Yang-Mills theories in curved space-times

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YANG-MILLS THEORIES IN CURVED SPACE-TIMES

by

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"If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in."

R.P.Feynman, The Character of Physical Law
(1964 Messenger Lectures)

CONTENTS

ABSTRACT

Multi-instanton solutions of four dimensional $\mathbb{H}P^1$ models are sought, and a singular two instanton solution in flat Euclidean space-time is constructed. Non-singular multi-instanton solutions can be constructed if a gravitational field is introduced, as first pointed out by Gürsey et al . Their method is developed, and in the process a formalism for the construction of an (anti) self-dual SU(2) Yang-Mills field tensor in curved space-times is exhibited. Demanding that a potential for the SU(2) field exists implies that, for a space of non-zero scalar curvature, Einstein's field equations must be satisfied, and conditions on the Weyl tensor are found. It is shown how the formalism relates to the work of Charap and Duff . Finally the method is applied to the four dimensional complex projective space and the four dimensional manifold consisting of the outer product of two two spheres.

DECLARATION

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Ph.D. Thesis by Brian Patrick Dolan

The work for this thesis was carried out at the Department of Mathematics, University of Durham, Durham, U.K., between October 1978 and July 1981. This thesis has not been submitted for any other degree.

The latter part of chapter two was done in collaboration with Dr. D.B.Fairlie and Dr.W.J.Zakrzewski, and is claimed as original. Chapters three, four, five and six are also claimed as original, except where otherwise indicated. The first part of chapter three is published in Journal of Physics A[23] and most of chapters four and five is available as a Durham University pre-print[24]. Where other authors have done similar work, they have been acknowledged in the text.

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CHAPTER 1

INTRODUCTION

At least three of the four forces of nature presently known, electromagnetism and the strong and weak nuclear forces, seem to be described by gauge theories, of the type first enunciated by Yang and Mills[63] and Shaw[56]. Gauge theories of the fourth force, gravity have also been developed, Utiyama[60], Kibble[47]. In particular the symmetry of the strong nuclear force is widely believed to be SU(3) - chromodynamics. The evidence for SU(3) of colour is manifold, though indirect[17]. The most compelling evidence is,

(i) The ratio, R, of the amplitudes for $e^+e^- \longrightarrow$(hadrons) over $e^+e^- \longrightarrow$ (leptons), depends on the number of quarks and their charges. For energies below charmed threshold, with only three flavours of quark, i = 1,2,3 with charges e_i,

$$R = \sum_{i=1}^{3} e_i^2 = \begin{cases} 2/3, & \text{no colour} \\ 2, & \text{three colours.} \end{cases} \quad (1)$$

Experiment favours the coloured case.

(ii) The rate for the π^0 to decay into two photons, again depends on the number of quarks in the pion, their charges and the direction of their isospins in isospin space. The amplitude is proportional to

$$\sum_{u,d} I_3 e^2 = \begin{cases} 1/6, & \text{no colour} \\ 1/2, & \text{three colours} \end{cases} \quad (2)$$

Experiment favours the coloured case.

(iii) With only one type of quark for each flavour, the Pauli exclusion principle forbids three quarks of the same flavour to be in the same state. Thus the Δ^{++} should not exist. However, if each quark can come in three different colour states, they can all have the same spin without violating the exclusion principle.

Thus, in any attempt to understand the colour force, it is very important to analyse the Yang-Mills equations for a non-abelian SU(3) gauge theory. Unfortunately, explicit solutions are difficult to find, though Atiyah et al[4] have given a procedure for implicitly constructing all solutions for which the field tensor is (anti) self-dual . These solutions are topologically non-trivial, a fact which owes its existence to the four dimensional nature of the world in which we live. Since the topological charge density can be written as a total divergence, it depends only on the value of the gauge fields at very large distances from the origin, i.e. on the "surface at infinity", S^3. The topological charge is the winding number of the map from S^3 to the gauge group, given by the fields at infinity. It is a remarkable fact, that for <u>any</u> simple lie group G

$$\pi_3(G) \approx \mathbb{Z} \tag{3}$$

Thus the maps fall into topologically inequivalent classes, labelled by the integers, $\mathbb{Z}$[1].

In order to try and understand SU(3) better, it is useful to examine the case of SU(2). Here, explicit solutions are known[6,44], the 'tHooft solutions. These solutions are localised in both Euclidean space and time, and so are called "instantons". Since instantons have non-zero action, they will contribute to the quantum mechanical functional integral for the Yang-Mills fields and it has been suggested that they may provide a mechanism for the confinement of quarks[14]. Indeed, for a simplified, two dimensional U(1) gauge theory, $\mathbb{C}P^1$, the functional integral can be explicitly evaluated and a logarithmic, confining potential between "instanton quarks" has been demonstrated[19,20]. To try and extend this to SU(2) in four dimensions, it is a very compelling step to consider quaternionic fields, and this approach has been considered by a number of authors[30, 38, 45, 46,52]. In particular Gürsey[36] has suggested an extension of Einstein's work on a generalised theory of gravitation[29,39]. Einstein considered a complex, Hermitian, metric whose real part was the usual $g_{\mu\nu}$ of four dimensional curved space-time and whose purely imaginary part was an electromagnetic field tensor, $F_{\mu\nu}$. He showed that, with certain conditions on the Christoffel symbols, the field equations for electrodynamics in a curved space-time were automatically satisfied. Gürsey[36] has proposed that this approach could be extended to SU(2) Yang-Mills in curved space-time by considering a quaternionic, Hermitian metric whose real part is the metric of space-time and whose purely quaternionic part is a SU(2) Yang-Mills field tensor.

From a completely different point of view, Charap and Duff[15] and Atiyah et al[5] have considered SU(2) Yang-Mills in a curved space-time, by taking Utiyama's O(4) gauge theory of gravity[60] and performing the decomposition $O(4) \approx SU(2) \times SU(2)$. They show that, provided $R_{\mu\nu} = 0$, the O(4) field tensor decomposes into a self-dual and an anti-self-dual SU(2) field. Other authors who have considered SU(2) Yang-Mills in curved space-times are Boutaleb-Joutei et al[8-13]. Pope and Yuille[55] and Gibbons and Pope[32].

In this work a method of implementing Gürsey's suggestion is developed, and it is shown that it is intimately related to the construction of Charap and Duff. In chapter 2, $\mathbb{C}P^n$ models in two dimensions and their extension to SU(2) invariant models in four dimensions, $\mathbb{H}P^n$ models, are reviewed, and a singular, two instanton configuration in $\mathbb{H}P^1$ is constructed. In chapter 3, $\mathbb{H}P^1$ is coupled to gravity, via Gürsey's quaternionic metric, and the non-singular, O(4) symmetric, multi-instanton solutions of Gürsey et al[37] are extended beyond the O(4) symmetric case. In the process, a method is developed for the construction of a quaternionic metric, whose purely quaternionic part automatically satisfies the Yang-Mills equations in the curved space-time described by its real part. This requires the introduction of quaternionic Vierbeins. In chapter 4, the methods developed for $\mathbb{H}P^1$ are extended to SU(2) Yang-Mills, and it is shown that the existence of a potential for the self-dual field constructed from the quaternionic Vierbeins, actually implies that Einstein's field equations, with a cosmological constant, are satisfied, provided the curvature scalar is non-zero. Further conditions on the Weyl tensor are also derived and it is shown that the construction is the same as that of Charap and Duff[15] except that $R \neq 0$. In chapter 5, the method is applied to $\mathbb{C}P^2$, a gravitational instanton, to yield a self-dual SU(2) field with non-integral topological charge and an anti-self-dual electromagnetic instanton, as in[5,32]. In chapter 6, U(1) fields over $S^2 \times S^2$ are considered from the same point of view and "dyons" are constructed. Finally, in chapter 7, the main results are summarised and the possible extension to the SU(3) of nature is dicussed.

Appendix A sets up notation, by way of a review of quaternions and their relationship to SU(2), and appendix B contains the explicit evaluation of some integrals encountered in chapter 2.

All references are collected together at the end, in alphabetical order, and are referred to in the text by superscript, e.g.[5]. Equations appearing in current chapters are referred to by their numbers in round brackets, e.g.(42), while equations appearing in remote chapters are denoted by round brackets with the chapter number, followed by the equation number in that chapter, e.g. (3.42), means equation 42 of chapter 3.

CHAPTER 2

PROJECTIVE SPACE MODELS IN TWO AND FOUR DIMENSIONS

The complex projective space, $\mathbb{C}P^n$, is the space of all complex lines passing through a point (e.g. the origin) of $\mathbb{C}^{n+1}$. It can be represented by identifying some of the points of $\mathbb{C}^{n+1}$ in the following manner

Let

$$z = \begin{bmatrix} z_o \\ z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{bmatrix} \tag{1}$$

be a complex vector in $\mathbb{C}^{n+1}$, $z_i \in \mathbb{C}$, $i = o, \ldots, n$. The complex line through the origin is given by cz for some z and all $c = |c|e^{i\alpha}$, where $\alpha \in \mathbb{R}$. Then all the points on the same line are identified, and we can represent each such line by a subset of its points. We choose to normalise the representatives of each line so that

$$z^{\dagger}z = \sum_{i=o}^{n} \bar{z}_i z_i = 1 \tag{2}$$

(here $\bar{z}_i$ denotes complex conjugate on any complex number, and $\dagger$ denotes Hermitian conjugate on any matrix i.e. transpose followed by complex conjugation).

Given the normalisation (2), there is still a phase degeneracy in the choice of z. Any $z \in \mathbb{C}^{n+1}$ which obeys (2) is in the same complex line through the origin of $\mathbb{C}^{n+1}$ as $e^{i\alpha}z$ (where α is real) which also satisfies (2), and thus must be identified with z for the construction of $\mathbb{C}P^n$.

Thus $\mathbb{C}P^n$ can be thought of as the set of all complex (n+1)-plets, z, satisfying (2) such that, any two (n+1)-plets differing by an overall U(1) factor are identified. To calculate the dimension of $\mathbb{C}P^n$, we note that z has $2(n+1)$ degrees of freedom. Equation (2) removes one degree of freedom and the identification of z's differing by an overall $U(1)$ factor removes another, giving $\mathbb{C}P^n$ a real dimension of $2n$.

$\mathbb{C}P^n$ can also be thought of as the coset space $SU(n+1)/(SU(n)\times U(1))$ where $SU(n+1)$ is the special, unitary group, which can be represented by the set of all $(n+1) \times (n+1)$ complex matrices, M, for which $M^\dagger M = 1_{(n+1)\times(n+1)}$ and $\det M = 1$. This can be seen by thinking of the elements of $SU(n)$ as being $n \times n$ submatrices (the dotted submatrix below) embedded in the matrix representation of $SU(n+1)$

$$M = \begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0n} \\ u_{10} & u_{11} & \cdots & u_{1n} \\ \vdots & \vdots & & \vdots \\ u_{n0} & u_{n1} & \cdots & u_{nn} \end{bmatrix} \tag{3}$$

Taking the coset space $SU(n+1)/SU(n)$ means that all elements of $SU(n+1)$ which differ solely by a $SU(n)$ submatrix, as shown, are identified. The $(0,0)$ component of the unitarity condition $M^\dagger M = 1$ is equation (2), if we take $z_i = u_{io}$. The u_{oi} components are fixed, since given u_{io} and any element of $SU(n)$, u_{oi} are given by the unitarity condition. Factoring out a $U(1)$ component from z_i then gives $\mathbb{C}P^n$. As a check on the dimensions, note that the real dimension of $SU(n)$ is n^2-1, so $SU(n+1)/(SU(n) \times U(1))$ has dimension

$$(n+1)^2 - 1 - [(n^2-1)+1] = 2n \tag{4}$$

as before.

We now construct a field theory in two dimensions, where the fields are $\mathbb{C}P^n$ valued functions of $\mathbb{R}^2$ (as first developed by Eichenherr[27] and Golo and Perelomov[34]).

$$z : \mathbb{R}^2 \longrightarrow \mathbb{C}P^n$$

The U(1) freedom in the representation of z will be used as a gauge freedom.

Define

$$D_\mu z = \partial_\mu z - (z^\dagger \partial_\mu z) z \tag{5}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$, x^μ are co-ordinates on the underlying space $\mathbb{R}^2$, $\mu = 1,2$. Then take the Lagrangian density to be

$$\mathcal{L}(x_1,x_2) = \frac{1}{2}(D_\mu z)^\dagger D_\mu z \tag{6}$$

where the summation convention is used over repeated indices, μ. Note that, in Euclidean space-time, there is no distinction between covariant and contravariant indices.

Since, at each point of $\mathbb{R}^2$, z is only defined up to a U(1) factor, we can perform the local phase (gauge) transformation,

$$z(x_1,x_2) \longrightarrow e^{i\alpha(x_1,x_2)} z(x_1,x_2) \tag{7}$$

where $\alpha(x_1,x_2)$ is a real function of $\mathbb{R}^2$.

Note that $D_\mu z$ is covariant and $\mathcal{L}(x_1,x_2)$ is invariant under such a phase transformation.

For the action

$$S = \frac{1}{2}\int d^2x\,(D_\mu z)^\dagger D_\mu z \tag{8}$$

to be finite, z must be a constant vector (to within a, possibly direction dependent, phase factor) as $|x| \to \infty$. This phase factor gives a mapping from the circle at infinity into U(1).

The winding number of this map is given by

$$Q = \lim_{|x|\to\infty} \left(\frac{-i}{2\pi}\right) \oint_{S^1} (z^{+}\partial_\mu z)\, dx_\mu \tag{9}$$

where the integral is taken round the circle, radius $|x|$, centred on the origin.

Using Stokes's theorem, this is

$$\begin{aligned} Q &= -\frac{i}{2\pi}\int d^2x\; \varepsilon_{\mu\nu}\,\partial_\mu z^{+}\partial_\nu z \\ &= -\frac{i}{2\pi}\int d^2x\; \varepsilon_{\mu\nu}(D_\mu z)^{+} D_\nu z \end{aligned} \tag{10}$$

where $\varepsilon_{01} = -\varepsilon_{10} = 1$ is the antisymmetric tensor in two dimensions. Note in passing that the action, (8), is invariant as a functional under conformal transformations of the variables (x_1, x_2), and so the equations of motion and the field theory as a whole are invariant under such transformations. Thus we can equally well take the two dimensional space-time to be S^2 (conformally compactified $\mathbb{R}^2$) and (10) is the winding number of the map

$$z : S^2 \longrightarrow \mathbb{C}P^2$$

The equations of motion that one obtains by varying z in (8) are

$$D_\mu D_\mu z + 2\mathcal{L}(x_1,x_2)z = 0 \tag{11}$$

and it is well known[27,34] that these are satisfied by taking z to be of the form $z_i = \zeta_i/|\zeta|$ where $|\zeta|^2 = \zeta^{+}\zeta$, $i = 0,\dots,n$, with $\zeta_i(x)$ analytic, except for isolated poles, in the complex variable $x = x_1 + ix_2$ Such forms of z automatically saturate the lower bound on S

$$S \geqslant \pi\,|Q| . \tag{12}$$

For example, in $\mathbb{C}P^1$ write

$$z = \frac{e^{i\alpha}}{\sqrt{1+w\bar{w}}}\begin{bmatrix}1\\ w\end{bmatrix} \tag{13}$$

where w is a single, complex function of position. Then the solutions with winding number k are given by taking w to be a ratio of k^{th} degree polynomials in x (all the roots of one polynomial must be different from all the roots of the other, though multiple roots may occur within each polynomial)

$$w(x) = \prod_{s=1}^{k} \frac{(x-a_s)}{(x-b_s)} \tag{14}$$

where a_s and b_s , $s=1,\ldots,k$, are complex constants. A solution with winding number (-k) may be obtained from (14) by replacing x with $\bar{x}$. (14) is the k instanton solution of $\mathbb{C}P^1$. It has 4k-1 parameters (the -1 is due to global gauge freedom). For $\mathbb{C}P^1$ this exhausts all the solutions of the equations of motion (11).

For $\mathbb{C}P^n$ $(n \geq 2)$, solutions to the equations of motion have been found which do not saturate the inequality (12)[21,22]. For such solutions, the action is stationary, though it is not a minimum, but a saddle point. The solutions found in[21,22] exhaust all the solutions of $\mathbb{C}P^n$.

<u>O(3) σ - model in two dimensions</u>

The O(3) σ- model in two dimensions, is a field theory in which the fields are represented by real, three vectors of unit magnitude, which rotate under global O(3) rotations in field space.

$$\phi(x_1,x_2) = \begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\end{bmatrix} \qquad \phi^T\phi = 1 \tag{15}$$

(here ϕ^T denotes the transpose of ϕ).

The Lagrangian density is taken to be,

$$\mathcal{L}(x_1,x_2) = \frac{1}{2\lambda}\,\partial_\mu \phi^T \partial_\mu \phi \qquad (\lambda = \text{real constant}) \tag{16}$$

with the constraint $\phi^T\phi = 1$. While the topological charge density is

$$p(x_1,x_2) = \frac{1}{8\pi}\,\varepsilon_{abc}\,\varepsilon_{\mu\nu}\,\partial_\mu \phi^a \partial_\nu \phi^b \phi^c \tag{17}$$

where ε_{abc} is the completely antisymmetric tensor in three indices, $\varepsilon_{123} = 1$ etc.

As for the $\mathbb{C}P^n$ models, the action obtained by integrating (16) over space-time is invariant as a functional under conformal transformations of (x_1,x_2), and so we can take the space-time to be S^2 rather than $\mathbb{R}^2$. Thus $\phi : S^2 \to S^2$ and the integral of (17) is just the winding number of this map.

It is well known[19], that this model is equivalent to the $\mathbb{C}P^1$ model described above if we make the identifications

$$\phi^a = z^+ \sigma^a z \tag{18}$$

where σ^a, a= 1,2,3 are the Pauli matrices.

Then

$$\phi^1 = \frac{2\,\mathrm{Re}\,w}{(1+w\bar{w})} \qquad \phi^2 = \frac{2\,\mathrm{Im}\,w}{(1+w\bar{w})} \qquad \phi^3 = \frac{(1-w\bar{w})}{(1+w\bar{w})} \tag{19}$$

$$w = \frac{\phi^1 + i\phi^2}{(1+\phi^3)} \tag{20}$$

In terms of $w = w_1 + iw_2$, the Lagrangian density for $\mathbb{C}P^1$ is

$$\mathcal{L} = \frac{1}{2}\,\frac{\partial_\mu \bar{w}\,\partial_\mu w}{(1+w\bar{w})^2} \tag{21}$$

which, with (20) is identical to (16), with $\lambda = 4$.

In terms of w, the topological charge density of $\mathbb{C}P^1$ is

$$\rho(x_1, x_2) = -\frac{i}{2\pi}\,\varepsilon_{\mu\nu}\,\frac{\partial_\mu \bar{w}\,\partial_\nu w}{(1 + w\bar{w})^2}$$

$$= \frac{1}{\pi}\det\begin{bmatrix} \partial_1 w_1 & \partial_1 w_2 \\ \partial_2 w_1 & \partial_2 w_2 \end{bmatrix} \Big/ (1 + w\bar{w})^2 \tag{22}$$

Writing (17) as

$$\rho(x_1, x_2) = \frac{1}{4\pi}\det\begin{bmatrix} \partial_1 \phi^1 & \partial_2 \phi^1 & \phi^1 \\ \partial_1 \phi^2 & \partial_2 \phi^2 & \phi^2 \\ \partial_1 \phi^3 & \partial_2 \phi^3 & \phi^3 \end{bmatrix} \tag{23}$$

we find that (23) is identical to (22) using (19) and (20). Thus the $O(3)$ σ- model in two dimensions corresponds to the $\mathbb{C}P^1$ model, with the identifications (19).

$\mathbb{H}P^n$ Models in Four Dimensions

The quaternionic projective space, $\mathbb{H}P^n$, is defined in the same way as the complex projective space, $\mathbb{C}P^n$, with "complex" replaced with "quaternionic". (For a summary of the properties of quaternions, see appendix A and reference[38]. $\mathbb{H}P^n$ is the space of all quaternionic lines passing through a point (e.g. the origin) in $\mathbb{H}^{n+1}$, where a quaternionic line through the origin is the set $\{ q h : \forall h \in \mathbb{H} \}$ for some $q \in \mathbb{H}^{n+1}$. Given any such line we choose an element of unit norm to represent it

$$q = \begin{bmatrix} q_0 \\ \vdots \\ q_n \end{bmatrix} \tag{24}$$

where $q_i \in \mathbb{H}$, i=o...n and

$$\frac{1}{2}\mathrm{Tr}(q^\dagger q) = \sum_{i=0}^{n} \frac{1}{2}\mathrm{Tr}(q_i^\dagger q_i) = 1 \tag{25}$$

(here, and throughout this work, quaternions are thought of as being represented by 2x2 matrices - see appendix A).

The choice of a unit norm q to represent a line is not unique, since $q g$ will also do, where g is a quaternion of unit magnitude (it can be represented as $g = e^{i\alpha_a \sigma_a}$ where α_a, $a=1,2,3$ are real). g is thus an element of SU(2). Thus there is a SU(2) phase (gauge) freedom in our choice of q. q has 4(n+1) degrees of freedom, (25) removes one and the phase freedom removes another three, giving $\mathbb{H}P^n$ a real dimension of 4n.

Just as for $\mathbb{C}P^n$, $\mathbb{H}P^n$ can be thought of as the Grassmanian Sp(n+1)/(Sp(n)xSp(1)) where Sp(n+1) is the group of all (n+1)x(n+1) quaternionic matrices, N, for which $N^\dagger N = 1_{2(n+1)\times 2(n+1)}$. The dimension of Sp(n) is n(2n+1), so the dimension of Sp(n+1)/(Sp(n)xSp(1)) is

$$(n+1)[2(n+1)+1] - [n(2n+1)+3] = 4n \tag{26}$$

in agreement with the previous analysis. The Sp(1) factor is the SU(2) gauge freedom, Sp(1) $\approx$ SU(2).

We now construct a field theory in four dimensions where the fields are $\mathbb{H}P^n$ valued functions on $\mathbb{R}^4$ (as in ref [38])

$$q : \mathbb{R}^4 \longrightarrow \mathbb{H}P^n \tag{27}$$

Define

$$D_\mu q = \partial_\mu q - q(q^\dagger \partial_\mu q) \tag{28}$$

and

$$F_{\mu\nu} = (D_\mu q)^\dagger D_\nu q - (D_\nu q)^\dagger D_\mu q \tag{29}$$

note that $D_\mu q$ is covariant under local SU(2) gauge transformations $q \to q\, g(x)$ and that $F_{\mu\nu} \to g^{-1} F_{\mu\nu} g$ under such gauge transformations. Also, $F_{\mu\nu} = - F^{+}_{\mu\nu}$ is purely quaterionic.

Then we can set up a field theory using the Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2} \operatorname{Tr} (F_{\mu\nu} F^{\dagger}_{\mu\nu}) \tag{30}$$

This Lagrangian density integrates up to give an action whose functional form is conformally invariant, and so is a natural choice. Other candidates would be

$$\mathcal{L}(x) = (D_\mu D_\nu q)^{\dagger} D_\mu D_\nu q \tag{31}$$

or

$$\mathcal{L}(x) = [(D_\mu q_a)^{\dagger} D_\nu q_b - (D_\nu q_b)^{\dagger} D_\mu q_a][(D_\mu q_a)^{\dagger} D_\nu q_b - (D_\nu q_b)^{\dagger} D_\mu q_a] \tag{32}$$

where we sum over a,b=o,...,n, which label the components of the vector q. However, we choose to analyse (30), since it proves to be analogous to the SU(2) Yang-Mills Lagrangian density.

Define

$$A_\mu = - A^{\dagger}_\mu = q^{\dagger} \partial_\mu q \tag{33}$$

then (30) is the Lagrangian density for a SU(2) Yang-Mills theory with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{34}$$

(the Yang-Mills coupling constant is set equal to one).

The $\mathbb{H}P^n$ fields, for a given n, therefore form a subset of the possible SU(2) Yang-Mills fields i.e. those that can be written in the form (33).

The topological charge density is taken to be

$$p(x) = - \frac{1}{2} \operatorname{Tr} (F_{\mu\nu} {}^{*}F_{\mu\nu}) \tag{35}$$

where

$${}^{*}F_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \tag{36}$$

is the dual of $F_{\mu\nu}$. $\varepsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor in four indices.

The topological charge is the normalised integral over $\mathbb{R}^4$ of p(x),

$$Q = -\frac{1}{16\pi^2}\int d^4x \, \mathrm{Tr}\,(F_{\mu\nu}{}^*F_{\mu\nu}) \tag{37}$$

For the action to be finite, q must tend to a constant to within a, possibly direction dependent, SU(2) phase factor as $|x| \to \infty$

$$q \xrightarrow[|x|\to\infty]{\lim} q_0\, g\left(\frac{x}{|x|}\right) \tag{38}$$

where q_0 is a constant vector.

