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The Motley of Mathematics : A study of Wittgenstein's
Philosophy of Mathematics.

by Paul Michael Severn

Submitted for the Degree of Master of Arts at the
University of Durham.

June 1990.

In this thesis I try to examine Wittgenstein's philosophy of mathematics, both against the background of early twentieth century foundational studies and against the background of Wittgenstein's general philosophical position. I have tried to explain Wittgenstein's objections to the foundational programmes and to show that they are consistent with and understandable in terms of, Wittgenstein's general philosophical outlook.

In chapters one to four I discuss Wittgenstein's remarks on the mainstream foundational schools: logicism, intuitionism, formalism and strict finitism; and try to explain that in addition to the technical difficulties inherent in each, on Wittgenstein's view they are pointless endeavours, as mathematics has no need of foundations.

In chapter five I discuss proof, and elaborate upon various points made in preceding chapters.

In chapters six to eight I discuss the connections between the philosophy of mathematics, and Wittgenstein's other central concerns (language games and forms of life, scepticism and rule following, and philosophy of mind) drawing parallels and trying to gain a fuller understanding of the former in terms of the latter.

Chapter nine concludes my discussion, with an attempt to evaluate the significance of Wittgenstein's contribution; and to assess the principal objections levelled against it.

Whilst not wholeheartedly embracing Wittgenstein's position, I have frequently found myself defending him, and I do this because I think all too many of the criticisms of Wittgenstein derive from misunderstanding. The principal causes of misunderstanding are:

(i) an over selective reading of Wittgenstein and a failure to understand individual remarks in their wider context.

(ii) a failure to appreciate Wittgenstein's distinction between mathematics and philosophy; leading to a muddle between Wittgenstein's mathematical and his philosophical remarks, and much unfair criticism.

I have particularly tried to establish Wittgenstein's views on central questions, and to refute criticisms which derive from misunderstandings of these views.

The Motley of Mathematics

A Study of Wittgenstein's Philosophy of Mathematics

by

Paul Michael Severn

Submitted for the degree of Master of Arts at the
University of Durham, following a period of supervised study
in the Department of Philosophy.

June 1990

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Preface.

In this thesis I have tried to examine Wittgenstein's philosophy of mathematics, both against the background of early twentieth century foundational studies, and against the background of Wittgenstein's general philosophical position.

I have tried to explain Wittgenstein's objections to the foundation programmes, and to show that these are consistent with; and understandable in terms of, Wittgenstein's general philosophical outlook.

Whilst not wholeheartedly embracing Wittgenstein's position, I have frequently found myself defending him, and I do this because all too many of the criticisms of Wittgenstein derive from misunderstanding. The principal causes of these misunderstandings are:

(i) An over selective reading of Wittgenstein and a failure to understand individual remarks in their wider context.

(ii) A failure to distinguish between Wittgenstein's mathematical comments about mathematical statements, and his philosophical comments about mathematical statements. Wittgenstein saw the two as entirely separate, but critics have often muddled the two: reading philosophical comments as mathematical comments, and misunderstanding Wittgenstein as a result.

I have particularly tried to establish correctly

Wittgenstein's views on central questions and to refute criticisms which derive from what I believe to be a misunderstanding of these views.

In chapters one to four I discuss Wittgenstein's remarks on the mainstream foundational schools: logicism, intuitionism, formalism and strict finitism; and try to explain that in addition to the technical difficulties inherent in each, on Wittgenstein's view they are pointless endeavours, as mathematics has no need of foundations.

In chapter five I discuss proof, and elaborate upon various points made in preceding chapters.

In chapters six to eight I discuss the connections between the philosophy of mathematics and Wittgenstein's other central concerns; (language games and forms of life, scepticism and rule following, and philosophy of mind.) drawing parallels and trying to gain a fuller understanding of the former in terms of the latter.

Chapter nine concludes my discussion with an attempt to evaluate the significance of Wittgenstein's contribution; and to assess the principal objections levelled against it.

Biographical Note.

Ludwig Wittgenstein was born in Vienna on 26th April 1889, of a family of Jewish descent. His father was a prominent industrial figure and young Ludwig was educated at home until he was fourteen. After three years of school at Linz, he studied engineering at Berlin University and in 1908 he registered as a research student at Manchester University.

Whilst designing a propeller shaft, his interests shifted from engineering to mathematics, and then to the philosophical foundations of mathematics. In 1911 Wittgenstein visited Frege at Jena University and was advised to study under Russell at Cambridge. He did this for five terms at Trinity College between 1912 and 1913.

The following passage from Russell is famous. "At the end of his first term at Cambridge he came to me and said: "Will you please tell me whether I am a complete idiot or not?" I replied, "My dear fellow, I don't know. Why are you asking me?" He said, "Because if I am a complete idiot I shall become an aeronaut; but if not, I shall become a philosopher." I told him to write me something during the vacation on some philosophical subject and I would then tell him whether he was a complete idiot or not. At the beginning of the following term he brought me the fulfilment of this suggestion. After reading only one sentence, I said to him: "No, you must not become an aeronaut." " (Russell 1957 pp26-27).

After Cambridge Wittgenstein lived in isolation in Norway, and at the outbreak of the first world war he enlisted as a volunteer in the Austrian Artillery. Throughout both these periods he recorded his philosophical thoughts in notebooks, and it was from these notes that grew the only philosophical book that Wittgenstein published in his lifetime: the Tractatus Logico Philosophicus published in English in 1922.

After the war, Wittgenstein gave away the fortune he had inherited from his father and gave up philosophy, believing it an attempt to say the unsayable, and after training as a teacher in Vienna he taught in a remote Austrian village from 1920 until 1926. After that he spent some time as a monastic gardener and considered entering the religious life. He also assisted in designing a house for his sister in Vienna.

During this period he became acquainted with Schlick, Carnap and Waismann, later to form the Vienna Circle; and legend has it that it was when Wittgenstein heard Brouwer lecture on the foundations of mathematics, that he decided to return to philosophy. He returned to Cambridge in 1929 and successfully submitted his Tractatus as a Ph.D. thesis.

He became a research fellow at Trinity College, and became greatly involved in both teaching and writing. Indeed the early thirties were the most philosophically productive years of Wittgenstein's life and it is from this period that the writings on the philosophy of mathematics come.

In 1939 Wittgenstein succeeded Moore as professor of philosophy at Cambridge, but before he could take up his chair, the second world war broke out. During this period he worked as a medical orderly at Guy's Hospital, London; and in a laboratory at the Royal Victoria Infirmary at Newcastle-upon-Tyne.

After the war he returned to his duties as a professor, but he disliked the formal academic routine, and the artificiality of university life; and he resigned his chair in 1947 after only two years. He described the life as professor of philosophy as "a living death." (Kenny p12)

He embarked on a series of travels both to Ireland where he finished the Philosophical Investigations, and to America where he stayed with Norman Malcolm at Cornell. Deteriorating health took him back to England and it was discovered that he had incurable cancer. He spent the last two years of his life staying with friends in Oxford and Cambridge. During this period he managed a last piece of work on epistemology, published posthumously as: On Certainty in 1969.

He died at the home of his Cambridge doctor on 29th April 1951. His last words were: "Tell them I've had a wonderful life."

Bibliographic Note.

Wittgenstein's work on the philosophy of mathematics is mainly to be found in the Lectures on the Foundations of Mathematics, the Remarks on the Foundations of Mathematics and the second part of Philosophical Grammar.

The Lectures are an edited version of notes taken by R.G. Bosanquet, Norman Malcolm, Rush Rhees and Yorick Smythies at Wittgenstein's two hourly, twice weekly lectures of the Lent and Easter terms of 1939. The four sets of notes at Cora Diamond's (the editor's) disposal were both edited by their authors, and complete to different degrees. Diamond writes: "No single version was taken as the basic text. Rather each passage is based on a comparison of all the available versions of that passage. Where two or more versions agreed in some point, I normally took them to be correct in that respect." (LFM p8).

Much of the text probably is (or is very close to) what Wittgenstein actually said, but because he did not write or correct it; and because of the discursive style of Wittgenstein's lecturing, great care must be taken in identifying any particular passage with Wittgenstein's definitive position. However, read in the light of the Remarks and the Investigations, the Lectures are a valuable and important source.

The Remarks are a selection from Wittgenstein's

manuscripts, found in five separate notebooks, from which the five sections of the Remarks derive. Part I is the earliest, written in 1937 it bears many similarities to the Investigations. Indeed it was probably Wittgenstein's original intention to incorporate his ideas on logic and mathematics into the Investigations.

Part II derives from the period October 1939 to April 1940 and is a discussion of Wittgenstein's position relative to Russell. There is much discussion of logic, derivability in mathematics and the nature of proof.

Part III and Part IV are taken from manuscripts of 1942 and 1943. Much of these parts can be seen as preliminary studies for the second section of part V; but in part IV Wittgenstein discusses topics relating to Brouwer and Intuitionism, particularly the law of the excluded middle.

Part V was written in two sections. The first (remarks 1-16) in June 1941, and the second in the spring of 1944; but as they occur in the same notebook, it is probable that Wittgenstein regarded them as belonging together. The material concerns the relation between mathematical and empirical propositions, calculation and experiment and a fresh treatment of consistency in the first section. A discussion of rule following and the concepts of proof and inference is found in the second.

The editors have numbered Wittgenstein's individual remarks, but "the division into separate 'remarks', here

indicated by a space between them, is Wittgenstein's own. With few exceptions we have not interfered with their order.

Sometimes, however, especially at the end of Part III and of Part IV, we have brought together remarks on the same topic that occurred in different places in the manuscripts." (RFM pviii)

The editors also suggest that at some stage a more extensive version of Wittgenstein's own manuscripts may be published, providing the scholar with the material omitted from the Remarks, but to date this has not been done.

The Philosophical Grammar is Wittgenstein's own Big Typescript which bore the parenthetical remark: "My book might be called Philosophical Grammar." It was completed in 1933, and so part of it is contemporary with the dictation of the Blue Book, and in fact many of the ideas in the Grammar are fuller discussions of ideas in the Blue Book. Many passages in the Grammar also appear in the Philosophical Remarks and the Philosophical Investigations.

Part II, which concerns us most, is from the second part of the Big Typescript and considers the problems of generality, proof, inference and cardinal numbers. Rush Rhees edited Wittgenstein's manuscript which is repeatedly changed, re-ordered and overwritten in 1969. Anthony Kenny first translated it into English in 1974.

I Logicism.

In discussing Wittgenstein's philosophy it is customary to distinguish between an early period (1912-1920) and a later period (1930s and 1940s); but this distinction is not appropriate for the philosophy of mathematics as Wittgenstein did almost all his work in this area during the later period.

Having said this however, there are remarks about logicism in both the early and the later writings; and these are of such a different character, care must be taken to separate the two.

Firstly the discussion of logicism in the Tractatus will be examined, but in order to understand this it is necessary to go back to Frege and the origins of logicism. Frege's work was motivated by the inadequate accounts of the foundations of mathematics that existed in the late nineteenth century. He was particularly hostile toward psychologism: the view that the nature of concepts such as truth, validity even knowledge were mental and subjective and he sought to set arithmetic on a mind-independent foundation.

Frege particularly rejected J.S Mill's account that arithmetical propositions state empirical facts, and arithmetical laws are inductive inferences from these facts.

He also rejected Kant's view that the truths of arithmetic are synthetic a priori. Frege wanted to show arithmetical truths to be analytic, although it is not the case that Frege saw Kant as a great rival. Frege writes in a very neo-

Kantian style and accepts Kant's account of geometry, but Frege had his own views on the foundations of arithmetic which were a profound break with anything that had gone before. (It is a point of scholarly debate to what extent Frege's views were anticipated by Leibniz)

The first step in giving arithmetic a mind-independent, non-empirical, analytic foundation was made in 1879 when Frege published his Begriffsschrift. This short work introduced a formal language for mathematics, designed to eliminate ambiguity and to express arithmetical statements in purely logical terms. Frege hoped to show that all arithmetical statements could be expressed in logical terms, and that arithmetical theorems could then be derived from the axioms of logic alone; although this task was not fully undertaken until the Grundgesetze of 1893.

Apart from the introduction of a formal language there were two other vital elements to the Begriffsschrift. The first was the idea of multiple generality allowing quantification over several variables or predicates, and many placed predicates to represent logical and mathematical functions of several variables. The logics of Aristotle and Boole could not accommodate these, but they are vital for the expression of concepts in higher mathematics.

The second was the introduction of the relation that Russell and Whitehead were to call the ancestral. Without going into great detail, Frege showed how to express the

statement "x is an ancestor of y", using only the two place predicate Pxy meaning "x is a parent of y" (the successor relation). He applied his idea to define $N = \{0,1,2,3,\dots\}$ the natural numbers (ie. finite cardinals) as 0 and the cardinal numbers reachable from it by finitely many steps of the successor relation. This meaning that all the cardinal numbers possess every property that belongs to 0 and is successor-hereditary.

Furthermore this relation enabled Frege to show that the principle of mathematical induction is a purely logical law, not based on any special "mathematical intuition" as Kant thought. Frege's ancestral was independently constructed by Peano in his Arithmetices Principia of 1889 and called Axiom V.

The mathematical community accepted Frege's new work very badly. Frege's symbolism was criticised as "a monstrous waste of space" (Currie p42) and another said that apart from Frege's introduction of notation for generality, "the new logic did not go beyond that of Boole". (ibid.) Frege however was not deterred and in 1884 he published the Grundlagen; which defined natural numbers in purely logical terms. Also included were other details of how functions were to be defined, matters of syntax etc. and the spadework had been done for Frege's last major work: Die Grundgesetze der Arithmetik (1893). Frege begins an explicit demonstration of logicism by using the logic of

Begriffsschrift to show how arithmetic could be derived from purely logical axioms and definitions alone. The text was highly symbolic and difficult to follow and went largely unnoticed until Russell read it in 1901. Russell discovered a paradox in Frege's work which he communicated to Frege in a private letter of 1902, and published in his *The Principles of Mathematics* in 1903.

Frege regarded functions as incomplete or "unsaturated", requiring the argument of the function to complete them. But consider two functions $f(x)$ and $g(x)$ that have the same values for the same arguments: Frege could not write $f(x) = g(x)$ for this is to treat functions as complete objects which they are not. So he introduced the idea of the *Werthverlauf* of a function, which is the "course of values" that the function takes, and is an object complete in itself. He then says that if two functions have the same values for the same arguments they have the same *Werthverlauf*; and this is Frege's axiom V. It is this axiom that leads to Russell's paradox, for it allows the introduction of a concept "not holding of itself". To then ask; does this concept hold of itself, leads to paradox. Formally in Frege's own notation, as there is no modern notation to express Frege's *Werthverlauf*. (expressed here $\hat{f}(f)$)

$$\forall x \quad \bar{\forall} : \quad \dot{\exists} f(\varepsilon) = \dot{\exists} g(\varepsilon) \longleftrightarrow \forall x (f(x) = g(x))$$

in the case when f, g are concepts F, G

$$\text{we have:} \quad \dot{\exists} F(\varepsilon) = \dot{\exists} G(\varepsilon) \longleftrightarrow \forall x (Fx \leftrightarrow Gx)$$

$$\text{Define } x \in y \text{ as } \exists F (y = \dot{\exists} F(\varepsilon) \wedge F(x))$$

$$\begin{aligned} \text{then } x \in \dot{\exists} G(\varepsilon) &\longleftrightarrow \exists F (\dot{\exists} G(\varepsilon) \longleftrightarrow \dot{\exists} F(\varepsilon) \wedge Fx) \\ &\longleftrightarrow \exists F (\forall x (Gx \leftrightarrow Fx) \wedge Fx) \\ &\longleftrightarrow Gx. \end{aligned}$$

Substituting $x \notin x$ for Gx and $\dot{\exists}(\varepsilon \notin \varepsilon)$ for x

$$\dot{\exists}(\varepsilon \notin \varepsilon) \in \dot{\exists}(\varepsilon \notin \varepsilon) \longleftrightarrow \dot{\exists}(\varepsilon \notin \varepsilon) \notin \dot{\exists}(\varepsilon \notin \varepsilon)$$

Which is a contradiction.

(from R. Black's University of Nottingham
Philosophy of Mathematics Notes.)

Russell's paradox caused Frege "great surprise ... and consternation" (Van Heijenoort p127), and led Frege to introduce a new axiom Vb. This introduction however was unacceptably ad hoc; and further led to other self referential paradoxes.

In Principia Mathematica Russell and Whitehead introduce a theory of types to avoid the Russellian paradox. Individuals are of type zero, classes of individuals type one and so on. Then Russell says we can only meaningfully speak of classes belonging to this hierarchy, and in particular classes which are members of themselves do not exist in the hierarchy and Russell's paradox cannot be framed.

With this extra apparatus Russell is able to define the natural numbers in logical terms without running into paradox; but because of the theory of types individuals of type zero form sets of individuals of type one and sets of sets of type two; which are identified as the natural numbers. Russell identifies n with the set of all sets with n members. But this raises a problem, for if the set of natural numbers is to be infinite, there have to be infinitely many sets of type one, and hence infinitely many individuals. The best Russell can do, is postulate this as the axiom of infinity. (Similar axioms had previously been employed by Zermelo, asserting the existence of an infinite collection of sets.) but Russell's axiom signals a departure from a purely logical foundation.

Further, the so called simple theory of types is found to lead to certain semantic paradoxes; stemming from the fact that Russell, in drawing up his simple theory had to speak of the meaning of signs. To obviate these difficulties Russell developed the ramified theory of types'; but in doing so he had to introduce two further axioms. The Axiom of Choice and the Axiom of Reducibility.

The latter is more controversial and states that for every non-elementary function, there is an equivalent elementary or predicative function. The details of this are unimportant, but this axiom is certainly not an axiom of logic, and as Ramsey says: it would be "a happy accident" if it were true! Hence many of the theorems of Principia Mathematica are stated: "if Ax. Inf. and/or Ax. Red. then theorem", but many see these proofs, which rest on such dubious conditionals, not to be proofs at all.

In the Tractatus I think four distinct criticisms of logicism can be singled out. The first at 4.1273 attacks Frege's and Russell's definition of successor and ancestral as circular. For in order to express the general proposition: "b is a successor of a" Wittgenstein says we have to give the general term in the series:

$$a R b$$

$$(\exists x) a R x . x R b.$$

$$(\exists x, y) a R x . x R y . y R b \quad \text{etc.}$$

and this requires the use of a variable which is the symbol

for a formal concept. But earlier in 4.1272 Wittgenstein says numbers themselves are formal concepts not functions or classes; and so this use of the variable in the definition of successor introduces both the primitive concept of number, and particular numbers, which is viciously circular.

