Hadronic interactions in the bag model

Graeme Thomas Fairley

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IN THE

BAG MODEL

by

Graeme Thomas Fairley

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CONTENTS

PREFACE

The work presented in this thesis was carried out in the Department of Mathematics of the University of Durham between October 1974 and June 1976 under the supervision of Professor E.J. Squires.

This material has not been submitted previously for any degree in this or any other university. It is claimed to be original except for chapter one, section one of chapter five and other places where explicitly referenced. Chapters two and three are based on two papers published by the author in collaboration with E.J. Squires and chapters four, five and six contain unpublished work by the author.

I would like to thank Professor Squires most sincerely for his continued guidance and encouragement. I would also like to thank the Science Research Council for a research studentship.

ABSTRACT

The object of this thesis is to investigate the predictions of the MIT bag model for hadronic scattering. Chapter one provides an introduction to the model, describes the results of the MIT group and presents the zeroth order classical scattering solution of Wu et Al. In chapter two we show how to relate this to experiment and explain why the model needs a quantum treatment and the inclusion of quark-quark interactions to make it realistic. In chapter three we try to improve the naive quantum-mechanical model of chapter two. In chapter four we consider explicit models for the quark-quark interaction and in chapter five we show how these effects may help the bag model to predict the correct form of the nucleon-nucleon interaction. In chapter six we consider a modified bag model without sharp boundaries and attempt to discover the scattering properties of this model. Chapter seven consists of a summary and conclusion.

CHAPTER ONE

1. Introduction

Considerable experimental evidence has accumulated in recent years in support of the idea that hadrons are composite. The most popular model of composite hadrons is the quark model, but so far all attempts to isolate and observe the quarks have failed. This means that some of the required properties of quarks are contradictory; for example, deep inelastic scaling suggests that quarks are light and essentially non-interacting, whereas the non-appearance of quarks can most readily be explained if they are massive and have strong interactions.

Numerous approaches to the problem of quark confinement have been developed recently, all of which assume that quarks may be classified in triplets of su(3) colour, that hadrons are colour singlets and that the sea of $\bar{q}q$ pairs carries no quantum numbers.

One of these approaches, the MIT bag model, describes hadrons as composite systems with their internal structure being associated with quark and gluon field variables. Unlike ordinary field theory, where we hang field variables on all points of space, the fields describe only the substructure of an extended object and so we hang the field variables only on the subset of points which are inside of the object. We call this set of points a "bag". As usual we associate the quantized amplitudes of the fields with the creation and annihilation operators for particles. However these "particles" will be present only inside a hadron since the operators are constructed from fields which exist only in the interior of a hadron, so we have guaranteed quark confinement. There is an analogy here with phonon fields and spin wave fields.

To construct a set of equations which mathematically describes such a situation, Chodos et Al. (1) take a Lagrangian density

$$\mathcal{L} = \mathcal{L}_{\text{fields}} - B$$

with Lagrangian $L = \int d^3r \; \mathcal{L} \; \theta(R(t) - x)$

where R (t) describes the region of space occupied by the bag. So the fields supply the kinetic energy of the system and the potential energy is given by B times the volume. Thus the model is covariant.

As an example, a bag in one space dimension containing a free scalar field ϕ would have

$$L = \int_{z_0(t)}^{z_1(t)} dz \left\{ \partial_\mu \phi^+ \partial^\mu \phi - B \right\}$$

where z_0 and z_1 are the end-points of the bag. The equations of motion for ϕ, zo and z_1 would then be obtained by requiring $\int dt\, L$ to be stationary with respect to arbitrary variations of ϕ, z_0 and z_1.

The above Lagrangian formalism turns out to be inadequate to describe fermion fields. However the dynamics can be given in terms of the energy-momentum tensor by requiring local conservation, i.e.

$$\partial_\mu T^{\mu\nu} = 0$$

where $$T^{\mu\nu} = \left(T^{\mu\nu}_{\text{fields}} + B g^{\mu\nu} \right) \theta\left(R(t) - x \right)$$

so $$\partial_\mu T^{\mu\nu} = n_\mu \left(T^{\mu\nu}_{\text{fields}} + B g^{\mu\nu} \right) \delta\left(R(t) - x \right)$$

where n_μ is the normal to the space-time surface swept out by R (t) and is such that $n_\mu n^\mu = -1$

So on the surface $X = R(t)$ we require $n_\mu T^{\mu\nu} = 0$

This gives us the boundary conditions.

With our example of a free scalar field,

$$T^{\mu\nu} = \partial^\mu\phi^+\partial^\nu\phi - g^{\mu\nu}(\partial_\omega\phi^+\partial^\omega\phi - B)$$

so $n_\mu T^{\mu\nu} = 0 \quad \Rightarrow$ either (a) $n_\mu\partial^\mu\phi = 0$ and $|\partial\phi|^2 - B = 0$

or (b) ϕ = constant and $-|\partial\phi|^2 - B = 0$

Either choice leads to a Lorentz covariant theory. We will normally work with the Dirichlet boundary conditions (b).

A bag containing fermion quarks interacting via coloured gauge fields can be obtained from

$$\mathcal{L} = -\tfrac{1}{4}\,\underline{F}_{\mu\nu}\cdot\underline{F}^{\mu\nu} + \tfrac{1}{2}i\,\bar\psi\,\gamma\cdot\overleftrightarrow{\partial}\psi + g\bar\psi\,\partial\cdot\underline{A}\cdot\underline{G}\,\psi - B$$

the standard quark-gluon Lagrangian with the extra B term.

The equations of motion are found to be

$$\mathcal{D}^\mu_{ij}\,F_{j\,\mu\nu} = -g\,\bar\psi\,G_i\,\gamma_\nu\,\psi$$

$$i\gamma\cdot\partial\,\psi_i + g\,(\partial\cdot\underline{A}\cdot\underline{G})_{ij}\,\psi_j = 0 \quad .$$

with boundary conditions:

$$n_\mu F_j^{\mu\nu} = 0$$

$$i\gamma\cdot n\,\psi = \psi$$

$$-\tfrac{1}{4}\underline{F}_{\mu\nu}\cdot\underline{F}^{\mu\nu} + \tfrac{1}{2}n\cdot\partial\,\bar\psi\psi - B = 0$$

on the surface.

The MIT group propose this as a realistic model of a hadron. De Grand et Al. (2) have solved these equations of motion for the case of a static spherical bag and have calculated the energies of the low lying hadron states to first order in $g^2/4\pi$ Fitting their free parameters B, $g^2/4\pi$, Ms and Zo, where Ms is the mass of the strange quark and Zo a constant related to zero-point energies, to the masses of the Ω^-, Δ, N and ω they produce quite a good spectrum of the $J = \frac{3}{2}, \frac{1}{2}$ baryons and the J = 1, 0 mesons. (see chapter five). They also obtain magnetic moments, weak decay constants and charge radii in reasonable agreement with experiment. Thus it seems that the bag model is quite good for predicting the static properties of hadrons. This thesis will be concerned with the bag model predictions for the scattering properties of hadrons. We start with the simplest system, the classical scalar bag in one space dimension.

2. The one-dimensional scalar bag

This system has been solved exactly. Following ref. (1), we find that the Lagrangian,

$$L = \int_{z_1(t)}^{z_2(t)} dz \left\{ \partial_\mu \phi^\dagger \partial^\mu \phi - B \right\} \tag{1.1}$$

gives rise to a free field equation,

$$\partial^2 \phi = 0 \tag{1.2}$$

together with the boundary conditions, on $z = z_1, z_2$,

$$\phi = 0 \tag{1.3}$$

$$\left|\frac{\partial \phi}{\partial t}\right|^2 - \left|\frac{\partial \phi}{\partial z}\right|^2 = -B \tag{1.4}$$

If we define light cone variables

$$\tau = t + z$$
$$x = t - z \tag{1.5}$$

then eqn. (1.2) becomes $\dfrac{\partial^2 \phi}{\partial \tau \partial x} = 0$ which has solutions

$$\phi(\tau, x) = f(\tau) + g(x) \tag{1.6}$$

The boundary conditions (1.3) and (1.4) become

$$f(\tau_i(x)) + g(x) = 0 \tag{1.7}$$

$$\dot{f}^{\dagger}(\tau_i(x))\, g'(x) + g'^{\dagger}(x)\, \dot{f}(\tau_i(x)) = -\tfrac{1}{2} B \tag{1.8}$$

where $\dot{f} = \frac{df}{d\tau}$, $g' = \frac{dg}{dx}$ and $\tau = \tau_i(x)$, $i = 1,2$ is the equation of an end-point.

Differentiating (1.7) w.r.t x we get

$$\dot{f}(\tau_i(x)) \frac{d\tau_i}{dx} + g'(x) = 0 \tag{1.9}$$

and substituting this into (1.8) we obtain

$$\frac{d\tau_i}{dx} = \frac{4}{B} \left| g'(x) \right|^2 \tag{1.10}$$

This means that $\frac{d\tau_1(x)}{dx} = \frac{d\tau_2(x)}{dx}$ so that the length of the bag measured along the τ direction is constant. Similarly if we denote the end-points by $x = x_i(\tau)$ and repeat the above procedure, this time differentiating (1.7) w.r.t. τ we find that the bag "length" in the x direction is also constant. We shall see that these conditions determine the scattering.

We now consider the collision in the CM frame of two identical bags with velocities $\pm v_0$ whose end-points meet at $t = 0$:

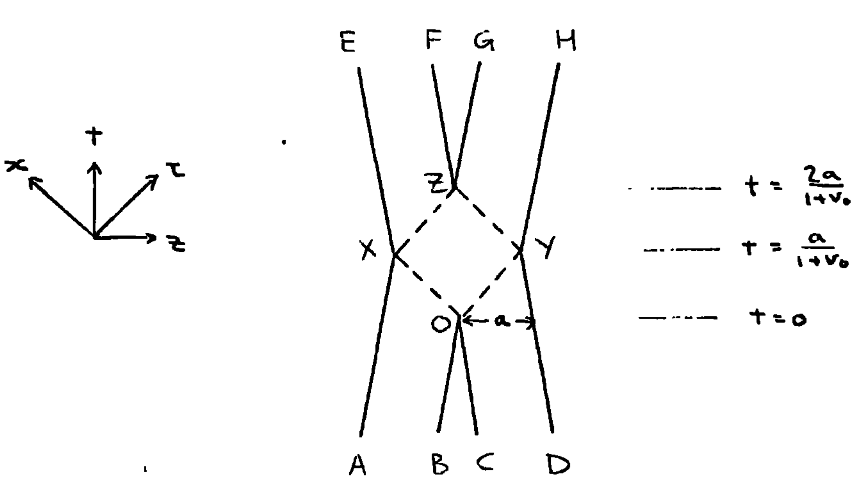


Fig. 1.1

The constant length conditions mean that YH is parallel to AX and XE is parallel to DY. At point Z the bags are free to move apart again as above, resulting in elastic scattering in which the fields acquire only phase shifts. From ref. (3),

$$
\begin{array}{llll}
\text{if} & \phi = f_1(\tau) + g_1(x) & & \text{in AXOB} \\
 & \phi = f_2(\tau) + g_2(x) & & \text{in COYD} \\
\text{then} & \phi = f_2(\tau) + g_1(x) & & \text{in OXZY} \\
 & \phi = f_1(\tau - b) + g_1(x) & & \text{in GZYH} \\
 & \phi = f_2(\tau) + g_2(x-b) & & \text{in EXZF}
\end{array}
$$

So we see that in fact the two bags pass through each other, rather than bounce back. This is obvious if we construct the above diagram for incident bags of different lengths.

The diagram clearly shows that the time taken for the bags to move through each other is less than if they had not interacted. Thus we have an attractive force. At least the model will give the correct sign for the nuclear force! In principle these interactions involve no free parameters (once B has been fitted to the nucleon mass, say.), so if we can calculate them we have a crucial test of the model. This will be considered in chapter two.

There is one peculiarity in the above system. At time $t = \frac{2a}{1+V_0}$ we have an ambiguity. The bags can remain together and oscillate:

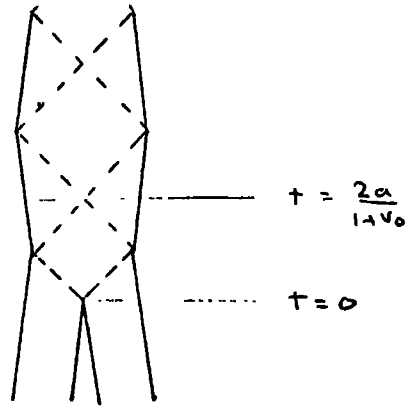


Fig. 1.2

The solution is not unique even if all the initial conditions are specified. This is because at point Z we have:

(i) $\dot{\phi} = 0$

(ii) eqn. (1.8) is satisfied for both possible solutions

(iii) two surfaces of discontinuity in the partial derivatives intersect.

In classicial mechanics this type of non-uniqueness is resolved by requiring that for sufficiently small time intervals the action is a minimum, not merely an extremal. This does not seem to be possible in this case.

If we reverse the process in Fig. 1.2 by reflecting in the line t = 0 we obtain a fission process. Thus we see that there are scattering solutions, fusion solutions and fission solutions for the wave equation (1.2) with boundary conditions (1.3) and (1.4). In the next chapter we shall investigate the scattering solutions.

CHAPTER TWO

We now attempt to estimate the strength of the interaction described in the previous chapter and compare it to the known strength of the strong interaction, as seen, for example, in the binding energy of the deuteron.

1. The classical binding energy

Consider first a static bag with end-points at $z = \pm \frac{l}{2}$ containing set of N complex scalar fields ϕ_α, where α labels the type of quark. So ,

$$L = \int_{-l/2}^{l/2} dz \left\{ \sum_\alpha \left(\partial_\mu \phi_\alpha^\dagger \partial^\mu \phi_\alpha \right) - B \right\} \tag{2.1}$$

This gives equations of motion

$$\partial^2 \phi_\alpha = 0 \tag{2.2}$$

with boundary conditions:

$$\phi_\alpha = 0 \tag{2.3}$$

$$\sum_{\alpha=1}^{N} \left\{ \left| \frac{\partial \phi_\alpha}{\partial t} \right|^2 - \left| \frac{\partial \phi_\alpha}{\partial z} \right|^2 \right\} = -B \tag{2.4}$$

at $z = \pm \frac{l}{2}$

The ground state solution of (2.2) satisfying (2.3) is

$$\phi_\alpha = A_\alpha e^{-i\frac{\pi t}{l}} \cos \frac{\pi z}{l} \tag{2.5}$$

The charge normalisation condition for a single particle to be associated with each field ϕ_α is

$$i \int_{-l/2}^{l/2} dz \left\{ \phi_\alpha^\dagger \frac{\partial \phi_\alpha}{\partial t} - \frac{\partial \phi_\alpha^\dagger}{\partial t} \phi_\alpha \right\} = 1 \tag{2.6}$$

From this condition we obtain $|A_\alpha|^2 = \frac{1}{\pi}$, so choosing the phase we can write

$$\phi_\alpha = \frac{1}{\sqrt{\pi}} e^{-\frac{\lambda \pi t}{\ell}} \cos \frac{\pi z}{\ell} \qquad (2.7)$$

The non-linear boundary condition (2.4) can be used to determine the length of the bag. We obtain

$$\ell = \sqrt{\frac{N\pi}{B}} \qquad (2.8)$$

The energy, i.e. the rest mass, of the bag is given by

$$M = \int_{-\ell/2}^{\ell/2} dz \left\{ \sum_\alpha \left(\left|\frac{\partial \phi_\alpha}{\partial t}\right|^2 + \left|\frac{\partial \phi_\alpha}{\partial z}\right|^2 \right) + B \right\}$$

$$= 2\sqrt{N\pi B} \qquad (2.9)$$

Putting $N = 3$ and $M = M_N$, the mass of a nucleon, this equation gives us the value of the bag constant B. We ignore at this stage the fact that this model does not distinguish between the N and the Δ.

If we put $N = 6$ we can have a bag with the quantum numbers of the deuteron with mass $M_d = 2\sqrt{6\pi B}$

The resulting binding energy is

$$\Delta E = 2M_N - M_d = (2 - 2^{1/2}) M_N \simeq 500 \text{ MeV.} \qquad (2.10)$$

This calculation can be done with spinor quark fields in three dimensions. Chodoset Al. (4) find that $M \propto N^{\frac{3}{4}}$ and so we have

$$\Delta E = (2 - 2^{3/4}) M_N \simeq 300 \text{ MeV.} \qquad (2.11)$$

In the classical bag model these states would exist. It is clear that they do not resemble the deuteron since the binding energy is far too large and the radius too small.

