NAVIER–STOKES EQUATIONS ON THE β-PLANE

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A thesis presented for the degree of
Doctor of Philosophy

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June 2012

Dedicated to

my parents, my wife, my brothers and my sisters

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Abstract

Mathematical analysis has been undertaken for the vorticity formulation of the two dimensional Navier–Stokes equation on the β-plane with periodic boundary conditions. This equation describes the flow of fluid near the equator of the Earth. The long time behaviour of the solution of this equation is investigated and we show that, given a sufficiently regular forcing, the solution of the equation is nearly zonal. We use this result to show that, for sufficiently large β, the global attractor of this system reduces to a point. Another result can be obtained if we assume that the forcing is time-independent and sufficiently smooth. If the forcing lies in some Gevrey space, the slow manifold of the Navier–Stokes equation on the β-plane can be approximated with $O(\varepsilon^{n/2})$ accuracy for arbitrary $n = 0, 1, \cdots$, as well as with exponential accuracy.

Declaration

The work in this thesis is based on research carried out at the Numerical Analysis Group, the Department of Mathematical Sciences, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text. Chapters 3 and 4, are based on published work [2], done in collaboration with Dr Djoko Wirosoetisno, the Department of Mathematical Sciences, University of Durham, England.

Acknowledgements

All praise and thanks are firstly due to the Almighty who has given us the health and knowledge.

I wish to express my deepest gratitude to my supervisor Dr Djoko Wirosoetisno for his guidance, suggestions, help, regular supervision and patience. Undoubtedly, Dr. Wirosoetisno's efforts have enabled me to complete my work in the thesis. I would like also to thank the other staff members of the department in their various contexts for the support they have provided during my doctoral studies.

I am indebted to my father, my mother, my mother in law, my father in law, my brothers and my sisters for their support and encouragement. Naturally, it is my wife Luma who deserves the biggest thanks for supporting me during my study.

Finally, I would like to thank the Iraqi Cultural Attaché for their financial support and otherwise.

Contents

Chapter 1

Introduction

The incompressible Navier–Stokes equations are a set of nonlinear partial differential equations that describe the flow of fluid. They model weather, ocean currents, and movements of air, along with many other fluid flow phenomena. Furthermore, these equations describe the evolution of the velocity field $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$ at a point $\boldsymbol{x} = \boldsymbol{x}(x_1, \cdots, x_n) \in \mathbb{R}^n$ and time $t \in \mathbb{R}$, where $n = 2, 3$ is the space dimension. These equations can be written as:

$$\begin{aligned} &\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla p = \mu \Delta \boldsymbol{v} + f_{\boldsymbol{v}}, \\ &\nabla \cdot \boldsymbol{v} = 0. \end{aligned} \tag{1.0.1}$$

where $f_{\boldsymbol{v}}$ is the external body force, μ is the viscosity coefficient and p is the unknown pressure. For a review of the physical background and the derivation of the Navier–Stokes equations see, e.g., [12, 14]. It is well known that the two dimensional Navier–Stokes equation has been subject of a significant number of studies and its basic mathematical properties (existence, uniqueness, regularity, etc.) are now well understood, i.e., have global, in time, unique strong solution (see e.g. [16, 48, 59]). However, the solution of the problem of global regularity for $n = 3$ is still open to debate.

As a tool to understand various geophysical flows, it is often desirable to include the effect of planetary rotation, but a constant rotation rate (the so-called f-plane approximation) has no effect on the dynamics when periodic boundary conditions are used. To feel the effect of rotation, it is necessary to use the β-plane approximation,[1] which treats a region of the earth's surface as being locally flat. In this case, the variation of the Coriolis parameter F with latitude is approximated by:

$$F = f_0 + \beta y, \tag{1.0.2}$$

where f_0 is the value of F at the mid-latitude of the region and β the latitudinal gradient of F at that same latitude. The formula (1.0.2) is used to investigate both equatorial and mid-latitude phenomena (see e.g. [37, 50]). In the case of equatorial β-plane approximation we have

$$F = \beta y. \tag{1.0.3}$$

In this thesis we work on the two dimensional Navier–Stokes equation on β-plane near the equator of earth with periodic boundary conditions in x and y directions, with symmetry assumptions on the velocity. For our purpose in this work we deal with a vorticity form of equation (1.0.1). The vorticity form of the incompressible two dimensional Navier–Stokes equation represents a popular approach for the study of steady and unsteady two dimensional viscous flows. The equivalence between the vorticity form with the original primitive variable form of the viscous incompressible problem is well established for steady and unsteady state equations [27]. For the equivalence between the vorticity form and original primitive variable form for Navier–Stokes equations on the β-plane see Lemmata A.1.1 and A.1.2.

Simple physical arguments and numerical studies [47, 63] suggest that a rotation rate that varies as βy tends to force the solution to become more zonal, but to

[1]The β-plane approximation was first introduced into meteorological literature by Rossby C.G. et al (1939).

our knowledge no rigorous mathematical proof has been proposed. In this research the solution of the vorticity form of two dimensional Navier–Stokes equations on β-plane is demonstrated to be nearly zonal. This aim is achieved by splitting the solution into fast mode and slow mode with zero-frequency, and proving that the fast mode is small. The main difficulty with this method is to bound the nonlinear term, i.e., the energy transfer from slow mode to fast one. This energy transfer can be made easy to handle by a resonance between two fast and slow modes; in this case fast-fast-slow resonance is obtained. Because our equation is a PDE with an infinite number of modes, we have infinitely many near resonances as well, where the difference between two fast frequencies is small but not zero [61, 68]. A key part of the approach is an estimate involving near resonances in our equation (cf. Lemma 3.2.1).

In the past three decades, developments in dynamical system theory in fluid mechanics have contributed significantly to the understanding of complicated long time behaviour demonstrated by fluid flows. In addition, a mathematical approach to the finite dimensional behaviour in turbulence is presented by the theory of global attractor, estimates of its dimensions and inertial manifolds [10, 11, 19]. Foias and Prodi [18] were the first to investigate the long-term behaviour of the solution of the two dimensional Navier–Stokes equations. The global attractor, $\mathcal{A}$, of two dimensional Navier–Stokes equations was first obtained for a bounded domain by O. Laydyzhenskaya [35]. Thereafter, Temam and Foias [20] proved the finite dimensionality of the attractor in the sense of the Hausdorff dimension, $\dim_H$. Still later Temam [57] proved that:

$$\dim_H(\mathcal{A}) \leq c_1\, G, \tag{1.0.4}$$

where $G := |f|_{L^2}/(\mu^2\lambda_1)$ is the Grashof number and c_1 is a constant dependent on the domain. The sharp estimate, founded by Constantin et al [10], for the Hausdorff dimension of the global attractor $\mathcal{A}$ is:

$$\dim_H(\mathcal{A}) \le c\, G^{2/3}\, (1 + \log G)^{1/3}, \tag{1.0.5}$$

with periodic boundary conditions. The bound in (1.0.5) can be applied to our rotating case, but it does not take into account the effect of the rotation. Using our bounds on the fast mode, we show that the dimension of $\mathcal{A}$ is zero for sufficiently large β, reducing the long-time dynamics to a single steady (and stable) flow determined completely by the forcing f. This is to be contrasted with the situation for smaller (but still large) β, where the solution, although nearly zonal, evolves in time even though $\partial_t f = 0$.

One of the main methods of simplifying a dynamical system with two time scales is by reducing its dimension; this reduction of the dynamical system is called slow manifold. A slow manifold is approximately an invariant submanifold[2] of the state space of this system near which the dynamic is slow; its dimension is the number of slow variables, and these are defined by constraints slaving fast variables to slow ones [66–68]. This manifold is parameterized by a small number of system variables, knowing these variables suffices to approximate the full system state [65]. In this thesis a slow manifold means a manifold in phase space on which the normal velocity is small; if the normal velocity is zero, we have an exact slow manifold. To approximate a slow manifold for our equation with $O(\varepsilon^{n/2})$ accuracy, for arbitrary $n = 0, 1, \cdots$, the solution is truncated into low and high mode. The first part (low mode) which is a finite dimensional system whose size depends on ε can be made small with order of $\varepsilon^{n/2}$ by carefully balancing the truncation size and the estimates of the finite part (see Lemma 5.3.4). By using Gevrey Regularity for the solution, the high mode part can be made small of order $\varepsilon^{n/2}$ as well (see Lemma 5.3.2).

[2]Approximately invariant in the sense that trajectories are attracted to thin neighborhoods of this submanifold [67].

Therefore the total error is also small with $O(\varepsilon^{n/2})$ accuracy (see Lemma 5.3.4 and Theorem 5.3.3. We can approximate a slow manifold for the same equation up to an error that scales exponentially in ε as $\varepsilon \to 0$ by using the same method as above (see Lemmata 5.3.5 and 5.3.7 and Theorem 5.3.6).

Over all the aims of this thesis are to:

(i) Prove that the solution of the vorticity form of two dimensional Navier–Stokes equation on β-plane is nearly zonal;

(ii) Prove that the Hausdorff dimension of the global attractor of this equation is zero; and

(iii) Approximate the slow manifold for this equation with order of $\varepsilon^{n/2}$ accuracy and with exponential accuracy.

To our knowledge no rigorous mathematical proofs for the above aims have been proposed. The structure of the thesis starts with Chapter 2, in which the Navier–Stokes equation on β-plane is described, the vorticity form for this equation is derived and the bounds for H^{-1}, L^2 and H^m norms for the solution are found. These are used later in the thesis.

Chapter 3 is devoted to defining the zonal and non-zonal components for the solution of Navier–Stokes equation on β-plane, finding the bound of L^2 norm for the normal component of the solution of linear problem, finding the bounds for nonlinear term as well as L^2 and H^m bounds for the normal component of the solution of nonlinear problem. We describe some technical tools to define the attractor $\mathcal{A}$ for the vorticity form Navier–Stokes equation on β-plane in Chapter 4 and then prove that the Hausdorff dimension of the attractor, $\dim_H(\mathcal{A})$, is equal to zero.

Finally, in Chapter 5 Gevrey space is defined and Gevrey regularity reviewed for use in the research equation and then the slow manifold with $O(\varepsilon^{n/2})$ accuracy and with exponential accuracy is approximated.

Chapter 2

Preliminary Estimates

This chapter is divided into four sections. In section 2.1 we give a brief review of some notation adopted in the thesis. We introduce the Navier–Stokes equations on the β-plane in section 2.2. In section 2.3 we derive the vorticity form of the equations. Finally we give the H^{-1}, L^2 and H^m bounds for the solution of our equation (vorticity form).

2.1 Notation

Let $\mathcal{M} := [0, L_1] \times [-L_2/2, L_2/2]$ be a bounded set. We denote by $C(\mathcal{M})$ the set of all continuous functions on $\mathcal{M}$, and define $C^r(\mathcal{M})$, $r \in \mathbb{Z}_+$, as the space of all functions on $\mathcal{M}$ which are continuously differentiable up to order r, i.e.

$$C^r(\mathcal{M}) = \big\{u : D^\alpha u \in C(\mathcal{M}) \text{ for all } |\alpha| \le r\big\}, \tag{2.1.1}$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, and α_1, α_2 are non-negative integers with $|\alpha| = \alpha_1 + \alpha_2$, and D^α is defined as follows

$$D^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}}. \tag{2.1.2}$$

Another important function space is the Lebesgue space $L^p(\mathcal{M})$, $1 \leq p < \infty$, which consists of all Lebesgue measurable functions $u : \mathcal{M} \to \mathbb{R}$, with

$$\int_{\mathcal{M}} |u(\boldsymbol{x})|^p \, \mathrm{d}\boldsymbol{x} < \infty. \tag{2.1.3}$$

$L^p(\mathcal{M})$ is a Banach space for the norm

$$|u|_{L^p} := \Big(\int_{\mathcal{M}} |u(\boldsymbol{x})|^p \, \mathrm{d}\boldsymbol{x} \Big)^{1/p}. \tag{2.1.4}$$

For $p = \infty$, $L^\infty(\mathcal{M})$ is the space of all functions on $\mathcal{M}$ which are measurable and essentially bounded; it is also Banach space for the norm

$$\begin{aligned} |u|_{L^\infty} &= \operatorname{ess} \sup_{\boldsymbol{x} \in \mathcal{M}} |u(\boldsymbol{x})| \\ &= \inf \big\{ \sup_{\boldsymbol{x} \in S} |u(\boldsymbol{x})| : S \subset \overline{\mathcal{M}}, \text{with } \mathcal{M} \backslash S \text{ of measure zero} \big\}. \end{aligned} \tag{2.1.5}$$

If $p = 2$, then $L^2(\mathcal{M})$ is a Hilbert space under the inner product

$$(u, v)_{L^2} = \int_{\mathcal{M}} u(\boldsymbol{x}) \, v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}. \tag{2.1.6}$$

In addition we define the Sobolev space H^s as follows

$$H^s = \big\{ u : D^\alpha u \in L^2(\mathcal{M}), \text{ for all } 0 \leq |\alpha| \leq s \big\}. \tag{2.1.7}$$

The Sobolev space H^s is a Hilbert space for the inner product

$$(u, v)_{H^s} = \sum_{|\alpha| \leq s} (D^\alpha u, D^\alpha v)_{L^2}, \tag{2.1.8}$$

and the norm associated with H^s is defined by

$$|u|_{H^s} = \Big(\sum_{|\alpha| \leq s} \int_{\mathcal{M}} |D^\alpha u(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} \Big)^{1/2}. \tag{2.1.9}$$

We denote by $\dot{H}^s_{per}(\mathcal{M})$ ($\dot{L}^p_{per}(\mathcal{M})$) the Sobolev spaces (Lebesgue space) of all functions u with periodic boundary condition on $\mathcal{M}$, and $\int_{\mathcal{M}} u(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = 0$. For simplicity, we will use $H^s(\mathcal{M})$ ($L^p(\mathcal{M})$) for $\dot{H}^s_{per}(\mathcal{M})$ ($\dot{L}^p_{per}(\mathcal{M})$) from here on. We define now function spaces depending on time and space. Let X be a Banach space, we denote by $C([0,T];X)$ the space of all continuous functions, u, from $[0,T]$ into X. $C([0,T];X)$ is a Banach space with the norm

$$|u|_{C(0,T;X)} := \sup_{t\in[0,T]} |u(t)|_X. \tag{2.1.10}$$

Furthermore we define the Lebesgue space, $L^p(0,T;X)$, consists of all functions $u(t)$ that take values in X for almost every $t \in [0,T]$, such that the L^p norm of $u(t)$ is finite. $L^p(0,T;X)$ is a Banach space with the norms

$$\begin{aligned} |u|_{L^p(0,T;X)} &:= \Big(\int_0^T |u(t)|_X^p \, \mathrm{d}t \Big)^{1/p}, \\ |u|_{L^\infty(0,T;X)} &:= \operatorname{ess} \sup_{t\in(0,T)} |u(t)|_X. \end{aligned} \tag{2.1.11}$$

Note that $C([0,T];X)$ is dense in $L^p(0,T,X)$. We recall now the Leibniz formula. Let $f, g \in H^s$. Then their product, fg, is also in H^s, and

$$D^\alpha(fg) = \sum_{0\le|\beta|\le|\alpha|} \binom{|\alpha|}{|\beta|} D^\beta f \, D^{\alpha-\beta} g, \tag{2.1.12}$$

where

$$\binom{|\alpha|}{|\beta|} = \frac{|\alpha|!}{|\beta|!(|\alpha|-|\beta|)!}, \tag{2.1.13}$$

is the usual binomial coefficient. For our purpose in this work we need to recall the Sobolev embedding theorem.

Theorem 2.1.1 (Sobolev embedding theorem).[1] Suppose that $u \in H^s(\mathcal{M})$ then

1. If $s < m/2$ then $u \in L^{2m/(m-2s)}(\mathcal{M})$, and there exists a constant c such that

$$|u|_{L^{2m/(m-2s)}} \le c\, |u|_{H^s}. \tag{2.1.14}$$

[1] See e.g. [1, 48, 59].

2. If $s = m/2$ then $u \in L^p(\mathcal{M})$ for every $1 \leq p < \infty$, and for each p there exists a constant $c = c(\mathcal{M})$ such that

$$|u|_{L^p} \leq c\,|u|_{H^s}. \tag{2.1.15}$$

We shall use in our work the following Sobolev interpolation inequality.[2]

Lemma 2.1.2 If $u \in H^s(\mathcal{M})$, then there exist a constant $c = c(\mathcal{M})$

$$|u|_{H^s} \leq c\,|u|_{H^l}^{(k-s)/(k-l)}\,|u|_{H^k}^{(s-l)/(k-l)}, \tag{2.1.16}$$

for $0 \leq l < s < k$.

Lemma 2.1.3 (Agmon Inequality).[3] Let Ω be a bounded subset of $\mathbb{R}^1$, there exists a constant c depending only on Ω such that

$$|u|_{L^\infty} \leq c\,|u|_{L^2}^{1/2}\,|\nabla u|_{L^2}^{1/2}, \quad \forall\, u \in H^1(\Omega). \tag{2.1.17}$$

If Ω is a bounded subset of $\mathbb{R}^2$, there exists a constant $c = (\Omega)$ such that

$$|u|_{L^\infty} \leq c\,|u|_{L^2}^{1/2}\,|\nabla^2 u|_{L^2}^{1/2}, \quad \forall\, u \in H^2(\Omega) \tag{2.1.18}$$

Lemma 2.1.4 If $r \in [2, \infty)$, then there exists a constant c such that, for any $u \in H^1$ we have

$$|u|_{L^r} \leq c\,|u|_{L2}^{1-\sigma}\,|\nabla u|_{L^2}^{\sigma}, \tag{2.1.19}$$

where $\sigma = (r-4)/r$.

Lemma 2.1.5 (Young's inequality). For $a, b \geq 0$ and any $\epsilon > 0$ we have

$$a\,b \leq \frac{1}{p}\,(a\,\epsilon)^p + \frac{1}{q}\,(\frac{b}{\epsilon})^q. \tag{2.1.20}$$

where $1 < p, q < \infty$ with $1/p + 1/q = 1$.

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[2]For the proof we refer to [48, Ch. 6].

[3]See [59, p. 52]

Lemma 2.1.6 (Hölder's inequality). Let $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$. If $f \in L^p(\mathcal{M})$ and $g \in L^q(\mathcal{M})$, then $f\,g \in L^1(\mathcal{M})$ with

$$|f\,g|_{L^1} \leq |f|_{L^p}\,|g|_{L^q}. \tag{2.1.21}$$

Note that if $p = q = 2$, then this is the Cauchy-Schwarz inequality. Furthermore, we shall need in our work the generalised form of Hölder's inequality with three exponents. Let $1 \leq p, q, r \leq \infty$. If $f \in L^p(\mathcal{M}), g \in L^q(\mathcal{M})$, and $h \in L^r(\mathcal{M})$ with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$$

then $f\,g\,h \in L^1(\mathcal{M})$ with

$$|f\,g\,h|_{L^1} \leq |f|_{L^p}\,|g|_{L^q}\,|h|_{L^r}.$$

Lemma 2.1.7 (Poincaré inequality). If $u \in H^1(\mathcal{M})$, then there exists a positive constant c_0 such that

$$c_0\,|u|_{L^2} \leq |\nabla u|_{L^2}, \tag{2.1.22}$$

where $c_0 = \frac{2\pi}{\sqrt{L_1 L_2}}$.

Theorem 2.1.8 The Sobolev space spaces $H^s(\mathcal{M})$, equipped with appropriate norms, satisfy

(i) H^s is a Banach space, separable and reflexive [1].

(ii) $H^{s+1}(\mathcal{M})$ is compactly embedded in $H^s(\mathcal{M})$ [48].

2.2 Statement of the problem

In dimensional form, the two-dimensional Navier–Stokes equations read

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \beta y\,\boldsymbol{v}^{\perp} + \nabla p = \mu \Delta \boldsymbol{v} + f_{\boldsymbol{v}}. \tag{2.2.1}$$

Here $\boldsymbol{v} = (u, v)$ is the velocity, $\boldsymbol{v}^\perp = (-v, u)$, p is the pressure obtained by enforcing the incompressibility constraint $\nabla \cdot \boldsymbol{v} = 0$ and μ is the viscosity coefficient. In what follows, we will work with the dimensionless form

$$\begin{aligned} &\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \frac{Y}{\varepsilon} \boldsymbol{v}^\perp + \nabla p = \mu \Delta \boldsymbol{v} + f_{\boldsymbol{v}}, \\ &\nabla \cdot \boldsymbol{v} = 0. \end{aligned} \tag{2.2.2}$$

Note that we have replaced βy in (2.2.1) by $Y(y)/\varepsilon$, where Y is the periodic extension of

$$Y(y) = \begin{cases} +1 & \text{if } y = -L_2/2, \\ y & \text{if } y \in (-L_2/2, L_2/2]. \end{cases} \tag{2.2.3}$$

In (2.2.2), the parameter ε is called the Rossby number; the strong rotation limit corresponds to the regime when ε tends to zero. We work with $\boldsymbol{x} = (x, y) \in \mathcal{M}$, with periodic boundary conditions in both directions, i.e.,

$$\begin{aligned} u(x, y + L_2, t) &= u(x, y, t) \text{ and } u(x + L_1, y, t) = u(x, y, t), \\ v(x, y + L_2, t) &= v(x, y, t) \text{ and } v(x + L_1, y, t) = v(x, y, t). \end{aligned} \tag{2.2.4}$$

Furthermore we assume the following symmetry on the velocity

$$u(x, -y, t) = u(x, y, t) \quad \text{and} \quad v(x, -y, t) = -v(x, y, t). \tag{2.2.5}$$

In addition we make the simplifying assumption that the initial condition, $\boldsymbol{v}(\cdot, 0) = \boldsymbol{v}_0(\cdot)$, and $f_{\boldsymbol{v}}$ have zero average over $\mathcal{M}$,

$$\int_{\mathcal{M}} \boldsymbol{v}_0(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = 0 \qquad \text{and} \qquad \int_{\mathcal{M}} f_{\boldsymbol{v}}(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x} = 0;$$

the condition on $f_{\boldsymbol{v}}$ ensures that the zero average of $\boldsymbol{v}(\boldsymbol{x}, t)$ is preserved under the time evolution. Note also that periodicity and (2.2.5) imply that

$$v(x, -L_2/2, t) = v(x, L_2/2, t) = 0. \tag{2.2.6}$$

Equation (2.2.2) describes the movement of the fluid near the equator of the earth.

2.3 Vorticity form of the Navier–Stokes equations on the β-plane

In this section we recast (2.2.2) into an alternative form[4] in terms of streamfunction ψ and vorticity ω. This form provides a better understanding of the physical mechanisms driving the flow than the primitive variable form in terms of $\boldsymbol{v}$. Moreover the vorticity form is useful for numerical work. Let us now find a vorticity form of our equation. Taking $\nabla^{\perp}\cdot$ of the equation (2.2.2), where $\nabla^{\perp}\cdot\boldsymbol{v} = -\partial_y u + \partial_x v =: \omega$, we have

$$\nabla^{\perp}\cdot\partial_t\boldsymbol{v} + \nabla^{\perp}\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{v}) + \frac{1}{\varepsilon}\nabla^{\perp}\cdot(Y\boldsymbol{v}^{\perp}) + \nabla^{\perp}\cdot\nabla p = \mu\nabla^{\perp}\cdot\Delta\boldsymbol{v} + \nabla^{\perp}\cdot f_{\boldsymbol{v}}. \quad (2.3.1)$$

We compute every term separately. The first term is

$$\begin{aligned}\nabla^{\perp}\cdot\partial_t\boldsymbol{v} &= (-\partial_y,\partial_x)\cdot(\partial_t u,\partial_t v) = -\partial_y\partial_t u + \partial_x\partial_t v\\ &= -\partial_t\partial_y u + \partial_t\partial_x v = \partial_t\nabla^{\perp}\cdot\boldsymbol{v} = \partial_t\omega.\end{aligned} \quad (2.3.2)$$

The second term is

$$\begin{aligned}\nabla^{\perp}\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{v}) &= (-\partial_y,\partial_x)\cdot[(u\partial_x + v\partial_y)(u,v)]\\ &= (-\partial_y,\partial_x)\cdot(u\,\partial_x u + v\,\partial_y u, u\,\partial_x v + v\,\partial_y v)\\ &= -\partial_y u\,\partial_x u - u\,\partial_{xy}u - \partial_y v\,\partial_y u - v\,\partial_{yy}u\\ &\qquad + \partial_x u\,\partial_x v + u\,\partial_{xx}v + \partial_x v\,\partial_y v + v\,\partial_{xy}v\\ &= -\partial_y u(\partial_x u + \partial_y v) + \partial_x v(\partial_x u + \partial_y v)\\ &\qquad + u\,\partial_x(\partial_x v - \partial_y u) + v\partial_y(\partial_x v - \partial_y u)\\ &= \boldsymbol{v}\cdot\nabla(\nabla^{\perp}\cdot\boldsymbol{v}) = \boldsymbol{v}\cdot\nabla\,\omega,\end{aligned} \quad (2.3.3)$$

since $\partial_x u + \partial_y v = 0$. Next,

$$\begin{aligned}\mu\nabla^{\perp}\cdot\Delta\boldsymbol{v} &= \mu\Delta\nabla^{\perp}\cdot\boldsymbol{v}\\ &= \mu\Delta\omega.\end{aligned} \quad (2.3.4)$$

[4]We call this form the vorticity form.

We also have

$$\nabla^\perp \cdot \nabla p = 0. \tag{2.3.5}$$

Finally

$$\nabla^\perp \cdot (Y \boldsymbol{v}^\perp) = Y\, \nabla^\perp \cdot \boldsymbol{v}^\perp + \boldsymbol{v}^\perp \cdot \nabla^\perp Y = vY', \tag{2.3.6}$$

where Y' is taken in the distribution sense. Gathering (2.3.2), (2.3.3), (2.3.4), (2.3.5) and (2.3.6) with $f := \nabla^\perp \cdot f_{\boldsymbol{v}}$. Then (2.3.1) becomes

$$\partial_t \omega + \boldsymbol{v} \cdot \nabla \omega + \frac{1}{\varepsilon} vY' = \mu \Delta \omega + f. \tag{2.3.7}$$

with initial condition $\omega(\cdot, 0) = \nabla^\perp \cdot \boldsymbol{v}(\cdot, 0)$. By our assumption on $\boldsymbol{v}$, the integral of ω over $\mathcal{M}$ is zero; similarly, Δ^{-1} is defined uniquely by the zero-integral condition. The symmetry (2.2.5) implies that

$$\omega(x, -y, t) = -\omega(x, y, t) \tag{2.3.8}$$

and

$$\omega(x, -L_2/2, t) = \omega(x, L_2/2, t) = 0. \tag{2.3.9}$$

Now $Y'(y) = 1 - L_2\, \delta(y - L_2/2)$, where δ is the Dirac distribution. Using the fact that $v(x, \pm L_2/2, t) = 0$, we replace $Y'v$ by v in (2.3.7) and write

$$\partial_t \omega + \boldsymbol{v} \cdot \nabla \omega + \frac{1}{\varepsilon} v = \mu \Delta \omega + f. \tag{2.3.10}$$

This is the form that we will be mostly working with. Since $\boldsymbol{v} \in L^2(\mathcal{M})$, then by Helmholtz–Hodge Decomposition Theorem A.0.10, the vector field $\boldsymbol{v}$ can be decomposed into

$$\boldsymbol{v} = \nabla^\perp \psi + \nabla \mathbf{V}. \tag{2.3.11}$$

Moreover, the streamfunction ψ and $\mathbf{V}$ are unique, up to an additive constant. Since $\mathbf{V}$ does not depend on x, y, so we can take $\mathbf{V} = 0$,

$$\boldsymbol{v} = \nabla^\perp \psi. \tag{2.3.12}$$

From this we have

$$\omega = \nabla^{\perp} \cdot \boldsymbol{v} = \nabla^{\perp} \cdot \nabla^{\perp}\psi = \Delta\psi, \tag{2.3.13}$$

and

$$\psi = \Delta^{-1}\omega, \tag{2.3.14}$$

where Δ^{-1} is defined uniquely by the condition

$$\int_{\mathcal{M}} \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0. \tag{2.3.15}$$

Then $\boldsymbol{v} = (u, v) = \nabla^{\perp}\Delta^{-1}\omega$; that is

$$\begin{aligned} u &= -\partial_y \Delta^{-1}\omega, \\ v &= \partial_x \Delta^{-1}\omega. \end{aligned} \tag{2.3.16}$$

We define now a linear operator $L = \partial_x \Delta^{-1}$. It is an antisymmetric operator, for any ω, $\breve{\omega}$,

$$\begin{aligned} (L\omega, \breve{\omega})_{L^2} &= \int_{\mathcal{M}} \partial_x\, \Delta^{-1}\omega(\boldsymbol{x})\, \breve{\omega}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathcal{M}} \partial_x\, \psi(\boldsymbol{x})\, \Delta\breve{\psi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathcal{M}} \partial_x\, \Delta\psi(\boldsymbol{x})\, \breve{\psi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\ &= -\int_{\mathcal{M}} \Delta\psi(\boldsymbol{x})\, \partial_x\, \breve{\psi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\ &= -\int_{\mathcal{M}} \omega(\boldsymbol{x})\, \partial_x\, \Delta^{-1}\breve{\omega}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\ &= -(\omega, L\breve{\omega})_{L^2}. \end{aligned} \tag{2.3.17}$$

In addition let $A = -\Delta$, which is called the Stokes operator. The operator A is positive, self-adjoint, invertible, and its inverse is compact with the property[5]

$$(A\omega, \omega)_{L^2} = |\nabla\omega|^2_{L^2}. \tag{2.3.18}$$

Finally we define the bilinear operator

$$\boldsymbol{v} \cdot \nabla\breve{\omega} = \nabla^{\perp}\psi \cdot \nabla\breve{\omega} = (\nabla^{\perp}\Delta^{-1}\omega) \cdot \nabla\breve{\omega} =: B(\omega, \breve{\omega}), \tag{2.3.19}$$

[5]This means that A is coercive.

with the property[6]

$$\begin{aligned}(B(\omega^\sharp,\omega),\omega)_{L^2} &= \int_{\mathcal{M}} (\nabla^\perp\psi^\sharp\cdot\nabla\omega)\,\omega\,\mathrm{d}\boldsymbol{x}\\ &= \int_{\mathcal{M}} (-\partial_y\psi^\sharp\,\partial_x\omega + \partial_x\psi^\sharp\,\partial_y\omega)\,\omega\,\mathrm{d}\boldsymbol{x}\\ &= \int_{\mathcal{M}} (-\partial_y(\psi^\sharp\,\partial_x\omega) + \psi^\sharp\,\partial_{xy}\omega + \partial_x(\psi^\sharp\,\partial_y\omega) - \psi^\sharp\,\partial_{xy}\omega)\,\omega\,\mathrm{d}\boldsymbol{x}\\ &= \int_{\mathcal{M}} -\psi^\sharp\,\partial_x\omega\,\partial_y\omega + \psi^\sharp\,\partial_y\omega\,\partial_x\omega\,\mathrm{d}\boldsymbol{x}\\ &= 0.\end{aligned} \tag{2.3.20}$$

Thus (2.3.10) will take the following functional form

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} + \frac{1}{\varepsilon}L\omega + \mu A\omega + B(\omega,\omega) = f. \tag{2.3.21}$$

For more about the equivalence between the vorticity form and the primitive variables form for the Navier–Stokes equation on the β-plane see Lemmata A.2.8 and A.1.2.

2.4 H^{-1}, L^2 and H^m bounds for the solution of the equation

The estimates derived in this section are standard from the theory of 2d NSE (see, e.g., [48, 58]), with very minor modifications to handle the Coriolis term. We gather them here for later use.[7] Note that here, c is a generic positive constant depending only on $\mathcal{M}$ whose value may not be the same each time it appears, while numbered constants such as c_1 have fixed value. In this thesis $L^\infty_t L^q_{\boldsymbol{x}} := L^\infty((0,\infty); L^q(\mathcal{M}))$ and $L^\infty_t H^s_{\boldsymbol{x}} := L^\infty((0,\infty); H^s(\mathcal{M}))$.

[6] This property means that B conserves energy.

[7] For the existence and uniqueness of the solution of Navier–Stokes equation on the β-plane see the Appendix, Lemma A.2.8. For more regular solutions for the 2d NSE with no β see, e.g., [14, 16, 48, 58] .

Lemma 2.4.1 Let $\boldsymbol{v}_0 \in L^2(\mathcal{M})$ and $f \in L_t^\infty H_{\boldsymbol{x}}^{-2}$. Then for all $t \geq 0$ there exists a solution $\omega(t) \in H^{-1}(\mathcal{M})$ of (2.3.21) with $\omega(0) = \omega_0 = \nabla^\perp \cdot \boldsymbol{v}_0$ and

$$|\omega(t)|_{H^{-1}}^2 \leq I'(|\boldsymbol{v}_0|_{L^2}, |f|_{L_t^\infty H_{\boldsymbol{x}}^{-2}}; \mu), \tag{2.4.1}$$

Moreover there exist $t_0(|\boldsymbol{v}_0|_{L^2}, |f|_{L_t^\infty H_{\boldsymbol{x}}^{-2}}; \mu)$, $I(|f|_{L_t^\infty H_{\boldsymbol{x}}^{-2}}; \mu)$ and $K(|f|_{L_t^\infty H_{\boldsymbol{x}}^{-2}}; \mu)$ such that for all $t \geq t_0$,

$$|\omega(t)|_{H^{-1}}^2 \leq I(|f|_{L_t^\infty H_{\boldsymbol{x}}^{-2}}; \mu), \tag{2.4.2}$$

and

$$\int_t^{t+1} |\omega(\tau)|_{L^2}^2 \, \mathrm{d}\tau \leq K(|f|_{L_t^\infty H_{\boldsymbol{x}}^{-2}}; \mu), \tag{2.4.3}$$

where

$$\begin{aligned} I' &= |\boldsymbol{v}(0)|_{L^2}^2 \, \mathrm{e}^{-c_0^2 \mu t/2} + \frac{1}{c_0^2 \, \mu^2} \, |f|_{L_t^\infty H_{\boldsymbol{x}}^{-2}}^2 (1 - \mathrm{e}^{-c_0^2 \mu t/2}), \\ I &= \frac{2}{c_0^2 \, \mu^2} \, |f|_{L_t^\infty H_{\boldsymbol{x}}^{-2}}^2, \\ K &= \frac{c}{c_0^2 \, \mu^3} \, |f|_{L_t^\infty H_{\boldsymbol{x}}^{-2}}^2. \end{aligned}$$

Proof. Multiplying (2.3.21) by $\psi = -\Delta^{-1}\omega$ in L^2, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\omega|_{H^{-1}}^2 + \mu \, |\omega|_{L^2}^2 - \frac{1}{\varepsilon} \, (L\omega, \Delta^{-1}\omega)_{L^2} - (B(\omega,\omega), \Delta^{-1}\omega)_{L^2} = -(f, \Delta^{-1}\omega)_{L^2}. \tag{2.4.4}$$

Calculating the terms on the left-hand side of (2.4.4), the antisymmetric term is

$$(L\omega, \Delta^{-1}\omega)_{L^2} = \int_{\mathcal{M}} v \, \psi \, \mathrm{d}\boldsymbol{x} = \int_{\mathcal{M}} \partial_x \psi \, \psi \, \mathrm{d}\boldsymbol{x} = \frac{1}{2} \int_{\mathcal{M}} \partial_x(\psi^2) \, \mathrm{d}\boldsymbol{x} = 0. \tag{2.4.5}$$

Next, the nonlinear term is

$$\begin{aligned} (B(\omega,\omega), \Delta^{-1}\omega)_{L^2} &= \int_{\mathcal{M}} (v \cdot \nabla\omega) \, \psi \, \mathrm{d}\boldsymbol{x} \\ &= \int_{\mathcal{M}} (\nabla^\perp\psi \cdot \nabla\omega) \, \psi \, \mathrm{d}\boldsymbol{x} \\ &= \int_{\mathcal{M}} (-\partial_y\psi \, \partial_x\omega + \partial_x\psi \, \partial_y\omega) \, \psi \, \mathrm{d}\boldsymbol{x} \\ &= \int_{\mathcal{M}} [\partial_y(\partial_x(\psi) \, \omega)\psi - \partial_x(\partial_y(\psi) \, \omega)\psi] \, \mathrm{d}\boldsymbol{x} \\ &= \int_{\mathcal{M}} (-\partial_x\psi \, \partial_y\psi \, \omega + \partial_x\psi \, \partial_y\psi \, \omega) \, \mathrm{d}\boldsymbol{x} = 0. \end{aligned} \tag{2.4.6}$$

By using Cauchy-Schwarz inequality and Young's inequality (2.1.20), the right-hand side of (2.4.4) can be majorized by

$$|-(f,\Delta^{-1}\omega)_{L^2}| \le |f|_{H^{-2}}\,|\omega|_{L^2} \le \frac{1}{2\mu}\,|f|^2_{H^{-2}} + \frac{\mu}{2}\,|\omega|^2_{L^2}. \tag{2.4.7}$$

Thus, (2.4.4) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}|\omega|^2_{H^{-1}} + \mu\,|\omega|^2_{L^2} \le \frac{1}{\mu}|f|^2_{H^{-2}}. \tag{2.4.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}|\omega|^2_{H^{-1}} + \frac{\mu}{2}\,|\omega|^2_{L^2} + \frac{\mu}{2}\,|\omega|^2_{L^2} \le \frac{1}{\mu}|f|^2_{H^{-2}}. \tag{2.4.9}$$

Applying Poincaré inequality on the second term in the left-hand side

$$\frac{\mathrm{d}}{\mathrm{d}t}|\omega|^2_{H^{-1}} + \frac{c_0^2\,\mu}{2}\,|\omega|^2_{H^{-1}} + \frac{\mu}{2}\,|\omega|^2_{L^2} \le \frac{1}{\mu}|f|^2_{H^{-2}}. \tag{2.4.10}$$

Multiplying by $\mathrm{e}^{\nu t}$, $\nu = c_0^2\,\mu/2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\nu t}\,|\omega|^2_{H^{-1}}) + \frac{\mu}{2}\,\mathrm{e}^{\nu t}\,|\omega|^2_{L^2} \le \frac{1}{\mu}\,\mathrm{e}^{\nu t}\,|f|^2_{H^{-2}}. \tag{2.4.11}$$

Integrating from 0 to t, and multiplying by $\mathrm{e}^{-\nu t}$, we obtain

$$\begin{aligned} |\omega(t)|^2_{H^{-1}} &\le |\omega(0)|^2_{H^{-1}}\,\mathrm{e}^{-\nu t} + \frac{1}{c_0^2\,\mu^2}\,|f|^2_{L^\infty_t H^{-2}_{\boldsymbol{x}}}(1-\mathrm{e}^{-\nu t}) \\ &\le |\boldsymbol{v}(0)|^2_{L^2}\,\mathrm{e}^{-\nu t} + \frac{1}{c_0^2\,\mu^2}\,|f|^2_{L^\infty_t H^{-2}_{\boldsymbol{x}}}(1-\mathrm{e}^{-\nu t}) = I', \end{aligned} \tag{2.4.12}$$

and so there is a time $t_0(|\boldsymbol{v}_0|_{L^2}, |f|_{L^\infty_t H^{-2}_{\boldsymbol{x}}};\mu)$, which we can take as [8]

$$t_0(|\boldsymbol{v}_0|_{L^2}, |f|_{L^\infty_t H^{-2}_{\boldsymbol{x}}};\mu) = \max\Big(\frac{-1}{c_0^2\,\mu}\,\ln\Big(\frac{|f|^2_{L^\infty_t H^{-2}_{\boldsymbol{x}}}}{c_0^2\,\mu^2\,|\boldsymbol{v}(0)|_{L^2}}\Big), 0\Big). \tag{2.4.13}$$

For all $t \ge t_0$ we have

$$|\omega(t)|^2_{H^{-1}} \le \frac{2}{c_0^2\,\mu^2}\,|f|^2_{L^\infty_t H^{-2}_{\boldsymbol{x}}} = I. \tag{2.4.14}$$

We now integrate (2.4.8) from t to $t+1$

$$\begin{aligned} \int_t^{t+1} |\omega(\tau)|^2_{L^2}\,\mathrm{d}\tau &\le \frac{1}{\mu}\Big\{\frac{1}{\mu}\,|f|^2_{L^\infty_t H^{-2}_{\boldsymbol{x}}} + \frac{2}{c_0^2\,\mu^2}\,|f|^2_{L^\infty_t H^{-2}_{\boldsymbol{x}}}\Big\} \\ &\le \frac{c}{c_0^2\,\mu^3}\,|f|^2_{L^\infty_t H^{-2}_{\boldsymbol{x}}} = K. \end{aligned} \tag{2.4.15}$$

□

[8] See [48, p. 311].