This argument seems to hold so long as numbers are not identified with functions or classes; and Wittgenstein's own understanding of "there are two objects which ..." as " $(\exists x,y) \dots$ " is accepted. But is doubtful that the expression "there are two objects which ..." when used in any other way is "nonsensical", and it is hard to assess whether this objection is really fatal.

The next two criticisms rest on Russell's axioms of reducibility and infinity that I have already mentioned; and are perhaps clearer. To validate the logicist thesis the axiom of reducibility must be a logically true proposition. Wittgenstein denies this, although not directly or specifically. He cites it as an alleged example, when he attacks Russell's notion of logical propositions in general.

Russell held propositions to be logically true in virtue of their generality. In Principia Mathematica he wrote: logic "is distinguished from various special branches of mathematics mainly by its generality." And again in Our Knowledge of the External World he spoke of: "self-evident general propositions" (p66). Hence for Russell the problem of whether to accept the axiom of reducibility boils down to

a question of its generality.

Wittgenstein rejected this whole notion of generality. He said a logically true proposition has no content or subject matter, but is true in virtue of its form. This can be clarified by Wittgenstein's distinction between accidental and essential validity at 3.34: "Accidental features are those that result from the particular way in which the proposition is produced. Essential features are those without which the proposition could not express its sense." (TLP 3.34)

So logically true propositions must be tautologous, and as (for example) the axiom of reducibility is not tautologous it cannot be a logical truth. "The general validity of logic might be called essential, in contrast with the accidental validity of such propositions as 'all men are mortal'. Propositions like Russell's axiom of reducibility are not logical propositions, and this explains our feeling that even if they were true, their truth could only be the result of a fortunate accident." (TLP 6.1232)

The axiom of infinity is an assumption which may be stated: if n is any inductive cardinal number, there is at least one class of individuals having n terms. Russell considered this to be true in some possible worlds and false in others. Whether it is true of our actual world we cannot tell. Wittgenstein seemed to share this view at some time saying it was an empirical question ("Sache der Physik") to

determine the number of extant things.

However later in the Tractatus this opinion had been abandoned in favour of the view that the axiom must be either a logical tautology or a contradiction. If the axiom is not an empirical truth or falsehood, it must derive its truth value from its logical form. Wittgenstein does not say whether he thinks the axiom a tautology or a contradiction, but either way the difficulties involved and the apparently blind acceptance of the axiom which is required marks "the decisive lacuna in the Frege-Russell derivation of arithmetic from logic" (A. Church. Review of Schmidt).

It is worth noting that at the time of the Tractatus, Wittgenstein was not troubled by the concept of infinity or the use of the expression, "an infinite number of names." (TLP 5.535) In later works he was greatly troubled by the use of the concept of infinity, and rejected the above expression as meaningless.

A last objection found in the Tractatus is to Frege and Russell's concept of proof. "What is a proof?" and "how are proofs to be recognized?" are questions that recur throughout Wittgenstein's philosophy of mathematics and will be dealt with in detail in chapter five. At this point it is sufficient to note that the Tractarian view of proof is that it is "merely a mechanical expedient to facilitate the recognition of tautologies in complicated cases." (TLP 6.1262). Wittgenstein writes that the ancient conception of

deriving theorems from axioms by logical proofs is wrong; for every theorem is a tautology, and in no need of proof.

"Every proposition is its own proof." (TLP 6.1265)

It is important to distinguish proofs within logic: manipulation of symbols to yield logical truths; and proofs by logic that establish contingent truths from contingent premises. Wittgenstein only disagrees with the traditional conception of the former, as there is not inferential order or precedence in logic. All tautologies are at the same level.

I do not see this as a head on criticism of Frege, who I think would have probably accepted this view. There is more a difference of emphasis between the two, and Frege may well have agreed tautologies have no need of proof theoretically speaking, but in practice have to be re-expressed in terms of simpler tautologies. Indeed this is the very idea of the logicist programme.

Whereas the critique of logicism to be found in the Tractatus and early writings considers the technicalities of logicism and argues against particular details, or methods'; the critique in the later writings is from a much broader perspective. Untroubled by details, the whole project is attacked as resting on a "ramifying series of misunderstandings" (Baker and Hacker 1976 p282). Two main points can be singled out. The first concerns the whole idea of a reduction of mathematics to logic, and the second

concerns the nature of proof and will be dealt with here, only briefly. It is important to note that the later criticisms are entirely independent of the earlier ones, and the epistemic status of Russell's axioms is irrelevant.

Wittgenstein attacks the direction of fit of the reduction of mathematics to logic. Our reduction of arithmetical formulae to logic depends on arithmetic, not explains it! Consider Wittgenstein's frequent example. In Russell's notation $5 + 7 = 12$ is written as :

$$(\exists 5x)\phi(x) \ \& \ (\exists 7x)\psi(x) \ \& \ Ind(\phi, \psi) \Rightarrow (\exists 12x)(\phi(x) \vee \psi(x))$$

Russell claims this tautology explains the correct use of the formula $5 + 7 = 12$, but in contrast Wittgenstein denies this, because it is only my ability to understand the arithmetic formula, that allows me to write the Russellian proof. The proof correlates with the formula, but does not explain it. Wittgenstein says:

"In order to be able to write down the proposition (the Russellian proof) I have to know that $5 + 7 = 12$... because arithmetic is used in constructing the proposition."

(WWK p35)

"Tautology is an application of the calculus not its expression."

(WWK p106)

i.e. The new Russellian proof is redundant, and adds nothing to what we already know.

Crispin Wright considers an example of the reduction of a multiplication table to logic - "It is not what happens in such a logic that justifies the multiplication table; it is what happens in the multiplication table which determines whether a particular proof in such a logic has been carried out properly." (Wright 1980 p132)

So Wittgenstein claims that any reduction of arithmetic to logic presupposes arithmetic and is thus redundant. Further to this, the notion of any reduction at all is questioned. Wittgenstein is suspicious of the alleged fundamentality of logic, and does not regard the translation of arithmetical formulae into logical tautologies as providing anything more fundamental or basic. Russell's calculus is an auxiliary or alternative calculus for expressing arithmetic with "ifs" and "thens"; and that is all. "The Russellian method is just one method like any." (LFM p262) For if we were to do a calculation by Principia, and the same calculation by the ordinary method, and obtain a different result; there is nothing to say which answer is correct. Turing said to Wittgenstein "It is just like any other pairs of ways of counting" (LFM p261) and the point is that neither is more basic or fundamental.

Wittgenstein's second major objection to logicism is to the logicist notion of proof. I do not intend to go into details here, but two points can be made. Whereas Frege and Russell regarded proof as a particular concept, like a fixed

set of rules; Wittgenstein regarded proof as a grammatical category, a family concept. A Merkmal definition of proof, as Frege and Russell conceived is not adequate: as there is a variety of mathematical practice, and a variety of techniques for proving and deriving. For Wittgenstein mathematics is a "motley".

Secondly Wittgenstein argues that Russell's proofs are not surveyable. This will be dealt with later, but loosely, Wittgenstein means that Russell's proofs cannot be convincingly checked. Any doubt about the validity of the Russellian proof that three million plus four million is seven million opens the question as to whether such a proof can be given at all.

To summarize: Wittgenstein offers two distinct critiques of logicism: the earlier dealing with the explicit details of the programme, the later as part of a more general conception of the nature of mathematics. Both seem fairly fatal to the original logicist programme, and indeed the later considerations, if accepted, undermine the whole logicist ideology and make any attempted rebuilding of logicism (such as the one at the end of Wright's book: Frege's Conception of Numbers as Objects) impossible.

II Intuitionism

Both logicism and formalism developed from the worry that mathematics has no firm epistemic foundation. Both schools looked for some kind of justification for mathematics. Frege and the logicians tried to reduce mathematics to logic; that is to derive theorems from definitions and logically obvious axioms. Hilbert and the formalists tried to give justification in terms of consistency proofs.

Through the work of Russell, Gödel, Church and others, both of these projects were shown to have serious flaws. In the light of these difficulties the intuitionists, rather than seek an alternate foundation, declared mathematics not in need of justification or foundation at all.

For the intuitionist, there is no need of foundation, for mathematics is a wholly mental activity; the subject matter of which is intuited, non-perceptual mental objects and constructions. Heyting said of Brouwer's 1907 programme: "It consisted in the investigation of mental mathematical construction as such, without reference to questions regarding the nature of the constructed objects, such as whether these objects exist independently of our knowledge of them." For in the realm of mental mathematics, to exist is synonymous with to be constructed.

So if the need for foundations has been disposed of, and mathematics can be known by an intuiting of mental objects; then it follows that mathematics is a languageless activity.

Anything written or spoken of mathematics simply serves to communicate a description or representation of mental entities and is entirely separate from these entities.

These initial steps of reasoning have very far reaching consequences, and intuitionist mathematics is, in important respects, very different from classical mathematics. The most important differences are about logic, and the notion of infinity. Au fond, these two disputes are one, but considering them separately sheds light on the crucial aspects of intuitionism, and the aspects upon which Wittgenstein makes most of his comments.

Every intuitionist proposition p , is a report of a mental construction. As Heyting put it: "I have effected a construction A in my mind." Similarly an intuitionist negation $\text{not-}p$ effectively says "I have effected a construction B which deduces a contradiction from the supposition that construction A were brought to an end." Furthermore "I have not effected a construction... is of no worth or interest and there is a construction that nobody has been able to effect, is no more than an empty promise from the intuitionist standpoint.

Hence it follows that a proof in intuitionistic logic of $(p \ \& \ q)$ is a proof of p , together with a proof of q . A proof of $(p \vee q)$ is a proof of p or a proof of q . And " a proof of $(p \rightarrow q)$ is a construction of which we can recognize that, applied to any proof of p , it yields a proof of q ." (Dummett

(1) 1977 p 12) That is, the intuitionist implication is not truth functional. In other words the intuitionist implication is not a truth functional mapping from truth values to truth values, but a mapping from proofs to proofs.

In classical logic we may suppose p and ask if q follows, but in intuitionist logic this is nonsensical for a proof of the implication must begin with a proof of p . There can be no doubt about the status of p !

But there is an objection, first cited by Menger in 1930, that this rejection of truth functional logic leads to a tensed concept of "true". For example define:

k is the greatest prime for which $(k-2)$ is also prime or $k = 1$ if such a number does not exist.

Now it is not known whether the sequence of twin primes terminates, so intuitionists reject this definition of k , as it is not well-defined. But suppose the problem is solved at time t , and suppose there are infinitely many twin primes.

Then $k = 1$. Was $k = 1$ before t , or not?

The intuitionists reject the query as misconceived, as before t there was no k , it had not been constructed, so we cannot ask of its value. Hence the statement " $k=1$ " does not become true at a particular time t , after being something else before; because before, it was meaningless. It is only possible to ask of k once it has been constructed and Menger's objection is avoided.

Intuitionists also reject the law of the excluded middle.

For the classical mathematician $(p \vee \sim p)$ is a truth of logic, but for the intuitionist this means I have constructed a proof of p , or I have a proof that a contradiction arises from supposing p .

But in many cases we do not have a proof of either, so the intuitionist will not assert $(p \vee \sim p)$ as a general law.

In the above example, the classical mathematician would assert $k = 1$ or $k \neq 1$; but without a proof either way the intuitionist will not assert this.

The rejection of the law of the excluded middle is very closely connected with the intuitionist concept of infinity.

Classically the existential and universal quantifiers are seen as an extension of disjunction and conjunction respectively, and it is legitimate to quantify over an infinite totality. A quantified statement has a truth value independently of there being any available method for evaluating it.

This is quite contrary to the intuitionist position, which only assents to universally quantified statements if there is an effective method for checking (proving) the truth value of each constituent statement. Methods involving unlimited time or energy are not effective; but, for example, proofs derived from the principle of mathematical induction are accepted.

To summarize, the intuitionist criticises the classical mathematician as having failed to accept or realize that his

account of quantification over finite domains, cannot be extended to domains over which there is no effective decision procedure. The central mistake is one of omission: a failure to realise that infinite quantification needs a different account; and that mathematical objects exist only in virtue of some specific construction, and do not exist unspecified in some Platonic realm.

The philosophy underlying these technicalities is that: unless one rejects the classical theory of truth, it is incorrect to regard the meaning of a proposition as residing in its truth conditions. For Brouwer; mathematical statements have sense and truth value in virtue of mental activity (constructions). For Dummett the theory of meaning residing in truth conditions is replaced by a theory based upon assertability conditions, Wittgenstein's idea of meaning as use, and an ability to determine or recognize the validity of a given proof.

Brouwer's claim is a relatively weak one, amounting to little more than the central claim of intuitionism that mathematical entities are mental entities, and the truth or falsity of relations between them depends on whether such relations have been constructed.

Dummett's claim is much stronger for it urges a move away from Frege/Tractarian type explanations based upon truth conditions, and suggests that a general account of meaning and the logical operators must be based on assertability

conditions.

The correspondance theory of truth which gives that a proposition p is true if it corresponds with the relevant facts in an appropriate way, is replaced by the redundancy theory of truth which claims that asserting " p is true" simply equates to asserting " p ". That is, Dummett questions the legitimacy of a notion of truth that is not epistemically constrained.

Furthermore Dummett claims that "the strongest arguments for intuitionism seem to be quite independent of the objectivity of mathematical proof...The strongest arguments come from the insistence that the general form of explanation of meaning, and hence of the logical operators in particular, is a statement not of the truth-conditions but of the assertability-conditions." (Dummett (3) 1978 p184)

Clearly the issues raised here are complex and contentious, but a thorough exanination is beyond the scope of this thesis and it is sufficient to bear them in mind as we proceed to examine the views of Wittgenstein.

As well as differences, there are many broad similarities between the intuitionist school and Wittgenstein's position; although it is frequently the case that Wittgenstein shares a view with the intuitionists for wholly different reasons or to a wildly different degree. Hence what follows is an examination of the motivation for and the extent of certain views common to both, rather than the assessment of a head-

on debate.

Firstly, Wittgenstein agrees with the intuitionists that mathematics has no need of foundation. For Wittgenstein mathematics has a foundation in ordinary human practice and needs; that people use mathematics and perform calculations the way they do, is ample foundation. It is the successful application of mathematics that ultimately justifies it.

"What does mathematics need a foundation for? It no more needs one, I believe, than propositions about physical objects ... what is called foundations are no more the foundations of mathematics for us than the painted rock in the support of a painted tower" (RFM V 13) Here Wittgenstein agrees with the intuitionists about the lack of need for justification; but not because of a claim that mathematics is a mental activity, but because it is already rooted in daily practice.

Secondly, Wittgenstein's remarks on the law of the excluded middle are ambivalent. In the Philosophical Grammar Wittgenstein argues against Brouwer's view as muddled and misconceived, but in the Remarks Wittgenstein expresses worries about the unrestricted use of law of the excluded middle, and seems to accept the intuitionist rejection of the law, but for reasons of his own.

In the Tractatus there is a theory of meaning based upon truth conditions and conjunctions of atomic propositions. In later works, particularly the Philosophical Investigations, this view is rejected in favour of the view that an expression

derives its meaning from its application or use within a particular language game. Having said this, Wittgenstein regarded mathematical propositions as a special case and argued that for them there could be no distinction between sense and truth value. Mathematical propositions are verified in a different way to non-mathematical propositions, and so "The verification is not a mere token of truth, but determines the sense of the proposition." (PG p459)

Two criticisms of intuitionism are consequent upon this. The first is that the intuitionists are muddled or under a misconception to try to ground mathematics in the experience of mental construction. For mathematical propositions are true in virtue of their grammatical form, not in virtue of any mental or material observation or construction. In his eleventh lecture on the foundations of mathematics Wittgenstein is at pains to distinguish between the results of a mathematical calculation, and the results of an experiment. The way we treat the two is essentially different and reinforces the point that in trying to reduce mathematics to reports of mental activity (experiments with mental entities) the intuitionists are fundamentally wrong.

The second consequence is a rejection of Brouwer's account of the law of the excluded middle. The propositions that Brouwer sees without truth value are seen by Wittgenstein as nonsense. For example define π' as equal to π , except that three consecutive zeros replace the first three

consecutive sevens in the decimal expansion of π . Brouwer maintained neither $\pi = \pi'$ nor $\pi \neq \pi'$. (It is not known whether the decimal expansion of π contains three consecutive sevens.)

Wittgenstein on the other hand saw a proposition such as $\pi = \pi'$ as nonsensical, since π' is a different grammatical structure from the real number π . π' is not even a number in the same sense as π , for I cannot compare the two, and comparability is a fundamental characteristic of number. Although the rule for the expansion of π' is unambiguous, it still leaves π' fundamentally different from π or $\sqrt{2}$, and there is no sense to propositions like the above. (cf PG p476)

This example is not different from the other types of case where Brouwer claims the law of the excluded middle does not hold, and so Wittgenstein is unable to agree with Brouwer's account of the matter.

The Philosophical Grammar was written between 1931 and 1934 and the Remarks on the Foundations of Mathematics between 1937 and 1944, and in the later work Wittgenstein's views on the excluded middle have changed; and there is a deep questioning of the completely unrestricted use of the law. Following Wittgenstein, consider whether the sequence 777 appears in the decimal expansion of π . This question is equivalent to asking whether a person trained to write down numbers in accordance with an expansion rule for π , would

ever write the sequence 777.

Wittgenstein does not reject the question as nonsense, but he will not assert that the person will or will not write the sequence, because to claim this is an extra rule or premise. For there is nothing in the expansion rule itself, prescribing or proscribing the sequence.

Wittgenstein compares this to the poet who is asked "whether the hero of his poem has a sister or not - when, that is he has not decided anything about it". (RFM IV 9) Further, even the claim that an omniscient God would know is rejected, for there is nothing for God to know. The expansion rule limits God as it limits us, and Wittgenstein claims even God can only determine mathematical questions by mathematics.

The crucial point is that it is not clear how a person expanding π will or ought to interpret the expansion rule. At any stage there is a choice how to interpret the rule. In reply to the objector who says: "if you want to remain in accord with the rules you must go this way", Wittgenstein replies "Not at all, I call this 'accord' or the meaning of the rule". When further pressed that this involves a change of meaning of accord, Wittgenstein says: "No - who says what change and remain the same mean here?" (RFM I 113)

And also the remark that: "However queer it sounds, the further expansion of an irrational number is a further expansion of mathematics." That is, in expanding an irrational number we are not discovering or revealing some

external mathematical reality, but we are actually inventing : further expanding mathematics. This sounds queer for it is not the way we are accustomed to think, we tend to think of mathematics as external and to be discovered, but Wittgenstein argues that this is wrong.