In this instance, the topological charge can be expressed as a surface integral[6].

$$Q = -\frac{1}{4\pi^2}\varepsilon^{\mu\alpha\beta\gamma}\int d^4x\, \partial_\mu \mathrm{Tr}\left[\tfrac{1}{2}(q^\dagger\partial_\alpha q)\partial_\beta(q^\dagger\partial_\gamma q) + \tfrac{1}{3}(q^\dagger\partial_\alpha q)(q^\dagger\partial_\beta q)(q^\dagger\partial_\gamma q)\right] \tag{39}$$

$$= -\frac{1}{4\pi^2}\varepsilon^{\mu\alpha\beta\gamma}\lim_{|x|\to\infty}\int d^3\sigma\, n_\mu \partial_\mu \mathrm{Tr}\left[\tfrac{1}{2}(q^\dagger\partial_\alpha q)\partial_\beta(q^\dagger\partial_\gamma q) + \tfrac{1}{3}(q^\dagger\partial_\alpha q)(q^\dagger\partial_\beta q)(q^\dagger\partial_\gamma q)\right]$$

where S^3 is the sphere with radius $|x|$ centred on the origin, n_μ is the unit outward normal to this sphere and $d^3\sigma$ is its volume element. Since $q^\dagger\partial_\mu q \to g^{-1}\partial_\mu g$ as $|x| \to \infty$ this becomes

$$Q = \lim_{|x|\to\infty}\frac{1}{24\pi^2}\int_{S^3} d^3\sigma\; \varepsilon^{\mu\alpha\beta\gamma} n_\mu \mathrm{Tr}\left[g^{-1}\partial_\alpha g\; g^{-1}\partial_\beta g\; g^{-1}\partial_\gamma g\right]. \tag{40}$$

As $g \in SU(2) \approx S^3$, this is the winding number of the map $g: S^3 \to S^3$.

The $\mathbb{H}P^n$ construction is exactly the form used by Atiyah et al[4] in constructing self-dual solutions of SU(2) Yang-Mills. Their construction exhausts all the self-dual solutions, but is however implicit, taking the form of conditions on q. Explicit solutions have been given by 'tHooft and subsequently conformally extended by Jackiw, Nohl and Rebbi[44]. They show that the lower bound

$$S \geqslant 8\pi^2 |Q| \tag{41}$$

where the action, S, is the integral of (30), is saturated by q's of the form

$$q_s = \frac{\rho_s (x^\dagger + a_s)^{-1}}{|v|} \quad (\text{no sum over } s) \tag{42}$$

where q_s, $s=0,\ldots,n$ are the components of q, x is the quaternion labelling position (see appendix A) ρ_s is real and

$$|v| = \sum_{s=0}^{n} \frac{\rho_s^2}{|x^\dagger + a_s|^2} \tag{43}$$

is a normalisation factor. These are n instanton configurations. Anti-instantons are obtained by sending $x^\dagger \to x$. The form (42), however, does not exhaust all possible self-dual configurations.

The $\mathbb{H}P^1$ Model

To examine the properties of these models, let us first of all restrict ourselves to the simplest case, that of $\mathbb{H}P^1$, where the field q is simply a two component quaternion unit vector $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$.

Writing this as

$$q = \frac{1}{|v|}\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \frac{1}{|v|} v \tag{44}$$

where v has any magnitude, $|v| = \frac{1}{2}\mathrm{Tr}\, v^\dagger v$, we can use the SU(2) freedom to rotate v_0 so that it is real at every point x (the gauge is now fixed) so that

$$q = \frac{1}{\sqrt{v_0^2 + \frac{1}{2}\mathrm{Tr}\,(v_1^\dagger v_1)}}\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \frac{1}{\sqrt{1 + v_0^{-2}\frac{1}{2}\mathrm{Tr}(v_1^\dagger v_1)}}\begin{bmatrix} 1 \\ v_1/v_0 \end{bmatrix} \tag{45}$$

where v_0 is now real, but v_1 is still quaterionic.

Let $u = u_i e_i = v_0^{-1} v_1$ be a single quaternionic function of position, then

$$q = \frac{1}{\sqrt{1 + u_i u_i}}\begin{bmatrix} 1 \\ u \end{bmatrix} \tag{46}$$

and now in terms of u

$$A_\mu = \frac{1}{2}\,\frac{u^+\partial_\mu u - \partial_\mu u^+ u}{(1+u_i u_i)} \tag{47}$$

and

$$F_{\mu\nu} = \frac{(\partial_\mu u^+ \partial_\nu u - \partial_\nu u^+ \partial_\mu u)}{(1+u_i u_i)^2} \tag{48}$$

Thus

$$\mathcal{L} = -\frac{1}{2}\,\frac{\mathrm{Tr}\left[\partial_\mu u^+ \partial_\nu u - \partial_\nu u^+ \partial_\mu u\right]^2}{(1+u_i u_i)^4} \tag{49}$$

The equations of motion for $\mathbb{H}P^1$ are obtained by varying $\mathcal{L}$ with respect to u. They are most easily obtained by writing u out in its components, then

$$\mathcal{L} = 2\,\frac{\left[\partial_\mu u_i \partial_\nu u_j - \partial_\nu u_i \partial_\mu u_j\right]^2}{(1+u_k u_k)^4} \tag{50}$$

and, varying u_i, we find that

$$\begin{aligned}
&\{\Box u_k (\partial_\mu u_i)(\partial_\mu u_i) + \partial_\rho\partial_\mu u_i \partial_\mu u_i \partial_\rho u_k - \partial_\rho \partial_\mu u_k \partial_\mu u_i \partial_\rho u_i - \partial_\mu u_k \partial_\mu u_j \Box u_j\} \\
&\qquad - 4\,\frac{\partial_\rho (u_m u_m)}{(1+u_\ell u_\ell)}\{(\partial_\mu u_i)(\partial_\mu u_i)\partial_\rho u_k - \partial_\mu u_k \partial_\mu u_j \partial_\rho u_j\} \\
&\qquad + \frac{2u_k}{(1+u_\ell u_\ell)}\{(\partial_\mu u_i \partial_\nu u_j)^2 - (\partial_\mu u_i \partial_\nu u_i)^2\} = 0 .
\end{aligned} \tag{51}$$

The O(5) σ - Model in Four Dimensions

Just as for $\mathbb{C}P^1$ in two dimensions, where there is a correspondence with the O(3) σ -model, there is a similar correspondence between $\mathbb{H}P^1$ in four dimensions and the O(5) σ -model [30]. The O(5) σ -model has fields which are real, five component unit vectors, $\underline{\phi}(x_0,..,x_3)$ with $\phi^T\phi = 1$. The $\underline{\phi}$ rotate under global O(5) rotations, and the Lagrangian density takes the following form,

$$\mathcal{L}(x) = \frac{1}{2\lambda}\left(\partial_\mu \phi_a \partial_\nu \phi_b - \partial_\nu \phi_a \partial_\mu \phi_b\right)^2 \tag{52}$$

where ϕ_a ,a=o,..,4 are the components of $\underline{\phi}$ and μ,ν,a and b are summed over. The topological charge density is

$$\rho(x) = \frac{1}{8}\,\varepsilon_{abcde}\,\varepsilon_{\mu\nu\rho\sigma}\,\partial_\mu \phi_a \partial_\nu \phi_b \partial_\rho \phi_c \partial_\sigma \phi_d\, \phi_e \tag{53}$$

$$= \frac{4!}{8} \det \begin{bmatrix} \phi_0 & \partial_0\phi_0 \ldots & \partial_3\phi_0 \\ \phi_1 & \partial_0\phi_1 \ldots & \partial_3\phi_1 \\ \vdots & & \vdots \\ \phi_4 & \partial_0\phi_4 \ldots & \partial_3\phi_4 \end{bmatrix}$$

If we make the following identifications, for $q \in \mathbb{H}P^1$

$$\phi_i = q^+ \begin{pmatrix} 0 & e_i^+ \\ e_i & 0 \end{pmatrix} q \qquad \phi_4 = q^+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} q \tag{54}$$

where i=o,...,3 and the entries in the 2x2 matrices are themselves 2x2 complex matrices, then

$$\phi_i = \frac{2u_i}{1+u_j u_j} \qquad \phi_4 = \frac{1+u_j u_j}{1-u_k u_k} \tag{55}$$

$$u_i = \frac{\phi_i}{1+\phi_4} \qquad u_j u_j = \frac{1-\phi_4}{1+\phi_4} \qquad 1+u_j u_j = \frac{2}{1+\phi_4}$$

where again i=o,...3,only, and

$$\partial_\mu u_i = \frac{\partial_\mu \phi_i}{1+\phi_4} - \phi_i \frac{\partial_\mu \phi_4}{(1+\phi_4)^2} \tag{56}$$

Now substitution of (55) and (54) into the $\mathbb{H}P^1$ Lagrangian density, together with $\phi_i\phi_i = 1-\phi_4^2 \Rightarrow \phi_i\partial_\mu\phi_i = -\phi_4\partial_\mu\phi_4$, i=o,...3, shows that these Lagrangians are identical, with λ =4.

Similarly, writing the $\mathbb{H}P^1$ topological charge density as

$$p(x) = -\operatorname{Tr}\left[\varepsilon_{\mu\nu\rho\sigma}\frac{\partial_\mu u^+\partial_\nu u\,\partial_\rho u^+\partial_\sigma u}{(1+u_iu_i)^4}\right]$$

$$= 2\frac{\varepsilon_{\mu\nu\rho\sigma}}{(1+u_iu_i)^4}\det\begin{bmatrix}\partial_\mu u_0 \ldots \partial_\sigma u_0\\ \vdots \\ \partial_\mu u_3 \ldots \partial_\sigma u_3\end{bmatrix} \tag{57}$$

one finds, using the properties of determinants, that this is identical to the O(5) σ- model topological charge density. Thus the $\mathbb{H}P^1$ model corresponds to the O(5) σ- model in a similar fashion to the way $\mathbb{C}P^1$ corresponds to the O(3) σ- model.

<u>Instantons in $\mathbb{H}P^1$</u>

In the light of $\mathbb{C}P^1$ models, where solutions are given by $w(x_1,x_2)$ being a function of either x_1+ix_2 or x_1-ix_2, but not both, with isolated poles and zeros, one's first guess for solutions of $\mathbb{H}P^1$ might be

$$u = \frac{1}{\rho}x^\dagger \tag{58}$$

where $x = x_ie_i$ is a quaternion, labelling position, and ρ is a real constant, with dimensions of length. Indeed, this satisfies the equations of motion (51). It is in fact a self-dual solution, since $\partial_\mu u = \frac{1}{\rho}e_\mu^\dagger$, $\mu=0,\ldots,3$, so (48) gives

$$F_{\mu\nu} = \frac{e_\mu e_\nu^+ - e_\nu e_\mu^+}{\rho^2(1+|x|^2/\rho^2)^2}$$

$$= \frac{2i\,\eta^{(+)a}_{\mu\nu}\,\sigma^a}{\rho^2(1+|x|^2/\rho^2)^2} \tag{59}$$

when $\eta^{(+)a}_{\mu\nu}$ is the self-dual 'tHooft tensor (see Appendix A).

A more general solution is

$$u = s(x^\dagger+a_1)(x^\dagger+a_2)^{-1} \tag{60}$$

where a_1 and a_2 are constant quaternions, with dimensions of length, and s is a real dimensionless constant. This is also self-dual, since

$$\partial_\mu u = (s-u)\, e_\mu^\dagger (x^\dagger + a_2)^{-1} \tag{61}$$

$$\Leftrightarrow \quad F_{\mu\nu} = \frac{(x^\dagger + a_2)^{\dagger -1} (e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger)(x^\dagger + a_2)^{-1} |s-u|^2}{(1 + u_i u_i)^2} \tag{62}$$

which is again manifestly self-dual. It has the same topological charge and action as (58) since it is merely a conformal transformation of (58).

The topological charge of (58) (and so of (60)) is one, since

$$\begin{aligned} Q &= \frac{1}{4\pi^2} \int \frac{d^4x}{\rho^4} \; \frac{\mathrm{Tr}\left[\eta^{(+)a}_{\mu\nu}\, \eta^{(+)b}_{\mu\nu}\, \sigma^a \sigma^b\right]}{(1 + |x|^2/\rho^2)^4} \\ &= \frac{6}{\pi^2 \rho^4} \int_0^\infty dR \int_0^\pi d\theta \int_0^\pi d\phi \int_0^{2\pi} d\psi \; \frac{R^3 \sin^2\theta \sin\phi}{(1 + R^2/\rho^2)^4} \\ &= 1 \end{aligned} \tag{63}$$

where we have used polar co-ordinates (R, θ, ϕ, ψ) on $\mathbb{R}^4$, with $R^2 = |x|^2$. Equation (60) is, in fact, the single instanton in Jackiw, Nohl and Rebbi's construction[44], and, noting that if u(x) is a solution of $\mathbb{H}P^1$, then $u^{-1}(x)$ gives exactly the same Lagrangian density and topological charge density and so is an equivalent configuration, (58) is seen to be simply the 'tHooft single instanton.

Again, guided by the $\mathbb{C}P^1$ model, a natural choice for a possible two instanton solution might be

$$u = {x^\dagger}^2/\rho^2 \tag{64}$$

However, we find that this does not give a self-dual field, nor indeed does it even satisfy the equations of motion (51).

A more general configuration would be

$$u = \frac{1}{\rho^2}(x^+ + a)(x^+ + b) \tag{65}$$

with a and b constant quaternions. One can try varying the values of a, b and ρ to see if there are any values for which they make the action stationary. Without loss of generality, we can move the origin to (a+b)/2 and rotate the time axis so that it passes through a and b. Then a = -b is real. Now let us see how the action varies as a function of the dimensionless parameter a/ρ, and how it compares with the topological charge.

With (65) and a = -b, real,

$$\partial_\mu u = \frac{1}{\rho^2}(x^+ e_\mu^+ + e_\mu^+ x^+) \tag{66}$$

giving

$$p(x) = -\operatorname{Tr}\left[\varepsilon_{\mu\nu\rho\sigma}\frac{(\partial_\mu u^+ \partial_\nu u\, \partial_\rho u^+ \partial_\sigma u)}{(1+u_i u_i)^4}\right]$$

$$= \frac{3.2^8}{\rho^8}\cdot\frac{t^2(t^2+r^2)}{\{1+\frac{1}{\rho^4}[(t+a)^2+r^2][(t-a)^2+r^2]\}^4} \tag{67}$$

where $t = x_0,\ r^2 = x_1^2 + x_2^2 + x_3^2$. Then

$$Q = \frac{96}{\rho^8\pi^2}\int_{-\infty}^{\infty}dt\int_0^{\infty}r^2dr\int_0^{\pi}\sin\theta\, d\theta\int_0^{2\pi}d\phi\,\frac{t^2(t^2+r^2)}{\{1+\frac{1}{\rho^4}[(t+a)^2+r^2][(t-a)^2+r^2]\}^4}$$

$$= \frac{3.2^8}{\rho^8\pi}\int_0^{\infty}dt\int_0^{\infty}dr\,\frac{t^2r^2(t^2+r^2)}{\{1+\frac{1}{\rho^4}[(t+a)^2+r^2][(t^2-a^2)+r^2]\}^4} \tag{68}$$

$$= 2.$$

(for details of the integral, see appendix B)

The value of the Lagrangian density is

$$\mathcal{L}(x) = -\frac{1}{2}\,\mathrm{Tr}\left[\frac{(\partial_\mu u^\dagger \partial_\nu u - \partial_\nu u^\dagger \partial_\mu u)(\partial_\mu u^\dagger \partial_\nu u - \partial_\nu u^\dagger \partial_\mu u)}{(1+u_i u_i)^4}\right]$$

$$= p(x) + \frac{2^7}{\rho^8}\cdot\frac{\tau^4}{\{1+\frac{1}{\rho^4}[(t+\alpha)^2+\tau^2][(t-\alpha)^2+\tau^2]\}^4} \qquad (69)$$

giving

$$S = 16\pi^2 + 2^{10}\pi\int_0^\infty d\tilde{t}\int_0^\infty d\tilde{\tau}\,\frac{\tilde{\tau}^6}{\{1+[(\tilde{t}+\tilde{\alpha})^2+\tilde{\tau}^2][(\tilde{t}-\tilde{\alpha})^2+\tilde{\tau}^2]\}^4} \qquad (70)$$

where $\tilde{t}=t/\rho$, $\tilde{\tau}=\tau/\rho$, $\tilde{\alpha}=\alpha/\rho$ are dimensionless. (For details of (69) see appendix B).

Upon performing one integration in (70) (see appendix B) we obtain

$$S = 16\pi^2 + 80\pi^2\int_0^\infty \frac{v^3\,dv}{[(v-\tilde{\alpha}^2)^2+1]^{1/2}\,[(v+\tilde{\alpha}^2)^2+1]^{7/2}} \qquad (71)$$

which was evaluated numerically, see graph on the next page

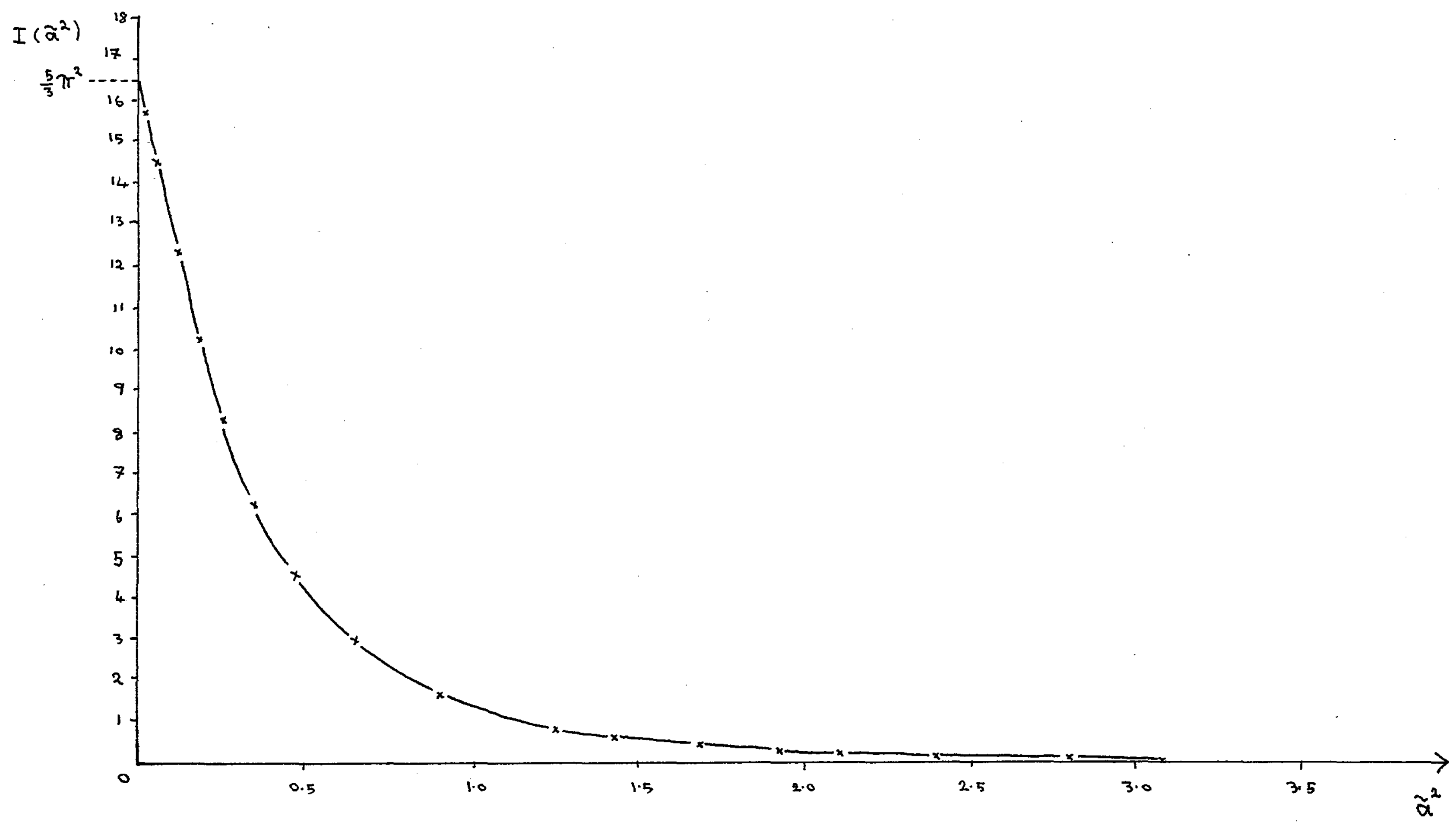
A measure of how close the configuration (65) is to a solution of the equations of motion is given by

$$I(\tilde{\alpha}^2) = S(\tilde{\alpha}^2) - 16\pi^2 \qquad (72)$$

with self-duality for $I(\tilde{\alpha}^2)=0$. The graph of $I(\tilde{\alpha}^2)$ as a function of $\tilde{\alpha}^2$ is shown on the next page

We see that the action mon~~t~~otonically approaches its lower bound in the two instanton sector as $\tilde{\alpha}^2\to\infty$. Furthermore, it approaches this limit very rapidly, being within 0.7% of it for $\tilde{\alpha}^2=1$. Thus we have a two instanton solution, only in certain limits:

Deviation of $S(\tilde{a}^2)$ from Lower Bound



(i) ρ fixed, $a \to \infty$ i.e. the instantons are of finite size, but $I(\tilde{a}^2)$ represents a repulsive interaction which sends them infinitely far apart.

(ii) a fixed, $\rho \to 0$ i.e. the instantons are at a finite separation, but their size shrinks to zero.

Case (i) has been analysed, in a slightly different form, by Neinast and Stack[52]. I am grateful to Werner Nahm for pointing out the interpretation(ii). Nahm has coined the phrase "virtual stationary points" for such configurations[51].

One expects that such configurations would contribute to functional integrals, since the action is finite, and therefore must be taken into account in any attempt to quantise SU(2) Yang-Mills theory. However, it has not proved possible to perform the functional integral, using (65) with $\tilde{a} \to \infty$, as a stationary point, due to the singular nature of the fields.

Indeed, for any finite integer $k > 1$, the configuration

$$u = (x^{\dagger})^k / \rho^k \tag{73}$$

has finite action and topological charge k, and so such configurations will contribute to functional integrals. The action and topological charge are most easily calculated using spherical polar coordinates.

$$x = R(\cos\theta + \hat{\underline{n}}\sin\theta) \tag{74}$$

where $R^2 = x_\mu x_\mu$, and $\hat{\underline{n}} = \cos\phi\, e_1 + \sin\phi\cos\psi\, e_2 + \sin\phi\sin\psi\, e_3$ and $0 \leq R < \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq \pi$, $0 \leq \psi \leq 2\pi$.

Since $\hat{\underline{n}}^2 = -1_{2\times 2}$, we have de Moivre's theorem for quaternions,

$$x^k = R^k(\cos k\theta + \hat{\underline{n}}\sin k\theta). \tag{75}$$

Then, with (73)

$$F_{R\theta} = -\frac{2k^2}{\rho}\cdot\frac{(R/\rho)^{2k-1}}{[1+(R/\rho)^{2k}]^2}\,\hat{\underline{n}}$$

$$F_{R\phi} = -\frac{2k}{\rho}\cdot\frac{(R/\rho)^{2k-1}\sin k\theta}{[1+(R/\rho)^{2k}]^2}\cdot\left(\cos k\theta\,\hat{\underline{n}}_\phi + \frac{\sin k\theta}{\sin\phi}\,\hat{\underline{n}}_\psi\right)$$

$$F_{R\psi} = -\frac{2k}{\rho}\cdot\frac{(R/\rho)^{2k-1}\sin k\theta}{[1+(R/\rho)^{2k}]^2}\cdot\left(\cos k\theta\,\hat{\underline{n}}_\psi - \sin k\theta\sin\phi\,\hat{\underline{n}}_\phi\right)$$

(76)

$$F_{\theta\phi} = \frac{2k\,(R/\rho)^{2k}\sin k\theta}{[1+(R/\rho)^{2k}]^2}\cdot\left(\sin k\theta\,\hat{\underline{n}}_\phi - \frac{\cos k\theta}{\sin\phi}\,\hat{\underline{n}}_\psi\right)$$

$$F_{\theta\psi} = \frac{2k\,(R/\rho)^{2k}\sin k\theta}{[1+(R/\rho)^{2k}]^2}\cdot\left(\sin k\theta\,\hat{\underline{n}}_\psi + \cos k\theta\sin\phi\,\hat{\underline{n}}_\phi\right)$$

$$F_{\phi\psi} = -\frac{2\,(R/\rho)^{2k}\sin^2 k\theta\sin\phi\,\hat{\underline{n}}}{[1+(R/\rho)^{2k}]^2}$$

where $\hat{\underline{n}}_\phi = \frac{\partial}{\partial\phi}\hat{\underline{n}}$ etc.

With (75) one obtains the action and topological charge

$$S = 8\pi^2\cdot\frac{3}{k}\left\{\frac{k^4}{2} + 2k^2 + \frac{2}{\pi}\int_0^{\pi/2}\frac{\sin^4 k\theta}{\sin^2\theta}\,d\theta\right\} \qquad (77)$$

$$Q = k$$

The action is manifestly finite, for finite k, since the integrand in the last term is bounded in the finite region of integration. However, the only one of these configurations which is self-dual is k=1. It seems probable that a similar situation will arise for $k>2$ as did for $k=2$ i.e. that self-duality will be achieved in the limit of the instantons becoming infinitely far apart. The calculation has not been performed for general k, however, due to the complications arising from the quaternionic nature of the variables. Configurations of topological charge $-k$ are obtained by sending $x \to x^{\dagger}$ or indeed changing the sign of any odd number of the components of x. All of these configurations, therefore will contribute to the functional integral, though it is not clear how to perform the calculation.

