other situations, with the children being told that there was a game which they would be playing with the experimenter. This game did not involve the use of the toy, but rather was played on an enlarged grid (6 x 6 instead of 5 x 5) which was supplied to each child printed on a piece of paper. It should be noted here that none of the children had ever played or seen the game before.

Each child was first given an explanation of the rules of the game. They were told that both they and the experimenter would initially place a number of boats on the grid in any arrangement they wished. This was to be done 'secretly'. The types of boats allowed were the same as those used in the previous task, except that there would be more of them

permitted on the larger board. Thus, the child was to position two air-craft carriers which took up 3 squares each, three submarines which took up 2 squares each, and four destroyers which took up 1 square each. The experimenter 'pretended' to make an arrangement also, but in fact always used the standard pattern pictured in diagram 9e.

The grids were labelled 1-6 down the left hand side and A-F across the top. This was used for reference purposes when 'shots' were taken.

The child was then told that each side would alternate in the firing of shots. This took place by calling out one square, such as "C 3", as a shot on the enemy navy. The enemy would then have to correctly tell his opponent whether or not they missed (the shot landed in the water), or not one of the boats. In addition, if a hit was made, the enemy had to reveal which type of boat it was. As a bonus, when a boat was sunk, that is, all sections of it hit, another shot was awarded. The winner of the game is simply the person who first sinks all of the boats in the enemy navy.

In order that the child did not learn any strategy about shooting from the experimenter's attempts at hits, a standard random pattern of shots was used by the experimenter. This is reproduced in diagram 9e.

9 42 Scoring

Since the experimenter was using a random sequence of shots, for our purposes it was not important who actually won the game. Rather, subjects were scored on two items. The first score was given for the arrangement which they used in the initial placement of their ships. These 10 arrangements were ranked in order from 1-10 (1 being the 'best') by a blind scorer. The main concern was the effectiveness of the placement in terms of 'defending' against the enemy's use of strategy. Since this game clearly involves a great deal of luck, the only use of skill which

a player can demonstrate is in the initial placement of boats, and in the following up of hits made on a portion of a submarine or an aircraft carrier. This is done by calling shots on squares which are adjacent to those where hits have already been made on one of the two types of boats mentioned above. This will increase the chance of 'sinking' that boat and earning the important bonus shot which can be used to explore further. These bonus shots, in a game where the shooting must begin by guessing where a boat may be, can be seen to often make the difference between victory and defeat. Therefore, the initial arrangement is usually more effective if the boats are spread out. This will prevent a simple sequence of shots to hit them all (as in the arrangement shown in Table I in the appendix.) In addition, the enemy may be confused by placing some of the larger boats horizontally as well as vertically on the board. If all of the ships are placed in one direction in sequence, and this is discovered by the opponent, the chance of his making a second hit once a boat is found increases. Examples of this are given in two 'good' and two 'poor' placements illustrated in the appendix, Tables H and I.

The second score given to each child is based upon the shots which they take. This is an objective number which is calculated in the following way. The number of opportunities which the child had to follow-up a hit on the experiment's boats was recorded. Then, the number of chances actually taken in making an effective follow-up shot was counted and the overall percentage of chances made was calculated. The higher the percentage, the better the strategy which was employed and the greater the chances of winning the game.* As was explained, the follow-up

* Interestingly, although the final "won-lost" result was not considered for reasons mentioned earlier, the 4 subjects who had a strategy % above 70% all won, while the 6 subjects scoring below 70% only contained 3 winners

is represented by the child shooting on a vertical or horizontal where a boat of 2 or 3 spaces has already been hit. In addition, scores are also given where an air-craft carrier has been hit twice and needs only one more hit to sink it. By knowing the first two places where it was hit, the child has a 50% chance of sinking it on his next turn since there are only two places where it can be

B	C	D	E	F	G	
A	"Y"	X	X	"Z"	H	X = hits already made on the air-craft carrier
N	M	L	K	J	I	

In the diagram above, shots "Y" or "Z" would be given credit, while shots "A" - "N" would not.

The combination of good placement and effective strategy cannot guarantee a win in a game such as this where luck plays such a large part in the making of hits initially and on exploratory shots. However, as with good playing in certain card games, over a period of time players with better strategies will be found to win consistently more games than their opponents who may have an inferior approach to the playing of the game.

9.43 Results and Discussion

Table 9d shows the rank order of the battleship placements and the percentage scored on the 'hit' strategy for the ten subjects. They are listed directly across from the scores they earned on the previous task so that each subject may be individually identified. The game playing scores seem to correspond very well to the performances on the perceptual task. In fact, there is a correlation of $r_s = .96$ between the 'gradient' of play strategy as measured by the 1-5 ranking that each child was exposed to, and the level of his/her placement of ships in the actual battleships game. This result is statistically significant at $p < .001$. In addition, there is an internal consistency between the way the ships were placed, and

the percentage scores on the 'hits' strategy. The correlation between them is very similar to the one found above, in this case $r_s = .97$ which is also significant at $p < .001$

These results may best be interpreted in terms of a progression of tasks from strategies used to perceive an arrangement of boats in the toy, to the formulating of a pattern of boats in a personal arrangement, finally to the use of a second strategy 'attacking' the enemy's configuration within a competitive game. In each of these tasks there was a spectrum of more to less sophisticated arrangements. These were able to be identified and ranked by a blind scorer, and in addition were found to selectively affect a group of subjects yoked to them. Further, while playing a game with paper and pencil, it was found that the various levels of the subjects' perceptions of these relationships corresponded to the efficiency of the strategies used by them in playing.

This last factor is most directly related to our interest in education. It will be argued in the next section of the thesis that mathematics teaching in the primary and middle school which will both be remembered and used by the children must involve itself at a fundamental level with the child's conceptualisation of relationships. This cannot be accomplished if the teaching is limited to a series of demonstrations and examples presented in an instructional mode. Rather, children must have a certain amount of time to discover the dynamics of these relationships in terms which they can fit into their perceptual structure of reality * The patterns, relationships and rules which are taught in the classroom must not merely be applied as a set of formulae to memorise.

* That is, the way they perceive things to operate, i.e. randomly, in structured patterns, by rules, or whatever.

Instead, the context within which the material is taught should permit the child to use information he or she has acquired to develop methods of solving mathematical problems which have not been previously seen and therefore might otherwise go unsolved.

9.5 Summary

This chapter contains three separate experimental tasks which are related by the common theme of exploring and producing arrangements in a toy and with paper and pencil. In the first task, there were 16 children who were divided up into two groups of play and yoked. They were each given an experience on a specially designed toy which was intended to reveal the patterns embedded within it. Scores based on interview rankings showed that the yoked group failed to attain as high a level of conceptualisation on the patterns as their play-mates. Initial and final numerical scores showed the same trend and correlated highly with interview rankings, but did not show statistical significance on their own. The second task used 10 children, again divided up into two groups, play and yoked. The play group explored the toy which was re-programmed to conceal a number of 'battleships' as defined by the game. This play was recorded and ranked in the order of its effectiveness in revealing the patterns of the ships. Scores of the subjects yoked to these sequences correlated with their performance on a task of remembering where the ships were in the arrangement. Further, a third task found that when subjects constructed their own battleships arrangement, that scores based upon the effectiveness of the arrangement were also closely related to the effectiveness of the strategies which they employed while playing the game. These scores also correlated positively with their perception of the arrangement found in the previous task. Some educational implications are discussed briefly.

	A	B	C	D	E
1	R	B	R	B	R
2	B	G	G	G	B
3	R	G	Y	G	R
4	B	G	G	G	B
5	R	B	R	B	R

Initial pattern used on the toy

	A	B	C	D	E
1	Y	B	R	B	Y
2	B	R	G	R	B
3	R	G	Y	G	R
4	B	R	G	R	B
5	Y	B	R	B	Y

Pattern not used on the toy because of the difficulty it presented in a pilot study.

Diagram 9a

Group 2 YOKED

Sex	Initial score	Final score	Concept rank
M	25	25	4
F	20	19	9
M	13	15	7
F	10	14	11
M	6	9	15
F	13	12	14
F	11	12	12
M	8	6	16
average			13 3
average			14 0
average			11 0

(Yoked to →)

Group 1 PLAY

Sex	Initial score	Final score	Concept rank
(1) M	25	25	1
(2) F	25	25	2
(3) F	25	25	3
(4) M	25	25	5
(5) M	14	18	6
(6) F	10	14	10
(7) M	10	12	8
(8) M	9	11	13
average			17 9
average			19 4
average			6 0

Table 9b

	A	B	C	D	E
1	R	R	R	B	Y
2	B	G	B	G	Y
3	B	R	R	R	B
4	Y	B	G	B	G
5	Y	G	Y	Y	B

Pattern used as an introduction to the boats

	A	B	C	D	E
1			G		
2		R		Y	Y
3		R			
4		R		Y	G
5	G			Y	

Pattern used in the experimental session

Diagram 9c

Table 9d

Group 1 PLAY

Sex	Chart score (out of 10)	Organisation of play (1-5) seen	Arrangement of ships (1-10)	Game strategy % scores
F	6	1	2	100%
M	7	2	3	75%
M	7	3	6	55%
M	1	4	9	17%
F	1	5	10	17%

Group 2 YOKED

Sex	Chart score (out of 10)	Organisation of play (1-5) seen	Arrangement of ships (1-10)	Game strategy % scores
F	6	1	1	75%
M	7	2	4	71%
M	5	3	5	67%
M	1	4	7	43%
F	0	5	8	29%

	A	B	C	D	E	F
1	G					Y
2		R	R	R		Y
3		G				
4	G		Y		R	
5			Y		R	
6	Y	Y			R	

Placement of the experimenter's boats

	A	B	C	D	E	F
1	9	22	21	10	16	26
2	13	31	1	30	19	4
3	29	14	15	32	25	11
4	36	3	24	5	35	18
5	28	33	8	34	17	23
6	20	7	12	27	2	6

'Random' order of shots made by the experimenter on the enemy's boats

Diagram 9e

Part IV Educational Applications

Chapter Ten Classroom Use of the Mathematical Toys

10.0 Introduction

In this chapter, I will discuss the results of using the Binary and the base four toys as teaching aids in two different schools. Previous chapters demonstrated that there is some empirical support for the hypothesis presented on mathematical learning. However, it remains the task of this section of the thesis to examine how these toys and the experimental information gathered about them can be used in actual classroom situations.

