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ABSTRACT

A complete theoretical overview of the acoustics of the kettledrum is taken. Starting with the well understood vibrations of a stretched circular membrane in a vacuum, the thesis goes on to investigate the deformation produced when the drum is struck and the subsequent behaviour of the system with time. The vibration of the membrane is modified by mass loading, by the hemispherical kettle, by the membrane's stiffness and by the air loading present; the latter is accounted for using a Green function technique. These effects produce significant changes to the frequencies of a freely vibrating system. The results are compared with those obtained experimentally, with good agreement. The most significant conclusions are that the air loading modifies the frequencies of vibration of those modes having only radial nodes to match the harmonic series, thereby creating a distinct sensation of musical pitch, and that the fundamental mode of vibration is damped completely.

ACOUSTICS OF THE ORCHESTRAL KETTLEDRUM

Thomas Arnold Packer

Master of Science Thesis, University of Durham

1993

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DECLARATION

None of the material contained in this Thesis has been submitted for any degree in this nor any other University.

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T A Packer

April 1993

To

Janet

without whose support

this thesis would have not been possible

Introduction

This Thesis is concerned with the acoustical properties of the orchestral kettledrum. The usual kettledrum, often referred to by its Italian name "timpano" (plural "timpani") in musical scores, consists of a hemispherical bowl of fibreglass or copper over whose open part a membrane is stretched, normally of calfskin or a plastic¹ material. By means of screw attachments or a foot pedal mechanism the tension of this membrane can be altered. The drums are available in several standard sizes, the most common being a pair of diameters seventy-one centimetres and sixty-four centimetres (28 and 25 in). Others are used in more ambitious scores with diameters of seventy-six, sixty-six and fifty-eight centimetres (30, 26 and 23 in) to supplement this usual pair. Sadie (1984) gives more details of the construction of these instruments.

On striking the drum, off centre, at about a quarter of the way between the rim and the centre, with a suitable stick, often having a soft, felt covered head, the instrument produces a musical note of identifiable pitch, the actual pitch depending on the size of the drum and the tension of the membrane, enabling the drums to contribute to the orchestral colour and harmony. The range of the usual pair is shown in figure I.1.

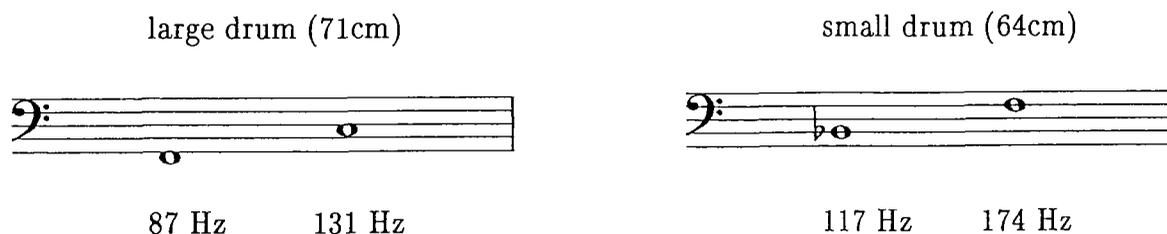


Figure I.1 Range of the orchestral kettledrums

¹ Mylar or ICI Melinex (Polyethene Terephthalate): see Appendix B.

The musical rôle, history and performing techniques of these instruments are adequately covered by Blades (1973a and 1975).

At the bottom of the hemispherical shell there is a small hole (of radius approximately one centimetre) to equalize the pressure inside and outside of the drum in the event of a change in atmospheric conditions. It is suspected by some authors that this produces a significant effect on the sound produced (eg Benade 1976a) whilst others maintain that it has little or no effect (eg Rossing 1982a). However it is apparent that the cavity itself, as well as contributing to the aesthetic appearance of the drum, affects the sound produced.

This thesis attempts to explain how the membrane will behave when struck and to investigate the effects of air damping, the bowl and the hole mentioned above to give a theoretical sound spectrum. Comparison with the harmonic series may then give an insight into the tone colour to be expected and to establish the nominal pitch of the drum, often the source of debate amongst musicians, eg Berlioz (1844).

Rayleigh (1945a) produced expressions for the behaviour of a stretched membrane; much of his theory still stands and has been utilised here. Certain misconceptions on his part led to an error in his treatment of a mass loaded membrane (Rayleigh 1945b) and this has been perpetuated by others (eg Morse 1948a). The correct treatment has been suggested by Kornhauser and Mintzer (1953a). Benade (1976b) has produced tables of frequencies based on experimental results, although he does not give any procedural details. More recent research by Christian et al (1984a), which concentrates especially on the problems associated with air damping, also gives tables of frequencies, based on both practical observations and theoretical calculations. Rossing (1982b) and Fletcher and Rossing (1991a) have produced tables showing the effect of the size of the bowl of the instrument.

In short, the mathematical treatment of the vibrations of a stretched membrane in a vacuum is well understood; the more specialised problems encountered with kettledrums are not so well documented. This thesis is intended to address these problems and to bring together the work of other authors on the subject.

Section 1 deals with the vibration of a stretched membrane not subject to any restrictions or damping forces. Section 2 introduces a suggested function to define the initial displacement that the membrane might suffer when struck with a typical drumstick.

Section 3 takes into account the effects of the cavity, assuming some kind of average displacement of the membrane; various treatments of this cavity problem are advanced, each with its own assumptions and conditions. Section 4 deals with the air damping of the system. In Section 5, consideration is given to other effects: the mass loading effect of the drum stick and the stiffness of the membrane. Section 6 deals with the time development of the system and radiation effects.

The whole thesis should be read bearing in mind the discussion of Section 7 which is based on observations made by performers and comments on the sensations of pitch, together with experimental results showing the frequencies of the important modes of vibration.

Section 1 Vibration of a stretched membrane in a vacuum

This section deals with the modes of vibration of a simple, circular membrane, not subject to any restraining forces and firmly fixed around its perimeter.

1.1 Derivation of the wave equation for a stretched membrane and of the normal modes of vibration².

Although polar co-ordinates will be used to specify points on the membrane's surface later, the initial wave equation is simpler to derive using cartesian co-ordinates, in which the displacement y is expressed as

$$y = y(x, z, t) \quad \text{where } x, z \text{ defines a point on the surface} \\ \text{and } t \text{ defines the instant in time.}$$

The membrane will be assumed to have negligible stiffness, to be perfectly elastic, to be of uniform thickness and to be subject to small displacements. If σ is the area density (in kg m^{-2}) and T the tension (in N m^{-1}) to which the membrane is stretched at its edge, then the net force acting on the element $dS (= dx.dz)$ due to the tension along the sides dz is

$$Tdz \left[\left[\frac{\partial y}{\partial x} \right]_{x+dx} - \left[\frac{\partial y}{\partial x} \right]_x \right]$$

which reduces to

$$T \left[\frac{\partial^2 y}{\partial x^2} \right] dx dz$$

² Similar calculations are to be found in Raleigh (1945a) and Kinsler and Frey (1962a).

and that due to the tension acting along dx is

$$T \left[\frac{\partial^2 y}{\partial z^2} \right] dx dz$$

(refer to fig 1.1)

(The tension will be distributed uniformly throughout the membrane, so that the material on opposite sides of a line segment dl will be pulled apart with a force $T \cdot dl$ Newtons).

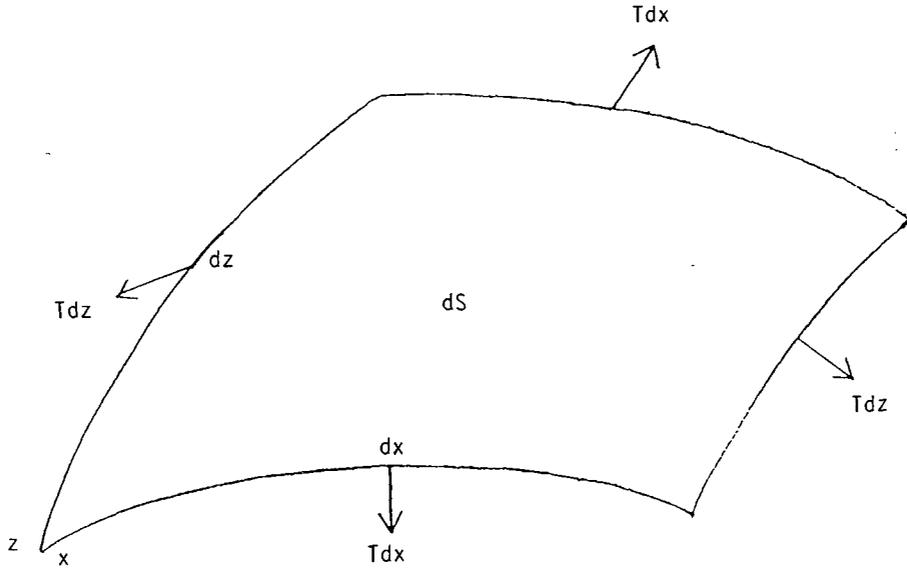


Figure 1.1 Element dS under tension T

This restoring force will equal the product of mass and acceleration, the mass being $\sigma dx dz$ and acceleration $\frac{d^2 y}{dt^2}$. Hence

$$T \left[\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2} \right] dx dz = \sigma dx dz \frac{d^2 y}{dt^2} \quad (1.1)$$

or

$$\frac{d^2 y}{dt^2} = c^2 \left[\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2} \right] \quad (1.2)$$

where
$$c = \sqrt{\frac{T}{\sigma}} \quad (1.3)$$

c in fact being the speed of propagation of transverse waves in the material.

Polar co-ordinates may be used with advantage when dealing with this circular membrane. Equation (1.2) may be expressed more generally as

$$\frac{d^2y}{dt^2} = c^2 \nabla^2 y \quad (1.4)$$

In the case of the circular membrane having a fixed boundary of radius a , it is necessary to express ∇^2 in terms of the polar co-ordinates, r and θ , where $x = r \cos\theta$ and $z = r \sin\theta$. By partial differentiation with respect to r and θ , the operator ∇^2 is found to be:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \quad (1.5)^3$$

Substitution into (1.4) gives

$$\frac{dy^2}{dt^2} = c^2 \left[\frac{\partial^2 y}{\partial r^2} + \frac{1}{r} \frac{\partial y}{\partial r} + \frac{1}{r^2} \frac{\partial^2 y}{\partial \theta^2} \right] \quad (1.6)$$

The boundary condition imposed on this equation is that the displacement must be zero at the circumference, ie $y = 0$ when $r = a$.

Assuming that y is a harmonic function of time (ie y is proportional to $\cos(\omega t - \epsilon)$) and writing $\frac{\omega}{c}$ as k

$$\frac{\partial^2 y}{\partial r^2} + \frac{1}{r} \frac{\partial y}{\partial r} + \frac{1}{r^2} \frac{\partial^2 y}{\partial \theta^2} + k^2 y = 0 \quad (1.7)$$

³ Similar calculations are to be found in Raleigh (1945a) and Kinsler and Frey (1962a).

Whatever y 's relationship with r and θ it can be expressed as a Fourier series

$$y = y_0 + y_1 \cos(\theta + \epsilon_1) + y_2 \cos^2(\theta + \epsilon_2) \dots \quad (1.8)$$

in which y_0, y_1 , etc are functions of r but not θ .

Substitution of (1.8) into (1.7) gives:

$$\sum \left[\frac{d^2 y_n}{dr^2} + \frac{1}{r} \frac{dy_n}{dr} + \left[k^2 - \frac{n^2}{r^2} \right] y_n \right] \cos n(\theta + \epsilon_n) = 0 \quad (1.9)$$

for all integral $n > 0$.

Multiplying both sides by $\cos n(\theta + \epsilon_n)$ and integrating with respect to θ between limits of 0 and 2π , each term must vanish separately, and thus it is possible to determine y_n as a function of r .

$$\frac{d^2 y_n}{dr^2} + \frac{1}{r} \frac{dy_n}{dr} + \left[k^2 - \frac{n^2}{r^2} \right] y_n = 0 \quad (1.10)^4$$

The solution of (1.10) involves two distinct functions of r each multiplied by an arbitrary constant. One of these functions becomes infinite when r becomes zero, and hence the corresponding particular function may be excluded as it does not satisfy the prescribed conditions⁵.

The other solution satisfying (1.10) is Bessel's function of order n (denoted by $J_n(kr)$) and may be expressed in several ways⁶.

⁴ It makes no difference whether $\cos n(\theta + \epsilon_n)$ is included in y_n or not.

⁵ Rayleigh (1945c) gives examples of this situation.

⁶ The properties of Bessel Functions are appended to this thesis, Appendix A.