Excluding the singular configurations, it appears that $\mathbb{H}P^1$ does not have the rich topological structure of $\mathbb{C}P^1$. However, Gürsey et al[37], Jafarizadeh et al[45] and Kafiev[46] have shown that, for $u(x)=x^{\dagger k}/\rho^k$, coupling $\mathbb{H}P^1$ to gravity, with the metric

$$g_{\mu\nu} = \frac{1}{4\rho^{2k-2}} \frac{\mathrm{Tr}\,[\partial_\mu x^k \partial_\nu x^{\dagger k} + \partial_\nu x^k \partial_\mu x^{\dagger k}]}{[1 + \frac{1}{\rho^{2k}}(x^k)_i (x^k)_i]^2} \tag{78}$$

gives a self-dual solution of $\mathbb{H}P^1$, in the curved space-time described by (78), with winding number k. In the next chapter, it is shown that this can be done for any quaternionic polynomial of degree k, and in the process a method is developed for constructing a self-dual SU(2) Yang-Mills field over any space-time with a given metric.

CHAPTER 3

MULTI-INSTANTON SOLUTIONS OF $\mathbb{HP}^1$ IN CURVED SPACE-TIMES

The problem of finding multi-instanton solutions to $\mathbb{HP}^1$, as exemplified in the last chapter, has been circumvented by Gürsey et al[37], Jafarizadeh et al.,[45] and Kafiev[46]. These authors extend the conformal invariance of the Lagrangian (2.49) to invariance under general co-ordinate transformations by introducing a metric, and constructing $\mathbb{HP}^1$ in a curved space-time. In this way they construct spherically symmetric, k instanton, $\mathbb{HP}^1$ configurations. In this chapter, their method is extended to more general, non-spherically symmetric configurations.

The Lagrangian density for $\mathbb{HP}^1$ in curved space-time, with metric $g_{\mu\nu}$, is taken to be (in naturalised units, $\hbar = c = 1$).

$$\mathcal{L} = \frac{1}{4K}\sqrt{g}\,(R-2\Lambda) - \frac{1}{2}\sqrt{g}\,\mathrm{Tr}\{g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}\} \tag{1}$$

where

$$F_{\mu\nu} = \frac{\partial_\mu u^\dagger \partial_\nu u - \partial_\nu u^\dagger \partial_\mu u}{(1+u_i u_i)^2} \tag{2}$$

Here Λ is a cosmological constant and $K=4\pi G$, while G is the gravitational constant. R is the curvature scalar obtained from $g_{\mu\nu}$ (the conventions are those of reference[41] except that here the metric has signature (++++)), and $g = \det g_{\mu\nu}$.

By varying the metric in $\mathcal{L}$, we obtain Einstein's field equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R-2\Lambda) = 2K\,T_{\mu\nu} \tag{3}$$

where the energy momentum tensor for the field $F_{\mu\nu}$ is $T_{\mu\nu}$, given by

$$T_{\mu\nu} = 2\,\mathrm{Tr}\{F_{\mu\rho}F_{\nu\lambda}g^{\rho\lambda} - \frac{1}{4}g_{\mu\nu}g^{\alpha\beta}g^{\rho\sigma}F_{\alpha\rho}F_{\beta\sigma}\} \tag{4}$$

Since the energy momentum tensor is traceless ($g^{\mu\nu}T_{\mu\nu}=0$), it is necessary that $R = 4\Lambda$ in order that equation (3) be satisfied. Thus, if the $\mathbb{H}P^1$ are the only fields present, apart from gravity, a cosmological constant is necessary in order to satisfy Einstein's equations (unless the metric is such that R=0).

The expressions for the dual of $F_{\mu\nu}$, the $\mathbb{H}P^1$ action and the topological charge are modified from the flat space-time definitions. Define

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 \text{ for any even permutation of } 0123 \\ -1 \text{ for any odd permutation of } 0123 \\ \;\;0 \text{ otherwise} \end{cases} \tag{5}$$

for <u>any</u> curvilinear co-ordinate system. Then $\varepsilon^{\mu\nu\alpha\beta}$ is not a tensor, but a tensor density of weight -1. The correct tensor is $\frac{1}{\sqrt{g}}\varepsilon^{\mu\nu\alpha\beta}$ Thus the dual of $F^{\mu\nu}$ becomes

$$*F^{\mu\nu} = \frac{1}{2}\frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}} F_{\rho\sigma} \tag{6}$$

So the $\mathbb{H}P^1$ action and topological charge become, respectively,

$$S = -\frac{1}{2}\int \mathrm{Tr}\{F_{\mu\nu}F^{\mu\nu}\}\sqrt{g}\, d^4x \tag{7}$$

$$Q = -\frac{1}{16\pi^2}\int \mathrm{Tr}\{{}^*F_{\mu\nu}F^{\mu\nu}\}\sqrt{g}\, d^4x \tag{8}$$

Here the integration is over the manifold described by $g_{\mu\nu}$ and all Greek indices are raised and lowered with the metric.

Writing the $\mathbb{H}P^1$ field tensor (2) as a Yang-Mills field, derivable from a potential, as in (2.34) we again define

$$A_\mu = \frac{1}{2}\frac{u^\dagger\partial_\mu u - \partial_\mu u^\dagger u}{(1+u_i u_i)} \tag{9}$$

i.e. (2.47) is unchanged by the introduction of a metric. The Euler-Lagrange equations obtained from (1) by varying A_μ are now

$$\partial_\mu \{ \sqrt{g}\, F^{\mu\nu} \} = \sqrt{g}\, [F^{\mu\nu}, A_\mu] \tag{10}$$

By restricting ourselves to A_μ of the form (9), any solution of (10) is a solution of $\mathbb{H}P^1$.

The authors in references[37, 45, 46] show explicitly that $u = (x^+/\rho)^k$ is a solution of (10), with topological charge k, provided that the metric is of the form

$$g_{\mu\nu} = \frac{\rho^2}{4} \frac{\mathrm{Tr}\{ \partial_\mu (\frac{x^+}{\rho})^k \partial_\nu (\frac{x}{\rho})^k + \partial_\nu (\frac{x^+}{\rho})^k \partial_\mu (\frac{x}{\rho})^k \}}{[1 + \frac{1}{2} \mathrm{Tr} (\frac{x^+}{\rho})^k (\frac{x}{\rho})^k]} \tag{11}$$

where ρ is a constant, with dimensions of length (in[37,45 and 46] $\rho = 1$). This metric describes S^4 wrapped round itself k times, and the resulting field, $F_{\mu\nu}$ is spherically symmetric. In what follows, this result will be extended to field configurations which are not spherically symmetric. Consider the metric

$$g_{\mu\nu} = \frac{\rho^2}{2} \frac{\mathrm{Tr}(\partial_\mu u\, \partial_\nu u^+)}{(1 + u_i u_i)^2} \tag{12}$$

where u is any quaternionic function of x, which is differentiable, so as to give a non-singular, continuously differentiable metric, $g_{\mu\nu}$. The requirement of non-singularity of the metric excludes the meron configurations of de Alfaro, Fubini and Furlan[2] in which

$$u = x^+/|x| \tag{13}$$

gives a meron via (9) (though their space-time is S^4 not that given by (12)). To see why this is excluded, write x in spherical polars, as in (2.74), giving

$$u = \cos\theta - \hat{n}\sin\theta \tag{14}$$

so $g_{R\nu} = 0$ for all ν and det g = 0 everywhere.

For well-behaved u(x), metric (12) and potential (9), equation (10) can be simplified in the following manner. Construct Vierbeins for the metric (9)

$$h_{i\mu} = \rho \frac{\partial_\mu u_i}{(1 + u_j u_j)} \tag{15}$$

in terms of which

$$g_{\mu\nu} = h_{i\mu} h_{i\nu} \qquad h_{i\mu} h_j{}^{\mu} = \delta_{ij} \tag{16}$$

Here, as in everything that follows, Greek indices represent curvilinear co-ordinates and must be raised and lowered using the metric, while Roman indices label locally flat co-ordinates. Since the metric has signature (++++) there is no distinction between co-variant and contravariant Roman indices.

Now construct the quaternions

$$h_\mu = h_{i\mu} e_i \tag{17}$$

Thus

$$A_\mu = \frac{1}{2\rho} (u^+ h_\mu - h_\mu^+ u) \tag{18}$$

$$F_{\mu\nu} = \frac{1}{\rho^2} (h_\mu^+ h_\nu - h_\nu^+ h_\mu) \tag{19}$$

Using the properties of quaternions, given in appendix A, (18) and (19) can be written as

$$A_\mu = \frac{1}{\rho} i \sigma^a \eta^{(-)a}_{ij} u_i h_{j\mu} \tag{20}$$

$$F_{\mu\nu} = \frac{2}{\rho^2} i \sigma^a \eta^{(-)a}_{ij} h_{i\mu} h_{j\nu} \tag{21}$$

Then, using the commutation properties of the η symbols (A.13) and (16) the left hand side of (10) becomes

$$\begin{aligned}[F^{\mu\nu}, A_\nu] &= \frac{8i}{\rho^3} \sigma^a \eta^{(-)a}_{ij} h_i{}^\mu u_j \\ &= -\frac{8}{\rho^2} A^\mu \end{aligned} \tag{22}$$

Now consider the left hand side of equation (10)

$$\begin{aligned} &\partial_\mu \{\sqrt{g}\, g^{\mu\rho} g^{\nu\sigma} (\partial_\rho u^+ \partial_\sigma u - \partial_\sigma u^+ \partial_\rho u) / (1+u_i u_i)^2\} \\ &= \frac{\partial_\mu \{\sqrt{g}\, g^{\mu\rho} g^{\nu\sigma} (\partial_\rho u^+ \partial_\sigma u - \partial_\sigma u^+ \partial_\rho u)\}}{(1+u_i u_i)^2} - 2\sqrt{g}\, F^{\mu\nu} \frac{\partial_\mu (1+u_i u_i)}{(1+u_j u_j)} \\ &= \frac{\partial_\mu \{\sqrt{g}\, g^{\mu\rho} g^{\nu\sigma} (\partial_\rho u^+ \partial_\sigma u - \partial_\sigma u^+ \partial_\rho u)\}}{(1+u_i u_i)} - \frac{8}{\rho^2} A^\mu \sqrt{g} \end{aligned} \tag{23}$$

where (15), (16) and (21) have been used. Thus we see a remarkable cancellation between the non-abelian term in the equations of motion (10) and the derivative of the denominator of $F^{\mu\nu}$, reducing (10) to

$$\partial_\mu \{\sqrt{g}\, g^{\mu\rho} g^{\nu\sigma} (\partial_\rho u^+ \partial_\sigma u - \partial_\sigma u^+ \partial_\rho u)\} = 0 \tag{24}$$

now, use the identity (see e.g. ref [61])

$$\frac{1}{\sqrt{g}} \partial_\mu \{\sqrt{g}\, B^{\mu\nu}\} = B^{\mu\nu}{}_{;\mu} \tag{25}$$

for any anti-symmetric tensor $B^{\mu\nu} = -B^{\nu\mu}$. Here a semi-colon denotes co-variant differentiation with the connection induced by the metric. Hence (24) becomes

$$g^{\mu\alpha} (\partial_\alpha u^+ \partial_\nu u - \partial_\nu u^+ \partial_\alpha u)_{;\mu} = 0 \tag{26}$$

since the metric is covariantly constant.

The Christoffel symbols induced by the metric are

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\alpha} (g_{\alpha\rho,\sigma} + g_{\alpha\sigma,\rho} - g_{\rho\sigma,\alpha}) \tag{27}$$

$$= \frac{\partial_\rho \partial_\sigma u_i \partial^\mu u_i}{(1+u_k u_k)^2} - \frac{2}{(1+u_k u_k)} \left(\delta^\mu_\rho u_i \partial_\sigma u_i + \delta^\mu_\sigma u_i \partial_\rho u_i - g_{\rho\sigma} u_i \partial^\mu u_i \right)$$

Then, writing (26) as

$$2 i \sigma^a \eta^{(-)a}_{ij} g^{\mu\alpha} \{ \partial_\mu (\partial_\alpha u_i \partial_\nu u_j) - \Gamma^\rho_{\alpha\mu} \partial_\rho u_i \partial_\nu u_j - \Gamma^\rho_{\nu\mu} \partial_\alpha u_i \partial_\rho u_j \} = 0 \qquad (28)$$

it is straightforward to check, using (15, (16) and (27), that (28) is identically satisfied. Thus any u(x), provided it gives a continuous, differentiable, invertible metric, via (12), will satisfy equation (10). This is an extension of the work of references [37,45] and [46] where only the special cases $u = \left(\frac{x^\dagger}{\rho}\right)^k$, for integral k, were proved to satisfy (10), using spherical polar co-ordinates. In the more general case, however u(x) could, for example, be a quaternionic polynomial in x of the form

$$u(x) = (x^\dagger - a_1) b_1^{-1} (x^\dagger - a_2) \ldots (x^\dagger - a_k) b_k^{-1} + \chi(x^\dagger) \qquad (29)$$

where a_i, b_i $i = 1,\ldots,k$ are quaternionic constants with dimension of length and $\chi(x^\dagger)$ is a polynomial in $x^\dagger$ of degree less than or equal to k-1. Polynomials of the form (29) are homotopic to $(x^\dagger/\rho)^k$ as has been shown by Eilenberg and Niven[28].

In fact, one can go much further and prove that the above construction for $F_{\mu\nu}$ in the metric (12) is (anti) self-dual. The proof is quite simple and proceeds as follows. We have, by definition

$$\begin{aligned} {}^*F^{\mu\nu} &= \frac{1}{2} \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}} F_{\rho\sigma} \\ &= \frac{1}{\rho^2} i \sigma^a \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}} \eta^{(-)a}_{ij} h_{i\rho} h_{j\sigma} \end{aligned} \qquad (30)$$

using (21). Now, from the properties of determinants

$$\frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}} = \pm \frac{\varepsilon^{\mu\nu\rho\sigma}}{h} = \pm h_i^{\ \mu} h_j^{\ \nu} h_k^{\ \rho} h_l^{\ \sigma} \varepsilon^{ijkl} \tag{31}$$

where

$$h = \det(h_{i\mu}) = [\det(h_i^{\ \mu})]^{-1} = \pm\sqrt{g} \tag{32}$$

and the sign ambiguity emerges because (16) only determines det $h_{i\mu}$ up to a sign. Now, putting (31) into (30) and using (16) yields

$$\begin{aligned} {}^*F^{\mu\nu} &= \pm i \frac{\sigma^a}{\rho^2} \eta^{(-)a}_{ij} \varepsilon_{rskl} h_r^{\ \mu} h_s^{\ \nu} h_k^{\ \rho} h_l^{\ \sigma} h_{i\rho} h_{j\sigma} \\ &= \mp F^{\mu\nu} \end{aligned} \tag{33}$$

since $\eta^{(-)a}_{ij}$ is anti-self-dual in (i,j). If, instead of (19), $F_{\mu\nu}$ had been defined as

$$\begin{aligned} F_{\mu\nu} &= \frac{1}{\rho^2} (h_\mu h_\nu^+ - h_\nu h_\mu^+) \\ &= \frac{2 i \sigma^a}{\rho^2} \eta^{(+)a}_{ij} h_{i\mu} h_{j\nu} \end{aligned} \tag{34}$$

the role of the $(\pm)$ sign in (33) would have been interchanged, since $\eta^{(+)a}_{ij}$ is self-dual. Using similar methods, involving the properties of the η symbols and contraction of Vierbein indices, the action (7) and topological charge (8) are found to be

$$\begin{aligned} S &= -\frac{1}{2\rho^4} \int \sqrt{g}\, \mathrm{Tr}\{ (h_\mu^+ h_\nu - h_\nu^+ h_\mu)(h^{\mu +} h^\nu - h^{\nu +} h^\mu) \} \sqrt{g}\, d^4x \\ &= \frac{4}{\rho^4} \eta^{(-)a}_{ij} \eta^{(-)a}_{kl} \int \sqrt{g} (h_{i\mu} h_{j\nu} h_k^{\ \mu} h_l^{\ \nu}) d^4x \\ &= \frac{48}{\rho^4} \int \sqrt{g}\, d^4x \end{aligned} \tag{35}$$

$$Q = \mp \frac{1}{8\pi^2} S = \mp \frac{6}{\pi^2 \rho^4} \times (\text{Volume of space-time}) \tag{36}$$

Note, in passing, that the analysis from equation (30) through to (36) does not depend on the Vierbeins, $h_{i\mu}$, being of the form (15), and will hold for any metric, $g_{\mu\nu}$, not just those of the form (12), provided $F_{\mu\nu}$ is of the form (21). The potential A_μ will not be given by (18) in the general case. Thus, given any metric, $g_{\mu\nu}$, an (anti) self-dual SU(2) field tensor can be constructed using (34) (self-dual if $\det(h_{i\mu}) > 0$). The only degree of freedom between $g_{\mu\nu}$ and $F_{\mu\nu}$ is that of ρ, a scale. This is only to be expected since Yang-Mills is scale invariant, whereas gravity is not. In general, however, the existence of a potential, A_μ, for a field of the form (34) is a more complicated question, and will be deferred until the next chapter. For the moment, let us restrict ourselves to $\mathbb{H}P^1$, where the Vierbeins are of the form (15) for which the potential is given by (9).

To calculate the topological charge of the configuration (29) use will be made of the theorem of Eilenberg and Niven[28] that the polynomial given by (29) is homotopic to $(x^\dagger/\rho)^k$. Since Q is a topological invariant it is invariant under homotopic deformations of the fields and it suffices to calculate Q for $u = (x^\dagger/\rho)^k$ and this will be the same as that of a general k^{th} degree polynomial. The calculation proceeds as for chapter one, except that now the metric must also be included. Write (c.f(2.74)).

$$x = R(\cos\theta + \hat{n}\sin\theta) \tag{37}$$

Then the metric (11) is diagonal and is given by

$$g_{\mu\nu} = \frac{(R/\rho)^{2k-2}}{[1+(R/\rho)^{2k}]^2} \begin{bmatrix} k^2 & & & 0 \\ & k^2 R^2 & & \\ & & R^2 \sin^2 k\theta & \\ 0 & & & R^2 \sin^2 k\theta \sin^2 \phi \end{bmatrix} \tag{38}$$

$$\sqrt{\det g} = \frac{k^2 \rho^3 (R/\rho)^{4k-1}}{[1+(R/\rho)^{2k}]^4} \sin^2 k\theta \sin\phi \tag{39}$$

In order to see what space this metric describes, let us make use of the invariance under general co-ordinate transformations of the action obtained by integrating (1). Since the whole of the above construction is invariant under general co-ordinate transformations, let us simply choose co-ordinates $\left(\frac{x'^{+}}{\rho}\right)$ =u, then the metric (12) is, locally, just the standard metric on S^4, obtained by embedding S^4 in $\mathbb{R}^5$ and using the standard, flat metric in $\mathbb{R}^5$, except that now,

$$0 \leq R' < \infty, \quad 0 \leq \theta' \leq k\pi, \quad 0 \leq \phi' \leq \pi, \quad 0 \leq \psi' \leq \pi \tag{40}$$

where

$$\begin{aligned} x' &= R'(\cos\theta' + \hat{\underline{n}}'\sin\theta') \\ \hat{\underline{n}}' &= \cos\phi' e_1 + \sin\phi'\cos\psi' e_2 + \sin\phi'\sin\psi' e_3 \end{aligned} \tag{41}$$

Thus $\theta' = k\theta$, and S^4 is wrapped round itself k times. Thus the simple interpretation of (38) is that it is a metric for S^4 wrapped round itself k times. This space, however, is not a manifold, and if one believes that the space-time in which we live is a four dimensional manifold (which may, or may not, be the case) the physical significance of these configurations is obscure.

The topological charge, (36), is

$$Q = \frac{6}{\pi^2 \rho} \int_0^\infty dR \int_0^\pi d\theta \int_0^\pi d\phi \int_0^{2\pi} d\psi \; \frac{k^2 (R/\rho)^{4k-1}}{[1 + (R/\rho)^{2k}]^4} \cdot \sin^2 k\theta \sin \phi$$

$$= k. \tag{42}$$

The sign is positive, since with u a function of $x^\dagger$ only, $\det h < 0$. The k anti-instanton configuration would be obtained by taking u to be a function of x only, whence $\det h > 0$ (or, alternatively, use (34) rather than (19)). This is exactly the same value for Q as was obtained for similar forms of u, but in flat space-time, in chapter two. In curved space-times, however, the action is given by

$$S = \pm 8\pi^2 Q \tag{43}$$

since $F_{\mu\nu}$ is (anti)self-dual, and so does not depend upon the parameters in u(x).

For the two instanton case, Q can be evaluated explicitly, without relying on homotopy arguments. As in chapter two take

$$u = \frac{1}{\rho^2}(x^\dagger + a)(x^\dagger - a) \tag{44}$$

where a is real, with dimensions of length.

Then

$$\partial_\mu u = \frac{1}{\rho^2}(x^\dagger e_\mu^\dagger + e_\mu^\dagger x^\dagger) \tag{45}$$

and using co-ordinates (t, r, θ, ψ) where $r^2 = (x_1^2 + x_2^2 + x_3^2)$ with $-\infty < t < \infty, 0 \leqslant r < \infty$, $0 \leqslant \phi \leqslant \pi$, $0 \leqslant \psi \leqslant 2\pi$, $x = t + \hat{n} r$, the metric is diagonal

$$g_{\mu\nu} = \frac{4}{\rho^2} \cdot \frac{1}{\left\{1 + \frac{[(t+a)^2+r^2][(t-a)^2+r^2]}{\rho^4}\right\}^2} \begin{bmatrix} t^2+r^2 & & & \\ & t^2+r^2 & & 0 \\ & & t^2r^2 & \\ 0 & & & t^2r^2\sin^2\phi \end{bmatrix} \tag{46}$$

So $\sqrt{g}$ is easily evaluated to be

$$\sqrt{g} = \frac{16(t^2+r^2)t^2r^2\sin\phi}{\rho^4\{1+[(t+a)^2+r^2][(t-a)^2+r^2]/\rho^4\}^4} \tag{47}$$

Putting (45) into (36), with a positive sign, since (44) gives a self-dual field, yields

$$Q = \frac{96}{\pi^2}\int_{-\infty}^{\infty} dt \int_0^{\infty} dr \int_0^{\pi} d\phi \int_0^{2\pi} d\psi \frac{(t^2+r^2)t^2r^2\sin\phi}{\{\rho^4+[(t+a)^2+r^2][(t-a)^2+r^2]\}^4} \tag{48}$$

This is identical to the topological charge for the field configuration (44) in a flat space time, which was evaluated in chapter two (2.68), and appendix B. There it was shown that $Q = 2$. In the presence of gravity, however, the interaction between the instantons, $I(\tilde{a}^2)$ in equation (2.72), is exactly balanced by the gravitational field, represented by the metric, making S independent of $\tilde{a} = a/\rho$, so $S = 16\pi^2$.

To summarise so far:

Although

$$F_{\mu\nu} = \frac{\partial_\mu u^\dagger \partial_\nu u - \partial_\nu u^\dagger \partial_\mu u}{(1+u_i u_i)^2} \tag{49}$$

does not solve the equations of motion in flat space-time, for u(x) an arbitrary function of x or $x^\dagger$, it does if we introduce a gravitational field

$$g_{\mu\nu} = \frac{\rho^2}{2} \frac{\mathrm{Tr}\,(\partial_\mu u^\dagger \partial_\nu u)}{(1+u_i u_i)^2} \tag{50}$$

In particular, taking u to be a k^{th} degree polynomial in $x^\dagger$ (or x) yields a self-dual (anti-self-dual) field configuration, with topological charge $\pm k$.

Once the space-time is chosen, i.e. the metric (50) is given, there is only one degree of freedom in our choice of $F_{\mu\nu}$, that of the scale ρ. It is natural to ask whether or not it is possible to extend the above idea to a more general $F_{\mu\nu}$, e.g. if we take

$$g_{\mu\nu} = \frac{1}{2\rho^{2k-2}} \cdot \frac{\mathrm{Tr}\,(\partial_\mu x^{\dagger k} \partial_\nu x^k)}{(1+|x|^{2k}/\rho^{2k})^2} \tag{51}$$

and $F_{\mu\nu}$ of the form (49), with u an arbitrary, k^{th} degree polynomial, then is $F_{\mu\nu}$ self-dual in the space described by (51)?

