# *Chebyshev series approximation on complex domains*

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# CHEBYSHEV SERIES APPROXIMATION ON COMPLEX DOMAINS

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## ABSTRACT

This thesis is an account of work carried out at the Department of Mathematics, Durham University, between October 1979 and September 1982. A method of approximating functions in regions of the complex plane is given. Although it is not, in general, a near minimax approximation it is shown that it can give good results. A review of approximation in the complex plane is given in Chapter 1. Chapter 2 contains the basic properties of Chebyshev polynomials and the Chebyshev series, together with methods for calculating the coefficients in the series. The maximum error, over a complex domain, of a truncated Chebyshev series is investigated in Chapter 3 and Chapter 4 shows how the Bessel functions of the first and second kinds of integer order could be approximated over the entire complex plane. Numerical calculations were performed on the NUMAC IBM370.168 computer.

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# CHAPTER 1

## INTRODUCTION TO COMPLEX APPROXIMATION

The theory of best minimax polynomial approximations is well developed for real valued functions, see Davis (1963), but less well developed for complex valued functions. Algorithms, such as the Remez algorithm for approximation on closed intervals and the iterative scheme described by Ellacott and Williams (1976) for closed regions of the complex plane, do exist to calculate the required polynomials. Opfer (1978) introduced an algorithm based on Kolmogorov's Criterion and Barrodale (1977) has investigated the solution of the best polynomial approximation problem for a number of different norms and suggested that the $\ell_\infty$ norm,

$$||g(z)||_\infty = \max_{1 \leq t \leq M} |g(z_t)|,$$

be replaced by the star norm $\ell_*$, defined by

$$||g(z)||_* = \max_{1 \leq t \leq M} (\max(|Re(g(z_t))|, |Im(g(z_t))|)),$$

thus changing the non-linear problem of finding best $\ell_\infty$ approximations to the linear one of finding best $\ell_*$ polynomials. Glashoff and Roleff (1981) reformulate the Chebyshev approximation problem in the complex plane as a problem of linear optimisation in the presence of infinitely many constraints, and show that algorithms exist for its solution.

However, these algorithms are all relatively complex and expensive in computation time, and require the recomputation of all the coefficients when the degree of the approximation is changed. So in many circumstances other approximations, which are easier to produce and nearly as accurate, are used in preference. An approximation, which has a maximum error less than ten times the maximum error of the minimax approximation, over the same region and of the same degree, is called a near-best or near-minimax approximation.

Near-best polynomial approximations have been found for a restricted set of regions on the complex plane. Geddes and Mason (1975) showed that the truncated Taylor series and the Fourier projection, interpolation at the (n+1)th roots of unity, are near-minimax approximations on the unit disc. The truncated Chebyshev series was shown to be a near-best approximation for a family of ellipses by Geddes (1978), and Mason (1981) showed that an interpolating projection existed that was near-minimax on a circular annulus.

For more arbitrary regions of the complex plane, no such results exist. On closed intervals of the real line, Chebyshev series approximation has been extensively used, Luke (1975, 1977), Clenshaw (1962), Coleman (1980) and the special function section of the NAG subroutine library, because by truncating a Chebyshev series for a function on the interval [-1, 1] we obtain, in many cases, a close approximation to the minimax polynomial of the same degree. The error in truncating a rapidly convergent Chebyshev series $\sum_{K=0}^{\infty} a_K T_K(X)$ after (n+1) terms is dominated by the first neglected term; this, being proportional to $T_{n+1}(X)$, has the equioscillation property characteristic of the error in the minimax polynomial approximation of degree n.

For a domain D of the complex plane we can ask what monic polynomial of degree n will minimise

$$\max_{Z \in D} | Z^n + a_{n-1} z^{n-1} + \ldots\ldots + a_1 z + a_0 |$$

The polynomial $\mathcal{J}_n^D(Z)$ which satisfies this condition is called the Chebyshev polynomial of degree n for the domain D. The success of Chebyshev expansions of the real axis suggests that by truncating an expansion of the form

$$f(z) = \sum_{K=0}^{\infty} b_K \mathcal{J}_K^D(z) \tag{1.1}$$

we could hope to obtain a near-best polynomial approximation for a function f in the domain D. When D is the real interval [-1,1], $\mathcal{T}_n^{\mathcal{D}}$ is $2^{1-n}$ times the classical Chebyshev polynomial $T_n$, whereas if D is the unit disc $\mathcal{T}_n^{\mathcal{D}}(z) = z^n$, in which case the right hand side of (1.1) is simply the Taylor series for f. For other domains the properties of the polynomials $\mathcal{T}_n^{\mathcal{D}}(z)$ are not in general known. Explicit expressions for a few polynomials $\mathcal{T}_n^{\mathcal{D}}(z)$ of low degree have been found,Geiger and Opfer (1977),and algorithms exist Elliot (1978) and Opfer (1978) which will produce numerical values of the coefficients for a given domain D and degree n, but without a better understanding of the properties of the polynomials $\mathcal{T}_n^{\mathcal{D}}(z)$ it is not feasible to obtain the coefficients of the expansion (1.1).

Therefore another expansion similar to (1.1), whose coefficients can be more rapidly calculated, would be preferable even if it did not converge as fast as the series in (1.1).

One possibility would be a series in terms of the Faber polynomials related to the compact set of the complex plane on which the function being expanded would be evaluated. Curtiss (1971) gives a good introduction to Faber polynomials and the Faber series, including a study of the convergence of the Faber series. Ellacott (1981a) gives algorithms for calculating the Faber polynomials and the Faber coefficients and for evaluating the Faber series. Also, Ellacott (1981b) has given algorithms for producing rational approximations using the Faber transform.

Expanding in a series of Faber polynomials would seem preferable to finding the Chebyshev polynomials for a particular region, since they are all related to the function $\phi(w)$ which maps the complement of the region conformally onto the exterior of the unit circle. If however, a known set of orthogonal polynomials was chosen, it would reduce the amount of computation time required and the coefficients could be calculated in the same manner as the expansion of functions of a real variable. The resulting series would probably converge slower than the Faber series of the region as it would not be "tailored" to that specific region. The ease of producing such a series could be the deciding factor, particularly if the series converges quickly enough so that the extra accuracy could be achieved by truncating the series after slightly more coefficients. For these reasons the Chebyshev series has been chosen in this thesis for investigation as an approximation over complex regions.

CHAPTER 2

# CHEBYSHEV POLYNOMIALS AND CHEBYSHEV SERIES

## §1 BASIC PROPERTIES OF CHEBYSHEV POLYNOMIALS

In this section we shall give a brief summary of the properties of Chebyshev polynomials of the first kind. For further details the reader is referred to Rivlin (1974).

The Chebyshev polynomial of the first kind is defined by

$$T_N(z) = \text{Cos}\,(N\Theta) \qquad (2.1)$$

where

$$Z = \text{Cos}\,\Theta.$$

Normally $\Theta$ is taken to be real so that $z \in [-1,1]$, but in the following chapters we shall allow z to take any value in bounded regions of the complex plane. The variable x will be used when the argument is restricted to [-1,1]. A closely related set of polynomials, the shifted Chebyshev polynomials of the first kind, are defined as follows

$$T_N^*(z) = T_N(2z-1) \qquad (2.2)$$

The properties of these polynomials can easily be derived from those we shall give for $T_N(z)$.

The following recurrence relationship can be obtained from the connection with trignometric functions:

$$T_{N+1}(z) = 2zT_N(z) - T_{N-1}(z), \; N \geqslant 1 \qquad (2.3)$$

with $\quad T_0(z) = 1, \quad T_1(z) = z$

It is clear from (2.3) that $T_N(z)$ is a polynomial of degree N, with real coefficients and for $N \geqslant 1$ the leading coefficient is $2^{N-1}$. Moreover, if N is even then $T_N(z)$ is an even function and if N is odd then $T_N(z)$

is an odd function. An alternative expression for $T_N(z)$, which is more useful in some circumstances than (2.1), is obtained by solving the second order recurrence equations given in (2.3). They yield

$$T_N(z) = \tfrac{1}{2}\left\{(z + \sqrt{z^2 - 1})^N + (z - \sqrt{z^2-1})^N\right\}, \qquad (2.4)$$

where the branch of $\sqrt{z^2-1}$ to be chosen is determined by the conditions

$$|z + \sqrt{z^2-1}| > 1 \quad \text{for } z \notin [-1,1]$$

and

$$\text{Arg}\,(\sqrt{z^2-1}) = \pi/2 \quad \text{for } z \in [-1,1]$$

As $\text{Cos}\, N\theta$ has N zeros in $[0,\pi)$ given by

$$\theta_k = \frac{(2k-1)\pi}{2N}, \; k = 1,2, \ldots\ldots N \qquad (2.5)$$

all the zeros $x_k^{(n)}$ of $T_N(z)$ lie in the interval [-1,1] and they are symmetrically distributed about the origin.

This property will now be used to obtain a bound on the absolute value of $T_N(z)$ on a sector D of the unit circle in which

$$|\text{Arg}\,\zeta| \leqslant \Theta \leqslant \pi/2$$

and we shall show that the bound is obtained. Since the Chebyshev polynomials have real coefficients it is obvious that

$$|T_N(z)| = |T_N(\bar{z})|,$$

where $\bar{z}$ is the complex conjugate of z. Therefore it is only necessary to consider the sector $0 \leqslant \arg\zeta \leqslant \Theta$ . Furthermore the Maximum Modulus Theorem shows that the maximum occurs on the boundary, i.e. on the radius $\zeta = r\text{Exp}(i\Theta)$, $0 \leqslant r \leqslant 1$, or on the arc $\zeta = \text{Exp}(i\theta)$, $0 \leqslant \theta \leqslant \Theta$ .

The zeros of the even polynomial $T_{2m}(\zeta)$ may be labelled as $\pm x_k$ (k = 1,2,....m). So we can write

$$T_{2m}(\zeta) = 2^{2m-1}\zeta\prod_{K=1}^{m}(\zeta^2 - \chi_K^2)$$

and therefore on the unit circle

$$|T_{2m}(e^{i\theta})|^2 = 2^{4m-2}\prod_{K=1}^{m}(1 + \chi_K^4 - 2\chi_K^2 \cos 2\theta).$$

Clearly, for $0<\theta<\pi/2$, this is a monotonically increasing function of $\theta$. The maximum value on the arc therefore occurs at $\zeta = \text{Exp}(i\Theta)$ and as $T_{2m}(1) = 1$ we have

$$|T_{2m}(e^{i\theta})| \geq 1 \quad \text{for } 0\leq\theta\leq\pi/2 .$$

On the ray $\zeta = re^{i\Theta}$,

$$|T_{2m}(re^{i\Theta})| = 2^{4m-2}\prod_{K=1}^{m}(r^4 + \chi_K^4 - 2r^2\chi_K^2 \cos 2\Theta).$$

If $\pi/4 \leq \Theta \leq \pi/2$ each term in the product is positive and monotonically increasing with r. On the other hand, if $0 \leq \Theta \leq \pi/4$ the $k^{th}$ term in the product decreases, from the value $\chi_k^4$ when r = o, to a positive minimum at $r = \chi_k(\cos 2\Theta)^{\frac{1}{2}}$ and thereafter increases with r. It follows that the product takes its maximum value at r = o or r = 1.

Since

$$|T_{2m}(o)| = 1 \leq |T_{2m}(e^{i\Theta})|,$$

we have, for the sector D, the bound

$$|T_{2m}(\zeta)| \leq |T_{2m}(e^{i\Theta})|.$$

The Chebyshev polynomial $T_{2m+1}(\zeta)$ has a zero at the origin and may be written as

$$T_{2m+1}(\zeta) = \zeta 2^{2m}\prod_{K=1}^{m}(\zeta^2 - y_K^2)$$

where $\pm y_K$ (K = 1,2...m) are the other zeros. Arguments analogous to those used above show that once again the Maximum modulus on the sector D is attained at the point $\zeta = \text{Exp}(i\Theta)$.

We have therefore shown that on the sector $D = \{\zeta; |\zeta| \leq 1, |\text{Arg}\,\zeta| \leq \Theta\}$, where $0 \leq \Theta \leq \pi/2$,

$$|T_N(\xi)| \leq |T_N(e^{i\Theta})| \qquad (2.6)$$

The Chebyshev polynomial of degree N may be written as

$$T_N(\xi) = \tfrac{1}{2}(W^N + W^{-N}),$$

where

$$W(\xi) = \xi + (\xi^2 - 1)^{\frac{1}{2}}$$

Therefore, on the sector D,

$$|T_N(\xi)| \leq \tfrac{1}{2}(\rho^N + \rho^{-N}), \qquad (2.7)$$

where

$$\rho = |e^{i\Theta} + (e^{2i\Theta} - 1)^{1/2}|$$

It is also useful to note that since $\rho \geq 1$, for $0 \leq \Theta \leq \pi/2$,

$$|T_{N+k}(\xi)| \leq \tfrac{1}{2}(\rho^N + \rho^{-N})\rho^k \qquad (2.8)$$

in the sector D.

The relationship with trignometric functions is used to prove two of the most important properties of Chebyshev polynomials, namely their orthogonality over [-1,1] with respect to the weight function $(1-x^2)^{-\frac{1}{2}}$, and the discrete orthogonality. Specifically,

$$\int_{-1}^{1} T_j(x)\, T_k(x)(1-x^2)^{-\frac{1}{2}}\, dx = \begin{cases} \pi & j = k = 0 \\ 0 & j \neq k \\ \pi/2 & j = k \neq 0 \end{cases} \qquad (2.9)$$

and if $\Theta_K = K\frac{\pi}{N}$, $K = 0,1,\ldots,N$

and $y_K = \cos\Theta_K$, $N > 0$, $S \leq N$ and $K = 0,1,\ldots,N$

$$\sum_{K=0}^{N}{}'' T_r(y_K)\, T_s(y_K) = \begin{cases} N, & r = 2pN+S,\ S = 0 \text{ or } N \\ 0, & r \neq 2pN \pm S \\ N/2 & r = 2pN \pm S,\ S \neq 0 \text{ or } N \end{cases} \qquad (2.10)$$

for $p = 0, 1, \ldots\ldots$

where the double prime on the summation indicates that the first and last terms are halved.

Equation (2.9) is proved by the change of variable $x = \cos\theta$, and it then becomes a standard trignometic identity. The discrete orthogonality can be proved using the following:

(1) $T_r(x)T_s(x) = \frac{1}{2}\left\{ T_{r+s}(x) + T_{|r-s|}(x) \right\}$

(2) $T_r(y_k) = \cos(r\, k\pi/N)$

and

(3) $$\sum_{k=0}^{N}{}'' \cos(j k\pi/N) = \begin{cases} N & j = 2Np \\ 0 & j \neq 2Np \end{cases} \quad p = 0, 1, \ldots \qquad (2.11)$$

## §2 CHEBYSHEV SERIES

We define the Chebyshev coefficients of a function, f(z), by

$$a_j = \frac{2}{\pi}\int_{-1}^{1} f(x)T_j(x)\,(1-x^2)^{-\frac{1}{2}}\,dx \qquad (2.12)$$

$$j = 0, 1 \ldots$$

We call $\sum_{j=0}^{\infty}{}' a_j T_j(z)$ the Chebyshev series for f(z), the prime on summation indicates that the first term is to be halved. The series may or may not converge.

Definition

The Chebyshev ellipse $\mathcal{E}_\rho$, $\rho > 1$ is defined by

$$\mathcal{E}_\rho = \left\{ \omega \mid \omega = \tfrac{1}{2}(z+z^{-1}),\ |z| = \rho \right\} \qquad (2.13)$$

For values of the argument lying on a Chebyshev ellipse we have the following lemma.

LEMMA 1

Let $z \in \mathcal{E}_\rho$, $\rho > 1$, then $\exists\ W \in \{ W \mid |W| = \rho \}$

such that

$$T_N(z) = \tfrac{1}{2}(W^N + W^{-N})$$

Proof.

From the definition of $\mathcal{E}_\rho$ $\exists$ $W \in \{W \mid |W| = \rho\}$

such that

$$z = \tfrac{1}{2}(W + W^{-1})$$

substituting for z in (2.[illegible]) gives

$$T_N(z) = \tfrac{1}{2}\Big[\Big(\frac{W+W^{-1}}{2} + \sqrt{\frac{W^2 - 2 + W^{-2}}{4}}\Big)^N + \Big(\frac{W+W^{-1}}{2} + \sqrt{\frac{W^2 - 2 + W^{-2}}{4}}\Big)^{-N}\Big]$$

$$= \tfrac{1}{2}\Big[\Big(\frac{W+W^{-1}}{2} + \frac{(W-W^{-1})}{2}\Big)^N + \Big(\frac{W+W^{-1}}{2} + \frac{(W-W^{-1})}{2}\Big)^{-N}\Big]$$

$$= \tfrac{1}{2}(W^N + W^{-N}) \qquad (2.14)$$

We shall now prove a theorem on the convergence of the Chebyshev series of a function.

Theorem 1

Let f(z) be analytic in the interior of $\mathcal{E}_\rho$, $\rho > 1$, ~~and continuous on the boundary,~~ but not in the interior of any $\mathcal{E}_{\rho'}$ with $\rho' > \rho$, and continuous on $\mathcal{E}_\rho$

Then

$$f(z) = {\sum_{j=0}^{\infty}}' a_j T_j(z),$$

where $a_j$ is defined by (2.12).

The series converges absolutely and uniformly on any closed set in the interior of $\mathcal{E}_\rho$ and diverges exterior to $\mathcal{E}_\rho$.

Moreover

$$\operatorname*{Lim\,Sup}_{j\to\infty} |a_j|^{1/j} = \frac{1}{\rho}$$

Proof

For u in the interior of $\mathcal{E}_\rho$

$$f(u) = \frac{1}{2\pi i}\int_{\mathcal{E}_\rho} \frac{f(z)}{z-u}\,dz \qquad (2.15)$$

But since

$$(z-u)T_n(u) = z\,T_n(u) - \tfrac{1}{2}(T_{|n-1|}(u) + T_{n+1}(u)),$$

we have

$$\frac{1}{z-u} = \sum_{j=0}^{\infty}{}' \frac{2\,T_j(u)}{(z+\sqrt{z^2-1})^j\sqrt{z^2-1}} \tag{2.16}$$

and the series converges absolutely at any $u$ in the interior of $\mathcal{E}_\rho$.

So substituting (2.16) in (2.15) gives

$$f(u) = \frac{1}{2\pi i}\int_{\mathcal{E}_\rho} \sum_{j=0}^{\infty}{}' \frac{2\,T_j(u)\,f(z)}{(z+\sqrt{z^2-1})^j\sqrt{z^2-1}}\,dz. \tag{2.17}$$

Interchanging integration and summation yields

$$f(u) = \sum_{j=0}^{\infty}{}' a_j\,T_j(u) \tag{2.18}$$

where
$$a_j = \frac{1}{\pi i}\int_{\mathcal{E}_\rho} \frac{f(z)}{(z+\sqrt{z^2-1})^j\sqrt{z^2-1}}\,dz \tag{2.19}$$

Now replace $u$ in (2.16) with the real variable $x \in [-1, 1]$.

That gives

$$\frac{1}{z-x} = \sum_{j=0}^{\infty}{}' \frac{2\,T_j(x)}{(z+\sqrt{z^2-1})^j\sqrt{z^2-1}}$$

and from the orthogonality of the Chebyshev polynomials we get

$$\frac{2}{\pi}\int_{-1}^{1} \frac{T_j(x)}{(z-x)(1-x^2)^{1/2}}\,dx = \frac{2}{\sqrt{z^2-1}\,(z+\sqrt{z^2-1})^j}.$$

So

$$\begin{aligned} a_j &= \frac{1}{2\pi i}\int_{\mathcal{E}_\rho} f(z)\,\frac{2}{\pi}\int_{-1}^{1} \frac{T_j(x)}{(1-x^2)^{1/2}}\,\frac{1}{(z-x)}\,dx\,dz \\ &= \frac{1}{\pi^2 i}\int_{-1}^{1} \frac{T_j(x)}{(1-x^2)^{1/2}}\int_{\mathcal{E}_\rho} \frac{f(z)}{(z-x)}\,dz\,dx \\ &= \frac{2}{\pi}\int_{-1}^{1} \frac{T_j(x)\,f(x)}{(1-x^2)^{1/2}}\,dx \end{aligned}$$

To prove the absolute and uniform convergence on closed sets in the interior of $\mathcal{E}_\rho$ we chose a $\rho'$ such that $1<\rho'<\rho$. Then $f(z)$ is analytic in and on $\mathcal{E}_{\rho'}$ and we may write

$$a_j = \frac{1}{\pi^2 i}\int_{\mathcal{E}_{\rho'}} f(\omega)\,.\int_{-1}^{1} \frac{T_j(x)}{(\omega-x)(1-x^2)^{1/2}}\,dx\,d\omega.$$

If we let

$$Q_j(w) = \int_{-1}^{1} \frac{T_j(x)}{(w-x)(1-x^2)^{1/2}}\, dx \tag{2.20}$$

Then

$$|a_j| \leq \frac{1}{\pi^2} \mathcal{L}(\mathcal{E}_{\rho'}) \max_{w\in\mathcal{E}_{\rho'}} |f(w)| \max_{w\in\mathcal{E}_{\rho'}} |Q_j(w)|, \tag{2.21}$$

where $\mathcal{L}(\mathcal{E}_{\rho'})$ is the length of $\mathcal{E}_{\rho'}$.

To bound $|Q_j(w)|$ we make the change of variable $w = \frac{1}{2}(z+z^{-1})$, where $z = \rho' e^{i\psi}$. This yields

$$\begin{aligned} Q_j(w) &= 2\int_{-1}^{1} \frac{T_j(x)}{(1-x^2)^{1/2}(z+z^{-1}-2x)}\, dx \\ &= \frac{2}{z}\int_{-1}^{1} \frac{T_j(x)}{(1-x^2)^{1/2}\left\{1-2\frac{x}{z}+\frac{1}{z^2}\right\}}\, dx \end{aligned} \tag{2.22}$$

Putting $x = \cos\theta$ in (2.22) gives

$$Q_j(w) = \frac{2}{z}\int_0^{\pi} \frac{\cos j\theta}{\left(1-\frac{2\cos\theta}{z}+\frac{1}{z^2}\right)}\, d\theta .$$

Now

$$\frac{\sin\theta}{z}\left\{\frac{1}{1-\frac{2\cos\theta}{z}+\frac{1}{z^2}}\right\} = \sum_{M=1}^{\infty} \frac{\sin(M\theta)}{z^M} .$$

The last series converges uniformly and absolutely for $0 \leq \theta \leq \pi$ and for all $|z| \geq \rho'' > 1$

Hence $$Q_j(w) = \sum_{M=1}^{\infty} \frac{2}{z^M} \int_0^{\infty} \frac{\cos(j\theta)\sin(M\theta)}{\sin\theta}\, d\theta,$$

but from Gradshteyn and Ryzhik (1965) page 366 we have

$$\int_0^{\pi} \frac{\cos(n\theta)\sin(M\theta)}{\sin\theta}\, d\theta = \begin{cases} 0 & M \leq n \\ \pi & M > n \text{ if } M+n \text{ is odd} \\ 0 & M > n \text{ if } M+n \text{ is even} \end{cases}$$

So

$$\begin{aligned} Q_j(w) &= \frac{2\pi}{z^{j+1}} \sum_{k=0}^{\infty} z^{-2k} \\ &= \frac{2\pi}{z^{j+1}(1-z^{-2})} \end{aligned}$$

Therefore $$\max_{w\in\mathcal{E}_{\rho'}} |Q_j(w)| = \frac{2\pi(\rho')^{-j}}{(\rho' - \rho'^{-1})} \tag{2.23}$$

Combining (2.21) with (2.23) we get

$$|a_j| \leq K(\rho')^{-j} \qquad (2.24)$$

where K is independent of j.

Therefore

$$\lim_{j\to\infty} \sup |a_j|^{1/j} \leq \frac{1}{\rho'} .$$

Since $\rho' < \rho$ is arbitrary, it follows that

$$\lim_{j\to\infty} \sup |a_j|^{1/j} \leq \frac{1}{\rho} .$$

However from (2.14) we have for $w \in \mathcal{E}_{\rho'}$

$$\lim_{j\to\infty} \sup |T_j(w)|^{1/j} = \rho' < \rho$$

So the series converges uniformly and absolutely inside every $\mathcal{E}_{\rho'}$, $\rho' < \rho$.

Suppose now that

$$\lim_{j\to\infty} \sup |a_j|^{1/j} < \frac{1}{\rho}$$

Then the series would converge uniformly and absolutely in the interior of a larger ellipse $\mathcal{E}_{\rho''}$, $\rho'' > \rho$, and provide an analytic continuation of f(z) there. This is a contradiction, because by hypothesis $\mathcal{E}_\rho$ is the largest ellipse in which f(z) is analytic. So

$$\lim_{j\to\infty} \sup |a_j|^{1/j} = \rho^{-1}$$ and the series diverges

in the exterior of $\mathcal{E}_\rho$.

## §3 NUMERICAL CALCULATION OF CHEBYSHEV COEFFICIENTS

The calculation of the coefficients in (2.12) can be approached in a number of ways. We will give here three methods, the first uses the discrete orthogonality property of Chebyshev polynomials and the other two use differential equations satisfied by the function being expanded.

### §3.1 COLLOCATION METHOD

If we define the coefficients $a_j^{(N)}$, $j = 0, 1, \ldots, N$ such that the polynomial $\sum_{j=0}^{N}{}'' a_j^{(N)} T_j(z)$ collocates f(z) at the (N+1) points given by

$$y_k = \cos k\pi/N \quad , \quad k = 0, 1, \ldots\ldots N \quad ,$$

we can write

$$f(y_k) - \sum_{j=0}^{N}{}'' a_j^{(N)} T_j(y_k) = 0 \qquad \text{For } k = 0, 1, \ldots\ldots, N \ .$$

So

$$\sum_{k=0}^{N}{}'' f(y_k) T_r(y_k)$$
$$= \sum_{k=0}^{N}{}'' T_r(y_k) \sum_{j=0}^{N}{}'' a_j^{(N)} T_j(y_k)$$
$$= \sum_{j=0}^{N}{}'' a_j^{(N)} \sum_{k=0}^{N}{}'' T_r(y_k) T_j(y_k)$$
$$= \left(\frac{2}{N}\right)^{-1} a_r^{(N)} \quad , \qquad \text{for } r = 0, 1, \ldots\ldots, N$$

because of the orthogonality property (2.10).

We shall now show the relationship between the coefficients $a_r^{(N)}$ and the Chebyshev coefficients $a_r$. If the function f(x) has a convergent Chebyshev series on [-1,1] then we can write

$$a_r^{(N)} = \frac{2}{N} \sum_{k=0}^{N}{}'' \sum_{j=0}^{\infty}{}' a_j T_j(y_k) T_r(y_k) \tag{2.25}$$
$$= \frac{2}{N} \sum_{j=0}^{\infty}{}' a_j \sum_{k=0}^{N}{}'' T_j(y_k) T_r(y_k)$$
$$= a_r + \sum_{p=1}^{\infty} \left( a_{2pN-r} + a_{2pN+r} \right)_{r=0,1,\ldots\ldots,N} .$$

This shows that if the coefficients converge rapidly to zero ~~that~~ $a_r^{(N)}$ is a good approximation to $a_r$ for $r < N$ and $a_N^{(N)}$ is a good approximation to $2a_N$.

The error incurred when using these 'approximate' coefficients is

$$f(z) - \sum_{j=0}^{N}{}'' a_j^{(N)} T_j(z) = \sum_{j=0}^{N}{}' a_j T_j(z) - \sum_{j=0}^{N}{}'' a_j^{(N)} T_j(z) + \sum_{j=N+1}^{\infty} a_j T_j(z)$$
$$= \sum_{j=0}^{N-1} \sum_{p=1}^{\infty} \left[ a_{2pN-j} + a_{2pN+j} \right] T_j(z)$$
$$+ \sum_{p=1}^{\infty} a_{(2p+1)N} T_N(z) + \sum_{k=N+1}^{\infty} a_k T_k(z)$$
$$= \sum_{k=N+1}^{\infty} a_k \left[ T_k(z) - T_{|N-(k+N) \bmod 2N|}(z) \right] \tag{2.26}$$

If the approximation is to be used in a region such that

$$|T_j(z)| \leqslant \sigma^j \quad , \quad \sigma \geqslant 1$$

then given that f(z) has a convergent Chebyshev series we can write the error, caused by truncating the Chebyshev series after $(N+1)$ terms, is bounded by

$$\sum_{k=N+1}^{\infty} |a_k| \sigma^k \quad ,$$

and the error in approximating f(z) by the collocation polynomial of same degree is bounded by $\sum_{k=N+1}^{\infty}{}' |a_k| [\sigma^k + \sigma^N]$ .

So the collocation polynomial has a larger error bound, but it is less than twice the bound for the truncated Chebyshev series.

## §3.2 LANCZOS' $\tau$ -METHOD

If a function satisfies a differential equation of order q, which has only polynomial coefficients, then substituting a polynomial of degree n for the function will leave a residual polynomial $R_N(z)$ of degree N say. If the boundary conditions are met, only (n+1 -q) coefficients in the approximating polynomial are independent. So in constructing $R_N(z)$ we may impose (n+1-q) conditions upon its coefficients to give a unique approximation of degree n.

The conditions chosen by Lanczos (1938) are that the residual polynomial be a linear combination of $m$ Chebyshev polynomials, where $m = N+q-n$. For example if $m = 1$ then the polynomial would be $\tau T_N(z)$, while if $m = 2$ it would be $\tau_1 T_N(z) + \tau_2 T_{N-1}(z)$. The values of the coefficients $\tau$ are automatically determined by solving the resultant differential equation. When $\tau$ is small we may expect the polynomial to be a good approximation to the function.

## §3.3 CLENSHAW'S METHOD

This method was introduced by Clenshaw (1957) for functions satisfying a linear differential equation with polynomial coefficients. By substituting separate Chebyshev series for the function and each of its derivatives into the differential equation and then equating coefficients of $T_j(z)$, it is possible to produce recurrence relations for the coefficients.

To equate coefficients we must be able to write the coefficients of $Z^p g(z)$, $p=0,1\cdots\cdots$, in terms of the coefficients of $g(z)$. To solve the subsequent relations we need to be able to relate the coefficients of $g'(z)$ to those for $g(z)$. If we define $C_R(g(z))$ to be the $R^{th}$ Chebyshev coefficient of $g(z)$ for $R>0$, and twice this for $R=0$, then the required relations are:

$$C_R(z^p g(z)) = \frac{1}{2^p}\sum_{s=0}^{p}\binom{p}{s} C_{|R-p+2s|}(g(z)) \tag{2.27}$$

and

$$2RC_R(g'(z)) = \{C_{|R-1|}(g(z)) - C_{R+1}(g(z))\} \tag{2.28}$$

The differential equation yields an infinite system of equations in the unknown coefficients and a number of initial or boundary conditions, which can be solved directly using (2.28). However, it is possible to use (2.28) repeatedly to eliminate the coefficients of the derivatives and then solve a system of equations in the coefficients of the function only. For example suppose f(z) satisfies the O.D.E.

$$zf''(z) - f'(z) + z^2 f(z) = 0, \tag{2.29}$$

with the conditions

$$f(0) = 1, \; f(1) = 0,$$

then if we let

$$C_R(f''(z)) = a''_R$$
$$C_R(f'(z)) = a'_R$$
$$C_R(f(z)) = a_R$$

and equate coefficients of $T_R(z)$ in (2.29) we get

$$\frac{1}{2}\left[a''_{|R-1|} + a''_{R+1}\right] - a'_R + \frac{1}{4}\left[a_{|R-2|} + 2a_R + a_{R+2}\right] = 0 \quad (2.30)$$

$R = 0, 1, \ldots\ldots$

Subtracting the $(R + 2)^{th}$ equation from $R^{th}$ and using (2.28) to eliminate the coefficients of $f''(z)$ gives

$$Ra'_R + (R+2)a'_{R+2} - 2(R+1)a_{R+1} + \frac{1}{4}\left[a_{|R-2|} + a_R - a_{R+2} - a_{R+4}\right] = 0 \quad (2.31)$$

$R = 0, 1, \ldots\ldots$

which can be rearranged to give

$$a'_R + a'_{R+2} - 4a_{R+1} + \frac{1}{4(R+1)}\left[a_{|R-2|} + a_R - a_{R+2} - a_{R+4}\right] = 0 \quad (2.32)$$

$R = 0, 1, \ldots\ldots$

The coefficients from the expansion of first derivative can similarly be eliminated to yield the recurrence relations

$$2(R-1)a_{R+1} + 2(R+5)a_{R+3} + \frac{1}{4}\left\{\frac{2(a_R - a_{R+4})}{(R+1)(R+3)} + \frac{a_{|R-2|}}{(R+1)} + \frac{a_{R+6}}{(R+3)} - \frac{2(R+2)a_{R+2}}{(R+1)(R+3)}\right\} = 0 \quad (2.33)$$

$R = 0, 1, \ldots$

The boundary conditions yield the equations

$$\left.\begin{aligned} \sum_{j=0}^{\infty}{}' (-1)^j a_{2j} &= 1 = f(0) \\ \sum_{j=0}^{\infty}{}' a_j &= 0 = f(1) \end{aligned}\right\} \quad (2.34)$$

To solve systems of equations similar to (2.33) with conditions similar to (2.34) we find solutions from (2.33) using backward recurrence from the equation having $a_{N-1}$ as its lowest order coefficient, with $a_N$ and higher order coefficients given arbitrary values, for various values of $N$. A linear combination of a sufficient number of these solutions can be taken to satisfy the equations (2.34) and any equations of (2.33) not used in the backward recurrence. For instance in the above example we could use backward recurrence from $R = N+1$ to $R = 2$ to find an initial independent solution of the recurrence formulae and this would leave equations R = 0 and 1 to be satisfied as well as the two equations in (2.34). Therefore we would need a linear combination of four independent solutions. However in many cases we shall have only one or two outstanding conditions to satisfy.

## §3.4 CHOICE OF METHOD

The methods require quite different information about a function to be known. The first requires the evaluation of the function at $(N+1)$ points. The accuracy of the coefficients depends on the accuracy of these function evaluations and therefore, this method is useful only if the evaluation method used is inferior to the Chebyshev series either in the speed of calculation or in the region over which it can be used. The other methods require the knowledge of a differential equation satisfied by the function and a small number of values of the function or its derivatives. If these values are known exactly, then the accuracy of the coefficients calculated using Clenshaw's method is restricted only by round-off errors if $N$ is very large. The coefficients given in later chapters were calculated using Clenshaw's method. Clenshaw's and Lanczos' methods are further compared by Fox (1962).

## CHAPTER 3

## COMPARISON OF BEST POLYNOMIAL APPROXIMATION WITH TRUNCATED CHEBYSHEV SERIES OF SAME DEGREE

### §1 THE RELATIVE FACTOR

Chebyshev series approximation has been used extensively for continuous functions of a real variable. Schonfelder (1978, 1980), Schonfelder et al (1980), Razaz & Schonfelder (1980) and Shepherd & Laframboise (1981) are just a few examples. The error in this type of approximation, in comparison to the error in the best polynomial approximation of the same degree, was investigated by Powell (1967). He showed that on the interval [-1,1] the maximum norm of the error caused by using a truncated Chebyshev series of degree N was within a relative factor $\nu_N$ , the N'th Lebesque constant, of the norm of the minimax error. This can be written as

$$\| e_N(z) \|_I \leqslant (1 + \nu_N) \| E_N^I(z) \|_I \qquad (3.1)$$

where

$E_N^I(z)$ is the error at $z$ of the best polynomial approximation on [-1,1] of order N to the function $f(z)$, and $e_N(z)$ is the error at $z$ of the truncated Chebyshev series of $f(z)$. The norm is defined by

$$\| f(z) \|_I = \max_{z \in [-1,1]} | f(z) |$$ and the Lebesque constant by

$$\nu_N = \frac{1}{(2N+1)} + \frac{2}{\pi} \sum_{t=1}^{N} \frac{1}{t} \operatorname{Tan}\left( \frac{\pi t}{2N+1} \right)$$

This result was extended by Geddes (1978) to Chebyshev ellipses, with the space of functions restricted to those functions which are analytic within the ellipse and continuous onto the boundary. In this chapter a bound on the relative factor will be obtained when the interval

[-1,1] is replaced by any closed, bounded domain D, which encloses the interval [-1,1].

$\Gamma_D$ is defined to be the boundary of D. The set of points in the interior of $\mathcal{E}_\rho$ will be denoted $\mathcal{I}(\mathcal{E}_\rho)$ and the closure of $\mathcal{I}(\mathcal{E}_\rho)$ as $\bar{\mathcal{I}}(\mathcal{E}_\rho)$. We shall now define $\mathcal{E}_R$ to be the smallest Chebyshev ellipse which contains D.

So

$$\bar{\mathcal{I}}(\mathcal{E}_\rho) \supset D \quad \text{if and only if} \quad \rho \geq R \quad .$$

The space of functions analytic within D and continuous onto the boundary $\Gamma_D$ will be denoted $\mathbb{A}(D)$. We shall assume all functions to be approximated belong to this space. The norm $\|\cdot\|_D$ is defined as follows:

$$\|F(z)\|_D = \max_{z \in \Gamma_D} |F(z)| \quad . \tag{3.2}$$

For a given function f(z) and a region D the error in the best approximation of order N is

$$E_N^D(z) = f(z) - \sum_{j=0}^{N}{}' b_j T_j(z) \tag{3.3}$$

where the coefficients $b_j$ ( $j = 0, 1, \ldots, N$ ) are the solution of the minimisation problem

$$\min_{a_0, a_1, \ldots, a_N} \left\| f(z) - \sum_{j=0}^{N}{}' a_j T_j(z) \right\|_D \tag{3.4}$$

This minimisation problem has a unique solution, see Davis (1963). The error in the truncated Chebyshev series of degree N will be denoted by

$$e_N(z) = f(z) - \sum_{j=0}^{N}{}' c_j T_j(z) \quad . \tag{3.5}$$

From (2.12), the coefficients $c_j$, ( $j = 0, 1, \ldots, N$) above are given by

$$c_j = \frac{2}{\pi} \int_{-1}^{1} \frac{f(t)\, T_j(t)}{(1 - t^2)^{1/2}}\, dt \tag{3.6}$$

As f(z) belongs to $\mathbb{A}(D)$, but not necessarily to $\mathbb{A}(\mathcal{I}(\mathcal{E}_R))$ the series may not converge for all $z \in \bar{\mathcal{I}}(\mathcal{E}_R)$. Subtracting equation (3.5) from (3.3) yields

$$E_N^D(z) - e_N(z) = \sum_{j=0}^{N}{}' (c_j - b_j) T_j(z) \tag{3.7}$$

which is a polynomial of degree N. From (3.5) and (3.6) it can be shown that

$$(c_j - b_j) = \frac{2}{\pi} \int_{-1}^{1} \frac{E_N^D(t)\; T_j(t)}{(1-t^2)^{1/2}} \, dt$$

Substituting the above into (3.7) and taking the norm over D of both sides we see that

$$\| E_N^D(z) - e_N(z) \|_D = \frac{2}{\pi} \max_{z \in \Gamma_D} \left| \sum_{j=0}^{N}{}' \int_{-1}^{1} \frac{E_N^D(t)\, T_j(t)\, dt\, T_j(z)}{(1-t^2)^{1/2}} \right|$$

$$\leq \frac{2}{\pi} \max_{z \in \mathcal{E}_R} \left| \sum_{j=0}^{N}{}' \int_{-1}^{1} \frac{E_N^D(t)\, T_j(t)\, dt\, T_j(z)}{(1-t^2)^{1/2}} \right| \tag{3.8}$$

The inequality stems from the maximum modulus principle. We shall assume R>1, since R = 1 has already been dealt with by Powell.

## §2 UPPER BOUNDS ON THE RELATIVE FACTOR

We shall now find upper bounds for the right hand side of (3.8) by obtaining bounds on the integral independent of $z$. By elementary manipulation of summation, integration and modulus signs we have

$$\left| \sum_{j=0}^{N}{}' \int_{-1}^{1} \frac{E_N^D(t)\; T_j(t)\; T_j(z)}{(1-t^2)^{1/2}} \, dt \right| \leq \| E_N^D(z) \|_I \int_{-1}^{1} \frac{\left| \sum_{j=0}^{N}{}' (T_j(t)\, T_j(z)) \right| dt}{(1+t^2)^{1/2}} \tag{3.9}$$

It will now be shown that if $z \in \mathcal{E}_R$ and N>0 then

$$\int_{-1}^{1} \frac{\left| \sum_{j=0}^{N}{}' T_j(t)\, T_j(z) \right| dt}{(1-t^2)^{1/2}} \leq \frac{R^{N-1}(R+1)(1+R^{-(2N+1)})}{(1-R^{-1})^2} \tag{3.10}$$

By definition

$$T_j(z) = \cos(j \cos^{-1}(z)),$$

and as $z \in \mathcal{E}_R$

$$z = \tfrac{1}{2}\left(Re^{i\theta} + R^{-1}e^{-i\theta}\right), \quad \theta \in (-\pi, \pi]. \tag{3.11}$$

This means that

$$\cos^{-1}(z) = -\theta + i\sigma,$$

where $\sigma = \ln R > 0.$

The left hand side of (3.10) can now be written as

$$\int_0^{\pi} \left| \sum_{j=0}^{N}{}' \cos j\psi \; \cos j(-\theta + i\sigma) \right| d\psi . \tag{3.12}$$

Equation (3.12) can be rearranged using property 1 of (2.11) and

$$\sum_{j=0}^{N}{}' \cos j\phi = \frac{\frac{1}{2}\sin(N+\frac{1}{2})\phi}{\sin\frac{1}{2}\phi}, \tag{3.13}$$

to give

$$\frac{1}{4}\int_0^{\pi} \left| \frac{\sin(N+\frac{1}{2})(\psi-\theta+i\sigma)}{\sin\frac{1}{2}(\psi-\theta+i\sigma)} + \frac{\sin(N+\frac{1}{2})(\psi+\theta-i\sigma)}{\sin\frac{1}{2}(\psi+\theta-i\sigma)} \right| d\psi \tag{3.14}$$

$$= \frac{R^N}{2}\int_0^{\pi} \left( \frac{U_N(\theta,R,\psi)}{U_0(\theta,R,\psi)} \right)^{\frac{1}{2}} d\psi \tag{3.15}$$

where

$$\begin{aligned} U_j(\theta,R,\psi) = \{ & \cos^2 j\psi \, (1 + 2R^{-2(j+1)}\cos 2(j+1)\theta + R^{-4(j+1)}) \\ & - 2\cos j\psi \cos(j+1)\psi \, (R^{-1}(1+R^{-(4j+2)})\cos\theta + R^{-(2j+1)}\cos(2j+1)\theta) \\ & + \cos^2(j+1)\psi \, (R^{-2}(1 + 2R^{-2j}\cos j\theta + R^{-4j})) \} \\ & j = 0, N. \end{aligned}$$

Equation (3.15) was obtained by writing each of the sines in (3.14) in terms of exponentials and taking the modulus. $U_j(\theta,R,\psi)$ are even, $2\pi$- periodic functions of $\theta$ therefore only values of $\theta$ between $0$ and $\pi$ are of interest. The inequality (3.10) comes from noting that

$$U_N(\theta,R,\psi) \leq \{ (1 + R^{-2(N+1)})\cos N\psi + R^{-1}(1+R^{-2N})\cos(N+1)\psi \}^2$$

and

$$U_0(\theta,R,\psi) \geq (1 - R^{-1})^4 .$$

Therefore

$$\frac{R^N}{2}\int_0^{\pi} \left( \frac{U_N(\theta,R,\psi)}{U_0(\theta,R,\psi)} \right)^{\frac{1}{2}} d\psi \leq \frac{R^N}{2(1-R^{-1})^2} \int_0^{\pi} \left| (1+R^{-2(N+1)})\cos N\psi + R^{-1}(1+R^{-2N})\cos(N+1)\psi \right| d\psi \tag{3.16}$$

$$\leq \frac{R^N}{2(1-R^{-1})^2} \left\{ (1+R^{-2(N+1)}) \int_0^{\pi} |\cos N\psi| d\psi + R^{-1}(1+R^{-2N}) \int_0^{\pi} |\cos(N+1)\psi| d\psi \right\}$$

$$= \frac{R^N(1+R^{-2})(1+R^{-(2N+1)})}{(1-R^{-1})^2}, \quad N > 0. \tag{3.17}$$

We have now obtained the bound given in (3.10), but there is an upper bound on (3.14), which is better than (3.17) for R<6, in terms of an elliptic integral. We first note that

$$\frac{1}{4}\int_0^{\pi} \left| \frac{\sin(N+\frac{1}{2})(\psi-\theta+i\sigma)}{\sin\frac{1}{2}(\psi-\theta+i\sigma)} + \frac{\sin(N+\frac{1}{2})(\psi+\theta-i\sigma)}{\sin\frac{1}{2}(\psi+\theta-i\sigma)} \right| d\psi$$

$$\leq \frac{1}{2}\int_0^{\pi} \left| \frac{\sin(N+\frac{1}{2})(\psi+i\sigma)}{\sin\frac{1}{2}(\psi+i\sigma)} \right| d\psi \tag{3.18}$$

$$= \frac{R^N}{2}\left(\frac{1+R^{-2(2N+1)}}{1+R^{-2}}\right)^{\frac{1}{2}} \int_0^{\pi} \left(\frac{1-L\cos(2N+1)\psi}{1-M\cos(\psi)}\right)^{\frac{1}{2}} d\psi,$$

where

$$L = \{\cosh(2N+1)\sigma\}^{-1} \tag{3.19}$$

and $$M = \{\cosh\sigma\}^{-1}$$

To obtain the upper bounds in terms of an elliptic integral we shall now prove the following Lemma.

LEMMA 2

For all $x, y \in [0,1)$ and k a positive integer,

$$I_k(x,y) = \int_0^{\pi}\left(\frac{1-x\cos(2k+1)\psi}{1-y\cos\psi}\right)^{\frac{1}{2}} d\psi \leq \int_0^{\pi}\frac{d\psi}{(1-y\cos\psi)^{1/2}} \tag{3.20}$$

Proof

We shall show that the derivative of $I_k$ with respect to $x$ is non-positive

$$\frac{\partial}{\partial x} I_k(x,y) = -\frac{1}{2}\int_0^{\pi}\frac{\cos(2k+1)\psi \; d\psi}{(1-y\cos\psi)^{1/2}(1-x\cos(2N+1)\psi)^{1/2}} \tag{3.21}$$

Since $0 \leq x < 1$, we have that

$$\cos(2k+1)\psi \leq \frac{\cos(2k+1)\psi}{(1-x\cdot\cos(2k+1)\psi)^{1/2}}, \quad 0 \leq \psi \leq \pi.$$

So

$$\frac{\partial}{\partial x} I_k(x,y) \leq -\frac{1}{2}\int_0^{\infty} \frac{\cos(2k+1)\psi \, d\psi}{(1-y\cos\psi)^{1/2}}$$

$$= \frac{1}{(1+y)^{1/2}} \int_0^{\infty} \frac{\cos(2k+1)2\psi' \, d\psi'}{(1-2y\sin^2\psi'/(1+y))^{1/2}}$$

$$= \frac{(-1)^{2k+1}}{(1+y)^{1/2}} \frac{\pi}{2} \sum_{j=2k+1}^{\infty} \frac{(2j-1)!!\,(y/2)^j (1+y)^{-j}}{(j+2k+1)!\,(j-2k-1)!}$$

$$\leq 0$$

and the lemma is proved.

Since L and M in (3.19) are both positive and less than one we have

$$\int_0^{\pi} \left| \sum_{j=0}^{N'} \cos j\psi \cos j(-\sigma+i\sigma) \right| d\psi \leq \frac{R^N(1+R^{-2(2N+1)})^{1/2}}{2(1+R^{-2})^{1/2}} \int_0^{\pi} \frac{d\psi}{(1-M\cos\psi)}$$

$$= \frac{R^N(1+R^{-2(2N+1)})^{1/2}}{(1+R^{-1})} \int_0^{\pi/2} \frac{d\psi}{(1-k^2\sin^2\psi)^{1/2}} \qquad (3.23)$$

where $k^2 = (\cosh(\sigma/2))^{-2} = 4R(1+R)^{-2}$.

From (3.10) and (3.23) we have,

$$\| E_N^D - \mathcal{C}_N(z) \|_D \leq \frac{2}{\pi} \| E_N^D(z) \|_I R^N A,$$

where $A = \min\left( \frac{(1+R^{-1})(1+R^{-(2N+1)})}{(1-R^{-1})^2}, \frac{(1+R^{-2(2N+1)})^{1/2}}{(1+R^{-1})} \int_0^{\pi/2} \frac{d\psi}{(1-k^2\sin^2\psi)^{1/2}} \right)$

Therefore, we have

$$\| \mathcal{C}_N(z) \|_D \leq \| E_N^D(z) \|_D + \frac{2}{\pi} \| E_N^D(z) \|_I R^N A$$

$$\leq \left( 1 + \frac{2}{\pi} R^N A \frac{\| E_N^D(z) \|_I}{\| E_N^D(z) \|_D} \right) \| E_N^D(z) \|_D$$

So we have an upper bound on the Relative factor of

$$\frac{2R^N A}{\pi} \frac{\| E_N^D(z) \|_I}{\| E_N^D(z) \|_D}. \qquad (3.24)$$

In (3.24) $\| E_N^D(z) \|_I \leq \| E_N^D(z) \|_D$ and in many cases $\| E_N^D(z) \|_I \ll \| E_N^D(z) \|_D$. So we can say

$$\| \mathcal{C}_N(z) \|_D \leq \left( 1 + \frac{2R^N A}{\pi} \right) \| E_N^D(z) \|_D$$

## §3 NUMERICAL COMPARISON OF BOUNDS

The integral given by (3.12) was calculated numerically using a

Gauss-Konrad integration routine from the NAG library, for various values of $\theta$, $\sigma$ and N. Some of these results are produced in Table 3.1. From these results it can be seen that for $N\sigma$ greater than 1 the maximum value, as $\theta$ varies, appears to be reached at or near $\theta = \pi/2$. This is in contrast to the case $\sigma = 0$, where the maximum is obtained at $\theta = 0$, for all N.

The maximum values, obtained above for each value of N and $\sigma$, are compared with the bounds (3.10) and (3.23) in Table 3.2. It can be seen that (3.23) is a closer bound than (3.10) in all the cases given here, but (3.10) improves and (3.23) worsens as $\sigma$ increases. The elliptic integral in (3.23) was calculated using a NAG special function routine.

## 4 LOWER BOUND ON THE RELATIVE FACTOR FOR $J_0(z)$ OVER D

Given

$$D = \{ z \mid |z| \leq 8, \quad 0 \leq \operatorname{Arg} z \leq \pi/12 \}$$

&

$$\gamma = 8 e^{i\pi/24}$$

we define

$$A_N(z) = J_0(z) - \sum_{k=0}^{N}{}' a_{2k} T_{2k}(z/\gamma)$$

$$C_N^{\infty}(z) = J_0(z) - \sum_{k=0}^{N}{}' c_{2k}^{\infty} \cdot (z/\gamma)^{2k}$$

and

$$C_N^{*}(z) = J_0(z) - \sum_{k=0}^{N}{}' c_{2k}^{*} (z/\gamma)^{2k},$$

where the coefficients are given by

$$a_{2k} = \frac{2}{\pi} \int_{-1}^{1} \frac{J_0(\gamma t)\, T_{2k}(t)\,dt}{(1-t^2)^{1/2}}$$

analytic on $[-1,1]$ · $\Leftrightarrow J_0(\gamma t) \sim \sum_0^{\infty}{}' a_{2k} T_{2k}(t)$, $t \in [-1,1]$, $z = \gamma t$

$\Rightarrow J_0(z) \sim \sum_0^{\infty}{}' a_{2k} T_{2k}(z/\gamma)$

and

$c_{2k}^{\infty}$ and $c_{2k}^{*}$, $k = 0, 1, \ldots, N$ minimise

$\| C_N^{\infty} \|_{\infty}$ and $\| C_N^{*} \|_{*}$ respectively over D.

Then from Barrodale (1977) it can be shown that

$$\sqrt{2}\,\frac{\|A_N\|_*}{\|C_N^*\|_*} \geq \frac{\|A_N\|_\infty}{\|C_N^\infty\|_\infty} \geq \frac{1}{\sqrt{2}}\,\frac{\|A_N\|_*}{\|C_N^*\|_*}$$

So by calculating the best polynomial approximation in the star norm and the maximum error, in the star norm, of the truncated Chebyshev series, we can obtain upper and lower bounds on the relative factor. Numerical results for $J_0(z)$ on D for N = 5(15)15 are given below.

| N | Lower Bound | Upper Bound |
|---|---|---|
| 5 | 5.41 | 10.82 |
| 10 | 36.55 | 73.09 |
| 15 | 231.08 | 462.16 |

Although the results show that the truncated Chebyshev series approximation to $J_0(z)$ is not a near-best approximation on D, the lost accuracy can be retrieved by only slightly increasing the degree of the approximation.