In accordance with the usual notation for Bessel functions the expression for the vibration of a normal component may be written as:

$$y = P J_n(kr) \cos n(\theta + \epsilon) \cos (\omega t + \gamma) \quad (1.11)$$

the boundary condition requiring that

$$J_n(ka) = 0 \quad (1.12)$$

The complete solution for y is obtained by combining all the possible solutions embodied in (1.11) with all permissible values of n and k . As a function of r and θ , y can be expressed within the limits of the circle radius a in the series

$$y = \sum_i \sum_n J_n(k_{n,i}r) \phi_{n,i} \cos n\theta \quad (1.13)$$

where $(k_{n,i}a)$ is the i^{th} zero of $J_n(k_{n,i}a)$
 $n = 1, 2, 3$ etc.

For every integral value of n there is a series of values for k given by (1.12) and for each of them $\phi_{n,i}$ is arbitrary. The establishment of these constants can be achieved in the following manner. The energy of the motion is equal to

$$\frac{1}{2} \rho \int_0^a \int_0^{2\pi} \dot{y}^2 r dr d\theta$$

where ρ is the density of the membrane

and when expressed by means of normal co-ordinates it can only involve their squares, it follows that the product of any two terms in (1.13) becomes zero when integrated over the area of the membrane. Multiplying both sides of (1.13) by $J_n(k_{n,i}r) \cos(n\theta)r$ and integrating

$$\int_0^a \int_0^{2\pi} y J_n(k_{n,i}r) \text{Cos}(n\theta) r dr d\theta = \int_0^a \int_0^{2\pi} \phi_{n,i} [J_n(k_{n,i}r)]^2 \text{Cos}^2(n\theta) r dr d\theta \quad (1.14)$$

the right-hand side reduces to

$$\pi \phi_{n,i} \int_0^a [J_n(k_{n,i}r)]^2 r dr \quad (1.15)$$

the integral $\int_0^a [J_n(k_{n,i}r)]^2 r dr$ is found to be⁷ $\frac{a^2}{2} [J_{n+1}(k_{n,i}a)]^2$. Hence the values of the coefficients $\phi_{n,i}$ ($n = 1, 2, 3 \dots$ $i = 1, 2, 3 \dots$) can be determined from

$$\phi_{n,i} = \frac{\int_0^a \int_0^{2\pi} y J_n(k_{n,i}r) \text{Cos}(n\theta) r dr d\theta}{\frac{\pi a^2}{2} J_{n+1}^2(k_{n,i}a)} \quad (1.16)$$

The displacement of a membrane, then, can be expressed as the sum of a set of normal modes, whose amplitudes are easily determined.

1.2 Determination of the form and frequency of the normal modes⁸

Consideration of equation (1.11), rewritten here for a specific mode, (n, i) shows the character of each mode.

$$y_{n,i} = \phi_{n,i} J_n(k_{n,i}r) \text{Cos } n\theta \text{ Cos } \omega t \quad (1.17) \equiv (1.11)$$

⁷ See Appendix A

⁸ Similar calculations, extending to annular membranes are to be found in Gray and Matthews (1895a)

Each element of the membrane will exhibit simple harmonic motion of period $\frac{2\pi}{\omega}$.

The frequency of oscillation is therefore

$$f = \frac{\omega}{2\pi} = \frac{kc}{2\pi} = \frac{k_{n,i}}{2\pi} \sqrt{\frac{T}{\sigma}}, \quad (1.18)$$

see (1.3) & (1.7).

The amplitude of this element is

$$A = \phi_{n,i} J_n(k_{n,i}r) \cos n\theta \quad (1.19)$$

which vanishes, and the element accordingly remains at rest if

$$J_n(k_{n,i}r) = 0 \quad (1.19a)$$

or if

$$\cos n\theta = 0 \quad (1.19b)$$

The first is satisfied not only at $r = a$ but also when

$$r = \frac{k_{n,i}a}{k_{n,i}}, \quad r = \frac{k_{n,2a}}{k_{n,i}}, \quad r = \frac{k_{n,(i-1)a}}{k_{n,i}} \quad (1.20a)$$

Consequently there is a series of $(i-1)$ nodal circles, concentric with the fixed boundary.

The second equation (1.19b) is satisfied when

$$\theta = \frac{\pi}{2n}, \quad \theta = \frac{3\pi}{2n}, \quad \theta = \frac{(4n-1)\pi}{2n} \quad (1.20b)$$

giving rise to n nodal diameters, dividing the membrane into $2n$ equal sectors,

all vibrating in precisely the same way, alternate sectors moving in opposite directions. Figure (1.2) gives a representation of the motion experienced by such membrane. (Some excellent photographs, showing some of the normal modes, obtained by sprinkling sand on to a timpano are appended to Rossing (1982a)).

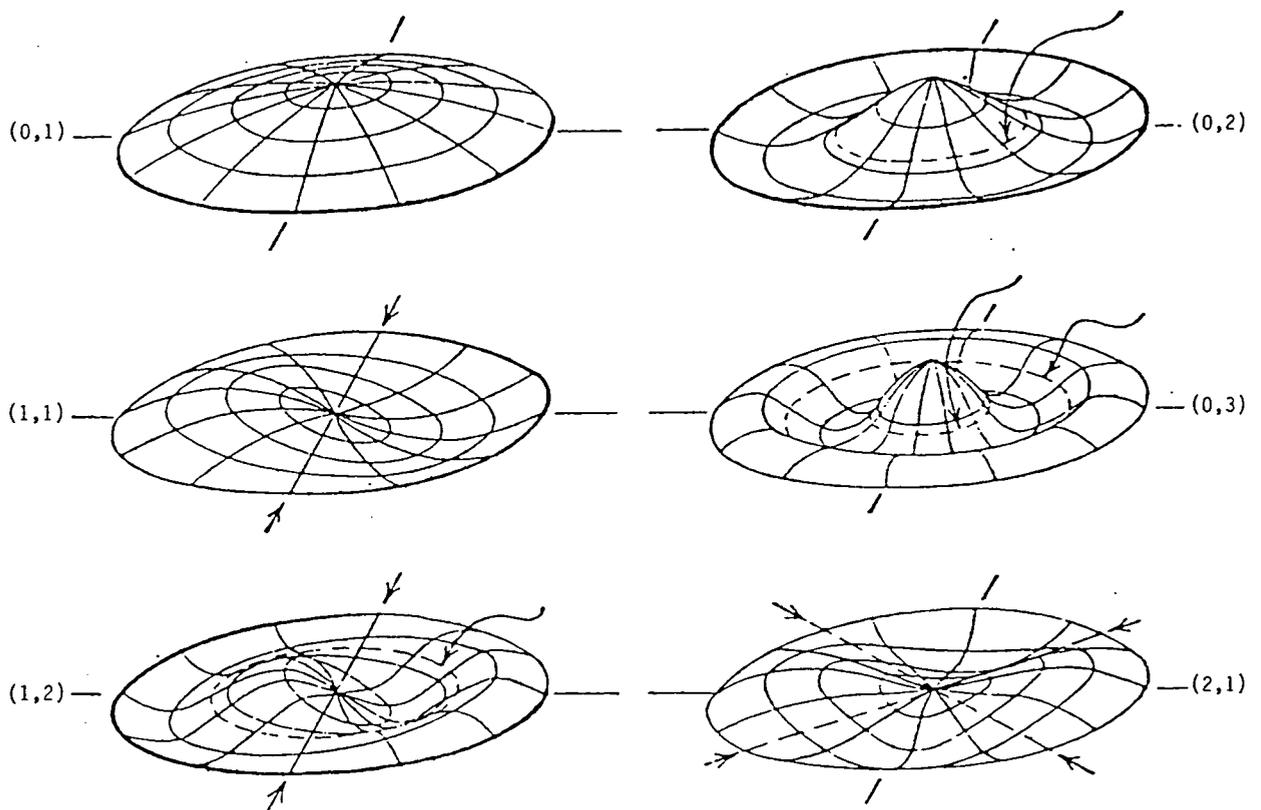


Figure 1.2. Shapes of some of the normal modes of vibration of a circular membrane (Arrows point to the nodal lines)

Having ascertained that the displacement of a membrane can be expressed in terms of its modes, it has been shown that it is then possible to determine the natural frequency of each of these modes and their associated shapes. The relative amplitudes of the modes will depend on the initial displacement of the membrane, dealt with in Section 2.

Section 2 Deformation of a membrane under a drumstick

This section deals with the shape of deformation caused by striking the drum at a point other than its centre.

2.1 The displacement at any point on the surface of a membrane due to a deformation at a specified position using the method of images.

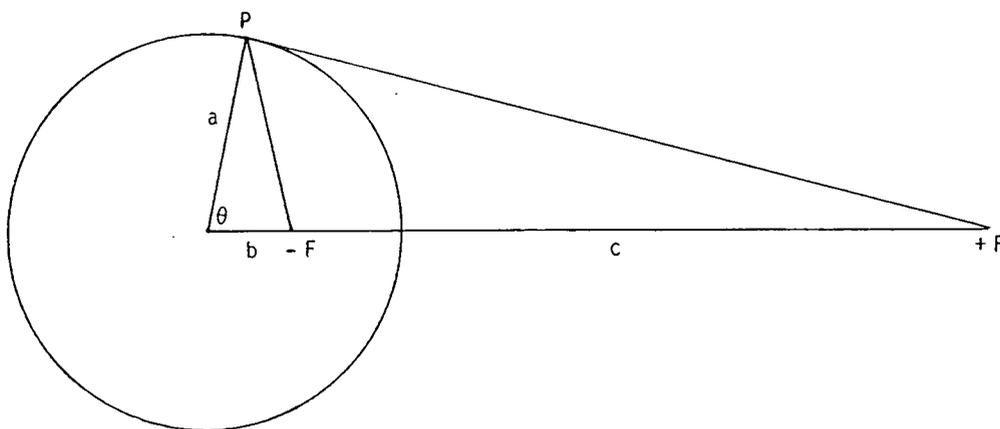


Figure 2.1 A membrane of radius a struck with a force at a distance b from the centre

For an infinite membrane, the pressure, $P(\mathbf{r})$, is given by:

$$\nabla^2 \psi(\mathbf{r}) = -P(\mathbf{r}) \tag{2.1}$$

where $\psi(\mathbf{r})$ is the displacement at \mathbf{r} due to $P(\mathbf{r})$.

For a point force F at \mathbf{b} :

$$\nabla^2 \psi(\mathbf{r}) = -F\delta(\mathbf{r} - \mathbf{b}) \tag{2.2}$$

the solution of which gives ψ as

$$\psi(r) = \frac{-F}{2\pi T} \ln(r - b) \quad (2.3)$$

where T is the tension of the membrane as defined in Section 1.1.

For a finite membrane, the method of images can be used to obtain an expression for the displacement, ψ , in terms of the initial force, $-F$. The method follows the usual pattern employed in electrostatics problems of this nature (Bleaney and Bleaney (1989)). The method of images enables the boundary condition at the perimeter to be met: an image force $+F$ is introduced outside the membrane at a distance c from the centre to yield zero displacement at the edge. Figure 2.1 shows this situation.

The displacement due to a point force $-F$ evaluated at the boundary point P is

$$\psi_- = \frac{F}{2\pi T} \ln(a^2 + b^2 - 2ab \cos \theta)^{\frac{1}{2}} + A_- \quad (2.4)$$

and that due to $+F$ at P is

$$\psi_+ = \frac{F}{2\pi T} \ln(a^2 + c^2 - 2ac \cos \theta)^{\frac{1}{2}} + A_+ \quad (2.5)$$

where A_- and A_+ are arbitrary constants.

The net displacement is therefore

$$\psi = \psi_+ + \psi_- = \frac{F}{2\pi T} \ln \left[\frac{a^2 + b^2 - 2ab \cos \theta}{a^2 + c^2 - 2ac \cos \theta} \right]^{\frac{1}{2}} + A_- + A_+ \quad (2.6)$$

The requirement is that P, being situated on the boundary, must have zero displacement. To achieve this, the ratio

$$\frac{a^2 + b^2 - 2ab \cos \theta}{a^2 + c^2 - 2ac \cos \theta}$$

must be independent of θ ; that is to say

$$\frac{a^2 + b^2 - 2ab \cos \theta}{a^2 + c^2 - 2ac \cos \theta} = k' \quad (2.7)$$

where k' is a constant.