This, however, is not surprising since we do not expect to obtain the deuteron in a classical calculation. The classical deuteron would have the neutron and proton stationary at the deepest part of the interaction potential. This would be at the centre, i.e. zero separation, unless there is a hard core. Note that in a standard nuclear physics calculation of the deuteron the hard core plays very little role. It is quantum mechanics, not repulsive forces, that give the deuteron a large radius.

2. A Quantum Mechanical Model

In three dimensions if we view the interaction as a simple fusion then fission of two bags and if we assume that the fissioning of the six-quark bag is a slow process on the time scale associated with the motion of the massless quarks, then we may adopt a Born-Oppenheimer picture where the mass of a deformed six quark bag is viewed as a potential. In this way we obtain a potential which, when the relative separation R is zero, has $V(0) = \Delta E \simeq 300$ Mev. At a relative separation of $R = 2R_N$ where R_N is the nucleon radius, the potential is zero.

To turn from a classical treatment to a quantum mechanical one we insert this potential into a Schrödinger equation and calculate the energy eigenvalues.

In one dimension, if x is the separation of the centres of the two bags this equation is

$$-\frac{1}{2\left(\frac{M_N}{2}\right)}\frac{d^2\psi}{dx^2} + V(x)\,\psi(x) = E\,\psi(x) \qquad (2.12)$$

Using conservation of energy we can relate the potential $V(x)$ to the relative velocity $2v(x)$ by

$$\frac{1}{2}\left(\frac{M_N}{2}\right)(2v_0)^2 = \frac{1}{2}\left(\frac{M_N}{2}\right)(2v(x))^2 + V(x)$$

$$\Rightarrow \qquad 2v(x) = \left[4v_0^2 - \frac{4V(x)}{M_N}\right]^{1/2} \qquad (2.13)$$

From Fig. 1.1 the time taken for the two bags to pass through each other is $\frac{2a}{1+v_0}$. This is related to the relative velocity by

$$\frac{2a}{1+v_0} = \int_{-a}^{a} \frac{dx}{2v(x)}$$

$$= \int_{-a}^{a} \frac{dx}{2\sqrt{v_0^2 - \frac{V(x)}{M_N}}} \tag{2.14}$$

This gives us information about the "average", potential. Note that the potential is velocity dependent. Since the deuteron is a low velocity system, $v_0 \ll 1$, (2.14) simplifies to

$$2a = \sqrt{M_N} \int_{0}^{a} \frac{dx}{\sqrt{-V(x)}} \tag{2.15}$$

In principle the calculation could be done for the collision of three-dimensional bags. Unfortunately there is no simple method in 3-d analagous to the use of constant length conditions in 1-d. One thing we can say is that if we regard a as the diameter of a nucleon and consider a collision with zero impact parameter then Fig. 1.1 must hold up to t = $\frac{a}{1+v_0}$ even in 3-d. This is because the points on the spheres diametrically opposite the point of impact cannot know about the collision until they intersect the light cone coming from the point of impact.

So as a working hypothesis we assume that (2.15) holds in 3-d, i.e.,

$$2a = \sqrt{M_N} \int_{0}^{a} \frac{dR}{\sqrt{-V(R)}} \tag{2.16}$$

where R is the spatial separation of the centres of the two bags.

To proceed we must know the shape of the potential. In principle, if we could attach an unambiguous meaning to the position of the individual nucleon bags during the collision, this could be determined from the scattering solution. However this is not possible so we must guess the shape. We recall that the classical solution corresponds to putting the particles at the centre of the potential, in which case V (R = 0) would be equal to the classical binding energy,

$$V(0) = (2 - 2^{3/4}) M_N \simeq 300 \text{ MeV}$$

Then parametrizing the shape by

$$V(R) = -V(0)\left(1 - \frac{R}{a}\right)^{2P} , \quad 0 < R < a , \qquad (2.17)$$

inserting in (2.16) and integrating, we find

$$(1 - p) = \frac{1}{2\sqrt{2 - 2^{3/4}}} , \quad \text{or} \quad p \simeq \frac{1}{9}$$

We can now use this potential in the Schrodinger equation (2.11) and perform a variational calculation of the binding energy.

We use a trial wave function $\psi = Ae^{-bR}$ to minimise $\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$ with respect to b, where $H = -\frac{1}{M_N}\nabla^2 + V(R)$ and $\langle\psi|\psi\rangle = 4\pi \int_0^a R^2 dR\, A^2 e^{-2bR}$.

The resulting binding energy is approximately 125 Mev.
(The correct deuteron binding energy (~ 2 Mev) would be obtained from this calculation if we had $V(0)$ = 60 Mev.)

Now due to the greater freedom in 3-d it is likely that $t = \frac{a}{1+V_0}$ is not equal to half of the interaction time as it was in 1-d, i.e., we expect that more can happen after this time in 3-d, than in 1-d. Thus we expect that (2.16) gives an overestimate of the average depth of the potential. To allow for this we can rewrite (2.16) as

$$2a = \lambda \sqrt{M_N} \int_0^a \frac{dR}{\sqrt{-V(R)}} \qquad (2.18)$$

To estimate λ we suppose that the average of the potential is reduced by the same factor as $V(o)$ in going from one to three dimensions, i.e. by a factor of $\frac{2-2^{3/4}}{2-2^{1/2}} \simeq 0{\cdot}55$. Then we obtain $\lambda \doteq 1.35$, $p \simeq 0.35$ and the binding energy is about 70 Mev.

Obviously the model as it stands is far too naive to give a decent answer.

3. Quark-quark interactions

One obvious source of error lies in the fact that in the above model the N and the Δ are degenerate. To break this degeneracy we must include quark-quark interactions which in turn will contribute to the deuteron binding energy. To investigate the possible effects of this we consider a spin and isospin dependant interaction between quarks, given by the interaction Hamiltonian

$$H = \sum_{i \neq j} \left\{ A\,\underline{\tau}_i.\underline{\tau}_j + B\,\underline{\sigma}_i.\underline{\sigma}_j + C(\underline{\tau}_i.\underline{\tau}_j)(\underline{\sigma}_i.\underline{\sigma}_j) \right\} \qquad (2.19)$$

where A, B and C will in general be functions of the separations of the quarks, but we assume them to be constant over a distance of the order of the deuteron radius.

We can calculate the matrix elements of this operator for the proton, neutron, delta and deuteron states, denoted by $|P\rangle$ $|N\rangle$, $|\Delta\rangle$ and $|D\rangle$ respectively, using the wavefunctions (5):

$$|P\rangle = \frac{1}{\sqrt{18}} |2p{\uparrow}n{\downarrow}p{\uparrow} + 2p{\uparrow}p{\uparrow}n{\downarrow} + 2n{\downarrow}p{\uparrow}p{\uparrow} - p{\uparrow}p{\downarrow}n{\uparrow} - p{\uparrow}n{\uparrow}p{\downarrow}$$
$$- p{\downarrow}n{\uparrow}p{\uparrow} - n{\uparrow}p{\downarrow}p{\uparrow} - n{\uparrow}p{\uparrow}p{\downarrow} - p{\downarrow}p{\uparrow}n{\uparrow}\rangle$$

$$|N\rangle = \frac{1}{\sqrt{18}} |-2n{\uparrow}p{\downarrow}n{\uparrow} - 2n{\uparrow}n{\uparrow}p{\downarrow} - 2p{\downarrow}n{\uparrow}n{\uparrow} + p{\uparrow}n{\downarrow}n{\uparrow} + n{\uparrow}p{\uparrow}n{\downarrow} \qquad (2.20)$$
$$+ n{\downarrow}p{\uparrow}n{\uparrow} + n{\uparrow}n{\downarrow}p{\uparrow} + p{\uparrow}n{\uparrow}n{\downarrow} + n{\downarrow}n{\uparrow}p{\uparrow}\rangle$$

$$|\Delta^{++}\rangle = |p{\uparrow}p{\uparrow}p{\uparrow}\rangle$$

where p and n refer to quarks and $\uparrow$, $\downarrow$ denote spin up and spin down.

Now the deuteron is an I = 0, S = 1 state, antisymmetric with respect to interchange of nucleons, so

$$|D\rangle = \frac{1}{\sqrt{2}}\left\{ |P\rangle|N\rangle - |N\rangle|P\rangle \right\}$$

$$\Rightarrow \langle D|H|D\rangle = \langle P,N|H|P,N\rangle - \langle N,P|H|P,N\rangle \tag{2.21}$$

To calculate the matrix elements we need the following results:

$$\begin{aligned}
\langle p|\tau_x|p\rangle &= \langle\uparrow|\sigma_x|\uparrow\rangle = 0\\
\langle p|\tau_x|n\rangle &= \langle\uparrow|\sigma_x|\downarrow\rangle = 1\\
\langle n|\tau_x|p\rangle &= \langle\downarrow|\sigma_x|\uparrow\rangle = 1\\
\langle n|\tau_x|n\rangle &= \langle\downarrow|\sigma_x|\downarrow\rangle = 0\\
\\
\langle p|\tau_y|p\rangle &= \langle\uparrow|\sigma_y|\uparrow\rangle = 0\\
\langle p|\tau_y|n\rangle &= \langle\uparrow|\sigma_y|\downarrow\rangle = -i\\
\langle n|\tau_y|p\rangle &= \langle\downarrow|\sigma_y|\uparrow\rangle = i\\
\langle n|\tau_y|n\rangle &= \langle\downarrow|\sigma_y|\downarrow\rangle = 0\\
\\
\langle p|\tau_z|p\rangle &= \langle\uparrow|\sigma_z|\uparrow\rangle = 1\\
\langle p|\tau_z|n\rangle &= \langle\uparrow|\sigma_z|\downarrow\rangle = 0\\
\langle n|\tau_z|p\rangle &= \langle\downarrow|\sigma_z|\uparrow\rangle = 0\\
\langle n|\tau_z|n\rangle &= \langle\downarrow|\sigma_z|\downarrow\rangle = -1
\end{aligned} \tag{2.22}$$

And from these we obtain,

$$\begin{aligned}
\langle pp|\underline{\tau}_1\cdot\underline{\tau}_2|pp\rangle &= \langle\uparrow\uparrow|\underline{\sigma}_1\cdot\underline{\sigma}_2|\uparrow\uparrow\rangle = 1\\
\langle nn|\underline{\tau}_1\cdot\underline{\tau}_2|nn\rangle &= \langle\downarrow\downarrow|\underline{\sigma}_1\cdot\underline{\sigma}_2|\downarrow\downarrow\rangle = 1\\
\langle pn|\underline{\tau}_1\cdot\underline{\tau}_2|pn\rangle &= \langle\uparrow\downarrow|\underline{\sigma}_1\cdot\underline{\sigma}_2|\uparrow\downarrow\rangle = -1\\
\langle np|\underline{\tau}_1\cdot\underline{\tau}_2|np\rangle &= \langle\downarrow\uparrow|\underline{\sigma}_1\cdot\underline{\sigma}_2|\downarrow\uparrow\rangle = -1\\
\langle pn|\underline{\tau}_1\cdot\underline{\tau}_2|np\rangle &= \langle\uparrow\downarrow|\underline{\sigma}_1\cdot\underline{\sigma}_2|\downarrow\uparrow\rangle = 2\\
\langle np|\underline{\tau}_1\cdot\underline{\tau}_2|pn\rangle &= \langle\downarrow\uparrow|\underline{\sigma}_1\cdot\underline{\sigma}_2|\uparrow\downarrow\rangle = 2
\end{aligned} \tag{2.23}$$

The matrix elements of H can be written,

$$\langle X|H|X\rangle = 3A\langle X|\underline{I}_1\cdot\underline{I}_2|X\rangle + 3B\langle X|\underline{\sigma}_1\cdot\underline{\sigma}_2|X\rangle + 3C\langle X|(\underline{I}_1\cdot\underline{I}_2)(\underline{\sigma}_1\cdot\underline{\sigma}_2)|X\rangle \tag{2.24}$$

where $|X\rangle$ stands for $|P\rangle$, $|N\rangle$ or $|\Delta\rangle$, and

$$\begin{aligned}\langle D|H|D\rangle = {} & 3A\langle P|\underline{I}_1\cdot\underline{I}_2|P\rangle\langle N|N\rangle \\ & + 3A\langle P|P\rangle\langle N|\underline{I}_1\cdot\underline{I}_2|N\rangle \\ & + 9A\langle P,N|\underline{I}_1\cdot\underline{I}_4|P,N\rangle \\ & + 3B\langle P|\underline{\sigma}_1\cdot\underline{\sigma}_2|P\rangle\langle N|N\rangle \\ & + 3B\langle P|P\rangle\langle N|\underline{\sigma}_1\cdot\underline{\sigma}_2|N\rangle \\ & + 9B\langle P,N|\underline{\sigma}_1\cdot\underline{\sigma}_4|P,N\rangle \\ & + 3C\langle P|\underline{I}_1\cdot\underline{I}_2\ \underline{\sigma}_1\cdot\underline{\sigma}_2|P\rangle\langle N|N\rangle \\ & + 3C\langle P|P\rangle\langle N|\underline{I}_1\cdot\underline{I}_2\ \underline{\sigma}_1\cdot\underline{\sigma}_2|N\rangle \\ & + 9C\langle P,N|\underline{I}_1\cdot\underline{I}_4\ \underline{\sigma}_1\cdot\underline{\sigma}_4|P,N\rangle \\ & - 9A\langle P,N|\underline{I}_1\cdot\underline{I}_4|N,P\rangle \\ & - 9B\langle P,N|\underline{\sigma}_1\cdot\underline{\sigma}_4|N,P\rangle \\ & - 9C\langle P,N|\underline{I}_1\cdot\underline{I}_4\ \underline{\sigma}_1\cdot\underline{\sigma}_4|N,P\rangle\end{aligned} \tag{2.25}$$

Finally, using wave functions (2.20) with results (2.23) in expressions (2.24) and (2.25) we obtain

$$\begin{aligned}\langle\Delta|H|\Delta\rangle &= 3A + 3B + 3C \\ \langle P|H|P\rangle &= -3A - 3B + 15C \\ \langle D|H|D\rangle &= -9A - 5B + \tfrac{65}{3}C\end{aligned} \tag{2.26}$$

and since H is SU(3) symmetric we have

$$\langle P|H|P\rangle = \langle N|H|N\rangle$$

If we attribute the Δ - N mass difference to this interaction we have

$$M_\Delta - M_N = 6A + 6B - 12C$$

The contribution of this interaction to the deuteron binding energy is then

$$\Delta E_{INT} = 3A - B + \frac{25}{3} C$$

There is nothing more we can say without further assumptions about the relative magnitudes of A, B and C. For example if we assume that the quark-quark interaction is mediated by coloured vector gluons, then according to De-Grand et Al. (2), the dominant contribution comes from the spin-spin interaction, i.e. $H = B \sum_{i \neq j} \underline{\sigma}_i \cdot \underline{\sigma}_j$

and $M_\Delta - M_N = 6B$

$$\Delta E_{INT} = -B$$

This gives us a contribution to the binding energy of - 50 Mev, so this interaction does bring our crude estimates much closer to the observed value.

We might worry at this stage about the effects of the quark-quark interaction on the simple scattering picture of fig. 1.1. This problem will be discussed in detail in chapter four.

4. Conclusion

Quite apart from these "technical", problems of doing the calculation in three dimensions and of taking into account quark-quark interactions, our procedure so far is inadequate at a more fundamental level. We have solved a classical scattering problem and used the result in a Schrodinger equation to find the bound states. This is unsatisfactory as the scattering problem and the original free states should also be treated quantum mechanically, i.e. we should obtain a wave equation for the energy eigenstates directly from the original lagrangian. This problem will be tackled in chapter three.

Note that in this model the deuteron will appear most of the time as it does in conventional nuclear physics, i.e. as a proton plus a neutron. The fact that it spends some time as a six-quark bag is the mechanism responsible for the nuclear force. Support for this picture comes from a paper by Frankfurt and Strikman (6), who use the parton model to examine the effects of small inter-nucleon distances in the deuteron. From the parton viewpoint the spacetime picture of nucleon-nucleon interactions as multimeson exchange looks doubtful at small distances and they argue that at distances comparable with the nucleon size the deuteron can be described not as a system of two nucleons but as a system of six quarks (called the kneading effect). The repulsive core in this model is a result of the kneading of quarks from different nucleons and has the same origin as the cutoff of transverse momenta for the partons in deep inelastic scattering. They estimate that the probability of the quarks being in a kneaded configuration is about 5%.