## TABLE 3.1

| | $\theta \backslash \sigma$ | 0.05 | 0.1 | 0.2 | 0.3 | 0.4 |
|---|---|---|---|---|---|---|
| N=5 | 0 | 3.14 | 3.33 | 4.17 | 5.85 | 8.81 |
| | $\pi/6$ | 2.71 | 3.01 | 4.13 | 6.10 | 9.36 |
| | $\pi/3$ | 2.71 | 3.05 | 4.29 | 6.42 | 9.89 |
| | $\pi/2$ | 2.71 | 3.06 | 4.33 | 6.51 | 10.30 |
| N=10 | 0 | 3.78 | 4.73 | 9.90 | 2.44(+1) | 6.32(+1) |
| | $\pi/6$ | 3.40 | 4.81 | 1.11(+1) | 2.72(+1) | 6.91(+1) |
| | $\pi/3$ | 3.51 | 5.05 | 1.17(+1) | 2.89(+1) | 7.31(+1) |
| | $\pi/2$ | 3.68 | 5.19 | 1.19(+1) | 2.93(+1) | 7.43(+1) |
| N=20 | 0 | 5.33 | 1.12(+1) | 7.07(+1) | 4.89(+2) | 3.45(+3) |
| | $\pi/6$ | 5.62 | 1.32(+1) | 8.22(+1) | 5.49(+2) | 3.77(+3) |
| | $\pi/3$ | 5.95 | 1.39(+1) | 8.67(+1) | 5.66(+2) | 3.99(+3) |
| | $\pi/2$ | 6.02 | 1.40(+1) | 8.80(+1) | 5.74(+2) | 4.05(+3) |

TABLE 3.2

| | $\sigma$ | Num.Int. | (3.10) | (3.23) |
|---|---|---|---|---|
| N=5 | | | | |
| | 0.05 | 3.14 | 1.66(+3) | 3.86 |
| | 0.10 | 3.33 | 4.62(+2) | 4.00 |
| | 0.20 | 4.33 | 1.67(+2) | 5.56 |
| | 0.30 | 6.51 | 1.2(+2) | 8.51 |
| | 0.40 | 10.30 | 3.8(+2) | 13.40 |
| N=10 | | | | |
| | 0.05 | 3.78 | 1.83(+3) | 4.54 |
| | 0.10 | 5.19 | 6.42(+2) | 6.23 |
| | 0.20 | 1.19(+1) | 4.15(+2) | 1.51(+1) |
| | 0.30 | 2.93(+1) | 5.22(+2) | 3.82(+1) |
| | 0.40 | 7.43(+1) | 8.39(+2) | 9.90(+1) |
| N=20 | | | | |
| | 0.05 | 6.02 | 2.52(+3) | 7.13 |
| | 0.10 | 1.40(+1) | 1.59(+3) | 1.70(+1) |
| | 0.20 | 8.80(+1) | 3.03(+3) | 9.70(+1) |
| | 0.30 | 5.74(+2) | 1.05(+3) | 7.65(+2) |
| | 0.40 | 4.05(+3) | 4.59(+4) | 5.40(+3) |

# CHAPTER 4

## APPROXIMATION OF $J_N(Z)$ AND $Y_N(Z)$ OVER THE ENTIRE COMPLEX PLANE

### §1 INTRODUCTION TO BESSEL FUNCTION APPROXIMATION

For the Bessel function $J_N(z)$ it is possible in principle to use the Taylor expansion within a disc $|z| \leq R$ and an asymptotic expansion for $|z| > R$. This approach suffers from cancellation errors and the need for an increasingly large number of terms in the Taylor series as $|z|$ increases, as well as from the limitation on the accuracy achievable from an asymptotic expansion for a given value of R.

Polynomial approximations for Bessel functions of real argument were obtained by Allen (1954, 1956). Ardill and Moriarty (1977) produced approximations for $J_0$ and $J_1$, by extrapolating Allen's approximations to complex arguments. Because the accuracy of such an approximation decreases rapidly as arg z is increased, Ardill and Moriarty (1977) divided the first quadrant into three regions in which they used extrapolations of polynomial approximations for $J_0$ and $J_1$, valid on the real axis, for the modified Bessel functions $I_0$ and $I_1$, or for the Kelvin functions ber and bei (see Abramowitz and Stegun (1965), p.379). In view of the nature of these approximations there is no basis for a theoretical appraisal of the error involved. The error can be assessed, as for Allen's approximations for real arguments, only by tabulation. Furthermore, the numerical results given by Ardill and Moriarty (1977) show that the truncation error is substantially greater in some regions than in others; in the authors' own words "...considerable improvement ... could be made by devising a new polynomial approximation for the region given approximately by $60° \leq \arg z \leq 80°$". It follows that this approach is not a useful one if the objective is a routine which will calculate $J_N(z)$ to a specified accuracy. Approximations, based on Chebyshev series, for Bessel functions of real argument have been obtained by Clenshaw (1962), Luke (1975) and Coleman (1980).

Our approach is to divide the complex plane into a number of sectors and, in the absence of an expansion of the form (1.1) we use in each sector an approximation

$$f(z) \simeq \sum_{k=0}^{n}{}' a_k T_k (g(z/\gamma)), \qquad (4.1)$$

the complex constant $\gamma$ being chosen so that $g(z/\gamma)$ is real on the central ray of the sector, where $g(\omega)$ is a function chosen to improve convergence for a particular function f(z). Our expansions for $J_N(Z)$ and the generation of their coefficients are described in Section 2. The study in Section 3, of the truncation error requires an upper-bound for $|T_n(z)|$ on a sector; such a bound was established in Chapter 2. Section 4 describes the coefficient tables and provides a guide to the accuracy achievable. In section 5 expansions for $Y_0(z)$ and $Y_1(z)$ are produced.

## §2. EXPANSIONS FOR $J_N(z)$

Throughout this chapter we shall concentrate on values of Z in the first quadrant, $0 \leq \arg z \leq \frac{\pi}{2}$, only, since symmetry relations can be used to deduce the values of $J_N(z)$ elsewhere. It is convenient to consider separately two regions of the complex plane, an inner region $|z| \leq R$ and an outer region $|z| > R$, each of which will be mapped into the unit disc. It will be assumed that N is a positive integer, but the extension to non-integral orders is straightforward.

### §2.1 The inner region

Clenshaw (1966) tabulated coefficients for expansions of the form

$$J_n(X) = \frac{1}{n!}\left(\frac{X}{2}\right)^n \sum_{r=0}^{\infty}{}' a_{2r} \, T_{2r}(X/8) \tag{4.2}$$

on the interval [-8,8], for n = 0 and 1. Coefficients of the corresponding expansions for n = 2, 3, ......, 10 were calculated by Coleman (1980) and are incorporated in a routine which calculates $J_n(X)$ to a specified accuracy.

If x is replaced by the complex variable z the expansion (4.2) still converges. Since $z^{-n} J_n(z)$ is a regular function of z it can be shown (Rivlin (1974) p.143) that, for any $\rho>1$,

$$|a_{2r}| \leqslant \frac{2M(\rho)}{\rho^{2r}}$$

where $M(\rho)$ is an upper bound on $|2^n n! \, z^{-n} J_n(z)|$ for $|z| = 8\rho$. Furthermore

$$|T_{2r}(\zeta)| \leqslant \tfrac{1}{2}\left[(1+\sqrt{2})^{2r} + (1-\sqrt{2})^{2r}\right]$$

for $|\zeta| \leqslant 1$, the bound being attained when $\zeta = \pm i$. It follows that

$$|a_{2r} T_{2r}(z/8)| \leqslant \frac{2M(\rho)(1+\sqrt{2})^{2r}}{\rho^{2r}}$$

and since we may take $\rho > (1+\sqrt{2})$ the convergence of the series is established. There is nevertheless a substantial deterioration in the rate of convergence as arg z increases. For example, if the series is truncated to give a polynomial of degree 20, the first neglected term increases by a factor of 1.3 x $10^8$ as arg z increases from 0° to 90°, while $|z| = 8$.

It was shown in Chapter 2 that in the sector $|\zeta| \leqslant 1$, $|\arg \zeta| \leqslant \theta$

$$|T_n(\zeta)| \leqslant |T_n(e^{i\theta})|.$$

The right-hand side of this inequality is a monotonically increasing function of $\theta$ for 0° < $\theta$ < 90° so the smaller the angular range about the real axis in which we use a given number of terms of the expansion (4.2) the less significant the loss of accuracy.

The advantages of a Chebyshev expansion on the real interval [-1,1] may be retained for any particular ray in the complex plane by writing

$$J_n(z) = \frac{1}{n!}\left(\frac{z}{2}\right)^n \sum_{r=0}^{\infty}{}' a_{2r} T_{2r}(z/\gamma)$$
$$= \frac{1}{n!}\left(\frac{z}{2}\right)^n V(z/\gamma) \qquad (4.3)$$

where $\gamma$ is a complex number having the same argument as Z, and such that $0 \leq z/\gamma \leq 1$. The function V in eqn. (4.3) satisfies the differential equation

$$tV'' + (2n + 1)V' + \gamma^2 tV = 0 \qquad (4.4)$$

where $V(o) = 1$, $V'(o) = 0$ and the prime denotes differentiation with respect to $t = z/\gamma$. We now seek a solution of this equation of the form

$$V(t) = \sum_{r=0}^{\infty}{}' a_{2r} T_{2r}(t) = \sum_{r=0}^{\infty}{}' a_{2r} T^*_r(t^2) \qquad (4.5)$$

Clenshaw's method applied to eqn (4.4) gives the recurrence relations

$$a'_{2r-1} = a'_{2r+1} + 4r\,a_{2r} \qquad (4.6)$$

$$a_{2r-2} = a_{2r+2} - \frac{4r}{\gamma^2}\left[a'_{2r-1} + a'_{2r+1} + 4n\,a_{2r}\right] \qquad (4.7)$$

which may be solved as indicated in Chapter 2, starting with

$$a_{2N} = 1, \quad a_{2r} = 0 \quad \text{for} \quad r > N$$

$$a'_{2r+1} = 0 \quad \text{for} \quad r \geq N$$

for some sufficiently large N. The results are normalised using the initial condition $V(o) = 1$.

We have used the method for $|\gamma| = 8$ and $\arg\gamma = 7.5°\ (15°)\ 82.5°$, thus providing a separate expansion for each 15° sector in the first quadrant. The accuracy of these expansions is discussed in Section 3.

The generalized hypergeometric function can be defined by the series

$$ {}_pF_q(\alpha_1, \alpha_2, \ldots \alpha_p; \beta_1, \beta_2, \ldots \beta_q; z) $$
$$ = \sum_{K=0}^{\infty} \left[ \prod_{h=1}^{p} (\alpha_h)_K z^K \Big/ \prod_{h=1}^{q} (\beta_h)_K K! \right], $$
$$ (\alpha)_K = \Gamma(\alpha+K) / \Gamma(\alpha), $$

$$ p \leqslant q \text{ or } p = q + 1 \text{ and } |z| < 1, $$

where $\Gamma(z)$ is the gamma function. The general theory of the expansion of these functions in a series of the same kind is described in Luke (1969). It is useful in this context, because both Bessel functions and Chebyshev polynomials can be written as hypergeometric functions;

$$ T_n(x) = {}_2F_1(-n, n; \tfrac{1}{2}; \tfrac{1-x}{2}) $$
$$ J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} {}_0F_1(\nu+1; -z^2/4) $$

In Luke (1977) are recurrence relations equivalent to (4.6) and (4.7), and in Luke (1975) a generalised case of the expansion

$$ V(t) = \sum_{j=0}^{\infty}{}' b_j T_j(t^2) \qquad (4.8) $$

is given. Initially this appears to be a superior method of expansion, because it can be used not only in its original sector, but also in a sector at right angles to it. The use of symmetry relations means that if, as before, the quadrant is divided into 15° sectors we need only find expansions in the three sectors for which

$$ |\arg \gamma| \leq \pi/4 . $$

However, the series in (4.8) converge slightly slower than those in (4.5) and therefore the time required to calculate the Bessel function to the required accuracy is increased.

§2.2 The outer region

The Bessel function $J_n(z)$ may be expressed as

$$J_n(z) = \frac{1}{2}[H_n^{(1)}(z) + H_n^{(2)}(z)]$$

in terms of the Hankel functions of the first and second kinds which have the asymptotic forms

$$H_n^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z - \frac{1}{2}n\pi - \frac{1}{4}\pi)}$$

$$H_n^{(2)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{-i(z - \frac{1}{2}n\pi - \frac{1}{4}\pi)}$$

as $z \to \infty$. Any ray in the region $|z| \gg |\gamma|$ may be mapped onto [0,1] by the transformation $t = \gamma/z$ and the function

$$U^{(2)}(t) = \left(\frac{\pi z}{2}\right)^{1/2} \exp[i(z - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)]\, H_n^{(2)}(z)$$

satisfies the differential equation

$$t^2 u'' + 2(t + i\gamma)u' + (\tfrac{1}{4} - n^2)u = 0 \qquad (4.10)$$

where the prime denotes differention with respect to the real variable t, and $u^{(2)}(0) = 1$. Similarly extraction of the asymptotically dominant term in $H_n^{(1)}(z)$ yields a function $u^{(1)}(t)$ which satisfies a differential equation differing from (4.10) only in the sign of $i\gamma$.

Clenshaw's method, used to find a solution of the form

$$u^{(i)}(t) = \sum_{r=0}^{\infty}{}' a_r^{(i)} T_r^*(t) \quad , \qquad i = 1, 2 \qquad (4.11)$$

for eqn. (4.10), gives the recurrence relations

$$[r(r+1) + \tfrac{1}{4} - n^2]a_{r-1} = (\tfrac{1}{4} - n^2)a_r - \tfrac{1}{4}(r+1)\,[7a'_{r+1} + a'_{r+3} + 4(a'_{r+2} + a'_r)] - 8i\gamma(r+1)a_{r+1} \qquad (4.12)$$

$$a'_{r-1} = 4r\,a_r + a'_{r+1} \qquad (4.13)$$

When these, and the corresponding equations for the Chebyshev coefficients of $u^{(2)}(t)$, are solved the results may be combined to evaluate the Bessel functions of the first and second kinds, since

$$J_n(z) = (2\pi z)^{-1/2}\left[u^{(1)}(t)\,\mathrm{Exp}\{i\alpha(z)\} + u^{(2)}(t)\,\mathrm{Exp}\{-i\alpha(z)\}\right] \qquad (4.14)$$

and

$$Y_n(z) = -i\,(2\pi z)^{-\frac{1}{2}} \left[ u^{(1)}(t)\, \mathrm{Exp}\left\{ i\alpha(z) \right\} - u^{(2)}(t)\,\mathrm{Exp}\left\{ -i\alpha(z) \right\} \right] \quad (4.15)$$

where

$$\alpha\ (z) = z - \tfrac{1}{2} n\pi - \tfrac{1}{4}\pi \ .$$

The expansions in (4.11) are particular cases of series discussed by Luke (1977, p.88). In the real case the differential equations (4.10) has only one solution of the required form and the coefficients can be found, as described in Chapter 2, by backward recurrence from an arbitrary starting point and the use of the initial condition. When $\mathrm{Im}\,\gamma > 0$ there is no longer just one solution, since (4.10) is also satisfied by

$$W(t) = \begin{cases} e^{2i\gamma/t}\, u(-t) & ,\ t > 0 \\ 0 & ,\ t \leqslant 0 \end{cases} .$$

It follows that any solution of equations (4.12) and (4.13) corresponds to a linear combination of $u(t)$ and $W(t)$. The unwanted solution $W(t)$ may be removed by solving a boundary value problem in the manner described in Chapter 2. The known value of $J_n(\gamma)$ is used in addition to the condition $u(0) = 1$. In practice the extent to which the unwanted solution $W(t)$ enters will depend on the value of $\mathrm{Im}\,\gamma$. In producing the tables we solved the recurrence systems with two different starting values and decided by comparison of the results whether to solve an initial - or boundary - value problem.

## §3 ERROR BOUNDS

A real - or complex-valued function f of a real variable $x$ may, under suitable conditions, be expressed on the interval [-1,1] as a Chebyshev series

$$f(x) = \sum_{r=0}^{\infty}{}' \, b_r\, T_r(x)$$

where

$$b_r = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_r(x)}{\sqrt{1-x^2}}\,dx \quad .$$

In particular, if f is differentiable to all orders on [-1,1] and $|f^{(r)}(x)| \leq Q_r$ on that interval, the work of Elliott (1963) shows that

$$|b_r| \leq \frac{Q_r}{2^{r-1}\,r!}$$

To apply the result it is convenient to write equation (4.5) as

$$2^n\, n!\, z^{-n} J_n(z) = f(x) = \sum_{r=0}^{\infty}{}' a_{2r} T_r(x)$$

where

$$x = 2t^2 - 1 = 2(z/\gamma)^2 - 1$$

Then

$$\frac{d^r f}{dx^r} = 2^n\, n!\left[\frac{\gamma}{2}\right]^{2r}\left[\frac{1}{z}\frac{d}{dz}\right]^r\left[\frac{J_n(z)}{z^n}\right]$$

$$= \frac{(-1)^r\, 2^n\, n!}{z^{n+r}}\left[\frac{\gamma}{2}\right]^{2r} J_{n+r}(z)$$

(See Watson (1944) p.18). Watson (1944) p.49 establishes the bound

$$|J_m(z)| \leq \frac{|z|^m}{2^m m!}\exp|I_m z| \qquad (4.16)$$

Thus if $|z| \leq |\gamma|$ and arg z = arg $\gamma = \phi$ then

$$|a_{2r}| \leq \frac{2\, n!\, |\gamma|^{2r}}{2^{4r}\, r!\, (n+r)!}\exp(|\gamma|\cdot|\sin\phi|) . \qquad (4.17)$$

Replacing (4.16) by the inequality

$$|J_m(z)| \leq \frac{|z|^m}{2^m m!}\exp\left(\frac{1}{4}\frac{|z|^2}{m+1}\right)$$

we obtain the alternative bound

$$|a_{2r}| \leq \frac{2\,|\gamma|^{2r}\, n!}{2^{4r}\, r!\, (n+r)!}\exp\left(\frac{1}{4}\frac{|\gamma|^2}{(n+r+1)}\right) \qquad (4.18)$$

From Luke (1977) p.77 we know that

$$a_{2r} = \frac{2(-1)^r \gamma^{2r} n!}{2^{4r}\, r!\,(n+r)!}\ {}_1F_2\left(r+\tfrac{1}{2};\, n+r+1,\, 2r+1;\, -\gamma^2/4\right)$$

where $\quad {}_1F_2\,(a;\,b,\,c;\,z) \;=\; \sum_{K=0}^{\infty} \frac{\Gamma(a+k)\,\Gamma(b)\,\Gamma(c)\,z^K}{\Gamma(a)\,\Gamma(b+k)\,\Gamma(c+k)\,K!}$ .

so

$$|a_{2r}| \leqslant \frac{2\,n!\,|\gamma|^{2r}}{2^{4r}\,r!(n+r)!} \sum_{K=0}^{\infty} \frac{(n+r)!\,r!\,(2r+2k)!\,|\gamma|^{2K}}{(r+k)!\,(r+n+k)!\,(2r+k)!\,K!\,16^K}$$

The ratio of successive terms in this series is

$$\frac{(2r+2k)(2r+2k-1)\,|\gamma|^2}{16(r+k)(r+n+k)(2r+k)k} < \frac{|\gamma|^2}{4(2r+k)k}$$

since $n > -\frac{1}{2}$ , and therefore

$$|a_r| < \frac{2\,n!\,|\gamma|^{2r}}{2^{4r}\,r!(n+r)!}\left[1 + \sum_{K=1}^{\infty}\left\{\frac{|\gamma|^2}{(4(2r+1)}\right\}^K \frac{1}{K!}\right]$$

$$= \frac{n!\,.\,2\,.\,|\gamma|^{2r}}{2^{4r}\,r!(n+r)!}\,\exp\left[\frac{1}{4}\,\frac{|\gamma|^2}{(2r+1)}\right] \qquad (4.19)$$

We now have three upper bounds on the coefficients, which differ only in the argument of the exponential. It can easily be seen that (4.19) is a lower bound than (4.18) if $r>n$ and vice versa if $n>r$ , whilst (4.17) is a lower upper bound than both if

$$|\sin\phi| < \frac{|\gamma|}{4.\ \mathrm{Max}(r+n+1,\ 2r+1)}$$

Table 4.1 gives some comparisons of these bounds with the values of $|a_{2r}|$ when $n = 1$ .

Suppose that the Chebyshev series for $2^n\,n!\,Z^{-n}J_n(z)$ on the ray $\arg\gamma = \phi$ is truncated to give a polynomial of degree of $2\bar{m}$ and the resulting approximation is used in the domain

$$D = \{\,z;\ |z| \leqslant |\gamma|\ ,\ \ \phi - \Theta \leqslant \arg Z \leqslant \phi + \Theta\,\},$$

where $\Theta$ is given. Then the truncation error is

$$E_{m,n}(z) = \sum_{r=m+1}^{\infty} a_{2r}\,T_{2r}\,(z/\gamma)$$

and its magnitude is bounded by

$$2\,.\,n!\ A_{m,n} \sum_{r=m+1}^{\infty} \frac{|\gamma|^{2r}}{4^{2r}}\,\frac{|T_{2r}(e^{i\Theta})|}{r!(n+r)!}$$

with

$$A_{m,n} = \mathrm{Min}\left(\mathrm{Exp}\,(|\gamma|\sin\phi),\ \mathrm{Exp}\left(\frac{1}{4}\cdot\frac{|\gamma|^2}{2m+3}\right),\ \mathrm{Exp}\left(\frac{1}{4}\cdot\frac{|\gamma|^2}{n+m+2}\right)\right)$$

Since

$$|T_{2r}(e^{i\theta})| \leq \frac{1}{2}(\rho^{2m+2} + \rho^{-(2m+2)})\rho^{2(r-m-1)}$$

with $\rho = | e^{i\theta} + \sqrt{e^{(2i\theta)} - 1} |$

and

$$\frac{(m+1)!\,(n+m+1)!}{(m+1+k)!(n+m+1+k)!} \leq \frac{1}{(m+2)^k\,(n+m+2)^k},$$

it follows that

$$|E_{m,n}(z)| \leq \frac{n!\,A_{m,n}}{(m+1)!(n+m+1)!} \cdot \left(\frac{|\gamma|}{4}\right)^{2(m+1)} \frac{(\rho^{2m+2} + \rho^{-(2m+2)})}{1 - R_{m,n}}$$

where $$R_{m,n} = \frac{|\gamma|^2 \rho^2}{16(m+2)(m+n+2)} \tag{4.20}$$

When $R_{m,n} << 1$ the truncation error is dominated by the 1st neglected term. For example, when $|\gamma| = 8$ and $\theta = 7.5°$, $R_{22,0} \simeq 0.014$ and $R_{32,0} \simeq 0.007$.

The corresponding bound for the expansion (4.8) is

$$|E^{*}_{m,n}(z)| \leq \frac{|\gamma|^{2m+2}}{2^{3m+3}} \frac{n!\,B_{m,n}}{(m+1)!(m+n+1)!} \frac{\{\delta^{m+1} + \delta^{-(m+1)}\}}{1 - P_{m,n}}$$

where

$$\delta = |\,\mathrm{Exp}\,(2i\theta) + \sqrt{\mathrm{Exp}(4i\theta) - 1}\,| \geq 1$$

$$P_{m,n} = \frac{|\gamma|^2 \delta}{16(m+2)(m+n+2)},$$

and

$$B_{m,n} = \mathrm{Exp}\,\{\,|\gamma|^4/(64(m+2)(m+n+2)(m+n+3))\,\} \tag{4.21}$$

For $\theta = \pi/24$ the ratio of these bounds is approximately $1:(1.6)^{m+1}$, which indicates why the previous expansion is preferred.

## §3.1 Estimation of the required number of terms

We wished to write a program capable of calculating the Bessel function $J_n(z)$, in the inner region, to a user specified accuracy, less than some maximum attainable accuracy. To do this efficiently we require a method of calculating the smallest number of terms needed in the truncated Chebyshev series. It is assumed that the degree of the truncated series would be at least 2 $M_o$, where $M_o + 1 > |\gamma|$, and that $\rho < 4$ and $|\gamma| > 1$. With these assumptions we have for $M \geq M_o$,

$$\frac{|z/2|^n}{n!}\,|E_{m,n}(z)| \leq \frac{\text{Exp}(|\gamma|/8)}{(m+1)!}\left(\frac{\rho|\gamma|}{4}\right)^{2(m+1)}\frac{(0\cdot 5|\gamma|)^n}{(m+n+1)!}\left(\frac{1+\rho^{-4|\gamma|}}{1-\rho^2/16}\right), \quad (4.22)$$

where the left hand side is the error in approximation of $J_n(z)$. Now set $x = m + 1$ and note that

$$\sqrt{2\pi}\,\text{Exp}\,(-x + 1/12x)\; x^{x+\frac{1}{2}} \geq \Gamma(x+1) \geq \sqrt{2\pi}\,\text{Exp}(-x)\,x^{x+\frac{1}{2}}$$
$$x > 0.$$

We can now write

$$\frac{1}{n!}\left|\frac{z}{2}\right|^n |E_{x-1,n}(z)| \leq \frac{1}{2}\left(\frac{\rho|\gamma|e}{4}\right)^{2x}\frac{1}{(x+n)^{x+n+\frac{1}{2}}}\cdot\frac{K_n}{x^{x+\frac{1}{2}}}$$

where

$$K_n = \frac{e^{|\gamma|/8}\,(1+\rho^{-4|\gamma|})}{\pi\,(1-\rho^2/16)}\left(\frac{|\gamma|e}{2}\right)^n$$

The bound in the last equation is a monotonic decreasing function of $x$. So if the accuracy required is $\frac{1}{2} \times 10^{-N}$, and we find $Y$ such that

$$\frac{1}{2} \times 10^{-N} = \frac{1}{2}\left(\frac{\rho|\gamma|e}{4}\right)^{2Y}\frac{K_n}{(Y+n)^{Y+n+\frac{1}{2}}\;Y^{Y+\frac{1}{2}}},$$

then by setting

$$M = [\,Y\,] + 1 \geq M_o$$

we have

$$\frac{|z/2|^n}{n!}\,|E_{M,n}(z)| \leq \frac{1}{2} \times 10^{-N}$$

The value of $Y$ can be obtained iteratively using

$$y_{j+1} = \frac{\left(2y_j - \frac{1}{2}\log_e y_j - (n+\frac{1}{2})\log_e(n+y_j) + \frac{1}{2}\left\{\frac{y_j}{n+y_j}\right\} + Nk_1 + k_2 + k_3\right)}{\left\{\log_e\{y_j(y_j+n)\} - 2\log_e\left\{\frac{\rho|\gamma|}{4}\right\} + \frac{1}{2y_j} + \frac{1}{2(y_j+n)}\right\}} \quad (4.23)$$

where

$$k_1 = \log_e 10, \quad k_2 = \frac{1}{2} + \frac{|\gamma|}{8} - \log_e\left\{\frac{\pi(1-\rho^2/16)}{(1+\rho^{-4|\gamma|})}\right\}$$

and $k_3 = n + n \log_e (|\gamma|/2)$.

Furthermore, if $y > \frac{1}{2}$ and $y_j > (\rho|\gamma|/4)$, the iteration converges to $y$ from above, so one iteration gives an upper bound on $y$, which can be used to determine the number of terms required.

Examples:

(1) $|\gamma| = 8$ $\rho^2 = 2.03$, $n = 0$

| N | $y_0$ | $y_1$ | $y_2$ |
|---|---|---|---|
| 4 | 10 | 10.7 | 10.7 |
| 24 | 20 | 23.63 | |
| 40 | 28 | 31.65 | |

(2) $|\gamma| = 2$ $\rho^2 = 1$, $n = 0$

| N | $y_0$ | $y_1$ | $y_2$ |
|---|---|---|---|
| 16 | 10 | 9.09 | 9.07 |

The initial value $y_0$ was chosen to be $\{|\gamma| + N/2\}$ as this satisfies the condition $y_j > |\gamma|$ and uses N/2 as an estimate of the variation of the number of coefficients required with N. Substituting values $[y_1 - 1]$ and $[y_1]$ from the first example into (4.22) gives the following results:

| $[y_1]$ | Bound using $[y_1-1]$ | Bound using $[y_1]$ | Required Accuracy |
|---|---|---|---|
| 10 | $0.2 \times 10^{-3}$ | $0.13 \times 10^{-4}$ | $0.5 \times 10^{-4}$ |
| 23 | $0.18 \times 10^{-23}$ | $0.25 \times 10^{-25}$ | $0.5 \times 10^{-24}$ |
| 31 | $0.30 \times 10^{-39}$ | $0.24 \times 10^{-41}$ | $0.5 \times 10^{-40}$ |

From these results we can see that $M = [y_1]$ is the smallest integer which reduces the bound on the error to the required level and therefore, we do not have to iterate to convergence. They also show that for M large, adding an extra coefficient gives an extra 2 decimal place accuracy. So although the relative factor of a truncated Chebyshev series to the best polynomial approximation of the same degree may be large, the extra accuracy can be produced by adding just one or two coefficients to the series. The extra evaluation time is more than offset against the difficulty in calculating best polynomial approximations.

## §3.2 Asymptotic estimates for the outer region

The dependence of the rate of convergence in the inner region, on $|\gamma|$ is evident from the inequality (4.20). By decreasing $|\gamma|$ the rate of convergence in the inner region is increased, but at the expense of a slower rate in the outer region. The bound (4.20) applies to the inner region $|z| \leq |\gamma|$ only. For $|z| > |\gamma|$ we do not have useful bounds for the coefficients of the expansions (4.11), but their asymptotic forms have been investigated by Miller (1966) and by Luke (1977). The latter reference gives two slightly different asymptotic estimates:

$$a_r^{(3+\epsilon)/2} \sim \frac{4(-1)^r (\pi/3)^{1/2} (2r^2 \epsilon \gamma i)^{1/6}}{r\,\Gamma(\frac{1}{2}-n)\,\Gamma(\frac{1}{2}+n)} \operatorname{Exp}\left\{ S(2P-3) + R/S \right\} \qquad (4.24)$$

and

$$a_r^{(3+\epsilon)/2} \sim \frac{4(-1)^r (\pi/3)^{1/2} (2r^2 \epsilon \gamma i)^{1/6}}{r\,\Gamma(\frac{1}{2}-n)\,\Gamma(\frac{1}{2}+n)} \operatorname{Exp}\left\{ \tfrac{2}{3}\epsilon\gamma i - 3S + \left(R + \tfrac{4\gamma^2}{15}\right)S^{-1} \right\} \qquad (4.25)$$

where

$$S = (2r^2 \epsilon \gamma i)^{1/3}, \qquad y = (2r\gamma\epsilon i/r)^{1/3}$$

$$P = \tfrac{1}{2}\left\{ 1 - y/\sinh(y) \right\}, \qquad R = n^2 - \tfrac{1}{9}$$

and $\epsilon = \pm 1$.

Tables 4.2 and 4.3 contain comparisons of our calculated coefficients with these estimates, in every case the estimate (4.24) is better than (4.25), but both are remarkably accurate. The value $|\gamma| = 8$ was chosen after experimentation with a number of different values. No compelling reason was found to change the value from that chosen by Clenshaw (1962) for the real case.

## §4 The coefficient tables for $J_\nu(z)$

The coefficients of the Chebyshev expansions (4.5) and (4.11), for $J_0(z)$ and $J_1(z)$ with $|\gamma| = 8$, were calculated in quadruple precision on an IBM 370 computer, and the results were rounded to double-precision form. For each Bessel function there are six tables, one corresponding to each of the rays arg $\gamma$ = 7.5° (15°) 82.5°. In each table the top set of figures corresponds to the expansion (4.5) and the two lower sets are the coefficients $a_r^{(1)}$ and $a_r^{(2)}$ to be used for $|z|>8$.

The number of coefficients tabulated in each case, and the number of decimal digits to which they are quoted, were determined by the requirement that a certain accuracy be achievable in the calculation of the corresponding Bessel function in double precision arithmetic. In the inner region the Chebyshev series converges rapidly and the accuracy is limited only by rounding error. Table 4.4 shows the accuracy achievable for $|z| \leq 8$ with the tabulated coefficients. As arg $\gamma$ increases the low order coefficients increase in magnitude but $\sum_{r=0}^{\infty}{}' (-1)^r a_{2r} = 1$ in all cases; the resulting loss of significant figures reduces the achievable accuracy as shown in the table.

In the outer region the truncation error, rather than the rounding error, is the decisive factor. The truncation error increases as arg is increased and we maintain the achievable accuracy (Table 4.5) at a fixed level for $0 < \arg z \leqslant 75°$ by increasing the number of coefficients retained in each of the series (4.11). For $75° < \arg z \leqslant 90°$ we have accepted a lower level of accuracy as the convergence of the series has then become very slow. The relative errors quoted in Table 4.5 are not applicable in the vicinity of the zeros of $J_0$ and $J_1$, all of which lie on the real axis. The tabulated coefficients are sufficiently accurate to permit calculations with absolute error no greater than $\frac{1}{2} \times 10^{-14}$ near the zeros.

The coefficients of the Chebyshev expansion (4.5), for $J_0(z)$, $J_1(z)$, ..., $J_{10}(z)$ with $|\gamma| = 8$ are also tabulated in quadruple precision form.

## §5 Expansions for $Y_0(z)$ and $Y_1(z)$ for $|z| \leq |\gamma|$

In the outer region the Bessel functions of the second kind can be calculated from the Hankel functions. $Y_0(z)$ and $Y_1(z)$ have singularities at the origin, but as the form of the singularity is known in each case, we can subtract it and expand the remaining auxiliary function, which is analytic.

For $Y_0(z)$ and $Y_1(z)$ we choose

$$V_0(z) = Y_0(z) - \frac{2}{\pi} \log_e\left(\frac{z}{2}\right) . J_0(z) \tag{4.26}$$

and

$$\left(\frac{z}{2}\right) V_1(z) = Y_1(z) + \frac{2}{\pi}\left(\frac{1}{z} - \log_e\left(\frac{z}{2}\right) . J_1(z)\right) \tag{4.27}$$

Putting $z = \gamma t$ we find that these auxiliary functions satisfy the non-homogenous differential equations

$$t V_0'' + V_0' + \gamma^2 t V_0 = \frac{4\gamma}{\pi} J_1(\gamma t)$$

and

$$t^2 V_1'' + 3 t V_1' + \gamma^2 t^2 V_1 = \frac{4}{\pi}\left(1 - J_0(\gamma t) + J_2(\gamma t)\right) \tag{4.28}$$

where the dash represents differentiation with respect to $t$. Since we already have the expansions of the Bessel functions of the first kind we can now find the coefficients in the expansions

$$V_i(z) = \sum_{j=0}^{\infty}{}' \left(b_{i,2j}\right) T_{2j}(t) \qquad i = 0, 1 \tag{4.29}$$

using Clenshaw's method. The recurrence relations obtained are

$$b_{0,2r} = b_{0,2r+4} + \frac{2}{\pi}\left(a_{1,2r} - a_{1,2r+4}\right) - \frac{4(r+1)}{\gamma^2}\left(b'_{0,2r+3} + b'_{0,2r+1}\right) \qquad r = 0, 1 \ldots \tag{4.30}$$

and

$$b_{1,2r} = b_{1,2r+4} - b_{1,2r+2} - b_{1,2r} - \frac{16}{\gamma^2\pi}(a_{0,2r} - a_{0,2r+2})$$
$$+ \frac{1}{2}(a_{2,2r-2} + a_{2,2r} - a_{2,2r+2} - a_{2,2r+4})\pi^{-1}$$
$$- 4/\gamma^2 \left( (r+1)\, b'_{1,2r-1} + (2r+1)\, b'_{1,2r+1} + r\, b'_{1,2r+3} \right) \qquad r=0,1\ldots \qquad (4.31)$$

where

$a_{i,j}$ is the coefficient of $T_j(t)$ in the expansion for $J_i(z)$ given by (4.5).

As previously we have

$$b'_{i,2r+1} = 2(2r+2)\, b_{i,2r+2} + b'_{i,2r+3} \qquad i=0,1..$$

We now set the coefficients $b_{i,2k}$ to zero for $k > N$ and recur backwards using the known values of $a_{i,j}$. The solution obtained is a solution of the differential equation of the form

$$W_K(z) = V_K(z) + A\,k!\,\left(\frac{2}{z}\right)^k J_K(z)$$

The value of A can be found from the initial conditions

$$V_0(0) = \frac{2\chi}{\pi}, \quad V_1(0) = \frac{2\chi - 1}{\pi}$$

where

$$\chi = \text{Euler's Constant} = 0.57721\ldots$$

To get the value of the coefficients of $V_k(z)$ we can subtract the appropriate multiple of the coefficients for the Bessel function of the first kind from those obtained by recurrence.

The coefficients in the expansion (4.29) are tabulated in quadruple precision form for $i = 0$ and 1. The coefficients for $Y_0(z)$ and $Y_1(z)$ converge to zero at a similar rate to those for $J_0(z)$ and $J_1(z)$.

TABLE 4.1

Values of $|a_{2r}|$ for $J_1(z)$ with $\gamma = 8 \exp(i\phi)$, and upper bounds (4.17), (4.18) and (4.19).

| $\phi$ | r | $\|a_{2r}\|$ | (4.17) | (4.18) | (4.19) |
|---|---|---|---|---|---|
| 0 | 0 | 3.2(-1) | 2.0 | 6.0(+3) | 1.8(+7) |
| | 5 | 7.3(-3) | 2.4(-2) | 2.3(-1) | 1.0(-1) |
| | 10 | 7.4(-9) | 1.4(-8) | 5.5(-8) | 3.1(-8) |
| | 20 | 1.2(-26) | 1.8(-26) | 3.7(-26) | 2.6(-26) |
| | 30 | 8.2(-49) | 1.1(-48) | 1.7(-48) | 1.4(-48) |
| $\pi/24$ | 0 | 3.3(-1) | 5.7 | 6.0(+3) | 1.8(+7) |
| | 5 | 7.6(-3) | 6.7(-2) | 2.3(-1) | 1.0(-1) |
| | 10 | 7.6(-9) | 4.1(-8) | 5.5(-8) | 3.1(-8) |
| | 20 | 1.2(-26) | 5.9(-26) | 3.7(-26) | 2.6(-26) |
| | 30 | 8.2(-49) | 3.0(-48) | 1.7(-48) | 1.4(-48) |
| $11\frac{\pi}{24}$ | 0 | 6.1(+1) | 5.6(+3) | 6.0(+3) | 1.7(+7) |
| | 5 | 6.9(-2) | 6.6(+1) | 2.3(-1) | 1.0(-1) |
| | 10 | 2.7(-8) | 4.0(-5) | 5.5(-8) | 3.1(-8) |
| | 20 | 2.5(-26) | 4.9(-23) | 3.7(-26) | 2.6(-26) |
| | 30 | 1.4(-48) | 2.9(-45) | 1.7(-48) | 1.4(-48) |

TABLE 4.2

Values of $|a_r^{(1)}|$ for the Hankel function $H_0^1(z)$, and the estimates $E_1$ and $E_2$, obtained from (4.24) and (4.25) respectively.

| $\phi$ | r | $\|a_r^{(1)}\|$ | $E_1$ | $E_2$ |
|---|---|---|---|---|
| 7.5° | 20 | 3.39(-22) | 3.43(-22) | 3.85(-22) |
| | 30 | 5.11(-29) | 5.13(-29) | 5.49(-29) |
| | 40 | 4.49(-35) | 4.50(-35) | 4.72(-35) |
| 82.5° | 20 | 1.67(-23) | 1.71(-23) | 1.53(-23) |
| | 30 | 4.80(-31) | 4.88(-31) | 4.57(-31) |
| | 40 | 8.88(-38) | 8.99(-38) | 8.60(-38) |

TABLE 4.3

Values of $|a_r^{(2)}|$ for the Hankel function $H_0^2(z)$, and estimates $E_1$ and $E_2$, obtained from (4.24) and (4.25) respectively.

| $\phi$ | r | $\|a_r^{(2)}\|$ | $E_1$ | $E_2$ |
|---|---|---|---|---|
| 7.5° | 20 | 1.22(-21) | 1.21(-21) | 1.33(-21) |
| | 30 | 3.69(-28) | 3.64(-28) | 3.84(-28) |
| | 40 | 6.10(-34) | 6.07(-34) | 6.29(-34) |
| 82.5° | 20 | 1.24(-16) | 1.20(-16) | 1.08(-16) |
| | 30 | 8.79(-21) | 8.65(-21) | 8.14(-21) |
| | 40 | 1.84(-24) | 1.82(-24) | 1.75(-24) |

TABLE 4.4

Maximum absolute error in using (4.5) with the tabulated coefficients for $|z| \leq 8$

| arg z | n = 0 | n = 1 |
|---|---|---|
| 0°-15° | 0.5(-14) | 0.5(-14) |
| 15°-30° | 1.0(-14) | 1.0(-14) |
| 30°-45° | 0.5(-13) | 0.5(-13) |
| 45°-60° | 1.0(-13) | 2.0(-13) |
| 60°-75° | 0.5(-12) | 0.5(-12) |
| 75°-90° | 0.5(-12) | 0.5(-12) |

TABLE 4.5

Maximum relative error in using (4.11) with the tabulated coefficients for $|z| \geq 8$

| arg z | n = 0 | n = 1 |
|---|---|---|
| 0°-15° | 0.25(-14) | 0.25(-14) |
| 15°-30° | 0.25(-14) | 0.25(-14) |
| 30°-45° | 0.25(-14) | 0.25(-14) |
| 45°-60° | 0.5(-14) | 0.5(-14) |
| 60°-75° | 0.5(-14) | 0.5(-14) |
| 75°-90° | 0.1(-13) | 0.1(-13) |

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# APPENDIX A

## Tables of Coefficients for the Chebyshev expansions

A: For $|z| \leq 8$

$$J_n(z) = \frac{1}{n!}\left(\frac{z}{2}\right)^n \sum_{r=0}^{\infty}{}' a_{ar}\, T_{ar}(t)$$

with

$$t = \left(\frac{z}{8}\right) \mathrm{Exp}(-i\phi), \qquad \phi = 7.5°(15°)82.5°,$$

$n = 0, 1$.

B: For $|z| > 8$

$$J_n(z) = \tfrac{1}{2}[\, H_n^{(1)}(z) + H_n^{(2)}(z)]$$

$$Y_n(z) = \tfrac{1}{2}[\, H_n^{(1)}(z) - H_n^{(2)}(z)]$$

where

$$H_n^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \mathrm{Exp}\{i\alpha(z)\} \sum_{r=0}^{\infty}{}' a_r^{(1)}\, T_r^*(t)$$

and

$$H_n^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \mathrm{Exp}\{-i\alpha(z)\} \sum_{r=0}^{\infty}{}' a_r^{(2)}\, T_r^*(t)$$

with

$$\alpha(z) = z - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi$$

and

$$t = \left(\frac{8}{z}\right) \mathrm{Exp}(i\phi) \qquad \phi = 7.5°(15°)82.5°$$

$n = 0, 1$.

COEFFICIENTS FOR $J_0(Z)$ $\phi$ = 7.5

| r | | $a_{2r}$ |
|---|---|---|
| 0 | 0.40795869055667140+00 | -0.56442646929690940-01 |
| 1 | 0.76307829250131400-01 | -0.61302046672716610-01 |
| 2 | 0.29061006595818790+00 | -0.2091143556144969C+00 |
| 3 | -0.41389519311709360+00 | -0.49959512745050740-01 |
| 4 | 0.14405373906065260+00 | 0.91255766225367240-01 |
| 5 | -0.22191868433940320-01 | -0.29516142867063310-01 |
| 6 | 0.15821011678673100-02 | 0.47897681004629970-02 |
| 7 | -0.5923415745504D-05 | -0.47863710713778OD-03 |
| 8 | -0.93656954962600-05 | 0.3222570625865 4D-04 |
| 9 | 0.98007182750 8D-06 | -0.1527106203697D-05 |
| 10 | -0.58849005869D-07 | 0.51370384202D-07 |
| 11 | ·0.24841843117D-08 | -0.11659304865D-08 |
| 12 | -0.791666853D-10 | 0.130027227D-10 |
| 13 | 0.19713529D-11 | 0.2198956D-12 |
| 14 | -0.3898610D-13 | -0.1571964D-13 |
| 15 | 0.61288D-15 | 0.46989D-15 |
| 16 | -0.749D-17 | -0.991D-17 |
| 17 | 0.7D-19 | 0.16D-18 |

| r | | $a_r^{(1)}$ |
|---|---|---|
| 0 | 0.1997209327893000D+01 | -0.1520487127942132D-C1 |
| 1 | -0.15163747475273 6D-02 | -0.75491091447177OD-02 |
| 2 | -0.11839260114959D-03 | 0.56461738916240-04 |
| 3 | 0.27392266708 9D-05 | 0.29782675602 1D-05 |
| 4 | 0.8231549978D-07 | -0.15861416551D-06 |
| 5 | -0.1014119058 4D-07 | -0.937045841D-09 |
| 6 | 0.231287498D-09 | 0.662368054D-09 |
| 7 | 0.39197085D-10 | -0.40313221D-10 |
| 8 | -0.4908361D-11 | -0.1388649D-11 |
| 9 | 0.126666D-12 | 0.497639C-12 |
| 10 | 0.393276D-13 | -0.405715C-13 |
| 11 | -0.65915C-14 | -0.12920D-14 |
| 12 | 0.3370D-15 | 0.7854D-15 |
| 13 | 0.5995D-16 | -0.987D-16 |
| 14 | -0.1641D-16 | 0.154D-17 |
| 15 | 0.168D-17 | 0.181D-17 |
| 16 | 0.6D-19 | -0.38D-18 |
| 17 | -0.5D-19 | 0.3D-19 |
| 18 | 0.1D-19 | 0.0D+00 |

| r | | $a_r^{(2)}$ |
|---|---|---|
| 0 | 0.2001223542594185D+01 | 0.1561761594331604D-02 |
| 1 | 0.47484187103883D-03 | 0.782249853130666D-02 |
| 2 | -0.13720222605498D-03 | 0.9262723570900D-05 |
| 3 | -0.6574471362D-07 | -0.442096449887D-05 |
| 4 | 0.20479952939D-06 | 0.19221566758D-07 |
| 5 | -0.3117148217C-08 | 0.12009103702D-07 |
| 6 | -0.315134366D-09 | -0.406005409D-09 |
| 7 | 0.51121957D-10 | -0.58999420D-10 |
| 8 | 0.4067520D-11 | 0.6416037D-11 |
| 9 | -0.798944D-12 | 0.192784D-12 |
| 10 | 0.121137D-13 | -0.965576D-13 |
| 11 | 0.108301D-13 | 0.61567D-14 |
| 12 | -0.13761D-14 | 0.10111D-14 |
| 13 | -0.4638D-16 | -0.2443C-15 |
| 14 | 0.3671D-16 | 0.1065D-16 |
| 15 | -0.431D-17 | 0.441D-17 |
| 16 | -0.29D-18 | -0.100D-17 |
| 17 | 0.18D-18 | 0.4D-19 |
| 18 | -0.2D-19 | 0.2D-19 |

# COEFFICIENTS FOR $J_0(Z)$ $\phi$ = 22.5

| r | | $a_{2r}$ |
|---|---|---|
| 0 | 0.1862388136525494D+01 | 0.1648612285982524D+00 |
| 1 | 0.1453638662513095D+01 | -0.1254446663870422D+00 |
| 2 | 0.9954877760365573D+00 | -0.1034422910850202D+01 |
| 3 | -0.6836306791980399D+00 | -0.5932359624072628D+00 |
| 4 | -0.9210399403143698D-01 | 0.2761514578005225D+00 |
| 5 | 0.5432515528670650D-01 | -0.1247638183934976D-01 |
| 6 | -0.5312699741457737D-02 | -0.4654474585672789D-02 |
| 7 | -0.31338728717416D-04 | 0.636229272782623D-03 |
| 8 | 0.34287393291279D-04 | -0.26013530174550D-04 |
| 9 | -0.2215242645225D-05 | -0.459929886723D-06 |
| 10 | 0.477911496614D-07 | 0.824014419477D-07 |
| 11 | 0.9377246843D-09 | -0.31346016043D-08 |
| 12 | -0.855594194D-10 | 0.404926857D-10 |
| 13 | 0.21497692D-11 | 0.8482018D-12 |
| 14 | -0.1304485D-13 | -0.4496504D-13 |
| 15 | -0.36612D-15 | 0.80228D-15 |
| 16 | 0.1328D-16 | -0.464D-17 |
| 17 | -0.18D-18 | -0.9D-19 |

| r | | $a_r^{(1)}$ |
|---|---|---|
| 0 | 0.1993513105429556D+01 | -0.1383201838361587D-01 |
| 1 | -0.331819644425562D-02 | -0.681426224169510D-02 |
| 2 | -0.7344144303041D-04 | 0.10221388905934D-03 |
| 3 | 0.373712009005D-05 | 0.31890016651D-06 |
| 4 | -0.7508276339D-07 | -0.14026220050D-06 |
| 5 | -0.3398019934D-08 | 0.7710909403D-08 |
| 6 | 0.557125722D-09 | -0.98096302D-10 |
| 7 | -0.32126299D-10 | -0.28528229D-10 |
| 8 | -0.112145D-12 | 0.3684397D-11 |
| 9 | 0.270613D-12 | -0.222595D-12 |
| 10 | -0.360021D-13 | -0.51698D-14 |
| 11 | 0.23566D-14 | 0.33255D-14 |
| 12 | 0.862D-16 | -0.4810D-15 |
| 13 | -0.508D-16 | 0.359D-16 |
| 14 | 0.827D-17 | 0.122D-17 |
| 15 | -0.74D-18 | -0.92D-18 |
| 16 | -0.1D-19 | 0.17D-18 |
| 17 | 0.2D-19 | -0.2D-19 |
| 18 | -0.4D-20 | 0.0D+00 |

| r | | $a_r^{(2)}$ |
|---|---|---|
| 0 | 0.2005317773826089D+01 | 0.1496870848194241D-01 |
| 1 | 0.254232929699090D-02 | 0.756832055889608D-02 |
| 2 | -0.12017042742964D-03 | 0.8074522795418D-04 |
| 3 | -0.348285851733D-05 | -0.342209864342D-05 |
| 4 | 0.14370960389D-06 | -0.19381878407D-06 |
| 5 | 0.13221804724D-07 | 0.8303710072D-08 |
| 6 | -0.638103037D-09 | 0.1061644026D-08 |
| 7 | -0.96834157D-10 | -0.62761881D-10 |
| 8 | 0.7491277D-11 | -0.9690533D-11 |
| 9 | 0.1023817D-11 | 0.1025303D-11 |
| 10 | -0.1533926D-12 | 0.1084678D-12 |
| 11 | -0.104736D-13 | -0.241857D-13 |
| 12 | 0.39048D-14 | -0.6596D-15 |
| 13 | -0.6666D-16 | 0.6285D-15 |
| 14 | -0.9772D-16 | -0.4222D-16 |
| 15 | 0.1264D-16 | -0.1393D-16 |
| 16 | 0.160D-17 | 0.308D-17 |
| 17 | -0.66D-18 | 0.7D-19 |
| 18 | 0.39D-19 | -0.127D-18 |
| 19 | 0.21D-19 | 0.18D-19 |
| 20 | -0.5D-20 | 0.2D-20 |
| 21 | 0.0D+00 | -0.1D-20 |