Rearranging equation (2.7),

$$a^2 + b^2 - 2ab \cos \theta = k'(a^2 + c^2) - 2k'ac \cos \theta \quad (2.8)$$

or

$$a^2 + b^2 - k'(a^2 + c^2) + (2k'ac - 2ab)\cos \theta = 0 \quad (2.9)$$

for which the coefficient of $\cos \theta$ must itself be zero, i.e.

$$k' = \frac{b}{c} \quad (2.10)$$

then since

$$a^2 + b^2 - k'(a^2 + c^2) = 0 \quad (2.9a)$$

it follows that

$$a^2 = cb \quad (2.11)$$

hence

$$k' = \frac{b^2}{a^2} \quad (2.12)$$

the displacement, ψ at the boundary therefore follows as

$$\psi = \frac{F}{2\pi T} \ln \left[\frac{b}{a} \right] + A_+ + A_- \quad (2.13)$$

and choosing $A_+ + A_-$ as $\frac{-F}{2\pi T} \ln \frac{b}{a}$ gives zero at the boundary. The general displacement at a point (r, θ) will be

$$\psi(r, \theta) = \frac{F}{2\pi T} \ln \left[\left[\frac{r^2 + b^2 - 2br \cos \theta}{r^2 + c^2 - 2rc \cos \theta} \right]^{\frac{1}{2}} \frac{a}{b} \right] \quad (2.14)$$

$$\text{with } c = \frac{a^2}{b}.$$

It has been intimated by Morse (1948b) and Kornhauser and Mintzer (1953b) that a point force or one whose radius of contact is $\leq F/2\pi T$, would cause the membrane to break. Thus the expression obtained above is only valid if the force is taken over a finite area. Fortunately this is exactly what happens when the skin is struck with a typical drumstick. The shape of a common stick is shown in figure (2.2)

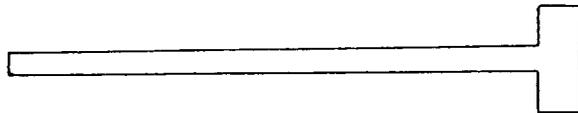


Figure 2.2 Usual form of a kettledrumstick

The displacement given in equation 2.14 is hence invalid for a point within the area of contact; the displacement for such a point will be equal to the initial displacement caused by the stick's blow. Outside the area of contact the displacement is given by (2.14); a discontinuity is introduced at the edge of the area of contact.

2.2 Deformation of a membrane deriving from an impulsive force

In the foregoing description of Section 2.1 it has been assumed that the membrane is released from a static position, from which it is released to perform the vibrations described in Section 1. The case of the stick producing an impulse rather than this static deformation will not be considered for intuitive reasons; the mass of a stick is considerably greater than the mass of a typical membrane (which, after all, has been taken as massless in the treatment of Section 1) and practical experience of playing these instruments which shows that the skin produces a definite upward force to the drumstick, which assists playing technique. This effect will give rise to a mass-loading of the membrane for its first half cycle. (Section 5).

Section 3. Considerations of the effect of the cavity upon the modes of vibration

This Section attempts to suggest ways in which the cavity behind the membrane in a real situation may adapt the behaviour of the drum.

3.1 Membrane coupled to a Helmholtz cavity.

Treating the kettle as a Helmholtz resonator, and examining the coupling between the membrane and this cavity produces expressions to amend the natural frequencies of Section 1. Damping can also be introduced; damping may occur due to the energy losses associated with the movement of air through the small hole at the bottom. Figure 3.1 shows such an arrangement. Opinions differ as to the importance of this hole in the overall sound spectrum (Benade (1976a), Rossing (1982a))

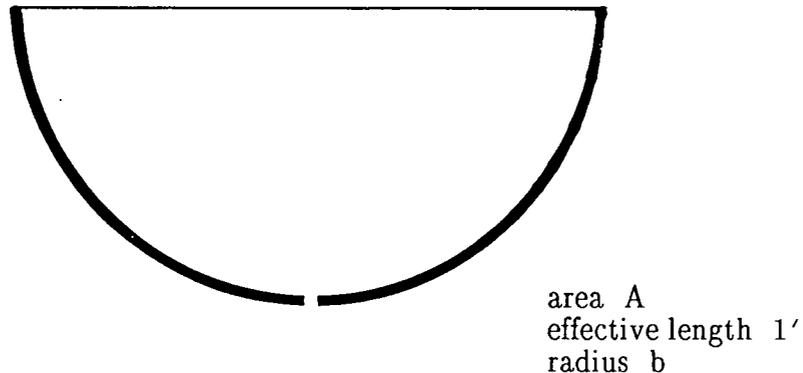


Figure 3.1 Cross Section of a kettledrum

The membrane's movement is denoted by y , and that of the air through the hole by x .

Rayleigh (1945d) gives the expression for the effective length of the air plug of a Helmholtz resonator in the case of a small circular aperture in a thin wall as

$$l' = \frac{A}{2b} \quad (3.1)$$

where A is the area of cross section of the hole,
 b is its radius,
 l' is the effective length.

Helmholtz (1912) and Wood (1953) give details of the principles involved in the resonance of such a cavity, such as the one under consideration here. The equation of motion of the air plug is

$$\rho_a A l' \frac{d^2x}{dt^2} + \rho_a A D \frac{dx}{dt} = -\rho_a \frac{c^2 A}{v} \left[A x - \int \psi ds \right] \quad (3.2)$$

where ρ_a = density of air
 x = displacement of air plug
 ψ = displacement of membrane
 D = damping constant
 c = velocity of sound in air.

The flow of air through the hole is damped by viscous forces acting at the opening; these are accounted for by the $D \frac{dx}{dt}$ term in (3.2).

Defining an average displacement, y for the membrane of radius a

$$y = \frac{1}{\pi a^2} \int \psi ds \quad (3.3)$$

and from (3.3) and (3.2)

$$\frac{d^2x}{dt^2} + D \frac{dx}{dt} + \frac{c^2 A x}{v l'} = \frac{c^2}{v l'} \pi a^2 y \quad (3.4)$$

the equation of motion of the membrane is

$$\rho_m h \frac{d^2 \psi}{dt^2} = -T \nabla^2 \psi + \frac{\rho_a c^2}{v} \left[Ax - \int \psi ds \right] \quad (3.5)$$

where h = thickness of membrane
 ρ_m = density of membrane

ψ can be expressed as the sum of the normal modes of the membrane (Section 1)

$$\psi = \sum_i a_i \phi_i \quad (3.6)$$

where ϕ_i is a mode
 a_i is the amplitude.

As an approximation

$$\int \psi ds = \sum_i a_i \int \phi_i ds \simeq a_0 \int \phi_0 ds \quad (3.7)$$

The last approximation deriving from the fact that the fundamental ϕ_0 is the only mode without nodes, (apart from the perimeter) hence only ϕ_0 is coupled to the resonator in this approximation. By considering the fundamental alone, and averaging the equation of motion over the area of the membrane gives:

$$\frac{d^2 y}{dt^2} + \left[\omega_m^2 + \frac{\rho_a c^2}{\rho_m} + \frac{\pi a^2}{v l^2} \right] y = \frac{\rho_a c^2 A x}{\rho_m v h} \quad (3.8)$$

where ω_m = frequency of fundamental.

The resonant frequency of the cavity, according to Helmholtz (1912) and Wood (1953) is

$$\frac{c^2 A}{v l^2} = \omega_c^2 \quad (3.9)$$

Two constants, α and β are defined, such that

$$\alpha = \frac{\pi a^2}{A} \quad \text{and} \quad \beta = \frac{\rho_a l'}{\rho_m h}$$

using these in (3.4) and (3.8) and trying solutions of the form $x = x_0 e^{i\omega t}$, $y = y_0 e^{i\omega t}$ the following identities are obtained:

$$-\omega^2 x_0 + i\omega D x_0 + \omega_c^2 x_0 = \alpha \omega_c^2 y_0 \quad (3.10)$$

$$-\omega^2 y_0 + (\omega_m^2 + \alpha\beta\omega_c^2) y_0 = \beta\omega_c^2 x_0. \quad (3.11)$$

For this solution to be non-trivial, the value of the determinant

$$\begin{vmatrix} \omega^2 - i\omega D - \omega_c^2 & \alpha\omega_c^2 \\ \beta\omega_c^2 & \omega^2 - (\omega_m^2 + \alpha\beta\omega_c^2) \end{vmatrix}$$

must equal zero, i.e.

$$(\omega^2 - i\omega D - \omega_c^2)(\omega^2 - (\omega_m^2 + \alpha\beta\omega_c^2)) - \alpha\beta\omega_c^4 = 0. \quad (3.12)$$

Its solution leads to the two normal mode frequencies.

For the lossless case ($D = 0$) an examination of equation (3.12) shows that one normal mode frequency will be greater than the larger of ω_c and $\sqrt{\omega_m^2 + \alpha\beta\omega_c^2}$, whilst the other will be less than the smaller of these.

The frequency of the fundamental will therefore be altered by a factor of $\left[1 + \frac{\alpha\beta\omega_c^2}{\omega_m^2}\right]^{\frac{1}{2}}$ if the damping factor is taken as zero. (The resonant frequency of the cavity is found to be about 25Hz.)

The justification for considering only the fundamental in the approximation of (3.7) was that any one of the modes with nodes will not produce a comparable volume change. Certainly all modes having diametrical nodes will produce no volume change whatsoever. This gives rise to the fact that the fundamental is not only the most seriously affected by the cavity in terms of a change in natural frequency, but must also be the most seriously affected by the damping effect of the air moving through the small hole.

3.2 Membrane coupled with a closed cavity.

In this treatment the kettle is considered as being an airtight cavity. This approximation will be better for the higher modes that involve a volume change that is small compared to the volume of the cavity.

If the speed of transverse waves in the membrane is considerably less than the speed of sound in air, then the compression and expansion of the enclosed air will be more or less the same over the whole extent of the membrane, and will depend on the average displacement of the membrane.

When the membrane is displaced from equilibrium to a shape determined by ψ , the volume of the vessel is diminished by

$$\Delta V = \int_0^a \int_0^{2\pi} \psi r dr d\theta \quad (3.13)$$

If the equilibrium volume is V_0 and density ρ_0 then when the alternations of pressure within the vessel are large enough to be adiabatic, the excess pressure inside the vessel will be

$$P = - \left[\frac{\rho_0 c_a^2}{V_0} \right] \int_0^a \int_0^{2\pi} \psi r dr d\theta \quad (3.14)$$

where c_a is the speed of sound in air.

(The minus sign originates from the fact that the pressure is in the opposite direction to the displacement of the membrane.)

The equation of motion of the membrane is, therefore

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = \nabla^2 \psi - \left[\frac{\rho_0 c_a^2}{V_0 T} \right] \int_0^a \int_0^{2\pi} \psi r dr d\theta \quad (3.15)$$

where c is the speed of transverse waves in the membrane.

For simple harmonic vibrations

$$\psi = Y(r, \theta) e^{-i\omega t}$$

$$\nabla^2 Y + \frac{\omega^2}{c^2} Y = \left[\frac{\rho_0 c_a^2}{V_0 T} \right] \int_0^a \int_0^{2\pi} Y r dr d\theta \quad (3.16)$$

If $Y = \text{Sin}(n\theta)J_n\left(\frac{\omega r}{c}\right)$, $n > 0$, the integral on the right-hand side will be zero (owing to the integration over θ), and the solution that satisfies the boundary conditions $Y = 0$ at $r = a$ will be the solutions derived in Section 1, with the corresponding allowed frequencies. The presence of the airtight cavity does not, then, according to this treatment, appear to affect the normal modes having one or more diametrical nodes (i.e. $n > 0$).

If n is zero then the integral on the right-hand side of (3.16) is not zero. The solution without the integral is $J_0\left(\frac{\omega r}{c}\right)$ so trying the solution $Y = J_0\left(\frac{\omega r}{c}\right) - J_0\left(\frac{\omega a}{c}\right)$ which satisfies the boundary condition, the integral reduces to

$$\left[\frac{\rho_0 c_a^2}{V_0 T}\right] \int_0^a \int_0^{2\pi} Y r dr d\theta = \left[\frac{\chi}{a^2}\right] \left[2 \left[\frac{c}{\omega a}\right] J_1\left[\frac{\omega a}{c}\right] - J_0\left[\frac{\omega a}{c}\right]\right] \quad (3.17)$$

$$\text{where } \chi = (\pi \rho_0^2 c_a^2 a^4 / V_0 T)$$

substituting this into equation (3.16) and using the properties of Bessel functions, (see Appendix A), gives:

$$J_0\left(\frac{\omega a}{c}\right) = \frac{\chi}{\left(\frac{\omega a}{c}\right)^2} \left[J_0\left(\frac{\omega a}{c}\right) - \frac{2c}{\omega a} J_1\left(\frac{\omega a}{c}\right)\right] = \frac{-\chi}{\omega^2 a^2} J_2\left(\frac{\omega a}{c}\right) . \quad (3.18)$$

This equation determines the allowed values of the frequency for those normal modes that are independent of θ (i.e. having no nodal diameters).

