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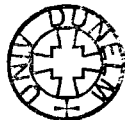
Confinement and the Infrared Behaviour of the Gluon Propagator

A thesis submitted for the degree of
Doctor of Philosophy
by

Kirsten Büttner

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University of Durham
Department of Physics
September 1996



28 MAY 1997

Das schönste Glück des denkenden Menschen ist,
das Erforschliche erforscht zu haben und
das Unerforschliche ruhig zu verehren.

Goethe, Maximen und Reflexionen

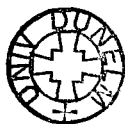
Abstract

We investigate the infrared behaviour of the gluon propagator in Quantum Chromodynamics (QCD). A natural framework for such a non-perturbative study is the complex of Schwinger-Dyson equations (SDE).

The possible infrared behaviour of the gluon, found by self-consistently solving the approximate boson SDE, is studied analytically. We find that only an infrared enhanced gluon propagator, as singular as $1/p^4$ as $p^2 \rightarrow 0$, is consistent and demonstrate why softer solutions, that others have found, are not allowed. Reassuringly the consistent, enhanced infrared behaviour is indicative of the confinement of quarks and gluons, implying, for example, *area-law behaviour* of the Wilson loop operator and forbidding a Källén-Lehmann spectral representation of both quark and gluon propagators. We then briefly consider the implications of these results for models of the pomeron.

The enhancement of the gluon propagator does however introduce infrared divergences in the SDE and these need to be regularised. So far *model* forms of the enhanced gluon propagator have been used in studies of dynamical chiral symmetry breaking and hadron phenomenology. Though very encouraging results have been obtained, one might hope to use the gluon propagator obtained directly from non-perturbative QCD to calculate hadron observables.

We therefore attempt to eliminate the infrared divergences in the SDEs in a self-consistent way, entirely within the context of the calculational scheme. To do this we introduce an infrared regulator λ in the truncated gluon SDE in quenched QCD. We find that this regulator is indeed determined by the equation and bounded by the QCD-scale Λ_{QCD} . Thus it is possible to perform the regularisation within the SDEs. However, we have not been able to choose $\lambda < \Lambda_{QCD}$.



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This thesis is dedicated to my family,
To Mami and Papi
and David
with much love.

Declaration

I declare that no material presented in this thesis has previously been submitted for a degree at this or any other university.

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- *Should the Pomeron and imaginary parts be modelled by two gluons and real quarks?*
Kirsten Büttner and M.R. Pennington, Phys. Lett. **B356**, 354 (1995).
- *Solutions to the Schwinger Dyson equation for the gluon and their implications for quark confinement*
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Chapter 1

Introduction

Elementary Particle Physics has provided us with a comprehensive theory of particle interactions, the *Standard Model* which, with the exception of gravity, embodies all the fundamental interactions of nature. Quantum Chromodynamics (QCD) is one of the three pillars of the Standard Model, being the accepted theory of the strong force; the other two are the theory of the weak and electromagnetic forces. Nevertheless QCD is distinguished from the other interactions forming the Standard Model, by describing only the interaction of elementary particles (quarks and gluons) which so far have not been observed experimentally.

The success of QCD is that, using perturbation theory, it predicts and explains the high energy phenomena of the strong interaction, from jet production in e^+e^- collisions to scaling violations in deep inelastic scattering, where quarks and gluons behave as though they are free. Yet away from the high energy region of such processes, perturbation theory fails and quarks and gluons are confined inside hadrons, the strongly interacting particles we actually observe in our detectors. Unfortunately, confinement and consequently properties of hadrons, such as their masses, lifetimes, decay modes and interactions, are not yet understood on the fundamental level of QCD. However, if QCD is to be regarded as the theory of the strong interactions, there clearly should be the possibility of describing hadronic properties in terms of the parameters of the theory itself. For this, it is necessary to understand the behaviour of basic quantities, such as the propagators and coupling,

not only in the high energy region, but also at low energies.

This thesis documents a non-perturbative study of the low energy behaviour of the gluon which can provide us with a better understanding of how confinement really happens within QCD. The structure of the thesis is as follows:

Chapter 2 contains a general introduction to QCD explaining in particular the property of confinement and what can be learnt about it from the behaviour of the gluon propagator. In chapter 3 we introduce the Schwinger-Dyson equation (SDE) approach to studying QCD non-perturbatively and derive the field equation for the gluon – the underlying tool in all the calculations carried out in this study.

Extensive work has previously been performed studying the gluon SDE, however different low energy (infrared) behaviours have been proposed in the literature and clarification is needed. For this reason, in chapter 4, we investigate which infrared behaviour of the gluon can be found as a self-consistent solution to the SDE. We determine that the low energy behaviour of the gluon is enhanced and discuss the implications for confinement. We also study the consequences of this infrared behaviour of the gluon for the modelling of the pomeron in terms of dressed gluon exchange.

However, the infrared enhanced gluon propagator which we find to be the only consistent solution to the truncated SDE leads to infrared divergences in the SDEs for both the gluon and the quark and these need to be regularised. To continue our study of the infrared behaviour of the gluon we address the problem of regularising these divergences in chapter 5. Introducing an infrared regulator in the gluon SDE we demonstrate that this regulator is determined by the equation studied providing us with a possibility of performing the regularisation within the calculational scheme of the SDE. We then highlight the problems that remain open and deserve to be addressed in future studies.

Finally, in chapter 6 we summarise all the results obtained in this study and describe how we can apply them to hadron phenomenology.

Chapter 2

QCD – a Quantum gauge theory

In this chapter we give an introduction to Quantum Chromodynamics (QCD), which is generally believed to be the theory of the strong interaction. We start with a brief review of how QCD developed historically from strong-interaction studies. In section 2.2 we define the QCD Lagrangian, set up the path integral formalism and give the QCD generating functional, which defines the quantum field theory of QCD. We then go on to describe two remarkable properties of QCD: asymptotic freedom and confinement. Since the purpose of this thesis is to study how confinement really happens within QCD, we shall explain this property in some detail.

2.1 Introduction

Historically, QCD originated as a development of the quark model. In the early 1960s many strongly interacting particles were observed, which could be classified according to the representations of what today we would call flavour $SU(3)_F$. The light mesons occur only in $SU(3)_F$ singlets and octets, the light baryons in singlets, octets and decuplets.

Gell-Mann [1] and Zweig [2] recognized in 1964 that this symmetry would naturally arise if hadrons were composite objects, made up of more fundamental constituents. Gell-Mann called these *quarks* and they belong to the fundamental representation of $SU(3)_F$. By postulating mesons to be composites of a quark and an antiquark and baryons to be

made up of three quarks this simple model reproduced the observed spectrum, predicting mesons and baryons in the following $SU(3)_F$ representations:

$$\text{Meson: } 3 \otimes \bar{3} = 8 \oplus 1$$

$$\text{Baryon: } 3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

However this naive quark model could not explain why only these combinations of quarks and antiquarks were observed and, in particular, why quarks did not appear as free particles in detectors.

Both these phenomena were to be explained by *colour*. It was realised that the lowest mass spin $\frac{3}{2}$ baryons, (e.g. Δ^{++}) made from three fermionic spin $\frac{1}{2}$ quarks could give rise to a totally symmetric state, which is of course forbidden for fermions. This spin-statistics crisis was resolved by the introduction of a new quantum number, colour [3]. Allowing each species of quark to come in any of three colours removes the unwanted symmetry. The new theory was based on the symmetry group colour $SU(3)_C$. Furthermore the nonexistence of free quarks can be explained by demanding all physical asymptotic states be colourless, i.e. only singlet representations of colour $SU(3)_C$ appear in $3_C \otimes 3_C \otimes 3_C$ (baryons) and $3_C \otimes \bar{3}_C$ (mesons). This is known as the confinement hypothesis: quarks carry colour and therefore they are confined within colour-singlet bound states.

In the late 1960s the famous SLAC deep inelastic lepton-nucleon scattering (DIS) experiment revealed that hadrons are really made up of more fundamental constituents. The DIS cross-sections satisfy Bjorken scaling which could be successfully interpreted by Feynman's parton model [4] (1969). It only took one step to identify these partons with quarks, which appear to be free at short distances.

In 1973 Fritzsche et al. [5], Gross and Wilczek [6] and Weinberg [7] extended the global $SU(3)_C$ colour model to a gauge theory, QCD, in which quarks are assumed to be bound together by exchanging gluons. These gluons are in the adjoint representation of $SU(3)_C$. They carry colour and therefore interact not only with the quarks but also among themselves. It was then discovered by Politzer [8] and Gross and Wilczek [9] that, in such a non-Abelian gauge theory, the coupling constant decreases at short distances

and increases at large distances. This explained the success of the parton-model and the fact that free quarks have not been observed, leading to the belief that QCD is indeed the correct theory of the strong interaction.

2.2 QCD, the Theory and its Gauge Invariance

2.2.1 The QCD Lagrangian

Quantum chromodynamics is a non-Abelian gauge theory composed of fermionic fields in the fundamental representation and gluonic fields in the adjoint representation of the group $SU(3)_C$. It's Lagrangian is defined as:

$$L^{QCD} [\bar{\psi}^f, \psi^f, \bar{\omega}, \omega, g_0, m_f] = L_{invar} + L_{gauge-fix} + L_{ghost} \quad (2.1)$$

which is a function of the quark fields ψ^f , gluon fields A_μ and ghost fields ω , and the parameters g_0 and m_f , where g_0 is the QCD (*strong*) coupling constant and m_f are the quark masses.

L_{invar} is the basic Lagrangian of QCD, invariant under local $SU(3)$ transformations and has the usual Yang-Mills form:

$$L_{invar} = \sum_{f=1}^{N_f} \bar{\psi}^f (iD^\mu \gamma_\mu - m_f) \psi^f - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} \quad . \quad (2.2)$$

With the covariant derivative:

$$D^\mu = \partial^\mu - ig_0 A_a^\mu T^a$$

and the field strength tensor:

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g_0 f_{abc} A_b^\mu A_c^\nu \quad .$$

$T^a = \frac{1}{2}\lambda^a$ are the Gell-Mann λ -matrices. Their commutator,

$$[T^a, T^b] = if^{abc} T^c \quad ,$$

defines the structure constants f^{abc} of the $SU(3)$ algebra.

It is now easy to show that L_{invar} as given in Eq. (2.2) is invariant under local gauge transformations of the form:

$$\begin{aligned}\psi^f(x) &\longrightarrow U(x)\psi^f(x) \quad , \\ A_\mu(x) &\longrightarrow U(x)A_\mu(x)U^{-1}(x) + \frac{i}{g_0}U(x)\partial_\mu U^{-1}(x) \quad ,\end{aligned}$$

where

$$U(x) = \exp(i\theta^a(x)T^a)$$

with $\theta^a(x)$ the space time dependent parameters of the local $SU(3)_C$ gauge transformation $U(x)$.

Quantization of QCD requires the two extra terms $L_{gauge-fix}$ and L_{ghost} in the Lagrangian, Eq. (2.1). There are many different gauge-fixing terms possible, the most common choices being:

$$\text{a.) } L_{gauge-fix} = -\frac{\xi}{2}(\partial_\mu A_a^\mu)^2 \quad , \quad 1 < \xi < \infty \quad (2.3)$$

defining the set of *covariant* gauges, and

$$\text{b.) } L_{gauge-fix} = -\frac{\xi}{2}(n_\mu A_a^\mu)^2 \quad , \quad \xi \rightarrow \infty \quad (2.4)$$

where n_μ is a fixed vector defining the *axial* gauges.

In axial gauges there is no need for ghost fields. However, in the covariant gauges we must add the ghost Lagrangian:

$$L_{ghost} = (\partial_\mu \bar{\omega}_a)(\partial^\mu \delta_{ab} - g_0 f_{abc} A_b^\mu)\omega_c \quad . \quad (2.5)$$

2.2.2 Path Integral Formalism

In this section we introduce Feynman's path integral formalism [10], which is the framework we will use in chapter 3 to derive the field equations (Schwinger-Dyson equations) of QCD. Furthermore using path integrals, the motivation for $L_{gauge-fix}$ and L_{ghost} terms in the QCD Lagrangian can be explained. For a detailed account see e.g. Ref. [11].

In the path integral formalism, the transition amplitude between initial and final state is expressed as the sum, or rather the functional integral, over all the possible paths in phase space connecting the initial and final states, weighted by the exponential of the action for each particular path.

Scalar Field Theory

For purposes of illustration we will consider a scalar field theory, defined by the generating functional:

$$Z[J] = N^{-1} \int [d\phi] \exp \left\{ i \int d^4x [L(\phi) + J(x)\phi(x)] \right\} \quad , \quad (2.6)$$

where N is a normalisation factor which ensures $Z[0] = 1$,

$L(\phi)$ is the Lagrangian density,

$J(x)$ is a source term and

$\int [d\phi]$ represents the functional integral over all classical field configurations $\phi(x)$.

Generally in quantum field theory we are interested in the n-point Green's functions. These $G^{(n)}$ correspond to probability amplitudes connecting n external states via the interactions of the theory. In general, there are two possible processes, those where all of the external states contribute to one single extended interaction (connected diagrams); and those where two or more subsets of the external states are involved in simultaneous but independent interactions. The n-point Green's functions are the vacuum expectation values of time-ordered products of fields:

$$G^{(n)}(x_1, x_2, \dots, x_n) = \langle 0 | T(\phi(x_1), \phi(x_2), \dots, \phi(x_n)) | 0 \rangle \quad . \quad (2.7)$$

Differentiation of Eq. (2.6) with respect to the source J brings down a factor of $i\phi$ from the exponent, giving :

$$\frac{1}{Z} \frac{\delta Z[J]}{i\delta J(x)} = \phi(x) \quad . \quad (2.8)$$

Thus we can obtain these Green's functions from the generating functional by func-

tional differentiation:

$$G^{(n)}(x_1, x_2, \dots, x_n) = (-i)^n \frac{\delta^{(n)}}{\delta J(x_1)\delta J(x_2)\dots\delta J(x_n)} Z[J] \Big|_{J=0} . \quad (2.9)$$

So far the Green's functions $G^{(n)}$ include contributions from both connected and disconnected vacuum to vacuum diagrams. However, the interactions of a field theory are described by the connected Green's functions only. As we shall see, we can always write $G^{(n)}$ in the form of sums and products of connected Green's functions $G_{conn}^{(n)}$. It is therefore desirable to introduce a new functional $W[J]$, defined by

$$Z[J] = \exp W[J] \quad (2.10)$$

which generates only the connected Green's functions. Thus connected Green's functions are given by:

$$G_{conn}^{(n)}(x_1, x_2, \dots, x_n) = (-i)^n \frac{\delta^{(n)}}{\delta J(x_1)\delta J(x_2)\dots\delta J(x_n)} W[J] \Big|_{J=0} . \quad (2.11)$$

Taking repeated derivatives of $W[J]$ one can find the following relationship between Z and W :

$$\frac{\delta^2 W}{\delta J(x_1)\delta J(x_2)} = \frac{1}{Z^2} \frac{\delta Z}{\delta J(x_1)} \frac{\delta Z}{\delta J(x_2)} - \frac{1}{Z} \frac{\delta^2 Z}{\delta J(x_1)\delta J(x_2)} .$$

Taking $J = 0$ this gives:

$$G_{conn}^{(2)}(x_1, x_2) = G^{(2)}(x_1, x_2)$$

as expected since the propagator is a two-point connected diagram.

For the four point Green's function, we find

$$G_{conn}^{(4)}(x_1, x_2, x_3, x_4) = \left\{ G^{(2)}(x_1, x_2)G^{(2)}(x_3, x_4) + \text{permutations} \right\} - G^{(4)}(x_1, x_2, x_3, x_4)$$

which is shown diagrammatically in Fig. (2.1).

We should note that there is a more fundamental subset of connected Green's functions, the proper vertices $\Gamma^{(n)}$ (or one particle irreducible Green's functions). These are

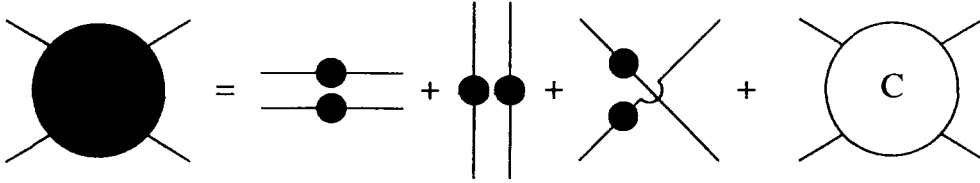


Figure 2.1: Relationship between the full and connected Green's functions.

A long line represents a full propagator, a circle a full Green's function and a C inside a circle denotes a connected Green's function.

diagrams which cannot be split into two parts by cutting a single propagator. Hence one can construct any connected graph from just the propagator $G_{conn}^{(2)}$ and the proper vertices $\Gamma^{(n)}(x_1, x_2, \dots, x_n)$. We would like to find the generating functional for these $\Gamma^{(n)}$, which is often called the effective action $\Gamma[\hat{\phi}]$. First we define $\hat{\phi}$, the *classical field*, by

$$\hat{\phi} = \frac{\delta W[J]}{\delta J(x)}, \quad (2.12)$$

where $\hat{\phi}$ can be interpreted as the expectation value of the quantum field ϕ in the presence of a source J .

The effective action $\Gamma[\hat{\phi}]$ is then the functional Legendre transform of $W[J]$

$$i\Gamma[\hat{\phi}] = W[J] - i \int d^4x J(x)\hat{\phi}(x) \quad (2.13)$$

and we have:

$$\frac{\delta \Gamma[\hat{\phi}]}{\delta \hat{\phi}} = -J(x) \quad (2.14)$$

We obtain the proper vertices from the effective action $\Gamma[\hat{\phi}]$ by functional differentiation

$$\Gamma_{conn}^{(n)}(x_1, x_2, \dots, x_n) = \frac{\delta^{(n)} \Gamma[\hat{\phi}]}{\delta \hat{\phi}(x_1) \delta \hat{\phi}(x_2) \dots \delta \hat{\phi}(x_n)} \Bigg|_{J=0} \quad (2.15)$$

With

$$G_{conn}^{(2)}(x, y) = -\frac{\delta^2 W}{\delta J(x) \delta J(y)} = -\frac{\delta \hat{\phi}(x)}{\delta J(y)},$$

$$\Gamma^{(2)}(x, y) = \frac{\delta^2 \Gamma}{\delta \hat{\phi}(x) \delta \hat{\phi}(y)} = -\frac{\delta J(x)}{\delta \hat{\phi}(y)}$$

which can be obtained by differentiating Eq. (2.12) and Eq. (2.14) with respect to J and $\hat{\phi}$ appropriately. This gives the following functional identity:

$$\int d^4 y G_{\text{conn}}^{(2)}(x, y) \Gamma^{(2)}(y, z) = \int d^4 y \frac{\delta \hat{\phi}(x)}{\delta J(y)} \frac{\delta J(y)}{\delta \hat{\phi}(z)} = \delta^{(4)}(x - z) \quad (2.16)$$

and $\Gamma^{(2)}(x, y)$ is the inverse of the full propagator $G^{(2)}$.

Taking the derivative of Eq. (2.16) with respect to J again, we obtain the following relationship between $W[J]$ and $\Gamma[\hat{\phi}]$:

$$\begin{aligned} \int d^4 y \frac{\delta^3 W}{\delta J(x) \delta J(u) \delta J(y)} \frac{\delta^2 \Gamma}{\delta \hat{\phi}(y) \delta \hat{\phi}(z)} = \\ \int d^4 y \frac{\delta^2 W}{\delta J(x) \delta J(y)} \int d^4 y' G_{\text{conn}}^{(2)}(u, y') \frac{\delta^3 \Gamma}{\delta \hat{\phi}(y) \delta \hat{\phi}(z) \delta \hat{\phi}(y')} \quad , \end{aligned} \quad (2.17)$$

where we have used the fact that:

$$\frac{\delta}{\delta J(u)} = \int d^4 y' \frac{\delta \hat{\phi}(y')}{\delta J(u)} \frac{\delta}{\delta \hat{\phi}(y')} = - \int d^4 y' G_{\text{conn}}^{(2)}(u, y') \frac{\delta}{\delta \hat{\phi}(y')} \quad .$$

Simplifying Eq. (2.17) further, we find:

$$\begin{aligned} \frac{\delta^3 W}{\delta J(x) \delta J(y) \delta J(z)} = \\ - \int d^4 x' d^4 y' d^4 z' G_{\text{conn}}^{(2)}(x, x') G_{\text{conn}}^{(2)}(y, y') G_{\text{conn}}^{(2)}(z, z') \frac{\delta^3 \Gamma}{\delta \hat{\phi}(x') \delta \hat{\phi}(y') \delta \hat{\phi}(z')} \quad , \end{aligned} \quad (2.18)$$

where we see that the diagram for the connected Green's functions can be built from a tree-structure of propagators $G_{\text{conn}}^{(2)}$ and proper vertices $\Gamma^{(n)}$. Graphically this is shown in Fig. (2.2).

Fermions and Path Integral Formalism

Thus far we have only considered scalar fields. Now we want to include fermions, which have Green's functions that are antisymmetric in their indices and therefore require us

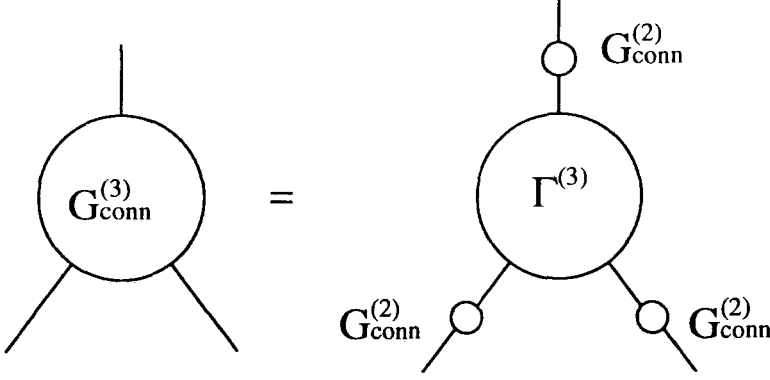


Figure 2.2: The relationship between G_{conn} and Γ for the 3-point Green's function.

to perform the functional integral over anticommuting fields. This can be done by using Grassmann (anticommuting) variables for the fermion fields $\bar{\psi}$, ψ and their sources $\bar{\eta}$, η .

A fermion field theory is then defined by the generating functional:

$$Z[\eta, \bar{\eta}] = N^{-1} \int [d\psi, d\bar{\psi}] \exp \left\{ i \int d^4x \left[L(\psi, \bar{\psi}) + \bar{\eta}_a(x)\psi_a(x) + \bar{\psi}_a(x)\eta_a(x) \right] \right\}, \quad (2.19)$$

where again N is a normalisation factor ensuring that $Z[0, 0] = 1$.

Differentiating the above with respect to η brings down a factor $-i\bar{\psi}$ from the exponent, whereas differentiation with respect to $\bar{\eta}$ gives a factor of $i\psi$. Now we can generate the n -point Green's functions from the generating functional in the same way as before. In general we have:

$$\begin{aligned} G^{(n)}(x_1, \dots, x_n, y_1, \dots, y_n) &= \langle 0 | T(\psi_a(x_1), \dots, \psi_a(x_n) \psi_b(y_1), \dots, \psi_b(y_n)) | 0 \rangle \\ &= (-i)^{2n} \frac{\delta^{(2n)}}{\delta\eta_b(y_n) \dots \delta\eta_b(y_1) \delta\bar{\eta}_a(x_n) \dots \delta\bar{\eta}_a(x_1)} Z[\eta, \bar{\eta}] \Big|_{\eta=\bar{\eta}=0} \end{aligned} \quad (2.20)$$

The generating functionals for both connected Green's functions, $W[\eta, \bar{\eta}]$, and proper vertices, $\Gamma[\bar{\psi}, \psi]$, can be introduced as before. Here, because of the anticommuting nature of the fields, we have as the parallel of Eq. (2.12) and (2.14):

$$\begin{aligned}\psi(x) &= \frac{\delta W}{i\delta\bar{\eta}(x)}, & \bar{\psi}(x) &= \frac{\delta W}{-i\delta\eta(x)}, \\ \eta(x) &= -\frac{\delta\Gamma}{\delta\bar{\psi}(x)}, & \bar{\eta}(x) &= \frac{\delta\Gamma}{\delta\psi(x)}.\end{aligned}\tag{2.21}$$

2.2.3 The Generating Functional of QCD

Here we introduce the QCD generating functional with a source J_μ for the gluon field, anticommuting sources $\bar{\eta}$ and η for the quark-antiquark fields and anticommuting $\bar{\epsilon}$ and ϵ sources for the ghost fields.