Lemma 2.4.2 Let $\boldsymbol{v}_0 \in L^2(\mathcal{M})$ and $f \in L^\infty_t H^{-1}_{\boldsymbol{x}}$. Then for all $t \geq 0$ there exist a solution $\omega(t) \in L^2(\mathcal{M})$ of (2.3.21). Moreover there exist $t_1(|\boldsymbol{v}_0|_{L^2}, |f|_{L^\infty_t H^{-1}_{\boldsymbol{x}}}; \mu)$, $I_0(|f|_{L^\infty_t H^{-1}_{\boldsymbol{x}}}; \mu)$ and $K_0(|f|_{L^\infty_t H^{-1}_{\boldsymbol{x}}}; \mu)$ such that, for all $t \geq t_1$,

$$|\omega(t)|^2_{L^2} \leq I_0(|f|_{L^\infty_t H^{-1}_{\boldsymbol{x}}}; \mu) \tag{2.4.16}$$

and

$$\int_t^{t+1} |\omega(\tau)|^2_{H^1}\,\mathrm{d}\tau \leq K_0(|f|_{L^\infty_t H^{-1}_{\boldsymbol{x}}}; \mu), \tag{2.4.17}$$

where

$$\begin{aligned} I_0 &= \frac{c}{c_0^2\,\mu^3}\,|f|^2_{L^\infty_t H^{-2}_{\boldsymbol{x}}} + \frac{1}{\mu}\,|f|^2_{L^\infty_t H^{-1}_{\boldsymbol{x}}} \\ K_0 &= \frac{1}{\mu}\Big\{\frac{c}{c_0^2\,\mu^3}\,|f|^2_{L^\infty_t H^{-2}_{\boldsymbol{x}}} + \frac{2}{\mu}\,|f|^2_{L^\infty_t H^{-1}_{\boldsymbol{x}}}\Big\}. \end{aligned} \tag{2.4.18}$$

Proof. Multiplying (2.3.21) by ω in L^2, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\omega|^2_{L^2} + \mu\,|\omega|^2_{H^1} + \frac{1}{\varepsilon}\,(L\omega,\omega)_{L^2} + (B(\omega,\omega),\omega)_{L^2} = (f,\omega)_{L^2}. \tag{2.4.19}$$

The antisymmetric term,

$$\begin{aligned} (L\omega,\omega)_{L^2} &= \int_{\mathcal{M}} v\,\omega\,\mathrm{d}\boldsymbol{x} = \int_{\mathcal{M}} -v\,\partial_y u + v\partial_x v \mathrm{d}\boldsymbol{x} \\ &= \int_{\mathcal{M}} -v\,\partial_y u\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\mathcal{M}} \partial_x v^2 \mathrm{d}\boldsymbol{x} \\ &= -\int_{\mathcal{M}} v\,\partial_y u\,\mathrm{d}\boldsymbol{x} = \int_{\mathcal{M}} u\,\partial_y v\,\mathrm{d}\boldsymbol{x} \\ &= -\int_{\mathcal{M}} u\,\partial_x u\,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\int_{\mathcal{M}} \partial_x u^2 \mathrm{d}\boldsymbol{x} = 0, \end{aligned} \tag{2.4.20}$$

because the periodic boundary conditions and $\nabla \cdot \boldsymbol{v} = 0$. The nonlinear term, by (2.3.20), is

$$(B(\omega,\omega),\omega)_{L^2} = 0. \tag{2.4.21}$$

The right-hand side of (2.4.19), by applying Hölder's inequality (2.1.21) and Young's inequality (2.1.22) inequality, becomes

$$|(f,\omega)_{L^2}| \leq |f|_{H^{-1}}\,|\omega|_{H^1} \leq \frac{1}{2\mu}\,|f|^2_{H^{-1}} + \frac{\mu}{2}\,|\nabla\omega|^2_{L^2}. \tag{2.4.22}$$

Hence, (2.4.19) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}|\omega|_{L^2}^2 + \mu\,|\omega|_{H^1}^2 \le \frac{1}{\mu}|f|_{H^{-1}}^2. \tag{2.4.23}$$

We integrate (2.4.23) from τ to t, where $t-1 \le \tau < t$, giving

$$|\omega(t)|_{L^2}^2 + \int_\tau^t |\omega(s)|_{H^1}^2\,\mathrm{d}s \le |\omega(\tau)|_{L^2}^2 + \frac{1}{\mu}|f|_{L_t^\infty H_{\boldsymbol{x}}^{-1}}^2. \tag{2.4.24}$$

Neglecting the second term of the left-hand side and integrating again with respect to τ from $\tau = t$ to $\tau = t+1$, we have

$$|\omega(t)|_{L^2}^2 \le \int_t^{t+1} |\omega(\tau)|_{L^2}^2\,\mathrm{d}\tau + \frac{1}{\mu}\,|f|_{L_t^\infty H_{\boldsymbol{x}}^{-1}}^2. \tag{2.4.25}$$

For $t \ge t_1(|\boldsymbol{v}_0|_{L^2}, |f|_{L_t^\infty H_{\boldsymbol{x}}^{-1}}; \mu) = t_0(|\boldsymbol{v}_0|_{L^2}, |f|_{L_t^\infty H_{\boldsymbol{x}}^{-2}}; \mu) + 1$, by using (2.4.15), we obtain

$$|\omega(t)|_{L^2}^2 \le \frac{c}{c_0^2\,\mu^3}\,|f|_{L_t^\infty H_{\boldsymbol{x}}^{-2}}^2 + \frac{1}{\mu}\,|f|_{L_t^\infty H_{\boldsymbol{x}}^{-1}}^2 = I_0. \tag{2.4.26}$$

We now integrate (2.4.23) from t to $t+1$ and using the results in (2.4.15), we have

$$\int_t^{t+1} |\omega(\tau)|_{H^1}^2\,\mathrm{d}\tau \le \frac{1}{\mu}\Big\{\frac{c}{c_0^2\,\mu^3}\,|f|_{L_t^\infty H_{\boldsymbol{x}}^{-2}}^2 + \frac{2}{\mu}\,|f|_{L_t^\infty H_{\boldsymbol{x}}^{-1}}^2\Big\} = K_0. \tag{2.4.27}$$

□

Lemma 2.4.3 Let $m \ge 1$, $\boldsymbol{v}_0 \in L^2(\mathcal{M})$ and $f \in L_t^\infty H_{\boldsymbol{x}}^{m-1}$. Then for all $t \ge 0$ there exist a solution $\omega(t) \in H^m(\mathcal{M})$ of (2.3.21). Moreover there exist $t_m(|\boldsymbol{v}_0|_{L^2}, |f|_{L_t^\infty H_{\boldsymbol{x}}^{m-1}}; \mu)$, $I_m(|f|_{L_t^\infty H_{\boldsymbol{x}}^{m-1}}; \mu)$ and $K_m(|f|_{L_t^\infty H_{\boldsymbol{x}}^{m-1}}; \mu)$ such that, for all $t \ge t_m$

$$|\omega(t)|_{H^m}^2 \le I_m(|f|_{L_t^\infty H_{\boldsymbol{x}}^{m-1}}; \mu) \tag{2.4.28}$$

and

$$\int_t^{t+1} |\omega(\tau)|_{H^m}^2\,\mathrm{d}\tau \le K_m(|f|_{L_t^\infty H_{\boldsymbol{x}}^{m-1}}; \mu), \tag{2.4.29}$$

where

$$\begin{aligned} I_m &= \frac{c'(m)}{\mu}\,|f|_{L_t^\infty H_{\boldsymbol{x}}^{m-1}}^2 + (1 + \frac{c(m)}{\mu}\,I_0)\,K_{m-1} \\ K_m &= \frac{1}{\mu}\,\{\frac{c'(m)}{\mu}\,|f|_{L_t^\infty H_{\boldsymbol{x}}^{m-1}}^2 + \frac{c(m)}{\mu}\,I_0\,K_{m-1}\}. \end{aligned} \tag{2.4.30}$$

Proof. We set $|\omega|_m := \big(\sum_{|\alpha|=m} |D^\alpha \omega|_{L^2}^2\big)^{1/2}$, where α is a multi-index. It is proved in [1, p. 184] that $|\omega|_{H^m}$ is equivalent to $|\omega|_m$; that is, there exist positive constants c_1 and c_2 such that

$$c_1\,|\omega|_m \le |\omega|_{H^m} \le c_2\,|\omega|_m. \tag{2.4.31}$$

Now fixed a multi-index α and multiply (2.3.21) by $D^{2\alpha}\omega$ in L^2,

$$\begin{aligned}(\frac{\mathrm{d}\omega}{\mathrm{d}t}, D^{2\alpha}\omega)_{L^2} + \frac{1}{\varepsilon}\,(L\omega, D^{2\alpha}\omega)_{L^2} + \mu\,(A\omega, D^{2\alpha}\omega)_{L^2}\\ + (B(\omega,\omega), D^{2\alpha}\omega)_{L^2} = (f, D^{2\alpha}\omega)_{L^2}.\end{aligned} \tag{2.4.32}$$

The anti-symmetric term vanishes, i.e.,

$$(L\omega, D^{2\alpha}\omega)_{L^2} = 0. \tag{2.4.33}$$

Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|D^\alpha\omega|_{L^2}^2 + \mu\,|\nabla D^\alpha\omega|_{L^2}^2 \le |(f, D^{2\alpha}\omega)_{L^2}| + |(B(\omega,\omega), D^{2\alpha}\omega)_{L^2}|. \tag{2.4.34}$$

By using Hölder's inequality (2.1.21) and Young's inequality (2.1.22), the first term of the right-hand side of (2.4.34) can be majorized by

$$|(f, D^{2\alpha}\omega)_{L^2}| \le |f|_{H^{m-1}}\,|\omega|_{H^{m+1}} \le \frac{c'(m)}{\mu}\,|f|_{H^{m-1}}^2 + \frac{\mu}{4}\,|\omega|_{H^{m+1}}^2. \tag{2.4.35}$$

The second term in right-hand side of (2.4.34) is

$$\begin{aligned}\big(B(\omega,\omega,), D^{2\alpha}\omega\big)_{L^2} &= \sum_{|\alpha|=m}\int_{\mathcal{M}} \boldsymbol{v}\cdot\nabla\omega\, D^{2\alpha}\omega\,\mathrm{d}\boldsymbol{x}\\ &= \sum_{|\alpha|=m}(-1)^{|\alpha|}\int_{\mathcal{M}} D^\alpha(\boldsymbol{v}\cdot\nabla\omega)\,D^\alpha\omega\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{2.4.36}$$

Using Leibniz formula (2.1.12),

$$\begin{aligned}(B(\omega,\omega), D^{2\alpha}\omega)_{L^2} = \sum_{|\alpha|=m}(-1)^{|\alpha|}\int_{\mathcal{M}} \boldsymbol{v}\cdot\nabla\, D^\alpha\omega\, D^\alpha\omega\,\mathrm{d}\boldsymbol{x}\\ + \sum_{1\le|\beta|\le|\alpha|}(-1)^{|\alpha|}\binom{|\alpha|}{|\beta|}\int_{\mathcal{M}} D^\beta\boldsymbol{v}\cdot(\nabla D^{\alpha-\beta}\,\omega)\,D^\alpha\omega\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{2.4.37}$$

We see that, for each α, the first term of the right-hand side of (2.4.37) vanishes, i.e.,

$$(B(\omega, D^\alpha\omega), D^\alpha\omega)_{L^2} = 0. \tag{2.4.38}$$

Now to estimate (2.4.37), we need only to estimate the second term of the right hand side of (2.4.37),

$$\begin{aligned}
|(B(\omega,\omega), D^{2\alpha}\omega)_{L^2}| &= |\sum_{1\le|\beta|\le|\alpha|}\binom{|\alpha|}{|\beta|}\int_{\mathcal{M}}(D^\beta \boldsymbol{v})\cdot\nabla(D^{\alpha-\beta}\omega)\,D^\alpha\omega\,\mathrm{d}\boldsymbol{x}| \\
&\le c\sum_{1\le|\beta|\le|\alpha|}|D^\beta\,\boldsymbol{v}|_{L^4}\,|\nabla\,D^{\alpha-\beta}\omega|_{L^4}\,|D^\alpha\omega|_{L^2} \\
&\le c\sum_{1\le|\beta|\le|\alpha|}|D^\beta\,\nabla^{-1}\omega|_{H^{1/2}}\,|D^{\alpha-\beta}\,\nabla\omega|_{H^{1/2}}\,|D^\alpha\omega|_{L^2} \\
&\le c(m)\sum_{l=1}^{m}|\omega|_{H^{l-1/2}}\,|\omega|_{H^{m-l+3/2}}\,|\omega|_{H^m},
\end{aligned} \tag{2.4.39}$$

where we have used Sobolev inequalities for the second and third lines, and where $l := |\beta|$ in the last line. Using the interpolation inequality

$$|u|_{H^s} \le c\,|u|_{H^l}^{(k-s)/(k-l)}\,|u|_{H^k}^{(s-l)/(k-l)}, \tag{2.4.40}$$

for $0 \le l < s < k$. The terms on the right-hand side of (2.4.39) become

$$\begin{aligned}
|\omega|_{H^{l-1/2}} &\le c\,|\omega|_{L^2}^{(2m-2l+3)/(2m+2)}\,|\omega|_{H^{m+1}}^{(2l-1)/(2m+2)}, \\
|\omega|_{H^{m-l+3/2}} &\le c\,|\omega|_{L^2}^{(2l-1)/(2m+2)}\,|\omega|_{H^{m+1}}^{(2m-2l+3)/(2m+2)}.
\end{aligned} \tag{2.4.41}$$

Followed by Cauchy-Schwarz and summing over α, the nonlinear term becomes

$$\begin{aligned}
|(B(\omega,\omega), D^{2\alpha}\omega)_{L^2}| &\le c(m)\,(|\omega|_{L^2}\,|\omega|_{H^m}\,|\omega|_{H^{m+1}}) \\
&\le \frac{c(m)}{\mu}\,|\omega|_{L^2}^2\,|\omega|_{H^m}^2 + \frac{\mu}{4}\,|\omega|_{H^{m+1}}^2.
\end{aligned} \tag{2.4.42}$$

Gathering (2.4.34), (2.4.35) and (2.4.42), we obtain the following differential inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\omega|_{H^m}^2 + \mu\,|\omega|_{H^{m+1}}^2 \le \frac{c'(m)}{\mu}\,|f|_{H^{m-1}}^2 + \frac{c(m)}{\mu}\,|\omega|_{L^2}^2\,|\omega|_{H^m}^2. \tag{2.4.43}$$

We integrate (2.4.43) between τ and $t+1$ with $t<\tau<t+1$, where $t \geq t_m(|\boldsymbol{v}_0|_{L^2}, |f|_{L_t^\infty H_{\boldsymbol{x}}^{m-1}};\mu) = t_0(|\boldsymbol{v}_0|_{L^2}, |f|_{L_t^\infty H_{\boldsymbol{x}}^{m-2}};\mu)+1$, giving

$$\begin{aligned} |\omega(t+1)|_{H^m}^2 \leq |\omega(\tau)|_{H^m}^2 &+ \frac{c'(m)}{\mu}\,|f|_{L_t^\infty H_{\boldsymbol{x}}^{m-1}}^2 \\ &+ \frac{c(m)}{\mu}\,|\omega|_{L_t^\infty L_{\boldsymbol{x}}^2}^2 \int_t^{t+1} |\omega(s)|_{H^m}^2\,\mathrm{d}s. \end{aligned} \tag{2.4.44}$$

Integrating again with respect to τ from t to $t+1$

$$\begin{aligned} |\omega(t+1)|_{H^m}^2 &\leq \int_t^{t+1} |\omega(\tau)|_{H^m}^2\,\mathrm{d}\tau + \frac{c'(m)}{\mu}\,|f|_{L_t^\infty H_{\boldsymbol{x}}^{m-1}}^2 \\ &\qquad + \frac{c(m)}{\mu}\,|\omega|_{L_t^\infty L_{\boldsymbol{x}}^2}^2 \int_t^{t+1} |\omega(\tau)|_{H^m}^2\,\mathrm{d}\tau \\ &\leq \frac{c'(m)}{\mu}\,|f|_{L_t^\infty H_{\boldsymbol{x}}^{m-1}}^2 + (1+\frac{c(m)}{\mu}\,|\omega|_{L_t^\infty L_{\boldsymbol{x}}^2}^2) \int_t^{t+1} |\omega(\tau)|_{H^m}^2\,\mathrm{d}\tau. \end{aligned} \tag{2.4.45}$$

Neglecting now the first term of the left hand side of (2.4.43) and integrating between t and $t+1$

$$\mu \int_t^{t+1} |\omega(\tau)|_{H^{m+1}}^2\,\mathrm{d}\tau \leq \frac{c'(m)}{\mu}\,|f|_{L_t^\infty H_{\boldsymbol{x}}^{m-1}}^2 + \frac{c(m)}{\mu}\,|\omega|_{L_t^\infty L_{\boldsymbol{x}}^2}^2 \int_t^{t+1} |\omega(\tau)|_{H^m}^2\,\mathrm{d}\tau. \tag{2.4.46}$$

For $m=1$

$$|\omega(t+1)|_{H^1}^2 \leq \frac{c'(1)}{\mu}\,|f|_{L_t^\infty L_{\boldsymbol{x}}^2}^2 + (1+\frac{c(1)}{\mu}\,|\omega|_{L_t^\infty L_{\boldsymbol{x}}^2}^2) \int_t^{t+1} |\omega(\tau)|_{H^1}^2\,\mathrm{d}\tau. \tag{2.4.47}$$

By using (2.4.16) and (2.4.17) we have

$$|\omega(t+1)|_{H^1}^2 \leq \frac{c'(1)}{\mu}\,|f|_{L_t^\infty L_{\boldsymbol{x}}^2}^2 + (1+\frac{c(1)}{\mu}\,I_0)\,K_0 = I_1, \tag{2.4.48}$$

and

$$\int_t^{t+1} |\omega(\tau)|_{H^2}^2\,\mathrm{d} \leq \frac{1}{\mu}\Big\{\frac{c'(1)}{\mu}\,|f|_{L_t^\infty L_{\boldsymbol{x}}^2}^2 + \frac{c(1)}{\mu}\,I_0\,K_0\Big\} = K_1. \tag{2.4.49}$$

For $m=n$ we have

$$|\omega(t+1)|_{H^n}^2 \leq \frac{c'(n)}{\mu}\,|f|_{L_t^\infty H_{\boldsymbol{x}}^{n-1}}^2 + (1+\frac{c(n)}{\mu}\,I_0)\,K_{n-1} = I_n, \tag{2.4.50}$$

and

$$\int_t^{t+1} |\omega(\tau)|_{H^{n+1}}^2\,\mathrm{d}\tau \leq \frac{1}{\mu}\Big\{\frac{c'(n)}{\mu}\,|f|_{L_t^\infty H_{\boldsymbol{x}}^{n-1}}^2 + \frac{c(n)}{\mu}\,I_0\,K_{n-1}\Big\} = K_n. \tag{2.4.51}$$

Finally, for $m = n+1$ we have

$$
\begin{aligned}
|\omega(t+1)|^2_{H^{n+1}} &\le \frac{c'(n+1)}{\mu}\,|f|^2_{L^\infty_t H^n_{\boldsymbol{x}}} + (1 + \frac{c(n+1)}{\mu}\,|\omega|^2_{L^\infty_t L^2_{\boldsymbol{x}}}) \int_t^{t+1} |\omega(\tau)|^2_{H^{n+1}}\,\mathrm{d}\tau \\
&\le \frac{c'(n+1)}{\mu}\,|f|^2_{L^\infty_t H^n_{\boldsymbol{x}}} + (1 + \frac{c(n+1)}{\mu}\,I_0)\,K_n = I_{n+1},
\end{aligned}
\tag{2.4.52}
$$

as well as

$$
\int_t^{t+1} |\omega(\tau)|^2_{H^{n+2}}\,\mathrm{d}\tau \le \frac{1}{\mu}\Big\{\frac{c'(n+1)}{\mu}\,|f|^2_{L^\infty_t H^n_{\boldsymbol{x}}} + \frac{c(n+1)}{\mu}\,I_0\,K_n\Big\} = K_{n+1}. \tag{2.4.53}
$$

□

Chapter 3

Bounds on the Non-zonal component

This chapter is devoted to show the non-zonal component of the solution of the vorticity form of Navier–Stokes equation on β-plane is small, for the linear and nonlinear problems. In Section 3.1 we expand our equation in Fourier series, define the fast and slow variables and prove that the L^2 bound for the fast variable of the solution for the linear problem is small. We show in Section 3.2 the L^2 bound for the non-zonal component of the solution for nonlinear problem of our equation is of order $\varepsilon^{1/2}$. Finally, in Section 3.3 we prove that the bound in Sobolev spaces for the fast variable of the solution of our equation is of $O(\varepsilon^{1/2})$.

3.1 Fourier Expansion

To prove that the solution, ω, of the research equation is nearly zonal, we need to split ω into zonal component with zero-frequency, and non-zonal component with frequency and prove the non-zonal component is small. Motivated by this we expand the solution, ω, and the external force, f, in Fourier series as follows

$$\begin{aligned} \omega(\boldsymbol{x},t) &= \textstyle\sum_{\boldsymbol{k}} \omega_{\boldsymbol{k}}(t)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \mathrm{i}\Omega_{\boldsymbol{k}} t/\varepsilon}, \\ f(\boldsymbol{x},t) &= \textstyle\sum_{\boldsymbol{k}} f_{\boldsymbol{k}}(t)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \end{aligned} \tag{3.1.1}$$

where

$$\omega_{\boldsymbol{k}}(t) = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \omega(\boldsymbol{x},t)\, \mathrm{e}^{\mathrm{i}\Omega_{\boldsymbol{k}} t/\varepsilon}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}, \tag{3.1.2}$$

and $\boldsymbol{k} = (k_1, k_2) \in \mathbb{Z}_L := \big\{(2\pi l_1/L_1, 2\pi l_2/L_2) : (l_1, l_2) \in \mathbb{Z}^2\big\}$, $\boldsymbol{x} = (x, y) \in \mathcal{M}$ and $\Omega_{\boldsymbol{k}} = -k_1/|\boldsymbol{k}|^2$ denotes the wave frequency. Since ω and f have zero integrals over $\mathcal{M}$, $\omega_{\boldsymbol{k}} = 0$ and $f_{\boldsymbol{k}} = 0$ when $\boldsymbol{k} = 0$. Here and in what follows sums over wavenumbers are understood to be taken over $\mathbb{Z}_L$.

Let us now do some computations which are useful later. Since

$$\psi = \Delta^{-1}\omega = -\textstyle\sum_{\boldsymbol{k}} \dfrac{1}{|\boldsymbol{k}|^2}\, \omega_{\boldsymbol{k}}\, \mathrm{e}^{-\mathrm{i}\Omega_{\boldsymbol{k}} t/\varepsilon}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{3.1.3}$$

and $\boldsymbol{v} = (u, v) = (-\partial_y\psi, \partial_x\psi)$, we can write

$$\begin{aligned} u(\boldsymbol{x},t) &= \textstyle\sum_{\boldsymbol{k}} \dfrac{\mathrm{i}k_2}{|\boldsymbol{k}|^2}\, \omega_{\boldsymbol{k}}\, \mathrm{e}^{-\mathrm{i}\Omega_{\boldsymbol{k}} t/\varepsilon}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \\ v(\boldsymbol{x},t) &= \textstyle\sum_{\boldsymbol{k}} \dfrac{-\mathrm{i}k_1}{|\boldsymbol{k}|^2}\, \omega_{\boldsymbol{k}}\, \mathrm{e}^{-\mathrm{i}\Omega_{\boldsymbol{k}} t/\varepsilon}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}. \end{aligned} \tag{3.1.4}$$

From this we can write the nonlinear term $B(\omega, \omega^\sharp)$ as follows

$$\begin{aligned} \boldsymbol{v}\cdot\nabla\omega^\sharp &= u\,\partial_x\omega^\sharp + v\,\partial_y\omega^\sharp \\ &= \Big(\textstyle\sum_{\boldsymbol{j}} \dfrac{\mathrm{i}j_2}{|\boldsymbol{j}|^2}\, \omega_{\boldsymbol{j}}\, \mathrm{e}^{-\mathrm{i}\Omega_{\boldsymbol{j}} t/\varepsilon}\, \mathrm{e}^{\mathrm{i}\boldsymbol{j}\cdot\boldsymbol{x}}\Big)\Big(\textstyle\sum_{\boldsymbol{k}} \mathrm{i}k_1\, \omega^\sharp_{\boldsymbol{k}}\, \mathrm{e}^{-\mathrm{i}\Omega_{\boldsymbol{k}} t/\varepsilon}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\Big) \\ &\quad + \Big(\textstyle\sum_{\boldsymbol{j}} \dfrac{-\mathrm{i}j_1}{|\boldsymbol{j}|^2}\, \omega_{\boldsymbol{j}}\, \mathrm{e}^{-\mathrm{i}\Omega_{\boldsymbol{j}} t/\varepsilon}\, \mathrm{e}^{\mathrm{i}\boldsymbol{j}\cdot\boldsymbol{x}}\Big)\Big(\textstyle\sum_{\boldsymbol{k}} \mathrm{i}k_2\, \omega^\sharp_{\boldsymbol{k}}\, \mathrm{e}^{-\mathrm{i}\Omega_{\boldsymbol{k}} t/\varepsilon}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\Big) \\ &= \textstyle\sum_{\boldsymbol{j},\boldsymbol{k}} \dfrac{j_1 k_2 - j_2 k_1}{|\boldsymbol{j}|^2}\, \omega_{\boldsymbol{j}}\, \omega^\sharp_{\boldsymbol{k}}\, \mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol{j}} + \Omega_{\boldsymbol{k}}) t/\varepsilon}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{j}+\boldsymbol{k})\cdot\boldsymbol{x}} \\ &= \textstyle\sum_{\boldsymbol{j},\boldsymbol{k}} \dfrac{\boldsymbol{j}\wedge\boldsymbol{k}}{|\boldsymbol{j}|^2}\, \omega_{\boldsymbol{j}}\, \omega^\sharp_{\boldsymbol{k}}\, \mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol{j}} + \Omega_{\boldsymbol{k}}) t/\varepsilon}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{j}+\boldsymbol{k})\cdot\boldsymbol{x}}, \end{aligned} \tag{3.1.5}$$

where $\boldsymbol{j} \wedge \boldsymbol{k} := j_1 k_2 - j_2 k_1$. Also we define the form $(B(\omega, \omega^\sharp), \omega^\flat)$ as follows

$$
\begin{aligned}
(B(\omega, \omega^\sharp), \omega^\flat)_{L^2} &:= (u\, \partial_x \omega^\sharp + v\, \partial_y \omega^\sharp, \omega^\flat)_{L^2} \\
&= \int_{\mathcal{M}} \sum\nolimits_{\boldsymbol{jkl}} \frac{\boldsymbol{j} \wedge \boldsymbol{k}}{|\boldsymbol{j}|^2}\, \omega_{\boldsymbol{j}}\, \omega^\sharp_{\boldsymbol{k}}\, \overline{\omega^\flat_{\boldsymbol{l}}}\, \mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol{j}} + \Omega_{\boldsymbol{k}} - \Omega_{\boldsymbol{l}})t/\varepsilon}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{j}+\boldsymbol{k}-\boldsymbol{l})\cdot \boldsymbol{x}}\, \mathrm{d}\boldsymbol{x} \\
&= \sum\nolimits_{\boldsymbol{jkl}} \frac{\boldsymbol{j} \wedge \boldsymbol{k}}{|\boldsymbol{j}|^2}\, \omega_{\boldsymbol{j}}\, \omega^\sharp_{\boldsymbol{k}}\, \overline{\omega^\flat_{\boldsymbol{l}}}\, \mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol{j}} + \Omega_{\boldsymbol{k}} - \Omega_{\boldsymbol{l}})t/\varepsilon} \int_{\mathcal{M}} \mathrm{e}^{\mathrm{i}(\boldsymbol{j}+\boldsymbol{k}-\boldsymbol{l})\cdot \boldsymbol{x}}\, \mathrm{d}\boldsymbol{x} \qquad (3.1.6) \\
&= |\mathcal{M}| \sum\nolimits_{\boldsymbol{jkl}} \frac{\boldsymbol{j} \wedge \boldsymbol{k}}{|\boldsymbol{j}|^2}\, \omega_{\boldsymbol{j}}\, \omega^\sharp_{\boldsymbol{k}}\, \overline{\omega^\flat_{\boldsymbol{l}}}\, \mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol{j}} + \Omega_{\boldsymbol{k}} - \Omega_{\boldsymbol{l}})t/\varepsilon}\, \delta_{\boldsymbol{j}+\boldsymbol{k}-\boldsymbol{l}} \\
&= \sum\nolimits_{\boldsymbol{j}+\boldsymbol{k}=\boldsymbol{l}} B_{\boldsymbol{jkl}}\, \omega_{\boldsymbol{j}}\, \omega^\sharp_{\boldsymbol{k}}\, \overline{\omega^\flat_{\boldsymbol{l}}}\, \mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol{j}} + \Omega_{\boldsymbol{k}} - \Omega_{\boldsymbol{l}})t/\varepsilon}
\end{aligned}
$$

where

$$B_{\boldsymbol{jkl}} = (B(\mathrm{e}^{\mathrm{i}\boldsymbol{j}\cdot\boldsymbol{x}}, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}), \mathrm{e}^{\mathrm{i}\boldsymbol{l}\cdot\boldsymbol{x}}) = |\mathcal{M}|\, \frac{\boldsymbol{j} \wedge \boldsymbol{k}}{|\boldsymbol{j}|^2}\, \delta_{\boldsymbol{j}+\boldsymbol{k}-\boldsymbol{l}} \qquad (3.1.7)$$

and

$$\delta_{\boldsymbol{j}+\boldsymbol{k}-\boldsymbol{l}} = \begin{cases} 1 & \text{if } \boldsymbol{j}+\boldsymbol{k} = \boldsymbol{l} \\ 0 & \text{if } \boldsymbol{j}+\boldsymbol{k} \neq \boldsymbol{l}. \end{cases} \qquad (3.1.8)$$

Now let us compute the eigenvalues of the operators L and A in our equation (2.3.21). Since $L = \partial_x \Delta^{-1}$, it follows from the Fourier series that

$$L\omega = \sum\nolimits_{\boldsymbol{k}} \frac{-\mathrm{i}k_1}{|\boldsymbol{k}|^2}\, \omega_{\boldsymbol{k}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \qquad (3.1.9)$$

and from this we can find that the eigenvalue of L is $-\mathrm{i}k_1/|\boldsymbol{k}|^2$. In the same way we can find that the eigenvalue of A is $|\boldsymbol{k}|^2$.

It's easy to see that (2.3.21) in Fourier series can be written as

$$\frac{\mathrm{d}\omega_{\boldsymbol{k}}}{\mathrm{d}t} + \sum\nolimits_{\boldsymbol{j}+\boldsymbol{l}=\boldsymbol{k}} B_{\boldsymbol{jlk}}\, \omega_{\boldsymbol{j}}\, \omega_{\boldsymbol{l}}\, \mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol{j}} + \Omega_{\boldsymbol{l}} - \Omega_{\boldsymbol{k}})t/\varepsilon} + \mu\, |\boldsymbol{k}|^2\, \omega_{\boldsymbol{k}} = f_{\boldsymbol{k}}\, \mathrm{e}^{\mathrm{i}\Omega_{\boldsymbol{k}} t/\varepsilon}. \qquad (3.1.10)$$

3.1.1 L^2 Bound for the linear problem

Split our solution ω into a slow part $\bar{\omega}$, for which $\Omega_{\boldsymbol{k}} = 0$, and the remaining fast part $\tilde{\omega} := \omega - \bar{\omega}$, viz.,

$$\begin{aligned} \bar{\omega}(\boldsymbol{x}, t) &= \textstyle\sum_{k_1=0} \bar{\omega}_{\boldsymbol{k}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \\ \tilde{\omega}(\boldsymbol{x}, t) &:= \textstyle\sum_{k_1\neq 0} \tilde{\omega}_{\boldsymbol{k}}(t)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \mathrm{i}\Omega_{\boldsymbol{k}} t/\varepsilon}, \end{aligned} \tag{3.1.11}$$

where

$$\tilde{\omega}_{\boldsymbol{k}} = \begin{cases} \omega_{\boldsymbol{k}} & \text{if } k_1 \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.1.12}$$

and

$$\bar{\omega}_{\boldsymbol{k}} = \begin{cases} \omega_{\boldsymbol{k}} & \text{if } k_1 = 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.1.13}$$

We note that, also having zero integrals over $\mathcal{M}$, $\tilde{\omega}$ and $\bar{\omega}$ are orthogonal in H^m for $m = 1, 2, \cdots$. Now we shall find L^2 norm for the normal component, $\tilde{\omega}$, of the solution of linear equation

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} + \frac{1}{\varepsilon} L\omega + \mu\, A\, \omega = f. \tag{3.1.14}$$

Theorem 3.1.1 Assume that the initial data $\boldsymbol{v}(0) \in L^2(\mathcal{M})$ and that the forcing is bounded as $|f|_{L_t^\infty H_{\boldsymbol{x}}^2} + |\partial_t f|_{L_t^\infty L_{\boldsymbol{x}}^2} \leq \infty$. Then there exists a time $T_0(|\boldsymbol{v}(0)|_{L^2}, |f|_{L_t^\infty H_{\boldsymbol{x}}^2}, |\partial_t f|_{L_t^\infty L_{\boldsymbol{x}}^2}; \mu)$ and a constant $N_0(|f|_{L_t^\infty H_{\boldsymbol{x}}^2}, |\partial_t f|_{L_t^\infty L_{\boldsymbol{x}}^2}; \mu)$ such that, for all $t \geq T_0$,

$$\begin{aligned} &|\tilde{\omega}(t)|_{L^2}^2 \leq \varepsilon\, N_0, \\ &\mu\, \mathrm{e}^{-\nu(t+t')} \int_t^{t+t'} \mathrm{e}^{\nu\tau}\, |\nabla\tilde{\omega}(\tau)|_{L^2}^2\, \mathrm{d}\tau \leq 2\,\varepsilon\, N_0. \end{aligned} \tag{3.1.15}$$

Proof.