The constant χ is a measure of the relative importance of the cavity with respect to the tension, as a restoring force on the membrane. The limiting case $\chi = 0$ is

the one studied in Section 1. Morse (1948c) gives a table showing how the frequency varies with χ . The presence of the vessel tends to raise the frequencies. Sen (1990) has produced a similar expression for allowed frequencies; his result is identical to that of Morse (1948c).

Morse (1948d) gives an approximate formula for the frequency, valid for small values of χ as

$$f \simeq f_0 \left[1 + \frac{\rho_0 c_a^2 T}{8\pi^3 V_0 \sigma^2 (f_0)^4} \right] \quad (3.19)$$

where f_0 is the frequency
when χ is 0

σ is the area density
of the membrane.

Kinsler & Frey (1962a) in a similar treatment of this problem define a constant

$$\alpha = \frac{\pi a^4 \gamma P_0}{TV_0} \quad (3.20)$$

where P_0 is initial pressure
 γ is ratio of heat
capacities of air

which is again an expression for the importance of the cavity.

A similar expression is arrived at for ω

$$J_0 \left[\frac{\omega a}{c} \right] = -\alpha J_2 \left(\frac{\omega a}{c} \right) \frac{1}{\omega^2 a^2} c^2 \quad (3.21)$$

from which it is evident that χ and α are equivalent.

Table 3.1 shows the frequency modification factors for various values of χ and different modes, derived from (3.19).

χ	Frequency modification
0	1.0
0.5	1.041 86
1	1.082 11
2	1.158 40
3	1.229 97
4	1.297 60
5	1.361 87
6	1.423 25
8	1.538 68
10	1.646 03

Table 3.1 Frequency modification due to a Helmholtz cavity

χ	Frequency modification			
	(0,1)	(0,2)	(0,3)	(0,4)
0	1.0	1.0	1.0	1.0
0.5	1.029 39	1.001 080	1.000 145 2	1.000 026 6
1	1.037 74	1.002 220	1.000 326 7	1.000 079 9
2	1.111 69	1.004 553	1.000 726 1	1.000 186 5
3	1.162 51	1.007 000	1.001 089 1	1.000 293 1
4	1.210 32	1.009 561	1.001 488 4	1.000 399 6
5	1.254 60	1.012 293	1.001 887 1	1.000 506 2
6	1.295 10	1.015 139	1.002 287 1	1.000 612 8
8	1.364 67	1.021 285	1.003 122 1	1.000 825 9
10	1.450 16	1.028 115	1.004 029 5	1.001 103 9

Table 3.2 Frequency modification due to a closed cavity

3.3 The effect of the size of the cavity.

Experimental evidence shows that reducing the size of the cavity produces significant effects on the frequencies of each of the modes. Rossing (1982c) shows that reducing the volume of the cavity by a quarter produces little effect, whilst reducing the volume to a half, and then to only one-quarter of its original size produces a dramatic shift in frequency. The volume was altered by part filling the drum's cavity with water. Christian et al (1984a) give measured results which are reproduced in Section 7, together with a tentative theory as to the importance of the cavity volume⁹. Figure 3.2 shows the effect of altering the volume for some of the modes. (From Fletcher and Rossing (1991b).)

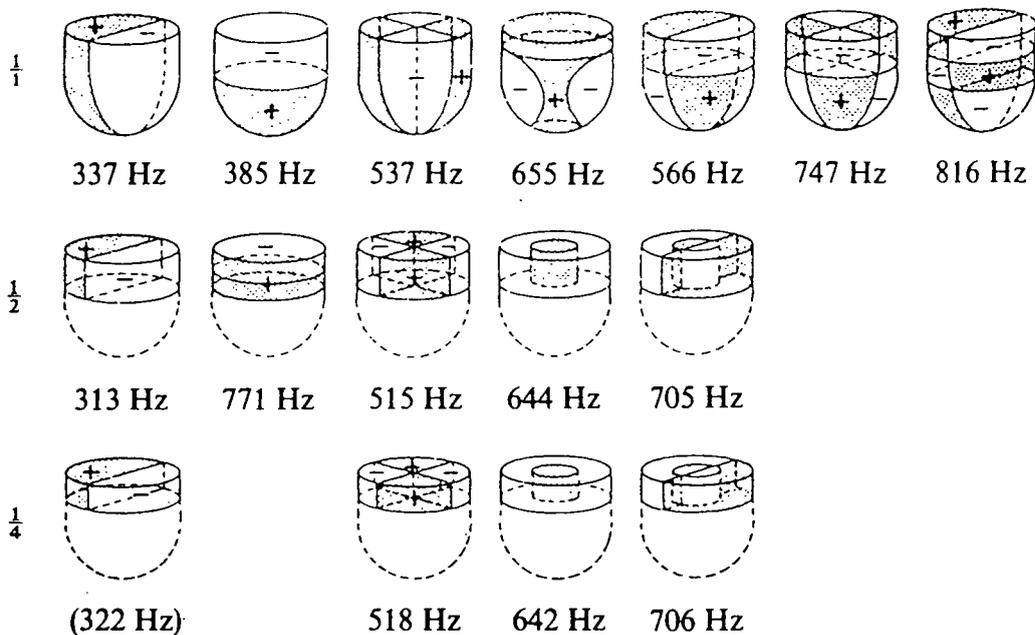


Figure 3.2 Normal modes for three different air volumes

⁹ See Section 4.2.

3.4 The effect of the kettle as a baffle.

The fact that the kettle only exists on one side of the membrane effectively prevents any sound from being radiated from the underside of the membrane. The $(0, 1)$ mode will therefore act as monopole source, radiating sound waves outwards in a hemispherical manner from the upper surface of the drum. Without the kettle, it would, of course act a dipole source. The other modes, however will act as multipole sources; the $(1, 1)$ mode for example will be a dipole source, the $(2, 1)$ mode a quadropole source. This action of the kettle has several implications. In the first place, the sound radiated from the underside of a membrane without the kettle as a baffle will be exactly half a cycle out of phase with that radiating from the top, thereby causing interference, and the sound would not be as efficiently radiated in the plane of the membrane. As it is, the kettle effectively cuts off the sound from the underside, thereby leaving the sound from the upper side to radiate well. Another acoustical characteristic is that sound from a monopole source radiates with great efficiency, which implies that the vibration giving rise to this monopole source will die away much more quickly than that from dipoles or quadropoles. This suggests that the fundamental $(0, 1)$ mode will not have a long time duration.

3.5 General considerations.

The fundamental frequency of the membrane from Section 1 is

$$\omega_m = \frac{2.404}{a} \sqrt{\frac{T}{\sigma}} \quad (3.22)$$

(2.404 is the first zero of J_0)

hence

$$\frac{1}{T} = \frac{1(0.77\pi)^2}{\omega_m^2 a^2 \sigma} \quad (3.23)$$

The parameter χ has been defined as

$$\chi = \frac{\pi \rho_o c^2 a^4}{V_o T} \quad (3.24)$$

and so

$$\chi = \frac{\pi \rho_o c^2 a^4}{V_o} \cdot \frac{2.404^2}{\omega_m^2 a^2 \sigma} \quad (3.25)$$

$$\chi = \frac{\rho_o c^2 \pi a^2}{\rho_m V_o h} \cdot \frac{2.404^2}{\omega_m^2} \quad (3.26)$$

ρ_m is density of membrane ($= \frac{\sigma}{h}$)

$$\chi = 2.404^2 \alpha \beta \frac{\omega_c^2}{\omega_m^2} \quad (3.27)$$

$$\chi = 5.85 \alpha \beta \frac{\omega_c^2}{\omega_m^2} \quad (3.28)$$

α and β are defined in Section 3.1.

The theory of Section 3.1 gives the frequency as

$$f = \frac{\omega_m}{2\pi} \left[1 + \alpha \beta \frac{\omega_c^2}{\omega_m^2} \right]^{\frac{1}{2}} \quad (3.29)$$

in the closed cavity limit.

The frequency in this case has modified the cavity-less value by a factor of

$$\left[1 + \alpha \beta \frac{\omega_c^2}{\omega_m^2} \right]^{\frac{1}{2}} \quad \text{or} \quad \left[1 + \frac{\chi}{5.85} \right]^{\frac{1}{2}}. \quad \text{Comparison of this with the results of}$$

Section 3.2 shows reasonable agreement at least for the lower values of χ . Table 3.1 shows the modification to the frequency according to the theory of Section 3.1; it should be compared with the results shown in Table 3.2.

Other coupling effects will be experienced by the system; the higher frequencies of the drum will couple with the spherical modes of the cavity. These modes can be expressed by means of spherical Bessel functions, but the relatively low amplitude of most of the affected modes does not render a full treatment worthwhile at this stage.

It has been shown in this section that the frequencies of all modes that have no nodal diameters are affected by the cavity. A constant χ has been defined, giving the relative importance of the cavity to the system. In all cases for which $\chi > 0$ the cavity has the effect of raising the frequencies of these modes. In addition there is a damping factor due to the viscous effect of the air being forced through the small hole at the bottom of the cavity by the vibrating membrane. This factor again only affects the modes with no diametrical nodes, reducing their amplitude; it is suspected that since this effect must depend on the volume of air the membrane displaces that this damping will be far more apparent on the fundamental than on any other modes. By far the greatest effect of the cavity, however, is as a baffle, causing the fundamental $(0, 1)$ mode to be a monopole sound source, and therefore die away quickly. However, air damping, (Section 4) can be shown to have a far greater impact on the frequencies of the modes of vibration, as well as affecting their amplitudes and decay times.

Section 4. Air Loading

This section deals with the modifications to the normal modes due to air loading, using a Green Function technique.

4.1 Modelling the Air Loading using Green's Functions.

It is suspected that air loading alters the normal mode frequencies by a greater degree than any other factor. Rayleigh (1945e) recognised that the frequency ratios actually obtained from a kettledrum deviated significantly from the ideal ratios of a membrane vibrating in a vacuum, derived in Section 1. His suspicions were confirmed by Benade (1976b) who found that the first ten components of the sound spectrum had frequencies closely adhering to the harmonic series, without the fundamental. A discussion of the importance of the harmonic series, and its rôle in giving rise to a sensation of pitch is to be found in Section 7.

To estimate the effect of air loading, the effective mass of loading a piston in an infinite baffle is given by Kinsler and Frey (1962b) as

$$m = \rho_0 c_a \frac{\pi a^2}{\omega} \chi \quad (4.1)$$

where χ is the piston reactance function
 ρ_0 is the equilibrium density of air
 c_a is the speed of sound in air

for frequencies $f < \frac{c_a}{4\pi a}$ $\chi \approx \frac{16}{3} \frac{fa}{c_a}$

so $m = \frac{8}{3} \rho_0 a^3$. This is equivalent to the air in the form of a disc, with the radius, a , of the piston and thickness $\frac{8a}{3\pi}$. At higher frequencies the air load decreases as

$\frac{1}{f^2}$, so above 500Hz the air load becomes negligible. The effective mass loading on a rigid baffled piston would therefore shift the frequencies by as much as 40 per cent.

A more exact determination of the effects of air loading can be made by applying the Green Function technique¹⁰. The kettledrum is considered as a rigid cylindrical kettle of length L and radius a , capped by the membrane, and with a rigid bottom, except for the small hole mentioned in Section 3. Christian et al (1984b) give details of this technique.

The incremental pressure P of the air satisfies the wave equation

$$\left[\nabla^2 \frac{1}{c_a^2} - \frac{\partial^2}{\partial t^2} \right] P = 0 . \quad (4.2)$$

For a given normal mode the pressure may be written as

$$P = P(\mathbf{r})e^{-i\omega t} \quad (4.3)$$

with

$$\left[\nabla^2 + \frac{\omega^2}{c_a^2} \right] P(\mathbf{r}) = 0 . \quad (4.4)$$

Air loading inside the kettle may be accounted for using a Green Function, suitable for solving the Neumann Problem (Dettman (1984a)). For points inside the kettle the Green Function, $G_{in}(r, \theta, z, r', \theta', z')$ with (r, θ, z) being general cylindrical co-ordinates and (r', θ', z') the co-ordinates of the displaced membrane giving rise to the pressure, satisfies the equations (e.g. Pozrikidis (1992a))¹¹:

¹⁰ Appendix C gives some properties of Green's Functions.