Wittgenstein provides further elucidation by explaining that to think of the expansion as a row of numbers gives a false picture. Applying the law of the excluded middle forces us to picture the expansion as one of two possible rows, but Wittgenstein says these pictures cannot be applied here - they are unsurveyable. "We use the false picture of a completed expansion, and this forces us to ask unanswerable question." (RFM IV 9)

It is illegitimate to think of infinite series like finite ones. For a finite series the negation of "it must not occur", is "it must occur"; but for an infinite sequence the negation of "it must not occur" is "it can occur". (RFM IV 18) There is a basic disanalogy between our understanding and handling of finite and infinite sequences.

It is in views such as these that Wittgenstein stands firmly opposed to the Platonist and broadly in accord with the intuitionist, over the law of the excluded middle. However for the intuitionist problems about the expansion of π derive from the non effectiveness of the expansion; and for Wittgenstein they derive from the indeterminacy of rule application and the consequent non-objectivity of the

expansion of π .

I shall now turn to the subject of proof, and make a few remarks about Wittgenstein's view as related to the intuitionists, although a detailed examination of proof will be dealt with later. Wittgenstein shares with the intuitionists a distrust of non-constructive proofs. That is, he rejects the realist account of mathematics and proofs dependent on pre-existent entities. However Wittgenstein's constructionism is more extreme than the intuitionists, and this - as we shall see - derives from his view of proof as a modifying concept, (see later) changing the status of a statement proved. That is changing its sense as distinct from its truth value.

For associated reasons Wittgenstein also rejected the notion of the objectivity of proof. Intuitionists question classical proofs, but they accept a proof as objective if it is in accord with intuitionist logic. However Wittgenstein rejects not only the idea of objective mathematical truth, but also the idea of objective proof; because he holds a proof does not compel acceptance. We are free to accept or reject a proof at any stage.

Finally Wittgenstein attacks Brouwer's psychologism and his doctrine of the parallelism of the will. These are devices introduced to try to avoid the criticisms that if mathematics is based upon intuiting one's own mental states, then firstly it cannot be objective for there is no way to

intuit another's mental states. At best mathematics is intersubjective, at worst no more than a solitaire. Secondly it is impossible to explain or resolve mathematical dispute.

If a person can only intuit and report his own mental contents, there can be no scope for disagreement or dispute.

This is not only absurd, but is contrary to everyday experience!

Heyting replies to these difficulties by accepting them, but arguing they do not in fact cause a problem. "We are generally convinced that other people have thoughts similar to our own ... in this respect mathematics does not essentially differ from other subjects". And "slight divergences of opinion can be expected ... [but] they are in no way alarming". (Heyting: "Disputation" in Benacerraf and Putnam p61).

In other words Heyting argues that solipsistic doubts are as much a problem for any discipline as they are for intuitionist mathematics. And he further claims that there is no real problem.

This seems inadequate for intuitionist mathematics is about mental entities rather than the material entities of most discourse. Further, Heyting's line "We are generally convinced ..." rings hollow in the ears of the critic. But Brouwer's approach is to argue for intersubjectivity on account of the parallelism of the will. That is to say he argues that everybody intuits self-evident mental

constructions in the same way; and hence there can be no fundamental disputes. The disputes that there are, amount to no more than misunderstanding and breakdown of communication.

The essential debate here is over the Cartesian account of epistemology and theory of truth, and its development by Brentano. Largely discredited now, there is no doubt that the burden of proof lies with the intuitionist.

Wittgenstein's reaction to all of this is clear and succinct: "When the intuitionists speak of 'basic intuition' is this a psychological process? If so how does it come into mathematics?" (PG p322). For Wittgenstein, mathematics is ultimately a language game governed by linguistic rules; and any appeal to psychological or mental criteria is misplaced and misconceived.

To conclude : Wittgenstein concurs with the intuitionists in only the broadest of senses, and a closer examination reveals not only (relatively minor) differences of opinion, but also a profoundly different method of approach and underlying philosophy. The discussion here is not exhaustive, but chapters to follow on proof and strict finitism particularly will serve to elaborate and clarify Wittgenstein's position relative to intuitionism.

III Formalism.

Both intuitionism and formalism have their roots in the Kantian notion that theorems of mathematics follow from axioms in accordance with the laws of logic, but are not actually principles of the logic themselves. It is this common foundation in Kant, that accounts for the similarities between the two schools; and their respective attitudes towards transfinite mathematics that accounts for the differences.

To elaborate a little, both intuitionism and formalism reject the realist or Platonic idea that mathematical entities exist, or that mathematical theorems are true independently of human thought. Both are committed to the views that mathematical entities exist in virtue of some human action, and theorems are true in virtue of having been proved.

Furthermore, both schools understand "proof" as a finite proof. That is, the well-formed-formulae of the system must be expressible in terms of a finite alphabet; and it must be possible to effectively determine whether a finite sequence of formulae is a proof of its last member. Hence, in theory at least, mathematics is mechanically checkable, and there are no tacit assumptions about quantification over infinite domains.

As a result of this non-realist approach to ontology and finitist approach to proof, both schools see the so called foundation problems in mathematics to have been removed. As a finite activity with constructed entities there is no

further need for foundation.

However dispute arises over the precise nature of mathematical entities, and the validity or otherwise of classical mathematics. Intuitionists regard mathematical entities as mental constructions, and mathematical statements as descriptions or reports of these constructions. Formalists on the other hand see these entities as simply symbols (on paper); and theorems do nothing more than stipulate how the symbols may be combined, substituted etc..

Formalists deny that geometrical points, lines and the like can be mentally visualized (constructed) and hence they only have the properties they do in virtue of definitions. This view strips mathematics of any semantic content, and regards it as no more than mechanical operations with signs that are essentially meaningless.

Such a view is highly problematic, since a gap between syntax and semantics raises the question of what makes a system with axioms and rules mathematics? (Is snakes and ladders mathematical?) And further how is applied mathematics to be explained? Having raised these questions I shall not attempt to answer them, as there are more powerful arguments for rejecting formalism; and such a discussion would take us too far from the Wittgenstein theme.

The major dispute is over the status of classical, non-finite mathematics. Intuitionists reject this as being founded upon illegitimate assumptions about infinite domains,

but formalists wish to retain classical non-finite mathematics. In Hilbert's famous words: "No one will ever be able to expel us from the paradise that Cantor has created for us." It is this divergence of opinion over the infinite that account for the differences between the two schools.

Hilbert's programme attempted to formalise mathematical systems, and then by applying finite methods of analysis to the systems show them to be consistent. That is to validate classical mathematics by showing it to be consistent.

The formalisation of the system consists in listing all the undefined concepts in the system, listing the assumptions of the system (axioms) and listing the inference rules of the system. This is certainly not an easy task, but once it has been achieved the system is given fixed boundaries and can be surveyed by appeal to the formalisation alone.

Once a formalisation is complete, it itself can be investigated, and this gives rise to a metatheory. In particular metamathematics is the study of formalisations of mathematical systems. An example of this crucial distinction is that whereas " $2 + 3 = 5$ " is an ordinary statement of mathematics (arithmetic), the statement " $2 + 3 = 5$ is an arithmetical formula" does not express an arithmetical fact, but belongs to the metatheory.

This distinction is so important, because provided that the formalisation of the system is finite, the metatheory will be analysable by finite methods - ie in accordance with

intuitionist principles. Hence the problems of transfinite mathematics are reduced to finite problems in metamathematics, via the formalisation of the system.

The last stage of Hilbert's programme was to show that the formalisations were both complete and consistent. These are proofs in the metatheory. Completeness demonstrates that every proposition provable in the theory is true, and that every true proposition is provable in the theory. Consistency shows that a system cannot contain as theorems two propositions, one of which is the formal negation of the other.

The methods of proving consistency are varied, but they have the common feature of finitism; and hence they can be grounded in perceptual objects and the possibility of inconsistency is removed. So the formalists can retain transfinite mathematics, whilst only using finitary (intuitive) methods in their proofs and calculations.

Hilbert published his results in a series of papers. In 1900 he offered an axiomatisation of the set of real numbers, and in 1904 he published an attempt to prove the consistency of arithmetic. ("On the foundations of logic and arithmetic" reprinted in Benacerraf and Putnam pp129-138) Further papers in 1917 and the twenties reflected the development of formalist-foundational studies.

In 1931 a young mathematician at Vienna university published a paper entitled "On formally undecidable

propositions of Principia Mathematica and related systems."

Although Kurt Gödel's paper was largely unintelligible to most mathematicians and philosophers of the day, it has come to be regarded as one of the most important advances in logic in modern times; and in particular completely undermining the formalist programme as it was originally conceived.

Gödel showed how to map the propositions and proofs of a given system into the natural numbers, so that each proposition had its own label or Gödel number. Then Gödel showed that metamathematical statements can be construed as relations between Gödel numbers, and hence the metatheory is "arithmetized". It is in fact mirrored within the theory itself.

Then following the idea of the liar paradox, and Richard's paradox but avoiding their errors, Gödel was able to construct a formula G representing the metastatement "The formula G is not demonstrable".

As a result of this Gödel was able to show that consistency could not be established by reasoning in the metatheory that was mirrored in the theory itself. That is the proof of consistency of arithmetic and all higher mathematics cannot be achieved by finite methods; for there will always be a G-statement which is true but not provable in the system, and thus all (finite) formalisations are essentially incomplete.

Note that the theory does not say that proofs of

consistency, or Gödel statements are in a mystical realm, forever beyond our reach. It only says such proofs are impossible by finitistic methods. (Gentzen provided a proof of the consistency of arithmetic in 1936, but his proof is not finitistic, and cannot be mapped onto the formalisation of arithmetic.)

It is clear from Gödel's theorem that the formalist programme as it was originally conceived is impossible, and cannot be a candidate for any serious foundational account of mathematics.

In considering Wittgenstein's criticism of formalism, I shall examine four main points of disagreement; and also comment upon Wittgenstein's anomalous and perhaps even bizarre remarks about Gödel's theorem, which have provoked so much discussion amongst Wittgenstein's commentators and critics.

Firstly Wittgenstein objects to the idea that mathematical signs are meaningless signs, and that mathematics is no more than a game of sign-manipulation. It is simply not the case that the basic signs of logic and mathematics are meaningless. " \sim ", " \forall ", "and" etc. possess a meaning, and in order to function within a proof they must have a meaning outside the axiomatic system. "The "constants" must already have meaning in the language. In this way it is essential that " \forall " and " \sim " already possess a familiar application, and the construction of a proof in Principia Mathematica gets its importance, its sense from this." (RFM. II 34)

Furthermore, as I have already hinted, if mathematics is a game played with meaningless symbols, there is a large problem of how to differentiate it from other games, and to give it its particular importance and significance. For Wittgenstein, the significance of mathematics is derived from its application. "I want to say it is essential to mathematics that its signs are also employed in mufti. It is the use outside mathematics, and so the meaning of the signs that makes the sign game into mathematics." (RFM IV 2) That is the signs must have semantic content, or mathematics ceases to be mathematics.

Secondly, Wittgenstein was opposed to the narrow and rigid concept of proof in the formalist system. For the formalist, a proof is just a mechanical sequence leading from axioms to theorem; but for Wittgenstein a proof is a rule of language or grammar: an instance of a family resemblance concept. There is no single concept of proof.

This objection is part of a more general complaint that there is no science of proof and no metamathematics. On the one hand formalism constructs a theory where there is no need for one - mathematics already has a foundation in the way that it is used. Hilbert's metatheory gives no information about foundational problems. A metatheory "cannot give us information about the foundations of mathematics." (PG p 296)

On the other hand the notion of metamathematics is

misplaced. Hilbert's metatheory is not a higher theory, but simply a different sort of theory. In theorizing about axioms and formalisations we operate in a different calculus not a metacalculus. "What Hilbert does is mathematics and not metamathematics. It is another calculus just like any other one". (WWK p 121) There is no hierarchy of calculi, just separate individual ones.

Wittgenstein gives a chess example. "I can play with the chessmen according to certain rules. But I can also invent a game in which I play with the rules themselves. The pieces in my game are now the rules of chess, and the rules of this game are, say, the laws of logic. In that case I have yet another game and not a metagame." (PR p 319)

Waismann objected to this. (WWK p133) For by considering the rules of chess I can find whether I can force mate in six moves from a given position (say). Similarly by considering the rules of arithmetic I can obtain information about possibilities within arithmetic. This theory is Hilbert's metatheory.

Wittgenstein replied that the analogy was invalid, for a proof that I can mate in six consists in actually doing it in symbolism, or on a chessboard. The proof of the possibility consists of moves within the game. It is not a metagame and hence Waismann's objection does not hold. The alleged metagame is either a separate game, or just the game itself. (cf PR p 326-7)

Lastly Wittgenstein considers the question of consistency and consistency proofs. His view is summed up: "Mathematicians nowadays make so much fuss about proofs of consistency of axioms. I have the feeling that if there were a contradiction in the axioms of a system it wouldn't be such a great misfortune." (PG p 303)

In short Wittgenstein considers the search for consistency proofs as pointless, and he particularly wants to dispel the ideas that: (a) a calculus with a contradiction is in some way essentially defective. (b) Consistency proofs are needed for a system, and a system without them is somehow insecure. (c) A system with a hidden contradiction is just as bad as a system with a revealed one; and (d) the discovery of a contradiction forces some kind of reparation - it cannot be coherently ignored.

These ideas which most mathematicians accept without question, stem from the fact that truth values are only preserved in consistent systems. In an inconsistent system false theorems can be derived from true premises. This would be a disaster, for the inconsistent system would not truly describe the intended structure, and in fact would not truly describe anything at all. But this does not worry Wittgenstein as he regards it as a mistake to consider mathematics as a descriptive activity. Rather it is a collection of rules and conventions. But what if these are inconsistent or contradictory? Well, that doesn't matter

either.

Wittgenstein draws an analogy with games. Consider noughts and crosses as an example. The second player can always force at least a draw, so the first player can never win. However this does not spoil the game for children who do not know this, or do not know how to do it. (force a draw) A hidden contradiction does not matter.

Even if an inconsistency comes to light this need not ruin the game, it just makes it rather tedious. Wright imagines a society who play an inconsistent game, but are always able to agree when inconsistent rules are appealed to.

A player might concede the inconsistency and might say, "Yes, playing the game is a matter of good sense: you have to be careful who you play with" (Wright 1980 p 300)

Note, people who play such a game are not necessarily irrational, nor is the game in some way defective. Further there is no obligation to change or amend the rules of the game. It is not incoherent to leave them unaltered.

The analogy seems to support Wittgenstein's point, but does it really work? To me it does not seem convincing. Firstly it can be argued that mathematics is a descriptive, realist activity; and so must be truth preserving and hence consistent. This undercuts the whole position. Alternatively if we accept anti-realism, the analogy still seems weak for mathematics has other goals than amusement. If the end product is to validate certain inferences, and if

these inferences are important (say in engineering etc.) the analogy with games breaks down. It is crucial that inference patterns of applied mathematics are truth preserving, and it is not possible to think of theorems as simply moves or positions in a game. An inconsistent system is in grave danger of being useless.

This danger is only realised if the inconsistency cannot be contained, and Wittgenstein tries to show ways in which an inconsistency might be contained. He develops discussions of arithmetic allowing division by zero, and systems of measurement with elastic ("dough") rulers. Such systems may be unusable for "ordinary" purposes, but "perhaps usable for other ones." (RFM II 78) Unfortunately Wittgenstein gives no examples of useful inconsistent systems, and no general discussion of how inconsistencies might be identified and contained. His discussion is far from convincing and it remains the case that inconsistent systems are more than likely to yield spurious results.

The discussion above could be much developed, for the Wittgensteinian can claim his opponent makes illicit assumptions about objectivity of mathematics and rule following. Such a discussion is not appropriate here, and I conclude by remarking that it is unfortunate that much of Wittgenstein's critique of formalism is by analogy, and hence so difficult to evaluate. However, as the original conception of formalism has been undermined by Gödel's

theorem, the degree of success of Wittgenstein's critique is not of particular philosophical importance. It is more a question of assessing Wittgenstein's ability to analyse and comment; and this is a question that does not concern me here.

Turning to Wittgenstein's remarks on Gödel's theorem we find a fragmentary account in the Remarks on the Foundations of Mathematics and in the Notebooks. Wittgenstein does not follow carefully through the steps of Gödel's proof, and he has been accused of muddling statements in the system, with metastatements and correlating statements. His remarks on the interpretation of the theorem appear to be trivial or uninteresting misinterpretations; and the comment: "My task is not to talk about (e.g.) Gödel's proof, but to by-pass it" (RFM V 16) has confirmed Wittgenstein's critics in their view that he did not know what he was talking about, and proved acutely embarrassing for Wittgenstein's disciples.

Ross Anderson complained that Wittgenstein misunderstood the significance of Gödel's theorem, particularly as Wittgenstein rejected Gödel as an ally in the argument against formalism. For if Wittgenstein had fully understood Gödel, he could not have done other than welcome his theorem. Gödel himself wrote to Abraham Robinson that Wittgenstein "advance[d] a completely trivial and uninteresting misinterpretation" of the results. (Shanker p89)

The drawback of such assessment is twofold. Firstly it

is a great insult to Wittgenstein's intellect and mastery of his subject, and secondly it ignores Wittgenstein's more general criticisms of formalism examined above.

With these in mind Wittgenstein's remarks can be interpreted not as questioning the validity or originality of Gödel's proof, but as questioning it's philosophical significance. For Wittgenstein, philosophy and mathematics are wholly separate disciplines which have nothing to say to each other; and since the validity of formalism is a philosophical question and Gödel's theorem is a mathematical theorem the two could not be related. "It might justly be asked what importance Gödel's proof has for our work. For a piece of mathematics cannot solve a problem of the sort that troubles us." (RFM V 19)

Seen in this light, Wittgenstein is not critical of Gödel's theorem as such, but of the way it has been used in philosophy. As a work of mathematics, Gödel's theorem is of no consequence to Wittgenstein's philosophical concerns; and this helps to explain why Wittgenstein does not interpret or elaborate Gödel's results.

There are also other reasons why Wittgenstein would have not embraced Gödel's theorem. Firstly Wittgenstein does not want to say that Hilbert's programme is impossible (Gödel's theorem), but that it is unintelligible in that it arises from a spurious demand for foundations, where no such demand exists. In other words, formalism is a school to be

dissolved not refuted; and in as much as Gödel failed to see this, Wittgenstein's lack of enthusiasm can be understood.

Furthermore, although Gödel's theorem ruined Hilbert's original programme, it did nothing to halt the development of metamathematics and non-finitary consistency proofs. For Wittgenstein this was a hopeless misunderstanding of the philosophy of mathematics, catalysed by Gödel's results, and hence the theorem can only be seen as a step backwards and not a great advance.