Multiplying (3.1.14) by $\tilde{\omega}$ in L^2, and use $\omega = \tilde{\omega} + \bar{\omega}$, we have

$$(\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\omega} + \bar{\omega}), \tilde{\omega})_{L^2} + \frac{1}{\varepsilon}(L(\tilde{\omega} + \bar{\omega}), \tilde{\omega})_{L^2} + \mu(A(\tilde{\omega} + \bar{\omega}), \tilde{\omega})_{L^2} = (f, \tilde{\omega})_{L^2}. \tag{3.1.16}$$

The second term is vanishing because L is antisymmetric operator. Applying the orthogonality between $\tilde{\omega}$ and $\bar{\omega}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\omega}|_{L^2}^2 + \mu\,|\nabla\tilde{\omega}|_{L^2}^2 = (\tilde{f},\tilde{\omega})_{L^2}. \tag{3.1.17}$$

Using the Poincaré inequality (2.1.22) on the left-hand side,and multiplying by $\mathrm{e}^{\nu t}$, $\nu = c_0^2\,\mu$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\nu t}\,|\tilde{\omega}|_{L^2}^2) + \mu\,\mathrm{e}^{\nu t}\,|\nabla\tilde{\omega}|_{L^2}^2 \le 2\mathrm{e}^{\nu t}\,(\tilde{f},\tilde{\omega})_{L^2}. \tag{3.1.18}$$

We integrate the right-hand side from 0 to t by parts,

$$\begin{aligned}\int_0^t \mathrm{e}^{\nu\tau}\,(\tilde{f},\tilde{\omega})_{L^2}\,\mathrm{d}\tau &= |\mathcal{M}|\,{\textstyle\sum_{\boldsymbol{k}}'}\int_0^t \tilde{f}_{\boldsymbol{k}}(\tau)\,\overline{\tilde{\omega}_{\boldsymbol{k}}(\tau)}\,\mathrm{e}^{\mathrm{i}\Omega_{\boldsymbol{k}}\tau/\varepsilon+\nu\tau}\,\mathrm{d}\tau\\ &= -\mathrm{i}\,\varepsilon\,|\mathcal{M}|\,{\textstyle\sum_{\boldsymbol{k}}'}\frac{1}{\Omega_{\boldsymbol{k}}}\,\big[\tilde{f}_{\boldsymbol{k}}\,\overline{\tilde{\omega}_{\boldsymbol{k}}}\,\mathrm{e}^{\mathrm{i}\Omega_{\boldsymbol{k}}t/\varepsilon+\nu\tau}\big]_0^t\\ &\qquad + \mathrm{i}\,\varepsilon\,|\mathcal{M}|{\textstyle\sum_{\boldsymbol{k}}'}\frac{1}{\Omega_{\boldsymbol{k}}}\int_0^t \mathrm{e}^{\mathrm{i}\Omega_{\boldsymbol{k}}\tau/\varepsilon}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\big[\mathrm{e}^{\nu\tau}\,\tilde{f}_{\boldsymbol{k}}\,\overline{\tilde{\omega}_{\boldsymbol{k}}}\big]\,\mathrm{d}\tau,\end{aligned} \tag{3.1.19}$$

where the prime on the sums indicates that the resonant terms (i.e. those with $\Omega_{\boldsymbol{k}} = 0$) are excluded. Defining the operator I_Ω by

$$\mathsf{I}_\Omega\tilde{f}(\boldsymbol{x},t) := {\textstyle\sum_{\boldsymbol{k}}'}\frac{1}{\mathrm{i}\Omega_{\boldsymbol{k}}}\,f_{\boldsymbol{k}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} = \mathrm{i}{\textstyle\sum_{\boldsymbol{k}}'}\frac{|\boldsymbol{k}|^2}{k_1}\,f_{\boldsymbol{k}}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{3.1.20}$$

which being the restricted inverse of L is also antisymmetric, we can write

$$\begin{aligned}\int_0^t \mathrm{e}^{\nu\tau}\,(\tilde{f},\tilde{\omega})_{L^2}\,\mathrm{d}\tau &= \varepsilon\,(\mathsf{I}_\Omega\tilde{f},\tilde{\omega})_{L^2}(t)\,\mathrm{e}^{\nu t} - \varepsilon(\mathsf{I}_\Omega\tilde{f},\tilde{\omega})_{L^2}(0)\\ &\quad - \varepsilon\int_0^t \big[\nu(\mathsf{I}_\Omega\tilde{f},\tilde{\omega})_{L^2} + (\partial_\tau\mathsf{I}_\Omega\tilde{f},\tilde{\omega})_{L^2} + (\mathsf{I}_\Omega\tilde{f},\partial_\tau^l\tilde{\omega})_{L^2}\big]\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau.\end{aligned} \tag{3.1.21}$$

where

$$\begin{aligned}&\partial_t^l\omega := \mathrm{e}^{-tL/\varepsilon}\,\partial_t(\mathrm{e}^{tL/\varepsilon}\omega)\\ \Rightarrow\qquad &\partial_t^l\tilde{\omega} := \partial_t\tilde{\omega} + \frac{1}{\varepsilon}\,L\tilde{\omega} = -\mu A\tilde{\omega} + \tilde{f}.\end{aligned} \tag{3.1.22}$$

Using (3.1.20), the endpoint terms can be bounded as

$$\begin{aligned}|(\mathsf{I}_\Omega\tilde{f},\tilde{\omega})_{L^2}| &= |{\textstyle\sum_{\boldsymbol{k}}'}\frac{1}{\Omega_{\boldsymbol{k}}}\,\tilde{f}_{\boldsymbol{k}}\,\overline{\tilde{\omega}_{\boldsymbol{k}}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot t}| = |{\textstyle\sum_{\boldsymbol{k}}'}\frac{|\boldsymbol{k}|^2}{k_1}\,\tilde{f}_{\boldsymbol{k}}\,\overline{\tilde{\omega}_{\boldsymbol{k}}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}|\\ &\le c\,{\textstyle\sum_{\boldsymbol{k}}'}|\boldsymbol{k}|^2\,|\tilde{f}_{\boldsymbol{k}}|\,|\tilde{\omega}_{\boldsymbol{k}}| \le c\,{\textstyle\sum_{\boldsymbol{k}}}|\boldsymbol{k}|^2\,|\tilde{f}_{\boldsymbol{k}}|\,|\tilde{\omega}_{\boldsymbol{k}}| \le c\,|\nabla\tilde{f}|_{L^2}\,|\nabla\tilde{\omega}|_{L^2}.\end{aligned} \tag{3.1.23}$$

We bound now the terms in the integrand. First

$$\begin{aligned}|(\partial_\tau \mathsf{I}_\Omega \tilde f, \tilde\omega)_{L^2}| &= |(\partial_\tau \tilde f, \mathsf{I}_\Omega \tilde\omega)_{L^2}| \le c\,|\partial_\tau \tilde f|_{L^2}\,|\Delta\tilde\omega|_{L^2} \\ &\le \frac{1}{2}\,|\partial_\tau \tilde f|^2_{L^2} + \frac{1}{2}\,|\Delta\tilde\omega|^2_{L^2}.\end{aligned} \tag{3.1.24}$$

The last term in (3.1.21) is

$$(\mathsf{I}_\Omega \tilde f, \partial^l_\tau \tilde\omega)_{L^2} = -(\mathsf{I}_\Omega \tilde f, \mu\, A\tilde\omega)_{L^2} + (\mathsf{I}_\Omega \tilde f, \tilde f)_{L^2}, \tag{3.1.25}$$

where we used the definition of ∂^l_τ. Noting that $(I_\Omega \tilde f, \tilde f) = 0$,[1] we bound the last term in (3.1.25) by

$$|(\mathsf{I}_\Omega \tilde f, \mu\,\Delta\tilde\omega)_{L^2}| \le \mu\, c\,|\Delta \tilde f|_{L^2}\,|\Delta\tilde\omega|_{L^2}. \tag{3.1.26}$$

Now (3.1.25) becomes

$$\begin{aligned}|(\mathsf{I}_\Omega \tilde f, \partial^l_\tau \tilde\omega)_{L^2}| &\le c\,\mu\,|\Delta \tilde f|_{L^2}\,|\Delta\tilde\omega|_{L^2} \\ &\le \frac{\mu}{2}\,|\Delta \tilde f|^2_{L^2} + \frac{\mu}{2}\,|\Delta\tilde\omega|^2_{L^2}.\end{aligned} \tag{3.1.27}$$

Thus the integral in (3.1.21) is bounded as

$$\begin{aligned}&\Big|\int_0^t \big[\nu(I_\Omega \tilde f, \tilde\omega)_{L^2} + (\partial_\tau I_\Omega \tilde f, \tilde\omega)_{L^2} + (I_\Omega \tilde f, \partial^l_\tau \tilde\omega)_{L^2}\big]\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau\Big| \\ &\quad\le \int_0^t \Big[\frac{\mu}{2}\,|\nabla\tilde\omega|^2_{L^2} + \frac{\mu}{2}\,|\nabla \tilde f|^2_{L^2} + (\frac{1}{2}+\frac{\mu}{2})\,|\Delta\tilde\omega|^2_{L^2} + \frac{1}{2}\,|\partial_\tau \tilde f|^2_{L^2} + \frac{\mu}{2}\,|\Delta \tilde f|^2_{L^2}\Big]\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau\end{aligned} \tag{3.1.28}$$

Collecting the terms (3.1.23) and (3.1.28), we obtain

$$\begin{aligned}|\tilde\omega(t)|^2_{L^2} &\le c\,\mathrm{e}^{-\nu t}\,|\tilde\omega(0)|^2_{L^2} + c\,\varepsilon \sup_{0\le t'\le t} |\nabla f(t')|\,|\nabla\omega(t')| \\ &\quad + \Big\{c(\mu)\,\varepsilon \int_0^t |\Delta\omega|^2_{L^2} + |\partial_\tau f|^2_{L^2} + |\nabla^2 f|^2_{L^2}\Big\}\mathrm{e}^{\nu(\tau-t)}\,\mathrm{d}\tau.\end{aligned} \tag{3.1.29}$$

Now there exist a time $T_0(|\boldsymbol{v}(0)|_{L^2}, |f|_{L^\infty_t H^2_{\boldsymbol{x}}}, |\partial_t f|_{L^\infty_t L^2_{\boldsymbol{x}}}; \mu)$ such that, for $t \ge T_0$,

$$|\tilde\omega(t)|^2_{L^2} \le \varepsilon\, N_0(|f|_{L^\infty_t H^2_{\boldsymbol{x}}}, |\partial_t f|_{L^\infty_t L^2_{\boldsymbol{x}}}; \mu), \tag{3.1.30}$$

and

$$\mu\,\mathrm{e}^{-\nu(t+t')} \int_t^{t+t'} \mathrm{e}^{\nu\tau}\,|\nabla\tilde\omega(\tau)|^2_{L^2}\,\mathrm{d}\tau \le 2\,\varepsilon\, N_0. \tag{3.1.31}$$

□

[1] I_Ω is the restricted inverse of the antisymmetric operator L which is also antisymmetric.

3.2 L^2 bound for the nonlinear problem

In this section we find L^2 bound for $\tilde{\omega}$ for the equation (2.3.21). The development in this Section largely follows that in [61]. For later convenience, we define the operator ∂_t^* by, for any ω for which it makes sense,

$$\begin{aligned} &\partial_t^*\omega := \mathrm{e}^{-tL/\varepsilon}\,\partial_t(\mathrm{e}^{tL/\varepsilon}\omega) \\ \Rightarrow \qquad &\partial_t^*\tilde{\omega} := \partial_t\tilde{\omega} + \frac{1}{\varepsilon}\,L\tilde{\omega} = -\tilde{B}(\omega,\omega) - \mu A\tilde{\omega} + \tilde{f}, \end{aligned} \tag{3.2.1}$$

Furthermore, we have

$$\partial_t\bar{\omega} = \bar{f} - \mu\,A\,\bar{\omega} - \bar{B}(\omega,\omega), \tag{3.2.2}$$

where

$$\tilde{B}(\omega,\omega^\sharp) = \textstyle\sum_{j_1+k_1\neq 0} B_{\boldsymbol{jkl}}\,\omega_{\boldsymbol{j}}\,\omega^\sharp_{\boldsymbol{k}}\,\mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol{j}}+\Omega_{\boldsymbol{k}})t/\varepsilon} \tag{3.2.3}$$

and

$$\bar{B}(\omega,\omega^\sharp) = \textstyle\sum_{j_1+k_1=0} B_{\boldsymbol{jkl}}\,\omega_{\boldsymbol{j}}\,\omega^\sharp_{\boldsymbol{k}}\,\mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol{j}}+\Omega_{\boldsymbol{k}})t/\varepsilon} \tag{3.2.4}$$

In addition we need the following lemmata

Lemma 3.2.1 For any $\boldsymbol{j},\boldsymbol{k}$ and $\boldsymbol{l}\in\mathbb{Z}_L$ with $j_1k_1\neq 0$ and $l_1=0$, we have

$$B_{\boldsymbol{jkl}} + B_{\boldsymbol{kjl}} = -l_2\,(\Omega_{\boldsymbol{j}}+\Omega_{\boldsymbol{k}})\,|\mathcal{M}| \tag{3.2.5}$$

Proof. First let $\boldsymbol{l}=\boldsymbol{j}+\boldsymbol{k}$, since $l_1=0$ then $j_1=-k_1$. From (3.1.7) we have

$$B_{\boldsymbol{jkl}} = |\mathcal{M}|\,\frac{(\boldsymbol{j}\wedge\boldsymbol{k})}{|\boldsymbol{j}|^2} \quad\text{and}\quad B_{\boldsymbol{kjl}} = |\mathcal{M}|\,\frac{(\boldsymbol{k}\wedge\boldsymbol{j})}{|\boldsymbol{k}|^2}.$$

Furthermore, we have

$$\begin{aligned} B_{\boldsymbol{jkl}} + B_{\boldsymbol{kjl}} &= |\mathcal{M}|(\frac{\boldsymbol{j}\wedge\boldsymbol{k}}{|\boldsymbol{j}|^2} + \frac{\boldsymbol{k}\wedge\boldsymbol{j}}{|\boldsymbol{k}|^2}) \\ &= |\mathcal{M}|(\frac{j_1k_2-k_1j_2}{|\boldsymbol{j}|^2} + \frac{k_1j_2-k_2j_1}{|\boldsymbol{k}|^2}). \end{aligned}$$

Since $j_1 = -k_1$,

$$
\begin{aligned}
B_{\boldsymbol{jkl}} + B_{\boldsymbol{kjl}} &= |\mathcal{M}| \, \big(\frac{j_1\, k_2 + j_1\, j_2}{|\boldsymbol{j}|^2} + \frac{k_1\, j_2 + k_1\, k_2}{|\boldsymbol{k}|^2} \big) \\
&= |\mathcal{M}| \big(\frac{j_1}{|\boldsymbol{j}|^2} (k_2 + j_2) + \frac{k_1}{|\boldsymbol{k}|^2} (k_2 + j_2) \big) \\
&= |\mathcal{M}| \, (\frac{j_1}{|\boldsymbol{j}|^2} + \frac{k_1}{|\boldsymbol{k}|^2}) (k_2 + j_2) \\
&= -|\mathcal{M}| \, (\Omega_{\boldsymbol{j}} + \Omega_{\boldsymbol{k}}) (k_2 + j_2) \\
&= -|\mathcal{M}| \, (\Omega_{\boldsymbol{j}} + \Omega_{\boldsymbol{k}}) \, l_2.
\end{aligned}
$$

□

Lemma 3.2.2 For any $\omega, \omega^\flat$ and $\omega^\sharp \in H^1(\mathcal{M})$, we have the bound

$$
|(B(\omega, \omega^\flat), \omega^\sharp)_{L^2}| \le c \, |\nabla^{-1}\omega|_{L^2}^{1/2} \, |\nabla\omega|_{L^2}^{1/2} \, |\omega^\sharp|_{H1} \, |\omega^\flat|_{L^2}. \tag{3.2.6}
$$

Proof. By (3.1.7)

$$
\begin{aligned}
|(B(\omega, \omega^\sharp), \omega^\flat)_{L^2}| &= |\mathcal{M}| \textstyle\sum_{\boldsymbol{j}+\boldsymbol{k}=\boldsymbol{l}} B_{\boldsymbol{jkl}} \, |\omega_{\boldsymbol{j}}| \, |\omega^\sharp_{\boldsymbol{k}}| \, |\omega^\flat_{\boldsymbol{l}}| \\
&\le c \textstyle\sum_{\boldsymbol{j}+\boldsymbol{k}=\boldsymbol{l}} \dfrac{|\omega_{\boldsymbol{j}}|}{|\boldsymbol{j}|} \, |\boldsymbol{k}| |\omega^\sharp_{\boldsymbol{k}}| \, |\omega^\flat_{\boldsymbol{l}}| \\
&\le c \displaystyle\int_{\mathcal{M}} \phi(\boldsymbol{x}) \, \chi(\boldsymbol{x}) \, \vartheta(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \\
&\le c \, |\nabla^{-1}\omega|_{L^p} \, |\nabla\omega^\sharp|_{L^q} \, |\omega^\flat|_{L^r},
\end{aligned} \tag{3.2.7}
$$

where

$$
\phi(\boldsymbol{x}) = \textstyle\sum_{\boldsymbol{j}} \dfrac{|\omega_{\boldsymbol{j}}|}{|\boldsymbol{j}|} \, \mathrm{e}^{\mathrm{i}\boldsymbol{j}\cdot\boldsymbol{x}}, \quad \chi(\boldsymbol{x}) = \textstyle\sum_{\boldsymbol{k}} |\boldsymbol{k}| \, |\omega^\sharp_{\boldsymbol{k}}| \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \vartheta(\boldsymbol{x}) = \textstyle\sum_{\boldsymbol{l}} |\omega^\flat_{\boldsymbol{l}}| \, \mathrm{e}^{\mathrm{i}\boldsymbol{l}\cdot\boldsymbol{x}}, \tag{3.2.8}
$$

and $1/p + 1/q + 1/r = 1$. Choose $p = \infty$ and $q = r = 2$ we have

$$
|(B(\omega, \omega^\sharp), \omega^\flat)_{L^2}| \le c \, |\nabla^{-1}\omega|_{L^2}^{1/2} \, |\nabla\omega|_{L^2}^{1/2} \, |\omega^\sharp|_{H^1} \, |\omega^\flat|_{L^2}, \tag{3.2.9}
$$

where we used Agmon inequality (2.1.18). □

Lemma 3.2.3 For any $\omega, \omega^\sharp \in H^1(\mathcal{M})$ we have the bound

$$
|B(\omega, \omega^\sharp)|_{L^2} \le c \, |\nabla^{-1}\omega|_{L^2}^{1/2} \, |\omega|_{H^1}^{1/2} \, |\omega^\sharp|_{H^1}. \tag{3.2.10}
$$

Proof. By (3.1.7)

$$\begin{aligned}|B(\omega,\omega^\sharp)|_{L^2}^2 &= \int_{\mathcal{M}} |\boldsymbol{v}\cdot\nabla\omega^\sharp|^2\,\mathrm{d}\boldsymbol{x}\\ &\le c\,|\boldsymbol{v}|_{L^\infty}^2\,|\omega^\sharp|_{L^2}^2.\end{aligned}$$

Using the Agmon inequality (2.1.18) we have

$$|B(\omega,\omega^\sharp)|_{L^2} \le c\,|\nabla^{-1}\omega|_{L^2}^{1/2}\,|\omega|_{H^1}^{1/2}\,|\omega^\sharp|_{H^1}. \tag{3.2.11}$$

□

Lemma 3.2.4 Let u and $v \in H^2(\mathcal{M})$ have zero integrals and are L^2 orthogonal,

$$(u,v)_{L^2} = 0; \tag{3.2.12}$$

and let $w = u + v$. Then the following Agmon inequality holds,

$$|u|_{L^\infty} \le c\,|\nabla w|_{L^2}\Big(\log\frac{|\Delta w|_{L^2}}{c_0\,|\nabla w|_{L^2}} + 1\Big)^{1/2} \tag{3.2.13}$$

Before the proof, we note that that the interpolation inequality

$$|\nabla w|_{L^2}^2 \le c\,|w|_{L^2}\,|\Delta w|_{L^2} \tag{3.2.14}$$

can be written as

$$\begin{aligned} &2\log|\nabla w|_{L^2} \le \log|w|_{L^2} + \log|\Delta w|_{L^2} + \log c\\ \Leftrightarrow\quad &\log|\nabla w|_{L^2} - \log|w|_{L^2} \le \log|\Delta w|_{L^2} - \log|\nabla w|_{L^2} + \log c\\ \Leftrightarrow\quad &\log\frac{|\nabla w|_{L^2}}{c_0\,|w|_{L^2}} \le \log\frac{|\Delta w|_{L^2}}{c_0\,|\nabla w|_{L^2}} + \log c;\end{aligned} \tag{3.2.15}$$

which can be used to simplify, e.g., $|w|_{L^\infty}\,|\nabla w|_{L^\infty}$ when bounded using (3.2.13).

Proof. For most of this proof, up to (3.2.18) below, we follow ([14], Lemma 7.1) exactly. For conciseness, we put $L_1 = L_2 = 1$ but keep the Poincaré constant c_0. With $\kappa > 0$, we expand u in Fourier series

$$u(\boldsymbol{x}) = \textstyle\sum_{|\boldsymbol{k}|<\kappa} u_{\boldsymbol{k}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + \sum_{|\boldsymbol{k}|\ge\kappa} u_{\boldsymbol{k}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} =: u^<(\boldsymbol{x}) + u^>(\boldsymbol{x}), \tag{3.2.16}$$

and analogously for v and w. Then

$$\begin{aligned}|u|_{L^\infty} &= \sup_{\boldsymbol{x}} |\textstyle\sum_{\boldsymbol{k}} u_{\boldsymbol{k}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}| \le \sum_{|\boldsymbol{k}|<\kappa} |u_{\boldsymbol{k}}| + \sum_{|\boldsymbol{k}|\ge\kappa} |u_{\boldsymbol{k}}| \\ &=: \textstyle\sum^< |\boldsymbol{k}|^{-1}\, |\boldsymbol{k}|\, |u_{\boldsymbol{k}}| + \sum^> |\boldsymbol{k}|^{-2}\, |\boldsymbol{k}|^2\, |u_{\boldsymbol{k}}| \\ &\le \big(\textstyle\sum^< |\boldsymbol{k}|^{-2}\big)^{1/2} \big(\sum^< |\boldsymbol{k}|^2\, |u_{\boldsymbol{k}}|^2\big)^{1/2} \\ &\qquad + \big(\textstyle\sum^> |\boldsymbol{k}|^{-4}\big)^{1/2} \big(\sum^> |\boldsymbol{k}|^4\, |u_{\boldsymbol{k}}|^2\big)^{1/2}.\end{aligned} \tag{3.2.17}$$

Now on the right-hand side,

$$\textstyle\sum^< |\boldsymbol{k}|^{-2} \le c\, \log\kappa$$

and

$$\textstyle\sum^> |\boldsymbol{k}|^{-4} \le c/\kappa^2,$$

so fixing

$$\kappa = |\Delta w|_{L^2} / (c_0\, |\nabla w|_{L^2}), \tag{3.2.18}$$

the lemma follows from

$$\begin{aligned}|u|_{L^\infty} &\le c\, |\nabla u^<|_{L^2} \Big(\log \frac{|\Delta w|_{L^2}}{c_0\, |\nabla w|_{L^2}}\Big)^{1/2} + c\, |\Delta u^>|_{L^2} \frac{|\nabla w|_{L^2}}{|\Delta w|_{L^2}} \\ &\le c\, |\nabla w|_{L^2} \Big(\log \frac{|\Delta w|_{L^2}}{c_0\, |\nabla w|_{L^2}}\Big)^{1/2} + c\, |\nabla w|_{L^2}.\end{aligned} \tag{3.2.19}$$

□

With the above Lemmata we can prove now the main result of this Chapter: the normal component of the solution of the research equation is of $O(\varepsilon^{1/2})$.

Theorem 3.2.5 Assume that the initial data $\boldsymbol{v}(0) \in L^2(\mathcal{M})$ and that the forcing is bounded as $|f|_{L^\infty_t H^2_{\boldsymbol{x}}} + |\partial_t f|_{L^\infty_t L^2_{\boldsymbol{x}}} \le \infty$. Then there exists a time $T_0(|\boldsymbol{v}(0)|_{L^2}, |f|_{L^\infty_t H^2_{\boldsymbol{x}}}, |\partial_t f|_{L^\infty_t L^2_{\boldsymbol{x}}}; \mu)$ and $M_0(|f|_{L^\infty_t H^2_{\boldsymbol{x}}}, |\partial_t f|_{L^\infty_t L^2_{\boldsymbol{x}}}; \mu)$ such that, for all $t \ge T_0$,

$$\begin{aligned}&|\tilde\omega(t)|^2_{L^2} \le \varepsilon\, M_0, \\ &\mu\, \mathrm{e}^{-\nu(t+t')} \int_t^{t+t'} \mathrm{e}^{\nu\tau}\, |\nabla\tilde\omega(\tau)|^2_{L^2}\, \mathrm{d}\tau \le \varepsilon\, M_0.\end{aligned} \tag{3.2.20}$$

Proof. Multiplying (2.3.21) by $\tilde{\omega}$ in L^2, and use $\omega = \tilde{\omega} + \bar{\omega}$, we have

$$\begin{aligned}(\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\omega}+\bar{\omega}),\tilde{\omega})_{L^2} + \frac{1}{\varepsilon}(L(\tilde{\omega}+\bar{\omega}),\tilde{\omega})_{L^2} + \mu(A(\tilde{\omega}+\bar{\omega}),\tilde{\omega})_{L^2} \\ + (B(\tilde{\omega}+\bar{\omega},\tilde{\omega}+\bar{\omega}),\tilde{\omega})_{L^2} = (f,\tilde{\omega})_{L^2}.\end{aligned} \tag{3.2.21}$$

The second term is vanishing because L is an antisymmetric operator (2.3.17), and since $\tilde{\omega}$ and $\bar{\omega}$ are orthogonal and $(B(\tilde{\omega},\tilde{\omega}),\tilde{\omega})_{L^2} = (B(\bar{\omega},\tilde{\omega}),\tilde{\omega})_{L^2} = 0$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\omega}|_{L^2}^2 + \mu\,|\nabla\tilde{\omega}|_{L^2}^2 + (B(\bar{\omega},\bar{\omega}),\tilde{\omega})_{L^2} + (B(\tilde{\omega},\bar{\omega}),\tilde{\omega})_{L^2} = (\tilde{f},\tilde{\omega})_{L^2}. \tag{3.2.22}$$

Now because

$$\begin{aligned}(B(\tilde{\omega},\omega),\omega)_{L^2} &= (B(\tilde{\omega},\tilde{\omega}+\bar{\omega}),\tilde{\omega}+\bar{\omega})_{L^2} \\ &= (B(\tilde{\omega},\tilde{\omega}),\tilde{\omega})_{L^2} + (B(\tilde{\omega},\tilde{\omega}),\bar{\omega})_{L^2} + (B(\tilde{\omega},\bar{\omega}),\tilde{\omega})_{L^2} \\ &\qquad + (B(\tilde{\omega},\bar{\omega}),\bar{\omega})_{L^2} = 0,\end{aligned} \tag{3.2.23}$$

which implies

$$\big(B(\tilde{\omega},\bar{\omega}),\tilde{\omega}\big)_{L^2} = -\big(B(\tilde{\omega},\tilde{\omega}),\bar{\omega}\big)_{L^2}. \tag{3.2.24}$$

Furthermore

$$\begin{aligned}(B(\bar{\omega},\bar{\omega}),\tilde{\omega})_{L^2} &= \textstyle\sum_{\boldsymbol{jkl}} B_{\boldsymbol{jkl}}\,\bar{\omega}_{\boldsymbol{j}}\,\bar{\omega}_{\boldsymbol{k}}\,\overline{\tilde{\omega}_{\boldsymbol{l}}}\,\mathrm{e}^{\mathrm{i}\Omega_{\boldsymbol{l}}t/\varepsilon} \\ &= \textstyle\sum_{\boldsymbol{jkl}} \frac{\boldsymbol{j}\wedge\boldsymbol{k}}{|\boldsymbol{j}|^2}\,\bar{\omega}_{\boldsymbol{j}}\,\bar{\omega}_{\boldsymbol{k}}\,\overline{\tilde{\omega}_{\boldsymbol{l}}}\,\mathrm{e}^{\mathrm{i}\Omega_{\boldsymbol{l}}t/\varepsilon} = 0\end{aligned} \tag{3.2.25}$$

because $\boldsymbol{j}\wedge\boldsymbol{k} = j_1k_2 - j_2k_1 = 0$. Therefore (3.2.22) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\omega}|_{L^2}^2 + \mu\,|\nabla\tilde{\omega}|_{L^2}^2 + \mu\,|\nabla\tilde{\omega}|_{L^2}^2 - 2(B(\tilde{\omega},\tilde{\omega}),\bar{\omega})_{L^2} = 2(\tilde{f},\tilde{\omega})_{L^2}. \tag{3.2.26}$$

Using the Poincaré inequality (2.1.22) on the left-hand side, and multiplying by $\mathrm{e}^{\nu t}$, $\nu = c_0^2\,\mu$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\nu t}\,|\tilde{\omega}|_{L^2}^2) + \mu\,\mathrm{e}^{\nu t}\,|\nabla\tilde{\omega}|_{L^2}^2 \le 2\,\mathrm{e}^{\nu t}\,(B(\tilde{\omega},\tilde{\omega}),\bar{\omega})_{L^2} + 2\mathrm{e}^{\nu t}\,(\tilde{f},\tilde{\omega})_{L^2}. \tag{3.2.27}$$

We integrate from 0 to t by parts, the last term of the right-hand side as in Theorem 3.1.1, becomes

$$\begin{aligned}\int_0^t \mathrm{e}^{\nu\tau}\,(\tilde{f},\tilde{\omega})_{L^2}\,\mathrm{d}\tau &= \varepsilon\,(\mathsf{I}_\Omega\tilde{f},\tilde{\omega})_{L^2}(t)\,\mathrm{e}^{\nu t} - \varepsilon(\mathsf{I}_\Omega\tilde{f},\tilde{\omega})_{L^2}(0) \\ &\quad - \varepsilon\int_0^t \big[\nu(\mathsf{I}_\Omega\tilde{f},\tilde{\omega})_{L^2} + (\partial_\tau\mathsf{I}_\Omega\tilde{f},\tilde{\omega})_{L^2} + (\mathsf{I}_\Omega\tilde{f},\partial_\tau^*\tilde{\omega})_{L^2}\big]\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau.\end{aligned} \tag{3.2.28}$$

The endpoint terms can be bounded as

$$\begin{aligned} |(\mathsf{I}_\Omega\tilde{f},\tilde{\omega})_{L^2}| &= |\textstyle\sum_{\boldsymbol{k}}'\frac{1}{\Omega_{\boldsymbol{k}}}\,\tilde{f}_{\boldsymbol{k}}\,\overline{\tilde{\omega}_{\boldsymbol{k}}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot t}| = |\textstyle\sum_{\boldsymbol{k}}'\frac{|\boldsymbol{k}|^2}{k_1}\,\tilde{f}_{\boldsymbol{k}}\,\overline{\tilde{\omega}_{\boldsymbol{k}}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}| \\ &\le c\textstyle\sum_{\boldsymbol{k}}'|\boldsymbol{k}|^2\,|\tilde{f}_{\boldsymbol{k}}|\,|\tilde{\omega}_{\boldsymbol{k}}| \le c\sum_{\boldsymbol{k}}|\boldsymbol{k}|^2\,|\tilde{f}_{\boldsymbol{k}}|\,|\tilde{\omega}_{\boldsymbol{k}}| \le c\,|\nabla\tilde{f}|_{L^2}\,|\nabla\tilde{\omega}|_{L^2}. \end{aligned} \tag{3.2.29}$$

We bound now the terms in the integrand. First

$$|(\partial_\tau\mathsf{I}_\Omega\tilde{f},\tilde{\omega})_{L^2}| = |(\partial_\tau\tilde{f},\mathsf{I}_\Omega\tilde{\omega})_{L^2}| \le c\,|\partial_\tau\tilde{f}|_{L^2}\,|\Delta\tilde{\omega}|_{L^2}. \tag{3.2.30}$$

The last term in (3.2.28) is

$$(\mathsf{I}_\Omega\tilde{f},\partial_\tau^*\tilde{\omega})_{L^2} = -(\mathsf{I}_\Omega\tilde{f},\tilde{B}(\omega,\omega))_{L^2} - (\mathsf{I}_\Omega\tilde{f},\mu\,A\tilde{\omega})_{L^2} + (\mathsf{I}_\Omega\tilde{f},\tilde{f})_{L^2}, \tag{3.2.31}$$

where we used the definition of ∂_τ^*. Noting that $(I_\Omega\tilde{f},\tilde{f})=0$, we bound the second term of the right-hand side of (3.2.31) by

$$|(\mathsf{I}_\Omega\tilde{f},\mu\,\Delta\tilde{\omega})_{L^2}| \le \mu\,c\,|\Delta\tilde{f}|_{L^2}\,|\Delta\tilde{\omega}|_{L^2}; \tag{3.2.32}$$

and using Sobolev and interpolation inequalities,[2]

$$\begin{aligned} |(I_\Omega\tilde{f},\tilde{B}(\omega,\omega))_{L^2}| &\le c\,|\nabla\tilde{f}|_{L^2}\,|\nabla\tilde{B}(\omega,\omega)|_{L^2} \\ &\le c\,|\nabla\tilde{f}|_{L^2}\,(|\tilde{B}(\nabla\omega,\omega)|_{L^2} + |\tilde{B}(\omega,\nabla\omega)|_{L^2}) \\ &\le c\,|\nabla\tilde{f}|_{L^2}\,|\omega|_{L^4}\,|\nabla\omega|_{L^4} + c\,|\nabla\tilde{f}|_{L^2}\,|\nabla^{-1}\omega|_{L^\infty}\,|\Delta\omega|_{L^2} \\ &\le c\,|\nabla\tilde{f}|_{L^2}\,|\nabla\omega|_{L^2}\,|\Delta\omega|_{L^2}. \end{aligned} \tag{3.2.33}$$

Now (3.2.31) becomes

$$|(\mathsf{I}_\Omega\tilde{f},\partial_\tau^*\tilde{\omega})_{L^2}| \le c\,\mu\,|\Delta\tilde{f}|_{L^2}\,|\Delta\tilde{\omega}|_{L^2} + c\,|\nabla\tilde{f}|_{L^2}\,|\nabla\omega|_{L^2}\,|\Delta\omega|_{L^2}. \tag{3.2.34}$$

[2] Note that $|D^2\omega|_{L^2} = |\Delta\omega|_{L^2} = |\nabla^2|_{L^2}$, see [14].

