Twist-spun knots

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July 1984

Submitted for the degree of Doctor of Philosophy

University of Durham

ABSTRACT

We attempt to answer two questions; for q greater than one, when is a simple (2q-1)-knot the branched cyclic cover of another such knot? and, for q sufficiently large to ensure the existence of an appropriate classification theorem, when is a (2q)-knot the m-twist-spin of such a knot? The methods used will be mainly algebraic, including some arising from the theory of projective modules over an integral group ring. The work is original except where references indicate otherwise; part of chapter 1 has been published previously as [St].

INTRODUCTION

In this thesis we consider two related questions; when is an odd-dimensional simple knot the branched cyclic cover of another knot? and when is an even-dimensional knot the m-twist-spin of an odd-dimensional one? The second question has already been considered by R.A.Litherland in [Li] in order to show that we may use a construction of R.H.Fox called "rolling" in conjunction with the twist-spinning construction to obtain knots which cannot be obtained by simply twist-spinning a knot; in our case the question is motivated by work of Cherry Kearton on spun knots in [K3]. We work in the piecewise linear category throughout; and all embeddings and isotopies will be assumed to be locally flat. The methods used are mostly algebraic, making use of the classification theorems of Kearton and Kojima ([K1],[Ko],[K2]); unfortunately the geometric investigations have been less successful (see chapter 5 especially).

In the first chapter we obtain an algebraic condition for an odd-dimensional simple knot to be the m-fold branched cyclic cover of another, and give two sufficient conditions on the module of a knot which ensure that the knot may only be the m-fold branched cyclic cover of finitely many knots for any value of m. In the second we describe Zeeman's definition of the twist-spinning construction, and we show how to calculate the Alexander modules of a twist-spun knot via an exact sequence which is due to Milnor. We apply this to the twist spinning of the simple knots whose modules may be considered as modules over certain Dedekind domains, described by Eva Bayer in [Ba]; and, in the special case of knots whose modules are modules over the ring of integers of some cyclotomic field, we calculate the modules in such a way as to be able to compare them.

Chapter 3 attacks the second question stated above. Necessary conditions for a module to belong to an m-twist spun simple knot are derived, and we introduce two special classes of finite knot modules in the hope of being able to identify exactly which knots whose modules fall into these classes may be obtained by the twist-spinning process. We also investigate the Levine, or torsion, pairing associated with a twist-spun knot, and show that in each of the special cases we consider, this never provides any further obstruction to a knot's being twist-spun.

Chapter 4 introduces some algebraic machinery to tackle the most sensitive obstruction to a knot's being twist-spun, namely that the projection of its order ideal to the ring $Z[t,t^{-1}]/(t^m-1)$ must be principal and generated by a self-conjugate element; and we give two examples to show that this obstruction is not vacuous.

In chapter 5 we investigate the case of a simple even-dimensional knot whose modules are Z-torsion free, and attempt to analyze the geometry of the situation; the results are incomplete, all following from the simplest necessary condition derived in chapter 3. In the appendix we have collected together some algebraic results which assist various calculations, referenced as (Ri) or (Ai) in the text, where i denotes an integer.

Due to the printer used to produce this thesis, some of the notation is slightly non-standard; in particular, the capital letters Q and Z will always be used to denote the field of rational numbers and the integers respectively.

It only remains to express my thanks; to my supervisor, Cherry Kearton, for suggesting these two questions, and for the support and encouragement he has given me throughout my stay in Durham; to Steve Wilson, who patiently introduced me to much of the number theory, especially that used in chapter 4; and to the computing services at Durham and Newcastle for the use of their facilities, which I have made use of for various calculations, and in producing this document.

CONTENTS

1 BRANCHED CYCLIC COVERS OF SIMPLE KNOTS

1.1 DEFINITIONS

Let $\underline{k}:S^n \longrightarrow S^{n+2}$ denote an n-knot, N the interior of a regular neighbourhood of $\underline{k}(S^n)$, and K the exterior of the knot, which will be $S^{n+2}\backslash N$. $\underline{k}(S^n)$ has a trivial normal bundle, so the closure of N is homeomorphic to $S^n \times B^2$. By Alexander duality, $H_*(K)\cong H_*(S^1)$; so we may form the infinite cyclic cover $\tilde{K}$ of K corresponding to the kernel of the Hurewicz homomorphism $\pi_1(K) \longrightarrow H_1(K)\cong Z$; and the finite cyclic cover $\tilde{K}_m$ corresponding to the kernel of the composition of this map with the natural surjection $Z \rightarrow Z_m$. To construct the m-fold branched cyclic cover K_m, we glue the m-fold cyclic cover of N, branched along $\underline{k}$, into $\tilde{K}_m$; the branch set then gives us a natural embedding $\underline{k}_m:S^n \rightarrow K_m$. In the first chapter we will be interested in the case where K_m is another (n+2)-sphere, so that $\underline{k}_m$ is a knot.

The invariants of the knot which will interest us (and, in the simple case, classify the knot type) will be associated with the infinite cyclic cover $\tilde{K}$, together with the homeomorphism t generating the group of covering translations. t will also be used (ambiguously) to denote the generator of the ring $\Lambda = Z[t,t^{-1}]$ of Laurent polynomials and the induced homomorphism of chain groups and homology groups of the various covering spaces of K, taking t to be the identity on the knot itself. The Alexander modules of k are the homology groups of $\tilde{K}$, considered as Λ-modules via the action of t.

To determine whether K_m is a sphere, we use the following short exact sequence of chain groups;

$$0 \longrightarrow C_*(\tilde{K}) \xrightarrow{t^m-1} C_*(\tilde{K}) \longrightarrow C_*(\tilde{K}_m) \longrightarrow 0$$

which leads to Milnor's long exact sequence of homology modules; ([M1])

$$\cdots \longrightarrow H_q(\tilde{K}) \xrightarrow{t^m-1} H_q(\tilde{K}) \longrightarrow H_q(\tilde{K}_m) \longrightarrow H_{q-1}(\tilde{K}) \longrightarrow \cdots$$

$$\longrightarrow H_1(\tilde{K}) \xrightarrow{t^m-1} H_1(\tilde{K}) \longrightarrow H_1(\tilde{K}_m) \longrightarrow H_0(\tilde{K}) \longrightarrow H_0(\tilde{K})$$

If K_m is a sphere, then $\tilde{K}_m$ must be a homology S^1, by Alexander duality; so $1-t^m$ must act as an automorphism of $H_q(\tilde{K})$ for all q>0. Conversely, if $1-t^m$ does so act, then K_m, which can be obtained from K by attaching a two-handle (killing the first homology group), and an (n+2)-handle, will be a homology sphere. If we have $\pi_1(K) \cong Z$, then the fundamental group of $\tilde{K}_m$ will also be infinite cyclic, and thus be killed by attaching the two-handle; so that K_m will be a simply connected homology sphere, hence a homotopy sphere. Using the generalised Poincare conjecture, we then have the following (probably already known) result;

Proposition 1/Provided $n+2 \geq 5$ and $\pi_1(K) \cong Z$, then the m-fold branched cyclic cover $\underline{k}_m$ of $\underline{k}$ will be a genuine knot if and only if $1-t^m$ is an automorphism of $H_q(\tilde{K})$ for q>0.//

Suppose now we have the situation where $\underline{k}$ is the m-fold branched cyclic cover (b.c.c.) of a knot $\underline{l}$. The infinite cyclic cover $\tilde{L}$ of $\underline{l}$ is the same as $\tilde{K}$; but it is acted on by a group of covering translations generated by a homeomorphism u, which we may choose so that $u^m=t$. We will similarly use u to denote induced maps, and as a generator of $Z[u,u^{-1}]$. In attempting to characterise which knots $\underline{k}$ can arise as branched cyclic covers, we shall be trying to construct suitable candidates for $\underline{l}$. We shall therefore work with classes of knots which have been classified by algebraic information, the best known examples being the so-called simple knots.

1.2 SIMPLE KNOTS

We have already noted that the exterior of a knot is a homology S^1; by Levine's unknotting theorem, if $\pi_i(K) \cong \pi_i(S^1)$ for $i \leq q$, where $\underline{k}$ is either a (2q-1)- or a (2q)-knot, then $\underline{k}$

is unknotted, i.e. $\underline{k}(S^n)$ bounds a disc in S^{n+2}. We say $\underline{k}$ is simple if $\pi_i(K)\cong\pi_i(S^1)$ for $i<q$. We deduce that $\pi_i(\tilde{K})=0$ for $i<q$; then the first Alexander module of $\underline{k}$ which may be non-trivial is $H_q(\tilde{K})\cong\pi_q(K)$, by the Hurewicz theorem. Additional information which enables us to calculate the higher dimensional Alexander modules and to classify the odd-dimensional knots is summed up in two duality pairings; one on the torsion-free parts F_k of $H_k(\tilde{K})$; and the other on the torsion parts, denoted T_k. Both are described by Levine in [L2]. We shall write the two pairings as follows;

$$\langle\, ,\rangle : F_k \times F_{n-k+1} \longrightarrow Q(t)/\Lambda$$

$$[\, ,] : T_k \times T_{n-k} \longrightarrow Q/Z$$

Suppose A is any Λ-module with Λ acting on the left by $(\lambda,a)\longmapsto\lambda a$. We define the conjugate module $\bar{A}$ to have the same underlying abelian group structure as A, but with Λ acting on the right by $(a,\lambda)\longmapsto\bar{\lambda}a$, where this bar denotes the involution of Λ induced by $t\longmapsto t^{-1}$ (and also the involution inherited by quotient modules of Λ). The above pairings then give duality isomorphisms;

$$\overline{F}_k \cong \mathrm{Hom}(F_{n-k+1}, Q(t)/\Lambda)$$

$$\text{by } x \longmapsto (y \longmapsto \langle x,y\rangle)$$

$$\text{and } \overline{T}_k = \mathrm{Hom}(T_{n-k}, Q/Z)$$

$$\text{by } a \longmapsto (b \longmapsto [a,b])$$

These duality isomorpisms show that $H_q(\widetilde{K})$ is the only non-trivial Alexander module if n=2q-1 is odd, when it must also be torsion-free; if n=2q, then $H_q(\widetilde{K})$ and $H_{q+1}(\widetilde{K})$ are the only two possibly non-zero modules, with the latter necessarily torsion-free.

The torsion pairing is complicated to describe in general; a simpler description for the case where k is a fibred even-dimensional knot will be used in chapter 2. The description of the torsion-free pairing given by Blanchfield will be used later in this chapter when we come to calculate the pairing of an odd-dimensional simple knot given the pairing of its branched cyclic cover; since odd-dimensional simple knots are classified by their module and pairing ([Kl],[T]), this will effectively enable us to deduce the type of such a knot from that of its branched cyclic cover.

1.3 RELATIONSHIP BETWEEN THE MODULES AND PAIRINGS

We return to the situation we had at the end of section (1.1), where $\underline{k}$ is the m-fold branched cyclic cover of $\underline{l}$. We may take their infinite cyclic covers to be both $\tilde{K}$, with corresponding generators of the groups of covering translations given by t and u respectively, where $t=u^m$. The Alexander modules of $\underline{l}$ will then be the same as those for $\underline{k}$, but considered as modules over $Z[u,u^{-1}]$ via the induced action of u. Suppose that $\underline{k}$ and $\underline{l}$ are both odd-dimensional simple knots (if one is, they clearly both are); and denote their non-zero modules by A_t and A_u respectively. We may write down their Blanchfield pairings $\langle,\rangle_t$ and $\langle,\rangle_u$ as follows.

Take two elements a,b of $A_t=H_q(\tilde{K})$. A_t is a Λ-torsion module, so there exists a non-zero $\underline{p}(t)\in\Lambda$ such that $\underline{p}a=0$. Choose a triangulation of $\tilde{K}$ induced by a triangulation of the exterior of $\underline{l}$ (so that it will be acted on by u); and let C_q be the group of q-chains, and $\tilde{C}_{q+1}$ the group of (q+1)-chains in the dual triangulation. As $\underline{p}a=0$, we may choose an element $\tilde{a}$ of $\tilde{C}_{q+1}$ whose boundary represents $\underline{p}a$; we also choose $\beta\in C_q$ representing b. Then we define ([B],[Kl]);

$$\langle a,b\rangle_t = \Big(\sum_{i=-\infty}^{\infty} I(\tilde{a},t^i\beta)t^i\Big)/\underline{p}(t)\in Q(t)/Z[t,t^{-1}]$$

$$\langle a,b\rangle_u = (\sum_{i=-\infty}^{\infty} I(\tilde{a},u^i \tilde{b})u^i)/\underline{p}(u^m) \in Q(u)/Z[u,u^{-1}]$$

where I(,) denotes intersection numbers of chains. We now group the second infinite sum into groups of powers of u having the same value modulo m. Let M be a complete set of representatives of the integers modulo m, for instance $\{0,1,\ldots,m-1\}$. Then we have;

$$\langle a,b\rangle_u = (\sum_{k\in M} u^k \sum_{i=-\infty}^{\infty} I(\tilde{a},u^{im+k}\tilde{b}))(u^{im}))/\underline{p}(u^m)$$

$$= \sum u^k (\sum I(\tilde{a},t^i(u^k\tilde{b}))(u^m)^i)/\underline{p}(u^m)$$

$$= \sum_{k\in M} u^k . \theta\langle a,u^k b\rangle_t$$

where $\theta : Q(t)/Z[t,t^{-1}] \longrightarrow Q(u)/Z[u,u^{-1}]$ is defined by

$$f(t) \longmapsto f(u^m).$$

1.4 WHICH ODD-DIMENSIONAL SIMPLE KNOTS ARE BRANCHED CYCLIC COVERS?

Given such a knot $\underline{k}$ and a homeomorphism u of $\tilde{K}$ such that $u^m=t$, we now know what the module and pairing structure would have to be of a knot whose m-fold b.c.c. was $\underline{k}$, and whose infinite cyclic cover (when identified with $\tilde{K}$) had u as a generator of the group of covering translations. If these new structures satisfy the Levine axioms, given below, then we know that we can construct a knot $\underline{l}$ with these invariants. The m-fold b.c.c. of $\underline{l}$ would then have the same module and pairing as $\underline{k}$, and so be equivalent by the classification theorem given below;

<u>Theorem 2/</u>([Kl])Given any odd-dimensional simple knot $\underline{k}: S^{2q-1} \hookrightarrow S^{2q+1}$, the module A and pairing $\langle , \rangle$ satisfy the Levine axioms as follows;

(L1) A is a finitely generated, Λ-torsion module.

(L2) Multiplication by (1-t) is an automorphism of A.

(L3) $\langle , \rangle$ is $(-1)^{q+1}$-Hermitian, that is;

$$\langle a,b\rangle = (-1)^{q+1}\overline{\langle b,a\rangle}$$

& $t\langle a,b\rangle = \langle ta,b\rangle$.

(L4) $\langle , \rangle$ is non-singular, so that the adjoint map;

$$\overline{A} \longrightarrow \mathrm{Hom}\,(A, Q(t)/\Lambda) \text{ given by}$$

$$a \longmapsto (x \longmapsto \langle x,a\rangle)$$

is an isomorphism.

Furthermore, these conditions characterize the modules and pairings which can arise, together with the condition that the signature of the corresponding quadratic pairing (as in [T]) must be divisible by 16 if q=2. //

To derive the algebraic conditions for $\underline{k}$ to be the m-fold b.c.c. of a knot, we shall need to use the following trick repeatedly;

<u>Lemma 3/</u>Suppose that M is a set of integers having distinct values modulo m, and we are given that;

$$\left(\sum_{k\in M}\sum_{i=-\infty}^{\infty} a_{k+im}\, u^{k+im}\right)/\underline{p}(u^{m})=0 \text{ in } Q(u)/Z[u,u^{-1}]$$

Then, for all $k\in M$;

$$\left(\sum_{i=-\infty}^{\infty} a_{k+im}\, t^{i}\right)/\underline{p}(t) =0 \text{ in } Q(t)/Z[t,t^{-1}]$$

<u>Proof</u> Since $\underline{p}(u^{m}) \mid \sum_{i=-\infty}^{\infty} b_i u^{i}$

$$\Leftrightarrow \underline{p}(u^{m}) \mid \sum_{i=k \pmod{m}} b_i u^{i} \quad \forall k\in Z \;//$$

If the module A_u and pairing $\langle,\rangle_u$ defined above satisfy the Levine conditions, we may construct a knot $\underline{l}$ corresponding to them. Let $\underline{k}'$ denote the m-fold b.c.c. of $\underline{l}$, which is a spherical knot by proposition 1, noting that $1-u^{m}$ is an automorphism of A_u. The Alexander module of $\underline{k}'$ is clearly A_t; and if $\langle,\rangle_t'$ is its Blanchfield pairing, we have;

$$\sum_{k=0}^{m-1} u^{k}\,\theta\langle a,u^{k} b\rangle_t' = \langle a,b\rangle_u = \sum_{k=0}^{m-1} u^{k}\,\theta\langle a,u^{k} b\rangle_t$$

for all $a,b\in A_t$. Using lemma 3 for k=0 we see that $\langle a,b\rangle_t=\langle a,b\rangle_t'$; so $\underline{k}$ and $\underline{k}'$ are equivalent, and $\underline{k}$ does arise as an m-fold b.c.c. The condition we require is as follows;

Theorem 4/ $(A_u,\langle,\rangle_u)$ satisfies the Levine conditions (and hence $\underline{k}$ is an m-fold b.c.c.) if and only if u acts as an isometry of $(A_t,\langle,\rangle_t)$ with $u^m=t$.