More specifically, a set of four general question areas have been developed and are listed below. These were not discussed in advance with the teachers in the classrooms where the toys were introduced. Rather, they provided the experimenter with a basis for evaluating the usefulness of the toys. The form of this evaluation follows from the general thrust of the entire thesis in that it is not limited to a series of psychometric tests administered at the conclusion of the educational innovation in order to decide whether or not it was successful.

Although I was not present either when the toys were introduced to the students, or during most of the time they played with them in the classroom, I was able to visit the schools more extensively after the children had seen the toys in order to present them with some related problems to solve. Neither the children nor the teacher knew of this in advance.

The questions to which this chapter is addressed are the following. First, will play with the toys in the classroom be a successful enterprise in itself? Will the children be curious enough to explore the toy when no one is ensuring that they must (as in

the experimental situations) and will it hold their interest over a period of days?

Second, can this 'free play' be profitably combined with more formal instruction by the teacher? Will the toys lend themselves to some of the written work which the children may be doing in their maths text books, for example?

Third, will the play be a beneficial experience when the children are presented with problems to solve which only relate in an indirect way to the operation of the toy? The experimental studies revealed that play was a more useful way of encountering the toy than a passive, yoked, or combined experience. However, will periods of free play alone be sufficient in a classroom situation where the children must extrapolate from the toy without any instructions from the teacher?

Finally, will play with the toy encourage the children to think about the subject area in a wider context and perhaps enable them to make discoveries on their own? While it would defeat the purpose of this question if the experimenter himself inquired about this in advance of the periods of play (by alerting the teacher or pupils about the need or possibilities to do so), it is possible to express an interest after the play had occurred. This I did, giving the children an opportunity to show me what they learned from the toy as well as giving me their opinions in more general terms.

I have divided this chapter into two sections. The first deals with the use of the Binary toy as part of the classroom process of instruction. For this I was able to go back to the same school (over two years later) where the initial observations (reported in Chapters Two and Three) were carried out

The second section makes use of the base four toy in the Middle school where experiments had been carried out one year earlier. Both

sessions extended over a period of a few weeks. I spoke extensively to the teachers involved both before and after the toys were used. In addition, I was able to personally conduct several lessons within the classroom in both the primary and Middle school in order to accumulate the data presented in this chapter. It should be stressed that every effort was made to be fair and objective. However, because I was more interested in what was possible and not just what was likely to occur in a classroom, and also because of the nature of the questions posed in this introduction, work in the classroom did not take place under controlled or experimental conditions. Therefore, although it may be argued that the use of another school or teacher or class would have produced different results, to do so would be to misinterpret the point of the exercise.

10.10 Section One The Binary Toy

10.11 Play and More Formal Classroom Instruction

I met with the headmaster of the primary school and showed him the Binary toy which had been developed as a direct result of my interest in mathematics teaching and of time spent in his school. He told me that Mrs. G, who normally taught the Alpha (or top) group of the eleven year olds had left the school and that he now took this group for maths. He was interested in the toy and related to me in several conversations how classrooms in his early days of teaching conformed to very rigid structures. Maths lessons were dominated by reciting tables, memorising certain rules for division, etc., and learning how to handle money.

In the early part of the 1960's, certain changes began to take place in education, and he was able to attend a series of seminars on the newly emerging Modern maths in the primary schools. One way of approaching this material was titled "Maths in the environment". Here children were

encouraged to use the basic skills learned in the classroom and apply them to the world around them. For example, the concept of average was demonstrated when a heavy rain brought out large numbers of worms from the ground in the spring time. To the delight of the children they were able to measure the worms outside and then calculate the length of the average worm seen by various individuals, all of the girls, the entire class, etc. Likewise, a visit to the farm provided information to answer the question "How much milk do cows give?"

Slowly, as the new maths became more accepted, this type of teaching became the norm. But the problem still remained of integrating materials from the outside world into the classroom in a way which would enable the children to form generalisations about them.

For example, one child who was asked the meaning of 'circumference' after the class had used a leaf and a book as an example reported that it was the distance around any book or leaf! Clearly, then, the point made earlier that children must have several examples of concepts available to them so that they can generalise beyond triangles, or wooden squares, or whatever, is well taken.

By way of this brief background, I learned that the headmaster was not new to the ideas of progressive mathematics. However, much of the time in his classroom was spent in instruction and demonstration. On the surface, I found many similarities with the original analysis made from Mrs. G's classroom. It was encouraging to find that the model presented was still a very good indication of the basic processes which occurred in the school.

The headmaster, (hereafter Mr. H), willingly agreed to introduce the toy to the class especially as it related to certain material that he was teaching at the time. We both agreed that it would be very

interesting to see how much the children would learn from playing with the toy and whether or not any practical difficulties might arise. I left the toy with Mr H and returned to the school about two weeks later. I was very pleased to find that the toy was available in the classroom for any child to play with or use. At that time they were studying modular arithmetic (base 5) which preceded a section in their books on the binary system (base 2).

Mr. H told me that children were initially permitted to explore the toy one or two at a time in a small room away from the rest of the class. Doing this, he explained, avoided the problems caused when a large number of children crowd around any new object when it is first presented in class.

Perhaps because of the 'electronic' aspect of the toy, I found both in this classroom and in using the toy more generally, that children were very careful with it and that they always ensured that it was switched off when they were finished with it. Had they not, the battery would have quickly run down since the number 00 shows even when all of the lights are 'off', if the main switch is left on.

Mr. H did not immediately use the toy as part of a very specific lesson on base 2, but rather he let the children play with it first, and only then used it in front of the classroom. While interviewing the children, I found that all of them thought that this was a good idea, and that they especially enjoyed the opportunity of 'messing about' with the toy before being presented with a lesson

Another point of interest is that Mr H reported that the boys were the more 'aggressive' players on the toy and that overall they spent more time with it than the girls. Further, some of the more extroverted boys spent the longest periods of time with it even though they were not

necessarily the best mathematicians in the class. This evidence relates to the study made by Littlejohn (1977) reported earlier, and it deserves further investigation.

In summary, then, the toy was introduced to the children in very general terms and then they were permitted to see it for a period of time in a separate room where they were either alone or in small groups of 2-3. Finally, when this had been completed, the toy was placed in the classroom where individual children could play with it and relate it to the more formal instruction on base 2. It was also available to be used in small groups while the children were working in the classroom (see photo 10A).

I would now like to report several uses to which the toy was put by the students as a result of their own perceptions of base 2.

10.12 Play and Original Discoveries

After two weeks or so passed and all of the children had an opportunity to play with the toy in the classroom, I returned to conduct one or two lessons and present them with a few problems to solve. Before doing this, I spoke to Mr. H in his office and he told me that some of the children had developed quite a few interesting activities with the toy on their own. In addition, there were some quite unexpected games carried out on the basis of information learned from the toy. I was able to circulate around the classroom and speak to the children about these things.

By far the most common 'discovery' made was that each light on the toy lit up a number which was double the number which the light one place over to the right lit. Children reported noticing this when they placed each lettered switch 'on' singularly and found the pattern 1,2,4,8, etc. Because of this relationship, several games were developed. The first was called 'doubling up' and was played by switching the lights, A,B,C, etc

PHOTOS



PLAY WITH THE TOY WITHIN THE CLASSROOM
PHOTO 10 A



THE GAME OF 'COUNTING UP'
PHOTO 10 B

'on' one at a time and watching how the numbers on the digital read-out would double. The second game followed from this and was called 'counting up'. Here one child, usually surrounded by a small group of friends, would operate the switches in such a way that the digital read out would count up in base 10 from 1 to 63 (see photo 10B) If a mistake was made, the group would call out and occasionally the 'player' would forfeit his role to another. All in all it was a most active enterprise having an air of hectic competition about it as each child tried to speed along with greatest accuracy.

The final game in this series was played in a much more settled atmosphere. Since the children were all familiar with the small electronic pocket calculators available, and indeed one or two children even brought them to school, they soon wondered whether or not the toy could be used in the same way. One group of pupils discovered that within limits it could indeed be shown to add up certain numbers below 63. However, there were limitations and children were able to explain that numbers could only be added together on the toy when each of them singularly was able to be displayed using a completely different set of switches. Interestingly, the children were using notions of intersecting and non-intersecting sets of lights (A-F) although I did not pursue this with them at the time. Thus, I was shown how the number 18, (switches B and E), could be added to the number 9, (switches A and D) and it would yield 27 (switches A, B, D and E) But, 7, (switches A, B and C), could not be added to 12,, (switches C and D), without 'carrying over' to switch E, since both 7 and 12 have the need for switch C (which equals 4) in common.

10.13 Making Blocks and Patterns

One particularly interesting discovery was made by a boy described by Mr. H as being extremely shy. His desk was nearest to the teacher's and he rarely raised his hand or shouted out in class although his maths

work was well above average.

Mr. H reported to me that this boy asked him for some celo-tape while he was playing with a set of Dienes blocks during play-time. Interestingly, the school owned Dienes blocks sets in base 3, 4, 5 and 6, but not in base 2. What this child did was to take a large number of the single or units blocks and tape them together thus making a base 2 set which corresponded to each of the place values, 1 to 32 as on the toy. Further, when I spoke to him about it (see photo 10C) he explained to me that he noticed the following pattern the first shape (1 block) was a cube, the second shape (2 blocks) made a line, and the third shape (4 blocks) made a flat square. The next shape, which contained 8 blocks, made a larger cube, and then a line, a flat square and so on. The photo shows him pointing to the long line shape which is made up of 16 small blocks. In addition, when I asked him if this would hold true of any base he reported that it would, although as he put it, "the lines and squares would be longer and would grow much more quickly." Quite an insightful discovery

The usefulness of these celo-taped Dienes blocks did not end there. One group of boys, while playing with them, noticed that any series of shapes put together "in order"* would always have one small block missing at the corner. Photo 10D shows the group proudly displaying the missing corner when all of the pieces were assembled. After a few questions, they were able to point out that the number of small blocks in any piece made up in this way would equal the base to the power of the number of small blocks less one, or, $\text{base}^x - 1$, which in this case was $2^6 - 1$, or 63.