CHAPTER THREE

1. The quantization of one-dimensional bags

In the quantization procedure given in ref. (1) the deuteron binding energy would not differ greatly from the classical value found in chapter two, section one. Certainly there is no way whereby a term corresponding to a kinetic energy associated with the relative separation could arise. This is basically because the length of the bag is not, in this treatment, a quantum variable. We propose to modify the Lagrangian to remedy this "defect".

We consider first a one-dimensional bag, containing a single complex scalar field, described by the lagrangian

$$L = \int_{z_1(t)}^{z_2(t)} dz \left\{ \left|\frac{\partial\phi}{\partial t}\right|^2 - \left|\frac{\partial\phi}{\partial z}\right|^2 - B \right\} \tag{3.1}$$

We are not interested in the classical equations of motion or boundary conditions but we choose to impose the condition

$$\phi(z,t) = 0 \qquad \text{at} \quad z = z_1, z_2 \tag{3.2}$$

This allows us to put

$$\phi(z,t) = \sum_n A_n \sin \frac{n\pi}{l}\left(z - z_0 + \frac{l}{2}\right) \tag{3.3}$$

where $l = Z_2 - Z_1$ is the length of the bag,

and $Z_0 = \dfrac{Z_2 + Z_1}{2}$ is the centre of the bag.

Then (3.1) becomes

$$L = \langle \dot\phi | \dot\phi \rangle - \langle \phi' | \phi' \rangle - B\ell$$

$$= \sum_n \sum_m \Big\{ \dot A_m^* \dot A_n \langle e_m | e_n \rangle - \frac{n\pi}{\ell^2} \dot A_m^* A_n \dot\ell \langle e_m | (z-z_0) | f_n \rangle$$

$$- \frac{m\pi}{\ell^2} A_m^* \dot A_n \dot\ell \langle f_m | (z-z_0) | e_n \rangle - \frac{n\pi}{\ell} \dot A_m^* A_n \dot z_0 \langle e_m | f_n \rangle$$

$$- \frac{m\pi}{\ell} A_m^* \dot A_n \dot z_0 \langle f_m | e_n \rangle + \frac{nm\pi^2}{\ell^4} A_m^* A_n \dot\ell^2 \langle f_m | (z-z_0)^2 | f_n \rangle$$

$$+ \frac{2nm\pi^2}{\ell^3} A_m^* A_n \dot\ell \dot z_0 \langle f_m | (z-z_0) | f_n \rangle + \frac{nm\pi^2}{\ell^2} A_m^* A_n \dot z_0^2 \langle f_m | f_n \rangle$$

$$- A_n^* A_m \frac{nm\pi^2}{\ell^2} \langle f_m | f_n \rangle \Big\} - B\ell \tag{3.4}$$

where $|e_n\rangle = \sin \frac{n\pi}{\ell}(z - z_0 + \frac{\ell}{2})$

$|f_n\rangle = \cos \frac{n\pi}{\ell}(z - z_0 + \frac{\ell}{2})$

and the scalar product is defined by

$$\langle a | b \rangle = \int_{z_0-\ell/2}^{z_0+\ell/2} dz\, a^*(z)\, b(z) \quad .$$

We define momenta Π_n, Π_ℓ and Π_{z_0}, conjugate to A_n, l and Z_0 in the usual way, Thus,

$$\Pi_n = \sum_m \Big\{ \dot A_m^* \langle e_m | e_n \rangle - \frac{m\pi}{\ell^2} A_m^* \dot\ell \langle f_m | (z-z_0) | e_n \rangle - \frac{m\pi}{\ell} A_m^* \dot z_0 \langle f_m | e_n \rangle \Big\} \tag{3.5}$$

$$\Pi_\ell = \sum_m \sum_n \Big\{ -\frac{n\pi}{\ell^2} \dot A_m^* A_n \langle e_m | (z-z_0) | f_n \rangle - \frac{m\pi}{\ell^2} A_m^* \dot A_n \langle f_m | (z-z_0) | e_n \rangle$$

$$+ \frac{2nm\pi^2}{\ell^4} A_m^* A_n \dot\ell \langle f_m | (z-z_0)^2 | f_n \rangle + \frac{2nm\pi^2}{\ell^3} A_m^* A_n \dot z_0 \langle f_m | (z-z_0) | f_n \rangle \Big\} \tag{3.6}$$

$$\Pi_{z_0} = \sum_m \sum_n \Big\{ -\frac{n\pi}{\ell} \dot A_m^* A_n \langle e_m | f_n \rangle - \frac{m\pi}{\ell} A_m^* \dot A_n \langle f_m | e_n \rangle$$

$$+ \frac{2nm\pi^2}{\ell^3} A_m^* A_n \dot\ell \langle f_m | (z-z_0) | f_n \rangle + \frac{2nm\pi^2}{\ell^2} A_m^* A_n \dot z_0 \langle f_m | f_n \rangle \Big\} \tag{3.7}$$

Now using (3.5) in (3.6) and (3.7) we obtain

$$\Pi_\ell = \sum_n \sum_m \left\{ -\frac{n\pi}{\ell^2} A_n \Pi_m \langle e_m | (z - z_0) | f_n \rangle \quad + \quad h.c. \right\} \tag{3.8}$$

$$\Pi_{z_0} = \sum_n \sum_m \left\{ -\frac{n\pi}{\ell} A_n \Pi_m \langle e_m | f_n \rangle \quad + \quad h.c \right\} \tag{3.9}$$

from which we see that Π_ℓ and Π_{z_0} are not independant variables and so the length of the bag is not an independant quantum variable. At this stage it becomes necessary to make some approximations.

We make the assumption that, in the first few states, only the lowest modes will be significantly excited and therefore we can truncate the series (3.3). This has an interesting effect on the structure of the equations. To investigate this put $A_n = 0$ for $n > N$

Then

$$\Pi_\ell = \sum_{m=1}^{\infty} \sum_{n=1}^{N} \left\{ -\frac{n\pi}{\ell^2} A_n \Pi_m X_{mn} \quad + \quad h.c. \right\}$$

$$= \sum_{m=1}^{N} \sum_{n=1}^{N} \left\{ -\frac{n\pi}{\ell^2} A_n \Pi_m X_{mn} + h.c. \right\} + \sum_{m=N+1}^{\infty} \sum_{n=1}^{N} \left\{ -\frac{n\pi}{\ell^2} A_n \Pi_m X_{mn} + h.c. \right\} \tag{3.10}$$

where we have used the notation $X_{mn} = \langle e_m | (z - z_0) | f_n \rangle$

However from (3.5)

$$\Pi_m = \frac{\ell}{2} \dot{A}_m^{*} - \dot{\ell} \sum_p \frac{p\pi}{\ell^2} A_p^{*} X_{pm} - \dot{z}_0 \sum_p \frac{p\pi}{\ell} A_p^{*} \langle f_p | e_m \rangle$$

which implies that for $m > N$,

$$\Pi_m = -\dot{\ell} \sum_{p=1}^{N} \frac{p\pi}{\ell^2} A_p^{*} X_{pm} - \dot{z}_0 \sum_{p=1}^{N} \frac{p\pi}{\ell} A_p^{*} \langle f_p | e_m \rangle \tag{3.11}$$

So inserting (3.11) into (3.10) we obtain

$$\Pi_{\ell} = \sum_{m=1}^{N}\sum_{n=1}^{N}\left\{ -\frac{n\pi}{\ell^2} A_n \Pi_m X_{mn} + h.c. \right\}$$
$$+ \dot{\ell} \sum_{m=N+1}^{\infty}\sum_{n=1}^{N}\sum_{p=1}^{N}\left\{ \frac{np\pi^2}{\ell^4} A_p^* A_n X_{pm} X_{mn} + h.c. \right\}$$
$$+ \dot{z}_0 \sum_{m=N+1}^{\infty}\sum_{n=1}^{N}\sum_{p=1}^{N}\left\{ \frac{np\pi^2}{\ell^3} A_p^* A_n \langle f_p | e_m \rangle X_{mn} + h.c. \right\} \tag{3.12}$$

In the same way a similar expression can be obtained for Π_{z_0}. These show that the velocities $\dot{l}$ and $\dot{Z}_0$ can now be expressed in terms of momenta and co-ordinates and so the lagrangian is no longer singular (7). Thus we can continue with canonical quantization.

The integrals in (3.4) are straightforward and we obtain

$$L = \sum_n \left\{ \frac{\ell}{2}\dot{A}_n^*\dot{A}_n + \frac{\dot{\ell}}{4}\left(\dot{A}_n^* A_n + A_n^*\dot{A}_n\right) + \frac{\dot{\ell}^2}{4\ell}\left(1+\frac{n^2\pi^2}{6}\right)A_n^* A_n + \frac{\dot{z}_0^2}{2\ell} n^2\pi^2 A_n^* A_n - \frac{n^2\pi^2}{2\ell} A_n^* A_n \right\}$$
$$+ \sum_{\substack{m,n \\ (m+n\ \text{odd})}} \left\{ -\frac{2nm}{n^2-m^2} A_m^* \dot{A}_n \dot{z}_0 - \frac{2nm}{m^2-n^2}\dot{A}_m^* A_n \dot{z}_0 - \frac{4nm}{(m+n)^2(m-n)^2} A_m^* A_n \frac{\dot{\ell}\dot{z}_0}{\ell} \right\}$$
$$+ \sum_{\substack{m,n \\ (m+n\ \text{even} \\ m\neq n)}} \left\{ \frac{mn}{n^2-m^2} A_m^* \dot{A}_n \dot{\ell} + \frac{mn}{m^2-n^2}\dot{A}_m^* A_n \dot{\ell} + \frac{2nm(m^2+n^2)}{(m+n)^2(m-n)^2} A_m^* A_n \frac{\dot{\ell}^2}{\ell} \right\}$$
$$- B\ell \tag{3.13}$$

To consider the ground state we truncate the series (3.3) after only one term, and we write $A_1 = X + iY$ where X and Y are real. Then

$$L = \frac{\ell}{2}\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{\dot{\ell}^2}{4\ell}\left(1+\pi^2/6\right)\left(X^2+Y^2\right) + \frac{\dot{\ell}}{2}\left(X\dot{X} + Y\dot{Y}\right)$$
$$+ \frac{\pi^2}{2\ell}\left(X^2+Y^2\right)\dot{z}_0^2 - \frac{\pi^2}{2\ell}\left(X^2+Y^2\right) - B\ell \tag{3.14}$$

In this approximation the motion of the centre of mass of the bag decouples from the motion of the fields, i.e. we have no $\dot{Z}o\,\dot{X}$ terms in (3.14), so we can choose to work in the bag rest frame and put $\dot{Z}o = 0$

The conjugate momenta are now given by

$$\pi_x = \ell\dot{X} + \tfrac{1}{2}X\dot{\ell}$$

$$\pi_y = \ell\dot{Y} + \tfrac{1}{2}Y\dot{\ell}$$

$$\pi_\ell = \tfrac{1}{2}\left(1+\tfrac{\pi^2}{6}\right)(X^2+Y^2)\frac{\dot{\ell}}{\ell} + \tfrac{1}{2}\left(X\dot{X}+Y\dot{Y}\right) \qquad (3.15)$$

Solving these equations for $\dot{X}$, $\dot{Y}$ and $\dot{\ell}$ we obtain

$$\dot{X} = \left(\frac{2}{a\ell} - \frac{1}{b\ell}\frac{Y^2}{X^2+Y^2}\right)\pi_x + \frac{1}{b\ell}\frac{XY}{X^2+Y^2}\pi_y - \frac{2}{b}\frac{X}{X^2+Y^2}\pi_\ell$$

$$\dot{Y} = \frac{1}{b\ell}\frac{XY}{X^2+Y^2}\pi_x + \left(\frac{2}{a\ell} - \frac{1}{b\ell}\frac{X^2}{X^2+Y^2}\right)\pi_y - \frac{2}{b}\frac{Y}{X^2+Y^2}\pi_\ell$$

$$\dot{\ell} = -\frac{2}{b}\frac{X}{X^2+Y^2}\pi_x - \frac{2}{b}\frac{Y}{X^2+Y^2}\pi_y + \frac{4\ell}{b(X^2+Y^2)}\pi_\ell \qquad (3.16)$$

where $a = \dfrac{1+\pi^2/3}{1+\pi^2/6}$ and $b = 1+\pi^2/3$

The Hamiltonian is given by

$$H = \dot{X}\pi_x + \dot{Y}\pi_y + \dot{\ell}\pi_\ell - L \qquad (3.17)$$

Substituting equations (3.16) into (3.14) and (3.17) we obtain,

$$H = \frac{1}{2\ell}\left(\pi_x^2+\pi_y^2\right) + \frac{1}{2\ell(1+\pi^2/3)(X^2+Y^2)}\left(X^2\pi_x^2 + 2XY\pi_x\pi_y + Y^2\pi_y^2\right)$$

$$+ \frac{2\ell}{(1+\pi^2/3)(X^2+Y^2)}\pi_\ell^2 + \frac{2}{(1+\pi^2/3)(X^2+Y^2)}\left(X\pi_x\pi_\ell + Y\pi_y\pi_\ell\right)$$

$$+ \frac{\pi^2}{2\ell}(X^2+Y^2) + B\ell \qquad (3.18)$$

To obtain the energy eigenvalues of this Hamiltonian we wish to solve $H\psi = E\psi$. To obtain a wave equation we make the usual substitutions $\pi_x = -i\frac{\partial}{\partial x}$ etc., taking care with the ordering of the operators to ensure that H is hermitian. To simplify the form of the wave equation we use polar co-ordinates defined by $X = r\cos\theta$, $Y = r\sin\theta$. Then the Hamiltonian (3.18) can be written in the form

$$H = H_o + H' \tag{3.19}$$

where

$$H_o = -\frac{1}{2\ell}\nabla^2 + \frac{\pi^2 r^2}{2\ell} + B\ell \tag{3.20}$$

$$H' = -\frac{2}{(1+\pi^2/3)}\left[\ell\frac{\partial}{\partial\ell} - \frac{1}{2}r\frac{\partial}{\partial r}\right]\frac{1}{\ell r^2}\left[\ell\frac{\partial}{\partial\ell} - \frac{1}{2}r\frac{\partial}{\partial r}\right] \tag{3.21}$$

We see that Ho is just the Hamiltonian of a two-dimensional harmonic oscillator. The correction term H' is positive definite and so increases the energy of the eigenstates. We shall see that typically it gives a small correction.

The charge normalization condition (2.6) now becomes

$$\ell(\dot{X}Y - \dot{Y}X) = 1 \tag{3.22}$$

This is a first class constraint (7) and is used to restrict the space of possible eigenstates to those satisfying

$$\langle\psi|\ell(\dot{X}Y - \dot{Y}X)|\psi\rangle = \langle\psi|\psi\rangle$$

or,

$$\langle\psi|\, i\frac{\partial}{\partial\theta}\,|\psi\rangle = \langle\psi|\psi\rangle \tag{3.23}$$

It follows that a bag containing one particle will have wave-function

$$\psi(r,\theta,\ell) = e^{-i\theta}\psi(r,\ell) \tag{3.24}$$

The lowest eigenstate of the Hamiltonian, in which the wave-function is independent of θ represents the "empty bag".

We now wish to investigate the consequences of this quantization on the binding energy of the deuteron.

2. The binding energy of two quantum bags

We consider a prototype model of the deuteron as follows. We assume our world contains two scalar fields ϕ_1 and ϕ_2. A bag containing a single quark corresponding to one of these fields will represent a nucleon and a bag containing both fields will represent the deuteron. The generalization of the previous section to a world with two fields is straightforward and leads to a Hamiltonian

$$H = H_0 + H' \tag{3.25}$$

where

$$H_0 = -\frac{1}{2\ell}\left(\nabla_1^2 + \nabla_2^2\right) + \frac{\pi^2}{2\ell}\left(r_1^2 + r_2^2\right) + B\ell \tag{3.26}$$

$$H' = -\frac{2}{(1+\pi^2/3)}\left[\ell\frac{\partial}{\partial\ell} - \frac{1}{2}r_1\frac{\partial}{\partial r_1} - \frac{1}{2}r_2\frac{\partial}{\partial r_2} - \frac{1}{2}\right]\frac{1}{\ell\,(r_1^2+r_2^2)}\left[\ell\frac{\partial}{\partial\ell} - \frac{1}{2}r_1\frac{\partial}{\partial r_1} - \frac{1}{2}r_2\frac{\partial}{\partial r_2} - \frac{1}{2}\right] \tag{3.27}$$

We find eigenstates by a variational calculation using factorised wave-functions $\psi(r,\theta)\,\psi(\ell)$ where the $\psi(r,\theta)$ are taken to be the eigenfunctions of (3.26). This turns out to give a reasonable approximation since the correction due to H' is small. We use

$$\psi(\ell) = \ell^{\beta} e^{-\frac{1}{2}\alpha\ell^2} \tag{3.28}$$

where α and β are variational parameters.