Thus the integral in (3.2.28) is bounded as

$$\begin{aligned}
&\Big|\int_0^t \big[\nu(I_\Omega\tilde f,\tilde\omega)_{L^2} + (\partial_\tau I_\Omega \tilde f,\tilde\omega)_{L^2} + (I_\Omega\tilde f,\partial_\tau^*\tilde\omega)_{L^2}\big]\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau\Big| \\
&\qquad\le c\int_0^t \big[\nu\,|\nabla\tilde f|_{L^2}\,|\nabla\tilde\omega|_{L^2} + |\partial_\tau\tilde f|_{L^2}\,|\Delta\tilde\omega|_{L^2} + \mu\,|\Delta\tilde f|_{L^2}\,|\Delta\tilde\omega|_{L^2} \\
&\qquad\qquad + |\nabla\tilde f|_{L^2}\,|\nabla\omega|_{L^2}\,|\Delta\omega|_{L^2}\big]\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau \\
&\qquad\le c\int_0^t \mu\,|\nabla\tilde\omega|_{L^2}^2 + \mu\,|\Delta\tilde f|_{L^2}^2 + |\partial_\tau\tilde f|_{L^2}^2 + |\Delta\tilde\omega|_{L^2}^2 + \mu\,|\Delta\tilde f|_{L^2}^2 + \mu\,|\Delta\tilde\omega|_{L^2}^2 \\
&\qquad\qquad + |\Delta\omega|_{L^2}\,|\nabla\omega|_{L^2}\,|\nabla\tilde f|_{L^2}\big\}\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau \\
&\qquad\le c\int_0^t \big\{(1+\mu)\,|\Delta\tilde\omega|_{L^2}^2 + \mu\,|\Delta\tilde f|_{L^2}^2 + |\partial_\tau\tilde f|_{L^2}^2 + |\Delta\omega|_{L^2}\,|\nabla\omega|_{L^2}\,|\nabla\tilde f|_{L^2}\big\}\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau
\end{aligned} \tag{3.2.35}$$

We now treat the penultimate term in (3.2.27). First we write

$$\begin{aligned}
(B(\tilde\omega,\tilde\omega),\bar\omega)_{L^2} &= \textstyle\sum_{\boldsymbol{jkl}} B_{\boldsymbol{jkl}}\,\tilde\omega_{\boldsymbol j}\,\tilde\omega_{\boldsymbol k}\,\overline{\omega_{\boldsymbol l}}\,\mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol j}+\Omega_{\boldsymbol k})t/\varepsilon} \\
&= \frac12 \textstyle\sum_{\boldsymbol{jkl}} (B_{\boldsymbol{jkl}} + B_{\boldsymbol{kjl}})\,\tilde\omega_{\boldsymbol j}\,\tilde\omega_{\boldsymbol k}\,\overline{\omega_{\boldsymbol l}}\,\mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol j}+\Omega_{\boldsymbol k})t/\varepsilon}
\end{aligned} \tag{3.2.36}$$

and then note that $B_{\boldsymbol{jkl}} + B_{\boldsymbol{kjl}} = 0$ in the resonant case, i.e. when $\Omega_{\boldsymbol j} + \Omega_{\boldsymbol k} = 0$. Motivated by (3.2.36) and Lemma 3.2.1, we introduce the bilinear symmetric operator B_Ω by

$$\begin{aligned}
(B_\Omega(\tilde\omega^\sharp,\tilde\omega^\flat),\bar\omega)_{L^2} :&= \frac{\mathrm{i}}{2}\textstyle\sum'_{\boldsymbol{jkl}} \dfrac{B_{\boldsymbol{jkl}} + B_{\boldsymbol{kjl}}}{\Omega_{\boldsymbol j}+\Omega_{\boldsymbol k}}\,\tilde\omega^\sharp_{\boldsymbol j}\,\tilde\omega^\flat_{\boldsymbol k}\,\overline{\omega_{\boldsymbol l}}\,\mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol j}+\Omega_{\boldsymbol k})t/\varepsilon} \\
&= \frac{|\mathcal M|}{2\mathrm{i}}\textstyle\sum'_{\boldsymbol{jkl}} l_2\,\tilde\omega^\sharp_{\boldsymbol j}\,\tilde\omega^\flat_{\boldsymbol k}\,\overline{\omega_{\boldsymbol l}}\,\mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol j}+\Omega_{\boldsymbol k})t/\varepsilon}
\end{aligned} \tag{3.2.37}$$

for any $\tilde\omega^\sharp$, $\tilde\omega^\flat$ and $\bar\omega$, where the prime on the sum again indicates that resonant terms (for which $\Omega_{\boldsymbol j} + \Omega_{\boldsymbol k} = 0$) are omitted. We note that, thanks to lemma 3.2.1, the resonant terms are also absent in $(B(\tilde\omega,\tilde\omega),\bar\omega)_{L^2}$. Integrating by parts, we have

$$\begin{aligned}
&\int_0^t \mathrm{e}^{\nu\tau}\,(B(\tilde\omega,\tilde\omega),\bar\omega)_{L^2}\,\mathrm{d}\tau \\
&= \frac12\textstyle\sum'_{\boldsymbol{jkl}} \displaystyle\int_0^t (B_{\boldsymbol{jkl}} + B_{\boldsymbol{kjl}})\,\tilde\omega_{\boldsymbol j}\,\tilde\omega_{\boldsymbol k}\,\overline{\omega_{\boldsymbol l}}\,\mathrm{e}^{\nu\tau-\mathrm{i}(\Omega_{\boldsymbol j}+\Omega_{\boldsymbol k})\tau/\varepsilon}\,\mathrm{d}\tau. \\
&= \frac{\mathrm{i}\,\varepsilon}{2}\textstyle\sum'_{\boldsymbol{jkl}} \dfrac{B_{\boldsymbol{jkl}} + B_{\boldsymbol{kjl}}}{\Omega_{\boldsymbol j}+\Omega_{\boldsymbol k}}\Big\{\tilde\omega_{\boldsymbol j}(t)\,\tilde\omega_{\boldsymbol k}(t)\,\overline{\omega_{\boldsymbol l}(t)}\,\mathrm{e}^{\nu t-\mathrm{i}(\Omega_{\boldsymbol j}+\Omega_{\boldsymbol k})t/\varepsilon} \\
&\qquad - \tilde\omega_{\boldsymbol j}(0)\,\tilde\omega_{\boldsymbol k}(0)\,\overline{\omega_{\boldsymbol l}(0)}\Big\} - \frac{\mathrm{i}\,\varepsilon}{2}\textstyle\sum'_{\boldsymbol{jkl}}\displaystyle\int_0^t \dfrac{B_{\boldsymbol{jkl}} + B_{\boldsymbol{kjl}}}{\Omega_{\boldsymbol j}+\Omega_{\boldsymbol k}}\,\mathrm{e}^{-\mathrm{i}(\Omega_{\boldsymbol j}+\Omega_{\boldsymbol k})\tau/\varepsilon} \\
&\qquad\qquad \frac{\mathrm{d}}{\mathrm{d}\tau}[\tilde\omega_{\boldsymbol j}\,\tilde\omega_{\boldsymbol k}\,\overline{\omega_{\boldsymbol l}}\,\mathrm{e}^{\nu\tau}]\,\mathrm{d}\tau.
\end{aligned} \tag{3.2.38}$$

By (3.2.37) we have

$$\int_0^t \mathrm{e}^{\nu\tau}\big(B(\tilde\omega,\tilde\omega),\bar\omega\big)_{L^2}\,\mathrm{d}\tau = \varepsilon\,\mathrm{e}^{\nu t}\big(B_\Omega(\tilde\omega,\tilde\omega),\bar\omega\big)_{L^2}(t) - \varepsilon\,\big(B_\Omega(\tilde\omega,\tilde\omega),\bar\omega\big)_{L^2}(0)$$
$$- \varepsilon\int_0^t \mathrm{e}^{\nu\tau}\big[\nu\big(B_\Omega(\tilde\omega,\tilde\omega),\bar\omega\big)_{L^2} + 2\,\big(B_\Omega(\partial_\tau^*\tilde\omega,\tilde\omega),\bar\omega\big)_{L^2}$$
$$+ \big(B_\Omega(\tilde\omega,\tilde\omega),\partial_\tau\bar\omega\big)_{L^2}\big]\,\mathrm{d}\tau. \tag{3.2.39}$$

For the last term in the integrand, we use the fact that

$$\bar B(\tilde\omega,\bar\omega) = \bar B(\bar\omega,\tilde\omega) = \bar B(\bar\omega,\bar\omega) = 0 \tag{3.2.40}$$

to write

$$\partial_\tau\bar\omega = -\bar B(\tilde\omega,\tilde\omega) - \mu\,A\bar\omega + \bar f \tag{3.2.41}$$

and estimate, using $H^1\subset L^\infty$ for $\bar f$ and Lemma 3.2.4 for L^∞ estimates,

$$\begin{aligned}
&|(B_\Omega(\tilde\omega,\tilde\omega),\partial_\tau\bar\omega)_{L^2}|\\
&\quad= |(B_\Omega(\tilde\omega,\tilde\omega),\bar f)_{L^2} - (B_\Omega(\tilde\omega,\tilde\omega),\mu\,A\bar\omega)_{L^2} - (B_\Omega(\tilde\omega,\tilde\omega),B(\tilde\omega,\tilde\omega))_{L^2}|\\
&\qquad\le c\,|\tilde\omega|_{L^2}\,|\partial_y\tilde\omega|_{L^2}\,|\bar f|_{L^\infty} + c\,\mu\,|\tilde\omega|_{L^4}\,|\partial_y\tilde\omega|_{L^4}\,|\Delta\bar\omega|_{L^2}\\
&\qquad\qquad + c\,|\tilde\omega|_{L^\infty}\,|\partial_y\tilde\omega|_{L^2}\,|B(\tilde\omega,\tilde\omega)|_{L^2}\\
&\qquad\le c\,|\tilde\omega|_{L^2}\,|\partial_y\tilde\omega|_{L^2}\,|\bar f|_{L^\infty} + c\,\mu\,|\tilde\omega|_{L^4}\,|\partial_y\tilde\omega|_{L^4}\,|\Delta\bar\omega|_{L^2}\\
&\qquad\qquad + c\,|\tilde\omega|_{L^\infty}\,|\partial_y\tilde\omega|_{L^2}\,|\nabla^{-1}\tilde\omega|_{L^\infty}\,|\nabla\tilde\omega|_{L^2}\\
&\le c\,|\tilde\omega|\,|\nabla\tilde\omega|\,|\bar f'| + c\,\mu\,|\tilde\omega|^{1/2}\,|\nabla\tilde\omega|\,|\Delta\omega|^{3/2} + c\,|\nabla\omega|^3\,|\tilde\omega|\,\Big(\log\frac{|\Delta\omega|}{c_0|\nabla\omega|} + c'\Big),
\end{aligned}\tag{3.2.42}$$

where we used the Sobolev interpolation theorem. For the term involving $(B_\Omega(\partial_\tau^*\tilde\omega,\tilde\omega),\bar\omega)$, we bound, using

$$\partial_\tau^*\tilde\omega + \tilde B(\omega,\omega) + \mu\,A\tilde\omega = \tilde f \tag{3.2.43}$$

and the one-dimensional Agmon inequality

$$|\bar\omega|_{L^\infty} \le c\,|\bar\omega|^{1/2}\,|\bar\omega'|^{1/2}, \tag{3.2.44}$$

$$
\begin{aligned}
&|(B_\Omega(\partial_\tau^*\tilde\omega,\tilde\omega),\bar\omega)_{L^2}| \\
&\quad = |(B_\Omega(\tilde f,\tilde\omega),\bar\omega)_{L^2} - (B_\Omega(\mu\Delta\tilde\omega,\tilde\omega),\bar\omega)_{L^2} - (B_\Omega(\tilde B(\omega,\omega),\tilde\omega),\bar\omega)_{L^2}| \\
&\quad \le c\,|\partial_y\tilde f|_{L^2}\,|\tilde\omega|_{L^2}\,|\bar\omega|_{L^\infty} + c\,|\tilde f|_{L^2}\,|\partial_y\tilde\omega|_{L^2}\,|\bar\omega|_{L^\infty} \\
&\qquad + \mu\,c\,|\Delta\tilde\omega|_{L^2}\,|\tilde\omega|_{L^2}\,|\bar\omega'|_{L^\infty} + |\tilde B(\omega,\omega)|_{L^2}\,|\tilde\omega|_{L^2}\,|\bar\omega'|_{L^\infty} \\
&\quad \le c\,|\nabla\tilde f|_{L^2}\,|\tilde\omega|_{L^2}\,|\bar\omega'|_{L^2} + c\,|\tilde f|_{L^2}\,|\nabla\tilde\omega|_{L^2}\,|\bar\omega'|_{L^2} \\
&\qquad + \mu\,c\,|\tilde\omega|_{L^2}\,|\bar\omega'|_{L^2}^{1/2}\,|\Delta\tilde\omega|_{L^2}^{3/2} + c\,|\nabla^{-1}\omega|_{L^\infty}\,|\nabla\omega|_{L^2}\,|\tilde\omega|_{L^2}\,|\bar\omega'|_{L^\infty} \\
&\quad \le c\,|\nabla\tilde f|_{L^2}\,|\tilde\omega|_{L^2}\,|\bar\omega'|_{L^2} + c\,|\tilde f|_{L^2}\,|\nabla\tilde\omega|_{L^2}\,|\bar\omega'|_{L^2} \\
&\qquad + \mu\,c\,|\tilde\omega|_{L^2}\,|\bar\omega'|_{L^2}^{1/2}\,|\Delta\omega|_{L^2}^{3/2} + c\,|\omega|_{L^2}^{3/2}\,|\nabla\omega|_{L^2}^{3/2}\,|\bar\omega''|_{L^2}^{1/2}\left(\log\frac{|\nabla\omega|}{c_0\,|\omega|}+1\right)^{1/2}.
\end{aligned}
\tag{3.2.45}
$$

Finally we bound

$$
\begin{aligned}
|(B_\Omega(\tilde\omega,\tilde\omega),\bar\omega)_{L^2}| &\le c\,|\tilde\omega|_{L^2}\,|\partial_y\tilde\omega|_{L^2}\,|\bar\omega|_{L^\infty} \\
&\le c\,|\tilde\omega|_{L^2}\,|\nabla\tilde\omega|_{L^2}\,|\bar\omega|_{L^2}^{1/2}\,|\bar\omega'|_{L^2}^{1/2}.
\end{aligned}
\tag{3.2.46}
$$

Now the integral in (3.2.39) is bounded as

$$
\begin{aligned}
&|\int_0^t \mathrm{e}^{\nu\tau}\,(B(\tilde\omega,\tilde\omega),\bar\omega)_{L^2}\,\mathrm{d}\tau| \\
&\quad \le c\,\varepsilon\,[|\tilde\omega|_{L^2}\,|\nabla\tilde\omega|_{L^2}\,|\bar\omega|_{L^2}^{1/2}\,|\bar\omega'|_{L^2}^{1/2}](t)\,\mathrm{e}^{\nu t} + [|\tilde\omega|_{L^2}\,|\nabla\tilde\omega|_{L^2}\,|\bar\omega|_{L^2}^{1/2}\,|\bar\omega'|_{L^2}^{1/2}](0) \\
&\quad + \varepsilon\int_0^t\Big\{c\,|\nabla f|_{L^2}\,|\nabla\omega|_{L^2}\,|\tilde\omega|_{L^2} + c\,|\tilde f|_{L^2}\,|\nabla\omega|_{L^2}^2 + c\,\mu\,|\omega|_{L^2}^{1/2}\,|\nabla\omega|_{L^2}\,|\Delta\omega|_{L^2}^{3/2} \\
&\qquad\qquad + c\,|\omega|_{L^2}\,|\nabla\omega|_{L^2}^{5/2}\,|\Delta\omega|_{L^2}^{1/2}\left(\log\frac{|\Delta\omega|_{L^2}}{c_0\,|\nabla\omega|_{L^2}}+c'\right)\Big\}\,\mathrm{e}^{\nu\tau}\,\mathrm{d}\tau.
\end{aligned}
\tag{3.2.47}
$$

Putting together (3.2.29), (3.2.35) and (3.2.47), we have

$$
\begin{aligned}
&|\tilde\omega(t)|_{L^2}^2 + \mu\int_0^t |\nabla\tilde\omega(\tau)|_{L^2}^2\,\mathrm{e}^{\nu(\tau-t)}\,\mathrm{d}\tau \le \mathrm{e}^{-\nu t}\,|\tilde\omega(0)|_{L^2}^2 \\
&\qquad + c_2\,\varepsilon\,(1+\mathrm{e}^{-\nu t})\sup_{0\le t'\le t}\{|\nabla\tilde f|_{L^2}\,|\nabla\tilde\omega|_{L^2} + |\omega|_{L^2}^{3/2}\,|\nabla\omega|_{L^2}^{3/2}\} \\
&+ c_3(\mu)\,\varepsilon\int_0^t\Big\{|\Delta\tilde f|_{L^2}^2 + |\partial_\tau\tilde f|_{L^2}^2 + |\nabla\tilde f|_{L^2}\,|\nabla\omega|_{L^2}\,|\Delta\omega|_{L^2} + |\Delta\omega|_{L^2}^2(1+|\nabla\omega|_{L^2}) \\
&\qquad\qquad + |\omega|_{L^2}\,|\nabla\omega|_{L^2}^{5/2}\,|\Delta\omega|_{L^2}^{1/2}\left(\log\frac{|\Delta\omega|_{L^2}}{c_0|\nabla\omega|_{L^2}}+c'\right)\Big\}\,\mathrm{e}^{-\nu(\tau-t)}\,\mathrm{d}\tau.
\end{aligned}
\tag{3.2.48}
$$

We now shift the origin of time so that $t = 0$ corresponds to t_2 in Lemma 2.4.3. The hypothesis that $|f|_{L_t^\infty H_{\boldsymbol{x}}^2} + |\partial_t f|_{L_t^\infty L_{\boldsymbol{x}}^2} \le \infty$ then implies that both the endpoints and the integral in (3.2.48) are bounded uniformly for all $t > 0$, independently of the initial data provided that $\boldsymbol{v} \in L^2$ initially. Rewriting the bound in (3.2.48) as

$$\begin{aligned} |\tilde\omega(t)|_{L^2}^2 + \mu \int_0^t |\nabla\tilde\omega(\tau)|_{L^2}^2 \exp^{\nu(\tau-t)}\, \mathrm{d}\tau \le \mathrm{e}^{-\nu t}\, |\tilde\omega(0)|_{L^2}^2 \\ + \frac{\varepsilon}{2} M_0(|f|_{L_t^\infty H_{\boldsymbol{x}}^2}, |\partial_t f|_{L_t^\infty L_{\boldsymbol{x}}^2}; \mu). \end{aligned} \tag{3.2.49}$$

Now there exists a time $T_0(|\boldsymbol{v}(0)|_{L^2}, |f|_{L_t^\infty H_{\boldsymbol{x}}^2}, |\partial_t f|_{L_t^\infty L_{\boldsymbol{x}}^2}; \mu)$ such that, for all $t \ge T_0$,

$$\begin{aligned} &|\tilde\omega(t)|_{L^2}^2 \le \varepsilon\, M_0, \\ &\mu\, \mathrm{e}^{-\nu(t+t')} \int_t^{t+t'} \mathrm{e}^{\nu\tau}\, |\nabla\tilde\omega(\tau)|_{L^2}^2\, \mathrm{d}\tau \le \varepsilon\, M_0. \end{aligned} \tag{3.2.50}$$

□

3.3 Bounds in Sobolev Spaces

The following two Theorems shows that the H^s, $s = 1, 2, \cdots$, bounds for $\tilde\omega$ scale as $\sqrt{\varepsilon}$.

Theorem 3.3.1 Let the initial data $\boldsymbol{v}(0) \in L^2(\mathcal{M})$ and the forcing be bounded as

$$K_1(f) := |f|_{L_t^\infty H_{\boldsymbol{x}}^3} + |\partial_t f|_{L_t^\infty H_{\boldsymbol{x}}^1} \le \infty. \tag{3.3.1}$$

Then there exists a time $T_1(|\boldsymbol{v}(0)|_{L^2}, K_1; \mu)$ and $M_1(K_1; \mu)$ such that

$$\begin{aligned} &|\nabla\tilde\omega(t)|_{L^2}^2 \le \varepsilon\, M_1, \\ &\mu\, \mathrm{e}^{-\nu(t+t')} \int_t^{t+t'} \mathrm{e}^{\nu\tau}\, |\nabla^2\tilde\omega(\tau)|_{L^2}^2\, \mathrm{d}\tau \le \varepsilon\, M_1. \end{aligned} \tag{3.3.2}$$

for all $t \ge T_1$

Proof. Multiply (2.3.21) by $A\tilde\omega$ in L^2 to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\tilde\omega|_{L^2}^2 + \mu\,|\Delta\tilde\omega|_{L^2}^2 + (B(\omega,\omega), A\tilde\omega)_{L^2} = (\tilde f, A\tilde\omega)_{L^2}, \tag{3.3.3}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\nu t}\,|\nabla\tilde{\omega}|_{L^2}^2) + \mu\,\mathrm{e}^{\nu t}\,|\Delta\tilde{\omega}|_{L^2}^2 \le 2\,\mathrm{e}^{\nu t}\,(B(\omega,\omega),\Delta\tilde{\omega})_{L^2} - 2\,\mathrm{e}^{\nu t}\,(\tilde{f},\Delta\tilde{\omega})_{L^2}. \tag{3.3.4}$$

As in L^2 case, we integrate from 0 to t,

$$\begin{aligned} \mathrm{e}^{\nu t}\,|\nabla\tilde{\omega}(t)|_{L^2}^2 - |\nabla\tilde{\omega}(0)|_{L^2}^2 + \mu \int_0^t |\Delta\tilde{\omega}|_{L^2}^2\,\mathrm{e}^{\nu t}\,\mathrm{d}\tau \\ \le 2\int_0^t \bigl\{(B(\omega,\omega),\Delta\tilde{\omega})_{L^2} - (\tilde{f},\Delta\tilde{\omega})_{L^2}\bigr\}\,\mathrm{e}^{\nu t}\,\mathrm{d}\tau. \end{aligned} \tag{3.3.5}$$

The forcing term gives

$$\begin{aligned} \int_0^t \mathrm{e}^{\nu t}\,(\tilde{f},\Delta\tilde{\omega})_{L^2}\,\mathrm{d}\tau = \varepsilon(I_\Omega\tilde{f},\Delta\tilde{\omega})_{L^2}(t)\,\mathrm{e}^{\nu t} - \varepsilon(I_\Omega\tilde{f},\Delta\tilde{\omega})_{L^2}(0) \\ + \varepsilon\int_0^t \bigl[\nu(I_\Omega\tilde{f},\Delta\tilde{\omega})_{L^2} + (\partial_\tau I_\Omega\tilde{f},\Delta\tilde{\omega})_{L^2} + (I_\Omega\tilde{f},\Delta\partial_\tau^*\tilde{\omega})_{L^2}\bigr]\,\mathrm{e}^{\nu t}\,\mathrm{d}\tau, \end{aligned} \tag{3.3.6}$$

which can be bounded as in the L^2 case as follow. The endpoint terms

$$|(I_\Omega\tilde{f},\Delta\tilde{\omega})_{L^2}| \le c\,|\Delta\tilde{f}|_{L^2}\,|\Delta\tilde{\omega}|_{L^2}. \tag{3.3.7}$$

We now bound the terms in the integrand. First

$$|\nu(I_\Omega\tilde{f},\Delta\tilde{\omega})_{L^2}| \le c\,\mu\,|\nabla^2\tilde{f}|_{L^2}\,|\nabla^2\tilde{\omega}|_{L^2} \le c\,(\mu\,|\nabla^2\tilde{f}|_{L^2}^2 + \mu\,|\nabla^2\tilde{\omega}|_{L^2}^2). \tag{3.3.8}$$

The second term is

$$|(\partial_t I_\Omega\tilde{f},\Delta\tilde{\omega})_{L^2}| \le c\,|\nabla\partial_t\tilde{f}|_{L^2}^2 + c\,|\nabla^3\tilde{\omega}|_{L^2}^2. \tag{3.3.9}$$

The last term

$$\begin{aligned} |(I_\Omega\tilde{f},\Delta\partial_t^*\tilde{\omega})_{L^2}| &\le |(I_\Omega\tilde{f},\mu\,\Delta^2\tilde{\omega})_{L^2}| + |(I_\Omega\tilde{f},\Delta\tilde{B}(\omega,\omega))_{L^2}| \\ &\le c\,(\mu\,|\nabla^3\tilde{f}|_{L^2}^2 + \mu\,|\nabla^3\tilde{\omega}|_{L^2}^2 + |\nabla^3\tilde{\omega}|_{L^2}\,|\nabla\tilde{\omega}|_{L^2}\,|\Delta\tilde{f}|_{L^2}). \end{aligned} \tag{3.3.10}$$

Now

$$\begin{aligned} -2\int_0^t (\tilde{f},\Delta\tilde{\omega})_{L^2}\,\mathrm{e}^{\nu t}\,\mathrm{d}\tau \le \varepsilon\,c\,\bigl[|\Delta\tilde{f}|_{L^2}\,|\Delta\tilde{\omega}|_{L^2}\bigr](t)\mathrm{e}^{\nu t} + \varepsilon\,\bigl[|\Delta\tilde{f}|_{L^2}\,|\Delta\tilde{\omega}|_{L^2}\bigr](0) \\ + \varepsilon\,c\int_0^t \bigl\{(1+\mu)|\nabla^3\tilde{\omega}|_{L^2}^2 + \mu\,|\nabla^3\tilde{f}|_{L^2}^2 + |\nabla\partial_\tau\tilde{f}|_{L^2}^2 \\ + |\nabla^3\omega|_{L^2}\,|\nabla\omega|_{L^2}\,|\Delta\tilde{f}|_{L^2}\bigr\}\,\mathrm{e}^{\nu t}\,\mathrm{d}\tau. \end{aligned} \tag{3.3.11}$$

For the nonlinear term, we use the fact that $B(\bar{\omega},\bar{\omega}) = 0$ to write

$$(B(\omega,\omega), A\tilde{\omega})_{L^2} = (B(\bar{\omega},\tilde{\omega}), A\tilde{\omega})_{L^2} + (B(\tilde{\omega},\bar{\omega}), A\tilde{\omega})_{L^2} + (B(\tilde{\omega},\tilde{\omega}), A\tilde{\omega})_{L^2}, \quad (3.3.12)$$

and using

$$(B(\omega^\sharp,\omega), A\tilde{\omega})_{L^2} = (B(\nabla\omega^\sharp,\tilde{\omega}), \nabla\tilde{\omega})_{L^2}, \quad (3.3.13)$$

previously used in L^2 bound, we bound

$$\begin{aligned}
|(B(\tilde{\omega},\tilde{\omega}), A\tilde{\omega})_{L^2}| &= |(B(\nabla\tilde{\omega},\tilde{\omega}), \nabla\tilde{\omega})_{L^2}| \le c\,|\tilde{\omega}|_{L^2}\,|\nabla\tilde{\omega}|_{L^4}^2 \\
&\le c\,|\tilde{\omega}|_{L^2}\,|\nabla\tilde{\omega}|_{L^2}\,|\Delta\tilde{\omega}|_{L^2} \\
&\le \frac{\mu}{4}\,|\Delta\tilde{\omega}|_{L^2}^2 + \frac{c}{\mu}\,|\tilde{\omega}|_{L^2}^2\,|\nabla\tilde{\omega}|_{L^2}^2 \\
&\le \frac{\mu}{4}\,|\Delta\tilde{\omega}|_{L^2}^2 + \frac{c}{\mu}\,|\nabla\omega|_{L^2}^2\,|\nabla\tilde{\omega}|_{L^2}^2
\end{aligned}$$

$$|(B(\bar{\omega},\tilde{\omega}), A\tilde{\omega})_{L^2}| = |(B(\bar{\omega}',\tilde{\omega}), \partial_y\tilde{\omega})_{L^2}| \le c\,|\bar{\omega}|_{L^\infty}\,|\nabla\tilde{\omega}|_{L^2}^2 \quad (3.3.14)$$

$$\begin{aligned}
|(B(\tilde{\omega},\bar{\omega}), A\tilde{\omega})_{L^2}| &\le c\,|\Delta\tilde{\omega}|_{L^2}\,|\nabla^{-1}\tilde{\omega}|_{L^\infty}\,|\bar{\omega}'|_{L^2} \le c\,|\Delta\tilde{\omega}|_{L^2}\,|\nabla\tilde{\omega}|_{L^2}\,|\bar{\omega}'|_{L^2} \\
&\le \frac{\mu}{4}\,|\Delta\tilde{\omega}|_{L^2}^2 + \frac{c}{\mu}\,|\nabla\tilde{\omega}|_{L^2}^2\,|\bar{\omega}'|_{L^2}^2. \\
&\le \frac{\mu}{4}\,|\Delta\tilde{\omega}|_{L^2}^2 + \frac{c}{\mu}\,|\nabla\tilde{\omega}|_{L^2}^2\,|\nabla\omega|_{L^2}^2.
\end{aligned}$$

The nonlinear term becomes

$$\begin{aligned}
&2\int_0^t (B(\omega,\omega), \Delta\tilde{\omega})_{L^2}\,\mathrm{e}^{\nu t}\,\mathrm{d}\tau \\
&\qquad \le c_2(\mu)\int_0^t \Big\{\big[|\tilde{\omega}|_{L^\infty} + |\nabla\omega|_{L^2}^2\big]\,|\nabla\tilde{\omega}|_{L^2}^2 + \frac{\mu}{2}\,|\Delta\tilde{\omega}|_{L^2}^2\Big\}\,\mathrm{e}^{\nu t}\,\mathrm{d}\tau.
\end{aligned} \quad (3.3.15)$$

After moving the $|\Delta\tilde{\omega}|^2$ to the left-hand side, a factor of ε can be obtained by pulling the square bracket outside the integral and using (3.2.50). Collecting, we have

$$\begin{aligned}
&|\nabla\tilde{\omega}(t)|_{L^2}^2 + \frac{\mu}{2}\int_0^t \mathrm{e}^{\nu(\tau-t)}\,|\Delta\tilde{\omega}|_{L^2}^2\,\mathrm{d}\tau \\
&\quad\le \mathrm{e}^{-\nu t}\,|\nabla\tilde{\omega}(0)|_{L^2}^2 + c\,\varepsilon\,\sup_{t'>0}|\Delta\tilde{f}(t')|_{L^2}\,|\Delta\tilde{\omega}(t')|_{L^2} \\
&\qquad + c\,\varepsilon\int_0^t \big\{(1+\mu)\big(|\nabla^3\omega|_{L^2}^2 + |\nabla^3\tilde{f}|_{L^2}^2\big) \\
&\qquad\quad + |\nabla\partial_\tau\tilde{f}|_{L^2}^2 + |\nabla\omega|_{L^2}^2\,|\Delta\tilde{f}|_{L^2}^2\big\}\mathrm{e}^{\nu(\tau-t)}\,\mathrm{d}\tau \\
&\qquad + \varepsilon\,c_3(\mu)\,M_0\,\sup_{t'>0}\big\{|\tilde{\omega}(t')|_{L^\infty} + |\nabla\omega(t')|_{L^2}^2\big\}.
\end{aligned} \quad (3.3.16)$$

Arguing as in the L^2 case, $f \in L_t^\infty H_{\boldsymbol{x}}^3$ and $\partial_t \tilde{f} \in L_t^\infty H_{\boldsymbol{x}}^1$ gives us an $O(\sqrt{\varepsilon})$ bound for $\tilde{\omega}(t)$ in $L_t^\infty H_{\boldsymbol{x}}^1$ uniform for large t. □

Theorem 3.3.2 Let the initial data $\boldsymbol{v}(0) \in L^2(\mathcal{M})$ and the forcing be bounded as

$$K_s(f) := |f|_{L_t^\infty H_{\boldsymbol{x}}^{s+2}} + |\partial_t f|_{L_t^\infty H_{\boldsymbol{x}}^s} \leq \infty. \tag{3.3.17}$$

Then there exists a time $T_s(|\boldsymbol{v}(0)|_{L^2}, K_s; \mu)$ and $M_s(K_s;\mu)$ such that

$$\begin{aligned} &|\nabla^s \tilde{\omega}(t)|_{L^2}^2 \leq \varepsilon\, M_s, \\ &\mu\, \mathrm{e}^{-\nu(t+t')} \int_t^{t+t'} \mathrm{e}^{\nu\tau}\, |\nabla^{s+1}\tilde{\omega}(\tau)|_{L^2}^2 \,\mathrm{d}\tau \leq \varepsilon\, M_s. \end{aligned} \tag{3.3.18}$$

for all $t \geq T_s$

Proof. Multiply (2.3.21) by $A^s\tilde{\omega}$ in L^2, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla^s\tilde{\omega}|_{L^2}^2 + \mu\,|\nabla^{s+1}\tilde{\omega}|_{L^2}^2 \leq (\tilde{f}, A^s\tilde{\omega})_{L^2} - (B(\omega,\omega), A^s\tilde{\omega})_{L^2}. \tag{3.3.19}$$

Applying Poincaré on the left-hand side and multiplying by $\mathrm{e}^{\nu t}$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\nu t}\,|\nabla^s\tilde{\omega}|_{L^2}^2) + \mu\,\mathrm{e}^{\nu t}\,|\nabla^{s+1}\tilde{\omega}|_{L^2}^2 \leq 2\,\mathrm{e}^{\nu t}\,(\tilde{f}, A^s\tilde{\omega})_{L^2} - 2\,\mathrm{e}^{\nu t}\,(B(\omega,\omega), A^s\tilde{\omega})_{L^2}. \tag{3.3.20}$$

Now integrate the resulting equation in time from 0 to t. We bound the forcing term

$$\begin{aligned} \int_0^t (\tilde{f}, A^s\tilde{\omega})_{L^2}\,\mathrm{e}^{\nu t}\,\mathrm{d}\tau = {}& \varepsilon\,(I_\Omega\tilde{f}, A^s\tilde{\omega})_{L^2}(t)\,\mathrm{e}^{\nu t} - \varepsilon\,(I_\Omega\tilde{f}, A^s\tilde{\omega})_{L^2}(0) \\ & - \varepsilon\int_0^t \left\{\nu(I_\Omega\tilde{f}, A^s\tilde{\omega})_{L^2} + (\partial_\tau I_\Omega\tilde{f}, A^s\tilde{\omega})_{L^2} + (I_\Omega\tilde{f}, \partial_\tau^* A^s\tilde{\omega})_{L^2}\right\} \end{aligned} \tag{3.3.21}$$

We bound now the endpoint terms

$$|(I_\Omega\tilde{f}, A^s\tilde{\omega})_{L^2}| \leq c\,|\nabla^{s+1}\tilde{f}|_{L^2}\,|\nabla^{s+1}\tilde{\omega}|_{L^2}. \tag{3.3.22}$$

the bound of the integrand

$$|\nu(I_\Omega\tilde{f}, A^s\tilde{\omega})_{L^2}| \leq c\,\mu\,|\nabla^{s+1}\tilde{f}|_{L^2}\,|\nabla^{s+1}\tilde{\omega}|_{L^2} \leq c\,\mu\,|\nabla^{s+1}\tilde{f}|_{L^2}^2 + \frac{\mu}{4}\,|\nabla^{s+1}\tilde{\omega}|_{L^2}^2. \tag{3.3.23}$$

The second term

$$|(\partial_t I_\Omega \tilde{f}, A^s \tilde{\omega})_{L^2}| \leq c\,|\nabla^s \partial_t \tilde{f}|_{L^2}\,|\nabla^{s+2}\tilde{\omega}|_{L^2}. \tag{3.3.24}$$

To bound the last term we need to bound the following term, with $0 \leq |\beta| = r \leq s = |\alpha|$,

$$\begin{aligned}|(I_\Omega \tilde{f}, A^s \tilde{B}(\omega,\omega)_{L^2}| &\leq c \sum_{\alpha\beta} |D^\beta \boldsymbol{v}|_{L^\infty}\,|D^{\alpha-\beta}\nabla\omega|_{L^2}\,|\nabla^{\alpha+2}\tilde{f}|_{L^2} \\ &\leq c(s)\sum_{r=0}^{s} |\nabla^{r+1}\omega|_{L^2}\,|\nabla^{s-r+1}\omega|_{L^2}\,|\nabla^{s+2}\tilde{f}|_{L^2} \\ &\leq c(s)\,|\nabla^{s+1}\omega|_{L^2}^2\,|\nabla^{s+2}\tilde{f}|_{L^2}.\end{aligned} \tag{3.3.25}$$

Then the last term

$$\begin{aligned}|(I_\Omega \tilde{f}, \partial_t^* A^s \tilde{\omega})_{L^2}| &= |(I_\Omega \tilde{f}, -A^s \tilde{B}(\omega,\omega) - \mu\, A^s\tilde{\omega} + A^s \tilde{f})_{L^2}| \\ &\leq c(s)|\nabla^{s+1}\omega|_{L^2}^2\,|\nabla^{s+2}\tilde{f}|_{L^2} + c\,\mu\,|\nabla^{s+2}\tilde{\omega}|_{L^2}\,|\nabla^{s+2}\tilde{f}|_{L^2}.\end{aligned} \tag{3.3.26}$$

We bound the nonlinear term

$$(B(\omega,\omega), A^s\tilde{\omega})_{L^2} = (B(\tilde{\omega},\tilde{\omega}), A^s\tilde{\omega})_{L^2} + (B(\bar{\omega},\tilde{\omega}), A^s\tilde{\omega})_{L^2} + (B(\tilde{\omega},\bar{\omega}), A^s\tilde{\omega})_{L^2} \tag{3.3.27}$$

as follow. The first term

$$\begin{aligned}|(B(\tilde{\omega},\tilde{\omega}), A^s\tilde{\omega})_{L^2}| &\leq c\sum_{\alpha\beta} |D^\beta \nabla^{-1}\tilde{\omega}|_{L^4}\,|D^{\alpha-\beta}\nabla\tilde{\omega}|_{L^4}\,|D^\alpha\tilde{\omega}|_{L^2} \\ &\leq c(s)\,|\tilde{\omega}|_{L^2}\,|\nabla^s\tilde{\omega}|_{L^2}\,|\nabla^{s+1}\tilde{\omega}|_{L^2} \\ &\leq \frac{\mu}{4}\,|\nabla^{s+1}\tilde{\omega}|_{L^2}^2 + \frac{c(s)}{\mu}\,|\tilde{\omega}|_{L^2}^2\,|\nabla^s\tilde{\omega}|_{L^2}^2.\end{aligned} \tag{3.3.28}$$

We bound now the second term, with $|\alpha| = s$ and $1 \leq |\beta| = r \leq s$,

$$\begin{aligned}|(B(\bar{\omega},\tilde{\omega}), A^s\tilde{\omega})_{L^2}| &\leq c\sum_{\alpha\beta} |D^\beta \bar{\boldsymbol{v}}|_{L^\infty}\,|D^{\alpha-\beta}\nabla\tilde{\omega}|_{L^2}\,|D^\alpha\tilde{\omega}|_{L^2} \\ &\leq c(s)\sum_{r=1}^{s} |\nabla^r\bar{\omega}|_{L^2}\,|\nabla^{s-r+1}\tilde{\omega}|_{L^2}\,|\nabla^s\tilde{\omega}|_{L^2} \\ &\leq c(s)\,|\nabla^s\bar{\omega}|_{L^2}\,|\nabla^s\tilde{\omega}|_{L^2}^2.\end{aligned} \tag{3.3.29}$$

Finally, we bound the last term as, where now $0 \leq |\beta| = r \leq s = |\alpha|$,

$$\begin{aligned}
|(B(\tilde{\omega}, \bar{\omega}), A^s \tilde{\omega})_{L^2}| &\leq c \sum_{\alpha\beta} |D^\beta \tilde{\boldsymbol{v}}|_{L^4} \, |D^{\alpha-\beta} \nabla \bar{\omega}|_{L^2} \, |D^\alpha \tilde{\omega}|_{L^4} \\
&\leq c(s) \sum_{r=0}^{s} |\tilde{\omega}|_{H^{r-1/2}} \, |\bar{\omega}|_{H^{s-r+1}} \, |\tilde{\omega}|_{H^{s+1/2}} \\
&\leq c(s) \sum_{r=0}^{s} |\tilde{\omega}|_{H^{r-1}}^{1/2} \, |\tilde{\omega}|_{H^r}^{1/2} \, |\bar{\omega}|_{H^{s-r+1}} \, |\tilde{\omega}|_{H^s}^{1/2} \, |\tilde{\omega}|_{H^{s+1}}^{1/2} \qquad (3.3.30) \\
&\leq \frac{\mu}{4} |\nabla^{s+1} \tilde{\omega}|_{L^2}^2 + c(s,\mu) \sum_{r=0}^{s} |\nabla^{r-1} \tilde{\omega}|_{L^2}^{2/3} \, |\nabla^r \tilde{\omega}|_{L^2}^{2/3} \, |\nabla^s \tilde{\omega}|_{L^2}^{2/3} \, |\nabla^{s-r+1} \bar{\omega}|_{L^2}^{4/3} \\
&\leq \frac{\mu}{4} |\nabla^{s+1} \tilde{\omega}|_{L^2}^2 + c(s,\mu) \, |\nabla^s \tilde{\omega}|_{L^2}^2 \, |\nabla^{s+1} \bar{\omega}|_{L^2}^{4/3}
\end{aligned}$$

Collecting the terms and moving the $|\nabla^{s+1}\tilde{\omega}|_{L^2}^2$ to the left hand side of of the main inequality (3.3.20), the right-hand side depends at most on $|\nabla^s \tilde{\omega}|_{L^2}^2$, which is of $O(\varepsilon)$ in L_t^2 from the step $s-1$, and on $|\nabla^{s+1}\bar{\omega}|_{L^2}^2$. The Theorem follows by the same argument used to obtain Lemma 2.4.3 and Theorem 3.3.1. □

Chapter 4

Stability and the Global Attractors

In this chapter we use the results obtained in last chapter to prove that $dim_H \mathcal{A} = 0$, where $dim_H \mathcal{A}$ is the Hausdorff dimension of the attractor of Navier–Stokes equation on β-plane. In section 4.1 we give some notation and auxiliary results related to the existence and uniqueness of the solution of Navier–Stokes equations on the β-plane and the existence of the global attractor ($\mathcal{A}$) for this equation. In section 4.2 we define the fractal and the Hausdorff dimensions and prove that $dim_H \mathcal{A} = 0$.