* "in order" meaning the 1 block shape added to the 2 block shape added to the 4 block shape etc.

PHOTOS



EXPLANATION OF BLOCKS CONSTRUCTION

PHOTO 10 C



"THE MISSING PIECE"

PHOTO 10 D

When I asked one of the boys how he had come to first notice this relationship he reported that he could never quite make a 'completed' shape out of the smaller blocks when he used them in this way and so he wondered why.

10.14 Play and Problem Solving

When I was given the opportunity to conduct a lesson in front of the entire class, rather than attempt to teach them about the place value system in base 2, I decided to present them with a problem to solve. It was stated as follows.

Suppose that you were the Minister of Transport in an imaginary country, and that it was your job to design the number plates which appear on cars. Let us also say that you have chosen to use only letters and not numbers. (I then made sure that all of the children knew that there were 26 letters in the alphabet) The problem is this: you have 1,000 cars in this country. How many spaces must you allow on each licence plate (that is, how many letters would appear on each assuming that the same amount of letters appeared on all of the licence plates), in order to assure that no two cars would have the same licence plate, and also, since it was important to economise, to make sure that only the minimum number was chosen which would be sufficient to register the 1,000 cars?

After I presented the problem, the children gave out 'Ohhs' and 'Ahhs' indicating that it seemed to them to be a very complicated problem. I stressed that I was more concerned with the ways they chose to work out the problem than with the answer itself. Thus, wild and unsubstantiated guessing out loud was discouraged. I then circulated around the room to see how the pupils set about to solve the problem.

I should mention here that earlier in the day I randomly handed out 10 of the maths tests which were used in the experiment reported in chapter eight and shown in appendix Tables F and G. Nine of them were

returned to me completed, and the marks they received were quite high, averaging 19.3 out of 20. Five of the children received perfect scores of 20, and the other scores were two 19's and two 18's. Clearly then, at least in the short term, the children did understand the concept of place value in base 2. The test results showed no differences for boys or girls although the boys did finish more quickly.

After collecting the tests, I waited a few minutes and then asked if anyone had gotten an answer to the problem. Since no hands went up I re-phrased the question to make it a bit more direct. I asked the children how many cars could be registered if only one space was permitted on each number plate. They all knew the answer to that, it was 26. Then I asked how many different number plates could be issued if there were two letters on each. The children went back to work and shortly one boy answered 676. I asked him how he had gotten this and he said by multiplying 26×26 . Children were now beginning to understand what was happening, and I once again altered the problem slightly so that only 2 letters (or in this case colours) would be permitted in each space on the licence plate. Red and green were suggested as the colours. One group of boys began to explore the toy at their desks (photo 10E) and declared that they had found a way to predict the number of cars possible given any number of spaces by using the place values as represented on the toy. Thus, 4 spaces would yield $2 \times 2 \times 2 \times 2$ or 16 (the value of the fifth position on the toy). Substituting 15 colours was no problem for one boy with a calculator as he shows the result of multiplying out 15 four times: 50625 (see photo 10F).

I then wrote on the board that x = the number of digits, letters, or colours permitted on the licence plate and that y = the number of spaces permitted. "What", I asked, "would be the formula for finding out the number of cars?"

PHOTOS



WORKING ON THE LICENCE PLATE PROBLEM

PHOTO 10 E



DISPLAYING THE CORRECT ANSWER : 50625

PHOTO 10 F

Several groups wrote "X to the power of Y," or even X^Y . One boy noticed that it could also be interpreted as $(X + 1)^Y$ if blank spaces were to be permitted. For example, he explained, the letters AB in a 3 spaced plate could be seen as AB- or -AB or even A-B. A dash was suggested as a space holder for those people who wanted these 'personalised' number places. Borrowing his friend's calculator, he showed that 531,441 cars could be registered using four places with the letters of the alphabet and a dash (i.e. $(26 + 1)^4 = 531,441$).

10.20 Section Two The Base Four Toy

10.21 Introduction

In contrast to the primary school, the Middle school was more informal and operated on an open-plan arrangement. This was evidenced by the construction of the school, the attitudes of the teachers and head, and also by the curriculum which included project activities as a large part of the teaching. In addition, unlike the primary school, children had no set seats during the maths lessons. Since I had spent quite a bit of time in the school (as reported in Chapter Seven), my presence there was not at all disruptive to the daily routine.

Perhaps in part because of these differences, the specialist maths teacher in the Middle school saw the base four toy much more as an aid to those children who had difficulty in abstracting mathematical concepts. She found that it was very important to have the opportunity to work with individual children and that devices or toys which could attract children's interests would be very useful in this regard

Most of the time spent in maths classes was taken up by children working on their own out of the Fletcher's or SMP programme. This contrasts with the Alpha and Beta books which were being used by the primary school mentioned earlier. Even still, the teacher did on occasion present

the entire class with a lesson, explanation, or problem which was projected onto the wall from a piece of video equipment. This was preferred to a small portable blackboard which was not easily placed so that the entire class could see it

When I returned to the school (after having left the base four toy there earlier in the week), I was permitted to work individually and in large groups with several of the classes. What is reported below is a session with one of the 11-12 year old groups. It should be mentioned that the school was divided up into four year groups with three sections in each group. For this time period the ability groupings were made according to the requirements of the French lessons which were conducted with some of the other fourth year children at the same time of the day. Because of this, the maths teacher reported to me that this group was "a bit of a mixed bag mathematically".

10 22 Solving Problems with the Toy

The children arrived in the classroom and spread themselves out around the room. Since this classroom was much more informal, I was able to sit on one of the desks on the side of the room and speak from there. There wasn't the same need to 'go to the front' of the classroom in this sense, and indeed, except perhaps for the placement of the teacher's desk there really wasn't any front of the room.

I asked how many children had played with the toy and most hands went up. Children then reported how they had noticed that when two buttons were pressed one of the lights lit on the top of the toy and cries of "red plus red makes yellow!", and "green and red equals blue!" were heard.

I then asked the children to consider a problem as follows

This problem is based on the fact that there are four year groups in the school (1-4). Because of this, and the small size of the lunch facilities, lunch-times had to be 'staggered' with one group going first, another going second, etc., overlapping during the entire lunch hour. A rota was therefore established whereby during the first week of school the 1st years ate first, the 2nd years ate second, the 3rd years third and so on. The next week of school, the 2nd years ate first, the 3rd years ate second, and the 1st years ate last. The next week found the 3rd years eating first and so on. I told the children to suppose that they were the head of the school and therefore responsible for scheduling. Since one of the year groups occasionally had to eat lunch first (or last) because of the time required to go to and from the swimming pool, it was important to know in advance who was going to eat first on which day. To simplify the problem, I told the class to imagine that the luncheon rota changed every day instead of every week and that all days referred to in the problem were school days, that is, no Saturdays or Sundays. All problems presented assumed that the first years began the rota. I then asked who would be eating lunch first on the 6th day of school.

Almost immediately, someone answered "the second years". I then asked how that answer was arrived at, and the pupil shrugged his shoulders. Interestingly, in each of the three classes where this problem was presented, it was clear that the child knew the correct answer, but he or she had difficulty reporting how it was gotten. At this point, I told the pupils that I was interested to know if there was a method of doing this type of problem, because this would be important if the following was considered: suppose, I asked, you wanted to make a swimming schedule far in advance. How would you know which year group was eating when on the 152nd day of school? Again the familiar Ohh's and Ahh's and the children went to work. In a little while one girl raised her hand and said that the answer was the fourth years. When I asked how this was done she reported that she did it by dividing 152 by 4, and that it came

out even. However, she wasn't quite sure why that meant the fourth years were eating or even why one should divide by 4 except that there were four different year groups.

I then said, suppose that there were seven years in the school, that is, the school had expanded and you were all going to be here for three more years. There were some "Boo's" and "Oh no's" and a buzz filled the room as the children began to write and talk at once.

Shortly, another girl showed me her paper which had written on it:
$$\begin{array}{r} 21 \text{ r}5 \\ 7 \overline{) 152} \end{array}$$
 Clearly, she was on the trail of something, and when she raised her hand she said, "the fifth years would be eating first that day." After she had explained how she did it, most of the children understood what was happening. One boy even commented that the answer to the division problem (21) was not important to the question, but only represented the number of complete cycles when all of the year groups had eaten. The bell rang and I asked any interested students to work on this for the next day to see whether or not they could develop a formula or rule for finding out this information for any number of years and days.

The next day I was presented with a piece of paper (by two girls who had worked on the problem together). I have duplicated it in diagram 10G without alteration except to darken some of the writing which was done in pencil and to border the 'rule' they developed to make it easier to find on the paper. This very clear exposition of the principal convinced me that some of the children were ready for the next step. The next day I asked them to convert the number of days (in this case 152) to base four. Most of the children were able to do this, some referring to the toy for guidance, and it was agreed that the answer was 2120_4 . "Why should we be interested in doing this?", I asked. Slowly, a few children began to think about the toy and one boy gave the following explanation. "Since we converted 152 into base 4 by dividing by multiples

D I A G R A M 1 0 G

"X is your number and
you divide that by which-
ever base you are using
and if you get a remainder
2, the 2nd year goes 1st.
But if you don't get a
remainder the highest
class goes 1st."



DIAGRAM 10 G

X is your number and you divide that by which ever base you are using and if you get remainder 2, the 2nd year goes 1st. But if you don't get a remainder the highest class goes 1st.