The "deuteron" state requires a single particle in each field, so following (3.24) we take as its wave-function

$$\psi_{11} = r_1 e^{-i\theta_1} e^{-\frac{\pi}{2}r_1^2} r_2 e^{-i\theta_2} e^{-\frac{\pi}{2}r_2^2} \, \ell^\beta e^{-\frac{1}{2}\alpha\ell^2} \tag{3.29}$$

The "nucleon" states have just one field excited so we put

$$\psi_{10} = r_1 e^{-i\theta_1} e^{-\frac{\pi}{2}r_1^2} \, e^{-\frac{\pi}{2}r_2^2} \, \ell^\beta e^{-\frac{1}{2}\alpha\ell^2} \tag{3.30}$$

Finally we need a wave-function for the "empty-bag",

$$\psi_{00} = e^{-\frac{\pi}{2}r_1^2} \, e^{-\frac{\pi}{2}r_2^2} \, \ell^\beta e^{-\frac{1}{2}\alpha\ell^2} \tag{3.31}$$

The energies of these states are given by $E = \langle H_0\rangle + \langle H'\rangle$

Now $$\langle H_0\rangle = \frac{\langle\psi|H_0|\psi\rangle}{\langle\psi|\psi\rangle}$$

$$= \frac{\langle\psi(\ell)|\left(B\ell + (n_1+n_2)\frac{\pi}{\ell}\right)|\psi(\ell)\rangle}{\langle\psi(\ell)|\psi(\ell)\rangle} \tag{3.32}$$

where $n_i = 1$ or 2 depending on whether the i^{th} field is in the ground state or first excited state. This just comes from the spectrum of the pair of two-dimensional oscillators in (3.26)

To simplify H' we write $r_1 = R\cos\phi$, $r_2 = R\sin\phi$,

Then $$H' = \frac{-2}{(1+\pi^2/3)}\left[\ell\frac{\partial}{\partial\ell} - \frac{1}{2}R\frac{\partial}{\partial R} - \frac{1}{2}\right]\frac{1}{\ell R^2}\left[\ell\frac{\partial}{\partial\ell} - \frac{1}{2}R\frac{\partial}{\partial R} - \frac{1}{2}\right] \tag{3.33}$$

and since H' contains no derivatives w.r.t. θ_1, θ_2 or ϕ we have

$$\langle H' \rangle = \frac{\langle \psi | H' | \psi \rangle}{\langle \psi | \psi \rangle}$$

$$= \frac{\int r_1 dr_1 \, d\theta_1 \, r_2 dr_2 \, d\theta_2 \, dl \, \psi^* H' \psi}{\int r_1 dr_1 \, d\theta_1 \, r_2 dr_2 \, d\theta_2 \, dl \, \psi^* \psi}$$

$$= \frac{\int R^3 dR \, dl \, \psi^* H' \psi}{\int R^3 dR \, dl \, \psi^* \psi} \tag{3.34}$$

i.e. the angular integrations cancell

So in calculating $\langle H' \rangle$ using (3.34) we can use, instead of (3.29) - (3.31),

$$\psi_{11} = R^2 e^{-\frac{\pi}{2}R^2} \, l^\beta e^{-\frac{\alpha}{2}l^2}$$

$$\psi_{10} = R \, e^{-\frac{\pi}{2}R^2} \, l^\beta e^{-\frac{\alpha}{2}l^2}$$

$$\psi_{00} = e^{-\frac{\pi}{2}R^2} \, l^\beta e^{-\frac{\alpha}{2}l^2} \tag{3.35}$$

Using (3.35) in (3.32) and (3.34) and integrating we obtain, for ψ_{00},

$$\langle H' \rangle_{00} = \frac{2\pi}{1+\pi^2/3} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1/2)} \left(1 + \frac{1}{4\beta}\right) \alpha^{1/2}$$

$$\langle H_0 \rangle_{00} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1/2)} \left(B\alpha^{-1/2} + \frac{2\pi}{\beta} \alpha^{1/2} \right)$$

So $$E_{00} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1/2)} \left\{ B\alpha^{-1/2} + \frac{2\pi}{1+\pi^2/3} \left(1 + \frac{5+4\pi^2/3}{4\beta} \right) \alpha^{1/2} \right\} \tag{3.36}$$

We now require $\frac{\partial E_{00}}{\partial \alpha} = \frac{\partial E_{00}}{\partial \beta} = 0$ to minimize the energy.

Now $\frac{\partial E_{00}}{\partial \alpha} = 0$ implies that

$$\alpha = \frac{B(1+\pi^2/3)}{2\pi\left(1 + \frac{5+4\pi^2/3}{4\beta}\right)} \tag{3.37}$$

So $$E_{00} = 2\sqrt{\frac{2}{1+\pi^2/3}}\;\frac{\Gamma(\beta+1)}{\Gamma(\beta+1/2)}\left(1+\frac{5+4\pi^2/3}{4\beta}\right)^{1/2}\sqrt{\pi B} \qquad (3.38)$$

Numerically this is found to have a minimum at $\beta \simeq 1.2$, when

$$\langle H\rangle_{00} = 3.60\sqrt{\pi B}$$

$$\langle H_0\rangle_{00} = 3.15\sqrt{\pi B}$$

$$\langle H'\rangle_{00} \simeq 0.45\sqrt{\pi B}$$

Exactly the same calculations are used to evaluate the energies in states ψ_{10} and ψ_{11} The results are shown in the following table:

		α	β	$\langle H\rangle$	$\langle H_0\rangle$	$\langle H'\rangle$	E_0
Empty bag	ψ_{00}	0.57	1.2	3.60	3.15	0.45	2.83
one-quark bag	ψ_{10}	0.61	2.2	4.02	3.67	0.35	3.46
two-quark bag	ψ_{11}	0.65	3.0	4.46	4.16	0.28	4.00

All energies are in units of $\sqrt{\pi B}$

The α's are in units of B/π

The last column refers to the <u>exact</u> eigenvalues of H_0 obtained by minimizing $E_0 = Bl + (n_1+n_2)\frac{\pi}{l}$ w.r.t. l.

We see that the corrections due to H', i.e. due to quantum fluctuations in the length of the bag are of the order of 10%.

We assume that the observed mass is equal to the energy above the ground state (empty bag), Thus

$$M_N \doteq \langle H\rangle_{10} - \langle H\rangle_{00}$$

$$M_D = \langle H\rangle_{11} - \langle H\rangle_{00}$$

If we ignore the quantum fluctuations in the bag length then the binding energy obtained from the values in the last column of the table is

$$\Delta E = 2M_N - M_D \simeq 0.09\sqrt{\pi B} \simeq \frac{1}{5} M_N$$

Including the effects of quantizing the length, i.e. using the values of $\langle H \rangle$ obtained by the variational calculation we obtain

$$\Delta E = 2M_N - M_D \simeq 0$$

So the quantum fluctuations in the lengths have the effect of reducing the binding energy of the two-particle bound state. This is just the effect we were seeking.

There is one interesting consequence of the above treatment. If our world contains k fundamental fields then the empty bag has approximate rest mass E_0 (as in the final column of the table) given by

$$(E_0)_{0\ldots 0} = B\ell + \frac{k\pi}{\ell}$$

$$= 2\sqrt{k\pi B}$$

on minimizing w.r.t. l.

The single particle bag has lowest energy,

$$(E_0)_{1,0\ldots 0} = 2\sqrt{(k+1)\pi B}$$

So if the rest mass of the "nucleon", is identified, as above, with the difference $(E)_{1,0\ldots 0} - (E)_{0,\ldots 0}$ then it becomes

$$M_N \simeq 2\sqrt{\pi B}\left[(k+1)^{1/2} - k^{1/2}\right]$$

This depends on k. This means that, for example, the mass of a hadron which contains no charmed quarks depends on whether charmed quarks exist or not. Whether this is desirable or not is, at the moment, a philosophical question. If we do not find it acceptable then some other method for subtracting the zero-point energies will be required.

3. The method of quantization

When this calculation was first attempted we used the Feynman path integral method (8) to obtain a wave equation.

A wave function $\psi(x,t)$ satisfies an integral equation

$$\psi(x_2,t_2) = \int_{-\infty}^{\infty} dx_1(t)\, K(x_2,t_2;x_1,t_1)\, \psi(x_1,t_1) \tag{3.39}$$

where the amplitude for a system to go from state a to state b, K (b, a) can be written as a sum over all paths, i.e.

$$K(b,a) = \int_a^b e^{\frac{i}{\hbar} S[b,a]}\, \mathcal{D}x(t)$$

where $$S[b,a] = \int_{t_a}^{t_b} dt\, L(x,\dot{x},t)$$

Feynman showed that in a lagrangian where $\dot{x}$ only appears up to the second degree, i.e. when the path integral is a Gaussian then the kernel K is always proportional to the classical action,

i.e. $$K(b,a) = F(t_b,t_a)\, e^{\frac{i}{\hbar} S_{cl.}} \tag{3.40}$$

Now our bag Lagrangian (3.1) gives a Gaussian integral and so we expected to be able to use this method.

The wave equation is obtained from (3.39) by considering an infinitesinal time interval $t_2 - t_1 = \varepsilon$ in which case

$$\psi(x_2,t_2) \simeq \psi(x_2,t_1) + \varepsilon \frac{\partial\psi}{\partial t}(x_2,t_1)$$

$$\psi(x_1,t_1) \simeq \psi(x_2,t_1) - h \frac{\partial\psi}{\partial x}(x_2,t_1) + \frac{h^2}{2} \frac{\partial^2\psi}{\partial x^2}(x_2,t_1)$$

and $$S_{cl.} \simeq \varepsilon L_{cl} \tag{3.41}$$

Inserting these approximations into (3.40) and (3.39), doing the Gaussian integrals w.r.t. h and expanding to first order in ε (second order in h) we obtain a wave equation, i.e. an equation involving $\frac{\partial\psi}{\partial x}$, $\frac{\partial^2\psi}{\partial x^2}$ and $\frac{\partial\psi}{\partial t}$.

Using expansion (3.3) our bag lagrangian is

$$L_{cl} = L_{cl}\,(A_n, \dot{A}_n, \ell, \dot{\ell})$$

and by the above method we obtain a wave equation for the bag states. However the wave equation obtained is not the same as that given by (3.20) and (3.21), obtained by canonical quantization. Thus we have to look a bit more closely at the path integral method. One obvious possible problem is that the coefficients of $\dot{X}^2$, $\dot{Y}^2$ and $\dot{\ell}^2$ in (3.14) are not constants. To see what effect this has we consider a lagrangian.

$$L = \frac{m}{2}\dot{x}^2 + \frac{1}{2}f(x)\dot{y}^2 \tag{3.42}$$

and we ask under what conditions on $f(x)$ will the path integral method give the same wave equation as canonical quantization. In the latter method

$$\pi_x = m\dot{x}$$

$$\pi_y = f(x)\dot{y}$$

$$H = \frac{1}{2m}\pi_x^2 + \frac{1}{2f(x)}\pi_y^2$$

So $$E\psi = -\frac{1}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{1}{2f(x)}\frac{\partial^2\psi}{\partial y^2} \tag{3.43}$$

is our wave equation

For the path integral, let

$$x(t+\varepsilon) = X \qquad y(t+\varepsilon) = Y$$

$$\dot{x} = \frac{\delta}{\varepsilon} \qquad \dot{y} = \frac{h}{\varepsilon} \tag{3.44}$$

i.e. $$x(t) = X - \delta + \frac{\delta t}{\varepsilon}$$

In the lagrangian we replace $f(x)$ by its average value in the interval and write $S_{cl} \simeq \varepsilon L_{cl}$.

Now $$\bar{f} = \frac{1}{\varepsilon}\int_0^\varepsilon f(x(t))\,dt \tag{3.45}$$

So $$\bar{f} = \bar{f}\,(X, \delta, \varepsilon)$$

And $$S_{cl} \simeq \frac{m}{2\varepsilon}\delta^2 + \frac{\bar{f}}{2\varepsilon}h^2 \tag{3.46}$$

Equation (3.39) becomes

$$\psi(x,y,t+\varepsilon) = A\int_{-\infty}^{\infty} d\delta\, d\eta\, e^{\frac{im\delta^2}{2\varepsilon}} e^{\frac{i\bar{f}\eta^2}{\varepsilon}} \psi(x-\delta, y-\eta, t) \tag{3.47}$$

$$\Rightarrow \psi + \varepsilon\frac{\partial\psi}{\partial t} = A\int_{-\infty}^{\infty} d\delta\, d\eta\, e^{\frac{im\delta^2}{2\varepsilon}} e^{\frac{i\bar{f}\eta^2}{2\varepsilon}} \Big\{ \psi - \delta\frac{\partial\psi}{\partial x} - \eta\frac{\partial\psi}{\partial y} + \delta^2\frac{\partial^2\psi}{\partial x^2} + 2\delta\eta\frac{\partial^2\psi}{\partial x\partial y} + \eta^2\frac{\partial^2\psi}{\partial y^2}\Big\}$$

and since $\bar{f}$ is a function of X, δ and ε only we can do the η integration to give

$$\psi + \varepsilon\frac{\partial\psi}{\partial t} = \sqrt{\pi}A\int_{-\infty}^{\infty} d\delta\, e^{\frac{im\delta^2}{2\varepsilon}} \left\{ \left(\psi - \delta\frac{\partial\psi}{\partial x} + \delta^2\frac{\partial^2\psi}{\partial x^2}\right)\sqrt{\frac{2i\varepsilon}{\bar{f}}} + \frac{1}{2}\frac{\partial^2\psi}{\partial x^2}\left(\frac{2i\varepsilon}{\bar{f}}\right)^{3/2}\right\}$$

To obtain the wave equation we integrate and equate terms of the same order in ε . In order for the wave equation to be identical to (3.43) we require

$$\text{(i)}\quad \sqrt{\pi}A\int_{-\infty}^{\infty} d\delta\, e^{\frac{im\delta^2}{2\varepsilon}}\left(\frac{2i\varepsilon}{\bar{f}}\right)^{1/2} = 1$$

$$\text{(ii)}\quad \sqrt{\pi}A\int_{-\infty}^{\infty} d\delta\, e^{\frac{im\delta^2}{2\varepsilon}}\,\delta\left(\frac{2i\varepsilon}{\bar{f}}\right)^{1/2} = 0$$

$$\text{(iii)}\quad \sqrt{\pi}A\int_{-\infty}^{\infty} d\delta\, e^{\frac{im\delta^2}{2\varepsilon}}\,\delta^2\left(\frac{2i\varepsilon}{\bar{f}}\right)^{1/2} = \frac{i\varepsilon}{2m}$$

$$\text{(iv)}\quad \sqrt{\pi}A\int_{-\infty}^{\infty} d\delta\, e^{\frac{im\delta^2}{2\varepsilon}}\left(\frac{2i\varepsilon}{\bar{f}}\right)^{3/2} = \frac{i\varepsilon}{2f}$$

One condition which follows immediately from these four is that $(\bar{f})^{-1/2}$ must be an even function of δ . This is already enough to rule out the bag lagrangian (3.14). This has a term $\frac{\ell}{2}\dot{X}^2$ which is equivalent to having $f(x) = x$ in (3.42). Then

$$\bar{f} = \frac{1}{\varepsilon}\int_0^{\varepsilon} x(t)\, dt$$

$$= \frac{1}{\varepsilon}\int_0^{\varepsilon}\left(X - \delta + \frac{\delta t}{\varepsilon}\right)dt$$

$$= X - \delta/2$$

$$\Rightarrow \quad (\bar{f})^{-1/2} = \sqrt{2}\,(2X - \delta)^{-1/2}$$

which is not an even function of δ

This explains why the path integral method does not reproduce the same wave equation as the canonical quantization method for our bag model.