4.1 Notation and Auxiliary Results

A semidynamical system consists of a triplet (X, T, ϕ) where X is called the phase space or state space which contains all possible states $x \in X$ of the system. Time T is either continuous ($T = \mathbb{R}_+$) or discrete ($T = \mathbb{Z}_+$). Finally a map $\phi : X \to X$, is the evolution map such that

$$\phi(t)x_0 = x(t), \quad \text{for all} \quad t \in T, \tag{4.1.1}$$

where x_0 is the state of the system at time $t = 0$ and $x(t)$ is the state of the system at time t. Clearly $\phi(0)x_0 = x_0$ for all $x_0 \in X$. We will denote a semidynamical system by $(X, \{\phi(t)\}_{t\in T})$. We say that $(X, \{\phi(t)\}_{t\in T})$ is a dynamical system if $T = \mathbb{R}$ or $T = \mathbb{Z}$. If the system is autonomous, we obtain:

$$x(s+t) = \phi(s+t)x_0 = \phi(s)\phi(t)x_0. \tag{4.1.2}$$

The orbit or trajectory at a point x_0 is the set

$$\xi(x_0) = \cup_{t\in T}\phi(t)x_0. \tag{4.1.3}$$

If an initial-value problem is well posed[1] for all $t \geq 0$, this allows us to define, in the phase space X, the semigroup $\{S(t)\}_{t\geq 0}$, i.e., the family of operators

$$S(t) : X \to X$$

depending on a real parameter $t \geq 0$ (time) and satisfying the following identities

$$\begin{aligned} &S(t)S(s) = S(s)S(t) = S(s+t), \quad \forall s,t \in T \\ &S(0) = I \quad (I \text{ is identity operator}). \end{aligned} \tag{4.1.4}$$

See e.g. [48, 49, 59].

Definition 4.1.1 [2] A set $B \in X$ is said to be an invariant set for the semigroup $\{S(t)\}_{t\geq 0}$ if

$$S(t)B = B, \qquad t \geq 0. \tag{4.1.5}$$

Definition 4.1.2 A set $B \in X$ is said to be attracting set for a semigroup $\{S(t)\}_{t\geq 0}$ if for each bounded set $B_0 \in X$ we have

$$\operatorname{dist}(S(t)B_0, B) \to 0 \quad \text{as } t \to \infty, \tag{4.1.6}$$

where $\operatorname{dist}(A, B)$ is the Hausdorff-semi distance between two sets,

$$\operatorname{dist}(A, B) = \sup_{x\in A}\inf_{y\in B}|x - y|. \tag{4.1.7}$$

[1]Well posed means that a solution exists, the solution is unique and the solution depends continuously on the data.

[2]See e.g. [59, p. 19]

Note that if $\text{dist}(A, B) = 0$ then $A \subset B$. The existence of a bounded absorbing set is an important step to prove the existence of the global attractor, so we need the following definition.

Definition 4.1.3 We say that a set $B \in X$ is an absorbing set for the semigroup $\{S(t)\}_{t\geq 0}$ if for each bounded set $B_0 \in X$ there exists a time $t_1(B_0) > 0$ such that

$$S(t)B_0 \subset B \quad \text{for all} \quad t \geq t_1(B_0), \tag{4.1.8}$$

i.e., the orbits of all bounded sets eventually enter and do not leave B [6, p. 37]. Clearly, any absorbing set is an attracting set. Another property needed for proving the existence of the global attractor is some kind of compactness of the semigroup.

Definition 4.1.4 A semigroup $\{S(t)\}_{t\geq 0}$ is said to be dissipative if it possesses a compact absorbing set.

Furthermore, the long time dynamics of a system are captured in limit sets which are particular type of invariant sets and are mapped into themselves under the evolution equation, i.e., for a bounded set $B \subset X$ the ω-limit set of a set B consist of all limit points of the orbit of B,

$$\omega(B) = \{y : \exists\, t_n \to \infty,\, x_n \in B \text{ with } S(t_n)x_n \to y\}. \tag{4.1.9}$$

This can also be characterized as

$$\omega(B) = \cap_{t\geq 0}\overline{\cup_{s\geq t} S(s)B}, \tag{4.1.10}$$

see e.g. [48, 49]. It was proven in [48, p. 265] that if, for some for $t_0 > 0$, the set $\overline{\cup_{t\geq t_0} S(t)B}$ is compact, then $\omega(B)$ is nonempty, compact, and invariant, where B is a bounded set and $B \subset X$.

Definition 4.1.5 [3] An attractor is a set $\mathcal{A} \subset X$ that enjoys the following properties

[3] See [6, p. 19] and [48, p. 268]

1. $\mathcal{A}$ is the maximal compact invariant set, $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geq 0$.

2. $\mathcal{A}$ is the minimal set that attracts all bounded sets

$$\operatorname{dist}(S(t)B, \mathcal{A}) \to 0 \qquad \text{as } t \to \infty, \tag{4.1.11}$$

where the distance in (4.1.11) is understood to be the semidistance (4.1.7).

The following Lemma shows $\mathcal{A}$ coincides with the limit set $\omega(B)$ where the existence of the global attractor is proved provided that $S(t)$ is dissipative and B is an absorbing set.

Lemma 4.1.6 If $S(t)$ is dissipative and B is a compact absorbing set then there exists a global attractor $\mathcal{A} = \omega(B)$. If X is connected then so is $\mathcal{A}$, and if the flow is injective, i.e.,

$$\text{if} \quad S(t)u_0 = S(t)v_0 \quad \text{for some} \quad t > 0 \quad \text{then} \quad u_0 = v_0, \tag{4.1.12}$$

then

$$S(t)\mathcal{A} = \mathcal{A} \tag{4.1.13}$$

is satisfyed for all $t \in \mathbb{R}$. Furthermore, $\mathcal{A}$ is the maximal compact invariant set in X.

Proof. See [48, p.269]. □

If it exists then the global attractor is unique (see e.g. [49,59]): suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are two global attractors. Then, since $\mathcal{A}_2$ is bounded, it is attracted by $\mathcal{A}_1$,

$$\operatorname{dist}(S(t)\mathcal{A}_2, \mathcal{A}_1) \to 0 \quad \text{as } t \to \infty. \tag{4.1.14}$$

But $\mathcal{A}_2$ is invariant, $S(t)\mathcal{A}_2 = \mathcal{A}_2$, and so $\operatorname{dist}(\mathcal{A}_2, \mathcal{A}_1) = 0$. The argument is symmetric, so $\operatorname{dist}(\mathcal{A}_1, \mathcal{A}_2) = 0$, from which it follows that $\mathcal{A}_1 = \mathcal{A}_2$.

The following Lemma shows the existence and uniqueness of the weak and strong solutions of the voricity form of two dimensional Navier–Stokes equations on the β-plane.

Lemma 4.1.7 (i) (**Weak solution**). If $\boldsymbol{v}_0 \in L^2$ and $f \in L^\infty((0,T);H^{-1})$ then there exists a unique solution of the vorticity form of Navier–Stokes equation on the β-plane

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} + \frac{1}{\varepsilon} L\omega + B(\omega,\omega) + \mu A\omega = f, \tag{4.1.15}$$

that satisfies

$$\omega \in L^\infty((0,T);L^2) \cap L^2((0,T);H^1), \qquad \forall T > 0, \tag{4.1.16}$$

and in fact $\omega \in C^0([0,T];L^2)$.

(ii) (**Strong solution**). If $\boldsymbol{v}_0 \in L^2$ and $f \in L^\infty((0,T);L^2)$ then there exists a unique solution of (A.2.7) that satisfies

$$\omega \in L^\infty((0,T);H^1) \cap L^2((0,T);H^2), \qquad \forall T > 0, \tag{4.1.17}$$

and in fact $\omega \in C^0([0,T];H^1)$.

Proof. See the Appendix. □

The results in Lemma 4.1.7 show that, when f is independent of time t, we can define a C^0 semigroup $\{S(t)\}_{t\geq 0}$ of solution operator

$$S(t)\omega_0 = \omega(\cdot,t) \quad \forall t \geq 0, \tag{4.1.18}$$

satisfying the following properties

$$\begin{aligned} &S(0) = I, \\ &S(s)S(t) = S(s+t). \end{aligned} \tag{4.1.19}$$

Furthermore, we can define semidynamical systems on L^2, $(L^2, \{S_{L^2}(t)\}_{t\geq 0})$, and on H^1, $(H^1, \{S_{H^1}(t)\}_{t\geq 0})$.

By Lemmata 2.4.2, and 2.4.3 we can find absorbing sets in L^2 and H^1, and since H^1 is compactly embedded in L^2, this gives a compact absorbing set in L^2 and guarantees the existence of a global attractor $\mathcal{A}_L$ for the semigroup on L^2. In the same way, by Lemma (2.4.3) and Theorem 2.1.8 ii, we can define global attractor $\mathcal{A}_H$ in H^1. We will summarize these results in the following Lemma

Lemma 4.1.8 (i) The vorticity form of $2D$ Navier–Stokes equations on β-plane have global attractors in L^2 and in H^1.

(ii) If $f \in L^\infty((0,T);L^2)$ then $\mathcal{A}_L = \mathcal{A}_H$.

Proof. We can follow the same way as in [48, ch. 10] to prove this Lemma. □

4.2 Attractor Dimension

An important attribute of an attractor is its dimension, that is, the number of orthogonal coordinate axes in the space in which it exists. That space is necessarily a subspace of the function space of the solution. The practical use of this attribute, say dim($\mathcal{A}$), lies in its relation to the number of degrees of freedom of the solution, (e.g., the number, say n, needed to parameterize the attractor), $n \leq 2\mathrm{dim}(\mathcal{A})+1$ [16]. Before we introduce our result, let us recall the definitions of the fractal and the Hausdorff dimensions.

Definition 4.2.1 Let $\overline{X}$ be compact, the fractal dimension of X is defined by

$$dim_f X = \limsup_{\epsilon\to 0} \frac{\ln N_X(\epsilon)}{\ln(1/\epsilon)}, \tag{4.2.1}$$

where $N_X(\epsilon)$ is the smallest number of balls of radius ϵ necessary to cover X, and we allow the limit in (4.2.1) to be infinity. The Hausdorff dimension is based on an approximation of the d-dimensional volume of a space X by a covering of a finite balls of radius not larger than ϵ. Note that here, unlike with the fractal dimension, we can take balls with arbitrarily small radii less than ϵ. The following definition gives us the best approximation of the volume using such a covering of balls with radii $\leq \epsilon$ (see e.g. [16, 48, 59]).

Definition 4.2.2 The Hausdorff dimension of a compact set X, denoted by $dim_H(X)$, is defined by

$$dim_H(X) = \inf\{d > 0 : \mathcal{H}^d(X) = 0\}, \tag{4.2.2}$$

where

$$\mathcal{H}^d(X) = \lim_{\epsilon\to 0} \mu(X, d, \epsilon) \tag{4.2.3}$$

and

$$\mu(X, d, \epsilon) = \inf\left\{\textstyle\sum_i r_i^d : r_i \le \epsilon \,\text{and}\, X \subseteq \cup_i B(x_i, r_i)\right\} \tag{4.2.4}$$

where $B(x_i, r_i)$ are balls with radius $r_i \le \epsilon$, which are covering X.

In the non-rotating case, the Hausdorff dimension, $dim_H(\mathcal{A})$, of the attractor for Navier–Stokes equation, is bounded by

$$\dim_H \mathcal{A} \le c(\mathcal{M})\, G^{2/3}(1 + \log G)^{1/3}, \tag{4.2.5}$$

where in our notation the Grashof number is

$$G := |\nabla^{-1} f|_{L^2}/\mu^2. \tag{4.2.6}$$

The rotation not posing any extra essential difficulty, the usual analysis, e.g. [14] carries over essentially line-by-line to our case, giving the bound (4.2.5) also for the rotating case.

As discussed in the introduction, and following our results that the flow becomes more zonal ("ordered") as $\varepsilon \to 0$, we expect the dimension of the attractor to decrease as $\varepsilon \to 0$. In this section, we use a simple computation similar to that used for Theorem 3.2 to show that $\dim_H \mathcal{A} = 0$ for ε sufficiently small.

Theorem 4.2.3 Let the forcing f be time independent, $\partial_t f = 0$, and assume the hypotheses of Theorem 3.2.5 that

$$|\nabla^2 f|_{L^2} < \infty. \tag{4.2.7}$$

Then there exists an $\varepsilon_*(|\nabla^2 f|; \mu)$ such that, for all $\varepsilon < \varepsilon_*$,

$$dim_H \mathcal{A} = 0. \tag{4.2.8}$$

Since $\mathcal{A}$ is connected, (4.2.8) implies that $\mathcal{A}$ consists of a single point. In turbulence parlance, the smallness of ε demanded by Theorem 4.2.3 implies that the Rhines scale is so large that it overwhelms the entire spectral range, rendering the dynamics trivial. Analogous to the Kolmogorov and Kraichnan scales in homogeneous isotropic turbulence, the Rhines scale $1/(k_\beta)$ is a length scale at which the effect of differential planetary rotation balances that of the nonlinearity. Rhines [47] defined $k_\beta = \sqrt{\beta/(2U)}$ for some typical velocity scale U, which we can take here to be $|\boldsymbol{v}|_{L^2}$, but alternate definitions have been proposed [62]; our bound in Theorem 3.2.5 suggests that velocity norms up to H^2 may play some role.

A general result related to ours is described in [6, Ch. 18], where the trajectory attractor $\mathcal{A}_\epsilon$ of a dynamical system depending on t/ϵ (formally, in our case $\mathcal{A}_\epsilon$ would simply be the attractor $\mathcal{A}$ for $\varepsilon > 0$) converges weakly to the attractor $\mathcal{A}_0$ of the corresponding averaged system. Formally averaging our equations following this construction (which does not apply directly to our case, in which the oscillations have an infinite number of frequencies which accumulate at zero), we obtain the purely zonal Navier–Stokes equation, whose dynamics is trivial and whose attractor thus has dimension zero. This is of course consistent with our results: strong convergence at finite ε of $\mathcal{A}$ to a point (which becomes zonal as $\varepsilon \to 0$).

Proof. Fix a solution $\omega(t)$ of (2.3.21) that lives in $\mathcal{A}$, so the bounds (3.3.18) hold for all t. We consider a nearby solution $\omega(t) + \phi(t)$. The linearized evolution equation for ϕ is then

$$\begin{aligned} \partial_t\phi &= -(\nabla^\perp\Delta^{-1}\omega)\cdot\nabla\phi - (\nabla^\perp\Delta^{-1}\phi)\cdot\nabla\omega(t) - \frac{1}{\varepsilon}\partial_{\boldsymbol{x}}\Delta^{-1}\phi + \mu\,\Delta\phi \\ &= -B(\omega,\phi) - B(\phi,\omega) - \frac{1}{\varepsilon}L\phi - \mu A\phi =: \mathcal{L}(t)\phi. \end{aligned} \tag{4.2.9}$$

Multiplying this by ϕ in L^2 and noting that $(B(\omega,\phi),\phi) = 0$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\phi|_{L^2}^2 + \mu\,|\nabla\phi|_{L^2}^2 &= (B(\phi,\phi),\omega)_{L^2} \\ &= (B(\phi,\phi),\bar{\omega})_{L^2} + (B(\phi,\phi),\tilde{\omega})_{L^2}. \end{aligned} \tag{4.2.10}$$

For the first term, we split $\phi = \bar{\phi} + \tilde{\phi}$ in analogy with $\omega = \bar{\omega} + \tilde{\omega}$ to get

$$(B(\phi,\phi),\bar{\omega})_{L^2} = (B(\tilde{\phi},\tilde{\phi}),\bar{\omega})_{L^2} \tag{4.2.11}$$

using the fact that $B(\bar{\phi},\bar{\phi}) = 0$ and all tilde-bar-bar terms vanish.

Using Poincaré inequality in (4.2.10) gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\nu t}|\phi|_{L^2}^2) + \mu\,\mathrm{e}^{\nu t}\,|\phi|_{L^2}^2 \le 2\,\mathrm{e}^{\nu t}(B(\tilde{\phi},\tilde{\phi}),\bar{\omega})_{L^2} + 2\,\mathrm{e}^{\nu t}(B(\phi,\phi),\tilde{\omega})_{L^2}, \tag{4.2.12}$$

which integrates to

$$\begin{aligned} |\phi(t)|_{L^2}^2\,\mathrm{e}^{\nu t} &+ \mu\int_0^t |\nabla\phi|_{L^2}^2\,\mathrm{e}^{\nu t}\,\mathrm{d}\tau \\ &\le |\phi(0)|_{L^2}^2 + 2\int_0^t \{(B(\tilde{\phi},\tilde{\phi}),\bar{\omega})_{L^2} + (B(\phi,\phi),\tilde{\omega})_{L^2}\}\,\mathrm{e}^{\nu t}\,\mathrm{d}\tau. \end{aligned} \tag{4.2.13}$$

We bound the last term of the integrand using

$$\begin{aligned} |(B(\phi,\phi),\tilde{\omega})_{L^2}| &\le c\,|\nabla^{-1}\phi|_{L^\infty}\,|\nabla\phi|_{L^2}\,|\tilde{\omega}|_{L^2} \\ &\le c_4|\nabla\phi|^2\,|\tilde{\omega}|_{L^2}. \end{aligned} \tag{4.2.14}$$

The other term needs to be integrated by parts,

$$\begin{aligned} \int_0^t (B(\tilde{\phi},\tilde{\phi}),\bar{\omega})_{L^2}\,\mathrm{e}^{\nu t}\,\mathrm{d}\tau &= \varepsilon(B_\Omega(\tilde{\phi},\tilde{\phi}),\bar{\omega})_{L^2}(t)\,\mathrm{e}^{\nu t} - \varepsilon(B_\Omega(\tilde{\phi},\tilde{\phi}),\bar{\omega})_{L^2}(0) \\ - \varepsilon\int_0^t \big\{\nu(B_\Omega(\tilde{\phi},\tilde{\phi}),\bar{\omega})_{L^2} &+ (B_\Omega(\tilde{\phi},\tilde{\phi}),\partial_\tau\bar{\omega})_{L^2} + 2(B_\Omega(\partial_\tau^*\tilde{\phi},\tilde{\phi}),\bar{\omega})_{L^2}\big\}\mathrm{e}^{\nu t}\,\mathrm{d}\tau \end{aligned} \tag{4.2.15}$$

where $\partial_t^*\phi = -B(\omega,\phi) - B(\phi,\omega) - \mu A\phi$. We bound the endpoint terms using

$$|(B_\Omega(\tilde{\phi},\tilde{\phi}),\bar{\omega})_{L^2}| \le c_5\,|\tilde{\phi}|^2\,|\bar{\omega}'|_{L^\infty}. \tag{4.2.16}$$

It remains to bound the integrand in (4.2.15):

$$\begin{aligned} |(B_\Omega(\tilde{\phi},\tilde{\phi}),\bar{\omega})_{L^2}| &\le c\,|\partial_y\tilde{\phi}|_{L^2}\,|\tilde{\phi}|_{L^4}\,|\bar{\omega}|_{L^4} \\ &\le c\,|\nabla\tilde{\phi}|^2\,|\bar{\omega}|_{L^4} \end{aligned} \tag{4.2.17}$$

$$\begin{aligned} |(B_\Omega(\tilde{\phi},\tilde{\phi}),\partial_t\bar{\omega})_{L^2}| &\le c\,|\partial_y\tilde{\phi}|_{L^2}\,|\tilde{\phi}|_{L^{10}}\,|\partial_t\bar{\omega}|_{L^{5/2}} \\ &\le c\,|\nabla\tilde{\phi}|^2\,|\partial_t\bar{\omega}|_{L^{5/2}} \end{aligned} \tag{4.2.18}$$

Recalling (4.2.9) for the last term in (4.2.15), we bound

$$\begin{aligned} |(B_\Omega(\tilde{B}(\phi,\omega),\tilde{\phi}),\bar{\omega})_{L^2}| &\le c\,|\tilde{B}(\phi,\omega)|_{L^2}\,|\tilde{\phi}|_{L^{10}}\,|\bar{\omega}'|_{L^{5/2}} \\ &\le c\,|\nabla^{-1}\phi|_{L^\infty}\,|\nabla\omega|_{L^2}\,|\tilde{\phi}|_{L^{10}}\,|\bar{\omega}'|_{L^{5/2}} \\ &\le c\,|\nabla\phi|^2\,|\bar{\omega}'|_{L^{5/2}}\,|\nabla\omega|_{L^2} \end{aligned} \tag{4.2.19}$$

$$\begin{aligned} |(B_\Omega(\tilde{B}(\omega,\phi),\tilde{\phi}),\bar{\omega})_{L^2}| &\le c\,|\tilde{B}(\omega,\phi)|_{L^2}\,|\tilde{\phi}|_{L^{10}}\,|\bar{\omega}'|_{L^{5/2}} \\ &\le c\,|\nabla^{-1}\omega|_{L^\infty}\,|\nabla\phi|_{L^2}\,|\tilde{\phi}|_{L^{10}}\,|\bar{\omega}'|_{L^{5/2}} \\ &\le c\,|\nabla\phi|^2\,|\bar{\omega}'|_{L^{5/2}}\,|\nabla\omega|_{L^2} \end{aligned} \tag{4.2.20}$$

$$\begin{aligned} |(B_\Omega(\Delta\tilde{\phi},\tilde{\phi}),\bar{\omega})_{L^2}| &\le |(B_\Omega(\nabla\tilde{\phi},\nabla\tilde{\phi}),\bar{\omega})_{L^2}| + |(B_\Omega(\partial_y\tilde{\phi},\tilde{\phi}),\bar{\omega}')_{L^2}| \\ &\le c\,|\nabla\tilde{\phi}|^2\,|\bar{\omega}'|_{L^\infty} + c\,|\nabla\tilde{\phi}|_{L^2}\,|\tilde{\phi}|_{L^{10}}\,|\bar{\omega}''|_{L^{5/2}} \\ &\le c\,|\nabla\tilde{\phi}|^2|\bar{\omega}''|_{L^{5/2}}. \end{aligned} \tag{4.2.21}$$

Collecting, (4.2.13) now implies

$$\begin{aligned} |\phi(t)|^2_{L^2}(1-\varepsilon\,c_5\,|\bar{\omega}'(t)|_{L^\infty}) + \int_0^t \Big\{\mu - \varepsilon\,N(\tau) - c_4\,|\tilde{\omega}(\tau)|_{L^2}\Big\}|\nabla\phi|^2_{L^2}\,\mathrm{e}^{\nu(\tau-t)}\,\mathrm{d}\tau \\ \le \mathrm{e}^{-\nu t}\,|\phi(0)|^2_{L^2}(1+\varepsilon\,c_5\,|\bar{\omega}'(0)|_{L^\infty}), \end{aligned} \tag{4.2.22}$$

where

$$N(t) := c_6\big\{\mu\,|\bar{\omega}''|_{L^{5/2}} + |\bar{\omega}'|_{L^{5/2}}\,|\nabla\omega|_{L^2} + |\partial_t\bar{\omega}|_{L^{5/2}} + |\bar{\omega}|_{L^4}\big\}(t). \tag{4.2.23}$$

By Lemma (2.4.3), $f \in H^2$ implies that $\omega \in H^3$ with a uniform bound in t since we are already on the attractor, and by Theorem 3.2.5 we can find an ε_* so small that, for $\varepsilon < \varepsilon_*$

$$\sup_{t>0}\big\{\varepsilon\,N(t) + c_4\,|\tilde{\omega}(t)|_{L^2}\big\} \le \mu. \tag{4.2.24}$$

If we further require that ε_* also satisfies

$$\varepsilon_* c_5\,\sup_{t>0}|\bar{\omega}'(t)|_{L^\infty} \le 1, \tag{4.2.25}$$

these and (4.2.22) then imply that

$$|\phi(t)|^2_{L^2} \le C(\cdots)\,\mathrm{e}^{-\nu t}|\phi(0)|^2_{L^2}, \tag{4.2.26}$$

in other words, all phase space volumes contract and thus the global attractor has dimension zero. □

It is clear from the above proof that our solution $\omega(t)$ is linearly stable. Since (4.2.9) only differents by $B(\phi, \phi)$ from the nonlinear system, the fact that $(B(\phi, \phi), \phi)_{L^2} = 0$ implies that stability also holds under the same hypotheses for the full nonlinear system.

Chapter 5

Higher-Order Estimates

5.1 Slow Manifold

In this chapter we use the results obtained in the last chapters to construct a slow manifold for the Navier–Stokes equation on β-plane with order of $\varepsilon^{n/2}$ accuracy for arbitrary $n \in \mathbb{N}$, as well as with exponentially accuracy a slow manifold for the same equation is approximated. In section 5.2 a brief introduction concerning Gevrey space is given, along with auxiliary results about Gevrey regularity for our equation. In section 5.3 a slow manifold for our equation with order of $\varepsilon^{n/2}$ and exponential accuracy is approximated.

5.2 Gevrey space

The set of all functions in the domain of $\mathrm{e}^{\sigma A^{1/2}}$, $D(\mathrm{e}^{\sigma A^{1/2}})$ for each $\sigma > 0$, is called Gevrey space and denoted by G_σ. We say that $\omega \in G_\sigma$, if

$$|\omega|_\sigma := |\mathrm{e}^{\sigma A^{1/2}} \omega|_{L^2} < \infty, \tag{5.2.1}$$

where $|\omega|_\sigma$ is the norm of ω in Gevrey space. In Fourier space we can write $e^{\sigma A^{1/2}}\,\omega$ as

$$e^{\sigma A^{1/2}}\,\omega = \sum_{\boldsymbol{k}} e^{\sigma \boldsymbol{k}}\,\omega_{\boldsymbol{k}}\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{5.2.2}$$

Also G_σ is a Hilbert space under the inner product

$$(\omega,\omega^\sharp)_\sigma = (e^{\sigma A^{1/2}}\,\omega, e^{\sigma A^{1/2}}\,\omega^\sharp)_{L^2}, \quad \text{for} \quad \omega,\omega^\sharp \in G_\sigma. \tag{5.2.3}$$

From (5.2.1) and (5.2.2), we can see

$$|e^{\sigma A^{1/2}}\,\omega|_{L^2}^2 = |\mathcal{M}| \sum_{\boldsymbol{k}} e^{2\sigma|\boldsymbol{k}|}\,|\omega_{\boldsymbol{k}}|^2 < \infty. \tag{5.2.4}$$

Thus the norm of the elements in Gevrey space is given by:

$$|\omega|_\sigma^2 = |e^{\sigma A^{1/2}}\,\omega|_{L^2}^2 = |\mathcal{M}| \sum_{\boldsymbol{k}} e^{2\sigma|\boldsymbol{k}|}\,|\omega_{\boldsymbol{k}}|^2, \quad \text{for}\,\omega \in G_\sigma. \tag{5.2.5}$$

The set $D(A^{1/2}\,e^{\sigma A^{1/2}})$ is also Gevrey space and a Hilbert space under the inner product

$$((\omega,\omega^\sharp))_\sigma = (A^{1/2}\,e^{\sigma A^{1/2}}\,\omega, A^{1/2}\,e^{\sigma A^{1/2}}\,\omega^\sharp)_{L^2}, \quad \text{for}\,\omega,\,\omega^\sharp \in D(A^{1/2}\,e^{\sigma A^{1/2}}). \tag{5.2.6}$$

Also the associated norm is given by:

$$\|\omega\|_\sigma^2 = |A^{1/2}\,e^{\sigma A^{1/2}}\,\omega|_{L^2}^2 = |\mathcal{M}| \sum_{\boldsymbol{k}} |\boldsymbol{k}|^2\,e^{2\sigma|\boldsymbol{k}|}\,|\omega_{\boldsymbol{k}}|^2, \tag{5.2.7}$$

(see e.g., [16] , [21]). For our purpose in this chapter we need the following regularity results for our equation, and you can find the proof in the Appendix.

Lemma 5.2.1 If ω, $\omega^\sharp$ and $\omega^\flat$ are given in $D(e^{\sigma A^{1/2}}\,A)$, for some $\sigma > 0$, then $B(\omega,\omega^\sharp) \in D(e^{\sigma A^{1/2}})$ and

$$|\big(B(\omega,\omega^\sharp), A\omega^\flat\big)_\sigma| \le c\,\|\omega\|_\sigma\,\|\omega^\sharp\|_\sigma\,|A\omega^\flat|_\sigma. \tag{5.2.8}$$

Lemma 5.2.2 If $f \in L^\infty(\mathbb{R}_+; G_\sigma)$, for some $\sigma > 0$. Then there exists a time $T_\sigma(|\omega(0)|_{H^1}, |f|_\sigma; \mu)$ such that

$$|A^{1/2}\,e^{\sigma_2 A^{1/2}}\,\omega(t_1)|_{L^2} \le K_\sigma(|f|_\sigma; \mu), \tag{5.2.9}$$

for all $t_1 \ge T_\sigma$ where $\sigma_2 = \sigma_1(T_\sigma) = \min(\sigma, T_\sigma)$

5.3 Slow manifold approximation for the Navier–Stokes equation on β-plane

Consider the following fast/slow dynamical system for $X \in \mathbb{R}^p$ and $Y \in \mathbb{R}^q$

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} + \frac{1}{\varepsilon}\, LX &= M(X,Y) \\ \frac{\mathrm{d}Y}{\mathrm{d}t} &= N(X,Y), \end{aligned} \tag{5.3.1}$$

where M and N are vector-valued polynomial functions of their arguments, L is skew-hermitian non-singular $p \times p$ matrix and ε is a small parameter that represents the ratio of time scales. We look for slow solutions[1] of (5.3.1)

$$X = U(Y;\varepsilon) \tag{5.3.2}$$

for some function U, so that the fast variable X is slaved to the slow variable Y. In this case the equations in (5.3.1) become

$$DU(Y;\varepsilon)\, N(U(Y;\varepsilon),Y) + \frac{1}{\varepsilon} LU(Y;\varepsilon) - M(U(Y;\varepsilon),Y) = 0, \tag{5.3.3}$$

and

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = N(U(Y;\varepsilon),Y), \tag{5.3.4}$$

where DU is the derivative, called Fréchet derivative,[2] of U with respect to Y. It is shown in [69] that U is a slow manifold. Now if $DU = 0$, applying the fixed point theorem pointwise gives us the existence of a unique solution of (5.3.3), for ε sufficiently small, but when $DU \neq 0$, we apply the iteration

$$U^0 = 0, \qquad U^{n+1} = \varepsilon\, L^{-1}\big\{M(U^n(Y;\varepsilon),s) - DU^n(Y;\varepsilon)\, N(U^n(Y;\varepsilon),Y)\big\}, \tag{5.3.5}$$

with Banach's fixed-point theorem to find a unique solution. For the Navier–Stokes equation on β-plane we cannot apply the above method to find a slow manifold

[1] A slow solution means that the fast variable X evolves entirely on a slow timescale without fast oscillations on the $O(1/\varepsilon)$ timescale.