¹¹ Kreisig (1988b) gives the derivation of the operator ∇^2 for cylindrical co-ordinates.

$$\begin{aligned} \left[\nabla^2 + \frac{\omega^2}{c^2} \right] G_{\text{in}}(\mathbf{r}, \mathbf{r}') &= \left[\frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial}{\partial r} \right] + \frac{1}{r^2} \frac{\partial}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} \right] G_{\text{in}}(\mathbf{r}, \mathbf{r}') \\ &= -4\pi \delta \left[\frac{r - r'}{r} \right] \delta(\theta - \theta') \delta(z - z') \end{aligned} \quad (4.5)$$

where δ is the Dirac Delta Function

and from the general properties of Green's functions, eg Pozrikidis (1992b) and Dettmann (1984b)

$$G_{\text{in}}(r, \theta, z, r', \theta', z') = G_{\text{in}}(r' \theta' z', r, \theta, z) \quad (4.6)$$

and Pozrikidis (1992c)

$$\left. \frac{\partial G_{\text{in}}(\mathbf{r}, \mathbf{r}')}{\partial r'} \right|_{r=a} = \left. \frac{\partial G_{\text{in}}(\mathbf{r}, \mathbf{r}')}{\partial z'} \right|_{z'=0, L} \quad (4.7)$$

Equation 4.7 is a boundary condition required for a Green Function which can be used to relate the pressure $P(r')$ inside the enclosure to the boundary values of the normal pressure derivative.

$G_{\text{in}}(\mathbf{r}, \mathbf{r}')$ may be calculated using eigenfunction expansions for the air inside the kettle. Using the cylindrical Bessel function of Section 1, but whose derivatives vanish at $r = a$:

$$\left. \frac{a}{b_{n,i}} \frac{\partial}{\partial r} J_n(b_{n,i}r) \right|_{r=a} = J'_n(b_{n,i}a) = 0 \quad (4.8)$$

and the differential equation of Section 1 (equation (1.10))

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + b_{n,i}^2 - \frac{n^2}{r^2} \right] J_n(b_{n,i}r) = 0 \quad (4.9)$$

and given the recurrence relation for J_n (Appendix I, equation (A–xii)), it can be shown that

$$\int_0^a r J_n^2(b_{n,i}r) dr = \frac{a^2}{2} \left[1 - \frac{n^2}{b_{n,i}^2 a^2} \right] J_n^2(b_{n,i}a). \quad (4.10)^{12}$$

Because $\frac{1}{\sqrt{2\pi}} e^{in\theta} J_n(b_{n,i}r)$ is complete in (r, θ) space

$$\frac{1}{2\pi} \sum_{n=-\infty}^{n=\infty} e^{in(\theta-\theta')} \sum_{i=1}^{\infty} \frac{2J_n(b_{n,i}r)J_n(b_{n,i}r')}{\left[a^2 - \frac{n^2}{b_{n,i}^2} \right] J_n^2(b_{n,i}a)} = \delta(\theta-\theta') \delta\left(\frac{r-r'}{r}\right) \quad (4.11)$$

hence $G_{in}(r, r')$ may be expressed as the left-hand side of (4.11) with an extra factor $g_{n,i}$ in the sum, provided that

$$\left[\frac{d}{dz^2} + \frac{\omega^2}{c^2} - b_{n,i}^2 \right] g_{n,i}(z, z') = -4\pi \delta(z - z') \quad (4.12)$$

and

$$g_{n,i}(z, z') = g_{n,i}(z', z) \quad (4.13)$$

and

$$\left. \frac{d}{dz} g_{n,i}(z, z') \right|_{z'=0,L} = 0 \quad (4.14)$$

so that equations 4.5, 4.6, 4.7 are satisfied.

¹² Morse (1948e) gives a similar relationship.

Equations 4.12 to 4.14 are satisfied by (from Morse and Ingard (1968a))

$$g_{n,i}(z, z') = \left[-\frac{4\pi}{\gamma_{n,i}} \text{Sin}(\gamma_{n,i}L) \right] \text{Cos}(\gamma_{n,i}z_{<}) \text{Cos}(\gamma_{n,i}[L-z_{>}]) \quad (4.15)$$

where $z_{>}$ is the greater of z and z' and $z_{<}$ the lesser, and

$$\gamma_{n,i} = \sqrt{\frac{\omega^2}{c^2} - b_{n,i}^2} \quad \text{for} \quad b_{n,i} < \frac{\omega}{c} \quad (4.16)$$

$$\gamma_{n,i} = \sqrt{b_{n,i}^2 - \frac{\omega^2}{c^2}} \quad \text{for} \quad b_{n,i} > \frac{\omega}{c}.$$

The Green function inside the kettle, $G_{n,i}(r, \theta, z, r', \theta', z')$ is therefore given by

$$G_{in}(r, r') = -4\pi \sum_{n=-\infty}^{\infty} \frac{e^{in(\theta - \theta')}}{2\pi} \sum_{i=1}^{\infty} \frac{2J_n(b_{n,i}r)J_n(b_{n,i}r')}{\left[a^2 - \frac{m^2}{b_{n,i}^2} \right] J_n^2(b_{n,i}a)} \frac{\text{Cos}(\gamma_{n,i}z_{<}) \text{Cos}[\gamma_{n,i}(L-z_{>})]}{\gamma_{n,i} \text{Sin}(\gamma_{n,i}L)}. \quad (4.17)$$

To model the outside air loading will require specific assumptions about the surroundings. In this treatment the reflections of sound waves from the walls of the room will be ignored and, in addition, a plane, rigid baffle will be assumed to surround the membrane in its plane. This is easily the most un-physical assumption of this section, but it does lead to greater simplicity. Christian et al (1984b) are currently engaged in calculations that do not involve this assumption, but as yet their investigations are incomplete. Preliminary results, however, indicate that there is good agreement with this simplified model.

The Green Function $G_{\text{out}}(\mathbf{r}, \theta, z, \mathbf{r}', \theta', z')$ for points outside the kettle must have outgoing waves that vanish at infinity and vanishing normal derivatives on the membrane's plane, $z = L$. A suitable function is equation (4.18) as shown by Morse and Ingard (1968a):

$$G_{\text{out}}(\mathbf{r}, \mathbf{r}') = \frac{\exp \frac{i\omega}{c} \left[r^2 + r'^2 - 2rr' \cos(\theta - \theta') + (z - z')^2 \right]^{\frac{1}{2}}}{\left[r^2 + r'^2 - 2rr' \cos(\theta - \theta') + (z - z')^2 \right]^{\frac{1}{2}}} + \frac{\exp \frac{i\omega a}{c} \left[r^2 + r'^2 - 2rr' \cos(\theta - \theta') + (z - z' - 2L)^2 \right]^{\frac{1}{2}}}{\left[r^2 + r'^2 - 2rr' \cos(\theta - \theta') + (z - z' - 2L)^2 \right]^{\frac{1}{2}}}. \quad (4.18)$$

An expression for $P(\mathbf{r})$, the pressure inside or outside the kettle enclosure may now be established. Multiplying equation (4.4) by $G_{\text{in}}(\mathbf{r}, \mathbf{r}')$ and replacing \mathbf{r}, θ, z with \mathbf{r}', θ', z'

$$G_{\text{in}}(\mathbf{r}, \mathbf{r}') \left[\nabla^2 + \frac{\omega^2}{c^2} \right] P(\mathbf{r}) = 0 \quad (4.19)$$

and multiplying (4.5) by $P(\mathbf{r}')$

$$P(\mathbf{r}') \left[\nabla^2 + \frac{\omega^2}{c^2} \right] G_{\text{in}}(\mathbf{r}, \mathbf{r}') = -\frac{4\pi}{r} P(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') \delta(\theta - \theta') \delta(z - z'). \quad (4.20)$$

Subtracting (4.19) from (4.20)

$$\frac{4\pi}{r'} P(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') \delta(\theta - \theta') \delta(z - z') = P(\mathbf{r}') \left[\nabla^2 + \frac{\omega^2}{c^2} \right] G_{\text{in}}(\mathbf{r}, \mathbf{r}') - G_{\text{in}}(\mathbf{r}, \mathbf{r}') \left[\nabla^2 + \frac{\omega^2}{c^2} \right] P(\mathbf{r}') \quad (4.21)$$

re-arranging and integrating over the kettle enclosure:

$$\begin{aligned}
 & 4\pi \int \int \int \delta(\mathbf{r}-\mathbf{r}')(\theta-\theta')(z-z')P(\mathbf{r}')d\mathbf{r}d\theta dz \\
 & = \int \int \int_{\text{enclosure}} \mathbf{r}' \left[P(\mathbf{r}')\nabla^2 G_{\text{in}}(\mathbf{r},\mathbf{r}') - G_{\text{in}}(\mathbf{r},\mathbf{r}')\nabla^2 P(\mathbf{r}') \right] d\mathbf{r}' d\theta' dz' .
 \end{aligned} \tag{4.22}$$

The left-hand side reduces to $4\pi P(\mathbf{r}')$, hence:

$$P(\mathbf{r}') = -\frac{1}{4\pi} \int \int \int_{\text{enclosure}} \mathbf{r}' \left[P(\mathbf{r}')\nabla^2 G_{\text{in}}(\mathbf{r},\mathbf{r}') - G_{\text{in}}(\mathbf{r},\mathbf{r}')\nabla^2 P(\mathbf{r}') \right] d\mathbf{r}' d\theta' dz' . \tag{4.23}$$

Applying the second form of Green's theorem (Kreisig (1988c), Morse and Ingard (1968b))

$$P(\mathbf{r}') = -\frac{1}{4\pi} \int \int_{\text{surface}} \left[P(\mathbf{r}')\frac{\partial}{\partial n} G_{\text{in}}(\mathbf{r},\mathbf{r}') - G_{\text{in}}(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial n} P(\mathbf{r}') \right] \mathbf{r}' d\mathbf{r}' d\theta' \tag{4.24}$$

where $\frac{\partial}{\partial n}$ is the directional derivative normal to the surface, S , of the membrane.

Using the boundary conditions, given by equation (4.7), this reduces to:

$$P(\mathbf{r}') = -\frac{1}{4\pi} \int_0^a \int_0^{2\pi} G_{\text{in}}(\mathbf{r},\mathbf{r}') \frac{\partial P}{\partial z}(\mathbf{r}', \theta', L_-) \mathbf{r}' d\theta' dr' \tag{4.25}$$

where $P(\mathbf{r}', \theta', L_-)$ is the pressure on the kettle side of the membrane.

A similar calculation for the pressure outside the kettle can be derived,

substituting r, θ, z for r', θ', z' and using $G_{\text{out}}(\mathbf{r}, \mathbf{r}')$ from equation (4.18):

$$P(\mathbf{r}) = -\frac{1}{4\pi} \int_0^a \int_0^{2\pi} G_{\text{out}}(\mathbf{r}, \mathbf{r}') \frac{\partial P}{\partial z} (r', \theta', L_+) r' d\theta' dr' \quad (4.26)$$

where $P(r', \theta', L_+)$ is the pressure on the outside of the membrane.

Christain et al (1984b) extend these expressions for the pressure without the kettle and baffle (i.e. an isolated air-loaded membrane). The results are consistent, insofar as the expressions show that in the plane of the membrane a wave from the underside exactly cancels one from the upper surface, which is of course clearly the case.

4.2 Determination of the normal modes.

From equation (1.10) and including the effects of the pressure both inside and outside, the displacement ψ of the membrane satisfies the equation

$$\sigma \frac{d^2\psi}{dt^2} = -\sigma\omega^2\psi = T\nabla^2\psi + P(r') + P(\mathbf{r}) \quad (4.27)$$

hence

$$\dot{\psi} = \psi(r, \theta) e^{-i\omega t}$$

and

$$\ddot{\psi} = -\omega^2\psi(r, \theta) e^{-i\omega t} \quad (4.28)$$

[σ and T are the area density of the membrane and the tension respectively, as defined in Section 1.]

Linear fluid dynamics gives

$$\rho_0 \ddot{\psi} = -\rho_0 \omega^2 \psi = -\left. \frac{\partial P}{\partial z}(\mathbf{r}, \theta, z, t) \right|_{z=L} \quad (4.29)$$

where ρ_0 is the equilibrium density of air.