The QCD generating functional is defined as:

$$Z[\bar{\eta}, \eta, J_\mu, \bar{\epsilon}, \epsilon] = \int [d\bar{\phi}, d\phi, dA, d\bar{\omega}, d\omega] e^{i\sigma},\tag{2.22}$$

where σ is:

$$\sigma = S_{QCD} + \int d^4x \left\{ \bar{\psi}^f \eta^f + \bar{\eta}^f \psi^f + A_{\mu a} J_a^\mu + \bar{\omega}_a \epsilon_a + \bar{\epsilon}_a \omega_a \right\}$$

and S_{QCD} is the gauge-fixed action for QCD, given by:

$$S_{QCD}[\bar{\psi}^f, \psi^f, \bar{\omega}, \omega, g_0, m_f] = \int d^4x L_{QCD}[\bar{\psi}^f, \psi^f, \bar{\omega}, \omega, g_0, m_f].$$

A normalisation factor N ensuring that $Z[0] = 1$ is understood.

2.3 Asymptotic Freedom

As mentioned earlier, to a high momentum probe in DIS, quarks appear as freely moving, non-interacting particles, with a coupling which is effectively small. This property of QCD is called asymptotic freedom and it is this, which explains why perturbative QCD can be used successfully to describe high-energy, large momentum-transfer cross-sections.

Let us consider the effective quark-gluon vertex, which we calculate perturbatively. While to lowest order this is just the bare coupling of L_{QCD} , to calculate higher orders we have to include loop corrections (see Fig.(2.3)).

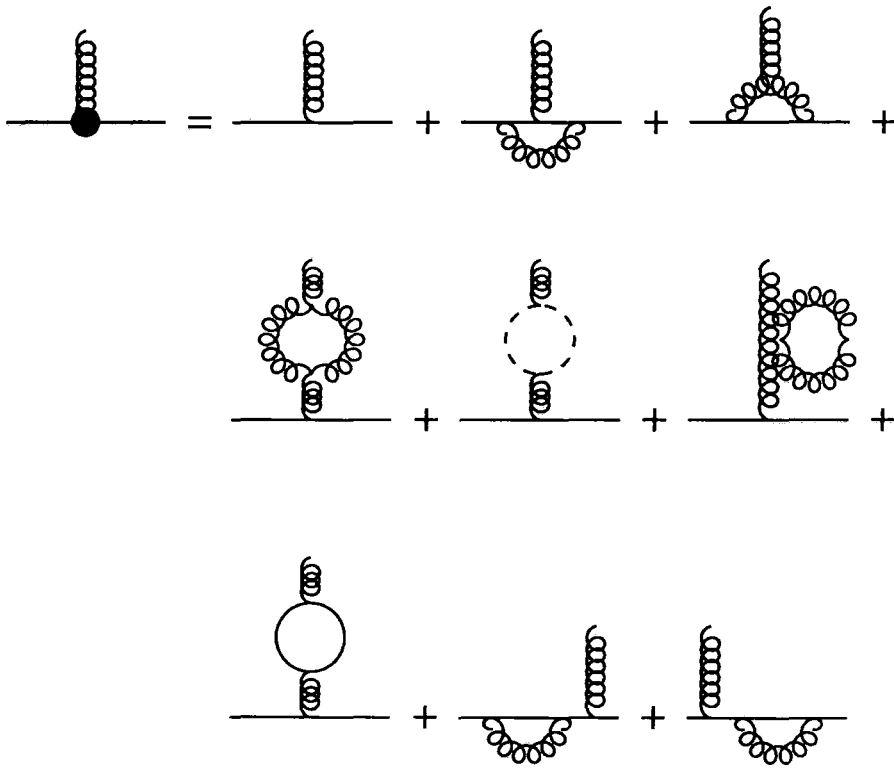


Figure 2.3: The effective quark-gluon coupling \bar{g} to one loop in perturbation theory. The dashed line represents the ghosts.

Each of these loops involves one totally unconstrained momentum, which has to be integrated over. These integrals are divergent because of the behaviour of the integrand at high virtual momenta (ultraviolet divergences). In order to give a meaning to the integrals, one introduces an ultraviolet cut-off κ^2 . Keeping leading logarithms only, the effective quark-gluon coupling, $\bar{g}(Q^2)$, becomes:

$$\bar{g}(Q^2) = g_0 - \beta_0 \frac{g_0^3}{32\pi^2} \ln\left(\frac{Q^2}{\kappa^2}\right) + O\left(g_0^5 \ln^2\left(\frac{Q^2}{\kappa^2}\right)\right) \quad , \quad (2.23)$$

where g_0 is the bare coupling which appears in L_{QCD} Eq. (2.1),

Q^2 is the incoming gluon momentum and

κ^2 is the ultraviolet cutoff introduced to make the loop-integrals finite.

Explicit calculation shows

$$\beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f \quad , \quad (2.24)$$

where N_c is the number of colours and

n_f is the number of quark flavours.

The coupling diverges as the ultraviolet cutoff $\kappa^2 \rightarrow \infty$ and hence Eq. (2.23) has no physical meaning. We need to renormalise $\bar{g}(Q^2)$, i.e. make it independent of κ^2 . This can be done by defining its value at some momentum scale $Q^2 = \mu^2$. We then find:

$$\bar{g}(Q^2) = \bar{g}(\mu^2) - \frac{\beta_0}{32\pi^2} \bar{g}^3(\mu^2) \ln\left(\frac{Q^2}{\mu^2}\right) + O\left(\bar{g}^5(\mu^2) \ln^2\left(\frac{Q^2}{\mu^2}\right)\right) \quad . \quad (2.25)$$

The renormalisation group equation, which is based on the fundamental observation [12] that a physical quantity cannot depend upon our (arbitrary) choice of μ^2 , the renormalisation scale, gives the evolution of \bar{g} under a change of μ^2 :

$$\mu \frac{\partial g(\mu)}{\partial \mu} = \beta(g(\mu)) = -\frac{\beta_0}{16\pi^2} \bar{g}^3 + O(\bar{g}^5) \quad .$$

This differential equation can be solved to give:

$$\alpha_S(Q^2) = \frac{\alpha_S(\mu^2)}{1 + \beta_0/(4\pi)\alpha_S(\mu^2) \ln(Q^2/\mu^2)} \quad , \quad (2.26)$$

where we have defined $\alpha_S(Q^2) \equiv \bar{g}^2(Q^2)/4\pi$.

That $\alpha_S(Q^2)$ is now independent of μ^2 can be seen by taking the inverse of the above:

$$\frac{1}{\alpha_S(Q^2)} - \frac{\beta_0}{4\pi} \ln(Q^2) = \frac{1}{\alpha_S(\mu^2)} - \frac{\beta_0}{4\pi} \ln(\mu^2) = C \quad ,$$

where C is a constant independent of Q^2 and μ^2 , which we define to be:

$$C = -\frac{\beta_0}{4\pi} \ln(\Lambda_{QCD}^2) \quad .$$

Thus the running coupling constant becomes:

$$\alpha_S(Q^2) = \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda_{QCD}^2)} \quad . \quad (2.27)$$

Λ_{QCD} is the momentum scale characteristic of the theory which has to be determined from experiment.

It is important to notice here that at large momenta, $Q^2 \gg \Lambda_{QCD}^2$ the running coupling constant α_S becomes smaller and smaller, so that well above Λ_{QCD}^2 perturbation theory gives a reliable approximation to QCD. Correspondingly α_S increases as Q^2 becomes smaller so that, although Eq. (2.27) will not be valid, nevertheless, it is plausible to suppose that α_S may become very strong for sufficiently small Q^2 , confining quarks permanently into hadrons. However, without going beyond perturbation theory, we cannot know the behaviour of the coupling at small momenta at all. The momentum dependence of α_S is sketched in Fig. (2.4).

To understand the physical meaning of the momentum dependence of the coupling let us contrast the effects of *charge screening* in both QED and QCD. In the case of QED, we know that at large distances, the effective coupling constant α gets smaller. This is because any charged particle is surrounded by a cloud of electron-positron virtual pairs which tend to screen the charge of the particle. At smaller distances, and at higher energies, a probe can penetrate through this virtual cloud, and hence the QED coupling constant gets larger at short distances.

As we have seen in QCD the situation is precisely the opposite. Having colour instead of electric charge means that, at large distances the presence of a cloud of virtual particles creates an antiscreening effect. Note that it is the quantity β_0 , Eq. (2.24), which

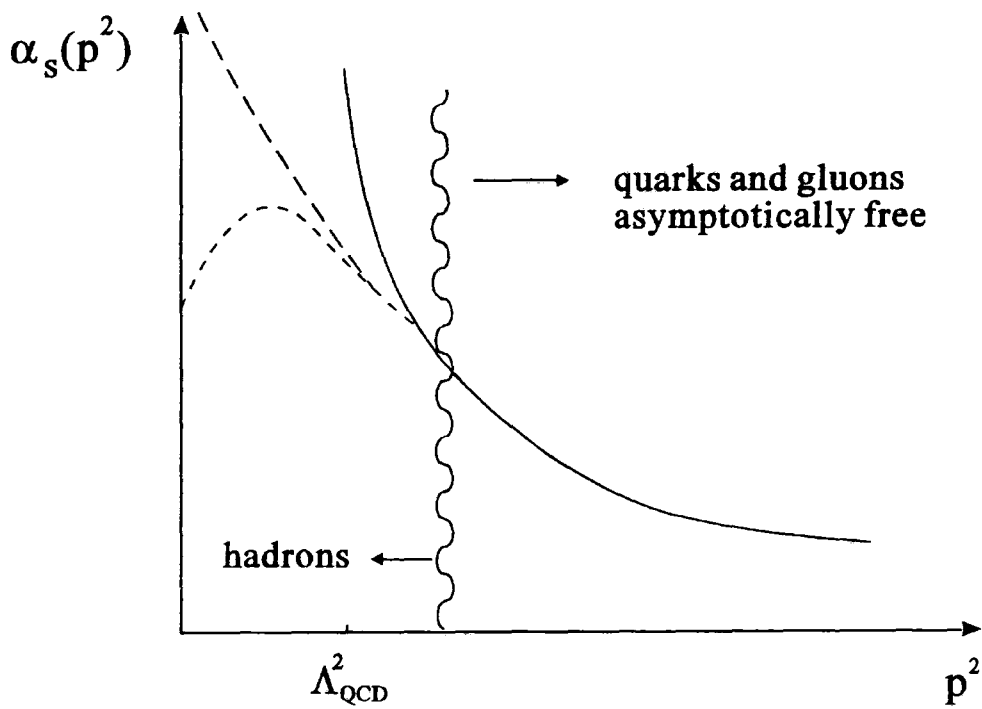


Figure 2.4: The running coupling α_s . Here the solid line represents the perturbative result, and the dashed lines indicate that we do not know for certain what the behaviour of the coupling is at low momenta where perturbation theory is inapplicable.

determines the running of the coupling. The term proportional to n_f comes about from the fermion loop contribution to the effective quark-gluon vertex and has qualitatively the same effect as in QED in that it tends to enhance the coupling constant at short distances (screening). However, the contribution due to the gluon self-coupling (proportional to N_C) is of the opposite sign and tends to decrease the strength of the interaction at short distances (antiscreening). It is this which makes QCD crucially different from QED.

To conclude this short discussion we stress again that asymptotic freedom implies that at high energies perturbative QCD is theoretically consistent. Higher order calculations have been performed for many processes, and good agreement has been found with experiment. For a recent review of perturbative QCD see e.g. Ref. [13] and references therein.

2.4 Confinement

The flip side of asymptotic freedom is that at smaller and smaller energies, the coupling constant becomes increasingly large. This implies that the quarks bind more tightly together, giving rise to confinement. This property of the strong interaction is well known empirically, for quarks and gluons have not been observed as free particles in nature, however some fundamental questions are still open:

- How does confinement happen within QCD?
- How can hadronic properties be understood in terms of QCD?

To answer these questions it is necessary to formulate a non-perturbative framework in which we can study the low energy properties of QCD. In particular we have to understand the behaviour of the basic quantities of the theory, e.g. propagators and couplings, not only in the high energy region, where perturbation theory is reliable, but also in the low energy region. This thesis will discuss one non-perturbative framework which permits

such a study: the Schwinger-Dyson equation approach to QCD, which we will introduce in chapter 3.

In the following we first give a brief qualitative discussion of quark confinement. Then we introduce a more formal criterion for confinement which has been derived in the context of lattice gauge theory. Finally, we discuss what we can learn about confinement by studying the behaviour of the gluon propagator, which will be of particular importance for the rest of this thesis.

2.4.1 Intuitive Picture of Confinement

One can obtain an intuitive idea about the nature of confinement by picturing quarks as being bound by *strings*, as first proposed by Nambu in 1974 [14], or tubes of colour flux. Let us imagine the quark and antiquark inside a meson to be held together by a string. At short distances, i.e. distances much smaller than the size of a meson, the string is slack and the quarks move as if they are free. The potential between them is just the well-known Coulombic one:

$$V_{short}(r) \propto \frac{1}{r} \quad ,$$

where r is the separation between the quarks. However, if we imagine trying to move the quarks apart from each other, then as the separation between them becomes bigger, the string gets stretched. Hence the total energy of the quark-antiquark system is linearly proportional to the distance. This means quarks are confined by a linearly rising long-distance potential:

$$V_{long}(r) \propto Kr \quad ,$$

where K is a constant, which is often referred to as the string tension.

To achieve this *string model* of confinement it has been suggested that the QCD vacuum is analogous to the ground state of a superconductor [15], so that the properties of a quark in the physical vacuum are analogous to a magnetic monopole in a superconducting medium. To illustrate this picture, let us imagine placing a magnetic monopole-antimonopole pair ($m\bar{m}$) inside a superconductor. Due to the Meissner effect, which tries

to eliminate magnetic fields from the superconductor, the field caused by the $m\bar{m}$ -pair is channeled into a thin flux tube extending from the monopole to the antimonopole. If we try to separate the $m\bar{m}$ -pair, we find that the energy required is proportional to the distance between them. A single monopole in a superconducting medium has a flux tube, or *string* running from it to the boundary of the superconductor where the magnetic field can escape. If the medium fills all space the energy of the monopole is infinite. Hence we can say that monopoles are confined and cannot exist as single particles. This picture has been further investigated in the context of monopole condensation [16] and the formalism of dual QCD [17].

In Fig. (2.5) we depict the colour force lines between a quark and an antiquark. Imagining the QCD vacuum to be a chromomagnetic superconductor, the colour (electric) flux is confined to a string-like configuration joining the quark-antiquark pair.

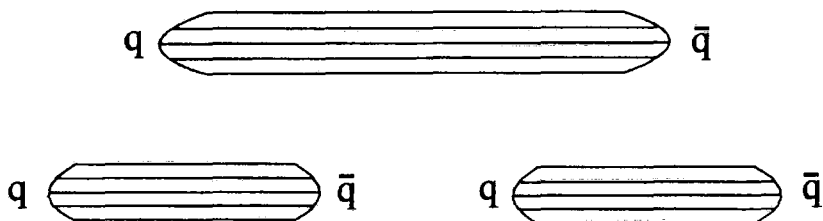


Figure 2.5: Lines of force between a quark and an antiquark. When the quarks are separated, the string breaks producing a further quark- antiquark pair. (Figure taken from Cheng and Li [18])

It is worth mentioning here that adding the two contributions to the interquark potential, V_{short} and V_{long} from the string or flux tube model, we can reproduce heavy quark spectra rather well. A non-relativistic quantum mechanical approach to the spectroscopy of heavy quark flavours, where bound states of heavy quarks (quarkonium) are described in analogue to the e^+e^- -system (positronium), gives detailed information about the form of the quark-antiquark interaction potential. The simplest potential model consistent

with experimental data is the so-called funnel-potential, given by:

$$V(r) = -\frac{4}{3} \frac{\alpha_S}{r} + ar$$

which is exactly of the form $V_{short} + V_{long}$, we have described. For a review of QCD potential models see e.g. Ref. [19].

2.4.2 Wilson Criterion for Confinement

In 1974, Wilson [20] proposed that QCD be defined on a Euclidean hypercubical lattice of space-time points in order to calculate effects that lie beyond the reach of perturbation theory, such as the confinement of quarks. Once the gauge theory of QCD is formulated on a finite, four-dimensional lattice, one can, in principle, calculate the basic properties of the low-energy, strong interaction spectrum numerically. Rough qualitative agreement between the theory and experimental data have been already obtained. The only apparent limitation facing lattice gauge theory is the available computer power.

In this section, however, we shall not go into the technical details of lattice gauge theory, instead we shall only be interested in setting up the Wilson loop operator, because this gives us a criterion for confinement.

Let us start by defining the link operator between two neighbouring sites of the lattice by:

$$w(y, x; C) = P_C \exp \left\{ ig \int_x^y A(z) dz \right\} \quad . \quad (2.28)$$

This is the integral of the gauge field along a curve C connecting two lattice sites x and y , where P_C indicates that the exponential is to be path ordered along C .

A gauge transformation of this link operator is given by:

$$w(y, x; C) \rightarrow U^{-1}(y)w(y, x; C)U(x) \quad .$$

The Wilson loop is a gauge invariant operator built with this string operator Eq. (2.28)

$$W(C) = \text{tr} [w(x, x; C)] = P_C \text{tr} \left[\exp \left\{ ig \oint_C A_\mu dx^\mu \right\} \right] \quad , \quad (2.29)$$

where the trace is taken over colour indices and where we take the integral of the gauge field around a closed loop C in space time.

Now let us consider a heavy quark-antiquark pair to be taken around a rectangular loop with width R in one spatial direction and length T in the time direction, in the limit of large T (as in Fig. (2.6)).

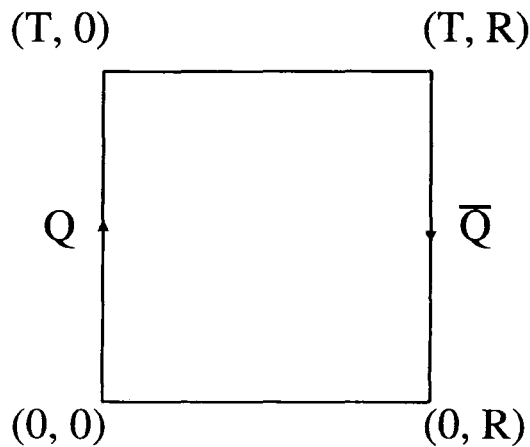


Figure 2.6: Quarks taken around the Wilson loop

One can show that the Wilson loop operator is related to the interquark potential, $V(r)$, by:

$$W(C) \sim \exp \{-TV(r)\} \quad . \quad (2.30)$$

For a derivation of this equation see e.g. Ref. [21].

As we have argued before, confinement implies that the interquark potential grows without bound

$$V(r) \rightarrow \infty \quad \text{as } r \rightarrow \infty$$

and we shall assume linear growth here. So with $V(r) = Kr$, we find:

$$W(C) \sim \exp \{-KTR\} \sim \exp \{-KA_C\} \quad , \quad (2.31)$$

where A_C is the area enclosed by a rectangular loop C .

This is the Wilson criterion. An *area law* behaviour of the Wilson loop gives confinement. We should note here, that in Eq. (2.31) the quarks are treated merely as external colour sources and do not appear in the condition on $W(C)$. Thus using the Wilson loop it is possible to study confinement in a pure gluon theory.

In the strong coupling approximation of lattice gauge theory one can demonstrate that the Wilson loop does obey an area law and hence quarks are confined in lattice gauge theory (see Ref. [18]). However the continuum limit of the theory is not well defined, preventing us from rigorously proving quark confinement for QCD.

We shall come back to *area-law behaviour* of the Wilson loop in the next section, where we will discuss what we can learn about confinement by studying the behaviour of the gluon propagator in the low-momentum (infrared) region.

2.4.3 Confinement and the Gluon Propagator

The gluon propagator, $\Delta_{\mu\nu}(p^2)$, is gauge dependent and as such is not experimentally observable. However its infrared behaviour has important implications for quark confinement: (i) gluons control the interquark dynamics and we expect that their propagators have a peculiar low-energy behaviour so that gluons confine quarks by having very strong long range interactions, (ii) gluons themselves are confined particles and we expect this to be apparent from their propagator.

In this section, we shall discuss two different infrared behaviours of the gluon propagator and their implications for confinement. These are:

- an infrared enhanced gluon, as singular as $1/p^4$ for $p^2 \rightarrow 0$
- an infrared softened gluon, not having a mass pole in the propagator

After introducing the Schwinger-Dyson approach to QCD in chapter 3 we will come back to these different gluon propagators and investigate whether it is possible to derive them from non-perturbative QCD.

a.) $1/p^4$ -Behaviour of the Gluon

One obtains an intuitive feeling for the meaning of a gluon propagator as singular as $1/p^4$ for small p^2 by considering the *one-gluon exchange* contribution to the interquark potential, i.e. neglecting multi-gluon exchanges, which due to the self-coupling of the gluons are present in QCD. It is well known that the one-gluon exchange contribution to the potential is related to the 3-dimensional Fourier transform of the time-time component of the gauge boson propagator $\Delta_{00}(p^2)$ by:

$$V(r) = -C_F g^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Delta_{00}(k^2) e^{-i\mathbf{k}\cdot\mathbf{r}} \quad , \quad (2.32)$$

where \mathbf{k} and \mathbf{r} are three-vectors in momentum and configuration space respectively.

Therefore the long distance behaviour of the potential is determined by the behaviour of the propagator near $k = 0$. A gluon proportional to k^{-a} leads to an interquark potential that behaves like

$$V(r) \propto r^{a-3}$$

and we see that a linearly rising, confining interquark potential comes about from a $1/p^4$ infrared behaviour of the gluon propagator.

Most importantly, West [22] derived a relation between the Wilson loop operator, Eq. (2.29), and the full gluon propagator. He proved that

$$W(C) \leq \exp \left\{ -\frac{g^2}{2} \oint dx^\mu \oint dy^\nu \Delta_{\mu\nu}^{ab}(x, y) \delta_{ab} \right\} \quad .$$

He showed that if in any gauge the gluon propagator is as singular as $1/p^4$ the Wilson loop operator falls exponentially to zero with the damping factor proportional to the area of the loop

$$W(C) \leq \exp \{-A_C\} \quad .$$

This area law behaviour is, as we have discussed in the previous section, a signal for confinement.

West's result is so important because it relates the gauge dependent propagator to a gauge invariant quantity, which itself is related to the interquark potential. Thus, if we

could rigorously prove that the full gluon propagator in QCD indeed has a $1/p^4$ infrared behaviour in any gauge, we would be able to demonstrate quark confinement.

Furthermore it should be noted that a gluon propagator which is as singular as $1/p^4$ when $p \rightarrow 0$, does not only confine quarks, but can also be shown to correspond to gluons being confined themselves.

Based on Lorentz invariance, causality and the assumption that there exist stable single particle states one can derive the Källén-Lehmann spectral representation of the propagator [23]:

$$\Delta(p^2) = \int d\mu^2 \frac{\rho(\mu^2)}{p^2 - \mu^2} \quad ,$$

where $\Delta(p^2)$ is the scalar coefficient of the propagator, $\Delta_{\mu\nu}$, and $\rho(\mu^2)$ is the spectral weight function, defined by:

$$\rho(\mu^2) = (2\pi)^3 \sum_n \delta^{(4)}(\mu - p_n) |\langle 0 | \phi(0) | n \rangle|^2 \quad ,$$

where $|n\rangle$ are eigenstates of the energy – momentum operators P^μ and the energy of all the intermediate states $p_n^0 > 0$. If we assume that the field ϕ creates a single particle state of mass m as well as creating multiparticle states, we can separate out the one particle state contribution to the spectral weight function $\rho(\mu^2)$. This is proportional to $\delta^{(4)}(\mu^2 - m^2)$. Thus the propagator has a single pole at the physical mass of the particle and a more complicated structure for momenta beyond the threshold energy of multi-particle production.