Proof. Firstly the "only if" part. For $\langle,\rangle_u$ to satisfy (L3) we must have, for all $a,b\in A_t$;

$$\langle a,b\rangle_u = e\overline{\langle b,a\rangle}_u \quad (e=(-1)^{q+1})$$

$$\Leftrightarrow \sum_{k=0}^{m-1} u^k\theta\langle a,u^k b\rangle_t = e\sum_{k=0}^{m-1} u^{-k}\theta\overline{\langle b,u^k a\rangle}_t$$

$$= \sum_{k=0}^{m-1} u^{-k}\theta\langle u^k a,b\rangle_t \qquad \text{as } \langle,\rangle_t \text{ is e-Hermitian}$$

$$\Leftrightarrow \langle a,u^k b\rangle_t = t^{-1}\langle u^{m-k}a,b\rangle_t \quad \text{for all } 0\leqslant k<m \text{ by lemma 3}$$

$$\Leftrightarrow \langle a,u^k b\rangle_t = \langle u^{-k}a,b\rangle_t \qquad \text{for all } 0\leqslant k<m$$

$$\Leftrightarrow u \text{ is an isometry of } \langle,\rangle_t .$$

Conversely, A is clearly finitely generated; and 1-u is an automorphism since;

$$(1-u).(1+u+\ldots+u^{m-1})=1-u^m=1-t.$$

We have already proved the first half of (L3) above; for the second half we have;

$$\langle ua,b\rangle_u = \sum_{k=0}^{m-1} u^k\theta\langle ua,u^k b\rangle_t$$

$$= \sum_{k=0}^{m-1} u^k \theta\langle a, u^{k-1} b\rangle_t \quad \text{as } u \text{ is an isometry}$$

$$= u.\sum_{k=-1}^{m-2} u^k \theta\langle a, u^k b\rangle_t = u.\langle a,b\rangle_u .$$

Finally we prove (L4) in two parts;

<u>(i) The adjoint map is injective.</u>

Suppose $\langle a,b\rangle_u = 0$ for all a in A_u. Then;

$$\sum_{k=0}^{m-1} u^k \theta\langle a, u^k b\rangle_t = 0 \text{ for all a in } A_t$$

$\Rightarrow \langle a,b\rangle_t = 0$ for all a in A_t (putting k=0 in lemma 3)

$\Rightarrow$ b = 0 as required, since $\langle , \rangle_t$ is non-singular

<u>(ii) The adjoint map is surjective.</u>

Suppose $a \longmapsto f_a(u)$ is a u-linear map

$$A_u \longrightarrow Q(u)/Z[u,u^{-1}]$$

If $\Delta(t)$ is the Alexander polynomial of <u>k</u>, then $\Delta(u^m)$ will kill A_u, and we may write;

$$f_a(u) = [\sum_{k=0}^{m-1} u^k f_a^{(k)}(u^m)]/\Delta(u^m)$$

for some integer polynomials $f_a^{(k)}$.

Then $a \longmapsto f_a^{(0)}(t)/\Delta(t)$ is a t-linear map

$$A_t \longrightarrow Q(t)/\Lambda$$

So, as $<,>_t$ is non-singular, there is an x in A_t such that;

$$<a,x>_t = f_a^{(0)}(t)/\Delta(t)$$

Then we have;

$$<a,x>_u = \sum_{k=0}^{m-1} u^k.\theta<a,u^k x>_t$$

$$= \sum u^k.\theta<u^{-k} a,x>_t \quad \text{as u is an isometry}$$

$$=[\sum u^k f_{u\,a}^{(0)}(u^m)]/\Delta(u^m)$$

$$=[\sum_{k=0}^{m-1} u^k f_a^{(k)}(u^m)]/\Delta(u^m) \text{ as } a \longmapsto f_a(u) \text{ is u-linear}$$

$$= f_a \text{ as required.}$$

We have proved that if u is an isometry of $<,>_t$, then $<,>_u$ is a genuine Blanchfield pairing; and this suffices to prove the theorem for q>2. Suppose now that $\underline{k}$ is a 3-knot. As the Levine conditions depend only on the value of q modulo 2, the above proof shovs that $<,>_u$ is the Blanchfield pairing of a 7-knot $\underline{l}_7$, whose m-fold cover $\underline{k}_7$ has pairing $<,>_t$. Any lift of a 2-connected Seifert surface of $\underline{l}_7$ to the complement of

$\underline{k}_7$ will be a Seifert surface for $\underline{k}_7$. It follows that the quadratic pairings associated with the pairings $\langle,\rangle_u$ and $\langle,\rangle_t$ will have the same signature, namely that of the surface; so it follows that $\langle,\rangle_u$ is the Blanchfield pairing of a 3-knot, completing the proof. //

Ideas used in the lemma and the above proof enable us to define a set of maps;

$$\nu_\kappa : Q(u)/Z[u,u^{-1}] \longrightarrow Q(t)/Z[t,t^{-1}] \qquad \kappa \in Z_m$$

Suppose we had an element of $Q(u)/Z[u,u^{-1}]$, expressed as the quotient of two integral polynomials $f(u)/d(u)$. We form the polynomial;

$$\delta(u) = \prod d(\zeta u)$$

where the product runs over all the mth roots of unity ζ. The coefficients of δ, being symmetric functions of the roots of the polynomial $1-x^m$, are all integers; and since $\delta(\zeta u)=\delta(u)$ for any mth root of unity ζ, $\delta(u)$ must be of the form $\Delta(u^m)$ where Δ is an integral polynomial. So we have;

$$f(u)/d(u) = f(u).\prod d(\zeta u)/\Delta(u^m) \quad \text{(product over } \zeta^m=1, \zeta\neq 1)$$

$$= [\sum_{k=0}^{m-1} u^k \phi_k(u^m)]/\Delta(u^m) \text{ say,}$$

and we define $\nu_\kappa(f(u)/d(u)) = \phi_\kappa(t)/\Delta(t)$, where κ is the image of k under the natural projection $Z \longrightarrow Z_m$. Adapting the proof of lemma 3, we see that ν_K is well-defined. So, by the calculation at the end of section three, we see that we may write $<,>_t$ in terms of $<,>_u$ as;

$$<a,b>_t = \nu_0 <a,b>_u$$

where 0 denotes the identity element of Z_m.

This map will also be useful in calculating the torsion pairing of a twist-spun knot in the third chapter.

1.5 HOW MANY DIFFERENT KNOTS IS K AN M-FOLD B.C.C. OF?

We may translate this question into an algebraic one by means of the following theorem;

Theorem 5/Suppose u and v are two isometries of $(A_t,<,>_t)$ with $u^m=t=v^m$. Then $(A_u,<,>_u)$ and $(A_v,<,>_v)$ are isometric, and hence correspond to equivalent knots, if and only if u and v are conjugate by an isometry of $(A_t,<,>_t)$.

Proof We write both $<,>_u$ and $<,>_v$ with values in $Q(u)/Z[u,u^{-1}]$ Then;

$$\phi:(A_u,<,>_u) \longrightarrow (A_v,<,>_v) \text{ is an isometry}$$

$$\Longleftrightarrow <a,b>_u = <\phi a,\phi b>_v \text{ for all } a,b \in A_u$$

$$\Leftrightarrow \sum_{k=0}^{m-1} u^k \theta\langle a, u^k b\rangle_t = \sum_{k=0}^{m-1} u^k \theta\langle \phi a, v^k \phi b\rangle_t$$

$$\Leftrightarrow \langle a, u^k b\rangle_t = \langle \phi a, v^k \phi b\rangle_t \quad \text{for all } a,b, 0 \leqslant k < m \text{ by lemma 3}$$

$$\Rightarrow \phi \text{ is an isometry of } \langle,\rangle_t \qquad (k=0)$$

and taking k=1, we see that $u=\phi^{-1}v\phi$, since $\langle,\rangle_t$ is non-singular. //

So an equivalent algebraic question is to ask how many conjugacy classes of elements u with $u^m=t$ there are in the isometry group of $\langle,\rangle_t$. In certain cases we can show this number is finite; the following proposition arises from Jonathan Hillman's work in [H1];

<u>Proposition 6/</u>If a simple (2q-1)-knot <u>k</u>, $q \geqslant 2$ has the same Alexander polynomial and minimal polynomial, then it may only be the m-fold branched cyclic cover of finitely many distinct knots.

<u>Proof</u> Let M be the Alexander module of <u>k</u>. We shall in fact show that the group of automorphisms Aut(M) contains only finitely many elements with $u^m=t$, which we will do by considering successively more complex forms for M. We firstly note that our polynomial condition is equivalent to insisting that the second Alexander polynomial (Δ_1 in Jonathan Hillman's notation) is equal to one.

Suppose first that $M \cong \Lambda/(p(t))^j$ where p is irreducible; we shall prove the proposition in this case by induction on j. The case j=1 follows from lemma 4 of [H1], since Aut(M) is just the group of units U(M) ($M^\times$ in Hillman's notation), considering M with its natural ring structure inherited from Λ. Suppose we have proved the proposition for j=h, and $u_1,\ldots,u_s \in \Lambda$ satisfy;

$(\pi_h(u_i))^m = \pi_h(t)$ $(\pi_j : \Lambda \dashrightarrow \Lambda/(p(t))^j$ the natural projection)

with the $\pi_h(u_i)$ being the only mth roots of t in $\mathrm{Aut}(\Lambda/p^h)$. Then if $\pi_{h+1}(u)$ is to be an mth root of t in $\mathrm{Aut}(\Lambda/p^{h+1})$, we must have

$$u=u_i+f(t)(p(t))^h \quad \text{for some } i \text{ and } f(t).$$

Suppose that;

$$u_i^m=t+g(t)(p(t))^h$$

Then

$$\begin{aligned} u^m &= u_i^m+m.u_i^{m-1}f(t)p(t)^h \\ &= t+p(t)^h(g(t)+m.u_i^{m-1}f(t)) \\ &= t \pmod{p^{h+1}} \end{aligned}$$

for $\pi_{h+1}(u)$ to be such an mth root. If a suitable f can be found, then it will be uniquely determined (mod p(t)); so there is at most one choice for u (mod p^{h+1}) for each i, and the number of mth roots of t in $\mathrm{Aut}(\Lambda/p^{h+1})$ is no greater than the number in $\mathrm{Aut}(\Lambda/p^{h})$.

Suppose now that $M \cong \Lambda/(f(t))$, where f is a composite polynomial. Any element of $\mathrm{Aut}(M) \cong U(\Lambda/f(t))$ will be uniquely determined by its projections onto $U(\Lambda/(p(t))^{j})$ for all the factors $(p(t))^{j}$ of f which are powers of irreducible polynomials; and the projections of mth roots of t must also be mth roots of t. By the first case there are only finitely many of these in each primary factor; so there are only finitely many in Aut(M).

Finally, we cover the general case where we only know that $\Delta_1(M)=1$. By Crowell's result, the annihilator ideal of M in Λ is principal, generated by f(t), say. By theorem 2 of [H1], Aut(M) contains $U(\Lambda/f)$ as a subgroup of finite index, and is itself Abelian. Any two roots of t in the same coset will then have a quotient which is an mth root of unity in $U(\Lambda/f)$; and since the above reasoning may be readily adapted to show that there are only finitely many of these (the same number as there are mth roots of t, in fact), there are only finitely many mth roots of t in each of the finite number of cosets, which completes the proof. //

We may derive another finiteness condition from the work of E. Bayer and F. Michel [BM];

Proposition 7/If a simple (2q-1)-knot k, $q \geqslant 2$, has Alexander module M, annihilated by a squarefree minimal polynomial, then it may only be the m-fold branched cyclic cover of finitely many knots.

Proof Suppose k were the m-fold b.c.c. of a knot l with minimal polynomial $\mu(u)$, where $u^m=t$. Over the rationals, the Alexander module of l tensored with Q splits up as a direct sum of terms;

$$\oplus_i N_i \cong \oplus_i Q[u,u^{-1}]/(f_i(u))$$

where the f_i's are powers of irreducible polynomials over Q, since $Q[u,u^{-1}]$ is a P.I.D. If μ, which is a lowest common multiple in $Z[u,u^{-1}]$ of the f_i's, is not squarefree, then one of these polynomials, say f_k, must be a non-trivial power $(g(t))^h$.

Now the Alexander module of k tensored with Q will be $Q\otimes M=\oplus N_i$, considered as a $Q[t,t^{-1}]$-module in the obvious way, by the reasoning of section 3 of this chapter. We claim that the module N_k thus considered will have a minimal polynomial with a square factor. We define, as in the last section;

$$G(t)=G(u^m)=\prod_{\zeta^m=1} g(\zeta u)$$

and we distinguish two cases;

(i) g has no pair of roots a,b with $a^m=b^m$

In this case, since G(t) is a polynomial of the same degree as g(u), whose roots are the mth powers of the roots of g, we deduce that G has no repeated roots. Define;

$$N^{(i)} = \{x \in N_j : g(u)^i x=0\}$$

$$\cong Q[u,u^{-1}]/(g(u))^{\min(i,h)} \quad \text{as a } Z[u,u^{-1}]\text{-module.}$$

Now $N^{(1)}$ will certainly be annihilated by G(t); so, as deg(g)=deg(G), the only way it can have a square-free minimum polynomial is if it is isomorphic as a $Q[t,t^{-1}]$-module to $Q[t,t^{-1}]/(G(t))$. So $N^{(1)}$ is a cyclic module over this ring, generated by x, say. Pick $y \in N^{(2)}$ such that g(u)y=x. Now y is annihilated by $G(t)^2$, but not by G(t), as x was a generator; so $N^{(2)}$ must have a minimal polynomial, over $Q[t,t^{-1}]$, with a square factor.

(ii) g has a pair of roots a,b with $a^m=b^m$

This time we must have $a=\zeta b$, where ζ is some non-trivial mth root of unity; then a will be a root of the polynomial $G(u^m)/g(u)$, and we must have $g(u)^2|G(u^m)$, since g is irreducible. So the polynomials;

$$G(u^m) \text{ and } \frac{d}{du}G(u^m) = mu^{m-1}G'(u^m)$$

must have a common root, an mth power of which will be a common root of $G'(t)$ and $G(t)$; so $G(t)$ must have a square factor. Again $\deg(g)=\deg(G)$, and $G(t)$ annihilates

$$N^{(2)} \cong Q[u,u^{-1}]/(g(u))^2 \quad \text{(as a } Z[u,u^{-1}]\text{-module)};$$

so, by considering its dimension as a Q-vector-space, the minimum polynomial over $Q[t,t^{-1}]$ must have a square factor.

Thus we conclude that if $\underline{k}$ has a squarefree minimum polynomial, which is an invariant of the rational homology module, then any knot whose m-fold b.c.c is $\underline{k}$ will also have a squarefree minimum polynomial. How many choices are there for this minimum polynomial $\mu(u)$? Since $\prod \mu(\zeta u)$ must annihilate the module, μ must be a factor of $\Delta(u^m)$, where Δ is the Alexander polynomial of $\underline{k}$; so there are only finitely many possible choices. The rank of the isometric structure corresponding to $\underline{l}$ will be the same as that for $\underline{k}$ (being dim $Q \otimes M$); so, by [BM], there are only finitely many isometric structures (which all have unit determinant) corresponding to each choice of μ; and hence finitely many knots altogether wich may have $\underline{k}$ as their m-fold b.c.c. //

It is natural to ask whether any simple knot may be the m-fold b.c.c. of infinitely many distinct knots; however, we have been unable make any further progress on this question.

2 TWIST SPINNING

2.1 DEFINITION

We revert to the notations at the beginning of the first chapter, where $\underline{k}:S^n \hookrightarrow S^{n+2}$ was any n-knot. The twist spinning construction was described by Zeeman in [Z]; and the description which follows is taken from this paper.

Pick a point $x_0 \in \underline{k}(S^n)$; since our embeddings are locally flat, x_0 has a closed neighbourhood $X \cong B^{n+2}$ such that $(X, X \cap \underline{k}(S^n))$ is an unknotted ball pair. Removing the interior of X leaves us with another ball pair, (D^{n+2}, D^n), whose boundary is unknotted. We consider ∂D^{n+2} as $B^n \times S^1$, where $\partial B^n \times S^1$ is identified to $\partial B^n = \partial D^n$; this enables us to parametrize the space as pairs (x,ϕ), where $x \in B^n$ and $\phi \in S^1$, so that $(x,\phi)=(x,\phi')$ whenever $x \in \partial B^n$. We parametrize D^2 by polar coordinates (r,θ).

We now form two (n+3)-dimensional 'solid torus pairs' by;

$$Y = \partial(D^{n+2}, D^n) \times D^2 = (\partial D^{n+2} \times D^2, \partial D^n \times D^2)$$

$$Z = (D^{n+2}, D^n) \times \partial D^2 = (D^{n+2} \times \partial D^2, D^n \times \partial D^2)$$

so $\partial Y = \partial Z = (\partial D^{n+2}, \partial D^{n}) \times \partial D^{2}$; and we may parametrize points in these boundaries by triples (x,ϕ,θ) derived from the parametrizations for (D^{n+2}, D^{n}) and D^{2}. We join Y and Z along their boundaries by the map;

$$f: \quad \partial Y \longrightarrow \partial Z$$
$$(x,\phi,\theta) \longmapsto (x,\phi+m\theta,\theta)$$

This map is a map of pairs, ie. $f(\partial D^{n} \times \partial D^{2}) = \partial D^{n} \times \partial D^{2}$; and by Zeeman's lemma 4, the result is a smooth pair of spheres (S^{n+3}, S^{n+1}), which we may consider as a knot $\underline{l}: S^{n+1} \hookrightarrow S^{n+3}$, which is the m-twist spin of $\underline{k}$. Zeeman's main theorem then goes as follows;

Theorem 8/Provided $m \neq 0$, there is a bundle;

$$(K_m \backslash B^{n+2}) \longrightarrow (S^{n+3} \backslash \underline{l}(S^{n+1})) \longrightarrow S^{1}$$

with group Z_m, whose generator may be taken to be induced by the covering translation of the m-fold branched cyclic cover K_m of $\underline{k}$. Further, the closure F of the fibre is a smoothly embedded surface bounded by the knot $\underline{l}(S^{n+1})$. Finally, S^{1} acts on S^{n+3} in such a way as to leave S^{n+1} setwise fixed, rotating it once about an unknotted S^{n-1}. //

So $\underline{l}$ is a fibred knot, with fibre F; its infinite cyclic cover $\tilde{L}$ is homeomorphic to $F \times \mathbb{R}$, whence $H_i(\tilde{L}) \cong H_i(F)$ for all i. Suppose now that $\underline{k}$ is once more a simple (2q-1)-knot;

Milnor's exact sequence shows that $H_i(F)=0$ for $i \neq q, q+1$, and reduces to;

$$0 \longrightarrow H_{q-1}(F) \longrightarrow H_q(\tilde{K}) \xrightarrow{t^m-1} H_q(\tilde{K}) \longrightarrow H_q(F) \longrightarrow 0$$

which enables us to calculate the Alexander modules of $\underline{1}$ as the cokernel and kernel of the map t^m-1.