[2] See the Appendix

$U(\bar{\omega}, \bar{f}; \varepsilon)$ because $\tilde{\omega}$ and $\bar{\omega}$ are infinite dimensional variables. In this section we follow [60, 61, 68] to find the slow manifold $U^*(\bar{\omega}^<, f^<; \varepsilon)$ [3], where the normal velocity is small, such that the bound of $(\tilde{\omega} - U^*(\bar{\omega}^<, f^<; \varepsilon))$ is of $O(\varepsilon^{n/2})$, i.e., we can approximate $\tilde{\omega}$ by $U^*(\bar{\omega}^<, f^<; \varepsilon)$ with order of $\varepsilon^{n/2}$ accuracy. Let us now describe the idea of the proof. We truncate the solution $\tilde{\omega}$ into a low-mode truncation, $\tilde{\omega}^<$, and prove that this part is small with $O(\varepsilon^{n/2})$, and into a high-mode truncation, $\tilde{\omega}^>$. Due to Gevrey regularity, $\tilde{\omega}^>$ is small with order of $\varepsilon^{n/2}$ as well see Lemmata 5.3.2 and 5.3.4. By combining these two results together we obtain that the bound of $\hat{\omega}$ is small of order of $\varepsilon^{n/2}$. In the same way we can approximate $\tilde{\omega}$ by $U^*(\bar{\omega}^<, \bar{f}^<; \varepsilon)$ up to an error that scales exponentially in ε as $\varepsilon \to 0$ (see Lemmata 5.3.7 and 5.3.5). Given now a fixed $\kappa > 0$, split ω into $\omega^<$ and $\omega^>$, where $\omega^<$ is the low-mode truncation of ω and $\omega^>$ is the high-mode truncation of ω and they are defined as

$$\omega^<(\boldsymbol{x}, t) = \mathsf{P}^< \omega(\boldsymbol{x}, t) := \sum_{|\boldsymbol{k}|<\kappa} \omega_{\boldsymbol{k}}(t)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \Omega_{\boldsymbol{k}} t/\varepsilon)}, \tag{5.3.6}$$

$$\omega^>(\boldsymbol{x}, t) = \mathsf{P}^> \omega(\boldsymbol{x}, t) := \sum_{|\boldsymbol{k}|\geq\kappa} \omega_{\boldsymbol{k}}(t)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \Omega_{\boldsymbol{k}} t/\varepsilon)}. \tag{5.3.7}$$

In the same way, define the fast and slow variables as $\tilde{\omega} = \tilde{\omega}^< + \tilde{\omega}^>$ and $\bar{\omega} = \bar{\omega}^< + \bar{\omega}^>$. It is easy to see that $\mathsf{P}^<$ and $\mathsf{P}^>$ are orthogonal projections in H^s and commute with the operators A and L, i.e., $\mathsf{P}^< A = A\mathsf{P}^<$, $\mathsf{P}^> A = A\mathsf{P}^>$, $\mathsf{P}^< L = L\mathsf{P}^<$ and $\mathsf{P}^> L = L\mathsf{P}^>$. The following Lemma shows that the low-mode $\omega^<$ satisfies a "reverse Poincaré" inequality.

Lemma 5.3.1 For any $s \geq 0$,

$$|\nabla\omega^<|_{H^s} \leq \kappa\, |\omega^<|_{H^s} \tag{5.3.8}$$

[3] U^* is a finite dimensional manifold in a phase space and lives in the same space as $\tilde{\omega}$ lives i.e., $(\bar{\omega}, U^*)_{L^2} = 0$.

Proof.

$$
\begin{aligned}
|\nabla\omega^{<}|_{H^s}^2 &\le |\mathcal{M}| \sum_{|\boldsymbol{k}|<\kappa} |\boldsymbol{k}|^{2s+2}\,|\omega_{\boldsymbol{k}}|^2 \\
&\le |\mathcal{M}|\,\kappa^2 \sum_{|\boldsymbol{k}|<\kappa} |\boldsymbol{k}|^{2s}\,|\omega_{\boldsymbol{k}}|^2 \\
&= \kappa^2\,|\omega^{<}|_{H^s}^2.
\end{aligned} \tag{5.3.9}
$$

□

Furthermore if $\omega \in G_\sigma(\mathcal{M})$, then the exponential decay of its Fourier coefficient implies that $\omega^{>}$ is exponential small. The following Lemma shows that

Lemma 5.3.2 For all m, we have

$$
|\nabla\omega^{>}|_{L^2} \le C_m\,\kappa^{-m}\|\omega\|_\sigma. \tag{5.3.10}
$$

Proof. From (5.2.7), we have

$$
\begin{aligned}
\|\omega\|_\sigma^2 = |A^{1/2}\,\mathrm{e}^{\sigma A^{1/2}}\omega|_{L^2}^2 &= |\mathcal{M}| \sum_{\boldsymbol{k}} \mathrm{e}^{2\sigma|\boldsymbol{k}|}\,|\boldsymbol{k}|^2\,|\omega_{\boldsymbol{k}}|^2 \\
&= |\mathcal{M}| \sum_{s,\boldsymbol{k}} \frac{(2\sigma|\boldsymbol{k}|)^s}{s!}\,|\boldsymbol{k}|^2\,|\omega_{\boldsymbol{k}}|^2 \\
&= |\mathcal{M}| \sum_{s} \frac{(2\sigma)^s}{s!} \sum_{\boldsymbol{k}} |\boldsymbol{k}|^s\,|\boldsymbol{k}|^2\,|\omega_{\boldsymbol{k}}|^2 \\
&\ge |\mathcal{M}| \sum_{s} \frac{(2\sigma)^s}{s!} \sum_{|\boldsymbol{k}|\ge\kappa} |\boldsymbol{k}|^s\,|\boldsymbol{k}|^2|\omega_{\boldsymbol{k}}|^2 \\
&\ge |\mathcal{M}| \sum_{s} \frac{(2\sigma)^s}{s!}\,\kappa^s \sum_{|\boldsymbol{k}|\ge\kappa} |\boldsymbol{k}|^2\,|\omega_{\boldsymbol{k}}|^2 \\
&= \sum_{s} \frac{(2\sigma)^s}{s!}\,\kappa^s\,|\nabla\omega^{>}|_{L^2}.
\end{aligned} \tag{5.3.11}
$$

It follows that

$$
|\nabla\omega^{>}|_{L^2} \le C_m\,\kappa^{-m}\|\omega\|_\sigma \quad \text{for every} \quad m = s \tag{5.3.12}
$$

□

Note that the above Lemmata can be applied with the slow, $\bar{\omega}$, and fast, $\tilde{\omega}$, parts separately. The main result of this chapter is given by the following Theorem.

Theorem 5.3.3 Assume that the regularity in Lemmata 2.4.1, 2.4.2, 2.4.3 and 5.2.2 hold. Let $\boldsymbol{v}_0 \in L^2(\mathcal{M})$ and $f \in G_\sigma(\mathcal{M})$ be given, with $\partial_t f = 0$. Then there exist $\varepsilon_*(f)$ and time $T_*(|\boldsymbol{v}_0|_{L^2}, |f|_{G_\sigma})$ such that for $\varepsilon \le \varepsilon_*$ and time $t \ge T_*$ we can approximate the fast variable $\tilde{\omega}$ by a function $U^*(\bar{\omega}^<(t), f^<; \varepsilon)$ of the slow variable $\bar{\omega}$ as

$$|\tilde{\omega}(t) - U^*(\bar{\omega}^<(t), f^<; \varepsilon)|_{L^2} \le \varepsilon^{n/2} K_*(|f|_{G_\sigma}; \sigma). \tag{5.3.13}$$

where $n = 0, 1, \cdots$ and K_* is a continuous function of its first argument.

Before we start the proof of Theorem 5.3.3, we need to find a uniform bound for our slow manifold $U^*(\bar{\omega}^<, f^<; \mu)$ and prove that the remainders $\mathcal{R}^*$ and $\hat{\mathcal{Q}}$ are small with order of $\varepsilon^{n/2}$,

$$\begin{aligned} \mathcal{R}^* &:= \mathsf{P}^<[(DU^*)\mathfrak{D}^*] + \frac{1}{\varepsilon} LU^* + \mu AU^* + \tilde{B}^<(\bar{\omega}^< + U^*, \bar{\omega}^< + U^*) - \tilde{f}^<, \\ \hat{\mathcal{Q}} &:= -\tilde{B}^<(\omega^<, \omega^>) - \tilde{B}^<(\omega^>, \omega) + (1 - \mathsf{P}^<)[(DU^*)\mathfrak{D}^*], \end{aligned} \tag{5.3.14}$$

where DU^* is the derivative, called Fréchet derivative, of U^* with respect to $\bar{\omega}^<$ and

$$\mathfrak{D}^* = -\bar{B}^<(\bar{\omega}^< + U^*, \bar{\omega}^< + U^*) - \mu A\bar{\omega}^< + \bar{f}^<. \tag{5.3.15}$$

Lemma 5.3.4 Let $s > 1$ and $\gamma > 0$ be fixed. Given $\bar{\omega}^< \in H^s(\mathcal{M})$ and $f \in H^s(\mathcal{M})$ with $\partial_t f = 0$, there exists $\varepsilon_{**}(|\bar{\omega}|_{H^s}, |f|_{H^s}, \gamma)$ such that for $\varepsilon \le \varepsilon_{**}$ one can find $\kappa(\varepsilon)$ and $U^*(\bar{\omega}^<, f^<; \varepsilon)$ that make the remainder function

$$\mathcal{R}^*(\bar{\omega}^<, f^<; \varepsilon) := \mathsf{P}^<[(DU^*)\mathfrak{D}^*] + \frac{1}{\varepsilon} LU^* + \mu AU^* + \tilde{B}^<(\bar{\omega}^< + U^*, \bar{\omega}^< + U^*) - \tilde{f}^< \tag{5.3.16}$$

of order $\varepsilon^{n/2}$,

$$|\mathcal{R}^*|_s \le c\, |f|_s\, \varepsilon^{n/2} \tag{5.3.17}$$

and

$$|\hat{\mathcal{Q}}|_{L^2} \le c\, \varepsilon^{n/2} \left[K_\sigma\, |\omega|_2 + ((|\bar{\omega}^<|_2 + \gamma)^2 + \mu\,(|\bar{\omega}^<|_2 + \gamma) + |f|_2)^2\right] \tag{5.3.18}$$

Proof. Firstly we construct U^* iteratively, and we shall do that by solving (5.3.19) with $\mathcal{R}^* = 0$, i.e.,

$$\frac{1}{\varepsilon} LU^* = -\mathsf{P}^<[(DU^*)\mathfrak{D}^*] - \mu AU^* - \tilde{B}^<(\bar{\omega}^< + U^*, \bar{\omega}^< + U^*) + \tilde{f}^<, \tag{5.3.19}$$

and use Banach's fixed-point Theorem.[4] Taking $U^0 = 0$,[5] we find the correction U^1 satisfies

$$\frac{1}{\varepsilon} LU^1 = -\tilde{B}^<(\bar{\omega}^< + U^0, \bar{\omega}^< + U^0) + \tilde{f}^< = \tilde{f}^<, \tag{5.3.20}$$

as well as for $n = 1, 2, \cdots$, let

$$\frac{1}{\varepsilon} LU^{n+1} = -\mathsf{P}^<[(DU^n)\mathfrak{D}^n] - \tilde{B}^<(\bar{\omega}^< + U^n, \bar{\omega}^< + U^n) - \mu\, AU^n + \tilde{f}^<, \tag{5.3.21}$$

with

$$\mathfrak{D}^n = -\bar{B}^<(\bar{\omega}^< + U^n, \bar{\omega}^< + U^n) - \mu A\bar{\omega}^< + \bar{f}^<. \tag{5.3.22}$$

where $U^{n+1} \in$ range L, $n = 0, 1 \cdots$, for uniqueness. Since the right hand side of (5.3.20) and (5.3.21) do not lie in Ker$L$, so U^1 and U^{n+1} are well defined. Furthermore, U^n lives in the same space as $\tilde{\omega}^<$, i.e., $(U^n, \bar{\omega})_{L^2} = 0$.

To bound U^{n+1}, $n = 1, 2, ...$, we need to define the complex neighborhood of $\bar{\omega}^<$ in the space $\mathsf{P}^< H^s(\mathcal{M})$. For any $\gamma > 0$, the complex γ-neighborhood of $\bar{\omega}^<$ in $\mathsf{P}^< H^s(\mathcal{M})$ (denoted by $\mathcal{N}_\gamma$) is defined in Fourier series as

$$\mathcal{N}_\gamma = \left\{\bar{\omega}^\flat : \bar{\omega}^\flat(\boldsymbol{x}) = \textstyle\sum_{\boldsymbol{k} \leq \kappa} \bar{\omega}^\flat_{\boldsymbol{k}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \quad \text{with} \quad \textstyle\sum_{|\boldsymbol{k}| \leq \kappa} |\boldsymbol{k}|^{2s}\, |\bar{\omega}^\flat_{\boldsymbol{k}} - \bar{\omega}_{\boldsymbol{k}}|^2 \leq \gamma^2\right\} \tag{5.3.23}$$

In addition we need to define the norm of a function g of $\bar{\omega}^<$. Let $\rho > 0$ be fixed. For any function g of $\bar{\omega}^<$ let

$$|g(\bar{\omega}^<)|_{s;n} := \sup_{V \in \mathcal{N}_{\gamma - n\rho}(\bar{\omega}^<)} |g(V)|_s, \tag{5.3.24}$$

which is meaningful for $n \in \{0, ..., \lfloor \gamma/\rho \rfloor =: n_*\}$, when $\mathcal{N}_{\gamma - n\rho}$ is non-empty. Note that for $m \leq n$ we have

$$|\cdot|_{s;n} \leq |\cdot|_{s;m}. \tag{5.3.25}$$

Furthermore, we have

$$|\bar{\omega}^<|_{s;0} \leq |\bar{\omega}^<|_s + \gamma. \tag{5.3.26}$$

[4]See the Appendix.

[5]$U^0 = 0$ corresponding to the leading order slow manifold $\tilde{\omega}$, see Ch. 3 Section 3.1.1.

In addition we need the following Banach algebra property (see e.g. [1]), for $s > 1$

$$|\omega\,\omega^\sharp|_s \leq c\,|\omega|_s\,|\omega^\sharp|_s. \tag{5.3.27}$$

We find now uniform bounds for U^n valid for all $n \in \{1, \cdots, n_*\}$. First

$$\frac{1}{\varepsilon}\,LU^1 = \tilde{f}^< \Rightarrow \quad U^1 = \varepsilon\,L^{-1}\tilde{f}^< \tag{5.3.28}$$

and the bound of U^1 is

$$\begin{aligned} |U^1|^2_{s;1} &= \varepsilon^2\,|L^{-1}\tilde{f}^<|^2_{s;1} \\ &= \varepsilon^2\,|L^{-1}\tilde{f}^<|^2_s \\ &= \varepsilon^2\,|\sum_{\substack{k_1\neq 0\\ |\boldsymbol{k}|<\kappa}} \frac{|\boldsymbol{k}|^2}{k_1}\,\tilde{f}_{\boldsymbol{k}}|^2_s \\ &\leq c\,\varepsilon^2 \textstyle\sum_{|\boldsymbol{k}|<\kappa} |\boldsymbol{k}|^{2(2+s)}\,|\tilde{f}_{\boldsymbol{k}}|^2 \\ &\leq c\,\varepsilon^2\,\kappa^4 \textstyle\sum_{|\boldsymbol{k}|<\kappa} |\boldsymbol{k}|^{2s}\,|\tilde{f}_{\boldsymbol{k}}|^2 \\ &\leq c\,\varepsilon^2\,\kappa^4\,|\tilde{f}^<|^2_s. \end{aligned} \tag{5.3.29}$$

Hence [6]

$$|U^1|_{s;1} \leq c\,\varepsilon\,\kappa^2\,|\tilde{f}^<|_s \tag{5.3.30}$$

where we used the reverse Poincaré inequality (5.3.8). We derive now iterative estimates for $|U^n|_{s;n}$. Recall that we have for $n = 1, 2, ...$

$$\frac{1}{\varepsilon}\,LU^{n+1} = -\mathsf{P}^<[(DU^n)\mathfrak{D}^n] - \tilde{B}^<(\bar{\omega}^< + U^n, \bar{\omega}^< + U^n) - \mu\,AU^n + \tilde{f}^<, \tag{5.3.31}$$

which implies

$$U^{n+1} = -\varepsilon\,L^{-1}\left\{\mathsf{P}^<[(DU^n)\mathfrak{D}^n] - \tilde{B}^<(\bar{\omega}^< + U^n, \bar{\omega}^< + U^n) - \mu\,AU^n + \tilde{f}^<\right\}, \tag{5.3.32}$$

[6] We use the bound $|f|_s$ instead the bound $|f|_{s;n}$ because f does not depend on $\bar{\omega}$, see (5.3.24).

with the bound

$$|U^{n+1}|_{s;n+1} \le c\,\varepsilon\,\big\{|L^{-1}\,\mathsf{P}^{<}[(DU^n)\mathfrak{D}^n]|_{s;n+1} + |L^{-1}\tilde{B}^{<}(\bar{\omega}^{<}+U^n,\bar{\omega}^{<}+U^n)|_{s;n+1} + |L^{-1}\,\mu\,AU^n|_{s;n+1} + |L^{-1}\tilde{f}^{<}|_{s;n+1}\big\}$$
$$\le c\,\varepsilon\,\kappa^2\big\{|\mathsf{P}^{<}[(DU^n)\mathfrak{D}^n]|_{s;n+1} + |\tilde{B}^{<}(\bar{\omega}^{<}+U^n,\bar{\omega}^{<}+U^n)|_{s;n} + \mu\,|AU^n|_{s;n} + |\tilde{f}^{<}|_{s}\big\}, \tag{5.3.33}$$

where we used (5.3.25) in the second and third terms of the right hand side. Furthermore, for the first term we need to apply the Cauchy's integral formula (5.3.34), so we can estimate (DU^n) only in $\mathcal{N}_{\gamma-(n+1)\rho}$ and not in $\mathcal{N}_{\gamma-n\rho}$. We bound now the terms in the right-hand side separately. The first can be bounded by a technique based on Cauchy's integral formula:[7] Let $\mathcal{N}_\gamma(z_0)\subset\mathbb{C}$ be the complex γ-neighborhood of z_0. For $\phi:\mathcal{N}_\gamma(z_0)\to\mathbb{C}$ analytic and $\rho\in(0,\gamma)$, we can bound $|\phi'|$ in $\mathcal{N}_{\gamma-\rho}(z_0)$ by $|\phi|$ in $D_\gamma(z_0)$ as

$$|\phi'\cdot z|_{\mathcal{N}_{\gamma-\rho}(z_0)} \le \frac{1}{\rho}\,|\phi|_{\mathcal{N}_\gamma(z_0)}|z|_{\mathbb{C}}. \tag{5.3.34}$$

For the proof of this formula see [44]. Now, by (5.3.20), U^1 is an analytic function of the finite-dimensional variable $\bar{\omega}^{<}$, so assuming that U^n is analytic in $\bar{\omega}^{<}$, we can regard the Fréchet derivative DU^n as an ordinary derivative. Taking for ϕ' in (5.3.34) the derivative of U^n in the direction $\mathfrak{D}^n$, we have

$$\begin{aligned}|\mathsf{P}^{<}[(DU^n)\mathfrak{D}^n]|_{s;n+1} &\le |(DU^n)\mathfrak{D}^n|_{s;n+1}\\ &\le \frac{1}{\rho}\,|U^n|_{s;n}\,|\mathfrak{D}^n|_{s;n}.\end{aligned} \tag{5.3.35}$$

In addition we need the following bound for $\mathfrak{D}^n$

$$\begin{aligned}|\mathfrak{D}^n|_{s;n} &\le c\,(|\nabla(\bar{\omega}^{<}+U^n)|^2_{s;n} + \mu\,|\nabla^2\bar{\omega}^{<}|_{s;n} + |\bar{f}^{<}|_s)\\ &\le c\,(\kappa^2|\bar{\omega}^{<}+U^n|^2_{s;n} + \mu\,\kappa^2\,|\bar{\omega}^{<}|_{s;n} + |\bar{f}^{<}|_s).\end{aligned} \tag{5.3.36}$$

The second term of the right-hand side of (5.3.33) can be bounded as

$$|\tilde{B}^{<}(\bar{\omega}^{<}+U^n,\bar{\omega}^{<}+U^n)|_{s;n} \le c\,\kappa^2\,|(\bar{\omega}^{<}+U^n)|^2_{s;n}. \tag{5.3.37}$$

[7]See the Appendix

where we used Lemma 5.3.8 in (5.3.36) and (5.3.37). Substituting (5.3.35), (5.3.36), (5.3.37) in (5.3.33), we obtain

$$\begin{aligned} |U^{n+1}|_{s;n+1} \le \frac{c\,\varepsilon}{\rho}\,\kappa^2\,|U^n|_{s;n}(\kappa^2|\bar{\omega}^< + U^n|^2_{s;n} + \mu\,\kappa^2\,|\bar{\omega}^<|_{s;n} + |\bar{f}^<|_s) \\ + c\,\varepsilon\,\kappa^2(\kappa^2\,|\bar{\omega}^< + U^n|^2_{s;n} + \mu\,\kappa^2\,|U^n|_{s;n} + |\tilde{f}^<|_s). \end{aligned} \tag{5.3.38}$$

Take $\rho = \varepsilon^{1/12}$ and $\kappa = \varepsilon^{-1/12}$, then (5.3.38) becomes

$$\begin{aligned} |U|_{s;n+1} &\le c_1\,\varepsilon^{7/12}\,|U^n|_{s;n}(|\bar{\omega}^< + U^n|^2_{s;n} + \mu\,|\bar{\omega}^<|_{s;n} + \varepsilon^{1/6}\,|\bar{f}^<|_s) \\ &\quad + c_2\,\varepsilon^{2/3}\,(|\bar{\omega}^< + U^n|^2_{s;n} + \mu\,|U^n|_{s;n} + \varepsilon^{1/6}\,|\tilde{f}^<|_s) \\ &\le c_1\,\varepsilon^{7/12}\,|U^n|_{s;n}(|\bar{\omega}^<|^2_{s;0} + \mu\,|\bar{\omega}^<|_{s;0} + |\bar{f}^<|_s) \\ &\quad + c_2\,\varepsilon^{7/12}\,(|\bar{\omega}^<|^2_{s;0} + \mu\,|U^n|_{s;0} + |\tilde{f}^<|_s) \end{aligned} \tag{5.3.39}$$

Take ε small enough such that

$$(c_1 + c_2)\,\varepsilon^{7/12}(|\bar{\omega}^<|^2_{s;0} + \mu\,|\bar{\omega}^<|_{s;0} + |f|_s) \le \min\{1, |\bar{\omega}^<|_s\}, \tag{5.3.40}$$

and we claim that

$$|U^n|_{s,n} \le (c_1 + c_2)\,\varepsilon^{7/12}(|\bar{\omega}^<|^2_{s;0} + \mu\,|\bar{\omega}^<|_{s;0} + |f^<|_s), \tag{5.3.41}$$

then from (5.3.40) and (5.3.41) we have for $m = 0, ..., n$ for some $n < n_*$,

$$|U^m|_{s,m} \le (c_1 + c_2)\,\varepsilon^{7/12}(|\bar{\omega}^<|^2_{s;0} + \mu\,|\bar{\omega}^<|_{s;0} + |f|_s) \le \min\{1, |\bar{\omega}^<|_s\}, \tag{5.3.42}$$

which implies

$$\begin{aligned} &|U^m|_{s,m} \le |\bar{\omega}^<|_s \le |\bar{\omega}^<|_{s;0} \\ &\text{and} \\ &|U^m|_{s,m} \le 1. \end{aligned} \tag{5.3.43}$$

Using all these, we have

$$\begin{aligned} |U^{n+1}|_{s;n+1} &\le c_1\,\varepsilon^{7/12}\,(|\bar{\omega}^<|^2_{s;0} + \mu\,|\bar{\omega}^<|_{s;0} + |f^<|_s) \\ &\quad + c_2\,\varepsilon^{7/12}\,(|\bar{\omega}^<|^2_{s;0} + \mu\,|\bar{\omega}^<|_{s;0} + |f^<|_s) \\ &\le (c_1 + c_2)\,\varepsilon^{7/12}\,(|\bar{\omega}^<|^2_{s;0} + \mu\,|\bar{\omega}^<|_{s;0} + |f^<|_s), \end{aligned} \tag{5.3.44}$$

This proves (5.3.41) and (5.3.43) for $n = 0, \cdots, n_*$. Now we find the bound for the remainders $\mathcal{R}^*$ and $\hat{\mathcal{Q}}$.

Recall that the remainder $\mathcal{R}^n$ for $n = \{0, 1, ..., \lfloor \gamma/\rho \rfloor =: n_*\}$,

$$\mathcal{R}^n := \mathsf{P}^< [(DU^n)\mathfrak{D}^n] + \frac{1}{\varepsilon} LU^n + \tilde{B}(\bar{\omega}^< + U^n, \bar{\omega}^< + U^n) + \mu\, AU^n - \tilde{f}^< \tag{5.3.45}$$

and

$$\frac{1}{\varepsilon} LU^{n+1} = -\mathsf{P}^< [(DU^n)\mathfrak{D}^n] - \tilde{B}^< (\bar{\omega}^< + U^n, \bar{\omega}^< + U^n) - \mu\, AU^n + \tilde{f}^<. \tag{5.3.46}$$

Adding the last two equations, we have

$$\mathcal{R}^n := \frac{1}{\varepsilon} L(U^n - U^{n+1}). \tag{5.3.47}$$

Since $U^0 = 0$ and $\tilde{B}^< (\bar{\omega}^<, \bar{\omega}^<) = 0$, then for $n = 0$ the remainder (5.3.45) is

$$\mathcal{R}^0 := -\tilde{f}^<. \tag{5.3.48}$$

Moreover

$$\begin{aligned}\mathcal{R}^{n+1} := \mathsf{P}^< [(DU^{n+1})\mathfrak{D}^{n+1}] &+ \frac{1}{\varepsilon} LU^{n+1} \\ &+ \tilde{B}^< (\bar{\omega}^< + U^{n+1}, \bar{\omega}^< + U^{n+1}) + \mu\, AU^{n+1} - \tilde{f}^<.\end{aligned} \tag{5.3.49}$$

Now we simplify every term in the right-hand side of (5.3.49) separately, by using (5.3.47), the first term

$$\begin{aligned}\mathsf{P}^< [(DU^{n+1})\mathfrak{D}^{n+1}] &= \mathsf{P}^< [D(U^n - \varepsilon\, L^{-1}\, \mathcal{R}^n)(\mathfrak{D}^n + \delta\mathfrak{D}^n)] \\ &= \mathsf{P}^< [(DU^n)\mathfrak{D}^n] + \mathsf{P}^< [DU^n\, \delta\mathfrak{D}^n] - \varepsilon\, \mathsf{P}^< [(DL^{-1}\mathcal{R}^n)\, \mathfrak{D}^{n+1}].\end{aligned} \tag{5.3.50}$$

The second term

$$\begin{aligned}&\tilde{B}^< (\bar{\omega}^< + U^{n+1}, \bar{\omega}^< + U^{n+1}) \\ &\quad = \tilde{B}^< (\bar{\omega}^< + U^n - \varepsilon\, L^{-1}\mathcal{R}^n, \bar{\omega} + U^n - \varepsilon\, L^{-1}\mathcal{R}^n) \\ &\quad = \tilde{B}^< (\bar{\omega}^< + U^n, \bar{\omega}^< + U^n) - \varepsilon\, \tilde{B}^< (\bar{\omega}^< + U^n, L^{-1}\mathcal{R}^n) \\ &\qquad - \varepsilon\, \tilde{B}^< (L^{-1}\mathcal{R}^n, \bar{\omega}^< + U^{n+1}).\end{aligned} \tag{5.3.51}$$

Hence

$$\begin{aligned}\mathcal{R}^{n+1} &= \mathsf{P}^< [(DU^n)\mathfrak{D}^n] + \frac{1}{\varepsilon} LU^n + \tilde{B}^< (\bar{\omega}^< + U^n, \bar{\omega}^< + U^n) + \mu\, AU^n - \tilde{f}^< - \mathcal{R}^n \\ &\quad + \mathsf{P}^< [DU^n\, \delta\mathfrak{D}^n] - \varepsilon\, \mathsf{P}^< [(DL^{-1}\mathcal{R}^n)\, \mathfrak{D}^{n+1}] - \varepsilon\, \tilde{B}^< (\bar{\omega}^< + U^n, L^{-1}\mathcal{R}^n) \\ &\quad - \varepsilon\, \tilde{B}^< (L^{-1}\mathcal{R}^n, \bar{\omega}^< + U^{n+1}) - \mu\, \varepsilon\, AL^{-1}\mathcal{R}^n \\ &= \mathsf{P}^< [DU^n\, \delta\mathfrak{D}^n] - \varepsilon\, \mathsf{P}^< [(DL^{-1}\mathcal{R}^n)\, \mathfrak{D}^{n+1}] - \varepsilon\, \tilde{B}^< (\bar{\omega}^< + U^n, L^{-1}\mathcal{R}^n) \\ &\quad - \varepsilon\, \tilde{B}^< (L^{-1}\mathcal{R}^n, \bar{\omega}^< + U^{n+1}) - \mu\, \varepsilon\, AL^{-1}\mathcal{R}^n,\end{aligned} \tag{5.3.52}$$

where

$$\begin{aligned}&\delta\mathfrak{D}^n = \mathfrak{D}^{n+1} - \mathfrak{D}^n \\ &= -\bar{B}^< (\bar{\omega}^< + U^{n+1}, \bar{\omega}^< + U^{n+1}) - \mu\, A\bar{\omega}^< + \bar{f} + \bar{B}^< (\bar{\omega}^< + U^n, \bar{\omega}^< + U^n) + \mu\, A\bar{\omega}^< - \bar{f}^< \\ &= -\bar{B}^< (\bar{\omega}^< + U^n - \varepsilon\, L^{-1}\mathcal{R}^n, \bar{\omega}^< + U^n - \varepsilon\, L^{-1}\mathcal{R}^n) + \bar{B}^< (\bar{\omega}^< + U^n, \bar{\omega}^< + U^n) \\ &= -\bar{B}(\bar{\omega}^< + U^n, \bar{\omega}^< + U^n) + \bar{B}(\bar{\omega}^< + U^n, \varepsilon\, L^{-1}\mathcal{R}^n) \\ &\qquad + \bar{B}^< (\varepsilon\, L^{-1}\mathcal{R}^n, \bar{\omega}^< + U^{n+1}) + \bar{B}^< (\bar{\omega}^< + U^n, \bar{\omega}^< + U^n) \\ &= \varepsilon\, \bar{B}^< (\bar{\omega}^< + U^n, L^{-1}\mathcal{R}^n) + \varepsilon\, \bar{B}^< (L^{-1}\mathcal{R}^n, \bar{\omega}^< + U^{n+1}),\end{aligned} \tag{5.3.53}$$

and the bound of $\delta\mathfrak{D}^n$ is

$$\begin{aligned}|\delta\mathfrak{D}^n|_{s;n+1} &\le c\, \varepsilon\, |\nabla(\bar{\omega}^< + U^n)|_{s;n+1}\, |\nabla\, L^{-1}\mathcal{R}^n|_{s;n+1} \\ &\qquad + c\, \varepsilon\, |\nabla(\bar{\omega}^< + U^{n+1})|_{s;n+1}\, |\nabla\, L^{-1}\mathcal{R}^n|_{s;n+1} \\ &\le c\, \varepsilon\, \kappa^4\, |(\bar{\omega}^< + U^n)|_{s;n+1}\, |\mathcal{R}^n|_{s;n+1} \\ &\qquad + c\, \varepsilon\, \kappa^4\, |(\bar{\omega}^< + U^{n+1})|_{s;n+1}\, |\mathcal{R}^n|_{s;n+1} \\ &\le c\, \varepsilon\, \kappa^4\, |\bar{\omega}^<|_{s,0}\, |\mathcal{R}^n|_{s;n+1}.\end{aligned} \tag{5.3.54}$$

Therefore, the bound of $\mathcal{R}^{n+1}$ becomes

$$
\begin{aligned}
|\mathcal{R}^{n+1}|_{s;n+2} \le\, & |DU^n|_{s;n+1}|\delta\mathfrak{D}^n|_{s;n+1} - \varepsilon\,|(DL^{-1}\mathcal{R}^n)|_{s;n+2}\,|\mathfrak{D}^{n+1}|_{s;n+1} \\
& - \varepsilon\,|\tilde{B}(\bar{\omega}^< + U^n, L^{-1}\mathcal{R}^n)|_{s;n+1} - \varepsilon\,|\tilde{B}^<(L^{-1}\mathcal{R}^n, \bar{\omega}^< + U^{n+1})|_{s;n+1} \\
& \qquad - \mu\,\varepsilon\,|AL^{-1}\mathcal{R}^n|_{s;n+1} \\
\le\, & \frac{c\,\varepsilon\,\kappa^4}{\rho}\,|U^n|_{s;n}\,|\bar{\omega}^<|_{s;0}\,|\mathcal{R}^n|_{s;n+1} + \frac{c\,\varepsilon\,\kappa^2}{\rho}\,|\mathcal{R}^n|_{s;n+1}|\mathfrak{D}^{n+1}|_{s;n+1} \\
& + c\,\varepsilon\,\kappa^4\,|\bar{\omega}^< + U^n|_{s;n}\,|\mathcal{R}^n|_{s;n+1} + \mu\,\varepsilon\,\kappa^4\,|\mathcal{R}^n|_{s;n+1} \\
\le\, & \frac{c\,\varepsilon\,\kappa^4}{\rho}\,|U^n|_{s;n}\,|\bar{\omega}^<|_{s;0}\,|\mathcal{R}^n|_{s;n+1} \\
& + \frac{c\,\varepsilon\,\kappa^2}{\rho}\,|\mathcal{R}^n|_{s;n+1}(\kappa^2\,|\bar{\omega}^< + U^{n+1}|^2_{s;n+1} + \mu\,\kappa^2\,|\bar{\omega}^<|_s + |\bar{f}^<|_s) \\
& + c\,\varepsilon\,\kappa^4\,|\bar{\omega}^< + U^n|_{s;n}\,|\mathcal{R}^n|_{s;n+1} + \mu\,\varepsilon\,\kappa^4\,|\mathcal{R}^n|_{s;n+1} \\
\le\, & \frac{c\,\varepsilon\,\kappa^4}{\rho}\,|\bar{\omega}^<|_{s;0}\,|\mathcal{R}^n|_{s;n+1} \\
& + \frac{c\,\varepsilon\,\kappa^4}{\rho}\,|\mathcal{R}^n|_{s;n+1}(|\bar{\omega}^<|^2_{s;0} + \mu\,|\bar{\omega}^<|_{s;0} + |\bar{f}^<|_s) \\
& + c\,\varepsilon\,\kappa^4\,|\bar{\omega}^<|_{s;0}\,|\mathcal{R}^n|_{s;n+1} + \mu\,\varepsilon\,\kappa^4\,|\mathcal{R}^n|_{s;n+1} \\
\le\, & c\,\varepsilon^{7/12}\,(|\bar{\omega}^<|^2_{s;0} + \mu|\bar{\omega}^<|_{s;0} + |f^<|_s + \mu)\,|\mathcal{R}^n|_{s;n+1}.
\end{aligned}
\tag{5.3.55}
$$

If ε is small enough, such that

$$
c\,\varepsilon^{1/12}(|\bar{\omega}|^2_{s;0} + \mu|\bar{\omega}|_{s;0} + |f|_s + \mu) \le 1, \tag{5.3.56}
$$

then we have, for $n = 0, 1, ..., n_* - 1$

$$
|\mathcal{R}^{n+1}|_{s;n+2} \le \varepsilon^{1/2}\,|\mathcal{R}^n|_{s;n+1}. \tag{5.3.57}
$$

By (5.3.48), we have

$$
|\mathcal{R}^1|_{s;0} = |f|_s. \tag{5.3.58}
$$

Taking now $n = n_* - 1$ we get the following bound

$$
|\mathcal{R}^{n_*-1}|_s \le |\mathcal{R}^{n_*-1}|_{s;n_*} \le c\,|f|_s\,\varepsilon^{n/2}. \tag{5.3.59}
$$

Setting $U^* = U^{n_*-1}$ and taking as ε_{**} the largest value that satisfies $\varepsilon < 1$, (5.3.41) and (5.3.56). The result is

$$|\mathcal{R}^*|_s \le c\,|f|_s\,\varepsilon^{n/2}. \tag{5.3.60}$$

Now to find $\hat{\mathcal{Q}}$ we need to find the following bound

$$\begin{aligned} |\tilde{B}^<(\omega^<,\omega^>)|_{L^2} + |\tilde{B}^<(\omega^>,\omega)|_{L^2} &\le c\,|\nabla\omega^<|_{L^2}\,|\nabla\omega^>|_{L^2} + c\,|\nabla\omega|_{L^2}\,|\nabla\omega^>|_{L^2} \\ &\le c\,|\nabla\omega^>|_{L^2}\,|\nabla\omega|_{L^2} \\ &\le C_m\,\kappa^m\,K_\sigma\,|\nabla\omega|_{L^2} \\ &\le C_m\,\varepsilon^{n/2}\,K_\sigma\,|\nabla\omega|_{L^2}, \end{aligned} \tag{5.3.61}$$

where we used Lemmata 5.2.2 and 5.3.2 with $m = 6n + \frac{1}{12}$. In addition we need the following bound

$$\begin{aligned} |\mathsf{P}^>[(DU^*)\mathfrak{D}^*]|_{L^2} &\le c\,\kappa^{-m}\,|(DU^*)\mathfrak{D}^*|_{1;n_*} \\ &\le c\,\kappa^{-m}\,\frac{1}{\rho}\,|U^*|_{1;n_*}\,|\mathfrak{D}^*|_{1;n_*} \\ &\le c\,\varepsilon^{n/2}\,(|\bar{\omega}^<|_{2;0}^2 + \mu\,|\bar{\omega}^<|_{2;0} + |f|_2)^2. \end{aligned} \tag{5.3.62}$$