In general this could be a difficult question to answer, so let us first of all consider the simpler case of the instantons being strung out along the time axis,

$$u = (x^\dagger + a_1) \dots (x^\dagger + a_k) \tag{52}$$

where a_i, $i=1\ldots,k$ are real constants (for simplicity, ρ has been set equal to unity). Let us use a co-ordinate system with $x'=x^k$, so that the metric and SU(2) field tensor take the simple form,

$$g'_{\mu\nu} = \frac{\delta_{\mu\nu}}{(1+|x'|^2)^2} \tag{53}$$

$$\begin{aligned} F'_{\mu\nu} &= \frac{\partial'_\mu u^+ \partial'_\nu u - \partial'_\nu u^+ \partial'_\mu u}{(1+u_i u_i)^2} \\ &= \left(\frac{\partial x^\rho}{\partial x'^\mu} \cdot \frac{\partial x^\sigma}{\partial x'^\nu}\right) \frac{(\partial_\rho u^+ \partial_\sigma u - \partial_\sigma u^+ \partial_\rho u)}{(1+u_i u_i)^2} \end{aligned} \tag{54}$$

(here $\partial_\rho = \frac{\partial}{\partial x^\rho}$, $\partial'_\rho = \frac{\partial}{\partial x'^\rho}$). Because of the form of the metric, (53), the self-duality conditions are the same as the flat space-time self-duality conditions and take the form

$$\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} \frac{\partial u_i^+}{\partial x'^\rho} \frac{\partial u_j^+}{\partial x'^\sigma} \eta^{(+)\alpha}_{ij} = \frac{\partial u_i^+}{\partial x'^\mu} \frac{\partial u_j^+}{\partial x'^\nu} \eta^{(+)\alpha}_{ij} \tag{55}$$

In order to eliminate confusing minus signs, we have used the notation $u^+ = u_i^+ e_i = u_i e_i^+$, $u = u_i e_i = u_i^+ e_i^+$ in (55). This enables us to work with x rather than x^+. Contracting both sides of (55) with $\eta^{(+)\alpha}_{kl}$ gives the following condition on the form that u^+ can take

$$\begin{aligned} &\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} \left\{ \frac{\partial u_k^+}{\partial x'^\rho} \frac{\partial u_l^+}{\partial x'^\sigma} - \frac{\partial u_l^+}{\partial x'^\rho} \frac{\partial u_k^+}{\partial x'^\sigma} + \varepsilon_{ijkl} \frac{\partial u_i^+}{\partial x'^\rho} \frac{\partial u_j^+}{\partial x'^\sigma} \right\} \\ &= \frac{\partial u_k^+}{\partial x'^\mu} \frac{\partial u_l^+}{\partial x'^\nu} - \frac{\partial u_l^+}{\partial x'^\mu} \frac{\partial u_k^+}{\partial x'^\nu} + \varepsilon_{ijkl} \frac{\partial u_i^+}{\partial x'^\mu} \frac{\partial u_j^+}{\partial x'^\nu} \end{aligned} \tag{56}$$

Equations (56) represent nine differential equations which u^+ must satisfy for (54) to be self-dual.

To check whether or not (52) satisfies (56), rewrite u^+ as

$$u^+ = x^k + b_{k-1}x^{k-1} + \dots + b_1 x + b_0 \tag{57}$$

where

$$\begin{aligned} b_{k-r} &= \sum_{i_1<i_2<\dots<i_r} a_{i_1}\dots a_{i_r} \qquad (1 \le r \le k) \\ b_k &= 1 \end{aligned} \tag{58}$$

i.e. b_{k-r} is the symmetric product of r a_i's (b_r real). Using spherical polar co-ordinates

$$x = R(\cos\theta + \hat{\underline{n}} \sin\theta) \tag{59}$$

we have

$$u^+ = \sum_{r=0}^{k} b_r R^r (\cos r\theta + \hat{\underline{n}} \sin r\theta) \tag{60}$$

Thus

$$\begin{aligned} u_0^+ &= \sum_{r=0}^{k} b_r R^r \cos r\theta \\ u_1^+ &= \cos\phi \sum_{r=0}^{k} b_r R^r \sin r\theta \\ u_2^+ &= \sin\phi \cos\psi \sum_{r=0}^{k} b_r R^r \sin r\theta \\ u_3^+ &= \sin\phi \sin\psi \sum_{r=0}^{k} b_r R^r \sin r\theta \end{aligned} \tag{61}$$

$$\begin{aligned} x_0' &= R^k \cos k\theta \\ x_1' &= \cos\phi\, R^k \sin k\theta \\ x_2' &= \sin\phi \cos\psi\, R^k \sin k\theta \\ x_3' &= \sin\phi \sin\psi\, R^k \sin k\theta \end{aligned} \tag{62}$$

$$
\begin{aligned}
R &= \left[(x_0')^2 + (x_1')^2 + (x_2')^2 + (x_3')^2 \right]^{\frac{1}{2k}} \\
\theta &= \frac{1}{k} \arccos \left[\frac{x_0'}{(x_0'^2 + x_1'^2 + x_2'^2 + x_3'^2)^{1/2}} \right] \\
\phi &= \arccos \left[\frac{x_1'^2}{(x_1'^2 + x_2'^2 + x_3'^2)^{1/2}} \right] \\
\psi &= \arccos \left[\frac{x_2'^2}{(x_2'^2 + x_3'^2)^{1/2}} \right]
\end{aligned} \tag{63}
$$

Now it is a straightforward, though tedious, application of the chain rule to check equations (56). One finds, using (61), (62) and (63),

$$
\begin{aligned}
\frac{\partial u_0^+}{\partial x_0'} &= \frac{1}{k R^k} \sum_r r b_r R^r \cos(k-r)\theta \\
\frac{\partial u_1^+}{\partial x_1'} &= \frac{1}{k R^k} \cos^2\phi \sum_r r b_r R^r \cos(k-r)\theta + \frac{\sin^2\phi}{R^k \sin k\theta} \cdot \sum_r b_r R^r \sin r\theta \\
\frac{\partial u_2^+}{\partial x_2'} &= \frac{1}{k R^k} \sin^2\phi \cos^2\psi \sum_r r b_r R^r \cos(k-r)\theta \\
&\quad + \frac{1}{R^k \sin k\theta} \cdot (\cos^2\phi \cos^2\psi + \sin^2\psi) \sum_r b_r R^r \sin r\theta \\
\frac{\partial u_3^+}{\partial x_3'} &= \frac{1}{k R^k} \sin^2\phi \sin^2\psi \sum_r r b_r R^r \cos(k-r)\theta \\
&\quad + \frac{1}{R^k \sin k\theta} \cdot (\cos^2\phi \sin^2\psi + \cos^2\psi) \sum_r b_r R^r \sin r\theta \\
\frac{\partial u_0^+}{\partial x_1'} &= -\frac{\partial u_1^+}{\partial x_0'} = \frac{1}{k R^k} \cos\phi \sum_r r b_r R^r \sin(k-r)\theta \\
\frac{\partial u_0^+}{\partial x_2'} &= -\frac{\partial u_2^+}{\partial x_0'} = \frac{1}{k R^k} \sin\phi \cos\psi \sum_r r b_r R^r \sin(k-r)\theta \\
\frac{\partial u_0^+}{\partial x_3'} &= -\frac{\partial u_3^+}{\partial x_0'} = \frac{1}{k R^k} \sin\phi \sin\psi \sum_r r b_r R^r \sin(k-r)\theta \\
\frac{\partial u_1^+}{\partial x_2'} &= \frac{\partial u_2^+}{\partial x_1'} = \frac{\sin\phi \cos\phi \cos\psi}{R^k} \sum_r b_r R^r \left\{ \frac{r}{k} \cos(k-r)\theta - \frac{\sin r\theta}{\sin k\theta} \right\} \qquad (64) \\
\frac{\partial u_1^+}{\partial x_3'} &= \frac{\partial u_3^+}{\partial x_1'} = \frac{\sin\phi \cos\phi \sin\psi}{R^k} \sum_r b_r R^r \left\{ \frac{r}{k} \cos(k-r)\theta - \frac{\sin r\theta}{\sin k\theta} \right\} \\
\frac{\partial u_2^+}{\partial x_3'} &= \frac{\partial u_3^+}{\partial x_2'} = \frac{\sin^2\phi \sin\psi \cos\psi}{R^k} \sum_r b_r R^r \left\{ \frac{r}{k} \cos(k-r)\theta - \frac{\sin r\theta}{\sin k\theta} \right\}
\end{aligned}
$$

It is convenient at this point to define the three quantities

$$
\begin{aligned}
C_k(b_i;R,\theta) &= \frac{1}{k}\sum_{r=1}^{k} r\, b_r R^r \cos(k-r)\theta \\
S_k(b_i;R,\theta) &= \frac{1}{k}\sum_{r=1}^{k} r\, b_r R^r \sin(k-r)\theta \\
T_k(b_i;R,\theta) &= \sum_{r=1}^{k} b_r R^r \frac{\sin r\theta}{\sin k\theta}
\end{aligned}
\tag{65}
$$

Now let us put (64) into the nine equations (56). For example, consider (56) for the case $\mu=0,\ \nu=1, k=0, l=1$ (the analysis for the other eight cases proceeds in a similar fashion). In this case (56) gives

$$
\frac{\partial u_0^+}{\partial x_0'}\frac{\partial u_1^+}{\partial x_1'} - \frac{\partial u_2^+}{\partial x_2'}\frac{\partial u_3^+}{\partial x_3'} + \left(\frac{\partial u_0^+}{\partial x_1'}\right)^2 + \left(\frac{\partial u_2^+}{\partial x_3'}\right)^2 + 2\left(\frac{\partial u_2^+}{\partial x_0'}\frac{\partial u_3^+}{\partial x_1'} - \frac{\partial u_3^+}{\partial x_0'}\frac{\partial u_2^+}{\partial x_1'}\right)
= 0. \tag{66}
$$

Using (64) and a little trigonometry, (66) becomes

$$
\begin{aligned}
&\frac{1}{R^{2k}}\left(C_k^2 - T_k^2 + S_k^2\right)\cos^2\phi = 0 \\
\Leftrightarrow\quad & C_k^2 - T_k^2 + S_k^2 = 0 .
\end{aligned}
\tag{67}
$$

Using the forms for C_k, T_k and S_k in (65), (67) is

$$
\sum_{r=1}^{k}\sum_{r'=1}^{k} b_r b_{r'} R^{r+r'}\left\{\frac{rr'\cos(r-r')\theta}{k^2} - \frac{\sin r\theta\sin r'\theta}{\sin^2 k\theta}\right\} = 0. \tag{68}
$$

Since (68) must be satisfied for all values of R and θ, each term must vanish separately, giving

$$
b_r b_{r'}\left[rr'\sin^2 k\theta\cos(r-r')\theta - k^2\sin r\theta\sin r'\theta\right] = 0 \tag{69}
$$

for all θ and $r, r' = 1, \dots, k$. This can only be satisfied if $b_r = 0$; $r = 1 \dots, k-1$. b_0 is thus the only arbitrary constant, since b_k was fixed to unity. (Changing b_k amounts to changing the scale, i.e. changing ρ). The only function u(x) of the form (52) that is allowed is thus identical to the function used in forming the metric (51) modulo an additive constant. The freedom between $F_{\mu\nu}$ and $g_{\mu\nu}$ carried by b_0 amounts simply to a translation of the origin of the x' co-ordinates. In general, a full conformal transformation of x' will leave $F_{\mu\nu}$ self-dual, i.e. take $v = (a\,u + b)\,(cu + d\,)^{-1}$, where a, b, c and d are constant quaternions, then $F_{\mu\nu}$ of the form (49), with u replaced by v will be (anti)self-dual in the metric (50). But this is the only freedom between $F_{\mu\nu}$ and $g_{\mu\nu}$. We cannot perform conformal transformations on the original, unprimed x .

In this chapter, the work of references [37, 45 and 46] on $\mathbb{H}P^1$ models in curved space-times has been extended from O(4) symmetric solutions, $u = (x^+/\rho)^k$, to solutions with no particular symmetry, $u(x^+)$ any polynomial in x^+. On the way, an interesting result was uncovered, the self-duality of $F_{\mu\nu}$ of the form (34), in the background metric (16). This result is in no way dependent on the $\mathbb{H}P^1$ structure of the fields and is a general result for any SU(2) Yang- Mills field. However the existence of a potential for $F_{\mu\nu}$ constructed from (34) is, in general, a tricky problem. In the remaining part of this work, $\mathbb{H}P^1$ models are abandoned, and the more interesting problem of SU(2) Yang-Mills in curved space-time will be examined. The analysis will be based on equation (34).

CHAPTER FOUR

SELF-DUAL SU(2) FIELDS IN CURVED SPACE-TIMES

In the previous chapter, $\mathbb{H}P^1$ fields in curved space-times were considered. These fields are a special case of SU(2) fields, namely SU(2) fields of the form (3.2) and (3.9). In this chapter (3.2) and (3.9) will not be assumed, but more general SU(2) fields will be examined, in the presence of a gravitational field. First let us summarise the salient formulae for SU(2) Yang-Mills coupled to gravity (3.1), (3.3), (3.4), (3.6), (3.7), (3.8) and (3.10). The action is[15]

$$S = \frac{1}{4K}\int_M d^4x\sqrt{g}\,(R-2\Lambda) + \text{surface term} - \frac{1}{2e^2}\int_M d^4x\sqrt{g}\,\mathrm{Tr}\{F^{\mu\nu}F_{\mu\nu}\} \tag{1}$$

The quantities are as in chapter three, except that e, the Yang-Mills coupling constant is no longer taken to be unity. The integrals are over the whole manifold, M , endowed with the metric, $g_{\mu\nu}$. The surface term is only present if the manifold has a boundary, or is non-compact (for details, see reference[31]). In what follows, only compact manifolds, without boundary will be considered, and so the surface term will not be present.

By varying the metric in (1) we obtain Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R-2\Lambda) = 2K\,T_{\mu\nu} \tag{2}$$

where the energy momentum tensor, $T_{\mu\nu}$, is

$$T_{\mu\nu} = 2\,\mathrm{Tr}\left(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{3}$$

$F_{\mu\nu}$ is derivable from a potential ($A_\mu = -i\sigma_\alpha A^\alpha_\mu$)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,. \tag{4}$$

Varying A_μ in (1) gives

$$\partial_\mu\{\sqrt{g}\, F^{\mu\nu}\} = \sqrt{g}\,[F^{\mu\nu}, A_\nu] \tag{5}$$

The topological charge for the Yang-Mills field is

$$Q = -\frac{1}{16\pi^2}\int_M \mathrm{Tr}\,({}^*F^{\mu\nu}\, F_{\mu\nu})\sqrt{g}\, d^4x \tag{6}$$

If $F_{\mu\nu}$ is (anti) self-dual

$$F^{\mu\nu} = \pm\frac{1}{2}\frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}F_{\rho\sigma} \tag{7}$$

the Yang-Mills equations are automatically satisfied, due to the Bianchi identities for $F_{\mu\nu}$. If (7) holds, then $T_{\mu\nu} = 0$ and so, from (2) $R_{\mu\nu} = \Lambda g_{\mu\nu}$, which is the definition of an Einstein space[54].

It was shown in the last chapter (3.33) how, given any metric (not necessarily Einstein), provided it is non-singular, a self-dual Yang-Mills field tensor can be constructed from the Vierbeins $h_\mu = h_{i\mu}e_i$ (deth>o) via

$$F^{(+)}_{\mu\nu} = \frac{\lambda}{2}(h_\mu h_\nu^\dagger - h_\nu h_\mu^\dagger) \tag{8}$$

λ is a real constant with dimensions of $(\text{length})^{-2}$ ($\lambda = \frac{2}{\rho^2}$ in the notation of chapter 3). The superscript $^{(+)}$ denotes self-duality, as opposed to anti-self-duality, which will be represented by the superscript $^{(-)}$. Anti-self-dual fields may be obtained either by switching the $\dagger$ from the second to the first factors on the right hand side (8) or by choosing the Vierbeins with the opposite orientation ($\det h < 0$). As in equation (3.36), the topological charge of the configuration (8) is

$$Q = \pm\frac{3\lambda^2}{2\pi^2}\int_M \sqrt{g}\, d^4x \tag{9}$$

The above construction is, however, of no use unless a potential satisfying (4) can be found. For a general manifold M, which admits a Riemannian metric, a potential, A_μ, need not necessarily exist. In chapter three, only fields of the form (3.2), were considered, since for these a potential can easily be found, (3.9). In this chapter, we face the more difficult question, what conditions must the metric satisfy in general for an $F_{\mu\nu}$ of the form (8) to be derivable from a potential? To find these conditions let us <u>assume</u> that an A_μ satisfying (5) exists and see what this implies about the metric.

From (5)

$$\partial_\mu\{\sqrt{g}\, F^{(+)\mu\nu}\} = -\sqrt{g}\,[A^{(+)}_\mu, F^{(+)\mu\nu}]$$

$$\Leftrightarrow \quad i\eta^{(+)c}_{ij}\,\partial_\mu\{h\, h_i{}^\mu h_j{}^\nu\} = 2i\, h\, \varepsilon^{abc}\, \eta^{(+)b}_{ij}\, A^{(+)a}_\mu\, h_i{}^\mu h_j{}^\nu \tag{10}$$

$$\Leftrightarrow \quad \varepsilon^{cab}\,\eta^{(+)c}_{ij}\, h_{k\nu}\,\partial_\mu\{h\, h_i{}^\mu h_j{}^\nu\} = 2h\,\{\eta^{(+)b}_{ik}\, A^{(+)a}_\mu\, h_i{}^\mu - \eta^{(+)a}_{ik}\, A^{(+)b}_\mu\, h_i{}^\mu\}$$

multiplying both sides of (10) by $\eta^{(+)b}_{kl}$ and using the properties of the η symbols, this becomes

$$(\eta^{(+)a}_{ki}\,\delta_{lj} - \eta^{(+)a}_{kj}\,\delta_{il} + \eta^{(+)a}_{lj}\,\delta_{ki} - \eta^{(+)a}_{il}\,\delta_{kj})\, h_{k\nu}\,\partial_\mu(h\, h_i{}^\mu h_j{}^\nu)$$
$$= -4h\, A^{(+)a}_\mu\, h_l{}^\mu - 2h\,\varepsilon^{abc}\,\eta^{(+)c}_{li}\, A^{(+)b}_\mu\, h_i{}^\mu \tag{11}$$

Using the middle line of (10) to eliminate the second term with $A^{(+)b}_\mu$ on the right hand side of (11) and contracting with $h_{l\rho}$ finally yields

$$A^{(+)a}_\rho = \{\eta^{(+)a}_{kj}\,\delta_{il} + \eta^{(+)a}_{li}\,\delta_{kj} + \eta^{(+)a}_{ij}\,\delta_{kl} - \eta^{(+)a}_{ki}\,\delta_{lj} - \eta^{(+)a}_{lj}\,\delta_{ki}\}\, h_{l\rho}\, h_{k\nu}\,\frac{\partial_\mu(h\, h_i{}^\mu h_j{}^\nu)}{4h} \tag{12}$$

In quaternion notation, this is

$$A^{(+)}_\rho = \frac{1}{8}\{[h_\nu(\partial_\rho h^{\nu+}) - (\partial_\rho h^\nu)h_\nu^+] + g_{\rho\nu}[h^\mu(\partial_\mu h^{\nu+}) - (\partial_\mu h^\nu)h^{\mu+}]$$
$$+ (h^\mu h_\nu^+ - h_\nu h^{\mu+})\,\mathrm{Re}[h_\rho(\partial_\mu h^{\nu+})]\} \tag{13}$$

Here, a term proportional to ($\frac{\partial_\mu \det h}{\det h} + h_{j\nu}\partial_\mu h_j^{\ \nu}$) has been dropped by use of the identity $\ln \det h_{i\mu} = \mathrm{Tr} \ln h_{i\mu}$

As an example of the use of (13), consider the simple case of the four dimensional sphere, S^4, where

$$g_{\mu\nu} = \frac{\delta_{\mu\nu}}{(1+|x|^2)^2} \tag{14}$$

(the dimensions are set to unity). Then $h_{i\mu}$ can be taken to be

$$h_{i\mu} = \frac{\delta_{i\mu}}{(1+|x|^2)}$$

giving

$$h_\mu = \frac{e_\mu}{(1+|x|^2)} \qquad h^\mu = (1+|x|^2)\, e_\mu$$
$$\partial_\mu h^\nu = 2x_\mu e_\nu \tag{15}$$

Then the first term of (13) vanishes (as it must, since it is conformally invariant and S^4 is conformally flat) and the last two terms reinforce, giving

$$A^{(+)}_\rho = i\sigma^a \eta^{(+)a}_{\mu\rho} \frac{x_\mu}{(1+|x|^2)} \tag{16}$$

From which

$$F^{(+)}_{\mu\nu} = \frac{2i\sigma^a \eta^{(+)a}_{\mu\nu}}{(1+|x|^2)^2} \tag{17}$$

thus $\lambda = 2$ in equation (8). A similar result holds true for $F^{(-)}_{\mu\nu}$. The topological charge of (17) is found from (6) to be Q = 1. The configuration (16), (17) is, in fact, the single instanton of Belavin et al[6].

Equation (13) can be expressed succintly in the following manner. Consider the second and third terms on the right hand side of (13) and use $h^\alpha = g^{\mu\alpha} h_\mu$ to find

$$g_{\rho\nu}\left[h^{\mu}(\partial_{\mu}h^{\nu+})-(\partial_{\mu}h^{\nu})h_{\mu}^{+}\right]+(h^{\mu}h_{\nu}^{+}-h_{\nu}h^{\mu+})\,\mathrm{Re}(h_{\rho}\partial_{\mu}h^{\nu+})$$
$$=\left[h^{\mu}\partial_{\mu}h_{\rho}^{+}-(\partial_{\mu}h_{\rho})h^{\mu+}\right]+(h^{\mu}h^{\nu+}-h^{\nu}h^{\mu+})\,\mathrm{Re}(h_{\rho}\partial_{\mu}h_{\nu}^{+})$$
$$+(h^{\mu}h_{\alpha}^{+}-h_{\alpha}h^{\mu+})g_{\rho\nu}\partial_{\mu}g^{\nu\alpha}+(h^{\mu}h_{\nu}^{+}-h_{\nu}h^{\mu+})\,\mathrm{Re}(h_{\rho}h_{\alpha}^{+})\partial_{\mu}g^{\nu\alpha}\,. \tag{18}$$

Since $g_{\nu\rho}\partial_{\mu}g^{\nu\alpha}=-\partial_{\mu}g_{\nu\rho}\,g^{\mu\alpha}$ and $\mathrm{Re}(h_{\rho}h_{\alpha}^{+})=g_{\rho\alpha}$, the last two terms on the right hand side of (18) reinforce to give

$$g_{\rho\nu}\left[h^{\mu}\partial_{\mu}h^{\nu+}-(\partial_{\mu}h^{\nu})h^{\mu+}\right]+(h^{\mu}h_{\nu}^{+}-h_{\nu}h^{\mu+})\,\mathrm{Re}(h_{\rho}\partial_{\mu}h^{\nu+})$$
$$=h^{\mu}\partial_{\mu}h_{\rho}^{+}-(\partial_{\mu}h_{\rho})h^{\mu+}+(h^{\mu}h^{\nu+}-h^{\nu}h^{\mu+})\,\mathrm{Re}(h_{\rho}\partial_{\mu}h_{\nu}^{+})$$
$$-2(h^{\mu}h^{\nu+}-h^{\nu}h^{\mu+})\partial_{\mu}g_{\nu\rho}\,. \tag{19}$$

Now, the last term on the right hand side of (19) is

$$-(h^{\mu}h^{\nu+}-h^{\nu}h^{\mu+})(\partial_{\mu}g_{\nu\rho}-\partial_{\nu}g_{\mu\rho})$$
$$=2(h^{\mu}h^{\nu+}-h^{\nu}h^{\mu+})\,\Gamma_{\mu,\nu\rho} \tag{20}$$

Where

$$\Gamma_{\mu,\nu\rho}=g_{\mu\alpha}\Gamma^{\alpha}_{\nu\rho} \tag{21}$$

and $\Gamma^{\alpha}_{\nu\rho}$ are the usual Christoffel symbols for $g_{\mu\nu}$ on M.[41] Equation (19) can be further simplified, since

$$(h^{\mu}h^{\nu+}-h^{\nu}h^{\mu+})\partial_{\mu}g_{\nu\rho}=(h^{\mu}h^{\nu+}-h^{\nu}h^{\mu+})\partial_{\mu}(h_{i\nu}h_{i\rho})$$
$$=(h^{\mu}h^{\nu+}-h^{\nu}h^{\mu+})(\partial_{\mu}h_{i\nu})h_{i\rho}+\left[h^{\mu}\partial_{\mu}h_{\rho}^{+}-(\partial_{\mu}h_{\rho})h^{\mu+}\right]$$
$$=(h^{\mu}h^{\nu+}-h^{\nu}h^{\mu+})\,\mathrm{Re}(h_{\rho}\partial_{\mu}h_{\nu}^{+})+\left[h^{\mu}\partial_{\mu}h_{\rho}^{+}-(\partial_{\mu}h_{\rho})h^{\mu+}\right]. \tag{22}$$

Where, in the first step, use has been made of $h^{\nu}h_{i\nu}=e_i$. So the first two terms on the right hand side of (19) are of the same form as the third, reducing (19) to

$$g_{\rho\nu}\left[h^{\mu}\partial_{\mu}h^{\nu+}-(\partial_{\mu}h^{\nu})h^{\mu+}\right]+(h^{\mu}h_{\nu}^{+}-h_{\nu}h_{\mu}^{+})\,\mathrm{Re}(h_{\rho}\partial_{\mu}h^{\nu+})$$
$$=(h_{\mu}h^{\nu+}-h^{\nu}h_{\mu}^{+})\,\Gamma^{\mu}_{\nu\rho}\,. \tag{23}$$

Putting (23) into the expression for $A^{(+)}_{\rho}$,(13), one finds

$$A^{(+)}_{\rho} = \frac{1}{8}\{ [h_{\mu}\partial_{\rho}h^{\mu +} - (\partial_{\rho}h^{\mu})h^{+}_{\mu}] + (h_{\mu}h^{\nu +} - h^{\nu}h^{+}_{\mu})\Gamma^{\mu}_{\nu\rho}\}. \tag{24}$$

Denoting covariant differentiation of a contravariant tensor by

$$h^{\mu}{}_{;\rho} = \partial_{\rho}h^{\mu} + \Gamma^{\mu}_{\nu\rho}h^{\nu} \tag{25}$$

the following, beautifully simple, form emerges for $A^{(+)}_{\rho}$,

$$\begin{aligned} A^{(+)}_{\rho} &= \frac{1}{8}\{ h_{\mu}(h^{\mu}{}_{;\rho})^{+} - h^{\mu}{}_{;\rho}h^{+}_{\mu}\} \\ &= \frac{1}{4} h_{\mu}(h^{\mu}{}_{;\rho})^{+} \end{aligned} \tag{26}$$

Where, in the second step, we have used $\mathrm{Re}[h_{\mu}(h^{\mu}{}_{;\rho})^{+}] = 0$, since $g_{\alpha\beta}$ is covariantly constant.