2.2 FOX'S FORMULA

The two duality pairings;

$$\langle , \rangle : F_k \times F_{(2q+1)-k} \longrightarrow Q(t)/\Lambda$$

$$[,] : T_k \times T_{2q-k} \longrightarrow Q/Z$$

enable us to make some deductions about the two non-trivial Alexander modules of our twist-spun simple knot (and, indeed, about any even-dimensional simple knot). The existence of the non-singular torsion pairing for k=q+1 shows that $H_{q+1}(F)$ is always torsion-free; then the pairing on the torsion-free part gives us the duality isomorphism;

$$\overline{H_{q+1}(F)} \cong \mathrm{Hom}(F_q, Q(t)/\Lambda) \cong \mathrm{Hom}(H_q(K), Q(t)/\Lambda)$$

In particular, if $H_q(F)$ is a Z-torsion module, then $H_{q+1}(F)=0$, and our even-dimensional knot only has one non-zero module; we call such a knot a <u>finite simple (2q)-knot</u>. We may compute the order of $H_q(F)$ by a formula of

Fox proved by Claude Weber in [We], noting that his proof, given for classical knots, only uses Milnor's sequence, together with the properties of the Alexander module which come from the Levine axioms; and these both apply to any odd-dimensional simple knots. The theorem then goes as follows; if Δ is the middle-dimensional Alexander polynomial of k (ie. a generator of the order ideal of $H_q(\tilde{K})$), then the order of $H_q(F)$ is given by the formula;

$$|H_q(F)| = |R(t^m-1,\Delta(t))|$$

where R(,) denotes the resultant (see appendix A). If the resultant is zero, we interpret this to mean that the module is infinite; this will happen if and only if t^m-1 and $\Delta(t)$ have a common factor, by (R2).

In general, the rational invariants of k can yield no more information about the structure of the torsion submodule of $H_q(F)$, as the following example shows;

Example I/Let $M = \Lambda/(t^2-t+1) \oplus \Lambda/(t^2-3t+1)$ and

$$N = \Lambda/(t^2-t+1)(t^2-3t+1)$$

be the Alexander modules of two odd-dimensional simple knots (which can only be (2q-1)-knots for some odd q, from the form of the Alexander polynomials ([L1]); however, in the other dimensions we could use $M\oplus M$ and $N\oplus N$). Tensoring with the rationals yields two isomorphic modules, so the rational invariants are the same; however the qth Alexander modules of the 6-twist spins will be respectively;

$$M/(1-t^6)M \cong \Lambda/(t^6-1,t^2-t+1) \oplus \Lambda/(t^6-1,t^2-3t+1) \qquad (A3)$$
$$\cong \Lambda/(t^2-t+1) \qquad \oplus \Lambda/(t^6-1,t^2-3t+1)$$
$$N/(1-t^6)N \cong \Lambda/(t^6-1,(t^2-t+1)(t^2-3t+1)) \qquad (A3)$$
$$\cong \Lambda/(t^2-t+1)((t^2+t+1)(t+1)(t-1),t^2-3t+1)$$

So the orders of the torsion submodules are respectively $|R(t^6-1,t^2-3t+1)|$ and $|R((t^6-1)/(t^2-t+1),t^2-3t+1)|$ by A1, which differ by a factor of;

$|R(t^2-t+1,t^2-3t+1)|=|R(t^2-t+1,2t)|=4$.

<u>Example II</u>/Claude Weber uses the example of the (3,3,3)-pretzel knot; its Alexander polynomial is $7t^2-13t+7$, which is irreducible; so its rational homology is cyclic as a $Q[t,t^{-1}]$-module. He calculates the homology of the two-fold cover to be $Z_3 \oplus Z_9$. However, if we look at the two-fold cover of a knot whose Alexander module is cyclic with the same Alexander polynomial, then, by lemma A3, we may write the homology of the two-fold cover as;

$$\Lambda/(7t^2-13t+7,t^2-1) \cong \Lambda/(7t^2-13t+7,t+1)(7t^2-13t+7,t-1) \qquad (A4)$$
$$\cong \Lambda/(27,t+1)(1,t-1) \cong \Lambda/(27,t+1)$$

and the homology group is Z_{27}. We note that these knots are capable of having distinct homology groups, because their common order $\Delta(-1)$ is not square-free ($\Delta(t)=7t^2-13t+7$); and this observation also applies to the other example given in [We], of the two knots 6_1 and 9_{46} whose common Alexander polynomial is $2t^2-5t+2$. Now, for a quadratic Alexander polynomial $\Delta(t)$, the discriminant is $\Delta(-1)$ (modulo a sign); and as Levine proves in section 31 of [L3], this discriminant's being square-free is necessary and sufficient to ensure that the ring $\Lambda/(\Delta(t))$ is a Dedekind domain. In the case of a knot whose module is annihilated by a polynomial $d(t)$ such that $\Lambda/(d(t))$ is a Dedekind domain, we have the positive result of the next section.

2.3 TWIST SPINNING DEDEKIND KNOTS

Proposition 9/Suppose that $R=\Lambda/(d(t))$ is a Dedekind domain, and that the knot module $A=H_q(\tilde{K})$ of the odd-dimensional simple knot $\underline{k}$ is a module of rank r over R, which is necessarily Z-torsion-free because of the Levine axioms. Then the homology module $H_q(F)$ of the m-twist spin of $\underline{k}$ is a direct sum of r copies of $\Lambda/(t^m-1,d(t))$, provided $d\nmid t^m-1$.

Proof By the structure theorem for torsion-free modules over Dedekind domains ([C], page 413), we can write A as a direct sum;

$$A \cong R \oplus R \oplus \ldots \oplus I$$

where there are r summands, and I is an ideal of R determined uniquely up to its ideal class. From Milnor's exact sequence;

$$H_q(F) \cong A/(t^m-1)A \cong R/(t^m-1)R \oplus R/(t^m-1) \oplus \ldots \oplus I/(t^m-1)I$$

$$\cong \Lambda/(t^m-1,d(t)) \oplus \Lambda/(t^m-1,d(t)) \oplus \ldots \oplus I/(t^m-1)I$$

so it only remains to prove that;

$$I/(t^m-1)I \cong R/(t^m-1)R$$

Now, by corollary 3 on page 411 of [C], we may multiply I by a unit in the field of fractions of R to get a new ideal I' in the same ideal class as I, but with $I'+(t-1)=R$ (so I' must be an integral ideal). Then we have;

$$R/(t^m-1)R \cong I'/(I' \cap (t^m-1)R) \quad \text{(isomorphism theorem)}$$

$$\cong I'/(t^m-1)I' \quad \text{by (A2), as desired. //}$$

The class of cyclotomic polynomials provides examples of polynomials d(t) such that $\Lambda/(d(t))$ is Dedekind; in what follows we shall calculate the modules of twist-spins of knots having cyclotomic minimal polynomials, in such a way as to be able to compare the results. If we denote the nth cyclotomic polynomial, whose roots are the primitive nth roots of unity, by $\Phi_n(t)$, there are some restrictions we must place on n, and on the rank of the module, to ensure that there are (2q-1)-dimensional simple knots whose modules are annihilated by Φ_n, by the results of [L1]. Firstly, in order that $\Phi_n(1)=1$, we must ensure that n is not a prime power. If q is odd, this will suffice, as Levine's results show. If q is even, we must ensure that $\Delta(-1)$ is a square, where Δ is the Alexander polynomial, by theorem 1 of [L1]. As we see in example 5.2 of [Ba], $\Phi_n(-1)=1$ provided n is not of the form $2p^i$; if it is, $\Phi_n(-1) = \Phi_{p^i}(1) = p$, so any module annihilated by Φ_n must be of even rank over $\Lambda/(\Phi_n)$, if it is to belong to such a knot. The other conditions which Levine derives do not apply to us, since we are living in the PL category.

In any event, proposition 9 shows us that if <u>k</u> has an Alexander module A of rank r over $\Lambda/(\Phi_n)$, its m-twist spin will have as its qth Alexander module a direct sum of r copies of;

$$\Lambda/(t^m-1,\Phi_n(t)) \quad \text{provided } \Phi_n \nmid t^m-1,$$

ie. provided $n \nmid m$. Denoting this summand by $A_{n,m}$, we may derive the following properties;

(i) If I is any ideal of Λ, and we have t^k-1 and $t^h-1 \in I$, then;

$$t^k-1 - t^{k-h}(t^h-1) = t^{k-h}-1 \in I$$

so we may apply Euclid's algorithm to deduce that $t^{(k,h)}-1 \in I$ where (k,h) denotes the highest common factor of k and h.
So $A_{n,m} = A_{n,(n,m)}$; and we shall assume from now on that m divides n.

(ii) We may use the remainder theorem to deduce that there exists a polynomial $f \in \Lambda$ such that;

$$(t-1)f(t)+\Phi_k(t) = \Phi_k(1) = \begin{cases} 1 \text{ for } k \text{ not a prime power} \\ p \text{ for } k=p^i,\ p \text{ prime.} \end{cases}$$

Now the mth power of a primitive (km)th root of unity is clearly a primitive kth root of unity; so $\Phi_{km}(t) \mid \Phi_k(t^m)$, and there exists a $g \in \Lambda$ such that;

$$(t^m-1)f(t^m)+\Phi_k(t^m) = (t^m-1)f(t^m)+\Phi_{km}(t)g(t)$$

$$= \begin{cases} 1, & k \text{ not a prime power} \\ p, & k=p^i. \end{cases}$$

So we have;

<u>Lemma 10/</u> $1 \in (\Phi_n(t), t^m-1)$ if n/m is not a prime power

$p \in (\Phi_n(t), t^m-1)$ if $n/m=p^i$.//

So $A_{n,m}=0$ if n/m is not a prime power; let us assume that $n=p^i m=p^k m'$, where p is prime, $p \nmid m'$. Then the primitive nth roots of unity are precisely those complex numbers whose (p^k)th powers are primitive (m')th roots of unity, but whose (p^{k-1})st powers are not. So we may write;

$$\Phi_n(t) = \Phi_{m'}(t^{p^k})/\Phi_{m'}(t^{p^{k-1}})$$

$$=[\Phi_{m'}(t)]^{p^k}/[\Phi_{m'}(t)]^{p^{k-1}} \pmod p$$

$$=[\Phi_{m'}(t)]^{(p-1)p^{k-1}}$$

Now, let $t^{m'}-1 = \Phi_{m'}(t)\Psi_{m'}(t)$. We claim;

<u>Lemma 11/</u> $\Phi_{m'}$ is coprime to $\Psi_{m'}$ (mod p) (ie. $(p,\phi_{m'},\Psi_{m'})=\Lambda$)

<u>Proof</u> $\Phi_{m'}(t) = \prod_{d|m'} (t^{d}-1)^{\mu(m'/d)}$

where μ is the Moebius function. So $\Psi_{m'}(t)$ is a quotient of factors of the form $(t^{d}-1)$ for d strictly dividing m'; as $p\nmid m'$, m'/d cannot be a power of p. So there exist polynomials $f_d, g_d \in \Lambda$ such that;

$$(t^{d}-1)f_d(t) + \Phi_m(t)g_d(t) = q_d$$

where q_d is either one or a prime not equal to p, by lemma 10. Thus $(p,t^{d}-1,\Phi_{m'})=\Lambda$ for all proper divisors d of m'. So

$$(p,\Psi_{m'},\Phi_{m'}) \supset \prod_{\substack{d|m' \\ d\neq m'}} (p,t^{d}-1,\Phi_{m'}) = \Lambda. \;//$$

We may then compute the ideal;

$$(\Phi_n(t),t^{m}-1) = (p,\Phi_n(t),t^{m}-1) \qquad \text{(lemma 9)}$$

$$= (p,(\Phi_{m'})^{(p-1)p^{k-1}},(t^{m'}-1)^{p^{k-l}})$$

$$= (p,(\Phi_{m'})^{(p-1)p^{k-1}},(\Phi_{m'}\Psi_{m'})^{p^{k-l}})$$

Now, by lemma 11 there exist polynomials $f,g,h \in \Lambda$ such that;

$$fp + g\Phi_{m'} + h\Psi_{m'} = 1.$$

Raising to the $((p-1)p^{k-1})$st power and grouping all the terms divisible by p together gives us polynomials f',g',h' in Λ with

$$f'p + g'(\Phi_{m'})^{(p-1)p^{k-1}} + h'(\Psi_{m'})^{(p-1)p^{k-1}} = 1.$$

Multiplying by $(\Phi_{m'})^{p^{k-l}}$ shows us that this element lies in our ideal (as $(p-1)p^{k-1} \geqslant p^{k-l}$ because $l>0$); so in fact;

$$(\Phi_n(t), t^m - 1) = (p, (\Phi_{m'}(t))^{p^{k-l}}).$$

To summarise;

$$A_{n,m} = A_{n,(n,m)}.$$

If $m|n$; $A_{n,m} = 0$ if n/m is not a prime power,

and $A_{n,m} = \Lambda/(p, \Phi_{m'}(t)^{p^j})$ if n/m is a power of a prime p,

and j is the highest power of p dividing $m = p^j m'$.

These ideas, together with lemma A1, allow us to calculate the absolute value of the resultant of two cyclotomic polynomials as follows;

Proposition 12/Let $n > m \geqslant 1$ be two integers. Then;

$$|R(\Phi_n,\Phi_m)| = \begin{cases} 1 & \text{if } n/m \text{ is not a prime power} \\ p^{\phi(m)} & \text{if } n/m \text{ is a power of a prime } p \end{cases}$$

where $\phi(m) = \deg \Phi_m(t)$ is Euler's function.

<u>Proof</u> If $m|n$, then we know by lemma 9 that $(\Phi_m,\Phi_n) \supset (t^m-1,\Phi_n)$ which contains p if $n/m=p^h$, and contains 1 if n/m is not a prime power. So if m divides n, but m/n is not a prime power, then $|R(\Phi_m,\Phi_n)|=1$, using lemma A1. If $n=mp^h$, we have;

$$\Phi_n(t) = \begin{cases} \Phi_m(t^{p^h}) & \text{if } p|m \\ \Phi_m(t^{p^h}) / \Phi_m(t^{p^{h-1}}) & \text{if } p\nmid m. \end{cases}$$

In either case, since $\Phi_m(t^{p^k})=(\Phi_m(t))^{p^k} \pmod p$, we find that $\Phi_m|\Phi_n \pmod p$, so;

$$(\Phi_n,\Phi_m) = (p,\Phi_n,\Phi_m) = (p,\Phi_m)$$

and $|R(\Phi_n,\Phi_m)| = |R(p,\Phi_m)| = p^{\phi(m)}$ by (R1).

Finally, if m does not divide n, so that $(m,n)\neq m$, we use the fact that;

$$|R(t^m-1,\Phi_n)| = |A_{n,m}| = |A_{n,(n,m)}| = |R(t^{(m,n)}-1,\Phi_n)|$$

So, as Φ_m is a factor of t^m-1 but not of $t^{(m,n)}-1$, it must have unit resultant with Φ_n, by the multiplicative property of the resultant (R3). //

This result can also be deduced from theorem 25.26 and isomorphism 25.28 of [CR]. Although the results of these calculations differ from those in [S], we only part company on the third line from the bottom of page 29; in his notation, $m=p^k n'$, p does not divide n', and;

$$\frac{p^{\phi(m)/\phi(p^i)}}{p^{\phi(m)/\phi(p^{i+1})}} = p^{(p-1)p^{k-1}\phi(n')[(p-1)/p^i(p-1)]}$$

$$= p^{(p-1)p^{k-i-1}\phi(n')} = p^{\phi(n)} \text{ as above.}$$

In fact, using properties (R5) and (R6), it is possible to show that if $n > m \geqslant 1$, then $R(\Phi_n,\Phi_m) > 0$ unless n=2 and m=1, thus giving us the actual values of these resultants.

3 CHARACTERIZING FINITE TWIST-SPUN KNOTS.

3.1 ELEMENTARY IDEALS

Suppose that A is an Alexander module of a knot. From Levine's first axiom, A is a finitely-generated Λ-module; so we may find an exact sequence of the form;

$$\Lambda^n \xrightarrow{f} \Lambda^r \longrightarrow A \longrightarrow 0$$

and f may be specified by an r by n matrix (a_{ij}) over Λ. Since A always is a Λ-torsion module, n must be greater than or equal to r. We define the elementary ideals E_k of A to be the ideals of Λ generated by the (r-k)x(r-k) minors of (a_{ij}); and these are invariants of the module. Now suppose A is the qth Alexander module of a simple (2q-1)-knot; the qth Alexander module of the m-twist spin of this knot will be given by the Milnor exact sequence as $B \cong A/(1-t^m)A$. Now A may be given a square presentation matrix over Λ (for instance, a Seifert matrix plus or minus t times its transpose); then B can be presented by the exact sequence;

$$\Lambda^{2r} \xrightarrow{g} \Lambda^r \longrightarrow B \longrightarrow 0$$

where g is given by the matrix;

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} & 1-t^m & & & \\ \vdots & & & \vdots & & 1-t^m & & 0 \\ \vdots & & & \vdots & & & \ddots & \\ a_{r1} & \cdots & \cdots & a_{rr} & & 0 & & 1-t^m \end{pmatrix}$$

Then we may write the elementary ideals F_i of B by;

$$F_0 = E_0 + (1-t^m)E_1 + \cdots\cdots + (1-t^m)^r$$

$$F_1 = E_1 + (1-t^m)E_2 + \cdots (1-t^m)^{r-1} \qquad \text{etc.}$$

The conditions that we will be able to set on a finite knot module in order that it may be the module of a twist-spun knot will all follow from the next proposition;

Proposition 13/Suppose that B is the qth Alexander module of an m-twist spun simple (2q-1)-knot $\underline{k}$. Then B must be annihilated by $1-t^m$; and if π is the projection map $\Lambda \longrightarrow \Lambda/(t^m-1)$, and F_0 the zeroth elementary ideal (order ideal) of B, then πF_0 is a principal ideal, generated by the image of the Alexander polynomial $\Delta(t)$ of $\underline{k}$.

Proof The first part follows immediately from Milnor's exact sequence; the second from the fact that the order ideal E_0 of the Alexander module of $\underline{k}$ is principal and generated by the Alexander polynomial, together with the above formula for

F_o. //

<u>Corollary 14</u>/Denote by ϕ_d the projections $\Lambda \longrightarrow \Delta/(\Phi_d)$, where Φ_d is the dth cyclotomic polynomial, and by N_d the compositions of ϕ_d with the norm map into the integers; when we apply this map to ideals in the ring of cyclotomic integers, the result will be taken to be the positive integer generating the norm of the ideal. Then, with other notations as above, if B is to be the qth Alexander module of an m-twist spun knot, we must have;

$$|B| = \prod_{d|m} N_d(F_0)$$

<u>Proof</u> By proposition 13, πF_o must be principal and generated by $\pi(\Delta(t))$; so since each ϕ_d can be factored through π, $\phi_d F_o$ must be principal and generated by $\phi_d(\Delta(t))$ for each d|m. Then;

$$\prod_{d|m} N_d(F_0) = \prod_{d|m} N_d(\Delta(t)).$$

But $N_d(\Delta(t)) = R(\Phi_d, \Delta(t))$ (A8)

$$\Longrightarrow \prod_{d|m} N_d(F_0) = \prod_{d|m} |R(\Phi_d, \Delta(t))|$$

$$= |R(t^m - 1, \Delta(t))| \quad \text{(R3)}$$

$$= |B| \text{ by Fox's formula. //}$$

It would be nice to use Crowell's result that the annihilator ideal of k is principal in a similar fashion. However, example II in section (2.2) gives us two knots whose annihilator ideals must both be $(7t^2-13t+7)$ as this polynomial is irreducible; but the annihilator ideals of the two-twist spins are respectively $(9,t+1)$ and $(27,t+1)$, whose projections into $\Lambda/(t^2-1)$ cannot be the same, since their projections into the quotient ring $\Lambda/(t+1)$ are different. In fact, only the ideal $\pi(27,t+1)$ is generated by the image of the polynomial $7t^2-13t+7$.