One can show that any physical asymptotic state must have this Källén-Lehmann spectral representation of the propagator. However, the propagator of a confined particle does not have a spectral representation. Analytic structures of the gluon propagator which do not allow this spectral representation correspond to confined particles and it has been shown that the infrared enhanced propagator is one such structure.

We conclude this section by stressing once more, that a $1/p^4$ behaviour of the full gluon propagator in QCD would demonstrate confinement. Gluons confine quarks by having very strong long range interactions and are themselves confined by not having a Källén-Lehmann representation that any physical asymptotic state must have.

In contrast, it is sometimes argued that $\Delta(p^2)$ must be less singular than $1/p^2$ to ensure that gluons themselves do not propagate over large distances and we shall briefly discuss this in the following section.

b.) Absence of a Mass Pole in the Propagator

Another sufficient condition for confinement is that there should be only colour singlet on-shell states. Then a propagator, which does not have any *mass* singularity on the real positive p^2 axis will effectively confine the particle, in the sense that it can never be on mass-shell, and thus never be observed as a real, asymptotic particle. Here again the analytic structure of the propagator, i.e. the absence of a mass singularity, corresponds to the absence of a Källen-Lehmann representation. For a discussion of this see [24].

Chapter 3

Schwinger-Dyson Equation Approach to QCD

3.1 What are the Schwinger-Dyson Equations?

The Schwinger-Dyson equations (SDEs) [25] are coupled integral equations, which interrelate the Green's functions of a field theory. It is well known that a field theory is completely defined when all of its Green's functions are known and hence solving these integral equations provides us with a solution of the theory. Unfortunately, the SDE's are impossible to solve exactly since they build an infinite tower of coupled, non-linear integral equations, one for each Green's function. In general, for a non-Abelian field theory, the $(n + 1)$ and the $(n + 2)$ point function enter the equation for the n -point function and we can write the full hierarchy of equations symbolically as

$$\begin{aligned} \Gamma_2 &= F[\Gamma_2, \Gamma_3, \Gamma_4] \\ &\cdot \\ &\cdot \\ \Gamma_n &= F[\Gamma_2, \dots, \Gamma_n, \Gamma_{n+1}, \Gamma_{n+2}] \quad , \end{aligned} \tag{3.1}$$

where F stands for the relevant combination of Green's functions Γ .

Thus truncations are unavoidable in any study based on SDE's. This means that the

tower of equations must be limited to some number m , where m is the maximum number of legs on any Green's function included in the self-consistent solution of the equation. An ansatz must then be made for the omitted n -point functions. An approximation is required, which incorporates the main properties of the theory, including the various global and local symmetries, and the known perturbative behaviour in the weak coupling limit. Once the SDE's have been truncated in a self-consistent way, they provide us with a non-perturbative approximation to the field theory.

It is important to note that the SDEs are non-perturbative in nature, where *non-perturbative* means more than just a resummation of all orders in perturbation theory. Since the SDEs are the field equations of the quantum field theory, they contain all its dynamics. So any inherently non-perturbative effects in the theory, not accessible to a perturbative expansion, are included in the SDEs. One example of such an effect is fermion mass generation through dynamical symmetry breaking. We know that in perturbation theory the corrected fermion mass is proportional to the bare mass, appearing in the Lagrangian, and hence a theory which is originally massless remains so at each order in perturbation theory. However, dynamical mass generation has been shown to happen [26] provided the coupling is larger than some critical value, and this can be studied in the continuum using the SDEs.

In this thesis, we study the gluon propagator and we derive its SDE in detail in section 3.2. This derivation is somewhat mathematical, but is included because of the importance of the SDE to the research of this thesis. As we shall see explicitly, the gluon SDE involves not only the full gluon propagator, but the full triple and quartic gluon interactions too, as well as the propagator and coupling for the quarks and the ghosts. These in turn satisfy equations which involve yet higher point functions and so on. In order to make a study of the gluon propagator tractable simplifying assumptions are clearly necessary. We are going to describe the approximations used and discuss their justification in chapter 4. Here we shall just point out that using the Slavnov-Taylor identities [27] of the theory we can truncate the SDE in a natural hierarchical way, as was first done by Baker, Ball and Zachariasen [28]. The Slavnov-Taylor identities are a property of gauge theories and, for our example of the gluon propagator, allow the triple

gluon vertex to be constrained in terms of the gluon propagator. This allows us to model the behaviour of the propagator without having to solve an infinity of equations.

We first derive the gluon SDE using the path integral formalism which we have introduced earlier and then turn to the relevant Slavnov-Taylor identities we will need for the truncation of this equation.

3.2 Derivation of the Schwinger-Dyson Equations

In this section we derive the Schwinger-Dyson equation of the gluon propagator of QCD in a covariant gauge, i.e. in an environment where ghosts are present. We start from the QCD generating functional, which we introduced in section 2.2.2, Eq. (2.22). Consider the functional derivative of the QCD generating functional with respect to the gluon field. Since the particle fields vanish at infinity the integral of the derivative must vanish too.

$$\int [d\bar{\psi}, d\psi, dA, d\bar{\omega}, d\omega] \frac{\delta}{\delta A_\rho^b(y)} e^{i\sigma} = \int [d\bar{\psi}, d\psi, dA, d\bar{\omega}, d\omega] \left[\frac{\delta S}{\delta A_\rho^b(y)} + J_\mu(x) \right] e^{i\sigma} = 0 .$$

The expression in square brackets can be taken outside the integral by replacing every occurrence of a field by a derivative with respect to its source (see Eq. (2.8)).

$$\left\{ \frac{\delta S_{QCD}}{\delta A_\rho^b(y)} \left(-\frac{\delta}{i\delta\eta}, \frac{\delta}{i\delta\bar{\eta}}, \frac{\delta}{i\delta J_\mu}, -\frac{\delta}{i\delta\epsilon}, \frac{\delta}{i\delta\bar{\epsilon}} \right) + J_\rho(x) \right\} Z[\bar{\eta}, \eta, J_\mu, \bar{\epsilon}, \epsilon] = 0 \quad , \quad (3.2)$$

where

$$\begin{aligned} \frac{\delta S_{QCD}}{\delta A_\rho^b(y)} [\bar{\psi}^f, \psi^f, A_\mu, \bar{\omega}, \omega] &= \bar{\psi}^f g_0 \gamma_\rho \psi^f T^b + [\square g_{\mu\rho} - (1 - \xi) \partial_\rho \partial_\mu] A_b^\mu + g_0 f_{abc} A_c^\mu (\partial_\mu A_\rho^a) \\ &\quad - g_0 f_{abc} A_c^\mu (\partial_\rho A_\mu^a) + g_0 f_{abc} \partial^\mu (A_\mu^a A_\rho^c) + g_0^2 f_{acd} f^{aeb} A_\mu^c A_\rho^d A_e^\mu \\ &\quad + (\partial_\rho \bar{\omega}_a) g_0 f_{abc} \omega_c \quad , \end{aligned}$$

so that Eq. (3.2) becomes:

$$\left\{ \begin{aligned} & [\square g_{\mu\rho} - (1 - \xi)\partial_\rho\partial_\mu] \frac{\delta}{i\delta J_\mu^b} - g_0 \frac{\delta}{i\delta\eta} \gamma_\rho \frac{\delta}{i\delta\bar{\eta}} T^b + g_0 f_{abc} \frac{\delta}{i\delta J_\mu^c} (\partial_\mu \frac{\delta}{i\delta J_a^\rho}) \\ & - g_0 f_{abc} \frac{\delta}{i\delta J_\mu^c} (\partial_\rho \frac{\delta}{i\delta J_a^\mu}) + g_0 f_{abc} \partial^\mu (\frac{\delta}{i\delta J_a^\mu} \frac{\delta}{i\delta J_c^\rho}) + g_0^2 f_{acd} f^{aeb} \frac{\delta}{i\delta J^{\mu d}} \frac{\delta}{i\delta J^{\rho e}} \frac{\delta}{i\delta J_{\mu c}} \\ & - g_0 f_{abc} (\partial_\rho \frac{\delta}{i\delta \epsilon^a}) \frac{\delta}{i\delta \bar{\epsilon}^c} + J_\rho(x) \end{aligned} \right\} Z[\bar{\eta}, \eta, J_\mu, \bar{\epsilon}, \epsilon] = 0 \quad . \quad (3.3)$$

Eq. (3.3) is in principle the SDE of the gluon propagator. However, as previously explained, Z is the generating functional for disconnected Green's functions as well as connected ones. In order to relate physical quantities we must express Eq. (3.3) in terms of the generating functional for connected Green's functions only, W . This is defined by,

$$Z[\bar{\eta}, \eta, J_\mu, \bar{\epsilon}, \epsilon] = e^{W[\bar{\eta}, \eta, J_\mu, \bar{\epsilon}, \epsilon]} \quad .$$

Some useful relations are,

- $$\frac{\delta}{i\delta J_\mu} e^{W[\bar{\eta}, \eta, J_\mu, \bar{\epsilon}, \epsilon]} = \frac{\delta W}{i\delta J_\mu} e^W \quad ,$$
- $$\frac{\delta}{i\delta\eta} \gamma_\mu \frac{\delta}{i\delta\bar{\eta}} e^W = \frac{\delta}{i\delta\eta} \left[\gamma_\mu \frac{\delta W}{i\delta\bar{\eta}} e^W \right] = \frac{\delta}{i\delta\eta} \left(\gamma_\mu \frac{\delta W}{i\delta\bar{\eta}} \right) e^W + \frac{\delta W}{i\delta\eta} \gamma_\mu \frac{\delta W}{i\delta\bar{\eta}} e^W \quad ,$$
- $$\frac{\delta}{i\delta J_\mu} \partial_\mu \frac{\delta}{i\delta J_\rho} e^W = \frac{\delta}{i\delta J_\mu} \left(\partial_\mu \frac{\delta W}{i\delta J_\rho} \right) e^W + \frac{\delta W}{i\delta J_\mu} \partial_\mu \frac{\delta W}{i\delta J_\rho} e^W \quad ,$$
- $$\partial_\mu \frac{\delta}{i\delta J_\mu} \frac{\delta}{i\delta J_\rho} e^W = \partial_\mu \left[\frac{\delta}{i\delta J_\mu} \left(\frac{\delta W}{i\delta J_\rho} \right) e^W + \frac{\delta W}{i\delta J_\mu} \frac{\delta W}{i\delta J_\rho} e^W \right] \quad ,$$
- $$\begin{aligned} \frac{\delta}{i\delta J_\mu} \frac{\delta}{i\delta J_\rho} \frac{\delta}{i\delta J_\mu} e^W &= \frac{\delta}{i\delta J_\mu} \left[\frac{\delta}{i\delta J_\rho} \left(\frac{\delta W}{i\delta J_\mu} \right) e^W + \frac{\delta W}{i\delta J_\rho} \frac{\delta W}{i\delta J_\mu} e^W \right] \\ &= \frac{\delta}{i\delta J_\mu} \left(\frac{\delta}{i\delta J_\rho} \left(\frac{\delta W}{i\delta J_\mu} \right) \right) e^W + \frac{\delta W}{i\delta J_\mu} \frac{\delta}{i\delta J_\rho} \left(\frac{\delta W}{i\delta J_\mu} \right) e^W \\ &\quad + \frac{\delta}{i\delta J_\mu} \left(\frac{\delta W}{i\delta J_\rho} \frac{\delta W}{i\delta J_\mu} \right) e^W + \frac{\delta W}{i\delta J_\mu} \frac{\delta W}{i\delta J_\rho} \frac{\delta W}{i\delta J_\mu} e^W \quad , \end{aligned}$$

$$\left(\partial_\rho \frac{\delta}{i\delta\epsilon} \right) \frac{\delta}{i\delta\bar{\epsilon}} e^W = \partial_\rho \frac{\delta}{i\delta\epsilon} \left[\frac{\delta W}{i\delta\bar{\epsilon}} e^W \right] = \partial_\rho \frac{\delta}{i\delta\epsilon} \left(\frac{\delta W}{i\delta\bar{\epsilon}} \right) e^W + \partial_\rho \frac{\delta W}{i\delta\epsilon} \frac{\delta W}{i\delta\bar{\epsilon}} e^W ,$$

where the above space-time derivatives act on everything to the right.

Using the above results and dividing by Z , we can rewrite Eq. (3.3) in terms of W , to give:

$$\begin{aligned} & J_\rho(x) + [\square g_{\mu\rho} - (1 - \xi)\partial_\rho\partial_\mu] \frac{\delta W}{i\delta J_\mu^\rho} - g_0 \frac{\delta}{i\delta\eta} \left(\gamma_\rho \frac{\delta W}{i\delta\bar{\eta}} \right) T^b - g_0 \frac{\delta W}{i\delta\eta} \gamma_\rho \frac{\delta W}{i\delta\bar{\eta}} T^b \\ & + g_0 f_{abc} \left\{ \frac{\delta}{i\delta J_\mu^c} \left(\partial_\mu \frac{\delta W}{i\delta J_a^\rho} \right) + \frac{\delta W}{i\delta J_\mu^c} \partial_\mu \frac{\delta W}{i\delta J_a^\rho} - \frac{\delta}{i\delta J_\mu^c} \left(\partial_\rho \frac{\delta W}{i\delta J_a^\mu} \right) - \frac{\delta W}{i\delta J_\mu^c} \partial_\rho \frac{\delta W}{i\delta J_a^\mu} \right. \\ & \quad \left. + \partial^\mu \frac{\delta}{i\delta J_a^\mu} \left(\frac{\delta W}{i\delta J_c^\rho} \right) + \partial^\mu \frac{\delta W}{i\delta J_a^\mu} \frac{\delta W}{i\delta J_c^\rho} \right\} \\ & + g_0^2 f_{acd} f^{aeb} \left\{ \frac{\delta}{i\delta J^{\mu d}} \left(\frac{\delta}{i\delta J^{\rho e}} \left(\frac{\delta W}{i\delta J_{\mu c}} \right) \right) + \frac{\delta W}{i\delta J^{\mu d}} \frac{\delta}{i\delta J^{\rho e}} \left(\frac{\delta W}{i\delta J_{\mu c}} \right) \right. \\ & \quad \left. + \frac{\delta}{i\delta J^{\mu d}} \left(\frac{\delta W}{i\delta J^{\rho e}} \frac{\delta W}{i\delta J_{\mu c}} \right) + \frac{\delta W}{i\delta J^{\rho e}} \frac{\delta W}{i\delta J^{\rho e}} \frac{\delta W}{i\delta J_{\mu c}} \right\} \\ & - g_0 f_{abc} \left\{ \partial_\rho \frac{\delta}{i\delta\epsilon^a} \left(\frac{\delta W}{i\delta\bar{\epsilon}^c} \right) + \partial_\rho \frac{\delta W}{i\delta\epsilon^a} \frac{\delta W}{i\delta\bar{\epsilon}^c} \right\} = 0 \quad . \end{aligned} \quad (3.4)$$

We want relations between the connected proper vertices of the theory. These are given by derivatives of the effective action with respect to the fields in the limit of vanishing sources (Eq. (2.15)). Thus we now perform a Legendre transformation and express our equation in terms of the effective action:

$$W[\bar{\eta}, \eta, J_\mu, \bar{\epsilon}, \epsilon] = i\Gamma(\bar{\psi}, \psi, A_\mu, \bar{\omega}, \omega) + i \int d^4x [\bar{\psi}\eta + \bar{\eta}\psi + A_\mu J^\mu + \bar{\omega}\epsilon + \bar{\epsilon}\omega] \quad ,$$

satisfying:

$$\begin{aligned} A_\mu(x) &= \frac{\delta W}{i\delta J^\mu(x)} , & \psi(x) &= \frac{\delta W}{i\delta\bar{\eta}(x)} , & \bar{\psi}(x) &= -\frac{\delta W}{i\delta\eta(x)} , \\ \omega(x) &= \frac{\delta W}{i\delta\bar{\epsilon}(x)} , & \bar{\omega}(x) &= -\frac{\delta W}{i\delta\epsilon(x)} , \\ J_\mu(x) &= -\frac{\delta\Gamma}{i\delta A_\mu(x)} , & \eta(x) &= -\frac{\delta\Gamma}{i\delta\bar{\psi}(x)} , & \bar{\eta}(x) &= \frac{\delta\Gamma}{i\delta\psi(x)} , \\ \epsilon(x) &= -\frac{\delta\Gamma}{i\delta\bar{\omega}(x)} , & \bar{\epsilon}(x) &= \frac{\delta\Gamma}{i\delta\omega(x)} . \end{aligned}$$

Inserting this into Eq. (3.4) gives:

$$\begin{aligned}
\frac{\delta\Gamma}{\delta A_b^\mu(x)} &= [\square g_{\mu\rho} - (1 - \xi)\partial_\rho\partial_\mu] A_b^\mu + g_0\bar{\psi}\gamma_\rho\psi T^b - g_0\frac{\delta}{i\delta\eta}\left(\gamma_\rho\frac{\delta W}{i\delta\bar{\eta}}\right) T^b \\
&+ g_0 f_{abc} \left\{ \partial_\mu \frac{\delta^2 W}{\delta J_\mu^c \delta J_a^\rho} + A_c^\mu \partial_\mu A_\rho^a + \partial_\rho \frac{\delta^2 W}{\delta J_\mu^c \delta J_a^\mu} - A_c^\mu \partial_\rho A_\mu^a \right. \\
&\quad \left. - \partial^\mu \frac{\delta^2 W}{\delta J_a^\mu \delta J_c^\rho} + \partial^\mu A_\mu^a A_\rho^c \right\} \\
&+ g_0^2 f_{abc} f_{ade} \left\{ i \frac{\delta^3 W}{\delta J_d^\mu \delta J_e^\rho \delta J_\mu^c} - A_\mu^d \frac{\delta^2 W}{\delta J_e^\rho \delta J_\mu^c} - \frac{\delta^2 W}{\delta J_d^\mu \delta J_e^\rho} A_c^\mu \right. \\
&\quad \left. - A_\rho^e \frac{\delta^2 W}{\delta J_d^\mu \delta J_\mu^c} + A_\mu^d A_\rho^e A_c^\mu \right\} \\
&+ g_0 f_{abc} \left\{ \bar{\omega}_a \omega_c - \partial_\rho \frac{\delta}{i\delta\epsilon^a} \left(\frac{\delta W}{i\delta\bar{\epsilon}^c} \right) \right\} .
\end{aligned} \tag{3.5}$$

Finally we must take the limit of vanishing sources.

Consider the fermion term

$$\frac{\delta}{i\delta\eta_\alpha(x)} \left[(\gamma_\mu)_{\alpha\beta} \frac{\delta W}{i\delta\bar{\eta}_\beta(x)} \right] ,$$

where α, β are spinor indices.

We have,

$$\begin{aligned}
(\gamma_\mu)_{\alpha\beta} \frac{\delta^2 W}{\delta\eta_\alpha(x)\delta\bar{\eta}_\beta(x)} &= (\gamma_\mu)_{\alpha\beta} \int d^4 y \frac{\delta^2 W}{\delta\eta_\alpha(x)\delta\bar{\eta}_\beta(y)} \delta(x-y) \\
&= \text{tr} \left(\gamma_\mu \int d^4 y \frac{\delta^2 W}{\delta\eta(x)\delta\bar{\eta}(y)} \delta(x-y) \right) .
\end{aligned}$$

Furthermore,

$$-\delta_{\alpha\beta} \delta^4(x-y) = \frac{1}{i} \int d^4 z \frac{\delta^2 W}{\delta\eta_\alpha(x)\delta\bar{\eta}_\gamma(z)} \frac{\delta^2 \Gamma}{\delta\psi_\gamma(z)\delta\bar{\psi}_\beta(y)} \Big|_{\eta=\bar{\eta}=\psi=\bar{\psi}=0} .$$

Thus in the limit of vanishing fermion sources,

$$-g_0 \frac{\delta}{i\delta\eta} \left(\gamma_\rho \frac{\delta W}{i\delta\bar{\eta}} \right) T^b \rightarrow -ig_0 T^b \text{tr} \left(\gamma_\rho \left(\frac{\delta^2 \Gamma}{\delta\bar{\psi}\delta\psi} \right)^{-1} \right) .$$

Equivalently for the ghost term in the limit of vanishing ghost sources,

$$-g_0 f_{abc} \partial_\rho \frac{\delta}{i \delta \epsilon^a} \left(\frac{\delta W}{i \delta \bar{\epsilon}^c} \right) \rightarrow -i g_0 f_{abc} \text{tr} \left(\partial_\rho \left(\frac{\delta^2 \Gamma}{\delta \bar{\omega} \delta \omega} \right)^{-1} \right) .$$

Thus Eq. (3.5) becomes:

$$\begin{aligned} \frac{\delta \Gamma}{\delta A_b^\rho(x)} \Big|_{\eta=\bar{\eta}=\psi=\bar{\psi}=0} &= [\square g_{\mu\rho} - (1-\xi) \partial_\rho \partial_\mu] A_b^\mu - i g_0 T^b \text{tr} \left(\gamma_\rho \left(\frac{\delta^2 \Gamma}{\delta \bar{\psi} \delta \psi} \right)^{-1} \right) \\ &\quad - i g_0 f_{abc} \text{tr} \left(\partial_\rho \left(\frac{\delta^2 \Gamma}{\delta \bar{\omega} \delta \omega} \right)^{-1} \right) \\ &\quad + g_0 f_{abc} \left\{ \partial_\mu \frac{\delta^2 W}{\delta J_\mu^c \delta J_a^\rho} + A_c^\mu \partial_\mu A_\rho^a + \partial_\rho \frac{\delta^2 W}{\delta J_\mu^c \delta J_a^\mu} - A_c^\mu \partial_\rho A_\mu^a \right. \\ &\quad \quad \left. - \partial^\mu \frac{\delta^2 W}{\delta J_a^\mu \delta J_c^\rho} + \partial^\mu A_\mu^a A_\rho^c \right\} \\ &\quad + g_0^2 f_{abc} f_{ade} \left\{ i \frac{\delta^3 W}{\delta J_d^\mu \delta J_e^\rho \delta J_\mu^c} - A_\mu^d \frac{\delta^2 W}{\delta J_e^\rho \delta J_\mu^c} - \frac{\delta^2 W}{\delta J_d^\mu \delta J_e^\rho} A_c^\mu \right. \\ &\quad \quad \left. - A_\rho^e \frac{\delta^2 W}{\delta J_d^\mu \delta J_\mu^c} + A_\mu^d A_\rho^e A_c^\mu \right\} . \end{aligned} \quad (3.6)$$

Now take the derivative of the above with respect to $A_f^\nu(y)$ and set $A = 0$. We will look at the terms separately.

$$\begin{aligned} \frac{\delta}{\delta A_f^\nu(y)} [\square g_{\mu\rho} - (1-\xi) \partial_\rho \partial_\mu] A_b^\mu(x) &= [\square g_{\mu\rho} - (1-\xi) \partial_\rho \partial_\mu] \delta^4(x-y) \delta_{bf} g_\nu^\mu \\ &= [\Delta_{\nu\rho}^{(0)}(x,y)]^{-1} , \end{aligned} \quad (3.7)$$

where $\Delta_{\nu\rho}^{(0)}(x,y)$ is the bare gluon propagator.

$$\begin{aligned} \frac{\delta}{\delta A_f^\nu(y)} \left[-i g_0 T^b \text{tr} \left(\gamma_\rho \left(\frac{\delta^2 \Gamma}{\delta \bar{\psi}(x) \delta \psi(x)} \right)^{-1} \right) \right] \\ = i g_0^2 \int d^4 z_1 d^4 z_2 \text{tr} \left[T^b \gamma_\rho S(x, z_1) \Lambda_\nu^f(x, z_1, z_2) S(z_2, x) \right] . \end{aligned} \quad (3.8)$$