COEFFICIENTS FOR $J_0(Z)$ $\phi = 37.5$

| $r$ | $a_{2r}$ | |
|---|---|---|
| 0 | 0.67119273258907000D+01 | 0.83712702818187070D+01 |
| 1 | 0.71500895090149270D+01 | 0.59935320789535870D+01 |
| 2 | 0.62159845055422610D+01 | -0.19486400634296970D+00 |
| 3 | 0.91766949812533030D+00 | -0.24065140778222980D+01 |
| 4 | -0.56777356598961930D+00 | -0.31719376085575830D+00 |
| 5 | -0.52984967577101030D-01 | 0.92811127584290710D-01 |
| 6 | 0.10994998091648050D-01 | 0.53006984777873090D-02 |
| 7 | 0.33155441080985570D-03 | -0.96600009434642700D-03 |
| 8 | -0.64158538295280D-04 | -0.11644140366370D-04 |
| 9 | -0.19262278486D-07 | 0.32792409067350D-05 |
| 10 | 0.130984600823D-06 | -0.24389700433D-07 |
| 11 | -0.16816077796D-08 | -0.41353051819D-08 |
| 12 | -0.10371152882D-09 | 0.70597257860D-10 |
| 13 | 0.218593469D-11 | 0.205450364D-11 |
| 14 | 0.31222455D-13 | -0.53302500D-13 |
| 15 | -0.1057308D-14 | -0.327813D-15 |
| 16 | -0.1176D-17 | 0.17379D-16 |
| 17 | 0.239D-18 | -0.41D-19 |

| $r$ | $a_r^{(1)}$ | |
|---|---|---|
| 0 | 0.1990345351455046D+01 | -0.1163923411861265D-01 |
| 1 | -0.4845909318989980D-02 | -0.5696876791120940D-02 |
| 2 | -0.155174522924 9D-04 | 0.1206468529061000D-03 |
| 3 | 0.28191128922 5D-05 | -0.2117642443590D-05 |
| 4 | -0.14360704559D-06 | -0.17950366700D-07 |
| 5 | 0.4906623595D-08 | 0.5773310055D-08 |
| 6 | 0.6628038D-11 | -0.4767936730D-09 |
| 7 | -0.22669934D-10 | 0.263230620D-10 |
| 8 | 0.2807401D-11 | -0.516505D-12 |
| 9 | -0.230439D-12 | -0.119535D-12 |
| 10 | 0.111511D-13 | 0.232199D-13 |
| 11 | 0.4999D-15 | -0.27122D-14 |
| 12 | -0.2263D-15 | 0.2203D-15 |
| 13 | 0.379D-16 | -0.641D-17 |
| 14 | -0.449D-17 | -0.203D-17 |
| 15 | 0.36D-18 | 0.56D-18 |
| 16 | 0.0D+00 | -0.9D-19 |
| 17 | -0.1D-19 | 0.1D-19 |
| 18 | 0.2D-20 | -0.1D-20 |

| $r$ | $a_r^{(2)}$ | |
|---|---|---|
| 0 | 0.2009203666687719D+01 | 0.1321490775411509D-01 |
| 1 | 0.4541978994061490D-02 | 0.6745117980415220D-02 |
| 2 | -0.65149863654560D-04 | 0.13806779265349D-03 |
| 3 | -0.54281129349 3D-05 | 0.15640526452D-06 |
| 4 | -0.11938798791D-06 | -0.26193682934D-06 |
| 5 | 0.14755561664D-07 | -0.13890379393D-07 |
| 6 | 0.1500431342D-08 | 0.935095139D-09 |
| 7 | -0.65065841D-10 | 0.170509568D-09 |
| 8 | -0.20990882D-10 | -0.4971750D-11 |
| 9 | 0.4439555D-12 | -0.2815298D-11 |
| 10 | 0.4098006D-12 | 0.540714D-13 |
| 11 | -0.98505D-14 | 0.641445D-13 |
| 12 | -0.106568D-13 | -0.23368D-14 |
| 13 | 0.60373D-15 | -0.18494D-14 |
| 14 | 0.32894D-15 | 0.16112D-15 |
| 15 | -0.4228D-16 | 0.5855D-16 |
| 16 | -0.10070D-16 | -0.1091D-16 |
| 17 | 0.276D-17 | -0.157D-17 |
| 18 | 0.183D-18 | 0.678D-18 |
| 19 | -0.160D-18 | -0.2D-20 |
| 20 | 0.12D-19 | -0.36D-19 |
| 21 | 0.7D-20 | 0.5D-20 |
| 22 | -0.2D-20 | 0.1D-20 |

COEFFICIENTS FOR $J_0(Z)$ $\phi$ = 52.5

| $r$ | $a_{2r}$ | | |
|---|---|---|---|
| 0 | -0.246156076800721 0D+02 | , | 0.407081845197416 8D+02 |
| 1 | -0.136746804332633 8D+02 | , | 0.367508686017841 2D+02 |
| 2 | 0.420503239865833 9D+01 | , | 0.203677396915274 3D+02 |
| 3 | 0.586227597664479 1D+01 | , | 0.355483613819136 0D+01 |
| 4 | 0.132040346349086 4D+01 | , | -0.615534617425367 6D+00 |
| 5 | 0.120113620494861 2D-01 | , | -0.210714857647863 5D+00 |
| 6 | -0.193724906630377 1D-01 | , | -0.103180731906572 4D-01 |
| 7 | -0.131937658867522 3D-02 | , | 0.108686857430741 D-02 |
| 8 | 0.328820941236510 D-04 | , | 0.977692403701930 D-04 |
| 9 | 0.495612000248830 D-05 | , | 0.100890716327 0D-06 |
| 10 | 0.666210818440 D-07 | , | -0.182232882643 0D-06 |
| 11 | -0.494119407830 D-08 | , | -0.391067555900 D-08 |
| 12 | -0.143231305600 D-09 | , | 0.960901744 0D-10 |
| 13 | 0.114262290 D-11 | , | 0.386637610 D-11 |
| 14 | 0.813990500 D-13 | , | 0.87296 0D-15 |
| 15 | 0.42964 0D-15 | , | -0.136794 0D-14 |
| 16 | -0.1841 0D-16 | , | -0.1244 0D-16 |
| 17 | -0.236 0D-18 | , | 0.194 0D-18 |

| $r$ | $a_r^{(1)}$ | | |
|---|---|---|---|
| 0 | 0.198784112439379 1D+01 | , | -0.879111917261575 0D-02 |
| 1 | -0.603832318930778 0D-02 | , | -0.428132348205218 0D-02 |
| 2 | 0.419038436955 2D-04 | , | 0.110943187326070 D-03 |
| 3 | 0.690039947840 D-06 | , | -0.329275575214 0D-05 |
| 4 | -0.921764601900 D-07 | , | 0.983130586500 D-07 |
| 5 | 0.659074664300 D-08 | , | -0.196222507800 D-08 |
| 6 | -0.409571982 0D-09 | , | -0.921720090 D-10 |
| 7 | 0.223506830 D-10 | , | 0.195075790 D-10 |
| 8 | -0.955058 0D-12 | , | -0.220170 3D-11 |
| 9 | -0.207947 0D-13 | , | 0.206915 0D-12 |
| 10 | 0.105995 0D-13 | , | -0.169201 0D-13 |
| 11 | -0.17497 0D-14 | , | 0.110150 D-14 |
| 12 | 0.2272 0D-15 | , | -0.279 0D-16 |
| 13 | -0.256 0D-16 | , | -0.819 0D-17 |
| 14 | 0.248 0D-17 | , | 0.223 0D-17 |
| 15 | -0.18 0D-18 | , | -0.40 0D-18 |
| 16 | 0.0D+00 | , | 0.6 0D-19 |

| $r$ | $a_r^{(2)}$ | | |
|---|---|---|---|
| 0 | 0.201255826617313 1D+01 | , | 0.103962919011509 0D-01 |
| 1 | 0.630090516560386 0D-02 | , | 0.535312537605316 0D-02 |
| 2 | 0.181885576072 0D-04 | , | 0.159575311521380 D-03 |
| 3 | -0.391503313628 0D-05 | , | 0.460723587925 0D-05 |
| 4 | -0.347060124640 D-06 | , | -0.907773644100 D-08 |
| 5 | -0.146007728380 D-07 | , | -0.227517457000 D-07 |
| 6 | 0.124144321500 D-08 | , | -0.234291961800 D-08 |
| 7 | 0.310669679 0D-09 | , | 0.173963280 D-10 |
| 8 | 0.123824040 D-10 | , | 0.402265350 D-10 |
| 9 | -0.533118 2D-11 | , | 0.351320 4D-11 |
| 10 | -0.765457 5D-12 | , | -0.739264 3D-12 |
| 11 | 0.109230 8D-12 | , | -0.157782 9D-12 |
| 12 | 0.325606 0D-13 | , | 0.175800 0D-13 |
| 13 | -0.316230 D-14 | , | 0.686631 D-14 |
| 14 | -0.148926 D-14 | , | -0.64858 0D-15 |
| 15 | 0.15171 0D-15 | , | -0.33195 0D-15 |
| 16 | 0.7562 0D-16 | , | 0.3957 0D-16 |
| 17 | -0.1113 0D-16 | , | 0.1744 0D-16 |
| 18 | -0.401 0D-17 | , | -0.327 0D-17 |
| 19 | 0.984 0D-18 | , | -0.904 0D-18 |
| 20 | 0.192 0D-18 | , | 0.298 0D-18 |
| 21 | -0.89 0D-19 | , | 0.36 0D-19 |
| 22 | -0.43 0D-20 | , | -0.26 0D-19 |
| 23 | 0.76 0D-20 | , | 0.6 0D-21 |
| 24 | -0.7 0D-21 | , | 0.21 0D-20 |
| 25 | -0.5 0D-21 | , | -0.4 0D-21 |
| 26 | 0.2 0D-21 | , | -0.1 0D-21 |

COEFFICIENTS FOR $J_0(Z)$ $\phi = 67.5$

| $r$ | $a_{2r}$ | |
|---|---|---|
| 0 | -0.120232340040035 8D+03 | -0.66691018233263760+02 |
| 1 | -0.99170227887441290+02 | -0.39147826531356590+02 |
| 2 | -0.49482551698438320+02 | -0.14800978678558560+01 |
| 3 | -0.12834635332439320+02 | 0.63351619838680030+01 |
| 4 | -0.13924145634822310+01 | 0.23497334840795790+01 |
| 5 | 0.35873442631228920-01 | 0.36184555175082580+00 |
| 6 | 0.24623296955251350-01 | 0.25407683584997800-01 |
| 7 | 0.2598902940404090-02 | 0.2732861489688850-03 |
| 8 | 0.123854742046580-03 | -0.864737060665 3D-04 |
| 9 | 0.16233818774300-05 | -0.68164033686270-05 |
| 10 | -0.1284318859020-06 | -0.233223932872D-06 |
| 11 | -0.79995001110-08 | -0.26718216600-08 |
| 12 | -0.20551599040-09 | 0.93883769 1D-10 |
| 13 | -0.195512950-11 | 0.480153630-11 |
| 14 | 0.3750730-13 | 0.9592080-13 |
| 15 | 0.162150-14 | 0.75440-15 |
| 16 | 0.2590-16 | -0.890-17 |

| $r$ | $a_r^{(1)}$ | |
|---|---|---|
| 0 | 0.19861136347907940+01 | -0.54669931181814 7D-02 |
| 1 | -0.68543619874731 2D-02 | -0.26533949572118 2D-02 |
| 2 | 0.37216072860410-04 | 0.77118451024820-04 |
| 3 | -0.15826142644 0D-05 | -0.28499847995 8D-05 |
| 4 | 0.22986899400-07 | 0.12642389127D-06 |
| 5 | 0.87193215 6D-09 | -0.63725645 70D-08 |
| 6 | -0.16498739 3D-09 | 0.34872554 5D-09 |
| 7 | 0.17998231D-10 | -0.19751442D-10 |
| 8 | -0.1784492D-11 | 0.1076218D-11 |
| 9 | 0.173556D-12 | -0.46317D-13 |
| 10 | -0.169763D-13 | -0.2451D-15 |
| 11 | 0.16416D-14 | 0.4788D-15 |
| 12 | -0.1551D-15 | -0.964D-16 |
| 13 | 0.147D-16 | 0.154D-16 |
| 14 | -0.126D-17 | -0.225D-17 |
| 15 | 0.9D-19 | 0.32D-18 |
| 16 | -0.0D+00 | -0.4D-19 |

| $r$ | $a_r^{(2)}$ | |
|---|---|---|
| 0 | 0.20150499312310760+01 | 0.66635309632220 6D-02 |
| 1 | 0.76297366863711 3D-02 | 0.34553257558162 4D-02 |
| 2 | 0.10605258903314D-03 | 0.12976477119246D-03 |
| 3 | 0.11428837593 7D-05 | 0.65697840758 6D-05 |
| 4 | -0.16742326288D-06 | 0.38315240024D-06 |
| 5 | -0.32534134114D-07 | 0.17113269862D-07 |
| 6 | -0.40038573 80D-08 | -0.12110868 98D-08 |
| 7 | -0.22171255 9D-09 | -0.53541494 9D-09 |
| 8 | 0.52177757D-10 | -0.77367965D-10 |
| 9 | 0.15601458D-10 | 0.16985555D-11 |
| 10 | 0.10220395D-11 | 0.31926549D-11 |
| 11 | -0.5720293D-12 | 0.4330504D-12 |
| 12 | -0.1302844D-12 | -0.987470D-13 |
| 13 | 0.164144D-13 | -0.356915D-13 |
| 14 | 0.954190D-14 | 0.258140D-14 |
| 15 | -0.36883D-15 | 0.256325D-14 |
| 16 | -0.70180D-15 | -0.4415D-16 |
| 17 | 0.399D-17 | -0.19714D-15 |
| 18 | 0.5692D-16 | 0.685D-18 |
| 19 | -0.597D-18 | 0.16362D-16 |
| 20 | -0.5105D-17 | -0.450D-18 |
| 21 | 0.266D-18 | -0.1570D-17 |
| 22 | 0.486D-18 | 0.136D-18 |
| 23 | -0.638D-19 | 0.150D-18 |
| 24 | -0.458D-19 | -0.282D-19 |
| 25 | 0.119D-19 | -0.135D-19 |
| 26 | 0.37D-20 | 0.49D-20 |
| 27 | -0.19D-20 | 0.91D-21 |
| 28 | -0.16D-21 | -0.72D-21 |
| 29 | 0.26D-21 | 0.1D-22 |
| 30 | -0.3D-22 | 0.9D-22 |
| 31 | -0.3D-22 | -0.2D-22 |
| 32 | 0.1D-22 | -0.1D-22 |

COEFFICIENTS FOR $J_0(Z)$ $\phi$ = 82.5

| $r$ | $a_{2r}$ | |
|---|---|---|
| 0 | 0.1478633007790340D+03 | -0.1868581252354210D+03 |
| 1 | 0.1041576820539821D+03 | -0.1449343537676332D+03 |
| 2 | 0.3777371602900483D+02 | -0.6810072634367687D+02 |
| 3 | 0.7151068457825564D+01 | -0.1994217104163069D+02 |
| 4 | 0.5871894746374649D+00 | -0.3794599128627371D+01 |
| 5 | -0.2733681629252401D-01 | -0.4894460017760235D+00 |
| 6 | -0.1263329838543830D-01 | -0.4434411685005576D-01 |
| 7 | -0.161111416812855D-02 | -0.289444838751854D-02 |
| 8 | -0.12682368313316D-03 | -0.13765284525622D-03 |
| 9 | -0.7101047357656D-05 | -0.4705227601243D-05 |
| 10 | -0.300244837003D-06 | -0.1064111996390D-06 |
| 11 | -0.9908641501D-08 | -0.9276122284D-09 |
| 12 | -0.260274841 2D-09 | 0.409400746D-10 |
| 13 | -0.54945502D-11 | 0.23689420D-11 |
| 14 | -0.930979D-13 | 0.721757D-13 |
| 15 | -0.12406D-14 | 0.16058D-14 |
| 16 | -0.121D-16 | 0.284D-16 |

| $r$ | $a_r^{(0)}$ | |
|---|---|---|
| 0 | 0.1985232771500772D+01 | -0.185450999967658D-02 |
| 1 | -0.726843639817799D-02 | -0.89853335704178D-03 |
| 2 | 0.11205048918879D-03 | 0.2754903884657D-04 |
| 3 | -0.300955145505D-05 | -0.111295795257D-05 |
| 4 | 0.11225446153D-06 | 0.5604713539D-07 |
| 5 | -0.5246839364D-08 | -0.3349880947D-08 |
| 6 | 0.289389972D-09 | 0.229698742D-09 |
| 7 | -0.13228529D-10 | -0.17639901D-10 |
| 8 | 0.1269652D-11 | 0.1490587D-11 |
| 9 | -0.95903D-13 | -0.136739D-12 |
| 10 | 0.77137D-14 | 0.134747D-13 |
| 11 | -0.6490D-15 | -0.14144D-14 |
| 12 | 0.560D-16 | 0.1570D-15 |
| 13 | -0.48D-17 | -0.193D-16 |
| 14 | 0.39D-19 | 0.22D-17 |
| 15 | -0.3D-19 | -0.29D-18 |
| 16 | 0.0D+00 | 0.4D-19 |

| $r$ | $a_r^{(2)}$ | |
|---|---|---|
| 0 | 0.2016386488605358D+01 | 0.229775537603711D-02 |
| 1 | 0.835169553995907D-02 | 0.119637438016248D-02 |
| 2 | 0.16445031237631D-03 | 0.5050430885251D-04 |
| 3 | 0.634654576342D-05 | 0.328838920541D-05 |
| 4 | 0.37235206362D-06 | 0.31746053136D-06 |
| 5 | 0.25928705699D-07 | 0.42651973225D-07 |
| 6 | 0.639050288D-09 | 0.6927994043D-08 |
| 7 | -0.600617041D-09 | 0.1056296040D-08 |
| 8 | -0.237168859D-09 | 0.72957770D-10 |
| 9 | -0.47230421D-10 | -0.31181667D-10 |
| 10 | 0.181992D-13 | -0.140132703D-10 |
| 11 | 0.32628649D-11 | -0.17443959D-11 |
| 12 | 0.8104444D-12 | 0.6346405D-12 |
| 13 | -0.9278265D-13 | 0.2845294D-12 |
| 14 | -0.9034806D-13 | -0.21917D-14 |
| 15 | -0.602460D-14 | -0.2756387D-13 |
| 16 | 0.832652D-14 | -0.360423D-14 |
| 17 | 0.159220D-14 | 0.253840D-14 |
| 18 | -0.79206D-15 | 0.63277D-15 |
| 19 | -0.239865D-15 | -0.255793D-15 |
| 20 | 0.86190D-16 | -0.88850D-16 |
| 21 | 0.32508D-16 | 0.30429D-16 |
| 22 | -0.11253D-16 | 0.11793D-16 |
| 23 | -0.42381D-17 | -0.43386D-17 |
| 24 | 0.17310D-17 | -0.15017D-17 |
| 25 | 0.5196D-18 | 0.7082D-18 |
| 26 | -0.2944D-18 | 0.1726D-18 |
| 27 | -0.532D-19 | -0.1233D-18 |
| 28 | 0.5163D-19 | -0.1406D-19 |
| 29 | 0.229D-20 | 0.2144D-19 |
| 30 | -0.876D-20 | -0.62D-21 |
| 31 | 0.94D-21 | -0.349D-20 |
| 32 | 0.134D-20 | 0.69D-21 |

COEFFICIENTS FOR $J_1(Z)$ $\phi$ = 7.5

| r | | $a_{2r}$ |
|---|---|---|
| 0 | 0.3316508613065400D+00 , | 0.4859399743025670D-02 |
| 1 | -0.2802067447370602D+00 , | 0.8134684829761730D-01 |
| 2 | 0.3461112527660639D+00 , | -0.1488138765345434D-01 |
| 3 | -0.1669142496114551D+00 , | -0.5725533995454156D-01 |
| 4 | 0.3656935542269132D-01 , | 0.2926869361591609D-01 |
| 5 | -0.415029240471591D-02 , | -0.639288974678747D-02 |
| 6 | 0.225556965297570D-03 , | 0.810285502639704D-03 |
| 7 | 0.15209710883040D-05 , | -0.67265551812858D-04 |
| 8 | -0.12464984222240D-05 , | 0.3894514443776D-05 |
| 9 | 0.1090898095490D-06 , | -0.1622271914540D-06 |
| 10 | -0.5775251345500D-08 , | 0.48661474047D-08 |
| 11 | 0.219457462000D-09 , | -0.99130676380D-10 |
| 12 | -0.637623153D-11 , | 0.96680976D-12 |
| 13 | 0.14608098D-12 , | 0.1785835D-13 |
| 14 | -0.267651D-14 , | -0.110792D-14 |
| 15 | 0.39193D-16 , | 0.30552D-16 |
| 16 | -0.448D-18 , | -0.601D-18 |
| 17 | 0.4D-20 , | 0.9D-20 |

| r | | $a_r^{(1)}$ |
|---|---|---|
| 0 | 0.2007383366905616D+01 , | 0.4601490703152938D-01 |
| 1 | 0.3896804586714230D-02 , | 0.2292296248522280D-01 |
| 2 | 0.2014994918627 2D-03 , | -0.8900405356368D-04 |
| 3 | -0.3747379028520D-05 , | -0.4312553098740D-05 |
| 4 | -0.1130522280 1D-06 , | 0.2027990812 0D-06 |
| 5 | 0.1243152947 6D-07 , | 0.148793943 3D-03 |
| 6 | -0.255755564 0D-09 , | -0.797692249 0D-09 |
| 7 | -0.468658891 0D-10 , | 0.45860127D-10 |
| 8 | 0.5583867D-11 , | 0.1726258D-11 |
| 9 | -0.129957D-12 , | -0.565616D-12 |
| 10 | -0.449315D-13 , | 0.443428D-13 |
| 11 | 0.72691D-14 , | 0.15922D-14 |
| 12 | -0.3519D-15 , | -0.8713D-15 |
| 13 | -0.6775D-16 , | 0.1065D-15 |
| 14 | 0.1786D-16 , | -0.131D-17 |
| 15 | -0.178D-17 , | -0.199D-17 |
| 16 | -0.7D-19 , | 0.41D-18 |
| 17 | 0.6D-19 , | -0.3D-19 |
| 18 | -0.1D-19 , | 0.0D+00 |

| r | | $a_r^{(2)}$ |
|---|---|---|
| 0 | 0.1995237786481450D+01 , | -0.4670794665173130D-01 |
| 1 | -0.2153532661257880D-02 , | -0.2338268704211800D-01 |
| 2 | 0.228122745261270D-03 , | -0.2251934450293D-04 |
| 3 | 0.2816597227 5D-06 , | 0.6192748306950D-05 |
| 4 | -0.2646313693 5D-06 , | -0.1693788999 6D-07 |
| 5 | 0.338892316 9D-08 , | -0.1483734583 4D-07 |
| 6 | 0.981138453D-09 , | 0.454002406D-09 |
| 7 | -0.57357445D-10 , | 0.70069087D-10 |
| 8 | -0.4835022D-11 , | -0.7185141D-11 |
| 9 | 0.892607D-12 , | -0.2406758D-12 |
| 10 | -0.106927D-13 , | 0.1077711D-12 |
| 11 | -0.121157D-13 , | -0.64884D-14 |
| 12 | 0.14794D-14 , | -0.11442D-14 |
| 13 | 0.5686D-16 , | 0.2646D-15 |
| 14 | -0.4001D-16 , | -0.1060D-16 |
| 15 | 0.454D-17 , | -0.486D-17 |
| 16 | 0.34D-18 , | 0.107D-17 |
| 17 | -0.19D-18 , | -0.4D-19 |
| 18 | 0.2D-19 , | -0.3D-19 |

COEFFICIENTS FOR $J_1(Z)$ $\phi = 22.5$

| $r$ | | | $a_{2r}$ |
|---|---|---|---|
| 0 | 0.40374947400739900+00 | , | 0.29030589498529460+00 |
| 1 | -0.8813810817561304D-01 | , | 0.4743506827181057D+00 |
| 2 | 0.6394271028327639D+00 | , | 0.1027687902694894D-01 |
| 3 | -0.1195814266318472D+00 | , | -0.2610087927618932D+00 |
| 4 | -0.3773369287973807D-01 | , | 0.5821591694659590D-01 |
| 5 | 0.1055985377013672D-01 | , | -0.61293627327682OD-03 |
| 6 | -0.744253518531214D-03 | , | -0.828857922803210D-03 |
| 7 | -0.11931063836971D-04 | , | 0.85216933527843D-04 |
| 8 | 0.426034124101O4D-05 | , | -0.27846664665102D-05 |
| 9 | -0.2300544040736D-06 | , | -0.61450261154 3D-07 |
| 10 | 0.41678050944D-08 | , | 0.80190314303D-08 |
| 11 | 0.95756893910D-10 | , | -0.26733021810D-09 |
| 12 | -0.68907840 2D-11 | , | 0.2988029750-11 |
| 13 | 0.15591706D-12 | , | 0.6667477D-13 |
| 14 | -0.115575D-14 | , | -0.309846D-14 |
| 15 | -0.24464D-16 | , | 0.50679D-16 |
| 16 | 0.801D-18 | , | -0.263D-18 |
| 17 | -0.99D-20 | , | -0.52D-20 |

| $r$ | | | $a_r^{(1)}$ |
|---|---|---|---|
| 0 | 0.2018796704796675D+01 | , | 0.4230504848413659D-01 |
| 1 | 0.9532265270458230-02 | , | 0.2098377636981585D-01 |
| 2 | 0.1285296782799 8D-03 | , | -0.169526262381710-03 |
| 3 | -0.52953457340 2D-05 | , | -0.58359414058D-06 |
| 4 | 0.9360748744D-07 | , | 0.18527832676D-06 |
| 5 | 0.5052240130D-08 | , | -0.9453333411D-08 |
| 6 | -0.670794303D-09 | , | 0.103004349D-09 |
| 7 | 0.36916323D-10 | , | 0.34204738D-10 |
| 8 | 0.2155570-12 | , | -0.4235536D-11 |
| 9 | -0.311964D-12 | , | 0.246442D-12 |
| 10 | 0.402750D-13 | , | 0.65719D-14 |
| 11 | -0.25509D-14 | , | -0.37450D-14 |
| 12 | -0.1047D-15 | , | 0.5289D-15 |
| 13 | 0.563D-16 | , | -0.384D-16 |
| 14 | -0.899D-17 | , | -0.148D-17 |
| 15 | 0.73D-18 | , | 0.101D-17 |
| 16 | 0.1D-19 | , | -0.19D-18 |
| 17 | -0.2D-19 | , | 0.2D-19 |

| $r$ | | | $a_r^{(2)}$ |
|---|---|---|---|
| 0 | 0.1983144046239353D+01 | , | -0.4421028253377329D-01 |
| 1 | -0.8238685627909 04D-02 | , | -0.2224910385820393D-01 |
| 2 | 0.1944418038450 5D-03 | , | -0.139618700469370-03 |
| 3 | 0.49915740285 3D-05 | , | 0.46057792649OD-05 |
| 4 | -0.17598086096D-06 | , | 0.25361396310 7D-06 |
| 5 | -0.16410604625D-07 | , | -0.9605643249D-08 |
| 6 | 0.71344023 6D-09 | , | -0.1274469679D-08 |
| 7 | 0.113756062D-09 | , | 0.69913239D-10 |
| 8 | -0.8157977D-11 | , | 0.11230241D-10 |
| 9 | -0.1178261D-11 | , | -0.1112879D-11 |
| 10 | 0.1663028D-12 | , | -0.1249904D-12 |
| 11 | 0.122363D-13 | , | 0.262171D-13 |
| 12 | -0.42352D-14 | , | 0.8353D-15 |
| 13 | 0.5397D-16 | , | -0.6828D-15 |
| 14 | 0.10651D-15 | , | 0.4275D-16 |
| 15 | -0.1318D-16 | , | 0.1530D-16 |
| 16 | -0.180D-17 | , | -0.325D-17 |
| 17 | 0.70D-18 | , | -0.9D-19 |
| 18 | -0.4D-19 | , | 0.14D-19 |
| 19 | -0.2D-19 | , | -0.2D-19 |
| 20 | 0.6D-20 | , | -0.3D-20 |
| 21 | 0.0D+00 | , | 0.1D-20 |

COEFFICIENTS FOR $J_1$ (Z) $\phi$ = 37.5

| r | $a_{2r}$ | |
|---|---|---|
| 0 | -0.4381621831242275D+00 | , 0.2377738202865119D+01 |
| 1 | -0.2688522523108127D+00 | , 0.2574730924661963D+01 |
| 2 | 0.1471809508094292D+01 | , 0.1038934235972632D+01 |
| 3 | 0.4414432465670281D+00 | , -0.4525763182192840D+00 |
| 4 | -0.9344331075105422D-01 | , -0.9300501469646787D-01 |
| 5 | -0.1189301113931178D-01 | , 0.1375504626057259D-01 |
| 6 | 0.14756222334243 2D-02 | , 0.99603030861356 8D-03 |
| 7 | 0.5567967447799 5D-04 | , -0.1175855980268 73D-03 |
| 8 | -0.710349238840 4D-05 | , -0.1953024252153D-05 |
| 9 | -0.2598056514500D-07 | , 0.3317296245340D-06 |
| 10 | 0.1217319690450D-07 | , -0.1518404241 3D-08 |
| 11 | -0.1238957347D-09 | , -0.3551948597D-09 |
| 12 | -0.828613038D-11 | , 0.513053423D-11 |
| 13 | 0.15185262D-12 | , 0.15381743D-12 |
| 14 | 0.221351D-14 | , -0.351061D-14 |
| 15 | -0.65918D-16 | , -0.22537D-16 |
| 16 | -0.960D-19 | , 0.1027D-17 |
| 17 | 0.13D-19 | , -0.2D-20 |

| r | $a_r^{(1)}$ | |
|---|---|---|
| 0 | 0.2028791098155013D+01 | , 0.3591612354884225D-01 |
| 1 | 0.1443051867298572D-01 | , 0.1775074431952528D-01 |
| 2 | 0.306922281723 4D-04 | , -0.2044110009761 5D-03 |
| 3 | -0.409361250226D-05 | , 0.294099294513D-05 |
| 4 | 0.187724620520D-06 | , 0.2713166071D-07 |
| 5 | -0.5979962792D-08 | , -0.7302555923D-08 |
| 6 | -0.13039065D-10 | , 0.5739479310D-09 |
| 7 | 0.27168204D-10 | , -0.30487297D-10 |
| 8 | -0.3250009D-11 | , 0.543316D-12 |
| 9 | 0.260094D-12 | , 0.140155D-12 |
| 10 | -0.121490D-13 | , -0.263161D-13 |
| 11 | -0.6018D-15 | , 0.30150D-14 |
| 12 | 0.2537D-15 | , -0.2401D-15 |
| 13 | -0.417D-16 | , 0.647D-17 |
| 14 | 0.487D-17 | , 0.228D-17 |
| 15 | -0.39D-18 | , -0.61D-18 |
| 16 | 0.0D+00 | , 0.10D-18 |
| 17 | 0.0D+00 | , -0.1D-19 |

| r | $a_r^{(2)}$ | |
|---|---|---|
| 0 | 0.1971951375310728D+01 | , -0.3854849627726323D-01 |
| 1 | -0.1393142345701756D-01 | , -0.1950284067059785D-01 |
| 2 | 0.1001872131593 5D-03 | , -0.2293433297216 6D-03 |
| 3 | 0.747546335277D-05 | , -0.44157148929D-06 |
| 4 | 0.16168774400D-06 | , 0.32739760003D-06 |
| 5 | -0.17283851982D-07 | , 0.17332879857D-07 |
| 6 | -0.178932732 4D-08 | , -0.1037522455D-08 |
| 7 | 0.69189938D-10 | , -0.197340946D-09 |
| 8 | 0.23787157D-10 | , 0.4845001D-11 |
| 9 | -0.3952418D-12 | , 0.3142357D-11 |
| 10 | -0.4524883D-12 | , -0.455607D-13 |
| 11 | 0.86229D-14 | , -0.702992D-13 |
| 12 | 0.116235D-13 | , 0.21883D-14 |
| 13 | -0.59772D-15 | , 0.20121D-14 |
| 14 | -0.35772D-15 | , -0.16269D-15 |
| 15 | 0.4340D-16 | , -0.6380D-16 |
| 16 | 0.1104D-16 | , 0.1132D-16 |
| 17 | -0.288D-17 | , 0.175D-17 |
| 18 | -0.213D-18 | , -0.712D-18 |
| 19 | 0.169D-18 | , -0.2D-20 |
| 20 | -0.11D-19 | , 0.39D-19 |
| 21 | -0.8D-20 | , -0.5D-20 |
| 22 | 0.2D-20 | , -0.1D-20 |

COEFFICIENTS FOR $\mathcal{J}_1(Z)$ $\phi$ = 52.5

| $r$ | $a_{2r}$ | |
|---|---|---|
| 0 | -0.10940927246808720+02 | 0.3957315917957566D+01 |
| 1 | -0.3349127899161334D+01 | 0.5750793711332447D+01 |
| 2 | -0.1181457033599053D+01 | 0.4881541487591793D+01 |
| 3 | 0.9435637614053528D+00 | 0.1084139545280347D+01 |
| 4 | 0.2558724891775055D+00 | -0.5539580850155349D-01 |
| 5 | 0.7257413889462510-02 | -0.3196017931743417D-01 |
| 6 | -0.2432126123650240-02 | -0.1727141710520250-02 |
| 7 | -0.1713718681351 3D-03 | 0.114175034779520-03 |
| 8 | 0.2673751754594D-05 | 0.1081415081444 2D-04 |
| 9 | 0.4827760412270-06 | 0.4262404048660-07 |
| 10 | 0.6859834263 9D-08 | -0.1590186732180-07 |
| 11 | -0.3895900982D-09 | -0.3401666543D-09 |
| 12 | -0.111710540D-10 | 0.68394653D-11 |
| 13 | 0.7081440-13 | 0.2758958D-12 |
| 14 | 0.5370900-14 | 0.22780D-15 |
| 15 | 0.2399D-16 | -0.8401D-16 |
| 16 | -0.1060-17 | -0.75D-18 |

| $r$ | $a_r^{(1)}$ | |
|---|---|---|
| 0 | 0.2036798376608749D+01 | 0.2731520465201010D-01 |
| 1 | 0.1833185846808839D-01 | 0.1346254910222601D-01 |
| 2 | -0.685200862503 3D-04 | -0.190221643609530-03 |
| 3 | -0.1059451352260-05 | 0.470182470842D-05 |
| 4 | 0.123023594190-06 | -0.12755829863D-06 |
| 5 | -0.8254885221D-08 | 0.2340962569D-08 |
| 6 | 0.493157364D-09 | 0.117669603D-09 |
| 7 | -0.260569410-10 | -0.23315203D-10 |
| 8 | 0.959168D-12 | 0.2559386D-11 |
| 9 | 0.264198D-13 | -0.235753D-12 |
| 10 | -0.121633D-13 | 0.139479D-13 |
| 11 | 0.19676D-14 | -0.12096D-14 |
| 12 | -0.2521D-15 | 0.293D-16 |
| 13 | 0.281D-16 | 0.930D-17 |
| 14 | -0.269D-17 | -0.247D-17 |
| 15 | 0.19D-18 | 0.43D-19 |
| 16 | 0.0D+00 | -0.6D-19 |

| $r$ | $a_r^{(2)}$ | |
|---|---|---|
| 0 | 0.1962525523972734D+01 | -0.2998557135887398D-01 |
| 1 | -0.1877788245533377D-01 | -0.1524499621202601D-01 |
| 2 | -0.358797428598 3D-04 | -0.2586211701 3952D-03 |
| 3 | 0.518165803215D-05 | -0.643560379780D-05 |
| 4 | 0.4362625184 1D-06 | -0.100977896D-08 |
| 5 | 0.182827680 40D-07 | 0.26684890183D-07 |
| 6 | -0.1353252808D-08 | 0.2759792835D-08 |
| 7 | -0.352779116D-09 | -0.8394561D-11 |
| 8 | -0.15273691D-10 | -0.4424333D-10 |
| 9 | 0.5749189D-11 | -0.4063714D-11 |
| 10 | 0.8619019D-12 | 0.779185D-12 |
| 11 | -0.1128424D-12 | 0.174762D-12 |
| 12 | -0.356604D-13 | -0.178139D-13 |
| 13 | 0.31568D-14 | -0.74607D-14 |
| 14 | 0.160941D-14 | 0.64268D-15 |
| 15 | -0.15048D-15 | 0.35751D-15 |
| 16 | -0.8130D-16 | -0.3951D-16 |
| 17 | 0.1121D-16 | -0.1875D-16 |
| 18 | 0.4325D-17 | 0.3326D-17 |
| 19 | -0.1007D-17 | 0.978D-18 |
| 20 | -0.210D-18 | -0.306D-18 |
| 21 | 0.93D-19 | -0.40D-19 |
| 22 | 0.5D-20 | 0.28D-19 |
| 23 | -0.8D-20 | -0.4D-21 |
| 24 | 0.7D-21 | -0.22D-20 |
| 25 | 0.6D-21 | 0.4D-21 |
| 26 | -0.2D-21 | 0.1D-21 |

# COEFFICIENTS FOR $J_8(Z)$ $\phi$ = 67.5

| $r$ | $a_{2r}$ | |
|---|---|---|
| 0 | -0.21062112152594560+02 | -0.27543191701907170+02 |
| 1 | -0.19618062895952710+02 | -0.17287972760804510+02 |
| 2 | -0.10451550397097540+02 | -0.30917831418917030+01 |
| 3 | -0.26466325873531990+01 | 0.73004478697562500+00 |
| 4 | -0.28523013984809640+00 | 0.35508311729530770+00 |
| 5 | -0.53432671824450-03 | 0.53118086463053 7D-01 |
| 6 | 0.23912211970889C-02 | 0.35458698775011D-02 |
| 7 | 0.29764093920420-03 | 0.52218049920170-04 |
| 8 | 0.13417240674940-04 | -0.82827909037 1D-05 |
| 9 | 0.184524261950-06 | -0.63964807056D-06 |
| 10 | -0.10375428460-07 | -0.20744303350-07 |
| 11 | -0.6352672680-09 | -0.2364774360-09 |
| 12 | -0.15525173D-10 | 0.6547619D-11 |
| 13 | -0.144468D-12 | 0.330266D-12 |
| 14 | 0.23006D-14 | 0.62995D-14 |
| 15 | 0.983D-16 | 0.482D-16 |
| 16 | 0.15D-17 | -0.48D-18 |

| $r$ | $a_r^{(1)}$ | |
|---|---|---|
| 0 | 0.20423826622466760+01 | 0.17066779775250200-C1 |
| 1 | 0.21041129814451510-01 | 0.83957226168811 6D-02 |
| 2 | -0.147939414122670-03 | -0.133378619664030-03 |
| 3 | 0.2235421341140-05 | 0.411336112370D-05 |
| 4 | -0.23857238970-07 | -0.166875473370-06 |
| 5 | -0.11571646580-08 | 0.7969670301D-08 |
| 6 | 0.20292995 6D-09 | -0.420369494D-09 |
| 7 | -0.2142470 2D-10 | 0.23160073D-10 |
| 8 | 0.2077355D-11 | -0.1232505D-11 |
| 9 | -0.198654D-12 | 0.51540D-13 |
| 10 | 0.190571D-13 | 0.4045D-15 |
| 11 | -0.13329D-14 | -0.5475D-15 |
| 12 | 0.1748D-15 | 0.1081D-15 |
| 13 | -0.161D-15 | -0.170D-16 |
| 14 | 0.137D-17 | 0.248D-17 |
| 15 | -0.9D-19 | -0.35D-18 |
| 16 | 0.0D+00 | 0.5D-19 |

| $r$ | $a_r^{(2)}$ | |
|---|---|---|
| 0 | 0.19556772162547030+01 | -0.19048970227279180-01 |
| 1 | -0.22334150554248140-01 | -0.97226089821058 3D-02 |
| 2 | -0.174643454392940-03 | -0.206532519784060-03 |
| 3 | -0.172100011196D-05 | -0.88678881413 5D-05 |
| 4 | 0.19741967605D-06 | -0.48123495844D-06 |
| 5 | 0.37970803695D-07 | -0.21249665620D-07 |
| 6 | 0.4629282553D-08 | 0.1255613691D-08 |
| 7 | 0.267353409D-09 | 0.592851561D-09 |
| 8 | -0.54857193D-10 | 0.87333730D-10 |
| 9 | -0.18409935D-10 | -0.12434736D-11 |
| 10 | -0.12227410D-11 | -0.34187594D-11 |
| 11 | 0.5992116D-12 | -0.4862340D-12 |
| 12 | 0.1427909D-12 | 0.1008775D-12 |
| 13 | -0.1620144D-13 | 0.385474 6D-13 |
| 14 | -0.1019941D-13 | -0.240390D-14 |
| 15 | 0.30210D-15 | -0.271903D-14 |
| 16 | 0.74026D-15 | 0.2302D-16 |
| 17 | 0.227D-17 | 0.20711D-15 |
| 18 | -0.5964D-16 | 0.111D-17 |
| 19 | 0.93D-19 | -0.17645D-16 |
| 20 | 0.5340D-17 | 0.312D-18 |
| 21 | -0.229D-18 | 0.1643D-17 |
| 22 | -0.510D-18 | -0.127D-18 |
| 23 | 0.618D-19 | -0.153D-18 |
| 24 | 0.483D-19 | 0.279D-19 |
| 25 | -0.120D-19 | 0.143D-19 |
| 26 | -0.40D-20 | -0.49D-20 |
| 27 | 0.19D-20 | -0.99D-21 |
| 28 | 0.18D-21 | 0.74D-21 |
| 29 | -0.27D-21 | 0.0D+00 |
| 30 | 0.3D-22 | -0.9D-22 |
| 31 | 0.3D-22 | 0.2D-22 |
| 32 | -0.1D-22 | 0.9D-23 |

## COEFFICIENTS FOR $\mathcal{J}_1(Z)$ $\phi$ = 82.5

| r | $a_{2r}$ | |
|---|---|---|
| 0 | 0.43705618725051950D+02 | -0.41923771467787840D+02 |
| 1 | 0.28604219418342190D+02 | -0.31584658429505430D+02 |
| 2 | 0.91755271832928420D+01 | -0.13664310564945430D+02 |
| 3 | 0.15480330031503700D+01 | -0.35828118036049390D+01 |
| 4 | 0.12391565686220680D+00 | -0.60544156619452080D+00 |
| 5 | -0.1262998345261220D-02 | -0.69486323969685970D-01 |
| 6 | -0.1425105567103680D-02 | -0.5636788221570370D-02 |
| 7 | -0.174407541264000D-03 | -0.332025068590770D-03 |
| 8 | -0.127185649761980D-04 | -0.143707847908810D-04 |
| 9 | -0.6569688799650D-06 | -0.4513193171310D-06 |
| 10 | -0.2567930233330D-07 | -0.9513692255D-08 |
| 11 | -0.78638698370D-09 | -0.8329726110D-10 |
| 12 | -0.1924753230D-10 | 0.28195300D-11 |
| 13 | -0.38019760D-12 | 0.1597702D-12 |
| 14 | -0.605278D-14 | 0.461407D-14 |
| 15 | -0.7612D-16 | 0.9707D-16 |
| 16 | -0.70D-18 | 0.162D-17 |

| r | $a_r^{(1)}$ | |
|---|---|---|
| 0 | 0.20452491167887920D+01 | 0.58031920924712500D-02 |
| 1 | 0.22428260562066170D-01 | 0.28520784830632000L-02 |
| 2 | -0.19182056701972D-03 | -0.47825105695510D-04 |
| 3 | 0.4322430341500D-05 | 0.16135589452200D-05 |
| 4 | -0.14789697254D-06 | -0.74379389530D-07 |
| 5 | 0.65645643860D-08 | 0.42167215270D-08 |
| 6 | -0.35035578200D-09 | -0.27911457600D-09 |
| 7 | 0.21438523D-10 | 0.20899693D-10 |
| 8 | -0.14686850D-11 | -0.17327240D-11 |
| 9 | 0.109291D-12 | 0.156602D-12 |
| 10 | -0.86837D-14 | -0.152483D-13 |
| 11 | 0.7231D-15 | 0.15848D-14 |
| 12 | -0.618D-16 | -0.1745D-15 |
| 13 | 0.526D-17 | 0.202D-16 |
| 14 | -0.42D-18 | -0.246D-17 |
| 15 | 0.30D-19 | 0.31D-18 |
| 16 | 0.0D+00 | -0.4D-19 |

| r | $a_r^{(2)}$ | |
|---|---|---|
| 0 | 0.19520627426271020D+01 | -0.65353600348163400D-02 |
| 1 | -0.24225147172738180D-01 | -0.33431632636719800D-02 |
| 2 | -0.26465807685359D-03 | -0.79494560674590D-04 |
| 3 | -0.85728657339800D-05 | -0.43542583236900D-05 |
| 4 | -0.46310623549D-06 | -0.38551739322D-06 |
| 5 | -0.31162115952D-07 | -0.49308253522D-07 |
| 6 | -0.90663633D-09 | -0.78157967890D-08 |
| 7 | 0.63155782700D-09 | -0.11881337600D-08 |
| 8 | 0.256266867D-09 | -0.87827546D-10 |
| 9 | 0.52146566D-10 | 0.31958917D-10 |
| 10 | 0.50193300D-12 | 0.15019153200D-10 |
| 11 | -0.34103945D-11 | 0.19803078D-11 |
| 12 | -0.88271320D-12 | -0.6424836D-12 |
| 13 | 0.8743925D-13 | -0.3037824D-12 |
| 14 | 0.9512234D-13 | -0.98477D-15 |
| 15 | 0.730658D-14 | 0.2869353D-13 |
| 16 | -0.858334D-14 | 0.406359D-14 |
| 17 | -0.174955D-14 | -0.259425D-14 |
| 18 | 0.80352D-15 | -0.68596D-15 |
| 19 | 0.25790D-15 | 0.25796D-15 |
| 20 | -0.86561D-16 | 0.95040D-16 |
| 21 | -0.34668D-16 | -0.30496D-16 |
| 22 | 0.11276D-16 | -0.12560D-16 |
| 23 | 0.45152D-17 | 0.43543D-17 |
| 24 | -0.17419D-17 | 0.16032D-17 |
| 25 | -0.5571D-18 | -0.7150D-18 |
| 26 | 0.2984D-18 | -0.1865D-18 |
| 27 | 0.584D-19 | 0.1254D-18 |
| 28 | -0.527D-19 | 0.159D-19 |
| 29 | -0.295D-20 | -0.220D-19 |
| 30 | 0.902D-20 | 0.40D-21 |
| 31 | -0.87D-21 | 0.361D-20 |
| 32 | -0.139D-20 | -0.67D-21 |

APPENDIX B

Tables of Coefficients for the Chebyshev expansions

For $|z| \leqslant 8$

C: $$J_n(z) = \frac{1}{n!}\left(\frac{z}{2}\right)^n \sum_{r=0}^{\infty}{}' a_{ar} T_{ar}(t) \qquad n=0,1,\ldots,10$$

D: $$Y_0(z) = \frac{2}{\pi}\log_e\left(\frac{z}{2}\right)J_0(z) + \sum_{r=0}^{\infty}{}' b_{ar} T_{ar}(t)$$

and

E: $$Y_1(z) = \frac{2}{\pi}\left(\log_e\left(\frac{z}{2}\right)J_1(z) - \frac{1}{z}\right) + \frac{z}{2}\sum_{r=0}^{\infty}{}' b_{ar} T_{ar}(t),$$

with

$$t = \left(\frac{z}{8}\right)\mathrm{Exp}\,(-i\phi) \qquad \phi = 7.5°(15°)82.5°$$

COEFFICIENTS FOR $J_N(Z)$, N= 0

ARG($\phi$) = 7.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.40795869055667137640715028153693Q+00 | -0.56442646929690935200795128274590Q-01 |
| 0.76307829250131400361146778095084Q-01 | -0.61302046672716605147332674516919Q-01 |
| 0.29061006595818788085586515391498Q+00 | -0.20911435561449686363680202686931Q+00 |
| -0.41389519311709364289690627937354Q+00 | -0.49959512745050736116971017573850Q-01 |
| 0.14405373906065264429620605762981Q+00 | 0.91255766225367239948363268407435Q-01 |
| -0.22191868433940324530320446112275Q-01 | -0.29516142867063306110340315867003Q-01 |
| 0.15821011678673097559220170114149Q-02 | 0.47897681004629969864723074241209Q-02 |
| -0.59234157455037023995012675293566Q-05 | -0.47863710713778006789287843649731Q-03 |
| -0.93656954962602142941518489788133Q-05 | 0.32225706258653841087281828118680Q-04 |
| 0.98007182750822194752002275702969Q-06 | -0.15271062036972918272686508255232Q-05 |
| -0.58849005869019316334881884219035Q-07 | 0.51370384201749153937317288042167Q-07 |
| 0.24841843116601105419586447412207Q-08 | -0.11659304865213573041154924553111Q-08 |
| -0.79166685333981694744509040991190Q-10 | 0.13002722707254774191498280825128Q-10 |
| 0.19713528931142066795752587275933Q-11 | 0.21989564315413682332706849016329Q-12 |
| -0.38986101629433760169040062140733Q-13 | -0.15719635840571344149732102786680Q-13 |
| 0.61288305780992463777428485624698Q-15 | 0.46989454558630322083751642792260Q-15 |
| -0.74872296145083913456919480921153Q-17 | -0.99111434114507891007149844922754Q-17 |
| 0.65125722022767106932468459658214Q-19 | 0.16392989156702043206417304659115Q-18 |
| -0.24484306038263621355486069536909Q-21 | -0.22147117906606840778028263028522Q-20 |
| -0.39482281695962560448896899872804Q-23 | 0.24892188592384991764317178926069Q-22 |
| 0.10567641477407692569908964539183Q-24 | -0.23401714881381196783886748810376Q-24 |
| -0.15087853069717455406447080694267Q-26 | 0.18260278516002535548651669668619Q-26 |
| 0.16246331754032631900049937714552Q-28 | -0.11458225051944390195654405947447Q-28 |
| -0.14329199354165747403873789530818Q-30 | 0.52084078228743871177741921848437Q-31 |
| 0.10689014912284315842898385395235Q-32 | -0.89267514828895868537298747317694Q-34 |
| -0.68331426076520037960310045221448Q-35 | -0.12555686975918394598649293578940Q-35 |
| 0.37492633224211049946541724110212Q-37 | 0.17950167586171787447862828877555Q-37 |
| -0.17459403685946996851518989893939Q-39 | -0.15040125647625372237119683989617Q-39 |
| 0.66565776420248262321144030300722Q-42 | 0.98283442944539753342027380529403Q-42 |
| -0.18518818117174614520531911760205Q-44 | -0.53871044191808775224137666081112Q-44 |
| 0.17106286956069944153166782398420Q-47 | 0.25472928825176313443730862834340Q-46 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 0

### ARG($\phi$) = 22.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.18623881365254941206391184280746Q+01 | 0.16486122859825242144718307919130Q+00 |
| 0.14586386625180950972545797220381Q+01 | -0.12544466638704219695403027896923Q+00 |
| 0.99548777603655731948538246885643Q+00 | -0.10844229108502024582959366154897Q+01 |
| -0.68363067919803988183879835761588Q+00 | -0.59323596240726279387806001876372Q+00 |
| -0.92103894031436975124900736670351Q-01 | 0.27615145780052247518564222799368Q+00 |
| 0.54325155286706497009706705913268Q-01 | -0.12476381839349755533666766352444Q-01 |
| -0.53126997414577367734725624978496Q-02 | -0.46544745856727888381187414478990Q-02 |
| -0.31338728717415638055222588885682Q-04 | 0.63622927278262264692972383255147Q-03 |
| 0.34287393291279206654024854743379Q-04 | -0.26013530174549737200216869491038Q-04 |
| -0.22152426452254745284258733781922Q-05 | -0.45992988672321494631582857389451Q-06 |
| 0.47791149661373039954614575852049Q-07 | 0.82401441947674595322343130211135Q-07 |
| 0.98772468429146011554951907842518Q-09 | -0.31346016042881186165079671391767Q-08 |
| -0.85559418413663835305923031026596Q-10 | 0.40492685731528486918403002058275Q-10 |
| 0.21497692022316954524086941727099Q-11 | 0.84820179582633413317487408538878Q-12 |
| -0.18044845865653460488125005161979Q-13 | -0.44965043319601150272937439932755Q-13 |
| -0.36611959638375363485072844099656Q-15 | 0.80228194969170118787213386821454Q-15 |
| 0.13282565182822653728060423561167Q-16 | -0.46368682453174704019240608040720Q-17 |
| -0.17697985099743731846229593931446Q-18 | -0.89566498342508217565158876682037Q-19 |
| 0.73565583861815498696056484311028Q-21 | 0.23784602955667591021187110468363Q-20 |
| 0.13439277020255245432805184079622Q-22 | -0.24563383016281616649391002845311Q-22 |
| -0.27343095881597148765216409975695Q-24 | 0.75979310571844082254091676766722Q-25 |
| 0.22524555302524473310456212866351Q-26 | 0.13101506878535092988935606352495Q-26 |
| -0.53291005516586209617603948448949Q-29 | -0.21115269624392155974828639121683Q-28 |
| -0.86754651739648244417089788877729Q-31 | 0.14195300876382134298811178252330Q-30 |
| 0.11362604863590713575985169424467Q-32 | -0.26277104068594085415779274347173Q-33 |
| -0.63512437869492818751052544852765Q-35 | -0.40436324862400408223202596401835Q-35 |
| 0.93755737997489190974076797605123Q-38 | 0.43922566481873963909743193811676Q-37 |
| 0.13661645456438597289667311109739Q-39 | -0.20731565753726878313488826244435Q-39 |
| -0.12512496648563425010552281402351Q-41 | 0.24804524270781308073626942673081Q-42 |
| 0.50528791030627037833593076928280Q-44 | 0.34291931359358081498946228055910Q-44 |
| -0.49692211992199977997653744395681Q-47 | -0.26850205932367154274118374713516Q-46 |