Lastly Wittgenstein's attack on the theorem can be understood in terms of the Platonism which it provoked. Gödel's demonstration of the existence of true but formally undecidable propositions, can be regarded as one of the motivating forces behind the modern revival of mathematical Platonism. The elimination of such metaphysical confusion was one of Wittgenstein's main tasks in his work, and his opposition to anything that undermined that is of no great surprise.

To conclude, I have tried to show that the critical interpretation of Wittgenstein's remarks about Gödel's theorem is not the only possible one; and further it is only when the remarks are considered within the wider framework of Wittgenstein's philosophy of mathematics, that they can be properly understood.

IV Strict Finitism.

The three traditional schools in the philosophy of mathematics can each be associated with a particular ontological position, and some interpreters have thought that the essential disagreement between the schools is an ontological one. However such an interpretation raises two difficulties: It presupposes that mathematical procedures can be decided by prior ontological issues. (even when there is no clear way to resolve these issues.) And it avoids the more fundamental question about the objectivity of mathematical thought.

Wittgenstein particularly contributes to this second question. In the Remarks on the Foundations of Mathematics, Platonism stands opposed to various degrees of constructivism; as associated with intuitionism and formalism. Wittgenstein rejects all of these for reasons we have seen, but he rejects them principally because they are all attempts to give mathematics an objective external foundation. Platonism does this by grounding the sense of mathematics in the truth of statements concerning relations between abstract objects. Intuitionism and formalism do this by developing objective notions of proof: either by intuited logical procedures, (to which all mathematicians should agree) or by developing internal consistency proofs of formalised systems.

As an alternative, Wittgenstein is alleged to adopt an

extreme form of constructivism, which has been called anthropologism by Hao Wang and strict finitism by Georg Kreisel. However such an allegation is highly contentious and there is dispute as to whether Wittgenstein can really be identified as a strict finitist. In this chapter I shall examine the strict finitist position, and consider to what extent it is coincident with Wittgenstein's own views: arguing towards a conclusion that on a selective reading of Wittgenstein he does indeed appear to hold a strict finitist position, but on a wider reading which considers Wittgenstein's more general position this is not in fact the case.

Dummett claims that Paul Bernays foreshadowed the strict finitist position in his paper 'On Platonism in Mathematics' of 1935. Bernays rejects Platonism "which has been shown untenable by the antinomies, particularly those surrounding the Russell-Zermelo paradox" (Benacerraf and Putnam p 277).

He goes on to suggest a new approach to questions of mathematical methodology, and argues that the way mathematics is actually done should provide the philosophical foundations of mathematics.

Analysing the way mathematics is actually done, is equivalent to analysing the way that language is used by mathematicians. The strict finitist holds that the answers to our philosophical worries lie in a correct analysis of language, and the way we use it. The beliefs we hold about

mathematics are, or should be, explicable in terms of the way we use language. This coincides with Wittgenstein's view that "Grammar tells us what kind of object anything is." (PI 373) Rather than explain our practices in terms of them relating to, or describing some underlying reality; the best we can do is to "investigate how the application of the picture goes." (PI 374)

At the heart of the strict finitist position is the view that mathematical statements assert necessary truths, but are not statements of fact. Hence a proof of p is a demonstration that p is necessarily true, or alternatively must be true; and an explanation or understanding of why this is the case comes (and can only come) from an analysis of how mathematical truths are actually proved.

The strict finitist takes the paradigm mathematical statement to be "the X of Y is Z ", where the "is" here is read as the "is" of identity. It is assumed that virtually all mathematical statements can be expressed in this form, so it is only a matter of explaining how necessary truth is demonstrated for the paradigm case.

Call the statement "the X of Y is Z " S . The first step of the proof is to show S is true, and this is done by ordinary standard means. A trivial example is the demonstration that the number of letters in the word "Bismarck" is eight. That is, to show that the letters of the word (strictly a typical token of the word) can be put into

one-one correspondence with the first eight numerals.

Secondly, the necessity of S is argued from the inconceivability of doing the first part wrongly. By showing S to be true, a new concept is established and this eliminates the possibility that S be false. Wittgenstein says: "Yes, this is how it has to be; I must fix the use of my language in this way". (RFM II 30) Again "The mathematical Must is only another expression of the fact that mathematics forms concepts" (RFM V 46); where "to give a new concept", can only mean to introduce a new employment of a concept, a new practice." (RFM V 49)

In other words, establishing the truth of S, fixes a new rule of language, and the necessity of S is derived from compliance with the new rule. As an example examine the strict finitist proof of: "the opposite of OVER is REVO" (cf RFM III 51) The first stage after this proof is to examine a perspicuous token of OVER. (For a full discussion of perspicuity see the next chapter) so that it is possible to "read-off", as Wittgenstein puts it, the opposite of OVER.

Secondly we perform the experiment of reversing the letters of OVER, by putting the first last, the last first etc., and then thirdly we read-off the opposite of OVER as REVO establishing the truth of the claim.

The fourth stage is to argue from "the opposite of OVER is REVO" to "the opposite of OVER must be REVO". This is done by appeal to the perspicuity and surveyability of stage

three. In other words since we can see and explain how we read of the opposite of OVER with full clarity, it is impossible to conceive that this could be done in any other way. "When we say in a proof: "This must come out" - then this is not for reasons that we do not see" (RFM II 39)

Finally, the strict finitist appeals to perspicuity again in order to move from the particular to the general case. To move from a proof about a token of OVER to a general proof about the sequence OVER. This is not done by inductive reasoning, but by perspicuity: the particular OVER must have all the essential properties of the general OVER in order that they both be identified as OVERs, and hence what is attributable to the particular is attributable to the general.

Berkelian worries about abstract general ideas such as whether a particular OVER is written in red or blue ink, or how large it is; and how this can be related to an universal OVER are not relevant. For perspicuity is only concerned with the essential properties of OVER, those properties without which it would not be recognizable as an OVER. When we perceive a single perspicuous OVER, we see something general or universal, because our criterion for identifying OVERs is universal, in that it must be common to all OVERs.

"We must be sure we can exactly reproduce what is essential to the proof. It may for example be written in two different handwritings or colours. What goes to make the reproduction of a proof is not anything like an exact

reproduction of a shade of colour or a hand-writing." (RFM II 1)

Two extra notions need to be explicated here. The first is the concept of memorability. This enters the strict finitist proof at the second experimental stage. We have to tell that the results of the experiment are the ones we expected; and when subsequently referring to the results we have to be able to see how or why they are relevant. In other words they have to be reproducible. This is very closely linked to surveyability and will be returned to.

The second notion is that of vagueness. This idea is developed by Dummett in his paper: "Wang's Paradox", where he argues that if the strict finitist is going to introduce terms such as perspicuous and surveyable; he must accept that these predicates have a certain vagueness. Because although it is generally possible to distinguish between a perspicuous proof and a non-perspicuous one; it is generally impossible to give a non-arbitrary boundary. The point is illustrated by Wang's Paradox:

0 is small.

If n is small, $(n + 1)$ is small

therefore Every number is small.

An inductive argument with a ridiculous conclusion. Dummett argues that the strict finitist rejects the repeated application of modus ponens in the presence of a vague predicate, and so for the strict finitist the paradox cannot

arise, but there is no boundary such that k is small and $(k + 1)$ is not.

In trying to elucidate the idea of strict finitism I have quoted from Wittgenstein, and it would seem as though Wittgenstein's position can be largely identified with that of the strict finitist. But care is required for brief quotations from a collection of remarks are not conclusive!

What other arguments are there in favour of interpreting Wittgenstein as a strict finitist?

There are several key points where the finitists and Wittgenstein agree. Firstly there is general agreement on proof procedure. Without going into details, both Wittgenstein and the strict finitists thought that a proof gave a mathematical statement a new sense. It could be used in situations where formerly it was not applicable. Further the two agreed that any statement only has one proof, or more accurately, one type of proof pattern. "The proof must be our model, our picture, of how these operations have a result" (RFM II 24) and "eg. : this proof is a mathematical entity that cannot be replaced by any other." (RFM II 59)

Secondly both Wittgenstein and the strict finitists saw the laws of logic as laws of thought. The laws of logic correspond to conventions in language, but do not correspond to any facts. "The propositions of logic are 'laws of thought' because they bring out the essence of human thinking... the technique of thinking. They show what

thinking is." (RFM I 133) This is closely connected with the view that mathematics is created not discovered; and as such has no external objective foundations.

Thirdly Wittgenstein's views on the law of the excluded middle are coincident with the strict finitist's. I have already explained how Wittgenstein thought "(p \vee -p) is true" to be nonsensical because it is based on a misunderstanding of grammar. Similarly for the strict finitists, a proof of p (a demonstration that p is necessarily true) entails or is causally dependant upon \sim p being unthinkable. Not false but nonsensical. Hence the whole disjunction is nonsense.

Wittgenstein and the strict finitists agree on the role of consistency proofs as examined in chapter three. Like Wittgenstein, strict finitists do not favour contradictions, for they are not convenient for practical application; but they do not hold that a system containing a contradiction is essentially disordered. The strict finitist does not hold that a contradiction necessarily indicates any error in the proof of a proposition. Just as experiments can have different results, so two people could hold contradictory statements as necessarily true. The picture in RFM I 136 which leads to the conclusion that $4 \times 3 + 2 = 10$, does not destroy proof patterns of $4 \times 3 + 2 = 14$.

A last, and more general point is that Wittgenstein and the strict finitists share the same methodology. They both have the intention to describe mathematics as it is, and not

to prescribe how it should be; and both claim the philosophical questions surrounding mathematics have their solutions in the way that mathematics is actually done.

But there is a difficulty here, for despite the common claim to describe mathematics, there is only limited agreement about that description. Wittgenstein does not consider the paradigm mathematical statement to be: "The X of Y is Z"; nor does he speak of the five stage proof procedure that the strict finitists employ. Everywhere Wittgenstein talks of a motley, a family of practices and techniques, and rejects any attempt to demarcate mathematics from non-mathematics.

Certainly Wittgenstein's remarks can be used to support a strict finitist position, but he adopts a broader perspective imagining "a landscape gardener designing paths for the layout of a garden" (RFM I 166) I think Wittgenstein would further suggest that the strict finitists do not describe mathematics fully, as they do not allow for the endless possibility of new paths, and new forms of description.

Other points of disagreement between the strict finitists and Wittgenstein are over meaning. The strict finitists held that mathematical statements have meaning, in virtue of a fixed use and proof pattern of the paradigm: "the X of Y is Z". But Wittgenstein held that mathematical statements were meaningless-senseless. A mathematical statement is just an algorithm, a piece in a game, along

with all the other pieces, and has no meaning and cannot be said to be true or false. Essentially mathematics is not about anything, and this is the point of disagreement, for the strict finitists hold mathematical statements to be about something, and proofs of such statements show them to be necessarily true.

Lastly those wishing to interpret Wittgenstein as a strict finitist, particularly Kielkopf, have argued that Wittgenstein only accepts finite or elementary mathematics.

He defends his interpretation with a passage from the Blue Book: "If I wished to find out what sort of a thing mathematics is, I should be very content indeed to have investigated the case of finite cardinal arithmetic. For

(a) this would lead me to all the more complicated cases

(b) a finite cardinal arithmetic is not incomplete, it has no gaps which are then filled by the rest of arithmetic."

(BB p 20. Kielkopf p 177)

The point is meant to be that Wittgenstein is to be interpreted as a strict finitist as his main mathematical discussions are of elementary finite mathematics. He is "very content" to have investigated finite arithmetic; and his discussions of higher mathematics (Cantor, Dedekind, Godel etc.) are notoriously poor. He is supposedly less concerned about higher mathematics.

But this is a travesty! Firstly Wittgenstein is not

unconcerned with the results of higher mathematics. Admittedly he does not question them mathematically, but he questions their philosophical significance. Many of the philosophical points that Wittgenstein wants to elucidate can be derived from finite arithmetic just as well as from higher mathematics; and as Wittgenstein has no intention of discussing mathematics but philosophy, he is content with the simpler systems.

Furthermore, the simplest examples reveal more clearly the basic foundational problems of inference, rule following etc., that are Wittgenstein's primary concern. His preoccupation with elementary mathematics in no way suggests he thought only finite mathematics is really clear. (as Kreisel suggests) Indeed Wittgenstein accepts Cantor's theorem and the concept of infinity, provided that the concept is understood and used in conjunction with unending techniques; "The concept of infinite decimals in mathematical propositions are not concepts of series, but of the unlimited technique of expansion of series." (RFM IV 19) And again: "The licence to play language-games with cardinal numbers does not terminate." (RFM A II 5)

To conclude then, I think that the interpretation of Wittgenstein as a strict finitist is mistaken, for it relies on a selective reading of Wittgenstein, and fails to understand Wittgenstein's conception of mathematics as a motley, or family of practices. To a lesser degree, it fails

to understand the motivation behind Wittgenstein's preoccupation with elementary results. In fact the very idea that Wittgenstein would endorse any all-embracing theory of mathematics, or that he would even suggest there could be such a theory is a misunderstanding of what Wittgenstein thought foundational studies to be.

"We may not advance any kind of theory ... We must do away with all explanation, and description alone must take its place." (PI 109) There can be no "philosophical theory" of mathematics, for philosophy just describes mathematics; and as mathematics is always being expanded and invented, so the philosophy (descriptions) thereof must grow and expand.

As a postscript it is worthwhile to observe that Kielkopf tries to develop a critique of Wittgenstein by attributing strict finitism to him, and then criticising the metaphysics underlying strict finitism. Kielkopf suggests that the acceptance of necessary truths leads to undesirable epistemology and ontology; and that there are important mathematical statements (eg the principle of mathematical induction) that cannot be rephrased as "the X of Y is Z".

The details are not important, for it is sufficient to note that if Wittgenstein is not interpreted as a strict finitist, Kielkopf's critique, as it stands, does not apply.

That is not to say that criticisms of strict finitism are not criticisms of Wittgenstein, (indeed my chapter on proof will explore some of these) but that they are not necessarily so,

and need to be examined individually.

V Proof.

The purpose of this section is twofold. Firstly to conclude the previous sections, and in particular to make precise the many references to proof. Secondly to introduce the second strand of this thesis, which will aim to locate Wittgenstein's philosophy of mathematics within his general philosophical position. My discussion of proof will draw on Wittgenstein's views on language and rule following, and this will be groundwork for a more thorough examination of the continuity of Wittgenstein's later thought.

(a) Perspicuity and Surveyability.

"A mathematical proof must be perspicuous" (RFM II 1) is the bold opening of section two of Wittgenstein's Remarks on the Foundations of Mathematics, and an unambiguous statement of his position. But what does it mean? A perspicuous proof is one that can be taken in, comprehended as a whole, clearly viewed in its entirety. As opposed to proofs that are too long, complex or tortuous to be taken in. These are not perspicuous and do not count as proofs at all.

An alternative way of explaining this is to say a proof must be reproducible. Given a proof and a copy, "it must be possible to decide with certainty whether we really have the same proof twice over". (RFM II 1) Again a proof that is too long, complex or unwieldy to be reproduced with certainty cannot count as a proof.

Rather than give examples of perspicuous proofs,

Wittgenstein offers counter-examples from Russell's Principia Mathematica. Russell's notation is certainly adequate for elementary proofs of calculations such as $2 + 3 = 5$, or $2 \times 4 = 8$; but Wittgenstein asks : what of " $7034174 + 6594321 = 13628495$ ". (RFM II 3) There is no Russellian proof of this, for such an alleged proof would not be surveyable, (it would be too long to read) and could not be reproduced with any degree of certainty.

Again Wittgenstein asks how should we regard a Russellian proof of $10^{10} + 1 = 10^{10}$? Because of the length and complexity of such an alleged proof it should be disregarded as a proof, and no attempt should be made to show that a person producing such an alleged proof must have miscalculated. It would, in fact, not be possible to show this.

Wittgenstein considers a counter-argument to this: that by introducing arabic numerals and decimal notation, Russellian proofs can be hugely abbreviated, and otherwise unsurveyable proofs can be made accessible.

The reply to this is that the new proof is a wholly new entity, and not "a pale shadow of the unshortened one". (RFM V 19). The shortened proof tells what ought to come out of the unshortened proof; but this is at odds with the fact that the Russellian signs are primary, and the abbreviations are defined in terms of them. Hence the point remains that large calculations cannot be proved by an appeal to Russellian

symbolism, as it is unsurveyable.

So far then Wittgenstein has cited surveyability or perspicuity and reproducibility as criteria for valid proof.

A surveyable proof can be understood as cogent in its entirety; a reproducible proof can be followed step by step, in order to ensure that two allegedly identical proofs are in fact so.

But is there a gap here? It would seem quite possible to be able to understand and reproduce each stage in a proof, to comprehend each step, but not to be able to grasp the whole. In order to close this gap, and to enable decisions to be made about whether a proof is surveyable, Wittgenstein introduces the notion that causality plays no part in proof.

That is to say that it is not enough to simply understand the steps in a proof, and to accept the result as causally dependent on these; we must be able to reproduce "every step and the result." (RFM II 55) We must have a concept of how a certain procedure ought to result, not just how it will result.

Hence our naive understanding of reproducibility is enlarged so that we know why and how a certain procedure yields results rather than it simply does! The gap between understanding individual steps in a proof and understanding the whole proof is closed.

As a result of this there is no difference between checking the reasoning of a proof and checking that it has

been carried out properly. Because the "how" of the proof is known, the former collapses into the latter, for there is no doubt that the proof will be carried out properly.

(b) As a modifying concept.

Wittgenstein saw a proof as modifying our understanding: as establishing new connections between concepts where formerly there were none. The reason for this, Wittgenstein argues, is that unless a proof provides new connections, enables us to recognize new relevance of particular statements, then a proof has done nothing. As a result of a proof our understanding must develop, and the sense or significance attached to certain statements must change. Proof modifies our understanding of concepts.

This is highly counter-intuitive, for we tend to think of a proof as compelling, "bundling us along" from assumptions to conclusions. In a valid proof we have no option but to move from stage to stage, and we cannot choose to simply reject a stage in a proof as fancy takes us! But Wittgenstein is suggesting just this. At any stage in a proof we have the option whether to accept the next stage; or even to accept all the stages in a proof and reject the conclusion. This option derives from our choice whether to establish new concepts, new connections, new meanings and understandings at any stage.

Wittgenstein expresses this in several ways. "When I say a proof introduces a new concept, I meant something like:

a proof puts a new paradigm among the paradigms of language; like when someone mixes a special reddish-blue, somehow settles the special mixture of colours and gives it a name."