Here we have used,

$$\begin{aligned} & \frac{\delta}{\delta A_f^\nu(y)} \left(\frac{\delta^2 \Gamma}{\delta \bar{\psi}(x) \delta \psi(x)} \right)^{-1} \\ &= - \int d^4 z_1 d^4 z_2 \left(\frac{\delta^2 \Gamma}{\delta \bar{\psi}(x) \delta \psi(z_1)} \right)^{-1} \frac{\delta}{\delta A_f^\nu(y)} \frac{\delta^2 \Gamma}{\delta \bar{\psi}(z_1) \delta \psi(z_2)} \left(\frac{\delta^2 \Gamma}{\delta \bar{\psi}(z_2) \delta \psi(x)} \right)^{-1} \\ &= - \int d^4 z_1 d^4 z_2 S(x, z_1) g_0 \Lambda_\nu^f(x, z_1, z_2) S(z_2, x) \quad , \end{aligned}$$

where $S(x, z)$ is the full fermion propagator,

$\Lambda_\nu^f(x, z_1, z_2)$ is the full fermion gluon vertex function and

$$\Lambda_\nu^{f(0)}(x, z_1, z_2) = T^f \gamma_\nu \delta(x - z_1) \delta(z_1 - z_2) \quad .$$

Equivalently:

$$\begin{aligned} & \frac{\delta}{\delta A_f^\nu(y)} \left[-i g_0 f_{abc} \text{tr} \left(\partial_\rho \left(\frac{\delta^2 \Gamma}{\delta \bar{\omega}(x) \delta \omega(x)} \right)^{-1} \right) \right] \\ &= i g_0^2 \int d^4 z_1 d^4 z_2 \text{tr} \left[f_{abc} \partial_\rho B(x, z_1) \Lambda_\nu^{acf}(x, z_1, z_2) B(z_2, x) \right] \quad , \quad (3.9) \end{aligned}$$

where $B(x, z)$ is the full ghost propagator,

$\Lambda_\nu^{acf}(x, z_1, z_2)$ is the full ghost gluon vertex function and

$$\Lambda_\nu^{acf(0)}(x, z_1, z_2) = f^{acf} \partial_\mu \delta(x - z_1) \delta(z_1 - z_2) \quad .$$

Now we rewrite the gluon term in Eq. (3.6) introducing the bare three and four gluon vertices:

$$\begin{aligned} & g_0 f_{abc} \left\{ \partial_\mu \frac{\delta^2 W}{\delta J_\mu^c \delta J_a^\rho} + A_c^\mu \partial_\mu A_\rho^a + \partial_\rho \frac{\delta^2 W}{\delta J_\mu^c \delta J_a^\mu} - A_c^\mu \partial_\rho A_\mu^a - \partial^\mu \frac{\delta^2 W}{\delta J_a^\mu \delta J_c^\rho} + \partial^\mu A_\mu^a A_\rho^c \right\} \\ &= \int d^4 x_1 d^4 x_2 \frac{g_0}{2!} \Gamma_{\mu\sigma\rho}^{abc(0)}(x, x_1, x_2) \left(\frac{\delta^2 W}{\delta J_\mu^c(x_1) \delta J_\sigma^a(x_2)} + A_c^\mu(x_1) A_a^\sigma(x_2) \right) \quad , \end{aligned}$$

where

$$\begin{aligned} \Gamma_{\mu\sigma\rho}^{abc(0)}(x, x_1, x_2) &= f_{abc} \left\{ \delta_{\mu\sigma} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x_1} \right)_\rho \delta(x - x_2) \delta(x_1 - x_2) \right. \\ &\quad \delta_{\sigma\rho} \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right)_\mu \delta(x_1 - x) \delta(x_2 - x) \\ &\quad \left. \delta_{\mu\rho} \left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x} \right)_\sigma \delta(x_2 - x_1) \delta(x - x_1) \right\} \quad . \end{aligned}$$

Similarly,

$$\begin{aligned}
& g_0^2 f_{abc} f_{ade} \left\{ i \frac{\delta^3 W}{\delta J_d^\mu \delta J_e^\rho \delta J_\mu^c} - A_\mu^d \frac{\delta^2 W}{\delta J_e^\rho \delta J_\mu^c} - \frac{\delta^2 W}{\delta J_d^\mu \delta J_e^\rho} A_c^\mu - A_\rho^e \frac{\delta^2 W}{\delta J_d^\mu \delta J_\mu^c} + A_\mu^d A_\rho^e A_c^\mu \right\} \\
&= \int d^4 x_1 d^4 x_2 d^4 x_3 \frac{g_0^2}{3!} \Gamma_{\mu\sigma\tau\rho}^{bcde(0)}(x, x_1, x_2, x_3) \left(\frac{\delta^3 W}{\delta J_\mu^c(x_1) \delta J_\sigma^d(x_2) \delta J_\tau^e(x_3)} \right. \\
&\quad \left. + 3 \frac{\delta^2 W}{\delta J_\mu^c(x_1) \delta J_\sigma^d(x_2)} A_\tau^e(x_3) + A_c^\mu(x_1) A_d^\sigma(x_2) A_e^\tau(x_3) \right) ,
\end{aligned}$$

where

$$\begin{aligned}
\Gamma_{\mu\sigma\tau\rho}^{bcde(0)}(x, x_1, x_2, x_3) &= \left(f^{abc} f^{ade} [\delta_{\mu\tau} \delta_{\sigma\rho} - \delta_{\sigma\tau} \delta_{\mu\rho}] \right. \\
&\quad + f^{acd} f^{aeb} [\delta_{\mu\tau} \delta_{\sigma\rho} - \delta_{\tau\rho} \delta_{\mu\sigma}] \\
&\quad \left. + f^{ace} f^{adb} [\delta_{\sigma\tau} \delta_{\mu\rho} - \delta_{\mu\sigma} \delta_{\tau\rho}] \right) \delta(x - x_1) \delta(x_2 - x_3) \delta(x_1 - x_2) .
\end{aligned}$$

Using the functional identity:

$$\frac{\delta}{\delta A_f^\nu(y)} = \int d^4 y' \frac{\delta J_{\nu'}^{f'}(y')}{\delta A_f^\nu(y)} \frac{\delta}{\delta J_{\nu'}^{f'}(y')} = \int d^4 y' \frac{-\delta^2 \Gamma}{\delta A_{f'}^{\nu'}(y') \delta A_f^\nu(y)} \frac{\delta}{\delta J_{\nu'}^{f'}(y')}$$

and only keeping the gluon terms which will remain when we set $A = 0$ after we have taken the derivative with respect to $A_f^\nu(y)$, we find for the triple gluon term:

$$\begin{aligned}
& \frac{\delta}{\delta A_f^\nu(y)} \left[\frac{g_0}{2!} \Gamma_{\mu\sigma\rho}^{abc(0)}(x, x_1, x_2) \frac{\delta^2 W}{\delta J_\mu^c(x_1) \delta J_\sigma^a(x_2)} \right] \\
&= \frac{g_0}{2!} \int d^4 y' \Gamma_{\mu\sigma\rho}^{abc(0)}(x, x_1, x_2) \frac{\delta^3 W}{\delta J_{\nu'}^{f'}(y') \delta J_\mu^c(x_1) \delta J_\sigma^a(x_2)} \frac{-\delta^2 \Gamma}{\delta A_{f'}^{\nu'}(y') \delta A_f^\nu(y)} \\
&= \frac{g_0}{2!} \int d^4 x'_1 d^4 x'_2 \Gamma_{\mu\sigma\rho}^{abc(0)}(x, x_1, x_2) \Delta^{\mu\mu'}(x_1 - x'_1) \Delta^{\sigma\sigma'}(x_2 - x'_2) \Gamma_{\nu'\mu'\sigma'}^{fca}(x'_1, x'_2, y) \quad (3.10)
\end{aligned}$$

where $\Gamma_{\nu'\mu'\sigma'}^{fca}(x'_1, x'_2, y)$ is the full three gluon vertex function

and where we have used

$$\frac{\delta^3 W}{\delta J(x) \delta J(y) \delta J(z)} = - \int d^4 x' d^4 y' d^4 z' \Delta(x - x') \Delta(y - y') \Delta(z - z') \frac{\delta^3 \Gamma}{\delta A(x') \delta A(y') \delta A(z')} .$$

Similarly we obtain for the quartic gluon term:

$$\begin{aligned}
& \frac{\delta}{\delta A_f^\nu(y)} \left[\frac{g_0^2}{3!} \Gamma_{\mu\sigma\tau\rho}^{bcde(0)}(x, x_1, x_2, x_3) \left(\frac{\delta^3 W}{\delta J_\mu^c(x_1) \delta J_\sigma^d(x_2) \delta J_\tau^e(x_3)} + 3 \frac{\delta^2 W}{\delta J_\mu^c(x_1) \delta J_\sigma^d(x_2)} A^{\tau e}(x_3) \right) \right] \\
&= \frac{g_0^2}{3!} \int d^4 y' \Gamma_{\mu\sigma\tau\rho}^{bcde(0)}(x, x_1, x_2, x_3) \frac{\delta^4 W}{\delta J_{\nu'}^{f'}(y') \delta J_\mu^c(x_1) \delta J_\sigma^d(x_2) \delta J_\tau^e(x_3)} \frac{-\delta^2 \Gamma}{\delta A_{f'}^{\nu'}(y') \delta A_f^\nu(y)} \\
&+ \frac{g_0^2}{2!} \int d^4 y' \Gamma_{\mu\sigma\tau\rho}^{bcde(0)}(x, x_1, x_2, x_3) \frac{\delta^2 W}{\delta J_\mu^c(x_1) \delta J_\sigma^d(x_2)} \delta^{\tau\nu} \delta(x_3 - y) \delta_{ef} \\
&= \frac{g_0^2}{3!} \int d^4 x'_1 d^4 x'_2 d^4 x'_3 \Gamma_{\mu\sigma\tau\rho}^{bcde(0)}(x, x_1, x_2, x_3) \Delta^{\mu\mu'}(x_1 - x'_1) \Delta^{\sigma\sigma'}(x_2 - x'_2) \Delta^{\tau\tau'}(x_3 - x'_3) \\
&\quad \Gamma_{\nu\mu'\sigma'\tau'}^{fcae}(x'_1, x'_2, x'_3, y) \\
&+ \frac{g_0^2}{2!} \int d^4 x'_1 d^4 x'_2 d^4 x'_3 d^4 z d^4 z' \Gamma_{\mu\sigma\tau\rho}^{bcde(0)}(x, x_1, x_2, x_3) \Delta^{\mu\mu'}(x_1 - x'_1) \Delta^{\sigma\sigma'}(x_2 - x'_2) \\
&\quad \Delta^{\tau\tau'}(x_3 - x'_3) \Delta^{\alpha\alpha'}(z - z') \Gamma_{\alpha\nu\mu'}^{gfc}(z, y, x'_1) \Gamma_{\alpha'\sigma'\tau'}^{gae}(z', x'_2, x'_3) \\
&+ \frac{g_0^2}{2!} \Gamma_{\mu\sigma\tau\rho}^{bcde(0)}(x, x_1, x_2, x_3) \Delta_{cd}^{\mu\sigma}(x_1 - x_2) \quad , \tag{3.11}
\end{aligned}$$

where $\Gamma_{\nu\mu'\sigma'\tau'}^{fcae}(x'_1, x'_2, x'_3, y)$ is the full four gluon vertex function

and where we have used

$$\begin{aligned}
& \frac{\delta^4 W}{\delta J(w) \delta J(x) \delta J(y) \delta J(z)} \\
&= \int d^4 w' d^4 x' d^4 y' d^4 z' \frac{\delta^4 \Gamma}{\delta A(w') \delta A(x') \delta A(y') \delta A(z')} \\
&\quad \Delta(w - w') \Delta(x - x') \Delta(y - y') \Delta(z - z') \\
&+ \int d^4 u d^4 v d^4 w' d^4 x' d^4 y' d^4 z' \frac{\delta^3 \Gamma}{\delta A(u) \delta A(w') \delta A(x')} \frac{\delta^3 \Gamma}{\delta A(v) \delta A(y') \delta A(z')} \\
&\quad \Delta(w - w') \Delta(y - y') \Delta(x - x') \Delta(z - z') \Delta(u - v) \quad + \quad \text{cyclic permutations} \quad .
\end{aligned}$$

On adding up the separate contributions, Eq. (3.8) - (3.11), we obtain the SDE for the inverse gluon propagator. This equation is displayed diagrammatically in Fig. (3.1).

The SDE's for the quark and ghost propagator can be derived in a similar way, by taking the functional derivative of Eq. (2.22) with respect to the quark and ghost fields respectively.

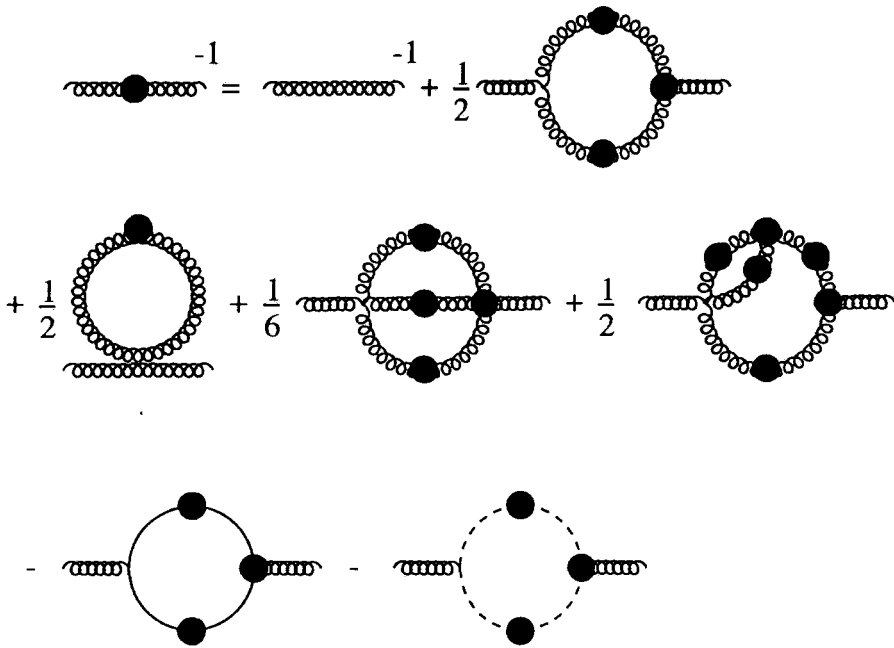


Figure 3.1: The Schwinger-Dyson equation for the gluon propagator. Here the solid line represents the quark propagator and the broken line the ghost propagator, the \bullet denote full quantities and stand for inclusion of all possible one particle irreducible diagrams.

Here we restrict ourselves to the study of quenched QCD, i.e. a world without quarks. This is reasonable since we expect the non-Abelian nature of QCD to be responsible for confinement.

3.3 Slavnov-Taylor Identities

A very important aspect of a gauge theory is that gauge invariance imposes relationships between Green's functions with different numbers of external legs. These relations, in general called Slavnov-Taylor identities, are exact, they have to be satisfied not only order by order in perturbative calculations but also, since they relate the full $(n + 1)$ point function to the full n -point functions, in any non-perturbative approach to QCD. Hence they can be used to truncate the SDEs. In the following we present the basic steps in the derivation of these Slavnov-Taylor identities, if one drops the fermion fields in the QCD Lagrangian, i.e. we are considering a pure gauge theory here.

The important thing to note first is that the gauge-fixed QCD Lagrangian, Eq. (2.1), is not invariant under a general gauge transformation, but instead it is invariant under the Becchi-Rouet-Stora (BRS) transformations. The variations of the different fields under these transformations are given below:

$$\begin{aligned}\delta A_a^\mu &= D_{ab}^\mu \omega^b \theta \quad , \\ \delta \bar{\omega}_a &= -\xi(\partial_\mu A_a^\mu) \theta \quad , \\ \delta \omega_a &= -1/2g f_{abc} \omega^b \omega^c \theta \quad ,\end{aligned}\tag{3.12}$$

where θ is an infinitesimal, constant real Grassmann number. This had to be introduced so that the BRS transformations do not alter the character of the fields, i.e. the transformed ω and $\bar{\omega}$ fields are still Grassmann variables.

It is now convenient to introduce two extra, new sources u_μ^a and v^a for the composite operators $D_{ab}^\mu \omega^b$ and $f_{abc} \omega^b \omega^c$ that appear in the BRS transformation. The generating functional becomes:

$$Z[J_\mu, \epsilon, \bar{\epsilon}, u_\mu, v] = \int [dA, d\omega, d\bar{\omega}] e^{i\sigma} \quad ,\tag{3.13}$$

where now σ is given by:

$$\sigma = S_{qQCD}(A, \omega, \bar{\omega}) + \int d^4x \left\{ A_{\mu a} J_a^\mu + \bar{\omega}_a \epsilon_a + \bar{c}_a \omega_a + u_\mu^a (D_{ab}^\mu \omega^a) + v^a \left(-1/2 f_{abc} \omega^b \omega^c \right) \right\}$$

and S_{qQCD} is the action for quenched QCD.

Next we perform a BRS (infinitesimal) transformation of the fields in Eq. (3.13). It can be shown that both the measure $[dA, d\omega, d\bar{\omega}]$ in the path integral formalism and the terms involving the newly introduced sources are invariant under such a transformation [11] and so, of course, is S_{qQCD} . The only terms which are not BRS invariant are the source terms for the gluon and ghost fields. However, the generating functional Z is BRS invariant. (Transforming the variables of integration does not change the value of the integral itself). Thus the contributions from the source terms must vanish. Performing the BRS transformations, Eq. (3.12), on the generating functional gives:

$$Z + \delta Z = \int [dA, d\omega, d\bar{\omega}] \exp \left\{ i\sigma + \int d^4x (J_a^\mu \delta A_{\mu a} + \delta \bar{\omega}_a \epsilon_a + \bar{c}_a \delta \omega_a) \right\} .$$

Expanding the exponential to first order in θ , we find:

$$Z + \delta Z = \int [dA, d\omega, d\bar{\omega}] e^{i\sigma} \left[1 + \int d^4x (J_a^\mu \delta A_{\mu a} + \delta \bar{\omega}_a \epsilon_a + \bar{c}_a \delta \omega_a) \right]$$

and hence:

$$0 = \int [dA, d\omega, d\bar{\omega}] e^{i\sigma} \int d^4x (J_a^\mu \delta A_{\mu a} + \delta \bar{\omega}_a \epsilon_a + \bar{c}_a \delta \omega_a) .$$

We can rewrite the above as a functional differential equation:

$$\theta \int d^4x \left\{ J_a^\mu(x) \frac{\delta}{\delta u_\mu^a(x)} + \bar{c}^a(x) \frac{\delta}{\delta v^a(x)} - \xi \epsilon(x) \left[\partial_\mu \frac{\delta}{\delta J_\mu^a(x)} \right] \right\} Z = 0 , \quad (3.14)$$

where we have used the following functional relations:

$$\begin{aligned} \frac{1}{Z} \frac{\delta Z [J_\mu, \epsilon, \bar{c}, u_\mu, v]}{i \delta u_\mu^a(x)} &= D_{ab}^\mu \omega^a = \frac{1}{\theta} \delta A_a^\mu , \\ \frac{1}{Z} \frac{\delta Z [J_\mu, \epsilon, \bar{c}, u_\mu, v]}{i \delta v(x)} &= -\frac{1}{2} f_{abc} \omega^b \omega^c = \frac{1}{\theta} \delta \omega_a . \end{aligned}$$

This equation is a Slavnov-Taylor identity. To derive identities for specific one particle irreducible Green's functions one proceeds exactly as we described in detail for the SDE. First one would perform the Legendre transform on Z to obtain the generating functional for the proper vertices. Then, by functionally differentiating the resulting expression with respect to the external sources and afterwards putting them to zero, one obtains the Green's functions one is interested in. We will not give the actual manipulations in obtaining specific Slavnov-Taylor identities here, they can be found in Ref. [27]. Instead we shall simply state some of the simplest identities.

First we introduce the Slavnov-Taylor identity for the full gluon propagator:

$$p_\mu \Pi_{ab}^{\mu\nu} = \frac{i}{\xi} p^\nu p^2 \delta_{ab} \quad , \quad (3.15)$$

where $\Pi^{\mu\nu}$ is the inverse gluon propagator.

This identity determines the tensor structure of the gluon and in a covariant gauge can be solved to give:

$$\Pi_{ab}^{\mu\nu}(p) = i\delta_{ab} \left[\frac{p^2}{G(p^2)} \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) + \frac{1}{\xi} p^\mu p^\nu \right]$$

or,

$$\Delta_{ab}^{\mu\nu}(p) = -i\delta_{ab} \left[\frac{G(p^2)}{p^2} \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) + \xi \frac{p^\mu p^\nu}{p^4} \right] \quad ,$$

where $G(p^2)$ is the gluon renormalisation function, containing the full non-perturbative content of the propagator, and is equal to 1 for the free gluon propagator.

We now separate the gluon propagator into a longitudinal and a transverse piece, where the transverse piece is defined to vanish when contracted with the external momentum, i.e.

$$\Delta^{\mu\nu}(p) = \Delta_T^{\mu\nu}(p) + \Delta_L^{\mu\nu}(p) \quad , \quad \text{where} \quad p_\mu \Delta_T^{\mu\nu}(p) = 0 \quad .$$

The Slavnov-Taylor identity, Eq. (3.15), thus only implies that the longitudinal part of the full gluon propagator, $\Delta_L^{\mu\nu}(p)$, is equal to that of the free propagator. The transverse piece, $\Delta_T^{\mu\nu}(p)$, is unconstrained by Eq. (3.15).

This last statement is in fact a general property of the Slavnov-Taylor identities. They only determine the longitudinal part of a $(n + 1)$ -point Green's function in terms of the n -point functions, their generic form being

$$p_1^\mu \Gamma_{(n+1)}^{\mu\nu\dots}(p_1, p_2, \dots) = F[\Gamma_{(n)}, \Gamma_{(n-1)}, \dots, \Gamma_2] \quad ,$$

where F stands for a combination of the lower Green's functions and where obviously the transverse part, as defined, is unconstrained.

If we now define the full ghost propagator to be

$$B_{ab}(q) = i \frac{H(q^2)}{q^2} \delta_{ab} \quad ,$$

and decompose the full ghost-gluon vertex, Λ^μ , as:

$$\Lambda_{abc}^\mu(p, q, r) = r^\nu \Lambda_{abc}^{\mu\nu}(p, q, r)$$

then the Slavnov-Taylor identity for the triple gluon vertex is:

$$q^\nu \Gamma_{\mu\nu\sigma}^{abc}(p, q, r) = H(q^2) \left\{ -\frac{1}{G(r^2)} (g^{\tau\sigma} r^2 - r^\tau r^\sigma) \Lambda_{abc}^{\tau\mu}(p, q, r) - \frac{1}{G(p^2)} (g^{\mu\tau} p^2 - p^\mu p^\tau) \Lambda_{abc}^{\tau\sigma}(r, q, p) \right\} . \quad (3.16)$$

Demanding that the longitudinal part of the vertex function should be free of kinematic singularities, Eq. (3.16) can be solved uniquely to determine the longitudinal part of the three-gluon vertex in terms of the ghost-gluon vertex and the ghost and gluon propagator, see Ref. [29].

It is worth stressing here that defining the transverse part of a Green's function as we demonstrated in Eq. (3.3) and demanding it to be free from kinematic singularities, the transverse piece Γ_T itself vanishes in the limit of external momenta becoming zero. This can be seen by taking the derivative of

$$p_i^{\mu_i} \Gamma_T^{\mu_1 \dots \mu_i \dots \mu_n}(p_1 \dots p_i \dots p_n) = 0$$

with respect to p_i^ν giving:

$$\Gamma_T^{\mu_1 \dots \nu \dots \mu_n} + p_i^{\mu_i} \frac{\partial}{\partial p_i^\nu} \Gamma_T^{\mu_1 \dots \mu_i \dots \mu_n} = 0 \quad .$$

If Γ_T is free of kinematic singularities, then, in the limit $p_i^{\mu_i} \rightarrow 0$, the second term vanishes and hence the transverse part of the vertex function vanishes when the external momenta approach zero. This result is very important for the SDE approach to QCD. It means that the longitudinal part of the vertex, which is determined by the Slavnov-Taylor identity, contains all the low-momentum (infrared) behaviour of the vertex. Hence truncating the infinite set of SDEs, Eq. (3.1), at some Γ_m , approximating Γ_{m+1} by its longitudinal part and setting $\Gamma_{m+2} = 0$, we get a closed set of equations. In non-perturbative studies, where we are interested in the infrared behaviour of the theory, this should be a valid approximation.