COEFFICIENTS FOR $J_N(Z)$ , N= 0

ARG(ϕ) = 37.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.67119273258906999397417562680239Q+01 | 0.83712702818187066160128862554041Q+01 |
| 0.71500895090149274249562302933488Q+01 | 0.59935320789535874704309890908769Q+01 |
| 0.62159845055422611610087734646016Q+01 | -0.19486400634296973199162619237438Q+00 |
| 0.91766948812533030008971126548282Q+00 | -0.24065140778222976905395753482710Q+01 |
| -0.56777356588961925012223189451426Q+00 | -0.31719376085575826201011612499029Q+00 |
| -0.52984967577101028658611440784450Q-01 | 0.92811127584290710768303757704639Q-01 |
| 0.10994988091648049146745669092850Q-01 | 0.53006984777873094100192858823990Q-02 |
| 0.33155441080985683550871115114717Q-03 | -0.96600009434642741994234559722918Q-03 |
| -0.64158538295280157832171771619448Q-04 | -0.11644140366370472243416450105202Q-04 |
| -0.19262278486119835822887559091950Q-07 | 0.32792409067351769904708187461371Q-05 |
| 0.13098460082289649995221225867198Q-06 | -0.24389700433340074222958066472352Q-07 |
| -0.16816077796311041189112778914050Q-08 | -0.41353051818623120210689188092807Q-08 |
| -0.10371152882321160186361693197984Q-09 | 0.70597257863178096934493061241108Q-10 |
| 0.21859346943753071283032864827525Q-11 | 0.20545036410632514002637938671880Q-11 |
| 0.31222454779154038507531746800889Q-13 | -0.53302500363141758467434181670466Q-13 |
| -0.10573078967372922849271977707796Q-14 | -0.32781289949432533628380676339244Q-15 |
| -0.11756726790979670550122091559254Q-17 | 0.17379430376588814964417894251052Q-16 |
| 0.23918281026749472894061443142955Q-18 | -0.40755209566834685198756324933495Q-19 |
| -0.11947502045502751672080653424766Q-20 | -0.27655685125209732758600102490881Q-20 |
| -0.26729840697905951875410589489261Q-22 | 0.20282438263448503305629880293553Q-22 |
| 0.26247196204098981414737826408936Q-24 | 0.21156346934994648642359041179737Q-24 |
| 0.12897963883569176515585530211121Q-26 | -0.27869637139173354839687197440611Q-26 |
| -0.25035651838797726597574836978744Q-28 | -0.47370861045896446416853047963321Q-29 |
| 0.11734694565831946831155271368519Q-31 | 0.19300680330736498144511520864952Q-30 |
| 0.12832433620734723751359383046365Q-32 | -0.41046820113458423420150726959960Q-33 |
| -0.45903276163115953174237946205508Q-35 | -0.73307874882075810998447859481484Q-35 |
| -0.35382723873725811024606758356275Q-37 | 0.37181894194269406039102168670998Q-37 |
| 0.24657413518293437964736301762449Q-39 | 0.13762270701873969701694544929432Q-39 |
| 0.36686604181049271304777475605924Q-42 | -0.13969378383478819261520215595429Q-41 |
| -0.68854303543706907044843314712174Q-44 | -0.27092262682857062761565080270755Q-46 |
| 0.75736308133371409863328094870638Q-47 | 0.29719079859615113795104140798596Q-46 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 0

### ARG($\phi$) = 52.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| -0.24615607680072097946409414783998Q+02 | 0.40708184519741684906894024764506Q+02 |
| -0.13674680433263381816930047837609Q+02 | 0.36750868601784119090822057617668Q+02 |
| 0.42050323986583392854024267002657Q+01 | 0.20367739691527432297883143363871Q+02 |
| 0.58622759766447914134576827477783Q+01 | 0.35548361381913597009511675099555Q+01 |
| 0.13204034634908637782731463139533Q+01 | -0.61553461742536762917630456224676Q+00 |
| 0.12011362049486119603651628069050Q-01 | -0.21071485764786350942543741438291Q+00 |
| -0.19372490663037713599494764213266Q-01 | -0.10318073190657241386059601834323Q-01 |
| -0.13193765886752324090956713249527Q-02 | 0.10868685743074080331045584156917Q-02 |
| 0.32882094123650831394692136588888Q-04 | 0.97769240370192918375983719587556Q-04 |
| 0.49561200024879495862816530660429Q-05 | 0.10089071632680748236592179984534Q-06 |
| 0.66621081843728297217183148014726Q-07 | -0.18223288264254629134116750106942Q-06 |
| -0.49411940775348074211939481683830Q-08 | -0.39106755594795768418695956250335Q-08 |
| -0.14323130562185905266344553318864Q-09 | 0.96090174415940859625911547790801Q-10 |
| 0.11426229129018497661619105639929Q-11 | 0.38663761072019751013073789910377Q-11 |
| 0.81399052533547393688503984541264Q-13 | 0.87295719916708646689766836918206Q-15 |
| 0.42963581930446140337461610709026Q-15 | -0.13679352041698177218734837566741Q-14 |
| -0.18413047890098872367751566818597Q-16 | -0.12441337372666640599612697442972Q-16 |
| -0.23556437667867229352963180294575Q-18 | 0.19433016360925339500623608205876Q-18 |
| 0.14780223743818327032278009370160Q-20 | 0.34422016606709071729170561679391Q-20 |
| 0.40956734504856370170797264766348Q-22 | -0.50882962561403372040902485809500Q-23 |
| 0.63981925565223277240096031917410Q-25 | -0.40558120586652640015269127140345Q-24 |
| -0.33618720134599604516805907738920Q-26 | -0.15617312296875415002587281808420Q-26 |
| -0.19927812351128876188982915416726Q-28 | 0.23097458034639807215285458498273Q-28 |
| 0.12653928079114571953761923102439Q-30 | 0.19179314075437628141322658676683Q-30 |
| 0.15148831459979218067536099487641Q-32 | -0.48291887511308268323169572633154Q-33 |
| -0.35457362648932105717005203942104Q-36 | -0.10136460884656706702109567689352Q-34 |
| -0.58150487026493486555445415501452Q-37 | -0.14094518049251148437476781919616Q-37 |
| -0.15975891547896140605983712801556Q-39 | 0.28576217592464116638248112392485Q-39 |
| 0.11831408057472102297251922652125Q-41 | 0.11729079678680638855149063611130Q-41 |
| 0.68633517028189059100077279083896Q-44 | -0.39214101947560129170821841454352Q-44 |
| -0.86383280198037442732074842918968Q-47 | -0.33970818794065285978908534066290Q-46 |

# COEFFICIENTS FOR $J_N(Z)$ , N= 0

## ARG($\phi$) = 67.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| -0.12023234004003584999818886107656Q+03 | -0.66691018233263761381657206590178Q+02 |
| -0.99170227887441285329135194491064Q+02 | -0.39147826531356588336958685234818Q+02 |
| -0.49482551698438323677964812571407Q+02 | -0.14800978678558559364104741152456Q+01 |
| -0.12834635332439317353249812196134Q+02 | 0.63351619838680025091559421924811Q+01 |
| -0.13924145634822307560273601121626Q+01 | 0.23497334840795793775590047916078Q+01 |
| 0.35873442631228915638088468993558Q-01 | 0.36184555175082579591418906704027Q+00 |
| 0.24623296955251348484432877021334Q-01 | 0.25407683584997795664451288720100Q-01 |
| 0.25989029404040905193856549537677Q-02 | 0.27328614896885497885059913797181Q-03 |
| 0.12385474204657920414516265382266Q-03 | -0.86478706066528572035583131906616Q-04 |
| 0.16233818774304976462790471703260Q-05 | -0.68164033686269323775951271638189Q-05 |
| -0.12848188590205171272287441105036Q-06 | -0.23322393287194919485736536781241Q-06 |
| -0.79995001109688431240715126441317Q-08 | -0.26718216601645247528390463650696Q-08 |
| -0.20551599036084747083254899649849Q-09 | 0.93883769053122440787827556762722Q-10 |
| -0.19551294992334729981372286345167Q-11 | 0.48015363459005741396881230966287Q-11 |
| 0.37507349404640345799651797934362Q-13 | 0.95920761478909471667874560717069Q-13 |
| 0.16214600690405924996084918798866Q-14 | 0.75440387851336339973278181017069Q-15 |
| 0.25894198253276515890314860610469Q-16 | -0.88611990878570833706219473950495Q-17 |
| 0.16963885835163051058784661923728Q-18 | -0.33091881056889251909482520124499Q-18 |
| -0.13151889963508586786701781619170Q-20 | -0.43198897528320881209066285578019Q-20 |
| -0.43155758831589959275961716074911Q-22 | -0.23825721607151014874315892698848Q-22 |
| -0.46903719427128928309181255754079Q-24 | 0.12866821474808433187918726739619Q-24 |
| -0.22018698090879407279084234728340Q-26 | 0.37606144136947100549130543087069Q-26 |
| 0.86281155469266477971682953927907Q-29 | 0.34553644478355585737498465900116Q-28 |
| 0.22713070854464166764224304073967Q-30 | 0.13950524503754715029219540715481Q-30 |
| 0.17873486006162661044327487865169Q-32 | -0.40963510166528041796508686348469Q-33 |
| 0.62652933706147687905142320609119Q-35 | -0.98027781360612562544596100114940Q-35 |
| -0.14150553808635658780091448512327Q-37 | -0.66805156657504333366626559124035Q-37 |
| -0.31024687126385932720181561031139Q-39 | -0.20507272069832033063755063045718Q-39 |
| -0.18487614673649109979228641554924Q-41 | 0.36405841380286782369569079512925Q-42 |
| -0.50084778511551406989439808273949Q-44 | 0.73617520664380580417787332142104Q-44 |
| 0.71176572747691937093496026753804Q-47 | 0.38683765499870789188474037233299Q-46 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 0

## ARG(Ø) = 82.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.14786330077903399910834413043463Q+03 | -0.18685812523542100076518581169684Q+03 |
| 0.10415768205398205162399654475226Q+03 | -0.14493435376763316555403379954426Q+03 |
| 0.37773716029004829631634647285402Q+02 | -0.68100726343676867881843240961907Q+02 |
| 0.71510684578255635647552427740306Q+01 | -0.19942171041630688504315865053873Q+02 |
| 0.58718947463746490892023107893163Q+00 | -0.37945991286273713731645640517225Q+01 |
| -0.27336816292524014223244975541634Q-01 | -0.48944600177602346303981707099740Q+00 |
| -0.12633298385438303996328684848139Q-01 | -0.44344116850055764209395580208655Q-01 |
| -0.16111141681285472669446158410552Q-02 | -0.28944488875185423952758626824804Q-02 |
| -0.12682388318315677499352037464151Q-03 | -0.13765284525622497163004271726241Q-03 |
| -0.71010473576556223110269068237272Q-05 | -0.47052276012432276254645764427057Q-05 |
| -0.30024483700336217893916277191359Q-06 | -0.10641119963921452180967194363728Q-06 |
| -0.99086415014185669218051899910643Q-08 | -0.92761222837942349840197804571025Q-09 |
| -0.26027484117930187788192176623065Q-09 | 0.40940074636336826992996865653823Q-10 |
| -0.54945502031913302315248827628845Q-11 | 0.23689419516543360236253155310313Q-11 |
| -0.93097909164864721426484987076695Q-13 | 0.72175746483862135156193005583481Q-13 |
| -0.12405514383185900794313961451409Q-14 | 0.16057793958388156658720457530673Q-14 |
| -0.12100809941441867001603700742549Q-16 | 0.28359465561914272852397225563504Q-16 |
| -0.61402516926354081139078013313073Q-19 | 0.41173060997307404969131501659123Q-18 |
| 0.54121172480611844676219493583295Q-21 | 0.49969014268740635137936784721376Q-20 |
| 0.19452840346739931358323830445185Q-22 | 0.51000584521037047207876926762789Q-22 |
| 0.31204514306599118157275391121056Q-24 | 0.43581643056407606329070172519971Q-24 |
| 0.36820345317120975520032057649888Q-26 | 0.30512590903047769797613238757884Q-26 |
| 0.35315464449440018810041928637612Q-28 | 0.16376144713659745783339920962069Q-28 |
| 0.28580566388241294610135183367357Q-30 | 0.50849901113658300951720520451353Q-31 |
| 0.19836067158649677864126489289905Q-32 | -0.16369074613715444505432167096866Q-33 |
| 0.11863994266786195581653997529741Q-34 | -0.42101148754094705704577267314263Q-35 |
| 0.60815557684675400071345237144200Q-37 | -0.41597029420894998273193569008155Q-37 |
| 0.26140923382011523100280946805280Q-39 | -0.30302663845261326002874852819372Q-39 |
| 0.88434651555956693190035115231847Q-42 | -0.18190421665068652999707007238636Q-41 |
| 0.18295586081000214890887670576021Q-44 | -0.93583279237706162808161717143004Q-44 |
| -0.27944315469870372074187423912818Q-47 | -0.41938207208324406336812682175833Q-46 |

COEFFICIENTS FOR $J_N(Z)$ , N= 1

ARG($\phi$) = 7.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.331650861306539976046003503441840Q+00 | 0.48593997430256699465375462423286Q- |
| -0.28020674473706018265923536641445Q+00 | 0.81346848297617297782801638913665Q- |
| 0.34611125276606388482375235700901Q+00 | -0.14881387653454337076133925474944Q- |
| -0.16691424961145506535924281886925Q+00 | -0.57255339954541558145714616435313Q- |
| 0.36569355422691324747952979491212Q-01 | 0.29268693615916085505841393253322Q- |
| -0.41502924047159137283095984284926Q-02 | -0.63928897467874703676258454980431Q- |
| 0.22555696529756961672302548945317Q-03 | 0.81028550263970382178848993942696Q- |
| 0.15209710883043568767233087953378Q-05 | -0.67265551812858283025707285895073Q- |
| -0.12464984222397633027336984161773Q-05 | 0.38945144437760507466499717506369 3Q- |
| 0.10908980954867918536642675121908Q-06 | -0.16222719145416547549337232664003Q- |
| -0.57752513455056210899292078824517Q-08 | 0.48661474047348637475283503149842Q- |
| 0.21945746198824051123807577467122Q-09 | -0.99130676381298998137907296734863Q- |
| -0.63762315331630816503414103493347Q-11 | 0.96680975881599084980880224383144Q- |
| 0.14608097533534666919616956854719Q-12 | 0.17858349957729693388065263762958Q- |
| -0.26765123039394560770827053789744Q-14 | -0.11078169548851953812394692101090Q- |
| 0.39192847989154024959016691973908Q-16 | 0.30551709794305953430172781216620Q- |
| -0.44768536013542468161935114956219Q-18 | -0.60144451407608299630220580340237Q- |
| 0.36434866106913209315345619239843Q-20 | 0.93427147672250877723963199358518Q- |
| -0.12481994664354918917576134007972Q-22 | -0.11907426190414492712693446677251 1Q- |
| -0.20760742168313878200135884612938Q-24 | 0.12670553847602525469228558332698Q- |
| 0.51707289101285615493893310481711Q-26 | -0.11310629158223537173740539538092Q- |
| -0.70005351582556064872079927516915Q-28 | 0.84001782864102962740933761465353Q- |
| 0.71838552811691688766668563004057Q-30 | -0.50253077895558263046292282992253Q- |
| -0.60565181415817746475042673214826Q-32 | 0.21772175366030197188671463146932Q- |
| 0.43284363698986344220638342081829Q-34 | -0.34728319521611394144474329079696Q- |
| -0.26560244327395905958704200000379Q-36 | -0.49631138636480988176999802523068Q- |
| 0.14011313698688281314218301537436Q-38 | 0.67562957481977781156810715312794Q- |
| -0.62815278995322935897115439202192Q-41 | -0.54396029802292208501872083462563Q- |
| 0.23077280119695161982292241620196Q-43 | 0.34251575582187551874989629396549Q- |
| -0.61838480605700457963625457230486Q-46 | -0.18121992924502119400041716420578Q- |
| 0.53861301998113532422615953596505Q-49 | 0.82829411533041324672473983544281Q- |

COEFFICIENTS FOR $J_N(Z)$ , N= 1

ARG(ϕ) = 22.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.40374947400739902338453870603654Q+00 | 0.29030589498529461840121335816053Q+ |
| -0.88138108175613036557239756955584Q-01 | 0.47435068271810566021069674229339Q+ |
| 0.63942710283276385088367676709279Q+00 | 0.10276879026948940920512851933717Q- |
| -0.11958142663184717566342473740303Q+00 | -0.26100879276189324358970757626359Q+ |
| -0.37733692879738068020066223777798Q-01 | 0.58215916946595901765042686098983Q- |
| 0.10559853770136716991430919076343Q-01 | -0.61293627327681952647610903719922Q- |
| -0.74425351853121363308124920938846Q-03 | -0.82885792280320990733827413627010Q- |
| -0.11931063836970950641432573874115Q-04 | 0.85216933527842977719908945573381Q- |
| 0.42603412410103943460304496234092Q-05 | -0.27846664665102053756186946649220Q- |
| -0.23005440407360849147791481113895Q-06 | -0.61450261154334234166598634270232Q- |
| 0.41678050943597051554066841494641Q-08 | 0.80190314302725333364238487329655Q- |
| 0.95756893912482312959153112951876Q-10 | -0.26733021810351527618112387123096Q- |
| -0.68907840222421640594904769064313Q-11 | 0.29880297475032918861014830487766Q- |
| 0.15591705610438350125370621535845Q-12 | 0.66674768179946522159070694899207Q- |
| -0.11557490280015465119282951651163Q-14 | -0.30984550287166174445521694821064Q- |
| -0.24463632089036369596417803159916Q-16 | 0.50678582961172157676640602187596Q- |
| 0.80106345720037499759306573339584Q-18 | -0.26256729611572430347944706552821Q- |
| -0.99083586616427445349623729529673Q-20 | -0.52286070629780640755188633381542Q- |
| 0.37349945500819304709776999755489Q-22 | 0.12764520561942215835658543281793Q- |
| 0.69819983790738622455537020424396Q-24 | -0.12352904547766827781678429921744Q- |
| -0.13225725214553030034213468710560Q-25 | 0.34971983515892016690402800742190Q- |
| 0.10284075796948478676365609390929Q-27 | 0.61399725030866895267611728643793Q- |
| -0.22438307259806460919297069164112Q-30 | -0.92990443815096385805089313296586Q- |
| -0.37061468256130320865899612850662Q-32 | 0.59354292870596098797177797276729Q- |
| 0.45939634291541491455250653105002Q-34 | -0.10196655907971881345962530320819Q- |
| -0.24496363093819362115183877887898Q-36 | -0.15879210418349082026705107546195Q- |
| 0.33740174576013422980584343265537Q-39 | 0.16415862419538616285330051206266Q- |
| 0.49657772946503430674320756815624Q-41 | -0.74210151285781260581944685691338Q- |
| -0.43483704480270018161843698004638Q-43 | 0.83229933411456819933551135062198Q- |
| 0.16874592744376149481713550737913Q-45 | 0.11604438138044624210987543731646Q- |
| -0.15618962244555390879283290021941Q-48 | -0.87200343259351583198351514446403Q- |

# COEFFICIENTS FOR $J_N(Z)$ , N= 1

## ARG($\phi$) = 37.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| -0.4381621831242274852144740253248 8Q+00 | 0.23777382028651191455818971645272Q+0 |
| -0.26885225231081266002910940730051Q+00 | 0.25747309246619626231005266475263Q+0 |
| 0.14718095080942915840056756433482Q+01 | 0.10389342359726319562215619881985Q+0 |
| 0.44144324656702805445101763453712Q+00 | -0.45257631821928395505836840434952Q+0 |
| -0.93443310751054222530709126050471Q-01 | -0.93005014696467869431995201960880Q-0 |
| -0.11893011139311777202518705869361Q-01 | 0.13755046260572593595389031777369Q-0 |
| 0.14756222334243170819510250677735Q-02 | 0.99603030861356795719005623160517Q-0 |
| 0.55679674477995456263297077881371Q-04 | -0.11758559802687314506146035693468Q-0 |
| -0.71034923884038081093562428977654Q-05 | -0.19530242521531124581780416637941Q-0 |
| -0.25980565144970626517837247558289Q-07 | 0.33172962453404536121438826535092Q-0 |
| 0.12173196904521103267030295313217Q-07 | -0.15184042413262830500117600999741Q-0 |
| -0.12389573472045318662097118819969Q-09 | -0.35519485968886486517597865633557Q-0 |
| -0.82861303784049343269097000935301Q-11 | 0.51305342309898421916665459550042Q-1 |
| 0.15185261680643643815882356277484Q-12 | 0.15381743493724142021368082990552Q-1 |
| 0.22135080235417658132480659310637Q-14 | -0.35106113611528249431002746014444Q-1 |
| -0.65918405516796886161574599591960Q-16 | -0.22536931882504213189018277529083Q-1 |
| -0.95849052629609285799464543488336Q-19 | 0.10265720166102173184045188547896Q-1 |
| 0.13411150333618504664879603246672Q-19 | -0.19711685136277977037606222148366Q-2 |
| -0.60185031358122344706491862827886Q-22 | -0.14756516523607960324735969513814Q-2 |
| -0.13613760026691481060727421809714Q-23 | 0.99373270561614902967744879127925Q-2 |
| 0.12379336496632654310361887688323Q-25 | 0.10327404303977696087227055587926Q-2 |
| 0.60804961116735779891100348067285 2Q-28 | -0.12625286598338979621045179025075Q-2 |
| -0.10893859755312475183785103196152Q-29 | -0.22207505610334869986166922610714Q-3 |
| 0.38595017523338083180659550172736Q-33 | 0.80742446581822942585441907037466Q-3 |
| 0.51680304267710195333088740105443Q-34 | -0.15852186429369414162776563947638Q-3 |
| -0.17391320845712136974401680745915Q-36 | -0.28469803800588745967701604002047Q-3 |
| -0.13280589021085557221900347302659Q-38 | 0.13671842300489378287411569485097Q-3 |
| 0.87804749239330421106467824338385Q-41 | 0.50114336895901910028456004593304Q-4 |
| 0.13110047074026216155753137812912Q-43 | -0.48166609312509968021457111384547Q-4 |
| -0.22999617792488802730649675157149Q-45 | -0.28241436574998738215630110995361Q-4 |
| 0.23728801940044736980131953242610Q-48 | 0.96251610974220821000762454885176Q-4 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 1

## ARG(ϕ) = 52.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| -0.10940927246808716129479366946389Q+02 | 0.39573159179575658160719671468391Q+0 |
| -0.83491278991613342749254616283719Q+01 | 0.57507937113324469463743150446817Q+0 |
| -0.11814570335990525524815512811297Q+01 | 0.48815414875917929001902841644347Q+0 |
| 0.94356376140535276469469889793886Q+00 | 0.10841395452803469481805616803058Q+0 |
| 0.25587248917750552546858028068995Q+00 | -0.55395808501553487531591549673767Q-0 |
| 0.72574138884625078316483206086081Q-02 | -0.31960179317434170523227349873777Q-0 |
| -0.24321261236502427573487062738691Q-02 | -0.17271417105202489800027426611846Q-0 |
| -0.17137186813513031482635816187172Q-03 | 0.11417503477951557347583972971305Q-0 |
| 0.26737517545941825886431190617516Q-05 | 0.10814150814441503845979551199974Q-0 |
| 0.48277604122690481382780149588591Q-06 | 0.42624040486572034975229623493027Q-0 |
| 0.68598342639081278651616621298922Q-08 | -0.15901867321818508449467437112872Q-0 |
| -0.38959009817263018784011872457180Q-09 | -0.34016665430631215551560972027989Q-0 |
| -0.11171053985580555859007085484820Q-10 | 0.68394652527480149719826791395192Q-1 |
| 0.70814439815526978123722295836021Q-13 | 0.27589583525122415740244499883531Q-1 |
| 0.53709002269942060570226490575051Q-14 | 0.22780434505152393894241069031530Q-1 |
| 0.28993807809274387842116612256904Q-16 | -0.84012340888282492526072016348504Q-1 |
| -0.10568038502553286487665414801138Q-17 | -0.75309417230874672378125317700421Q-1 |
| -0.13238628692475815324420615857334Q-19 | 0.10433337060800019353898155252635Q-1 |
| 0.73701024206502918041851855839560Q-22 | 0.18164087377541996842071949078884Q-2 |
| 0.20414544418567161256565784054564Q-23 | -0.21592471144066054847355047044603Q-2 |
| 0.33507232714118094960842434369708Q-26 | -0.19169516918978938656134100033092Q-2 |
| -0.15106955604823808629962447425567Q-27 | -0.72980966914057343680822126384247Q-2 |
| -0.87685562131398461427949642145405Q-30 | 0.98821830414667873828919719587461Q-3 |
| 0.51512571955283736321765020770391Q-32 | 0.80292681141736342373043821086972Q-3 |
| 0.60618476194210024288242227247953Q-34 | -0.18538048290341634732473396329818Q-3 |
| -0.93410512849783081364866831373596Q-38 | -0.38880029041187950907296876653479Q-3 |
| -0.21422322421356623626381144939564Q-38 | -0.54235263594353271099611910110661Q-3 |
| -0.57868297442938983906341167927955Q-41 | 0.10124651742292061270613564859295Q-4 |
| 0.40339872426686616599738885834214Q-43 | 0.40704413799829145177274129758802Q-4 |
| 0.22932691680283722938417314685591Q-45 | -0.12852221900804684674861491953276Q-4 |
| -0.26956804297955956014686251711005Q-48 | -0.10956139903635251256837250133292Q-4 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 1

### ARG($\phi$) = 67.5 DEGREES

| REAL PART | IMAGINARY PART |
| --- | --- |
| -0.21062112152594564669053666585503Q+02 | -0.27543191701907173044698521355362Q+0 |
| -0.19618062895952712895442082340300Q+02 | -0.17287972760804514799105385734199Q+0 |
| -0.10451550397097535860286217239061Q+02 | -0.30917831418917028023374396511769Q+0 |
| -0.26466325873531993719281743290462Q+01 | 0.73004478697562498072295132290189Q+0 |
| -0.28523013984809636983658492259600Q+00 | 0.35508311729530773623504403642207Q+0 |
| -0.53432671824445562063099204406652Q-03 | 0.53118086463053725117398885614330Q-0 |
| 0.28912211970888601754883088473253Q-02 | 0.35458698775011299090688929268442Q-0 |
| 0.29764093920420612277151002271519Q-03 | 0.52218048920171887019739849036804Q-0 |
| 0.13417240674837476152630302603391Q-04 | -0.82827909037130778959623603451413Q-0 |
| 0.18452426195142764065136040272766Q-06 | -0.63964807056049232429084520456664Q-0 |
| -0.10375428464636338612409160655577Q-07 | -0.20744303350006659981034416704508Q-0 |
| -0.63526726800831448623021311948901Q-09 | -0.23647743617114103131477376977710Q-0 |
| -0.15525173137111074406946019433247Q-10 | 0.65476186214586529227646650447866Q-1 |
| -0.14446750659750254534279293598156Q-12 | 0.33026588568828155434989900698979Q-1 |
| 0.23006341328401762201708208685077Q-14 | 0.62994757527479159324355721580748Q-1 |
| 0.98312667356936428391499501918831Q-16 | 0.48158664941240662972014038270232Q-1 |
| 0.15042128717555419636123325716617Q-17 | -0.48490409724200829658805698386195Q-1 |
| 0.95587859425419998948350170615573Q-20 | -0.17931914010878113095634591759522Q-1 |
| -0.64947036146494214783335326202223Q-22 | -0.22496874236790560308799535387124Q-2 |
| -0.21133085566725459837621655740407Q-23 | -0.12032181241517026311333856416615Q-2 |
| -0.22134237045671069348768001924158Q-25 | 0.57880347965764956038227824728839Q-2 |
| -0.10081732515544430198880468303404Q-27 | 0.16794348031416070719978905784691Q-2 |
| 0.35634422785272704991005992961784Q-30 | 0.14906762856961256734623989082861Q-2 |
| 0.93212544440583627997579077655326Q-32 | 0.58447712503372606844222137921304Q-3 |
| 0.71011012461943232331326329582230Q-34 | -0.15636081255731552868375521427330Q-3 |
| 0.24199982071127197329484645418415Q-36 | -0.37209550566376416671830074124986Q-3 |
| -0.50205656278667425486737197452662Q-39 | -0.24595932858353434566865671949768Q-3 |
| -0.10953067005061329671945753453793Q-40 | -0.73486622789130655942959318926882Q-4 |
| -0.63416061288043903311542572212320Q-43 | 0.12065359164542154779250805294458Q-4 |
| -0.16740043430295218532805644670895Q-45 | 0.24290164159668846627281101342191Q-4 |
| 0.22129381581571266166768866199305Q-48 | 0.12420363116390780829630611834630Q-4 |

COEFFICIENTS FOR $J_N(Z)$ , N= 1

ARG(ϕ) = 82.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.43705618725051947484347585682349Q+02 | -0.41923771467787835211152012152582Q+0 |
| 0.28604219418342189849641811316528Q+02 | -0.31584658429505431833073499337817Q+0 |
| 0.91755271882928422930782748337748Q+01 | -0.13664310564945434006043559906726Q+0 |
| 0.15480330031503695938222900049909Q+01 | -0.35828118036049386796983480939276Q+0 |
| 0.12391565686220676018195055839356Q+00 | -0.60544156619452080411930287547991Q+0 |
| -0.12629983452612194415691843738527Q-02 | -0.69486323969685972382058150831310Q-0 |
| -0.14251055671036787155002368900967Q-02 | -0.56367882215703729076145171601647Q-0 |
| -0.17440754126400095348040179606917Q-03 | -0.33202506859076857680301623416888Q-0 |
| -0.12718564976197552015411585408691Q-04 | -0.14370784790881258460241441695177Q-0 |
| -0.65696887996539806797364989244697Q-06 | -0.45131931713130223280064569538351Q-0 |
| -0.25679302333142161372360569704899Q-07 | -0.95136922545545306276052828034417Q-0 |
| -0.78638698373798369213913151686153Q-09 | -0.83297261073526140761185475363179Q-1 |
| -0.19247532307587339841244547499472Q-10 | 0.28195299878684876126591629996834Q-1 |
| -0.38019758145626080951814716454888Q-12 | 0.15977023686601266706686926058032Q-1 |
| -0.60527810491395520649490363170897Q-14 | 0.46140683881766576563809905634054Q-1 |
| -0.76117284960917883772717583941357Q-16 | 0.97067233240961046043292955667481Q-1 |
| -0.70493793349747147108740675573692Q-18 | 0.16229465614349770884640824549436Q-1 |
| -0.34664096911561194986692426062488Q-20 | 0.22352712971661768676031800804496Q-1 |
| 0.26567399391173611188597568451623Q-22 | 0.25793382684047941229111315518391Q-2 |
| 0.94323866817110571225096137822622Q-24 | 0.25087851324398685030361998266601Q-2 |
| 0.14527465276411891143781524708393Q-25 | 0.20477128509242162308538342335555Q-2 |
| 0.16431688648146850006693432846889Q-27 | 0.13727363898429389158176061385632Q-2 |
| 0.15117887938262157962194576207530Q-29 | 0.70775656841198186567859285647384Q-3 |
| 0.11750751252483752313320523937243Q-31 | 0.21315335941138441718438519413034Q-3 |
| 0.78435961171214034575939576118808Q-34 | -0.62267129972523794555480203928082Q-3 |
| 0.45182051324294693468773035376136Q-36 | -0.15886693043481200396514310072876Q-3 |
| 0.22338145913959951880665237919400Q-38 | -0.15191061873385387636325017389106Q-3 |
| 0.92751724293621062517802036056810Q-41 | -0.10697527857763015195664366514750Q-4 |
| 0.30372566015432926706052442270415Q-43 | -0.62104306249702564821317224328693Q-4 |
| 0.61162232282572711990769979224487Q-46 | -0.30924847649001085256018023632763Q-4 |
| -0.86687201040442531962537129095383Q-49 | -0.13426007538609068163431704239594Q-4 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 2

### ARG($\phi$) = 7.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.40308494020922546954424296204847Q+00 | -0.44411369904234416216618904541799Q- |
| -0.41754533166874937832199181561564Q+00 | 0.64152157300714423239290376259072Q- |
| 0.27828253128751248569728290188814Q+00 | 0.31137561976775172749260947063827Q- |
| -0.87079120395012042423141255796058Q-01 | -0.39802343304926248857882406334570Q- |
| 0.14214195953383715950657341129684Q-01 | 0.12981824691857978493150006147790Q- |
| -0.12819600218292729171396303128984Q-02 | -0.21892604752469197373218531280033Q- |
| 0.55505353194485932478041088599371Q-04 | 0.22961820696852244560607525527066Q- |
| 0.80583257252861197365513050414224Q-06 | -0.16368709368611071044041518304244Q- |
| -0.28625902194487790066369335237516Q-06 | 0.83304196737182864199327632255147Q- |
| 0.21630536553127718633811884136976Q-07 | -0.30990914751121996563797651356571Q- |
| -0.10257225370044115074023509129598Q-08 | 0.83917306806766046499660137653251Q- |
| 0.35472536562801654054849293474590Q-10 | -0.15502294953288898736300143883529Q- |
| -0.94755882736143169362064451285697Q-12 | 0.13348621125257055964696725284813Q- |
| 0.20108960551198824692169394930352Q-13 | 0.26450781982304070712665836602991Q- |
| -0.34328608245084331966969800857901Q-15 | -0.14531992901886071086577407986319Q- |
| 0.47051550274056085101326150282783Q-17 | 0.37216565340591311163924073199864Q- |
| -0.50469197665460385070169230984994Q-19 | -0.68703747302939723265782171408978Q- |
| 0.38585763240366174408498231716797Q-21 | 0.10061995844199233756583225790111Q- |
| -0.12098628098888397943992443879166Q-23 | -0.12138377857504921153851262128580Q- |
| -0.20677546327456728421450739138650Q-25 | 0.12264250425223904290062326204925Q- |
| 0.48178731611425609716495375314371Q-27 | -0.10422534144482073733275420787949Q- |
| -0.62027595311262583740084535676789Q-29 | 0.73849071413306284338410070722897Q- |
| 0.60799749727898954135060288135271Q-31 | -0.42211848224368161777489703605446Q- |
| -0.49094766440845593395557716773393Q-33 | 0.17468401906974201410347305029671Q- |
| 0.33675727113580166108917211917878Q-35 | -0.25998816514493800107448919159594Q- |
| -0.19867748295122840118868818383382Q-37 | -0.37699710122142967219194447565253Q- |
| 0.10091894502753782385543483484286Q-39 | 0.48986704466019979778716211408417Q- |
| -0.43618660937352871716367921089154Q-42 | -0.37957477794181475869860785949375Q- |
| 0.15462043280136651676547996990589Q-44 | 0.23061326419158153128935374620518Q- |
| -0.39960236966011166345280302559821Q-47 | -0.11792156837381972178741879260936Q- |
| 0.32875278299097385097425559741372Q-50 | 0.52157687076354394525516035949044Q- |

## COEFFICIENTS FOR $J_N(Z)$, N= 2

### ARG($\phi$) = 22.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.24936584518021976412813962164267Q+00 | -0.29353519835758802046088754012975Q- |
| -0.48504347400558459162727768269892Q+00 | 0.30938253579410447952678926023978Q+ |
| 0.38591261039947551737874911825984Q+00 | 0.16951629935542641407168030229095Q+ |
| -0.25633382668680016420793464047729Q-01 | -0.13549385384402128726628035276918Q+ |
| -0.17764277080409066591374853505914Q-01 | 0.19509924901564072810950619560378Q- |
| 0.33328563942349431750849451090666Q-02 | 0.19938567678700190577795763669853Q- |
| -0.18044504557718524414256979800066Q-03 | -0.24096410961405814468034331468697Q- |
| -0.42169089542006823478149774046732Q-05 | 0.19912632850071877465276070761758Q- |
| 0.92822840530724185923479160783481Q-06 | -0.53508261032342183273822606843672Q- |
| -0.43070394700630416761598172750252Q-07 | -0.13658043925360565074561620790397Q- |
| 0.66624042137477388886711370949483Q-09 | 0.14124872512190851017666078359967Q- |
| 0.16579038266351002054410839898490Q-10 | -0.41902635319658221776594313729740Q- |
| -0.10232141593114072670053738963755Q-11 | 0.40977634535197469612753699271398Q- |
| 0.21054188515146056053351664152511Q-13 | 0.96219203380949983726642554326405Q- |
| -0.13893478853506210575049510337677Q-15 | -0.39852988784155803189317522724838Q- |
| -0.30472442510789686658333130680304Q-17 | 0.60162293930268696794756513373892Q- |
| 0.90982941133498437938898883563759Q-19 | -0.28114226205787185060426730669198Q- |
| -0.10499539119290932429415794300036Q-20 | -0.57475376008820984801168348896922Q- |
| 0.36066945228527460341068290389300Q-23 | 0.12988556530063415279443046779422Q- |
| 0.68800051104053899312157548463954Q-25 | -0.11824343840415431290319905695458Q- |
| -0.12194414967362986875633984352221Q-26 | 0.30760556837203396074520286383337Q- |
| 0.89779464817653412863429455762135Q-29 | 0.54881446806274953237301940429515Q- |
| -0.18124349821857342009480952552233Q-31 | -0.78403435819833709723460971718314Q- |
| -0.30330742001556885643898092659593Q-33 | 0.47632022856581416675435694535575Q- |
| 0.35687536627306479492772403888998Q-35 | -0.76153764228139303324723263210883Q- |
| -0.18191277653803327321881145375307Q-37 | -0.11989082967850529867952611125355Q- |
| 0.23434636865930288645903354584303Q-40 | 0.11824413671145594348134746747624Q- |
| 0.34808418739678765566115142348364Q-42 | -0.51285255023547633181319643772999Q- |
| -0.29199783171390708840597123804351Q-44 | 0.54029665347192854642497418807737Q- |
| 0.10905507360909896609989051445165Q-46 | 0.75925301989857448052314998327695Q- |
| -0.95176448497391949215683657645075Q-50 | -0.54845233659949038516378210500106Q- |

## COEFFICIENTS FOR $J_N(Z)$ , N= 2

### ARG($\phi$) = 37.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| -0.74953051761511623986362630157393Q+00 | 0.31493950776626674477435873677681Q+0 |
| -0.11604411736034028293565233670991Q+01 | 0.10238644591262204444585976439551 8Q+0 |
| 0.45917698871457204064854958229220Q+00 | 0.77797987440623588402954566689515 7Q+0 |
| 0.22402456819623889651511768845330Q+00 | -0.12889838924111171355832189071017Q+0 |
| -0.25174863172514964306067460545658Q-01 | -0.37325330420036801730568475613223Q-0 |
| -0.40171400627983682733253590294246Q-02 | 0.34772868687132947762141287342793Q-0 |
| 0.34567842352909733608451283292577Q-03 | 0.29425595573124696072476248789301Q-0 |
| 0.14891545931972912539870663275633Q-04 | -0.25363166513818696259013345212884Q-0 |
| -0.14095966091596140189215546537368Q-05 | -0.49944148751196647924075393763458Q-0 |
| -0.84367935602192537801806804351055Q-08 | 0.60701017392192736231811359722265Q-0 |
| 0.20621506329285415015315498803936Q-08 | -0.15189172042210125264061325539141Q-0 |
| -0.16846924073265012341785062432258Q-10 | -0.55968679801069037553800424731115Q-1 |
| -0.12209195731651344144917164592972Q-11 | 0.69420495741880734641151169233545Q-1 |
| 0.19718912171740024795180283413710Q-13 | 0.21324017112899881194960596136447Q-1 |
| 0.29121124914074773028722854015873Q-15 | -0.43368034139247730090389441939854Q-1 |
| -0.77328560103721758289519677867936Q-17 | -0.28664150970003573354001490349138Q-1 |
| -0.13325768000540332811915551532217Q-19 | 0.11443206076980097763978148755639Q-1 |
| 0.14228945388846939790527756073265Q-20 | -0.17938157391515768663385480041702Q-2 |
| -0.57703443034957239269091450127002Q-23 | -0.14934799524715009686455239467089Q-2 |
| -0.13178968476423027427384359057786Q-24 | 0.92834977112275833825323616643052Q-2 |
| 0.11151500954041243880281813786475Q-26 | 0.95978240369723093603597343769647Q-2 |
| 0.54611037915713940148304255292429Q-29 | -0.10942028495626780311389519629934Q-2 |
| -0.90834646018198474116790259382628Q-31 | -0.19746322757847392961064528039793Q-3 |
| 0.23221572561980018206304521388456Q-34 | 0.64824498406073537262484949206081Q-3 |
| 0.40000194491745826267909312969293Q-35 | -0.11786778315934986457864893696309Q-3 |
| -0.12698727226283256126030430189510Q-37 | -0.21276224404177765902190095862218Q-3 |
| -0.96027666682404693367997097540993Q-40 | 0.96983935038620181150758097580711Q-4 |
| 0.60383652169415542810222822993594Q-42 | 0.35181646549320813021583855363001Q-4 |
| 0.90271633028939900067076630418121Q-45 | -0.32106500576013013278924197630937Q-4 |
| -0.14866792769620802000137777646987Q-46 | -0.29883739503320216661125385496643Q-4 |
| 0.14415856059568308988391601741863Q-49 | 0.60379989698039396493128507399441Q-4 |

COEFFICIENTS FOR $J_N(Z)$ , N= 2

ARG($\phi$) = 52.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| -0.49810229695014757620352087670975Q+01 | -0.15037270521279964482410042710935Q+ |
| -0.477844724370818781495626068981614Q+01 | 0.57950148249376936412268725349803Q+ |
| -0.11231462306332141330370531407769Q+01 | 0.17843846117929431831908532510290Q+ |
| 0.24254333252404589779571220497067Q+00 | 0.45726543745111917125603196414154Q+ |
| 0.81332940917732494736838104822222Q-01 | -0.36282398458974924279266803743093Q- |
| 0.30775016905636889509450970764500Q-02 | -0.83672730976178931497227724513757Q- |
| -0.54085795243008036965540644254909Q-03 | -0.47379088262564775959745181638865Q- |
| -0.38928978317916385560288936720420Q-04 | 0.21538714081130593964489949862818Q- |
| 0.37826084696641253543297810280562Q-06 | 0.21455568666621322632184680434644Q- |
| 0.85620422438632885042979157667928Q-07 | 0.12497840184077284260566102616823Q- |
| 0.12509708876289545565636166456040Q-08 | -0.25530533010086300021041259466575Q- |
| -0.56977991931770952528870893146707Q-10 | -0.54084635880092506961571491699432Q- |
| -0.16124193261076881086901667266775Q-11 | 0.90880273230822495252445932538134Q- |
| 0.81939740650425315818835101975490Q-14 | 0.36718234788948390347094338729934Q- |
| 0.66478901678563086709555338281691Q-15 | 0.46878331509950976715841278415013Q- |
| 0.36298899845976511409392896481347Q-17 | -0.97246101753540588089611570240564Q- |
| -0.11477902993575974387154403887537Q-18 | -0.85722889352035132302989781003093Q- |
| -0.14071936188020184797632685837603Q-20 | 0.10637993065025930122342262361335Q- |
| 0.70012077421071644110729790686567Q-23 | 0.18200346906245721156535814822151Q- |
| 0.19382262865583033005476314676931Q-24 | -0.17255361834892070137071407320611Q- |
| 0.32956863323176117616456899350902Q-27 | -0.17304328061460970420185103940613Q- |
| -0.12996014535718458465935896565582Q-28 | -0.65026404692435631841751128455731Q- |
| -0.73853326616182932708216733002268Q-31 | 0.81118017809695419890247414397093Q- |
| 0.40314675092211472256455928534769Q-33 | 0.64503069677714908275465152388596Q- |
| 0.46664030255248143257388071176831Q-35 | -0.13706861136716056723898675952414Q- |
| -0.39791936790084779706185806214373Q-39 | -0.28724219321641868028882506534380Q- |
| -0.15224229801510004279279040037241Q-39 | -0.40081411147563164338234101628915Q- |
| -0.40425921439742921187318175599319Q-42 | 0.69299747040476227748358054711630Q- |
| 0.26606916955491203790215217901422Q-44 | 0.27295390479368539941825227961411Q- |
| 0.14827413084961013706065617424111Q-46 | -0.81592370530087324974146121321038Q- |
| -0.16315569362780249978361478621252Q-49 | -0.68459335431997708071764747987381Q- |

## COEFFICIENTS FOR $J_N(Z)$ , N= 2

### ARG($\phi$) = 67.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| -0.44995534229269107853302059456158Q+01 | -0.14987282147406928911182450982170Q+0 |
| -0.61110083325701180234372434008260Q+01 | -0.94641264126085413311786307220146Q+0 |
| -0.36757421552573258552333586200573Q+01 | -0.19390281076041520307860197731536Q+0 |
| -0.90686699845936955578265133980126Q+00 | 0.11245657268195255702329714445702Q+0 |
| -0.94234951356679112634961056947006Q-01 | 0.93820237975435851929664425337311Q-0 |
| -0.13273060398763842073805064739360Q-02 | 0.13669544603964303552434583197207Q-0 |
| 0.61001161211701446485623588369253Q-03 | 0.86440523583876638740447574862971Q-0 |
| 0.61413523766638477316742115936043Q-04 | 0.14412354090356988972500033315384Q-0 |
| 0.26209471931651662311800661332026Q-05 | -0.14515845275738284868493135124299Q-0 |
| 0.36565842017204958879603295614452Q-07 | -0.11006291293812661069284216526437Q-0 |
| -0.15531101696408236435121406240641Q-08 | -0.33894365126973263747947197270293Q-0 |
| -0.93666706396240503356911953712056Q-10 | -0.38045963922569044655849905012182Q-1 |
| -0.21817006591280280006678801854810Q-11 | 0.85474214081982120509718955146212Q-1 |
| -0.19782003021341064896011854345170Q-13 | 0.42574136369385987769898032284051Q-1 |
| 0.26614698828587752618752512311880Q-15 | 0.77673119418978835720469878355651Q-1 |
| 0.11253162953264551119864674098257Q-16 | 0.57642360443643710501489002080788Q-1 |
| 0.16522419749031455843391993028427Q-18 | -0.50348191230640379666222359225397Q-1 |
| 0.10181744533558583749624242722231Q-20 | -0.18451282873822355328561545431848Q-2 |
| -0.61155410240200565655849989927468Q-23 | -0.22278317297371241301641030674482Q-2 |
| -0.19744909995542389926133848455525Q-24 | -0.11557073765193969887666605703030Q-2 |
| -0.19954188994359969077048315873891Q-26 | 0.49850959967641238530946385375490Q-2 |
| -0.88222975494377960027420820887340Q-29 | 0.14366917875978806128304318287737Q-2 |
| 0.28275243523762499281920105678299Q-31 | 0.12332484547083110508269986296523Q-3 |
| 0.73524296414463768831054797264234Q-33 | 0.46984464722049064624555608678343Q-3 |
| 0.54278527080407694131574343564776Q-35 | -0.11500689693764400708056155668525Q-3 |
| 0.17993690147996346694333608159714Q-37 | -0.27225143319484154885584201478130Q-3 |
| -0.34411643657669401982490333123053Q-40 | -0.17470646374106102551895450834472Q-3 |
| -0.74723876593930379524492458801080Q-42 | -0.50833049452608270615265170064758Q-4 |
| -0.42068336656446721217541811864557Q-44 | 0.77419481158396077986668933903921Q-4 |
| -0.10826360671916244437170219909273Q-46 | 0.15521136976744786294641051514745Q-4 |
| 0.13347306852420770458162373056944Q-49 | 0.77284438964657187350832411421362Q-4 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 2

### ARG(ϕ) = 82.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.21577630050059540153560286526739Q+02 | -0.16312562326469069320421825958462Q+ |
| 0.12952461486699565037709024321836Q+02 | -0.11946898576373331884686626287394Q+ |
| 0.36754973442446046874057122993758Q+01 | -0.47600020585592129837295361263633Q+ |
| 0.55299899330652664888912046006489Q+00 | -0.11229892884441387177722791201445Q+ |
| 0.41619863021846211417538296435140Q-01 | -0.16989701630007108114834741808635Q+ |
| 0.21271076128623024726990230799813Q-03 | -0.17509462216414392194801646926374Q- |
| -0.29119022132967329575680278087467Q-03 | -0.12832089953326133963189347144179Q- |
| -0.34374856208393831597049588211566Q-04 | -0.68770868659660298214304827329696Q- |
| -0.23340411166770530927556879028100Q-05 | -0.27284582771386587928010231633593Q- |
| -0.11182586081339670961699885776993Q-06 | -0.79192596587617789553051480856364Q- |
| -0.40618357896435009269489012518734Q-08 | -0.15615859112124649078143169675196Q- |
| -0.11597791367626047078400681706624Q-09 | -0.13528783601381119990710116213576Q- |
| -0.26566025069406584403685316862965Q-11 | 0.36406046951653031315422340507607Q- |
| -0.49294032281176354390305187426934Q-13 | 0.20242993524865955184132709740121Q- |
| -0.73992638304976658765153708829989Q-15 | 0.55561950689022425491549116100239Q- |
| -0.88081076585386892252341579614512Q-17 | 0.11083861509831164771348972264451Q- |
| -0.77633132834507698162833501897993Q-19 | 0.17592340656588119976608699495821Q- |
| -0.36977710609947574976479947316634Q-21 | 0.23044252493846029191023599353588Q- |
| 0.24881771747158141134693525474858Q-23 | 0.25342409458445191533533532925555Q- |
| 0.87339879511101687253303717475236Q-25 | 0.23540554395148570032254424725912Q- |
| 0.12936075586806783274989800030974Q-26 | 0.18388658266696097045496542337568Q- |
| 0.14048973099259056971137931142480Q-28 | 0.11824292304438070201093173723608Q- |
| 0.12419070461631291026286627096278Q-30 | 0.58652091484410240351052317836089Q- |
| 0.92853211218829721445160431070910Q-33 | 0.17142257189953688515439909518917Q- |
| 0.59694444508020998272300433872963Q-35 | -0.45653105755586483005352074258649Q- |
| 0.33161982395112164109676672190944Q-37 | -0.11560201616636927033661533778731Q- |
| 0.15832802646139366825239628264032Q-39 | -0.10709186857291252058999108141491Q- |
| 0.63576388056973810249565050959556Q-42 | -0.72979995680003681415697860697570Q- |
| 0.20171996219592952990168287903911Q-44 | -0.41018598394445250436045760980107Q- |
| 0.39562558341425562080918840980665Q-47 | -0.19789643077383578865807601644957Q- |
| -0.52175675426047644779776547462658Q-50 | -0.83314859764605237415660029059988Q- |

## COEFFICIENTS FOR $J_N(Z)$ , N= 3

## ARG($\phi$) = 7.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.50925250556809636866863983468064Q+00 | -0.84962239014809072050180230863410 |
| -0.46706664946481771879639241447521Q+00 | 0.35401932285209651108593574906193 |
| 0.21921699100951471687225449461052Q+00 | 0.42563943169020852964286012039038 |
| -0.51777732414787461646877862259929Q-01 | -0.27520433318044407863523037672753 |
| 0.67864454435374947598713827823504Q-02 | 0.67700959179137651280409885894388 |
| -0.50832508956174835785438257435275Q-03 | -0.93870796204083060243567114934585 |
| 0.18068197140903437994015463442855Q-04 | 0.84342549893439423567194681648357 |
| 0.37119689943525489175467582499808Q-06 | -0.52805500073885655386004535276244 |
| -0.87138842186415858918057336085730Q-07 | 0.24004921318739126558003027917165 |
| 0.58097647594262650343524979011044Q-08 | -0.80737831154086676181851915273460 |
| -0.24977140610268428408389566620628Q-09 | 0.19930484676214861883422001840589 |
| 0.79303823291570260898803801723312Q-11 | -0.33671148556354942057694597535576 |
| -0.19611849307878977921283427225497Q-12 | 0.25807165030278999214567015630635 |
| 0.38774032443955133254072360462529Q-14 | 0.54159237088354695085967731039627 |
| -0.61973819034473560999863447568482Q-16 | -0.26753050217177078850705795077563 |
| 0.79843426260864603818222815352738Q-18 | 0.63975808610423454295825456408641 |
| -0.80726193648016924362887681310088Q-20 | -0.11120361983846865713385667643794 |
| 0.58185859831429703337425327334591Q-22 | 0.15407407364367918785758594772261 |
| -0.16764739919615434069079080161075Q-24 | -0.17645057360865911143464954022131 |
| -0.29350254728067527141733820317434Q-26 | 0.16972460784183300688796704661704 |
| 0.64268204178565067998078001351337Q-28 | -0.13763907364534496068677001019777 |
| -0.78880384282497207438374505258741Q-30 | 0.93242939301778280619783205261932 |
| 0.74007571177293579357547207713388Q-32 | -0.51025507243051348364462597462445 |
| -0.57341788315585909341021168722879Q-34 | 0.20208568865732976818848184444281 |
| 0.37813261170437100795922476274845Q-36 | -0.28131847828406713162389385067375 |
| -0.21481425935726755394226619861129Q-38 | -0.41339504687226625787253834995792 |
| 0.10521343460164403289642432890823Q-40 | 0.51386157557610919711561258704451 |
| -0.43898559751401570076458819693292Q-43 | -0.38375799382761814636799684965641 |
| 0.15033405749679821199050236108014Q-45 | 0.22524998926650868277945432407254 |
| -0.37518181807483533990730790537165Q-48 | -0.11144335000837427287017700888653 |
| 0.29201870746858362668646429719765Q-51 | 0.47751270112884295797947019642587 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 3