From Section 1 (equation 1.14) the loaded displacement function, ψ , can be expressed as the sum of the normal mode displacements, i.e.:

$$\psi(\mathbf{r}, \theta) = \sum_{m=0}^{\infty} \sum_{s=1}^{\infty} \phi_{m,s} \psi'_{m,s}(\mathbf{r}, \theta) \quad \text{where } \psi' \text{ is the in vacuo displacement function.} \quad (4.30)$$

Substitution of the expression for the pressure inside the kettle enclosure and outside into (4.27), and bearing in mind that $c^2 = T/\sigma$ (equation (1.3))

$$\begin{aligned} \left[\nabla^2 + \frac{\omega^2}{c^2} \right] \psi(\mathbf{r}, \theta) &= \frac{1}{4\pi T} \int_0^a \int_0^{2\pi} G_{\text{in}}(\mathbf{r}, \mathbf{r}') \frac{\partial P}{\partial z}(\mathbf{r}', \theta', L_-, t) r' d\theta' dr' \\ &+ \frac{1}{4\pi T} \int_0^a \int_0^{2\pi} G_{\text{out}}(\mathbf{r}, \mathbf{r}') \frac{\partial P}{\partial z}(\mathbf{r}', \theta', L_+, t) r' d\theta' dr' . \end{aligned} \quad (4.31)$$

Using the relationship between $\psi(\mathbf{r}, \theta)$ and $\frac{\partial P}{\partial z}(\mathbf{r}, \theta, z, t)$ (equation (4.29)), and given that $P(\mathbf{r}', \theta', L_+) = -P(\mathbf{r}', \theta', L_-)$ gives

$$\begin{aligned} \left[\nabla^2 + \frac{\omega^2}{c^2} \right] \psi(\mathbf{r}, \theta) &= \frac{\rho_0 \omega^2}{4\pi T} \int_0^a \int_0^{2\pi} G_{\text{in}}(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}, \theta) r' d\theta' dr' \\ &- \frac{\rho_0 \omega^2}{4\pi T} \int_0^a \int_0^{2\pi} G_{\text{out}}(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}, \theta) r' d\theta' dr' . \end{aligned} \quad (4.32)$$

The original in vacuo displacement function, $\psi'(r, \theta)$ has the orthonormal property:

$$\int_0^a \int_0^{2\pi} \psi'_{p,q}(r, \theta) \psi'_{m,n}(r, \theta) r d\theta dr = \delta_{pm} \delta_{qn} \quad (4.33)$$

where δ is the Kronecker delta function.

So re-writing equation (4.32) for an individual mode, $\psi_{m,s}(r, \theta)$, substituting for $G_{in}(r, r')$ and $G_{out}(r, r')$ (equations (4.17) and (4.18)), multiplying both sides by $\psi'(r, \theta)r$ and integrating with respect to r and θ over the membrane's surface gives

$$\begin{aligned} \phi_{m,s} \left[\frac{\omega_{m,s}^2}{c^2} - k_{m,s}^2 \right] = \\ \frac{4\rho_0\omega_{m,s}^2}{T} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} \frac{k_{m,n} k_{m,q} \gamma_{m,n} \text{Cot}(\gamma_{m,n}L) \phi_{n,s}}{(k_{m,s}^2 - b_{m,n}^2)(k_{m,q}^2 - b_{m,n}^2) \left(1 - \frac{m^2}{b_{m,q}^2}\right)} \\ - \frac{\rho_0\omega^2}{2\pi T} \sum_{q=1}^{\infty} \phi_{n,s} I_{m,s,q} \end{aligned} \quad (4.34)$$

where $k_{m,s}$ is the s^{th} zero of $J_m(k_{m,s}a) = 0$

and

$$I_{m,s,q} = \int_0^a \int_0^{2\pi} \int_0^a \int_0^{2\pi} G_{out}(r, r') \psi_{m,s}(r, \theta) \psi'_{m,s}(r, \theta) r r' d\theta' dr' d\theta dr. \quad (4.35)$$

This result uses the series relationship given by Morse and Feshbach (1953a) and the expression for $G_{out}(r, r')$ given by Morse and Feshbach (1953b).

$I_{m,s,q}$ reduces to

$$I_{m,s,q} = \frac{4\pi k_{m,s} k_{m,q}}{a^2} \int_0^\infty \lambda \left[\lambda^2 - \frac{\omega_{m,s}^2}{c_a^2} \right]^{-\frac{1}{2}} \frac{J_m^2(a\lambda) d\lambda}{(\lambda^2 - k_{m,s}^2)(\lambda^2 - k_{m,q}^2)} \quad (4.36)$$

$$\text{where } \left[\lambda^2 - \frac{\omega_{m,s}^2}{c_a^2} \right] = -i \left[\frac{\omega_{m,s}^2}{c_a^2} - \lambda^2 \right]. \quad (4.37)$$

The solution of (4.34) give an infinite series of eigenmodes, from which the modified frequencies, $\omega_{m,s}$ can be calculated using numerical methods. Christian et al (1984b) derive similar relationships: they limit s, q and n to 1, 2, 3 and 4, estimate for $\omega_{m,s}$ based on the in vacuo value and pick out eigenvalues and hence the eigenfrequencies. Comparison and re-substitution for $\omega_{m,s}$ is continued until reasonable agreement (about 0.5%) is achieved. Numerical results obtained by these means for a 64cm drum are given in Table 4.1 below:

Mode	frequency (Hz)	ratio to (1,1) mode
(0,1)	138	0.80
(1,1)	172	1.00
(2,1)	261	1.52
(0,2)	291	1.69
(3,1)	344	2.00
(1,2)	390	2.27
(4,1)	427	2.48
(2,2)	471	2.74
(0,3)	511	2.97
(5,1)	506	2.94

Table 4.1 The calculated effect of air loading on the normal mode frequencies
(Tension: 5360 N/m)

From Table 4.1, it can be observed that if the fundamental (0,1) mode is ignored, the remainder of the calculated frequency ratios approximate well to the ideal ratios of the harmonic series, discussed in Section 7.1.

Furthermore, it is interesting to note that the calculated values of these frequency ratios for various values of L produce nearly harmonic ratios when $L/a = 1.2$. This implies that a kettle having internal volume $1.2\pi a^3$ is the ideal one necessary to produce a musically pleasing sound. Christian et al (1984c) have produced a tentative theory to account for this, but without any attempt to include the details of the kettle geometry.

Section 5 Other considerations

This section deals with other effects likely to alter the sound produced, namely the mass loading of the membrane and the effects of membrane stiffness.

5.1 Mass loading.

The original treatment of a vibrating membrane to which a small mass load is attached was first suggested by Rayleigh (1945b). He calculated the perturbation of the eigenfrequencies by considering the additional kinetic energy due to the mass load, assuming that the eigenfunction was unchanged by this load. This gave a paradoxical result, since the perturbed frequency derived is independent of the area of attachment. Indeed, a finite mass attached to a single point would cause the membrane to break (Morse (1948b)). Rayleigh's (1945b) solution gives frequencies that are lowered by the addition of the mass load.

Kornhauser and Mintzer (1953a) have produced a more rigorous solution to this problem. Starting with the equation of motion of the membrane

$$\alpha^2 y + k^2 y = 0 \tag{5.1}$$

where y is the displacement of the membrane from Section 1

$$k \leq \frac{\omega}{c}.$$

and the force on the membrane due to the attached mass, m , is given by

$$F = 2\pi b T \left. \frac{\partial y}{\partial r} \right|_{r=b} \tag{5.2}$$

where b = radius of attachment

so the equation of motion becomes

$$2\pi bT \left. \frac{\partial y}{\partial r} \right|_{r=b} = m \left. \frac{\partial^2 y}{\partial t^2} \right|_{r=b} = -\omega^2 m y \Big|_{r=b} \quad (5.3)$$

The other solution, is of course, the boundary condition

$$y \Big|_{r=a} = 0 \quad (5.4)$$

These two equations, (5.3) and (5.4), combined with (5.1) give a general solution of the form

$$y = AJ_0(kr) + BN_0(kr) \quad (5.5)$$

where N_0 is the Neumann function, or Bessel function of the second kind.

The eigenvalue, k , can be calculated from

$$\frac{\alpha x J_0(\alpha x) - \gamma \alpha^2 J_1(\alpha x)}{\alpha x N_0(\alpha x) - \gamma \alpha^2 N_1(\alpha x)} = \frac{J_0(x)}{N_0(x)} \quad (5.6)$$

where

$$\begin{aligned} x &= ka \\ \alpha &= \frac{b}{a} \\ \gamma &= \frac{2M}{m} \\ M &= \text{mass of membrane.} \end{aligned}$$

Thus the eigenfrequencies are dependent on the size of attachment of the mass, m , as expressed in the parameter, α . As $\alpha \rightarrow 0$, and the area of attachment therefore approaches a point, the left-hand side of (5.6) vanishes (because the numerator $\rightarrow 0$ and the denominator $\rightarrow -\infty$), so $J_0(x)/N_0(x)$ must vanish.

This gives $J_0(x) = 0$ whose solution gives the unperturbed eigenfrequencies, derived in Section 1. Numerical calculations show that the effect of the mass loading due to an object the size and mass of a typical drumstick raise the frequencies of the significant modes by about 2%

When the membrane is struck with a drumstick the stick will remain in contact with the skin until it passes through its equilibrium position after which the player removes the stick. This will have the effect of altering the behaviour of the first half-cycle, in raising the frequencies. However it is to be borne in mind that it is not infrequent to stretch the skin slightly on contact, thus lowering the frequencies, reported by Blades (1973c). The combined effect of the stretching and mass loading is not readily analysed, but confining attention to small forces of striking, thus rendering the stretching ineffective makes the problem simpler, and the expression quoted above can be utilised.

5.2 Membrane stiffness.

To account for the stiffness of the membrane, a plate like term can be added to the equation of motion of Section 1. Kinsler and Frey (1982c) give the equation of motion of a thin circular plate as

$$\frac{\partial^2 y}{\partial t^2} = \frac{h^2 E}{12\rho(1-\nu^2)} \nabla^2(\nabla^2 y) \quad (5.7)$$

where E = Young's modulus
 h = thickness
 ρ = density
 ν = Poisson's ratio.

Combining this with the equation of motion of the membrane (equation (1.4)) gives:

Section 6. Radiation effects and the time development of the system

In this section an attempt is made to explain the effects of sound radiation and therefore subsequent energy loss from the vibrating system, together with accounting for the decay times of the various normal modes.

6.1 Radiation effects.

There are four possible ways in which energy from the vibrating system may be lost: radiation of sound; mechanical loss in the membrane; viscothermal loss in the contained air in the cavity and mechanical loss in the kettle walls.

The second and fourth of these may be considered negligible, although the preference for calfskin rather than plastic heads, compared by Hardy and Ancell (1961), and the general feeling that drums equipped with copper kettles sound superior to their fibreglass counterparts may indicate that some loss may indeed take place in the membrane and the kettle walls, although the non uniformity of calfskin heads probably accounts for their superior tone. In particular, the visible line due to the animal's backbone is not terribly elastic, thus encouraging a nodal diameter (Shivas (1957a)).

The efficiency of sound production will depend on the net displacement of the surrounding air. With the exception of the fundamental this is very small for the kettledrum, giving it a very low efficiency of sound production for its overtones. The average displacement of a membrane provides one parameter for judging the efficiency for a particular mode. It may be defined as

$$\bar{\psi}_i = \int_S \psi_i \frac{(r, \theta)}{\pi a^2} dS \quad (6.1)$$

n will be 0 since for all
n > 0 there is no net displacement.

Since all parts of a ring-shaped element, $dS (= 2\pi r.dr)$, contained between r and $r + dr$ have the same displacement amplitude, ψ_i , it is possible to carry out the surface integration by merely integrating over r from 0 to a .

$$\bar{\psi}_i = \int_0^a \phi_{0,i} \frac{J_0(k_{0,i}r)}{\pi a^2} 2\pi r dr \quad (6.2)$$

which upon integration gives

$$\bar{\psi}_i = \frac{2\phi_{0,i}}{k_{0,i}a} J_1(k_{0,i}a) \quad (6.3)$$

Substitution shows that if the amplitude at the centre were equal for the first two modes (i.e. if $\phi_{0,1} = \phi_{0,2}$) then the fundamental displaces three times as much air as the second one.

Morse (1948f) gives an expression for the average displacement due to a constant force applied all over the surface;

$$\bar{\psi} = \int_{-\infty}^t F(\tau) \bar{\psi}_\delta(t - \tau) d\tau \quad (6.4)$$

where $F(\tau)$ is the applied impulsive force as a function of time and $\bar{\psi}_\delta(t)$ is the response of the membrane to this force.

This predicts that the fundamental mode will be quickly damped but the higher modes will persist. The lower the frequency of vibration the higher the acoustic efficiency, but in addition the higher the radiation damping.