CHAPTER FOUR

In chapter two, section three, we saw how the effects of quark-quark interactions could improve our estimates of the deuteron binding energy, but we did not consider the consequences of this for the simple scattering process described in chapter one. To do this we now add an interaction term to the Lagrangian and attempt to solve the resulting equations of motion. We shall write down an interaction between scalar fields without involving gluons or other exchange particles.

1. A simple interaction

The simplest choice is to make the interaction energy proportional to the product of the fields in the overlapping region. Although this is rather unrealistic it gives rise to nice equations of motion and allows us to see what might happen. So if ϕ_1 is the field of one bag, and ϕ_2 the field of the other, we write

$$\mathcal{H}_I = \lambda \left(\phi_1^+ \phi_2 + \phi_2^+ \phi_1 \right) \tag{4.1}$$

Then the total Lagrangian density becomes

$$\mathcal{L} = \partial_\mu \phi_1^+ \partial^\mu \phi_1 + \partial_\mu \phi_2^+ \partial^\mu \phi_2 - \lambda \left(\phi_1^+ \phi_2 + \phi_2^+ \phi_1 \right) - B \tag{4.2}$$

The equations of motion obtained from this Lagrangian are

$$\partial^2 \phi_1 = - \lambda \phi_2 \tag{4.3}$$

$$\partial^2 \phi_2 = - \lambda \phi_1$$

The boundary conditions are obtained by requiring $n_\mu T^{\mu\nu} = 0$ on the boundary, where $n_\mu n^\mu = -1$ (4.4)

Now $$T^{\mu\nu} = \sum_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_\alpha} \partial^\nu \phi_\alpha - g^{\mu\nu} \mathcal{L}$$

$$= \partial^\mu \phi_1^+ \partial^\nu \phi_1 + \partial^\mu \phi_1 \partial^\nu \phi_1^+ + \partial^\mu \phi_2^+ \partial^\nu \phi_2 + \partial^\nu \phi_2^+ \partial^\mu \phi_2$$
$$- g^{\mu\nu} (\partial_\omega \phi_1^+ \partial^\omega \phi_1 + \partial_\omega \phi_2^+ \partial^\omega \phi_2) + g^{\mu\nu} B$$
$$+ \lambda g^{\mu\nu} (\phi_1^+ \phi_2 + \phi_2^+ \phi_1)$$

So (4.4) becomes

$$n_\mu \partial^\mu \phi_1^+ \partial^\nu \phi_1 + n_\mu \partial^\mu \phi_1 \partial^\nu \phi_1^+ + n_\mu \partial^\mu \phi_2^+ \partial^\nu \phi_2 + n_\mu \partial^\mu \phi_2 \partial^\nu \phi_2^+$$
$$- n^\nu \partial_\mu \phi_1^+ \partial^\mu \phi_1 - n^\nu \partial_\mu \phi_2^+ \partial^\mu \phi_2 + n^\nu B + \lambda n^\nu (\phi_1^+ \phi_2 + \phi_2^+ \phi_1) = 0$$

The Dirichlet boundary conditions are obtained by putting

$$\partial^\nu \phi_1 = \beta_1 n^\nu , \quad \partial^\nu \phi_2 = \beta_2 n^\nu \quad \text{on the boundary, so} \tag{4.5}$$

we now have

$$- |\beta_1|^2 n^\nu - |\beta_2|^2 n^\nu + B n^\nu + \lambda n^\nu (\phi_1^+ \phi_2 + \phi_2^+ \phi_1) = 0$$

$$\Rightarrow \quad |\beta_1|^2 + |\beta_2|^2 = B + \lambda (\phi_1^+ \phi_2 + \phi_2^+ \phi_1)$$

$$\Rightarrow \quad \partial_\mu \phi_1^+ \partial^\mu \phi_1 + \partial_\mu \phi_2^+ \partial^\mu \phi_2 = - B - \lambda (\phi_1^+ \phi_2 + \phi_2^+ \phi_1) \tag{4.6}$$

Also, since $\partial^\nu \phi_1$ and $\partial^\nu \phi_2$ are proportional to n^ν on the boundary, ϕ_1 and ϕ_2 are constant on the boundary and using the argument of Wu et Al. (3) (see next chapter), this constant is zero, i.e.

$$\phi_1 = \phi_2 = 0 \qquad \text{on boundary} \tag{4.7}$$

We now consider a pair of one-quark bags moving towards each other and coming into contact at t = 0. We wish to solve equations (4.3) with boundary conditions (4.6) and (4.7) for $t > 0$.

The field of a stationary bag in its lowest excited state with end-points $z = 0, \ell$ is given by

$$\phi = \frac{1}{\sqrt{\pi}} e^{-ikt} \sin kz \qquad \text{where} \quad k = \frac{\pi}{\ell} = \sqrt{\pi B}$$

or
$$\phi = A \left\{ e^{-ik(t-z)} - e^{-ik(t+z)} \right\} \tag{4.8}$$

where $A = \dfrac{1}{2i\sqrt{\pi}}$

Taking a Lorentz transformation from the rest frame S to a frame S$'$ moving with velocity $+v$ with respect to S we obtain the field of a bag moving with velocity $-v$ in the lab frame (i.e. the centre of momentum frame)

So
$$z' = \gamma(z - vt)$$
$$t' = \gamma(t - vz)$$

or
$$z = \gamma(z' + vt') \tag{4.9}$$
$$t = \gamma(t' + vz')$$

$$\Rightarrow \; t - z = \gamma(1-v)(t' - z')$$
$$t + z = \gamma(1+v)(t' + z')$$

And so the field of bag 2, dropping primes on the lab frame variables, is

$$\phi_2 = A \left\{ e^{-ik\gamma(1-v)(t-z)} - e^{-ik\gamma(1+v)(t+z)} \right\} \tag{4.10}$$

Similarly a bag with end-points at $z = 0, -\ell$ at $t = 0$ boosted in the opposite direction is described by

$$\phi_1 = A \left\{ e^{-ik\gamma(1+v)(t-z)} - e^{-ik\gamma(1-v)(t+z)} \right\} \tag{4.11}$$

Transforming to light cone co-ordinates

$$\tau = t + z$$
$$x = t - z \tag{4.12}$$

and writing
$$\omega = k\gamma(1+v)$$
$$\omega' = k\gamma(1-v) \tag{4.13}$$

we have
$$\phi_1 = A\left\{e^{-i\omega x} - e^{-i\omega'\tau}\right\}$$
$$\phi_2 = A\left\{e^{-i\omega' x} - e^{-i\omega\tau}\right\} \tag{4.14}$$

The picture we have is

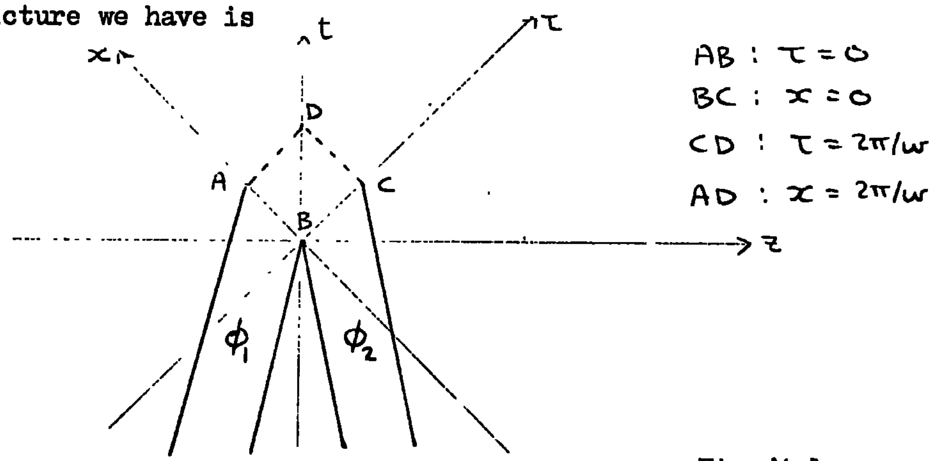


Fig. 4.1

The fields (4.14) describe the situation up to the lines AB and BC. For the square ABCD we want to solve equations (4.3) with

$$\phi_1 = A\left\{e^{-i\omega x} - 1\right\}$$
$$\phi_2 = 0 \qquad \text{on} \quad \tau = 0 \tag{4.15}$$

and
$$\phi_1 = 0$$
$$\phi_2 = A\left\{1 - e^{-i\omega\tau}\right\} \qquad \text{on} \quad x = 0 \,. \tag{4.16}$$

If we define

$$\Phi = \phi_1 + \phi_2$$
$$\Psi = \phi_1 - \phi_2 \tag{4.17}$$

then equations (4.3) become

$$\partial^2 \Phi = -\lambda \Phi$$
$$\partial^2 \Psi = +\lambda \Psi \qquad (4.18)$$

i.e.

$$\frac{\partial^2 \Phi}{\partial x \partial \tau} + \frac{\lambda}{4}\Phi = 0$$
$$\frac{\partial^2 \Psi}{\partial x \partial \tau} - \frac{\lambda}{4}\Psi = 0 \qquad (4.19)$$

with

$$\Phi = \Psi = \phi_1 \quad \text{on} \quad \tau = 0$$
$$\Phi = -\Psi = \phi_2 \quad \text{on} \quad x = 0$$

Equations (4.19) are solved using Riemann's Method (9) which tells us that given Φ on $x=0$ and on $\tau=0$ the solution of $\frac{\partial^2 \Phi}{\partial x \partial \tau} + \frac{\lambda}{4}\Phi = 0$ is

$$\Phi(x,\tau) = \left[\Phi . R\right]_{\substack{x=0\\ \tau=0}} + \int_0^{\tau}\left[R \frac{\partial \Phi}{\partial \tau'}\right]_{x'=0} d\tau' + \int_0^{x}\left[R \frac{\partial \Phi}{\partial x'}\right]_{\tau'=0} dx' \qquad (4.20)$$

where the Riemann function $R(x,\tau;\, x',\tau')$ satisfies

$$\frac{\partial^2 R}{\partial x \partial \tau} + \frac{\lambda}{4} R = 0$$

with

$$R = 1 \quad \text{on} \quad \tau = \tau' \qquad (4.21)$$
$$R = 1 \quad \text{on} \quad x = x'$$

To find R, let

$$y = x - x'$$
$$z = \tau - \tau'$$

then (4.21) becomes

$$\frac{\partial^2 R}{\partial y \partial z} + \frac{\lambda}{4} R = 0 \qquad (4.22)$$

with

$$R = 1 \quad \text{on } y = 0$$
$$R = 1 \quad \text{on } z = 0 \qquad (4.23)$$

Taking Laplace transforms of (4.22) with respect to y, we get

$$\int_0^\infty e^{-py} \frac{\partial^2 R}{\partial y \partial \gamma} dy \quad + \frac{\lambda}{4} \bar{R} \quad = 0$$

$$\Rightarrow \quad \left[\frac{\partial R}{\partial \gamma} e^{-py}\right]_0^\infty + p \int_0^\infty \frac{\partial R}{\partial \gamma} e^{-py} dy \quad + \frac{\lambda}{4} \bar{R} \quad = 0$$

$$\Rightarrow \qquad p \frac{\partial \bar{R}}{\partial \gamma} + \frac{\lambda}{4} \bar{R} \quad = 0$$

$$\Rightarrow \qquad \bar{R} = c\, e^{-\frac{\lambda}{4p} \gamma}$$

But on $\gamma = 0$, $\quad \bar{R} = \int_0^\infty e^{-p y} dy = \frac{1}{p}$

$$\Rightarrow \quad c = \frac{1}{p}$$

$$\Rightarrow \quad \bar{R} = \frac{1}{p} e^{-\frac{\lambda \gamma}{4p}}$$

and the inverse transform of this is $\quad R = J_0(\sqrt{\lambda y \gamma})$

So $\quad R(x, \tau \,;\, x', \tau') = J_0\left(\sqrt{\lambda (x - x')(\tau - \tau')}\right)$

(4.24)

So the Riemann function for Φ is

$$R_+ = J_0\left(\sqrt{\lambda (x - x')(\tau - \tau')}\right)$$

and since $J_0(ix) = I_0(x)$ the Riemann function for Ψ is

$$R_- = I_0\left(\sqrt{\lambda (x - x')(\tau - \tau')}\right) \tag{4.25}$$

Now (4.20) becomes

$$\Phi(\tau, x) = i\omega A \int_0^\tau d\tau' e^{-i\omega\tau'} J_0\left(\sqrt{\lambda x(\tau - \tau')}\right) - i\omega A \int_0^x dx' e^{-i\omega x'} J_0\left(\sqrt{\lambda \tau (x - x')}\right)$$

$$= i\omega A\, e^{-i\omega\tau} I_+(\tau, x) \;-\; i\omega A\, e^{-i\omega x} I_+(x, \tau)$$

(4.26)

where

$$I_+(a, b) \equiv \int_0^a ds\, e^{i\omega s} J_0\left(\sqrt{\lambda b s}\right)$$

Similarly we have

$$\Psi(\tau,x) = -i\omega A e^{-i\omega\tau} I_-(\tau,x) - i\omega A e^{-i\omega x} I_-(x,\tau) \tag{4.27}$$

where $I_-(a,b) \equiv \int_0^a ds\, e^{i\omega s} I_0(\sqrt{\lambda b s})$

To calculate $I_+(a,b)$ we use $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k}}{(k!)^2}$

i.e.

$$I_+(a,b) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{\lambda b}{4}\right)^k \int_0^a s^k e^{i\omega s} ds \tag{4.28}$$

Now

$$\int_0^a s^k e^{i\omega s} ds = e^{i\omega a}\left[\frac{a^k}{i\omega} - \frac{k a^{k-1}}{(i\omega)^2} + \dots + \frac{(-1)^k k!}{(i\omega)^{k+1}}\right] - \frac{(-1)^k k!}{(i\omega)^{k+1}}$$

$$= e^{i\omega a} \sum_{n=0}^{k} (-1)^n \frac{k!}{(k-n)!} \frac{a^{k-n}}{(i\omega)^{n+1}} - \frac{(-1)^k k!}{(i\omega)^{k+1}}$$

$$\Rightarrow I_+(a,b) = e^{i\omega a} \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{(-1)^{n+k}}{(i\omega)^{n+1}} \frac{a^{k-n}}{k!(k-n)!} \left(+\frac{\lambda b}{4}\right)^k - \sum_{k=0}^{\infty} \frac{\left(+\frac{\lambda b}{4}\right)^k}{k!\,(i\omega)^{k+1}} \tag{4.29}$$

and similarly

$$I_-(a,b) = e^{i\omega a} \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{(-1)^{n+k}}{(i\omega)^{n+1}} \frac{a^{k-n}}{k!(k-n)!} \left(-\frac{\lambda b}{4}\right)^k - \sum_{k=0}^{\infty} \frac{\left(-\frac{\lambda b}{4}\right)^k}{k!\,(i\omega)^{k+1}} \tag{4.30}$$

Substituting these expressions into (4.26) and (4.27) gives us Φ and Ψ, and hence ϕ_1 and ϕ_2 in ABCD. We now assume that λ is small and expand everything to first order in λ. So

$$I_+(a,b) = \frac{1}{i\omega}\left(e^{i\omega a} - 1\right) + \frac{\lambda b}{4} \frac{1}{(i\omega)^2}\left(e^{i\omega a} - 1\right) - \frac{\lambda b}{4} \frac{a}{i\omega} e^{i\omega a}$$

$$I_-(a,b) = \frac{1}{i\omega}\left(e^{i\omega a} - 1\right) - \frac{\lambda b}{4} \frac{1}{(i\omega)^2}\left(e^{i\omega a} - 1\right) + \frac{\lambda b}{4} \frac{a}{i\omega} e^{i\omega a} \tag{4.31}$$

$$\Rightarrow \quad \Phi(\tau,x) = A\left\{e^{-i\omega x} - e^{-i\omega\tau}\right\} + \frac{\lambda A}{4i\omega}\left\{x(1-e^{-i\omega\tau}) - \tau(1-e^{-i\omega x})\right\}$$

$$\Psi(\tau,x) = A\left\{e^{-i\omega x} - e^{-i\omega\tau} - 2\right\} + \frac{\lambda A}{4i\omega}\left\{x(1-e^{-i\omega\tau}) + \tau(1-e^{-i\omega x})\right\} - \frac{\lambda A\tau x}{2} \tag{4.32}$$

$$\Rightarrow \quad \phi_1(\tau,x) = A\left(e^{-i\omega x} - 1\right) + \frac{\lambda A}{4i\omega}x\left(1-e^{-i\omega\tau}\right) - \frac{\lambda A\tau x}{4}$$

$$\phi_2(\tau,x) = A\left(1-e^{-i\omega\tau}\right) - \frac{\lambda A}{4i\omega}\tau\left(1-e^{-i\omega x}\right) + \frac{\lambda A\tau x}{4} \tag{4.33}$$