### ARG($\phi$) = 22.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.34008587629008985463235816209296Q+00 | -0.23830192209057947871152746977233 |
| -0.59148516434357175168458871337294Q+00 | 0.15065120030892570429202581672065 |
| 0.24692759584089332027972554938454Q+00 | 0.18300609191966862080277639303607 |
| -0.23638415114586139635395766226027Q-02 | -0.78871748135575055848184695415196 |
| -0.94242676960611657070183718898293Q-02 | 0.82191525766942548574152555025045 |
| 0.13392367242480653248746036919323Q-02 | 0.19877481873571435680559862675343 |
| -0.57996491516045780438094538085516Q-04 | -0.89716396894178234356628719956988 |
| -0.16789965561354153603525047545804Q-05 | 0.62043361638035106932000808166786 |
| 0.27091069656607619647514810165606Q-06 | -0.14004305136984180759158353524035 |
| -0.11020210367203555539576629410251Q-07 | -0.39666890209873199139290758825148 |
| 0.14751700577529921764624193482457Q-09 | 0.34123924601555549116631781585658 |
| 0.39078842807973538796204320087073Q-11 | -0.91175673232489268165656446195156 |
| -0.21151362824174737227082297081182Q-12 | 0.78745281161598445972608056206396 |
| 0.39899380146550614886167880593223Q-14 | 0.19284057617662403675497368748059 |
| -0.23603483948875394596557380135883Q-16 | -0.72103773677288787920318001566167 |
| -0.53361714342230859270915905134987Q-18 | 0.10104834008462241606211159056287 |
| 0.14648231151088719803709774578091Q-19 | -0.42823351949280228901424690109038 |
| -0.15840762388593448035459485731768Q-21 | -0.89574902390320766038445023822362 |
| 0.49804389085677601098922461506602Q-24 | 0.18847398634397020597013912800880 |
| 0.96726067033612608506435971882357Q-26 | -0.16196076676939654793154960794858 |
| -0.16110998068100851264684102609735Q-27 | 0.38856175005492050445668119508011 |
| 0.11261792227224280182288587691136Q-29 | 0.70334699710567977855840337149938 |
| -0.21099395040356080229121511805235Q-32 | -0.95096621681225044758271511566120 |
| -0.35731980231755944316106768191017Q-34 | 0.55114302893202011251639646214537 |
| 0.40014119003600185971435581009223Q-36 | -0.82217933284076818086584172113640 |
| -0.19535140826680173859201031665833Q-38 | -0.13073399668433373545485189037316 |
| 0.23590192476832162687714119416489Q-41 | 0.12327728605542977958984935912675 |
| 0.35337493866134632348152632276648Q-43 | -0.51381740653004223795047905235750 |
| -0.28449275369035163235782242828833Q-45 | 0.50947156931277310902916852731496 |
| 0.10239958038830100416688838960588Q-47 | 0.72117326727814487956944730989856 |
| -0.84410025842273772860834226273101Q-51 | -0.50154909245760968510622143312971 |

# COEFFICIENTS FOR $J_N(Z)$, N= 3

## ARG($\phi$) = 37.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| -0.37538318311886753109629621978658Q+00 | -0.4952948326352266597746155615087l |
| -0.11794407586607311808837899279676Q+01 | 0.32044517808709056005550321189946 |
| 0.14109145231099988648005913630334Q+00 | 0.54412695818840855855272053579704 |
| 0.12603114550066699911558786855985Q+00 | -0.42817177964281707960143342109229 |
| -0.85885788977404122299717643622327Q-02 | -0.17816108677534487722400192635458 |
| -0.16765585224834512442119770065357Q-02 | 0.11524544382455622682267677028485 |
| 0.10800424029507104182206682174489Q-03 | 0.10981907766054843355131903879049 |
| 0.50806175339334735588312783116020Q-05 | -0.73785830524347327526359624491876 |
| -0.38054954318743118238722586558253Q-06 | -0.16106891091457236259162986677450 |
| -0.29057452894068815274395664162351Q-08 | 0.15221750891303057015038773741491 |
| 0.48173012763417035487687229261802Q-09 | -0.14611908256024532444195137846824 |
| -0.31873735298493839724168719445803Q-11 | -0.12226789316847215435737712496527 |
| -0.25053156471202868533820341052886Q-12 | 0.13182179559972015552135174233651 |
| 0.36058014669806962212360929800223Q-14 | 0.41315886921920062829377153549387 |
| 0.53661754191517423172879721845284Q-16 | -0.75666064745002276309997637407607 |
| -0.12847921502030396325344789611068Q-17 | -0.50988985178267778629358126346460 |
| -0.24870787277071617538104158355655Q-20 | 0.18113527598286554935743936534791 |
| 0.21489344344007092496377996122265Q-21 | -0.23049809735438414277769077899916 |
| -0.79161461578995935576966449569653Q-24 | -0.21562752308373163524544749900997 |
| -0.18234807787833405421434366046254Q-25 | 0.12429771381361677543892846306150 |
| 0.14419216196657195799651394988559Q-27 | 0.12768316092817074718943005180945 |
| 0.70262497813910822890177844368037Q-30 | -0.13633002649641131286001288687894 |
| -0.10904429608937896147422909602770Q-31 | -0.25073768141096667832774976177244 |
| 0.18568725583020640291524178606343Q-35 | 0.75035856350864229043465392377546 |
| 0.44695108678869723717037647133802Q-36 | -0.12672759264206387229486940594703 |
| -0.13420370041921706749672520350095Q-38 | -0.22981736469199130768829027607431 |
| -0.10046198322685886908880099810484Q-40 | 0.99668302655829522912484965831307 |
| 0.60218406066423915384371549635718Q-43 | 0.35761584061470942309182340435499 |
| 0.89969363955225030941036072949396Q-46 | -0.31064700766799898577109171542768 |
| -0.13961889782660211984386929210291Q-47 | -0.38323958109069933599321133004565 |
| 0.12748676841321267651728046942091Q-50 | 0.55079446406115994764862655203203 |

COEFFICIENTS FOR $J_N(Z)$ , N= 3

ARG($\phi$) = 52.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| -0.216679518864650079111865528220310+01 | -0.2552958181837170320704292394668З |
| -0.294632919427981926606908011990050+01 | -0.70459447082310676251090130511511 |
| -0.819413737894356018499659207448550+00 | 0.79627150460509440716089277213250 |
| 0.750830289500522681537922651285590-01 | 0.22997037349883032450393589535272 |
| 0.331330011186266002538687096739990-01 | 0.28587908980680440282605614312491 |
| 0.141765492556010779029902775966310-02 | -0.28972182656575530162178268937726 |
| -0.162053491833184786863570506769280-03 | -0.16756683990543734687778225159341 |
| -0.118263739594572176283044891406170-04 | 0.55281922571502653035537413965657 |
| 0.684044199847563991758293872995220-07 | 0.57950981438054641365540629848077 |
| 0.209121089585030933726455958530510-07 | 0.40948535413590644440394232090073 |
| 0.309006735853524426785638975764860-09 | -0.56936600586443293435177040252134 |
| -0.116551159267051276169805085863940-10 | -0.11893489893498042356831016512074 |
| -0.325010538124222958090562116730640-12 | 0.16983807464115525037181978828928 |
| 0.133181713451524505422810403346370-14 | 0.68682372940500930954798135615005 |
| 0.116225544775161934246745991706960-15 | 0.11115697046711689405315942776765 |
| 0.636926602112230472430035040661510-18 | -0.15965020615153011096636434073174 |
| -0.177439841861473730237022823751690-19 | -0.13817288517742388925301569034353 |
| -0.212843529576649194642781839042840-21 | 0.15488684724944383107555642393066 |
| 0.952407460040464022174139491334250-24 | 0.26039955866776834481295190213896 |
| 0.263499174690161110873574029776340-25 | -0.19437822493445456688275475480474 |
| 0.459409264377213361136981070266740-28 | -0.22421044709752438652662616015310 |
| -0.160818693139732370686343285498360-29 | -0.83056743263886368836689778476353 |
| -0.894722493531882289266584095434770-32 | 0.95969959038396162426034765481421 |
| 0.455601640827715012151073497924790-34 | 0.74703187138220317869568745803093 |
| 0.518280754692414587082026090861570-36 | -0.14660556767934282512681983032629 |
| -0.134478566846152572411522083145630-40 | -0.30697888958015970299463566959604 |
| -0.156735314125224373936930935872150-40 | -0.42749847407912067449926004939565 |
| -0.409043932898815135688309091666510-43 | 0.68805484543383989901348658193630 |
| 0.254883216700416412332697665982360-45 | 0.26557031075826293710937308606093 |
| 0.139281267998343513972080279619290-47 | -0.75326071294472578272128627189997 |
| -0.143777024068676307162047296211300-50 | -0.62218643888510859610141278351753 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 3

### ARG($\phi$) = 67.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.11693430530701552406002433680026Q+00 | -0.94908533553291926889810574263470 |
| -0.22417809065042875710705196776149Q+01 | -0.59515464307962076634658557492874 |
| -0.16613250940787737796704243681536Q+01 | -0.12308908081716627526855901505664 |
| -0.39971748638804349064008993011603Q+00 | 0.38167607146797907873458780785440 |
| -0.39591222979810385602531267866611Q-01 | 0.33008439831471370556822888887862 |
| -0.79218754586774303251138817646973Q-03 | 0.46965006470575256169646521026381 |
| 0.17518247032058274779362611295741Q-03 | 0.28113764263880218645770184154288 |
| 0.17288008041470808118961571663201Q-04 | 0.49349653740021292941526430142832 |
| 0.69952986114237041341408655530172Q-06 | -0.35156757724576811269567793187783 |
| 0.97192634796348782244060674075982Q-08 | -0.26220249033942731469492798473424 |
| -0.32490074215445587478950002509007Q-09 | -0.76820583842501076003267927584637 |
| -0.19327592715727881025273210610633Q-10 | -0.84369683498915137489723600060673 |
| -0.42985768790523241276747467009680Q-12 | 0.15727096759051054475241737711218 |
| -0.37890649741574615803484941955300Q-14 | 0.77439499493647480695613900851562 |
| 0.43689959785323456270706247601321Q-16 | 0.13536512143205719319070537627115 |
| 0.18293466372783473834145948199295Q-17 | 0.97448417427497285180485445298536 |
| 0.25813599716342829448427365014329Q-19 | -0.74586484741905417724624962941465 |
| 0.15426246643115957611966130545646 4Q-21 | -0.27106198715386705166508460413237 |
| -0.82528687281866195654888517636265Q-24 | -0.31540111451501227574565636588994 |
| -0.26453215187624806965274551309934Q-25 | -0.15875351937832931356578563466641 |
| -0.25825347295399728381534294617777Q-27 | 0.61764757498597415223637271407514 |
| -0.11088848732077670751878196562458Q-29 | 0.17688123851977881888789646682599 |
| 0.32378199000912788650350087716519Q-32 | 0.14699044567210975608931749267285 |
| 0.83725208828580657747672691283690Q-34 | 0.54444414527872654303459447660549 |
| 0.59951927460445977393722102293495Q-36 | -0.12241091214703585432669200587771 |
| 0.19343926980866046856515738959107Q-38 | -0.28834931909499854972459062465945 |
| -0.34213118027340395345988524777050Q-41 | -0.17978313940421202491755043683031 |
| -0.73965516495964502432062656319990Q-43 | -0.50970644474581507396220963512833 |
| -0.40521207768169185630526123493160Q-45 | 0.72210648977808843562949417005322 |
| -0.10172117994546187110744083369056Q-47 | 0.14419579042143780871043436695526 |
| 0.11723572682149307961451827403439Q-50 | 0.69964695733440197974688846032176 |

COEFFICIENTS FOR $J_N(Z)$ , N= 3

ARG($\phi$) = 82.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.13575900469681263094854577400065Q+02 | -0.83228132860143043128094704522056 |
| 0.74260978594819319685049727927416Q+01 | -0.59357243038778503156114061717056 |
| 0.18734990105628468231099551035865Q+01 | -0.21822076620690445595743542037313 |
| 0.25274236436432026465574654451976Q+00 | -0.46594399092199903080391718966231 |
| 0.17683204668119724240267223770665Q-01 | -0.63612870900208725048291968200711 |
| 0.21951205242093691446730799338583Q-03 | -0.59362829747544957612967680374721 |
| -0.81464490506594895552782893489411Q-04 | -0.39616528840312343502466411203065 |
| -0.93164202011210831260533841461468Q-05 | -0.19456696097687094772978163247199 |
| -0.59164865278790085046957189295997Q-06 | -0.71205403413879410694595600540360 |
| -0.26418431154009965708462718760082Q-07 | -0.19200444354397972223066498011515 |
| -0.89584303187714822408532063331023Q-09 | -0.35538568234861532270193078976442 |
| -0.23951776687349306923189383923534Q-10 | -0.30196890376565210432150358725738 |
| -0.51545109834101556736260902337467Q-12 | 0.66346034944260410379720373341865 |
| -0.90158948863212747495544145988602Q-14 | 0.362619038110869182144666485879833 |
| -0.12799995574903340137184703946706Q-15 | 0.94825086926193058591388939056546 |
| -0.14462982033429893297924266729357Q-17 | 0.17983715329914395315678835210443 |
| -0.12158147456825895765407174293631Q-19 | 0.27163593534610703578980956129186 |
| -0.56090393252331978319271121106524Q-22 | 0.33919048513474446716540990100221 |
| 0.33413434827099165004106299926370Q-24 | 0.35625878055722559598854124836971 |
| 0.11605660745207039236613969093339Q-25 | 0.31666739153983896377456492327922 |
| 0.16554585074753903364158190370335Q-27 | 0.23716311687073832646711458010719 |
| 0.17289549750477798792856905937503Q-29 | 0.14651619482518552390026576463440 |
| 0.14706725409553425250980019186523Q-31 | 0.70017832008660076927674055294686 |
| 0.10591911367691039353717601465591Q-33 | 0.19871335794095903925430899299327 |
| 0.65671655299119580507363909103302Q-36 | -0.48448655962631052754549237486530 |
| 0.35227488390307387442202130647481Q-38 | -0.12181298557566038493976461039438 |
| 0.16260678830070716787254865737623Q-40 | -0.10943186217701554612197268761751 |
| 0.63212569600875150804616250045993Q-43 | -0.72241897291348648907563097209059 |
| 0.19451819268327589805654164142973Q-45 | -0.39349488565269600452916705445038 |
| 0.37177404591943802590076286759559Q-48 | -0.18411205333787789927741077359729 |
| -0.45738016172347818943801868962146Q-51 | -0.75232438791108561631250850576772 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 4

## ARG(ϕ) = 7.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.61707465320865149158462059664872Q+00 | -0.11162423193963405419210071600016 |
| -0.47899266909901679450721553667617Q+00 | 0.11653252480512891955564119692594 |
| 0.17508560820112288506925497648103Q+00 | 0.43457152968904633600295173497176 |
| -0.33465244490471403660999568595588Q-01 | -0.19594588134833241709465157864110 |
| 0.36768244861877712050593385470246Q-02 | 0.39122508747162658638001449310619 |
| -0.23551375710519716957164694938427Q-03 | -0.46279147791174398849452049335692 |
| 0.70253335680836713756151966380339Q-05 | 0.36460626734307896219610558117743 |
| 0.17822176167867288429706136369577Q-06 | -0.20379559132556685909750134911683 |
| -0.31774102994679681144185589079522Q-07 | 0.83779540877131821000803293916941 |
| 0.18989616305575355841175466013849Q-08 | -0.25725010628764238433555288414573 |
| -0.74726089188694219963755258126763Q-10 | 0.58360328118374169257371459322768 |
| 0.21940135591934055424956326086021Q-11 | -0.90810674087170249189178826870733 |
| -0.50525012369727457632703918835801Q-13 | 0.62391288015555698050896192321046 |
| 0.93517918032201688558851868532484Q-15 | 0.13735886807696579704393839636471 |
| -0.14053781175433950937641967523629Q-16 | -0.61719072905522430905420316304699 |
| 0.17082254268020806594297386702620Q-18 | 0.13846345607410200500912461653265 |
| -0.16333985858468608702029822386828Q-20 | -0.22743227546343717072338989332961 |
| 0.11134678338825295811913149252484Q-22 | 0.29900835594250349208074176912041 |
| -0.29586870916270095550832982668110Q-25 | -0.32594499715672982013056733239620 |
| -0.52922231569654998587627064117057Q-27 | 0.29918099976638921820875256811441 |
| 0.10933659567461159154274038408959Q-28 | -0.23201907184683599379949498669116 |
| -0.12822338432397996545470212937089Q-30 | 0.15057413405620065570097641513132 |
| 0.11536782404829670605307856020688Q-32 | -0.79030980203692379929428493448523 |
| -0.85915067567035891272346259758402Q-35 | 0.30009305408709225574198196768933 |
| 0.54549936091789251970666370258561Q-37 | -0.39160874634687461146417979362009 |
| -0.29881873230115815162231753816472Q-39 | -0.58255340429185543242716412191701 |
| 0.14130676637375739180904968528082Q-41 | 0.69410099701233430290900228389769 |
| -0.56983405419129610364929887904321Q-44 | -0.50027691215049540912333171863788 |
| 0.18874231821179957395223511912640Q-46 | 0.28401886605845147913178150786837 |
| -0.45538440730657893020548210498488Q-49 | -0.13610721282700124756305794132673 |
| 0.33584499326722136235133375949795Q-52 | 0.56551980426602063287126540245506 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 4

## ARG($\phi$) = 22.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.47858339845587016575111073716333Q+00 | -0.35847880080382417788568569890391 |
| -0.60398352290372192075444818440342Q+00 | 0.40632428794241340157156783452084 |
| 0.16686671310962534204250806450641Q+00 | 0.16629085520708918341439311126107 |
| 0.40345292483443152819878830098438Q-02 | -0.49766228978439606117485686238686 |
| -0.54599958296902657607385931551026Q-02 | 0.39930764421894391102585095867109 |
| 0.62594895628520195833684291377430Q-03 | 0.13689949431864912037055080931727 |
| -0.22303871755186601890395851948738Q-04 | -0.39065293174135859343023948506874 |
| -0.74375409592514713567040461212137Q-06 | 0.23233331698105377436414273246902 |
| 0.95416511092663066265959827097328Q-07 | -0.44787001805626294222260790979126 |
| -0.34549010757675574265238001805361Q-08 | -0.13722855455804840086976691363690 |
| 0.40467415510471252652986161110957Q-10 | 0.10133317075579841544802066321847 |
| 0.11275241187728454178113936728650Q-11 | -0.24624481751371634772507775934581 |
| -0.54408768179534532450735646862331Q-13 | 0.18934026176918888576081649872867 |
| 0.94733620521586490822347936808130Q-15 | 0.48034252984740057810298659561047 |
| -0.50549494253955906945767499549168Q-17 | -0.16378486563169320587055261988978 |
| -0.11730705133561013754648955905740Q-18 | 0.21415725677342929483984053767426 |
| 0.29810397569169469443188598648929Q-20 | -0.82707787754565565932805977103269 |
| -0.30326156803101673086132273555546Q-22 | -0.17652586095207331527300040590205 |
| 0.87615839855007810460101209678226Q-25 | 0.34755278459372069147965657032196 |
| 0.17291106535658113621398305726830Q-26 | -0.28278333730192236382839722367796 |
| -0.27167393490286492253396894648523Q-28 | 0.62772607288472691330528617791105 |
| 0.18075318540894420124976071168793Q-30 | 0.11512085536535432206883422631772 |
| -0.31516590654169590205189505675429Q-33 | -0.14774859465916066435518648510637 |
| -0.53956065858586768965740091431130Q-35 | 0.81857224904708833701748903342447 |
| 0.57644606550824575210687542980848Q-37 | -0.11421196838474935790855873900194 |
| -0.27000794992421892848321668707222Q-39 | -0.18327913293503432305610207781976 |
| 0.30627909725211442080843686338935Q-42 | 0.16556396963851835725884044634448 |
| 0.46240912045606052799970146003952Q-44 | -0.66412829031121063709651579001610 |
| -0.35786702586617823244401408021218Q-46 | 0.62091148638872115048586062942334 |
| 0.12429939179327096891519231765562Q-48 | 0.88490360464235837516568810837560 |
| -0.96935100469648331206828051897891Q-52 | -0.59334086572687432172048695077262 |

COEFFICIENTS FOR $J_N(Z)$, N= 4

ARG($\phi$) = 37.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.18596588160846084309094512679226Q-01 | -0.83327456781790560048981191072186 |
| -0.10472813165080234206378655940024Q+01 | -0.23379409460784507546257516418254 |
| 0.22861130673403583133034143535860Q-01 | 0.38891503655067472812879085674954 |
| 0.76925556799089949340319790241190Q-01 | -0.14286325306712697278710371127870 |
| -0.33557411451626374023868589900716Q-02 | -0.95408818610942142823813271639459 |
| -0.80215233972674472394427750221481Q-03 | 0.45280242563351429654277595834360 |
| 0.40560003213306685681161103405422Q-04 | 0.47659805819213334908405515569687 |
| 0.20314863038278054137770465596451Q-05 | -0.26024717800668946227450502460502 |
| -0.12542025840556014666837386183948Q-06 | -0.60716746336200177495444405774914 |
| -0.11064181450750886837317709182936Q-08 | 0.46876617621800903037316614470047 |
| 0.13892236963532422654121118373402Q-09 | 0.97471498252537357004611771529370 |
| -0.74902431978003088282855914655543Q-12 | -0.33123550731520711449250499164713 |
| -0.64000382981119738235785867234274Q-13 | 0.31360205936665601104936473951791 |
| 0.82851810909758728164620685698416Q-15 | 0.99972584927366436385250274418122 |
| 0.12375099649773140426733646397718Q-16 | -0.16632040990425994850169468169542 |
| -0.26960075718497058723171199394135Q-18 | -0.11345106774640789016052704771030 |
| -0.56715828224072714334552468326877Q-21 | 0.36297308048243790110594255415072 |
| 0.41175150786585550139516190632497Q-22 | -0.37158818768928323924375676445762 |
| -0.13842387820378683226010717207477Q-24 | -0.39575847168963653286210650736910 |
| -0.32129274350128994447041821970772Q-26 | 0.21244245531527735315247877664500 |
| 0.23833200381048606904541909263522Q-28 | 0.21661759951264747608971649313377 |
| 0.11537287109204902126134490841740Q-30 | -0.21743494457732360842346240047560 |
| -0.16780081282800572964898498501303Q-32 | -0.40544100927353537115808182204488 |
| 0.16133733140509804883410636119110Q-36 | 0.11148187329989242148751333303105 |
| 0.64178896905790992470364690251291Q-37 | -0.17536106784218317662247602493017 |
| -0.18269610431381703756363969257346Q-39 | -0.31936600106966392042919728288923 |
| -0.13534635626020483130880298257246Q-41 | 0.13205510126486170795507352037558 |
| 0.77495636109654824166525796198628Q-44 | 0.46845368098875221406609068741662 |
| 0.11553119581033604062170251472072Q-46 | -0.38821534581559215861432687310234 |
| -0.16950459436444485726195934498425Q-48 | -0.58251291865902313649433300485941 |
| 0.14600885484873033958444159950309Q-51 | 0.65006354456736548999616528861831 |

## COEFFICIENTS FOR $J_{\nu}(Z)$ , N= 4

## ARG($\phi$) = 52.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| -0.674316049343086111839063045561930Q+00 | -0.26291341676897096640229140081480 |
| -0.193846673996853944905265051101240Q+01 | -0.10410272755675281411060305581265 |
| -0.592335937150422127492784452627160Q+00 | 0.39916376193758198657851175917905 |
| 0.239714444751003816570991745454640Q-01 | 0.12977804720043558884324304795512 |
| 0.157682784123955214764521114611790Q-01 | 0.30277141884742357301873331260589 |
| 0.715157473101798042661835967907470Q-03 | -0.11976768528914976635851117083853 |
| -0.587917941715060376530653227188660Q-04 | -0.69757013492491852547086596624727 |
| -0.433017659328330013680361444200000Q-05 | 0.17300020915226922485073314291293 |
| 0.132072520871013487296211875830820Q-07 | 0.19117669108231520650317366797943 |
| 0.629644381195940769145618295778590Q-08 | 0.15079100737933837614766970667741 |
| 0.931065558543292990849081763214470Q-10 | -0.15765879506999989760538123699161 |
| -0.297776783424170993974807587074970Q-11 | -0.32381332719630002149171751649436 |
| -0.817380050872461924173309771758390Q-13 | 0.39835900861026599368285284087458 |
| 0.270962262331807344237194639668550Q-15 | 0.16116595124438932458661503901991 |
| 0.256007899319607627893123292661930Q-16 | 0.30274676467340444573433841510132 |
| 0.140023083355007717240172715793510Q-18 | -0.33142010984589437929659727108294 |
| -0.347968654596960381236102566084130Q-20 | -0.28130837431868063598349335463571 |
| -0.408332082223706976167299175137970Q-22 | 0.28690261828368936245397622616026 |
| 0.165258894501858235107453832359160Q-24 | 0.47399344680835209688268176432528 |
| 0.456896608299403146554293610167230Q-26 | -0.27322360504227708170750262702342 |
| 0.810539134067720609295557149802260Q-29 | -0.37134018313482015620695320627512 |
| -0.254878747045782761687503319171810Q-30 | -0.13547524329257155625584437724038 |
| -0.138834165019169409419361537899280Q-32 | 0.14568537857486493310057992090527 |
| 0.661802670408783544790561248695630Q-35 | 0.11103812491615846988676409157209 |
| 0.740406150972394709324878829327610Q-37 | -0.20188322501461295774039235548459 |
| 0.188511779483920225753572201090550Q-41 | -0.42238500444041098639977615069091 |
| -0.208025986994127931174419987389950Q-41 | -0.58593252142294032781581871951920 |
| -0.533530342138452905091541283518560Q-44 | 0.88180322326020389086419555206732 |
| 0.315541681913616939051404400090920Q-46 | 0.33360738326654237044021637058908 |
| 0.169129358326308152296534898149500Q-48 | -0.89973622724197353612595862148582 |
| -0.164116302756977485229175473599950Q-51 | -0.73176324326869683943942789831000 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 4

### ARG($\phi$) = 67.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.17478308810061849594833468364952Q+01 | -0.66194758598597459543856475851872 |
| -0.81451244107349084752959623722662Q+00 | -0.41068919137690145718272612108368 |
| -0.87536947783269953602379997414406Q+00 | -0.82591853917344660800762182964988 |
| -0.20578643623327606002146209271395Q+00 | -0.16680959495309474153986863319611 |
| -0.19344070130668884008151426869896Q-01 | 0.13913724227485082576341646704694 |
| -0.44358069041949821486810380139798Q-03 | 0.19379629177747191486997735033417 |
| 0.61373074511090406296438527436609Q-04 | 0.10989287885121069186491975715268 |
| 0.59490346685145652447576302702390Q-05 | 0.19606401823644366683330741958572 |
| 0.22863284191338328246400186474230Q-06 | -0.10522755025430343106763309244622 |
| 0.31296575380685502381399149337765Q-08 | -0.77317419591320704186286848525878 |
| -0.84815184622302783172770734723515Q-10 | -0.21592121889232789927792895111167 |
| -0.49828590060961443768738531126488Q-11 | -0.23112022582694850348317639711192 |
| -0.10600687748105359961439338405128Q-12 | 0.36384575514272293447168754258605 |
| -0.90716133638246787980117319828966Q-15 | 0.17727624818029435085248755522649 |
| 0.90722982730813810143142598398643Q-17 | 0.29737712108072386055985444703670 |
| 0.37646215242868884468272127345504Q-18 | 0.20761191689326657707855296423502 |
| 0.51126045085775139077471353345781Q-20 | -0.14045941646176158486032102789351 |
| 0.29635295121127430306177011345921Q-22 | -0.50650973522644762394264706258036 |
| -0.14215662900131027264781995002076Q-24 | -0.56867571852340752506809503393893 |
| -0.45259194996881733882690665026370Q-26 | -0.27784910222795835267899756716588 |
| -0.42731147865953245801123781185082Q-28 | 0.98016706940191268100398434974134 |
| -0.17828256119295104315586809671403Q-30 | 0.27904022336114498223982253053442 |
| 0.47629207735237014918243098782943Q-33 | 0.22470988052046831713649118116499 |
| 0.12251829152061523705586244100378Q-34 | 0.80964039689815454223922197685256 |
| 0.85168555512182592205664516640740Q-37 | -0.16780270121148288672268754223810 |
| 0.26761777544667120227006334359403Q-39 | -0.39343392534244726163139429798784 |
| -0.43906583618681659487027139423517Q-42 | -0.23852578685130268957350077272058 |
| -0.94526112759914035895224981949803Q-44 | -0.65929186258990028428286548651990 |
| -0.50427838224905459493005678993593Q-46 | 0.87107045263259653563603792420786 |
| -0.12354561327896986402262314335323Q-48 | 0.17328877807787881999392630386195 |
| 0.13340812748734404862883277202692Q-51 | 0.81984853823596959338196585973842 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 4

### ARG($\phi$) = 82.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.98122292247818075407110420560794Q+01 | -0.49939468763089119938816864418543 |
| 0.48832354861099685524819165983418Q+01 | -0.34758105255632773289114962151392 |
| 0.11028910174367510904147561175444Q+01 | -0.11824930302905818102046366018994 |
| 0.13431951982640030883170246899945Q+00 | -0.22990349715530874614607155512091 |
| 0.87143496211064584183815281719839Q-02 | -0.28534639188487899037317498514712 |
| 0.13997349394348195614743773324404Q-03 | -0.24292381939137248494408125352582 |
| -0.27927636890605182623288025847636Q-04 | -0.14866350009098283006055144222095 |
| -0.31053041052811411580125929655642Q-05 | -0.67331227706390201337431243081123 |
| -0.18518525881004409104918159788508Q-06 | -0.22855444429099529823861539448674 |
| -0.77395561630068892908515044357180Q-08 | -0.57522055291281920056346591454779 |
| -0.24602039490695152607333914045855Q-09 | -0.10026046825515525470128333263163 |
| -0.61827845802677576602538615895829Q-11 | -0.83105966738023440241376553208248 |
| -0.12544170650210006493739696961146Q-12 | 0.15219340037548777728432249696751 |
| -0.20748344205751257170963245911014Q-14 | 0.81888196131404784201987820749980 |
| -0.27939457251182820209275218884916Q-16 | 0.20446783439118173708310906820989 |
| -0.30039996922462585466041011345648Q-18 | 0.36952360624357722633526357856376 |
| -0.24134006897302081300238199079291Q-20 | 0.53236032666123342988001889124282 |
| -0.10785727668468657528862656270440Q-22 | 0.63503605050550036250646666474350 |
| 0.57297902530709159204788946863691Q-25 | 0.63827737536700596846556843661887 |
| 0.19707262951691293808292859786752Q-26 | 0.54388005065971298526212239907721 |
| 0.27109674309251565800998203063078Q-28 | 0.39118281044915349604334950218645 |
| 0.27267056870161387207223405438325Q-30 | 0.23253380070455849461228732917042 |
| 0.22349322784755007962772743707569Q-32 | 0.10719801438885866629098741109729 |
| 0.15525575954896775082223107778313Q-34 | 0.29559852202312589321669281779444 |
| 0.92951885673193185419279612253250Q-37 | -0.66231573004925627882633328620308 |
| 0.48201989329767017037363660866913Q-39 | -0.16541428636695416901109739968881 |
| 0.21534425077380531316532338471303Q-41 | -0.14423806142699217190895795342110 |
| 0.81126004233814172262601102781812Q-44 | -0.92329936323118382813668100798005 |
| 0.24233021164315233840642291443170Q-46 | -0.48783625879867703740192006025696 |
| 0.45159891653534288880407522684542Q-49 | -0.22156129286009529837710376141338 |
| -0.51950778537904488178642440347368Q-52 | -0.87948052367414230767332603921507 |

## COEFFICIENTS FOR $J_N(Z)$, N= 5

### ARG($\phi$) = 7.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.71697914072940655358743725506113Q+00 | -0.1281096389441442504755468170444 |
| -0.47386022697282716563049859782970Q+00 | -0.6040193154996630409852476689589 |
| 0.14241000822124513469978699293594Q+00 | 0.4092611450399297525484680296178 |
| -0.22942102954627750376381887415771Q-01 | -0.1438795199536927488243701803695 |
| 0.21735212462738713192394363868938Q-02 | 0.2430579084793338806830369680232 |
| -0.12157912694591354508852595915341Q-03 | -0.2513922139405059099512782844472 |
| 0.30958160927312461489488614225558Q-05 | 0.1766161617479967594212758709870 |
| 0.90791005990642409076600361788274Q-07 | -0.8924323424893056552378000205756 |
| -0.13170916173648693904323192527625Q-07 | 0.3350361306922193432483877382704 |
| 0.71414678281173427867145611364911Q-09 | -0.9466900499609805005274864898814 |
| -0.25923272126626176198775258705894Q-10 | 0.1987074889594738886406187088804 |
| 0.70809469332575257506240448526595Q-12 | -0.2865098571635790357686375605412 |
| -0.15260319342339457193827018152257Q-13 | 0.1775480892996323401973075566070 |
| 0.26555888369042496294964012284251Q-15 | 0.4070931080694027583576494443714 |
| -0.37661221044883500514132163247608Q-17 | -0.1679301965259418263485206654739 |
| 0.43330139470785595892640321717940Q-19 | 0.3548836067972215284631452050285 |
| -0.39300755322157125335656254858107Q-21 | -0.5525744955237430313179847934012 |
| 0.25410247270471242011113817105137Q-23 | 0.6912204571291114622317925865522 |
| -0.62475343329925297792681888879026Q-26 | -0.7189212395147127307466521687052 |
| -0.11392523885996735011537255092424Q-27 | 0.6310582222764799395732641168046 |
| 0.22283904212773785077478549522770Q-29 | -0.4689187359509309494730860283120 |
| -0.25021610813047943440791944704873Q-31 | 0.2920483712710857437619572854871 |
| 0.21626872452540393437799926246575Q-33 | -0.1472674125667668399844377610251 |
| -0.15503792215307211064377192417985Q-35 | 0.5370264674295039067386542704930 |
| 0.94913020464894254857724757823214Q-38 | -0.6583129660080458426322210345786 |
| -0.50199261387064423797917935184703Q-40 | -0.9904230048223671033049880863297 |
| 0.22946865654543267788085145016494Q-42 | 0.1133187769390830125413977867107 |
| -0.89537870504448266688389759635979Q-45 | -0.7892388382898683537342446642861 |
| 0.28715083805539291671157205126166Q-47 | 0.4338599335730048859276198753392 |
| -0.67052186969410863999365125272740Q-50 | -0.2015870773878610635035480088045 |
| 0.46923818429544725569882622053428Q-53 | 0.8129599849565099729668785158397 |

## COEFFICIENTS FOR $J_\nu(Z)$ , N= 5

### ARG(ϕ) = 22.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.61582012433420544087143086184347Q+00 | -0.4249622641999541928644299794164 3 |
| -0.58343349628173330704266007465410Q+00 | -0.3231379325236084352698437132554 2 |
| 0.11788340079211215724294026554150Q+00 | 0.1448031241833580517080918036972 2 |
| 0.55114431191899834485904756538639Q-02 | -0.3333177582206879285137568080911 0 |
| -0.33810804366530376217085755938205Q-02 | 0.2142320526594883327715532654107 6 |
| 0.32509988741072670952114504187016Q-03 | 0.8988171039451795293534956277994 4 |
| -0.97347339053242940752565900864295Q-05 | -0.1899316648125463078562183867871 5 |
| -0.35925766887001947290237286617942Q-06 | 0.9908497870344766042566731540206 9 |
| 0.38412997117060888382217399353174Q-07 | -0.1653219544429968725387908996722 8 |
| -0.12529754199096076266020138004850Q-08 | -0.5387601310832791918468599099536 0 |
| 0.12962684468872675308852299710919Q-10 | 0.3490965069689931302551794828266 5 |
| 0.37653165423365109051489025444587Q-12 | -0.7777776331353001260686700300553 7 |
| -0.16406004710931099561085353213880Q-13 | 0.5361589084912883994917752033064 0 |
| 0.26519477025958821795692168548691Q-15 | 0.1401754739798015095364414780667 6 |
| -0.12833434962539365149233053981520Q-17 | -0.4394827731508630911748590800493 0 |
| -0.30468544147510900160849941440818Q-19 | 0.5385077989502379137477125761071 7 |
| 0.72094795341659825117788492771806Q-21 | -0.1903567859637340908711059079889 8 |
| -0.69231210956933489758852324864241Q-23 | -0.4136162667818810892547882036991 7 |
| 0.18446222343867152808710836554396Q-25 | 0.7652956100065988785807069316793 7 |
| 0.36932372130467720638983248404330Q-27 | -0.5912009415341317961373735763478 4 |
| -0.54919384021175787593228186921171Q-29 | 0.1217961871920571347456022636969 0 |
| 0.34857917153892919364824406367005Q-31 | 0.2260328558720564803247323998369 9 |
| -0.56711142575671630963031283454505Q-34 | -0.2761011004480708989631637489556 3 |
| -0.98059167978067165415336582134552Q-36 | 0.1465075927084813188623473045114 2 |
| 0.10016372200594681966882082484507Q-37 | -0.1916185359367250085711274867074 0 |
| -0.45086065675090317186329685555353Q-40 | -0.3101065530246274266859439833632 6 |
| 0.48134530731157193798330808692913Q-43 | 0.2688474246087974721133113654174 6 |
| 0.73202625326800317699746247565513Q-45 | -0.1039330292329238214354737328021 0 |
| -0.54543725582602672918138773426936Q-47 | 0.9177941154632585652831288867500 6 |
| 0.18303522105556231803122942271802Q-49 | 0.1316319611780607009862225642608 7 |
| -0.13524684808351802967372348147330Q-52 | -0.8520645422371296646840918181145 3 |

## COEFFICIENTS FOR $J_\nu(Z)$ , N= 5

### ARG($\phi$) = 37.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.34271673950163347231035737673710Q+00 | -0.96997600846342206562275998669492 |
| -0.90395145645904504079575341243072Q+00 | -0.20010106263287621584900699048914 |
| -0.24332167615711741973340595630920Q-01 | 0.28695528475334629442706664096804 |
| 0.49998478659224818611522032055104Q-01 | -0.36599080067565330240198263053891 |
| -0.14186604474643848213569897154760Q-02 | -0.55520468259598492105133906234942 |
| -0.42309091080541084254334064191816Q-03 | 0.20026752188886249285692876782968 |
| 0.17347393353851377306213391080437Q-04 | 0.23037880182879879602564535572771 |
| 0.91062882751462548428649117203691Q-06 | -0.10526470076387639862285311108995 |
| -0.47671039669559336403597319006625Q-07 | -0.25670358996101463835218623527372 |
| -0.46142139953847916911132687994639Q-09 | 0.16731566143544227154904424365605 |
| 0.46639403997328763238733161822253Q-10 | 0.18485270051813267883765069199952 |
| -0.20568169740169357679978548848568Q-12 | -0.10487591332221130417986276068500 |
| -0.19173217938208356309614082976415Q-13 | 0.87956765871607055183013939264241 |
| 0.22502493730692870953488001784666Q-15 | 0.28448730690346123678701991110462 |
| 0.33629084127059927188361924102595Q-17 | -0.43313414379833854058885931254798 |
| -0.67175780662330658806086383663502Q-19 | -0.29746229171361208816143622504195 |
| -0.15021647882146584728224300184942Q-21 | 0.86550824282389367863091671135096 |
| 0.94065741761849437918248891109544Q-23 | -0.70330548295066949916128271759833 |
| -0.28981427086313735482005493702998Q-25 | -0.86761261623173101256083530682410 |
| -0.67728077561812485649204244424999Q-27 | 0.43532400850248549441176642983467 |
| 0.47290291910170865176932086864137Q-29 | 0.440251746319280357977908466433849 |
| 0.22712985897226654629537788448812Q-31 | -0.41685131288108337968847344709817 |
| -0.31077866036750750463580637244007Q-33 | -0.78476726437269298322978283090087 |
| 0.98056591883695818529040791379658Q-38 | 0.19958683771471104547915481495776 |
| 0.11117563959062473154599691517676Q-37 | -0.29314408553573895839707189569523 |
| -0.30069488114383697187470214663688Q-40 | -0.53595955518948538697058814951029 |
| -0.22040763450304907755485113466757Q-42 | 0.21170976937587968419163609837721 |
| 0.12077744786114739398624158169293Q-44 | 0.74225129733534441407511066885997 |
| 0.17943104936990429284669869658933Q-47 | -0.58804257558962812083410259189938 |
| -0.24963558961590501760396491216548Q-49 | -0.10207215535770114618693572554814 |
| 0.20319543904364873830470737754352Q-52 | 0.93142766633216217154426800012190 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 5

### ARG($\phi$) = 52.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.19134774173865152867894503974156Q+00 | -0.2462179773326490749072602554555 |
| -0.13410036861085022606683030321361Q+01 | -0.1092591101505386532694648880667 |
| -0.43845603418680898540349596696354Q+00 | 0.2151361928310771474182727675070 |
| 0.61723344099748239624731835591553Q-02 | 0.7951238806590402162578290821500 |
| 0.83628449662742907965501530518177Q-02 | 0.2347439467425407440984691574691 |
| 0.38941724828587909271943702193489Q-03 | -0.5606991828633644345960859210470 |
| -0.24426146649227052561986376621723 4Q-04 | -0.3260736836138470922926634833196 |
| -0.18102382380689528814035564973803Q-05 | 0.6229870275795387719800956911727 |
| 0.19798766584757163921666068826589Q-08 | 0.7276089041580664875321350389222 |
| 0.22039112038813480231647834129789Q-08 | 0.6130824092806699791773259911015 |
| 0.32385885237303927509881225123713Q-10 | -0.5106187712249228261443447131873 |
| -0.89444335278397792082285264911973Q-12 | -0.1029192025466707955967206558716 |
| -0.24150785889810286212583154564851Q-13 | 0.1103236333531331772114324204379 |
| 0.64815422360608350759096065615807Q-16 | 0.4463787553233911690825692177286 |
| 0.66810843898091770639647078808109Q-17 | 0.9269586849727737286432345693968 |
| 0.36323083665441114503133301659175Q-19 | -0.8177762315830066610215497080587 |
| -0.81343852763051679999756901066615Q-21 | -0.6802424861385947855192974150398 |
| -0.93381340412760644772153161873733Q-23 | 0.6351848652884501291965822580982 |
| 0.34354871054664556888469183956357Q-25 | 0.1031285803341455630251236520685 |
| 0.94911484550258641248308930651720Q-27 | -0.4457111559829141113554734387714 |
| 0.17031444136464931770506196305733Q-29 | -0.7382743952332097545393351928542 |
| -0.48578743909211572065632292397634Q-31 | -0.2650728391529885546022393986148 |
| -0.25909913755614495647699509562557Q-33 | 0.2664056709511302371512678912589 |
| 0.11599071965002320372923680273074Q-35 | 0.1988708366925056500076718638071 |
| 0.12764681207709623193648329237417Q-37 | -0.3359484385002135783993679064153 |
| 0.89474368220430176750940843347451Q-42 | -0.7023201600212431011903020641900 |
| -0.33406772105841167518488778953415Q-42 | -0.9689902529575841851883177556232 |
| -0.84197635403236285158640135286311Q-45 | 0.1368955267763597293703712146947 |
| 0.47371904850503272461772572020089Q-47 | 0.5077778388774346063951335467005 |
| 0.24912839938172068348751578684156Q-49 | -0.1304696177755200509262107524625 |
| -0.22767390100986997949332285831625Q-52 | -0.1045038787150044286666245151528 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 5

### ARG($\phi$) = 67.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.23981379237132517651678309376418Q+01 | -0.4933200555032287624040337815021 9 |
| -0.20525674795803005230639017010004Q+00 | -0.3025770761176945301852439300839 4 |
| -0.51194189051474125210311990172914Q+00 | -0.5832525646946106640054796330338 7 |
| -0.11783901280303569235507495247674Q+00 | -0.1829002723579448938509129979287 5 |
| -0.10500271911249658774046290593610Q-01 | 0.6653200727616868570912151778529 2 |
| -0.25495321324383426263268840799403Q-03 | 0.9091448258547759482557505597046 8 |
| 0.24763307785399520504172435767361Q-04 | 0.4890193606978280161676313389846 0 |
| 0.23619067867422398562538461954382Q-05 | 0.8699482883102290992694709505288 7 |
| 0.86375897366343845473995894075233Q-07 | -0.3668026554301623141003832160971 7 |
| 0.11573140285259174918407815161511Q-08 | -0.2658897354161448728630964260464 8 |
| -0.26006557181715955841502576722893Q-10 | -0.7090814858768607918405597217475 9 |
| -0.15105450592156427261576262018224Q-11 | -0.7379757258334805757743921802313 3 |
| -0.30791385089310038253611344832790Q-13 | 0.9953920630986081570801608116049 6 |
| -0.25562927563562604519602401344136Q-15 | 0.4803018044479203736990069448576 9 |
| 0.22398653515025010501105501440846Q-17 | 0.7743485379087653973591725596433 7 |
| 0.92173892471594089605724009443909Q-19 | 0.5242787532301718962916723649355 7 |
| 0.12063536516977147054900456112827Q-20 | -0.3159046519020752679603506637603 5 |
| 0.67849484691602976594861482687328Q-23 | -0.1130985546093956698978032703481 2 |
| -0.29354428529449662659315802493393Q-25 | -0.1226668369181769669858354888174 0 |
| -0.92869223791201824147869710746236Q-27 | -0.5820634562549619308792177525839 4 |
| -0.84885366148441779760122510864205Q-29 | 0.1870632350071538206743232415556 5 |
| -0.34431947043564024345830517583079Q-31 | 0.5295897509384087315120590284267 5 |
| 0.84491575361621556843577567692772Q-34 | 0.4136629611669878477020364531953 6 |
| 0.21627059020978384782016328300234Q-35 | 0.1450669703879791208756317951776 5 |
| 0.14607248631959377750377825159495Q-37 | -0.2780553322141331044971580841808 3 |
| 0.44723917842882512800766843018665Q-40 | -0.6490673184653033745054776083362 2 |
| -0.68253755818347128492248362790094Q-43 | -0.3829229511314785791363631568816 8 |
| -0.14636224629711351429111469867286Q-44 | -0.1032421107263795965165340835036 9 |
| -0.76086249459202642169895219398768Q-47 | 0.1275164361514210624601326042369 1 |
| -0.18201660115155374344210634802803Q-49 | 0.2527728380297841958285438720424 1 |
| 0.18453475319867493876068106760685Q-52 | 0.1166799358779194474269010026751 3 |

COEFFICIENTS FOR $J_N(Z)$ , N= 5

ARG($\phi$) = 82.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.77349979465329318658546296981541Q+01 | -0.3333773794543033202078533182204O |
| 0.35075052598263660970318803005207Q+01 | -0.2269546804746748171385428694361l |
| 0.71443640519736975728515979971239Q+00 | -0.7168977840390038115927580333226 7 |
| 0.79098674164917196540425177630990Q-01 | -0.12766841129833068042824049968300 |
| 0.47635422928683202567967169104111Q-02 | -0.14503000552541207882153714433030 |
| 0.85057037674741627071044035892395Q-04 | -0.11338848564491021574635009275998 |
| -0.11064722576980576856811563465542Q-04 | -0.64024904622688522266296189188318 |
| -0.11998577899169309507249831446954Q-05 | -0.26889678760726523523917542463344 |
| -0.67433584450771308705753034016955Q-07 | -0.85075706289431982940497870704962 |
| -0.26479788437897489042152562565948Q-08 | -0.20067530116799382346078393854608 |
| -0.79195005488640832975447776794140Q-10 | -0.33036628169209694031425533556169 |
| -0.18771615506343912389516237387905Q-11 | -0.26627006678828183716080535920222 |
| -0.36018508382575450479477611315621Q-13 | 0.41322902176562134088424054770373 |
| -0.56496473798025474379797909202836Q-15 | 0.21916958512725268521762054207747 |
| -0.72343925527921217857359116190832Q-17 | 0.52359285649000008489629717777410 |
| -0.74182790674065784409172309471776Q-19 | 0.90365643127296627958175243504617 |
| -0.57063901756023131531684348247675Q-21 | 0.12442865762629000926192798331197 |
| -0.24712867953199784017560938568249Q-23 | 0.14206683857334947311902894521244 |
| 0.11783203234053184484372212586606Q-25 | 0.13689180881125465761860082471472 |
| 0.40158619960549328423124392379483Q-27 | 0.11200908316562191312083639072849 |
| 0.53342875845033502394389804656701Q-29 | 0.77487445341208910722474285286176 |
| 0.51739318135375228665958337607733Q-31 | 0.44382679251979921401725076996149 |
| 0.40917121152564182734512483890653Q-33 | 0.19761425011009313541383961161105 |
| 0.27450445228161537587158702453834Q-35 | 0.52978002051457576193702224869911 |
| 0.15888089856084433682882839977000Q-37 | -0.10946677236507294452981794496274 |
| 0.79736153651449972046375944495930Q-40 | -0.27167554237844459130698663625829 |
| 0.34512592410292499136373996343494Q-42 | -0.23013817276606964471805676181622 |
| 0.12611767901219927208604462626159Q-44 | -0.14297612472677462627865456524270 |
| 0.36599881383366475896982404340221Q-47 | -0.73343530257379338679838909221485 |
| 0.66540749249636374486452199874224Q-50 | -0.32361378570259356900254620438174 |
| -0.71734115361082564402661579165059Q-53 | -0.12488761173150582953980327891663 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 6

### ARG(ϕ) = 7.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.80676508810935050866389000355579Q+00 | -0.1378007166420151208520794776337 |
| -0.46088913551145985266485501920205Q+00 | -0.1887752502344100094366216473472 |
| 0.11784936564605370833193210254487Q+00 | 0.3741379269746265684564013119130 |
| -0.16439247295602547933852153917099Q-01 | -0.1085801913882408201952729215400 |
| 0.13702597724933993845292787810515Q-02 | 0.1594507115779471979929348458541 |
| -0.68004514384715340674986265768709Q-04 | -0.1467566346083485013964032026614 |
| 0.14984368957087634251930639081130Q-05 | 0.9312827535107674871910602157801 |
| 0.48907771938461336916799467250149Q-07 | -0.4296393175582088787349031779631 |
| -0.60147511071253810164258741782740Q-08 | 0.1484769745277494308895427355257 |
| 0.29871258699758837733144888272972Q-09 | -0.3886485412505714310540892781629 |
| -0.10067118833415732687586251703927Q-10 | 0.7590831404251032365512251956653 |
| 0.25714008298803007167080962317312Q-12 | -0.1019499898701420898425961229919 |
| -0.52085557979931708110624289056366Q-14 | 0.5729665754938953834117123698092 |
| 0.85533572528254522637681521551969Q-16 | 0.1360444774945455077329158189083 |
| -0.11484804084195065137036093104727Q-17 | -0.5191038258047875121298936718094 |
| 0.12543856070590710026338481175622Q-19 | 0.1037050671787863828986070941534 |
| -0.10820953383619076771267990889336Q-21 | -0.1535030023976028821279865944643 |
| 0.66529788222605675753767399130950Q-24 | 0.1831441107005321015808069328525 |
| -0.15181435765504198352168669757113Q-26 | -0.1821377413890566593583733847782 |
| -0.28169902121946040747083892676986Q-28 | 0.1531912531730357211280951956761 |
| 0.52326981151304620988392011896808Q-30 | -0.1092633679282680985315002098407 |
| -0.56361937864428499900027807519209Q-32 | 0.6541498768746480236514565905852 |
| 0.46872188032304092440963531114329Q-34 | -0.3173996988143244305076759997582 |
| -0.32392014252392090825629299740567Q-36 | 0.1113248514664907102608416205410 |
| 0.19144995405801748828637010710979Q-38 | -0.1284450086199378931493359180799 |
| -0.978833527085826466572984119947608Q-41 | -0.1952745113726117637628914731229 |
| 0.43300656747353276072902601418826Q-43 | 0.2149027872665612464743423840355 |
| -0.16365735297342243078434089688647Q-45 | -0.1448011708932715879573066982445 |
| 0.50869852603275634441680436450723Q-48 | 0.7715460783805448649998119924213 |
| -0.11507952003693125685828801035859Q-50 | -0.3479061048935461575403757247270 |
| 0.76523084257061105999162035958139Q-54 | 0.1362976204377546055094046925924 |