Now demanding than an $F^{(+)}_{\mu\nu}$ of the form (8) comes from (26) via (4) will produce a set of second order, partial differential equations for the Vierbeins which must be satisfied for a potential to exist. Not all manifolds will admit a metric which factorises into Vierbeins which meet these conditions.

Before applying (4) to find these conditions, it is instructive to stop and examine what is happening from the point of view of group theory. In terms of the η symbols, (26) and the corresponding equation for anti-self-dual fields, can be written as

$$A^{(\pm)}_{\rho} = \frac{i}{4}\sigma^{a}\eta^{(\pm)a}_{jk}h_{j\mu}(h_{k}{}^{\mu})_{;\rho} \tag{27}$$

Now the spin connection for a manifold, viewing the curvature from the point of view of an O(4) gauge theory, is defined as (see Weinberg [61] p 370 and Utiyama [60])

$$\Gamma_{\rho} = \frac{1}{2}\sigma_{ij}h_{i\mu}(h_{j}{}^{\mu})_{;\rho} \tag{28}$$

Where $\sigma_{ij} = -\sigma_{ji}$ are the generators of O(4), satisfying the following commutation relations,

$$[\sigma_{ij}, \sigma_{k\ell}] = \delta_{i\ell}\sigma_{jk} + \delta_{jk}\sigma_{i\ell} - \delta_{ik}\sigma_{j\ell} - \delta_{j\ell}\sigma_{ik} \tag{29}$$

Since $O(4) \approx SU(2) \times SU(2)$ as a group, so that the algebras decompose as $O(4) \approx SU(2) \oplus SU(2)$, the 4 x 4 matrices σ_{ij} can be decomposed into the direct sum of two 2 x 2 matrices. A faithful 4 x 4 representation of SU(2) x SU(2) is given by

$$\sigma_{ij} = \begin{bmatrix} \frac{i}{2}\sigma^{\alpha}\eta^{(+)\alpha}_{ij} & 0 \\ 0 & \frac{i}{2}\sigma^{\alpha}\eta^{(-)\alpha}_{ij} \end{bmatrix} \tag{30}$$

where each entry in (30) is a 2 x 2 submatrix. That (30) satisfies (29) can be checked by direct substitution, using the properties of the η symbols. Note that (30) are complex matrices, and so, strictly speaking, form a representation of SU(2) x SU(2) rather than O(4). Using (30) as the generators for an SU(2) x SU(2) gauge theory with spin connection (28), we see that

$$\Gamma_{\rho} = \begin{bmatrix} A^{(+)}_{\rho} & 0 \\ 0 & A^{(-)}_{\rho} \end{bmatrix} \tag{31}$$

Which expresses the fact that the (anti) self-dual SU(2) potential (26) is simply the spin connection, restricted to one of its SU(2) subgroups.

Equation (31) is identical to the starting point of Charap and Duff [15], though they restrict themselves to manifolds, M, for which the Ricci tensor vanishes, in order that Einstein's equations (2) be satisfied, without a cosmological constant, and the Riemann tensor is double self-dual [49]. In this work, this restriction has not been made, and SU(2) fields in an arbitrary $g_{\mu\nu}$ are being considered to see what restrictions

$g_{\mu\nu}$ must satisfy in order that a SU(2) connection, derived from (8) exists. We shall consider Einstein's equations later.

A note on gauge transformations may usefully be inserted here. The decomposition of $g_{\mu\nu}$ into Vierbeins, equation (3.16), is not unique, but only defined up to a local O(4) gauge transformation $h_{i\mu} \rightarrow O_i{}^j h_{j\mu}$ where $O_i{}^j \in O(4)$(or SO(4) if we restrict ourselves to $\det h > 0$), and $O_i{}^j$ can depend on position. A different choice of Vierbeins merely alters $F^{(+)}_{\mu\nu}$ obtained from (8) by a SU(2) gauge transformation $F^{(+)}_{\mu\nu} \longrightarrow g\, F^{(+)}_{\mu\nu}\, g^{-1}$ where $g \in SU(2)$, the SU(2) element obtained from $O_i{}^j$ via the decomposition (30).

Now we shall proceed to derive a set of second order, partial, differential equations which the Vierbeins must satisfy in order that an SU(2) potential derived from equation (8) exists. From (4), (8) and (26), one finds that

$$
\begin{aligned}
F^{(+)}_{\mu\nu} = \tfrac{1}{4}\{ & \partial_\mu (h_\rho \partial_\nu h^{\rho +}) + \partial_\mu (h_\lambda h^{\rho +})\, \Gamma^\lambda_{\rho\nu} + h_\lambda h^{\rho +} \partial_\mu \Gamma^\lambda_{\rho\nu} \\
& - \partial_\nu (h_\rho \partial_\mu h^{\rho +}) - \partial_\nu (h_\lambda h^{\rho +})\, \Gamma^\lambda_{\rho\mu} - h_\lambda h^{\rho +} \partial_\nu \Gamma^\lambda_{\rho\mu} \\
& + \tfrac{1}{4} [h_\rho \partial_\mu h^{\rho +}, h_\lambda \partial_\nu h^{\lambda +}] + \tfrac{1}{4} [h_\rho \partial_\mu h^{\rho +}, h_\lambda h^{\sigma +}]\, \Gamma^\lambda_{\sigma\nu} \\
& + \tfrac{1}{4} [h_\lambda h^{\sigma +}, h_\rho \partial_\nu h^{\rho +}]\, \Gamma^\lambda_{\sigma\mu} + \tfrac{1}{4} [h_\lambda h^{\rho +}, h_\sigma h^{\alpha +}]\, \Gamma^\lambda_{\rho\mu} \Gamma^\sigma_{\alpha\nu}
\end{aligned}
\tag{32}
$$

The commutators can be evaluated using the properties of the η symbols (here the notation $\mathrm{Vec}\,(h_\lambda h^+_\rho) = \frac{1}{2}(h_\lambda h^+_\rho - h_\rho h^+_\lambda)$ is used - see appendix A)

$$
\begin{aligned}
\tfrac{1}{4} [h_\rho \partial_\mu h^{\rho +}, h_\lambda \partial_\nu h^{\lambda +}] &= -\tfrac{i}{2} \sigma^c \varepsilon^{abc} \eta^{(+)a}_{ij} \eta^{(+)b}_{kl} h_{i\rho} (\partial_\mu h_j{}^\rho) h_{k\lambda} \partial_\nu h_l{}^\lambda \\
&= \tfrac{1}{2} \mathrm{Vec}\{ (\partial_\nu h_\lambda)(\partial_\mu h^\lambda)^+ - (\partial_\mu h_\lambda)(\partial_\nu h^\lambda)^+ - h^\rho h^{\lambda +} (\partial_\mu h_{i\rho})(\partial_\nu h_{i\lambda}) \\
&\qquad - h_\rho h^+_\lambda (\partial_\mu h_i{}^\rho)(\partial_\nu h_i{}^\lambda)\}
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
& \tfrac{1}{4} [h_\rho \partial_\mu h^{\rho +}, h_\lambda h^{\sigma +}] \\
&\quad = -\tfrac{i}{2} \sigma^c \varepsilon^{abc} \eta^{(+)a}_{ij} \eta^{(+)b}_{kl} h_{i\rho} (\partial_\mu h_j{}^\rho) h_{k\lambda} h_l{}^\sigma \\
&\quad = \tfrac{1}{2} \mathrm{Vec}\{ (\partial_\mu h^\sigma) h^+_\lambda + g^{\rho\sigma} (\partial_\mu h_\rho) h^+_\lambda - (\partial_\mu h_\lambda) h^{\sigma +} - g_{\rho\lambda} (\partial_\mu h^\rho) h^{\sigma +}\}
\end{aligned}
\tag{34}
$$

$$\frac{1}{4}[h_\lambda h^{\rho+}, h_\sigma h^{\alpha+}] = -\frac{i}{2}\sigma^c \varepsilon^{abc} \eta^{(+)a}_{ij} \eta^{(+)b}_{kl} h_{i\lambda} h_j{}^\rho h_{k\sigma} h_l{}^\alpha$$
$$= \frac{1}{2} \mathrm{Vec}\{ \delta^\rho_\sigma h_\lambda h^{\alpha+} + g_{\lambda\sigma} h^\alpha h^{\rho+} - \delta^\alpha_\lambda h_\sigma h^{\rho+} - g^{\alpha\rho} h_\lambda h^+_\sigma \} \tag{35}$$

Inserting (33), (34) and (35) into (32) gives (after some algebra!)

$$F^{(+)}_{\mu\nu} = \frac{1}{4} \mathrm{Vec}\{ h_\sigma h^{\rho+} [\partial_\mu \Gamma^\sigma_{\rho\nu} - \partial_\nu \Gamma^\sigma_{\rho\mu} + \Gamma^\sigma_{\alpha\mu} \Gamma^\alpha_{\rho\nu} - \Gamma^\alpha_{\rho\mu} \Gamma^\sigma_{\alpha\nu}]\}$$
$$= \frac{1}{4} h^\sigma h^{\rho+} R_{\sigma\rho\mu\nu} \tag{36}$$

A similar equation, deduced from $F^{(-)}_{\mu\nu}$ can be combined with (36) into the single equation,

$$F^{(\pm)}_{\mu\nu} = \frac{i\sigma^a}{4} \eta^{(\pm)a}_{ij} h_i{}^\sigma h_j{}^\rho R_{\sigma\rho\mu\nu} \tag{37}$$

Again, we make contact with Utiyama's O(4) gauge theory of gravity [60] and the work of Charap and Duff [15]. Utiyama defined an O(4) Lie Algebra valued tensor

$$F_{\mu\nu} = \frac{1}{2} \sigma_{ij} h_i{}^\sigma h_j{}^\rho R_{\sigma\rho\mu\nu}$$
$$= \begin{bmatrix} F^{(+)}_{\mu\nu} & 0 \\ 0 & F^{(-)}_{\mu\nu} \end{bmatrix} \tag{38}$$

Which decomposes under O(4) $\approx$ SU(2) x SU(2) into two SU(2) Lie Algebra valued tensors as shown in (38). Charap and Duff [15] use the decomposition (38) and, in addition <u>demand</u> that $R_{\sigma\rho\mu\nu}$ is double self-dual, so as to make $F^{(+)}_{\mu\nu}$ (anti) self-dual as an SU(2) field. In this work, constraints on $R_{\sigma\rho\mu\nu}$ have been <u>derived</u>, via (8) and (37) simply by requiring that a potential exist for $F^{(+)}_{\mu\nu}$. These constraints take the form (from (8) and (37))

$$\lambda \eta^{(\pm)a}_{ij} h_{i\mu} h_{j\nu} = \frac{1}{4} \eta^{(\pm)a}_{ij} h_i{}^\sigma h_j{}^\rho R_{\sigma\rho\mu\nu} \tag{39}$$

The Riemann tensor can of course be written out in terms of the Vierbeins and their first and second derivatives, yielding a set of second order, partial differential equations for $h_{i\mu}$.

Contracting the left hand side of (39) with both sides of (39) yields

$$\lambda \eta^{(\pm)a}_{ij} \eta^{(\pm)a}_{kl} h_{i\mu} h_{j\nu} h_k{}^\mu h_l{}^\nu = \frac{1}{4} \eta^{(\pm)a}_{ij} \eta^{(\pm)a}_{kl} h_i{}^\sigma h_j{}^\rho h_k{}^\mu h_l{}^\nu R_{\sigma\rho\mu\nu}$$

$$\Leftrightarrow \lambda (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \pm \varepsilon_{ijkl}) \delta_{ik}\delta_{jl} = \frac{1}{4} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \pm \varepsilon_{ijkl}) h_i{}^\sigma h_j{}^\rho h_k{}^\mu h_l{}^\nu R_{\sigma\rho\mu\nu} \qquad (40)$$

$$\Leftrightarrow \qquad \lambda = \frac{R}{4!}$$

Where the cyclicity of $R_{\mu\nu\rho\sigma}$ has been used to set $\varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = 0$. Thus the length scale, λ , is dictated by the curvature scalar, R. A not unexpected result! This also implies, however, that the scalar curvature must be a constant. So (8) can be written.

$$F^{(+)}_{\mu\nu} = \frac{R}{4!} \, \mathrm{Vec}\,(h_\mu h_\nu^\dagger) \qquad (41)$$

Yet more information can be gleaned from (39). Contracting both sides with themselves gives

$$\frac{R^2}{36} \eta^{(\pm)a}_{ij} \eta^{(\pm)a}_{kl} h_{i\mu} h_{j\nu} h_k{}^\mu h_l{}^\nu = \eta^{(\pm)a}_{ij} \eta^{(\pm)a}_{kl} h_i{}^\sigma h_j{}^\rho h_k{}^\alpha h_l{}^\beta R_{\sigma\rho\mu\nu} R_{\alpha\beta}{}^{\mu\nu}$$

$$\Leftrightarrow \qquad \frac{R^2}{3} = 2 R_{\sigma\rho\mu\nu} R^{\sigma\rho\mu\nu} \pm \frac{\varepsilon^{\sigma\rho\alpha\beta}}{\sqrt{g}} R_{\sigma\rho\mu\nu} R_{\alpha\beta}{}^{\mu\nu}$$

$$\Leftrightarrow \qquad \frac{R^2}{6} = \frac{1}{2} \left\{ R_{\sigma\rho\mu\nu} \pm \frac{\varepsilon_{\sigma\rho\alpha\beta}}{2\sqrt{g}} R^{\alpha\beta}{}_{\mu\nu} \right\} \left\{ R^{\sigma\rho\mu\nu} \pm \frac{\varepsilon^{\sigma\rho\lambda\tau}}{2\sqrt{g}} R_{\lambda\tau}{}^{\mu\nu} \right\} \qquad (42)$$

This is indeed a strongrestriction on the Riemann tensor for the manifold, and it all follows from (8) and the assumption that a potential exists!

Note that, if R = 0, (8) tells us nothing, since (40) shows that λ must vanish. Thus if we try and use (8) to construct a SU(2) field over a manifold with vanishing curvature scalar, (8) merely gives us $F_{\mu\nu} = 0$. If R = 0, (26) must be used as a starting point, and this is the approach of Charap and Duff[15].

The Riemann tensor can be decomposed into R, $R_{\mu\nu}$ and the Weyl tensor, $C^{\sigma}{}_{\rho\mu\nu}$, in the usual way,

$$R_{\rho\sigma\mu\nu} = C_{\rho\sigma\mu\nu} + \frac{1}{2}(g_{\rho\mu}R_{\sigma\nu} + g_{\sigma\nu}R_{\rho\mu} - g_{\rho\nu}R_{\sigma\mu} - g_{\sigma\mu}R_{\rho\nu}) - \frac{R}{6}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \tag{43}$$

where $C_{\rho\sigma\mu\nu}$ enjoys the same symmetries as $R_{\rho\sigma\mu\nu}$ but, in addition, contains no vestige of the Ricci tensor i.e.,

$$C^{\rho}{}_{\sigma\rho\nu} = 0. \tag{44}$$

Using (43), (42) becomes

$$R^2 - 4R_{\mu\nu}R^{\mu\nu} = \left(C_{\mu\nu\lambda\rho} \pm \frac{1}{2}\frac{\varepsilon_{\lambda\rho\sigma\gamma}}{\sqrt{g}}C_{\mu\nu}{}^{\sigma\gamma}\right)\left(C^{\mu\nu\lambda\rho} \pm \frac{1}{2}\varepsilon^{\lambda\rho\alpha\beta}C^{\mu\nu}{}_{\alpha\beta}\right). \tag{45}$$

There is onemore quadratic invariant in $R_{\mu\nu\rho\sigma}$ which has not yet been used, and that is[49]

$$\frac{1}{4}\frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}\cdot\frac{\varepsilon^{\alpha\beta\lambda\tau}}{\sqrt{g}}R_{\mu\nu\alpha\beta}R_{\rho\sigma\lambda\tau}. \tag{46}$$

We can derive an equation involving this quantity from(39) and the duality properties of $F^{(+)}_{\mu\nu}$. Writing (39) as

$$F^{(\pm)}_{\mu\nu} = \frac{6}{R}F^{(\pm)\rho\sigma}R_{\rho\sigma\mu\nu} \tag{47}$$

(anti) self-duality implies

$$
\begin{aligned}
F^{(\pm)}_{\mu\nu} &= \pm \frac{3}{R} \frac{\varepsilon^{\rho\sigma\alpha\beta}}{\sqrt{g}} F^{(\pm)}_{\alpha\beta} R_{\rho\sigma\mu\nu} \\
&= \pm \frac{18}{R^2} \frac{\varepsilon^{\rho\sigma\alpha\beta}}{\sqrt{g}} F^{(\pm)\lambda\tau} R_{\lambda\tau\alpha\beta} R_{\rho\sigma\mu\nu} \\
&= \frac{9}{R^2} \frac{\varepsilon^{\rho\sigma\alpha\beta}}{\sqrt{g}} \cdot \frac{\varepsilon^{\lambda\tau\gamma\delta}}{\sqrt{g}} F^{(\pm)}_{\gamma\delta} R_{\lambda\tau\alpha\beta} R_{\rho\sigma\mu\nu} .
\end{aligned}
\tag{48}
$$

Now, contracting both sides of (48) with $F^{(\pm)}_{\mu\nu}$ yields

$$
F^{(\pm)}_{\mu\nu} . F^{(\pm)\mu\nu} = \frac{9}{R^2} \frac{\varepsilon^{\rho\sigma\alpha\beta}}{\sqrt{g}} \cdot \frac{\varepsilon^{\lambda\tau\gamma\delta}}{\sqrt{g}} R_{\lambda\tau\alpha\beta} R_{\rho\sigma\mu\nu} F^{(\pm)}_{\gamma\delta} F^{(\pm)\mu\nu}
$$

$$
\Leftrightarrow \quad \frac{R^2}{48} = \frac{1}{64} \cdot \frac{\varepsilon^{\rho\sigma\alpha\beta}}{\sqrt{g}} \cdot \frac{\varepsilon^{\lambda\tau\gamma\delta}}{\sqrt{g}} \cdot R_{\lambda\tau\alpha\beta} R_{\rho\sigma\mu\nu} \eta^{(\pm)a}_{ij} \eta^{(\pm)a}_{kl} h_{i\gamma} h_{j\delta} h_k{}^{\mu} h_l{}^{\nu} \tag{49}
$$

$$
\Leftrightarrow \quad \frac{R^2}{3} = \frac{1}{4} \left\{ 2 \frac{\varepsilon^{\rho\sigma\alpha\beta}}{\sqrt{g}} \cdot \frac{\varepsilon^{\lambda\tau\mu\nu}}{\sqrt{g}} R_{\lambda\tau\alpha\beta} R_{\rho\sigma\mu\nu} \pm \frac{\varepsilon^{\rho\sigma\alpha\beta}}{\sqrt{g}} \frac{\varepsilon^{\lambda\tau\gamma\delta}}{\sqrt{g}} \frac{\varepsilon_{\gamma\delta\mu\nu}}{\sqrt{g}} R_{\lambda\tau\alpha\beta} R_{\rho\sigma}{}^{\mu\nu} \right\}
$$

$$
\Leftrightarrow \quad \frac{R^2}{6} = \left\{ \frac{1}{4} \frac{\varepsilon^{\rho\sigma\alpha\beta}}{\sqrt{g}} \cdot \frac{\varepsilon^{\lambda\tau\mu\nu}}{\sqrt{g}} R_{\lambda\tau\alpha\beta} R_{\rho\sigma\mu\nu} \pm \frac{1}{2} \frac{\varepsilon^{\rho\sigma\alpha\beta}}{\sqrt{g}} R_{\rho\sigma\mu\nu} R_{\alpha\beta}{}^{\mu\nu} \right\}
$$

Following Lanczos[49] (but beware of the factor of $\frac{1}{2}$ missing in equation (2.4) of that reference) define the following five quadratic invariants

$$
I_1 = R_{\alpha\beta} R^{\alpha\beta} \tag{50}
$$

$$
I_2 = R^2 \tag{51}
$$

$$
I_3 = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \tag{52}
$$

$$
K_1 = \frac{1}{2} \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}} R_{\mu\nu}{}^{\alpha\beta} R_{\rho\sigma\alpha\beta} \tag{53}
$$

$$
K_2 = \frac{1}{4} \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}} \frac{\varepsilon^{\alpha\beta\tau\lambda}}{\sqrt{g}} R_{\mu\nu\alpha\beta} R_{\rho\sigma\tau\lambda} \tag{54}
$$

Then (42) and (49) can be written

$$\frac{1}{6} I_2 = I_3 \pm K_1 \tag{55}$$

$$\frac{1}{6} I_2 = K_2 \pm K_1 \tag{56}$$

respectively. Subtracting (55) from (56) gives

$$I_3 = K_2 . \tag{57}$$

Now the relation between the five quadratic invariants derived by Lanczos in[49] (equation (5.5) of that reference) is

$$K_2 = I_3 - 4 I_1 + I_2 \tag{58}$$

which is true for any manifold which admits a Riemannian metric. Equation (57) is equivalent to the statement that the Riemann tensor is double-self-dual. Inserting (57) into (58) yields,

$$I_2 - 4 I_1 = 0 \tag{59}$$

which, in terms of the Ricci tensor and curvature scalar, (51) and (50), is

$$R^2 - 4 R_{\alpha\beta} R^{\alpha\beta} = 0 . \tag{60}$$

Equation (60), for a positive definite metric is true <u>if and only if</u> the metric is Einstein (Petrov[54]). This can be seen most easily by writing (60) as

$$(R_\mu{}^\nu - \frac{R}{4} \delta_\mu{}^\nu)(R_\nu{}^\mu - \frac{R}{4} \delta_\nu{}^\mu) = 0 . \tag{61}$$

Equation (61) must be satisfied at each point of space-time. At any point, we can choose the metric to be diagonal (though, of course, this cannot be done globally.) and, since the signature of the metric is (++++), each term in the sum on the left hand side of (61) is a perfect square with positive sign, therefore each term must vanish individually. Thus

$$R_{\mu\nu} = \frac{R}{4} g_{\mu\nu} \tag{62}$$

Equation (62) and the constraint that R is constant, (40), are the defining conditions for an Einstein metric. Not all manifolds admit an Einstein metric, but those that do are called Einstein spaces[54]. With (62), equation (45) gives a condition on the Weyl tensor

$$\left(C_{\mu\nu\lambda\rho} \pm \frac{1}{2} \frac{\varepsilon_{\lambda\rho\sigma\gamma}}{\sqrt{g}} C_{\mu\nu}{}^{\sigma\gamma} \right)\left(C^{\mu\nu\lambda\rho} \pm \frac{1}{2} \frac{\varepsilon^{\lambda\rho\alpha\beta}}{\sqrt{g}} C^{\mu\nu}{}_{\alpha\beta} \right) = 0. \tag{63}$$

The same reasoning that was applied to (61) tells us that, for positive definite metrics, with R ≠ 0,

$$C_{\mu\nu\lambda\rho} = \mp \frac{\varepsilon_{\lambda\rho\sigma\gamma}}{2\sqrt{g}} C_{\mu\nu}{}^{\sigma\gamma} \tag{64}$$

The upper (lower) sign applies for instantons (anti-instantons) constructed using (8). Thus, if R ≠ 0, we have

<u>For a self-dual field to exist, the Weyl tensor must be anti-self-dual</u>

<u>For an anti-self-dual field to exist the Weyl tensor must be self-dual</u>

Thus for R ≠ 0, the <u>only</u> manifolds that will admit <u>both</u> instantons <u>and</u> anti-instantons via equation (8) are conformally flat spaces.

For R = 0, equation (8) cannot be used, but one can use (31) and (38) as a starting point as in reference[15].