3.2 THE LEVINE PAIRING

Suppose that we have the Milnor exact sequence;

$$0 \longrightarrow H_{q+1}(F) \longrightarrow H_q(K) \xrightarrow{t^m-1} H_q(K) \xrightarrow{p} H_q(F) \longrightarrow 0$$

associated with the knot l which is the m-twist spin of the simple $(2q-1)$-knot k. From this we know the structure of the Alexander modules of l; the most obvious pieces of algebraic information to calculate next are the pairings on the Z-torsion and Z-torsion-free parts of these modules. The second of these will simply express the duality isomorphism between $H_{q+1}(F)$ and the free part of $H_q(F)$, as in (2.2); so it is to the first pairing which we now turn.

The definition of the torsion pairing is quite complex in general; but when we are dealing with twist spun knots, which must be fibred, it may be defined more simply in terms of a linking pairing on the Z-torsion elements of the homology of the fibre, by [L2, section 7], as follows. Let $a,b \in H_q(F)$ be two torsion elements, and suppose that n is an integer such that na=0. We may then choose chains $\tilde{a} \in C_{q+1}(F)$ such that $\partial\tilde{a}$ represents na, and $\tilde{b} \in \tilde{C}_q(F)$, the group of q-chains in the dual triangulation, representing b; and we define the torsion pairing by;

$$[a,b] = \frac{I(\tilde{a},\tilde{b})}{n} \in Q/Z$$

This definition has similarities with the definition of the Blanchfield pairing of $\underline{k}$, and it would seem natural to try and express the torsion pairing as;

$$[a,b] = \theta\langle x,y\rangle$$

where $\langle , \rangle$ denotes the Blanchfield pairing of $\underline{k}$, x and y are elements of $H_q(\tilde{K})$ with p(x)=a, p(y)=b, and θ is some map;

$$Q(t)/\Lambda \longrightarrow Q/Z$$

This cannot be done in general; we shall give an example below where, whatever our choice of x and y, their Blanchfield pairing is zero. If $\underline{1}$ turns out to have a finite knot module, however, which is the case where the module (if it has odd order) and pairing can classify the knot ([Ko]), we have the following result; for consistency of notation we write u for the covering translation of $\tilde{K}$, so that the pairing <,> takes values in $Q(u)/Z[u,u^{-1}]$;

Proposition 15/With notation as above, and with $H_q(F)$ finite, we have;

$$[a,b] = \epsilon \nu_0 \langle x,y\rangle \quad (0 \text{ denoting the identity of } Z_m)$$

where ν_0 is the map defined at the end of section (1.4), and ϵ is defined as follows. We define the map $e:Q(t) \longrightarrow Q$ by expressing any rational function as a sum of partial fractions, whose denominators are powers of irreducible polynomials, together with a rational polynomial; e is then defined by taking the sum of those terms whose denominators are not powers of (1-t), and setting t=1. Clearly $e(\Lambda) \subset Z$; so e induces a map $\epsilon : Q(t)/\Lambda \longrightarrow Q/Z$ as required.

Proof Let d(u) be the Alexander polynomial of $\underline{k}$. Then, as in (1.4), we define;

$$\Delta(u^m) = \delta(u) = \prod_{\xi^m=1} d(\xi u)$$

$\delta(u)$ must annihilate $H_q(\tilde{K})$, since it is divisible by d(u); so we may find chains $\mathfrak{s} \in C_{q+1}(\tilde{K})$, $\mathfrak{t} \in \tilde{C}_q(\tilde{K})$ such that $\partial\mathfrak{s}$ represents $\delta(u)x$, and $\mathfrak{t}$ represents y. Then we have

$$\langle x,y\rangle = \Big(\sum_{i=-\infty} I(\mathfrak{s},u^i\mathfrak{t})u^i\Big)/\delta(u)$$

Considering the exact sequences;

$$0 \longrightarrow C_{q+1}(\tilde{K}) \xrightarrow{u^m-1} C_{q+1}(\tilde{K}) \xrightarrow{p} C_{q+1}(F) \longrightarrow 0$$

$$0 \longrightarrow C_q(K) \xrightarrow{u^m-1} C_q(K) \xrightarrow{r} C_q(F) \longrightarrow 0$$

we see that $r(\mathfrak{t})$ represents b, and $p(\mathfrak{s})$ represents $\delta(u)a=\Delta(u^m)a=\Delta(1)a$. Now;

$$|\Delta(1)| = \left|\prod_{\zeta^m=1} d(\zeta)\right| = |R(d(u),u^m-1)|$$

$$= |H_q(F)| \quad \text{by Fox's formula}$$

$$\neq 0 \quad \text{by assumption.}$$

So we may write $[a,b] = I(p(\mathfrak{s}),r(\mathfrak{t})) \,/\, \Delta(1)$

Now, $u^{jm}\mathfrak{t}$ projects down to $r(\mathfrak{t})$ for all integers j, from the above sequence; so if any of these lifts intersect with $\mathfrak{s}$, this will give rise to intersections of $p(\mathfrak{s})$ with $r(\mathfrak{t})$; and the intersections corresponding to all the different lifts will give rise to all the intersections of the projections, without duplication. So we may write;

$$I(p(\mathfrak{s}),r(\mathfrak{t})) = \sum_{j=-\infty}^{\infty} I(\mathfrak{s},u^{jm}\mathfrak{t})$$

Then we have;

$$[a,b] = \left(\sum_{j=-\infty}^{\infty} I(\mathfrak{s},u^{jm}\mathfrak{t})\right) / \Delta(1)$$

$$= \epsilon\left(\sum_{j=-\infty}^{\infty} I(\mathfrak{s},u^{jm}\mathfrak{t})t^j\right) / \Delta(1)$$

$$= \epsilon \mathcal{D}_0\langle x,y\rangle \quad \text{as desired. //}$$

<u>Example III</u> Let $\underline{k}$ be a simple (2q-1)-knot with Alexander module $\Lambda/(1-t+t^2)^2=A$. The six-twist spin of this knot has qth Alexander module;

$$B \cong A/(1-t^6)A \cong \Lambda/((1-t+t^2)^2, 1-t^6) \qquad \text{(A3)}$$

$$\cong \Lambda/(1-t+t^2)((1-t+t^2),(1-t)(1+t)(1+t+t^2))$$

$$\cong \Lambda/(1-t+t^2)(3,1+t)(2,1+t+t^2) \qquad \text{(A4)}$$

and the torsion elements are precisely the multiples of $1-t+t^2$. Elements of A projecting to torsion elements must then be of the form $(1-t+t^2)x$ for x in A. But we must have;

$$\langle(1-t+t^2)x,(1-t+t^2)y\rangle = \langle(1-t^{-1}+t^{-2})(1-t+t^2)x,y\rangle$$
$$= \langle t^{-2}(1-t+t^2)^2x,y\rangle = \langle 0,y\rangle = 0$$

and there is no hope of deriving the pairing on the torsion part of B by such a straightforward approach. However, the next section will suggest methods to compute such torsion pairings, as well as further techniques for recognizing twist-spun knots

3.3 <u>RELATING DIFFERENT TWIST-SPINS OF THE SAME KNOT.</u>

Suppose that m and n are two integers, with n dividing m. We may construct the Milnor exact sequences corresponding to the n- and m-fold cyclic covers of $\underline{k}$ as below;

$$\begin{array}{ccccccccc}
0 \longrightarrow & H_{q+1}(\tilde{X}_n) & \longrightarrow & H_q(\tilde{X}) & \xrightarrow{t^n-1} & H_q(\tilde{X}) & \xrightarrow{p_n} & H_q(\tilde{X}_n) & \longrightarrow 0 \\
 & & & & & \downarrow i & & \downarrow j & \\
0 \longrightarrow & H_{q+1}(\tilde{X}_m) & \longrightarrow & H_q(\tilde{X}) & \xrightarrow{t^m-1} & H_q(\tilde{X}) & \xrightarrow{p_m} & H_q(\tilde{X}_m) & \longrightarrow 0
\end{array}$$

where i is defined as multiplication by $(t^m-1)/(t^n-1)$ and j is induced by i. This diagram clearly commutes; and any multiple of $(t^m-1)/(t^n-1)$ in $H_q(\tilde{X}_m)$ is the projection of an element i(x) which is a multiple of this polynomial in $H_q(\tilde{X})$. Since $j(p_n(x))=p_m(i(x))$, we see that j maps onto the module;

$$\frac{(t^m-1)}{(t^n-1)}.H_q(\tilde{X}_m)$$

If $H_q(\tilde{X}_n)$ is finite, then we may deduce that the order of this module divides the order of $H_q(\tilde{X}_n)$. If <u>k</u> has Alexander polynomial $\Delta(t)$, then we have;

$$\begin{aligned}
|H_q(\tilde{X}_n)| &= |R(t^n-1,\Delta(t))| \\
&= \prod_{d|n} |R(\Phi_d(t),\Delta(t))| \\
&= \prod_{d|n} [N_d(F_0)]
\end{aligned}$$

with the notation as in corollary 14. This gives us a simple numerical condition for determining whether a knot whose module is infinite, but not torsion-free, may be an m-twist spin if m is composite;

<u>Proposition 16/</u>If a simple even-dimensional knot $\underline{1}$ with qth Alexander module B whose order ideal is F_0 is to be an m-twist spin, then we must have;

$$\left|\frac{(t^m-1)}{(t^n-1)}.B\right| \quad \text{dividing} \quad \prod_{d|n} N_d(F_0) \qquad \text{for all } n|m$$

If B is finite, then these orders must be equal.

<u>Proof</u> The last remark follows from noting that when B is finite, $H_{q+1}(X_n) = H_{q+1}(X_m) = 0$ for all n dividing m; and the map j in the diagram above must be an injection by a generalisation of the 5-lemma. //

The cases n=m and n=1 correspond to corollary 14 and the fact that B must be annihilated by t^m-1 (t-1 is an automorphism of B) respectively. We may also derive the torsion pairing on $\tilde{X}_m$, and hence on the m-twist spin of $\underline{k}$, as follows;

<u>Proposition 17/</u>Let $a,b \in H_q(\tilde{X})$ map to torsion elements of $H_q(\tilde{X}_m)$, and let $[\,,\,]_m, [\,,\,]_n$ denote the torsion pairings on the images of $H_q(\tilde{X})$ under the maps p_m and p_n. Then we have;

$$[a,b]_n = \left[a, \frac{(t^m-1)}{(t^n-1)}.b\right]_m$$

<u>Proof</u> Denote by $p_n^m : \tilde{K}_m \longrightarrow \tilde{K}_n$ the natural projection map, so that $p_n^m p_m = p_n$; where, as usual, we are using the same notation for a map and for its induced map in chain groups and homology modules. For some non-zero integer k, $kp_m a=0$; so we may find a chain $\tilde{a}$ in $C_{q+1}(\tilde{K}_m)$ whose boundary represents this

element. Then $p_m^n \tilde{a}$ will have boundary representing $kp_n a$, proving our first assertion. If $\tilde{b} \in \tilde{C}_q(\tilde{K}_n)$ is a dual cycle representing $p_m b$, then $\tilde{b}$ will have m/n distinct lifts to $C_q(\tilde{K}_m)$, representing elements $p_m t^{ni} b$ for $0 \leq i < m/n$; and the total number of intersections of these with $\tilde{a}$ will be the same as the number of intersections of $\tilde{b}$ with $p_m^n \tilde{a}$. So, by definition of the Levine pairing in the fibred case;

$$[a,b]_m = \sum_{i=0}^{m/n-1} [a,t^{ni} b]_m$$

$$= [a,(\sum t^{ni})b]_m \quad \text{by linearity}$$

$$= [a, (t^m-1)/(t^n-1).b]_m . //$$

In example III, the 2- and 3-twist spins of the knot with module $\Lambda/((1-t+t^2)^2)$ have finite modules, these being $\Lambda/(9,1+t)$ and $\Lambda/(4,1+t+t^2)$ respectively; so we can work out the pairings $[,]_2$ and $[,]_3$ on X using proposition 15. Then we have;

$$[a,(1+t^2+t^4)b]_6 = [a,b]_2$$

$$[a,(1+t^3)b]_6 = [a,b]_3$$

and, since $1-t+t^2=(1+t^2+t^4)-t(1+t^3)$ we have;

$$[a,(1-t+t\ )b]_6 = [a,b]_2 - [a,tb]_3 ,$$

which is sufficient to determine the Levine pairing on

$H_q(\widetilde{K}_6)$, as all the torsion elements are multiples of $1-t+t^2$. Unfortunately, this method does not appear to be sufficient even to calculate the torsion pairing on the (6k)-twist spin of a knot with this module for $k>1$.

3.4 TWO CLASSES OF FINITE KNOT MODULES

In an attempt to find a reasonably wide spectrum of examples of knots where we can determine which are the twist-spun knots, we will look at two classes of finite Λ-modules which may be written in standard forms (as a sum of cyclic modules in each case), and where the pairings arising have been classified. The first is motivated by the form of the modules of twist-spun cyclotomic knots, as in section (2.3); and we shall see that, in a sense to be made precise later, the modules arising there are the only modules of that form which satisfy the conditions of proposition 16. The second class is the class of semisimple modules defined by Jonathan Hillman in [H2].

Class I: Modules Annihilated by Squarefree Integers n

In this case we may write our modules as direct sums of their p-primary components for all p dividing n. This decomposition will be orthogonal with respect to the torsion pairing, since if x is in the p-primary part (so is annihilated by p, as n is squarefree), and y is annihilated by another prime q, then the pairing $[x,y] \in Q/Z$ will be annihilated by both p and q, so must be zero. So we need only consider the case of modules annihilated by primes p; then our pairings will take values in $Z_p \subset Q/Z$; and the problem is equivalent to that of classifying Z_p-inner product spaces with isometries. This was achieved by John Milnor in [M2], as follows; since $Z_p[t,t^{-1}]$ is a principal ideal domain, we can split up our module as a direct sum of cyclic components of the form $\Lambda_p/(q(t)^m)$, where $\Lambda_p = Z_p[t,t^{-1}]$ and q is irreducible; and the splitting may be chosen so that any two components annihilated by q_1^m and q_2^n are orthogonal, unless $q_1 \sim \bar{q}_2$ and m=n, when the terms may be grouped together in pairs to form hyperbolic summands (where $q \sim r$ iff q is equal to a unit $\pm t^i$ times r). The pairings on these hyperbolic summands are uniquely determined by the module structure; and so are the pairings on the q(t)-primary parts of the module, for $q(t) \not\sim 1+t$, because of theorem 3.3 and example 1 in section 1 of [M2]. If $q(t) \sim 1+t$, then we may ignore the case where p=2, since 1-t=1+t (mod 2) could not be an automorphism. We may write the q-primary part of our module as $V^1 \oplus V^2 \oplus \ldots \oplus V^r$, where V^i is free over $Z_p[t]/((1+t)^i)$; this splitting may also be chosen to be orthogonal. Then a complete invariant for ϵ-symmetric pairings on V^i on which t acts as an isometry is given by a $(-1)^{i-1}\epsilon$-symmetric pairing on $V^i/(1+t)V^i$, which is a vector space over Z_p. Now the symmetric pairings on a

Z_p-vector space are classified by rank and determinant, which lies in $U(Z_p)/(U(Z_p))^2$; so there are only two of these for each rank; and an anti-symmetric one only exists for even ranks, when it will be unique. So we have the following result;

Proposition 18/Any finite knot module which is annihilated by a squarefree integer splits up as a direct sum, orthogonal with respect to the torsion pairing, of three types of summand;

(i) $\Lambda/(p,q(t)^i) \oplus \Lambda/(p,\overline{q(t)}^i)$ p prime, $q \not\sim \bar{q}$ irreducible

(ii) $\Lambda/(p,q(t)^i)$, $q \sim \bar{q}$ irreducible, degree$\neq$2

(iii) $[\Lambda/(p,(1+t)^i)]^r$ where this denotes a direct sum of r like terms.

Summands of types (i) and (ii) have unique symmetric and anti-symmetric pairings definable on them; and summands of type (iii) have a unique $(-1)^{i-1}$-symmetric pairing if and only if r is even, but two distinct pairings of the other symmetry whatever the value of r. //

We have seen in section (2.3) that the twist spins of cyclotomic knots have modules which, provided they are finite, are of this form. The resulting modules will all be of type (ii), except those resulting from $(2p^k)$-twist spinning a knot whose minimal polynomial is of the form

$\Phi_{2p^l}(t)$, l>k. Then, since all modules of type (ii) support unique pairings, Kojima's classification of odd finite simple even-dimensional knots ([Ko]) shows us that any such knot of dimension greater than 6 with module;

$$[\Lambda/(p,(\Phi_m(t))^{p^k r})] \qquad p \text{ an odd prime}, p\nmid m, m\neq 2$$

is the twist-spin of a cyclotomic knot. Modules of type (iii) above arise from $(2p^k)$-twist spinning cyclotomic knots with minimal polynomial $\Phi_{2p^l}(t)$, l>k; and in this case we must investigate which pairings can occur. We begin with the case of a 2-twist spun knot, and extend to the general case using proposition 17. The 2-twist spin of a knot with Alexander polynomial $\Delta(t)$ will have an Alexander module which may be considered as an Abelian group of order $|\Delta(-1)|$, on which t acts as multiplication by -1. To work out the pairing we use the following result, where 0 denotes the identity element of Z_2.