Formula

$$X - \text{Base} = r y.$$

Year's sitting	Number	Base
5th	152	7
7th	154	7
4th	152	4
2nd	154	4
2nd	68	3

$$\begin{array}{r} 152 \\ 7 \overline{) 152} \\ \underline{14} \\ 12 \\ \underline{14} \\ 2 \end{array}$$

$$\begin{array}{r} 154 \\ 7 \overline{) 154} \\ \underline{14} \\ 14 \\ \underline{14} \\ 0 \end{array}$$

$$X - \text{Base}$$

$$X \div \text{Base} = Y r$$

$$X =$$

$$X \div 7 = Y$$

$$\begin{array}{r} 152 \\ 7 \overline{) 152} \\ \underline{14} \\ 12 \\ \underline{14} \\ 2 \end{array}$$

$$\therefore 7$$

5

Base 7 = 152 = 5th year
Base 7 = 154 = 7th year

of four until we got to the units place, the last digit is the same as the remainder when the number is divided by 4!"

I told him to continue. After some thought he went on, "So, I suppose, that means that the last digit is the year group that eats first. A '1' is the first years, a '2' the second years, and so on."

I then asked what about the remainder of 0 as in the example. Unable to hold his excitement, one of his classmates called out, "That's the top year group!"

10 3 Conclusion

This chapter brings us full-circle from our initial aim of developing a kind of educational research which would enable us to provide concrete suggestions about what might go on in classrooms during periods of mathematical instruction.

As in the observational chapters, I have used a high degree of descriptive detail in an attempt to convey the feeling which was present in the classrooms. In doing this I have attempted to document the activities taking place there by using quotations, photographs, test results, and actual work sheets which were used by the children in solving the problems.

While it could be argued that part of the results may have been due to my presence in the classroom as a guest, it should be noted that the actual encounters with the toys took place under the direction of the regular classroom teachers and without any knowledge that I would appear in the classroom in the future. In addition, none of the teachers were told of the problems that I would be presenting.

Equally unexpected were some of the games discovered and developed while the children were playing with the toys. There was also the highly

creative constructions made from the Dienes blocks. Thus the toys did seem to generate interest in the classroom and Mr. H asked if it could be left as a permanent part of the school's equipment when I completed my studies.

Because of the methods I have chosen in reporting the results of my evaluation, this exercise has not been intended to prove anything in a strictly scientific sense. Rather, it has in large measure helped to provide affirmative answers regarding what may be possible in the classroom. In particular, even relatively formal classes were able to integrate aspects of play and autonomy in the learning of base 2 with a measure of success.

The final chapter follows with some concluding remarks and suggestions for further research.

Chapter Eleven Concluding Remarks

11.10 Implications for a Theory of Mathematical Instruction

In their introduction to Krutetskii's impressive volume of work done in the Soviet Union on the psychology of mathematical abilities in schoolchildren, Kilpatrick and Wirszup note the following:

"Consider the investigation of mathematical abilities. Nearly every recent research study on the topic has taken roughly the same form. The investigator assembles a battery of tests assumed to have some relation to mathematical abilities. He administers the tests to a sample of schoolchildren, obtaining a score on each test for each child. He then uses the statistical technique of factor analysis to determine, from the correlations between the test scores, how the tests are related. Tests whose scores are highly correlated are presumed to measure the same underlying ability, tests whose scores are uncorrelated are presumed to measure different abilities. The object is to account for the test scores in terms of a smaller number of factors, each of which represents a different ability. Different techniques of factor analysis can yield different configurations of abilities, but in each case the investigator is faced with the task of identifying each ability by inferring what is common to the tests that cluster together to produce the factor " 1976, page xi

This statement is followed, both in the introduction and the body of the volume itself, by a critical reminder of the severe limitations this type of research holds for actual mathematics teaching in schools. This theme is not an isolated one. I have attempted to present examples, both from an educational and psychological perspective, of how undue concern with testing and other 'measurable products' has tended to distract researchers from pursuing questions which relate more closely to actual educational practice.

Because of this, it was suggested at the beginning of the thesis that the methodology used to investigate mathematical learning in schools must deal with the 'social realities' found in the classroom. This interest in relating the processes of mathematical thought to the structure of the interactions which take place within the classroom is reflected by

the large periods of time spent in schools, both as an observer, and experimenter. In addition, the experiments performed on concept formation were designed to explore the various conditions which underlie the kinds of strategies and perspectives that children develop while they are playing with mathematical toys and solving problems.

In order to provide information that may be useful to practicing teachers in the classroom, research on thinking, learning, and conceptual development must be expressed in terms which can be

(a) . . . recognised by teachers as having a bearing on the kinds of day to day situations and problems which they deal with regularly in the classroom. These can vary from very general concerns such as ability grouping or discovery methods of learning to very specific concerns about presenting mathematical material to the less able students or providing project work in a specific mathematical topic.

(b) ... used by teachers in a way which will enable them to alter their actual classroom behaviour. It may be argued that research on mathematical learning which is very abstract has little chance of being integrated into the classroom unless it provides a basis for changing the ways teachers teach. Altering the ways in which teaching takes place may be accomplished in at least two ways, although undoubtedly there are others

The first way that this may be approached is by providing concrete suggestions which will enable teachers to re-structure their organisation of classroom activities. Using the present research as an example, this might include the introduction of certain mathematical toys in the classroom along with periods of free play with these toys. Games, discoveries, and small projects which develop from these activities may potentially replace short periods of what was much more formal and teacher-centered demonstrations in front of the entire class

A second way of changing teachers' behaviours also involves their re-structuring of classroom activities, but in a rather less direct manner. This process takes place by enabling teachers to gain certain insights into how children react to and are affected by the ways material is presented in the classroom. In other words, children's actual perceptions of what it is that they should be doing will depend to a certain extent on their perceptions of the teacher's expectations, what is and what is not legitimate mathematical knowledge, peer group pressures, etc. By providing teachers with information in a form which is designed to help them understand how children develop mathematical thought within a classroom context, I believe that teachers will be in a better position to identify and encourage such development. Simply stated, it is argued that teachers and learning are best served not by any elaborate theory of mathematics instruction, but rather when they are provided with 'dynamic models' of the processes involved in mathematical development and learning which represent an identifiable classroom reality.*

This, then, is an outline of the core of a theory of mathematical education for the primary school. I feel that it would be unwise to elaborate upon it at any great length at this time since it too should be grounded in the empirical findings of those involved in developing mathematical curriculums, training teachers, etc

However, from evidence developed in the present piece of work and also from research cited elsewhere in the thesis, I do believe that certain 'theorems' may be put forward. These are not meant to be rigid

* One very important aspect of this 'identifiable classroom reality' is that it must be expressed in terms which provide information to the teacher about the development of children's perceptions and behaviours in the classroom. It is within the dynamics of this milieu that the possibility resides of altering basic teaching methods and attitudes.

rules governing either mathematical learning or teaching. Rather, they are principles which I have found to re-occur in more than one area of the literature and which are potentially an effective method of affecting the ways that classrooms are organised and conducted.

Although these few theorems are in no way meant to reflect a complete or homogeneous theory, I believe that one may describe them under a single 'umbrella' statement. This is the notion that the learning of mathematics under any instructional system is inevitably affected by the teacher's perspective (or criteria) for determining what knowledge a child has acquired and is acquiring about mathematical relationships. Clearly, narrow definitions of learning that are supported by an assessment procedure which involves the repeating of rules of multiplication, division, place-value or whatever, will demand a certain type of classroom teaching in order to attain 'successful learning'. Other definitions of mathematical knowledge and criteria to ascertain its presence or absence^{''} will generate different methods of successful teaching.

In many ways this need to provide insight into these areas stems more from extensive observations and conversations with teachers, administrators, and mathematicians than from any particular educational or psychological theory. Enabling primary school teachers, many of whom do not have experience beyond the '0' level themselves, to recognise and encourage mathematical thought must be a prime goal of educational psychologists interested in this area *

What follows below will therefore be a set of propositions which reflect these premises.

* In this regard, Caleb Gattegno in "The common sense of teaching mathematics" (1974) is one of the few authors to devote a section in his book to 'Teaching mathematics to teachers'.

11.11 Theorems for Primary School Mathematics Teaching

(1) Notwithstanding the erroneous analogies which abound in the literature, teachers must be made aware of the fundamental differences which exist between the teaching of any school subject and control of the mechanical process of manufacture. This belief is widely held in some quarters and relates to the "I teach - you learn" section of the model of classroom teaching. It is not altogether profitable to see children as simply moving from states of learning, ($L_1 \rightarrow L_2$), through the application of periods of classroom instruction. As with any human in a social situation, children impose a meaning and structure upon the total set of interactions which take place in classrooms and teachers must be sensitive to the effects of their methods of presentation upon these perspectives. It should therefore not be assumed that children have a priori knowledge about what constitutes the discipline of mathematics, or indeed that it is initially separable from other kinds of activities and assignments which they are presented with in school. As Professor Geoffrey Matthews has pointed out in the summary to "Mathematics through School", (p.77):

"For infants 5 to 7, and indeed 'nursery' children, 3 to 5, it is very difficult to isolate mathematics from their other activities."

He continues on the same page. "In the Junior school, again mathematics is made meaningful by contact with the environment, for example the study of design or architecture."

It should be stressed that for young children up to the age of 11 or 12, even with separate time slots, text books, and other equipment, mathematics is not always seen as a meaningful enterprise on its own. It is suggested that this may be, in part, because it is not a subject which concerns itself with 'things' in the same way as geology or music. Rather, it is involved with ways of thinking about the relationships

between things, and thus is much more abstract.

(2) The second point follows from the first and underlines the interest in promoting mathematics study which will be part of the wider areas making up a child's world. To quote Matthews again,

"A general thread now becomes apparent: school mathematics must keep one eye on the outside world. The first reforms were carried out within the subject, and rightly so as there was so much to re-think. Idiotic problems about leaking cisterns and ditch-fillers, routines for factorisation, parroted 'theorems' of doubtful validity: these had obscured the nature of the subject and the big ideas had to be unearthed and updated." (page 78)

He concludes, "We are back finally where we should be talking about children. For mathematics is literally useless without people to create it, to use it and to enjoy it ..."