(RFM II 31) And again: "Do not look on a proof as a procedure which compels you, but as one which guides you - and what it guides is your conception of a particular situation".

(RFM III 30) And many others.

But there is a difficulty with all this. If a proof alters a proposition's sense, it does not prove what the proposition originally meant, and the notion of modification is pushed beyond explanation.

Wittgenstein is aware of this. He says: "Of course, some people would oppose this and say 'Then the proof of a proposition cannot ever be found, for if it has been found it is no longer the proof of this proposition.'" (RFM V 7)

Crispin Wright elaborates the point. (Wright 1980 pp44-46) Suppose a sentence S is proved and its sense is modified by the proof. The question then is: what did S mean before the proof? Suppose further that there is a statement T that expresses after the proof what S expressed beforehand. Now is T proved by the proof of S? If not, it was a mistake to accept the proof of S, for if it proved S it should also prove T. On the other hand if we accept the proof of S as a proof of T, T must now express what S now expresses, and cannot be what S formerly meant. That is, if T is proved by the proof of S, its sense will be modified by the proof and

will now express what S now expresses.

There is no way to express what a proof proved, and hence it must be impossible to prove anything. Wittgenstein's retort is that, "to say this is so far to say nothing at all". (RFM V 7) This point derives from Wittgenstein's challenge to our normal ways of thinking, in particular his views on understanding and rule following. Firstly there is no sense in which our understanding of a proof and its conclusions compel us to accept the proof. But even accepting this, the argument above still holds. It is based on what Wright calls the synonymy principle: "that a proof of any statement is eo ipso a proof of any synonymous statement". (Wright 1980 p45) and we instinctively believe this must be so.

But Wittgenstein's second point is that anti-realist rule following considerations oblige a new attitude to the principle. Synonymy only derives from a readiness to use two different expressions in the same context to mean the same thing - but we are free to choose. Wittgenstein can simply say that he is not prepared to accept Wright's statement T as expressing what S used to mean, or he can refuse to accept the proof of S as a proof of T. Our attitude to S and T is a free choice, depending on what we now choose to be synonymous.

So if the synonymy principle fails, Wright's reductio ad absurdum concerning S and T fails, and as Wittgenstein says: we have proved nothing. Once realist objectivity is rejected, the way is paved for Wittgenstein's view of proof

as a modifying concept. Much more could be said on this issue, but that would take us beyond the present scope, and I let my discussion stand as it is. (See remarks to follow on rule following and philosophy of mind.)

(c) As a rule of grammar.

Closely connected with the idea that a proof modifies concepts is the idea that a proof is a new rule of grammar.

We have seen that for Wittgenstein the mathematician "creates essence"; but not ex Nihilo. "He deposits what belongs to essence in the paradigms of language." (RFM I 32) That is, mathematics only has sense in the context of language, and in proving a particular result we agree to use language in a particular way. We fix a rule of grammar, or a convention for the use of language.

Clearly Wittgenstein does not see this in an individual, but as a communal activity. It is general agreement with a particular result that establishes it as a new rule; and in sophisticated societies where there are specialists in various fields, the agreement of the specialists is a condition for the adoption of the new rule.

Wittgenstein illustrates this in various ways. In lecture twenty-six he says "' $30 \times 30 = 90$ ' is not a statement about any reality, or about 30, but about our calculus. Taken by itself it has no meaning, but within a grammatical context we are able to use it. Compare how you verify ' $30 \times 30 = 900$ ' and 'I have 30 handkerchiefs' to see the difference."

(LFM p 250-251)

Again Wittgenstein discusses an example from John Wisdom, who when first told $3 \times 0 = 0$ disagreed; and wanted to say $3 \times 0 = 3$. For three cows multiplied by zero, are not multiplied at all, and are hence still three! Wittgenstein's point is that there is nothing intrinsically wrong with this, and it could be adopted. However as it conflicts with a previously adopted rule of grammar (the convention for multiplying by one) it is of little use, and the community does not gain by accepting it. (cf LFM p 135)

A more elaborate example comes from Dummett. We use Eratosthenes' Sieve to determine whether a given integer is prime, by trying to divide all smaller primes into the given integer. Suppose a fanatic devoted his life to proving the primality of a huge integer N , by means of the sieve. Suppose also we have a more powerful algebraic criterion by which we show N to be composite. We abandon the fanatic's result as it is unsurveyable and unreliable and accept the algebraic result as correct.

However in doing this the notion of primality has been changed. The algebraic criterion has replaced the sieve criterion, and the "sieve rule of grammar" has been replaced by an "algebraic rule of grammar." By adopting the algebraic proof that N is composite, we have implicitly accepted a new criterion, a new rule of grammar, and the sense of "prime" has been changed.

Similarly for perfect numbers which are equal to the sum of their factors (eg $6 = 1 + 2 + 3$). It is an unanswered question whether there are any odd perfect numbers. What is certain, is that if there were an odd perfect number it would be huge, and could not be shown to be perfect by calculating its factors and adding them up. A new criterion of perfection, a new rule of grammar would have to be introduced in the proof, and the term "perfect" would be given a new sense.

Without going into details, it is worthwhile to observe the connection between the notion of proof as a rule of grammar and the idea of the autonomy of grammar that Baker and Hacker attribute to Wittgenstein. It is an idea mainly developed in the Investigations, although remarks in the Philosophical Grammar and Zettel are closely related.

There are two main elements to the doctrine. Firstly that grammatical propositions do not answer to facts. "Grammar is not accountable to any reality. It is grammatical rules that determine meaning (constitute it) and so they themselves are not answerable to any meaning and to that extent are arbitrary". (PG p184) The explanation of the meaning of a proposition is given by showing (explaining) by example, how it is used. Proofs, like rules of grammar, establish new links new rules; and it is the "intralinguistic" explanation - for us the showing of what a proof does and how it changes our future mathematics - that

gives it its sense and meaning.

The second strand of the doctrine is that grammatical propositions are mutually independent. We construct new connections as and where we wish. We lay down new rules, as application demands; but the adoption or rejection of a particular rule has no bearing on whether a different rule (grammatical proposition) should be adopted. This point is harder to grasp, but since we adopt proofs by convention: to act or serve as a linguistic or grammatical norm, then there is no compulsion of any kind on us; and hence we (the community) are free to adopt rules (proofs) as we choose. Each one individually, and so rules (proofs) are mutually independent.

(d) The necessity of proof.

On the one hand we have seen that Wittgenstein claimed we are free to accept or reject any proof. "The laws of inference do not compel him to say or write such and such like rails compelling a locomotive." (RFM I 116) But on the other hand Wittgenstein claims proofs are necessarily true, and I am obliged to accept them. "Don't I have to find it (the proof) acceptable? Why do you say have to? Because at the end of the proof you say eg: Yes I have to accept this conclusion" (RFM I 33) Again, "A proof leads me to say this must be like this." (RFM V 30) And combining the two: "He must admit it - and all the time it is possible that he does not admit it!" (RFM I 51)

These conflicting views seem a clear case of paradox. The idea of being compelled by proof, and being able to choose to accept a proof, side by side. What is to be made of Wittgenstein here, is he talking nonsense?

If we accept a positivist/conventionalist account of the necessity of propositions, then it would seem yes. In the preface to Language, Truth and Logic, Ayer writes. [A priori propositions]" are distinguished also by being necessary, whereas linguistic rules are arbitrary. At the same time if they are necessary it is only because the relevant linguistic rules are pre-supposed ... in Russell's and Whithead's system of logic, it is a contingent, empirical fact that the sign should be given the meaning that it has; ... but given these rules, the a priori proposition $p \rightarrow (q \rightarrow q)$ is necessarily true." (P 23)

In other words, once we have agreed upon the meaning of our terms, and the way they are to be used, we cannot escape the fact that some propositions will be necessarily true, solely in virtue of syntax and semantics. Now Wittgenstein would agree to fixing meanings, rules for building propositions etc., but would not agree that we are then bound to accept necessary truths solely in virtue of these meanings and rules. i.e. Wittgenstein does not accept the positivist account of the necessity of propositions.

For Wittgenstein, necessity derives from communal practice. If a proposition is taken as true, and if

furthermore the community agree not to let anything count against it, then it becomes a necessary truth: ie necessary in virtue of the agreed linguistic and grammatical conventions. "To accept a proposition as unshakeably certain - I want to say - means to use it as a grammatical rule: this removes uncertainty from it." (RFM II 39)

The " must" of : I must accept the conclusion, does not derive from meanings or conjunctions of the terms themselves, but derives from our choice of employing axioms, terms, conjunctions in a particular way. If we choose to accept a proposition p as unquestionably true, then p is necessarily true, and there is nothing more to say.

But there is an obvious objection to all this - if necessity just derives from choice, anybody could choose anything, and any proposition could end up necessarily true!

"Then according to you, everybody could continue as he likes, and so infer anyhow." (RFM I 116) Surely Wittgenstein's account of necessity is not acceptable?

Wittgenstein's reply is to be found in his understanding of 'our form of life'. An idea we shall return to, but loosely we only choose conventions that yield correct mathematics; and correct mathematics is that which has application and is shown to work in everyday life. Our practices, although chosen, are not simply arbitrary: they are the ones that prove successful in application. The 'must' in mathematics ultimately derives from the way we

behave. Our surroundings, training, culture - our form of life - dictate how we do our mathematics, and this in turn dictates the conventions we use and the propositions we accept as necessary.

(e) Proof by induction.

To conclude this discussion by discussing Wittgenstein's comments on proof by induction is to break away from the course of the discussion so far; but Wittgenstein engages in a long discussion of proof by induction at the end of the Philosophical Grammar, and some mention of it must be made.

Despite the length of Wittgenstein's discussion the point he wants to establish is straightforward. The sense of the word "proof" in the two expressions "proof by induction" and "proof of a formula" is different. He says it is crucial to distinguish between proving a formula, where we imagine a sequence of algebraic manipulations terminating in the formula to be proved. And proof by induction, where the proposition to be proved does not occur in the proof itself.

Examining a proof by induction reveals that we prove a given formula for a given case, $n = 1$ say; then we prove that we can infer from the fact that the formula holds for an arbitrary $n = k$, to the fact that it holds for $n = k + 1$; and we conclude that the formula holds for all n . (by mathematical induction) The first two stages are proofs in the normal sense of proving a formula, but the last stage, the appeal to induction, is a spurious use of the word "proof".

He says there is no extra step, no concluding stage in the proof. "In this case here is no therefore. The proof is all there is, it is not a mere vehicle" (WWK p 112) "The equations do assert something (they don't prove anything in the sense in which they are proved)" (PG p 397) "The construction of the induction is not a proof, but a certain arrangement of proofs (a pattern in the sense of an ornament)" (PG p 399)

For Wittgenstein a proof by induction just gives us a particular insight, "allows us to see an infinite generality".

(WWK p 135) "We are not saying that when $f(1)$ holds and $f(c + 1)$ follows from $f(c)$, the proposition $f(x)$ is therefore true for all cardinal numbers; but "the proposition holds for all cardinal numbers" means "it holds for $X = 1$, and $f(c + 1)$ follows from $f(c)$ " " (PG p 406)

Perhaps the point seems pedantic, but if a proof is a new rule of grammar, a new conceptual link, deriving its necessity from agreed communal practice; then it cannot also be a type of proof pattern. The proof-of-a-formula sense and proof by induction must be kept separate. The expression "proof by induction" is no more than a convenient locution to explain what is meant, and what proof patterns are employed when a particular kind of mathematical formula is spoken of.

VI Language Games and Forms of Life.

Despite the importance of language games in his later philosophy, Wittgenstein, characteristically, does not give any definition of language game. In the Blue Book and early in the Philosophical Investigations he generally associates simple languages with language games, and emphasizes the process of learning a language.

"We can also think of the whole process of using words in (2) as one of these games by means of which children learn their native language. I will call these games "language-games", and will sometimes speak of a primitive language as a language game."(PI 7) In his later work Wittgenstein uses the term in a much wider sense, and it is only through Wittgenstein's examples and applications that we can see what he means, i.e. what his conception of a language game is.

A first insight comes from the theory of language in the Tractatus, and realizing what a language game is not! In the Tractatus we find a formal language system, explaining the construction of propositions and clearly differentiating sense from nonsense. The conception of the language game replaces this rigidity and formality by a more variable and flexible understanding of language used in context.

In the Tractatus, language is a picture of the world. Atomic propositions picture atomic facts, and logical complexes of atomic propositions represent possible states of affairs. The complex is a picture of reality, or of a

possible state of affairs: true if this state obtains, false if it does not. Propositions that do not picture a possible state of affairs are nonsense.

The important point in this brief account is that the Tractatus language is rigid, made up from elementary units that can only be conjoined in certain ways. There are no alternatives, and any propositions not accounted for by the so called picture theory are nonsense.

There are many reasons why Wittgenstein came to reject this theory, but they do not concern us here. The new idea was that language could be understood by analogy with a game, the variety of games reflecting the variety of ways in which language is used.

An early example of this occurs in a conversation between Wittgenstein and Waismann, recorded in: Wittgenstein and the Vienna Circle. From Frege we have the suggestion that arithmetic is just about signs, ink marks on paper; or that it is about something that the signs represent, namely numbers. Wittgenstein argues that this is a false dichotomy, and illustrates his point with a chess analogy. The game chess is not about chess pieces. "If I say: "Now I will make myself a queen with very frightening eyes, she will drive everyone off the board," you will laugh." (WWK p 104). On the other hand chess pieces do not symbolize or represent anything, they have no Bedeutung. The pieces in chess take on a meaning, only because of the rules of the game as a

whole. Similarly signs in arithmetic are only meaningful in the context of the rules and conventions of the language game arithmetic.

Wittgenstein develops his idea in the Philosophical Grammar, discussing at some length the comparison between arithmetic and a game; and he says: "What we do in games must correspond to what we do in calculating." (PG p 290)

There are two important points here: Firstly, on Wittgenstein's view, both arithmetic and games are governed by rules which are arbitrary. In chess or arithmetic there is no compulsion to adopt a particular system, and in particular, no justification derives from reference to reality. The rules are autonomous, and could be different, but this would change the game. For example if it was agreed that black should always start a game of chess, that would be perfectly satisfactory, but the game would not be chess.

Secondly it is the application of a system, a language game, that is all important. Our application of arithmetic dictates which rules we should adopt, and how we should use them to yield useful results. Crucially a game has no application, and so there is no place for affirmation and negation, truth and falsity. Wittgenstein says that the person who has been taught arithmetic as a game, and does the calculation $21 \times 8 = 168$ as a move in the game; is doing something very different, and has a different attitude, to the person who wants to know what 21×8 is, for practical

purposes. (cf PG p 290-291) The lack of application, and the absence of truth and falsity in a game, leads Wittgenstein to say "arithmetic isn't a game." (PG p 293) In arithmetic we are concerned with truth, and in games we are not concerned with abstract truth at all. Certainly "It would not occur to anyone to include arithmetic in a list of games played by human beings." (ibid) However, both are based on convention, and the nature of arithmetic can be clarified by drawing out its relationship to a game.

Wittgenstein's analogy between arithmetic and chess seems to successfully avoid the dichotomy that Frege posed, since it illustrates that arithmetic is not essentially about anything. It is a mistake to look for the subject matter of arithmetic just as it is clearly a mistake to look for the subject matter of chess. Frege is mistaken in asking: what is arithmetic about?

A final point made by Wittgenstein and amplified by Wittgenstein is that it is not sufficient to simply win a game, but it must also be possible to recognize a win. A true mathematical statement is worthless unless it can be shown to be true, so winning must be akin to proving a statement. This involves recognizing and interpreting a particular situation, and then proceeding by rules which govern the theory of the game, in an appropriate way.

Having said this much, we must not be misled, and think of language only in terms of an agreed calculus. There are

properties of formal systems (eg chess) that are not to be found in language, and a broader concept of language-game needs to be developed. The misleading features of the chess-type model are :

(i) it suggests a system of rules to cover all possible cases, whereas language does not have clear cut boundaries and rules.

(ii) it leads us to think of an ideal syntax, 'a logical form' that can be displayed; whereas there is no such form.

(iii) We cannot adapt the formal system to explain how we use radically different explanations for one and the same expression.

(iv) it suggests a potential gap between formal, 'intra-calculus' explanations of meaning, and everyday, ordinary explanations of linguistic practice. That is: it provokes a tension between a "proper" explanation of language based on formal rules etc.; and a "casual" explanation, when in fact this tension does not exist.

To avoid these misleading features, Wittgenstein emphasizes the analogy between the variety of linguistic practice, and the variety of games. Some games have strict rules, some games have loose rules. Some are competitive, some are not; some are played by teams, others not etc.. Even games governed by rules leave some questions open, as for example there is no rule in tennis to say how high one should throw the ball before serving. Our different uses of

language reflect the variety of game, and a different system of language -language game - is appropriate at different times. Compare doing arithmetic, telling jokes, praying etc..

To express this more succinctly, Wittgenstein develops the notion of family resemblance, initially for games, and for language games as a corollary. Wittgenstein asks: what defines "game", or what do all games have in common? He suggests various answers to this question, but for each one, also supplies a counter-example: ie a game that does not fit the Merkmal definition, the supposed necessary and sufficient conditions for an activity to be a game. As a result he concludes that there is no single property common to them all, only "similarities, relationships ... overlapping and criss-crossing; sometimes overall similarities, sometimes similarities of detail".(PI 66) It is these similarities that he calls family resemblance, and he says: "games form a family".(PI 67)

He anticipates an objection: "Don't say, "there must be something common, or they would not be called "games"" - but look and see whether there is anything common to all."(PI 66)

"What ties the ship to the Wharf is a rope, and the rope consists of fibres, but it does not get its strength from any fibre that runs through it from one end to the other, but from the fact that there is a vast number of fibres overlapping."(BB p87)

Wittgenstein gives a long list of the "multiplicity of language games":

"Giving orders and obeying them.

Describing the appearance of an object or giving its measurements.

Constructing an object from a description (or drawing).

Reporting an event.

Speculating about an event." etc.etc.. (PI 23)

In addition to family resemblance properties, the other crucial similarity between language and games is that both rely on rules. In as much as games are bounded by rules, but not everywhere; (it does not matter what colour sweaters golf players wear) so similarly language is bounded, but not dictated by rules. I have, for example, a degree of choice in replying to a given question. Furthermore, rule, like game, is a family resemblance term; and Wittgenstein does not think of rules as an inviolate canon from which there can be no deviation.

Given this new understanding of language, Wittgenstein is able to give a new criterion to distinguish between sense and nonsense. Wittgenstein says nonsense arises when a word or phrase is used outside a language game to which it is appropriate. Wittgenstein thinks this is the root of many philosophical difficulties: that questions are asked or suggestions made which fall outside the given language game, and are both misleading and senseless.