Chapter 4

Infrared Behaviour of the Gluon: An Analytical Calculation

In this chapter we study the possible infrared behaviour of the gluon propagator analytically, using the SDE. We concentrate on the two solutions proposed in the context of confinement (chapter 2.4.3):

- the infrared enhanced propagator, as singular as $1/p^4$ when $p^2 \rightarrow 0$
- the infrared softened propagator, less singular than $1/p^2$ when $p^2 \rightarrow 0$.

As discussed above, a gluon propagator which is as singular as $1/p^4$ when $p^2 \rightarrow 0$ indicates that the interquark potential rises linearly with separation and leads to an area law behaviour of the Wilson loop operator, often regarded as a signal for confinement. Gluons confine quarks by having strong, long range interactions and are themselves confined as they do not have a Källen-Lehmann spectral representation.

Alternatively a gluon that does not have a pole on the real, positive p^2 -axis describes a confined particle and has been suggested by Landshoff and Nachtmann [30] on purely phenomenological grounds. This is needed for their model of the pomeron in order to reproduce experiment.

Clearly, (Fig. (4.1)), the gluon propagator cannot both be more singular and less singular than $1/p^2$ as $p^2 \rightarrow 0$, but which is correct ?

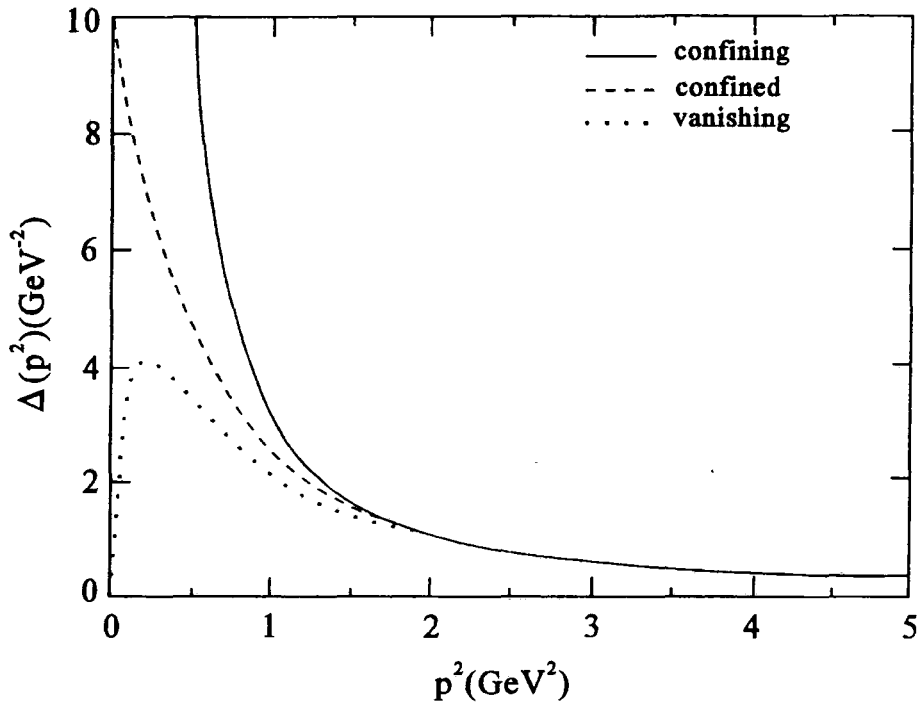


Figure 4.1: Possible behaviour of the gluon propagator $\Delta(p^2)$, which is the coefficient of the $g_{\mu\nu}$ or $\delta_{\mu\nu}$ component of $\Delta_{\mu\nu}(p)$.

- (a) *confining* gluon, $\Delta \sim (p^2)^{-2}$,
- (b) *confined* gluon, $\Delta \sim (p^2)^{-c}$ with c very small,
- (c) infrared vanishing gluon $\Delta \sim p^2$.

All are matched to the perturbative behaviour for p larger than a few GeV.

The SDEs provide the natural starting point for a non-perturbative investigation of this infrared behaviour of the gluon propagator. We shall derive a closed equation for the gluon renormalisation function, $G(p^2)$, following the procedure outlined in chapter 3 in both the axial and the Landau gauges. As mentioned before, in order to derive a closed equation for $G(p^2)$ and hence make a study of the infrared behaviour of the gluon possible, we must make approximations. We discuss in detail what these approximations are and how they are justified.

Extensive work has been previously performed in both the axial gauge [28], [31]–[33] and the Landau gauge [34]–[36]. (For a comprehensive review see Roberts and Williams [37].) A *confining* solution as singular as $1/p^4$ has been shown to exist in both gauges [28]–[32] and [34]–[36], whereas a *confined* solution for the gluon propagator, i.e. less singular than $1/p^2$, has only been claimed to exist in the axial gauge [33]. The purpose of the study presented in this chapter is to explore why these two different behaviours have been found. Fortunately, in studying just the infrared behaviour, there is no need to solve the SDE at all momenta. It is this that greatly simplifies our discussion and allows an analytic treatment.

In section 4.1 the axial gauge studies are reviewed and the possible, self-consistent solutions for the infrared behaviour of the gluon are reproduced analytically. We discuss the difficulties in justifying the approximations made in the axial gauge and then turn to covariant gauges and the Landau gauge in particular. We investigate the possibility of a gluon propagator less singular than $1/p^2$ when $p^2 \rightarrow 0$ in the Landau gauge in section 4.2. We find that this infrared softened, *confined* behaviour of the gluon propagator is inconsistent; only an infrared enhanced, *confining* gluon, as singular as $1/p^4$ when $p^2 \rightarrow 0$ is consistent with the truncated SDE. In section 4.3 we discuss the differing forms of the SDEs used in the axial and Landau gauge calculation to deduce these results. We then, in section 4.4, briefly review a third possible behaviour of the gluon, the infrared vanishing gluon (see Fig.(4.1)), proposed by Stingl et al. [38] in a completely different approach to solve the SDE. However, as we shall discuss, the infrared vanishing gluon propagator does not lead to quark confinement [39, 40] and hence the full gluon propagator in QCD cannot have this behaviour. An infrared behaviour of the gluon which is a consistent solution of

the SDE, and hence is possible in non-perturbative QCD, has implications not only for confinement, as we showed in chapter 2.4.3, but also for the modelling of the pomeron. This will be illustrated in section 4.5. In section 4.6 we state our conclusions.

4.1 Axial Gauge Calculation

In the axial gauge the gluon propagator is transverse to the gauge vector n_μ .

$$\text{Axial gauge formalism : } n_\mu A^{a\mu} = 0$$

Studies of the axial gauge SDE have the advantage that ghost fields are absent and thus, considering a pure gauge theory the gluon SDE relates the gluon propagator to the three- and four-gluon vertex functions only. This is displayed diagrammatically in Fig. (4.2).

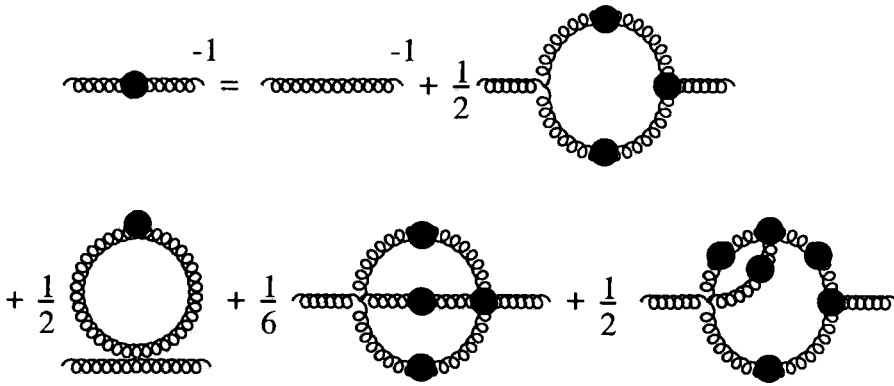


Figure 4.2: The Schwinger-Dyson equation for the gluon propagator in the axial gauge (neglecting quark loops).

Furthermore the four-gluon vertex terms, Fig. (4.2), may be projected out of the SDE. However the drawback of the axial gauge is that the gluon propagator depends not only on p^2 , but also on the unphysical gauge parameter γ , defined as

$$\gamma = \frac{(n \cdot p)^2}{n^2 p^2} .$$

Since the full gluon propagator, $\Delta_{\mu\nu}$, is transverse with respect to the axial gauge vector, i.e:

$$n^\mu \Delta_{\mu\nu} = 0 \quad , \quad (4.1)$$

the most general tensor structure for $\Delta_{\mu\nu}$ is:

$$\Delta_{\mu\nu}(p^2, \gamma) = -\frac{i}{p^2} \left[F(p^2, \gamma) M_{\mu\nu} + H(p^2, \gamma) N_{\mu\nu} \right] \quad , \quad (4.2)$$

with the tensors given by:

$$\begin{aligned} M_{\mu\nu} &= g_{\mu\nu} - \frac{p_\mu n_\nu + p_\nu n_\mu}{n \cdot p} + n^2 \frac{p_\mu p_\nu}{(n \cdot p)^2} \quad , \\ N_{\mu\nu} &= g_{\mu\nu} - \frac{n_\mu n_\nu}{n^2} \end{aligned}$$

and therefore the full axial gauge gluon propagator must depend on the two scalar functions, F and H . The full gluon vacuum polarisation $\Pi_{\lambda\mu}(p^2, \gamma)$ in the axial gauge is defined by:

$$\Pi^{\lambda\mu} \Delta_{\mu\nu} = g^\lambda_{\nu} - \frac{n^\lambda p_\nu}{n \cdot p} \quad ,$$

since contraction with n^ν must be zero from Eq. (4.1) and $\Pi^{\lambda\mu}$ has to be orthogonal to p_λ . Then its general form is:

$$\begin{aligned} \Pi^{\lambda\mu}(p^2, \gamma) &= \frac{ip^2}{F(p^2, \gamma) + H(p^2, \gamma) (n \cdot p)^2 H(p^2, \gamma) + p^2 n^2 F(p^2, \gamma)} \frac{1}{\left[F(p^2, \gamma) n^2 p^2 S^{\lambda\mu} + H(p^2, \gamma) (n \cdot p)^2 T^{\lambda\mu} \right]} \quad (4.3) \end{aligned}$$

with the tensors given by:

$$\begin{aligned} S^{\lambda\mu} &= g^{\lambda\mu} - \frac{p^\lambda p^\mu}{p^2} \quad , \\ T^{\lambda\mu} &= g^{\lambda\mu} - \frac{p^\lambda n^\mu + p^\mu n_\lambda}{n \cdot p} + p^2 \frac{n^\lambda n^\mu}{(n \cdot p)^2} \quad . \end{aligned}$$

The free quantities, $\Delta_{\mu\nu}^{(0)}$ and $\Pi^{\lambda\mu(0)}$, are obtained from Eq. (4.2) and Eq. (4.3) by substituting $F = 1$ and $H = 0$.

In principle, the SDE for the gluon propagator can be written as a set of two coupled equations for F and H . However, in all previous axial gauge studies it has been assumed

that any infrared singular part of the propagator has the same tensor structure as the free one (though importantly, as we shall see later, this contradicts the results of West [41]) and consequently $H(p^2, \gamma)$ has been neglected. Thus for $p^2 \rightarrow 0$ it is assumed that

$$\Delta_{\mu\nu}(p^2, \gamma) \rightarrow -\frac{i}{p^2} F(p^2, \gamma) M_{\mu\nu} = F(p^2, \gamma) \Delta_{\mu\nu}^{(0)}(p^2, \gamma) \quad (4.4)$$

which corresponds to the statement that as $p^2 \rightarrow 0$ the gluon vacuum polarisation, $\Pi_{\lambda\mu}(p^2, \gamma)$, becomes

$$\Pi^{\lambda\mu}(p^2, \gamma) \rightarrow \frac{ip^2}{F(p^2, \gamma)} S^{\lambda\mu} = \frac{ip^2}{F(p^2, \gamma)} \Pi^{\lambda\mu(0)}(p^2, \gamma) \quad . \quad (4.5)$$

Projecting the integral equation with $n_\mu n_\nu / n^2$ the loops involving the four gluon vertex, apart from the tadpole term, give an identically zero contribution. This is because of the tensor structure of the bare 4-gluon vertex, which is defined by:

$$\begin{aligned} \Gamma_{\mu\lambda\rho\sigma}^{abcd(0)} = & -ig_0^2 \left[f^{eab} f^{ecd} (g_{\mu\rho} g_{\lambda\sigma} - g_{\mu\sigma} g_{\lambda\rho}) \right. \\ & + f^{eac} f^{ebd} (g_{\mu\sigma} g_{\lambda\rho} - g_{\mu\lambda} g_{\rho\sigma}) \\ & \left. + f^{ead} f^{ebc} (g_{\mu\lambda} g_{\rho\sigma} - g_{\mu\rho} g_{\lambda\sigma}) \right] \quad , \end{aligned}$$

and the fact that the gluon propagator is transverse to the axial gauge vector:

$$n_\mu \Delta^{\mu\nu} = 0 = \Delta^{\mu\nu} n_\nu \quad .$$

For illustrative purposes consider:

$$n^\mu (g_{\mu\rho} g_{\lambda\sigma} - g_{\mu\sigma} g_{\lambda\rho}) \Delta^{\lambda\lambda'} \Delta^{\rho\rho'} \Delta^{\sigma\sigma'} = n^\mu \Delta_\mu^{\rho'} \Delta^{\lambda\lambda'} \Delta_\lambda^{\sigma'} - n^\mu \Delta_\mu^{\sigma'} \Delta^{\lambda\lambda'} \Delta_\lambda^{\rho'} = 0 \quad .$$

This holds similarly for the other tensor parts of the 4-gluon vertex.

Thus the relevant part of the SDE of Fig. (4.2) becomes:

$$\begin{aligned} \Pi_{\mu\nu} = \Pi_{\mu\nu}^{(0)} & - \frac{g^2}{2} \int \frac{d^4 k}{(2\pi)^4} \Gamma_{\mu\alpha\delta}^{(0)}(-p, k, q) \Delta^{\alpha\beta}(k) \Delta^{\gamma\delta}(q) \Gamma_{\beta\gamma\nu}(-k, p, -q) \\ & - \frac{g^2}{2} \int \frac{d^4 k}{(2\pi)^4} \Gamma_{\mu\nu\alpha\beta}^{(0)}(p, k, -k, -p) \Delta^{\alpha\beta}(k) \end{aligned} \quad (4.6)$$

where $q = p - k$. The last term is the tadpole contribution and all colour indices are implicitly included in the vertices. Once the full 3-gluon vertex is known, we have a closed equation for the gluon vacuum polarisation $\Pi_{\mu\nu}$.

As we discussed in detail in chapter 3.3 the Slavnov-Taylor identities impose a relationship between the 3-gluon vertex, the gluon propagator, the ghost propagator and the ghost-gluon vertex, Eq. (3.16). However, in the absence of ghosts the Slavnov-Taylor identity for the 3-gluon vertex, Eq. (3.16), reduces to the much simpler Ward-Takahashi identity, which constrains the vertex in terms of the vacuum polarisation only:

$$q_\nu \Gamma^{\mu\nu\rho}(p, q, k) = \Pi^{\mu\rho}(k) - \Pi^{\mu\rho}(p) \quad . \quad (4.7)$$

Separating $\Gamma^{\mu\nu\rho}$ into transverse and longitudinal parts, where the transverse part is defined to vanish when contracted with any external momentum, the Slavnov-Taylor identity exactly determines the longitudinal part [29] if it is to be free of kinematic singularities. One should note that this longitudinal part in general depends upon both the axial gauge gluon renormalisation functions F and H (see Kim and Baker [29]). Making the assumption $H(p^2, \gamma) = 0$ the gluon vacuum polarisation, Eq. (4.5), and hence also the longitudinal part of the vertex determined by Eq. (4.7) gets simplified drastically. Having made this approximation, Γ_L is given by:

$$\begin{aligned} \Gamma_{\mu\nu\rho}^L(p, q, k) &= g_{\mu\nu} \left(\frac{p_\rho}{F(p^2, \gamma)} - \frac{q_\rho}{F(q^2, \gamma)} \right) \\ &+ \frac{1}{p^2 - q^2} \left(\frac{1}{F(p^2, \gamma)} - \frac{1}{F(q^2, \gamma)} \right) (p_\nu q_\mu - g_{\mu\nu} p \cdot q)(p_\rho - q_\rho) \\ &+ \text{cyclic permutations} \quad . \end{aligned} \quad (4.8)$$

Ball and Chiu [29] showed that this longitudinal part is responsible for the dominant ultraviolet structure of the vertex. Moreover, as we have discussed in chapter 3.3, it is assumed that it entirely embodies the infrared behaviour, and so the transverse part can be neglected. This assumption is motivated by the fact that the transverse part (as defined) vanishes, when the external momenta approach zero.

Using the explicit expressions, Eq. (4.4) for $\Delta_{\mu\nu}$ and Eq. (4.8) for $\Gamma_{\mu\nu\rho}$ and multiplying with $n_\mu n_\nu/n^2$, we find in Euclidean space:

$$\begin{aligned}
-\frac{p^2}{F(p^2)}(1-\gamma) &= -p^2(1-\gamma) + \frac{g^2 C_A}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{n \cdot (k-q)}{n^2} \Delta_{(0)}^{\lambda\rho}(k) \Delta_{(0)}^{\lambda\sigma}(q) \\
&\quad \left\{ k \cdot n \left[\delta_{\rho\sigma} F(q^2) - \frac{F(q^2) - F(k^2)}{k^2 - q^2} (\delta_{\rho\sigma} k \cdot q - k_\rho q_\sigma) \right. \right. \\
&\quad \quad \left. \left. + \frac{F(k^2) - F(p^2)}{p^2 - k^2} \frac{F(q^2)}{F(p^2)} p_\sigma (p+k)_\rho \right] \right. \\
&\quad \left. - q \cdot n \left[\delta_{\rho\sigma} F(k^2) - \frac{F(q^2) - F(k^2)}{k^2 - q^2} (\delta_{\rho\sigma} k \cdot q - k_\rho q_\sigma) \right. \right. \\
&\quad \quad \left. \left. + \frac{F(q^2) - F(p^2)}{p^2 - q^2} \frac{F(k^2)}{F(p^2)} p_\rho (q+p)_\sigma \right] \right\} \\
&\quad + \frac{g^2 C_A}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{F(k^2)}{k^2} \left(2 + \frac{k^2 n^2}{(n \cdot k)^2} \right) . \tag{4.9}
\end{aligned}$$

This is the equation first found by Baker, Ball and Zachariasen [28] who studied its solution numerically. They came to the conclusion that the only consistent infrared behaviour for the function $F(p^2)$ is

$$F(p^2) \propto \frac{1}{p^2} \quad \text{as } p^2 \rightarrow 0$$

and that this is independent of γ as a numerical approximation.

Schoenmaker [32] simplified the BBZ equation (Eq. (4.9)) further by exactly setting $\gamma = 0$. Doing this the contribution of the tadpole diagram vanishes as we now demonstrate. Consider the tadpole term alone, which from Eq. (4.9) is:

$$\text{Tadpole} = \int \frac{d^4 k}{(2\pi)^4} \frac{F(k^2)}{k^2} \left(2 + \frac{k^2 n^2}{(n \cdot k)^2} \right) .$$

If we assume that $\gamma = 0$ the gluon renormalisation function depends only on the momentum p^2 and thus we can perform the angular integration of the integral above. As explained in more detail in Appendix A, Eq. (A.3), the four dimensional integration can be written as:

$$\int d^4 k = \int_0^\infty \frac{k^2 dk^2}{2} \int_0^\pi \sin^2 \psi d\psi \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \quad ,$$

where here we define the angle ψ to be the angle between \mathbf{k} and \mathbf{n} . Thus:

$$\text{Tadpole} = 2\pi \int \frac{dk^2}{(2\pi)^4} F(k^2) \int_0^\pi \sin^2 \psi \, d\psi \left(2 + \frac{k^2 n^2}{n^2 k^2 \cos^2 \psi} \right)$$

The tadpole term requires the integral

$$2 \int_0^\pi \sin^2 \psi \, d\psi + \int_0^\pi \frac{\sin^2 \psi}{\cos^2 \psi} \, d\psi \quad .$$

Clearly the second term in the integral diverges at $\psi = \pi/2$. Schoenmaker regulates this divergence by taking the principle value. Then

$$2 \int_0^\pi \sin^2 \psi \, d\psi + \int_0^\pi \frac{\sin^2 \psi}{\cos^2 \psi} \, d\psi \sim \left[\psi - \frac{1}{2} \sin 2\psi + \frac{\sin \psi}{\cos \psi} - \psi \right]_0^\pi = 0$$

and we find that the tadpole term vanishes.

Schoenmaker [32], moreover, approximates $F(q^2)$ by $F(p^2 + k^2)$, which since $q^2 = (p - k)^2$ should be exact in the infrared limit when either p or k is small. This allows the angular integrals to be performed analytically. Consequently, Schoenmaker found the following simpler equation:

$$\begin{aligned} p^2 \left(\frac{1}{F(p^2)} - 1 \right) = & \frac{g^2 C_A}{32\pi^2} \left\{ \int_0^{p^2} dk^2 \left[\left(-\frac{k^4}{12p^4} + \frac{5k^2}{2p^2} - \frac{2k^2}{3p^2 - k^2} \right) F_1 + \left(\frac{k^6}{24p^6} - \frac{1k^4}{4p^4} - \frac{1k^2}{4p^2} \right) F_2 \right. \right. \\ & + \left. \left(-\frac{1k^2}{6p^2} + \frac{2k^2}{3p^2 - k^2} \right) F_3 \right] + \int_{p^2}^\infty dk^2 \left[\left(-\frac{3}{4} - \frac{p^2}{4k^2} - \frac{2p^2}{3p^2 - k^2} \right) F_1 \right. \\ & \left. \left. + \left(-\frac{3k^2}{4p^2} + \frac{5}{12} - \frac{1p^2}{8k^2} \right) F_2 + \left(\frac{7p^2}{6k^2} + \frac{2p^2}{3p^2 - k^2} \right) F_3 \right] \right\} \quad , \quad (4.10) \end{aligned}$$

where

$$\begin{aligned} F_1 &= F(p^2 + k^2) \quad , \\ F_2 &= F(p^2 + k^2) - F(k^2) \quad , \\ F_3 &= \frac{F(p^2 + k^2)F(k^2)}{F(p^2)} \quad . \end{aligned}$$

In general, this equation has a quadratic ultraviolet divergence, which would give a mass to the gluon. Such terms have to be subtracted to ensure the masslessness condition,

$$\lim_{p^2 \rightarrow 0} \Pi_{\mu\nu} = 0 \quad , \text{ i.e. } \frac{p^2}{F(p^2)} = 0 \quad \text{for } p^2 \rightarrow 0 \quad , \quad (4.11)$$

is satisfied. This property can be derived generally from the Slavnov-Taylor identity [28] and always has to hold if the vertices are free of massless scalar singularities.

The complicated structure of the integral equation, Eq. (4.10), does not allow an exact analytic solution for the gluon renormalisation function, $F(p^2)$, to be found and most previous studies ([28], [33], [34] and [36]) solve the equation numerically. However, the possible asymptotic behaviour of $F(p^2)$ for both small and large p^2 can be investigated analytically.

We determine which infrared behaviour of $F(p^2)$ can give a self-consistent solution to the integral equation by taking a trial input function, $F_{in}(p^2)$, and substituting it into the right hand side of the equation. After performing the k^2 -integration, we obtain an output function $1/F_{out}(p^2)$ to be compared to the reciprocal of the input function. To do this, the gluon renormalisation function is approximated in the infrared region by a Laurent expansion in powers of p^2 and at large momenta by its bare form, i.e.

$$F(p^2) = \begin{cases} \sum_{n=0}^{\infty} a_n (p^2/\mu^2)^{n+\eta} & \text{for } p^2 < \mu^2 \\ 1 & \text{for } p^2 > \mu^2 \end{cases}, \quad (4.12)$$

where

$$\sum_{n=0}^{\infty} a_n = 1$$

to ensure continuity at $p^2 = \mu^2$. μ is the mass scale above which we assume perturbation theory applies. The exponent η can be negative to allow for an infrared enhancement. Eq. (4.12) is a sufficiently general representation for finding the dominant self-consistent infrared behaviour. Of course, the true renormalisation function is modulated by powers of logarithms of momentum, characteristic of a gauge theory. However, these do not qualitatively affect the dominant infrared behaviour and can be neglected. Indeed to make the presentation straightforward, we only need approximate $F(p^2)$ by its dominant infrared power $(p^2)^\eta$ for $p^2 < \mu^2$ to test whether consistency is possible and this is what we describe below. However, as we shall see, if η is negative then potential mass terms arise and these have to be subtracted. Only in this case do higher terms in Eq. (4.12) play a role too and it is necessary to consider other than the leading term in the low momentum input. Otherwise higher powers make no qualitative difference as we have checked. Consequently

we present only the results with the lowest powers in the representation, Eq. (4.12).