It is interesting how these feelings relate very closely to the maths in the environment technique mentioned by the headmaster of the primary school in the previous chapter. An important part of some of these activities, paradoxically, must be that they are not planned out in advance. Rather, the children themselves must be encouraged and permitted to make on their own some of the associations and connections with the outside world if mathematics is to become a genuine part of their lives.

(3) Another area which requires attention refers back to the suggestion made earlier that teachers should begin to place more emphasis upon the development of those strategies which are retained over long periods of time. This point should not be interpreted as having to do with a child's memory for a particular fact or formula, but rather it is asserting the idea that mathematical relationships which are understood will be able to be put to use at any time. A corollary to this is that methods and techniques which have become a part of the way children think should be

able to be used in situations where the exact form or context of the problem has been altered. Thus, material learned in mathematics should be applicable to problems found in science classes, after school activities, as well as out of the school environment entirely.

(4) Following from the hypothesis on mathematical learning, children should not be required to provide immediate verbalisations or explanations of their methods of discovery or exploration. It was found in Chapter Ten that many children who could solve the problems presented to them were only confused when they were asked for an immediate explanation of how they had accomplished it. A child's ability to understand a certain mathematical structure or relationship is not dependent upon his ability to verbalise it in a manner commensurate with adult expectations. To quote Dienes and Jeeves (1965).

"To impose an adult form of verbalisation on children while they are just learning new relationships is a dangerous procedure. It often has the effect of freezing the process in question at a stage in which it was when the verbalisation was attempted. An actively strategy-seeking or pattern-seeking kind of activity would in all probability lead to more effective learning situations for children than the rote-learning of associations." (Page 96)

(5) Finally, there is the goal of encouraging children to 'develop intuition', to help them become 'good guessers' and 'go beyond the information given' (Bruner). These may sound like very laudable goals in themselves, however, it is all too easy for the outside pressures of exams, standards, or 'covering the material' to force teachers into seeking very specific (and often artificially concrete) means of determining 'where children are' in their learning. Undoubtedly, much of this is caused by the 'lock-step' of the educational system which more often than not does not 'mesh' with the irregular, haphazard, and 'spurting' nature of the ways that children seem to learn in schools. These would require alteration at a different level of educational change.

But, even within individual classrooms, teachers have a certain amount of scope to make the kinds of activities which are presented^{as} springboards for further questioning. Only in this way will the psychological models of man as a 'curious' and 'problem-solving' animal hold any implication for learning in school. This is especially true in light of the evidence presented that children are very sensitive to what is genuine and what is not in the classroom. Because of this, what is being said here is that a successful theory of mathematical instruction probably depends as much upon the knowledge and attitudes of the teachers involved in schools than on any single detailed investigation of how children learn in the classroom.

11.20 Suggestions for Further Research

Suggestions for further research are discussed below in three sections. The first section is concerned with research carried out in the area of mathematical concept formation, especially as it relates to the development of strategies which children employ while conceptualising. The second section extends some of the remarks made earlier on personality and learning in the classroom, while the third section puts forward an interpretation of some of the important questions which surround the area of inquiry learning in the classroom.

11.21 Mathematical Conceptualisation and the Development of Strategies

The research which was carried out in Chapter Seven using the base four toy made the point that there appeared to be a relationship between the year of the pupils tested (either 2nd or 3rd) and their initial ability to notice relationships in the blocks task. In addition, the experiments reported in Chapter Nine attempted to relate the kinds of strategies which children used in both their exploration and formulation of a battleship arrangement to their perceptions of the patterns involved as it was affected by their pre-task experience. One way of combining

these two ideas has been suggested by a member of staff at the Durham Department of Psychology.* Work is presently underway to link one of the base four toys to an electronic event recorder. This would take place via a cable between the toy and the recorder and would permit a complete record to be made of the subject's exploration of the toy on punched paper tape. This tape could then be read through the computer facilities available and provide a complete profile of the organisation of search strategies including information on the exact pacing of the button pressing.

One specific suggestion would be to use this equipment to record the play sequences of 12, 11, 10 and 9 year olds (and so on to even younger children), in schools. Tape recorded interviews would then be made of the child's conceptual attainment in a manner similar to the four point scoring system developed in the experiments. These could then be compared across the age range to see if there is any support for the idea that older children are more likely to initially observe basic relationships such as the yellow rule mentioned in Chapter Seven.

In addition, the results of analysing the tape could be useful in exploring the suggestion that children proceed in their play through a series of inter-connected stages. While I was not able to formalise the nature of these stages in the experiments presented in this thesis, (nor was it the central concern of the experimental design at the time), there were suggestions from observations of children's play and post-experimental reporting from the subjects that the following outline of events often took place. Initially there is a period of exploration of the colour combinations which takes place quickly and in a "random" fashion as the

* Dr. Arthur Still

subject takes the opportunity of pushing the buttons for the pleasure of lighting the lights. In many subjects this is followed by a more complete sequence of exploring the 16 colour combinations in some systematic way. This is often accomplished by pushing down button A on the row or column, and then rapidly pushing buttons A, B, C, and D independently on the other set of buttons. Button B is then pressed replacing A on the first set and A, B, C, D is once again explored. This continues across all of the colours. It is here that sounds such as "Ah-ha!" or "I get it!" may emerge from the room. This period is then followed by a trial period where the emerging hypothesis is tested by making predictions as to what light will come up when two specific buttons are pressed. Interspersed between these roughly divided phases are often periods of "free play" when subjects report that they are not in pursuit of anything in particular, but rather are 'digesting' or merely 'messing about'.

Using a tape recording device with a microphone attachment, and varying the instructions to subjects (such as play, search, develop a rule, etc.), could prove to be useful in helping to determine (a) at what ages children begin to organise these patterns into coherent relationships, and (b) more generally, what are the stages which subjects go through as they develop these mathematical concepts.

11.22 Personality Research

It is not very interesting to report to teachers that "different children learn differently" unless this information can be put to use in their particular classrooms. One way of doing this is to follow up on the suggestions that mathematical ability is something quite separate from interest and enjoyment of playful learning.

In work cited elsewhere in this thesis, it was mentioned that

introverted children were not always as comfortable playing alone with the mathematical toys as extroverted children. In addition, even very clever children who are reported to be 'shy' or 'timid' can be found to feel insecure when they are placed in a classroom which does not have much guidance or structure provided by the teacher as to exactly what it is that they should be doing.

Future work should concentrate more closely on these problems, especially as they relate to possible variations in the way inquiry learning is organised in the classroom.

How is it that some children require a clear idea of their exact goals in order to successfully explore a mathematical environment? There are also the sex differences alluded to earlier which somehow seem more pronounced in the more formal schools visited.

These suggestions may be related to the work done on the intelligence creativity distinction as well as to work carried out by Hudson on the classifications of convergent and divergent thinkers. Future efforts in these directions should attempt to more closely identify those personality factors which may affect children's thinking strategies in mathematical situations as they are found in the classroom.

11.23 Learning by Inquiry

By and large, the aim of much of the current experimental work carried out in classrooms in this area of research attempts to compare a 'discovery' or 'inquiry' learning technique with a more directed or traditional approach. These efforts have provided us with precious small amounts of information about what actually takes place in the processes involved with the use of inquiry methods. For instance, are there any classroom procedures which are common to most similarly classified styles of teaching, and further, how do the (non-mathematical) primary

school teachers perceive and utilise materials which are available to them in modern mathematics courses?

I believe that it is important to conduct research which will begin to provide us with information about what effects some of the component stages of discovery (i.e., classifying, questioning, generating hunches, etc.) have on the overall process of learning mathematics. In order to do this, these elements must first be defined in a way which will enable researchers to identify their presence in classrooms.

Once this is done, we may begin to examine experimentally how some of these conditions affect mathematical learning in instructional settings. Only in this way, I believe, will the work carried out on mathematical learning provide a possible direction for teachers to follow.

References

Part I Overall Perspective to the Research

Introduction

1. Examples of this can be found in the works derived from Piaget, (see Schwebel & Ralph, (eds.) "Piaget in the Classroom" 1973), and from the work of Jerome Bruner. The philosophical basis for these notions can be traced (at least) back to Immanuel Kant who pointed out that man does not merely perceive the world about him, he orders it.
2. See Banesh Hoffman, "The Tyranny of Testing" 1962.

Chapter One The School as a Social Institution

- 1 Such a task would be beyond the scope and aside from the central concerns of the present work, for a recent selection of articles, however, see Eggleston, (ed.) "Contemporary Research in the Sociology of Education" 1974
2. Ellwood Cubberly, "Public School Administration" 1916.
3. Kimball, 1963, page 226.
- 4 Wertheimer, 1945, page 131.
5. ibid page 57.
- 6 Bartlett, 1958, page 111
- 7 Bruner (ed.) 1957, page 53.
8. ibid. page 67
9. Henry, 1965, page 286.
10. Jones, 1968, page 125.
- 11 Smith & Geoffrey 1968, page 50.
12. Leonard, 1968.
- 13 Jones, 1968, page 46.
14. For example, the influence of the work of Bloom; also see Chanan (ed.), 1973, especially page 93, the concept of product.

So much of the research done on education fails on these two points. But why should this be so? A close look reveals that the present day research paradigm will usually assume the following:

*that although it may be difficult, the domain of cognitive factors can be studied in the classroom

as an item separated from affective and social factors by a programme of carefully controlled research (see Bloom)
 * the notion that objective measurement of classroom learning is a requirement of the scientific methods of psychological research into education (footnote No. 2 above).

15. Young (ed.), 1971, page 75.

16. *ibid.* page 47.

Chapter Two The Research Design

1. The word 'paradigm' is as used by Kuhn (1970), to describe problem fields, research methods, and acceptable standards of solution and explanation.
2. For a good example of this, see Jackson, 1968.
3. See Bruner, 1957.