<u>Lemma 19</u>/ $\epsilon\gamma_0(f(t)/\Delta(t)) = [f(-1).(\Delta(-1)+1)/2] \;/\; \Delta(-1)$

<u>Proof</u> Let $\Delta(-t) = \sum_{-r}^{r} d_i t^i$, $d_r = d_{-r} \neq 0$,

and $f(t) = \sum_{-s}^{s} a_i t^i$. Then;

$$\epsilon\gamma_0(f(t)/\Delta(t)) = \epsilon\gamma_0[\;(f(t).\Delta(-t)) \;/\; (\Delta(t).\Delta(-t))\;]$$

$$= \epsilon\gamma_0 [\ (\sum t^i \sum_{j+k=i} a_j d_k) \ / \ (\Delta(t).\Delta(-t)) \ ]$$

$$= [\ (\sum a_j d_k) \ /\Delta(-1) \ ] \in Q/Z \text{ as } \Delta(1)=1$$

where the sum is taken over all values of j and k which are congruent to each other modulo 2. We note that twice the sum of the coefficients d_i for i even is equal to $\Delta(-1)+\Delta(1)$; and twice the sum over odd i is equal to $\Delta(-1)-\Delta(1)$. If we let $D=\Delta(-1)$, then modulo D we have;

$$2\times \sum_{j\equiv k(2)} a_j d_k = \sum a_j (D+(-1)^j) \qquad \text{(sum over } j\equiv k \pmod 2))$$

$$\therefore \sum a_j d_k = [(D+1)/2].\sum a_j (D+(-1)^j) \qquad \text{(D is odd)}$$

$$= [(D+1)/2].\sum a_j (-1)^j = f(-1).(D+1)/2$$

as claimed. //

If our knot to be twist-spun is a (2q-1)-knot with q even, and has minimal polynomial $\Phi_{2p^i}(t)$, then its rank over the ring $\Lambda/(\Phi_{2p^i})$ must be even, since $\Phi_{2p^i}(-1)=p$ is not an odd square. A skew-Hermitian form can always be defined on such a module using a hyperbolic form (see [Ba], definition 3.6). The two-twist spin will then have a torsion pairing which is unique up to isometry by proposition 18, $i=p^k$ being odd here. The only case where the module of the twist-spin of such a knot supports different pairings is when we start with a

(2q-1)-knot for q odd. Given a polynomial f(t) such that $f(t)=f(t^{-1})$, which represents a unit of $\Lambda/(\Phi_{2p^\iota})$, we may define the rank one Hermitian form;

$$<,>_f : \Lambda/(\Phi_{2p^\iota}) \times \Lambda/(\Phi_{2p^\iota}) \longrightarrow Q(t)/\Lambda$$

$$\text{by} \quad (\; x(t) \quad , y(t) \quad) \longrightarrow f x\bar{y}/\Phi_{2p^\iota}$$

where we have divided Φ_{2p^ι} by a power of t so that $\Phi_{2p^\iota}(t)=\Phi_{2p^\iota}(t^{-1})$. If our original knot has a Hermitian pairing which is a direct sum of $<,>_f$ with (r-1) copies of $<,>_1$, then the determinant of the torsion pairing of its two-twist spin will be given by

$$[\; (\Phi_{2p^\iota}(-1)+1)/2\;]^r .f(-1) = [\; (p+1)/2\;]^r .f(-1) \quad (\text{mod } p)$$

using lemma 19. Which of the two possible pairings we will obtain is determined by whether this is a square (mod p), hence by whether f(-1) is a square (mod p). We shall demonstrate that f may be chosen so that f(-1) has any non-zero value (mod p), so that both pairings can be realised by twist-spun knots.

$$\text{Define } u_k(t) = t^{-k}.(t^{2k+1}+1)/(t+1) = \sum_{-k}^{k} (-1)^i t^i$$

We claim this is a unit (mod Φ_{2p^ι}) provided p does not divide 2k+1; for then there will be an odd integer h such that $h(2k+1)=1$ (mod $2p^\iota$), and the inverse is given by;

$$t^k.(t^{h(2k+1)}+1)/(t^{2k+1}+1)$$

u_k is clearly self-conjugate; and $u_k(-1)=2k+1$ can be chosen to take any given non-zero value (mod p), as desired.

In the case of the $(2p^k)$-twist spin for $k<l$, we have, by proposition 17;

$$[a\ ,\ (t^{2p^k}-1)/(t^2-1).b]_{2p^k} = [a,b]_2$$

Working modulo p, $t^{2p^k}-1 = (t^2-1)^{p^k}$, so;

$$(t^{2p^k}-1)/(t^2-1) = (t+1)^{p^k-1}(t-1)^{p^k-1}.$$

But by Milnor's work, the isometry class of the pairing [,] on the module $\Lambda/(p,(t+1)^{p^k})$ is determined by the pairing $[a,(t+1)^{p^k-1}b]$; so, since (t-1) is an automorphism, the two pairings on the $(2p^k)$-twist spins correspond to the two distinct pairings on the 2-twist spins, and we have seen that both of these can arise.

By the results of section (2.3), $\Lambda/(p,(t+1)^{p^k})$ arises as the module of an m-twist spun knot with Alexander polynomial $\Phi_{2p^l}(t)$ provided that the highest common factor of m and $2p^l$ is $2p^k$. So this module arises from a $2p^kr$-twist spun knot, provided $p\nmid r$; and, by proposition 17;

$$[a\ ,\ (t^{2p^kr}-1)/(t^{2p^k}-1).b]_{2p^kr} = [a,b]_{2p^k}$$

Since $(t^{2p^k r}-1)/(t^{2p^k}-1)$ is a product of cyclotomic polynomials $\Phi_{2p^k s}$ for $p \nmid s>1$, it is an automorphism of $\Lambda/(p,(t+1)^{p^k})$ since $R(t+1,\Phi_{2p^k s})=1$. So once again the pairings on the $2p^k r$-twist spins are determined by the pairings on the $2p^k$-twist spin, and both possibilities can arise.

Class II: Semisimple Modules

We will use the following characterization of semisimple modules given in theorem 1 part (iii) of [H2], which says that a finite Λ-module M is semisimple if it is annihilated by an ideal of the form;

$$\mathrm{Ann}(M) = \prod_{i=1}^{r} (p_i^e, g_i(t))$$

where the p_i are primes, and each g_i is an irreducible factor (modulo p) of some cyclotomic polynomial. Furthermore, we insist that the maximal ideals (p_i,g_i) are all distinct. Then part (ii) of the same theorem tells us that such modules split up as a direct sum of cyclic modules of the form $\Lambda/(p^e,g)$, where g is irreducible (mod p^e); and any such direct sum must be a semisimple module, as by looking at the powers of the image of t in these finite summands, some of which must be equal, we see that g must be an irreducible factor (mod p^e) of t^n-1 for some n, and hence of some cyclotomic polynomial.

The decomposition of M can be rearranged into groups of summands annihilated by powers of different maximal ideals; and this coarser decomposition will be orthogonal with respect to the torsion pairing, as in section 2 of [H2]. If we write M as $M_1 \oplus M_2$ where M_1 is a sum of cyclic modules of the form $\Lambda/(p^e,t+1)$, and M_2 contains no such summands , then the usual involution (conjugation) induced by $t \mapsto t^{-1}$ will be non-trivial on M_2; the corollary to theorem 2 of [H2] shows that M_2 only supports one pairing up to isometry. On M_1, the involution is trivial; by corollary 1 of [H2] there can be at most one isometry class of anti-symmetric pairings on this module; and by corollary 2, there will be exactly 2^r symmetric ones, where r is the number of distinct irreducible summands; and these pairings are distinguished by the r determinants of the pairings on the summands made up of these different irreducible parts. Thus as in the case of modules annihilated by squarefree integers, we do not have to worry about the pairing when trying to determine which simple (2q)-knots may be twist spun for q even.

If q is odd, we must consider the pairing on submodules of the form $\Lambda/(p^e,t+1)$. These arise as the the modules of 2-twist spun knots with modules $\Lambda/(\Phi_{2p}^e)$ as;

$$\Lambda/(t^2-1,\Phi_{2p}^e) = \Lambda/(t+1,\Phi_{2p}^e) \quad \text{(as t-1 is an automorphism)}$$

$$= \Lambda/(t+1, \Phi_{2p}^{e}(-1))$$

$$= \Lambda/(t+1, p^{e})$$

To realise the two possible pairings, we need to find two self-conjugate polynomials f_1, f_2 representing units of $\Lambda/(\Phi_{2p}^{e})$ such that $f_1(-1)$ is a square (mod p^e), and $f_2(-1)$ is not, by lemma 19. Now;

$$f_i \text{ is a unit } (\text{mod } \Phi_{2p}^{e})$$

$$\Longleftrightarrow \exists h(t) \text{ such that } f_i(t)h(t)=1 \ (\text{mod } \Phi_{2p}^{e})$$

$$\Longleftrightarrow (f_i, \Phi_{2p}^{e}) = \Lambda$$

$$\Longleftrightarrow |R(f_i, \Phi_{2p}^{e})|=1 \quad (A1)$$

$$\Longleftrightarrow |R(f_i, \Phi_{2p})|=1 \quad (R3)$$

$$\Longleftrightarrow f_i \text{ is a unit } (\text{mod } \Phi_{2p})$$

Such f_i representing units (mod Φ_{2p}) were found while investigating pairings on the first class of finite knot modules; and $f_i(-1)$ will be a non-zero square (mod p^e) if and only if it is a non-zero square (mod p). Also, by using an exactly similar method to that used for the first class of modules, and the fact that $R(\Phi_p, \Phi_r)=1$ if p does not divide r, we see that these modules arise from (2r)-twist spins, provided (p,r)=1, and that both possible pairings can arise.

Having realised each cyclic module and pairing of this form as the module and pairing of a (2r)-twist spin for suitable r, we may obtain any semisimple module on which the involution is trivial, together with any pairing, by taking a connected sum of suitable odd-dimensional knots with modules $\Lambda/(\Phi_{2p_i}^{e_i})$ and forming its $(2r')$-twist spin, where r' is divisible by all the values of r corresponding to the summands, so long as none of these values are divisible by any of the different primes p_i.

3.5 FINITE CYCLIC KNOT MODULES

In this section we investigate a situation in which the necessary conditions of proposition 13 and corollary 14 are close to being sufficient.

Proposition 20/If I is an ideal of Λ such that Λ/I is finite, then Λ/I is the module of an m-twist spun knot if and only if;

(i) $t^m - 1 \in I$

(ii) $\pi_m I$ is principal and generated by the image $\pi_m(\Delta(t))$ of a symmetric polynomial Δ, where $\pi_m : \Lambda \longrightarrow \Lambda/(t^m-1)$ is the projection map.

and (iii) $|\Lambda/I| = \prod_{d|m} N_d I$

Proof The "only if" part follows from proposition 13 and corollary 14. Suppose now that all three conditions above are satisfied for some m. Since Λ/I is a knot module, (t-1) must act as an automorphism, so;

$$0 = \frac{\Lambda/I}{(t-1)\Lambda/I} \cong \frac{\Lambda}{I+(t-1)} \qquad (A3)$$

$$\therefore \Lambda = I+(t-1)$$

So $\pi_1(I)=\Lambda/(t-1)$; and if $\pi_m(I)$ is generated by $\pi_m(\Delta(t))$ we must have $\pi_1(\Delta(t))=\Delta(1)=\pm 1$. We may then define a Blanchfield pairing on $\Lambda/(\Delta(t))$ by $<[x],[y]> = x\bar{y}/\Delta(t)$, where [] denotes cosets in $\Lambda/(\Delta(t))$. So there exists a simple (2q-1)-knot with this module, at least for q odd. The module of its m-twist spin is then

$$\Lambda/(\Delta(t),t^m-1) \qquad \text{by (A3)}$$

Now, by (ii), $t^m-1 \in I$; and since $\pi_m I$ is generated by $\pi_m(\Delta(t))$, we must have $\Delta(t)+f(t)(t^m-1) \in I$ for some $f \in \Lambda$, whence $\Delta(t) \in I$ also. Then there is a projection map;

$$\Lambda/(\Delta(t),t^m-1) \longrightarrow \Lambda/I$$

but, by (iii);

$$|\Lambda/I| = |\prod_{d|m} N_d I|$$

$$= |\prod_{d|m} N_d (\Delta(t))| \qquad \text{(ii)}$$

$$= \left|\prod_{d|m} R(\Delta(t), \Phi_d(t))\right| \qquad \text{(A8)}$$

$$= |R(\Delta, t^m - 1)| \qquad \text{(R3)}$$

$$= |\Lambda/(t^m - 1, \Delta(t))|$$

so the above surjection, mapping a finite module to another of the same order, must be an isomorphism. //

In fact, condition (iii) is a consequence of the other two, as if they hold we have;

$$|\Lambda/I| = |\Lambda/(I+(t^m-1))| \qquad \text{by (i)}$$

$$= \left|\frac{\Lambda/(t^m-1)}{(I+(t^m-1))/(t^m-1)}\right| \qquad \text{(1st isomorphism theorem)}$$

$$= \left|\frac{\Lambda/(t^m-1)}{\pi_m I}\right|$$

$$= \left|\frac{\Lambda/(t^m-1)}{\pi_m(\Delta(t))}\right| \qquad \text{by (ii)}$$

$$= |\Lambda/((t^m-1), \Delta(t))| \qquad \text{(1st isomorphism theorem)}$$

$$= |R(\Delta(t), t^m-1)| \qquad \text{(A1)}$$

$$= \left|\prod_{m|d} N_d I\right| \qquad \text{as above.}$$

However, we shall retain this third condition for the present, as it will prove to be relevant when we discuss the projective class group of $\Lambda/(t^m-1)$ in chapter 4.

If Λ/I is the module of a (2q)-knot for q even, so that it supports a skew-symmetric Levine pairing, it is more difficult to decide whether the knot may be twist-spun, although the conditions of the proposition are still necessary. If $t^m-1 \in I$ where m is odd, and $\pi_m I=(\pi_m(\Delta(t)))$ where Δ is symmetric, we can show that $\Delta(1)=\pm 1$ as above. Considering the symmetry of Δ shows us that $\Delta(-1)-\Delta(1)$ is divisible by 4. If we add some multiple of $(t^m-1)(t^{-m}-1)$, which has value 4 when t=-1, to Δ, we can get a new symmetric polynomial $\Delta'(t)$ such that $\pi_m(\Delta')$ generates $\pi_m I$, $\Delta'(1)=\pm 1=\Delta'(-1)$. Then $R(\Delta',t-t^{-1})=\Delta(1)\Delta(-1)=1$; so $t-t^{-1}$ is a unit of $\Lambda/(\Delta')$, and we can define the skew-Hermitian pairing;

$$\Lambda/(\Delta') \times \Lambda/(\Delta') \longrightarrow Q(t)/\Lambda$$
$$([x]\ ,\ [y]) \longmapsto (t-t^{-1})x\bar{y}/\Delta'$$

so that there exists a (2q-1)-knot with this module and pairing, which has a (2q)-knot with module Λ/I as its m-twist spin provided $|R(\Delta(t),t^m-1)|=|\Lambda/I|$. If we can satisfy the conditions of proposition 20 for some even m and symmetric polynomial Δ, then we must have;

$$|R(\Delta,t-t^{-1})| = |R(\Delta,(t-1)(1+t^{-1}))| = |\Delta(1)\Delta(-1)|$$

$$= N_1(\Delta(t)).N_2(\Delta(t))$$

$$= N_1 I.N_2 I = N_2 I$$

as (1-t) is an automorphism of Λ/I, where $N_d I$ denotes the positive integer generating the norm of the image of I in the dth cyclotomic field, as in corollary 14. If $\Lambda/(\Delta)$ supported a skew-Hermitian pairing <,> which was non-singular, then it would have to be given by $<[x],[y]> = fx\bar{y}/\Delta$ where f(t), which may be taken to be of the form;

$$a_i(t^i - t^{-i})$$

projects to a skew-symmetric unit of $\Lambda/(\Delta)$. Clearly, $t-t^{-1}$ divides f(t); so $N_1 I . N_2 I = |R(t-t^{-1},\Delta)|$ divides $|R(f(t),\Delta(t))|$, which must be equal to one for f to project to a unit; so such an f can only exist if $N_2 I=1$, in which case we may take $f(t)=t-t^{-1}$. So Λ/I is the module of the m-twist spin of a (2q-1)-knot with module of the form $\Lambda/(\Delta(t))$ if and only if $N_2 I=1$ and the other conditions of proposition 20 are satisfied.

Considering Λ/I as an Abelian group with a skew-symmetric pairing, Wall's classification of quadratic pairings on finite Abelian groups in [Wa] shows that we may split it up as a sum of groups $Z_{p^i} \oplus Z_{p^i}$ with hyperbolic pairings, for p odd, together with a 2-group. Then if our knot is an m-twist spin, we have;

$$2^j (k)^2 = |\Lambda/I| = |R(\Delta,t^m -1)| = |\prod_{d|m} R(\Delta,\Phi_d)|$$

where k is an odd integer. Now, Φ_d is a symmetric polynomial for $d>2$, when $R(\Delta,\Phi_d)$ will be a square by (R5). So, since $R(\Delta,\Phi_1)=\Delta(1)=\pm 1$ and $R(\Delta,\Phi_2)=\Delta(-1)=4n+\Delta(1)$ for some n, so that $\Delta(-1)$ must be odd, we conclude that $|\Delta(-1)|$ is an odd square. If we normalize so that $\Delta(1)=1$, then $\Delta(-1)=4n+1$ must be an odd square, since $-(4n+1)$ cannot be a square for any n (as -1 is not a square (mod 4)). Then by theorem 2 of [L1], there is a simple (2q-1)-knot with Alexander polynomial Δ, whose m-twist spin has a module of the same size (both being equal to $|R(t^m-1,\Delta)|$) as Λ/I, and with order ideal having the same image in $\Lambda/(t^m-1)$. One would expect that with infinitely many choices possible for Δ, at least one should be the Alexander polynomial of a (2q-1)-knot whose m-twist spin has module Λ/I; but I am unable to think of a proof of this.

Proposition 20, and the discussion following, enable us to give some sufficient conditions for a finite cyclic Λ-module to support a non-singular torsion pairing;

<u>Proposition 21/</u>With notation as above, Λ/I supports a symmetric torsion pairing if conditions (i)-(iii) of proposition 20 are satisfied for some m. It will support a skew-symmetric pairing if in addition these conditions are satisfied for some odd m, or for an even m provided $N_1 I.N_2 I=1$

<u>Proof</u> Because in these circumstances, Λ/I is the module of an even-dimensional twist-spun knot of the appropriate dimension. //

4 IDEALS OF $Z(Z_m)$

4.1 PRELIMINARIES

In this chapter, we shall assume that our knot module is annihilated by t^m-1 for some m, so that we may consider it as a module over the ring $\Lambda_m = Z(Z_m) \cong \Lambda/(t^m-1)$. We define the projection ρ_m so that the following diagram commutes;

$$\begin{array}{ccc} \Lambda & & \\ {\scriptstyle \pi_m}\downarrow & \searrow^{\rho_m} & \\ \Lambda/(t^m-1) & \xrightarrow{\sim} & \Lambda_m \end{array}$$

Many studies have been made of the projective class group for various values of m (see [RU1],[G],[KM]); but in order to be able to use this work, we shall have to be able to identify which are the projective ideals in this group ring; we shall do this by comparing the index of ideals in Λ_m with the product of the indices of their projections into the cyclotomic rings of integers $\Lambda/(\Phi_d)$ for d dividing m.