The expressions (4.33) tell us what the fields are in the region

$$0 < x < 2\pi/\omega$$
$$0 < \tau < 2\pi/\omega$$

We now wish to see what happens for $\tau > 2\pi/\omega$ i.e. in the region beyond line CD in Fig. 4.1. We know that ϕ_1 and ϕ_2 must be zero on some line $x = x_1(\tau)$ passing through point C, i.e. the point $(\tau, x) = (\frac{2\pi}{\omega}, 0)$. We can make ϕ_1 and ϕ_2 zero on this line by a suitable choice of $\frac{\partial\phi_1}{\partial\tau}$ and $\frac{\partial\phi_2}{\partial\tau}$ on $x = 0$ using Riemann's method. We can then use the boundary condition (4.6) to determine the slope of $x_1(\tau)$

On $\tau = 2\pi/\omega$ we have, from (4.32),

$$\Phi = A\left(1 + \frac{\lambda\pi}{2i\omega^2}\right)\left(e^{-i\omega x} - 1\right)$$

$$\Psi = A\left(1 - \frac{\lambda\pi}{2i\omega^2}\right)\left(e^{-i\omega x} - 1\right) - \lambda A\frac{\pi}{\omega}x \tag{4.34}$$

Now Riemann's method tells us that for

$0 < x < \frac{2\pi}{\omega}$, $\tau > \frac{2\pi}{\omega}$ we have

$$\Phi = \int_0^x \left[R_+ \frac{\partial\Phi}{\partial x'}\right]_{\tau'=\frac{2\pi}{\omega}} dx' + \int_{2\pi/\omega}^{\tau} \left[R_+ \frac{\partial\Phi}{\partial\tau'}\right]_{x'=0} d\tau'$$

$$\Psi = \int_0^x \left[R_- \frac{\partial\Psi}{\partial x'}\right]_{\tau'=2\pi/\omega} dx' + \int_{2\pi/\omega}^{\tau} \left[R_- \frac{\partial\Psi}{\partial\tau'}\right]_{x'=0} d\tau'$$

$$\Rightarrow \Phi = -i\omega A\left(1+\frac{\lambda\pi}{2i\omega^2}\right) e^{-i\omega x} I_+\left(x, \tau-\frac{2\pi}{\omega}\right) + f(\tau,x)$$

$$\Psi = -i\omega A\left(1-\frac{\lambda\pi}{2i\omega^2}\right) e^{-i\omega x} I_-\left(x, \tau-\frac{2\pi}{\omega}\right) - \lambda A\frac{\pi}{\omega}x + g(\tau,x)$$

$$\Rightarrow \Phi = -A\left(1-e^{-i\omega x}\right) - \frac{\lambda A}{4i\omega}\tau\left(1-e^{-i\omega x}\right) + \frac{\lambda A}{4}x\left(\tau-\frac{2\pi}{\omega}\right) + f(\tau,x)$$

$$\Psi = -A\left(1-e^{-i\omega x}\right) + \frac{\lambda A}{4i\omega}\tau\left(1-e^{-i\omega x}\right) - \frac{\lambda A x}{4}\left(\tau+\frac{2\pi}{\omega}\right) + g(\tau,x) \tag{4.35}$$

$$\Rightarrow \phi_1 = -A\left(1-e^{-i\omega x}\right) - \lambda A\frac{\pi}{2\omega}x + \frac{f+g}{2}$$

$$\phi_2 = -\frac{\lambda A}{4i\omega}\tau\left(1-e^{-i\omega x}\right) + \frac{\lambda A}{4}\tau x + \frac{f-g}{2} \tag{4.36}$$

So f and g must be such that:

(a) $\frac{f+g}{2} = \frac{f-g}{2} = 0$ on $\tau = 2\pi/\omega$

(b) $\phi_1 = \phi_2 = 0$ on $x = x_1(\tau)$

(c) $\delta^2(f+g) = 0$ to $O(\lambda)$ (4.37)

$\delta^2(f-g) = -\lambda(f+g)$ to $O(\lambda)$

Using the fact that $x_1\left(\frac{2\pi}{\omega}\right)=0$ we see that these conditions are satisfied by

$$\frac{f+g}{2} = A\left(1-e^{-i\omega x_1(\tau)}\right) + \frac{\lambda\pi}{2\omega}A x_1(\tau)$$

$$\frac{f-g}{2} = \frac{\lambda A}{4i\omega}\tau_1(x)\left(1-e^{-i\omega x_1(\tau)}\right) - \frac{\lambda A}{4}\tau_1(x)x_1(\tau) \tag{4.38}$$

$$\Rightarrow \phi_1 = A\left(e^{-i\omega x} - e^{-i\omega x_1(\tau)}\right) - \lambda A\frac{\pi}{2\omega}\left(x - x_1(\tau)\right)$$

$$\phi_2 = -\frac{\lambda A}{4i\omega}\left\{\tau\left(1-e^{-i\omega x}\right) - \tau_1(x)\left(1-e^{-i\omega x_1(\tau)}\right)\right\} + \frac{\lambda A}{4}\left(\tau x - \tau_1(x)x_1(\tau)\right) \tag{4.39}$$

Now using (4.39) the boundary condition (4.6) becomes, to $O(\lambda)$,

$$|\partial\phi_1|^2 = -B$$

i.e. $$J = -\tfrac{1}{2}B \qquad (4.40)$$

where

$$J = \frac{\partial\phi_1^+}{\partial x}\frac{\partial\phi_1}{\partial\tau} + \frac{\partial\phi_1^+}{\partial\tau}\frac{\partial\phi_1}{\partial x}$$

From (4.39),

$$\frac{\partial\phi_1^+}{\partial x} = i\omega A^* e^{-i\omega x} - \lambda A^*\frac{\pi}{2\omega}$$

$$\frac{\partial\phi_1}{\partial\tau} = i\omega A\frac{dx_1}{d\tau}e^{-i\omega x_1(\tau)} + \lambda A\frac{\pi}{2\omega}\frac{dx_1}{d\tau}$$

So on $x = x_1(\tau)$ we have

$$J = -2\omega^2\frac{dx_1}{d\tau}|A|^2 - 2\lambda\pi|A|^2\frac{dx_1}{d\tau}\sin\omega x_1(\tau) \qquad (4.41)$$

Using (4.8), (4.13) and (4.41), equation (4.40) becomes

$$-\tfrac{1}{2}B = \frac{dx_1}{d\tau}\left\{-\frac{1}{2}\frac{\omega}{\omega'}B - \frac{1}{2}\lambda\sin\omega x_1(\tau)\right\}$$

$$\Rightarrow \quad \frac{dx_1}{d\tau} = \frac{\omega'}{\omega}\left\{1 + \frac{\lambda}{B}\frac{\omega'}{\omega}\sin\omega x_1(\tau)\right\}^{-1}.$$

$$\text{or} \quad \frac{dx_1}{d\tau} \simeq \frac{\omega'}{\omega}\left\{1 - \frac{\lambda}{B}\frac{\omega'}{\omega}\sin\omega x_1(\tau)\right\} \qquad (4.42)$$

and this is the slope of the boundary of the bag $x = x_1(\tau)$
The zeroth order term, $\frac{dx_1}{d\tau} = \frac{\omega'}{\omega}$ just gives us the slope of the boundary in Fig. 1.1, i.e. with no quark-quark interactions, as it should. This is a good check for the above calculation. The first order term adds a small oscillating motion to the boundary.
We can integrate equation (4.42) as follows:-

we have

$$\frac{dx_1}{d\tau} = \frac{\omega'}{\omega} + O(\lambda)$$

$$\Rightarrow \quad x_1(\tau) = \frac{\omega'}{\omega}\tau + O(\lambda)$$

$$\Rightarrow \quad \lambda\sin\omega x_1(\tau) = \lambda\sin\omega'\tau + O(\lambda^2)$$

and putting this back into (4.42), we obtain

$$\frac{dx_1}{d\tau} \simeq \frac{\omega'}{\omega}\left\{1 - \frac{\lambda}{B}\frac{\omega'}{\omega}\sin\omega'\tau\right\}$$

$$\Rightarrow \quad x_1(\tau) \simeq \frac{\omega'}{\omega}\tau \quad + \frac{\lambda}{B}\frac{\omega'}{\omega^2}\cos\omega'\tau \quad + \quad \text{constant}$$

but $\quad x_1\left(\frac{2\pi}{\omega}\right) = 0 \cdot$, so

$$x_1(\tau) = \frac{\omega'}{\omega}\left(\tau - \frac{2\pi}{\omega}\right) + \frac{\lambda}{B}\frac{\omega'}{\omega^2}\left(\cos\omega'\tau - \cos\frac{2\pi\omega'}{\omega}\right) \tag{4.43}$$

correct to $O(\lambda)$

We can do exactly the same calculation to find the motion of the bag in the region $x > 2\pi/\omega$.
We find that the boundary $\tau = \tau_2(x)$ is given by

$$\tau_2(x) = \frac{\omega'}{\omega}\left(x - \frac{2\pi}{\omega}\right) + \frac{\lambda}{B}\frac{\omega'}{\omega^2}\left(\cos\omega' x - \cos\frac{2\pi\omega'}{\omega}\right) \tag{4.44}$$

We can now extend Fig. 4.1 as follows:-

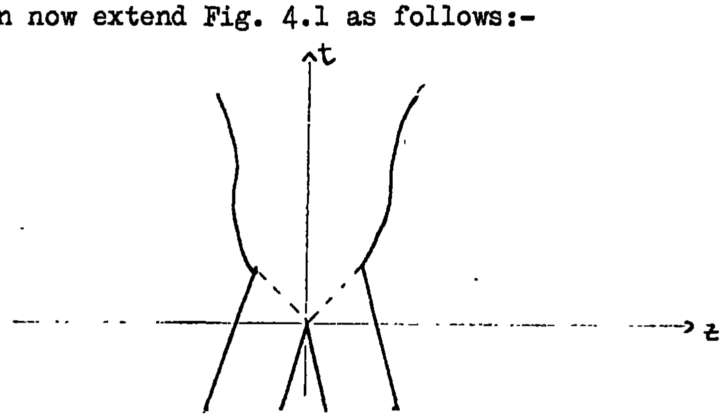


Fig. 4.2.

We do not know at the moment whether the bags split up again or not as they did in the simple scattering picture.

Note that at $x = \tau = \frac{2\pi}{\omega}$, $\phi_1 \neq 0$ and $\phi_2 \neq 0$ so they cannot split at this point, which they did before this interaction was introduced. We might think that they would split up again at a small distance, of order λ , from $x = \tau = \frac{2\pi}{\omega}$

So we put $x = \frac{2\pi}{\omega} + \varepsilon_1$, $\tau = \frac{2\pi}{\omega} + \varepsilon_2$ and require $\phi_1 = \phi_2 = 0$ using (4.39) with (4.43) and (4.44),

i.e.

$$0 = A\left\{e^{-i\omega\varepsilon_1} - 1\right\} - \lambda A \frac{\pi}{2\omega}\left(\frac{2\pi}{\omega} + \varepsilon_1\right)$$

$$0 = -\frac{\lambda A}{4i\omega}\frac{2\pi}{\omega}\left\{1 - e^{-i\omega\varepsilon_2}\right\} + \frac{\lambda A}{4}\left(\frac{2\pi}{\omega}\right)^2$$

and this implies that $\varepsilon_1 = i\lambda\frac{\pi^2}{\omega^3}$, so we cannot satisfy the boundary condition $\phi_1 = \phi_2 = 0$ at any point near $\tau = x = 2\pi/\omega$

If we analytically continue ϕ_1 and ϕ_2 given by (4.39) to the region $x > 2\pi/\omega$ we do not come to any more zeros of ϕ_1 or ϕ_2 so it seems that the bags cannot now split up but will oscillate as in Fig. 1.2. with a small sinusoidal perturbation of the boundary. So our interaction (4.1) forces them to stick together. We will now consider a more realistic type of interaction.

2. <u>A current-current interaction</u>

The standard way to write down an interaction between two charged fields is as a product of currents,

$$\mathcal{H}_I = \lambda\, j_1^{\mu} j_{2\mu} \tag{4.45}$$

where $j_1^{\mu} = i\,\phi_1^{+}\overleftrightarrow{\partial^{\mu}}\phi_1$

$$= i\left\{\phi_1^{+}\partial^{\mu}\phi_1 - \phi_1\partial^{\mu}\phi_1^{+}\right\} \tag{4.46}$$

So the Lagrangian density becomes

$$\begin{aligned}\mathcal{L} = {} & \partial_\mu\phi_1^{+}\partial^{\mu}\phi_1 + \partial_\mu\phi_2^{+}\partial^{\mu}\phi_2 - B \\ & + \lambda\left\{\phi_1^{+}\phi_2^{+}\partial_\mu\phi_1\partial^{\mu}\phi_2 - \phi_1^{+}\phi_2\partial_\mu\phi_1\partial^{\mu}\phi_2^{+}\right. \\ & \left. - \phi_2^{+}\phi_1\partial_\mu\phi_1^{+}\partial^{\mu}\phi_2 + \phi_1\phi_2\partial_\mu\phi_1^{+}\partial^{\mu}\phi_2^{+}\right\}\end{aligned} \tag{4.47}$$

The equations of motion obtained from this Lagrangian are

$$\partial^2\phi_1 = -\lambda\left\{2\phi_2\,\partial_\mu\phi_1\partial^\mu\phi_2^+ - 2\phi_2^+\partial_\mu\phi_1\partial^\mu\phi_2 + \phi_1\phi_2\,\partial^2\phi_2^+ - \phi_2^+\phi_1\,\partial^2\phi_2\right\}$$

$$\partial^2\phi_2 = -\lambda\left\{2\phi_1\,\partial_\mu\phi_2\partial^\mu\phi_1^+ - 2\phi_1^+\partial_\mu\phi_2\partial^\mu\phi_1 + \phi_1\phi_2\,\partial^2\phi_1^+ - \phi_1^+\phi_2\,\partial^2\phi_1\right\}$$

(4.48)

and the boundary conditions, obtained as in the previous section are,

$$\phi_1 = \phi_2 = 0 \qquad , \qquad (4.49)$$

$$\partial_\mu\phi_1^+\partial^\mu\phi_1 + \partial_\mu\phi_2^+\partial^\mu\phi_2 + \lambda\left\{\phi_1^+\phi_2^+\partial_\mu\phi_1\partial^\mu\phi_2 - \phi_2^+\phi_1\partial_\mu\phi_1^+\partial^\mu\phi_2\right.$$
$$\left. - \phi_1^+\phi_2\partial_\mu\phi_2^+\partial^\mu\phi_1 + \phi_1\phi_2\partial_\mu\phi_2^+\partial^\mu\phi_1^+\right\} + B = 0 \qquad (4.50)$$

The equations of motion (4.48) cannot be readily solved as in section 1, but working to $O(\lambda)$ we can put the zeroth order solutions in the right-hand-sides of (4.48).
Note that in one space dimension the Lagrangian density $\mathcal{L}$ has dimension $(\text{mass})^2$, so the field ϕ is dimensionless. This implies that the coupling constant λ is a dimensionless parameter.

The zeroth order solutions to be inserted in (4.48) are the solutions when there is no quark-quark interaction, i.e.

$$\phi_1^{(0)} = A_1\left(e^{-i\omega x} - 1\right)$$

$$\phi_2^{(0)} = A_2\left(1 - e^{-i\omega\tau}\right) \qquad (4.51)$$

for $0 < x < 2\pi/\omega$, $0 < \tau < 2\pi/\omega$.
Here $|A_1|^2 = |A_2|^2$ but A_1 and A_2 could differ by a phase so we keep the label on.