Chapter Three Initial Observations

1. This took place during free moments in the classroom and also over tea in the teachers' lounge during recess
2. See Glaser & Strauss, "The Discovery of Grounded Theory", 1967.
3. Smith & Geoffrey, 1968, page 260.
4. Most children went home for lunch although there was a small cafeteria for those who stayed.
5. SRA Science Research Associates copyright for the colour-coded step by step reading programme.
6. "Educational Psychology Its Scope and Limits", page 117.

Chapter Four A Model and Hypothesis of Mathematical Learning in the Classroom

1. Bruner, 1956 also Donaldson 1959 (in Wason & Johnson-Laird, 1968).
2. Torrence, 1962 and 1964
3. "How Children Fail"
4. Wertheimer, 1945
5. Harre' & Secord, "The Explanation of Social Behaviour" 1972 and also Harre', "Some Remarks on Rule as a Scientific Concept" and Toulmin, "Rules and Their Relevance for Understanding Human Behaviour" both of these are contained in "Understanding Other Persons", edited by Mischel, 1974.

6. See Nash, 1973 for support of this point.
7. Rommetveit, 1965.

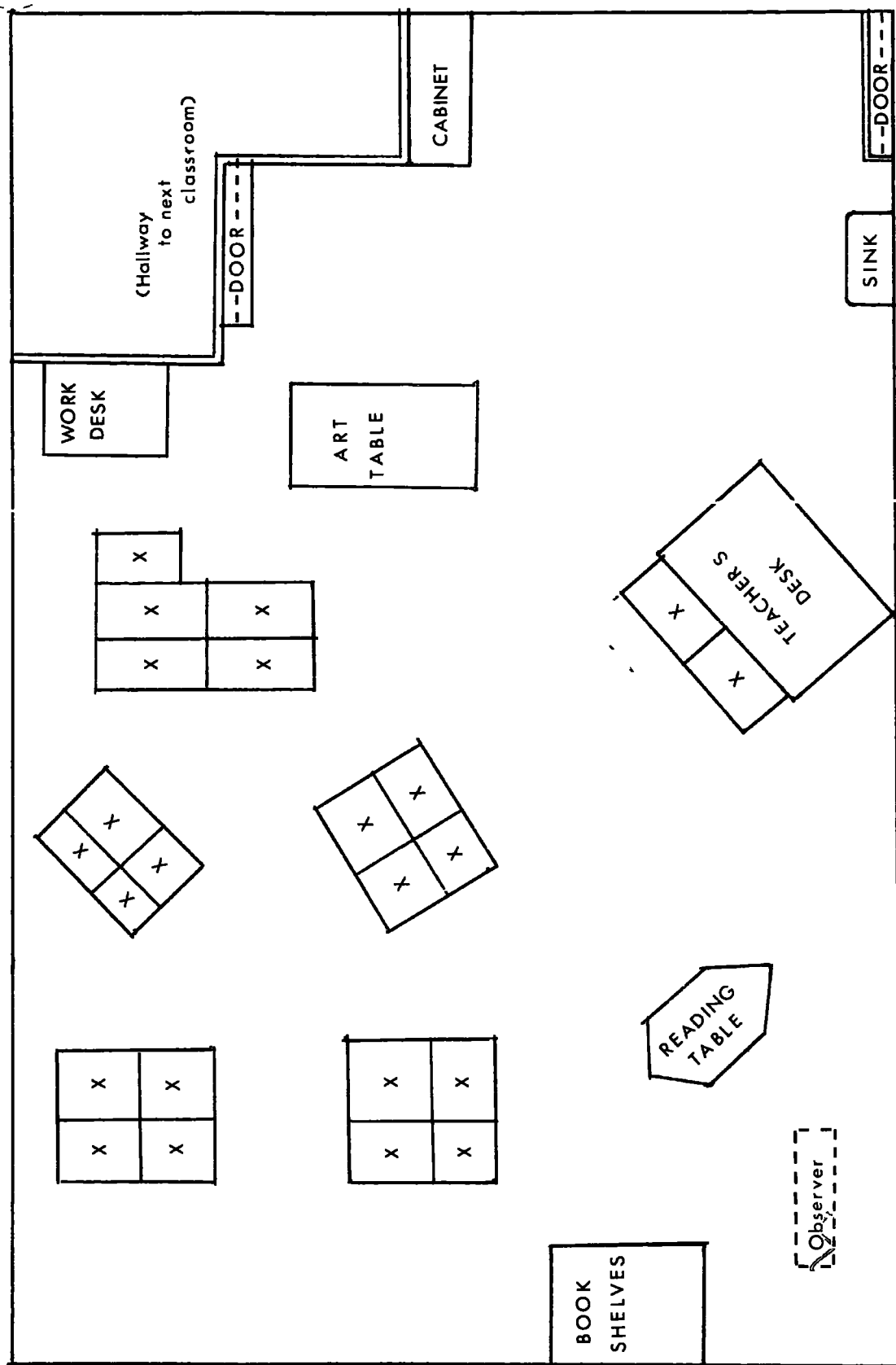
Chapter Five A Review of the Literature

1. In this situation, two sticks must first be connected together by telescoping one inside of the other before they are long enough to 'rake-in' a banana outside of the animal's cage.
2. See the work by Menzel (1972) for a further discussion of this issue.
3. To quote Bruner (in "The Relevance of Skill or the Skill of Relevance"):

"I have suggested that the human, species-typical way in which we increase our powers comes through converting external bodies of knowledge embodied in the culture into generative rules for thinking about the world and about ourselves "
4. Stated by Duncker (1945) as such: "given a human being with an inoperable stomach tumour, and rays which destroy organic tissue at sufficient density, by what procedure can one free him of the tumour by these rays and at the same time avoid destroying the healthy tissue which surrounds it?" (The 'solution' is to focus weaker rays through a lens so that they converge on the tumour.)
5. These 'closed systems' are useful from an experimental point of view since they enable one to more exactly identify the nature of the 'bits' of information which the subject acquires through a particular search activity. Although, most situations in day to day life are not so 'neat' as this.

Appendix

Table A



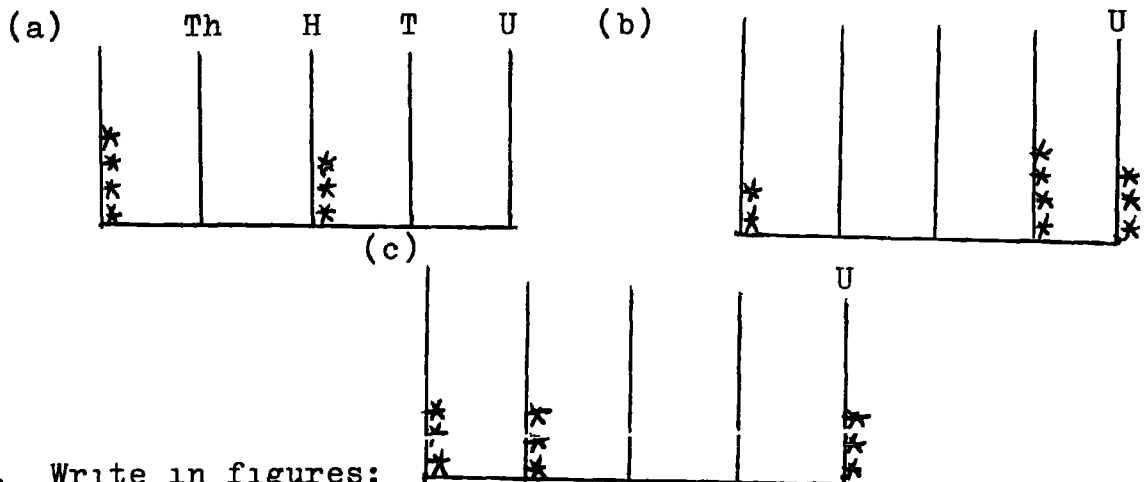
The Classroom

--DOOR--
X represents child's desk

1. Write in words and in figures the value of each of the columns M, N and P.

MILLIONS	P	TEN THOUSANDS	N	HUNDREDS	M	UNITS
----------	---	---------------	---	----------	---	-------

2. Write the number shown on each abacus pictured in figures and in words.



3. Write in figures:

- 1 (a) 15 thousand and sixty (b) 29 thousand 2 hundred
 2 (a) 17 thousand 2 hundred and eighty (b) 40 thousand and
 3 (a) 88 thousand and eight twenty
 (b) 63 thousand 4 hundred and ten

4. Write to the nearest thousand -

a) 9 750	:	14 280	:	39 500	:	3170
b) 27 480	:	52 910	:	28 320	:	17 620
c) 18249	:	31721	:	19259	:	21577

5. Write and complete: 100 = tens : 1000 = hundreds
 100000 = hundreds = tens : 1000 000 = hundreds
 1 000 000 = thousands 1000 000 = tens
 1 000 000 = hundreds = tens

6. Write in figures : 5 million : $\frac{1}{2}$ million : $\frac{1}{4}$ million
 $\frac{3}{4}$ million : $4\frac{1}{4}$ million : $3\frac{3}{4}$ million

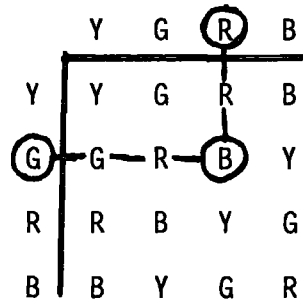
7. Which numbers in the box are:
 a) less than a hundred thousand
 b) more than $\frac{1}{2}$ million
 c) more than $\frac{1}{4}$ million
 d) more than $\frac{3}{4}$ million

501 402	:	94 060	:	236 890
800 500	:	450 732	:	

8. Which of the numbers is nearest to: a) $\frac{1}{2}$ million,
 b) $\frac{1}{4}$ million,
 c) $\frac{3}{4}$ million

Explanation of the sequence of photos (opposite) on the procedure in the blocks task.