Let $Z_{(p)} = \{a/b : a,b \in Z, (b,p)=1\}$ be the localisation of Z at the prime p. This has a metric defined by $d(a/b, c/e) = 1/p^i$, where i is the largest power of p dividing the numerator, in lowest terms, of $a/b-c/e$, provided this is not zero; the distance will be zero if it is. We define $Z_{\hat{(p)}}$ to be the completion of $Z_{(p)}$ with respect to this metric, which is the ring of p-adic integers. An ideal I in Λ_m will then be projective if and only if $Z_{(p)} \otimes I$ is free in $Z_{(p)}(Z_m)$ for all primes p, or equivalently if each $Z_{\hat{(p)}} \otimes I$ is free in $Z_{\hat{(p)}}(Z_m)$ (see [U], [RU1]). Then the index of $Z_{(p)} \otimes I$ in $Z_{(p)}(Z_m)$ will just be the p-component of the index of I in Λ_m. Suppose $Z_{(p)} \otimes I = (\alpha_p)$, where α_p is a polynomial over the ring $Z_{(p)}$. Then;

$$\left|\frac{Z_{(p)}(Z_m)}{Z_{(p)} \otimes I}\right| = \left|\frac{Z_{(p)}(Z_m)}{(\alpha_p(x))}\right| \qquad (x \text{ generates } Z_m)$$

$$= \left|\frac{Z_{(p)}[t]/(t^m-1)}{(\alpha_p(t))}\right|$$

$$= |R(\alpha_p, t^m-1)|$$

$$= \left|\prod_{d|m} R(\alpha_p, \Phi_d)\right|$$

which is the product of the norms of the images of $Z_{(p)} \otimes I$ in $Z_{(p)}[t]/(\Phi_d)$. Since localisation commutes with these projection maps, we have proved;

<u>Proposition 22/</u>Let n be the composition of the projection maps $\Lambda \longrightarrow \Lambda/(\Phi_d)$ with the norm maps into the integers. Then, for an ideal I to be projective in Λ_m we must have;

$$|\Lambda_m/I| = |\prod_{d|m} n_d(I)| \ .//$$

In fact we can reverse this proposition; to do this, we shall prove that any ideal of Λ_m contains a projective ideal with the same images in the rings $\Lambda/(\Phi_d)$. This time we look at the completion $\hat{Z}_{(p)}(Z_m)$ of Λ_m; if p does not divide $|\Lambda_m/I|$, then $\hat{Z}_{(p)}\otimes I=\hat{Z}_{(p)}(Z_m)$; at the other primes of Z we will show that $\hat{Z}_{(p)}\otimes I$ contains a free ideal L_p with the same projections into $\hat{Z}_{(p)}[t]/(\Phi_d)$. Then the intersection of I with the various L_p will be the projective ideal we are looking for, as in [U], page 504.

Over the p-adic integers, t^m-1 splits up as a product of distinct polynomials; by Hensel's lemma, there will be one for each factor in $Z_p[t]$, although these factors will not be distinct if p divides m. So the ideal $(1-t^m)$ in $\hat{Z}_{(p)}[t]$ is the intersection of the ideals $(f_i(t))$, as f_i runs through the p-adic factors of t^m-1. We may then break down the ring $\hat{Z}_{(p)}(Z_m)$ by a series of Cartesian squares, using (A5);

$$\begin{array}{ccc} R_{i+1} & \longrightarrow & \hat{Z_{(p)}}[t]/(f_{i+1}) \\ \downarrow & & \downarrow \\ R_i & \longrightarrow & \hat{Z_{(p)}}[t]/(g_i, f_{i+1}) \end{array} \qquad (*)$$

where $g_j = \prod_{i<j} f_i$, and $R_i = \hat{Z_{(p)}}[t]/(g_i)$.

We shall use these squares to build up an ideal contained in $\hat{Z_{(p)}}\otimes I$ which projects to the same ideals in $\hat{Z_{(p)}}[t]/(f_i)$; as the projections of $\hat{Z_{(p)}}\otimes I$ into $\hat{Z_{(p)}}[t]/(\Phi_d)$ will all be principal (since all ideals in cyclotomic fields are projective, hence locally free), these projections will be determined by the projections into the rings $\hat{Z_{(p)}}[t]/(f_i)$, essentially because elements in the top left of a Cartesian square are determined by their projections into the adjacent corners.

Since the rings $\hat{Z_{(p)}}[t]/(f_i)$ are all local, with maximal ideals $p\hat{Z_{(p)}}[t]/(f_i)$, we may construct the desired free ideal inside $\hat{Z_{(p)}}\otimes I$ by induction using the following lemma;

<u>Lemma 23/</u> Suppose
$$\begin{array}{ccc} R & \longrightarrow & R_1 \\ \downarrow & & \downarrow f \\ R_2 & \xrightarrow{g} & \bar{R} \end{array}$$

is a Cartesian square with all maps surjective, where $\bar{R}$ is a local ring, and that I is an ideal of $R \cong \{(r_1, r_2) \in R_1 \oplus R_2 : f(r_1) = g(r_2)\}$ which projects to principal ideals (α), (β) of R_1, R_2

respectively. Then I contains a principal ideal with the same projections.

Proof In order for I to project onto these ideals, it must contain elements $(\alpha, v\beta)$, $(u\alpha, \beta)$ for some $u, v \in R_1, R_2$. If u is a unit of $\overline{R}$ we may use the ideal generated by $(u\alpha, \beta)$; otherwise, we know that R contains an element (w,v) for some w in R_1, so that the ideal I contains the element;

$$(w,v)(u\alpha,\beta)-(\alpha,v\beta) = ((wu-1)\alpha,0)$$

since $\overline{R}$ is local, and u is not a unit, wu-1 must be a unit; so in fact I contains $(\alpha,0)$ and $(0,\beta)$ by further calculations; and hence it contains (α,β), which generates the required principal ideal. //

We apply this lemma to the squares (*), with the ideals (α), (β) being either the images of $Z_{(\hat{p})}(Z_m) \otimes I$, or the principal ideal, inside the image of this ideal, thrown up by the previous application of the lemma; and intersecting the ideals in the relevant localisations with I gives us the ideal indicated by the following proposition;

Proposition 24/Any ideal of $Z(Z_m)$ contains a projective ideal with the same set of images in the rings $Z[t]/(\Phi_d)$ for d dividing m. //

Corollary 25/If I is any ideal of finite index in Λ_m, then;

$$|\Lambda_m/I| \leq |\prod_{d|m} n_d(I)|$$

with equality only if I is projective.

<u>Proof</u> From the proposition, any ideal contains a projective ideal which satisfies the above inequality, with both sides in fact equal by proposition 22; and, by (A7), these two ideals will be the same if and only if the product of their "norms" are the same. //

I am indebted to Steve Wilson for the above result, which will generalize to other orders.

4.2 KNOT MODULES WHICH ARE SUMS OF CYCLIC MODULES

In this section we tackle the simplest generalisation of the situation in section (1.5), armed with the above results. We recall the conditions of proposition 20, which said that the module Λ/I belonged to an m-twist spun knot provided;

(i) $t^m-1 \in I$ (ie. t^m-1 annihilates Λ/I)

(ii) $\pi_m I$ is principal and generated by the image $\pi_m(\Delta(t))$ of a symmetric polynomial Δ, where $\pi_m:\Lambda \dashrightarrow \Lambda/(t^m-1)$ is the projection map.

and (iii) $|\Lambda/I| = \prod_{d|m} N_d I$

Given a knot module M which is a direct sum of cyclic modules $\Lambda/I_1 \oplus \Lambda/I_2 \oplus \ldots \oplus \Lambda/I_s$, this module will certainly belong to an m-twist spun knot if all of the summands do; so the first test we can perform is to look at all the different expressions of M as a direct sum of cyclic Λ-modules (using (A6)), and see whether each of the summands in any of these arrangements satisfies all the above conditions. If none of these summations show that M does belong to an m-twist spun knot, we have only been able to obtain negative results, so that the characterization of knot modules so arising is incomplete even in this simple case.

Suppose, therefore, that we have written M as a sum of cyclic modules as above, at least one of which fails to satisfy all the conditions above. If one fails to satisfy (i), then M will not be annihilated by t^m-1, so cannot arise from an m-twist spin by proposition 13. The same conclusion cannot be drawn if it fails to satisfy (ii), as later examples will show; but the corresponding result does hold in the case of condition (iii)'s being violated, provided (i) holds, as shown below;

<u>Proposition 26/</u>Suppose $t^m-1 \in I_1$, but $|\Lambda/I_1| \neq |\prod N_\alpha I_1|$. Then no knot with finite module of the form $\Lambda/I_1 \oplus \Lambda/I_2 \oplus \ldots \oplus \Lambda/I_s$ can ever be an m-twist spin for any collection of ideals $I_1, \ldots, I_s$ of finite index in Λ.

<u>Proof</u> $\left|\dfrac{\Lambda/(t^m-1)}{\pi_m I_1}\right| = |\Lambda/(I_1+(t^m-1))|$

$= |\Lambda/I_1|$ as $t^m-1 \in I_1$

So, by corollary 25, $|\Lambda/I_1| < \prod N_d I_1$. Now, if t^m-1 is not contained in each of the I_j, then our knot can certainly not be an m-twist spin; but if it is, then we have;

$$|\Lambda/I_j| = \left|\frac{\Lambda/(t^m-1)}{\pi_m I_j}\right| \text{ for } j>1, \text{ as above.}$$

Then $|\Lambda/I_1 \oplus \ldots \oplus \Lambda/I_s| = |\Lambda/I| \prod_{j>1} |\Lambda/I_j|$

$$\leqslant |\Lambda/I| \prod_{d|m} \prod_{j>1} N_d I_j$$

$$= |\Lambda/I| \prod_{d|m} N_d (\prod_{j>1} I_j)$$

$$< \prod_{d|m} N_d (\prod I_j)$$

So our knot cannot be an m-twist spin by corollary 14. //

<u>Example/</u> Let $J=(p,(\Phi_m(t))^i)$, where p does not divide m. Consider the projection J_d of J into $Z[\zeta_d]$, the ring of integers in the dth cyclotomic field. Unless d/m is a power of p, we can find an integer q such that;

$$f(t)\Phi_m(t) + g(t)\Phi_d(t) = q_d$$

for some integer polynomials f,g as in the proof of proposition 12. So J_α contains q_α, which is coprime to p; so $J_\alpha=Z[\zeta_\alpha]$, and $N_\alpha(J)=1$.

If $d=mp^k$, then $\Phi_d(t) = (\Phi_m(t))^{(p-1)p^{k-1}} \pmod p$.

So if $p^{k-1}(p-1)\leq i$, then $(\Phi_m(\zeta_d))^i = 0 \pmod p$ and $J_d=(p)$;

so $N_\alpha(J) = p^{\phi(d)} = p^{\phi(m)\phi(p^k)}$. If $(p-1)p^{k-1} > i$, then, because of the way ideals factorise in $Z[\zeta_\alpha]$ (see [La], page 27), we have;

$$J_d = (p,(\Phi_m(\zeta_d))^i) = (p,\Phi_m(\zeta_d))^i$$

and $(N_d J) = (N(p,\Phi_m(\zeta_d)))^i$ (N is the norm map)

$$\therefore N_d J = p^{i\phi(m)}.$$

Then $\prod_{d|n} N_d J = p^s$, where $s = \phi(m)[\sum_{j\in S_1}\phi(p^j) + \sum_{j\in S_2} i]$

and $S_1=\{k>0: mp^k|n,\ p^{k-1}(p-1)\leq i\}$, $S_2=\{k>0: mp^k|n,\ i<p^{k-1}(p-1)\}$

Now, $|\Lambda/I|=p^{i\phi(m)}$; so in order for I to satisfy the norm-product condition, we must have;

$$i = \sum_{j\in S_1}\phi(p^j) + \sum_{j\in S_2} i$$

$$= p^k + \sum_{j \in S_2} i$$

where k is the largest integer s.t. $mp^k | n$ and $p^{k-1}(p-1) \leq i$. But it is not hard to see that, in order for this equation to hold, we must have $i=p^k$; and then we must also ensure that mp^j does not divide n for $i<p^{j-1}(p-1)$, ie. for $j>k$.

Combining this example with proposition 26 shows that no knot module which is a sum of cyclic modules, one of which is $\Lambda/(p,\Phi_m^i)$ for $p\nmid m$ and i not a power of p, can belong to any twist spun knot. Taking m=2, we have already seen that if i is any power of p, this summand does arise from an $(2p^k r)$-twist spun knot, provided that $i=p^k$, and r is not divisible by p; and in these cases, either of the two possible pairings can be realised

In the case of a finite knot module M falling into one of the two classes of the last chapter, we will also run into the question of whether each of the alternative pairings on M can be realised by m-twist spun knots. The above result completes the proof that all the possible pairings can be so realised if M can, in the case that M is irreducible and so cyclic. If we have found a decomposition of M into cyclic summands satisfying the conditions of proposition 20, then we shall show that we can split off the irreducible summands which are annihilated by powers of (1+t), ie. those which may support more than one pairing, while leaving summands which still satisfy these conditions; so the different possible pairings will offer us no additional problems.

Suppose that one of the summands is of the form Λ/IJ, where I and J are two non-trivial ideals. Both classes of finite modules split up as sums of cyclic modules annihilated by the irreducible ideals which are factors of their order ideals; so we must have $\Lambda/IJ \cong \Lambda/I \oplus \Lambda/J$. Suppose that Λ/I is one of the irreducible modules which may support more than one pairing, so that I will be of the form $(p,(t+1)^{i})$ if M belongs to the first class, or $(p^{e},t+1)$ if M is semisimple. We have seen above that if Λ/I fails to belong to an m-twist spun knot then I will fail to satisfy the norm-product condition (iii) above; and then IJ will fail this condition too, as in proposition 26. Since Λ/IJ is part of a decomposition for M chosen so that each summand satisfies all three conditions of proposition 20, the module Λ/I, together with any permissible pairing on the direct sum of all the isomorphic components of M, arises from an m-twist spun knot. In particular, Λ/I satisfies (i)-(iii); it only remains to prove that Λ/J does too. It must be annihilated by $t^{m}-1$, as Λ/IJ is; so J contains $t^{m}-1$. Then;

$$|\Lambda/J| = |\Lambda/(J+(t^{m}-1))| = \left|\frac{\Lambda/(t^{m}-1)}{\pi_m J}\right|$$

and J satisfies the norm-product condition by corollary 25. I satisfies (ii), so $\pi_m I$ is generated by a self-conjugate element α; then we may write any self-conjugate generator of $\pi_m IJ$ as $\alpha\beta$, where β is itself self-conjugate, since this ideal is contained in $\pi_m I$. Then $(\beta) \subset \pi_m J$; and, if we denote by n_d the composition of the natural projection $\Lambda/(t^m-1) \longrightarrow \Lambda/(\Phi_d)$ with the norm map into the integers, then we have;

$$n_d(\alpha)n_d(\beta)=n_d(\alpha\beta)=N_d(IJ)=N_d I.N_d J=n_d(\alpha)N_d J \text{ for all } d|m,$$

so $N_d(J)=n_d(\beta)$ for all $d|m$, and $\pi_m J=(\beta)$ by (A7); so Λ/J does indeed satisfy all three conditions of proposition 20.

4.3 TWO EXAMPLES

It is usual to break down the study of $Cl(\Lambda_m)$ into two parts as follows. Define ϕ_d to be the natural projection $\Lambda_m \longrightarrow Z[\zeta_d]$ for $d|m$, where ζ_d is a primitive dth root of unity. The collection of all these maps gives us a homomorphism;

$$Cl(\Lambda_m) \longrightarrow \bigoplus_{d|m} Cl(Z[\zeta_d])$$

and we define the <u>reduced class group</u> $D(\Lambda_m)$ to be the kernel of this map. When we investigate self-conjugate ideals, two different approaches are used; in the cyclotomic fields the object of interest is the class group of the real subfield

$Q(\zeta_d+\zeta_d^{-1})$; but in the reduced class group it is usual to look at the subgroup D^+ generated by self-conjugate ideals, not worrying about whether prospective generators are themselves self-conjugate (see [KM]). Neither approach is ideal for our application; however, if an ideal of $Z[\zeta_d]$ can be generated by self-conjugate generators, then it will be generated by a single such element if its intersection with the real subfield is a principal ideal there (since then its original generators lie in the subfield, and can be expressed as multiples of this single generator). The following result gives an instance where the usual approach answers our question in the study of the reduced class group;

<u>Proposition 27/</u>If m is odd, then any principal self-conjugate ideal I in Λ_m can be generated by a self-conjugate element.