To conclude this discussion of the insights to be gained from, and the misleading features of Wittgenstein's analogy between language (in particular arithmetic) and games, a final point can be made. By comparing language (arithmetic) to a game, Wittgenstein in no way suggests that language is a trivial activity or a pastime. Rather he emphasizes the connections between our linguistic and non-linguistic activities. Indeed he says the two cannot be separated; language is a part of communal activity, a part of "a form of life". (PI 23)

A form of life is a general background, against which our linguistic activity makes sense, and of which it is a part.

In order to be able to fully use language, one has to master several techniques, to have an understanding beyond the machinery of the language, to partake in a form of life. In other words language presupposes broad agreements of definition, and basic shared natural and linguistic behaviour; which Wittgenstein calls a form of life.

It is in this context that it is possible to understand Wittgenstein's rather obscure remark: "If a lion could talk, we could not understand him." (PI p 223) Wittgenstein is not talking of linguistic or translational difficulties, but saying that since we do not share the same basic background; we do not share the lion's form of life, then we cannot share in his language games. We cannot fully communicate with him.

This raises the difficulty of whether individuals partake

in one, or many forms of life. Is there one for humans and a different one for lions, or one for British and perhaps a separate one for the Chinese, say? Wittgenstein is ambivalent on this issue; but would probably suggest that just as cultural differences, and cultural boundaries are vague, then also the term "form of life" is essentially vague.

The point is not important here, but what is important is that our common background, our form of life is given.

Certain features of language are explained by our form of life, and as such must be simply accepted, there can be no further explanation. "What has to be accepted, the given, is - so one could say - forms of life." (PI p 226) "If I have exhausted justifications, I have reached bedrock and my spade is turned. Then I am inclined to say: "This is simply what I do."" (PI 217)

This introduction of a form of life, as a given and necessary background to linguistic activity; strikes an immediate parallel with Wittgenstein's idea that mathematics is in no need of foundation. The projects of the logicians, intuitionists and formalists were not so much wrong as misguided, for mathematics has no need of foundation. It already has a foundation in our form of life.

This means the way we teach and learn mathematics, the way we actually do mathematics and essentially the way that we successfully apply mathematics, are all the foundations that there are, but they are all that are needed. Our

practices are embedded in the way we live, and beyond that there is nothing to say. The foundation of our mathematics is human technique, "a technique which is a fact of natural history". (RFM V 13)

The second point to be made is that mathematics like language is a communal activity, which can only function if there is broad agreement about what terms are to be used, how they are to be used, what is to count as proof etc. It is this agreement, which, although it is often intentional and deliberate, is part of our form of life. It is the given background, and basic convention without which mathematics, and language generally, cannot function.

"If language is to be a means of communication there must be agreement not only in definitions but also (queer as this may sound) in judgements." (PI 242) Our agreement must go beyond how terms and expressions are used, and must include beliefs and outlooks which are shared. This is our form of life and the foundation of our mathematics.

Furthermore, our language and our mathematics have successfully achieved that for which they were designed (or evolved) and the thought of changing mathematics or linguistic practices on a large scale is preposterous. It would change our whole lives, and the point is that mathematics and language are not somehow external activities that we indulge in, but are a central and vital part of the very life we live.

So to conclude, an understanding of Wittgenstein's

conception of language games and form of life enhances an understanding of some of the main themes in the philosophy of mathematics. The move away from a rigid Augustinian/Tractarian understanding of language to a looser theory based on language games, stressing the diversity of common practice; is reflected by a move away from an objective foundations of mathematics to a theory stressing the variety and motley of mathematical techniques.

The idea that language is based upon human agreement and application, governed by communal rules and gaining meaning from its use in context is reflected in the idea that validity or truth in mathematics only derives from communal application and practice. "I take the calculation to be correct because it is correct" (RFM V 4) is Wittgenstein's way of saying that the only criterion for mathematical correctness is that it is in accord with the relevant language game, and is understood in the appropriate context.

And lastly Wittgenstein's insistence that for language to function not only must there be linguistic agreement, but that there must also be a common form of life underlying the linguistic practice is reflected in the idea that a common form of life is an essential foundation for mathematics, and indeed this is the only foundation that is required. Our language has no need of justification, it is justified by the way we actually use it. Similarly mathematics has no need for foundation, for it is already founded on our everyday

application, our common practice, our form of life.

VII Scepticism and Rule Following.

Having examined Wittgenstein's discussion of mathematics as a language game, I now turn to examine some of his sceptical concerns which are intimately connected with rule following. The current debate in this field is very active, but it is not my intention to discuss this as such, but rather to draw out the connections with the philosophy of mathematics in a minimally controversial way. As in the previous chapter I shall be drawing parallels and similarities as they seem to occur, without suggesting any underlying motivation or structure. This will be left until the next chapter, when I hope to bring the themes together to give a broader understanding of Wittgenstein's essential philosophical outlook.

Firstly consider the question at the beginning of the Remarks on the Foundations of Mathematics: "How do I know that in working out the series $+ 2$ I must write "20004, 20006" and not "20004, 20008"?" (RFM I 3). Or as the same question appears in the Investigations: how can we be sure that a pupil has grasped the " $+2$ rule" since the set of his answers (of his continuation of the series) is finite. (cf. PI 185)

Having raised this question, Wittgenstein suggests that it is invalid. Having mastered the technique of $+2$; having developed the series to a reasonable extent; it is senseless to question within this language game whether the rule has been grasped, or why I must write "20004, 20006" and not

"20004, 20008". "Scepticism is not irrefutable, but obviously nonsensical when it tries to raise doubts where no questions can be asked." (TLP 6.51) This sentence from the Tractatus captures Wittgenstein's point perfectly, and indicates a theme or a worry running throughout Wittgenstein's work.

The point is the same as the remark that it is nonsensical to doubt one's own pain. "I can't be in error here; it means nothing to doubt whether I am in pain!" (PI 188) The two main points that Wittgenstein makes are firstly that doubt has to have grounds, and secondly that doubt presupposes the mastery of a (relevant) language game.

"One doubts on specific grounds" (OC 458) is not a point with which Descartes would have argued, and it is clear that the evil genius of the Meditations is supposed to provide the relevant grounds, but Wittgenstein would have regarded this as a frivolous suggestion. The grounds for doubt must have some credulity; the imaginability of not-p is not enough in itself to warrant the doubt of p.

If a pupil can competently continue the +2 series up to 100 or 500 (say), there is no reason to suppose that anything might go wrong after 20,000 or any other number! There is no genuine ground for doubt, just as I myself cannot doubt my own assertion "I am in pain."

Wittgenstein's second point is that doubt presupposes the mastery of a language game. Again the point comes from

Descartes, who despite claiming to doubt everything to be doubted; does not doubt the meanings of the words he uses.

It is not even the case that the meaning a word has, is a logical fact; it is only empirical. "And isn't it an empirical fact - that this word is used like this?" (OC 306)

However, Cartesian doubt taken to the extent of doubting the meaning of words refutes itself. If I am deceived about the meaning of words, I cannot even assert this. "If this deceives me, what does "deceive" mean any more?" (OC 507)

So in other words I must be able to play the +2 language game, I must have mastered the technique before I question why I write "20004, 20006" and not "20004, 20008", and then the question is senseless. Similarly to say "I doubt whether I am in pain", shows the speaker has not mastered the appropriate language game or is mad.

Although not uncontraversial, these two considerations lead to Wittgenstein's conclusion that, "When I obey a rule, I do not choose, I obey the rule blindly." (PI 219) The obedience is blind because at the lowest level there are no doubts to be refuted, and no justifications to be explained.

The application of the rule is all that is important. But the question still remains: what is it to follow a rule, or what constitutes following a rule?

Kripke interprets Wittgenstein as suggesting that there is no fact or substantive entity that shows a person follows a rule, Rather, a person's rule following is to be explained

by conformity to social practice in the relevant respects. That is, Kripke ascribes to Wittgenstein a community or social conception of rule following, deriving its normativeness from meaning and use generally. As we shall see this is a contentious interpretation of Wittgenstein, and McGinn, in particular, rejects this communitarian interpretation; but initially consider Kripke's interpretation.

Immediately it seems at odds with Wittgenstein's insistence that a person is free to choose whether, or how to apply a rule at any stage of a procedure. "The laws of inference do not compel him to say or write such and such like rails compelling a locomotive." (RFM I 116) But as I remarked in the section on the necessity of proof, the important point is that the compulsion or lack of choice is not derived from the verbal stipulation of the rule, but from the community's application of it.

To correctly follow a rule is to act in accordance with the consensus. This is not a consensus of opinion, but a consensus of use and as such is essentially changeable and non-dictatorial. The thoughts people think are irrelevant.

They all follow the rule if they make the same use of it, if they apply it in the same way.

This hangs together with the question of how to continue the series of cardinal numbers. Is there a criterion for the continuation - for a right and a wrong way - except that we do in fact continue them in that way, apart from a few cranks

who can be neglected' (LFM p 183) Wittgenstein's answer is clearly no!

But there is a difficulty here, of which Wittgenstein was aware. If everything is grounded in communal use or practice, where does this leave the notion of truth? "Then do you want to say that "being true" means being usable (or useful.)?" (RFM I 4) Wittgenstein did not, and goes on to write "But isn't there a truth corresponding to logical inference? Isn't it true that this follows from that?" (RFM I 5). Both Baker and Hacker, and Wright observe this tension between truth derived from use, and logical truth; and Wright asks: "Is there indeed a fundamental "disharmony of main themes" in Wittgenstein's later philosophy of mathematics, as suggested by Baker and Hacker?" (Wright 1980 p 329)

A "fundamental disharmony" seems like an exaggeration to me. Wittgenstein rejects a conventionalist account of meaning and truth, that is he rejects the idea that meaning and truth derive from something external to language or supra-linguistic. He is very firm that both meaning and truth come from our use of language; and our intra-linguistic grammatical agreements. Hence logical truth is no more than a grammatical agreement, a reflection of what we actually do; and thus not essentially different to any of our other agreements. Wittgenstein rejects any external or formal convention and understands meaning and truth to derive from casual, non stipulative agreements. A suggestion of

"fundamental disharmony" seems inappropriate.

In what has been said so far, no distinction has been drawn between what it is for a pupil to grasp or follow a rule, and what it is for me to grasp or follow a rule. The remarks above are applicable to the third person, but if attention is focused on the first person, a new problem arises. "In my own case at all events, I surely know that I mean such and such a series; it doesn't matter how far I have actually developed it." (PI 147)

The suggestion here that Wittgenstein makes in order to subsequently refute; is that my mathematical ability is essentially inner. I can surely know just by inspecting my own mental contents that I am correctly continuing the "+2" series, and any doubt of this seems out of place. I can know the simplest mathematical truths by inspection of my own mental contents, and as such this inspection is incorrigible.

This alleged incorrigibility derives from the supposition that I can introspect directly, without any intermediary, and as such there is no opportunity for error to be introduced.

But this suggestion immediately leads to the question: does my ability to do this simple mathematics constitute a counter-example to the so called private language argument?

Now it is not my intention to assess this major question directly; but to amplify it slightly to pave the way for a discussion of Kripke's very similar suggestion that the root problem of the philosophy of mathematics is that posed by the

private language argument.

Firstly, distinguish two senses of private. A sensation can be private if only I know of it, but could in principle tell someone else. For example, I may have a feeling or premonition that it will snow tomorrow, and contingently I have not told anybody. Hence my feeling is private. Alternatively I may have some sensation that nobody else has, and it may be impossible for me to communicate it to them: strictly for them to have it. Kenny calls this type of privacy "inalienable" (Kenny p 185) and Wittgenstein expresses his idea, "Another person can't have my pains". (PI 253) It is the second sense of private that Wittgenstein intends when he argues against the possibility of private languages.

The distinction can also be drawn by comparing ontological or epistemological privacy on the one hand, and logical privacy on the other. A person may not share an ontologically or epistemologically private sensation, but he cannot share a logically private sensation. It is logical privacy that is Wittgenstein's concern.

The suggestion that mathematical knowledge is in some way inner or private trades on this difference, and does not provide any direct counter-example to the private language argument. Contingently private or secret mathematics is irrelevant, and logically private or inalienable mathematics is incoherent.

However it seems to be this mistaken line of thought that

leads Kripke to suggest that "Wittgenstein regards the fundamental problem of the philosophy of mathematics and of the private language argument - the problem of sensation language - as at root identical" (Kripke p 20) Furthermore the root problem according to Kripke is given by Wittgenstein in paragraph 201 of the Investigations. "This was our paradox, no course of action could be determined by a rule, because every course of action can be made out to accord with the rule." (PI 201/Kripke p 7)

In other words, Kripke suggests Wittgenstein advocates a new scepticism about language; that I can never be sure that the meaning I intend by my present use of a word (current employment of the rule for the word) is the same meaning I intended by my past uses of the same word. (past employment of the same rule.) I do not wish to develop the point, but Kripke claims that the sceptical conclusions of the paradox seem "specially unnatural" (Kripke p 79) in the areas of mathematics and inner experience, and this is why he claims the paradox is at the root of Wittgenstein's concerns in both of these areas.

It is difficult to do Kripke justice without going to great length, and it is only my intention to try to understand Wittgenstein's philosophy of mathematics with reference to his other philosophical concerns; therefore I have only briefly stated the connection that Kripke is trying to make, and must now explain that the majority of opinion is against Kripke's

interpretation, and that the so called "fundamental problem" does not exist.

There are three main points to be made.

- (i) As McGinn points out, Kripke has misinterpreted and misunderstood Wittgenstein. Firstly he has ignored the words immediately following his quotation from Investigations paragraph 201. Wittgenstein says: "It can be seen there is a misunderstanding here". (PI 201) In other words Wittgenstein does not endorse the paradox Kripke attributes to him. Secondly and more importantly, the sceptical doubt that Kripke suggests goes outside and beyond a relevant language game, and so for Wittgenstein is nonsensical.
- (ii) Baker and Hacker claim Kripke misinterprets Wittgenstein's use of "private". Kripke understands private to mean in isolation from the community with regard to following or responding to a rule. Whereas Wittgenstein means that a rule followed privately, is one to whose expression only I have access. i.e. There is a (logically) private ostensive definition of the rule. Again Kripke's discussion seems to be based on dubious foundations!
- (iii) The reference Kripke makes to the beginning of the Remarks on the Foundations of Mathematics does not establish the connection that it is supposed to. Wittgenstein says: "How do I know that in working out the series +2 I must write "20004, 20006" and not 20004, 20008"? -(The question: "How do I know that this colour is 'red'?" is similar.)" (RFM I 3)

But there is a difference between "how do I know that this colour is red?" and "how do I know that this experience is seeing red? Wittgenstein himself emphasises the distinction in "Notes for Lectures on Private Experience and Sense Data," discussing the examples "seeing red" and "having a red visual impression". Kripke seems to confuse the two, understanding the first question as a problem about private languages, when in fact only the second relates to private language.

In criticising Kripke, my point is not to show that there is no connection between the private language argument and rule following, considerations in the philosophy of mathematics; but rather to show the connection is not to be found in Kripke's paradox. The point to be made is that there is no internal, private rule between the public rule "+2", and saying "20004, 20006"; just as there is no private ostensive definition intermediary between seeing red, and saying "I see red". (I have a red visual experience.)

Wittgenstein's concern is not about "the problem of sensations" and how I know I am following a rule, but rather to establish the non-primacy of the inner, the mental and the subjective. His intention is the refutation of idealism and scepticism - a point to which I shall return.

To conclude this discussion of Kripke's position, it should be observed that there are those who whilst admitting Kripke has misinterpreted Wittgenstein, and wrongly attributed a new form of scepticism to him: nevertheless suggest that he

(Kripke) has in fact developed a new and important form of scepticism. Some even suggest that Wittgenstein's writing, if properly developed, leads to Kripke's so called paradox.

This however also seems mistaken for Kripke's paradox is the very antithesis of Wittgenstein's position, for it raises doubts and questions where no doubts and questions can be meaningfully raised. Throughout his philosophy Wittgenstein laboured against spurious forms of scepticism: from the Tractatus to the Remarks and beyond, and the suggestion that Kripke's paradox could be consequent upon Wittgenstein is surely mistaken.

"The difficult thing here is not to dig down to the ground; no, it is to recognize the ground that lies before us as the ground." (RFM VI 31) In other words the solution to Kripke's so called paradox is to realise that it raises nonsensical doubts and that some things simply are; and have no further explanation. They are a part of our form of life.

Lastly I want to suggest, again without going into great detail, that there are other views on this issue: principally the arguments of McGinn who is critical of both Kripke and Wittgenstein, despite have some sympathy with Wittgenstein's understanding of language games and forms of life.

McGinn rejects Kripke's attribution of the sceptical paradox to Wittgenstein, and rejects the suggestion that the paradox is derivative to Wittgenstein's views. However McGinn does not agree that the paradox raises nonsensical

questions and considers it as a genuine difficulty.

Principally McGinn rejects Kripke's listing of the straight solutions to the paradox as exhaustive, he criticises Kripke's sceptical solution to the paradox as inadequate, and proposes an alternative straight solution to the paradox resisting any reference to the notion of communal agreement.

A problem has a sceptical solution if it concedes the sceptic's point, but shows that ordinary practice is not in need of the justification invited by the sceptic. A problem has a straight solution if it shows the scepticism unwarranted, by proving the doubted theses. McGinn offers a solution of the second type: arguing that there is "no compelling reason to depart from the natural idea that which concepts a person possesses depends simply upon facts about him: we can thus form a conception of someone possessing concepts and following rules without introducing other persons into our thought, at least so far as Kripke's arguments are concerned." (McGinn p 191)

This summary is brief, but without becoming too involved in questions about rule following, I hope I have suggested some parallels between Wittgenstein's philosophy of mathematics and his conception of rules; and also indicated that further clarification of Wittgenstein's essential concerns is a prerequisite to a full understanding of his philosophy of mathematics.

VIII The Philosophy of Mind.

In this chapter I am going to develop the hint at the end of the previous chapter, that it is Wittgenstein's rejection of the traditional approach to philosophy, in particular subjectivist egocentricism and subjectivist philosophy of mind that is central to the understanding of Wittgenstein's later thought. By subjectivist, I understand a school of thought originating in Descartes, and developed through Locke and Hume, which has come to dominate western philosophical thinking; and to which Wittgenstein was so strongly opposed.

The main idea that Wittgenstein rejects is the egocentric approach: that is the belief that the "inner" or the mental can be apprehended directly and with certainty; whereas the "outer" or external world can only be apprehended indirectly and knowledge of the outer is open to the possibility of error in the way that knowledge of the inner is not.