Putting (4.51) into the right-hand-side of (4.48) we obtain

$$\frac{\partial^2\phi_1}{\partial x\partial \tau} = -2\lambda\omega^2 A_1 |A_2|^2 e^{-i\omega x}(1-\cos\omega\tau)$$

$$\frac{\partial^2\phi_2}{\partial x\partial \tau} = 2\lambda\omega^2 A_2 |A_1|^2 e^{-i\omega\tau}(1-\cos\omega x) \tag{4.52}$$

These equations can be readily integrated to give

$$\phi_1 = -i\lambda\frac{\omega}{2\pi} A_1 e^{-i\omega x}(\tau - \tfrac{1}{\omega}\sin\omega\tau) + f_1(\tau) + g_1(x)$$

$$\phi_2 = i\lambda\frac{\omega}{2\pi} A_2 e^{-i\omega\tau}(x - \tfrac{1}{\omega}\sin\omega x) + f_2(\tau) + g_2(x) \tag{4.53}$$

where we have used $|A_1|^2 = |A_2|^2 = \frac{1}{4\pi}$

Now we require

$$\left.\begin{aligned}\phi_1 &= 0\\ \phi_2 &= A_2(1-e^{-i\omega\tau})\end{aligned}\right\} \quad \text{on} \quad x=0$$

$$\left.\begin{aligned}\phi_1 &= A_1(e^{-i\omega x}-1)\\ \phi_2 &= 0\end{aligned}\right\} \quad \text{on} \quad \tau=0$$

and these conditions determine f_1, g_1, f_2 and g_2

and (4.53) becomes

$$\phi_1 = A_1(e^{-i\omega x}-1) - i\lambda\frac{\omega}{2\pi} A_1 (e^{-i\omega x}-1)(\tau - \tfrac{1}{\omega}\sin\omega\tau)$$

$$\phi_2 = A_2(1-e^{-i\omega\tau}) + i\lambda\frac{\omega}{2\pi} A_2 (e^{-i\omega\tau}-1)(x - \tfrac{1}{\omega}\sin\omega x) \tag{4.54}$$

Note that at $x = \tau = \frac{2\pi}{\omega}$, $\phi_1 = \phi_2 = 0$ and that boundary condition (4.50) is also satisfied and so the bags can split again at this point.

On $\tau = 2\pi/\omega$

$$\phi_1 = A_1 \{ e^{-i\omega x} - 1 \} (1 - i\lambda)$$

$$\phi_2 = 0 \qquad (4.55)$$

So the equations of motion, together with the boundary conditions $\phi_1 = \phi_2 = 0$ on $x = x_1(\tau)$ are satisfied for $\tau > \frac{2\pi}{\omega}$ with

$$\phi_1 = A_1 (1 - i\lambda) \left(e^{-i\omega x} - e^{-i\omega x_1(\tau)} \right)$$

$$\phi_2 = 0 \qquad (4.56)$$

So now $\quad J = -\omega^2 |A_1|^2 \, 2 \cos \omega (x - x_1(\tau)) \cdot \frac{dx_1}{d\tau}$

and boundary condition (4.50) gives

$$-\tfrac{1}{2} B = -\tfrac{1}{2} B \frac{\omega}{\omega'} \frac{dx_1}{d\tau}$$

$$\Rightarrow \quad \frac{dx_1}{d\tau} = \frac{\omega'}{\omega} \qquad (4.57)$$

So the boundary now is exactly the same as it was without quark-quark interactions. The two bags just go through each other as before. The only difference is that the amplitude of the field is changed by a factor $(1 - i\lambda)$ to $O(\lambda)$. This just corresponds to a change of phase.

3. <u>Conclusion</u>

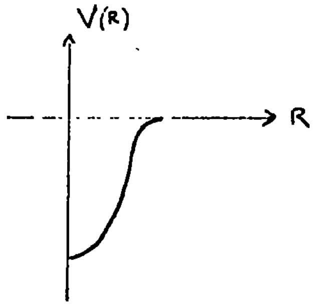
We have tried two possible types of interaction to investigate their effects. The first type is not really an interaction in the usual sense as it just corresponds to an off-diagonal mass term in the Lagrangian density. So although it gives an interesting result we do not take it seriously but just use it to show how the calculation can be done.

The second type of interaction considered is possible as a model of a physical process and so we do take it seriously. The result is surprising. It says that when quark-quark interactions are included in the bag the time taken for the scattering is not changed. This means that the "average", potential as in equation (2.16) remains the same, even although the classical potential, the potential at $R = 0$, is greatly effected by the interactions as shown in chapter two, section three. This will be discussed further in the next chapter.

CHAPTER FIVE

We now discuss the results of the previous three chapters and see what they imply about the nucleon-nucleon interaction.

We have seen that with no quark-quark interactions the potential will look like



and will give a large binding energy for the deuteron.

The effect of quark-quark interactions is to lower the classical binding energy $-V(o)$. The size of this effect depends on the model used. For the most general type of interaction discussed in chapter two the contribution to $V(o)$ could vary over quite a large range. We now consider the size of this contribution in two specific models; the MIT model and our model of chapter four, section two.

1. The MIT Model

In this model (2) the rest mass of a hadron has contributions from,

(a) the bag energy $\frac{4}{3}\pi R^3 B$

(b) the quark kinetic energies

(c) the first order quark-quark interaction, and

(d) the finite part of the zero-point energy.

These are calculated as follows:

(b) The strange quark is given a mass m_s to break $SU(3)$.

Solving the Dirac equation with the bag boundary conditions for a static sphere, the field energy is

$$E_q = \frac{nx}{R} + n_s \sqrt{m_s^2 + \frac{x_s^2}{R^2}}$$

where n = number of non-strange quarks

n_s = number of strange quarks

and x is the solution of $\tan x = \dfrac{x}{1 - mR - \sqrt{m^2R^2 + x^2}}$

We write this contribution as $E_q = \sum_i \frac{x_i(m_i)}{R}$

(c) The quark-quark interactions, mediated by an $SU(3)$ octet of coloured vector gluons, is calculated by analogy with electromagnetism. The "electric", part of the interaction is long-range and is assumed to be already included in the phenomenological bag term which gives quark confinement. The "magnetic", spin-spin interaction is given by

$$\Delta E = \sum_{i \neq j}' \Delta E_{ij}$$

where

$$\Delta E_{ij} = -\frac{g^2}{4\pi} \sum_a \lambda_i^a \sigma_i \cdot \lambda_j^a \sigma_j \frac{I(m_i, m_j)}{R}$$

$$= \frac{2g^2}{3\pi} \frac{1}{R} a_{ij} I(m_i, m_j)$$

and the a_{ij} are known for each hadron using group theory.

(d) Since the fields which occupy the hadron are quantized they will have a zero-point energy associated with them. Since this effect is divergent a cut off is introduced and the dependence of the zero-point energy on the cut off Ω is investigated. Cut-off dependent terms turn out to be proportional to R^3 and are used to renormalize B. Cut-off independent terms are found to be of the form $-Z_0/R$ where Z_0 is a positive constant which in principle can be calculated, although this is very difficult for a sphere, and so Z_0 is left as a free parameter.

So the rest mass of a hadron is now

$$M = \frac{4}{3}\pi R^3 B + \frac{1}{R}\left\{ -Z_0 + \sum_i x_i(m_i) + \frac{2g^2}{3\pi}\sum_{i\neq j} a_{ij}\, I(m_i, m_j) \right\} \tag{5.1}$$

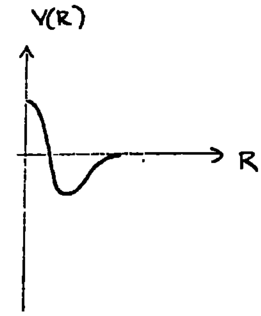
The non-linear boundary condition exists in order to balance the pressures locally at the surface. This is equivalent to minimizing M with respect to R. This leaves four free parameters : B, Z_0, m_s and $g^2/4\pi$ to be fitted to the data as mentioned in chapter one.

The energy of a six quark bag can now be calculated. The colour magnetic energy for an n-quark colour singlet nonstrange baryon is

$$\Delta E = \frac{4}{3}\frac{g^2}{4\pi}\left[n(n-6) + J(J+1) + 3I(I+1)\right]\frac{\mu}{R} \tag{5.2}$$

For n = 6 we can have (J, I) = (3, 0), (0, 3), (2, 1), (1, 2), (1, 0) and (0, 1). The lowest value of M occurs for the case $J = 1$, I = 0 which are the quantum numbers of the deuteron, and it turns out that $M_d = 2.29\ M_N$, so the deuteron is unbound, classically, in this model.

Johnson (10) then envisages a potential which is $+\ 0.29 M_N$ at R = 0, zero at R = $2R_N$ and has a region of attraction for R $< 2R_N$. The region of attraction exists because as two three-quark bags approach each other and begin to overlap they lose volume, and thus the energy decreases by $B\,\delta V$. If this region of attraction could be shown to be deep enough we would obtain a potential of the form that is usually associated with the nuclear force, i.e.



2. The one-dimensional scalar model

We now calculate the contribution of the interaction (4.45) to the classical binding energy $-V(0)$. This is given by

$$\Delta E = \int_{-\ell/2}^{\ell/2} dz\, \mathcal{H}_{INT.}$$

If we consider for simplicity two one-quark bags fusing to give a two-quark bag this becomes

$$\Delta E = -\lambda \int_{-\ell/2}^{\ell/2} \left\{ \phi_1^+\phi_2^+ \partial_\mu\phi_1 \partial^\mu\phi_2 - \phi_1^+\phi_2 \partial_\mu\phi_1 \partial^\mu\phi_2^+ - \phi_2^+\phi_1 \partial_\mu\phi_1^+ \partial^\mu\phi_2 + \phi_1\phi_2 \partial_\mu\phi_1^+ \partial^\mu\phi_2^+ \right\} \tag{5.4}$$

To $O(\lambda)$ we can use

$$\phi_1 = A_1 e^{-i\omega t}\cos\omega z$$
$$\phi_2 = A_2 e^{-i\omega t}\cos\omega z \tag{5.5}$$

where $\omega = \frac{\pi}{\ell}$, $|A_1|^2 = |A_2|^2 = \frac{1}{\pi}$ and $\ell = \sqrt{\frac{2\pi}{B}}$ is the length of a two-quark bag.

Inserting (5.5) into (5.4) we obtain

$$\Delta E = -\lambda \int_{-\ell/2}^{\ell/2} \frac{\omega^2}{\pi^2} \left\{ -2\cos^2\omega z + 2\cos^4\omega z - 2\cos^2\omega z \sin^2\omega z \right\}$$

$$= \lambda \frac{\omega^2}{\pi^2} \frac{\ell}{2}$$

$$= \frac{\lambda}{2} \sqrt{\frac{B}{2\pi}}$$

But in this model $M_N = 2\sqrt{\pi B}$, So

$$\Delta E = \lambda \frac{M_N}{4\sqrt{2}\pi}$$

$$\hat{=} \lambda . 50 \quad \text{MeV.}$$

So the interaction, calculated to $O(\lambda)$ has the effect of adding 50λ Mev to $-V(0)$ which was -500 Mev. Obviously this is not enough to give the large soft core of the MIT model, which would require $\lambda \sim 10$.

However if the result of chapter four section two, that the interaction does not change the time taken for the scattering, could be shown to hold to any order in λ, then as $-V(0)$ becomes positive the region of attraction would have to become deep and we would obtain the standard nuclear potential.

It is possible to check this point to $O(\lambda^2)$ by inserting the solutions (4.54) into the right-hand-sides of equations (4.48) and integrating. This has been done and we obtain, for $0 \leq x \leq \frac{2\pi}{\omega}$, $0 \leq \tau \leq \frac{2\pi}{\omega}$;

$$
\begin{aligned}
\phi_1 = {} & A_1\left(e^{-i\omega x}-1\right) - \frac{i}{2}\lambda A_1 \frac{\omega}{\pi}\left(e^{-i\omega x}-1\right)\left(\tau - \frac{1}{\omega}\sin\omega\tau\right) \\
& - \frac{1}{8}\lambda^2 A_1 \frac{\omega^2}{\pi^2}\left(e^{-i\omega x}-1\right)\left(\tau - \frac{1}{\omega}\sin\omega\tau\right)^2 \\
& + \frac{i}{4}\lambda^2 A_1 \frac{\omega^2}{\pi^2}\left\{\frac{1}{\omega}\sin\omega x - \frac{3}{2}x + \frac{i}{\omega}e^{-i\omega x}\left(1-\frac{e^{-i\omega x}}{4}\right) - \frac{3i}{4\omega}\right\}\left\{\frac{1}{\omega}\cos\omega\tau - \frac{1}{\omega}\right.
\end{aligned}
$$

and at the crucial point $x = \tau = 2\pi/\omega$ we obtain $\phi_1 = 0$. We can also show that $\phi_2 = 0$ and that the other boundary condition is satisfied to $O(\lambda^2)$ at this point. This means that the bags can still split at $x = \tau = 2\pi/\omega$ and we strongly suspect that this holds to all orders in λ. So in fact this model may give the desired form of the nuclear potential.

CHAPTER SIX

1. Boundary conditions and soft bags

In the original bag paper (1) the boundary conditions for a bag containing fermion fields cannot be obtained directly from the Lagrangian

$$\mathcal{L} = \tfrac{1}{2} i \bar{\psi} \gamma . \overleftrightarrow{\partial} \psi - m\bar{\psi}\psi - B$$

The problem is associated with the fact that only terms linear in the derivatives of ψ appear in the Lagrangian and is well known as the problem of solving the Dirac equation in a fixed finite region. This problem is resolved by first allowing the field to permeate all of spacetime and then proceeding to a limit in which the field becomes confined to the required region.

This method has been used by Wu et Al (3) as an alternative derivation of the boundary conditions for a scalar bag. In fact this method has to be used to obtain eqn. (1.3), i.e. $\phi = 0$ on the boundary, as the standard method only gives $\phi =$ constant on the boundary. The method consists of considering the action

$$S = \int_{bag} dV \{ \partial_\mu \phi^+ \partial^\mu \phi - B \} + \int_{not\text{-}bag} dV \{ \partial_\mu \Phi^+ \partial^\mu \Phi - M^2 \Phi^+ \Phi \} \tag{6.1}$$

obtaining boundary conditions from it, and taking the limit $M \to \infty$

Variation of S by changing ϕ and Φ and keeping the boundary fixed leads to

$$\partial^2 \phi = 0 \qquad \text{in bag} \tag{6.2}$$

$$(\partial^2 + M^2)\Phi = 0 \qquad \text{outside bag} \tag{6.3}$$

and

$$n_\mu \partial^\mu \phi = n_\mu \partial^\mu \Phi \qquad \text{on boundary.} \tag{6.4}$$

Continuity, i.e. $\phi = \Phi$ on boundary, together with (6.4) gives

$$\partial_\mu \phi = \partial_\mu \Phi \qquad \text{on boundary.} \tag{6.5}$$

Variation of the boundary gives the other boundary condition

$$\partial_\mu \phi^+ \partial^\mu \phi - B = \partial_\mu \Phi^+ \partial^\mu \Phi - M^2 \Phi^+ \Phi \qquad (6.6)$$

Putting (6.5) into (6.6) we obtain

$$|\phi|^2 = |\Phi|^2 = \frac{B}{M^2} \quad \text{on boundary,} \qquad (6.7)$$

When M is large the solution of (6.3) can be

written $\Phi = e^{Mj}$

where $|\partial_\mu j|^2 \sim -1$

Thus $|\partial_\mu \Phi|^2 \sim -M^2 |\Phi|^2$ (6.8)

The boundary conditions in the limit $M \to \infty$ are obtained from (6.7) and (6.8), i.e.

$$\phi = 0$$

$$|\partial_\mu \phi|^2 = -B \quad \text{on boundary}$$

A "soft", bag is now defined to be an extended object defined by the fields which are solutions to (6.2) and (6.3) subject to the boundary conditions for finite M. So the field is no longer zero outside the "boundary", but will fall off exponentially.

A static soft bag with "end-points", at $z = \pm \frac{\ell}{2}$ is described by the fields,

$$\phi = \left\{ \begin{array}{lll} \phi_1 = a e^{-i\omega t} e^{\Gamma z} & : & z < -\ell/2 \\ \phi_2 = b e^{-i\omega(t+z)} + c e^{-i\omega(t-z)} & : & -\frac{\ell}{2} < z < \frac{\ell}{2} \\ \phi_3 = d e^{-i\omega t} e^{-\Gamma z} & : & \frac{\ell}{2} < z \end{array} \right\} \qquad (6.9)$$

where $\Gamma^2 = M^2 - \omega^2$ (6.10)

and we assume $M, \Gamma \gg \omega$.