The topological charge of the field (8) can be expressed in terms of the topological invariants of the manifold, the Euler Characteristic χ, and the Hirzebruch signature τ, which, for a compact manifold without boundary are given by[26],

$$\chi = \frac{1}{128\pi^2}\int_M \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}\cdot\frac{\varepsilon^{\alpha\beta\gamma\delta}}{\sqrt{g}} R_{\mu\nu\alpha\beta} R_{\rho\sigma\gamma\delta}\sqrt{g}\, d^4x \tag{65}$$

$$\tau = \frac{1}{96\pi^2}\int_M \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}} R_{\mu\nu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta}\sqrt{g}\, d^4x . \tag{66}$$

Using (43) and (60), these can be re-expressed, for an Einstein space, as

$$\chi = \frac{1}{32\pi^2}\int_M \left(C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{R^2}{6}\right)\sqrt{g}\, d^4x \tag{67}$$

$$\tau = \frac{1}{96\pi^2}\int_M \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}} C_{\mu\nu\alpha\beta} C_{\rho\sigma}{}^{\alpha\beta}\sqrt{g}\, d^4x . \tag{68}$$

Together with (64), these give

$$2\chi \pm 3\tau = \frac{R^2}{96\pi^2}\int_M \sqrt{g}\, d^4x . \tag{69}$$

Thus the topological charge is, from (9), (40) and (69)

$$Q = \pm\frac{\chi}{2} + \frac{3}{4}\tau \tag{70}$$

where the upper (lower) sign is for instantons (anti-instantons). For R = 0 this result can still be obtained, using (38) and (6), without assuming (8).[32]

The results of this chapter so far can be summarised as follows. Starting from equation (8) (or its anti-self-dual counterpart) for a manifold with $R \neq 0$, a self-dual (anti-self-dual) potential exists <u>only</u> if the metric, $g_{\mu\nu}$, satisfies the following conditions;

a) <u>The metric is Einstein</u>

b) <u>The Weyl tensor is (anti) self-dual</u>

The topological charge of the configuration is given by equation (70) Note that condition (a) implies that, since $T_{\mu\nu}$ vanishes for an (anti) self-dual field configuration. Einstein's equations with a cosmological constant (2) must be satisfied for a potential to exist. Thus the very existence of a SU(2) potential implies that Einstein's equations are satisfied. This is in contrast to the R = 0 case, where consistency with Einstein's equations is an extra condition which must be put separately (Charap and Duff [15]).

The above analysis can be formulated quite neatly in terms of quaternions, by introducing a quaternionic metric [36], and taking the real part of the metric to be $g_{\mu\nu}$ and the pure quaternionic part to be an SU(2) $F_{\mu\nu}$. This is reminiscent of Einstein's work on a generalised theory of gravitation, where he considered a complex metric and took the real part to be $g_{\mu\nu}$ and the purely imaginary part to be a U(1), electromagnetic $F_{\mu\nu}$ [29, 39]. We construct the quaternionic metric as follows. Given a metric which satisfies both a) and b) above, choose Vierbeins, $h_{i\mu}$ ($\det h > 0$), for the metric, which can be considered as four quaternions $h_\mu = e_i h_{i\mu}$. Then construct the quaternionic metric

$$H_{\mu\nu} = h_\mu h_\nu^\dagger \tag{71}$$

and take

$$\begin{aligned} g_{\mu\nu} &= \mathrm{Re}\, H_{\mu\nu} \\ F_{\mu\nu} &= \frac{R}{4!}\, \mathrm{Vec}\, H_{\mu\nu} \end{aligned} \tag{72}$$

where the factor R is necessary on purely dimensional grounds. Then $F_{\mu\nu}$ is automatically self-dual in the metric $g_{\mu\nu}$. Assuming that a SU(2) potential, A_μ, exists for $F_{\mu\nu}$ gives

$$F_{\mu\nu} = \frac{6}{R} F^{\rho\sigma} R_{\rho\sigma\mu\nu} . \tag{73}$$

Using the cyclicity property of the Riemann tensor

$$R_{\rho\sigma\mu\nu} = R_{\rho\mu\sigma\nu} - R_{\sigma\mu\rho\nu} \tag{74}$$

(73) can be written as

$$F_{\mu\nu} = \frac{12}{R} F^{\rho\sigma} R_{\rho\mu\sigma\nu}. \tag{75}$$

Then, using (72), equation (36) and condition a) can be written as

$$\text{Vec } H_{\mu\nu} = \frac{12}{R} \text{Vec } H^{\rho\sigma} R_{\rho\mu\sigma\nu} \tag{76}$$

$$\text{Re } H_{\mu\nu} = \frac{4}{R} \text{Re } H^{\rho\sigma} R_{\rho\mu\sigma\nu}. \tag{77}$$

The potential for $F_{\mu\nu}$ is given by (26)

$$A_{\mu} = \frac{1}{4} h_{\nu} (h^{\nu +})_{;\mu}. \tag{78}$$

From the definition of the Weyl tensor (43), we see that (76) gives

$$\text{Vec } H^{\rho\sigma} C_{\rho\mu\sigma\nu} = 0 \tag{79}$$

which, together with $C^{\rho}{}_{\mu\rho\nu} = 0$, can be written

$$H^{\rho\sigma} C_{\rho\mu\sigma\nu} = 0. \tag{80}$$

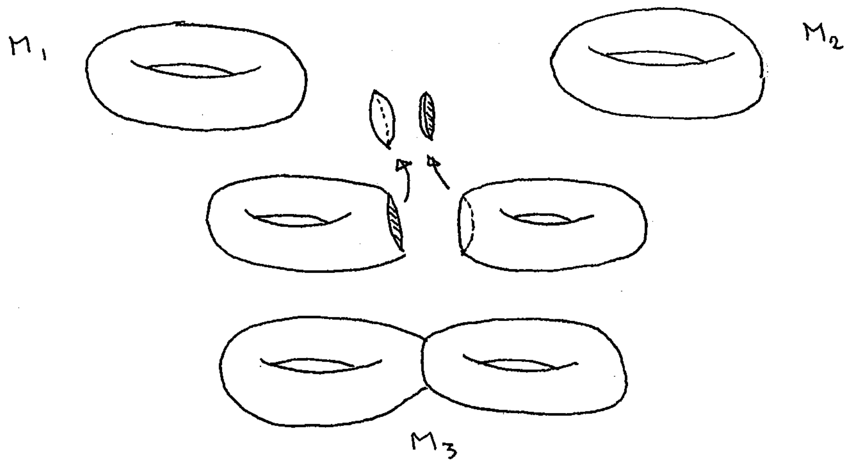
Equation (80) is the quaternionic form of condition b).

In general, the task of classifying all four dimensional Einstein spaces is a fascinating, but unsolved, problem[54, 43]. Some known examples of such spaces are[7, 43, 54], S^4, $S^2 \times S^2$, T^4, $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$, K^3, $\mathbb{C}P^2$. Of these spaces S^4 and T^4 are conformally flat[5, 7] and so (64) is automatically satisfied. Thus both instantons and anti-instantons exist over S^4, though all Einstein metrics on T^4 are flat, as can be seen from equation (67) and the fact that $\chi = 0$ for T^4 [7]. $S^1 \times S^3$ is also conformally flat, but it is known not to admit an Einstein metric[43]. $\mathbb{C}P^2$ admits an Einstein metric with non zero R and its Weyl tensor is either self-dual or anti-self-dual, depending on the orientation of the metric[5]. Thus either instantons or anti-instantons may be constructed, but not both. $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ admits an Einstein metric with non-zero R[53] but the Weyl tensor is neither self-dual nor anti-self-dual and so it will not admit an instanton structure via equation (8). $S^2 \times S^2$ has an Einstein metric with non-vanishing scalar curvature. Its Weyl tensor does not satisfy equation (64), but the manifold has other interesting properties which make it worthy of further study, and it will be considered in chapter six. K^3, Kummer's surface, has vanishing Ricci tensor and so is Einstein with vanishing scalar curvature. It has (anti) self-dual (depending on the orientation chosen) Riemann and Weyl tensors, [5]. A metric for K^3 has not been explicitly constructed, though implicit and approximate constructions have been given[33, 48].

Ths spaces $S^2 \times S^2$, K^3 and $\mathbb{C}P^2$ are of spacial interest since they constitute the space-time "foam" of Hawking[40]. In that reference it is conjectured that, since the gravitational action contains a dimensionful constant, large fluctuations in the topology of space-time will produce only small changes in the gravitational action, provided the fluctuations occur over distances small compared to the natural length of the action, the Planck length,

$$L_p = \sqrt{\frac{\hbar G}{c^3}} = 1\cdot 6 \times 10^{-33} \text{cm} \tag{81}$$

and thus would be expected to give important contributions to the gravitational action in any functional integral approach to quantum gravity. These ideas are closely allied to those of Wheeler [50]. Hawking's foam is made up of a topological sum of four dimensional "bubbles" consisting of $\mathbb{C}P^2$, $\mathbb{C}\bar{P}^2$ ($\mathbb{C}P^2$ with the opposite orientation), $S^2 \times S^2$ and K^3. The topological sum of two n dimensional manifolds is formed by removing an n dimensional ball from each and then gluing them together along the S^{n-1} boundaries of the resulting holes. In two dimensions one can draw a picture as shown



This picture represents the sum of two tori to form a pretzel. This operation is written as

$$M_1 \# M_2 \approx M_3 \tag{82}$$

(here $\approx$ means isomorphic to). Note that

$$M_1 \# S^n \approx M_1 \tag{83}$$

i.e. adding spheres changes nothing. From this point of view, it is interesting to find out as much as possible about $\mathbb{C}P^2$, $S^2 \times S^2$ and K^3. A metric for K^3 is not explicitly known, and so it will not be considered in this work. $\mathbb{C}P^2$ and $S^2 \times S^2$ will be considered in chapters five and six respectively.

CHAPTER 5

SU(2) AND U(1) FIELDS IN $\mathbb{C}P^2$

In this chapter the formalism of chapter four will be applied to $\mathbb{C}P^2$, a four dimensional Einstein space with constant scalar curvature. The question of SU(2) Yang-Mills over $\mathbb{C}P^2$ has also been considered by Atiyah et al [5] and Gibbons and Pope [32]. In chapter two, $\mathbb{C}P^n$ models were considered in which the fields were $\mathbb{C}P^n$ valued, and the space-time was a two dimensional sphere. In this chapter the space-time is four dimensional $\mathbb{C}P^2$ and the fields will be SU(2) (or U(1)) Yang-Mills fields. Considerations of the geometry of $\mathbb{C}P^2$ are greatly simplified by the use of complex co-ordinates and, for this reason, a brief diversion will be made to explain the concept of a complex, Kahler manifold (of which $\mathbb{C}P^2$ is an example) [3, 64].

Locally, a n-dimensional, complex manifold is a 2n-dimensional, real manifold, parameterised by n complex co-ordinates. Let $\{x^\mu\}$ $\mu = 1,\dots,2n$ be real co-ordinates and $\{z^\alpha\}, \alpha = 1,\dots,n$ be complex co-ordinates. Then a real line element on the manifold is given by

$$\begin{aligned} ds^2 &= g_{\mu\nu}\, dx^\mu dx^\nu \\ &= g_{\alpha\beta}\, dz^\alpha dz^\beta + g_{\bar\alpha\bar\beta}\, d\bar z^\alpha d\bar z^\beta + g_{\alpha\bar\beta}\, dz^\alpha d\bar z^\beta + g_{\bar\alpha\beta}\, d\bar z^\alpha dz^\beta . \end{aligned} \tag{1}$$

For ds^2 to be real, the complex metric must satisfy

$$g_{\alpha\beta} = \bar g_{\bar\alpha\bar\beta} \qquad\qquad g_{\alpha\bar\beta} = \bar g_{\bar\alpha\beta} \tag{2}$$

If it is possible to find a complex co-ordinate system for which $g_{\alpha\beta} = g_{\bar\alpha\bar\beta} = 0$, then (2) implies that $g_{\alpha\bar\beta}$ is Hermitian, as an n x n complex matrix. In this case the manifold itself is said to be Hermitian. $\mathbb{C}P^2$ is such a manifold. In a Hermitian co-ordinate system (1) reduces to

$$ds^2 = 2 g_{\alpha\bar\beta}\, dz^\alpha dz^{\bar\beta} \tag{3}$$

where we have adopted the notation $\bar{z}^{\beta} = z^{\bar{\beta}}$. One can think of $\{z^{\alpha}, z^{\bar{\alpha}}\}$ as 2n complex co-ordinates for a larger space and $\bar{z}^{\alpha} = z^{\bar{\alpha}}$ is then a condition which restricts us to lie in an n dimensional complex subspace.

Contravariant and covariant tensors can be defined on a Hermitian manifold in the usual manner, and indices raised and lowered using $g_{\alpha\bar{\beta}}$. Note that barred indices become unbarred upon raising or lowering and vice versa, since $g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$. Now let us define the 2-form

$$K = -2i\, g_{\alpha\bar{\beta}}\, dz^{\alpha} \wedge dz^{\bar{\beta}} . \tag{4}$$

This is real, since $g_{\alpha\bar{\beta}}$ is Hermitian. If it so happens that the exterior derivative of K vanishes (i.e. K is a <u>closed</u> 2-form)

$$\begin{aligned} dK = & -i(\partial_{\gamma} g_{\alpha\bar{\beta}} - \partial_{\alpha} g_{\gamma\bar{\beta}})\, dz^{\gamma} \wedge dz^{\alpha} \wedge dz^{\bar{\beta}} \\ & -i(\partial_{\bar{\gamma}} g_{\alpha\bar{\beta}} - \partial_{\bar{\beta}} g_{\alpha\bar{\gamma}})\, dz^{\bar{\gamma}} \wedge dz^{\alpha} \wedge dz^{\bar{\beta}} \\ = & \; 0 \end{aligned} \tag{5}$$

then the manifold is called a Kahler manifold and K is called the Kahler or fundamental 2-form. Condition (5) is equivalent to saying that the curl of the metric vanishes

$$\begin{aligned} \partial_{\gamma} g_{\alpha\bar{\beta}} - \partial_{\alpha} g_{\gamma\bar{\beta}} &= 0 \\ \partial_{\bar{\gamma}} g_{\alpha\bar{\beta}} - \partial_{\bar{\beta}} g_{\alpha\bar{\gamma}} &= 0 \end{aligned} \tag{6}$$

and so the metric can be derived from a potential

$$g_{\alpha\bar{\beta}} = \frac{\partial}{\partial z^{\alpha}} \frac{\partial}{\partial z^{\bar{\beta}}} \Phi(z, \bar{z}) . \tag{7}$$

Φ is called the Kahler potential. $\mathbb{C}P^2$ is a Kahler manifold.

As a consequence of (7), the usual expressions for the Christoffel symbols and Riemann tensor simplify to,

$$\begin{aligned} &\Gamma^{\alpha}_{\beta\gamma} = g^{\alpha\bar{\delta}} \partial_{\beta} g_{\gamma\bar{\delta}} && \Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} = g^{\delta\bar{\alpha}} \partial_{\bar{\beta}} g_{\delta\bar{\gamma}} \\ &R^{\alpha}{}_{\beta\gamma\bar{\delta}} = -\partial_{\bar{\delta}} \Gamma^{\alpha}_{\beta\gamma} && R^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}\delta} = -\partial_{\delta} \Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} \\ &R_{\alpha\bar{\beta}\gamma\bar{\delta}} = -R_{\bar{\beta}\alpha\gamma\bar{\delta}} = -R_{\alpha\bar{\beta}\bar{\delta}\gamma} = R_{\gamma\bar{\delta}\alpha\bar{\beta}} \end{aligned} \tag{8}$$

all other components vanish, due to (6). In particular, the cyclicity of the Riemann tensor is expressed as

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\alpha\bar{\delta}\gamma\bar{\beta}} \tag{9}$$

since $R_{\alpha\gamma\bar{\delta}\bar{\beta}}$ vanishes.

The Ricci tensor also simplifies nicely, using the identity

$$\Gamma^{\beta}_{\alpha\beta} = \partial_\alpha \log(\det g_{\gamma\bar{\delta}}) \tag{10}$$

one finds

$$R^{\gamma}{}_{\alpha\gamma\bar{\beta}} = R_{\alpha\bar{\beta}} = -\partial_\alpha \partial_{\bar{\beta}} \log(\det g_{\gamma\bar{\delta}}) \tag{11}$$

This elegant machinery can now be applied to $\mathbb{C}P^2$.

The general construction for $\mathbb{C}P^n$ was given in chapter two. $\mathbb{C}P^2$ can be considered as the space obtained by identifying all the points of $\mathbb{C}^3$ which differ only by a scalar multiple

$$(z_0, z_1, z_2) \sim c(z_0, z_1, z_2) \tag{12}$$

Where c is a complex scalar. Hence the name, "projective space". The points of $\mathbb{C}P^2$ away from $z_0 = 0$, can be parameterised by two complex variables of any magnitude

$$u_1 = z_1/z_0 \qquad u_2 = z_2/z_0 \ . \tag{13}$$

The points (u_1, u_2) also parameterise $\mathbb{C}^2 \approx \mathbb{R}^4$. The remaining points, $z_0 = 0$, are those which are obtained by identifying points in $\mathbb{C}^2 \subset \mathbb{C}^3$ which differ only by a scalar multiple

$$(z_1, z_2) \sim c(z_1, z_2) \tag{14}$$

These points constitute the manifold $\mathbb{C}P^1 \approx S^2$. Thus $\mathbb{C}P^2$ can be thought of as $\mathbb{R}^4$, compactified by adding a S^2 at infinity (in the same way as S^4 can be thought of as $\mathbb{R}^4$ compactified by adding a single point at infinity.) $\mathbb{C}P^2$ has been proposed as a gravitational instanton [32, 25], because it can be given a metric which satisfies Einstein's equations, ($R_{\mu\nu} = \Lambda g_{\mu\nu}$, Λ a constant) and it has finite gravitational action. References [5] and [32] also consider SU(2) Yang-Mills over $\mathbb{C}P^2$.

The metric is the Fubini-Study metric [26],

$$g_{\alpha\bar\beta} = \partial_\alpha \partial_{\bar\beta} \ln(1 + u^1\bar u^1 + u^2 \bar u^2) \tag{15}$$

(the scale has been set to unity.) In the form (15), $g_{\alpha\bar\beta}$ is manifestly a Kahler metric, with Kahler potential

$$\Phi = \ln(1 + |x|^2) \tag{16}$$

where $|x|^2 = u^1\bar u^1 + u^2\bar u^2$. Let $u^1 = y$, $u^2 = z$, then (15) is

$$g_{\alpha\bar\beta} = \begin{bmatrix} 1+z\bar z & -\bar y z \\ -y\bar z & 1+y\bar y \end{bmatrix} \Big/ (1+|x|^2)^2 \tag{17}$$

where α labels the rows and $\bar\beta$ the columns. The inverse of (17) is

$$g^{\bar\beta\alpha} = (1+|x|^2) \begin{bmatrix} 1+y\bar y & \bar y z \\ \bar z y & 1+z\bar z \end{bmatrix} = \bar g^{\beta\bar\alpha}. \tag{18}$$

Now we can use equations (18) to find the Christoffel symbols

$$\begin{aligned} \Gamma^y_{yy} &= -\frac{2\bar y}{(1+|x|^2)} & \Gamma^z_{zz} &= -\frac{2\bar z}{(1+|x|^2)} \\ \Gamma^y_{yz} = \Gamma^y_{zy} &= -\frac{\bar z}{(1+|x|^2)} & \Gamma^z_{yz} = \Gamma^z_{zy} &= -\frac{\bar y}{(1+|x|^2)} \\ \Gamma^y_{zz} &= 0 & \Gamma^z_{yy} &= 0 \end{aligned} \tag{19}$$

and their Hermitian conjugates. All others vanish.

For the Riemann tensor,

$$\begin{aligned} R_{y\bar y y\bar y} &= -\frac{2(1+z\bar z)^2}{(1+|x|^2)^4} & R_{z\bar z z\bar z} &= -\frac{2(1+y\bar y)^2}{(1+|x|^2)^4} \\ R_{y\bar y y\bar z} &= \frac{2(1+z\bar z)\bar y z}{(1+|x|^2)^4} & R_{z\bar z z\bar y} &= \frac{2(1+y\bar y)y\bar z}{(1+|x|^2)^4} \\ R_{y\bar y z\bar y} &= \frac{2(1+z\bar z)y\bar z}{(1+|x|^2)^4} & R_{z\bar z y\bar z} &= \frac{2(1+y\bar y)\bar y z}{(1+|x|^2)^4} \end{aligned} \tag{20}$$

$$R_{y\bar{y}z\bar{z}} = -\frac{(1+y\bar{y})(1+z\bar{z})+y\bar{y}z\bar{z}}{(1+|x|^2)^4} = R_{z\bar{z}y\bar{y}}$$

$$R_{y\bar{z}y\bar{y}} = \frac{2(1+z\bar{z})z\bar{y}}{(1+|x|^2)^4} \qquad R_{z\bar{y}z\bar{z}} = \frac{2(1+y\bar{y})y\bar{z}}{(1+|x|^2)^4}$$

$$R_{y\bar{z}y\bar{z}} = -\frac{2\bar{y}^2z^2}{(1+|x|^2)^4} \qquad R_{z\bar{y}z\bar{y}} = -\frac{2\bar{z}^2y^2}{(1+|x|^2)^4} \tag{20 contd.}$$

$$R_{y\bar{z}z\bar{y}} = -\frac{(1+z\bar{z})(1+y\bar{y})+y\bar{y}z\bar{z}}{(1+|x|^2)^4} = R_{z\bar{y}y\bar{z}}$$

$$R_{y\bar{z}z\bar{z}} = \frac{2(1+y\bar{y})\bar{y}z}{(1+|x|^2)^4} \qquad R_{z\bar{y}y\bar{y}} = \frac{2(1+z\bar{z})y\bar{z}}{(1+|x|^2)^4}$$

other components either can be found from the symmetries of the Riemann tensor (8), or they vanish. Note the interesting factorisation

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = -(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}) \tag{21}$$

(the ordering of the indices in (21) is crucial). This factorisation is not a general property of Kahler manifolds, but its form simplifies the calculations for $\mathbb{C}P^2$.

The Ricci tensor is, from (21),

$$R_{\alpha\bar{\beta}} = -R_{\alpha}{}^{\gamma}{}_{\gamma\bar{\delta}} = 3g_{\alpha\bar{\beta}} \tag{22}$$

showing that (17) is indeed an Einstein metric, with cosmological constant, $\Lambda = 3$. (22) could also have been obtained directly from (11), since $\det g_{\alpha\bar{\beta}} = (1+|x|^2)^{-3}$. From (22) the curvature scalar is

$$R = R_{\alpha}{}^{\alpha} + R^{\bar{\alpha}}{}_{\bar{\alpha}} = 12. \tag{23}$$

In the light of (23), note the similarity (and also the difference!) between (21) and the Riemann tensor for S^4,

$$R_{\mu\nu\rho\sigma} = \frac{R}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{24}$$

The Weyl tensor is defined by equation (4.43), in any co-ordinate system, and so in our complex co-ordinates we find, using (22) and (23)

$$C_{\alpha\bar{\beta}\gamma\bar{\delta}} = -g_{\alpha\bar{\beta}}\, g_{\gamma\bar{\delta}}$$
$$C_{\alpha\gamma\bar{\beta}\bar{\delta}} = g_{\alpha\bar{\delta}}\, g_{\gamma\bar{\beta}} - g_{\alpha\bar{\beta}}\, g_{\gamma\bar{\delta}} \,. \tag{25}$$

Note that $C^{\gamma}{}_{\alpha\gamma\bar{\delta}} + C^{\bar{\gamma}}{}_{\alpha\bar{\gamma}\bar{\delta}} = 0$, as it should be.