This is essentially Descartes' classical position, which progressed from certain knowledge of the self qua thinking thing, derived from reflection alone; to a knowledge of the outer world derived from reflection coupled with observation.

Furthermore this thesis in the philosophy of mind, permeates epistemology, metaphysics and ethics too, binding together a whole system which has proved so philosophically durable.

The general position originating in Descartes consists of two main theories. A theory of knowledge which does not concern us here, and a theory of meaning which can be

summarised in a single sentence from Locke, "... words in their primary or immediate signification stand for nothing but the ideas in the mind of him that uses them." (Essay Concerning Human Understanding III ii 2) In Locke's theory objects in the world are signified by ideas in the mind, and words signify these ideas. For Locke an idea is an "object of the understanding", and this means an object of consciousness or apprehension, rather than any special kind of mental activity.

Furthermore, words (in Locke's theory) can only signify ideas, so as a person speaks (or hears) there is a parallel mental process where the ideas signified by the words run through the user's consciousness. In a similar way, a person understands a sentence uttered by someone else because the words he hears produce a stream of ideas in his mind which are immediately (directly) perceived.

Such a theory certainly has difficulties; in particular if words only signify mental ideas then the question of exactly how words connect with the world is naturally posed.

But these difficulties need not concern us here, for Wittgenstein rejects the whole approach that distinguishes inner mental phenomena and outer physical phenomena: regarding the former as primary, and explaining the latter in terms of the former. The main reason for this rejection is that it uses the ego as the basic reference point, without first identifying it. The problem is to situate the ego.

Wittgenstein argues that an attempt to situate the ego in a particular body leads to solipsism; an attempt to 'spread' it among bodies collectively leads to idealism; and either way the egocentric approach refutes itself for it has to introduce physical bodies before it can get started. That is to say the egocentric approach which attempts to begin with the inner, the ego; cannot succeed for it has to introduce the outer, the physical, to individuate the ego! (See NLPESD p281)

Similarly Wittgenstein thought it incoherent to begin by identifying inner mental sensations, then to put a boundary around them, and argue that sensations apparently beyond the boundary must be reducible to sensations within the boundary.

For in doing this we have gone beyond experience. It is only possible to draw a boundary from inside, and then it is incoherent to talk of what is "outside" or beyond the boundary.

Ignoring these considerations leads to the use of an improperly individuated or unfixed ego as a reference point, which Pears calls "sliding-peg egocentricism." (Pears 1988 p233) (cf. fixed peg and sliding peg monetary exchange rates.) But if the ego is not pegged down, what justification is there for calling it mine? And without an individuated fixed ego, how can the Cartesian approach be adopted?

Also the private language argument is supposed to show

that even if the difficulties of the unindividuated ego could be resolved, without any external reference, language would be impossible. That is the necessary private language would be impossible; and mental sensations would be indistinguishable. A progression from the inner to the outer would be impossible.

Wittgenstein is more celebrated for his attack on the subjectivist theory of meaning. That is an attack on the theory that any verbal utterance must be accompanied by parallel mental activity, if it is to have meaning, or be understood. Typically Wittgenstein's attack is not a closely argued or formal one, but rather he makes a variety of separate points: "a wide field of thoughts, criss-cross in every direction." (PI p vii)

At the beginning of the Blue Book there is a discussion of what a person does in obeying the order; pick a red flower. On Locke's account a person compares the colours of flowers with his mental idea of red, until he finds one that matches, and then he picks it. But Wittgenstein says this cannot be correct, for how then could a person obey the analogous order: imagine a red patch? "You are not tempted in this case to think that before obeying you must have imagined a red patch to serve you as a pattern for the red patch which you were ordered to imagine." (BB p 3) Hence our understanding of the word "red" is not derived from having an inner patch of red to serve as a standard.

Later in the Blue Book (p 42) we are invited to perform the experiment: say "it is hot in this room" and mean "it is cold in this room". Can it be done? Wittgenstein suggests a negative reply, but clearly if we are being ironic we can say "hot" and mean "cold". Similarly we can imagine a group of people adopting a convention to transpose hot and cold in their speech, thus meaning "hot" by "cold" and vice-versa. However in both these cases meaning is not afforded by mental activity (a parallel flow of signifying ideas) but by the conventions of the language users. An ironic tone or the transposition convention gives the words their meaning.

Thirdly distinguish between the grammatical use of words such as pain, excitement, anxiousness and meaning and understanding which are all allegedly mental states, in the subjectivist theory. The first group of words can have temporal duration and degrees of intensity. It is perfectly reasonable to say the pain began at noon and went at 3pm, and that a particular mental state was present in between times, and not afterwards. But meaning and understanding are not like this, for I know what the word red means when I am not using the word "red", when I have no mental image of "red" and even when I am asleep! The remark "When do you know how to play chess? All the time? Or just when you are making a move?" (PI p 59(b)) illuminates the same point, and shows the error of attaching temporal predicates to pain and understanding (etc.) in the same way.

Similarly I can speak of an intense pain or a slight fear, but I cannot speak of slight meaning or intense understanding in the same way. The subjectivist account seems mistaken, and is rejected by Wittgenstein. Note however that this rejection does not entail that mental activity never accompanies meaning and understanding, but that it need not. "This, of course, does not mean that we have shown that peculiar acts of consciousness do not accompany the expressions of our thoughts! Only we no longer say that they must accompany them." (BB p 42) Meaning and understanding do not consist in a flow of ideas parallel to a flow of words. What then?

Firstly a distinction needs to be drawn between occurrent and dispositional mental states which is overlooked by those who regard meaning and understanding as parallel accompaniments to linguistic behaviour. Occurrent mental states are those immediately before the mind, with a temporal beginning and end, and varying intensity in between. Again toothache, or the sensation "seeing red" are good examples.

Dispositional mental states are more akin to ability or capacity such as understanding. Wittgenstein rejects the theory that meaning and understanding are occurrent mental states, but he does not simply substitute a dispositional account for this.

He stresses that meaning and understanding are family resemblance terms. Broadly, a person utters meaningfully,

and understands a language if he can use that language. That is, if he can make appropriate responses at appropriate times, make remarks in suitable context, paraphrase etc.. In general he must have mastered the technique of using the language: he must have grasped the rules of syntax and grammar, and be able to fully engage in conversation.

This mastery is an ability or capacity. An individual isolated act of understanding is not understanding unless it is an instance of a general ability. "To understand a sentence is to understand a language, and to understand a language means to be master of a technique." (P 199)

However the ability that Wittgenstein talks of is not an underlying mental state. To ascribe an ability to a person is to make a counterfactual statement about their behaviour.

In other words my ability to communicate in language L is grounded in the counterfactual that, if I am required to use language L I can so do.

Now whereas dispositions to act entail counterfactuals about behaviour, counterfactuals about behaviour do not entail dispositions to act, or the presence of any particular mental state. The actual mental states that a user of language has at any particular time, is for Wittgenstein, a scientific question not a philosophical one.

Now to re-focus on the philosophy of mathematics| in rejecting the egocentric approach, Wittgenstein stresses that both language and mathematics are communal activities.

Language is sustained by the way it is used, evolves according to use and is not simply a verbal expression of mental activity. Mathematics likewise is not a reflection of eternal truths stored in a Platonic heaven, nor is it a report of mental intuition. Rather it is a set of grammatical conventions adopted by the community on account of their application. The focus has been radically changed, and rather than invoke fanciful theories (philosophies) to explain our language and mathematics, Wittgenstein says we must simply see them as they are, for what they are. Describe them but not explain them for they have no need of the types of explanation we are prone to offer, indeed they have no need of explanation at all!

There is also a close connection between the Lockean theory of words standing for ideas, and the intuitionist theory of mathematical statements as reporting mental constructions, and as we have seen Wittgenstein rejects them both. Just as a word does not rely on a mental idea for its meaning, so neither does a mathematical statement rely on the possibility of effecting a particular mental construction. As words derive meaning from their use in a particular language game, so mathematical statements have meaning in virtue of adopted grammatical conventions and rules.

Similarly a proof progresses not on account of some mental or logical compulsion which obliges certain steps, but because a choice has been made to adopt a certain rule and

incorporate it into the system. Once this is done, there is then a counterfactual about my behaviour, that I will follow the rule as circumstances dictate, and my capacity to prove a proposition is analogous to my capacity to engage in language.

This is closely connected to Wittgenstein's remarks on the excluded middle, in the case of expanding infinite series.

For in the infinite case there is an uncertainty about what circumstances will dictate, which is not present in the finite case; so in the infinite case it is not clear how a rule should be interpreted and applied. "The totality of my dispositions is finite, being the dispositions of a finite being, that exists for a finite time" (Boghossian p509) so it is impossible to say how I might behave in circumstances not accounted for by the present set of dispositions about my behaviour. Whether I would ever write 777 in the decimal expansion of π is a question not covered by my present set of dispositions, so the answer is neither yes, nor no.

Lastly Wittgenstein says mathematics is a motley just as meaning and understanding are a motley. Rather than any singular experience that is understanding, there are a whole range of experiences linked by family resemblance.

"Then has "understanding" two different meanings here? I would rather say that these kinds of use of "understanding" make up its meaning, make up my concept of understanding.

For I want to apply "understanding" to all this." (PI

532)

Similarly we find:

"I should like to say: mathematics is a motley of techniques of proof". (RFM II 48)

That is, there is no essential grounds to mathematics, no defining quality and in particular, no underlying character that is fundamental. The logicians thought logic was the underlying fundamental; but Wittgenstein saw this as a misunderstanding, for arithmetic or the calculus are just as fundamental, for each is an autonomous grammatical system. There is no special form that a mathematical proposition takes, or any special way in which it is verified.

Wittgenstein's complaints against the classical schools of mathematics are not only about particulars, but also about their lack of scope: their limiting of mathematical procedures. The widest variety of techniques and grammatical conventions go to make up mathematics, and this is why Wittgenstein presents no anti-realist system. All he says is that mathematics is a family of procedures, in just the same way as the Lockean theory of meaning is replaced by no more than the assertion that meaning and understanding are family resemblance terms.

To conclude, I return to my original suggestion that Wittgenstein's rejection of subjectivist egocentrism, and his subsequent philosophy of mind is central to his whole philosophical outlook. The shift from the egocentric to the

communal, is paralleled by a shift from the theorizing and explaining obsession of science to the describing mode of Wittgenstein's later philosophy. "[philosophical problems] are solved not by giving new information, but by arranging what we have always known. Philosophy is a battle against the bewitchment of our intelligence by means of language". (PI 109)

Once this shift in approach and method has been made, many of the particular ideas in the philosophy of mathematics and elsewhere are no more than direct consequences of it. The anti-realist, community based approach, undercut traditional philosophy and it was Wittgenstein's intention to describe anew the activities whose "explanation" had been removed.

As the Platonic and subjectivist theories of language were replaced by the insistence that language is made up of a variety of procedures (language games) forming a family; so the realist schools were replaced by the insistence that mathematics is a family of procedures, none of which is most basic, in any relevant sense. These varieties of procedures in both language and mathematics are derived from human communal activity and agreement, and not any independent reality that various theorists have proposed.

This again is true in Wittgenstein's discussion of rules. The egocentric fixed rails, that lead us to follow rules are a fantasy. There are no rails, formulae, images in the mind

that guide behaviour in an unambiguous way, for the essential part of rule following is that it is communal. I cannot obey a rule privately.

Similarly an inner questioning about whether I am in pain, or whether I am following a rule correctly is nonsensical. Without an external (communal) reference against which I can confirm my own experiences; a questioning of these experiences is senseless. This consideration is at the core of the private language argument.

Lastly it is the inherently communal aspect of rule following that necessitates a sharp distinction between finite and non-finite mathematics. There is no justification for picturing infinite cases in the same way as finite cases, and the temptation to do so must be purged.

IX Conclusion: The Reception of Wittgenstein's Work.

When Wittgenstein's Remarks on the Foundations of Mathematics were published in 1956, they were not met with any great enthusiasm. In fact even those sympathetic towards the general line of thought, criticised the Remarks as muddled, confused, mathematically unsound and at best insignificant.

Alan Anderson claimed that for Wittgenstein mathematics means calculating procedures, and that "nothing Wittgenstein says would lead one to guess that abstract algebra, and the theory of games are part of mathematics." (Anderson p 482)

Further he says: "it is very doubtful whether this application of his method in the foundations of mathematics, will contribute substantially to his reputation as a philosopher." (ibid. p 490)

Kreisel criticises Wittgenstein for having a preoccupation with elementary mathematics, and for avoiding the difficult and genuinely interesting questions in foundational studies. He judges that the Remarks are "the surprisingly insignificant product of a sparkling mind." (Kreisel p 158)

Bernays makes similar objections and complains that "Wittgenstein argues as though mathematics existed almost solely for the purposes of housekeeping." (Bernays p 522) Also he criticises Wittgenstein for his non-realist approach and complains of a tendency to "dispute away the proper role of thinking - reflective intending in a behaviouristic

manner." (ibid. p 511) The conventionalist account that Bernays claims we are left with, cannot account for the stability of mathematics, cannot "in any way explain why these conceptual edifices are not continually collapsing." (ibid. p 527)

The criticisms here fall into two categories. Firstly that Wittgenstein is too simplistic, over-concerned with elementary mathematics, and unable to deal competently with the results of higher mathematics (eg Cantor's theorem, Godel's theorem.) Secondly that Wittgenstein's rejection of any objective mathematical reality leads to an unbounded laissez-faire, anything-goes attitude, which cannot account for the stability of mathematics. I shall consider each of these points in turn, but before doing so I think it is important to realize we are not dealing with Wittgenstein's definitive manuscript. The Remarks were not written as a book and were never intended for publication. They are a selection from five separate notebooks in which Wittgenstein recorded thoughts as they occurred to him, and hence it is not appropriate to criticise the Remarks as though they form a closely argued treatise. Some passages are inconclusive, some contradict others and some may even contain errors: but this is of little consequence if the work is properly regarded. I think the severity of some of Wittgenstein's critic's remarks arise, in part at least, from a failure to appreciate this.

The criticism of Wittgenstein based on the alleged simplicity of his account has three main strands. Firstly Wittgenstein discusses simple mathematics and gives elementary mathematical examples in the greater part of his work. He stresses the application and practical side of mathematics and seems hostile towards results of higher mathematics. The second strand extends this criticism to claiming that Wittgenstein's discussions of Gödel's theorem, Cantor's theorem and Dedekind's proof are muddled; and indicate that Wittgenstein simply did not understand them, or was unable to deal with them mathematically.

Lastly, on the point of simplicity, Wittgenstein has been labelled as a strict finitist, and Kielkopf has cited a passage from the Blue Book (BB p 20) in defence of his claim.

This last strand has been discussed and rebutted in my chapter on strict finitism, where I argued that a selective reading of Wittgenstein leads to such an identification; but that a wider reading of Wittgenstein indicates that this is an over-simplification, and to label Wittgenstein as a strict finitist is a mistake.

Against the other strands of the simplicity criticism, several points can be made. Firstly, Wittgenstein is not doing mathematics, he insists he is not looking for new results, but trying to clarify mathematics as it is now. His intent is to question the philosophical significance of certain results, rather than the results themselves; and to

eliminate philosophical confusion that sometimes accompanies the results. In as much as this can be done by considering simple examples, Wittgenstein concentrates on these; as complex mathematics would only confuse issues.

The same is true when Wittgenstein considers results of higher mathematics. He does not question mathematically, or have any intention of some (mathematical) refutation; but he questions the inferences drawn from the mathematics. Once this is understood it is easy to see why he does not dwell on mathematical technicalities, and to see that some of the criticisms of Wittgenstein's lack of mathematical ability are out of place.

The best examples of this are Wittgenstein's largely misunderstood remarks on Gödel's theorem, which I have considered already, in chapter three. A very similar type of example is found by considering Wittgenstein's remarks on Cantor's theorem.

Traditionally interpreted Cantor's diagonal procedure shows that the set of real numbers cannot be put into one-one correspondence with the set of rational numbers, and further that the set of real numbers is "bigger", ie. strictly has a higher cardinality, thus generating a whole family of sets of higher orders of infinity.

Wittgenstein rejects this interpretation as it regards the real numbers as objects, rather than as Wittgenstein claims, an unending technique. "The concepts of infinite

decimals in mathematical propositions not concepts of series, but of the unlimited technique of expanding series. We learn an endless technique..." (RFM IV 19). It is important to realise that Wittgenstein does not object to the concept of real numbers, or infinitely proceeding decimals so long as they are seen for what they are; and not regarded as completed entities or objects.

Wittgenstein also objects to the supposition that the real numbers can be ordered. The diagonal procedure invites an illegitimate picture of a real number as an endless row reaching into the far distance. Whereas it is legitimate to associate the picture $////$ with the numeral 4, this picture cannot be simply extended to the infinite case. "Is it really necessary here to conjure up a picture of the infinite (of the enormously big)? And how is the picture connected with the calculus? For its connection is not that of the picture $////$ with 4" (RFM A II 17)

This point again suggests that Wittgenstein does not think Cantor's proof should be discarded, nor as some have claimed, does he discount the infinite in mathematics; but he insists that the meaning of such terms be properly understood.

We must realise we are dealing with concepts and techniques of calculation, not objects - not completed entities that can be pictured as an endless row of numerals.

The notion of infinity is introduced into mathematics, not

derived from it. "Ought the word "infinite" to be avoided in mathematics? Yes; where it appears to confer a meaning upon the calculus, instead of getting one from it." (RFM A II 17)

It is perfectly legitimate to introduce, use and employ new techniques and concepts so long as they are seen as an invention, or as an extension of mathematics, and not a mathematical discovery. "Such employment is not: yet to be discovered, but: yet to be invented." (RFM A II 9)

Yet there is an objection to this reading of Wittgenstein who constantly stresses the application of mathematics; and his question: "What can the concept "non-denumerable" be used for?" (RFM A II 2) casts doubt on the legitimacy of the concept of the infinite.

Similarly, we can adopt the inequality $2^{x_0} > x_0$, "but what if we do say it, what are we to do next, in what practice is the proposition anchored?" (RFM A II 8) As we might say 10^{10} souls fit into a cubic centimetre, but in fact we do not say it "because it is of no use" (ibid) Is it then to be inferred that Wittgenstein wants to limit mathematics to the applied and the finite, as some have suggested; and to reject pure mathematics.