Continuity of ϕ and $\frac{\partial \phi}{\partial z}$ at $z = \pm \frac{\ell}{2}$ gives

$|a|^2 = |d|^2$ and $|b|^2 = |c|^2$

The mode with lowest energy is given by b = c in which case a = d and we have

$$\begin{aligned} \phi_1 &= a\, e^{-i\omega t} e^{\Gamma z} \\ \phi_2 &= A\, e^{-i\omega t} \cos \omega z \\ \phi_3 &= a\, e^{-i\omega t} e^{-\Gamma z} \end{aligned} \tag{6.11}$$

The boundary conditions

$$|\phi|^2 = \frac{B}{M^2}$$

$$|\partial_\mu \phi|^2 = -B + \frac{2B\omega^2}{M^2} \qquad \text{at} \quad z = \pm \ell/2$$

and the charge normalisation condition

$$i\int_{-\infty}^{-\ell/2} (\dot\phi_1^+ \phi_1 - \phi_1^+ \dot\phi_1) + i\int_{-\ell/2}^{\ell/2} (\dot\phi_2^+ \phi_2 - \phi_2^+ \dot\phi_2) + i\int_{\ell/2}^{\infty} (\dot\phi_3^+ \phi_3 - \phi_3^+ \dot\phi_3) = 1$$

can be used to determine $|a|^2$, $|A|^2$ and ω in terms of ℓ

2. The interactions of soft bags

We now wish to consider the effect of another bag approaching the first bag (6.11).

We work in the rest frame of the first bag, bag a, and obtain the fields of the second bag, bag b, by Lorentz transforming (6.11). If bag b comes in from $z = +\infty$ with velocity $-v$ then the part of the field which first interacts with bag a is obtained from boosting ϕ_1 in (6.11) and is

$$\phi^b = C\, e^{i\Omega t} e^{Kz} \tag{6.12}$$

where

$$\begin{aligned} i\Omega &= i\gamma\omega + v\gamma\Gamma \\ K &= i\gamma v\omega + \gamma\Gamma \end{aligned} \tag{6.13}$$

and C depends on the position of bag b at $t = 0$

When the bags are a reasonable distance apart the effect of bag b on bag a will be small and we assume that we can write the field of bag a as

$$\phi = \phi^a + \psi \tag{6.14}$$

Now ψ will have to satisfy

$$(\partial^2+M^2)\psi_1 = 0 \quad : \quad z < -\ell/2$$
$$\partial^2 \psi_2 = 0 \quad : \quad -\frac{\ell}{2} \le z \le \ell/2$$
$$(\partial^2+M^2)\psi_3 = 0 \quad : \quad \frac{\ell}{2} < z < \text{end-point of bag } b.$$

Solutions are

$$\psi_1 = b_1 e^{i\Omega t} e^{Kz}$$
$$\psi_2 = B_1 e^{i\Omega(t+z)} + B_2 e^{i\Omega(t-z)}$$
$$\psi_3 = b_2 e^{i\Omega t} e^{-Kz} + b_3 e^{i\Omega t} e^{Kz} \tag{6.15}$$

where b_3 depends on where bag b is at t = 0

Continuity of ψ and $\frac{\partial\psi}{\partial z}$ at $z = \pm\frac{\ell}{2}$ gives,

$$b_1 e^{-K\ell/2} = B_1 e^{-i\Omega\ell/2} + B_2 e^{i\Omega\ell/2}$$
$$K b_1 e^{-K\ell/2} = i\Omega B_1 e^{-i\Omega\ell/2} - i\Omega B_2 e^{i\Omega\ell/2}$$
$$b_3 e^{K\ell/2} + b_2 e^{-K\ell/2} = B_1 e^{i\Omega\ell/2} + B_2 e^{-i\Omega\ell/2}$$
$$K b_3 e^{K\ell/2} - K b_2 e^{-K\ell/2} = i\Omega B_1 e^{i\Omega\ell/2} - i\Omega B_2 e^{-i\Omega\ell/2}$$

or

$$\begin{bmatrix} e^{-i\Omega\ell/2} & e^{i\Omega\ell/2} & -e^{-K\ell/2} & 0 \\ i\Omega e^{-i\Omega\ell/2} & -i\Omega e^{+i\Omega\ell/2} & -K e^{-K\ell/2} & 0 \\ e^{i\Omega\ell/2} & e^{-i\Omega\ell/2} & 0 & -e^{-K\ell/2} \\ i\Omega e^{i\Omega\ell/2} & -i\Omega e^{-i\Omega\ell/2} & 0 & K e^{-K\ell/2} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ b_1 \\ b_2 \end{bmatrix} = b_3 e^{K\ell/2} \begin{bmatrix} 0 \\ 0 \\ 1 \\ K \end{bmatrix} \tag{6.16}$$

and this can be solved to give B_1, B_2, b_1, and b_2 in terms of b_3.

To see what (6.14) implies for the motion of bag a, we write the end-point of a nearest to b as

$$z_2(t) = \frac{\ell}{2} + \Delta z_2(t) \tag{6.17}$$

we know that $$\left|\phi_3^a\right|^2_{z=\ell/2} = \frac{B}{M^2} \tag{6.18}$$

and we use $$\left|\phi_3\right|^2_{z=\frac{\ell}{2}+\Delta z_2} = \frac{B}{M^2}$$ to find Δz_2 (6.19)

Now

$$|\phi_3|^2 = |\phi_3^a|^2 + 2\,\mathrm{Re}\, \phi_3^{+a}\psi_3 + O(\psi^2) \tag{6.20}$$

and $$\left|\phi_3^a\right|^2_{z=\frac{\ell}{2}+\Delta z_2} = \left|\phi_3^a\right|^2_{z=\ell/2} + \Delta z_2\left[\frac{\partial}{\partial z}\left|\phi_3^a\right|^2\right]_{z=\ell/2} + O(\Delta z_2^2)$$

$$\simeq \frac{B}{M^2} + \Delta z_2\left[\frac{\partial}{\partial z}\left|\phi_3^a\right|^2\right]_{z=\ell/2} \tag{6.21}$$

So (6.19) becomes

$$\frac{B}{M^2} = \frac{B}{M^2} + \Delta z_2\left[\frac{\partial}{\partial z}\left|\phi_3^a\right|^2\right]_{z=\frac{\ell}{2}} + 2\left[\mathrm{Re}\,\phi_3^{+a}\psi\right]_{z=\frac{\ell}{2}}$$

or,

$$\Delta z_2 = \frac{-2\left[\mathrm{Re}\,\phi_3^{+a}\psi_3\right]_{z=\ell/2}}{\left[\frac{\partial}{\partial z}\left|\phi_3^a\right|^2\right]_{z=\ell/2}} \tag{6.22}$$

Using (6.11) and (6.15) this becomes

$$\Delta z_2 = \frac{e^{\Omega\ell/2}}{\Gamma\left|a_2\right|^2}\,\mathrm{Re}\left\{e^{i\Omega t}e^{-i\omega t}\left(a_2^* b_2 e^{-K\ell/2} + a_2^* b_3 e^{K\ell/2}\right)\right\}$$

Inverting (6.16) we obtain $b_2 = b_3 e^{K\ell}$, so

$$\Delta z_2 = \frac{2e^{(1+\omega)\Omega\ell/2}}{\Gamma\left|a_2\right|^2}\,e^{\gamma v\Gamma t}\,\mathrm{Re}\left\{a_2^* b_3 e^{i(\gamma-1)\omega t}e^{i\gamma v\omega\ell/2}\right\} \tag{6.23}$$

= constant $\times\ e^{\gamma v\Gamma t}$ $\times$ oscillating function of t.

So the end-point $z = z_2$ oscillates about $z = \frac{\ell}{2}$ with increasing amplitude as the second bag moves nearer.

This method is only valid for small ψ, i.e. when the bags are some distance apart, and so is not very relevant to a discussion of hadronic interactions.

An alternative approach consists of writing the fields on either side of the end-point z_2 as

$$\phi_1 = \phi_1^a+\phi_1^b = A_1 e^{-i\omega(t+z)} + A_2 e^{-i\omega(t-z)} + B_1 e^{-i\Omega(t+z)} + B_2 e^{-i\Omega(t-z)}, \quad z \leq z_2$$

$$\phi_2 = \phi_2^a+\phi_2^b = a e^{-i\omega t}e^{-\Gamma z} + b e^{-i\Omega t}e^{Kz}, \quad z \geq z_2 \tag{6.24}$$

where Ω and K are given by (6.13)

We use initial conditions

$$z_2(0) = 0$$
$$\dot{z}_2(0) = 0 \qquad , \tag{6.25}$$

and continuity of $\phi^a, \phi^b, \frac{\partial\phi^a}{\partial z}$ and $\frac{\partial\phi^b}{\partial z}$ at $z = 0$ gives

$$\begin{aligned} A_1 &= \tfrac{a}{2}\left(1 - \tfrac{\Omega}{i\omega}\right) \\ A_2 &= \tfrac{a}{2}\left(1 + \tfrac{\Omega}{i\omega}\right) \\ B_1 &= \tfrac{b}{2}\left(1 + \tfrac{K}{i\omega}\right) \\ B_2 &= \tfrac{b}{2}\left(1 - \tfrac{K}{i\omega}\right) \quad . \end{aligned} \tag{6.26}$$

So at $(t, z) = (0, 0)$

$$\begin{aligned} \phi_1(0,0) &= A_1 + A_2 + B_1 + B_2 &&= a + b \\ \dot{\phi}_1(0,0) &= i\omega(A_1+A_2) + i\Omega(B_1+B_2) &&= i\omega a + i\Omega b \\ \ddot{\phi}_1(0,0) &= -\omega^2(A_1+A_2) - \Omega^2(B_1+B_2) &&= -a\omega^2 - b\Omega^2 \quad . \end{aligned} \tag{6.27}$$

The motion of the end-point for small t will be given by

$$\begin{aligned} z_2(t) &= z_2(0) + t\dot{z}_2(0) + \tfrac{1}{2}t^2\ddot{z}_2(0) + \ldots\ldots \\ &\simeq \tfrac{1}{2}t^2\ddot{z}_2(0) \end{aligned} \tag{6.28}$$

Now we know that $|\phi|^2_{t=0,z=0} = \frac{B}{M^2}$ and we wish to find $z_2(t)$ such that $|\phi|^2_{z=z_2} = \frac{B}{M^2}$

We can write, to $O(t^2)$,

$$\begin{aligned} |\phi|^2_{z=z_2} = {} & |\phi|^2_{0,0} + z_2\left[\tfrac{\partial}{\partial z}|\phi|^2\right]_{0,0} + t\left[\tfrac{\partial}{\partial t}|\phi|^2\right]_{0,0} + \tfrac{1}{2}z_2^2\left[\tfrac{\partial^2}{\partial z^2}|\phi|^2\right]_{0,0} \\ & + tz_2\left[\tfrac{\partial^2}{\partial z\partial t}|\phi|^2\right]_{0,0} + \tfrac{1}{2}t^2\left[\tfrac{\partial^2}{\partial t^2}|\phi|^2\right]_{0,0} \end{aligned} \tag{6.29}$$

and using (6.28) we see that the fourth and fifth terms are of higher order and can be dropped. Also, becuase of our initial conditions (6.25), we have

$$\left[\frac{\partial}{\partial t}|\phi|^2\right]_{0,0} = 0 \tag{6.30}$$

and so (6.29) gives,

$$\ddot{z}_2(0) = -\frac{\left[\frac{\partial^2}{\partial t^2}|\phi|^2\right]_{0,0}}{\left[\frac{\partial}{\partial z}|\phi|^2\right]_{0,0}} \tag{6.31}$$

To evaluate the expressions in (6.31) we need one useful relation obtained from (6.30) :

$$\left[\frac{\partial}{\partial t}|\phi|^2\right]_{0,0} = 0$$

$$\Rightarrow 2\,\mathrm{Re}\left[\phi^*\dot{\phi}\right]_{0,0} = 0$$

$$\Rightarrow \quad \mathrm{Re}\left\{(a+b)^*(i\omega a + i\Omega b)\right\} = 0$$

Without loss of generality we can take a to be real and write $b = b_r + i b_i$

$$\Rightarrow \quad \mathrm{Re}\left\{i\Omega a(b_r + ib_i) + i\omega a(b_r - ib_i) + i\Omega|b|^2\right\} = 0$$

and using (6.13) this becomes

$$ab_r\gamma v\Gamma - ab_i\gamma\omega + ab_i\omega + |b|^2\gamma v\Gamma = 0 \tag{6.32}$$

Now to evaluate (6.31):

$$\left[\frac{\partial^2}{\partial t^2}|\phi|^2\right]_{0,0} = -2\,\mathrm{Re}\left\{(a+b)(a\omega^2 + b\Omega^2)^*\right\} + 2\,|a\omega + b\Omega|^2$$

$$= \gamma^2\left[(ab_r + |b|^2)(\omega^2 - v^2\Gamma^2) + 2ab_i v\omega\Gamma - |b|^2(\omega^2 + v^2\Gamma^2)\right]$$

$$-\gamma\, 2\omega a(\omega b_r + v\Gamma b_i) \quad + \quad ab_r\omega^2$$

but from (6.32) $ab_r + |b|^2 = \dfrac{ab_i\omega}{v\Gamma} - \dfrac{ab_i\omega}{v\gamma\Gamma}$ so

$$\left[\frac{\partial^2}{\partial t^2}|\phi|^2\right]_{0,0} = \gamma^2\left[\frac{ab_i\omega^2}{v\Gamma} + ab_i v\omega\Gamma - |b|^2(\omega^2+v^2\Gamma^2)\right]$$
$$- \gamma\left[ab_i\omega v\Gamma + \frac{ab_i\omega^3}{v\Gamma} + 2ab_r\omega^2\right]$$
$$+ ab_r\omega^2 \tag{6.33}$$

Similarly, we find,

$$\left[\frac{\partial}{\partial t}|\phi|^2\right]_{0,0} = \frac{ab_i\omega}{v\gamma} - \left[\frac{ab_i\omega}{v} + a^2\Gamma + ab_r\Gamma\right] \tag{6.34}$$

so we can now write down $\ddot{z}_2(0)$ although it is not easy to see exactly what it means. However in the high energy limit, i.e. large γ we get

$$\ddot{z}_2(0) \simeq \frac{\gamma^2\left[\frac{ab_i\omega^2}{v\Gamma} + ab_i\omega v\Gamma - |b|^2(\omega^2+v^2\Gamma^2)\right]}{-\left[\frac{ab_i\omega}{v} + (a^2+ab_r)\Gamma\right]}$$

and using $\Gamma \gg \omega$ this reduces to

$$\ddot{z}_2(0) \simeq \frac{\gamma^2 v^2 |b|^2\Gamma}{a^2 + ab_r} \tag{6.35}$$

Now we know that to satisfy the boundary conditions we need:

$$|a|^2 = \frac{B}{M^2} \quad \text{and} \quad |b|^2 = \frac{B}{M^2}e^{-2\gamma\Gamma d}$$

where d is the distance between the bags at t = 0.

Thus $|b| < a$ and so $a^2 + ab_r > 0$ and hence $\ddot{z}_2(0)$ is always positive.

This suggests that there is an initial attraction between the two bags.

3. Conclusion

If we choose to believe that the MIT bag model arises as some limit of a more fundamental field theory, for example Creutz (11) has shown that the MIT bag can be obtained from the SLAC bag in this way, then in this theory the hadron will not have sharp boundaries and any calculation of interactions will have to take this into account. We have seen in this chapter that this is unlikely to be easy.

CHAPTER SEVEN

1. Summary

We have attempted to find out whether or not the MIT bag model is as successful in predicting the scattering properties of hadrons as it is in predicting their static properties. Although we have worked mostly in one space dimension for simplicity we have found that the model does possess some of the required properties. Using the deuteron as the canonical example of interacting nucleons we have shown that, taking quantum mechanical effects and quark-quark interactions into account, the bag model can reproduce a nucleon-nucleon potential of the required form. We have also tried to make the quantum mechanical treatment more complete and have shown that qualitatively this has the effect of reducing the binding energy as required. Finally we have found that for bags without sharp boundaries it is very difficult to obtain any conclusions about their scattering properties.

2. Conclusion

Any model of hadron structure and quark confinement which hopes to be successful must be able to say something about the strong interactions of hadrons. We hope that the above results have indicated that the MIT bag model has some chance. The next step must be to consider the collisions of three-dimensional quark/gluon bags. Some work in this direction has been done by Low (12) and he has shown that the model accounts qualitatively for the properties of constant total cross sections, zero real parts of scattering amplitudes and Feynman scaling.

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