## COEFFICIENTS FOR $J_\nu(Z)$ , N= 6

## ARG(ϕ) = 22.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.73941693799319228961106522299145Q+00 | -0.45971495114286007763486274352170 |
| -0.55180401914729574996363085620735Q+00 | -0.80479776234320073956900562516981 |
| 0.86318312103000633992046511930558Q-01 | 0.12483576774351821342803827812942 |
| 0.54382502621573430575577853728728Q-02 | -0.23374419361471786644166227334193 |
| -0.22054444878210194530871195651259Q-02 | 0.12374019193679886385256173857852 |
| 0.18263706617239145650712816288541Q-03 | 0.59386296477694136575075309886541 |
| -0.46722237531418936157292229435620Q-05 | -0.10030840830753316298240537266467 |
| -0.18617238102292422537462799875633Q-06 | 0.46599235930515123961537620118947 |
| 0.17108478687766381081311195318362Q-07 | -0.68043880471541829682885757677792 |
| -0.50762213563545302848008517837359Q-09 | -0.23305973971329560538443986491257 |
| 0.46758520652284039340202686071032Q-11 | 0.13469351254590762883082706067710 |
| 0.14065676883593647314003180503707Q-12 | -0.27701001864890544396071744140653 |
| -0.55898527572378171541145066119484Q-14 | 0.17224972850190811584449685634060 |
| 0.84307534733797442829638516981381Q-16 | 0.46218647893576294325489884079498 |
| -0.37180967588731086650773466698002Q-18 | -0.13415624637610936868959230383668 |
| -0.90062438769262620466710445185483Q-20 | 0.15464245199993946575365379856939 |
| 0.19941688098865125079249766782967Q-21 | -0.50232911394772528881879588348899 |
| -0.18131758796889015011207028205135Q-23 | -0.11090626445905494972724738214666 |
| 0.44701103851198128010420778283332Q-26 | 0.19357287722151146235931792731209 |
| 0.90672499423026645301242810932216Q-28 | -0.14233335988179936439343269871507 |
| -0.12798948096355301168670326352053Q-29 | 0.27289832919070749477427671199167 |
| 0.77658089929968644277993282954271Q-32 | 0.51196639361199554940067913418846 |
| -0.11817064506487098491077379610602Q-34 | -0.59662884332280296847808989684503 |
| -0.20620583492394106488828686482966Q-36 | 0.30374723840670816634872818597694 |
| 0.20178278281496036132684253933191Q-38 | -0.37317995163634399211174852462389 |
| -0.87413287826039312443309883788475Q-41 | -0.60868250157054892054540565110174 |
| 0.87995512896511146746270779997995Q-44 | 0.50727956222046226241474681553836 |
| 0.13473264754759400666350387190308Q-45 | -0.18924141422471322672348539284928 |
| -0.96789155059188882545316612794498Q-48 | 0.15809776124216565249523738998825 |
| 0.31415755621425619002455061015929Q-50 | 0.22809406016992515769116748844928 |
| -0.22026638667568696687844098357002Q-53 | -0.14271067803962783390192804577425 |

## COEFFICIENTS FOR $J_\nu(Z)$, N= 6

### ARG(φ) = 37.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.5979184751684620133332168138245 6Q+00 | -0.1013892518179387644833367074977 1 |
| -0.7784996649506597997274727704119 1Q+00 | -0.2928210868186101540715368945255 3 |
| -0.4294548039865470128945812302115 6Q-01 | 0.2181476191467659957889208817660 9 |
| 0.3414001148679391773117026064896 7Q-01 | 0.4965988711703338682827501627088 0 |
| -0.6213656294949805083708923913891 8Q-03 | -0.3441721537343446152208317222125 8 |
| -0.2402390749959612108270294725446 9Q-03 | 0.9667071365780869813874412668558 8 |
| 0.8181265753654655920893004941502 9Q-05 | 0.1208461062231277504864304066376 7 |
| 0.4453981141646398808453401523677 8Q-06 | -0.4721643246641439050958087863181 55 |
| -0.2018814051760476342444551725996 6Q-07 | -0.1186183575863373595236635143643 9 |
| -0.2079141245528184882262370401813 6Q-09 | 0.6682023193789946762915081088506 6 |
| 0.1758676176935743232329883802713 3Q-10 | 0.1203433352632046959943486270807 2 |
| -0.6353482826723525591674491393809 7Q-13 | -0.3742433623384224756089946343690 1 |
| -0.6493132706318662059394149351326 2Q-14 | 0.2801383482923266908555796602414 3 |
| 0.6956163723488670774298719404277 4Q-16 | 0.9174756984638019072598646406581 9 |
| 0.1037633237811106722438283225712 4Q-17 | -0.1286492129069486329536473983746 7 |
| -0.1912884400135490921244352785658 5Q-19 | -0.8858741275824287967246371435642 1 |
| -0.4476610726020642915107589816325 9Q-22 | 0.2363142307199432343526564855215 7 |
| 0.2465094703512426807205673984498 8Q-23 | -0.1494350906986424525431560966614 2 |
| -0.6987046763189283490761167960614 9Q-26 | -0.2185485349135016491155040819027 1 |
| -0.1642888882792969211394366536719 3Q-27 | 0.1028438463346490287875395302340 8 |
| 0.1083098437683669316250173155702 8Q-29 | 0.1030913419252173235959701092552 0 |
| 0.5155856729583797058501555967715 2Q-32 | -0.9235555578068841897642431852224 4 |
| -0.6659652207484673590627763754983 7Q-34 | -0.1749542263548004917503365939398 1 |
| -0.1657533203919995893047512928553 7Q-38 | 0.4138979357025249549602044466758 4 |
| 0.2233168571640140727122455591657 3Q-38 | -0.5689570754675904218230967995638 6 |
| -0.5750340323320036997405123983306 8Q-41 | -0.1043984921068071668152930868906 5 |
| -0.4169664281813708699666688458061 0Q-43 | 0.3946669005778534199529315322906 3 |
| 0.2190531566868321392885239844078 2Q-45 | 0.1367155534350045150469058533373 5 |
| 0.3239471836903993219273406510222 0Q-48 | -0.1037405673077492038508317793576 4 |
| -0.4285214326512015833273104004157 8Q-50 | -0.2016226704998205213971473405336 8 |
| 0.3301266456306431633202850821217 6Q-53 | 0.1556698109301467394822719107155 8 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 6

### ARG($\phi$) = 52.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.72880865190574268679251864008511Q+00 | -0.2245988121471590039733870109962 |
| -0.96425133112969332306567531815111Q+00 | -0.1051679339580582988027099723289 |
| -0.33361835159442340246772120900557Q+00 | 0.1211520972386752911219579565058 |
| -0.39220039156145451531825630278290Q-03 | 0.5182464977278785807357483209815 |
| 0.48066299068718551832987551154652Q-02 | 0.1716078182109518776692094520749 |
| 0.22573276078808161000325497535570Q-03 | -0.2880500806374434865637197669355 |
| -0.11238257038418484659226522482956Q-04 | -0.1663594456518147910243465722269 |
| -0.83636506418762052004298852364586Q-06 | 0.2492024483922612217372225335269 |
| -0.28142696275241310774390909064381Q-09 | 0.3086249694149842838870009391604 |
| 0.86525673536666457414188936236397Q-09 | 0.2706302860592917341385841940205 |
| 0.12575220341073990450268172917758Q-10 | -0.1864714311760061033205003394650 |
| -0.30431078333128292401326000942779Q-12 | -0.3683641401087365276006090832014 |
| -0.80786807888660165726994678427954Q-14 | 0.3474008172732221180042601644018 |
| 0.17520903794704418362416769645794Q-16 | 0.1405371958227968673710296234364 |
| 0.19886069656763866606086903924863Q-17 | 0.3130073996712365092491317078775 |
| 0.10714280370520276066182180575875Q-19 | -0.2308112718203153516781076848155 |
| -0.21807546224960095710274407319404Q-21 | -0.1880613210310189375787902232377 |
| -0.24492901858384869886254453639789Q-23 | 0.1616641266685536896891407773145 |
| 0.82284000250688778135442619613680Q-26 | 0.2579760846123214542848456592819 |
| 0.22714897333221194852725237124653Q-27 | -0.7976490085258743544183523253865 |
| 0.41041951631005733249448858259599Q-30 | -0.1694141267449092352595419310354 |
| -0.10704565248484143343946107311342Q-31 | -0.5983137121626064807857724190848 |
| -0.55912932553793273024180400363555Q-34 | 0.5641029790023972674420666477772 |
| 0.23575547002003021125449578013627Q-36 | 0.4125549587363961805506130558302 |
| 0.25525811763476865673300313593224Q-38 | -0.6492608608409678224169286812946 |
| 0.27807861757454375705148390663103Q-42 | -0.1356254677742285109158066881936 |
| -0.62378436261728790621511188375537Q-43 | -0.1858758804314631820337572301466 |
| -0.15450037385497971959287147954777Q-45 | 0.2473778095023851253719147720830 |
| 0.82868500499325788951305294026109Q-48 | 0.8998737577376612984262450064917 |
| 0.42772043870505575446623378726370Q-50 | -0.2206767816047640531677868139484 |
| -0.36878990471325826240149884323139Q-53 | -0.1741140963977439062505497935982 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 6

### ARG($\phi$) = 67.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.26686814968878098551887841638511Q+01 | -0.38566887899593543481131043492392 |
| 0.78344190259833645257475667331147Q-01 | -0.23391107953248144311102470020680 |
| -0.32292538088035792049451217961411Q+00 | -0.42966659709492329767249275191072 |
| -0.72928593590876668489275913799486Q-01 | -0.15852517335496444552537338613264 |
| -0.61624484835869074586556264483019Q-02 | 0.34933558805000733815923309545206 |
| -0.15257264783412751807465136932439Q-03 | 0.46922339442817807998490223225794 |
| 0.11113647007517654960210735373676Q-04 | 0.23980939455874832722294775482232 |
| 0.10446419017318243649827129474330Q-05 | 0.42065321833744243742467190080273 |
| 0.36420306870829659951916745774345Q-07 | -0.14360257633639851221839538802067 |
| 0.47573461176410952203207817121446Q-09 | -0.10282007540569263275635336221926 |
| -0.90239155539605360968670027433462Q-11 | -0.26230980647170245807111349212806 |
| -0.51868477432666022843953184657758Q-12 | -0.26506936327833174718157851245421 |
| -0.10146813534128707325341291686457Q-13 | 0.31002161574732930026119969395554 |
| -0.81698752532820199192667794266619Q-16 | 0.14826030352995616222852315930778 |
| 0.63268152510156339798508869414395Q-18 | 0.23005184179786545013189110899713 |
| 0.25835361350463755925242547218396Q-19 | 0.15108518112747853044125373357102 |
| 0.32626445596393749713102914094517Q-21 | -0.81620051818932454386731755523738 |
| 0.17812898331546827364286979354464Q-23 | -0.29025126027545644832112180755510 |
| -0.69873980349349170822723406591566Q-26 | -0.30445156731113085242411128191022 |
| -0.21975883929137357942203511506363Q-27 | -0.14037532498599880860542690802761 |
| -0.19465161070503496760879472617773Q-29 | 0.41275664426924290142191362094455 |
| -0.76806148996394754474596404542289Q-32 | 0.11624499924391237638826995448328 |
| 0.17373161044497604085357584971928Q-34 | 0.88148427342493473293395077000794 |
| 0.44262985360055269424960176250173Q-36 | 0.301046069931240396742227384936926 |
| 0.29070092432914705260599398129742Q-38 | -0.53524915524312592878913533326825 |
| 0.86774203134134549296923952438549Q-41 | -0.12442362340790862181691499357944 |
| -0.12350006180597162245485213538793Q-43 | -0.71481040393821279801595298208891 |
| -0.26383820729270324954745055137253Q-45 | -0.18808988300036882743511207596817 |
| -0.13373721587310016863216633381405Q-47 | 0.21765951183228411904752456701227 |
| -0.31254920859389443884919275507354Q-50 | 0.42999593030318774997823761987975 |
| 0.29808943754834403623852840982306Q-53 | 0.19376983186970346674239593811871 |

## COEFFICIENTS FOR $J_\nu(Z)$ , N= 6

### ARG($\phi$) = 82.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.64584092221188028552340685610685Q+01 | -0.23974937669811204548064557301272 |
| 0.26770998424946845036935829961222Q+01 | -0.15999201326218025089360214907410 |
| 0.49532144242220337184368949200128Q+00 | -0.47088670102653785094903161723906 |
| 0.50181717727637203427036067840617Q-01 | -0.77239123922754673535049245358143 |
| 0.28123341582207865707769231210398Q-02 | -0.80798254150276132694612483617170 |
| 0.52430051046393929415193963286143Q-04 | -0.58355997372349834647120258458809 |
| -0.48892307795184468956121478553307Q-05 | -0.30568272937352116868443013372650 |
| -0.51841936816051268917370766111635Q-06 | -0.11963452314698170621104491801058 |
| -0.27546723355146446428590534779156Q-07 | -0.35432504795850204635849456200963 |
| -0.10198355566130362696530598846601Q-08 | -0.78621750820313444448260036784148 |
| -0.28792420152776095660738152225744Q-10 | -0.12258919504338644054869398451337 |
| -0.64566823672422396943329654403846Q-12 | -0.95887309022033657535598792080733 |
| -0.11749704866514541744472586032596Q-13 | 0.12783518898239791145488128828380 |
| -0.17522489575848900193874925069082Q-15 | 0.66912825477448611410011102668206 |
| -0.21386316441855869046529149162578Q-17 | 0.15322819128240611777168088878110 |
| -0.209584098888510879433570836945748Q-19 | 0.25304363795000196854733494355366 |
| -0.15463022803048504581452200750894Q-21 | 0.33364996111982332439324005662975 |
| -0.64921526534518305191274241116932Q-24 | 0.36527463452070668357254896124227 |
| 0.27942593012517230458565867021654Q-26 | 0.33798862614968225334958424454515 |
| 0.94421459741907238483671624993016Q-28 | 0.26596962448870945271942907201894 |
| 0.12124984698756172756456453678313Q-29 | 0.17722793224895087736408622903220 |
| 0.11355292096120059856756880317380Q-31 | 0.97939162990825860274534107570517 |
| 0.86749684240994056202719252732028Q-34 | 0.42165620999321678539083366676771 |
| 0.56269666288943928484204326294901Q-36 | 0.10996662096849169512242760144136 |
| 0.31519652203709272099261003842970Q-38 | -0.21021548093890056980432627261695 |
| 0.15324505445793998275566982427042Q-40 | -0.51861362268575169934261335527270 |
| 0.64324877223416789374838597833587Q-43 | -0.42713667894365505983675052477353 |
| 0.22821012988003485059247487424566Q-45 | -0.25776649967215234542950560511858 |
| 0.64393299367210773058260377200490Q-48 | -0.12848580123192079244402545507344 |
| 0.11427215486278640734366537409242Q-50 | -0.55120636588251145883983950126820 |
| -0.11568409573653474904151691798186Q-53 | -0.20696570512003992931044896319682 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 7

### ARG($\phi$) = 7.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.88665053525695709821664032960456Q+00 | -0.14302587433459123412575197209746 |
| -0.44450430309116214029745547534549Q+00 | -0.28119796502019966586890435731479 |
| 0.99027016674659560919573209892910Q-01 | 0.33816170406033174365998517126072 |
| -0.12194285129914436396082726549395Q-01 | -0.83892872999271506905463273206294 |
| 0.90789939250908908928092113101901Q-03 | 0.10916035943177903635777370271745 |
| -0.40478345273411547169861443266056Q-04 | -0.90603357609524684962577164751264 |
| 0.78051568203016266291798854507995Q-06 | 0.52459436898970143272706567192580 |
| 0.27673004529168129288851670538218Q-07 | -0.22275304237591171555795735552998 |
| -0.29644020892467845113459975360786Q-08 | 0.71334881858772650444008168095279 |
| 0.13589273992049095694435080707637Q-09 | -0.17395041499018701297685616388557 |
| -0.42752993036088469921223240621076Q-11 | 0.31770104197449878974959704440649 |
| 0.10256893663101488782001994153247Q-12 | -0.39926844204143267486890247896549 |
| -0.19600775712210566774509166066264Q-14 | 0.20450011102538363639392292115801 |
| 0.30474709925016589277773755930300Q-16 | 0.50054641214661228307350022377313 |
| -0.38854684908765449939702893841613Q-18 | -0.17777448607244123096424511659308 |
| 0.40392474262579901534966456500912Q-20 | 0.33679849828253496638583474851277 |
| -0.33220460743053090952008523377684Q-22 | -0.47511286691556044551419178240082 |
| 0.19467750613033206046729574940427Q-24 | 0.54184893735316026760784529720936 |
| -0.41345150088356450953290706418443Q-27 | -0.51627571332963318528127694544875 |
| -0.77940165758503185804837480523773Q-29 | 0.41680882321448726796766720641214 |
| 0.13786364906882110721559690693281Q-30 | -0.28582556857216014785102709610974 |
| -0.14268919829818911216897277961268Q-32 | 0.16474360777379809369367817690803 |
| 0.11434246811124855347077887353634Q-34 | -0.77026004178718318181703328965969 |
| -0.76275018676544476294997779576619Q-37 | 0.26021811749411649983785176671060 |
| 0.43576857910745334563744105653734Q-39 | -0.28310420974997790330506171632966 |
| -0.21561567429900934507597983893771Q-41 | -0.43459968436394046312542793439380 |
| 0.92402253917718363533866466741362Q-44 | 0.46074817092988675831184796404767 |
| -0.33861961309185506797537201825382Q-46 | -0.30066856200842829758450621158326 |
| 0.10211066743432958166480730029203Q-48 | 0.15543295367101343005597846305886 |
| -0.22400507134798906229749081570421Q-51 | -0.68079102177762572421527088091320 |
| 0.14171956714886892215509990144347Q-54 | 0.25930916147768182637768563985671 |

COEFFICIENTS FOR $J_N(Z)$, N= 7

ARG($\phi$) = 22.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.84749605082875813998779687463946Q+00 | -0.47551549254326782349968011606494 |
| -0.51767905736368488285138240531923Q+00 | -0.11232703724051578979200425757025 |
| 0.65084771968519124975736466882291Q-01 | 0.10771850198414833867170756479256 |
| 0.49006287076922720600517790092045Q-02 | -0.17001197969398738115015863778062 |
| -0.14998587243086056579988481579817Q-02 | 0.75692309063034860956142508640716 |
| 0.10906245527960706458620217913989Q-03 | 0.40022318350886595760940709593911 |
| -0.24155661234486758255755511241262Q-05 | -0.56521409484930209819224149441226 |
| -0.10222977110163495338499703619618Q-06 | 0.23661638059528713597539380327024 |
| 0.82507774480551021983841482236139Q-08 | -0.30520968678819455698598070825520 |
| -0.22447797850684710361162993852820Q-09 | -0.10892256692851399621592254089402 |
| 0.18539140682612102957802958621308Q-11 | 0.56858113603397871940342191474537 |
| 0.57456978576674299474156989559835Q-13 | -0.10856605327467600231944023572481 |
| -0.20998526455834716951603261067802Q-14 | 0.61226948133927318315747465141227 |
| 0.29679703714110004906196687719143Q-16 | 0.16805052405390594807388729742036 |
| -0.11980905920097460727285612061969Q-18 | -0.45422508663574000872484928795276 |
| -0.29542226065281848121833530271092Q-20 | 0.49424820778658436153520149302739 |
| 0.61474670912852531754330143528492Q-22 | -0.14806548147502867557056927667680 |
| -0.53070329172271642942647066040834Q-24 | -0.33163338587079018799404653353843 |
| 0.12142791660950098893710510370081Q-26 | 0.54783395190963642998434095942989 |
| 0.24923837291854654659130766383687Q-28 | -0.38428366931540691150721820667231 |
| -0.33484691996813793927930930799352Q-30 | 0.68748672548266669332193027582388 |
| 0.19458934814311065341091122371678Q-32 | 0.13026069984937644994769785588573 |
| -0.27756718365014847348201943622432Q-35 | -0.14513880977317029958213155134707 |
| -0.48845465149940292096125794709562Q-37 | 0.71007959211552105567342113322885 |
| 0.45872442550435784546392998388602Q-39 | -0.82108696361858507904932833860944 |
| -0.19151717838689109884741571433849Q-41 | -0.13490166181586541828413015056002 |
| 0.18209866605302693754589679150818Q-44 | 0.10824261978115043634364727958340 |
| 0.28058534861287764279953884878736Q-46 | -0.39013091194993182832442699921837 |
| -0.19459102840393890932061045999209Q-48 | 0.30881748956212841485113603219867 |
| 0.61155050271100521579475733624074Q-51 | 0.44802413708251611133199513777072 |
| -0.40741371873723387874915324635961Q-54 | -0.27124890629599776451630928806276 |

COEFFICIENTS FOR $J_N(Z)$, N= 7

ARG(ϕ) = 37.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.79840790820707010652547660055424Q+00 | -0.101311383687535613585996843720 |
| -0.67434527056406032429108181429594Q+00 | -0.340716139767008419281161299545 |
| -0.49176574529689381052510396526442Q-01 | 0.170210364724313138954782807535 |
| 0.24252318107733975917951279238219Q-01 | 0.208403514741215428714458664850 |
| -0.26881957889417616575842630010118Q-03 | -0.224263046182497242399720173627 |
| -0.14456474317140137026782443460077Q-03 | 0.499053384670436442301051713312 |
| 0.41654671904946240215641271941497Q-05 | 0.676134035605226913055445382220 |
| 0.23345818231333431609472350366597Q-06 | -0.229730798565224124438673601209 |
| -0.93107696109679154376293639293117Q-08 | -0.588624226649946289862016527857 |
| -0.10002029342244149454725129940963Q-09 | 0.291683725701414539348731180772 |
| 0.72728301679441563030898810315655Q-11 | 0.684196987301925247084488566616 |
| -0.21509167919496024504724237279371Q-13 | -0.146902944834860787892234824063 |
| -0.24253692750975431832838162157455Q-14 | 0.987957255614796887474763266894 |
| 0.23859303663079852072092046930289Q-16 | 0.327105891428443604646948162397 |
| 0.35456495970459510028591958766407Q-18 | -0.424764146962781415033679778278 |
| -0.60661976386582105561530082453538Q-20 | -0.292332854215307190966474279097 |
| -0.14688578686920805297886923463194Q-22 | 0.719821535064404001590162170371 |
| 0.72188960531465301434282346191580Q-24 | -0.343869973636066020975500526073 |
| -0.18889341474872463047924562512329Q-26 | -0.616124761131705118909441424746 |
| -0.44662619765729030628392738981514Q-28 | 0.272757432610485120828935119283 |
| 0.27878514539158947295410967122617Q-30 | 0.270861970808974977371345681059 |
| 0.13142243646916331843385767885171Q-32 | -0.230217057361641743035205126897 |
| -0.16073963317014840041687656108286Q-34 | -0.437633511825632596095678790365 |
| -0.11947248793343852343814448798480Q-38 | 0.967796519472146223618626450821 |
| 0.50628435006532945236785686788101Q-39 | -0.124782880617109010687444008012 |
| -0.12434704600548708091409941570493Q-41 | -0.229729359266603359334489931931 |
| -0.89184758819701831761782455530350Q-44 | 0.832545579702951696655606957249 |
| 0.44991614062430543109712164751377Q-46 | 0.284890114239151232686717868850 |
| 0.66170876159700668379185940512206Q-49 | -0.207413078882863882711502675008 |
| -0.83426724868836495214598330648895Q-51 | -0.441178158504377729932226527877 |
| 0.60920878273401838744569777168485Q-54 | 0.295277432460493305819103387494 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 7

### ARG(ø) = 52.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.10805845587705430596482545684742Q+01 | -0.2036605463731195615529403372 89 |
| -0.71455466164237510857826262138897Q+00 | -0.98267946214688665779307181726 2 |
| -0.26040128727842228488808444141860Q+00 | 0.69676872111942938339765844093 7 |
| -0.27571317714417320814848259619819Q-02 | 0.35449929186568603429802681497 5 |
| 0.29401647915386156918236067693070Q-02 | 0.12464302436458864066158839777 7 |
| 0.13776634216917879849558674080515Q-03 | -0.15908257956199268500798812911 6 |
| -0.56010187553550806903295952088593Q-05 | -0.90920022380781678668617814071 9 |
| -0.41800005250114989730369758423369Q-06 | 0.10815146553886759101619542939 3 |
| -0.58472555797818629494612261490584Q-09 | 0.14259142611193279212921065765 9 |
| 0.37204874141962532549562502626530Q-09 | 0.12796371008738419405729078246 9 |
| 0.53302474883762926371159686262540Q-11 | -0.74924523866335645407376412473 8 |
| -0.11437235789001530275035903900267Q-12 | -0.14494056741812985965151820941 0 |
| -0.29844551553285805154922178093929Q-14 | 0.12126351112112464525531950927 1 |
| 0.52060171029971428198243485624971Q-17 | 0.49038287558714598365049563738 4 |
| 0.65796224903619507373851193188537Q-18 | 0.11479539822677839021030961876 8 |
| 0.35051959226210179111008642225901Q-20 | -0.72604593486837354372380824522 0 |
| -0.65313274252583875691615594832535Q-22 | -0.57928372884603776931781539977 9 |
| -0.71778966369081498916717136621209Q-24 | 0.46068402619555481012040024165 5 |
| 0.22110752304244249944714006825488Q-26 | 0.72262708069834379409869682528 5 |
| 0.60989716795503016111170838927503Q-28 | -0.14729596136766100394133467647 6 |
| 0.11055692524883226714333588711638Q-30 | -0.43688398223659210125920859389 8 |
| -0.26548864863343487143590087707706Q-32 | -0.15170994217340914653070762773 9 |
| -0.13582972943379099626749519684819Q-34 | 0.13463458124695285758974651845 4 |
| 0.54087449456297907980737365869257Q-37 | 0.96494685897501683457966158364 2 |
| 0.57627871863767255223760384476148Q-39 | -0.14182586731708618398675643217 4 |
| 0.82300352385280313618885572487650Q-43 | -0.29603506452857275382214827733 3 |
| -0.13179573656582221162809087081985Q-43 | -0.40259829744553784501896277756 2 |
| -0.32081027395168897036417223321291Q-46 | 0.50633859424328277751212604418 7 |
| 0.16435806742531699340447788131713Q-48 | 0.18068237073194309396148282369 2 |
| 0.83282928897075248281856230090333Q-51 | -0.42360439736571436120175940129 1 |
| -0.67862609840254306364229587102086Q-54 | -0.32928930299715645263632032344 6 |

## COEFFICIENTS FOR $J_{\nu}(Z)$ , N= 7

### ARG($\phi$) = 67.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.27757220557402990688168113244589Q+01 | -0.3125167141396498313000053645 39 |
| 0.21628220729771000203308663259539Q+00 | -0.1875520671222284795330062347 92 |
| -0.21572586791993218879984437357075Q+00 | -0.3276111964338080857152438064 61 |
| -0.47888825063255233790486375358588Q-01 | -0.1295055239707119989937659126 68 |
| -0.38417449605080035635072473758651Q-02 | 0.1971862847087875515047258102 15 |
| -0.95031857659289807256447790836800Q-04 | 0.2607819577316817771339190339 20 |
| 0.54217719800385241666612428448998Q-05 | 0.1268630797405859476877620962 07 |
| 0.50291467102024244808698404842641Q-06 | 0.2179469840561095754723554749 89 |
| 0.16745769743568661997796733640857Q-07 | -0.6163444987958036087396572620 34 |
| 0.21272982996446152112083430453548Q-09 | -0.4363616688549664744917234769 58 |
| -0.34558627349646436769108849543077Q-11 | -0.1066723739627632680912675983 93 |
| -0.19673724075954714287681301353715Q-12 | -0.1045858126364606660883694085 45 |
| -0.36990783751744152033399281138329Q-14 | 0.1071514281131341660263157614 01 |
| -0.28885723809553503263622180417731Q-16 | 0.5081993935131105628072029925 31 |
| 0.19920539794538959112477537560883Q-18 | 0.7599583939459183537176789230 84 |
| 0.80762236755645100998300175866586Q-20 | 0.4842878274632638731530328939 70 |
| 0.98528599897722977835582181740570Q-22 | -0.2359498777228373378694454278 21 |
| 0.52243652678512105438129210349385Q-24 | -0.8338055750719352224919278148 44 |
| -0.18669113195983391431107773739691Q-26 | -0.8467120039792222361398279568 48 |
| -0.58390948295527161848558704951923Q-28 | -0.3795573734033716680690998073 84 |
| -0.50166204665818462762337275869122Q-30 | 0.1025076489915366485936728969 26 |
| -0.19266560917133184751677760427848Q-32 | 0.2872749117319438175174154390 44 |
| 0.40302762123403962588495282377733Q-35 | 0.2116567962646268359059197778 11 |
| 0.10223176115049176444497272315968Q-36 | 0.7043498378236288433764165046 07 |
| 0.65335741094562545385947435475121Q-39 | -0.1164870509118747901063894649 15 |
| 0.19023900851116904593348292680310Q-41 | -0.2697165113790387386335190276 74 |
| -0.25310782324068450078617565913628Q-44 | -0.1509923481509947107911787169 92 |
| -0.53879707185266385606318669848610Q-46 | -0.3879523387319156041157684270 65 |
| -0.26646674470231544824130673276188Q-48 | 0.4215019538596538640454515594 92 |
| -0.60866372429084114029382852101808Q-51 | 0.8300013270898712125052217032 25 |
| 0.54709618562114124337551794850091Q-54 | 0.3653411464303757459520648642 91 |

COEFFICIENTS FOR $J_N(Z)$ , N= 7

ARG($\phi$) = 82.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.56111388441060315511084309626058Q+01 | -0.182024002738611291828505509953 |
| 0.21348010418955653994959362972693Q+01 | -0.119311079186459413448372433683 |
| 0.36119190693143105460364716986167Q+00 | -0.328330462749980093992173116036 |
| 0.33687272380229078129062639907608Q-01 | -0.498602812558346510079203602902 |
| 0.17623462721252877656041286796807Q-02 | -0.482924100141902940669587464247 |
| 0.33240121578708466056655686105865Q-04 | -0.323896463240259506362150313653 |
| -0.23535138368850414957745447199615Q-05 | -0.158156533800904691264867653344 |
| -0.24455620156274546189214104316973Q-06 | -0.579304702994036741892933479827 |
| -0.12321691759127232725845279984129Q-07 | -0.161232448012536158211293497553 |
| -0.43141910141269011244156224977066Q-09 | -0.337669497362627011313872250798 |
| -0.11532110872926472960418710236535Q-10 | -0.499938769651280732405359098681 |
| -0.24534464315326537800397893771188Q-12 | -0.379086316246370681063877097492 |
| -0.42452292490378379500293654454256Q-14 | 0.439148572092753872789108148213 |
| -0.60334010595574691196485668374444Q-16 | 0.227084301105649898297596075224 |
| -0.70337669837572590128014041901186Q-18 | 0.499314677296007861552558002049 |
| -0.66002467736610312065574694919341Q-20 | 0.790432749840772167885121389295 |
| -0.46780955868897130003319527235830Q-22 | 0.999768746880819752950082717824 |
| -0.19051043713450492619157292602870Q-24 | 0.105123855964283711174117181259 |
| 0.74398351846134580795974905347807Q-27 | 0.935517697423632644959952849212 |
| 0.24939744086255454384723266937771Q-28 | 0.709022147873708279737296145379 |
| 0.30995251242919495795207053801736Q-30 | 0.455679360915236946426730008680 |
| 0.28060292048726175583082061520497Q-32 | 0.243250704439898284086799446971 |
| 0.20731943114677931416623679676091Q-34 | 0.101371668940887265397631053774 |
| 0.13016017006701920894085972980736Q-36 | 0.257337949937023909122823488127 |
| 0.70634009840601383523695551365319Q-39 | -0.456467450033857366380570814711 |
| 0.33301085494168660671173827689262Q-41 | -0.111979543774130743869619154693 |
| 0.13567915718650329241513589411602Q-43 | -0.897388973790644072731999998469 |
| 0.46772546964820834988720506626757Q-46 | -0.526474205970465733079414305407 |
| 0.12841816193480049603565085742259Q-48 | -0.255199586630879526926134075725 |
| 0.22255623392462292347226445715331Q-51 | -0.106527766099528302492274725702 |
| -0.21198670390518689182774290393753Q-54 | -0.389451401572109874141519833934 |

## COEFFICIENTS FOR $J_\nu(Z)$, N= 8

## ARG($\phi$) = 7.5 DEGREES

| REAL PART | IMAGINARY PART |
| --- | --- |
| 0.95759866477106535702280604751455Q+00 | -0.14531167521678207037686034262090 |
| -0.42692134123091872170433115135486Q+00 | -0.34749254153383774671998635719911 |
| 0.84326414508050852399541519952233Q-01 | 0.30457407980661859262520650230201 |
| -0.93012538712166909059664584032293Q-02 | -0.66139163682128015654304506543204 |
| 0.62592089900115097931953475195652Q-03 | 0.77351177590467780331438939622679 |
| -0.25323704748731113257281190267446Q-04 | -0.58504768187462249516613593561643 |
| 0.43136295501460315878999050614097Q-06 | 0.31162119791821753014029277314088 |
| 0.16339950196487954026772089639625Q-07 | -0.12260614963295412298792435858107 |
| -0.15542843648742610537033159857896Q-08 | 0.36589394641877844357136370315434 |
| 0.66191204649973112930269027394254Q-10 | -0.83522620064934512396808717386923 |
| -0.19530826283722441613942082972665Q-11 | 0.14325689122153518754655604703160 |
| 0.44180462927166621852332440558710Q-13 | -0.16914102761094812839067643332279 |
| -0.79917049956956480363737637568504Q-15 | 0.79299484863058739002964192912322 |
| 0.11798348444541987783600922649404Q-16 | 0.19934955940330656892438986231548 |
| -0.14321052375796553970884944795893Q-18 | -0.66249769064311163421315375276199 |
| 0.14203929377059938843547403666838Q-20 | 0.11935929961018500348729347786743 |
| -0.11161808903689173071224380747040Q-22 | -0.16083594275913141849920825320677 |
| 0.62478632785863289953176101361239Q-25 | 0.17568639896019620509817993310109 |
| -0.12381585968572490147732822833505Q-27 | -0.16066418113252331769233794216507 |
| -0.23679566029878262917138720314006Q-29 | 0.12471179046710723921052323807968 |
| 0.39982206776516031223315431111546Q-31 | -0.82347595085204192622563409712312 |
| -0.39826625600191442727223025321660Q-33 | 0.45758655281018177908188068570459 |
| 0.30793989793850410428737053884014Q-35 | -0.20643290177463888096373621155616 |
| -0.19852982876002179948538705757294Q-37 | 0.67261796947811029258612587135758 |
| 0.10976069442825882997613208818001Q-39 | -0.69120734450630317983473362064145 |
| -0.52613734710206156062470484392381Q-42 | -0.10707176959139890540771711666763 |
| 0.21864837396171779838766422820036Q-44 | 0.10950593294582907103496409865389 |
| -0.77762274913619247543350831804136Q-47 | -0.69278251360105709932816552232368 |
| 0.22769319235433768405591915453844Q-49 | 0.34778326212961962840256792110569 |
| -0.48481544295004608784315594117814Q-52 | -0.14808505353820076227117354582382 |
| 0.29218708821230693750438977901098Q-55 | 0.54882153519126655459269531760336 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 8

### ARG($\phi$) = 22.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.94124601619466089828982017641775Q+00 | -0.47984791179505345772240945292058 |
| -0.48449039085460814234212468976108Q+00 | -0.13332509313405877030255267934388 |
| 0.50282852827693130256304794960422Q-01 | 0.93406412598466401583112007746792 9 |
| 0.42710764380632958475140893239049Q-02 | -0.12738694728747291196517330777144 |
| -0.10555146593276918554375086936339Q-02 | 0.48483161600312265873191809179329 |
| 0.68397841591133171810099457788826Q-04 | 0.27591071267258598126380695671168 |
| -0.13261152698573448814644454286998Q-05 | -0.33558739603375791499777976289544 4 |
| -0.58922959226501362977803298357221Q-07 | 0.12782318798035684349080192552149 |
| 0.42442903734740556385238879227333Q-08 | -0.14681973235506724366989019105125 |
| -0.10660256308929580041598146659748Q-09 | -0.54237230816693601968645765063910 |
| 0.79421595858104790620152851728382Q-12 | 0.25829487864804830936619628773543 |
| 0.25256487672475219607199223153001Q-13 | -0.46019615910996705028685575649969 |
| -0.85465399974175079556167864352377Q-15 | 0.23653006520379788106106358490767 |
| 0.11364240090872462140278102896245Q-16 | 0.66228254790206171158831350416834 |
| -0.42156620305644806347301683839270Q-19 | -0.16751918720371348410620060590691 |
| -0.10561494330550914349635851823814Q-20 | 0.17259383961214045164004152696774 |
| 0.20732342592350531582624228708816Q-22 | -0.47842628356915017421220423811317 |
| -0.17035873109008583476984172226465Q-24 | -0.10855667071784190461933688066320 |
| 0.36277101416543547193093650536540Q-27 | 0.17022964826955197132302471518197 |
| 0.75269329203563279537284568631424Q-29 | -0.11415041987297510548488602507705 |
| -0.96476262919876163615644147616873Q-31 | 0.19100735360093520730013179127122 |
| 0.53791185846001651774529010976682Q-33 | 0.36522571658189678625841358683132 |
| -0.72076208568677599013955221413816Q-36 | -0.38984817933110404294614504408939 |
| -0.12783226680605055840398807741581Q-37 | 0.18356153806837091573302349504960 |
| 0.11540690594997071997782765258982Q-39 | -0.20014046789882885649779087486653 |
| -0.46495365254405809830892674346773Q-42 | -0.33105533372626405787067407469619 |
| 0.41824276213262118875853577974451Q-45 | 0.25610539929200054403841442503386 |
| 0.64826890969859195600997752795238Q-47 | -0.89281686736784439728178550219012 |
| -0.43455563848677954464808603675557Q-49 | 0.67060013575186828601208783044513 |
| 0.13236510948604973532704731873229Q-51 | 0.97797603922381905995297414119909 |
| -0.83896342995482259668130781728075Q-55 | -0.57356072565099136403356714662957 |

COEFFICIENTS FOR $J_N(Z)$, N= 8

ARG(ϕ) = 37.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.95743429319235820528978554674261Q+00 | -0.99056962527153816169244473609479 |
| -0.58893586923288254345916899212294Q+00 | -0.36362506075793193727272868392628 |
| -0.49845542470162557537324931158402Q-01 | 0.13579774818486417173883245021845 |
| 0.17795143279552256078088818906653Q-01 | 0.25933690466867680094437470180248 |
| -0.10563135448759995267515908334495Q-03 | -0.15215663400856413963132724699841 |
| -0.91179189356333689392364692045371Q-04 | 0.27163440436143930019728540344995 |
| 0.22561106314149961071837488847562Q-05 | 0.39863703909700447300326422062958 |
| 0.12947677435962785681725486271748Q-06 | -0.11938376335931981948802849916332 |
| -0.46021951313630616285631427775320Q-08 | -0.30975190133467920743713920949988 |
| -0.50872438554805487646616598704019Q-10 | 0.13689627594165368619110184522889 |
| 0.32432333543040415948763672193704Q-11 | 0.37976935239412378222108437773337 |
| -0.78285635211766956725892659694178Q-14 | -0.62349628666646600055243891254126 |
| -0.98192056320765819808182877251611Q-15 | 0.37893198562018070298930879725965 |
| 0.89165268136359357578618697831107Q-17 | 0.12666723874401231258874332009461 |
| 0.13180775677456747655829221594688Q-18 | -0.15306250732812023308883400908197 |
| -0.21030557563511425065497119365738Q-20 | -0.10501980209116595768245834424500 |
| -0.52237604617550445246995042409809Q-23 | 0.24008614461508973546918389030169 |
| 0.23183386047082245499445848696216Q-24 | -0.82692273092567819034666673603079 |
| -0.56181960734392896805348287176859Q-27 | -0.19075315286896859680978422871206 |
| -0.13351130075453183992904857457111Q-28 | 0.79666281188215964659488536336778 |
| 0.79106062355917249804696843339311Q-31 | 0.78340926144432688629856822415052 |
| 0.36904786534613258666593400004355Q-33 | -0.63329238333724024787762838438808 |
| -0.42858193549043441621407460185373Q-35 | -0.12053422289515461912523358074884 |
| -0.50453933024802582692111674191350Q-39 | 0.25023247127870738068057988985235 |
| 0.12704033765569269628501281359286Q-39 | -0.30323896333689320587309113819984 |
| -0.29813181053480562967218462747827Q-42 | -0.56000079514620929170007219429959 |
| -0.21147847630344015853574769548405Q-44 | 0.19485473400223190894855167123559 |
| 0.10260103942565481465818669593928Q-46 | 0.65855117939981949364007402520666 |
| 0.14995375018744411350519734972307Q-49 | -0.46075838377646278147979617093709 |
| -0.18058825750888295085976516445178Q-51 | -0.10547949563883151467114450971567 |
| 0.12517582090351897926722957412904Q-54 | 0.62315511196977072145114512027895 |

## COEFFICIENTS FOR $J_\nu(Z)$ , N= 8

### ARG($\phi$) = 52.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.13206668380068808166699544419665Q+01 | -0.18491006007898918869576294036574 |
| -0.54225415132118554463607171188192Q+00 | -0.90877097052066620482434039773912 |
| -0.20784339449650788536720332786238Q+00 | 0.39985323230773487607272951865945 |
| -0.34561401818468862438472709941660Q-02 | 0.25207537780948039358604843577857 |
| 0.18902076258817583760368702286212Q-02 | 0.91373101176167418978972479963292 |
| 0.87767971875201788606094171813273Q-04 | -0.93110148174314151838093086521040 |
| -0.29776265093238908040532272697525Q-05 | -0.52536538373839183830996715499138 |
| -0.22262150770302491253800077953002Q-06 | 0.50075905561426299628412993934757 |
| -0.48915513771333258878198895372631Q-09 | 0.70615815098310966877126052087205 |
| 0.17228780823082845697784213025254Q-09 | 0.64127051549613147508516528314370 |
| 0.24276057629542471223781234095798Q-11 | -0.32552882756407194302767958047503 |
| -0.46648321015328227958533350128250Q-13 | -0.61635197396842567902219939422730 |
| -0.11962806465643616667002292773586Q-14 | 0.46078257631998532740190325230839 |
| 0.16647103669224135016496567058523Q-17 | 0.18624612564109657093419837133616 |
| 0.23758685438614973900731703699975Q-18 | 0.45178906266626806149575525229961 |
| 0.12493754876140269226162467582384Q-20 | -0.24984396088703900157542778994153 |
| -0.21445330016725615729229612512798Q-22 | -0.19517100296760365012751951403564 |
| -0.23066181219218979089740971371499Q-24 | 0.14421687065045626342570132542012 |
| 0.65393934348294591421007077176285Q-27 | 0.22240276847164391792773051720652 |
| 0.18023743497707459765454425739667Q-28 | -0.25178816822747754535287970544402 |
| 0.32683710327671423745877446007086Q-31 | -0.12419370911485414839211277666168 |
| -0.72687656103591830946225546932330Q-33 | -0.42393902330368641323495341871828 |
| -0.36434169347232816751903855935952Q-35 | 0.35520454954722839489458363503558 |
| 0.13734993267684665191300760156535Q-37 | 0.24956176414250275717226332426141 |
| 0.14403591269107890610465820896090Q-39 | -0.34335920200524381149561677426251 |
| 0.24831774574892943695116868411895Q-43 | -0.71615805633079145895917162168792 |
| -0.30893973686981590936277433908636Q-44 | -0.96563553189217402876009159862184 |
| -0.73910788294215233072675547340841Q-47 | 0.11509127015592797595599542778326 |
| 0.36233766449980175789538303603953Q-49 | 0.40298544151138070601248775965227 |
| 0.18030061388017137856645820892791Q-51 | -0.90466114595241669689489047409697 |
| -0.13906331877730908642349800627004Q-54 | -0.69299066645457249509431046791763 |

# COEFFICIENTS FOR $J_N(Z)$, N= 8

## ARG($\phi$) = 67.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.28075123091421335177989379213860Q+01 | -0.26034579935727398642125958322395 |
| 0.28351138365662222973167226231144Q+00 | -0.15472131695983158246201194683108 |
| -0.15074369147192290984639961475436Q+00 | -0.25694045467950479077708781384663 |
| -0.32948114373162691677826432369269Q-01 | -0.10423878900230564248788083382465 |
| -0.25131543207029416600644330500442Q-02 | 0.11791635126840991446804694361337 |
| -0.61382341881863290255599973434823Q-04 | 0.15377167911987994428630689591647 |
| 0.28292963252019390310363074004528Q-05 | 0.71333152487269084004235262565451 |
| 0.25929008068089268591423323455229Q-06 | 0.11952556215683605949853442539196 |
| 0.82601510025675873614119348948188Q-08 | -0.28510556989432406626476412896721 |
| 0.10190326775220603081623637346782Q-09 | -0.19977536973270630087653251235061 |
| -0.14350050960466027470284165173125Q-11 | -0.46870831067570332798605134977298 |
| -0.80970741600640941920359404846442Q-13 | -0.44572249164110035605960162326635 |
| -0.14653021657009812252389978260337Q-14 | 0.40352969433865363104563873167053 |
| -0.11099579379853819166098189329533Q-16 | 0.18992169493434442189344776930910 |
| 0.68622219489843275014388262247474Q-19 | 0.27404804946630290663010784857056 |
| 0.27635093402926568404315849618554Q-20 | 0.16952910159459677181165835155460 |
| 0.32605722325398705753628639583741Q-22 | -0.74884084397347242679041247119398 |
| 0.16799679532599068472441856520348Q-24 | -0.26307278045122289843576962796494 |
| -0.54923783120119699152410831443689Q-27 | -0.25888128976571540149995583084159 |
| -0.17089032653761279133721197888938Q-28 | -0.11289007320825196976585516650206 |
| -0.14253402828310676308419287870987Q-30 | 0.28103145637557560314976706040271 |
| -0.53310250806008514063378902393671Q-33 | 0.78393239554815342528016701218234 |
| 0.10344135955141504976340804274827Q-35 | 0.56162879033824713959754248611766 |
| 0.26129849606314735080935419495121Q-37 | 0.18221335997894918051462880667197 |
| 0.16261825753827027624424913066847Q-39 | -0.28103079284024160835311329882108 |
| 0.46211327408595690500847191026206Q-42 | -0.64826760564129853120421416972084 |
| -0.57604100465552391878026898162084Q-45 | -0.35386692021917662394173133101669 |
| -0.12220739130779775763913578789743Q-46 | -0.88823039626412636558792793123375 |
| -0.59002697361020476382182364288516Q-49 | 0.90783071912010785961836516085643 |
| -0.13178775368855747854943426382545Q-51 | 0.17821398036977581303371505046666 |
| 0.11183284840583881961070226404276Q-54 | 0.76663869649247991446143497240014 |

COEFFICIENTS FOR $J_N(Z)$ , N= 8

ARG($\phi$) = 82.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.50155418598567648146500601139670Q+01 | -0.14395645706499038085286698136265 |
| 0.17590903515957699006143605127206Q+01 | -0.92849752586101145683124462963580 |
| 0.27382334744147572439848859920403Q+00 | -0.23970584997203179792479865005296 |
| 0.23639279601028286437641950481563Q-01 | -0.33859409553430887074792890912112 |
| 0.11581666866202069765983413684430Q-02 | -0.30512957631036962532314787127745 |
| 0.21717011689949774266770824239101Q-04 | -0.19093095060473194536931664091109 |
| -0.12139656777004974757105657491452Q-05 | -0.87279394900814959579039726456902 |
| -0.12385840522859638888869107369899Q-06 | -0.30036344436948433390496220270489 |
| -0.59328234723695296394503092591336Q-08 | -0.78830194311094566250736134920968 |
| -0.19700570380386667602479372872625Q-09 | -0.15629153337660447734600416286759 |
| -0.49994798261878117138274639633531Q-11 | -0.22024049633187585182392548708469 |
| -0.10116432226834993680783301674229Q-12 | -0.16181059789236239757012847085203 |
| -0.16682970337928802706923150070411Q-14 | 0.16447360541249488745738785983475 |
| -0.22644246336100131363347615117559Q-16 | 0.84098055215226629054062257664803 |
| -0.25264968658149395522405102911822Q-18 | 0.17783458569598912202094251114627 |
| -0.22740725748074208085058507940858Q-20 | 0.27031304384343477898455800052714 |
| -0.15507165220799252099828931701512Q-22 | 0.32850613220889807750163106022012 |
| -0.61288683879883357903252626903986Q-25 | 0.33226352512863549580247280334974 |
| 0.21817317733537715850542719422452Q-27 | 0.28478933132918887270758189107540 |
| 0.72588712982651552235544765895356Q-29 | 0.20815630000230733548645394641401 |
| 0.87400628120907031516558606714194Q-31 | 0.12918945045124353374978507638046 |
| 0.76571372378321851434289745577263Q-33 | 0.66693987168344336099338269498976 |
| 0.54771503107271499420452543987456Q-35 | 0.26930529442353889869823399986371 |
| 0.33316949004853293340328507193159Q-37 | 0.66584310465682387557589513369958 |
| 0.17532628740334170592490446010789Q-39 | -0.10989283373074264822765268313073 |
| 0.80227776152006845657425433067443Q-42 | -0.26814874039876230978814902206310 |
| 0.31754966346370464475866957700193Q-44 | -0.20924417148917644856697026416403 |
| 0.10645252411815159697436582060167Q-46 | -0.11943170113429511543504137759941 |
| 0.28459800477884326559923702441234Q-49 | -0.56340705664632311273877506102927 |
| 0.48191828722439463817878849477694Q-52 | -0.22900323011003549286192419255468 |
| -0.43268077365239924944537737125339Q-55 | -0.81571189841737152543006113292942 |

COEFFICIENTS FOR $J_N(Z)$, N= 9

ARG(φ) = 7.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.10207229191335687423520311711623Q+01 | -0.14565576227903458559701710686234 |
| -0.40927172272696725342199428117045Q+00 | -0.39480008867590826128674824956516 |
| 0.72644653483007661816415890008482Q-01 | 0.27437124361500349160560760569406 |
| -0.72596378500539349614995025176434Q-02 | -0.53055918605613511142319236162031 |
| 0.44578374626267148591598163699029Q-03 | 0.56395019406422018958409541209740 |
| -0.16503183083552379070895549785479Q-04 | -0.39199093498599502721446637707421 |
| 0.25034872132604691736764332185943Q-06 | 0.19338375064689284206319212445248 |
| 0.10011888263833458822842039689594Q-07 | -0.70900281151119255019425071018344 |
| -0.85789883092459455880601559521499Q-09 | 0.19813034563690865019408242200220 |
| 0.34124821867357158724726974049495Q-10 | -0.42515783526842934144175381421762 |
| -0.94820810732064322011641166730131Q-12 | 0.68740268882403614468836487655002 |
| 0.20293108698762066496477519149076Q-13 | -0.76519764343332771686398158076950 |
| -0.34849104487188334194067685731805Q-15 | 0.32969491014899723386949475724316 |
| 0.48980775456472644913545796672434Q-17 | 0.84864802708421717438591459797476 |
| -0.56735057908047780002560760617657Q-19 | -0.26509665595277232500639033913114 |
| 0.53801360400918962000996654592274Q-21 | 0.45534169127653773483383158462649 |
| -0.40476823904861947722163393240497Q-23 | -0.58730396052910065008939421544970 |
| 0.21683991734942756851308933203969Q-25 | 0.61558007301890003397188322303091 |
| -0.40194144285616049127932999668566Q-28 | -0.54120254504968265010739635285272 |
| -0.77892320330003502163382156816331Q-30 | 0.40451802993936445299815817895618 |
| 0.12581854956536336301119704178505Q-31 | -0.25755327896244892400936831691496 |
| -0.12079470390768682309853907448329Q-33 | 0.13815618741437583436337545593515 |
| 0.90232479991611370155666208106017Q-36 | -0.60212969076319383673855751331497 |
| -0.56286526464382673468805475312406Q-38 | 0.18945454777748551844733753163373 8 |
| 0.30146170143373739569598692695270Q-40 | -0.18419582612110275029097853280541 |
| -0.14013304706783637287160760803063Q-42 | -0.28774528705049277527103498871138 |
| 0.56524251360753118573089983465606Q-45 | 0.28426562846827475640694957242393 |
| -0.19526817937073512723721070170109Q-47 | -0.17451447244947190087184032766036 |
| 0.55564883204431105647250920835373Q-50 | 0.85146800946361951981742010316552 |
| -0.11493041941806688272116382961088Q-52 | -0.35273105151887495484687363185155 |
| 0.66060359153142398965835552338762Q-56 | 0.12729143386650328505826813157397 |