I think not, for Wittgenstein's conception of application is quite broad. He thinks of application within the relevant mathematical context rather than within the world. "Is the question not really: what can this number be used for? True, that sounds queer. - But what it means is: what are

its mathematical surroundings?" (RFM AII 1) Similarly he suggests that so long as the more abstract areas of mathematics are connected to those with practical application, they can be called mathematics by extension. "Don't we call it "mathematics" only because e.g. there are transitions, bridges from the fanciful to non-fanciful applications?" (RFM V 25) And even if a new construction has no obvious practical application, it is not unreasonable to suppose one might turn up later. "May I not complete the construction of the form... and as it were prepare a form of language for possible employment." (RFM IV 40) Again, mathematics is a motley, and no strict boundary can be drawn around it.

A last defence of Wittgenstein against the simplicity criticism, is that his foundational interests are the very basics of mathematical inference, of procedures governed by rules and a rejection of any referential theory. These points can all be discussed by reference to the simplest examples; and indeed it is appropriate that Wittgenstein focuses his attention on the most elementary level where mathematics begins. This is not to say he is hostile toward advanced mathematics, but that his concern is with the philosophical not the mathematical.

The second major criticism made against Wittgenstein, principally by Dummett and Bernays, is his alleged extreme constructivism, and his rejection of the objectivity of mathematics. These two points, though distinct are

intimately related; and as I shall suggest, both derive from a similar misunderstanding of Wittgenstein.

Dummett understands Wittgenstein as suggesting that we are not compelled by any rules, and that we are free to choose which statements to include in, and exclude from our mathematics. Dummett ascribes to Wittgenstein the view that at each stage of a proof "we are making a new decision," "at each step we are free to choose to accept or reject the proof" and so "there is nothing which forces us to accept the proof." (Dummett 1959.)

Now this is completely at odds with Wittgenstein's view that proof compels a certain conclusion. "A proof leads me to say this must be like this." (RFM V 30); and it is precisely this that distinguishes mathematics from empirical sciences. (cf my section v (iv)). Dummett has misunderstood Wittgenstein, who far from claiming that we may infer whatever we wish, is claiming that there is no external mathematical reality or logical compulsion that dictates our mathematical procedures.

Dummett has confused Wittgenstein's rejection of one account of logical compulsion, with the rejection of compulsion itself. for Wittgenstein we are compelled by mathematics in virtue of the grammatical conventions we adopt, and more widely by our application of mathematics and form of life.

Similarly Bernays attacks Wittgenstein's account, as he

thinks it denies objectivity in mathematics by denying the existence of mathematical facts, about mathematical entities. He says Wittgenstein's theory "cannot in any way explain why these conceptual edifices are not continually collapsing". (Bernays p 527) Again there is confusion between Wittgenstein's rejection of a particular account of objectivity, and the rejection of objectivity itself.

Certainly Wittgenstein rejects the idea that mathematical propositions are about objective mathematical entities, or that there is any external compulsion governing our mathematical behaviour; but he does say that our behaviour and the way we actually do mathematics does compel us to give an unique, and objectively correct answer to a given mathematical problem. Objectivity is grounded in "a technique which is a fact of natural history." (RFM V 13)

But there are two difficulties here; firstly it may well be asked why it is a fact of natural history that we all do mathematics in the same way? Surely there must be a deeper explanation? Wittgenstein has two responses to this. Either he accepts the question as genuine, but regards it as a question for empirical science and not philosophy. The answer must reside in genetic make-up, brain structure etc., and so is not a philosophical difficulty. Or he rejects the question as the facts of natural history - of our form of life - are simply given, and should not be questioned. "We must do away with all explanation, and description alone must take is

place". (PI 109)

A Platonist is not likely to be happy with either of these responses, and may ask why human regularities should not be explained in terms of mathematical objects. But such a suggestion is fraught with metaphysical and epistemological difficulties, and given some sympathy towards Wittgenstein's understanding of our form of life, his answer does not seem outrageous.

The second difficulty in defending Wittgenstein against the compulsion/objectivity type distinctions of Dummett and Bernays is that there seems to be an inconsistency in Wittgenstein, who emphasizes that in mathematics we are both free to choose, creative, conventionalist, but also forced to conclusions by objective reality. How can we be both compelled by a proof, and free to accept or reject it at any stage. Surely Wittgenstein can't have it both ways?

A possible answer to this is that paths in mathematics are not created from nothing, but are, dependent on existent paths. "It forms even new rules: is always building new roads for traffic; by extending the network of the old ones." (RFM I 165) And so our past decisions compel our present ones, for our behaviour must be consistent. But this is inadequate: for Wittgenstein stresses that in any proof we are free to make new links, and new concepts regardless of what has gone before. We are not bound to fixed meanings or concepts, and indeed a proof modifies our concepts! We are

still faced with the dilemma.

The solution for Wittgenstein is to grasp both horns of the dilemma. At any stage we are free to create new links and introduce new concepts; but once this has been done, and the results have been accepted and agreed to by the community, then they become binding and compel future application. The central point is agreement, and a focus on the first person plural rather than the first person singular. We may choose any practices we like, but once they have been accepted, our agreement compels us to do certain things in certain ways. That we actually do this explains the objectivity of our mathematics, and further explanation, if it be required, lies beyond the scope of philosophy.

So far I have tried to defend Wittgenstein against the two most common criticisms of his philosophy of mathematics, and have argued that both stem from confusion and misunderstanding. Crucially a confusion about Wittgenstein's distinction between philosophy and mathematics, and a misunderstanding about his account of objectivity; which I hope I have disentangled!

But there are still two further points to be discussed. Firstly the fact that in the philosophical literature, there has been no careful study of Wittgenstein's Lectures on the Foundations of Mathematics, or of the second part of the Philosophical Grammar, subtitled: "On Logic and Mathematics."

In fact there has been little attention paid to these works

at all. Secondly it is clear that Wittgenstein's philosophy of mathematics has not been adopted or developed, so much so that in a recent book: New Directions in the Philosophy of Mathematics which specifically aimed to move away from traditional foundational debates, Wittgenstein hardly gets a mention. In the light of these facts, how can Wittgenstein be seen as having made a significant contribution to the philosophy of mathematics?

Firstly the Lectures on the Foundations of Mathematics (1976) and the Philosophical Grammar (translated into English 1974) are recent works compared to the Remarks (1964), and it is possible that they have not yet had time to sink in, and become a full part of Wittgenstein's accepted opera. It is further possible that in the next few years these works will be studied in depth. (Crispin Wright's lengthy book on the Remarks, appeared fourteen years after the Remarks themselves)

Secondly, however, the likelihood of this seems rather reduced by the belief that the content of the Lectures and of the Grammar is not essentially different to the content of the Remarks, and so does not justify extensive study. In the case of the Lectures I think there is some truth in this, but I also think the passages in the Lectures which record conversations between Wittgenstein and his pupils, help to illuminate Wittgenstein's thoughts. Turing particularly seems to ask many questions which one might have asked oneself, and

the subsequent discussions are an important aid to understanding Wittgenstein.

In the case of the Philosophical Grammar, I think there is important material not to be found in the Remarks - particularly a detailed discussion of proof which amplifies the treatment of proof in the Remarks; and secondly a fuller discussion of the problems which surround real numbers and the infinite, which is not to be found elsewhere. Hence I feel that both the Lectures and the Grammar deserve greater attention than they currently receive.

Considering the lack of development and general adoption of Wittgenstein's ideas, again the point can be made that they are still young, and have not had time to be fully digested.

We are still at a largely exegetical stage, and hopes for new development are perhaps premature.

Further it is true that at the current time, much work is being done to understand, develop and criticise Wittgenstein's more mainstream concerns, as found in the Investigations. The recent Kripke/McGinn debate over Wittgenstein's discussion of rules has rightly provoked much attention, and it is perhaps too much to expect great interest in the philosophy of mathematics at the same time. Indeed the Investigations are both temporally and conceptually prior to the Remarks, and it is unlikely that a full understanding of the latter can be gained without a full understanding of the former.

Lastly it has to be said that the general tone of Wittgenstein's work is critical and negative, and as such it does not readily invite expansion or development. The rejection of mathematics as referential, stating facts about mathematical entities, whether these entities be Platonic (logicism), mental (intuitionism) or ink marks on paper (formalism) ; was the rejection of a very deeply held conviction, and leads to the virtual collapse of pre-Wittgenstein foundational studies. But this negative and critical philosophizing is still an achievement. "Where does our investigation get its importance from, since it seems only to destroy everything interesting, that is, all that is great and important? (As it were all the buildings, leaving behind only bits of stone and rubble.) What we are destroying is nothing but houses of cards, and we are clearing up the ground of language on which they stand." (PI 118)

And this is not all. Wittgenstein does offer not an explanation, but a description of how we in fact do mathematics, and how our procedures are grounded in human behaviour and the structures of the empirical world. Wittgenstein's account avoids the Platonist conception of classical mathematics with its huge ontological commitment to sets, sets of sets etc.. And it also avoids abandoning much mathematical practice, which was the cost of ontological restraint (cf. intuitionism) Wittgenstein effectively finds a middle way between the two.

Certainly there are difficulties with Wittgenstein's account, but as I hope I have shown; there are also many advantages, and many of Wittgenstein's critics have failed fully to grasp the points that Wittgenstein was trying to make. Admittedly Wittgenstein's writing are sometimes abstruse and vague, but as Frege said of his own unpublished writings: they are not all gold, but there is gold in them.

Bibliography of Works Consulted.

1. Works by, or originating from Wittgenstein.

Lectures on the Foundations of Mathematics, ed Cora Diamond.
Harvester Press 1975. (LFM).

Ludwig Wittgenstein and the Vienna Circle. Conversations
recorded by Waismann. tr. J. Schulte and B. McGuinness
Blackwell 1979. (WWK).

Notes for Lectures on "Private Experience" and "Sense Data"
ed. R. Rhees. Philosophical Review 77. (1968) pp. 275-
320 (NLPESD).

On Certainty. ed G.E.M. Anscombe and G.H. von Wright. tr.
GEM Anscombe and D. Paul. Blackwell 1969 (OC).

Philosophical Grammar. ed. R. Rhees. tr. A.J.P. Kenny.
Blackwell 1953 (PI).

Philosophical Remarks. ed R. Rhees. tr. A. Hargreaves and
R. White. Blackwell 1975. (PR).

Remarks on the Foundations of Mathematics. ed. G.H. von
Wright, R. Rhees and G.E.M. Anscombe. tr. G.E.M. Anscombe.
Blackwell 1956. (RFM).

The Blue and Brown Books . Blackwell 1958 (BB).

Tractatus Logico Philosophicus. tr. D.F. Pears and B.
McGuinness, Routledge 1961. (TLP).

2. Other Works of General Relevance.

A. ANDERSON "Mathematics and the Language Game."
1958 reprinted in Benacerraf and Putnam
p 482.

A. AYER. Language, Truth and Logic. Gollancz
1946.

G. BAKER (1) " Philosophical Grammar: Critical
and Notice" Mind 1976.
P.M.S. HACKER (2) Understanding and Meaning. Blackwell
1980.
(3) Scepticism, Rules and Language.
Blackwell 1984.

P. BENACERRAF Philosophy of Mathematics. Prentice Hall
and 1964.
H. PUTHMAM (eds).

- P. BERNAYS " Comments on Ludwig Wittgenstein's Remarks on the Foundations of Mathematics." 1959. Reprinted in Benacerraf and Putnam p522
- M. BLACK A Companion to Wittgenstein's Tractatus. Cambridge 1964
- P. BOGHOSSIAN "The Rule-Following Considerations" Mind (vol 98) Oct. 1989 pp507-549
- C.S. CHIHARA "Mathematical Discovery and Concept Formation." Philosophical Review (vol 72). 1963.
- G. CURRIE Frege: An Introduction to His Philosophy. Brighton Harvester Press 1982.
- M. DUMMETT . (1) Elements of Intuitionism . Oxford 1977
 (2) "Reckonings" Encounter March 1977.
 (3) Truth and Other Enigmas. Duckworth 1978.
 (4) "Wittgenstein's Philosophy of Mathematics". 1959 reprinted in Benacerraf and Putnam. p 491.
- H.L. FINCH Wittgenstein: The Later Philosophy. New York 1971.
- R.J. FOGELIN "Wittgenstein and Intuitionism." American Philosophical Quarterly 1968.
- P. FRASCOLLA "The Constructivist Model in Wittgenstein's Philosophy of Mathematics". Revisita Philosophia 1980.
- G. FREGE (1) The Philosophical Writings of Gottlob Frege. eds. P Geach and M. Black. Cambridge 1952.
 (2) The Foundations of Arithmetic. tr. J. Austin. Blackwell.
- A.C. GRAYLING Wittgenstein. Oxford 1988.
- J. HAACK Deviant Logic. Cambridge 1964.
- J.VAN HEIJENOORT From Frege to Gödel. A Sourcebook in Mathematical Logic 1879-1931

- Harvard 1967.
- A.J.P. KENNY Wittgenstein. Penguin 1973.
- C. KIELKOPF Strict Finitism. Mouton 1970.
- V.H. KLENK Wittgenstein's Philosophy of Mathematics.
The Hague 1976..
- G. KREISEL "Wittgenstein's Remarks on the Foundations
of Mathematics" British Journal for the
Philosophy of Science. 1958-9.
- S. KRIPKE Wittgenstein on Rules and Private Language
Blackwell 1982.
- C. MCGINN Wittgenstein on Meaning. Blackwell 1984.
- E. NAGEL and
J. NEWMAN Gödel's Proof. New York 1958.
- D.F. PEARS The False Prison. Vol I Oxford 1987.
Vol II Oxford 1988.
- F. RAMSEY The Foundations of Mathematics and Other
Logical Essays. London. 1931.
- M.D. RESNIK Frege and the Philosophy of Mathematics
Cornell (New York) 1980.
- B. RUSSELL (1) Our Knowledge of the External World.
Allen and Unwin 1922.
(2) Portraits from Memory. Allen and
Unwin 1957.
(3) Principia Mathematica (with
A.N.Whitehead) Cambridge Vol I (1912) Vol
II (1912) Vol III (1913)
- S.G. SHANKER
(ed.) Gödel's Proof in Focus. Croom Helm 1988.
- T. TYMOCZKO
(ed.) New Directions in the Philosophy of
Mathematics. Birkhauser, Boston 1986.
- M. WRIGLEY "Wittgenstein's Philosophy of
Mathematics." Philosophical Quarterly
1977.
- C. WRIGHT (1) Frege's Conception of Numbers as
Objects. Aberdeen 1983.
(2) Wittgenstein on the Foundations of
Mathematics. Duckworth 1980.

ADDITIONAL BIBLIOGRAPHIC MATERIAL

- H. N. CASTANEDA "Arithmetic and Reality" Australasian Journal of Philosophy vol. 37 no. 2 1959 pp92-107
- A. CHURCH "Review of Schmidt" Journal of Symbolic Logic vol 17 1952 pp198-199
- G. FREGE Die Grundgesetze der Arithmetik excerpted and translated by M. Furth as The Basic Laws of Arithmetic Berkeley 1964
- D. A. T. GASKING "Mathematics and the World" Australasian Journal of Psychology and Philosophy vol. 18 no. 2 1940 pp97-116 Reprinted in Benacceraf and Putnam pp390-403
- A. HEYTING Intuitionism : an Introduction 3rd Revised Edition Amsterdam : North Holland Publishing Co.. 1971
- D. HILBERT (1) "Mathematische Probleme Vortrag gehalten auf dem Internationalen-Kongress zu Paris" 1900 Nachrichten von der Koniglichen Gesellschaft zu Gottingen. pp253-297
(2) "Uber die Grundlagen der Logik und der Arithmetik" 1904 Reprinted in Benacceraf and Putnam pp129-138
(3) "On the Infinite" 1925 Reprinted in van Heijenoort pp367-492
- N. MALCOLM Ludwig Wittgenstein : A Memoir O.U.P. 1958
- B. MCGUINESS Wittgenstein : A Life vol. 1 Duckworth 1988
- K. MENGER "Der Intuitionismus" Blatter Deutsch Philosophie vol 4 1930 pp311-325
- G. PEANO Arithmetices Principia novo methodo exposita Bocca Turin 1889
- F. RAMSEY "Mathematical Logic" in the Mathematical Gazette vol 13 no. 184 pp184-195 1926 Reprinted in The Foundations of Mathematics and other Logical Essays R.K.P. 1931
- B. RUSSELL The Principles of Mathematics Cambridge 1913

Errata

- P 7 last line: insert: (see Malcolm pp1-22 and Kenny pp1-18)
- P 16 line 12: insert: (Ramsey 1931 p65)
- P 19 line 11: reference should read: (Church p199)
- P 24 line 20: insert: (Heyting p1)
- P 25 line 17: insert: (Heyting p19)
- P 40 lines 2-4: should read: "No one shall be able to drive us from the paradise that Cantor has created for us" (Van Heijenoort p376)
- P 43 line 6: insert: (see "Die Widerspruchsfreiheit der Reinen Zahlentheorie" in Mathematische Annalen vol. 112 pp493-565)
- P 57 line 12: insert: (Dummett 1978 pp248-265)
- line 23: "An inductive argument" should read "A mathematical induction"
- P 75 lines 12-13: should read: the sign \rightarrow should be ...
- line 15: p23 should read p17
- P 77 line 4: insert: see footnote at the bottom.
insert: footnote, Gasking in his paper "Mathematics and the World" has argued that mathematical statements derive their meaning solely from convention and do not in any way depend on the nature of the World as Wittgenstein suggests. He gives examples of a builder measuring and tiling a room to support his view that "queer" mathematics which is based on "queer" conventions is indeed plausible. His examples seem rather thin and unconvincing to me and similarly Castaneda has argued in his paper "Arithmetic and Reality" that Gasking has traded on the isolation of counting from measuring of physical objects, which cannot be done. Our mathematical conventions must go hand in hand with our physical world view.
- P 78 line 2: here should read: there
- P117 line 22: pints should read: points
- P126 line 16: fourteen should read: twenty-four
- P130 line 3: should read: Wittgenstein's Lectures on the Foundations of Mathematics. ed. Cora Diamond. Harvester Press 1976
- line 14: (PI) should read: (PG)
- line 30: Mind 1976 should read: Mind 1976 vol. 85 no. 538 pp269-294
- last line: insert: (First Edition)
- P131 line 11: insert: pp17-34
- line 16: insert: vol. 50 no. 3 pp63-68
- line 25: insert: vol. 5 pp267-274
- line 28: Revisita Philosophia 1980 should read: Revisita di Filosofia 1980 vol. 71 pp297-306
- line 33: insert: 1953
- line 35: J HAACK should read: S HAACK
- P132 line 8: insert: vol. 9 no. 106 pp50-59
- lines 23-35: should read: (3) Principia Mathematica with A.N. WHITEHEAD. 3 vols. Cambridge C.U.P 1910-1913
- line 31: insert: vol. 27 no. 106 pp50-59