Adding (5.3.61) and (5.3.62) and using (5.3.26), we obtain

$$\begin{aligned} |\hat{\mathcal{Q}}|_{L^2} &\le c\,\varepsilon^{n/2}\,(K_\sigma\,|\nabla\omega|_{L^2} + |\bar{\omega}^<|_{2;0}^2 + \mu\,|\bar{\omega}^<|_{2;0} + |f|_2)^2 \\ &\le c\,\varepsilon^{n/2}\,[K_\sigma\,|\omega|_2 + ((|\bar{\omega}^<|_2 + \gamma)^2 + \mu\,(|\bar{\omega}^<|_2 + \gamma) + |f|_2)^2]. \end{aligned} \tag{5.3.63}$$

□

With the above results we give the proof of Theorem 5.3.3

Proof of Theorem 5.3.3 . Firstly, we prove that the low mode part is bounded with order of $\varepsilon^{n/2}$. Our equation in low-mode variable $\omega^<$ is

$$\partial_t\omega^< + \frac{1}{\varepsilon}\,L\omega^< + B^<(\omega,\omega) + \mu\,A\omega^< = f^<. \tag{5.3.64}$$

The nonlinear term can be written as

$$B^<(\omega,\omega) = B^<(\omega^< + \omega^>,\omega^< + \omega^>) = B^<(\omega^<,\omega^<) + B^<(\omega^<,\omega^>) + B^<(\omega^>,\omega), \tag{5.3.65}$$

and (5.3.64) becomes

$$\partial_t \omega^< + \frac{1}{\varepsilon} L\omega^< + B^<(\omega^<, \omega^<) + \mu A\omega^< - f^< = -B^<(\omega^<, \omega^>) - B^<(\omega^>, \omega) \qquad (5.3.66)$$

This equation for $\tilde{\omega}^<$ is

$$\partial_t \tilde{\omega}^< + \frac{1}{\varepsilon} L\tilde{\omega}^< + \tilde{B}^<(\omega^<, \omega^<) + \mu A\tilde{\omega}^< - \tilde{f}^< = -\tilde{B}^<(\omega^<, \omega^>) - \tilde{B}^<(\omega^>, \omega). \qquad (5.3.67)$$

Taking $\tilde{\omega}^< = U^*(\bar{\omega}^<, f^<; \varepsilon) + \hat{\omega}$, then the last equation becomes

$$\partial_t \hat{\omega} + \frac{1}{\varepsilon} L\hat{\omega} + \tilde{B}^<(\bar{\omega}^< + U^*, \hat{\omega}) + \tilde{B}^<(\hat{\omega}, \omega^<) + \mu A\hat{\omega} = -\mathcal{R}^* + \hat{\mathcal{Q}}, \qquad (5.3.68)$$

where

$$\mathcal{R}^* = \mathsf{P}^<[(DU^*)\mathfrak{D}^*] + \frac{1}{\varepsilon} LU^* + \mu AU^* + \tilde{B}(\bar{\omega}^< + U^*, \bar{\omega}^< + U^*) - \tilde{f}^< \qquad (5.3.69)$$

and

$$\hat{\mathcal{Q}} = -\tilde{B}^<(\omega^<, \omega^>) - \tilde{B}^<(\omega^>, \omega) + (1 - \mathsf{P}^<)[(DU^*)\mathfrak{D}^*]. \qquad (5.3.70)$$

Multiplying (5.3.68) by $\hat{\omega}$ in L^2, applying Poincaré inequality on the second term of the left-hand side and multiplying by $\mathrm{e}^{\nu t}$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\nu t} |\hat{\omega}|^2_{L^2}) + 2\mu \, \mathrm{e}^{\nu t} |\nabla \hat{\omega}|^2_{L^2} \le 2\mathrm{e}^{\nu t}(\tilde{B}^<(\hat{\omega}, \omega^<), \hat{\omega})_{L^2} \\ - 2\mathrm{e}^{\nu t}(\mathcal{R}^*, \hat{\omega})_{L^2} + 2\mathrm{e}^{\nu t}(\hat{\mathcal{Q}}, \hat{\omega})_{L^2}.\end{aligned} \qquad (5.3.71)$$

We bound now the last two term of the right-hand side

$$2|(\mathcal{R}^*, \hat{\omega})_{L^2}| \le \frac{c}{\mu} |\mathcal{R}^*|^2_{L^2} + \frac{\mu}{3} |\nabla \hat{\omega}|^2_{L^2}, \qquad (5.3.72)$$

$$2|(\hat{\mathcal{Q}}, \hat{\omega})_{L^2}| \le \frac{c}{\mu} |\hat{\mathcal{Q}}|^2_{L^2} + \frac{\mu}{3} |\nabla \hat{\omega}|^2_{L^2}. \qquad (5.3.73)$$

The nonlinear term can be written as

$$(\tilde{B}^<(\hat{\omega}, \omega^<), \hat{\omega})_{L^2} = (\tilde{B}^<(\hat{\omega}, \bar{\omega}^<), \hat{\omega})_{L^2} + (\tilde{B}^<(\hat{\omega}, U^*), \hat{\omega})_{L^2}, \qquad (5.3.74)$$

and the bound of the second term of the right-hand side of(5.3.74) is

$$|(\tilde{B}^<(\hat{\omega}, U^*), \hat{\omega})_{L^2}| \le c \, |\nabla U^*|_{L^2} \, |\nabla \hat{\omega}|^2_{L^2} \qquad (5.3.75)$$

Now (5.3.71) will be

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\nu t}\,|\hat{\omega}|_{L^2}^2) + \mu\,\mathrm{e}^{\nu t}\,|\nabla\hat{\omega}|_{L^2}^2 \le \frac{c}{\mu}\,\mathrm{e}^{\nu t}\,|\mathcal{R}^*|_{L^2}^2 + \frac{c}{\mu}\,\mathrm{e}^{\nu t}\,|\hat{\mathfrak{Q}}|_{L^2}^2 + 2\,\mathrm{e}^{\nu t}\,(\tilde{B}(\hat{\omega},\hat{\omega}),\bar{\omega}^{<})_{L^2}. \tag{5.3.76}$$

We integrate from T to $T+t$ and multiplying by $\mathrm{e}^{-\nu T}$, we obtain

$$\begin{aligned}&\mathrm{e}^{\nu t}\,|\hat{\omega}(T+t)|_{L^2}^2 - |\hat{\omega}(T)|_{L^2}^2 + \mu\int_T^{T+t}\mathrm{e}^{\nu(\tau-T)}\,|\nabla\hat{\omega}|_{L^2}^2\,\mathrm{d}\tau\\ &\quad\le \frac{c}{\mu}\int_T^{T+t}\mathrm{e}^{\nu(\tau-T)}\,(|\mathcal{R}^*|_{L^2}^2 + |\hat{\mathfrak{Q}}|_{L^2}^2)\,\mathrm{d}\tau + 2\int_T^{T+t}\mathrm{e}^{\nu(\tau-T)}\,(\tilde{B}(\hat{\omega},\hat{\omega}),\bar{\omega}^{<})_{L^2}\,\mathrm{d}\tau.\end{aligned} \tag{5.3.77}$$

The integral of the last term in the right hand side, gives us

$$\begin{aligned}&2\int_T^{T+t}\mathrm{e}^{\nu(\tau-T)}\,(\tilde{B}(\hat{\omega},\hat{\omega}),\bar{\omega}^{<})_{L^2}\,\mathrm{d}\tau\\ &\qquad= 2\,\varepsilon\,\mathrm{e}^{\nu t}\,(B_{\Omega^<}(\hat{\omega},\hat{\omega}),\bar{\omega}^{<})_{L^2}(T+t) - 2\,\varepsilon\,(B_{\Omega^<}(\hat{\omega},\hat{\omega}),\bar{\omega}^{<})_{L^2}(T)\\ &\qquad\quad - 2\,\varepsilon\int_T^{T+t}\mathrm{e}^{\nu(\tau-T)}\big\{\nu(B_{\Omega^<}(\hat{\omega},\hat{\omega}),\bar{\omega}^{<})_{L^2} + (B_{\Omega^<}(\hat{\omega},\hat{\omega}),\partial_\tau\bar{\omega}^{<})_{L^2}\\ &\qquad\qquad + 2(B_{\Omega^<}(\partial_\tau^*\hat{\omega},\hat{\omega}),\bar{\omega}^{<})_{L^2}\big\}\,\mathrm{d}\tau.\end{aligned} \tag{5.3.78}$$

The endpoint terms can be bound as

$$2\,|(B_{\Omega^<}(\hat{\omega},\hat{\omega}),\bar{\omega}^{<})_{L^2}| \le c\,|\hat{\omega}|_{L^2}^2\,|\bar{\omega}^{<''}|_{L^2}. \tag{5.3.79}$$

We bound now the terms in the integrand. First,

$$\begin{aligned}2\,\varepsilon\,\nu\,|(B_{\Omega^<}(\hat{\omega},\hat{\omega}),\bar{\omega})_{L^2}| &\le c\,\varepsilon\,\mu\,|\hat{\omega}|_{L^4}\,|\hat{\omega}|_{L^4}\,|\bar{\omega}^{<'}|_{L^2}\\ &\le c\,\varepsilon\,\mu\,|\nabla\hat{\omega}|_{L^2}^2\,|\bar{\omega}^{<'}|_{L^2}.\end{aligned} \tag{5.3.80}$$

We need ε to be small such that

$$c\,\varepsilon\,|\bar{\omega}^{<'}| \le \frac{1}{14}, \tag{5.3.81}$$

then

$$2\,\varepsilon\,\nu\,|(B_{\Omega^<}(\hat{\omega},\hat{\omega}),\bar{\omega}^{<})| \le \frac{\mu}{14}\,|\nabla\hat{\omega}|^2. \tag{5.3.82}$$

The second term in the integrand

$$\begin{aligned}2\,\varepsilon\,|(B_{\Omega^<}(\hat{\omega},\hat{\omega}),\partial_t\bar{\omega}^{<})| &\le c\,\varepsilon\,|\hat{\omega}|_{L^4}\,|\hat{\omega}|_{L^4}\,|\partial_y\partial_t\bar{\omega}^{<}|_{L^2}\\ &\le c\,\varepsilon\,|\nabla\hat{\omega}|_{L^2}^2\,|\partial_y\partial_t\bar{\omega}^{<}|_{L^2},\end{aligned} \tag{5.3.83}$$

we require ε to be

$$c\,\varepsilon\,|\partial_y\partial_t\bar{\omega}^{<}|_{L^2} \le \frac{\mu}{14}. \tag{5.3.84}$$

Then

$$2\,\varepsilon\,|(B_{\Omega^<}(\hat{\omega},\hat{\omega}),\partial_t\bar{\omega}^{<})| \le \frac{\mu}{14}\,|\nabla\hat{\omega}|^2. \tag{5.3.85}$$

The last term of the integrand, first

$$\partial_t^*\hat{\omega} = -\tilde{B}(\omega,\hat{\omega}) - \tilde{B}(\hat{\omega},U^*) - \tilde{B}(\hat{\omega},\bar{\omega}^{<}) - \mu A\hat{\omega} - \mathcal{R}^* + \hat{\mathcal{Q}}. \tag{5.3.86}$$

Now

$$\begin{aligned}
&4\,\varepsilon\,|(B_{\Omega^<}(\partial_t^*\hat{\omega},\hat{\omega}),\bar{\omega}^{<})| \\
&\le c\,\varepsilon\,\big\{|\tilde{B}^{<}(\omega,\hat{\omega})|_{L^2}\,|\hat{\omega}|_{L^4}\,|\bar{\omega}'^{<}|_{L^4} + |\tilde{B}(\hat{\omega},U^*)|_{L^2}\,|\hat{\omega}|_{L^4}\,|\bar{\omega}'^{<}|_{L^4} \\
&\qquad + |\tilde{B}^{<}(\hat{\omega},\bar{\omega}^{<})|_{L^2}\,|\hat{\omega}|_{L^4}\,|\bar{\omega}'^{<}|_{L^4} + \mu\,|\nabla\hat{\omega}|_{L^2}\,|\hat{\omega}|_{L^4}\,|\bar{\omega}''^{<}|_{L^4} \\
&\qquad + (|\hat{\mathcal{Q}}|_{L^2}^2 + |\mathcal{R}^*|_{L^2}^2)\,|\hat{\omega}|_{L^4}\,|\bar{\omega}'^{<}|_{L^4}\big\} \\
&\le c\,\varepsilon\big\{|\nabla^{-1}\omega|_{L^\infty}\,|\nabla\hat{\omega}|_{L^2}^2\,|\bar{\omega}''^{<}|_{L^2} + |\nabla^{-1}\hat{\omega}|_{L^\infty}\,|\nabla U^*|_{L^2}\,|\nabla\hat{\omega}|_{L^2}\,|\bar{\omega}''^{<}|_{L^2} \\
&\qquad + |\nabla^{-1}\hat{\omega}|_{L^\infty}\,|\nabla\bar{\omega}^{<}|_{L^2}\,|\nabla\hat{\omega}|_{L^2}\,|\bar{\omega}''^{<}|_{L^2} + \mu\,|\nabla\hat{\omega}|_{L^2}^2\,|\bar{\omega}''|_{L^2} \\
&\qquad + (|\hat{\mathcal{Q}}|_{L^2}^2 + |\mathcal{R}^*|_{L^2}^2)\,|\nabla\hat{\omega}|_{L^2}\,|\bar{\omega}''^{<}|_{L^2}\big\} \\
&\le c\,\varepsilon\big\{|\nabla\hat{\omega}|_{L^2}^2\,|\nabla\omega|_{L^2}\,|\bar{\omega}''^{<}|_{L^2} + |\nabla\hat{\omega}|_{L^2}^2\,|\nabla U^*|_{L^2}\,|\bar{\omega}''^{<}|_{L^2} + |\nabla\hat{\omega}|_{L^2}^2\,|\nabla\bar{\omega}^{<}|_{L^2}\,|\bar{\omega}''^{<}|_{L^2} \\
&\qquad + \mu\,|\nabla\hat{\omega}|_{L^2}^2\,|\bar{\omega}''^{<}|_{L^2}\big\} + \frac{c\,\varepsilon^2}{\mu}\,(|\hat{\mathcal{Q}}|_{L^2}^2 + |\mathcal{R}^*|_{L^2}^2)\,|\bar{\omega}''^{<}|_{L^2}^2 + \frac{\mu}{14}\,|\nabla\hat{\omega}|_{L^2}^2.
\end{aligned} \tag{5.3.87}$$

Now we require ε to be small such that

$$\begin{aligned}
c_3\,\varepsilon\,|\nabla\omega|_{L^2}\,|\bar{\omega}''^{<}|_{L^2} &\le \frac{\mu}{14} \\
c_4\,\varepsilon\,|\nabla U^*|_{L^2}\,|\bar{\omega}''^{<}|_{L^2} &\le \frac{\mu}{14} \\
c_3\,\varepsilon\,|\nabla\bar{\omega}^{<}|\,|\bar{\omega}''^{<}|_{L^2} &\le \frac{\mu}{14} \\
c_5\,\varepsilon\,|\bar{\omega}''^{<}|_{L^2} &\le \frac{\mu}{14}
\end{aligned} \tag{5.3.88}$$

Hence

$$|4\,\varepsilon(B_\Omega(\partial_t^*\hat{\omega},\hat{\omega}),\bar{\omega}^{<})_{L^2}| \le \frac{\mu}{2}\,|\nabla\hat{\omega}|_{L^2}^2 + \frac{c\,\varepsilon^2}{\mu}\,(|\hat{\mathcal{Q}}|_{L^2}^2 + |\mathcal{R}^*|_{L^2}^2)\,|\bar{\omega}''^{<}|_{L^2}^2. \tag{5.3.89}$$

Therefore the nonlinear term (5.3.78) becomes

$$2\int_T^{T+t} \mathrm{e}^{\nu(\tau-T)}\,(\tilde{B}(\hat\omega,\hat\omega),\bar\omega^{<})_{L^2}\,\mathrm{d}\tau$$
$$\le \mathrm{e}^{\nu t}\,\varepsilon\,|\hat\omega(T+t)|^2_{L^2}\,|\nabla^2\omega(T+t)|_{L^2} + \varepsilon\,|\hat\omega(T)|^2_{L^2}\,|\nabla^2\omega(T)|_{L^2}$$
$$+\frac{\mu}{2}\int_T^{T+t}\mathrm{e}^{\nu(\tau-T)}\,|\nabla\hat\omega|^2_{L^2}\mathrm{d}\tau + \frac{c\,\varepsilon}{\mu}\,|\nabla^2\omega|^2_{L^2}(\|\hat{\mathcal{Q}}\|^2_{L^2} + \|\mathcal{R}^*\|^2_{L^2})\,(\mathrm{e}^{\nu t}-1), \tag{5.3.90}$$

where $\|\mathcal{R}^*\|^2_{L^2} = \sup_{|\bar\omega^{<}|_{L^2}\le|\omega|_{L^2}}|\mathcal{R}^*|_{L^2}$ and similarly for $\|\hat{\mathcal{Q}}\|_{L^2}$. Substitute (5.3.90) in (5.3.77), we have

$$\mathrm{e}^{\nu t}\,|\hat\omega(T+t)|^2_{L^2} - |\hat\omega(T)|^2_{L^2} + \frac{\mu}{2}\int_T^{T+t}\mathrm{e}^{\nu(\tau-T)}\,|\nabla\hat\omega|^2_{L^2}\,\mathrm{d}\tau$$
$$\le \mathrm{e}^{\nu t}\,\varepsilon\,|\hat\omega(T+t)|^2_{L^2}\,|\nabla^2\omega(T+t)|_{L^2} + \varepsilon\,|\hat\omega(T)|^2_{L^2}\,|\nabla^2\omega(T)|_{L^2}$$
$$+\frac{c\,\varepsilon}{\mu}\,|\nabla^2\omega|^2_{L^2}(\|\hat{\mathcal{Q}}\|^2_{L^2} + \|\mathcal{R}^*\|^2_{L^2})\,(\mathrm{e}^{\nu t}-1). \tag{5.3.91}$$

which implies

$$(1-c_6\,\varepsilon\,|\nabla^2\omega(T+t)|^2_{L^2})\,|\hat\omega(T+t)|^2_{L^2} + \frac{\mu}{2}\int_T^{T+t}\mathrm{e}^{\nu(\tau-T)}\,|\nabla\hat\omega|^2_{L^2}\,\mathrm{d}\tau$$
$$\le \mathrm{e}^{-\nu t}\,(1+c_6\,\varepsilon\,|\nabla^2\omega(T)|^2_{L^2})\,|\hat\omega(T)|^2 + \frac{c\,\varepsilon}{\mu}\,|\nabla^2\omega|^2\,(\|\hat{\mathcal{Q}}\|^2_{L^2} + \|\mathcal{R}^*\|^2_{L^2}), \tag{5.3.92}$$

Taking ε small enough,

$$c_6\,\varepsilon^{1/2}\,|\nabla^2\omega|_{L^2} \le 1/2 \tag{5.3.93}$$

with the bounds of $\mathcal{R}^*$ and $\hat{\mathcal{Q}}$, form Lemma 5.3.4, we have

$$|\hat\omega(T+t)|^2_{L^2} \le 4\,\mathrm{e}^{-\nu t}\,\varepsilon\,|\hat\omega(T)|^2_{L^2}$$
$$+\frac{c}{\mu^2}\,[|\omega|^2_2\,K^2_\sigma + (|\omega|_2+\gamma)^2 + \mu^2\,(|\omega|_2+\gamma)^2 + |f|^2_2 + |f|^4_2]\,\varepsilon^{n^2/4}. \tag{5.3.94}$$

For sufficiently large t, we have

$$|\hat\omega(T+t)|^2_{L^2} \le \frac{c}{\mu^2}\,[|\omega|^2_2\,K^2_\sigma + (|\omega|_2+\gamma)^2 + (\mu^2\,|\omega|_2+\gamma)^2 + |f|^2_2 + |f|^4_2]\,\varepsilon^{n^2/4}. \tag{5.3.95}$$

We have then

$$|\tilde\omega - U^*|^2_{L^2} = |\tilde\omega^{>} + \tilde\omega^{<} - U^*|_{L^2} \le |\tilde\omega^{>}|^2_{L^2} + |\hat\omega|^2_{L^2} \le c\,\varepsilon^{n^2/4}\,K^2_*(|f|_{G_\sigma};\mu) + |\hat\omega|^2_{L^2}. \tag{5.3.96}$$

□

5.3.1 Exponential accuracy for the approximate slow manifold

In this subsection we will approximate the slow manifold for our equation with exponential accuracy. Note that if $\omega \in G_\sigma(\mathcal{M})$, then the exponential decay of its Fourier coefficient implies that $\omega^>$ is exponential small, the following Lemma shows that

Lemma 5.3.5 If $\omega \in G_\sigma(\mathcal{M})$, then $\omega^>$ is exponential small

$$|\nabla\omega^>|_{L^2} \le \mathrm{e}^{-\sigma\kappa}\,\|\omega\|_\sigma. \tag{5.3.97}$$

Proof. From (5.2.5), we have

$$\begin{aligned}
|\nabla\omega^>|_{L^2}^2 &= |\mathcal{M}| \sum_{|\boldsymbol{k}|\ge\kappa} |\boldsymbol{k}|^2\,|\omega_{\boldsymbol{k}}|^2 \\
&= |\mathcal{M}| \sum_{|\boldsymbol{k}|\ge\kappa} |\boldsymbol{k}|^2\,\mathrm{e}^{-2\sigma|\boldsymbol{k}|}\,\mathrm{e}^{2\sigma|\boldsymbol{k}|}\,|\omega_{\boldsymbol{k}}|^2 \\
&\le |\mathcal{M}|\,\mathrm{e}^{-2\sigma\kappa} \sum_{|\boldsymbol{k}|\ge\kappa} |\boldsymbol{k}|^2\,\mathrm{e}^{2\sigma|\boldsymbol{k}|}\,|\omega_{\boldsymbol{k}}|^2 \\
&\le |\mathcal{M}|\,\mathrm{e}^{-2\sigma\kappa} \sum_{|\boldsymbol{k}|} |\boldsymbol{k}|^2\,\mathrm{e}^{2\sigma\boldsymbol{k}}\,|\omega_{\boldsymbol{k}}|^2 \\
&\le \mathrm{e}^{-2\sigma\kappa}\,\|\omega\|_\sigma^2.
\end{aligned} \tag{5.3.98}$$

□

The above Lemma can be applied with the slow, $\bar{\omega}$, and fast, $\tilde{\omega}$, parts separately. The following Theorem shows that we can approximate a slow manifold for our equation with exponential accuracy.

Theorem 5.3.6 Assume that the regularity in Lemmata 2.4.1, 2.4.2, 2.4.3 and 5.2.2 hold. Let $\boldsymbol{v}_0 \in L^2(\mathcal{M})$ and $f \in G_\sigma(\mathcal{M})$ be given, with $\partial_t f = 0$. Then there exist $\varepsilon_*(f)$ and time $T_*(|\boldsymbol{v}_0|_{L^2}, |f|_{G_\sigma})$ such that for $\varepsilon \le \varepsilon_*$ and time $t \ge T_*$ we can approximate the fast variable $\tilde{\omega}$ by a function $U^*(\bar{\omega}^<(t), f^<; \varepsilon)$ of the slow variable $\bar{\omega}$ as

$$|\tilde{\omega}(t) - U^*(\bar{\omega}^<(t), f^<; \varepsilon)|_{L^2} \le \mathrm{e}^{(-\sigma\varepsilon^{-1/6})} K_*(|f|_{G_\sigma}; \sigma). \tag{5.3.99}$$

where K_* is a continuous function of its first argument. The proof of this Theorem is the same as the proof of Theorem 5.3.3 with a very minor changes, but we need to prove that the remainders $\mathcal{R}^*$ and $\hat{\mathcal{Q}}$ are exponentially small,

Lemma 5.3.7 Let $s > 1$ and $\gamma > 0$ be fixed. Given $\bar{\omega}^< \in H^s(\mathcal{M})$ and $f \in H^s(\mathcal{M})$ with $\partial_t f = 0$, there exist $\varepsilon_{**}(|\bar{\omega}|_{H^s}, |f|_{H^s}, \gamma)$ such that for $\varepsilon \le \varepsilon_{**}$ one can find $\kappa(\varepsilon)$ and $U^*(\bar{\omega}^<, f^<; \varepsilon)$ that make the remainder function

$$\mathcal{R}^*(\bar{\omega}^<, f^<; \varepsilon) := \mathsf{P}^<[(DU^*)\mathfrak{D}^*] + \frac{1}{\varepsilon} LU^* + \mu AU^* + \tilde{B}^<(\bar{\omega}^< + U^*, \bar{\omega}^< + U^*) - \tilde{f}^< \tag{5.3.100}$$

exponentially small in ε,

$$|\mathcal{R}^*|_{H^s} \le c\,|f|_s \exp(-\gamma/\varepsilon^{1/6}) \tag{5.3.101}$$

and

$$|\hat{\mathcal{Q}}|_{L^2} \le c\,\mathrm{e}^{-\sigma\kappa}\,[K_\sigma\,|\omega|_2 + \big((|\bar{\omega}^<|_2 + \gamma)^2 + \mu\,(|\bar{\omega}^<|_2 + \gamma) + |f|_2\big)^2] \tag{5.3.102}$$

Proof.

In the same way that used in Lemma 5.3.4 with $\rho = \varepsilon^{1/6}$ and $\kappa = \varepsilon^{-1/6}$ we have the following bound for $\mathcal{R}^{n+1}$

$$|\mathcal{R}^{n+1}|_{s;n+2} \le c\,\varepsilon^{1/6}\,(|\bar{\omega}^<|_s^2 + \mu|\bar{\omega}^<|_{s;0} + |f^<|_s + \mu)\,|\mathcal{R}^n|_{s;n+1}. \tag{5.3.103}$$

If ε is small enough, such that

$$c\,\varepsilon^{1/6}(|\bar{\omega}|_s^2 + \mu|\bar{\omega}|_{s;0} + |f|_s + \mu) \le \frac{1}{\mathrm{e}}, \tag{5.3.104}$$

then we have, for $n = 0, 1, ..., n_* - 1$

$$|\mathcal{R}^{n+1}|_{s;n+2} \le \frac{1}{\mathrm{e}}\,|\mathcal{R}^n|_{s;n+1}. \tag{5.3.105}$$

By (5.3.48), we have

$$|\mathcal{R}^1|_{s;0} = |f|_s. \tag{5.3.106}$$

Taking now $n = n_* - 1$ we get the following bound

$$|\mathcal{R}^{n_*-1}|_s \le |\mathcal{R}^{n_*-1}|_{s;n_*} \le c\,|f|_s \exp(-n_*) = c\,|f|_s \exp(-\gamma/\varepsilon^{1/6}). \tag{5.3.107}$$

Setting $U^* = U^{n_*-1}$ and taking as ε_{**} the largest value that satisfies $\varepsilon < 1$, we can get the result. Now to find $\hat{\mathcal{Q}}$ we need to find the following bounds

$$\begin{aligned} |\tilde{B}^{<}(\omega^{<},\omega^{>})|_{L^2} + |\tilde{B}^{<}(\omega^{>},\omega)|_{L^2} &\le c\,|\nabla\omega^{<}|_{L^2}\,|\nabla\omega^{>}|_{L^2} + c\,|\nabla\omega|_{L^2}\,|\nabla\omega^{>}|_{L^2} \\ &\le c\,|\nabla\omega|_{L^2}\,|\nabla\omega^{>}|_{L^2} \\ &\le c\,\mathrm{e}^{-\sigma\kappa}\,K_\sigma\,|\nabla\omega|_{L^2}. \end{aligned} \tag{5.3.108}$$

In addition we need the following bound

$$\begin{aligned} |\mathsf{P}^{>}[(DU^*)\mathfrak{D}^*]|_{L^2} &\le c\,\mathrm{e}^{-\sigma\kappa}\,|(DU^*)\mathfrak{D}^*|_{1;n_*} \\ &\le \frac{c}{\rho}\,\mathrm{e}^{-\sigma\kappa}\,|U^*|_{1;n_*}\,|\mathfrak{D}^*|_{1;n_*} \\ &\le c\,\varepsilon^{-1/6}\,\mathrm{e}^{-\sigma\kappa}\,(|\bar{\omega}^{<}|^2_{2;0} + \mu\,|\bar{\omega}^{<}|_{2;0} + |f|_2)^2. \end{aligned} \tag{5.3.109}$$

Adding (5.3.108) and (5.3.110) and using (5.3.26), we obtain

$$\begin{aligned} |\hat{\mathcal{Q}}|_{L^2} &\le c\,\varepsilon^{-1/6}\,\mathrm{e}^{-\sigma\kappa}\,[K_\sigma\,|\nabla\omega|_{L^2} + \big(|\bar{\omega}^{<}|^2_{2;0} + \mu\,|\bar{\omega}^{<}|_{2;0} + |f|_2\big)^2] \\ &\le c\,\varepsilon^{-1/6}\,\mathrm{e}^{-\sigma\kappa}\,[K_\sigma\,|\omega|_2 + \big((|\bar{\omega}^{<}|_2 + \gamma)^2 + \mu\,(|\bar{\omega}^{<}|_2 + \gamma) + |f|_2\big)^2]. \end{aligned} \tag{5.3.110}$$

$\square$

Chapter 6

Conclusions

In this thesis the two dimensional Navier–Stokes equation on the β-plane with periodic boundary conditions is studied. In Chapter 2, the research equation, the derived vorticity form and boundary conditions with the symmetry of the solution are introduced. In addition, using assumptions on the initial data and the forcing, the equivalence between the vorticity form and the original primitive variable form for the equation is proved. Finally, the H^{-1}, L^2 and H^m bounds for the solution are found.

Chapter 3 is devoted to proving the first aim of this thesis which is that the solution for the two dimensional Navier–Stokes equation on the β-plane is nearly zonal (for the linear and nonlinear problem). This aim was achieved by splitting the solution into the fast mode (non-zonal component) and the slow mode (zonal component), expanding the equation in a Fourier series and proving that the L^2 bound for the non-zonal component of the solution of the linear problem is $O(\varepsilon^{1/2})$. In addition, the H^1 and H^m bounds for the non-zonal component of the equation (nonlinear problem) were found. A resonance between fast and slow modes was key to finding the bound for the nonlinear term.

The second aim of this thesis is proved in Chapter 4. This aim is to prove that the Hausdorff dimension attractor of the equation is zero. We defined the global attractor for a semidynamical system generated by the Navier–Stokes equation on the β-plane, the Hausdorff dimension and the fractal dimension. It was found that the Hausdorff dimension of the attractor for the research equation is the same Hausdorff dimension of the attractor as found by Doering and Gibbon [14] for the vorticity form of the Navier–Stokes equation. However, by using our results, L^2, H^1 and H^m bounds for the non-zonal component for the solution, we proved that the Hausdorff dimension of the attractor for the research equation is zero.

The third aim of this thesis is proved in Chapter 5. We approximated, with $O(\varepsilon^{n/2})$ accuracy and exponential accuracy, the slow manifold for the research equation. This was achieved by truncating the equation to a finite dimensional system (low mode), and we proved that the finite system is small with $O(\varepsilon^{n/2})$ as well as exponentially small. In addition, by using the Gevrey regularity of the solution, it was shown that the ignored high modes are also small with order of $\varepsilon^{n/2}$ as well as exponentially small, so the total error of the slow manifold approximation is small with order of $\varepsilon^{n/2}$ and up to an error that scales exponentially in ε as $\varepsilon \to 0$.
Several mathematical and numerical Studies of the Navier–Stokes equation on rotating sphere have been done (see e.g., [5, 23, 29, 30]). The question is "can we find the same results that were obtained in Chapters 3, 4 and 5 for the Navier–Stokes equation on the fast rotating sphere? " We leave this for future work.

Appendix A

Basic and Auxiliary Results

Definition A.0.8 Let $F : X \to Y$ where X and Y are are normed vector spaces. We say that a linear transformation $D : X \to Y$ is a Fréchet derivative of F at x if for every $\epsilon > 0$ there is $\delta > 0$ such that it is the case that

$$|F(x+h) - F(x) - D(h)|_Y \leq \epsilon\, |h|_X, \tag{A.0.1}$$

for all $h \in X$ with $|h|_X \leq \delta$.

Theorem A.0.9 (Banach's fixed-point theorem)
Let (X, d) be a complete metric space and $f : X \to X$ be a map such that

$$d(f(x), f(x')) \leq c\, d(x, x')$$

for some $0 \leq c < 1$ and all x and x' in X. Then f has a unique fixed point in X. Moreover, for any $x_0 \in X$ the sequence of iterates $x_0, f(x_0), f(f(x_0)), \cdots$ converges to the fixed point of f.

Theorem A.0.10 (Helmholtz–Hodge Decomposition Theorem). Let $\boldsymbol{v} \in L^2(\Omega)$ and Ω is a bounded set with $\nabla \cdot \boldsymbol{v} = 0$. The vector $\boldsymbol{v}$ can be uniquely decomposed in the form

$$\boldsymbol{v} = \boldsymbol{u} + \nabla\phi, \tag{A.0.2}$$

with $\nabla \cdot \boldsymbol{u} = 0$.

A.1 The equivalence between original primitive variables form and vorticity form

Lemma A.1.1 There exists an equivalence between the original primitive variables Navier–Stokes equation on $\beta-$plane

$$\begin{aligned} &\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \frac{Y}{\varepsilon}\boldsymbol{v}^{\perp} + \nabla p = \mu\Delta\boldsymbol{v} + f_{\boldsymbol{v}}, \\ &\nabla\cdot\boldsymbol{v} = 0, \quad \boldsymbol{v}(\cdot, t=0) = \boldsymbol{v}_0 \end{aligned} \tag{A.1.1}$$

and a vorticity form

$$\begin{aligned} &\partial_t \omega + \boldsymbol{v}\cdot\nabla\omega + \frac{1}{\varepsilon} v Y' = \mu\Delta\omega + f, \\ &\Delta\psi = \omega, \\ &\omega|_{t=0} = \nabla^{\perp}\cdot\boldsymbol{v}_0. \end{aligned} \tag{A.1.2}$$

Proof.