To examine the duality properties of (25), define four real co-ordinates $\{x^{\mu}\}$, $\mu = 0, \ldots, 3$, for $\mathbb{C}P^2$ via

$$\begin{array}{lll} y = x^0 - i x^3 & x^0 = \frac{1}{2}(y+\bar{y}) & x^2 = \frac{1}{2}(z+\bar{z}) \\ z = x^2 - i x^1 & x^3 = \frac{i}{2}(y-\bar{y}) & x^1 = \frac{i}{2}(z-\bar{z}) \end{array} \tag{26}$$

Tensors can be transformed between real and complex co-ordinates using

$$\frac{\partial x^0}{\partial y} = \frac{\partial x^0}{\partial \bar{y}} = \frac{\partial x^2}{\partial z} = \frac{\partial x^2}{\partial \bar{z}} = \frac{1}{2}$$
$$\frac{\partial x^3}{\partial y} = -\frac{\partial x^3}{\partial \bar{y}} = \frac{\partial x^1}{\partial z} = -\frac{\partial x^1}{\partial \bar{z}} = i/2 \,. \tag{27}$$

For example, the Weyl tensor transforms as

$$C_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{\partial x^{\mu}}{\partial u^{\alpha}} \cdot \frac{\partial x^{\nu}}{\partial u^{\bar{\beta}}} \cdot \frac{\partial x^{\rho}}{\partial u^{\gamma}} \cdot \frac{\partial x^{\sigma}}{\partial u^{\bar{\delta}}}\, C_{\mu\nu\rho\sigma}$$
$$C_{\alpha\gamma\bar{\beta}\bar{\delta}} = \frac{\partial x^{\mu}}{\partial u^{\alpha}} \cdot \frac{\partial x^{\nu}}{\partial u^{\gamma}} \cdot \frac{\partial x^{\rho}}{\partial u^{\bar{\beta}}} \cdot \frac{\partial x^{\sigma}}{\partial u^{\bar{\delta}}}\, C_{\mu\nu\rho\sigma}. \tag{28}$$

Hence the (anti) self-duality equations for the Weyl tensor in real co-ordinates

$$C^{\mu\nu}{}_{\rho\lambda} = \pm \frac{1}{2} \frac{\varepsilon^{\mu\nu\alpha\beta}}{\sqrt{g}}\, C_{\alpha\beta\rho\lambda} \tag{29}$$

become, in complex co-ordinates (note that, since $g_{\mu\nu} = 2\, g_{\alpha\bar{\beta}} \frac{\partial u^{\alpha}}{\partial x^{\mu}} \frac{\partial u^{\bar{\beta}}}{\partial x^{\nu}}$, we have $\sqrt{g} = \sqrt{\det g_{\mu\nu}} = 4 \det g_{\alpha\bar{\beta}}$),

$$C^{\bar{y}z}{}_{\alpha\bar{\beta}} = \pm \frac{C_{y\bar{z}\alpha\bar{\beta}}}{\det g_{\gamma\bar{\delta}}} \qquad C^{\bar{z}y}{}_{\alpha\bar{\beta}} = \pm \frac{C_{z\bar{y}\alpha\bar{\beta}}}{\det g_{\gamma\bar{\delta}}}$$
$$C^{yz}{}_{\alpha\beta} = \mp \frac{C_{\bar{y}\bar{z}\alpha\beta}}{\det g_{\gamma\bar{\delta}}} \qquad C^{\bar{y}\bar{z}}{}_{\bar{\alpha}\bar{\beta}} = \mp \frac{C_{yz\bar{\alpha}\bar{\beta}}}{\det g_{\gamma\bar{\delta}}} \tag{30}$$
$$C^{y\bar{y}}{}_{\alpha\bar{\beta}} = \mp \frac{C_{z\bar{z}\alpha\bar{\beta}}}{\det g_{\gamma\bar{\delta}}}$$

Where $\alpha, \beta = 1,2$. With (25) and (17), the lower signs hold in (30). Thus, in these co-ordinates, the Weyl tensor is anti-self-dual and we will proceed to construct a self-dual Yang-Mills field in $\mathbb{C}P^2$, using the methods of chapter 4. First look for Vierbeins for the metric (17)

$$g_{\alpha\bar{\beta}} = (hh^{+})_{\alpha\bar{\beta}} \tag{31}$$

By inspection, h has the form

$$h_{\alpha\bar{a}} = \begin{bmatrix} 1 - i\frac{z\bar{z}}{|x|} & i\frac{z\bar{y}}{|x|} \\ i\frac{y\bar{z}}{|x|} & 1 - i\frac{y\bar{y}}{|x|} \end{bmatrix} \Big/ (1+|x|^2) \tag{32}$$

then (31) gives (17). Here, and henceforth, a,b,..., $\bar{a}$, $\bar{b}$,... label locally flat complex co-ordinates, i,j,... will label locally flat real co-ordinates, $\alpha, \beta, \ldots, \bar{\alpha}, \bar{\beta}, \ldots$ label curvilinear complex co-ordinates and $\mu, \nu, \ldots$ label curvilinear real co-ordinates.

From (4.8), a self-dual SU(2) Yang-Mills field is given by

$$F_{\mu\nu} = \frac{1}{4}\left(h_\mu h_\nu^{+} - h_\nu h_\mu^{+}\right) \tag{33}$$

where $R = 12, \lambda = \frac{R}{4!}$ have been used. The recipe for constructing h_μ is given in chapter four. Converting (33) into complex co-ordinates requires some care, however. From (27) and (33)

$$F_{yz} = \tfrac{1}{4}\{F_{02} - F_{31} + i(F_{01} + F_{32})\} = \tfrac{1}{16}\{(h_0 + ih_3)(h_2^{+} + ih_1^{+}) - (h_2 + ih_1)(h_0^{+} + ih_3^{+})\}$$

$$F_{\bar{y}\bar{z}} = \tfrac{1}{4}\{F_{02} - F_{31} - i(F_{01} + F_{32})\} = \tfrac{1}{16}\{(h_0 - ih_3)(h_2^{+} - ih_1^{+}) - (h_2 - ih_1)(h_0^{+} - ih_3^{+})\}$$

$$F_{\bar{y}z} = \tfrac{1}{4}\{F_{02} + F_{31} + i(F_{01} - F_{32})\} = \tfrac{1}{16}\{(h_0 - ih_3)(h_2^{+} + ih_1^{+}) - (h_2 + ih_1)(h_0^{+} - ih_3^{+})\}$$

$$F_{y\bar{z}} = \tfrac{1}{4}\{F_{02} + F_{31} - i(F_{01} - F_{32})\} = \tfrac{1}{16}\{(h_0 + ih_3)(h_2^{+} - ih_1^{+}) - (h_2 - ih_1)(h_0^{+} + ih_3^{+})\}$$

$$\begin{aligned} F_{y\bar{y}} &= \tfrac{1}{2} i F_{30} = \tfrac{1}{16}\{(h_0 + ih_3)(h_0^{+} - ih_3^{+}) - (h_0 - ih_3)(h_0^{+} + ih_3^{+})\} \\ F_{z\bar{z}} &= \tfrac{1}{2} i F_{12} = \tfrac{1}{16}\{(h_2 + ih_1)(h_2^{+} - ih_1^{+}) - (h_2 - ih_1)(h_2^{+} + ih_1^{+})\} \end{aligned} \tag{34}$$

In (34) $h_\mu = h_{i\mu} e_i$ where $h_{i\mu}$ are real and e_i are quaternions (2x2 matrices). Guided by (27) define

$$\begin{aligned} h_y &= \tfrac{1}{2}(h_0 + i h_3) \qquad & h_z &= \tfrac{1}{2}(h_2 + i h_1) \\ h_{\bar y} &= \tfrac{1}{2}(h_0 - i h_3) \qquad & h_{\bar z} &= \tfrac{1}{2}(h_2 - i h_1) \; . \end{aligned} \tag{35}$$

Note that $h_{\bar y} \neq (h_y)^\dagger$ and $h_y (h_y)^\dagger$ is <u>not</u> real. Then (34) become

$$\begin{aligned} F_{yz} &= \tfrac{1}{4}\{ h_y (h_{\bar z})^\dagger - h_z (h_{\bar y})^\dagger \} = -(F_{\bar y \bar z})^\dagger \\ F_{y\bar z} &= \tfrac{1}{4}\{ h_y (h_z)^\dagger - h_{\bar z} (h_{\bar y})^\dagger \} = -(F_{\bar y z})^\dagger \\ F_{y\bar y} &= \tfrac{1}{4}\{ h_y (h_y)^\dagger - h_{\bar y} (h_{\bar y})^\dagger \} \\ F_{z\bar z} &= \tfrac{1}{4}\{ h_z (h_z)^\dagger - h_{\bar z} (h_{\bar z})^\dagger \} . \end{aligned} \tag{36}$$

Now, let us define a basis of <u>complex</u> quaternions,

$$\begin{aligned} j_1 &= \tfrac{1}{2}(e_0 + i e_3) \qquad & j_2 &= \tfrac{1}{2}(e_2 + i e_1) \\ j_{\bar 1} &= \tfrac{1}{2}(e_0 - i e_3) \qquad & j_{\bar 2} &= \tfrac{1}{2}(e_2 - i e_1) \end{aligned} \tag{37}$$

In the ensuing calculations, the following multiplication table will be useful,

	$(j_1)^\dagger$	$(j_2)^\dagger$	$(j_{\bar 1})^\dagger$	$(j_{\bar 2})^\dagger$
j_1	j_1	$-j_{\bar 2}$	0	0
j_2	j_2	$j_{\bar 1}$	0	0
$j_{\bar 1}$	0	0	$j_{\bar 1}$	$-j_2$
$j_{\bar 2}$	0	0	$j_{\bar 2}$	j_1

Now

$$h_\alpha = \sqrt{2}\, h_{\alpha \bar b}\, j_{\bar b} \qquad h_{\bar \alpha} = \sqrt{2}\, h_{b \bar \alpha}\, j_b \tag{38}$$

Where $h_{\alpha\bar b}$ is given by (32), α labels rows $\bar b$ columns, and $h_{\alpha \bar b} \equiv \bar h_{b \bar \alpha}$

The factor $\sqrt{2}$ is inserted in (38) because of the normalisation of (37)

$\frac{\mathrm{Tr}}{2}\{j_1 (j_{\bar{1}})^\dagger\} = 1/2$ etc. With these definitions

$$g_{\alpha\bar{\beta}} = \frac{\mathrm{Tr}}{4}\{h_\alpha (h_\beta)^\dagger + h_{\bar{\beta}} (h_{\bar{\alpha}})^\dagger\} \tag{39}$$

and

$$\begin{aligned}
h_y &= \sqrt{2}\,\{(|x| - iz\bar{z})j_{\bar{1}} + iz\bar{y}\,j_{\bar{2}}\}/(1+|x|^2)|x| \\
h_{\bar{y}} &= \sqrt{2}\,\{(|x| + iz\bar{z})j_1 - i\bar{z}y\,j_2\}/(1+|x|^2)|x| \\
h_z &= \sqrt{2}\,\{i\bar{z}y\,j_{\bar{1}} + (|x| - iy\bar{y})j_{\bar{2}}\}/(1+|x|^2)|x| \\
h_{\bar{z}} &= \sqrt{2}\,\{-iz\bar{y}\,j_1 + (|x| + iy\bar{y})j_2\}/(1+|x|^2)|x|
\end{aligned} \tag{40}$$

Using (40) and (36) one finds, for the self-dual field

$$\begin{aligned}
F_{yz} &= F_{\bar{y}\bar{z}} = 0 \\
F_{y\bar{y}} &= \frac{1}{2(1+|x|^2)^2|x|^2}\{ie_3[z\bar{z}(y\bar{y}-z\bar{z}) - |x|^2] + 2j_2 y\bar{z}(z\bar{z}+i|x|) \\
&\qquad - 2j_{\bar{2}} z\bar{y}(z\bar{z} - i|x|)\} \\
F_{z\bar{z}} &= \frac{1}{2(1+|x|^2)^2|x|^2}\{-ie_3[y\bar{y}(z\bar{z}-y\bar{y}) - |x|^2] + 2j_2 y\bar{z}(y\bar{y}-i|x|) \\
&\qquad - 2j_{\bar{2}} z\bar{y}(y\bar{y} + i|x|)\} \\
F_{y\bar{z}} &= -(F_{\bar{y}z})^\dagger = \frac{1}{2(1+|x|^2)^2|x|^2}\{ie_3 z\bar{y}[z\bar{z} - y\bar{y} + 2i|x|] \\
&\qquad - 2j_2[z\bar{z}y\bar{y} + |x|^2 + i|x|(y\bar{y} - z\bar{z})] + 2j_{\bar{2}} z^2\bar{y}^2\}
\end{aligned} \tag{41}$$

It is straightforward to check that $F_{\alpha\bar{\beta}}$ given by (41) is self-dual in the sense of equations (30). The calculation of a potential for the field (41) proceeds via (4.26). The algebra is somewhat tedious, but eventually one arrives at the following expressions,

$$\begin{aligned}
A_y &= -(A_{\bar{y}})^\dagger = \frac{1}{2}\frac{(1+i/|x|)}{(1+|x|^2)}\left\{ie_3\bar{y}\left(\tfrac{1}{2} + \tfrac{z\bar{z}}{|x|^2}\right) + j_2\bar{z}\left(1 + \tfrac{z\bar{z}}{|x|^2}\right) + j_{\bar{2}}\tfrac{z\bar{y}^2}{|x|^2}\right\} \\
A_z &= -(A_{\bar{z}})^\dagger = -\frac{1}{2}\frac{(1+i/|x|)}{(1+|x|^2)}\left\{ie_3\bar{z}\left(\tfrac{1}{2} + \tfrac{y\bar{y}}{|x|^2}\right) + j_2\tfrac{y\bar{z}^2}{|x|^2} + j_{\bar{2}}\bar{y}\left(1 + \tfrac{y\bar{y}}{|x|^2}\right)\right\}.
\end{aligned} \tag{42}$$

As a check one can put (42) into

$$F_{\alpha\bar{\beta}} = \partial_\alpha A_{\bar{\beta}} - \partial_{\bar{\beta}} A_\alpha + [A_\alpha, A_{\bar{\beta}}] \tag{43}$$

to recover (41) (again, after some work!)

The topological charge of this configuration can be evaluated using (4.9)

$$Q = \frac{3}{8\pi^2} \int_M \sqrt{g}\, d^4x \, . \tag{44}$$

Since $\sqrt{g} = 4(1+|x|^2)^{-3}$, this integral can easily be evaluated, using polar co-ordinates $y = r_1 e^{i\theta_1}$, $z = r_2 e^{i\theta_2}$,

$$\begin{aligned} Q &= \frac{3}{2\pi^2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^\infty r_1\, dr_1 \int_0^\infty r_2\, dr_2 \frac{1}{(1+r_1^2+r_2^2)^3} \\ &= 3/4 \end{aligned} \tag{45}$$

The topological charge is not an integer! This agrees with the derivation in chapter four of the topological charge in terms of the Euler characteristic and the Hirzebruch Signature (4.70). For $\mathbb{C}P^2$, χ and τ can be calculated from (20) using (4.67) and (4.68) and are found to be 3 and -1 respectively (τ would have the opposite sign if the metric were chosen with the opposite orientation.. This would cause the Weyl tensor to be self-dual, and then only the anti-self-dual field would have a potential.) From (4.70)

$$Q = \pm \frac{\chi}{2} + \frac{3}{4}\tau \tag{46}$$

which gives Q = 3/4 for the self-dual field (upper-sign) and thus confirms (45). The fractional value of Q is due to the fact that it is not possible to put a SU(2) Yang-Mills field onto $\mathbb{C}P^2$ in a <u>globally</u> consistent manner, though (41) and (42) are perfectly satisfactory for a Yang-Mills field <u>locally</u>. For example, start at the origin and consider A_α spreading out over $\mathbb{C}P^2$. Equation (42) constrains the form of A_α. But $\mathbb{C}P^2$ is a compact manifold, and as we go towards $|x| \to \infty$, we are left the problem of matching up A_α on the S^2 at infinity. It cannot be done without having a discontinuous jump in A_α. An exactly analogous problem exists for fermion fields on $\mathbb{C}P^2$.

For fermions, one possible solution is to couple the spinor fields to a vector field, forming a generalised spin structure[42]. This will be discussed later.

For the SU(2) field above, we can make Q into an integer by using a trick similar to that which was used to construct multi-instantons in the $\mathbb{HP}^1$ model of chapter three,[37, 45, 46]. Let $u'^{\alpha} = (u^{\alpha})^2$, i.e.

$$y' = y^2 \qquad z' = z^2 \tag{47}$$

and consider the metric

$$g_{\alpha'\bar{\beta}'} = \partial_{\alpha'}\partial_{\bar{\beta}'} \ln (1 + u'^{\gamma}\bar{u}'^{\gamma}) \tag{48}$$

which is, locally, that of $\mathbb{C}P^2$ parameterised by y' and z'. Everything proceeds as before, except that $y' = r_1' e^{i\theta_1'}$, $z = r_2' e^{i\theta_2'}$ where $0 \leq \theta_1' \leq 4\pi$ and $0 \leq \theta_2' \leq 4\pi$, and globally the space is $\mathbb{C}P^2$ wrapped round itself four times. The integral in (45) is increased fourfold and Q = 3. However, the same criticism applies here as in chapter three, that the space described by (48) is not a manifold, but only a topological space, and the physical significance of this is not clear.

Atiyah et al[5] and Gibbons and Pope[32] both make the point that the anti-self-dual field constructed over $\mathbb{C}P^2$ obtained from the Christoffel connection reduces to a U(1) field. With this point in mind, let us examine the anti-self-dual field,

$$F_{\alpha\bar{\beta}} = \frac{\lambda}{2}\{ (h_{\bar{\alpha}})^{\dagger} h_{\bar{\beta}} - (h_{\beta})^{\dagger} h_{\alpha} \} \tag{49}$$

(Since we know, from chapter four, that no SU(2)) potential exists for this field, no restrictions have been put on λ). One finds, using

(40),

$$F_{y\bar{y}} = \frac{\lambda(1+z\bar{z})}{(1+|x|^2)^2} ie_3 \qquad F_{z\bar{z}} = \frac{\lambda(1+y\bar{y})}{(1+|x|^2)^2} ie_3$$

$$F_{y\bar{z}} = -\frac{\lambda z\bar{y}}{(1+|x|^2)^2} ie_3 \qquad F_{yz} = \frac{2\lambda(1-i|x|)}{(1+|x|^2)^2} j_{\bar{x}} \tag{50}$$

$$F_{z\bar{y}} = -\frac{\lambda \bar{z} y}{(1+|x|^2)^2} ie_3 \qquad F_{\bar{y}\bar{z}} = \frac{2\lambda(1+i|x|)}{(1+|x|^2)^2} j_x \; .$$

It is straightforward to show that the field (50) is anti-self-dual in the sense of (30). Note that only the components $F_{yz} = -(F_{\bar{y}\bar{z}})^{\dagger}$ prevent the field from being abelian. Let us, therefore, consider the abelian configuration

$$F_{y\bar{z}} = -\frac{\lambda z\bar{y}}{(1+|x|^2)^2} \qquad F_{z\bar{y}} = -\frac{\lambda y\bar{z}}{(1+|x|^2)^2}$$

$$F_{y\bar{y}} = \frac{\lambda(1+z\bar{z})}{(1+|x|^2)^2} \qquad F_{z\bar{z}} = \frac{\lambda(1+y\bar{y})}{(1+|x|^2)^2} \tag{51}$$

$$F_{yz} = F_{\bar{y}\bar{z}} = 0 .$$

By inspection, a U(1) potential exists for this field and is of the form

$$A_y = -\frac{\lambda\bar{y}}{2(1+|x|^2)} \quad A_{\bar{y}} = \frac{\lambda y}{2(1+|x|^2)} \quad A_z = -\frac{\lambda\bar{z}}{2(1+|x|^2)} \quad A_{\bar{z}} = \frac{\lambda z}{2(1+|x|^2)} \tag{52}$$

In fact $F_{\alpha\bar{\beta}}$ is a multiple of the Kahler 2-form for $\mathbb{C}P^2$

$$F_{\alpha\bar{\beta}} = \lambda\, \partial_\alpha \partial_{\bar{\beta}} \ln(1+|x|^2) \tag{53}$$

This is a general feature of Kahler manifolds, Since the Kahler 2-form is closed, it satisfies Maxwell's equations. This point is noted in reference[32]. The topological charge of (51), as a U(1) field is

$$Q_{U(1)} = \frac{1}{16\pi^2}\int_M {}^*F^{\mu\nu} F_{\mu\nu} \sqrt{g}\, d^4x = -\frac{\lambda^2}{2} \tag{54}$$

and the field has integral topological charge if λ is plus or minus the square root of twice a positive integer. The freedom to rescale

$F_{\alpha\bar{\beta}}$ by an arbitrary real factor is <u>not</u> present in the SU(2) case due to the non-linear nature of $F_{\alpha\bar{\beta}}$, equation (43). The U(1) bundle, with gauge field (53), over $\mathbb{C}P^2$ is in fact topologically S^5. It is the Hopf fibration of S^5 [32, 57].

It has been noted, by Hawking and Pope [42], that a U(1) field on $\mathbb{C}P^2$ can cure the problem of putting spinor fields on $\mathbb{C}P^2$ mentioned earlier. To see how this can happen, we need the index theorem for the Dirac operator on a compact manifold, without boundary, in the presence of a U(1) gauge field [26]. With no gauge fields present, the index theorem tells us that the number of positive helicity solutions (ν_+) of the Dirac equation minus the number of negative helicity solutions (ν_-) is related to the curvature of the manifold by

$$\begin{aligned} \nu_+ - \nu_- &= -\frac{1}{384\pi^2}\int_M {}^*R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\sqrt{g}\,d^4x \\ &= -\tau/8. \end{aligned} \tag{55}$$

For $\mathbb{C}P^2$, with the chosen orientation, $\tau = -1$ and so the right hand side of (55) is not an integer. This is a reflection of the fact that spinor fields cannot be defined on $\mathbb{C}P^2$ and may be taken as a proof of that statement. In the presence of a U(1) or SU(2) gauge field, however, (55) is modified to

$$\begin{aligned} (\nu_+ - \nu_-)_{U(1)} &= -\frac{1}{384\pi^2}\int_M {}^*R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\sqrt{g}\,d^4x + \frac{1}{16\pi^2}\int_M {}^*F^{\mu\nu}F_{\mu\nu}\sqrt{g}\,d^4x \\ &= -\tau/8 + Q_{U(1)} \end{aligned} \tag{56}$$

$$\begin{aligned} (\nu_+ - \nu_-)_{SU(2)} &= -\frac{1}{192\pi^2}\int_M {}^*R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\sqrt{g}\,d^4x - \frac{1}{16\pi^2}\int_M \mathrm{Tr}\,({}^*F^{\mu\nu}F_{\mu\nu})\sqrt{g}\,d^4x \\ &= -\tau/4 + Q_{SU(2)} \end{aligned} \tag{57}$$

respectively (the generators of SU(2) are normalised so that $\mathrm{Tr}\,(e_i e_j) = -2\delta_{ij}$.

For the self-dual SU(2) field derived earlier (41), $Q_{SU(2)} = 3/4$ and (57) becomes

$$(\nu_+ - \nu_-)_{SU(2)} = 1 \tag{58}$$

Thus, putting a SU(2) field on the manifold enables spinors to be defined on $\mathbb{C}P^2$ in a consistent fashion! However, we are still left with the problem of non-integral Q i.e. the gauge field does not match up.

For a U(1) field, (56) becomes, with (54)

$$(\nu_+ - \nu_-)_{U(1)} = 1/8 - \lambda^2/2 \tag{59}$$

so that a judicious choice of λ will lead to a consistent definition of spinor fields e.g. λ = 3/2 will do.

It is also possible to make $\nu_+ - \nu_-$ in (55) an integer without introducing gauge fields at all, using the method mentioned previously (47), (48), e.g. taking $y' = y^4$, $z' = z^4$ increases all integrals over the space by a factor of sixteen, making $\nu_+ - \nu_-$ an integer. Again, however, we are no longer dealing with a manifold.

In this chapter, the methods of chapter four have been applied to $\mathbb{C}P^2$, a four dimensional, compact, Einstein space, with non-zero scalar curvature. With the orientation on $\mathbb{C}P^2$ chosen so that the Weyl tensor is anti-self-dual, SU(2) Yang-Mills fields have been explicitly constructed, with topological charge 3/4, indicating a global obstruction to the field. The anti-self-dual field reduces to a U(1) field, and it has been demonstrated how this can cure the problem of putting spinors on $\mathbb{C}P^2$.

In the next chapter, the analysis of chapter four will be applied to another Einstein space with non-zero scalar curvature, $S^2 \times S^2$. The Weyl tensor for $S^2 \times S^2$ is neither self-dual nor anti-self-dual, however $S^2 \times S^2$ is a Kahler manifold and as such is worth studying from the point of view of U(1) fields.

CHAPTER SIX

U(1) INSTANTONS IN $S^2 x S^2$

In this chapter, U(1) instantons over $S^2 x\ S^2$ will be constructed. $S^2 x\ S^2$ is a compact, Einstein manifold with non-zero scalar curvature. The Weyl tensor is neither self-dual nor anti-self-dual and so SU(2) fields will not be considered. However, a Kahler metric exists for $S^2 x\ S^2$ and so the Kahler 2-form gives a non-trivial Maxwell field, as pointed out in the last chapter.

To construct an Einstein metric on $S^2 x\ S^2$, we simply take the direct sum of two S^2 metrics. The invariant line element on S^2 is

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \tag{1}$$

where $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ (the radius of the sphere is set to unity.) On $S^2 x\ S^2$ we take the line element to be

$$ds^2 = (d\chi^2 + \sin^2\chi\, d\psi^2) + (d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2}$$

where $0 \leq \chi \leq \pi$, $0 \leq \psi \leq 2\pi$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. That this is a Kahler metric can be seen by parameterising each S^2 by a complex co-ordinate

$$\begin{aligned} y &= |y| e^{i\phi} = \tan\frac{\theta}{2}\, e^{i\phi} \\ z &= |z| e^{i\psi} = \tan\frac{\chi}{2}\, e^{i\psi} . \end{aligned} \tag{3}$$

Then (2) becomes

$$ds^2 = 4\{(1+y\bar{y})^{-2} dy\, d\bar{y} + (1+z\bar{z})^{-2} dz\, d\bar{z}\} \tag{4}$$

In the form of equation (4), the metric is manifestly Kahler, since it is derivable from the Kahler potential

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \Phi = \partial_\alpha \partial_{\bar{\beta}} \{ 4 \ln[(1+y\bar{y})(1+z\bar{z})]\} . \tag{5}$$

This is a particular example of a more general result. S^2 is a Kahler mainfold with complex dimension one, and each term in (4) is a Kahler metric for $\mathbb{C}P^1$. The outer product of two Kahler manifolds is again a Kahler manifold, with Kahler metric the direct sum of the two Kahler metrics on the original two manifolds. The Kahler potential is just the sum of the two original Kahler potentials.