COEFFICIENTS FOR $J_N(Z)$ , N= 9

ARG(ϕ) = 22.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.1022589758522167073827484369241 7Q+01 | -0.47719089152783541260252735271934 |
| -0.45353997959298971265501805905428Q+00 | -0.14700158197519581583050440153546 |
| 0.39651342458687760976526395992957Q-01 | 0.81509675977249559556858685135954 |
| 0.36764632564828101771123341989795Q-02 | -0.97832550325261013538818562850273 |
| -0.76435215974503344024327435401911Q-03 | 0.32253534248810169013219288002115 |
| 0.44658607895997630516476690138975Q-04 | 0.19448637430636356567167718793972 |
| -0.76503373498260766446715567069393Q-06 | -0.20805276731522011822933281769357 |
| -0.35387614530761650313777943662975Q-07 | 0.72679618789203103409540640778628 |
| 0.23034139075923625264975777911998 7Q-08 | -0.74856710791798879485719442994335 |
| -0.53718556259606901830296415096271Q-10 | -0.28479394993261915443775374971153 |
| 0.36300124203259804568851569997671Q-12 | 0.12474934501477124271871980093728 |
| 0.11805819778895655633582673809862Q-13 | -0.20829965641608964170463050521962 |
| -0.37204017598679301560178667617512Q-15 | 0.97999682124919073847085775119391 |
| 0.46698843072610020799773704743027Q-17 | 0.27929582249237061794556928216558 |
| -0.15976871827816647974701842780769Q-19 | -0.66395790460512109624244249623040 |
| -0.40603096892075513215008399238413Q-21 | 0.64950836554408729236809179010561 |
| 0.75438961815922103837330167724896Q-23 | -0.16709774920329790787667362867722 |
| -0.59136578604962854597348179064365Q-25 | -0.38364772992307696044492791863989 |
| 0.11750275453030970612208074925681Q-27 | 0.57259922168480868485649312539398 |
| 0.24622358887350230498692206586313Q-29 | -0.36775263344479477116692596690442 |
| -0.30174748644982576385569940440338Q-31 | 0.57684418749181185247026865227558 |
| 0.16167814162242449841861110250331Q-33 | 0.11123103491493152311791505855823 |
| -0.20389721241579299534354748235659Q-36 | -0.11394816977750194881981862536103 |
| -0.36425818689968252173149074653450Q-38 | 0.51707933546218889164999105950787 |
| 0.31661168593544527379154689566608Q-40 | -0.53251110171783464951987807000121 |
| -0.12323919777102505167972135615617Q-42 | -0.88641221157759999100997350450853 |
| 0.10503920347854597831521153426575Q-45 | 0.66200175830490586084623204765822 |
| 0.16371408650893434307511300490958Q-47 | -0.22345603515861205780812622107100 |
| -0.10619431352893993471548779929204Q-49 | 0.15947674619303636749292886162642 |
| 0.31379971680086032304199921596907Q-52 | 0.23371884547162630737327460560371 |
| -0.18946256279155679959903574108904Q-55 | -0.13291099808970913259355570929650 |

COEFFICIENTS FOR $J_N(Z)$, N= 9

ARG($\phi$) = 37.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.108521518714986359716431242825460+01 | -0.95777318779619010847200024121095 |
| -0.51879934718479929553272237443727Q+00 | -0.37218220119290686955881971267275 |
| -0.48025790140501868861518170813133Q-01 | 0.11042618122229946935940758293929 |
| 0.13413746700318794177003978606839Q-01 | 0.26414302002829215648308472547390 |
| -0.28402260506634830565611725583076Q-04 | -0.10674512155113251747686016670426 |
| -0.59789975101214106951950727720934Q-04 | 0.15425555762210166251296855352153 |
| 0.12860161575802240562085934260199Q-05 | 0.24547167547924533036309638939485 |
| 0.75268825102523294915227595974734Q-07 | -0.65518033538740695610788969951896 |
| -0.24095237785455473687205138593795Q-08 | -0.17123375060938830497095556154147 |
| -0.27143407859938025157351196277812Q-10 | 0.68246453372627069332916620135332 |
| 0.15402942968986896588848396901165Q-11 | 0.21239859604881368742768522445519 |
| -0.30171903240694519356728720775128Q-14 | -0.28251247690263310319532421880506 |
| -0.42532671344409951903649774061934Q-15 | 0.15596931162172652218321238254608 |
| 0.35817930336106738023091930782788Q-17 | 0.52579529139395699316125384459883 |
| 0.52604954463498176185990076467997Q-19 | -0.59377334221275013561470328970804 |
| -0.78610664703174678236573463546900Q-21 | -0.40534944592530828237787740453320 |
| -0.19897763203663402067829485341777Q-23 | 0.86467119784116891036487464964831 |
| 0.80507388640712794041006100369380Q-25 | -0.19807826160193289777451417896038 |
| -0.18122043944835437817232012205784Q-27 | -0.63943488984458999660637680997792 |
| -0.43264430310713845892558833241417Q-29 | 0.25258614527361909763097120155913 |
| 0.24389057246987028616249488355763Q-31 | 0.24587641589193332654457714982521 |
| 0.11253918770968542425358514468985Q-33 | -0.18946994472310725663863977484572 |
| -0.12440360754792577042216635489773Q-35 | -0.36038679087543200139932245027724 |
| -0.19392859356259971091814984741151Q-39 | 0.70500880302599771068179850236628 |
| 0.34766532191314936035041941174052Q-40 | -0.80451833652947280245225243808211 |
| -0.78083569834085745263692172541547Q-43 | -0.14900054951012355320237937900364 |
| -0.54775632864857140093591107636631Q-45 | 0.49850593930095951616938158918901 |
| 0.25593166615164128526237251617316Q-47 | 0.16638099178502769452048959621878 |
| 0.37145943857774103031180478032250Q-50 | -0.11203597433021005075672411385189 |
| -0.42816139374226625147613374424708Q-52 | -0.27254415482183584366651266326620 |
| 0.28209345358739383715080090748057Q-55 | 0.14413508331530099505640224831164 |

# COEFFICIENTS FOR $J_{\nu}(Z)$ , N= 9

## ARG(ϕ) = 52.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.14901587876237078639644643242678Q+01 | -0.16855314662580261634478519045767 |
| -0.41935747691740330950925705827827Q+00 | -0.83835206548740054147888326099075 |
| -0.16912542253599059452467930380097Q+00 | 0.22183712698421688156526790934012 |
| -0.34823931773245912768779716661251Q-02 | 0.18503461772368255913963620946889 |
| 0.12656824283325412512843222023167Q-02 | 0.67945606914835688964547647172945 |
| 0.57978670566006744954801029711202Q-04 | -0.57160385123813877314791710725703 |
| -0.16696658455455617747040585967991Q-05 | -0.31789809003407271040477384732340 |
| -0.12497133429781679846192026999238Q-06 | 0.24427090698597791918341898063262 |
| -0.35017567726455584775520469151823Q-09 | 0.37045347132078442271668704778621 |
| 0.84864870124175846854673866197114Q-10 | 0.33775049839256807684683056714011 |
| 0.11741599757026453150464700386209Q-11 | -0.15098386793377358751551037801037 |
| -0.20376391312041416426509948212023Q-13 | -0.27971398388955066712105482988669 |
| -0.51351235934882269876637447252328Q-15 | 0.18804934021278505850435856600802 |
| 0.56283348592998806923837910218566Q-18 | 0.75963823335987010462052126011408 |
| 0.92354984473327268107343354317424Q-19 | 0.18898316283169913783535450075485 |
| 0.47877786671570704890166047881329Q-21 | -0.92752693890554192222452059532126 |
| -0.76116346642885189790080593536868Q-23 | -0.70937204899515318978913206278899 |
| -0.80143471083886086621873346253805Q-25 | 0.48894717761363245210517642048469 |
| 0.20983159555373565156988246868467Q-27 | 0.74143609957933808270638766358973 |
| 0.57787319916283497614720459722321Q-29 | -0.25943268012832166854445098633721 |
| 0.10458392802209250378149765494597Q-31 | -0.38358057059482718247494508845024 |
| -0.21650910547518187235715717130056Q-33 | -0.12868967491294405271330439007888 |
| -0.10634717427153137976054318884921Q-35 | 0.10208175535302543649260312883504 |
| 0.38040480182879044523722399634758Q-38 | 0.70328365774584104536085873216721 |
| 0.39271973981955606056882579796150Q-40 | -0.90772979074622420061164907601622 |
| 0.77866777500699107214184724686957Q-44 | -0.18918927196802006334864732223155 |
| -0.79155858882688467232509163004560Q-45 | -0.25273766337714179527616558418387 |
| -0.18614444376485889725597485316454Q-47 | 0.28620160016700776369208429907584 |
| 0.87466522371406139282689368442175Q-50 | 0.98357990974810945175946502077786 |
| 0.42752987534866217639797265958616Q-52 | -0.21173710760034335627883628703703 |
| -0.31258644494194354112416381893310Q-55 | -0.15986221322757718523689246358470 |

COEFFICIENTS FOR $J_N(Z)$ , N= 9

ARG(φ) = 67.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.28034474322191192608193328728190Q+01 | -0.22168113161764802686448039675227 |
| 0.31437732007530896934607561546251Q+00 | -0.13056373037273132576838685540433 |
| -0.10921749648625950896312724842387Q+00 | -0.20627408614340475317877152146504 |
| -0.23538711243335439179540181866114Q-01 | -0.83942942606640410352981936563336 |
| -0.17099788895775039989248377040687Q-02 | 0.73917011765269562352773029827696 |
| -0.40944438779461664435848115852173Q-04 | 0.95171256851767575578257753911374 |
| 0.15608669837079204926176351956846Q-05 | 0.42174403614562450408860759399778 |
| 0.14147427746469514158293022108193Q-06 | 0.68749121356512272483106309644459 |
| 0.43190723301963783514918403936564Q-08 | -0.14036189978173417209376458232879 |
| 0.51704560228225790745288045135541Q-10 | -0.97422090475217296443622542996218 |
| -0.63762847379053134580170257357905Q-12 | -0.21969510796320723848422157440046 |
| -0.35684093996780337870082557096429Q-13 | -0.20263602663466943144440441270413 |
| -0.62235832628429628481084041837072Q-15 | 0.16334784776688658525220529450233 |
| -0.45746062299811684071119861380747Q-17 | 0.76332301255886023525836020269285 |
| 0.25504533098548597290099342323586Q-19 | 0.10640604655280035751855542972636 |
| 0.10206883465563043155542036880182Q-20 | 0.63929255646737121352441696479633 |
| 0.11658892050574234114388341144133Q-22 | -0.25723622493695918750883418272077 |
| 0.58403563287322845053682570438405Q-25 | -0.89870587413964431000831259447947 |
| -0.17537189507411510580697441525761Q-27 | -0.85782613544627708503532014023758 |
| -0.54298019925813554725174836976316Q-29 | -0.36409355977478066253111339476316 |
| -0.44002894275054373203954687124595Q-31 | 0.83820489794358796915736661339038 |
| -0.16036649990360654811215671334478Q-33 | 0.23279156453701257930271111413308 |
| 0.28943863136944793706872777428400Q-36 | 0.16229255917354773751069893767741 |
| 0.72825814798311295605531477917760Q-38 | 0.51361259043849766427847960419543 |
| 0.44164985626135820301320262883108Q-40 | -0.74051497139181772286496912550755 |
| 0.12254829469855025684982835508145Q-42 | -0.17021028413411951197026437462458 |
| -0.14342258442334544405261526918478Q-45 | -0.90651572893514574671664532119642 |
| -0.30328852725337080725369002229257Q-47 | -0.22239761851124248205733353231595 |
| -0.14303057345016156655482094935250Q-49 | 0.21422229239532068611002609591837 |
| -0.31253235019473693992942880898498Q-52 | 0.41929535560653559381988676654213 |
| 0.25078662253834067163084730357517Q-55 | 0.17636779784355912179584888515659 |

## COEFFICIENTS FOR $J_N(Z)$ , N= 9

## ARG($\phi$) = 82.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.45778659072789435803243880792026Q+01 | -0.11750221819701168519828243218294 |
| 0.14865911988951922495747745377008Q+01 | -0.74694952932887756282107403006862 |
| 0.21407270736618212997212830790839Q+00 | -0.18148145861256552829324664063437 |
| 0.17190528410662748007631924748333Q-01 | -0.23946346828468768929276869118400 |
| 0.79126493965605218843812299238222Q-03 | -0.20166275657361019514539439253517 |
| 0.14599285289090907177866000783479Q-04 | -0.11821876051966164953545309570053 |
| -0.66293408719324839740283297516924Q-06 | -0.50785157478030162645431084729645 |
| -0.66522927825808089822811274261290Q-07 | -0.16477590281856967607776655379971 |
| -0.30366094068877039218135082219804Q-08 | -0.40906206517409217252675578801802 |
| -0.95874470163768310920473470576199Q-10 | -0.76987075071327321737086991058371 |
| -0.23155528512801822301739124945351Q-11 | -0.10348044356796781823183329200605 |
| -0.44667680475846809936170561665811Q-13 | -0.73654269259075211459263663529964 |
| -0.70353919802577003695169410034367Q-15 | 0.66245988770603706098404184861597 |
| -0.91379821424626194937826050237015Q-17 | 0.33521087090180482253367061302181 |
| -0.97752542298021561978953254907562Q-19 | 0.68269364808600618612002886641996 |
| -0.84534220094714737066362367388177Q-21 | 0.99795112002979000063433125050842 |
| -0.55536947925795873896040840423211Q-23 | 0.11670220886594546233543154434652 |
| -0.21314877223079242153626326680179Q-25 | 0.11370313712336180633335476270873 |
| 0.69454920565742270940222977862866Q-28 | 0.93990234785394433459433661377437 |
| 0.22945899058966714150834056417505Q-29 | 0.66335757568424907697473073106158 |
| 0.26792606017403100545991261488987Q-31 | 0.39803977856878961921443762540167 |
| 0.22738792245203343218290001209774Q-33 | 0.19893644721257648143801079203635 |
| 0.15762682839553318554361114948239Q-35 | 0.77907325196748499363687952535833 |
| 0.92988180632415942732054774593386Q-38 | 0.18771291219537093782831528280670 |
| 0.47495095554569864470476393241994Q-40 | -0.28899288697921946550294033575800 |
| 0.21112084222432560557817968807386Q-42 | -0.70159936933568687551485176184254 |
| 0.81245578481247655775878350756323Q-45 | -0.53346073020585476304811085461725 |
| 0.26505539772445594136616626066297Q-47 | -0.29645192111165589139948124066226 |
| 0.69047537490178655826353358349260Q-50 | -0.13619635246725904539952402427983 |
| 0.11429456439797865372249321677330Q-52 | -0.53940889092100702545067367899825 |
| -0.96892149560262123839875621498527Q-56 | -0.18732754308268549123272393345062 |

COEFFICIENTS FOR $J_N(Z)$, N=10

ARG($\phi$) = 7.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.10770779426106073890192043832212Q+01 | -0.14471617726677994164829168585454Q+0 |
| -0.39212966583325400750535674345730Q+00 | -0.42822693172595757977721559943020Q-0 |
| 0.63217179741744383728147191719876Q-01 | 0.24765213424044801892177336985751Q-0 |
| -0.57766794899547661487696606603912Q-02 | -0.43206573045827685773998282895837Q-0 |
| 0.32623043229235075669154216020844Q-03 | 0.42114593111328692199373708640293Q-0 |
| -0.11128648273865611237436324967911Q-04 | -0.27089968008436003028620257681448Q-0 |
| 0.15139848520002283027176413868088Q-06 | 0.12449474810360664978704310744351Q-0 |
| 0.63358108594102329667377990652636Q-08 | -0.42739393207804263528855321489781Q-0 |
| -0.49454316880328669096898436930727Q-09 | 0.11230807711054676115592435286881Q-0 |
| 0.18459489460356737865707045114105Q-10 | -0.22738519060877352783612498051743Q-1 |
| -0.48474284098336005334805655417731Q-12 | 0.34771037680945009470704147923679Q-1 |
| 0.98446594731482008322392190423090Q-14 | -0.36608836053076755707826611586407Q-1 |
| -0.16092574988635954642081991227612Q-15 | 0.14548228989709033084115748630499Q-1 |
| 0.21584244415346523125745667406023Q-17 | 0.38246403485211142502971379747897Q-1 |
| -0.23909029935970118910907688810859Q-19 | -0.11273968700904676704470255784396Q-1 |
| 0.21720066854263152721545196496415Q-21 | 0.18503519123490771051149676962184Q-2 |
| -0.15673082615050054274117119393181Q-23 | -0.22887453424008915733606621226772Q-2 |
| 0.80501395598236888611468356761614Q-26 | 0.23057539499173694923701488800925Q-2 |
| -0.13988665369483780824487652497639Q-28 | -0.19518350103661280837287638319327Q-2 |
| -0.27439783084603088637083456334797Q-30 | 0.14067528371550764518621536654098Q-2 |
| 0.42486511969182125404789613673033Q-32 | -0.86476004598322880177270084352663Q-3 |
| -0.39367209373285906562400771746547Q-34 | 0.44833905306905288982561962152829Q-3 |
| 0.28443362465281128612221320828633Q-36 | -0.18899112188947303819005768775349Q-3 |
| -0.17185740972647282211843368652655Q-38 | 0.57488754259333355761025717908944Q-3 |
| 0.89255031371157453272610224573363Q-41 | -0.52961039159847705513267306666401Q-4 |
| -0.40271650501305446811450701094978Q-43 | -0.83389340474374919543956016835359Q-4 |
| 0.15780448320322486421184122945988Q-45 | 0.79671770321043473046348405508266Q-4 |
| -0.52996713551233774079793198970871Q-48 | -0.47505880175707503010957889954265Q-4 |
| 0.14667329821418100086201404598724Q-50 | 0.22545343954568930493691392625562Q-5 |
| -0.29494577652425156628027794825673Q-53 | -0.90934226042688915493898962491234Q-5 |
| 0.16186533619042864589367328454462Q-56 | 0.31975785447339136437482027426421Q-5 |

COEFFICIENTS FOR $J_N(Z)$, N=10

ARG($\phi$) = 22.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.10934365869883215738394308304922Q+01 | -0.47028602446269440118915689945615Q+ |
| -0.42521461236315023627007955311524Q+00 | -0.15566301656672756046348216474698Q+ |
| 0.31819060565265267168071183275582Q-01 | 0.71602424661306323596915881812180Q- |
| 0.31542763944476711461019401953149Q-02 | -0.76714591660115322777735474948080Q- |
| -0.56709350574311854570514383181545Q-03 | 0.22148909866535180484739570299698Q- |
| 0.30159719201330213545099359232924Q-04 | 0.13996066148054386442657645937180Q- |
| -0.46015707246229465823701740070630Q-06 | -0.13376647042674452136864451628413Q- |
| -0.22017318292751463419500380330363Q-07 | 0.43145883711639812681855190560882Q- |
| 0.13079367033826251356243750667839Q-08 | -0.40091793666705322622005627536410Q- |
| -0.28463977632441842039992068840130Q-10 | -0.15645730759689277826822406831856Q- |
| 0.17530694536689453709873179018724Q-12 | 0.63466019804804903761488810487431Q- |
| 0.58153151991331585746944579237765Q-14 | -0.99698030677668387211514985389115Q- |
| -0.17150981648592178836950576946626Q-15 | 0.43105592882390350540053173161582Q- |
| 0.20384532499130557704954872545272Q-17 | 0.12480651957886549317103378075187Q- |
| -0.64532392780238604688709208229600Q-20 | -0.27989837324220134634599715035091Q- |
| -0.16613318327725543452309762656183Q-21 | 0.26062150414425241285394934214502Q- |
| 0.29301551513187396232117620799296Q-23 | -0.62402887418460114669138161693870Q- |
| -0.21958047473642545743166366089863Q-25 | -0.14482351658154859035216129810815Q- |
| 0.40808524177136503848660786259430Q-28 | 0.20622157878201833216783187477040Q- |
| 0.86294549510274699551258563531163Q-30 | -0.12707473004385527797409603267730Q- |
| -0.10131298417960660731707743332042Q-31 | 0.18723949489372310548225920747591Q- |
| 0.52244061160892101031525866621885Q-34 | 0.36387266537721670106359309018600Q- |
| -0.62125636604351486479613785149378Q-37 | -0.35834137852552893633384428818349Q- |
| -0.11173795483626256857449710871045Q-38 | 0.15691693477317070412427369090201Q- |
| 0.93639085759250130459588803809473Q-41 | -0.15288441968328379328017314198386Q- |
| -0.35254244552994616950521581637437Q-43 | -0.25599684525958349714597681406665Q- |
| 0.28511850610976042993105025370479Q-46 | 0.18479557799332439953018901551077Q- |
| 0.44670612584875300546018404878063Q-48 | -0.60456595263285699080420351167459Q- |
| -0.28068717767492755444456397018307Q-50 | 0.41050316306662456631194103872693Q- |
| 0.80533826700881921265531663978337Q-53 | 0.60439777526461537929395834428985Q- |
| -0.46372384893806546987618902678121Q-56 | -0.33358845033829961271679578818470Q- |

COEFFICIENTS FOR $J_N(Z)$, N=10

ARG($\phi$) = 37.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.11892731397061021040654756324953Q+01 | -0.92070712182592098061603414817541Q+ |
| -0.46082377180951198736839907799622Q+00 | -0.37235126135848637966983735515245Q+ |
| -0.45165739932707199292543477714636Q-01 | 0.912774936510891516017577756180436Q- |
| 0.10343784173674319839242823089156Q-01 | 0.24974876561291916107608167599555Q- |
| 0.79507749186446390767994724675178Q-05 | -0.77024577137716274268397855639896Q- |
| -0.40513259416953563357639481366671Q-04 | 0.90653826910150025246570868723397Q- |
| 0.76525533340996492925024774236890Q-06 | 0.15680875727619770982993471257640Q- |
| 0.45537336594143751690101859150321Q-07 | -0.37648906373435473209324875593666Q- |
| -0.13243833551223220538747463928333Q-08 | -0.98722401096745947386437086375846Q- |
| -0.15096685345703534113808439983121Q-10 | 0.35807397250830660866046807847830Q- |
| 0.77171652528247393555037980816695Q-12 | 0.12100160889017580884524260391945Q- |
| -0.12156334218876028024705413991347Q-14 | -0.13533740412746199432675615031962Q- |
| -0.19516712330693265724118699355359Q-15 | 0.68189324409909834515786520341586Q- |
| 0.15305493251388436393481737342540Q-17 | 0.23161500199510543895649978286785Q- |
| 0.22311578319588139056008532566540Q-19 | -0.24536994316450377063715577503711Q- |
| -0.31345371296322153212933411246127Q-21 | -0.16639683173760065235013111472695Q- |
| -0.80431383887768639269400963609724Q-24 | 0.33265357338027656563360596461776Q- |
| 0.29903245821330885972100599008225Q-25 | -0.42878122783181978180491684135084Q- |
| -0.62693296790304631866496038461793Q-28 | -0.22954801378603195901403579619828Q- |
| -0.15030726715889159520837011563137Q-29 | 0.85965255951477178039204037964469Q- |
| 0.807865928948466431706005408122002Q-32 | 0.82813885541594805183371996391602Q- |
| 0.36854718930376072903891613156706Q-34 | -0.60958298692728767195676509194550Q- |
| -0.38867074916762003688099378913567Q-36 | -0.11569332064387194220272824361761Q- |
| -0.73653499469989608896364535141621Q-40 | 0.21398055036140281096845725195902Q- |
| 0.10258158926465044574865126701273Q-40 | -0.23035345007292336547012618599439Q- |
| -0.22083059221419849490531625155202Q-43 | -0.42777008870140030223903118672166Q- |
| -0.15319139404231290368473338873328Q-45 | 0.13779724754099566569509672484470Q- |
| 0.69021955425802322147376347750959Q-48 | 0.45414013854402513048734340347319Q- |
| 0.99428687552751270426791507940520Q-51 | -0.29472207386639069570012329321266Q- |
| -0.10989264133111235764429538994789Q-52 | -0.75430765786330559529532446048065Q- |
| 0.68906997742274749733737821100824Q-56 | 0.36111617066616194965118257450540Q- |

## COEFFICIENTS FOR $J_N(Z)$ , N=10

### ARG($\phi$) = 52.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.16131901692933982020058639015388Q+01 | -0.15439212736918901332324849988329Q+ |
| -0.32925121694190330163095275667469Q+00 | -0.77415763657918279029279595027441Q+ |
| -0.13993035531835114601183784733108Q+00 | 0.11202972435901450391668133278642Q- |
| -0.32477100333648170948629780892145Q-02 | 0.13947594281469011753483699924640Q- |
| 0.87675427047145050806737058743826Q-03 | 0.51311143781636042535110904478778Q- |
| 0.39504441786992409929595299616255Q-04 | -0.36520882658532342937431039504513Q- |
| -0.97918432765922958778034720904758Q-06 | -0.19997994108134104885064675107526Q- |
| -0.73334710023407880688057358897999Q-07 | 0.12429779126227047852441180310511Q- |
| -0.23924209620577277766222412866763Q-09 | 0.20403426459801682898748947123482Q- |
| 0.44047742183406948178751261136048Q-10 | 0.18570993957155022926230610786376Q- |
| 0.59773999635131568020684186639394Q-12 | -0.74026278898078383207261331317165Q- |
| -0.94362568794462894070374595497220Q-14 | -0.13417029872131442977716707717604Q- |
| -0.23369752927692986482454342541443Q-15 | 0.81575011540892412976223261456772Q- |
| 0.19790524831005259460894762097039Q-18 | 0.32931082974101630348739273399272Q- |
| 0.38241075858849583826140549090876Q-19 | 0.83373331803903805291106730813360Q- |
| 0.19524909074511975140875204101316Q-21 | -0.36751282172291512821128944880172Q- |
| -0.28887091736785663835650503045987Q-23 | -0.27519401805042829973427544817706Q- |
| -0.29781886052901580270841998082916Q-25 | 0.17756087997486642841890546666815Q- |
| 0.72237079243055572287707626729999Q-28 | 0.26480508571246431838347298734745Q- |
| 0.19878240906129993364767420760478Q-29 | 0.86698360564704439086045727663089Q- |
| 0.35837360484424353113441861988638Q-32 | -0.12727735517113735941903850811010Q- |
| -0.69369306948066985023305391540348Q-34 | -0.41964393844151244608774565376343Q- |
| -0.33398516121575415608752136415887Q-36 | 0.31594037532879267414227608286185Q- |
| 0.11359424283166134017417206560414Q-38 | 0.21350122312764593977994122083172Q- |
| 0.11547164900803300405485156245012Q-40 | -0.25903484617452265166831964390329Q- |
| 0.25510943162098450077533937589163Q-44 | -0.53949243706967857363317030145751Q- |
| -0.21912110489853883337082128591615Q-45 | -0.71363331409637988600969460769551Q- |
| -0.50656867911546071622073659753358Q-48 | 0.76959629169699179435507740927286Q- |
| 0.22850126897083686270599679914753Q-50 | 0.25966240520705018274788479116694Q- |
| 0.10974225638919435684874321396094Q-52 | -0.53676787509358374721516988745198Q- |
| -0.76169062126935885500855521294288Q-56 | -0.39950709078496150627555314745569Q- |

COEFFICIENTS FOR $J_N(Z)$ , N=10

ARG($\phi$) = 67.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.27822907138463913982077228239064Q+01 | -0.19211850738764318831286117319236Q+0 |
| 0.32580066855842345192718396283317Q+00 | -0.11222345126969609690751959093825Q+0 |
| -0.81517601434128351486625630962365Q-01 | -0.16886843297843570184702308467810Q+0 |
| -0.17346194968614217143376700852984Q-01 | -0.68033252734030429613633911500232Q-0 |
| -0.12022018424009118429572782447224Q-02 | 0.48190253815463468175585909930113Q-0 |
| -0.28096257126468491302399908789547Q-04 | 0.61329385231121013017460389312020Q-0 |
| 0.90230018366711993716471965884418Q-06 | 0.26006533909254724720472649915863Q-0 |
| 0.80958625413888494762821598493514Q-07 | 0.41180617936743956242181187348924Q-0 |
| 0.23722903455583893267916837534972Q-08 | -0.72844930932444296167968487288868Q-0 |
| 0.27548520369482274784784077741642Q-10 | -0.50118494418729065294094909080112Q-1 |
| -0.30013872256094967813333178412419Q-12 | -0.10878570475951727749222044764682Q-1 |
| -0.16669346718516516649564482342204Q-13 | -0.97338177825989185336634979773048Q-1 |
| -0.28053780805678310593151879714947Q-15 | 0.70334161119495955003495937124837Q-1 |
| -0.20018077646409445146235950127044Q-17 | 0.32648772613562729731021026691739Q-1 |
| 0.10117738986138448080457622054619Q-19 | 0.44015944790935392325526145255116Q-1 |
| 0.40254237052003516289369140896282Q-21 | 0.25697299617608944753652829857738Q-2 |
| 0.44558803427369570093113741115989Q-23 | -0.94595287247135573635906636649411Q-2 |
| 0.21713708815461493035836976482057Q-25 | -0.32877419328717899362768592678456Q-2 |
| -0.60098274186338796587864559864839Q-28 | -0.30466136372478204540092384484482Q-2 |
| -0.18521392886501332093735154871307Q-29 | -0.12593126879328347638024896938318Q-2 |
| -0.14595161220914792130334097827895Q-31 | 0.26891482779783100245124983466398Q-3 |
| -0.51858284005809331672942661423354Q-34 | 0.74375073603579126121327254409973Q-3 |
| 0.87284902527443447414007744443399Q-37 | 0.50492564515150912624136340449118Q-3 |
| 0.21879706167675198333460340564104Q-38 | 0.15595394971255342244936048926047Q-3 |
| 0.12938184908698778721367959098851Q-40 | -0.21066331538343720110360088473163Q-4 |
| 0.35072427907614927879582394621915Q-43 | -0.48257804498128586104664533050125Q-4 |
| -0.38612746049153302910881766293428Q-46 | -0.25090791358394678576366476746410Q-4 |
| -0.81400786697637970641412370912304Q-48 | -0.60191991048523126286989588381375Q-4 |
| -0.37517263200483345273113356861754Q-50 | 0.54736708832091399121065888457811Q-5 |
| -0.80232072700061882663560246762198Q-53 | 0.10683396196357965851457116385584Q-5 |
| 0.60973356483030893412653042011155Q-56 | 0.43961298506966719755036932639659Q-5 |

## COEFFICIENTS FOR $J_N(Z)$ , N=10

## ARG($\phi$) = 82.5 DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.42447314438858021139768539021454Q+01 | -0.98334124575471919723004940647087Q+ |
| 0.12816185950022786480302670673817Q+01 | -0.61696510345942212007588790523339Q+ |
| 0.17157745371687049153830943040939Q+00 | -0.14149192886131537194456163290311Q+ |
| 0.12872522149607439830678476187445Q-01 | -0.17507432642116695670720268525378Q- |
| 0.55835887247162660114702881288113Q-03 | -0.13832386200317289901890398165733Q- |
| 0.10073534186631920691166692869692Q-04 | -0.76250525655960568692329827102811Q- |
| -0.37979524084068634629698613527629Q-06 | -0.30888592802055494189618835666205Q- |
| -0.37537419925526154983853718447047Q-07 | -0.94783344481441606245572624760200Q- |
| -0.16364772723703878577677766033556Q-08 | -0.22320172581098813248723308640775Q- |
| -0.49241535427625330889631346358770Q-10 | -0.39974879281919812397246662178058Q- |
| -0.11343973206188397919203464791644Q-11 | -0.51354242457369746175507841595217Q- |
| -0.20905295666558363970489745404086Q-13 | -0.35415960637573766224972180313220Q- |
| -0.31510189024237780595201108442400Q-15 | 0.28394096061830276096936639591218Q- |
| -0.39235245002541478782903652905210Q-17 | 0.14228957518635015326895721023317Q- |
| -0.40308123950144529339529001489088Q-19 | 0.27947787308485203846718520533009Q- |
| -0.33540523345508838684005531973484Q-21 | 0.39344829641267875848821222453201Q- |
| -0.21257041947446936231354841705603Q-23 | 0.44335968524975969670958912395662Q- |
| -0.79271557432924675688555316914276Q-26 | 0.41665618539189018229260641529583Q- |
| 0.23735454055847514179232386015774Q-28 | 0.33258089962189097469586745634458Q- |
| 0.77895478163617758143210152994354Q-30 | 0.22691701731190163068449207765829Q- |
| 0.88284331343533283843199434252525Q-32 | 0.13178276940464517975593295985025Q- |
| 0.72653356264619964407755637392408Q-34 | 0.63827807520755921114421630049285Q- |
| 0.48854145646350888618452057204284Q-36 | 0.24264220522136749138904946517078Q- |
| 0.27975424683552502864924514057543Q-38 | 0.57005087473485524126006460289620Q- |
| 0.13880587224784616649936733648024Q-40 | -0.82059700108623087145068438062453Q- |
| 0.59985432051894836742212767092778Q-43 | -0.19826086110294003030394191514909Q- |
| 0.22460891117575026417263733847949Q-45 | -0.14698426083363886378587189216027Q- |
| 0.71361800148885136683966260613486Q-48 | -0.79580705508134970881013554920667Q- |
| 0.18125655482174948734864600532207Q-50 | -0.35630400382439371429769289610840Q- |
| 0.29343207113616831655464794788975Q-53 | -0.13759059108864575839476319980849Q- |
| -0.23525597783363804989803641103955Q-56 | -0.46615512223085639938498020530894Q- |

COEFFICIENTS FOR $Y_N(Z)$ , N= 0

ARG(ϕ) = 7.5DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.28717212946796627430806347965 9Q-02 | 0.18917552628388886915523465849 3Q+00 |
| -0.31563716145249114616034536975 2Q+00 | 0.26503906409599366294013584035 4Q+00 |
| 0.33486523565728640051216631422 1Q+00 | 0.17897502548485781614555486613 25Q+00 |
| 0.15042790710065710871842335082 4Q+00 | -0.73521671581021489190503121429 7Q-01 |
| -0.11006483555403427345664797189 9Q+00 | -0.50326871859036836089010034278 4Q-01 |
| 0.22093794196679691106570597211 7Q-01 | 0.25997618374083483030925500619 6Q-01 |
| -0.18782807522105752352756558820 7Q-02 | -0.51034751445689865686276860282 3Q-02 |
| 0.19472513457572108539717342038 3Q-04 | 0.57325456129470973046904555917 6Q-03 |
| 0.11518579508109098893044272209 8Q-04 | -0.41975144620588072434172340354 1Q-04 |
| -0.13301387531873891394991560874 8Q-05 | 0.21237842446363040552189337755 1Q-05 |
| 0.84947880738805402527924774228 3Q-07 | -0.75426647359554116819188312914 5Q-07 |
| -0.37623125833937742138046507365 2Q-08 | 0.17965157864945650005983204409 5Q-08 |
| 0.12479946260033432567191440967 1Q-09 | -0.21189066242569198664457712481 3Q-10 |
| -0.32170540561990896362819856598 1Q-11 | -0.34438458578728258176262357163 5Q-12 |
| 0.65595417189976499947565274539 9Q-13 | 0.26165402493791876422726975371 9Q-13 |
| -0.10599401395788567580399916333 2Q-14 | -0.80734444970315593072667535663 5Q-15 |
| 0.13280088096753914667788471371 0Q-16 | 0.17480481776082825106466995430 6Q-16 |
| -0.11836157003451677706255602402 9Q-18 | -0.29589728864721558485499232026 6Q-18 |
| 0.45984167109472314649100287023 0Q-21 | 0.40823354020664598032720486539 3Q-20 |
| 0.73319615620423586697438136355 2Q-23 | -0.46775569529379516975123003902 8Q-22 |
| -0.20128723780704240816900464043 0Q-24 | 0.44767284962491753493647241209 1Q-24 |
| 0.29263388239867456744415421937 3Q-26 | -0.35520840010238173981037349429 3Q-26 |
| -0.32020699561465528378359397627 9Q-28 | 0.22645926660280483292367898126 2Q-28 |
| 0.28663560057098669538826108330 3Q-30 | -0.10456785972887979972372440485 2Q-30 |
| -0.21679931905053277852436976806 3Q-32 | 0.18338107201200659202546294087 2Q-33 |
| 0.14041165037474354692038469643 9Q-34 | 0.25659396093564383616551811138 8Q-35 |
| -0.77999828973720282915688977481 4Q-37 | -0.37258549205016053896036818301 6Q-37 |
| 0.36753044959545035508400479374 8Q-39 | 0.31608205471516401026549132478 8Q-39 |
| -0.14172557420881700435611613572 9Q-41 | -0.20891811511690919455301876596 2Q-41 |
| 0.39878865790352549469966717553 4Q-44 | 0.11574625368470108520846587235 7Q-43 |
| -0.37486895342617264253688006843 7Q-47 | -0.55290806542893660947890613222 7Q-46 |
| -0.45769851288949717635089041881 6Q-49 | 0.23020807188155353753146607720 6Q-48 |

## COEFFICIENTS FOR $Y_N(Z)$ , N= 0

### ARG($\phi$) = 22.5DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| -0.149982401841825042207821066617Q+01 | 0.107859591040152735387221625774Q+01 |
| -0.141357093233175097919703960627Q+01 | 0.148986082032130339624541018790Q+01 |
| 0.282951921154269806808589056208Q+00 | 0.133966569289799879053736615991Q+01 |
| 0.657552965009547521396763736648Q+00 | 0.126672405937401965979281578336Q+00 |
| 0.198291706529368157392915460618Q-01 | -0.248569829627057726857390461395Q+00 |
| -0.522518701407516862859822726742Q-01 | 0.194210168500914138659457696360Q-01 |
| 0.637179398047106469894454574630Q-02 | 0.474178746600371388992805121038Q-02 |
| -0.328634348978159485614814420281Q-05 | -0.783284338912632757302061535032Q-03 |
| -0.439092520757651666348547108820Q-04 | 0.360316803150178065168213046779Q-04 |
| 0.311469670381302876533120119137Q-05 | 0.554579750608498406761396553401Q-06 |
| -0.728051503366116481284280539682Q-07 | -0.119380750534273396772761336039Q-06 |
| -0.142712677403921807369950040297Q-08 | 0.482286911005137130473876802019Q-08 |
| 0.134851828657556199671937219998Q-09 | -0.661943242988190217983811484321Q-10 |
| -0.353932973478850986153770653065Q-11 | -0.134831224477900589647254954717Q-11 |
| 0.311877766002957957951895944222Q-13 | 0.755336107340111135024372308735Q-13 |
| 0.621967416488506945329328598647Q-15 | -0.139406218976751538861636227753Q-14 |
| -0.234810461498766217579620115046Q-16 | 0.838971365080555841659818271420Q-17 |
| 0.321559240901043744735354451325Q-18 | 0.160182372762271410187426277948Q-18 |
| -0.138364867392323959748222645644Q-20 | -0.438692734349015130211087780655Q-20 |
| -0.250668665976145974374741957813Q-22 | 0.463549011924315275878111807280Q-22 |
| 0.522919590406060196877774286877Q-24 | -0.147770094088740172304545740072Q-24 |
| -0.439280016942636565029332641684Q-26 | -0.253215861539776800269773816881Q-26 |
| 0.106736554307622893072702514977Q-28 | 0.416734871991164514710446161458Q-28 |
| 0.172917868679241206801702001513Q-30 | -0.284970575964110297693018241003Q-30 |
| -0.230577487360277615155178584082Q-32 | 0.540249891588032775311371188516Q-33 |
| 0.130833560937310589395468340261Q-34 | 0.828143464106095590455257989353 5Q-35 |
| -0.197344861622563937926334032317Q-37 | -0.913731401882880974859355400171Q-37 |
| -0.286655588122102846401879178211Q-39 | 0.437101783150790746689468155574Q-39 |
| 0.266204149094617817073086313546Q-41 | -0.533358782721014378644233062524Q-42 |
| -0.108806062887745602150902243316Q-43 | -0.735435554601903342305089007853Q-44 |
| 0.108952809466947606650166863959Q-46 | 0.583017946925049396861830701816Q-46 |
| 0.142879013281696972870821207824Q-48 | -0.205948927316980664622675106645Q-48 |

COEFFICIENTS FOR YJ(Z) , N= 0

ARG(φ) = 37.5DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| -0.107922180850610026963964637326Q+02 | -0.333324268974311560250880686962Q+01 |
| -0.988163426452300218561664667187Q+01 | -0.485318846118956569918482178475Q+00 |
| -0.467892581986132737538413725781Q+01 | 0.320586782202616462059376153913Q+01 |
| -0.717989813690795418116315725300Q-02 | 0.231138375488762331691777820764Q+01 |
| 0.610323799934335457685267403055Q+00 | 0.185697764512417610039951894297Q+00 |
| 0.435422950018807878197528969854Q-01 | -0.107448555939270962562695163217Q+00 |
| -0.135251065108790500047797078630Q-01 | -0.509014616775542648794833978378Q-02 |
| -0.343814423298775911184250099449Q-03 | 0.125414408649802251144933583053Q-02 |
| 0.874496121331639152789130244914Q-04 | 0.116965558658103692901872786620Q-04 |
| -0.139011163823896903424103396102Q-06 | -0.466953823081837792992535942891Q-05 |
| -0.193960340054676960306171341278Q-06 | 0.417758606606484211651215020718Q-07 |
| 0.274689911969373331712706233831Q-08 | 0.634061918897420083057736240819Q-08 |
| 0.163994041735505369725107525278Q-09 | -0.116140461991463290099257168600Q-09 |
| -0.366194998115547044306280417119Q-11 | -0.333658016058466586306759872511Q-11 |
| -0.518113847845265705306031247786Q-13 | 0.911664077428161326572993154683Q-13 |
| 0.184647250698689144543333006394Q-14 | 0.549902645466658894696224029104Q-15 |
| 0.179382676240224621132507214827Q-17 | -0.309663349107683685342262013836Q-16 |
| -0.434321720497827011904449601915Q-18 | 0.777853918633221669322134190596Q-19 |
| 0.225337779680624431328445006622Q-20 | 0.511145658470170493508707527559Q-20 |
| 0.502144294788007211580592248533Q-22 | -0.386252838062239940082631366364Q-22 |
| -0.506355441218174987134646899896Q-24 | -0.403272212885982151280620588842Q-24 |
| -0.248740707507716633635470461924Q-26 | 0.544963649857128012317334760885Q-26 |
| 0.496101382859511596194619205491Q-28 | 0.914401742421107605060446518772Q-29 |
| -0.252187963407082842421892591126Q-31 | -0.387393253620527405375990961652Q-30 |
| -0.260732923841807634184749949096Q-32 | 0.845203199486687922189804574837Q-33 |
| 0.950960010783038225449765792403Q-35 | 0.150676217767493445941643393574Q-34 |
| 0.735061871407931412803492699548Q-37 | -0.777490027601082226986859149853Q-37 |
| -0.520716535570809045597617942256Q-39 | -0.288593094932095386491103624381Q-39 |
| -0.773629478709789265413313016973Q-42 | 0.297921031509196806188617543013Q-41 |
| 0.148257045723851696789934974680Q-43 | 0.205349518158695416249384551169Q-46 |
| -0.166215108542638588883192344113Q-46 | -0.645848032264863330609385145345Q-46 |
| -0.245642346248618848497290110525Q-48 | 0.136022895874588912330551993222Q-48 |

COEFFICIENTS FOR $Y_N(Z)$ , N= 0

ARG($\phi$) = 52.5DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.27816776151510634359240149986 5Q+01 | -0.46057408575036116068651257727 6Q+02 |
| -0.40972634859487433692016750648 2Q+01 | -0.36647759941730628166998870019 4Q+02 |
| -0.10216263494685059120552546619 1Q+02 | -0.16270042762475951050871908975 6Q+02 |
| -0.63626029400229789581882199697 5Q+01 | -0.20212182880131063632987693907 3Q+01 |
| -0.12783064640710365302432737437 5Q+01 | 0.85932210472018630272295361513 5Q+00 |
| 0.11745268747885548771398594887 9Q-01 | 0.24184817357941869828668676921 6Q+00 |
| 0.24771666257157306344968955083 4Q-01 | 0.10908975553180198925833431040 7Q-01 |
| 0.16507313022194914526999552507 2Q-02 | -0.15190808296096051863598190548 7Q-02 |
| -0.51123820162249910028492402257 4Q-04 | -0.13335136134730461093259655772 2Q-03 |
| -0.71890980058356671474432198500 7Q-05 | 0.87606194995043664583681891450 7Q-07 |
| -0.93444107761120299584299665057 6Q-07 | 0.27780603039600453642735801277 3Q-06 |
| 0.78660602375449739243691322956 2Q-08 | 0.59587319843255927019210313593 6Q-08 |
| 0.22887102886101084086353964464 9Q-09 | -0.15944473750531207328994407707 9Q-09 |
| -0.19970573521590632570606793722 3Q-11 | -0.64078964702339219519456510268 8Q-11 |
| -0.13913145061861823387930165241 0Q-12 | 0.25590834794331109372865313312 4Q-15 |
| -0.72625435698418445553459166247 Q-15 | 0.24028556709130086397324933257 4Q-14 |
| 0.33162949766683608023063747678 3Q-16 | 0.21957395216570996995683896549 1Q-16 |
| 0.42736682260702125736448740030 6Q-18 | -0.35849475227388044657352928975 3Q-18 |
| -0.27977050825815139767835317607 2Q-20 | -0.63876996404501702323963134699 7Q-20 |
| -0.77534396338974524058250929775 1Q-22 | 0.10138890018954126363765474269 2Q-22 |
| -0.11875144442060679148638111340 4Q-24 | 0.78189003686602504073362074077 0Q-24 |
| 0.65919132260529542909335874175 2Q-26 | 0.30210227831607760141608368158 6Q-26 |
| 0.39339310371312718980159060405 2Q-28 | -0.46027449171613059712128360421 4Q-28 |
| -0.25625462248558709489089449976 6Q-30 | -0.38486162231227815398028247314 8Q-30 |
| -0.30844288713497673226846620394 5Q-32 | 0.99631403790874421369872521423 4Q-33 |
| 0.80511966197612463675166347884 8Q-36 | 0.20917774952726175213264612569 5Q-34 |
| 0.12152465462807924988956568796 8Q-36 | 0.29041986982489712303214890994 2Q-37 |
| 0.33557058413741401703914402498 3Q-39 | -0.60442616397087341749509429845 2Q-39 |
| -0.25319611086009536615651858197 6Q-41 | -0.24964059882103355072050671127 0Q-41 |
| -0.14776777363915266979380637417 3Q-43 | 0.84923241442396886633393398265 8Q-44 |
| 0.18980785927937078889200055847 3Q-46 | 0.73918215847070233380068662214 9Q-46 |
| 0.31990122030621154279785143025 5Q-48 | 0.58801035809668665565015019455 3Q-50 |

COEFFICIENTS FOR $Y_\nu(Z)$ , N= 0

ARG($\phi$) = 67.5DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.11946317547761506188103864799 6Q+03 | 0.22745838830428063815792695617 4Q+02 |
| 0.94357123981554036461338611646 0Q+02 | 0.93241834475990404438445325476 0Q+01 |
| 0.45686378922064310243432010121 8Q+02 | -0.78529784196677950742110955010 4Q+01 |
| 0.12019819225916612752540270292 6Q+02 | -0.81261932925855913876930166926 9Q+01 |
| 0.12851537297671818743871962712 3Q+01 | -0.27170376584873928084906463702 5Q+01 |
| -0.70643664042500587955091939090 4Q-01 | -0.42535959163563637710206283676 3Q+00 |
| -0.32678130402529730606293991228 6Q-01 | -0.30629531471813839675083087320 1Q-01 |
| -0.34926758175367772512659355757 5Q-02 | -0.23993033775256634730707810047 7Q-03 |
| -0.17084056484514003991380822832 9Q-03 | 0.12654753017892984946227657864 2Q-03 |
| -0.21629497042506976444462836236 2Q-05 | 0.10068463776903470456667183837 2Q-04 |
| 0.20255750890681101227641663532 3Q-06 | 0.35257078058463272995407321598 9Q-06 |
| 0.12701355234843368697337368353 0Q-07 | 0.40349755147280300314943016136 4Q-08 |
| 0.33297675122594605488346005540 3Q-09 | -0.15709285897337334221229078197 6Q-09 |
| 0.31919441532990830911305703695 6Q-11 | -0.80786382530174468454595008815 8Q-11 |
| -0.65923514279145076444592107926 4Q-13 | -0.16426914168429978521050947657 4Q-12 |
| -0.28623964184978261147975418227 5Q-14 | -0.13051247250191979933992635829 9Q-14 |
| -0.46431163761015252760432573508 4Q-16 | 0.16236730962613499195742960890 3Q-16 |
| -0.30744308487450269514368344000 8Q-18 | 0.60850799640408891539734362484 9Q-18 |
| 0.24980435772425171986347875062 0Q-20 | 0.80549053257710889737638200758 0Q-20 |
| 0.82208872995372326739912057623 3Q-22 | 0.44892726677016913926604724336 1Q-22 |
| 0.90472663550304032153693043004 6Q-24 | -0.25220863465724242812501328891 4Q-24 |
| 0.42897883496668529169975228232 6Q-26 | -0.73893833962767564129409873640 4Q-26 |
| -0.17391702903026710266115782410 1Q-28 | -0.68669075209662088825311192273 4Q-28 |
| -0.45877610635360901584980195398 4Q-30 | -0.27986372824282530587498718117 0Q-30 |
| -0.36477025507079702047992677136 4Q-32 | 0.84665691976972178238907235463 3Q-33 |
| -0.12900054180395342880254858105 8Q-34 | 0.20296897635420804689045397529 7Q-34 |
| 0.29917830225470820276129461756 3Q-37 | 0.13963889102202213180872340369 3Q-36 |
| 0.65694743979918207631559162021 9Q-39 | 0.43222155534566113999643847023 4Q-39 |
| 0.39491339064032957469668703359 4Q-41 | -0.78578254792067882631170935955 7Q-42 |
| 0.10782201967717639836989974612 1Q-43 | -0.15910888352842000034326718877 6Q-43 |
| -0.15656753793011397325817956705 6Q-46 | -0.84287524224664709446628594589 4Q-46 |
| -0.29403405800441232190784054355 6Q-48 | -0.20441871867763660889181472051 3Q-48 |

COEFFICIENTS FOR Yν(Z) , N= 0

ARG(φ) = 82.5DEGREES

| REAL PART | IMAGINART PART |
|---|---|
| -0.105542264171543564071520438503Q+03 | 0.169032416773011957509624836121Q+03 |
| -0.786454959742101137798108226126Q+02 | 0.133791944879590687355835253701Q+03 |
| -0.314062393504746738823798662881Q+02 | 0.664185803052271919867833761270Q+02 |
| -0.645617231729705975540108910410Q+01 | 0.208756985872904668262115650612Q+02 |
| -0.525897620533831067063786460931Q+00 | 0.426842613334153813691775459994Q+01 |
| 0.460025433876543768533964347604Q-01 | 0.588437584721592236767841336077Q+00 |
| 0.171279779866541465392643891199Q-01 | 0.565691928530921856999691876466Q-01 |
| 0.222954768191032223626178909764Q-02 | 0.388983345611003337422526856101Q-02 |
| 0.181899173757394841124506372776Q-03 | 0.193588612957359139021487214153Q-03 |
| 0.105624491738584652036684642917Q-04 | 0.688052032694014646891954911923Q-05 |
| 0.462151697221401511824997085012Q-06 | 0.160493583181161166957368438916Q-06 |
| 0.157401237840098723418452753118Q-07 | 0.139019743917576904551641636542Q-08 |
| 0.425548512696002523953044723605Q-09 | -0.688561152868893916693681704816Q-10 |
| 0.922344697059932542603491259717Q-11 | -0.401748264894704598017362687557Q-11 |
| 0.160083155775149914360304636191Q-12 | -0.124928157493799097298229956202Q-12 |
| 0.218015096161939951418285191851Q-14 | -0.283818509696517572876772260834Q-14 |
| 0.216753385426864853409838346719Q-16 | -0.511396210303886631918587047333Q-16 |
| 0.111198996315245818550660117690Q-18 | -0.756628091779054453756654081973Q-18 |
| -0.103012855256134166084231121283Q-20 | -0.934713600659930377647827746528Q-20 |
| -0.371845960655719469367960226927Q-22 | -0.970019178698256293113047412831Q-22 |
| -0.604728803138919725350217158798Q-24 | -0.841927940663335139546503884607Q-24 |
| -0.723687321526021144836591979543Q-26 | -0.598069506016744417938690829514Q-26 |
| -0.703654881490028781381384171013Q-28 | -0.325251680579870427887919029497Q-28 |
| -0.576954434515810638833185004931Q-30 | -0.101992478080088821193751760908Q-30 |
| -0.405448982636371774641061299170Q-32 | 0.338698681221392395622956437559Q-33 |
| -0.245391867206137431968235213445Q-34 | 0.873358609498524480738860608389Q-35 |
| -0.127214356857863649200323622058Q-36 | 0.871711232667847601568972581220Q-37 |
| -0.552676349759311455264891428418Q-39 | 0.641675668165550663357829763045Q-39 |
| -0.188832905672543734451483873709Q-41 | 0.389129270848897280184078410393Q-41 |
| -0.393840808489456309703426174283Q-44 | 0.202168566422885980982426010536Q-43 |
| 0.615103383532513096500352830331Q-47 | 0.914594100174680165314291277348Q-46 |
| 0.122679143902130988823571991299Q-48 | 0.361881072814514425761921993754Q-48 |