To illustrate the idea, let us take the trial infrared behaviour to be just

$$F(p^2) \propto \left(\frac{p^2}{\mu^2}\right)^\eta \quad (\text{i.e. } a_n = 0 \text{ for } n \geq 1) \quad .$$

Note that the masslessness condition, Eq. (4.11), restricts η to be less than 1. Furthermore we demand that in the high momentum region the solution of the integral equation matches the perturbative result, i.e. for $p^2 \rightarrow \infty$, we have $F(p^2) = 1$, modulo logarithms.

We now insert our trial input function in the left hand side of Schoenmaker's approximation, Eq. (4.10); there are then six integrals which should be calculated. We shall in the following use the shorthand notation:

$$\begin{aligned} I_i &= \int_0^{p^2} dk^2 \{F_i\} && \text{when } i = 1, 2, 3 && \text{and} \\ I_i &= \int_{p^2}^\infty dk^2 \{F_{i-3}\} && \text{when } i = 4, 5, 6 && . \end{aligned}$$

Furthermore we set

$$F(p^2 + k^2) = F(p^2) + k^2 \frac{d}{dp^2} F(p^2)$$

in those integrals for which $p^2 > k^2$, and

$$F(p^2 + k^2) = F(k^2) + p^2 \frac{d}{dk^2} F(k^2)$$

when $p^2 < k^2$, since we are taking p^2 , the external momentum in Eq. (4.10), to be small.

We introduce a cut-off Λ^2 to regularise the ultraviolet divergent integrals. It should be pointed out that the cut-off dependence can be removed in the standard way by a wavefunction renormalisation, and a renormalised version of Eq. (4.10) can be found in Ref. [33]. However, for the purpose of this study we need not consider renormalisation and hence shall not discuss it here.

Taking $\eta = -1$ for our trial input function (Eq.(4.1)), for example, i.e.

$$F_{in}(p^2) = A \frac{\mu^2}{p^2}$$

in Schoenmaker's approximation, Eq. (4.10), gives

$$p^2 \left(\frac{1}{F(p^2)} - 1 \right) = \text{const} \quad .$$

This violates the masslessness condition of Eq. (4.11) and so has to be mass renormalised. As explained above, now terms in $F(p^2)$ of higher order in p^2 will generate a contribution to the right hand side of the equation making it possible to find a self-consistent solution by these cancelling the explicit factor of 1. Consequently, we can approximate Eq. (4.12) by

$$F_{\text{in}}(p^2) = \begin{cases} A(\mu^2/p^2) + (p^2/\mu^2) & \text{if } p^2 < \mu^2 \\ 1 & \text{if } p^2 > \mu^2 \end{cases} , \quad (4.13)$$

We then find for the integrals I_i :

$$\begin{aligned} I_1 &= \frac{11}{144} A \mu^2 + \frac{533}{144} \frac{p^4}{\mu^2} - \frac{4}{3} \frac{p^4}{\mu^2} \int_0^{p^2} \frac{dk^2}{p^2 - k^2} , \\ I_2 &= \frac{433}{1440} A \mu^2 - \frac{19}{96} \frac{p^4}{\mu^2} , \\ I_3 &= \frac{7}{12} A \mu^2 - \frac{167}{72} \frac{p^4}{\mu^2} + \frac{4}{3} \frac{p^4}{\mu^2} \int_0^{p^2} \frac{dk^2}{p^2 - k^2} , \\ I_4 &= \left(\frac{3}{8} + \frac{31}{24} A \right) \mu^2 - \frac{3}{4} A \mu^2 \ln \left(\frac{\mu^2}{p^2} \right) - \left(\frac{1}{3} + \frac{7}{6} A \right) p^2 + \frac{5}{12} p^2 \ln \left(\frac{\Lambda^2}{\mu^2} \right) \\ &\quad + \left(\frac{17}{24} - \frac{1}{8} A \right) \frac{p^4}{\mu^2} - \frac{1}{4} \frac{p^4}{\mu^2} \ln \left(\frac{\mu^2}{p^2} \right) - \frac{3}{4} \Lambda^2 - \frac{4}{3} \frac{p^4}{\mu^2} \int_{p^2}^{\mu^2} \frac{dk^2}{p^2 - k^2} , \\ I_5 &= - \left(\frac{17}{48} A + \frac{3}{8} \right) + \frac{3}{4} A \mu^2 \ln \left(\frac{\mu^2}{p^2} \right) + \left(\frac{5}{12} A + \frac{5}{12} \right) p^2 \\ &\quad - \left(\frac{1}{16} A + \frac{1}{24} \right) \frac{p^4}{\mu^2} - \frac{1}{8} \frac{p^4}{\mu^2} \ln \left(\frac{\mu^2}{p^2} \right) , \\ I_6 &= \left(\frac{1}{4} - \frac{5}{36} A \right) \mu^2 + \frac{1}{2} \mu^2 \ln \left(\frac{\Lambda^2}{\mu^2} \right) - \frac{1}{6} p^2 - \left(\frac{1}{4} A + \frac{1}{12} \right) \frac{p^4}{\mu^2} \\ &\quad + \frac{7}{18} \frac{p^6}{\mu^4} + \frac{4}{3} \frac{p^4}{\mu^2} \int_{p^2}^{\mu^2} \frac{dk^2}{p^2 - k^2} . \end{aligned}$$

We see that the divergent integrals in I_1 and I_3 cancel and we get a finite result. The same is true for the remaining integrals in I_4 and I_6 . Furthermore we note that $I_1 - I_6$ in-

clude constant terms. These violate the masslessness condition and have to be subtracted.

We then find, adding the contributions from $I_1 - I_6$, after mass renormalisation:

$$\frac{1}{F_{out}(p^2)} = 1 + \frac{g^2 C_A}{32\pi^2} \left[\left(\frac{509}{288} - \frac{7}{16} A \right) \frac{p^2}{\mu^2} - \left(\frac{1}{12} + \frac{3}{4} A \right) - \frac{3}{8} \frac{p^2}{\mu^2} \ln \left(\frac{\mu^2}{p^2} \right) + \frac{5}{12} \ln \left(\frac{\Lambda^2}{\mu^2} \right) + \frac{7}{18} \frac{p^4}{\mu^4} \right] ,$$

where Λ is the ultraviolet cut-off introduced to make the integrals finite.

We should point out here that the above equation contains terms which are not included in Ref. [42] where this calculation was originally presented. This is because in Ref. [42] we did not cut off the infrared enhanced term $A \mu^2/p^2$ at the momentum scale μ^2 . This gives an additional contribution in the large momentum region. However these extra terms do not qualitatively alter the result. The ultraviolet divergent constant can be arranged to cancel the 1 and the infrared dominant part is $1/F_{out}(p^2) = p^2/\mu^2$. Thus we find self-consistency modulo logarithms. It is this result that Schoenmaker found [32] supporting the earlier result of BBZ [28]. However, importantly, self-consistency requires A , Eq. (4.13), to be negative as also found by Schoenmaker.

More recently, Cudell and Ross [33] have taken Schoenmaker's equation, Eq. (4.10), and investigated whether one can find self-consistency for a gluon renormalisation function which is less singular than $1/k^2$ for $k^2 \rightarrow 0$, i.e. which corresponds to *confined* gluons. The main motivation for their study being that this is the form of the full gluon propagator required for the Landshoff Nachtmann pomeron model [30]. (For a discussion of this model requirement and the consequences of the behaviour of the gluon for the modelling of the pomeron in terms of dressed gluon exchange we refer to chapter 4.5.)

The trial input function Cudell and Ross [33] use in their investigation is

$$F_{in}(p^2) \propto (p^2)^{1-c} ,$$

where c is small and positive to ensure a massless gluon, Eq. (4.11). Once more we want the integral equation for $\Pi_{\mu\nu}$ to agree with perturbation theory in the ultraviolet region,

but $F_{in}(p^2) \propto (p^2)^{1-c}$ grows for large momenta and hence spoils the ultraviolet behaviour. So to check whether this input function gives self-consistency in the infrared, we input the trial form :

$$F_{in}(p^2) = \begin{cases} (p^2/\mu^2)^{1-c} & \text{if } p^2 < \mu^2 \\ 1 & \text{if } p^2 > \mu^2 \end{cases} . \quad (4.14)$$

Inserting this into Eq. (4.10), we find :

$$\begin{aligned} I_1 &= \frac{283 - 161c}{144} \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} - \frac{2(2-c)}{3} \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} \int_0^{p^2} dk^2 \frac{1}{p^2 - k^2} , \\ I_2 &= \left(-\frac{161 - 66c}{480} - \frac{1}{24(5-c)} + \frac{1}{4(4-c)} + \frac{1}{4(3-c)} \right) \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} , \\ I_3 &= \left(-\frac{1}{6(3-c)} - \frac{1-c}{6(4-c)} - \frac{2}{3}\Psi(3-c) - \frac{2(1-c)}{3}\Psi(4-c) + \frac{2(2-c)}{3}\Psi(1) \right) \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} \\ &\quad + \frac{2(2-c)}{3} \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} \int_0^{p^2} dk^2 \frac{1}{p^2 - k^2} , \end{aligned}$$

where, for the calculation of I_3 , we have used

$$\begin{aligned} \int_0^{p^2} dk^2 \frac{(k^2)^{n-c}}{p^2 - k^2} &= - \int_0^{p^2} dk^2 \frac{(p^2)^{n-c} - (k^2)^{n-c}}{p^2 - k^2} + (p^2)^{n-c} \int_0^{p^2} dk^2 \frac{1}{p^2 - k^2} \\ &= - (p^2)^{n-c} [\Psi(n+1-c) - \Psi(1)] + (p^2)^{n-c} \int_0^{p^2} dk^2 \frac{1}{p^2 - k^2} \end{aligned}$$

and Ψ is the logarithmic derivative of the Gamma function which we define in Appendix B. Note that again, by combining the I_1 and I_3 , the logarithmic divergent integrals cancel, and we have a finite answer.

Similarly for I_4 to I_6 we find:

$$\begin{aligned} I_4 &= \left(\frac{3}{4(2-c)} + \frac{1}{4(1-c)} + \frac{3}{4} - \frac{1-c}{4c} - \frac{2}{3}\Psi(2-c) - \frac{2(1-c)}{3}\Psi(1-c) \right. \\ &\quad \left. - \frac{2(2-c)}{3}\Psi(1) \right) \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} + \left(\frac{3}{4} - \frac{3}{4(2-c)} \right) \mu^2 \\ &\quad + \left(\frac{2}{3(1-c)} - \frac{1}{4(1-c)} - \frac{3}{4} \right) p^2 + \left(\frac{1-c}{4c} - \frac{2(1-c)}{3c} \right) \frac{p^4}{\mu^2} - \frac{3}{4}\Lambda^2 \\ &\quad + \frac{5}{12} p^2 \ln \left(\frac{\Lambda^2}{\mu^2} \right) + \frac{2(2-c)}{3} \int_{p^2}^{\mu^2} dk^2 \frac{1}{p^2 - k^2} , \end{aligned}$$

$$\begin{aligned}
I_5 &= \left(\frac{3(1-c)}{4(2-c)} - \frac{5}{12} - \frac{1-c}{8c} \right) \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} - \frac{3(1-c)}{4(2-c)} \mu^2 + \frac{5}{12} p^2 + \frac{1-c}{8c} \frac{p^4}{\mu^2} , \\
I_6 &= \left(-\frac{7}{12(1-c)} - \frac{7(1-c)}{6(1-2c)} + \frac{2}{3} \Psi(3-2c) + \frac{2(1-c)}{3} \Psi(2-2c) \right. \\
&\quad \left. + \frac{2(2-c)}{3} \Psi(1) \right) \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} + \frac{1}{4(1-c)} \frac{(p^2)^c}{(\mu^2)^{-1+c}} + \frac{1-c}{2(1-2c)} \frac{(p^2)^{1+c}}{(\mu^2)^c} \\
&\quad + \frac{1}{2} p^2 \ln \left(\frac{\Lambda^2}{\mu^2} \right) - \frac{2(2-c)}{3} \int_{p^2}^{\mu^2} dk^2 \frac{1}{p^2 - k^2} ,
\end{aligned}$$

where, we have used the integral:

$$\begin{aligned}
\int_{p^2}^{\mu^2} dk^2 \frac{(k^2)^{n-c}}{p^2 - k^2} &= - \int_{p^2}^{\mu^2} dk^2 \frac{(p^2)^{n-c} - (k^2)^{n-c}}{p^2 - k^2} + (p^2)^{n-c} \int_{p^2}^{\mu^2} dk^2 \frac{1}{p^2 - k^2} \\
&= -\frac{1}{n-c} (\mu^2)^{n-c} + (p^2)^{n-c} [\Psi(n+1-c) + \Psi(1)] - (p^2)^{n-c} \int_{p^2}^{\mu^2} dk^2 \frac{1}{p^2 - k^2}
\end{aligned}$$

and again the divergent integrals cancel between I_4 and I_6 . Adding the contributions of the integrals I_i and using the relation, Eq. (B.3)

$$\Psi(z+1) = \Psi(z) + \frac{1}{z}$$

we find, after mass renormalisation:

$$\frac{1}{F_{out}(p^2)} = 1 + \frac{g^2 C_A}{32\pi^2} \left[D_1 + D_2 \left(\frac{\mu^2}{p^2} \right)^{1-c} + D_3 \left(\frac{p^2}{\mu^2} \right)^{1-c} + D_4 \left(\frac{p^2}{\mu^2} \right)^c + \dots \right],$$

where only the first few terms in the expansion for small p^2 have been collected in this equation so that

$$\begin{aligned}
D_1 &= \frac{5}{12(1-c)} - \frac{1}{3} + \frac{11}{12} \ln \frac{\Lambda^2}{\mu^2} , \\
D_2 &= \frac{1}{2(2-2c)} + \frac{1}{6} \ln \left(\frac{\Lambda^2}{\mu^2} \right) , \\
D_3 &= \frac{2407}{1440} - \frac{353c}{360} - \frac{3}{8c} - \frac{1}{24(5-c)} + \frac{1+2c}{12(4-c)} - \frac{7-8c}{12(3-c)} - \frac{7+c}{12(2-c)} - \frac{7}{12(1-c)} \\
&\quad + \frac{1-4c}{6(2-2c)} + \frac{3-7c}{6(1-2c)} + \frac{2}{3}(2-c)\Psi(1) - \frac{4}{3}(2-c)\Psi(1-c) + \frac{2}{3}(2-c)\Psi(1-2c) , \\
D_4 &= -\frac{7-9c}{24c} .
\end{aligned}$$

Again the 1 can be arranged to cancel with the constant term and the dominant infrared behaviour is indeed

$$\frac{1}{F_{out}(p^2)} \rightarrow \left(\frac{\mu^2}{p^2}\right)^{1-c} \quad \text{for } p^2 \rightarrow 0 \quad .$$

Hence a gluon propagator less singular than $1/p^2$ for $p^2 \rightarrow 0$ can be derived from Schoenmaker's equation as Cudell and Ross [33] have found. Note that the only contribution to the dominant infrared behaviour comes from I_6 . To check that once again terms in the gluon renormalisation function of higher order in p^2 do not qualitatively alter the result we calculate I_6 for the next term in Eq. (4.12). Thus with

$$F_{in}(p^2) = \begin{cases} (p^2/\mu^2)^{1-c} + a_1 (p^2/\mu^2)^{2-c} & \text{if } p^2 < \mu^2 \\ 1 & \text{if } p^2 > \mu^2 \end{cases}$$

the infrared dominant term in Eq. (4.10) becomes:

$$\left(\frac{1}{4(1-2c)} + \frac{1}{3-2c}a_1 + \frac{1}{4(2-c)}a_1^2 \right) \left(\frac{\mu^2}{p^2}\right)^{1-c} \quad .$$

This is positive for all a_1 and therefore higher order terms in the input function do not qualitatively change the behaviour.

Thus we see in the axial gauge that apparently both *confined* and *confining* solutions are possible for the gluon propagator. However, the singular *confining* behaviour must be an artefact of the approximation that one of the gluon functions, H , vanishes. West [41] has shown that in a gauge with only positive norm-states, i.e. where all the particles appearing are physical ones (no ghosts), the spectral functions have to obey certain positivity constraints. This leads to the conclusion that the full axial gauge gluon propagator cannot be more singular than $1/p^2$ in the infrared. Recall the general tensor structure of the gluon propagator, Eq. (4.2):

$$\Delta_{\mu\nu}(p^2, \gamma) = -\frac{i}{p^2} \left[F \left(g_{\mu\nu} - \frac{p_\mu n_\nu + p_\nu n_\mu}{n \cdot p} + n^2 \frac{p_\mu p_\nu}{(n \cdot p)^2} \right) + H \left(g_{\mu\nu} - \frac{n_\mu n_\nu}{n^2} \right) \right] \quad .$$

Therefore, though the axial gauge gluon renormalisation function F might be singular, as found by BBZ [28] and Schoenmaker [32], the neglected function H must cancel this

singularity in the infrared. This cancelation can occur even though F and H multiply different tensor structures, since BBZ study a particular projection of $\Pi_{\mu\nu}$, namely $n^\mu \Pi_{\mu\nu} n^\nu$, which projects the two tensor structures on to the same direction.

As an aside, we have in fact tried to find an explicit illustration of how this works in practice, but the complexity of the fully coupled two functions form of the gluon SDE has not allowed us to do this.

One should note, that this result by West does not only call into question the approximation of only considering one of the axial gauge renormalisation functions, but also, more importantly, makes it impossible to relate the behaviour of the gluon propagator to a gauge invariant (i.e. physical) quantity and prove confinement via the Wilson loop operator.

Moreover, the approximation of setting $\gamma = 0$ in the BBZ-equation, Eqs. (4.9,4.10), has been seriously questioned in Ref. [43]. Atkinson et al. pointed out that the gluon propagator, though it can be related to gauge independent, physical quantities, like the Wilson loop, is, of course, itself gauge dependent. There is no general argument excluding the dependence of F on the axial gauge parameter γ . Indeed it has been shown [43] that using a spectral ansatz to solve the axial gauge SDE a suppression of the γ -dependence leads to inconsistent results.

Because of the difficulty in justifying the neglect of one of the key gluon renormalisation functions in axial gauges, we turn our attention to covariant gauges and the Landau gauge in particular.

4.2 Landau Gauge Calculation

While the axial gauge boson propagator involves two renormalisation functions, $F(p^2, \gamma)$ and $H(p^2, \gamma)$, the advantage of Landau gauge studies is the much simpler structure of the gluon propagator, involving just the single renormalisation function, $G(p^2)$, so that:

$$\Delta_{\mu\nu}(p) = -i \frac{G(p^2)}{p^2} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \quad . \quad (4.15)$$

However other problems arise and the following approximations have to be made :

- In any covariant gauge, ghosts are necessary to keep the vacuum polarisation transverse and hence are present in the SDE of the gluon propagator, Fig. (3.1). However, in all previous studies [34, 36] the ghost loop diagram is only included in as much as to ensure the transversality of the gluon propagator, assuming that otherwise it does not affect the infrared behaviour of the propagator. This assumption is supported by the fact that in a one-loop perturbative calculation the ghost loop makes a numerically small contribution to $G(p^2)$. (Treating the ghosts as bare makes very little difference.)
- The 4-gluon terms cannot be eliminated as in the axial gauge and are simply neglected. This can be regarded as a first step in a truncation of the SDEs. Furthermore, it seems reasonable to expect that, the full 3-gluon vertex already contains the essence of the confinement mechanism. Brown and Pennington [36] found that this is indeed the case. Including only the 3-gluon vertex, the SDE results in an infrared enhanced, *confining* gluon propagator. as we will show.

With these assumptions, the SDE for the gluon propagator simplifies to Fig.(4.3) and we again find a closed integral equation for the gluon vacuum polarisation, $\Pi_{\mu\nu}$, once the full 3-gluon vertex is known.

$$\text{Gluon line with dot} \cdot (-1) = \text{Gluon line with ghost loop} \cdot (-1) + \frac{1}{2} \cdot \text{Gluon line with ghost loop and gluon loop}$$

Figure 4.3: The approximate Schwinger-Dyson equation for the gluon propagator in the Landau gauge

In the Landau gauge, the Slavnov-Taylor identity for the 3-gluon vertex involves the ghost self-energy, which is simply set to zero (equivalent to treating ghosts as bare), and

the proper ghost gluon vertex function, $\Lambda_{\mu\nu}$ (see Eq. (3.16)). However, in the limit of vanishing ghost momentum the ghost-gluon vertex is equal to the bare one. This result follows from the observation that, in the Landau gauge, the gluon propagator is transverse and thus, in the limit of vanishing ghost momentum [44]:

$$\Lambda_{\tau\mu}(p, q, r) (\Delta^{\tau\sigma}(r))^{-1} = (\Delta^{\mu\sigma}(r))^{-1} \quad \text{and} \quad \Lambda^{\tau\sigma}(r, q, p) (\Delta^{\mu\tau}(p))^{-1} = (\Delta^{\mu\sigma}(p))^{-1} .$$

This considerably simplifies the STI, making it possible to solve the identity for the 3-gluon vertex entirely in terms of $G(p)$ of Eq. (4.15). These simplifications should be valid in the infrared region and with them the STI has the same form as in the axial gauge and is given in Eq. (4.7). Once again approximating the full 3-gluon vertex by its longitudinal part, determined by the STI, and neglecting the transverse part of the vertex, we obtain a closed integral equation :

$$\Pi_{\mu\nu} = \Pi_{\mu\nu}^{(0)} - \frac{g^2}{2} \int \frac{d^4 k}{(2\pi)^4} \Gamma_{\mu\alpha\delta}^{(0)}(-p, k, q) \Delta^{\alpha\beta}(k) \Delta^{\gamma\delta}(q) \Gamma_{\beta\gamma\nu}(-k, p, -q) \quad , \quad (4.16)$$

where again the colour indices are implicit and $q = p - k$.

A scalar equation is obtained by projecting with

$$P^{\mu\nu} = \frac{1}{3p^2} (4p^\mu p^\nu - p^2 g^{\mu\nu}) \quad . \quad (4.17)$$

This projector has the advantage that the $g_{\mu\nu}$ term in Eq. (4.15), that is quadratically divergent in 4-dimensions, does not contribute. Thus we find

$$\begin{aligned} \frac{1}{G(p^2)} = 1 + \frac{g^2 C_A}{96\pi^4} \frac{1}{p^2} \int d^4 k \left[G(q^2) A(k^2, p^2) + \frac{G(k^2)G(q^2)}{G(p^2)} B(k^2, p^2) \right. \\ \left. + \frac{G(k^2) - G(p^2)}{k^2 - p^2} \frac{G(q^2)}{G(p^2)} C(k^2, p^2) + \frac{G(q^2) - G(k^2)}{q^2 - k^2} D(k^2, p^2) \right] , \quad (4.18) \end{aligned}$$

where

$$\begin{aligned} A(k^2, p^2) = & 48 \frac{(k \cdot p)^2}{k^2 p^2 q^2} - 64 \frac{(k \cdot p)}{k^2 q^2} + 16 \frac{(k \cdot p)^3}{k^2 p^2 q^4} - 12 \frac{1}{k^2} + 22 \frac{p^2}{k^2 q^2} - 42 \frac{(k \cdot p)^2}{k^2 q^4} \\ & - 10 \frac{p^4}{k^2 q^4} + 36 \frac{p^2 (k \cdot p)}{k^2 q^4} \quad , \end{aligned}$$

$$\begin{aligned}
B(k^2, p^2) &= -13\frac{p^2}{q^4} + 18\frac{p^2(k \cdot p)}{k^2 q^4} - 2\frac{(k \cdot p)^2}{k^2 q^4} - 4\frac{p^4}{k^2 q^4} + \frac{p^2(k \cdot p)^2}{k^4 q^4}, \\
C(k^2, p^2) &= 4\frac{(k \cdot p)^2}{k^2 q^2} + 6\frac{p^2(k \cdot p)}{k^2 q^2} + 6\frac{(k \cdot p)}{q^2} + 8\frac{p^2}{q^2}, \\
D(k^2, p^2) &= 12\frac{k^2}{q^2} - 48\frac{(k \cdot p)^2}{p^2 q^2} + 48\frac{(k \cdot p)^3}{k^2 p^2 q^2} + 24\frac{(k \cdot p)}{q^2} - 5\frac{p^2}{q^2} - 40\frac{(k \cdot p)^2}{k^2 q^2} + 9\frac{p^2(k \cdot p)}{k^2 q^2}.
\end{aligned}$$

Brown and Pennington [36] studied this equation numerically and found

$$G(p^2) = A \frac{\mu^2}{p^2} \quad \text{for } p^2 \rightarrow 0$$

to be a consistent solution. This result is in agreement with Mandelstam's study of the gluon propagator [34], which used a simpler approximation to the gluon SDE. We shall postpone a discussion of Mandelstam's approach to chapter 5, where we use this simpler equation as the basis of our calculations.