As Section 3 pointed out, the presence of the kettle acts as a baffle, and therefore the $(0, 1)$ mode acts effectively as a monopole source, and as such will lose its energy quickly. In the bass drum (this is an instrument which has two membranes of similar size to the kettledrum, mounted over each side of a shallow hollow wooden cylinder of depth approximately 17cm), which is not baffled in the same way, it is interesting to note that the fundamental $(0, 1)$ mode is present in no small degree, producing therefore an inharmonic series, and no definite pitch. Fletcher and Rossing (1991d) give an analysis of the bass drum. This comparison, however, shows that the effect of the kettle as a baffle has a considerable part in damping the fundamental.

On the other hand, the $(1, 1)$ mode will be a dipole source, and will therefore radiate sound at a power proportional to f^4 and is considerably less efficient, and so losing energy less quickly than the $(0, 1)$ mode. Thus the $(1, 1)$, $(2, 1)$, $(3, 1)$ and $(4, 1)$ modes will decay much more slowly than the others, especially the $(0, 1)$, $(0, 2)$ and $(0, 3)$. The decay times will decrease with frequency, so the $(1, 1)$ and $(2, 1)$ modes will be the most prominent sounds heard. The higher the tension, too, the shorter the decay time will be, so drums tuned to high pitches have a short time duration.

6.2 Time development.

A normal mode of vibration (with only outgoing sound radiation) will have a harmonic time dependence $e^{-i\omega t}$. The imaginary part, ω_i , provides an exponentially decreasing function of time (Goldstein (1980)) and is related to T_{60} , the time for a 60dB drop in the membrane's sound pressure, given by

$$T_{60} = -\frac{3}{\log_{10}e} \cdot \frac{1}{\omega_i} . \quad (6.5)$$

Using values calculated in Section 4 for ω , accounting for the air-loading of the membrane, the decay time, T_{60} , can be established for each mode. The (0, 1) and (0, 2) modes produce a value for T_{60} of less than 0.35s, less than the normal reverberation time of a typical concert room, and so their effect can be ignored. Christian et al (1984c) calculated values for T_{60} ; their results are shown for a 64cm drum in Table 6.1.

The equivalent measured values are given in the next Section (Table 7.1); the (3, 1), (4, 1) and (5, 1) modes have a shorter T_{60} than the calculated value, this is accounted for by the other mechanisms for energy losses discussed earlier.

mode	T_{60}
(0, 1)	0.06
(1, 1)	0.8
(2, 1)	1.8
(0, 2)	0.2
(3, 1)	4.6
(1, 2)	0.2
(4, 1)	10.5
(2, 2)	0.8
(0, 3)	0.1
(5, 1)	16.2

Table 6.1 Calculated decay times of the normal modes
(Tension = 5360 N/m)

Section 7. Observations

In this Section, the observed sound spectrum, based on experimental results, of the instrument is considered as well as the sensation of pitch that this produces.

7.1 Pitch¹³.

It need hardly be said that pitch produces a subjective response, and, of course, there is no guarantee that any two individuals will experience the same sensation. Indeed, many listeners even experience different sensations in each ear, when encountering a pure tone of constant amplitude. The two ears, reported by Burns (1982), can produce a sensation of up to a semitone difference.

This having been established, it is, nevertheless true that there is a strong correlation between the perceived pitch as a musical sound and the frequency of that sound's fundamental. All musical sounds are accompanied by a series of overtones, or partials, forming the harmonic series, in which the tones relate to the fundamental frequency, f by $2f$, $3f$, $4f$ etc. The harmonic series, based on C is given in Figure 7.1.



Figure 7.1 The harmonic Series

¹³ A fuller discussion of all aspects of pitch and perception is given by Campbell and Greated (1987a).

At first sight, it appears that the partials ($2f$, $3f$ etc) simply add 'tone colour' to the sound, although some musicians claim to hear each of the constituents distinctly (eg Messiaen (1956)). Randel (1986) maintains that for the psychological identification of pitch, a musical tone must be composed of harmonic partials. Furthermore, he suggests that even if the fundamental is missing entirely, the pitch perceived is that of the fundamental. Experiments by Fletcher (1924) and Wood (1962a) show that, if the fundamental is completely filtered out, the sensation of pitch is still that of the fundamental, in other words the ear appears to 'add' the fundamental back again. It is in this way that a small loudspeaker, for example, is able to give an impression of bass sounds. ~~On the other hand, if the filter is applied to non-musical sounds (i.e. those whose partials do not form a harmonic series), such as human speech or a snare drum, the sound becomes unrecognisable.~~ Further evidence is that suggested by Munrow (1976), that to most observers the recorder appears to sound an octave lower, (half the frequency) than is actually the case. Parncutt (1989) points out that complex tones (tones that form part of the harmonic series, but with the fundamental missing), produce a pitch that is ambiguous: usually the context of the harmony or melody of which the complex tone forms a part, suggests the pitch to the listener.

T.A.B.

In the example of the kettledrum, however, all observations made show that the fundamental is missing, and that the first overtone occurs at a frequency of 1.5 times that of the lowest partial present. Indeed, many musicians find that the pitch of a kettledrum is difficult to define, because of the relative strength of this first overtone (eg Blades (1973b,d)); this can be an aid when tuning the instruments in fifths¹⁴ (Blades (1973b)). Most would agree, however, that the nominal pitch of the drum is, in fact, that corresponding to the lowest sounding frequency; the ear does not, in this case, 'add' the missing fundamental of the harmonic series.

¹⁴ For a description of musical intervals see Murray Brown (1972).

This is contrary to the observations of Parncutt (1989) who states that usually, complex tones without highly audible higher harmonics, will be heard with the fundamental in place. Campbell and Greated (1987b) suggest that this is due to the relatively short decay times of the upper partials, leaving the brain with the longest sounding frequency as its only guide. There is, nevertheless, debate amongst musicians, since some perceive the pitch as sounding an octave lower, especially when played quietly (Rossing (1982e)).

7.2 The observed sound of the kettledrum.

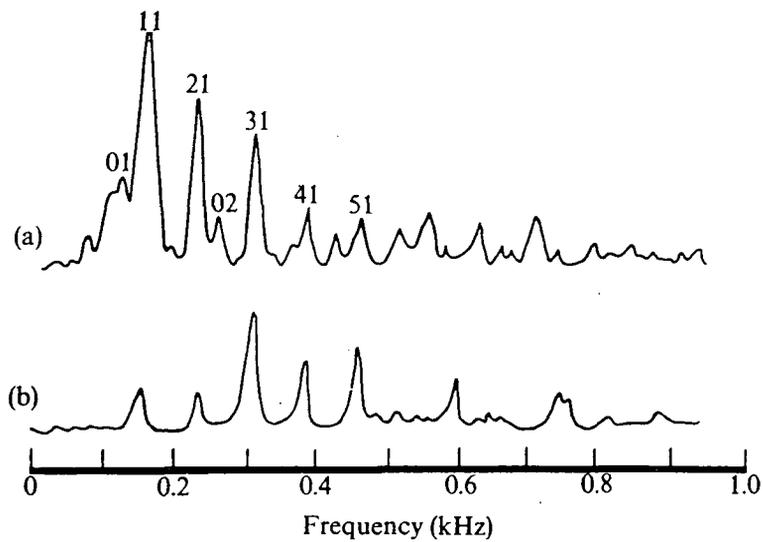
Rayleigh (1945e) identified the first four sounding frequency ratios as 1.00:1.50:1.88:2.00 and he ascribed the lowest sounding frequency to the (1, 1) mode. Later experiments by Benade (1976a) showed these ratios to be 1.00:1.504:1.742:2.000. In both instances they form part of the harmonic series with the fundamental missing. Rossing (1982f) repeated the experiment and found ratios of 1.00:1.50:2.00:2.44:2.90, nearly the harmonic sequence 2:3:4:5:6. These observations are consistent with those of performers (eg Shivas (1957b)).

The above results are in sharp contrast to the calculated mode frequencies of a vibrating membrane in vacuo, until the air-loading considerations of Section 4 are taken into account, and the fact that the (0, 1) mode is damped completely. It seems, from observations, that all the modes having no nodal radii play little or no part in the sound produced. Striking the drum in the centre, thereby preventing any modes from occurring with a node at that point (i.e. those with radial nodes), produces a pitchless thwack. The drum vibrates for a quarter cycle only; no further vibration takes place. Furthermore, placing a felt damper at the centre, encouraging the modes with radial nodes and discouraging the others, produces no detectable change in the tone quality of the instrument.

The most significant practical observations are, then, that the kettledrum produces a near-harmonic series of overtones with the fundamental missing, and that these overtones are due to the modes of vibration having only radial nodes.

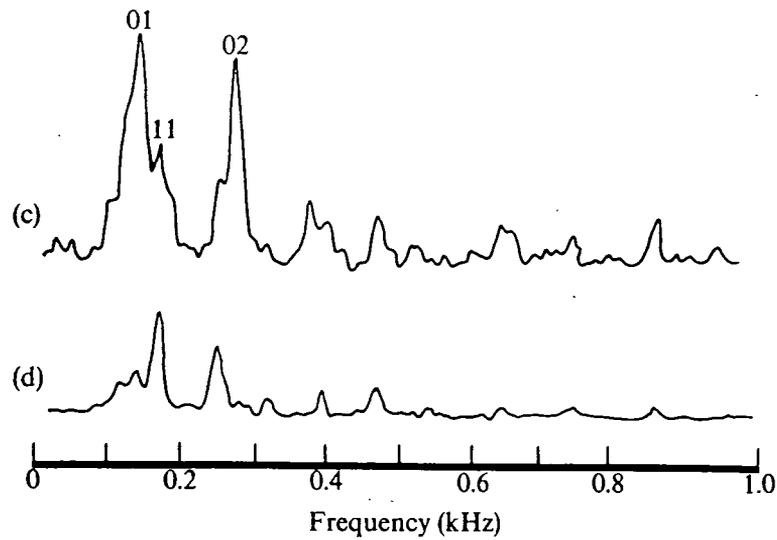
7.3 Experimental results.

Rossing and Kvistad (1976) have produced diagrams showing the sound spectrum produced by a 63cm diameter drum at various times after striking; their result is reproduced in Figure 7.2 below.



a) approximately 0.03s after striking at the normal playing position.

b) approximately 1s later.



c) approximately 0.03s after striking at the centre.

d) approximately 1s later.

Figure 7.2 Observed sound spectra of the kettledrum

From this evidence, it is possible to see that the (0, 1) and (0, 2) modes have a short time duration, compared with the others. Even when the drum is struck at the centre, thus exciting the (0, 1) and (0, 2) modes the (1, 1) and (2, 1) modes are more prominent after 1s.

Christian et al (1984d) undertook measurements to confirm the theoretical predictions of the air-loading calculations of the mode frequencies and decay times; their results are shown in Tables 7.1 and 7.2 for a 64cm drum.

Mode	Frequency (Hz)	Ratio to (1, 1) mode
(0, 1)	140	0.81
(1, 1)	172	1.00
(2, 1)	258	1.50
(0, 2)	284	1.65
(3, 1)	340	1.97
(1, 2)	344	2.00
(4, 1)	420	2.44
(2, 2)	493	2.86
(0, 3)	467	2.71
(5, 1)	501	2.91

Table 7.1 Observed frequencies of the normal modes
(Tension = 5360 N/m)

Mode	$T_{60}(s)$
(0, 1)	<0.3
(1, 1)	0.8
(2, 1)	1.7
(0, 2)	0.4
(3, 1)	2.7
(1, 2)	0.5
(4, 1)	1.7
(2, 2)	0.5
(0, 3)	<0.3
(5, 1)	2.6

Table 7.2 Observed time decays of the normal modes
(Tension = 5360 N/m)

Comparison of the experimental values for the frequencies of the normal modes is in good agreement with those calculated in Section 4. Furthermore, when the modes having relatively short time decays, T_{60} , are eliminated, the frequencies adhere closely to the harmonic series.