- (Photo 1) A block is initially placed on a designated area near the centre of the table. The experimenter sits on one side of the table, and the subject on the other. Each has a set of blocks which consists of the colours yellow, green, red and blue. In the photo, a green block is shown although the very beginning of any experimental task always found a yellow block in the centre.
- (Photo 2) The subject selects one of her four coloured blocks and places it next to the block already on the table. Here, the subject is seen choosing the red block.
- (Photo 3) The experimenter then places the block which results from those two colours on the table while verbalising the combination. In this case he would say, "Green plus red equals blue".



- (Photo 4) After a short time (a moment) has passed and the subject has seen the combination on the table, the experimenter removes the original block (the green one) and the block which the subject has placed (the red one) and the resultant block, (the blue one) becomes the new centre block and the cycle begins again. This continues until the subject can successfully predict all (or a percentage) of the sixteen colour combinations.

Table C

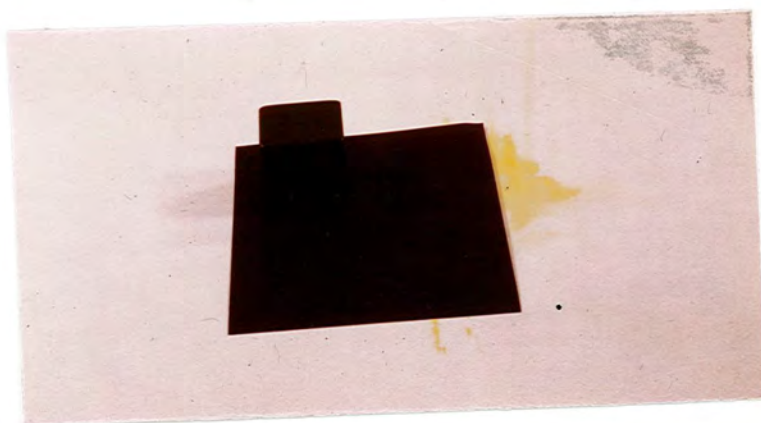


PHOTO 1

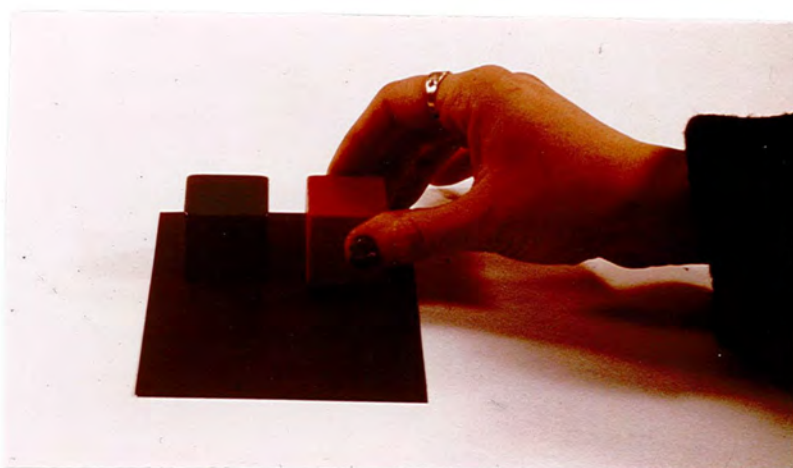


PHOTO 2

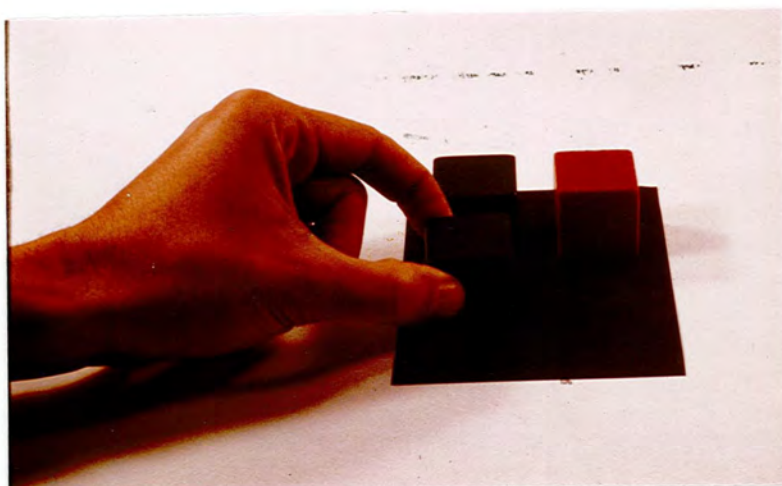


PHOTO 3

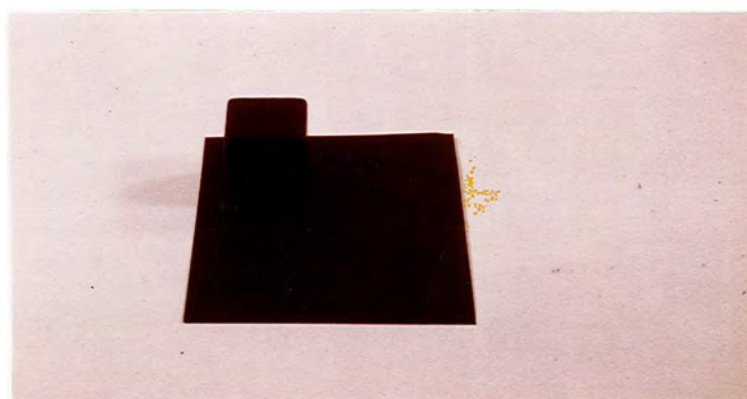


PHOTO 4

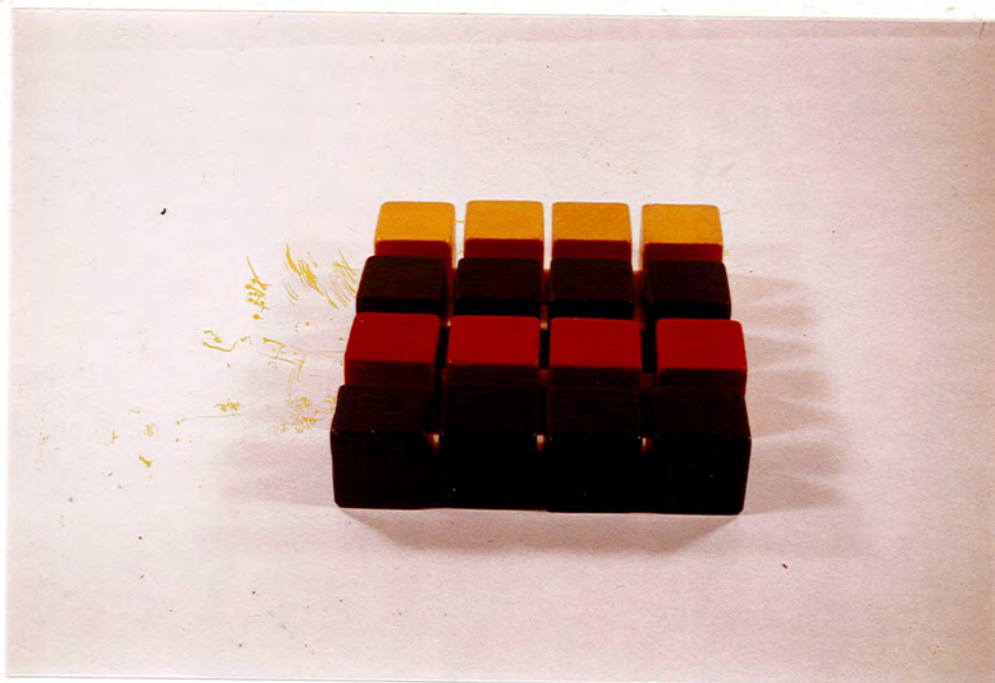
Do you remember what these colour combinations make? Do your best, but remember that this is not a test.

yellow with yellow?	_____	green with red?	_____
yellow with blue?	_____	yellow with red?	_____
blue with blue?	_____	blue with yellow?	_____
green with blue?	_____	red with blue?	_____
blue with red?	_____	red with red?	_____
green with green?	_____	red with yellow?	_____
blue with green?	_____	yellow with green?	_____
green with yellow?	_____	red with green?	_____

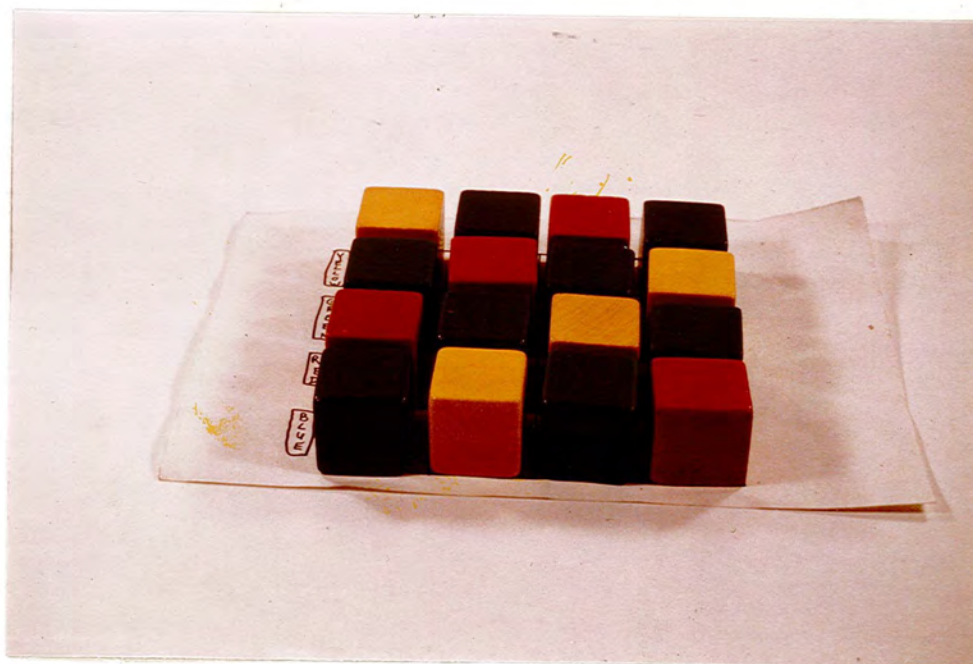
Table D The written test to measure the number of blocks combinations remembered (out of a possible 16)