The form of (2) most commonly found in the literature[32, 26] is obtained by setting $r = -\cos\chi$, then

$$ds^2 = (1-r^2)\, d\psi^2 + \frac{dr^2}{(1-r^2)} + d\theta^2 + \sin^2\theta\, d\phi^2 \tag{6}$$

where $-1 \leq r \leq 1$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \psi \leq 2\pi$. The form (2) will be used in what follows.

The Christoffel symbols obtained from (2) are

$$\begin{aligned} \Gamma^{\psi}_{\psi\chi} &= \Gamma^{\psi}_{\chi\psi} = \cot\chi & \qquad \Gamma^{\phi}_{\theta\phi} &= \Gamma^{\phi}_{\phi\theta} = \cot\theta \\ \Gamma^{\chi}_{\psi\psi} &= -\sin\chi\cos\chi & \qquad \Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta \end{aligned} \tag{7}$$

all others vanishing. The Riemann tensor is found to have the following non-zero components

$$\begin{aligned} R^{\psi}{}_{\chi\psi\chi} &= 1 & \qquad R^{\phi}{}_{\theta\phi\theta} &= 1 \\ R^{\chi}{}_{\psi\chi\psi} &= \sin^2\chi & \qquad R^{\theta}{}_{\phi\theta\phi} &= \sin^2\theta\,. \end{aligned} \tag{8}$$

From (8), the Einstein property of the metric (2) is obvious

$$R_{\mu\nu} = g_{\mu\nu} \tag{9}$$

thus R = 4. The Weyl tensor takes the form,

$$\begin{aligned} C_{\chi\psi\chi\psi} &= \tfrac{2}{3}\sin^2\chi & \qquad C_{\theta\phi\theta\phi} &= \tfrac{2}{3}\sin^2\theta \\ C_{\theta\psi\theta\psi} &= -\tfrac{1}{3}\sin^2\theta & \qquad C_{\chi\phi\chi\phi} &= -\tfrac{1}{3}\sin^2\chi \\ C_{\psi\phi\psi\phi} &= -\tfrac{1}{3}\sin^2\chi\sin^2\theta & \qquad C_{\chi\theta\chi\theta} &= -\tfrac{1}{3} \end{aligned} \tag{10}$$

plus components obtained by symmetry operations, all others vanishing. (Note that, as a consistency check $C^{\mu}{}_{\nu\mu\rho} = 0$). The Weyl tensor (10) is neither self-dual nor anti-self-dual, and so the method developed in chapter 4 will not yield a SU(2) field. However, the Kahler nature of the manifold makes it worthwhile to study U(1) fields in the metric (2).

Consider the Vierbeins

$$h_{\psi} = \sin\chi\, e_0 \quad h_{\chi} = e_1 \quad h_{\varphi} = \sin\theta\, e_2 \quad h_{\theta} = e_3 . \tag{11}$$

If we try and construct a SU(2) field, as in (4.26), it reduces to a U(1) field as follows.

Let

$$A_{\mu} = \frac{1}{4}\{h_{\nu} h^{+\nu}{}_{;\mu}\} \tag{12}$$

then one finds, using (11) and (7) that

$$A_{\chi} = 0 \quad A_{\psi} = -\frac{\cos\chi}{2} e_1 \quad A_{\theta} = 0 \quad A_{\phi} = -\frac{\cos\theta}{2} e_1 \tag{13}$$

which gives a self-dual field tensor,

$$\begin{aligned} F_{\chi\psi} &= \frac{\sin\chi}{2} e_1 \qquad F_{\theta\varphi} = \frac{\sin\theta}{2} e_1 \\ F_{\chi\varphi} &= F_{\chi\theta} = F_{\psi\varphi} = F_{\psi\theta} = 0 . \end{aligned} \tag{14}$$

An anti-self-dual field may be obtained by changing the relative sign of $F_{\chi\psi}$ and $F_{\theta\varphi}$ (or equivalently A_{ψ} and A_{ϕ}). This can be achieved by transferring the (†) in (12) from the second to the first factor on the right hand side. An alternative method of constructing an anti-self-dual field is to change the orientation of one of the S^2 factors of $S^2 \times S^2$. This amounts to changing the relative sign of the terms in brackets in equation (2).

The topological charge of the field (14) is (with the generator of U(1) normalised so that $e_1^2 = -1$)

$$Q = \frac{1}{16\pi^2}\int_0^\pi \sin\chi\, d\chi \int_0^{2\pi} d\psi \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi \; F_{\mu\nu}F^{\mu\nu} \qquad (15)$$

$$= 1 .$$

However, since $F_{\mu\nu}$ is linear in the potential A_μ, the whole configuration (13) and (14) can be multiplied by an arbitrary, real factor, n, and it remains self-dual. If we also perform a gauge transformation

$$A_\mu \rightarrow A_\mu + \partial_\mu a \qquad (16)$$

where

$$a = \frac{n}{2}(\psi + \phi) \qquad (17)$$

we obtain the field configuration

$$\begin{aligned} &A_\chi = A_\theta = 0 \qquad A_\psi = \frac{n}{2}(1-\cos\chi)e_1 \qquad A_\phi = \frac{n}{2}(1-\cos\theta)e_1 \\ &F_{\chi\psi} = \frac{n}{2}\sin\chi\, e_1 \qquad F_{\theta\phi} = \frac{n}{2}\sin\theta\, e_1 \\ &F_{\chi\phi} = F_{\chi\theta} = F_{\psi\theta} = F_{\psi\phi} = 0 . \end{aligned} \qquad (18)$$

This configuration has topological charge $Q = n^2$. The potential on each S^2 is like a Dirac monopole potential, for a monopole of charge n. It becomes singular on the south pole of each S^2 which can be seen by embedding each S^2 in $\mathbb{R}^3$ as follows. Let

$$\begin{aligned} v &= r\cos\theta & z &= r\cos\chi \\ w &= r\sin\theta\cos\psi & x &= r\sin\chi\cos\psi \\ u &= r\sin\theta\sin\psi & y &= r\sin\chi\sin\psi \end{aligned} \qquad (19)$$

where

$$u^2 + v^2 + w^2 = x^2 + y^2 + z^2 = r^2 = 1 \qquad (20)$$

since the spheres are both taken to be of unit radius. Writing the potential as a one-form, one finds (dropping the e_1)

$$A = A_\mu dx^\mu = \frac{n}{2}(1-\cos\chi)\,d\psi + \frac{n}{2}(1-\cos\theta)\,d\phi$$

$$= \frac{n}{2}\left\{\frac{x\,dy - y\,dx}{z+1}\right\} + \frac{n}{2}\left\{\frac{w\,du - u\,dw}{v+1}\right\} \tag{21}$$

Which is singular at $z = -1$ ($\chi = \pi$) and $u = -1$ ($\theta = \pi$), the south poles of the two spheres. This is the familiar Dirac string singularity, except that, in this case, they are not strings, since the spheres are of fixed radius. As is well known the singularity is illusory[62], and can be avoided by splitting the potential up into two parts on each sphere

$$A^\pm = \frac{n}{2}\left\{\frac{x\,dy - y\,dx}{z\pm 1}\right\} + \frac{n}{2}\left\{\frac{w\,du - u\,dw}{v\pm 1}\right\}$$

$$= \frac{n}{2}(\pm 1-\cos\chi)\,d\psi + \frac{n}{2}(\pm 1-\cos\theta)\,d\phi \tag{22}$$

A^+ is well defined away from the south pole of both spheres and A^- is well defined away from the north pole of both spheres. Furthermore, A^+ is related to A^- (away from both north poles and both south poles) by a gauge transformation

$$A^+ = A^- + n(d\psi + d\phi). \tag{23}$$

At the north pole of one sphere, but the south pole of the other, a non-singular potential is defined by taking the upper sign in one term and the lower sign in the other in (22), which is again related to $A^\pm$ by a gauge transformation. Thus four co-ordinate patches are needed in all to define a non-singular potential.

If we think of ψ as a compactified time co-ordinate and χ as a compactified radial co-ordinate (via $r = \tan\frac{\chi}{2}$), then (18) is like a "dyon" in that $F_{\chi\psi}$ can be thought of as an electric monopole and $F_{\theta\phi}$ as a magnetic monopole. For the field to be self-dual the electric and magnetic monopoles must have the same charge, n. Then $Q = n^2$ is like the Dirac quantisation condition in four dimensions.

We have now examined two compact Einstein manifolds with non-zero scalar curvature, $\mathbb{C}P^2$ and $S^2 \times S^2$. There is one other solution of Einstein's equations with positive, non-zero, scalar curvature which is explicitly known, and that is the Page metric on $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ [53]. This however, has neither self-dual nor anti-self-dual Weyl tensor, nor is it a Kahler manifold, and therefore will not be considered in this work.

CHAPTER SEVEN

CONCLUSION

In attempting to generalise U(1) invariant $\mathbb{C}P^n$ models in two dimensions, one is led, naturally, to $\mathbb{H}P^n$ models in four dimensions, which have SU(2) invariance[38]. This has led to the consideration of SU(2) Yang-Mills in curved space-times, viewed from a similar point of view to that of Einstein's generalised theory of gravitation[36]. A method was developed in chapter three for the construction of a Hermitian, quaternionic metric, which automatically yields a SU(2) Yang-Mills field tensor which is self-dual in the space described by the real part of the quaternionic metric (provided $R \neq 0$). In chapter four, it was shown that the very existence of a potential for the Yang-Mills field implies that Einstein's equations, with a cosmological constant, must be satisfied. Further, the relationship of the method to the SU(2) x SU(2) decomposition of Utiyama's O(4) gauge theory of gravity[60], as developed by Charap and Duff[15], was established. It was shown that, for $R \neq 0$, a self-dual (anti-self-dual) Yang-Mills field required anti-self-dual (self-dual) Weyl tensor.

In chapter five, the construction was applied to $\mathbb{C}P^2$, an important ingredient in Hawkings' space-time foam[40], to yield a self-dual Yang-Mills field with topological charge 3/4. U(1) fields were also discussed, from the point of view of Kahler geometry. Finally, U(1) fields over $S^2 \times S^2$ were discussed, yielding "dyon" type solutions.

For the case of S^4, an analysis of the metric and spin connection yields the single instanton of Belavin et al[6], and this has generalisations to multi-instantons over S^4[44], the 't Hooft solutions and their conformal extension. It is intriguing to ask whether or not a similar extension might exist for $\mathbb{C}P^2$. This question merits further investigation, though, if the answer is in the affirmative, it will not be easy to find. Compare the complexity of the single instanton over $\mathbb{C}P^2$ (5.41) to that of S^4 (4.17).

Another important question is, how to extend this construction to the real gauge symmetry of the strong interactions, SU(3), since this was our motivation for studying SU(2) in the first place - to obtain a better understanding of SU(3). The work so far has depended crucially on the relationship of quaternions to SU(2) and the fact that $O(4) \approx SU(2) \times SU(2)$, the later being an accident of low dimensional Lie group theory. SU(3) is not a sub-group of O(4), so it is not obvious that the same methods will work for SU(3). However, there is another accident of group theory, which could prove useful, and that is $O(6) \approx SU(4) \supset SU(3)$. An O(6) gauge theory could be used to describe the curvature of a six dimensional manifold. The simplest, compact, six dimensional manifold is the sphere S^6, which is also an Einstein space. Thus one could consider a SU(4) gauge theory over S^6, whose connection is simply the spin connection of S^6, and try reducing this to a field over S^4 by projecting out or compactifying two of the dimensions.

A metric for S^6 is

$$g_{\mu\nu} = \frac{\delta_{\mu\nu}}{(1+x^2)^2} \tag{1}$$

where $\mu, \nu = 1, \ldots, 6$, $x^2 = \sum_{\mu=1}^{6} x_\mu x_\mu$

A choice of Vierbeins is

$$h_{i\mu} = \frac{\delta_{i\mu}}{(1+x^2)} \tag{2}$$

$(i = 1, \ldots, 6)$.

The Christoffel symbols resulting from (1) are

$$\Gamma_{\mu,\rho\sigma} = \frac{2}{(1+x^2)^3} \left(\delta_{\rho\sigma} x_\mu - \delta_{\mu\rho} x_\sigma - \delta_{\mu\sigma} x_\rho \right). \tag{3}$$

Thus the spin connection is

$$\Gamma_\mu = \frac{1}{2}\sigma_{ij} h_i^{\ \nu} (h_{j\nu;\mu})$$

$$= \sigma_{ij} \frac{(\delta_{j\mu}\delta_{i\nu} - \delta_{i\mu}\delta_{j\nu})x_\nu}{(1+x^2)} \tag{4}$$

where $\sigma_{ij} = -\sigma_{ji}$ are the generators of O(6).

The O(6) field tensor is

$$F_{\mu\nu} = \frac{2(\delta_{j\nu}\delta_{i\mu} - \delta_{j\mu}\delta_{i\nu})\sigma_{ij}}{(1+x^2)^2} \tag{5}$$

The configuration (4) and (5) can be considered as a SU(4) field over S^6, with σ_{ij} being (possibly complex) generators of SU(4).

The concept of self-duality does not apply in six dimensions, but it is easy to check that, if one takes the six dimensional action to be

$$S = -\frac{1}{2}\int_{S^6} \mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu})\sqrt{g}\, d^4x \tag{6}$$

and varies the potential to obtain the six dimensional Yang-Mills equations

$$\partial_\mu\{\sqrt{g}\, F^{\mu\nu}\} + \sqrt{g}\,[\Gamma_\mu, F^{\mu\nu}] = 0 \tag{7}$$

then (4) and (5) satisfy (7).

Unfortunately, it does not seem possible to obtain a SU(4) field over S^4 from (4) and (5) in any simple way. If one tries to project down onto S^4, or integrate over two of the co-ordinates so as to eliminate them, one finds that nine of the generators are eliminated also, leaving an O(4) or SU(2) x SU(2) configuration over S^4, which is exactly that of Belavin et al[6]. Thus it remains unclear how to extend this formalism to SU(3).

Another field of study, which has not been touched upon in this thesis, is that of Yang-Mills - Higgs monopoles in curved space-times. Much work has been done on this subject[16, 18] and though the topic was investigated for this thesis, no further progress was made.

APPENDIX A - QUATERNIONS

Quaternions, or Hypercomplex numbers, were first considered by Sir William Hamilton in 1843 in attempting to understand the algebra of rotations. The quaternions have three basis elements e_1, e_2, e_3 which satisfy the algebra

$$e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1 \tag{A.1}$$

A matrix representation for such an algebra is given by the Pauli matrices,

$$e_1 = -i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad e_2 = -i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad e_3 = -i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{A.2}$$

it is thus identical to the algebra of SU(2). This representation is anti-hermitian $e_i^{+} = -e_i, i=1,2,3$ where $(+)$ means transpose followed by complex conjugation. If we supplement (A.2) with the unit 2 x 2 matrix $e_0 = 1_{2\times 2}$, then a general quaternion can be written as a linear combination of the four matrices $\{e_0, e_1, e_2, e_3\}$

$$p = p_i e_i \tag{A.3}$$

where p_i , $i=0,\dots,3$ are real numbers. The set of all such p form a skew symmetric field, the field of quaternions, denoted by $\mathbb{H}$. The magnitude of p is defined by

$$|p|^2 = \frac{1}{2}\,\mathrm{Tr}(pp^{+}) = p_i p_i \tag{A.4}$$

The set of quaternions of unit magnitude is identical to the group SU(2).

It is sometimes convenient to break quaternions up into their real and pure quaternionic parts (denoted by $\mathrm{Re}\,p$ and $\mathrm{Vec}\,p$ respectively).

$$\mathrm{Re}\,p = p_0 e_0 \qquad \mathrm{Vec}\,p = p_1 e_1 + p_2 e_2 + p_3 e_3 \tag{A.5}$$

Where there is no possibility of confusion, the unit matrix is omitted and we write $\mathrm{Re}\,p = p_0$.

Note that

$$(\mathrm{Vec}\,p)^{\dagger} = -\mathrm{Vec}\,p . \tag{A.6}$$

Any p obeys its own characteristic equation

$$p^2 - 2p_0 p + p p^{\dagger} = 0 \tag{A.7}$$

A useful property of the basis $\{e_i\}$ $i = 0,\ldots,3$ is the following (anti) self-duality property

$$\begin{aligned} e_m e_n^{\dagger} - e_n e_m^{\dagger} &= \tfrac{1}{2}\varepsilon_{mnrs}(e_r e_s^{\dagger} - e_s e_r^{\dagger}) \\ e_m^{\dagger} e_n - e_n^{\dagger} e_m &= -\tfrac{1}{2}\varepsilon_{mnrs}(e_r^{\dagger} e_s - e_s^{\dagger} e_r) \end{aligned} \tag{A.8}$$

Where ε_{mnrs} is the totally antisymmetric tensor (ε_{0123} = +1).

Equations A(8) can be written, using the symbols introduced by 't Hooft [59]

$$\begin{aligned} e_m e_n^{\dagger} - e_n e_m^{\dagger} &= -2 e_a \eta^{(+)a}_{mn} \\ e_m^{\dagger} e_n - e_n^{\dagger} e_m &= -2 e_a \eta^{(-)a}_{mn} \end{aligned} \tag{A.9}$$

Where the Euclidean η symbols are given by

$$\eta^{(\pm)a}_{mn} = \varepsilon_{0amn} \mp \delta_{ma}\delta_{0n} \pm \delta_{na}\delta_{0m} \tag{A.10}$$

$(m,n = 0,\ldots,3 \;;\; a = 1,\ldots,3)$.

The η symbols have the following useful properties

$$\eta^{(\pm)a}_{mn}\,\eta^{(\pm)a}_{kl} = \delta_{mk}\delta_{nl} - \delta_{nk}\delta_{ml} \pm \varepsilon_{mnkl} \tag{A.11}$$

$$\eta^{(\pm)a}_{mn}\,\eta^{(\pm)b}_{mk} = \varepsilon_{abc}\,\eta^{(\pm)c}_{nk} + \delta_{ab}\,\delta_{nk} \tag{A.12}$$

$$\varepsilon_{abc}\,\eta^{(\pm)b}_{mn}\,\eta^{(\pm)c}_{kl} = \eta^{(\pm)a}_{nl}\,\delta_{mk} + \eta^{(\pm)a}_{mk}\,\delta_{nl} - \eta^{(\pm)a}_{ml}\,\delta_{nk} - \eta^{(\pm)a}_{nk}\,\delta_{ml} \tag{A.13}$$

$$\eta^{(\pm)a}_{mn}\,\varepsilon_{nklr} = \pm\,\eta^{(\pm)a}_{kl}\,\delta_{mr} \pm \eta^{(\pm)a}_{rk}\,\delta_{ml} \pm \eta^{(\pm)a}_{lr}\,\delta_{km}\,. \tag{A.14}$$

In addition, the following properties sometimes prove useful

$$e_i\,p\,e_i = e_i^{+}\,p\,e_i^{+} = -2\,p^{+} \tag{A.15}$$

$$e_i\,p\,e_i^{+} = e_i^{+}\,p\,e_i = 4\,\mathrm{Re}\,p = 2(p+p^{+}) \tag{A.16}$$

$$p^{-1} = p^{+}/|p|^2 \tag{A.17}$$

Finally we note that points in four dimensional Euclidean space-time can be labelled by a single quaternion

$$x = x_i e_i = x_0 - i\,\underline{\sigma}.\underline{x} \tag{A.18}$$

For further details of quaternions and quaternionic valued functions see references [38] and [58].

APPENDIX B - EVALUATION OF TWO INTEGRALS

In calculating the topological charge and action of the $u = \frac{1}{\rho^2}(x^4+a)(x^4+b)$ configuration, equation (2.65) in chapter two, the following integrals were encountered

$$I_1 = \frac{1}{\rho^8}\int_0^\infty dt \int_0^\infty dr \, \frac{t^2 r^2 (t^2+r^2)}{\{1 + \frac{1}{\rho^4}[(t+a)^2+r^2][(t-a)^2+r^2]\}^4} \tag{2.68}$$

$$I_2 = \frac{1}{\rho^8}\int_0^\infty dt \int_0^\infty dr \, \frac{r^6}{\{1+\frac{1}{\rho^4}[(t+a)^2+r^2][(t-a)^2+r^2]\}^4} \tag{2.70}$$

First of all, absorb the factors of ρ, so as to make everything dimensionless by defining $\tilde{t} = t/\rho$, $\tilde{r} = r/\rho$, and subsequently drop the tildes. This is achieved simply by setting $\rho = 1$ in (2.68) and (2.70). Consider I_1, with the following change of variables

$$v = t^2 + r^2 \qquad w = t^2 - r^2 \tag{B.1}$$

where $0 \leq v < \infty$, $-v \leq w \leq v$.

Then, the Jacobian for the change of variables is

$$\left|\frac{\partial(t,r)}{\partial(v,w)}\right| = \frac{1}{4}(v^2 - w^2)^{-1/2} \tag{B.2}$$

and

$$t^2 r^2 = \frac{1}{4}(v^2 - w^2) \tag{B.3}$$

Hence

$$I_1 = \frac{1}{16}\int_0^\infty dv \int_{-v}^{v} dw \, \frac{(v^2-w^2)^{1/2}\, v}{[1+(v^2+a^4) - 2wa^2]^4} \tag{B.4}$$

Now, make the further change

$$z = w/v \qquad -1 \leq z \leq 1 \tag{B.5}$$

and define

$$\beta(v) = \frac{1}{2a^2 v}(1+v^2+a^4) \tag{B.6}$$

then

$$I_1 = \frac{1}{16}\cdot\frac{1}{(2a^2)^4}\int_0^\infty \frac{dv}{v}\int_{-1}^{1}\frac{(1-z^2)^{1/2}\,dz}{[\beta(v)-z]^4} \tag{B.7}$$

Consider, $(z = \cos\theta)$

$$\int_{-1}^{1}\frac{(1-z^2)^{1/2}}{(\beta-z)^4}\,dz = \int_0^\pi \frac{\sin^2\theta}{(\beta-\cos\theta)^4}\,d\theta \tag{B.8}$$

We have the standard integral (Ref[35] No. 331, 89b)

$$\int_0^\pi \frac{\sin^2\theta\,d\theta}{(\beta+b\cos\theta)^3} = \frac{\pi}{2}\cdot\frac{1}{(\beta^2-b^2)^{3/2}} \tag{B.9}$$

provided $\beta^2 > b^2$, which can be verified, for all v , from (B.6).
Differentiating (B.9) with respect to β , with b =-1, gives

$$\int_0^\pi \frac{\sin^2\theta\,d\theta}{(\beta-\cos\theta)^4} = \frac{\pi}{2}\cdot\frac{\beta}{(\beta^2-1)^{5/2}} . \tag{B.10}$$

Substituting (B.10) into (B.7) yields

$$I_1 = \frac{\pi}{2^5}\int_0^\infty \frac{[(1+a^4)+v^2]\,v^3\,dv}{[v^4+2(1-a^4)v^2+(1+a^4)^2]^{5/2}} \tag{B.11}$$

Let $x = v^2$, and this splits up into

$$I_1 = \frac{\pi}{2^6}\left\{\int_0^\infty \frac{x^2\,dx}{[x^2+2(1-a^4)x+(1+a^4)^2]^{5/2}} + (1+a^4)\int_0^\infty \frac{x\,dx}{[x^2+2(1-a^4)x+(1+a^4)^2]^{5/2}}\right\} \tag{B.12}$$

which again is a standard form (ref. [35] No. 213 5a and 5b) giving, finally

$$I_1 = \frac{\pi}{3.2^7} \tag{B.13}$$

Note that I_1 contains no dependence on $\tilde{a}$.

For I_2 (2.70) make the same substitutions, (B.1) leads to

$$I_2 = \frac{1}{32}\int_0^\infty dv \int_{-v}^{v} dw \, \frac{(v-w)^{5/2}(v+w)^{-1/2}}{[1+v^2+a^4-2wa^2]^4} \tag{B.14}$$

and (B.5) gives

$$I_2 = \frac{1}{32}\cdot\frac{1}{(2a^2)^4}\int_0^\infty \frac{dv}{v}\int_{-1}^{1} dz \, \frac{(1-z)^{5/2}(1+z)^{-1/2}}{[\beta(v)-z]^4} \tag{B.15}$$

The z integration can be performed (ref.[35] No. 131 4), yielding

$$\int_{-1}^{1} dz \, \frac{(1-z)^{5/2}(1+z)^{-1/2}}{(\beta-z)^4} = \frac{2^3 \, B(\frac{1}{2},\frac{7}{2})}{(\beta-1)^{1/2}(\beta+1)^{7/2}} \tag{B.16}$$

provided $\beta(v)^2 - 1 > 0$ which is again the case. Thus, I_2 has been forced into the form,

$$I_2 = \frac{5\pi}{2^6}\int_0^\infty \frac{v^3 \, dv}{[(v-a^2)^2+1]^{1/2}[(v+a^2)^2+1]^{7/2}} \tag{B.17}$$

An expression for this integral can be obtained in terms of derivatives of elliptic functions, but for the considerations of chapter two, a numerical evaluation proves to be more illuminating. I_2 is found to be monotonically decreasing with a, see graph between pages 21 and 22, Chapter 2.

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