<u>Proof</u> Suppose $I=(y)$; then, since I is self-conjugate, $u\bar{y}=y$, where u is a unit of Λ_m. Taking conjugates, $\bar{u}y=\bar{y}$; so $u\bar{u}=1$. The image of u in each of the rings $Z[\zeta_d]$ for $d|m$ must also have unit product with its conjugate; so, as in the proof of proposition 2.3 of [Ba], these images must all be roots of unity. So if p is the product of the orders of all its images in these rings, then $u^p=1$ in Λ_m. By [Hig], the only roots of unity in $Z(Z_m)$ are of the form $\pm x^i$, where x is a generator of Z_m; and projecting the equation $y=u\bar{y}$ down to $Z[\zeta_1]=Z$ shows that $u=+x^i$ for some i. Now, since m is odd, there exists a j such that $2j=i \pmod m$; then we have;

$$\overline{x^j y} = x^{-j}\bar{y} = x^{-j}x^{2j}y = x^j y$$

and this is the required self-conjugate generator. //

Now if m is a prime, then we know that $D(\Lambda_m)$ is zero by Rim's theorem ([M3], page 29); and if it is a non-trivial power of a prime p, then $D^+(\Lambda_m)=0$ if and only if p is a regular prime ([KM]). In either case, if the class number of the real subfield of $Z[\zeta_d]$ is one for all d dividing m, then we know that any knot module from either of our special classes which is annihilated by t^m-1, and whose cyclic summands satisfy the norm-product condition, belongs to an m-twist spun knot. In each case we can identify the cyclic modules satisfying the norm-product condition; they will be those which can arise as summands of modules of m-twist-spun knots; all these satisfy the condition by proposition 26. So, for the first class, if f(t) is a factor (modulo p) of a cyclotomic polynomial $\Phi_d(t)$, and f is coprime to Φ_d/f (mod p), then $\Lambda/(p,f)$ is a direct summand of $\Lambda/(p,\Phi_d)$ by (A6); and the latter module may arise from a (dr)-twist spun cyclotomic knot provided p does not divide r, from the work in section (2.3); and so $\Lambda/(p,f)$ must satisfy the norm-product condition with m=dr. If f divides Φ_d but is not coprime to Φ_d/f (mod p), then d must be divisible by p to give Φ_d a square factor (mod p). If g^i is a power of an irreducible (mod p) polynomial dividing both f and Φ_d/f, then $\Lambda/(p,g^i)$ will fail the norm-product condition for m divisible by d, using the method of the example in the last section; so $\Lambda/(p,f)$ must also fail, by proposition 26. The semisimple case will be susceptible to similar analysis; but we shall not burden the reader with the calculations. So we may completely solve the question of which modules and pairings in our two special classes which are annihilated by t^m-1 for m prime or a power of an regular prime arise from twist spun knots, provided that the real subfield of the mth cyclotomic field has class number 1; and thus the problem of which (2q)-knots having modules with these properties so arise for any $q>4$, remembering that the case where q was even presented no problems so long as m was odd, as we saw in the discussion leading up to proposition 21. All prime powers with $\phi(m)\leqslant 66$ satisfy these conditions; and if the generalized Riemann hypothesis is true, then any prime powers with $\phi(m)\leqslant 162$ do, by [vdL]; but the real subfield of $\Lambda/(\Phi_{163})$ has class number divisible by 4.

We now give two examples of rings Λ_m containing non-principal projective ideals, which lead to knots which fail to be twist-spun because of this fact.

<u>Example I/</u> Let p(t) be a polynomial of degree 7 which is irreducible modulo 3. $Z_3[t]/(p(t))$ is the field of order 3 , and its elements all satisfy $x^n-x=0$ where $n=3^7$; in particular we have, modulo 3;

$$p(t)\,|\,t^n - t = t(t-1)(t+1)\Phi_{1093}(t)\Phi_{2186}(t)$$

as $3^7-1=2.1093$, and 1093 is prime. Since p is of degree seven, it must divide one of the latter two factors modulo 3; and as the corresponding cyclotomic fields are the same, we assume without loss of generality that it divides Φ_{1093}. We consider the module Λ/I, where;

$$I = (3,p(t)\bar{p}(t)) = (3,p(t))(3,\bar{p}(t)) \text{ by (A4).}$$

There are 1092/7=156 different irreducible factors of Φ_{1093} modulo 3, which are all coprime modulo 3. So applying (A6) and (A4) we have;

$$\Lambda/(3,\Phi_{1093}(t)) \cong \oplus\Lambda/(3,p_i(t))$$

where the p_i are the irreducible factors. Now the module on the left comes from an m-twist spun knot for any m divisible by 1093 but not by 3, as we saw in section (2.3). Therefore, for these values of m, this module satisfies conditions (i) and (iii) of proposition 20; and proposition 26 shows that each summand must do so too (giving the promised example of a cyclic module satisfying (ii) which splits up into a direct sum of cylics not satisfying this condition); in particular, $\Lambda/I \cong \Lambda/(3,p) \oplus \Lambda/(3,\bar{p})$ does. We shall, however, show that the projection of the ideal I into $Z[\zeta]$, where ζ is a primitive 1093rd root of unity, is not principal, so I does not satisfy condition (ii).

The extension $Q(\zeta):Q$ is of degree $\phi(1093)=1092$; so the real subfield $Q(\zeta+\zeta^{-1})$ is of degree 546 over Q. There is, therefore, a unique subfield of degree 2, which must be $Q(\sqrt{1093})$, as 1093 is totally ramified in $Q(\zeta):Q$. The ring of integers in $Q(\sqrt{1093})$ is $Z[(1+\sqrt{1093})/2]$, since $1093\equiv 1 \pmod 4$; and the ideal (3) factorises into primes as;

$$(3) = (3,(1+\sqrt{1093})/2)(3,(1-\sqrt{1093})/2)$$

In $Z[\zeta]$ this ideal splits further as the product of all the ideals $(3,p_i(\zeta))$ by [La], page 27, or by lemma A4. The ideals $(3,p(\zeta))$ and $(3,\bar{p}(\zeta))$ lie over the same prime ideal in $Q(\sqrt{1093})$, since this field is totally real. So, without loss of generality, we may assume;

$$N_{Q(\zeta):Q(\sqrt{1093})}(3,p\bar{p}(\zeta)) = (3,(1+\sqrt{1093})/2))^{i}$$

for some i; and since;

$$N_{Q(\zeta):Q}(3,p\bar{p}(\zeta))^{14}$$

$$\text{and } N_{Q(\sqrt{1093}):Q}(3,(1+\sqrt{1093})/2) = (3)$$

we see that i=14. Now, if the ideal $(3,p\bar{p}(\zeta))$ were principal, then so would any norm of it be, generated by the norm of a generator. In particular, so would $(3,(1+\sqrt{1093})/2)^{14}$ be; and since $Z[(1+\sqrt{1093})/2]$ has class number 5 [B&S], this ideal is principal if and only if $(3,(1+\sqrt{1093})/2)$ is. This in turn is equivalent to the existence of integers x,y such that;

$$\frac{(x+y\sqrt{1093})}{2}\frac{(x-y\sqrt{1093})}{2}=\pm 3 \iff x^2-1093y^2=\pm 12$$

and we shall show in the appendix that no such integers can be found, so that I cannot satisfy condition (ii). Indeed, since I projects to a non-principal ideal of $Z[\zeta]$, the class of $\pi_m I$ lies outside $D(\Lambda_m)$, as desired.

<u>Example II/</u>For a module whose order ideal belongs to a non-zero class within the reduced class group, we take m=190. The cyclotomic fields $Q(\zeta_d)$ for $d|m$ all have real subfields with class number 1, by [vdL]; so provided our order ideal, containing $t^{190}-1$, can be generated by self-conjugate elements, all its projections into these cyclotomic fields will be principal, and generated by self-conjugate elements. We will implicitly be using Milnor's Meyer-Vietoris sequence (see [RU2],[M2]) for the Cartesian square;

$$\begin{array}{ccc} \Lambda/(t^{190}-1) \cong \Lambda_m & \xrightarrow{f_1} & Z[\zeta_m] \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle h_2} \\ R & \xrightarrow{h_1} & S \end{array}$$

where $R = \Lambda/(\Psi_{190})$, and $S = \Lambda/(\Phi_{190}, \Psi_{190})$ $(\Phi_m \Psi_m = t^m - 1)$

$$\cong \Lambda/(\Phi_{190}, (t-1)\Phi_2\Phi_5\Phi_{19})\,(\Phi_{190},\Phi_{10})\,(\Phi_{190},\Phi_{38})\,(\Phi_{190},\Phi_{95}) \quad (A4)$$

$$\cong \Lambda/(19,\Phi_{10}) \oplus \Lambda/(5,\Phi_{38}) \oplus \Lambda/(2,\Phi_{95}) \quad (A6)$$

We name these three components of S S_{19}, S_5, and S_2 respectively. The ideal we will look at in Λ_m will be the pullback of the ring R and an ideal $(\Delta(\zeta_m))$ in $Z[\zeta_m]$, where Δ

is some symmetric polynomial; that is the set of elements x in Λ_m such that $f_1(x) \in (\Delta(\zeta_m))$ (and $f_2(x) \in R$); and we denote this ideal by I. If I were principal, generated by an element δ, then $f_1(\delta)$ and $f_2(\delta)$ would generate $(\Delta(\zeta))$ and R respectively; so we should have;

$$f_1(\delta) = u_1 . \Delta(\zeta_m) \quad u \in U(Z[\zeta_m])$$

$$\text{and } f_2(\delta) = u_2 \in U(R)$$

Since the square commutes, we would have;

$$h_2(u_1)h_2(\Delta(\zeta_m)) = h_2 f_1(\delta) = h_1 f_2(\delta) = h_1(u_2)$$

$$\therefore \quad h_2(\Delta(\zeta_m)) = h_1(u_2).h_2(u_1) \in \text{Im } \psi$$

where $\psi : U(Z[\zeta_m])XU(R) \longrightarrow U(S)$

is given by $(v_1, v_2) \longrightarrow h_2(v_1)h_1(v_2)$

We shall show, however, that ψ is not surjective, and that if Δ is chosen suitably its projection to S will lie outside the image, so that I can not be principal; and we shall do this by examining the composition of ψ with the projection $S \longrightarrow S_2$.

Now, $\phi(95) = \deg(\Phi_{95}) = 72$; and the Galois group $\Gamma(Q(\zeta_{95}):Q)$ is

$$U(Z_{95}) \cong U(Z_5) \oplus U(Z_{19}) \cong Z_4 \oplus Z_{18} .$$

So the element sending a primitive root to its square has order at most 36; and we may easily check that it is exactly 36. When we work modulo 2, this element gives us the

Frobenius automorphism; and since it has order 36 it splits up the roots of unity into two orbits, whence Φ_{95} splits up as a product of two irreducible polynomials mod 2. Since $2^{18} \not\equiv -1 \pmod{95}$, the set of elements of the form $\zeta_{95}^{a} + \zeta_{95}^{-a}$ has just one orbit under this action; the characteristic polynomial for a generator $\zeta_{95}+\zeta_{95}^{-1}$ is thus irreducible mod 2, and the two factors of Φ_{95} (mod 2) must be mutually conjugate, being f and $\bar{f}$, say. So we may write;

$$S \cong \Lambda/(2,f) \oplus \Lambda/(2,\bar{f}) \cong V \oplus W \text{ say}$$

Let p be the projection $S \longrightarrow V$. Remembering that $S \cong \Lambda/(\Phi_{190}, \Phi_{95})$, we can see that $p\psi$ may be factored through $U(Z[\zeta_{190}]) \times U(Z[\zeta_{95}])$. Since $\Phi_{95} \equiv \Phi_{190}$ (mod 2), we may identify the images of these two unit groups; and then all the elements in the image of $p\psi$ come from units of $Z[\zeta_{190}]$ via the map ph_2. By Sinnott's result ([W], page 147), and the fact that the real subfield of $Z[\zeta_m]$ has class number one, the units of $Z[\zeta_m]$ are all cyclotomic units, which means that the image of ph_2 is generated by elements $\pm\zeta_m$, $1\pm\zeta_m$, $1\pm\zeta_m^5$ and $1\pm\zeta_m^{19}$, together with their Galois conjugates. Let T denote the image of ζ_m in V; we shall show that the images of these elements are all cubes of elements in V.

V is a finite field, so its unit group $V\backslash 0$ is cyclic of order $2^{36}-1$. T is of order dividing 190 in this group, so is certainly a cube, as three divides $2^{36}-1$ but not 190. We treat the other three images as follows.

<u>(i) 1+T.</u> Let T=RP, where R is a primitive 19th root of unity, and P a primitive 5th root. Then we have;

$$(1+T)^{(2^{36}-1)/3} = (1+RS)^{(4^{18}-1)/(4-1)}$$

$$= (1+RS)^{1+4+\ldots+4^{17}}$$

$$= (1+RS)(1+R^4S^4)\ldots(1+(RS)^{4^{17}}) \text{ (since we work mod 2)}$$

$$= (1+RS)(1+R^4S^{-1})(1+R^{16}S)\ldots(1+R^{4^{17}}S^{-1})$$

$$= S^kR^l(S+R^{-1})(S+R^4)(S+R^{-16})\ldots(S+R^{4^{17}})$$

for some integers k,l. But $-1,4,-16,\ldots,4^{17}$ form a complete set of non-zero residues (mod 19); so this expression is equal to;

$$S^kR^l\Phi_{19}(S) = S^kR^l(1+S+\ldots+S^{18})$$
$$= S^{k-1}R^l \quad \text{since } 1+S+S^2+S^3+S^4=0$$

Now this must be a cube root of unity; so k=1 and l=0, and 1+T must be a cube in V.

<u>(ii) $1+T^5$.</u> This will have the same order as 1+R. Now;

$$(1+R)^{1+4+\ldots+4^{17}} = (1+R)(1+R^4)\ldots(1+R^{4^{17}})$$

$$= R^{1+4+\ldots+4^{17}}(1+R^{-1})(1+R^{-4})\ldots(1+R^{-4^{17}})$$

and, since R is a 19th root of unity, the power of R on the right is equal to one. The numbers $1,4,\ldots,4^{17}$ do not form a complete set of residues modulo 19, as $4^{9}=1$; but, since the residue -1 is not among these, $1,4,\ldots,4^{17},-1,-4,\ldots,-4^{17}$ do form a complete set; so the square of the above expression is equal to $\Phi_{19}(1)=1$; and since squaring defines the Frobenius automorphism, the expression itself is equal to one, as desired.

(iii) $1+T^{19}$. This will have the same order as $1+S$; S is a primitive 5th root of unity, and Φ_5 is irreducible (mod 2); so the order of $1+S$ must divide the order of the unit group of $Z_2[\xi_5]$ which is 15; and since the order of the unit group of the finite field V is divisible by 27, $1+S$ must be a cube of a unit in V.

So we have shown that Im $p\psi \subset V^3$. But since h_2 is onto, we can find an element d of $Z[\xi_m]$ which maps to a cube of a unit in V, a unit which is not a cube in W, and the identity elements of S_{19} and S_5. Then $d\bar{d}$ gives a self-conjugate element of $Z[\xi_m]$ such that $h_2(d\bar{d})$ is a unit outside Im ψ. The ideal $(d\bar{d})$ factorises into ideals of the form (p_i,Δ_i), where p are rational primes, and the Δ_i are symmetric polynomials dividing Φ_m (mod p_i); all of these are principal, as mentioned above, and at least one, say (p,Δ), must have a symmetric generator mapping to a unit of S outside Im ψ. The pullback ideal I will then be given by $(p,\Delta(x))$, where x is the image of t in Λ_m; this projects onto R, since p cannot divide 190 if a generator of $(p,\Delta(\zeta_m))$ is to map onto a unit of S. Then the knot module $\Lambda/(p,\Delta(t))$ cannot arise from a twist-spun knot, since $(p,\Delta(x))$ is not principal in Λ_m. On the other hand, Λ/I is clearly annihilated by $t^{190}-1$; and $|\Lambda/I| = \prod N_\alpha I$, by corollary 25, using the fact that $(p,\Delta(t))$ is a factor of $(p,\Phi_m(t))$, which follows from (A4).

Modules which fail to belong to twist spun knots because of the projection of their order ideal's not being principal in Λ_m lead to examples where we have been unable to decide whether a module may belong to a twist spun knot. Suppose we have an ideal I whose projection to Λ_m is not principal, but such that Λ/I is a finite knot module annihilated by t^m-1 and satisfying the norm-product condition. By the Jordan-Zassenhaus theorem, the projective class group of Λ_m is finite; so there exists an integer n such that I^n projects to a principal ideal of Λ_m; and if this has no self-conjugate generator, then it is not hard to see that I^2 will have one (in the examples above we would take some n divisible by 5 and 3 respectively). Then a direct sum of n or 2n copies of Λ/I will be a module satisfying all the conditions of proposition 13 and corollary 14; but it is difficult to see how this module might be realised by a twist-spun knot. Certainly, it could not belong to the twist-spin of a knot whose module was a sum of cyclics, since each such summand could only give rise to a single summand of the finite module. It might be possible to use some of the results of [L1] to write down an admissible presentation matrix for the knot module of an odd-dimensional simple knot with the right twist spin; but I can think of no way of doing this.

5 TORSION FREE KNOTS

In this chapter $\underline{k}$ will be a simple $(2q-1)$-knot whose exterior has infinite cyclic cover $\tilde{K}$. If $\tilde{L}$ is the infinite cyclic cover of the exterior of its m-twist spin, then we have the Milnor exact sequence;

$$0 \longrightarrow H_{q+1}(\tilde{L}) \longrightarrow H_q(\tilde{K}) \xrightarrow{t^m-1} H_q(\tilde{K}) \longrightarrow H_q(\tilde{L}) \longrightarrow 0$$

We shall be interested in the case where $H_q(\tilde{L})$ turns out to be Z-torsion free. The simple even dimensional knots satisfying this condition have been classified for $q>4$ in [K2], as below;

Let $\mathcal{H}_i(\tilde{L}) = H_i(\tilde{L})/2.H_i(\tilde{L})$ for $i=q,q+1$

$\Pi_{q+1}(\tilde{L}) = \pi_{q+1}(\tilde{L})/2.\pi_{q+1}(\tilde{L})$,

and let $p_q(\tilde{L}): H_q(\tilde{L}) \longrightarrow \mathcal{H}_q(\tilde{L})$ denote the quotient map.

Then there is a short exact sequence of Γ-modules;

$$\xi(L):\ 0 \longrightarrow \mathcal{H}_q(\tilde{L}) \xrightarrow{\Omega} \Pi_{q+1}(\tilde{L}) \xrightarrow{H} \mathcal{H}_{q+1}(\tilde{L}) \longrightarrow 0$$

where $\Gamma = Z_2[t,t^{-1}]$, H is induced by the Hurewicz homomorphism, and Ω is induced by the map;

$$\omega : H_q(\tilde{L}) \longrightarrow \pi_{q+1}(\tilde{L})$$

which takes the homology class of a q-cycle, which will be homologous to a q-sphere Σ by the Hurewicz isomorphism theorem; and takes it into the homotopy equivalence class of the composition of the non-trivial element of $\pi_{q+1}(\Sigma)$ with the inclusion $\Sigma \hookrightarrow \tilde{L}$. There are also two non-singular hermitian pairings defined on these Γ-modules;

$$\langle , \rangle : \mathcal{H}_{q+1}(\tilde{L}) \times \mathcal{H}_q(\tilde{L}) \longrightarrow \Gamma_0/\Gamma$$

induced by the Blanchfield duality pairing, where Γ_0 is the field of fractions of Γ; and;

$$[,] : \Pi_{q+1}(\tilde{L}) \times \Pi_{q+1}(\tilde{L}) \longrightarrow \Gamma_0/\Gamma$$

which is related to the first pairing by;

$$[u, \Omega(v)] = \langle H(u), v \rangle \quad \text{for all } u \in \Pi_{q+1}(\tilde{L}),\ v \in \mathcal{H}_q(\tilde{L}).$$

All these pieces of information together define an algebraic object $(\xi(\tilde{L}), H_q(\tilde{L}), p_q, [,], \langle , \rangle)$ called an F-form; and Cherry Kearton in [K2] shows that two such knots are equivalent if and only if their F-forms are isometric, ie. their exact sequences ξ are connected by maps which make up a commutative diagram, which act as isometries of the two pairings, and which commute with the projection maps p_q. A special sort of F-form is singled out by him in [K3]; if the sequence ξ is split, so that there exist maps i,j such that;

$$0 \longrightarrow \mathcal{H}_{q+1}(\tilde{L}) \xrightarrow{i} \Pi_{q+1}(\tilde{L}) \xrightarrow{j} \mathcal{H}_q(\tilde{L}) \longrightarrow 0$$

is a short exact sequence with $j\Omega = id$, $Hi = id$; and if the image of i is self-annihilating under [,], then we say that this F-form is hyperbolic . In [K3] it is shown that any simple spun knot has a hyperbolic F-form; and that if a torsion free simple (2q)-knot has a hyperbolic F-form, and Alexander modules which could arise as the Alexander module of a simple (2q-1)-knot k, then the knot is equivalent to the spin of k, provided again that $q>4$. Unfortunately, the geometry has proved too intractable to give such a powerful characterization of m-twist spun knots; we can however show that a torsion-free m-twist spun knot which is Z-torsion free and simple will have a hyperbolic F-form provided m is odd by using the following purely algebraic result;

Proposition 28/If the annihilator ideal of $M=\Pi_{q+1}(\tilde{L})$ in Γ, which must be principal since Γ is a PID, is generated by a squarefree polynomial, then the F-form of a torsion-free simple knot 1 whose exterior has infinite cyclic cover $\tilde{L}$ will be hyperbolic.