THE 16 BLOCKS

E



AS PRESENTED TO THE SUBJECTS



WHEN PROPERLY ARRANGED

THE BINARY TOY

Part I of the 20 question Maths test

1	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input checked="" type="radio"/> A	<input type="radio"/> F <input checked="" type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A
2	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input checked="" type="radio"/> B <input type="radio"/> A	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A
3	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input checked="" type="radio"/> B <input checked="" type="radio"/> A	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A
4	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input checked="" type="radio"/> C <input checked="" type="radio"/> B <input type="radio"/> A	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A
5	<input type="radio"/> F <input type="radio"/> E <input checked="" type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A
6	<input type="radio"/> F <input type="radio"/> E <input checked="" type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A
7	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A	<input checked="" type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A
8	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A	<input checked="" type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A
9	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input checked="" type="radio"/> C <input checked="" type="radio"/> B <input checked="" type="radio"/> A	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A
10	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input checked="" type="radio"/> C <input checked="" type="radio"/> B <input checked="" type="radio"/> A	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A

Table F

THE BINARY TOY

Part II of the 20 question Maths test

1	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A	1.....	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A	6	23...
2	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A	2.....	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A	7	32..
3	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A	5.....	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A	8	24..
4	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A	10.....	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A	9	21..
5	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A	7.....	<input type="radio"/> F <input type="radio"/> E <input type="radio"/> D <input type="radio"/> C <input type="radio"/> B <input type="radio"/> A	10	53..

Table G

	A	B	C	D	E	F
1	G				G	
2				R	R	R
3	R	R	R			
4				Y	Y	
5		Y	Y			G
6	G			Y	Y	

	A	B	C	D	E	F
1					Y	
2	R	R	R	R	Y	G
3				R		
4		Y	Y	R	Y	Y
5	G				G	
6			G			

An example of two well-spaced arrangements

Table H

	A	B	C	D	E	F
1	R	R	R			
2	Y	Y				
3	R	R	R			
4	Y	Y				
5	G	G	G	G		
6	Y	Y				

	A	B	C	D	E	F
1	R	R	R	R	R	R
2	Y	Y	Y	Y	Y	Y
3						
4			G	G		
5		G				
6			G			

Two examples of poorly placed boats. Note how a simple pattern of shots would be very effective in destroying the enemy's ships

Table I

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APPENDIX J

The criteria for each of the scoring categories are given below along with sample protocols and explanations of each ranking. From these, one can see that the rankings represent a continuum from those subjects who were relying totally upon memory to those who used an operational method. The middle scores were given to those subjects between the two extremes, as they used some rules to aid what was largely a memory-dependent system.

Scoring

<u>Score</u>	<u>Criteria</u>
0	This score represents a failure on the part of the subject to produce an adequate number of correct combinations of blocks. In the text, this score never appeared since all of the subjects attained the required criteria.
1	This score was given to those subjects who simply memorised the 16 block combinations. They used no system, noticed no patterns, and usually were not able to explain how any two colours made a third beyond saying "I remember it that way". For them, putting together two colours to get a third did not make any sense save for the fact that it was defined in that way by the experimenter. Memory was the only possible strategy they could use to correctly predict the block combinations.
2	A score of two was awarded to those subjects who relied very heavily upon memory, but nevertheless noticed that the 16 combinations were not entirely random. For example, most of the subjects who were given this score noticed that the order of the two given blocks did not affect the third, or resultant block. Thus, colour A 'plus' colour B yields the same result as colour B 'plus' colour A. In addition, recipients of this score were able to explain that

when the yellow block combined with any other, then the result equaled the colour which appeared with the yellow block. However, when dealing with the results of combining the green, red or blue blocks, these subjects resorted to memory.

- 3 The score of 3 was in many ways the most varied. In order to attain it, the criteria for a score of two must also have been satisfied. In addition, the subjects were required to demonstrate that their predictions of the green, red and blue block combinations were not merely memorised. Rather, their workings were described by a rule or more usually, a series of rules. These often depended upon a specific placement of the green, red and blue blocks. Occasionally, subjects would divide the four blocks into two groups of two. The central difference between a score of 2 and 3 is that the non-yellow blocks were not merely memorised in the higher score.
- 4 A top score of four was given only if the subject was able to explain the workings of the blocks in one rule. Any singular rule which satisfied this criteria was accepted. To be consistent within the scoring, the complexity of the rule was not taken into account as long as a) it was able to be operationalised, and b) once operationalised, it correctly predicted all 16 combinations of blocks.

A critique of the scoring system and a further explanation of its use are explained on pages 84 and 91 - 92 in the body of the work.

Below are four protocols which represent 'types' of the four category scores. They are intended as examples of the verbal activity which was available to the scorer of the tape-recordings. Following the transcripts are the scores awarded to each subject and the reasons for giving these scores.

Subject A

Experimenter: "Very good. Now I would like you to tell me
how you did it."

Subject: "From the blocks."

E: "Yes, but how did you know what any two coloured blocks
equaled?"

S: "I remembered them."

E: "You memorised them . . .?"

S: "Yes."

E: "All of them?"

S: "Yes, I think so."

E: "O.K. Did you notice any patterns."

S: "No, not really."

E: "What does blue and green yield?"

S: "Yellow."

E: "Would green and blue equal the same as that?"

S: "No . . . or at least not always. (pause) Well, I'm not
really sure, perhaps it would."

E: "Would you be able to tell me what pink and yellow equal?"

S: "No."

E: "Why not?"

S: "Because that colour wasn't in the game."

E: "Thank you."

Subject B

E: "Could you explain to me how you did the task?"

S: "Well . . . mostly I just memorised them, you know. But
there were a few that were easier than others."

E: "Easier?"

S: "Yes, like the yellow block."

E: "Could you continue."

S: "Well . . . , when you have the yellow and any other one, it
equals the one which went with the yellow, if you understand"

E: "Anything else?"

S: "The blue block was the hardest. I had to just memorise all
of those combinations."

E: "Could you tell me what yellow and pink would equal?"

S: "Let's see . . . , yellow and pink would equal pink."

E: "What about pink and green?"
S: "I couldn't do that one... but it would be the same as green and pink."
E: "Why is that?"
S: "I'm not sure. But I did notice that they all were able to be sort of reversed like that."
E: "So green and blue are the same as blue and green?"
S: "Yes, that's right."
E: "What do they equal?"
S: "Err . . ., let's see, red, ... no, .. yellow."
E: "Which one?"
S: "I'm pretty sure that it's yellow."
E: "How did you know that?"
S: "Just from memory - although I suppose that I've forgotten a few of them already."

Subject C

E: "That's fine, all correct. Now I wonder if you'll explain to me how you did it."
S: "Well, first I noticed that yellow with any other colour makes yellow. Then I did the doubles, yellow doubled and red doubled both make yellow. So, I put those two colours together. The other two colours, the green and blue, both equal red when they are doubled. So that's how I did the yellows and the doubles."
E: "How did you know what the others made?"
S: "Well . . ., since I noticed that the order of the colours did not matter, that cut things down a bit. This left only red with green, blue with green and red with blue."
E: "Did you memorise those?"
S: "No, not really, -not totally anyway."
E: "How did you know them."
S: "Well, the red is on the left side, so the red with the green made this one, the blue, and the red with the blue make this one, the green. When the green and the blue are put together, they make the last one that is left, the yellow."
E: "Is that how you remembered all of the combinations?"
S: "Yes, sort of a system stemming from the order Y,R,G,B."
E: "If you didn't have the blocks in front of you, would you be able to predict the colours?"
S: "Not very well, I suppose. That is, except for the yellows."
E: "Yes."

Subject D

E: "Could you explain to me how you did that..."

S: "Well, I noticed this pattern which I used when you asked me the combinations."
E: "What sort of pattern?"
S: "Well, when the blocks are lined up like this, yellow first, then green, red and blue, you can find out what any two of them make by using the arrangement."
E: "Can you show me?"
S: "Yes, I think so. First, yellow and any other colour equals that colour. So, when you have yellow with any other colour, they stay where they are. Then, the next is green. Whenever you have green with any of the other colours, you move it over one, so that with green, yellow equals green, green equals red, red equals blue, and for blue .., for blue you go back to the beginning like a circle, to yellow. For the red one, you move it over two places, and for the blue one you move it over three places."
E: "Could you show me how you would do red and blue."
S: "Well, the red moves the colours along two places .., so, since the blue is here, you would move it over two, like this. One, two and it equals green."
E: "What about blue and red?"
S: "Well, I could work it out - but I know that it will equal green also."
E: "How?"
S: "Because the order doesn't matter."
E: "Could you tell me what purple and yellow would equal?"
S: "Since it is with yellow, it should equal purple."
E: "What about purple and green."
S: "I can't tell that one."
E: "That's fine."

Scoring and explanations

Subject A Score: 1

The subject memorised the blocks. Neither the identity element nor reversibility were noticed. Block combinations were soon forgotten. No patterns were noticed. The subject was not able to extrapolate the operation of the colours to one which did not appear in the game.

Subject B Score: 2

The subject commented that some colours operated differently than others. Yellow was explained as the identity. Reversibil-

ity was noticed. However, the majority of the combinations were memorised. The green, red and blue blocks were seen as distinct from yellow. Also, these were forgotten more quickly.

Subject C Score: 3

A system was given: the blocks were placed in an order (Y R G B), and explained by a series of rules. These rules included identity and reversibility. They went further to show a separate method for predicting the doubles and other red, green and blue combinations. Overall, the system operated under a number of rules, which, when combined, could yield any result. Some memory was required, however.

Subject D Score:4

Subject D arranged the blocks in the order Y G R B and explained all of the combinations from one rule. Beginning with the knowledge that 'yellow didn't affect change' (ie, was the identity), green, red and blue were related to the system. Reversibility was noticed. Except for the initial order of the blocks, (the same as appeared on the toy), no memory of individual combinations ~~was~~ required.