Again approximating $G(q^2)$ by $G(p^2 + k^2)$ allows us to perform the angular integrals in Eq. (4.18) analytically. Using the results of Appendix A we obtain:

$$\begin{aligned}
\int d^4 k G_1 A(k^2, p^2) &= 2\pi^2 \left(\int_0^{p^2} dk^2 G_1 \left\{ 6\frac{k^4}{p^4} - 10\frac{k^2}{p^2} + 5 \right. \right. \\
&\quad \left. \left. + \frac{k^2}{p^2 - k^2} \left[4\frac{k^4}{p^4} - \frac{47}{4}\frac{k^2}{p^2} + \frac{51}{4} - 5\frac{p^2}{k^2} \right] \right\} \right. \\
&\quad \left. + \int_{p^2}^{\Lambda^2} dk^2 G_1 \left\{ \frac{p^2}{k^2} + \frac{p^2}{p^2 - k^2} \left[-\frac{5}{4}\frac{p^2}{k^2} + \frac{5}{4} \right] \right\} \right) \\
\int d^4 k G_3 B(k^2, p^2) &= 2\pi^2 \left(\int_0^{p^2} dk^2 G_3 \frac{k^2}{p^2 - k^2} \left[-\frac{3}{4}\frac{k^2}{p^2} + \frac{21}{8} - \frac{15}{8}\frac{p^2}{k^2} \right] \right. \\
&\quad \left. + \int_{p^2}^{\Lambda^2} dk^2 G_3 \frac{p^2}{p^2 - k^2} \left[\frac{27}{4} - \frac{51}{8}\frac{p^2}{k^2} - \frac{3}{8}\frac{p^4}{k^4} \right] \right) \\
\int d^4 k \frac{G_1 - G_3}{p^2 - k^2} C(k^2, p^2) &= 2\pi^2 \left(\int_0^{p^2} dk^2 (G_1 - G_3) \frac{k^2}{p^2 - k^2} \left[6 + 2\frac{k^2}{p^2} \right] \right. \\
&\quad \left. + \int_{p^2}^{\Lambda^2} dk^2 (G_1 - G_3) \frac{p^2}{p^2 - k^2} \left[6 + 2\frac{p^2}{k^2} \right] \right)
\end{aligned}$$

$$\int d^4k \frac{G_2}{p^2} D(k^2, p^2) = 2\pi^2 \left(\int_0^{p^2} dk^2 G_2 \left\{ -3 \frac{k^6}{p^6} + 7 \frac{k^4}{p^4} - \frac{21}{4} \frac{k^2}{p^2} \right\} + \int_{p^2}^{\Lambda^2} dk^2 G_2 \left\{ -\frac{3}{2} + \frac{1}{4} \frac{p^2}{k^2} \right\} \right) . \quad (4.19)$$

where G_1 , G_2 and G_3 are combinations of the gluon renormalisation function at different momenta, which we define in the same way as we did for Schoenmaker's equation, Eq. (4.10), i.e. :

$$\begin{aligned} G_1 &= G(p^2 + k^2) \quad , \\ G_2 &= G(p^2 + k^2) - G(k^2) \quad , \\ G_3 &= \frac{G(k^2)G(p^2 + k^2)}{G(p^2)} . \end{aligned}$$

Again we have introduced an ultraviolet cut-off Λ^2 making all the integrals in Eq. (4.19) finite. Putting all this together we obtain the following equation:

$$\begin{aligned} \frac{1}{G(p^2)} &= 1 + \\ &\frac{g^2 C_A}{48\pi^2} \frac{1}{p^2} \left\{ \int_0^{p^2} dk^2 \left[G_1 \left(5 - 10 \frac{k^2}{p^2} + 6 \frac{k^4}{p^4} + \frac{k^2}{p^2 - k^2} \left(\frac{75}{4} - \frac{39}{4} \frac{k^2}{p^2} + 4 \frac{k^4}{p^4} - 5 \frac{p^2}{k^2} \right) \right) \right. \right. \\ &\quad \left. \left. + G_2 \left(-\frac{21}{4} \frac{k^2}{p^2} + 7 \frac{k^4}{p^4} - 3 \frac{k^6}{p^6} \right) + G_3 \left(\frac{k^2}{p^2 - k^2} \left(-\frac{27}{8} - \frac{11}{4} \frac{k^2}{p^2} - \frac{15}{8} \frac{p^2}{k^2} \right) \right) \right] \right. \\ &\quad \left. + \int_{p^2}^{\infty} dk^2 \left[G_1 \left(\frac{p^2}{k^2} + \frac{p^2}{p^2 - k^2} \left(\frac{29}{4} + \frac{3}{4} \frac{p^2}{k^2} \right) \right) + \right. \right. \\ &\quad \left. \left. + G_2 \left(-\frac{3}{2} + \frac{1}{4} \frac{p^2}{k^2} \right) + G_3 \left(\frac{p^2}{p^2 - k^2} \left(\frac{3}{4} - \frac{67}{8} \frac{p^2}{k^2} - \frac{3}{8} \frac{p^4}{k^4} \right) \right) \right] \right\} , \quad (4.20) \end{aligned}$$

Note that the integral equation has the usual ultraviolet divergences, but infrared divergences are also possible. The ultraviolet divergences can be handled in the standard way to give a renormalised function $G_R(p^2)$ — this will not be discussed here. However we have to make the potentially infrared divergent integrals finite in order to calculate the integrals¹. The infrared regularisation procedure proposed by Brown and Pennington [36] is to use the plus prescription of the theory of distributions, which is defined as

¹These divergences do not arise in an axial gauge when γ is set equal to zero as Schoenmaker does, Eq. (4.10).

follows:

$$\left(\frac{\mu^2}{k^2}\right)_+ = \frac{\mu^2}{k^2} \quad \text{for } \infty > k^2 > 0 \quad (4.21)$$

and in the neighbourhood of $k^2 = 0$ it is a distribution that satisfies:

$$\int_0^\infty dk^2 \left(\frac{\mu^2}{k^2}\right)_+ S(k^2, p^2) = \int_0^{p^2} dk^2 \frac{\mu^2}{k^2} [S(k^2, p^2) - S(0, p^2)] + \int_{p^2}^\infty dk^2 \frac{\mu^2}{k^2} S(k^2, p^2).$$

Simply taking

$$G_{in}(p^2) = A \left(\frac{\mu^2}{p^2}\right)_+$$

as an input function once again leads to a mass term and higher terms in the expansion Eq. (4.11) are necessary. Then we do have the chance of finding self-consistency for a gluon propagator as singular as $1/p^4$ and hence *confining* quarks. As before, we use the shorthand notation:

$$\begin{aligned} J_i &= \int_0^{p^2} dk^2 \{G_i\} & \text{when } i = 1, 2, 3 & \quad \text{and} \\ J_i &= \int_{p^2}^\infty dk^2 \{G_{i-3}\} & \text{when } i = 4, 5, 6 & \quad . \end{aligned}$$

With a trial input function of the form

$$G_{in}(p^2) = \begin{cases} A(\mu^2/p^2)_+ + (p^2/\mu^2) & \text{if } p^2 < \mu^2 \\ 1 & \text{if } p^2 > \mu^2 \end{cases}, \quad (4.22)$$

we find for the integrals J_i

$$\begin{aligned} J_1 &= \frac{77}{24} A \mu^2 - \frac{447}{8} \frac{p^4}{\mu^2} + 16 \frac{p^4}{\mu^2} \int_0^{p^2} \frac{dk^2}{p^2 - k^2}, \\ J_2 &= \frac{277}{120} A \mu^2 - \frac{25}{24} \frac{p^4}{\mu^2}, \\ J_3 &= -\frac{19}{4} A \mu^2 + \frac{641}{24} \frac{p^4}{\mu^2} - 16 \frac{p^4}{\mu^2} \int_0^{p^2} \frac{dk^2}{p^2 - k^2}, \\ J_4 &= \left(3 - \frac{9}{8} A\right) \mu^2 - 6A \mu^2 \ln\left(\frac{\mu^2}{p^2}\right) + \left(\frac{1}{4} + \frac{49}{4} A\right) p^2 - \frac{25}{4} p^2 \ln\left(\frac{\Lambda^2}{\mu^2}\right) \\ &\quad + \left(\frac{61}{4} + \frac{7}{8} A\right) \frac{p^4}{\mu^2} + \frac{7}{4} \frac{p^4}{\mu^2} \ln\left(\frac{\mu^2}{p^2}\right) + 16 \int_{p^2}^{\mu^2} \frac{dk^2}{p^2 - k^2}, \end{aligned}$$

$$\begin{aligned}
J_5 &= \frac{11}{8}A\mu^2 - \frac{3}{2}(1+A)p^2 + \left(\frac{3}{2} + \frac{1}{8}A\right) \frac{p^4}{\mu^2} + \frac{1}{4} \frac{p^4}{\mu^2} \ln\left(\frac{\mu^2}{p^2}\right) \quad , \\
J_6 &= \left(\frac{1825}{96}A - \frac{3}{8}\right) \mu^2 - \frac{6}{8}\mu^2 \ln\left(\frac{\Lambda^2}{\mu^2}\right) - \left(\frac{13}{2} + \frac{67}{4}A\right) p^2 + \left(\frac{3}{8}A - \frac{13}{2}\right) \frac{p^4}{\mu^2} \\
&\quad - \frac{3}{8} \frac{p^4}{\mu^2} \ln\left(\frac{\mu^2}{p^2}\right) - \frac{61}{24}A \frac{p^6}{\mu^4} - \frac{3}{32}A \frac{p^8}{\mu^6} - 16 \int_{p^2}^{\mu^2} \frac{dk^2}{p^2 - k^2} \quad .
\end{aligned}$$

Again the divergent integrals cancel each other so that the sum is free of divergences, as it should be. Adding the contributions of J_i we find, after mass renormalisation:

$$\begin{aligned}
\frac{1}{G_{out}(p^2)} &= 1 + \frac{g^2 C_A}{48\pi^2} \left[\left(-\frac{479}{24} + \frac{11}{8}\right) \frac{p^2}{\mu^2} + \frac{13}{8} \frac{p^2}{\mu^2} \ln\left(\frac{\mu^2}{p^2}\right) - \left(\frac{31}{4} + 6A\right) - \frac{25}{4} \ln\left(\frac{\Lambda^2}{\mu^2}\right) \right. \\
&\quad \left. - \frac{61}{24}A \frac{p^4}{\mu^4} - \frac{3}{32}A \frac{p^6}{\mu^6} \right] \quad .
\end{aligned}$$

The ultraviolet divergent constant can be arranged to cancel the 1 and, again, we find self-consistency. This is the result found numerically by Brown and Pennington [36] with a positive infrared enhancement to the gluon renormalisation function, i.e. $A > 0$.

Now we check whether it is possible in the Landau gauge, to find the behaviour Cudell and Ross [33] discovered using Schoenmaker's approximation in the axial gauge. With

$$G_{in}(p^2) = \begin{cases} (p^2/\mu^2)^{1-c} & \text{if } p^2 < \mu^2 \\ 1 & \text{if } p^2 > \mu^2 \end{cases}$$

we find:

$$\begin{aligned}
J_1 &= -\frac{689 - 348c}{24} \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} + 8(2-c) \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} \int_0^{p^2} \frac{dk^2}{p^2 - k^2} \quad , \\
J_2 &= \left(-\frac{757 - 72c}{120} + \frac{3}{5-c} - \frac{7}{4-c} + \frac{21}{4(3-c)} \right) \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} \quad , \\
J_3 &= \left(\frac{11(1-c)}{4} \Psi(5-c) + \frac{49 - 27c}{8} \Psi(4-c) + \frac{42 - 15c}{8} \Psi(3-c) + \frac{15}{8} \Psi(2-c) \right. \\
&\quad \left. - 8(2-c) \Psi(1) \right) \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} - 8(2-c) \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} \int_0^{p^2} \frac{dk^2}{p^2 - k^2} \quad ,
\end{aligned}$$

$$\begin{aligned}
J_4 &= \left(\frac{6}{2-c} + \frac{5+6c}{1-c} + \frac{1-c}{c} - \frac{29}{4} \Psi(2-c) + \left(-8 + \frac{29c}{4} \right) \Psi(1-c) \right. \\
&\quad \left. - \frac{3(1-c)}{4} \Psi(-c) + 8(2-c) \Psi(1) \right) \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} + \left(6 - \frac{6}{2-c} \right) \mu^2 - \frac{5+6c}{1-c} p^2 \\
&\quad - \frac{25}{4} p^2 \ln \left(\frac{\Lambda^2}{\mu^2} \right) - \frac{1-c}{c} \frac{p^4}{\mu^2} + 8(2-c) \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} \int_{p^2}^{\mu^2} \frac{dk^2}{p^2 - k^2} \\
J_5 &= \left(\frac{3}{2} + \frac{1-c}{4c} \right) \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} - \frac{3}{2} p^2 - \frac{1-c}{4c} \frac{p^4}{\mu^2} \quad , \\
J_6 &= \left(\frac{3}{4} \Psi(3-2c) - \frac{61+6c}{8} \Psi(2-2c) - \frac{70-67c}{8} \Psi(1-2c) - \frac{3(1-c)}{8} \Psi(-2c) \right. \\
&\quad \left. - 8(2-c) \Psi(1) \right) \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} - \left(\frac{3}{4(2-2c)} + \frac{3}{4} \ln \left(\frac{\Lambda^2}{\mu^2} \right) \right) \frac{(\mu^2)^{1-c}}{(p^2)^{-c}} \\
&\quad + \left(\frac{61+6c}{8(1-2c)} - \frac{3}{8} \right) \frac{(p^2)^{1+c}}{(\mu^2)^c} - \frac{70-67c}{16c} \frac{(p^2)^{2+c}}{(\mu^2)^{1+c}} - \frac{3(1-c)}{8(1+2c)} \frac{(p^2)^{3+c}}{(\mu^2)^{2+c}} \\
&\quad - 8(2-c) \frac{(p^2)^{2-c}}{(\mu^2)^{1-c}} \int_{p^2}^{\mu^2} \frac{dk^2}{p^2 - k^2} \quad .
\end{aligned}$$

Putting everything together we obtain, after mass renormalisation:

$$\frac{1}{G_{out}(p^2)} = 1 + \frac{g^2 C_A}{48\pi^2} \left[D_1 + D_2 \left(\frac{\mu^2}{p^2} \right)^{1-c} + D_3 \left(\frac{p^2}{\mu^2} \right)^{1-c} + D_4 \left(\frac{p^2}{\mu^2} \right)^c + \dots \right] ,$$

where

$$\begin{aligned}
D_1 &= - \left(\frac{3}{2} + \frac{5+6c}{1-c} + \frac{25}{4} \ln \left(\frac{\Lambda^2}{\mu^2} \right) \right) \quad , \\
D_2 &= - \left(\frac{3}{4(2-2c)} + \frac{3}{4} \ln \left(\frac{\Lambda^2}{\mu^2} \right) \right) \quad , \\
D_3 &= - \frac{1971}{60} + \frac{29c}{2} + \frac{37}{20c} + \frac{6-13c}{2(1-c)} + \frac{59-32c}{4(2-c)} + \frac{155-64c}{8(3-c)} \\
&\quad + \frac{127-49c}{8(4-c)} + \frac{23-11c}{4(5-c)} - \frac{125+61c}{8(1-2c)} - \frac{55+6c}{8(2-2c)} + \frac{3}{4(3-2c)} \\
&\quad - 8(2-c) \Psi(-2c) - 8(2-c) \Psi(1) \quad , \\
D_4 &= \frac{61+6c}{8(1-2c)} \quad .
\end{aligned}$$

Thus the dominant infrared behaviour is:

$$\frac{1}{G_{out}(p^2)} \rightarrow - \left(\frac{\mu^2}{p^2} \right)^{1-c}$$

and self-consistency is spoiled by a negative sign, since c is small and positive. Once again, as demonstrated in the axial gauge calculation, higher order terms in the Laurent expansion of $G_{in}(p^2)$, Eq. (4.12), do not qualitatively alter the result. This leads us to conclude that the infrared softened, *confined* behaviour of the gluon propagator is inconsistent with the approximate SDE in the Landau gauge.

4.3 “Confined” Gluons

A gluon propagator, which is less singular than $1/p^2$ for $p^2 \rightarrow 0$, and hence describes *confined* gluons appears to be a self-consistent solution only of the axial gauge SDE using Schoenmaker’s approximate integral Eq. (4.10). In the Landau gauge this behaviour of the gluon propagator is not possible : a minus-sign spoils self-consistency. We should therefore comment on the origin of this crucial minus sign.

Let us start by carefully looking at the approximate gluon SDE as depicted in Fig. (4.3). Working in Minkowski space we can write this as:

$$\Pi_{\mu\nu}(p) = \Pi_{\mu\nu}^{(0)}(p) - \Sigma_{\mu\nu} \quad ,$$

where

$$\Sigma_{\mu\nu} = \int \frac{d^4 k}{(2\pi)^4} \Gamma_{\mu\alpha\delta}^{(0)acd}(-p, k, q) \Delta^{\alpha\beta}(k) \Delta^{\gamma\delta}(q) \Gamma_{\beta\nu\gamma}^{bdc}(-k, p, -q) \quad .$$

In Minkowski space the propagator and vacuum polarisation are imaginary and we introduce the notation :

$$\Delta^{\alpha\beta}(k) = -i\tilde{\Delta}^{\alpha\beta}(k) \quad , \quad \Pi_{\mu\nu}(p) = i\tilde{\Pi}_{\mu\nu}(p) \quad .$$

Furthermore let us write the bare 3-gluon vertex as:

$$\Gamma_{\mu\alpha\delta}^{(0)acd}(-p, k, q) = -gf^{acd}\tilde{\Gamma}_{\mu\alpha\delta}^{(0)}(-p, k, q)$$

so that all the new quantities defined with a *tilde* contain no more factors of i or negative signs, but only the relevant tensor structures.

Recall that the $SU(3)$ structure constants, f_{abc} , are antisymmetric in the exchange of two of their indices and

$$C_A \delta_{ab} = \sum_{c,d} f_{acd} f_{bcd} \quad .$$

Using the above notation we find:

$$\tilde{\Pi}_{\mu\nu}(p) = \tilde{\Pi}_{\mu\nu}^{(0)}(p) + ig^2 C_A \int \frac{d^4 k}{(2\pi)^4} \tilde{\Gamma}_{\mu\alpha\delta}^{(0)}(-p, k, q) \tilde{\Delta}^{\alpha\beta}(k) \tilde{\Delta}^{\gamma\delta}(q) \tilde{\Gamma}_{\beta\nu\gamma}(-k, p, -q) \quad . \quad (4.23)$$

Note that this equation is true with the same assumptions in any gauge, and hence there should be no difference in sign between two gauges.

Let us now focus on the axial gauge calculation, where, recalling Eq. (4.5), the gluon vacuum polarisation $\tilde{\Pi}_{\mu\nu}$ is defined by:

$$\tilde{\Pi}_{\mu\nu} = \frac{p^2}{F(p^2)} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \quad .$$

and we obtain a scalar equation by multiplying Eq. (4.23) with the tensor

$$N^{\mu\nu} = \frac{n^\mu n^\nu}{n^2} \quad .$$

Eq. (4.23) then becomes:

$$\frac{p^2}{F(p^2)}(1-\gamma) = p^2(1-\gamma) + ig^2 C_A N^{\mu\nu} \int \frac{d^4 k}{(2\pi)^4} \tilde{\Gamma}_{\mu\alpha\delta}^{(0)}(-p, k, q) \tilde{\Delta}^{\alpha\beta}(k) \tilde{\Delta}^{\gamma\delta}(q) \tilde{\Gamma}_{\beta\nu\gamma}(-k, p, -q) \quad .$$

Now we perform a Wick rotation to Euclidean space using the conversion:

$$\begin{aligned} p_M^2 &\rightarrow -p_E^2 \\ d^4 k_M &\rightarrow i d^4 k_E \end{aligned}$$

Thus in Euclidean space the equation becomes:

$$\frac{p^2}{F(p^2)}(1-\gamma) = p^2(1-\gamma) + g^2 C_A N^{\mu\nu} \int \frac{d^4 k_E}{(2\pi)^4} \tilde{\Gamma}_{\mu\alpha\delta}^{(0)}(-p, k, q) \tilde{\Delta}^{\alpha\beta}(k) \tilde{\Delta}^{\gamma\delta}(q) \tilde{\Gamma}_{\beta\nu\gamma}(-k, p, -q) \quad .$$

We now perform the contractions under the integral. It is easiest first to multiply the vertex functions with the axial gauge vector n . For the bare vertex this gives:

$$n^\mu \tilde{\Gamma}_{\mu\alpha\delta}^{(0)}(-p, k, q) = n_\alpha(-p - k)_\delta + g_{\alpha\delta}(k - q) \cdot n + n_\delta(q + p)_\alpha \quad .$$

Note that due to the fact that the gluon propagator is transverse to the axial gauge vector, only the second term of the above gives a contribution to the SDE. The first term gives zero on multiplying with $\tilde{\Delta}^{\alpha\beta}(k)$ and similarly the last term vanishes on multiplying with $\tilde{\Delta}^{\gamma\delta}(q)$.