## COEFFICIENTS FOR $Y_N(Z)$ , N= 1

### ARG($\phi$) = 7.5DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.123748557806745811056135432336Q+00 | -0.515168267682041884177266904100Q-01 |
| -0.193838690455536387474256807103Q+00 | -0.610494832678543787921293439351Q-02 |
| -0.634618056986256598023429631538Q-01 | 0.676435364314541567768894428441Q-01 |
| 0.105760828190668091803609110273Q+00 | 0.180390199187001934207711382491Q-01 |
| -0.325932423164314776936268260171Q-01 | -0.226031413729368570106504904468Q-01 |
| 0.446152884151434668274525830978Q-02 | 0.633494552256200851427689143782Q-02 |
| -0.280844397053881951668433700629Q-03 | -0.923147544582503937655419202145Q-03 |
| -0.525893018132882161021067230126Q-06 | 0.843283216584284551488938784305Q-04 |
| 0.161145216994395102324930678646Q-05 | -0.524941639444993719960530163623Q-05 |
| -0.152856766553300750093316609584Q-06 | 0.231814714579735416035535860517Q-06 |
| 0.854648301634342315800685772603Q-08 | -0.730542134747220603219322849795Q-08 |
| -0.339294477936013940697600811272Q-09 | 0.155597457527462822510830937430Q-09 |
| 0.102305992687996096812899052888Q-10 | -0.160023801562510040897161976874Q-11 |
| -0.242096440468855133590423964792Q-12 | -0.286373390502693068011096071816Q-13 |
| 0.456536263195830152536756718730Q-14 | 0.187197230703066271313515432616Q-14 |
| -0.686175965714311671155256056347Q-16 | -0.531774044196844647042232373163Q-16 |
| 0.802889587240727866675653913956Q-18 | 0.107314548223636836161420762354Q-17 |
| -0.668818015404528756828163611320Q-20 | -0.170421574995371970661920717459Q-19 |
| 0.236529480318469956183961748183Q-22 | 0.221613526672372778454580002068Q-21 |
| 0.389486429654360978674936635945Q-24 | -0.240223356498914567026588351206Q-23 |
| -0.993423180737253136350800644806Q-26 | 0.218164410729927690245549543924Q-25 |
| 0.136855689871462335409146923331Q-27 | -0.164665261447144543031896582440Q-27 |
| -0.142636137684616794461307525376Q-29 | 0.100034598549146430880147648094Q-29 |
| 0.121990654606387512022111784894Q-31 | -0.440043642612906247892523027864Q-32 |
| -0.883630370712384978956953601878Q-34 | 0.717817926907847945625931199325Q-35 |
| 0.549132769671228875602534424085Q-36 | 0.102095389042727538146098749008Q-36 |
| -0.293189070193334715526711269838Q-38 | -0.141075259543392363205464118030Q-38 |
| 0.132960034454766960974985721680Q-40 | 0.114960678659966164743537759830Q-40 |
| -0.493915918014890342007552521647Q-43 | -0.731956429972298377856188054876Q-43 |
| 0.133825834568631859940836213739Q-45 | 0.391341827209621467559478584875Q-45 |
| -0.118580144177632238121203874371Q-48 | -0.180658248337403526701616084426Q-47 |
| -0.146225986660949306876362630425Q-50 | 0.727786943570877354246446781467Q-50 |

COEFFICIENTS FOR $Y_N(Z)$ , N= 1

ARG($\phi$) = 22.5DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| -0.24282576638639918683683289000Q+00 | -0.352681634483825046064836820031Q+00 |
| -0.581053479084877644991684664899Q+00 | -0.132128841713672614502911245256Q+00 |
| -0.302824695808903405217159243338Q+00 | 0.266733531216838924164990291598Q+00 |
| 0.146136562472512918417778962528Q+00 | 0.163219890999840252221514388141Q+00 |
| 0.264101624271980280833400171705Q-01 | -0.586749824735128397301684700748Q-01 |
| -0.111493645011921025726052782356Q-01 | 0.166456507941861115990395850473Q-02 |
| 0.934862756292923378408244483221Q-03 | 0.924621782378134271100847454228Q-03 |
| 0.106910845189902056295293234594Q-04 | -0.109386934708065166115350915234Q-03 |
| -0.567884046548716871600282628189Q-05 | 0.396681130838238994138425448285Q-05 |
| 0.332034760387430601141038095423Q-06 | 0.803281786685600381491013662806Q-07 |
| -0.647672277585484770185924835843Q-08 | -0.119120581029948827225290722686Q-07 |
| -0.143024622738837045956724406358Q-09 | 0.419138865544663377935110765910Q-09 |
| 0.110584635898943389753296089672Q-10 | -0.495981649808279736339699029161Q-11 |
| -0.260486138171834383328392373605Q-12 | -0.108191145827876199261075124440Q-12 |
| 0.202235356097057642050857568805Q-14 | 0.527851379530953799627376460392Q-14 |
| 0.421860782044955867304831146969Q-16 | -0.891166888824095853416444173829Q-16 |
| -0.143245241798367195553789422576Q-17 | 0.480003130650915589092232792156Q-18 |
| 0.181840855337160521490376791331Q-19 | 0.946196911254922161050281497942Q-20 |
| -0.708725092449964290927007161787Q-22 | -0.237717404125617122067695745887Q-21 |
| -0.131504137006064352405160459858Q-23 | 0.235136098152520977794428568353Q-23 |
| 0.255046012192962724451300479835Q-25 | -0.685420984793591509631444531484Q-26 |
| -0.202081035601418595191975241764Q-27 | -0.119658866822919756513887992049Q-27 |
| 0.452490717117115691216808987405Q-30 | 0.184870620215186997188991879046Q-29 |
| 0.744066888706386755962745451205Q-32 | -0.119953542146356296592946977991Q-31 |
| -0.938297183776031130941439561033Q-34 | 0.210923381889845618463269644157Q-34 |
| 0.507654818142582216725263539694Q-36 | 0.327298062802918533055491878558Q-36 |
| -0.714117920777066643467587809077Q-39 | -0.343496887549063650147959808378Q-38 |
| -0.104793987861206912852605089104Q-40 | 0.157316290188515794341736043277Q-40 |
| 0.930014481419320156059457677985Q-43 | -0.179866230490163910154683413053Q-43 |
| -0.365174727148313633165366709741Q-45 | -0.250168276529252043283346642995Q-45 |
| 0.344035554089943639643612423944Q-48 | 0.190259703528151649037340695141Q-47 |
| 0.454527528859774729511519974567Q-50 | -0.647541742051235062702875939918Q-50 |

## COEFFICIENTS FOR Yν(Z) , N= 1

### ARG(φ) = 37.5DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| -0.8566907957309969646221432510620+00 | -0.2785219112769970645474431700330+01 |
| -0.1583956896537442082710627183100+01 | -0.1993019862294305565886181101710+01 |
| -0.1480724804747765205209459727730+01 | -0.1940062127346534974901205113580+00 |
| -0.2787736374577424763475252519610+00 | 0.5008700752978752469722322203913 7Q+00 |
| 0.1091103908595226995443272183840+00 | 0.7888692996100318564780135073770-01 |
| 0.1179695430562723462685105645380-01 | -0.1682638662527931291400458576120-01 |
| -0.1891791230739023630336309094030-02 | -0.1092282187064981125651427911260-02 |
| -0.6512253778880546574784832932910-04 | 0.1578683504510402024330564463550-03 |
| 0.9961680597000344578148162016900-05 | 0.2329674060180062048608614123510-05 |
| 0.2329248836383304218396509634110-07 | -0.4842140873181934381144032033140-06 |
| -0.1842566530578813561589452796220-07 | 0.2750071065827431484761616133560-08 |
| 0.2060793030910288276188761357800-09 | 0.5554720793449398111305939113710-09 |
| 0.1334119280597749699890034429390-10 | -0.8564560549623192884646194766870-11 |
| -0.2578017795183440235239409131330-12 | -0.2540610702247982882794588156130-12 |
| -0.3734253138419585953298231549500-14 | 0.6079180177610309134439036431400-14 |
| 0.1164554155415979877961767718100-15 | 0.3850650055546165303345294321590-16 |
| 0.1562668010237743550211324606420-18 | -0.1848968817716050881574876252920-17 |
| -0.2460053323340299411750177470330-19 | 0.3809633396830600507805778516890-20 |
| 0.1144965105794855051748524093060-21 | 0.2753532417612117772094281715970-21 |
| 0.2580767631018212745138611142780-23 | -0.1907886614426904928566332339060-23 |
| -0.2406777866795391222768102592930-25 | -0.1985825970174352178870554233720-25 |
| -0.1182896243535529344884556393910-27 | 0.2487050964042922807868251994770-27 |
| 0.2173949515265429574594712262350-29 | 0.4332003318265017575234112144950-30 |
| -0.8452995988678178834935696240690-33 | -0.1631537074760250129969184401620-31 |
| -0.1056816580019274656854409360580-33 | 0.3283840507051109498334966645360-34 |
| 0.3623708941371487618921429132840-36 | 0.5887806835493170719552187495780-36 |
| 0.2775449297423596964623265430170-38 | -0.2874784177669174195240420042710-38 |
| -0.1864223708169025366067369274600-40 | -0.1057038214923937904998786715450-40 |
| -0.2781506373544115181938573866260-43 | 0.1032545976383728458849511624110-42 |
| 0.4976927641572702468634683501020-45 | 0.4909200093486995658374064887 7Q-47 |
| -0.5231466599395778613978953732620-48 | -0.2101755742162984340754029615330-47 |
| -0.7757735498265359263971086323950-50 | 0.4220552982991435086179793160130-50 |

COEFFICIENTS FOR $Y_N(Z)$ , N= 1

ARG(φ) = 52.5DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.77039364564606678340424723535 5Q+01 | -0.88387669210231734457698533286 4Q+01 |
| 0.43552648907298167651727236079 3Q+01 | -0.80447752435759287904716681623 8Q+01 |
| -0.30998927530331551628018773358 3Q+00 | -0.45648582685067140681268948791 0Q+01 |
| -0.11236388006780464942850085668 9Q+01 | -0.88150012449698223687403728773 3Q+00 |
| -0.26954673461072957076060341577 6Q+00 | 0.94244912163304639002312448012 4Q-01 |
| -0.53636740210260142116372276692 9Q-02 | 0.38427290695621738064423566875 8Q-01 |
| 0.32142740476910864742809317625 7Q-02 | 0.19997857224310208873899990908 8Q-02 |
| 0.22343950847712423799365597761 2Q-03 | -0.16405572510044436125995915013 8Q-03 |
| -0.43229291396238632170691887950 8Q-05 | -0.15169208381360629529666340344 9Q-04 |
| -0.71572114121910771121762398740 7Q-06 | -0.43096946437154635419244401206 4Q-07 |
| -0.99923688394313973648069234800 0Q-08 | 0.24682250435254851414368105521 7Q-07 |
| 0.62983257914881721137937076167 7Q-09 | 0.52959867921776046892770410795 0Q-09 |
| 0.18150252889374106203564644152 8Q-10 | -0.11504722760027050094995749303 2Q-10 |
| -0.12556695747604161214696789075 3Q-12 | -0.46372451574744273529458794957 2Q-12 |
| -0.92938325010579014458790239037 5Q-14 | -0.28900135845542069930578473315 2Q-15 |
| -0.49863597861481722583670667925 8Q-16 | 0.14920133714642188267246950600 3Q-15 |
| 0.19223234837160867196904535289 9Q-17 | 0.13452748466398180165516643236 7Q-17 |
| 0.24266302987759156942111540267 6Q-19 | -0.19421504339150953109828642327 5Q-19 |
| -0.14067529018257709296480132366 4Q-21 | -0.34018769265857184708346052580 1Q-21 |
| -0.38972290149632570114158115990 4Q-23 | 0.43560073420219852238835172687 2Q-24 |
| -0.63059556832855677738466201179 3Q-26 | 0.37242864295705102228620064109 2Q-25 |
| 0.29835282129945776013080586063 3Q-27 | 0.14237400292317784601421081062 2Q-27 |
| 0.17437937367667251659690054818 6Q-29 | -0.19825099547432992721118411087 5Q-29 |
| -0.10497247396576780756240620996 1Q-31 | -0.16221020486500204213967259602 7Q-31 |
| -0.12420246642385412787607744823 5Q-33 | 0.38472133139777634925790960128 3Q-34 |
| 0.22051499067690585211705037749 3Q-37 | 0.80708414694817484215281945671 1Q-36 |
| 0.45018994743310047534456739115 0Q-38 | 0.11252594029820426512351217530 6Q-38 |
| 0.12224997425418682683813506802 2Q-40 | -0.21528144288160061686498472572 2Q-40 |
| -0.86760628453191552649685561194 2Q-43 | -0.87094001503594710256304699814 9Q-43 |
| -0.49620318823229391159137121006 1Q-45 | 0.27965014403447601381936677887 2Q-45 |
| 0.59498848848768716113871493558 6Q-48 | 0.23952674309759876819359391885 9Q-47 |
| 0.10020100528393549395534008897 8Q-49 | 0.25833024974141518303895398827 4Q-51 |

COEFFICIENTS FOR $Y_N(Z)$ , N= 1

ARG(ϕ) = 67.5DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| 0.2556570665058648078319905748O4Q+02 | 0.16685992956761264808351964836 8Q+02 |
| 0.20653410746289695872181951086 1Q+02 | 0.10245423253593522752948776609 3Q+02 |
| 0.10281715117123220493936913239 1Q+02 | 0.12051028619194378661261819041 0Q+01 |
| 0.26528984222357982656929411831 3Q+01 | -0.10627840498943628737988586712 0Q+01 |
| 0.29157651716857062599774386164 2Q+00 | -0.42579114697332432653839453531 8Q+00 |
| -0.28659250397014417390856915310 6Q-02 | -0.64710196645208603474023464595 6Q-01 |
| -0.39359195269430675518950310074 8Q-02 | -0.44496413823592766772961993558 9Q-02 |
| -0.40993002218121186361937961238 7Q-03 | -0.59029228911288915360984814466 4Q-04 |
| -0.18980898338794524053425855663 8Q-04 | 0.12356849702921507554180277172 5Q-04 |
| -0.25770431837834533214511194223 1Q-06 | 0.96253837513000823823438672892 2Q-06 |
| 0.16610830261372366881798990811 5Q-07 | 0.31945825857582701463689197692 3Q-07 |
| 0.10237410398782048590514008337 9Q-08 | 0.36595643690049607781173000670 1Q-09 |
| 0.25523276432792826575815891095 6Q-10 | -0.11095599233973423898186016748 6Q-10 |
| 0.23983700875128247533328692231 9Q-12 | -0.56253962669907759644710368337 2Q-12 |
| -0.40871311360341941156952646777 1Q-14 | -0.10918157586154745678546096409 0Q-13 |
| -0.17536939393945912346291002911 0Q-15 | -0.84393514119852185681731529799 3Q-16 |
| -0.27246693222847662759435528926 9Q-17 | 0.89673606744071302590914918704 6Q-18 |
| -0.17506585577437318010353184394 9Q-19 | 0.33271980138286633212138748731 9Q-19 |
| 0.12435779958923438280244028224 7Q-21 | 0.42316143072003079433253923053 8Q-21 |
| 0.40576171516283912107436600077 1Q-23 | 0.22873441092984735418449545408 1Q-23 |
| 0.43023409840650455397466991003 5Q-25 | -0.11426750602367674715529036784 2Q-25 |
| 0.19793479115954613748460919558 0Q-27 | -0.33232096192332384050599231259 6Q-27 |
| -0.72289398828293586392454697820 8Q-30 | -0.29827015171068193960125695585 8Q-29 |
| -0.18946632242475014842230087483 9Q-31 | -0.11805142651261902507227989381 5Q-31 |
| -0.14581190945150013018161913884 1Q-33 | 0.32504907694809337377665517362 5Q-34 |
| -0.50130152496839149737719526030 3Q-36 | 0.77483363878677599015139558398 8Q-36 |
| 0.10670757441161619817542835390 4Q-38 | 0.51697160456426783403728973045 8Q-38 |
| 0.23313876630075831724905578906 7Q-40 | 0.15573459282431362967406559796 5Q-40 |
| 0.13614977048700650125540350571 6Q-42 | -0.26167764362591800269028434199 0Q-43 |
| 0.36217979821503676435247028121 6Q-45 | -0.52748922004343177318670691006 6Q-45 |
| -0.48895074969090973149923128214 2Q-48 | -0.27188575437202061612708376079 4Q-47 |
| -0.91456810934702647911125833695 2Q-50 | -0.64315207730224859934370664948 6Q-50 |

COEFFICIENTS FOR $Y_N(Z)$ , N= 1

ARG($\phi$) = 82.5DEGREES

| REAL PART | IMAGINARY PART |
|---|---|
| -0.31703692891868208599240603886OQ+02 | 0.38531513787878288137282579585 4Q+02 |
| -0.22649551509974526005751024301 2Q+02 | 0.29701038141244579287854602432 7Q+02 |
| -0.81244984162332938730944012906 7Q+01 | 0.13675512181016365910042700716 9Q+02 |
| -0.15022392952089790823958050494 5Q+01 | 0.38626929490859735662050636170 5Q+01 |
| -0.12487604877888649230082024308 9Q+00 | 0.70140901300071678300390236436 3Q+00 |
| 0.31586191203044773853825568795 7Q-02 | 0.85885833942138780168050560988 6Q-01 |
| 0.19742500695678244570457041637 4Q-02 | 0.73754653345075840591242910764 6Q-02 |
| 0.24613156067072876940799570215 2Q-03 | 0.45661010181432805149575386460 9Q-03 |
| 0.18580460346634509902319990250 5Q-04 | 0.20639503937169356641244951661 5Q-04 |
| 0.99398834305153719985565479044 8Q-06 | 0.67290758332291690593650547373 5Q-06 |
| 0.40151084414092039583092032571 2Q-07 | 0.14620223188854702245760984404 7Q-07 |
| 0.12673305934030783663054276648 7Q-08 | 0.12827186216268011185723018517 2Q-09 |
| 0.31890565631712227000305635437 2Q-10 | -0.47997241389306453711304999331 2Q-11 |
| 0.64611430686754271917715361492 1Q-12 | -0.27407055462181509007610449284 1Q-12 |
| 0.10527500577878868335519254743 3Q-13 | -0.80734206982098962992192266008 0Q-14 |
| 0.13520909342014119928233807967 8Q-15 | -0.17331914622893640586739405068 3Q-15 |
| 0.12756415969069704360448315089 8Q-17 | -0.29546664201105068522581404423 3Q-17 |
| 0.63448574910437848587677273268 9Q-20 | -0.41446790153864102662858969765 6Q-19 |
| -0.50968284167427652574794352351 2Q-22 | -0.48657174339678578947847274172 8Q-21 |
| -0.18168322776632445437325042676 4Q-23 | -0.48097377858402333047637934326 7Q-23 |
| -0.28359982445218823823258665284 1Q-25 | -0.39857223677842163049642080156 0Q-25 |
| -0.32521153516519313405268973618 4Q-27 | -0.27099626286144368983115840289 4Q-27 |
| -0.30322107424787599246372211674 6Q-29 | -0.14153514117182986105801249076 1Q-29 |
| -0.23871071639935910241649057561 5Q-31 | -0.43045697685179465430533497198 1Q-32 |
| -0.16128863838539516534198779747 9Q-33 | 0.12957714916336735169384269391 7Q-34 |
| -0.93990845605018204763756847664 5Q-36 | 0.33140323370409305677833145845 7Q-36 |
| -0.46984158931467073432545875909 3Q-38 | 0.32006505375808379109943392080 1Q-38 |
| -0.19713241630994780189817617756 1Q-40 | 0.22770172025175226340312292511 6Q-40 |
| -0.65183934334295397969764187442 5Q-43 | 0.13351552638348008647674147901 6Q-42 |
| -0.13232044408077683756821734984 6Q-45 | 0.67127016186691974965915679178 0Q-45 |
| 0.19165748633475299724717025654 6Q-48 | 0.29414434663597198247264321690 4Q-47 |
| 0.38006566208401458371831989700 7Q-50 | 0.11283988816448536464274076987 1Q-49 |

## APPENDIX C

### Fortran Subroutine for $J_0(z)$, $|z| \leq 8$

This subroutine can be used to calculate $J_0(z)$ for $|z| \leq 8$ to a user supplied accuracy. It can be adapted for different orders by changing, NU, DENOM and the coefficients in A.

```
      SUBROUTINE BESAC( NACC, Z, BESS, IFAIL )
C
C--------EXPLANATION OF VARIABLES-----------------------
C
C     NACC......INTEGER .
C                ON ENTRY CONTAINS THE NUMBER OF
C               DECIMAL PLACE ACCURACY REQUIRED .
C               MAXIMUM ACCURACY OF 28 TO 31 PLACES
C               DEPENDS ON ARG(Z) .
C               NACC SHOULD BE NO GREATER THAN 28 ,
C               FOR Z NEAR THE IMAGINARY AXIS .
C                UNCHANGED ON EXIT .
C
C     Z.........QUADRUPLE PRECISION COMPLEX .
C                ON ENTRY CONTAINS THE POINT AT WHICH
C               THE FUNCTION IS TO BE CALCULATED .
C               THE MODULUS OF Z MUST NOT BE GREATER
C               THAN 8.0Q0 .
C                UNCHANGED ON EXIT .
C
C     BESS......QUADRUPLE PRECISION COMPLEX .
C                UNSPECIFIED ON ENTRY .
C                ON SUCCESSFUL EXIT ( SEE IFAIL )
C               CONTAINS THE VALUE OF THE BESSEL
C               FUNCTION AT Z .
C
C     IFAIL.....INTEGER .
C                UNSPECIFIED ON ENTRY .
C                ON EXIT HAS ONE OF THE FOLLOWING VALUES
C
C      0........CALCULATION COMPLETED TO REQUIRED ACCURACY
C
C      1........REQUIRED ACCURACY TOO HIGH ,
C               CALCULATION COMPLETED TO MAXIMUM ACCURACY .
C
C      2........ABSOLUTE VALUE OF Z IS GREATER THAN EIGHT .
C               NO CALCULATION PERFORMED .
C
      IMPLICIT REAL*16(Q)
      COMPLEX*32 A, BESS, C, C1, T, Z, ZT
```

```
      REAL*16 ABSZ, ARGZT, COSIN, DENOM, PIFAC, X, Y
      REAL*4 FNACC, FNU, R0, R1, R2
      DIMENSION A(6, 28), COSIN(6)
C
C    NU........INTEGER .
C              THE ORDER OF THE BESSEL FUNCTION OF THE FIRST KIND .
C
C    DENOM.....QUADRUPLE PRECISION REAL .
C              SET TO NU FACTORIAL .
C
      DATA NU, DENOM / 0,  1.0Q0 /
C
C    PIFAC.....QUADRUPLE PRECISION REAL . SET TO PI/24 .
C
      DATA PIFAC / 1.308996938995747182692768076366 4Q-01 /
C
C    COSIN.....QUADRUPLE PRECISION REAL ARRAY OF DIMENSION 6
C              COSIN( K ) SET TO COS( PI * ( 2K-1 ) / 24 ) .
C
      DATA COSIN(1), COSIN(2), CCSIN(3), COSIN(4), COSIN(5), COSIN(6) /
     * 0.99144486137381041114455752692856 3Q+00 ,
     * 0.92387953251128675612818318939678 9Q+00 ,
     * 0.79335334029123516457977769615013 02Q+00 ,
     * 0.60876142900872063941609754289816 9Q+00 ,
     * 0.38268343236508977172845998403040 6Q+00 ,
     * 0.13052619222005159154840622789550 0Q+00 /
C
C    A.........QUADRUPLE PRECISION COMPLEX , TWO DIMENSIONAL ARRAY
C              OF DIMENSIONS  6 BY 28 ,
C               ROW K CONTAINS THE COEFFICIENTS IN THE EXPANSION OF THE
C              FUNCTION ALONG THE RAY WITH ARGUMENT PI * ( 2K-1 ) / 24 .
C
      DATA A(1,1), A(1,2), A(1,3), A(1,4), A(1,5), A(1,6), A(1,7) /
     *   (  0.40795869055667137640715028153693Q+00 ,
     *   -0.56442646929690935200795128274590Q-01 ) ,
     *   (  0.76307829250131400361146778095084Q-01 ,
     *   -0.61302046672716605147332674516919Q-01 ) ,
     *   (  0.29061006595818788085586515391498Q+00 ,
     *   -0.20911435561449686363680202686931Q+00 ) ,
     *   ( -0.41389519311709364289690627937354Q+00 ,
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     *    -0.49959512745050736116971017573850Q-01 ) ,
     *    (  0.14405373906065264429620605762981Q+00 ,
     *     0.91255766225367239943363268407435Q-01 ) ,
     *    ( -0.22191868433940324530320446112275Q-01 ,
     *    -0.29516142867063306110340315867003Q-01 ) ,
     *    (  0.15821011678673097559220170114149Q-02 ,
     *     0.47897681004629969864723074241209Q-02 ) /
      DATA A(1,8), A(1,9), A(1,10), A(1,11), A(1,12), A(1,13), A(1,14) /
     *    ( -0.59234157455037023995012675293566Q-05 ,
     *    -0.47863710713778006789287843649731Q-03 ) ,
     *    ( -0.93656954962602142941518489788133Q-05 ,
     *     0.32225706258653841087281828118680Q-04 ) ,
     *    (  0.98007182750822194752002275702969Q-06 ,
     *    -0.15271062036972918272686508255232Q-05 ) ,
     *    ( -0.58849005869019316334881884219035Q-07 ,
     *     0.51370384201749153937317288042167Q-07 ) ,
     *    (  0.24841843116601105419586447412207Q-08 ,
     *    -0.11659304865213573041154924553111Q-08 ) ,
     *    ( -0.79166685333981694744509040991190Q-10 ,
     *     0.13002722707254774191498280825128Q-10 ) ,
     *    (  0.19713528931142066795575258727593Q-11 ,
     *     0.21989564315413682332706849016329Q-12 ) /
      DATA A(1,15), A(1,16), A(1,17), A(1,18), A(1,19), A(1,20) /
     *    ( -0.38986101629433760169040062140733Q-13 ,
     *    -0.15719635840571344149732102786680Q-13 ) ,
     *    (  0.61288305780992463777428485624698Q-15 ,
     *     0.46989454558630322083751642792260Q-15 ) ,
     *    ( -0.74872296145083913456919480921153Q-17 ,
     *    -0.99111434114507891007149844922754Q-17 ) ,
     *    (  0.65125722022767106932468459658214Q-19 ,
     *     0.16392989156702043206417304659115Q-18 ) ,
     *    ( -0.24484306038263621355486069536909Q-21 ,
     *    -0.22147117906606840778028263028522Q-20 ) ,
     *    ( -0.39482281695962560448896899872804Q-23 ,
     *     0.24892188592384991764317178926069Q-22 ) /
      DATA A(1,21), A(1,22), A(1,23), A(1,24) /
     *    (  0.10567641477407692569908964539183Q-24 ,
     *    -0.23401714881381196783886748810376Q-24 ) ,
     *    ( -0.15087853069717455406447080694267Q-26 ,
     *     0.18260278516002535548651669668619Q-26 ) ,
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     *    (  0.162463317540326319000499377145520-28 ,
     *    -0.114582250519443901956544059474470-28 ) ,
     *    ( -0.143291993541657474038737895308180-30 ,
     *     0.520840782287438711777419218484370-31 ) /
      DATA A(1,25), A(1,26), A(1,27), A(1,28) /
     *    (  0.106890149122843158428983853952350-32 ,
     *    -0.892675148288958685372987473176940-34 ) ,
     *    ( -0.683314260765200379603100452214480-35 ,
     *    -0.125556869759183945986492935789400-35 ) ,
     *    (  0.374926332242110499465417241102120-37 ,
     *     0.179501675861717874478628288775550-37 ) ,
     *    ( -0.174594036859469968515189898939390-39 ,
     *    -0.150401256476253722371196839896170-39 ) /
      DATA A(2,1), A(2,2), A(2,3), A(2,4), A(2,5), A(2,6), A(2,7) /
     *    (  0.186238813652549412063911842807460+01 ,
     *     0.164861228598252421447183079191300+00 ) ,
     *    (  0.145863866251809509725457972203810+01 ,
     *    -0.125444666387042196954030278969230+00 ) ,
     *    (  0.995487776036557319485382468856430+00 ,
     *    -0.108442291085020245829593661548970+01 ) ,
     *    ( -0.683630679198039881838798357615880+00 ,
     *    -0.593235962407262793878060018763720+00 ) ,
     *    ( -0.921038940314369751249007366703510-01 ,
     *     0.276151457800522475185642227993680+00 ) ,
     *    (  0.543251552867064970097067059132680-01 ,
     *    -0.124763818393497555336676635244440-01 ) ,
     *    ( -0.531269974145773677347256249784960-02 ,
     *    -0.465447458567278883811874144789900-02 ) /
      DATA A(2,8), A(2,9), A(2,10), A(2,11), A(2,12), A(2,13) /
     *    ( -0.313387287174156380552225888856820-04 ,
     *     0.636229272782622646929723832551470-03 ) ,
     *    (  0.342873932912792066540248547433790-04 ,
     *    -0.260135301745497372002168694910380-04 ) ,
     *    ( -0.221524264522547452842587337819220-05 ,
     *    -0.459929886723214946315828573894510-06 ) ,
     *    (  0.477911496613730399546145758520490-07 ,
     *     0.824014419476745953223431302111350-07 ) ,
     *    (  0.987724684291460115549519078425180-09 ,
     *    -0.313460160428811861650796713917670-08 ) ,
     *    ( -0.855594184136638353059230310265960-10 ,
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     *     0.40492685731528486918403002058275Q-10 ) /
      DATA A(2,14), A(2,15), A(2,16), A(2,17), A(2,18), A(2,19) /
     *    (  0.21497692022316954524086941727099Q-11 ,
     *     0.84820179582633413317487408538878Q-12 ) ,
     *    ( -0.18044845865653460488125005161979Q-13 ,
     *    -0.44965043319601150272937439932755Q-13 ) ,
     *    ( -0.36611959638375363485072844099656Q-15 ,
     *     0.80228194969170118787213386821454Q-15 ) ,
     *    (  0.13282565182822653728060423561167Q-16 ,
     *    -0.46368682453174704019240608040720Q-17 ) ,
     *    ( -0.17697985099743731846229593931446Q-18 ,
     *    -0.89566498342508217565158876682037Q-19 ) ,
     *    (  0.73565583861815498696056484311028Q-21 ,
     *     0.23784602955667591021187110468363Q-20 ) /
      DATA A(2,20), A(2,21), A(2,22), A(2,23), A(2,24) /
     *    (  0.13439277020255245432805184079622Q-22 ,
     *    -0.24563383016281616649391002845311Q-22 ) ,
     *    ( -0.27343095881597148765216409975695Q-24 ,
     *     0.75979310571844082254091676766722Q-25 ) ,
     *    (  0.22524555302524473310456212866351Q-26 ,
     *     0.13101506878535092988935606352495Q-26 ) ,
     *    ( -0.53291005516586209617603948448949Q-29 ,
     *    -0.21115269624392155974828639121683Q-28 ) ,
     *    ( -0.86754651739648244417089788877729Q-31 ,
     *     0.14195300876382134298811178252330Q-30 ) /
      DATA A(2,25), A(2,26), A(2,27), A(2,28) /
     *    (  0.11362604863590713575985169424467Q-32 ,
     *    -0.26277104068594085415779274347173Q-33 ) ,
     *    ( -0.63512437869492818751052544852765Q-35 ,
     *    -0.40436324862400408223202596401835Q-35 ) ,
     *    (  0.93755737997489190974076797605123Q-38 ,
     *     0.43922566481873963909743193811676Q-37 ) ,
     *    (  0.13661645456438597289667311109739Q-39 ,
     *    -0.20731565753726878313488826244435Q-39 ) /
      DATA A(3,1), A(3,2), A(3,3), A(3,4), A(3,5), A(3,6) /
     *    (  0.67119273258906999397417562680239Q+01 ,
     *     0.83712702818187066160128862554041Q+01 ) ,
     *    (  0.71500895090149274249562302933468Q+01 ,
     *     0.59935320789535874704309890908769Q+01 ) ,
     *    (  0.62159845055422611610087734646016Q+01 ,
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     *    -0.19486400634296973199162619237438Q+00 ) ,
     *    (  0.91766948812533030008971126548282Q+00 ,
     *    -0.24065140778222976905395753482710Q+01 ) ,
     *    ( -0.56777356588961925012223189451426Q+00 ,
     *    -0.31719376085575826201011612499029Q+00 ) ,
     *    ( -0.52984967577101028658611440784450Q-01 ,
     *     0.92811127584290710768303757704639Q-01 ) /
      DATA A(3,7), A(3,8), A(3,9), A(3,10), A(3,11), A(3,12) /
     *    (  0.10994988091648049146745669092850Q-01 ,
     *     0.53006984777873094100192858823990Q-02 ) ,
     *    (  0.33155441080985683550871115114717Q-03 ,
     *    -0.96600009434664274199423455972291 8Q-03 ) ,
     *    ( -0.64158538295280157832171771619448Q-04 ,
     *    -0.11644140366370472243416450105202Q-04 ) ,
     *    ( -0.19262278486119835822887559091950Q-07 ,
     *     0.32792409067351769904708187461371Q-05 ) ,
     *    (  0.13098460082289649995221225867198Q-06 ,
     *    -0.24389700433340074222958066472352Q-07 ) ,
     *    ( -0.16816077796311041189112778914050Q-08 ,
     *    -0.41353051818623120210689188092807Q-08 ) /
      DATA A(3,13), A(3,14), A(3,15), A(3,16), A(3,17), A(3,18) /
     *    ( -0.10371152882321160186361693197984Q-09 ,
     *     0.70597257863178096934493061241108Q-10 ) ,
     *    (  0.21859346943753071283032864827525Q-11 ,
     *     0.20545036410632514002637938671880Q-11 ) ,
     *    (  0.31222454779154038507531746800889Q-13 ,
     *    -0.53302500363141758467434181670466Q-13 ) ,
     *    ( -0.10573078967372922849271977707796Q-14 ,
     *    -0.32781289949432533628380676339244Q-15 ) ,
     *    ( -0.11756726790979670550122091559254Q-17 ,
     *     0.17379430376588814964417894251052Q-16 ) ,
     *    (  0.23918281026749472894061443142955Q-18 ,
     *    -0.40755209566834685198756324933495Q-19 ) /
      DATA A(3,19), A(3,20), A(3,21), A(3,22), A(3,23) /
     *    ( -0.11947502045502751672080653424766Q-20 ,
     *    -0.27655685125209732758600102490881Q-20 ) ,
     *    ( -0.26729840697905951975410589489261Q-22 ,
     *     0.20282438263448503305629880293553Q-22 ) ,
     *    (  0.26247196204098981414737826408936Q-24 ,
     *     0.21156346934994648642359041179737Q-24 ) ,
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     *   ( 0.1289796388356917651558530211121Q-26 ,
     *   -0.2786963713917335483968719744061lQ-26 ) ,
     *   ( -0.2503565183879772659757483697874 4Q-28 ,
     *   -0.4737086104589644641685304796332lQ-29 ) /
      DATA A(3,24), A(3,25), A(3,26), A(3,27), A(3,28) /
     *   ( 0.1173469456583194683115527136851 9Q-31 ,
     *    0.1930068033073649814451152086495 2Q-30 ) ,
     *   ( 0.1283243362073472375135938304636 5Q-32 ,
     *   -0.4104682011345842342015072695996 0Q-33 ) ,
     *   ( -0.4590327616311595317423794620550 8Q-35 ,
     *   -0.7330787488207581099844785948148 4Q-35 ) ,
     *   ( -0.3538272387372581102460675835627 5Q-37 ,
     *    0.3718189419426940603910216867099 8Q-37 ) ,
     *   ( 0.2465741351829343796473630176244 9Q-39 ,
     *    0.1376227070187396970169454492943 2Q-39 ) /
      DATA A(4,1), A(4,2), A(4,3), A(4,4), A(4,5), A(4,6) /
     *   ( -0.2461560768007209794640941478395 8Q+02 ,
     *    0.4070818451974168490689402476450 6Q+02 ) ,
     *   ( -0.1367468043326338181693004783760 9Q+02 ,
     *    0.3675086860178411909082205761766 8Q+02 ) ,
     *   ( 0.4205032398658339285402426700265 7Q+01 ,
     *    0.2036773969152743229788314336387 1Q+02 ) ,
     *   ( 0.5862275976644791413457682747778 3Q+01 ,
     *    0.3554836138191359700951167509955 5Q+01 ) ,
     *   ( 0.1320403463490863778273146313953 3Q+01 ,
     *   -0.6155346174253676291763045622467 6Q+00 ) ,
     *   ( 0.1201136204948611960365162806905 0Q-01 ,
     *   -0.2107148576478635094254374143829 1Q+00 ) /
      DATA A(4,7), A(4,8), A(4,9), A(4,10), A(4,11), A(4,12) /
     *   ( -0.1937249066303771359949476421326 6Q-01 ,
     *   -0.1031807319065724138605960183432 3Q-01 ) ,
     *   ( -0.1319376588675232409095671324952 7Q-02 ,
     *    0.1086868574307408033104558415691 7Q-02 ) ,
     *   ( 0.3288209412365083139469213658888 8Q-04 ,
     *    0.9776924037019291837598371958755 6Q-04 ) ,
     *   ( 0.4956120002487949586281653066042 9Q-05 ,
     *    0.1008907163268074823659217998453 4Q-06 ) ,
     *   ( 0.6662108184372829721718314801472 6Q-07 ,
     *   -0.1822328826425462913411675010694 2Q-06 ) ,
     *   ( -0.4941194077534807421193948168383 0Q-08 ,
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     *    -0.39106755594795768418695956250335Q-08 ) /
      DATA A(4,13), A(4,14), A(4,15), A(4,16), A(4,17), A(4,18) /
     *    ( -0.14323130562185905266344553318864Q-09 ,
     *      0.96090174415940859625911547790801Q-10 ) ,
     *    (  0.11426229129018497661619105639929Q-11 ,
     *      0.38663761072019751013073789910377Q-11 ) ,
     *    (  0.81399052533547393688503984541264Q-13 ,
     *      0.87295719916708646689766836918206Q-15 ) ,
     *    (  0.42963581930446140337461610709026Q-15 ,
     *    -0.13679352041698177218734837566741Q-14 ) ,
     *    ( -0.18413047890098872367751566818597Q-16 ,
     *    -0.12441337372666640599612697442972Q-16 ) ,
     *    ( -0.23556437667867229352963180294575Q-18 ,
     *      0.19433016360925339500623608205876Q-18 ) /
      DATA A(4,19), A(4,20), A(4,21), A(4,22), A(4,23)  /
     *    (  0.14780223743818327032278009370160Q-20 ,
     *      0.34422016606709071729170561679391Q-20 ) ,
     *    (  0.40956734504856370170797264766348Q-22 ,
     *    -0.50882962561403372040902485809500Q-23 ) ,
     *    (  0.63981925565223277240096031917410Q-25 ,
     *    -0.40558120586652640015269127140345Q-24 ) ,
     *    ( -0.33618720134599604516805907738920Q-26 ,
     *    -0.15617312296875415002587281308420Q-26 ) ,
     *    ( -0.19927812351128876188982915416726Q-28 ,
     *      0.23097458034639807215285458498273Q-28 ) /
      DATA A(4,24), A(4,25), A(4,26), A(4,27), A(4,28) /
     *    (  0.12653928079114571953761923102439Q-30 ,
     *      0.19179314075437628141322658676683Q-30 ) ,
     *    (  0.15148831459979218067536099487641Q-32 ,
     *    -0.48291887511308268323169572633154Q-33 ) ,
     *    ( -0.35457362648932105717005203942104Q-36 ,
     *    -0.10136460884656706702109567689352Q-36 ) ,
     *    ( -0.58150487026493486555445415501452Q-37 ,
     *    -0.14094518049251148437476781919616Q-37 ) ,
     *    ( -0.15975891547896140605983712801556Q-39 ,
     *      0.28576217592466411663824811239248 5Q-39 ) /
      DATA A(5,1), A(5,2), A(5,3), A(5,4), A(5,5), A(5,6) /
     *    ( -0.12023234004003584995981888610765 6Q+03 ,
     *    -0.66691018233263761381657206590178Q+02 ) ,
     *    ( -0.99170227887441285329135194491064Q+02 ,
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     *    -0.39147826531356588336958685234818Q+02 ) ,
     *    ( -0.49482551698438323677964812571407Q+02 ,
     *    -0.14800978678558559364104741152456Q+01 ) ,
     *    ( -0.12834635332439317353249812196134Q+02 ,
     *     0.63351619838680025091559421924811Q+01 ) ,
     *    ( -0.13924145634822307560273601121626Q+01 ,
     *     0.23497334840795793775590047916078Q+01 ) ,
     *    (   0.35873442631228915638088468993558Q-01 ,
     *     0.36184555175082579591418906704027Q+00 ) /
      DATA A(5,7), A(5,8), A(5,9), A(5,10), A(5,11), A(5,12) /
     *    (   0.24623296955251348484432877021334Q-01 ,
     *     0.25407683584957795664451288720100Q-01 ) ,
     *    (   0.25989029404040905193856549537677Q-02 ,
     *     0.27328614896885497885059913797181Q-03 ) ,
     *    (   0.12385474204657920414516265382266Q-03 ,
     *    -0.86478706066528572035583131906616Q-04 ) ,
     *    (   0.16233818774304976462790471703260Q-05 ,
     *    -0.68164033686269323775951271638189Q-05 ) ,
     *    ( -0.12848188590205171272287441105036Q-06 ,
     *    -0.23322393287194919485736536781241Q-06 ) ,
     *    ( -0.79995001109688431240715126441317Q-08 ,
     *    -0.26718216601645247528390463650696Q-08 ) /
      DATA A(5,13), A(5,14), A(5,15), A(5,16), A(5,17), A(5,18) /
     *    ( -0.20551599036084747083254899649849Q-09 ,
     *     0.93883769053122440787827556762722Q-10 ) ,
     *    ( -0.19551294992334729981372286345167Q-11 ,
     *     0.48015363459005741396881230966287Q-11 ) ,
     *    (   0.37507349404640345799651797934362Q-13 ,
     *     0.95920761478909471667874560717069Q-13 ) ,
     *    (   0.16214600690405924996084918798866Q-14 ,
     *     0.75440387851336339973278181017069Q-15 ) ,
     *    (   0.25894198253276515890314860610469Q-16 ,
     *    -0.88611990878570833706219473950495Q-17 ) ,
     *    (   0.16963885835163051058784661923728Q-18 ,
     *    -0.33091881056889251909482520124499Q-18 ) /
      DATA A(5,19), A(5,20), A(5,21), A(5,22), A(5,23) /
     *    ( -0.13151889996350858678670178161917Q-20 ,
     *    -0.43198897528320881209066285578019Q-20 ) ,
     *    ( -0.43155758831589959275961716074911Q-22 ,
     *    -0.23325721607151014874315892698848Q-22 ) ,
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     *    ( -0.46903719427128928309181255754079Q-24 ,
     *     0.12866821474808433187918726739619Q-24 ) ,
     *    ( -0.22018698090879407279084234728340Q-26 ,
     *     0.37606144136947100549130543087069Q-26 ) ,
     *    (  0.86281155469266477971682953927907Q-29 ,
     *     0.34553644783555857374984659001160Q-28 ) /
      DATA A(5,24), A(5,25), A(5,26), A(5,27), A(5,28) /
     *    (  0.22713070854464166764224304073967Q-30 ,
     *     0.13950524503754715029219540715481Q-30 ) ,
     *    (  0.17873486006162661044327487865169Q-32 ,
     *    -0.40963510166528041796508686348469Q-33 ) ,
     *    (  0.62652933706147687905142320609119Q-35 ,
     *    -0.98027781360612562544596100114940Q-35 ) ,
     *    ( -0.14150553808635658780091448512327Q-37 ,
     *    -0.66805156657504333666265591240035Q-37 ) ,
     *    ( -0.31024687126385932720181561031139Q-39 ,
     *    -0.20507272069832033063755063045718Q-39 ) /
      DATA A(6,1), A(6,2), A(6,3), A(6,4), A(6,5), A(6,6) /
     *    (  0.14786330077903399910834413043463Q+03 ,
     *    -0.18685812523542100076518581169684Q+03 ) ,
     *    (  0.10415768205398205162399654475226Q+03 ,
     *    -0.14493435376763316555403379954426Q+03 ) ,
     *    (  0.37773716029004829631634647285402Q+02 ,
     *    -0.68100726343676867881843240961907Q+02 ) ,
     *    (  0.71510684578255635647552427740806Q+01 ,
     *    -0.19942171041630688504315865053873Q+02 ) ,
     *    (  0.58718947463746490892023107893163Q+00 ,
     *    -0.37945991286273713731645640517225Q+01 ) ,
     *    ( -0.27336816292524014223244975541634Q-01 ,
     *    -0.48944600177602346303981707099740Q+00 ) /
      DATA A(6,7), A(6,8), A(6,9), A(6,10), A(6,11), A(6,12) /
     *    ( -0.12633298385438303996328684848139Q-01 ,
     *    -0.44344116850055764209395580208655Q-01 ) ,
     *    ( -0.16111141681285472669446158410552Q-02 ,
     *    -0.28944488875185423952758626824804Q-02 ) ,
     *    ( -0.12682388318315677499352037464151Q-03 ,
     *    -0.13765284525622497163004271726241Q-03 ) ,
     *    ( -0.71010473576556223110269068237272Q-05 ,
     *    -0.47052276012432276254645764427057Q-05 ) ,
     *    ( -0.30024483700336217893916277191359Q-06 ,
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     *    -0.10641119963921452180967194363728Q-06 ) ,
     *    ( -0.99086415014185669218051899910643Q-08 ,
     *    -0.92761222837942349840197804571025Q-09 ) /
      DATA A(6,13), A(6,14), A(6,15), A(6,16), A(6,17), A(6,18) /
     *    ( -0.26027484117930187788192176623065Q-09 ,
     *      0.40940074636336826992996865653823Q-10 ) ,
     *    ( -0.54945502031913302315248827628845Q-11 ,
     *      0.23689419516543360236253155310313Q-11 ) ,
     *    ( -0.93097909164864721426484987076695Q-13 ,
     *      0.72175746483862135156193005583481Q-13 ) ,
     *    ( -0.12405514383185900794313961451409Q-14 ,
     *      0.16057793958388156658720457530673Q-14 ). ,
     *    ( -0.12100809941441867001603700742549Q-16 ,
     *      0.28359465561914272852397225563504Q-16 ) ,
     *    ( -0.61402516926354081139078013313073Q-19 ,
     *      0.41173060997307404969131501659123Q-18 ) /
      DATA A(6,19), A(6,20), A(6,21), A(6,22), A(6,23) /
     *    (   0.54121172480611844676219493583295Q-21 ,
     *      0.49969014268740635137936784721376Q-20 ) ,
     *    (   0.19452840346739931358323830445185Q-22 ,
     *      0.51000584521037047207876926762789Q-22 ) ,
     *    (   0.31204514306599118157275391121056Q-24 ,
     *      0.43581643056407606329070172519971Q-24 ) ,
     *    (   0.36820345317120975520032057649888Q-26 ,
     *      0.30512590903047769797613238757884Q-26 ) ,
     *    (   0.35315464449440018810041928637612Q-28 ,
     *      0.16376144713659745783339920962069Q-28 ) /
      DATA A(6,24), A(6,25), A(6,26), A(6,27), A(6,28) /
     *    (   0.28580566388241294610135183367357Q-30 ,
     *      0.50849901113658300951720520451353Q-31 ) ,
     *    (   0.19836067158649677864126489289905Q-32 ,
     *    -0.16369074613715444505432167096866Q-33 ) ,
     *    (   0.11863994266786195581653997529741Q-34 ,
     *    -0.42101148754094705704577267314263Q-35 ) ,
     *    (   0.60815557684675400071345237144200Q-37 ,
     *    -0.41597029420894998273193569008155Q-37 ) ,
     *    (   0.26140923382011523100280946805280Q-39 ,
     *    -0.30302663845261326002874852819372Q-39 ) /
      IFAIL = 0
C
```

```
C    CHECK THE MODULUS OF Z IS LESS THAN EIGHT .
       ABSZ = CQABS( Z )
       IF ( ABSZ .GT. 8.0Q0 ) GO TO 70
C
C    IF THE MODULUS IS LESS THAN 1.0Q-16 .
C    THE TAYLOR SERIES IS USED FOR THE CALCULATION ,
       IF ( ABSZ .LT. 1.0Q-16 ) GO TO 60
C
C    IF THE REAL AND IMAGINARY PARTS OF Z HAVE OPPOSITE SIGNS ,
C    J1 IS SET TO 1 .
       X = QREAL( Z )
       Y = QIMAG( Z )
       J1 = 0
       IF ( ( X*Y ) .LT. 0.0Q0 ) J1 = 1
C
C     THE NEXT FEW LINES ARE USED TO TRANSPOSE THE POINT Z INTO ZT
C    IN THE UPPER RIGHT QUADRANT BY TAKING THE ABSOLUTE VALUES OF
C    ITS REAL AND IMAGINARY PARTS .
       X = QABS( X )
       Y = QABS( Y )
       ZT = QCMPLX( X, Y )
C
C      NOW DECIDE WHICH SET OF COEFFICIENTS ARE REQUIRED .
C   I.E.  THE VALUE OF K SUCH THAT (2K-1)*PI/24 IS CLOSEST TO ARG(ZT) .
C     K IS POSITIVE AND NO GREATER THAN 6 .
C
C      FLOATING POINT DIVISION BY ZERO CHECK .
C     IF X IS ZERO THEN ARG(ZT) IS PI/2 , SO K IS 6 .
       IF ( X .NE. 0.0Q0 ) GO TO 10
       K = 6
       GO TO 20
    10 ARGZT = QATAN( Y /X )
       K = 1 + ( IQINT( ARGZT /PIFAC ) ) /2
C
C      CHECK  K  IS LESS THAN 7
C     IF NOT  K  IS SET EQUAL TO 6
       IF ( K .GT. 6 ) K = 6
C
    20 CONTINUE
C
```

```
C       CHECK NACC , IF NACC IS GREATER THAN 28 IFAIL
C      IS SET TO 1 AND NACC TO 32 .
C
       IF( NACC .GT. 28 ) IFAIL = 1
       IF( IFAIL .EQ. 1 ) NACC = 32
C
C       THE NEXT FEW LINES CALCULATE THE NUMBER OF COEFFICIENTS
C      REQUIRED FOR THE GIVEN ACCURACY .
       FNACC = FLOAT( NACC )
       FNU = FLOAT( NU )
       R0 =  8.0 + 0.5 * FNACC
       R1 = ALOG( R0 * ( R0 + FNU ) )
       R2 = ALOG( ( R0 + FNU ) /4.0 )
       R1 = ( R0 * 2.0 - ( 0.5 * R1 ) - ( FNU * R2 ) +
      1 ( 0.5 * R0 /( R0 + FNU ) ) + ( FNACC * 2.3025851 ) + 0.4909109 )
      2 /( R1 + ( 0.5 /R0 ) + ( 0.5 /( R0 + FNU ) ) - 2.0940008 )
       NUM = INT( R1 ) + 1
C
C       CALCULATION OF THE TRUNCATED CHEBYSHEV SERIES AT ZT
       T = ZT * QCMPLX( COSIN( K ), -COSIN( 7-K ) )
       T = 0.625Q-01 * T * T - 2.0Q0
       NUMM1 = NUM - 1
       BESS = T * A( K, NUM ) + A( K, NUMM1 )
       C = A( K, NUM )
       DO 30 J = 2, NUMM1
          C1 = C
          C = BESS
   30    BESS = T * C - C1 + A( K, NUM-J )
       BESS = 0.5Q0 * ( BESS -  C1  ) /DENOM
C
C       SYMMETRY RELATIONS ARE USED TC GET THE VALUE AT Z
C
       IF ( J1 .EQ. 1 ) BESS = QCONJG( BESS )
       IF ( NU .EQ. 0 ) GC TO 50
       Z2 = 0.5Q0 * Z
       DC 40 J = 1, NU
   40    BESS = Z2 * BESS
   50 CONTINUE
       RETURN
C
```

```
C  FOR SMALL MODULUS THE TERMS IN THE TAYLOR
C  SERIES GREATER THAN 1.0Q-32 ARE USED .
   60 BESS =  0.0Q0
      IF ( NU .EQ. 1 ) BESS = Z * 0.5Q0
      IF ( NU .EQ. 0 ) BESS = 1.0Q0
      RETURN
C
C  MODULUS OF Z TOO LARGE . NO CALCULATION
C  HAS BEEN PERFORMED .
   70 IFAIL = 2
      BESS = 0.0Q0
      RETURN
      END
```