Proof As Γ is a PID, M can be expressed as a direct sum $\oplus\,\Gamma/(f_i)$, where the f_i are polynomials in Γ; since this module is annihilated by a squarefree polynomial, the f_i can be chosen to be irreducible. Then it is clear that any submodule of M is in fact a direct summand, and the sequence $\zeta(\tilde{L})$ must split, so that;

$$M = \Omega(\mathcal{H}_q(\tilde{L})) \oplus N \text{ say, where } N \cong \mathcal{H}_{q+1}(\tilde{L}).$$

Choose a basis (a_i) for $\Omega(\mathcal{H}_q(L))$, so that the a_i generate the cyclic submodules $\Gamma/(f_i)$. By Blanchfield duality, we can extend this to a basis $(..,a_i,....,b_i,..)$ of M, where b_i generates $\Gamma/(f_i)$. On the subspace annihilated by an f_i such that $f_i \not\sim \bar{f}_i$, the pairing [,] must be hyperbolic by [M2]; on a subspace annihilated by an f_i such that $f_i \sim \bar{f}_i$, we may express the pairing by a matrix;

$$\begin{pmatrix} 0 & A \\ A & B \end{pmatrix}$$

where A is non-singular (as a matrix over the field $\Gamma/(f_i)$) since [,] is, and there are zeroes in the top left hand quadrant because;

$$\begin{aligned} [\Omega(x),\Omega(y)] &= \langle H\Omega(x),(y)\rangle \\ &= \langle 0,y\rangle \qquad \text{as } \mathfrak{S}(\tilde{L}) \text{ is exact} \\ &= 0 \qquad \text{for all } x,y \in \mathcal{H}_q(\tilde{L}) \end{aligned}$$

Our problem now is to change the basis for this submodule so that this matrix comes into hyperbolic form, ie. B becomes zero; this is equivalent to finding a matrix V such that;

$$B - VA - \overline{A^T V^T} = 0$$

Now, as M is the homotopy module of a knot, t-1 must act as an automorphism; so we must have $f_i(1)=1$, and f_i may be written in the form $t^r+..+1+..+t^{-r}$. Adding f_i to the diagonal elements of B if necessary, we may represent B as;

$$\begin{pmatrix} d_1 & b_{12} & b_{13} & \cdots & b_{1n} \\ \bar{b}_{12} & d_2 & b_{23} & \cdots & b_{2n} \\ \bar{b}_{13} & \bar{b}_{23} & d_3 & \cdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \bar{b}_{1n} & \bar{b}_{2n} & \cdots & \cdots & d_n \end{pmatrix}$$

where $d_i = \bar{d}_i$ has no constant term, and so can be written as $c_i + \bar{c}_i$. Then $B = C + \overline{C^T}$, where C is;

$$\begin{pmatrix} c_1 & b_{12} & b_{13} & \cdots & b_{1n} \\ & c_2 & b_{23} & \cdots & b_{2n} \\ & & c_3 & \cdots & \vdots \\ & 0 & & \ddots & \vdots \\ & & & & c_n \end{pmatrix}$$

Then we may set $V = CA^{-1}$, again considering these as over the finite field $\Gamma/(f_i)$. So the pairing [,] can be expressed in this hyperbolic form, with zeroes in the top left and bottom right quadrants of its matrix; and it is easy to see how this matrix may be used to define a splitting of the sequence $\mathfrak{J}(\tilde{L})$ which shows that the

F-form is hyperbolic. //

In the case of an m-twist spun knot for m odd, theorem 8 implies that $t^m=1$ when considered as a map of the fibre of the twist spin; and this fibre has the same homology and homotopy modules as the infinite cyclic cover. So $\pi_{q+1}(\tilde{L})$ is annihilated by t^m-1, which has no repeated factors modulo 2 since it is coprime to its derivative modulo 2. Unfortunately, of the conditions of proposition 13 and corollary 14, the condition that the module of an m-twist spun knot must be annihilated by t^m-1 is the only one that has any force in the torsion-free case; this is due to a result of Kervaire ([Ke], Lemme II.12) which shows that any torsion free Λ-module satisfying the Levine axioms possesses a square presentation matrix, so automatically has an order ideal which is principal; and the fact that this ideal must be generated by a self-conjugate element follows from the fact that it must be a Z-torsion free module with t-1 acting as an automorphism and annihilated by t^m-1. Also, the fact that the knot must be fibred also follows from this condition, using Browder and Levine's fibration theorem ([BL]), as the Alexander polynomial must be a product of cyclotomic polynomials dividing t^m-1, and thus monic.

We conjecture that the F-form of any Z-torsion-free twist-spun simple knot will always be hyperbolic, as is the case for spun simple knots ([K3]); but it seems rather hard to derive the F-form from the module and pairing of the original (2q-1)-knot $\underline{k}$. As we saw in the proof of proposition 28, it would suffice to show that the sequence $\zeta(L)$ split; and, as Γ is a PID, we would just have to express the constituent modules as sums of irreducible cyclics in order to discover whether or not this happens. The structure of the modules $\mathcal{H}_i$ is determined by Milnor's exact sequence; and it is not hard to see that if the module Π_{q+1} had the same annihilator ideal in Γ as these, then the sequence would have to split. In an attempt to find out how these ideals are related, we need to look at the geometry of the situation.

Firstly, we interpret Milnor's exact sequence associated with the m-fold cyclic cover the exterior of the knot $\underline{k}$, which has the form;

$$0 \longrightarrow H_{q+1}(\tilde{K}_m) \xrightarrow{d} H_q(\tilde{K}) \xrightarrow{t^m-1} H_q(\tilde{K}) \longrightarrow H_q(\tilde{K}_m) \longrightarrow 0$$

Elements of $H_q(\tilde{K}_m)$ can be represented by the projections of q-cycles in $H_q(\tilde{K})$; how can we represent an element a of $H_{q+1}(\tilde{K}_m)$ whose image d(a) is annihilated by t^m-1? The homotopy modules of K are trivial in dimensions below q; so by the Hurewicz theorem, we can represent d(a) by a spherical q-cycle $\tilde{a}$ such that $(t^m-1)\tilde{a}$ bounds a "cylindrical" (q+1)-chain homeomorphic to $S^q \times I$, since the two end spheres are homotopic. When we project down to $\tilde{K}_m$, the ends join up, and we get a "toroidal" (q+1)-cycle $\mathfrak{t}$, which may be seen to represent the desired element of $H_{q+1}(\tilde{K}_m)$.

To relate this to the sequnce $\xi(\tilde{L})$ associated with the m-twist spin $\underline{1}$ of $\underline{k}$, remember that the fibre F, which is homeomorphic to the branched cyclic coverK minus an open ball, of $\underline{1}$ is homotopy equivalent to $L \cong F \times \mathbb{R}$, so we may write this sequence as;

$$\xi(K_m):\ 0 \longrightarrow \mathcal{H}_q(K_m) \xrightarrow{\ \Omega\ } \Pi_{q+1}(K_m) \xrightarrow{\ H\ } \mathcal{H}_{q+1}(K_m) \longrightarrow 0$$

To find an element of $\Pi_{q+1}(K_m)$ projecting by H to the element of $\mathcal{H}_{q+1}$ represented by $\mathfrak{t}$, take a path $p: S^1 \longrightarrow \mathfrak{t} \cong S^q \times S^1$ running once round the S^1 factor; when we add the 2-handle to $\tilde{K}_m$ to form K_m, this path will bound a 2-disc; and if we perform an ambient surgery on $\mathfrak{t}$ whose core is this disc, we will get a (q+1)-sphere representing the same homology class, and which will define an appropriate element of $\Pi_{q+1}(K_m)$. This element is by no means unique; we could alter it by adding any element of $\Omega(\mathcal{H}_q(K_m))$, which would correspond to adding, in the ordinary homotopy group $\pi_{q+1}(K_m)$, an element represented by a homotopically non-trivial map from the (q+1)-sphere to a q-sphere in K_m. If we want to show that the sequence ξ splits, then we would need to choose this element in some canonical way for each generator of $\mathcal{H}_{q+1}(K_m)$; and then we would have to show that it was killed by the same elements of Γ.

Appendix A

Throughout this section Λ denotes $Z[t,t^{-1}]$, f,g,h denote polynomials in Λ, and I,J,K denote ideals of Λ. We define the resultant of two polynomials;

$$f(t) = a_r t^r + a_{r-1} t^{r-1} + \ldots + a_m t^m$$

and $$g(t) = b_s t^s + b_{s-1} t^{s-1} + \ldots + b_n t^n$$

where none of a_r, b_s, a_m, b_n is zero, to be the determinant;

$$\begin{vmatrix} a_r & & & b_s & & \\ a_{r-1} & a_r & & \vdots & b_s & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ \vdots & \vdots & a_r & b_{n+1} & \vdots & b_s \\ a_m & \vdots & \vdots & b_n & \vdots & \vdots \\ & a_m & \vdots & & b_n & \vdots \\ & & \ddots & & & \ddots \\ & & a_m & & & b_n \end{vmatrix}$$

where there are r+s-m-n columns and rows. We list some properties of the resultant, which we write R(f,g); for proofs see [vdW].

(R1) If $\{\alpha_i\},\{\beta_j\}$ are the sets of roots of f(t),g(t) (excluding zero), then;

$$R(f,g) = a_r^{s-n} b_s^{r-m} \prod_{i,j} (\alpha_i - \beta_j)$$

$$= a_r^{s-n} \prod g(\alpha_i)$$

(R2) R(f,g)=0 if and only if f and g have a common factor which is not a unit of Λ.

(R3) R(f,gh)=R(f,g).R(f,h).

(R4) There exist polynomials p,q such that;

$$p(t)f(t) + q(t)g(t) = R(f,g)$$

(R5) The resultant of two symmetric polynomials of even degree is a perfect square.

Proof If f,g are two such polynomials, we can write

$$t^{-n} f(t) = F(t+t^{-1}), \quad t^{-m} g(t) = G(t+t^{-1})$$

where n,m are integers, and F,G are integer polynomials. If the roots of g(t) are $\{\alpha_i\}$, we have;

$$R(f,g) = \prod f(\alpha_i) = \prod F(\alpha_i + \alpha_i^{-1})$$

But, since g is symmetric, the roots occur in reciprocal pairs; and the $\alpha_i + \alpha_i^{-1}$ are precisely the roots of G. So we have;

$$R(f,g) = (R(F,G))^2 \quad \text{as desired //}$$

<u>(R6)</u> $R(f,g) = (-1)^{pq} R(g,f)$, where $p=\deg f$, $q=\deg g$.

The following lemmas assist in the manipulation of Λ-modules;

<u>Lemma A1/</u>If at least one of f and g has first and last coefficients equal to ± 1, then we have;

$$|\Lambda/(f,g)| = |R(f,g)|$$

<u>Proof</u> [We] //

<u>Lemma A2/</u>If I and J are coprime ideals (ie. $I+J=\Lambda$), then $IJ=I \cap J$.

<u>Proof</u> If $i \in I, j \in J$, then $ij \in I$ and $ij \in J$; so $IJ \subset I \cap J$. Conversely, suppose $x \in I \cap J$. Since $I+J=\Lambda$, we can find $a \in I, b \in J$ such that $a+b=1$; then we have;

$$x = x(a+b) = ax+xb \in IJ \ //$$

<u>Lemma A3/</u>If $M=\Lambda/I$, then $M/J.M \cong \Lambda/I+J$

<u>Proof</u> Check that the sequence;

$$0 \longrightarrow I.(\Lambda/J) \longrightarrow \Lambda/J \longrightarrow \Lambda/(I+J) \longrightarrow 0$$

where the maps are either the obvious inclusion or surjection is exact; exactness at the middle term follows because;

$$\begin{aligned} x+(I+J)=(I+J) &\Leftrightarrow x=i+j \text{ for some } i \in I, j \in J \\ &\Leftrightarrow x+J=i+J \text{ for some } i \in I \\ &\Leftrightarrow x+J=i.(1+J) \quad I.(\Lambda/J) \ // \end{aligned}$$

<u>Lemma A4/</u>Suppose we have $I+J+K=\Lambda$; then $(I+J)(I+K)=I+JK$.

<u>Proof</u> $(I+J)(I+K) = I^2+IJ+IK+JK = I(I+J+K)+JK = I+JK$ //

We will use this lemma most often when I,J,K are principal ideals (f),(g),(h). In this case, $(f,g,h)=\Lambda$ if any two of the possible resultants of thes polynomials are coprime; for, by (R4), $R(f,g)$ must annihilate $\Lambda/(f,g)$; so in this case $\Lambda/(f,g,h)$ will be annihilated by two coprime integers.

Lemma A5/There is a Cartesian square;

$$\begin{array}{ccc} \Lambda/I\cap J & \xrightarrow{f_1} & \Lambda/I \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle h_2} \\ \Lambda/J & \xrightarrow{h_1} & \Lambda/I+J \end{array}$$

ie. $\Lambda/I\cap J \cong \{(x,y)\in\Lambda/I\oplus\Lambda/J : h_2x = h_1y\}$

Proof Define the maps by;

$$\begin{array}{lll} f_1: & x+(I\cap J) & \longmapsto x+I \\ f_2: & x+(I\cap J) & \longmapsto x+J \\ h_1: & x+J & \longmapsto x+(I+J) \\ h_2: & x+I & \longmapsto x+(I+J) \end{array}$$

Then there is an exact sequence;

$$0 \longrightarrow \Lambda/I\cap J \xrightarrow{(f_1,f_2)} \Lambda/I\oplus\Lambda/J \xrightarrow{p} \Lambda/I+J \longrightarrow 0$$

where $p(x+I,y+J)=h_2(x+I)-h_1(y+J)=x-y+(I+J)$, as may be checked. //

Corollary A6/If $I+J=\Lambda$, then $\Lambda/IJ \cong \Lambda/I \oplus \Lambda/J$

Proof A2 and A5. //

Lemma A7/If $I\subset J$ are ideals of finite index in $\Lambda_m=Z(Z_m)$, and $n_d I$ denotes the positive generator of the norm of the projection of I into $Z[\zeta_d]$, then $n_d I \geqslant n_d J$ for all d dividing m, and $\prod_{d|m} n_d I = \prod_{d|m} n_d I$ iff I=J.

Proof The first assertion is clear, as is the "if" part of the second. If $n_d I > n_d J$ for some d dividing m, we certainly have $n_d I \neq n_d J$; so there is an element in the projection of J into $Z[\zeta_d]$ which is not in the projection of I; and this element must be projected from an element of J not in I. //

Lemma A8/If N_d denotes the composition of the projection $\Lambda \longrightarrow Z[\zeta_d]$ with the norm map into the integers, then $N_d(f(t)) = R(f, \Phi_d)$

Proof By (R1), $R(f,\Phi_d) = \prod_{(a,d)=1} f(\zeta_d^a) = N_d(f(t))$. //

Finally, we prove the following numerical result which we make use of in chapter 4;

To Prove that $x^2-1093y^2=\pm 12$ has no solutions in the integers.

Proof If (u,v) were a solution, then so would $(\pm u, \pm v)$ be; so we may assume u,v>0. Then, $(u+v\sqrt{1093})(u-v\sqrt{1093})=\pm 12$; and since the largest value of $\pm u \pm v\sqrt{1093}$ is taken when both signs are positive, we can see that $u+v\sqrt{1093}>\sqrt{12}$. Let $f=33+\sqrt{1093}$, $f'=33-\sqrt{1093}$; $ff'=-4$, and if we had $u+v\sqrt{1093}>\sqrt{3}f$, then $s+t\sqrt{1093} = -(u+v\sqrt{1093})(33-\sqrt{1093})/2$ also has $s^2-1093t^2=\pm 12$, but;

(i) $s+t\sqrt{1093} < u+v\sqrt{1093}$

and (ii) $s+t\sqrt{1093} > -\sqrt{3}ff'/2 = \sqrt{12}$

so $s+t\sqrt{1093}$ would represent a smaller solution, again with $s,t>0$, since $s+t\sqrt{1093}>\sqrt{12}$ implies that this has the largest value among the numbers $\pm s\pm t\sqrt{1093}$. s and t must both be integers, since in order for u and v to satisfy the above equation, they must either both be even or both be odd. So, by continuing this process, we will find a solution (u,v) with;

$$\sqrt{12} < u+v\sqrt{1093} < \sqrt{3}.f$$

Finally, if $u^2-1093v^2=-12$, let $s+t\sqrt{1093} = (u+v\sqrt{1093})(33-\sqrt{1093})/2$; then $s^2-1093t^2=12$, and $(|s|,|t|)$ gives a solution (u,v) with $u,v>0$ and;

$$u^2-1093v^2=12 \implies u<v\sqrt{1093}$$

$$\text{and } \sqrt{12} < u+v\sqrt{1093} < \sqrt{3}.f \implies 2u < \sqrt{3}.f$$

$$\implies u < (\sqrt{3}/2).(33+\sqrt{1093}) < 34\sqrt{3} < 70$$

But $u^2=12 \pmod{1093}$; so $u=\pm 162 \pmod{1093}$, which gives us a contradiction. //

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