From primitive variables form into vorticity form, see Chapter 2 Section 2.3. From vorticity form into primitive variables, assume that ω and ψ are the solutions to (2.3.7) with data $\boldsymbol{v}_0$ and periodic boundary conditions. Now let us take $\boldsymbol{u} = \nabla^{\perp}\psi$, for all time $t > 0$. From this we see that $\nabla\cdot\boldsymbol{u} = 0$, since

$$\boldsymbol{u} = \nabla^{\perp}\psi \Longrightarrow \nabla\cdot\boldsymbol{u} = \nabla\cdot\nabla^{\perp}\psi = 0. \tag{A.1.3}$$

In addition,

$$\nabla^{\perp}\cdot\boldsymbol{u} = \nabla^{\perp}.\cdot\nabla^{\perp}\psi = \Delta\psi = \omega, \tag{A.1.4}$$

Hence, we can write (A.1.2) as

$$\partial_t\nabla^{\perp}\cdot\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\nabla^{\perp}\cdot\boldsymbol{u} + \frac{1}{\varepsilon}vY' = \mu\Delta(\nabla^{\perp}\cdot\boldsymbol{u}) + \nabla^{\perp}\cdot f_{\boldsymbol{v}}, \tag{A.1.5}$$

where the operator $\nabla^{\perp}$ commutes with ∂_t and Δ. In addition, the nonlinear term becomes

$$\begin{aligned}
(\boldsymbol{u}\cdot\nabla)\omega &= (\boldsymbol{u}\cdot\nabla)\nabla^{\perp}\cdot\boldsymbol{u} \\
&= (\boldsymbol{u}\cdot\nabla)\nabla\times\boldsymbol{u} \\
&= \nabla\times[(\nabla\times\boldsymbol{u})\times\boldsymbol{u}] \\
&= \nabla\times[(\boldsymbol{u}.\nabla)\boldsymbol{u}-\nabla(\frac{1}{2}\boldsymbol{u}^2)] \\
&= \nabla\times[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}] \\
&= \nabla^{\perp}\cdot[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}].
\end{aligned} \tag{A.1.6}$$

Furthermore, since $\nabla\cdot\boldsymbol{u}=\nabla^{\perp}\cdot\boldsymbol{u}^{\perp}=0$, we have

$$\begin{aligned}
Y'\,v &= Y'\,v + Y\,\nabla^{\perp}\cdot\boldsymbol{u}^{\perp} \\
&= Y'\,v + \nabla^{\perp}\cdot(Y\boldsymbol{u}^{\perp}) - Y'\,v \\
&= \nabla^{\perp}\cdot(Y\,\boldsymbol{u}^{\perp}).
\end{aligned} \tag{A.1.7}$$

Then the vorticity form equation can be written as

$$\nabla^{\perp}\cdot(\partial_t\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\frac{Y}{\varepsilon}\boldsymbol{u}^{\perp}-\mu\Delta\boldsymbol{v}-f_{\boldsymbol{u}})=0, \tag{A.1.8}$$

and

$$\partial_t\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\frac{Y}{\varepsilon}\boldsymbol{u}^{\perp}-\mu\Delta\boldsymbol{v}-f_{\boldsymbol{v}}=\nabla q, \tag{A.1.9}$$

for some scalar function q. The next step is to show that $\boldsymbol{u}$ have the same initial condition as $\boldsymbol{v}$, i.e., $\boldsymbol{u}_0=\boldsymbol{v}_0$

$$\boldsymbol{u}_0=\lim_{t\to 0^+}\boldsymbol{u}(t)=\lim_{t\to 0^+}\nabla^{\perp}\psi(t)=\nabla^{\perp}\lim_{t\to 0^+}\psi(t). \tag{A.1.10}$$

Let ψ_0 denote the solution of the equation

$$\Delta\psi_0=\omega_0\Longrightarrow\psi_0=\Delta^{-1}\omega_0. \tag{A.1.11}$$

Since $\omega_0=\omega|_{t=0+}$, then the well-posedness of last equation implies that

$$\lim_{t\longrightarrow 0^+}\psi(t)=\psi_0, \tag{A.1.12}$$

so that

$$\boldsymbol{u}|_{t=0}=\nabla^{\perp}\psi_0. \tag{A.1.13}$$

But, by (A.1.1)

$$\boldsymbol{v}_0 = \nabla^{\perp}\psi_0. \tag{A.1.14}$$

Hence, we obtain

$$\boldsymbol{u}_0 = \boldsymbol{v}_0. \tag{A.1.15}$$

□

Lemma A.1.2 Problem 2 is equivalent to problem 1.

Problem 1:

$$\begin{aligned}
&\text{For } f_{\boldsymbol{v}} \in L^2((0,T);H^{-1}) \text{ and } \boldsymbol{v}_0 \in L^2 \text{ find}\\
&\boldsymbol{v} \in L^2((0,T);H^1) \cap C([0,T];L^2) \text{ such that}\\
&\partial_t \boldsymbol{v} + \frac{Y}{\varepsilon}\boldsymbol{v}^{\perp} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} - \mu\,\Delta\boldsymbol{v} = f_{\boldsymbol{v}} \quad \text{in } H^{-1}\\
&\boldsymbol{v}|_{t=0} = \boldsymbol{v}_0.
\end{aligned} \tag{A.1.16}$$

Problem 2:

$$\begin{aligned}
&\text{For } f \in L^2((0,T);H^{-2}) \text{ and } \omega_0 \in H^{-1} \text{ find}\\
&\omega \in L^2((0,T);L^2) \cap C([0,T];H^{-1}) \text{ and}\\
&\psi \in L^2((0,T);H^2) \cap C([0,T];H^1) \text{ such that}\\
&\partial_t \omega + \frac{Y'}{\varepsilon} v + \nabla^{\perp}\psi.\nabla\omega - \mu\,\Delta\omega = f \quad \text{in } H^{-2}\\
&\Delta\psi = \omega \quad \text{in } L^2\\
&\omega|_{t=0} = \omega_0 \quad \text{in } H^{-1},
\end{aligned} \tag{A.1.17}$$

Proof. See [27] □

A.2 Existence and uniqueness of the vorticity form of Navier–Stokes equation on β-plane

Definition A.2.1 Let A be a linear operator from a normed space $(X, |\cdot|_X)$ into a normed space $(Y, |\cdot|_Y)$. We say that A is bounded if there exists a constant M

such that

$$|Ax|_Y \le M\,|x|_X \qquad \forall\, x \in X. \tag{A.2.1}$$

We denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operator from X into Y and define the operator norm of any $A \in \mathcal{L}$ by

$$|A|_{\mathcal{L}} = \sup_{x\neq 0} \frac{|Ax|_Y}{|x|_X} = \sup_{|x|_X=1} |Ax|_X. \tag{A.2.2}$$

Definition A.2.2 (Dual Space)

The space of all linear functional on a Banach space X is called the dual space and denoted by X^*.

Note that X^* is itself a Banach space when equipped with the norm

$$|f|_{X^*} = |f|_{\mathcal{L}(X,\mathbb{R})} \qquad \forall\, f \in X^*. \tag{A.2.3}$$

Definition A.2.3 (Weak convergence)

A sequence x_n, in a Banach space X, converges weakly to x in X, written $x_n \rightharpoonup x$, if $f(x_n)$ converges to $f(x)$ for every $f \in X^*$.

Definition A.2.4 (Weak-* convergence)

Let X be a Banach space. A sequence $f_n \in X^*$ converges weakly-* to f, written $f_n \rightharpoonup^* f$, if

$$f_n(x) \to f(x) \quad \forall\, x \in X. \tag{A.2.4}$$

Theorem A.2.5 (Alaoglu weak-* compactness)

Let f_n be a bounded sequence in X^*, where X is a separable Banach space. Then f_n has a weakly-* convergent subsequence.

Lemma A.2.6 (**Reflexive weak compactness**) [48, p. 106] Let X be a reflexive Banach space and x_n a bounded sequence in X. Then x_n has a subsequence that converges weakly in X.

Lemma A.2.7 Suppose that

$\omega \in L^2((0,T); H^1(\mathcal{M}))$ and $\frac{\mathrm{d}\omega}{\mathrm{d}t} \in L^2((0,T); H^{-1}(\mathcal{M}))$. Then

(i) ω is continuous from $[0,T]$ into L^2, with

$$\sup_{t\in[0,T]} |\omega(t)|_{L^2} \leq C\,(|\omega|_{L^2((0,T);H^1)} + |\frac{\mathrm{d}\omega}{\mathrm{d}t}|_{L^2((0,T);H^{-1})}), \tag{A.2.5}$$

and

(ii)

$$\frac{\mathrm{d}}{\mathrm{d}t}|\omega|^2_{L^2} = 2 < \frac{\mathrm{d}\omega}{\mathrm{d}t}, \omega > \tag{A.2.6}$$

for almost every $t \in [0,T]$

Proof. Similar to the proof of Theorem 7.2 in [48, p. 191]. □

Lemma A.2.8 (i) (**Weak solution**). If $\boldsymbol{v}_0 \in L^2$ and $f \in L^\infty((0,T); H^{-1})$ then there exists a unique solution of the vorticity form of Navier–Stokes equation on the β-plane

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} + \frac{1}{\varepsilon}\,L\omega + B(\omega,\omega) + \mu\,A\omega = f, \tag{A.2.7}$$

that satisfies

$$\omega \in L^\infty((0,T); L^2) \cap L^2((0,T); H^1), \qquad \forall\, T > 0, \tag{A.2.8}$$

and in fact $\omega \in C^0([0,T]; L^2)$.

(ii) (**Strong solution**). If $\boldsymbol{v}_0 \in L^2$ and $f \in L^\infty((0,T); L^2)$ then there exists a unique solution of (A.2.7) that satisfies

$$\omega \in L^\infty((0,T); H^1) \cap L^2((0,T); H^2), \qquad \forall\, T > 0, \tag{A.2.9}$$

and in fact $\omega \in C^0([0,T]; H^1)$.

Proof.

We shall utilize a Galerkin approximation of the vorticity form of Navier–Stokes equation on β-plane. Define the Galerkin projection $\mathbb{P}_n$ into the first n Fourier modes, by

$$\mathbb{P}_n\omega = \omega_n := \sum_{j=1}^{n} \alpha_j(t)\, w_j, \tag{A.2.10}$$

where $\{w_j\}$ is the set of orthonormal basis for L^2. Define a sequence of approximative solutions, ω_n, and the equation (2.3.21) in ω_n is

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_n + \frac{1}{\varepsilon}\, L\omega_n + \mathbb{P}_n B(\omega_n, \omega_n) + \mu\, A\omega_n = \mathbb{P}_n f \tag{A.2.11}$$

(i) Multiplying (A.2.11) by ω_n in L^2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\omega_n|_{L^2}^2 + \mu\, |\omega_n|_{H^1}^2 \leq \frac{c}{\mu}|f|_{H^{-1}}^2, \tag{A.2.12}$$

where $|\mathbb{P}_n f|_{H^{-1}} \leq |f|_{H^{-1}}$. Integrating from 0 to t, we obtain

$$|\omega_n(t)|_{L^2}^2 + \mu \int_0^t |\omega_n(\tau)|_{H^1}^2 \mathrm{d}\tau \leq |\omega_n(0)|_{L^2}^2 + \frac{c}{\mu}|f|_{L^\infty((0,\infty);H^{-1})}^2. \tag{A.2.13}$$

Since $|\omega_n(0)|_{L^2} \leq |\omega(0)|_{L^2}$, we have the uniform bounds in n for $\omega_n(t)$,

$$\sup_{t\in[0,T]} |\omega_n(t)|_{L^2}^2 \leq K = |\omega(0)|_{L^2}^2 + \frac{|f|_{L^\infty((0,\infty);H^{-1})}^2}{\mu}, \tag{A.2.14}$$

and

$$\int_0^T |\omega_n(\tau)|_{H^1}^2 \mathrm{d}\tau \leq K/\mu. \tag{A.2.15}$$

Then we have $\omega_n \in L^\infty((0,T);H^{-1}) \bigcap L^2((0,T);L^2)$. By using Alaoglu weak-* compactness Theorem(A.2.5) we can take a subsequence $\{\omega_{n_j}\}$ such that

$$\omega_{n_j} \rightharpoonup^* \omega \qquad \text{in} \quad L^\infty((0,T);L^2)$$

and by Lemma A.2.6

$$\omega_{n_j} \rightharpoonup \omega \qquad \text{in} \quad L^2((0,T);H^1),$$

with

$$\omega \in L^\infty((0,T);L^2) \bigcap L^2((0,T);H^1).$$

By Lemma A.1.2 and follow the same method as used in [48, Ch. 9], [58, p. 23] and [56, p. 252] we can find a uniform bound for $\frac{\mathrm{d}\omega_n}{\mathrm{d}t}$ in $L^2((0,T);H^{-1})$, where

$$\frac{\mathrm{d}\omega_n}{\mathrm{d}t} = -\frac{1}{\varepsilon}\, L\omega_n - \mathbb{P}_n B(\omega_n,\omega_n) - \mu\, A\omega_n + \mathbb{P}_n f. \tag{A.2.16}$$

Hence by Lemma A.2.7 we find that $\omega \in C^0([0,T],L^2)$. Furthermore, the solution depends continuously on the initial data ω_0 (see e.g., [48, Ch. 9], [58]). For uniqueness see the above references.

In the same way we can prove (ii). □

A.3 Gevrey regularity

We follow [21] to prove the following Lemmata.

Lemma A.3.1 Let ω, $\omega^\sharp$, $\omega^\flat$ be given in $D(A\,\mathrm{e}^{\sigma A^{1/2}})$, $\sigma > 0$. Then $B(\omega^\sharp,\omega^\flat)$ belong to G_σ and we have

$$|\big(B(\omega^\sharp,\omega^\flat),A\omega\big)_\sigma| \le c\,\|\omega^\sharp\|_\sigma\,\|\omega^\flat\|_\sigma\,|A\omega|_\sigma. \tag{A.3.1}$$

Proof. By (3.1.6), we have

$$(B(\omega^\sharp,\omega^\flat),\omega)_{L^2} = \sum_{\boldsymbol{j}+\boldsymbol{k}=\boldsymbol{l}} B_{\boldsymbol{jkl}}\,\omega^\sharp_{\boldsymbol{j}}\,\omega^\flat_{\boldsymbol{k}}\,\overline{\omega_{\boldsymbol{l}}}, \tag{A.3.2}$$

where $\boldsymbol{j},\boldsymbol{k}$ and $\boldsymbol{l} \in \mathbb{Z}_L$. Also

$$\begin{aligned}(\mathrm{e}^{\sigma A^{1/2}}\,B(\omega^\sharp,\omega^\flat),\mathrm{e}^{\sigma A^{1/2}}\,A\omega)_{L^2} &= \sum_{\boldsymbol{j}+\boldsymbol{k}=\boldsymbol{l}} B_{\boldsymbol{jkl}}\,\omega^\sharp_{\boldsymbol{j}}\,\omega^\flat_{\boldsymbol{k}}\,\overline{\omega_{\boldsymbol{l}}}\,|\boldsymbol{l}|^2\,\mathrm{e}^{2\sigma|\boldsymbol{l}|}\\ &= \sum_{\boldsymbol{j}+\boldsymbol{k}=\boldsymbol{l}} B_{\boldsymbol{jkl}}\,\omega^\sharp_{\boldsymbol{j}}\,\mathrm{e}^{\sigma|\boldsymbol{j}|}\,\omega^\flat_{\boldsymbol{k}}\,\mathrm{e}^{\sigma|\boldsymbol{k}|}\,\overline{\omega_{\boldsymbol{l}}}\,|\boldsymbol{l}|^2\,\mathrm{e}^{\sigma|\boldsymbol{l}|}\\ &= \sum_{\boldsymbol{j}+\boldsymbol{k}=\boldsymbol{l}} B_{\boldsymbol{jkl}}\,\omega^{*\sharp}_{\boldsymbol{j}}\,\omega^{*\flat}_{\boldsymbol{k}}\,\overline{\omega^*_{\boldsymbol{l}}}\,|\boldsymbol{l}|^2,\end{aligned} \tag{A.3.3}$$

where $\omega^* = \sum_{\boldsymbol{l}} \omega^*_{\boldsymbol{l}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{l}\cdot\boldsymbol{x}}, \quad \omega^*_{\boldsymbol{l}} = \mathrm{e}^{\sigma|\boldsymbol{l}|}\, \omega_{\boldsymbol{l}}$ and the same thing for $\omega^{*\sharp}$ and $\omega^{*\flat}$. Hence

$$
\begin{aligned}
|(\mathrm{e}^{\sigma A^{1/2}}\, B(\omega^\sharp, \omega^\flat), \mathrm{e}^{\sigma A^{1/2}}\, A\omega)_{L^2}| &\le |\mathcal{M}| \sum_{\boldsymbol{j}+\boldsymbol{k}=\boldsymbol{l}} \frac{|\omega^{*\sharp}_{\boldsymbol{j}}|}{|\boldsymbol{j}|}\, |\boldsymbol{k}|\, |\omega^{*\flat}_{\boldsymbol{k}}|\, |\boldsymbol{l}|^2\, |\omega^*_{\boldsymbol{l}}|\, \delta_{\boldsymbol{j}+\boldsymbol{k}-\boldsymbol{l}} \\
&= \int_{\mathcal{M}} \phi(\boldsymbol{x})\, \psi(\boldsymbol{x})\, \xi(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\
&\le c\, |\phi(\boldsymbol{x})|_{L^p}\, |\psi(\boldsymbol{x})|_{L^q}\, |\xi(\boldsymbol{x})|_{L^r}
\end{aligned} \tag{A.3.4}
$$

where

$$
\phi(\boldsymbol{x}) = \sum_{\boldsymbol{j}} \frac{|\omega^{*\sharp}_{\boldsymbol{j}}|}{|\boldsymbol{j}|} \mathrm{e}^{\mathrm{i}\boldsymbol{j}\cdot\boldsymbol{x}}, \qquad \psi(\boldsymbol{x}) = \sum_{\boldsymbol{k}} |\omega^{*\flat}_{\boldsymbol{k}}|\, |\boldsymbol{k}|\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}
$$

$$
\xi(\boldsymbol{x}) = \sum_{\boldsymbol{l}} |\boldsymbol{l}|^2\, |\omega^*_{\boldsymbol{l}}|\, \mathrm{e}^{\mathrm{i}\boldsymbol{l}\cdot\boldsymbol{x}},
$$

with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Now putting $q = r = 2$ and $p = \infty$ plus the embedding $H^2 \subset L^\infty$, we obtain the following result

$$
\big(B(\omega^\sharp, \omega^\flat), A\omega\big)_\sigma| \le c\, \|\omega^\sharp\|_\sigma\, \|\omega^\flat\|_\sigma\, |A\omega|_\sigma. \tag{A.3.5}
$$

□

Lemma A.3.2 If $f \in L^\infty(\mathbb{R}_+; G_\sigma)$, for some $\sigma > 0$. Then there exists a time $T_\sigma(|\omega(0)|_{H^1}, |f|_\sigma; \mu)$ such that

$$
|A^{1/2}\, \mathrm{e}^{\sigma_2 A^{1/2}}\, \omega(t_1)|_{L^2} \le K_\sigma(|f|_\sigma; \mu), \tag{A.3.6}
$$

for all $t_1 \ge T_\sigma$ where $\sigma_2 = \sigma_1(T_\sigma) = \min(\sigma, T_\sigma)$

Proof. We set $\sigma_1(t) = \min(\sigma, t)$ and multiplying equation (2.3.21) by $A\omega$ in Gevrey space

$$
(\partial_t\omega, A\omega)_{\sigma_1} + \frac{1}{\varepsilon}\, (L\omega, A\omega)_{\sigma_1} + (B(\omega, \omega), A\omega)_{\sigma_1} + (A\omega, A\omega)_{\sigma_1} = (f, A\omega)_{\sigma_1}. \tag{A.3.7}
$$

Let us now compute every term in last equation separately,

$$
(\partial_t\omega, A\omega)_{\sigma_1} = (\mathrm{e}^{\sigma_1 A^{1/2}} \partial_t\omega, \mathrm{e}^{\sigma_1 A^{1/2}} A\omega)_{L^2} = (A^{1/2}\mathrm{e}^{\sigma_1 A^{1/2}} \partial_t\omega, A^{1/2}\mathrm{e}^{\sigma_1 A^{1/2}} \omega)_{L^2} \tag{A.3.8}
$$

and because

$$\partial_t(A^{1/2}\mathrm{e}^{\sigma_1 A^{1/2}}\omega(t)) = A^{1/2}\mathrm{e}^{\sigma_1 A^{1/2}}\partial_t\omega(t) + \partial_t\sigma_1(t)\mathrm{e}^{\sigma_1 A^{1/2}}A\omega(t). \tag{A.3.9}$$

Thus

$$\begin{aligned}(\partial_t\omega, A\omega)_{\sigma_1} &= \big(\partial_t(A^{1/2}\mathrm{e}^{\sigma_1 A^{1/2}}\omega(t)) - \sigma_1'(t)\mathrm{e}^{\sigma_1 A^{1/2}}A\omega(t), A^{1/2}\mathrm{e}^{\sigma_1 A^{1/2}}\omega(t)\big)\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(A^{1/2}\mathrm{e}^{\sigma_1 A^{\frac{1}{2}}}\omega(t))^2 - \sigma_1'(t)\big(\mathrm{e}^{\sigma_1 A^{1/2}}\omega(t), \mathrm{e}^{\sigma_1 A^{1/2}}A^{1/2}\omega(t)\big)\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{\sigma_1}^2 - \sigma_1'(t)\big(A\omega(t), A^{1/2}\omega(t)\big)_{\sigma_1}\\ &\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{\sigma_1}^2 - \big(A\omega(t), A^{1/2}\omega(t)\big)_{\sigma_1}\\ &\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{\sigma_1}^2 - \frac{\mu}{4}\,|A\omega|_{\sigma_1}^2 - \frac{1}{\mu}\|\omega\|_{\sigma_1}^2.\end{aligned} \tag{A.3.10}$$

Also

$$(L\omega, A\omega)_{\sigma_1} = 0 \tag{A.3.11}$$

And by Lemma(A.3.1), we have

$$|\big(B(\omega,\omega), A\omega\big)_{\sigma_1}| \leq c\,\|\omega\|_{\sigma_1}^2\,|A\omega|_{\sigma_1}. \tag{A.3.12}$$

Putting all terms together

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{\sigma_1}^2 - \frac{\mu}{4}\,|A\omega|_{\sigma_1}^2 - \frac{1}{\mu}\,\|\omega\|_{\sigma_1}^2 + \mu\,|A\omega|_{\sigma_1}^2 \leq |f|_{\sigma_1}\,|A\omega|_{\sigma_1} - \|\omega\|_{\sigma_1}^2\,|A\omega|_{\sigma_1}. \tag{A.3.13}$$

Using the Poincaré inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{\sigma_1}^2 + \mu\,|A\omega|_{\sigma_1}^2 \leq \frac{4}{\mu}\,|f|_{\sigma_1}^2 + \frac{4}{\mu}\,\|\omega\|_{\sigma_1}^4 + \frac{2}{\mu}\,\|\omega\|_{\sigma_1}^2. \tag{A.3.14}$$

Neglecting the second term of left-hand side of (A.3.14), then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{\sigma_1}^2 &\leq \frac{4}{\mu}\,|f|_{\sigma_1}^2 + \frac{2}{\mu}\,\|\omega\|_{\sigma_1}^2 + \frac{4}{\mu}\,\|\omega\|_{\sigma_1}^4\\ \frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{\sigma_1}^2 &\leq \frac{4}{\mu}\,|f|_{\sigma_1}^2 + c_1 + \frac{4}{\mu}\,\|\omega\|_{\sigma_1}^4.\end{aligned} \tag{A.3.15}$$

Now let

$$G = 1 + \|\omega\|_{\sigma_1}^2. \tag{A.3.16}$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{\sigma_1}^2 &\leq \frac{4}{\mu}\big(|f|_{\sigma_1} + \|\omega\|_{\sigma_1}^2\big)^2 + \frac{4}{\mu}\left(1 + \|\omega\|_{\sigma_1}^2\right)^2 \\ &= \Big(\frac{4}{\mu}\,|f|_{\sigma_1}^2 + c_1 + \frac{4}{\mu}\Big)\left(1 + \|\omega\|_{\sigma_1}^2\right)^2.\end{aligned} \tag{A.3.17}$$

Now if we put

$$K = \frac{4}{\mu}\,|f|_{\sigma}^2 + c_1 + \frac{4}{\mu}. \tag{A.3.18}$$

Thus

$$\frac{\mathrm{d}G}{\mathrm{d}t} \leq K\,G^2, \tag{A.3.19}$$

this implies to

$$\frac{\mathrm{d}G}{G^2} \leq K\,\mathrm{d}t. \tag{A.3.20}$$

By integration from 0 to t we have

$$G(t) \leq G(0)\,(1 - KtG(0))^{-1} \tag{A.3.21}$$

and if we take $t \leq \frac{1}{2KG(0)} = \frac{1}{2K}(1 + |A^{1/2}\omega(0)|^2)^{-1} = T_1(A^{1/2}\omega(0))$, then we have

$$G(t) = 1 + |A^{1/2}\,\mathrm{e}^{\sigma_1(t)A^{1/2}}\,\omega(t)|_{L^2}^2 \leq 2\,G(0) = 2 + 2|A^{1/2}\omega(0)|^2. \tag{A.3.22}$$

Therefore $\omega(t)$ is in $D(A^{1/2}\,\mathrm{e}^{\sigma_1(t)A^{1/2}})$ and (A.3.22) holds for $t \in (0, T_1)$. In particular

$$|\;A^{1/2}\,\mathrm{e}^{\sigma_1(T_1)A^{1/2}}\,\omega(T_1)|_{L^2}^2 \leq (2 + 2|A^{1/2}\omega(0)|^2). \tag{A.3.23}$$

By Lemma 2.4.3 we have

$$|\;A^{1/2}\;\omega(t)|_{L^2}^2 \leq K_1 \quad \text{for} \quad \text{all}\, t \geq 0, \tag{A.3.24}$$

And then we can apply the argument above at any time $t_1 > 0$ and find that

$$|\;A^{1/2}\,\mathrm{e}^{\sigma_2 A^{1/2}}\,\omega(t)|_{L^2}^2 \leq (2 + 2K_1^2) \quad \text{for} \quad \text{all}\, t \geq T_\sigma \tag{A.3.25}$$

where $\sigma_2 = \sigma_1(T_\sigma) = \min(T_2, \sigma)$ and $T_\sigma = \frac{1}{2K}(1 + K_1^2)^{-1}$. Hence

$$|\;A^{1/2}\,\mathrm{e}^{\sigma_2 A^{1/2}}\,\omega(t)|_{L^2}^2 \leq K_\sigma(|f|_\sigma; \mu) \quad \text{for} \quad \text{all}\, t \geq T_\sigma. \tag{A.3.26}$$

□

A.4 Cauchy integral formula

Let $f(z)$ be an analytic function on a simply connected domain D. Let $z_0 \in D$, and let C simple closed curve in D encircling z_0. Then

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(z)}{z - z_0} \, \mathrm{d}z \tag{A.4.1}$$

In addition the nth derivative of $f(z)$ at $z = z_0$ is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z \tag{A.4.2}$$

Definition A.4.1 Cauchy inequality

Suppose that $f(z)$ is analytic function on and inside the disc $|z - z_0| = R$, $0 < R < \infty$. Then

$$|f^{(n)}(z_0)| \leq \frac{M\, n!}{R^n}, \quad n = 1, 2, \cdots \tag{A.4.3}$$

where M is a constant and $|f(z)| \leq M$ on and inside the disc.

Bibliography

[1] R. Adams, "Sobolev Spaces," Academic Press, New York, 1975.

[2] M. Al-Jaboori and D. Wirosoetisno, *Navier–Stokes equation on the β-plane*. DCDS-B, 16 (2011), 687-701.

[3] A. Babin, A. Mahalov and B. Nicolaenko, *Regularity and integrability of 3d Euler and Navier–Stokes equations for rotating fluid*, Asymp. Anal., 15 (1997), 103-150.

[4] M. Brin and G. Stuck, "Introduction to Dynamical System," Cambridge University Press, 2002.

[5] C. Cao, M. Rammaha and E. Titi, *The Navier–Stokes equation on the rotating 2-D sphere Gevrey regularity and asymptotic degrees of freedom*, Z. angew. Math. Phys., 50 (1999), 341-360.

[6] V. V. Chepyzhov and M. I. Vishik, "Attractors for Equations of Mathematical Physics," American Math. Soc., 2002.

[7] ———, *Trajectory Attractors for Dissipative 2D Euler and Navier–Stokes Equations*, Russian J. Math. Phys. 15 (2008), 156-170.

[8] J. Chemin, B. Desjardins, I. Gallagher and E. Grenier, "Mathematical Geophysics: An Introduction to Rotating Fluids and the Navier–Stokes Equations," Oxford University Press, 1995.

[9] P. Constantin, C. Foias, *Global Lyaponouv Exponents, Kaplan-Yorke Formulas and the Dimension of the Attractors for* 2*-D Navier–Stokes Equations*, Communications on Pure and Applied Mathematics, 38 (1985), 1-27.

[10] P. Constantin, C. Foias and R. Temam, *On the dimension of the attractors in two dimensional turbulence*, Physica, D 30 (1988), 284-296.

[11] P. Constantin, C. Foias and R. Temam, *Attractor Representing Turbulent Flows*, Memoir of the American Mathematical Society, 53 (1983).

[12] A. Chorin and J. Marsden, "A Mathematical Introduction to Fluid Mechanics, Third Edition," Springer-Verlag New York, 1990.

[13] S. Danilov and D. Gurarie, *Scaling, spectra and zonal jets in beta-plane turbulence*, Physics of Fluids, 16 (2004), 2592-2603

[14] C. R. Doering and D. Gibbon, "Applied Analysis of the Navier–Stokes Equations," Cambridge University Press, 1995.

[15] Weinan E , *Convergence of Fourier Methods for the Navier-Stokes Equations*, SIAM J. Numer. Anal., 30 (1990), 650-674.

[16] C. Foias, O. Manley, R. Rosa and R. Temam, "Navier–Stokes Equations and Turbulence," Cambridge University Press, 2004.

[17] C. Foias, O. Manley, R. Temam and Y. M. Treve, *Asymptotic analysis of the Navier–Stokes equations*, Physica, D 9 (1983), 157-188.

[18] C. Foias, G. Prodi, *Sur le comportement global des solutions non-stationnaires des équations de Navier–Stokes en dimension* 2, Rend. Sem. Mat. Univ. Padova, 39 (1967), 1-34.

[19] C. Foias, G.R. Sell and R. Temam, *Inertial Manifold for Nonlinear Evolutionary Equations*, J. Diff. Eq., 73 (1988) 309-353.

[20] C. Foias, R. Temam, *Some analytic and geometric properties of the solutions of the evolution Navier–Stokes equations*, J. Math. Pures Appl. 58 (1979), 339-368.

[21] C. Foias and R. Temam, *Gevrey class regularity for the solutions of the Navier–Stokes equations*, J. Funct. Anal., 87 (1989), 359-369.

[22] I. Gallagher and L. Saint-Raymond, *Mathematical study of the β-plane model: equatorial waves and convergence results*, Mém. Soc. Math. France, 107 (2006), pp. vi+116.

[23] M. Ganesh, Q. T. Le Gia and I. H. Sloan, *A pseudospectral quadrature method for Navier–Stokes equations on rotating spheres*, Math. Comp., 80 (2011), 1397-1430.

[24] D. Gong, H. Song, and C. Zhong, *Attractors for nonautonomous two-dimensional space periodic Navier–Stokes equations*, J. Math. Phys. 50, 102706 (2009).

[25] D. Gottlieb and S. Orszag, "Numerical Analysis of Spectral Methods: Theory and Applications," SIAM, 1977.

[26] M. Griebel, T. Dornseifer and T. Neunhoeffer, "Numerical Simulation in Fluid Dynamics: A Practical introduction," SIAM, 1998.

[27] J. L. Guermond, L. Quartapelle, *Equivalence of $u-p$ and $\zeta-\psi$ formulations of the time-dependent Navier-Stokes equations*, Int. J. Numer. Methods in Fluids, 18 (1994) 471-487.

[28] G. Guevremont, W. G. Habashi and M. M. Hafez, *Finite element solution of the Navier–Stokes equations by a velocity-vorticity method*, Int. J. Numer. Methods in Fluids, 11 (1990), 661-675.

[29] A.A. Ilyin, *Navier–Stokes equations on the rotating sphere. A simple proof of the attractor dimension estimate*, Nonlinearity, 7 (1994), 31-39.

[30] A.A. Ilyin and A. N. Filatov, *On unique solvability of the Navier–Stokes equations on the two dimensional sphere*, Soviet Math. Dokl., 38(1989) 9-13.

[31] N. Ju, *On the global stability of a temporal discretization scheme for the Navier-Stokes equations*, SIAM J. Numer. Anal., 22 (2002), 577-597.

[32] D. Kopriva, "Implementing Spectral Methods for Partial Differential Equations," Springer , 2009.

[33] H. Kreiss and J. Lorenz, *On the existence of slow manifolds for problems with different timescales*, Philosophical Transactions: Physical Sciences and Engineering, 346 (1994), 159-171.

[34] O. A. Ladyzhenskaya, "The Mathematical Theory of Viscous Incompressible Flow," Gordon and Breach Science Publishers, 1969.

[35] O. A. Ladyzhenskaya, *A dynamical system generated by the Navier–Stokes equations*, Zapiski Nauchnykh Seminarov LOMI, 27 (1972), 91-115. English translation in J. of Soviet Math., 3 (1975) 485-479.

[36] Y. Lee and L.M. Smith, *Stability of Rossby waves in the β-plane approximation*, Physica, D 179(2003), 53-91.

[37] F. B. Lipps, *A note on the β-plane approximation*, Tellus, 16 (1964), 535-537.

[38] G. Lukaszewicz, *Pullback attractors and statistical solutions for 2-D Navier–Stokes equations*, DCDS-B, 9 (2008), 643-659.

[39] W. Lyton, "Introduction to the Numerical Analysis of Incompressible Viscous Flows," SIAM, 2008.

[40] A. Majda, "Introduction to PDEs and Waves for the Atmosphere and Ocean," American Mathematical Society, 2006.

[41] A. Majda and A. L. Bertozzi, "Vorticity and Incompressible Flow," Cambridge University Press, 2002.

[42] A. Majda and X. Wang, "Non-linear dynamics and statistical theories for basic geophysical flows," Cambridge University Press, 2006.

[43] J. Marsden, L. Sirovich and S. Antman, "An Introduction to Applied Non-Linear Dynamical Systems and Chaos," 2 ed., Springer-Verlag New York, 2003.

[44] K. Matthies, *Time-Averaging under Fast Periodic Forcing of Parabolic Partial Differential Equations: Exponential Estimates*, J. Diff. Eq., 174 (2001), 133-180.

[45] R. Peyret, "Spectral Methods for Incompressible Viscous Flow," Springer, 2002.

[46] Y. Pinchover and J. Rubinstein, "An Introduction to Partial Differential Equations," Cambridge University Press, 2005.

[47] P. B. Rhines, *Waves and turbulence on a beta-plane*, J. Fluid Mech., 69 (1975), 417-443.

[48] J. Robinson, "Infinite-Dimensional Dynamical System," Cambridge University Press, 2001.

[49] ———, "Dimensions, Embeddings, and Attractors," Cambridge University Press, 2011.

[50] C. G. Rossby et al., *Relation between variations in the intensity of the zonal circulation of the atmosphere and the displacement of the semi-permanant centers of action*, Journal of Marine Research, 2 (1939), 38-55.

[51] S. Schochet, *Fast singular limits of hyperbolic PDEs*, J. Diff. Eq., 114 (1994), 476-512.

[52] T. G. Shepherd, *Non-ergodicity of inviscid two-dimensional flow on a β-plane and on the surface of a rotating sphere*, J. Fluid Mech., 184 (1987), 289-302.

[53] A. B. Siddiqi, "Applied Functional Analysis," Marcel Dekker, 2004.

[54] A. M. Stuart and A. R. Humphries, "Dynamical Systems and Numerical Analysis, " Cambridge University Press, 1988.

[55] L. Tartar, "An Introduction to Navier–Stokes Equations and Oceanography," Springer-Verlag Berlin Heidelberg, 2006.

[56] R. Temam, "Navier–Stokes Equations, Theory and Numerical Analysis," 3 ed., North-Holland, Amsterdam, 1985.

[57] R. Temam, "Infinite dimensional dynamical systems in fluid mechanics." *Nonlinear Functional Analysis and Its Applications* (F. Browder, ed.) (Proc. Sympos. Pure Math. 45), pp. 413-425. Amer. Math. Soc., Providence, RI, 1986.

[58] ———, "Navier–Stokes Equations and Nonlinear Functional Analysis," SIAM, 2 ed., 1995.

[59] ———, "Infinite-Dimensional Dynamical Systems in Mechanics and Physics," Applied Mathematical Sciences 68, Sipringer-Verlag, New York, 1997.

[60] R. Temam and D. Wirosoetisno, *Averaging of Differential Equations Generating Oscillations and an Application to Control*, Appl. Math. Optim., 46 (2002), 313-330.

[61] ———, *Stability of the slow manifold in the primitive equations*, SIAM J. Math. Anal., 42 (2010), 427-458. arXiv:0808.2878.

[62] G. K. Vallis, "Atmospheric and Oceanic Fluid Dynamics," Cambridge University Press, 2006.

[63] G. K. Vallis and M. E. Maltrud, *Energy spectra and coherent structures in forced two-dimensional and beta-plane turbulence*, J. Fluid Mech., 228 (1991), 321-342.

[64] G. K. Vallis and M. E. Maltrud, *Generation of Mean flows and Jets on a Beta Plane and over Topography*, J. Phys. Oceanogr., 23 (1994) 1346-1362.

[65] N. G. Van Kampen, *Elimination of fast variables*, Phys. Rep., 124 (1985), 69-160.

[66] J. Vanneste, *Asymptotics of a Slow Manifold*, SIAM J. Appl. Dyn. Syst., 7 (2008), 1163-1190.

[67] T. Warn, O. Bokhove, T. G. Shepherd and G. K. Vallis, *Rossby number expansions, slaving principles, and balance dynamics*, Quarterly Journal of the Royal Meteorological Society, 121 (1995), 723-739.

[68] D. Wirosoetisno, *Exponentially accurate balance dynamics*, Advances in differential equations, 9 (2004), 177-196.

[69] D. Wirosoetisno, T. Shepherd, and R. Temam, *Free gravity waves and balance dynamics*, J. Atmos. Sci., 59 (2002), 3382-3398.

[70] S. Wiggins, "Introduction to Applied Nonlinear Dynamical System and Chaos," 2 ed., Springer, 2003.