Conclusions

In Section 1, the normal modes of vibration of a stretched circular membrane in a vacuum were established. Clearly, the origin of the sound produced by a kettledrum must lie in the nature of these modes. In Section 2 the initial displacement function of the membrane was discussed; combining the results of these Sections will give the relative amplitude and frequency of each of the modes, simply by combining the relevant equations: (1.16), (1.18), (1.19) and (2.14) i.e.

$$A = \phi_{n,i} J_n(k_{n,i} r) \cos \theta \quad (1.19)$$

where

$$\phi_{n,i} = \int_0^a \int_0^{2\pi} \frac{y J_n(k_{n,i} r) \cos(n\theta) r dr d\theta}{\frac{\pi a^2}{2} J_{n+1}^2(k_{n,i} a)} \quad (1.16)$$

and

$$y = \frac{F}{2\pi T} \ln \left[\left(\frac{r^2 + b^2 - 2br \cos \theta}{r^2 + c^2 - 2rc \cos \theta} \right)^{\frac{1}{2}} \frac{a}{b} \right] \quad (2.14)$$

$$f = \frac{k_{n,i}}{2\pi} \sqrt{\frac{T}{\sigma}} \quad (1.18)$$

The subsequent vibration of the membrane, will, however, be altered considerably by several factors. Section 3 examined the effect that the cavity might have, by considering it as a Helmholtz resonator, a closed cavity and as a baffle. It seems likely that by far the greatest effect it has, is as a baffle. It also appears that the presence of the small hole in the bottom of the kettle (necessary to equalise the equilibrium pressures inside and outside of the kettle in the event of a change in atmospheric conditions, or a change in temperature of the enclosed air) has no real effect on the sound produced, contrary to the assertions of Benade (1976a). In any case its effect will only be felt by those modes involving a change in the enclosed volume of the cavity; these modes, as shown in Sections 5 and 7 play little part in the sound produced by the instrument: again the fundamental (0, 1) mode suffers most!

The effect as a Helmholtz resonator, or as a closed cavity will again only affect the modes involving a change in internal volume (i.e. those having circular nodes) in raising their frequencies slightly, but since, as stated earlier, these modes are not great in amplitude (because of air-loading and the membrane's resistance to shear) this can be ignored. Incidentally, these modes are the only ones likely to be affected by the viscous damping due to the small hole at the bottom of the cavity, again, therefore these amplitudes are reduced. Instead, the effect as a baffle is far more important insofar as it removes the $(0, 1)$ mode completely. Experimental evidence supports this view, as does the fact that for the bass drum (similar in size to the kettledrum, but without this baffle) the $(0, 1)$ mode is present.

Having established the importance of the kettle, the damping due to air-loading has been estimated in Section 4. The effect of the large mass of air surrounding the membrane depresses the frequencies considerably, as well as causing the amplitude of vibration to die away with time. The assumption that the kettle is cylindrical in shape, and that an infinite plane baffle exists in the plane of the membrane is easily the most unphysical aspect of this thesis; nevertheless the results calculated for the frequency modifications due to air-loading are in good agreement with those obtained experimentally. The ignoring of the reflections of sound from the walls of the room is not so invalid: the typical reverberation time is sufficiently great as to cause little modification to the equations derived. Thus the frequencies and relative amplitudes of the modes can be calculated by numerical means using equation (4.34):

$$\phi_{m,s} \left[\frac{\omega_{m,s}^2}{c^2} - k_{m,s}^2 \right] =$$

$$\frac{4\rho_0\omega_{m,s}^2}{T} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} \frac{k_{m,n} k_{m,q} \gamma_{m,n} \text{Cot}(\gamma_{m,n}L) \phi_{n,s}}{(k_{m,s}^2 - b_{m,n}^2)(k_{m,q}^2 - b_{m,n}^2) \left(1 - \frac{m^2}{b_{m,q}^2}\right)}$$

$$- \frac{\rho_0\omega^2}{2\pi T} \sum_{q=1}^{\infty} \phi_{n,s} I_{m,s,q} \quad (4.34)$$

Other effects that might modify the frequencies are the mass loading of the membrane due to the drumstick and the stiffness of the membrane itself. The mass loading lasts for only the first half cycle, and will raise the frequencies slightly. This is often more than countered by the lowering of the frequencies due to the initial stretching of the membrane, observed especially when the drum is played loudly (Blades (1973c)). The membrane stiffness raises the frequencies by about 0.5%, as shown in Section 5. A far greater effect, however, is the resistance to shear on all those modes having circular nodes, preventing them from contributing significantly to the sound.

The time development and radiation effects of the system predict the time decay of each of the modes, discussed in Section 6, again showing that the fundamental (0, 1) mode is effectively damped. These predictions are in good agreement with the results obtained experimentally, except for higher frequencies; it is suspected that more complex factors such as the loss of energy in the membrane itself and in the kettle walls may account for this.

After a brief discussion on the human sensation of pitch, in Section 7, it has been established that the kettledrum only sounds at a definite pitch because of the fact

that the normal mode frequencies that are clearly audible form part of a harmonic series. The calculated frequencies for those modes having only radial nodes do indeed form part of a harmonic series, when air-loading is considered, thus the definite pitch of the instrument must be attributed to the air-loading of the membrane, together with the very short time duration of the other modes, and the complete damping or absence of the fundamental, $(0, 1)$ mode. The other modes (i.e. those having circular nodes) contribute only to the transient, or impact noise of the drum. These predictions are, again, in good agreement with experimental results.

This thesis, then, provides a complete theoretical overview into the various mechanisms that give rise to the sound spectrum of the orchestral kettledrum, and accounts for the distinctive tone, and sometimes elusive pitch of these noble instruments.

APPENDIX A PROPERTIES OF BESSEL FUNCTIONS

References:

Rayleigh (1945b)

Gray & Matthews (1895b)

Kinsler & Frey (1982b)

Morse & Feshbach (1953a,b)

A.1 General Properties

The series for the ordinary Bessel function follows directly from the differential equation (see Section 1).

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{2 \cdot (2n+2)} + \frac{x^4}{2 \cdot 4 (2n+2)(2n+4)} - \frac{x^6}{2 \cdot 4 \cdot 6 (2n+2)(2n+4)(2n+6)} \dots \right] \quad (\text{A-i})$$

From which the following properties may be deduced

$$J_0'(x) = -J_1(x) \quad (\text{A-ii})$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \quad (\text{A-iii})$$

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x) \quad (\text{A-iv})$$

When n is integer, $J_n(x)$ may be expressed by the definite integral

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \text{Cos}(x \text{Sin } w - nw) dw \quad (\text{A-v})$$

which is Bessel's original form. From this expression it is evident that J_n and its differential coefficients with respect to x are always less than unity.

General properties (the first three are repeated for convenience):

$$J_0'(x) = -J_1(x) \quad (\text{A-ii})$$

$$2nJ_n'(x) = J_{n-1}(x) - J_{n+1}(x) \quad (\text{A-iii})$$

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x) \quad (\text{A-iv})$$

$$\int xJ_0(x)dx = xJ_1(x) \quad (\text{A-vi})$$

$$\int xJ_0^2(x)dx = \frac{x^2}{2} [J_0^2(x) + J_1^2(x)] \quad (\text{A-vii})$$

$$\int J_1(x)dx = -J_0(x) \quad (\text{A-viii})$$

General solution of $J_n(x) = 0$.

The k^{th} root is given by

$$x = a - \frac{m-1}{8a} - \frac{4(m-1)(2m-31)}{38a^2} - \frac{32(m-1)}{15} \frac{(83m^2 - 982m + 3779)}{(8a)^5} + \dots \quad (\text{A-ix})$$

$$\text{where } a = \frac{\pi}{4}(2n - 1 + 4k) \\ m = 4n^2 .$$

It seems probable that between every pair of successive real roots of J_n there is exactly one real root of J_{n+1} . It does not appear that this has been strictly proved.

A.2 Solution of the integral

$$\int_0^a J_n(kr)J_n(k'r)rdr$$

where k and k' are different.

Gray & Matthews (1895b) show that

$$\int_0^a J_n(kr)J_n(k'r)rdr = 0 \tag{A-x}$$

if k, k' are different roots of $J_n(ar) = 0$
 if they are different roots of $J'_n(ar) = 0$
 if they are different roots of
 $AxJ'_n(ar) + BJ_n(ar) + 0$
 where A & B are independent of r

if $k = k'$, then the integral becomes

$$\int_0^a J_n^2(kr)rdr = \frac{a}{2k} \left[ka(J'_n(ka))^2 - J_n(ka)J'_n(ka) - ka J_n(ka)J''_n(ka) \right] \tag{A-xi}$$

which gives

$$\int_0^a J_n^2(kr)rdr = \frac{a^2}{2} \left[J_n'^2(ka) + \left[1 - \frac{n^2}{k^2 a^2} \right] J_n^2(ka) \right] \tag{A-xii}$$

where $J_n(ka)$ is not necessarily zero.

If (as in the case of the use of this integral in Section 1) $J_n(ka)$ is zero

$$\int_0^a J_n^2(kr)rdr = \frac{a^2}{2} J_n'^2(ka) \tag{A-xiii}$$

from (iv) and (iii)

$$\frac{a^2}{2} J_n'^2(ka) = \frac{a^2}{2} J_{n+1}^2(ka) \tag{A-xiv}$$

and this is the form used in Section 1.

A.3 Bessel function of the second kind, or Neumann functions

The Bessel function of the second kind, N_n , is a second particular solution of Bessel's differentiation equation. It can be expressed as

$$N_n(x) = \frac{2}{\pi} J_n(x) \left[\ln\left(\frac{x}{2}\right) + \gamma \right] + \frac{x}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m (h_m + h_{m+n}) x^{2m}}{2^{2m+n} m! (m+n)!} - \frac{x}{\pi} \sum_{m=0}^n \frac{(n-m-1)! x^{2m}}{2^{2(m-n)} m!} \quad (\text{A-xv})$$

where

$$h_m = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m}$$

$$h_0 = 0$$

$$\gamma = \text{Euler's Constant.}$$

or

$$N_n(x) = \lim_{\nu \rightarrow n} \frac{1}{\sin \nu \pi} \left[J_\nu(x) \cos \nu \pi - J_{-\nu}(x) \right] \quad (\text{A-xvi})$$

It can be shown that

$$N_{-n}(x) = (-1)^n N_n(x) \quad (\text{A-xvii})$$

and that

$$N_0(x) \rightarrow -\infty \text{ as } x \rightarrow 0. \quad (\text{A-xviii})$$

A4 Further identities

Morse and Feshbach (1953) state that

$$\frac{e^{inv}}{v} = \int_0^{\infty} \frac{J_0(\lambda v)}{(\lambda^2 - u^2)} \lambda d\lambda \quad (\text{A-xix})$$

and

$$e^{iv\theta} J_\nu(kR) = \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(kr) J_{m+\nu}(kr') \quad (\text{A-xx})$$

where r , r' and $R (= (r^2 + r'^2 - 2rr' \cos\phi)^{\frac{1}{2}})$ are the sides of a triangle, with ϕ being the angle between r and r' , and θ the angle between R and r' .

APPENDIX B POLYETHENETEREPHTHALATE

Polyetheneterephthalate is made by reacting together ethanediol, $\text{CH}_2\text{OHCH}_2\text{OH}$ and terephthalic acid, (benzene-1, 4-dicarboxylic acid, $\text{C}_6\text{H}_4(\text{CO}_2\text{H})_2$) to give the 'condensation polymer' of formula $[\text{O}_2\text{CC}_6\text{H}_4\text{CO}_2(\text{CH}_2)_2\text{O}]_n$. The polymer is a typical polyester.

Date:

Density	1 650 kgm^{-3}
Tensile Strength	350 MPa
Thermal Expansivity	$12 \times 10^{-6} \text{ K}^{-1}$
Effect of humidity	Negligible
Young's Modulus	3.5 GPa
Poisson's Ratio	0.49
Flexural Strength	400 MPa

Typical drumhead:

thickness	$1.9 \times 10^{-4} \text{ m}$
radius	0.32 m (small drum) 0.355 m (large drum)
area density	0.262 kgm^{-2}

Data kindly supplied by I.R. Watson Esq BA, Senior Chemist, Downe House.

APPENDIX C GREEN FUNCTIONS

References:

Ffowes Williams (1992)

Green (1828)

Pierce (1981)

Dettman (1984c)

Pozrikidis (1992d)

The Green's Function derives its name from the work of George Green (1828) in connection with Laplace's equation to produce solutions to electrostatic and magnetostatic boundary value problems. Subsequently this class of functions has seen many uses in fluid dynamics problems.

A Green function may be defined as

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi} \ln|\mathbf{r} - \mathbf{r}'| + H(\mathbf{r}, \mathbf{r}') \quad (\text{C-i})$$

where H is harmonic inside a smooth surface, S , enclosing a simply connected domain, D

$$G(\mathbf{r}, \mathbf{r}') \text{ is continuous in } D \cup S \text{ except at } \mathbf{r}' \text{ in } D \quad (\text{C-ii})$$

$$G(\mathbf{r}, \mathbf{r}') = 0 \text{ when } \mathbf{r} \text{ is on } S \quad (\text{C-iii})$$

it can be shown that

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r}) \quad (\text{C-iv})$$

and

$$(\nabla^2 + k^2) G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}') \quad (\text{C-v})$$

where δ is the Dirac delta function.

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