Approximating $\tilde{\Gamma}_{\beta\nu\gamma}(-k, p, -q)$ by its longitudinal part determined by the Slavnov-Taylor identity (see Eq. (4.8)) and contracting this with the axial gauge vector gives:

$$\begin{aligned} n^\nu \tilde{\Gamma}_{\beta\nu\gamma}(-k, p, -q) &= g_{\gamma\beta} \left(\frac{k \cdot n}{F(k^2)} - \frac{q \cdot n}{F(q^2)} \right) \\ &+ \frac{1}{k^2 - p^2} \left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)} \right) k \cdot n p_\beta (k + p)_\gamma \\ &+ \frac{1}{p^2 - q^2} \left(\frac{1}{F(p^2)} - \frac{1}{F(q^2)} \right) - q \cdot n p_\gamma (p + q)_\beta \\ &+ \frac{1}{q^2 - k^2} \left(\frac{1}{F(q^2)} - \frac{1}{F(k^2)} \right) (q_\beta k_\gamma - g_{\beta\gamma} q \cdot k) (k \cdot n - q \cdot n) \\ &+ \text{terms that vanish when contracted with } \tilde{\Delta}^{\alpha\beta}(k) \text{ or } \tilde{\Delta}^{\gamma\delta}(q) \quad . \end{aligned}$$

Writing the propagators as:

$$\tilde{\Delta}^{\alpha\beta}(k) = F(k^2) \tilde{\Delta}_{(0)}^{\alpha\beta}(k) \quad \text{and} \quad \tilde{\Delta}^{\gamma\delta}(q) = F(q^2) \tilde{\Delta}_{(0)}^{\gamma\delta}(q)$$

we find BBZ's integral Eq. (4.9). This equation is Bose-symmetric (as it should be) and can therefore be rewritten as :

$$\begin{aligned} \frac{p^2}{F(p^2)} (1 - \gamma) &= p^2 (1 - \gamma) \\ &- \frac{g^2 C_A}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{n \cdot (k - q)}{n^2} \Delta_{(0)}^{\alpha\beta}(k) \Delta_{(0)}^{\alpha\gamma}(q) 2k \cdot n K_{\beta\gamma}(k, p) \quad , \end{aligned} \quad (4.24)$$

where

$$\begin{aligned} K_{\beta\gamma}(k, p) &= \delta_{\beta\gamma} F(q^2) - \frac{F(q^2) - F(k^2)}{k^2 - q^2} (\delta_{\beta\gamma} k \cdot q - k_\beta q_\gamma) \\ &+ \frac{F(k^2) - F(p^2)}{p^2 - k^2} \frac{F(q^2)}{F(p^2)} (p + k)_\beta p_\gamma \quad . \end{aligned}$$

However, taking the starting equation of Schoenmaker's paper (Eq (3.5) of Ref. [32]) we find:

$$\begin{aligned} \frac{p^2}{F(p^2)}(1-\gamma) &= p^2(1-\gamma) \\ &- \frac{i}{2}g^2C_A \int \frac{d^4k}{(2\pi)^4} \frac{n \cdot (k-q)}{n^2} \varsigma_{\beta\gamma}(k,q)[-2k \cdot n K_{\beta\gamma}(k,p)] \quad , \end{aligned} \quad (4.25)$$

where

$$\varsigma_{\beta\gamma}(k,q) = (i)^2 \Delta_{(0)}^{\alpha\beta}(k) \Delta_{(0)}^{\alpha\gamma}(q) \quad .$$

Schoenmaker formulates his equation in Minkowski space. Performing a Wick rotation to transform to Euclidean space, we find that Schoenmaker's equation, Eq. (4.25), becomes :

$$\begin{aligned} \frac{p^2}{F(p^2)}(1-\gamma) &= p^2(1-\gamma) \\ &+ \frac{g^2C_A}{2} \int \frac{d^4k}{(2\pi)^4} \frac{n \cdot (k-q)}{n^2} \Delta_{(0)}^{\alpha\gamma}(k) \Delta_{(0)}^{\alpha\beta}(q) 2k \cdot n K_{\gamma\beta}(k,p) \end{aligned} \quad (4.26)$$

which differs from Eq. (4.24) by a crucial minus sign.

We therefore see that in the axial gauge using BBZ's integral equation for the gluon propagator, and simplifying the angular dependence in the way Schoenmaker does in order to make an analytical discussion of the infrared behaviour of the propagator possible, yields an integral equation very similar to the one found by Brown and Pennington [36] in the Landau gauge.

Setting the gluon renormalisation function F equal to 1 in the right hand side of the approximate gluon SDE, we recover the one-loop perturbative contribution of the gluon loop to the vacuum polarisation. In this way the perturbative calculation can be used to double check the sign of the equation, since we know that the coefficient of the simple pole of F has to be positive in order to get the right contribution to β_0 (negative). Both the BBZ equation, Eq. (4.24), and the Brown-Pennington equation, Eq. (4.18), lead to the correct perturbative behaviour at large momenta.

In contrast Schoenmaker's own equation, Eq. (4.10, 4.26), which is the starting point for the study of Cudell and Ross [33] for instance, has an incorrect additional minus sign. This should have been heralded by the self-consistent enhanced gluon of Eq. (4.13)

having a negative sign using Schoenmaker's equation. In an axial gauge this sign should have been a little worrying for a wavefunction renormalisation of a state with positive definite norm. An infrared softened, *confined* gluon is only found with the incorrect sign in Schoenmaker's equation. Correcting this error, which Cudell and Ross later confirmed, the axial gauge SDE does not allow a self-consistent solution of the gluon propagator less singular than $1/p^2$ for $p^2 \rightarrow 0$, and hence a *confined* gluon cannot be found in either gauge.

We also mention the related study by Alekseev [45] who comes to the same conclusion using a different argument from the one presented above. Alekseev's starting point is the renormalised version of Schoenmaker's approximation, i.e. the equation Cudell and Ross studied (which, as we have shown, has a wrong sign). However, Alekseev investigates the possibility of an infrared softened, *confined* behaviour of the gluon propagator, which he describes by the power series:

$$F(p^2) = \left(\frac{p^2}{\mu^2}\right)^{1-c} \left(\alpha_0 + \alpha_1 \left(\frac{p^2}{\mu^2}\right) + \alpha_2 \left(\frac{p^2}{\mu^2}\right)^2 + \dots \right) , \quad (4.27)$$

where $0 < c < 1$.

Note that it is assumed that the ultraviolet, i.e. perturbative behaviour of the propagator does not alter the behaviour in the infrared. By studying the renormalised SDE Alekseev effectively only calculates the integrals up to the renormalisation scale μ^2 , assuming that the ultraviolet terms exactly cancel each other. Implicitly Alekseev's ansatz for the infrared behaviour, which grows for large momenta, is cut off at this scale and hence does not spoil the ultraviolet, perturbative behaviour. However, the crucial difference between Alekseev's study and the one discussed in chapter 4.1 is that not only the dominant infrared behaviour is matched, but Alekseev demands the SDE be solved exactly by his trial function, Eq. (4.27). This clearly is a much stronger self-consistency constraint.

Analytic calculation of the integrals in the SDE gives:

$$\frac{1}{F(p^2)} = 1 + C \left\{ P_1(p^2) + \frac{1}{F(p^2)} P_2(p^2) + \alpha_0 \left(\frac{p^2}{\mu^2}\right)^{1-c} \left[\Delta(c) + O\left(\frac{p^2}{\mu^2}\right) \right] \right\} ,$$

where P_1 and P_2 are integer power series in p^2 and $\Delta(c)$ is a dimensionless function.

This, together with the assumed form of $F(p^2)$, Eq. (4.27), has the following structure:

$$P(p^2) + \left(\frac{p^2}{\mu^2}\right)^{1-c} Q(p^2) + \left(\frac{p^2}{\mu^2}\right)^{c-1} R(p^2) = 0 \quad , \quad (4.28)$$

where again $P(p^2)$, $Q(p^2)$ and $R(p^2)$ are integer power series with some coefficients p_n, q_n, r_n with $n = 0, 1, 2, \dots$ correspondingly. Alekseev then concludes that for non-integer values of c the theorem of uniqueness for power series demands $p_n = q_n = r_n = 0$ for all n and therefore the value of the exponent of the infrared dominant behaviour should be determined by the characteristic equation $\Delta(c) = 0$. This equation is shown not to have solutions in the interval $0 < c < 1$, and hence an infrared softened gluon is found to be inconsistent.

However, we should stress that it is not clear that a single power series for the gluon renormalisation function $F(p^2)$, Eq. (4.27), should be enough to solve the SDE at all momenta. Thus it is questionable whether Alekseev's study really excludes an infrared softened gluon as a consistent solution to the gluon SDE (with an incorrect sign). However, as we demonstrated before, correcting the sign error in Schoenmaker's approximation leads to $R(p^2) \neq 0$ in Eq.(4.28) which undoubtedly makes an infrared softened solution for the gluon inconsistent.

4.4 An Infrared Vanishing Gluon Propagator

We should also discuss the related work of the group of Stingl et al. [38]. They too start from an approximate, but larger, set of SDEs, which is then to be solved self-consistently. However, the method employed is completely different. The philosophy [38] is to obtain the solution of these equations as power series in the coupling, as in perturbation theory, and to include non-perturbative effects by letting each Green's function depend upon a spontaneously generated mass scale, $b(g^2)$.

$$\Gamma = \Gamma^{(0)}(b^2(g^2)) + \sum_{n=1}^{\infty} g^{2n} \Gamma^{(n)}(b^2(g^2)) \quad . \quad (4.29)$$

Now each of the $\Gamma^{(n)}$ contains an additional nonanalytic dependence on the coupling through b .

We note that, as a formal power series in g^2 , Eq. (4.29) is still basically a weak-coupling solution: it is useful only if the running coupling remains reasonably small over the whole momentum range. Stingl et al. argue that this assumption does not contradict established knowledge, since what is truly known about the behaviour of α_S (see chapter 2.3) is only its decrease at very large momenta. It is conceivable that non-perturbative, low-energy effects, instead of being caused by an increase in the effective coupling constant, as widely believed, are due to non-perturbative terms in the Green's functions which become dominant at low momenta.

Stingl et al. assume each Green's function to be of the form:

$$\Gamma^{(n)}(b^2(g^2)) = \Gamma^{(n)_{pert}} + \Gamma^{(n)_{nonp}}(b^2(g^2)) \quad .$$

Furthermore it is demanded that the theory remains asymptotically free. Thus for large momenta:

$$\Gamma^{(n)} \rightarrow \Gamma^{(n)_{pert}} \quad .$$

Once an ansatz for the non-perturbative zeroth-order vertices $\Gamma^{(0)}$ is chosen it is required that this is self-consistent with the SDEs. That is, inserting this set of $\Gamma^{(0)}$ into the loop integrals constituting the SD-functional F (see Eq. (3.1)) one should not only generate the first order corrections $g^2\Gamma^{(1)}$, as one would in ordinary perturbation theory, but should also reproduce the non-perturbative part $\Gamma^{(0)_{nonp}}$ of the input. Thus the self-consistency requirement is:

$$g^2 F \left[\Gamma^{(0)} \right] = \Gamma^{(0)_{nonp}} + g^2 \Gamma^{(1)} + O(g^4) \quad .$$

In the Ansatz Stingl et al. chose, the non-perturbative zeroth-order terms take the form of nonleading powers of momentum p^2 .

One further restriction is made which simplifies the ghost-sector considerably. Stingl et al. demand that the ghost self-energy at zeroth order equals the perturbative one and find that then self-consistency requires the ghost-gluon vertex to be purely perturbative

too, so that

$$B^{(0)}(p) = B^{(0)pert}(p) = \frac{i}{p^2} \quad \text{and} \quad \Lambda_{\mu\nu}^{(0)} = \Lambda_{\mu\nu}^{(0)pert} = g_{\mu\nu} \quad .$$

Stingl et al. now investigate the coupled SDE for the gluon propagator and 3-gluon vertex. In this study the only diagrams that survive in the SDE for the gluon are the 3-gluon term, the tadpole diagram and the ghost loop, which is perturbative, because all 4-gluon terms are of higher order in g^2 and fermions are neglected. The equation for the 3-gluon vertex function is reduced to one involving itself, the gluon propagator and the perturbative ghost loops. These two coupled equations are shown diagrammatically in Fig. (4.4).

The gluon vacuum polarisation is assumed to be of the form:

$$\Pi^{(0)}(p^2) = p^2 + \frac{b^4}{p^2} \quad ,$$

where $\Pi^{(0)}(p^2)$ is defined by

$$\Pi_{\mu\nu}^{(0)}(p^2) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \Pi^{(0)}(p^2) + \frac{1}{\xi} p_\mu p_\nu$$

and the 3-gluon vertex, $\Gamma_{\mu\nu\rho}^{(0)}(p, k, q)$, depends on 9 parameters, each of which multiplies a ratio of p^2 and/or k^2 and/or q^2 . One solution is found to the set of coupled SDE (Fig. (4.4)) in the Landau gauge. This yields the infrared vanishing gluon propagator of the form (see Fig. (4.1)):

$$\Delta_{\mu\nu}(p^2) = \frac{p^2}{p^4 + b^4} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \quad , \quad (4.30)$$

representing *confined* gluons because there are no poles on the timelike real axis, and thus, as we have discussed in chapter 2.4.3, an asymptotically detectable gluon does not exist. Furthermore, if we continue the gluon propagator to Minkowski space we find that it has two complex poles and hence allows the interpretation of the gluon as an unstable excitation which fragments into hadrons before observation (in a time of the order of $1/b$).

However, apart from these desirable properties of the infrared vanishing gluon there is one major drawback. Recalling the masslessness condition, Eq. (4.11):

$$\lim_{p^2 \rightarrow 0} \Pi_{\mu\nu} = 0 \quad , \text{ i.e. } \quad \frac{p^2}{G(p^2)} = 0 \quad \text{for } p^2 \rightarrow 0 \quad ,$$

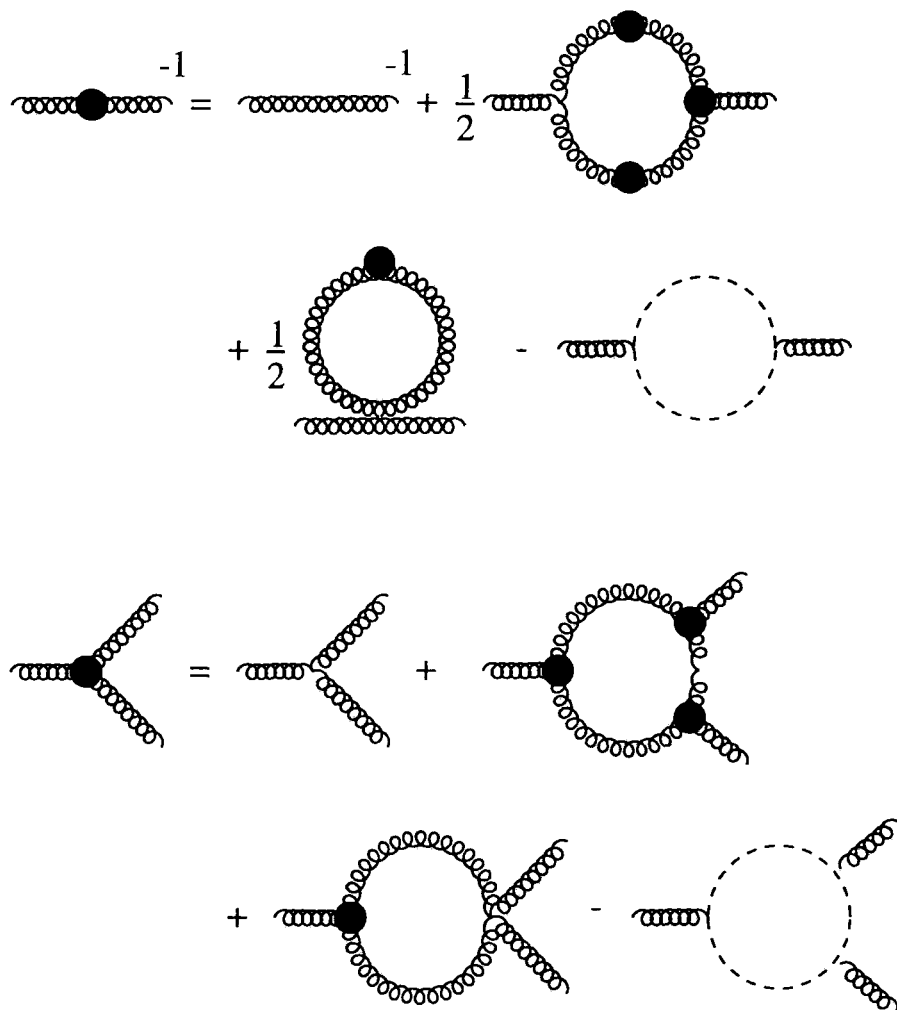


Figure 4.4: Set of coupled SDEs studied by Stingl et al. [38]. Here the \bullet stand for the zeroth order vertices, including their non-perturbative piece, i.e. $\Gamma^{(0)} = \Gamma^{(0)_{pert}} + \Gamma^{(0)_{nonp}}$, and by a bare vertex we mean the purely perturbative one $\Gamma^{(0)_{pert}}$.

it is easy to see that the infrared vanishing gluon, Eq. (4.30), which when related to the tensor structure of the boson propagator in the Landau gauge, Eq. (4.15), gives a gluon renormalisation function of the form

$$G(p^2) = \frac{p^4}{p^4 + b^4} \quad ,$$

grossly violates this condition, giving a mass to the gluon. In general, gluon masses can only arise in 4-dimensions if the vertex functions have dynamical singularities themselves. Otherwise the Slavnov-Taylor identities sufficiently constrain the vertex functions to require the inverse of the gluon propagator to vanish at $p \rightarrow 0$. Such singularities cannot arise in perturbation theory [29] and hence such an ansatz cannot reduce to the perturbative one in the asymptotically free region. In addition, it is not clear how physically to interpret these singularities in the vertex. They correspond to coloured massless scalar states which, of course, are unphysical. Not only do the vertices of Stingl et al. have these massless singularities but self-consistency can only be found if the 3-gluon vertex is complex, when conventional understanding of its singularity structure would lead us to expect it to be real for momenta which in Minkowski space are spacelike. However, the infrared vanishing gluon propagator, Eq. (4.30), is judged reasonable by the authors on the grounds that the unphysical features in the solution for the 3-gluon vertex are believed to be produced by the crude approximations made in the study.

It is interesting that the form in Eq. (4.30) has also been suggested by a number of other studies. Zwanziger [46] argued that in order to completely eliminate Gribov copies, and hence uniquely fix the Landau gauge in lattice studies, one must introduce new ghost fields into QCD, in addition to the Faddeev-Popov ghosts needed in the continuum. Analysing the lattice action thus obtained implies that the gluon propagator vanishes in the infrared. Furthermore there have been a number of lattice simulations of the gluon [47] which also provide some support for this form. (Note that these studies do not make use of the modifications proposed by Zwanziger). However, due to the finite lattice size problem, lattice studies of the gluon propagator cannot yet give reliable information about the infrared behaviour.

It is worth stressing here that in order to decide conclusively whether the gluon prop-

agator in QCD can be infrared vanishing, one has to investigate its phenomenological implications, i.e. determine whether it supports dynamical chiral symmetry breaking and quark confinement. These properties of QCD can be studied using the fermion SDE which depends on the behaviour of the gluon propagator. A fuller description of this and the application of SDE studies to hadron phenomenology will appear in chapter 6. Here we only mention that such a study has been recently carried out by Hawes, Roberts and Williams [39] and Alkofer and Bender [40] for the gluon propagator proposed by Stingl et al. [38], investigating whether this infrared vanishing gluon, Eq. (4.30), can confine quarks. Different methods to test quark confinement were employed by the two groups of authors, however the results obtained are similar.

To determine if the quark propagator found as a solution to the truncated SDE represents a confined particle, Hawes et al. [39] adopt a method commonly used in lattice QCD to estimate bound-state masses. They Fourier transform the scalar part of the quark propagator. The large time behaviour of this Fourier transform then indicates asymptotic states of massive deconfined quarks. In contrast, Alkofer and Bender [40] analytically continue the quark propagator they obtain to timelike momenta. As we discussed in chapter 2.4, a pole on the timelike axis signals a free asymptotic state.

Both Hawes et al. [39] and Alkofer and Bender [40] find unconfined quarks. They therefore also conclude that the full gluon propagator in QCD cannot vanish in the infrared region.

4.5 Consequences for the Modelling of the Pomeron

It has long been understood that at high energies total cross-sections for hadronic processes are controlled by cross-channel pomeron exchange [48, 49], where the pomeron is believed to be a colour singlet with vacuum quantum numbers. Low and Nussinov [49] proposed a QCD-inspired model for the pomeron in terms of two gluon exchange and Landshoff and Nachtmann [30] set up an explicit framework for phenomenological calculations of the resulting cross-sections. A key requirement of their model is that the

dressed gluon propagator, $\Delta(p^2)$, should not have the singularity of the bare massless boson $\sim 1/p^2$ as $p^2 \rightarrow 0$, but should be infrared softened.

However, as we have shown previously, the solution of the SDE for the gluon propagator in QCD does not support an infrared softened behaviour. Only the *confining* gluon behaviour of $\Delta(p^2) \sim 1/p^4$ is consistent with the truncated gluon SDE.

How does this infrared behaviour of the gluon affect the pomeron of Landshoff and Nachtmann [30]?

In the Landshoff-Nachtmann model the pomeron corresponds to two gluons. Since pomeron exchange is a soft process the QCD coupling is not small and the gluons must be non-perturbatively dressed. When the pomeron couples to hadrons, these two gluons couple to single (free) quarks with the other quarks in each initial state hadron being spectators (Fig. (4.5a)).

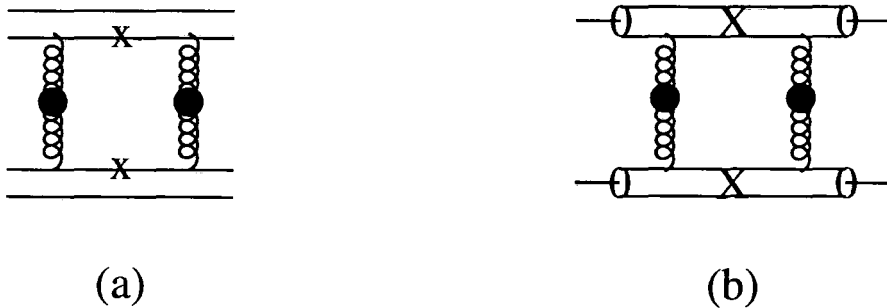


Figure 4.5: Diagrammatic representation of the pomeron in meson-meson scattering: (The lines marked with an X are on-shell in the determination of the total cross-section.)
 (a) Exchange of a gluon pair between two quarks (Landshoff-Nachtmann model),
 (b) Exchange of a gluon pair between two hadrons.

In this way the forward hadronic scattering amplitude is viewed as essentially quark-quark scattering (Fig. (4.5a)). By the optical theorem, the total cross-section σ^{tot} is then related to the imaginary part of this forward elastic quark scattering amplitude $\tilde{F}(s)$. When the two particles in the initial state have equal mass the optical theorem takes the

form:

$$\sigma^{tot} = \frac{Im \tilde{F}(s)}{2q \sqrt{s}} \quad ,$$

where q is the centre-of-mass momentum of the initial state particles and s is the centre-of-mass energy.

In order to generate an imaginary part Landshoff and Nachtmann put the two intermediate state quark lines on mass-shell. By Cutkosky's rule (see e.g. Ref. [50]) this introduces two δ -functions, one for each quark propagator put on-shell, which then allow the angular integrals of the box diagram, (Fig. (4.5a)), to be calculated. Here we only state the result of this calculation:

$$\sigma^{tot} \propto \int_0^\infty dk^2 \Delta(k^2)^2 \quad . \quad (4.31)$$

A detailed derivation of the above can be found in the Appendix of Ref. [30].

Landshoff and Nachtmann's belief in an infrared softened, rather than enhanced, gluon rests on Eq. (4.31), since the total cross-section, σ^{tot} , obviously has to be finite. This can be achieved either with a suitably regularised infrared enhanced gluon or, more easily, with an infrared softened gluon propagator. However, as we now explain we do not believe the issue of whether the integral

$$\int_0^\infty dk^2 \Delta(k^2)^2$$

is finite or not is relevant to the finiteness of total cross-sections. In the Landshoff-Nachtmann model, we described above, the quarks in the initial state hadrons are viewed as essentially free particles. The fact that these quarks can be on mass-shell and hence have poles in their propagators, as an electron or pion, is a crucial, key assumption for their picture, (Fig. (4.5a)), and the subsequent phenomenology. However, quarks are confined particles; their propagators are likely entire functions and the elastic quark amplitude has no imaginary part. As we have mentioned earlier only an infrared enhanced gluon propagator has been shown to produce a confined light quark propagator [37]. It is then the bound state properties of hadrons that are the essential ingredients of total cross-sections. It is the intermediate hadrons that have to be on-shell (Fig. (4.5b)) and not the confined quarks. Confinement requires that hadronic amplitudes are not merely the result of free quark interactions. Only for hard short distance processes is such a

perturbative treatment valid. In soft physics, the bound state nature of light hadrons has to be considered to compute observables.

Subsequently, Pichowsky and Lee [51] have studied a pomeron-exchange model of exclusive electroproduction of ρ -mesons, in which the pomeron couples to non-perturbatively dressed quarks which are confined within hadrons. They use a model form of the full quark propagator, S , which has been developed in SDE studies [37, 52] using the infrared enhanced, *confining* gluon propagator. This $S(p^2)$ is an entire function, i.e. the quark propagator has no poles in the complex momentum plane, and hence represents confined quarks. It is shown that by modelling the photon- ρ -meson-pomeron vertex by a non-perturbative quark loop, one obtains predictions for ρ -meson electroproduction that are in good agreement with experiment. It should be pointed out here that the Landshoff-Nachtmann pomeron model has been applied earlier to study exclusive ρ -meson electroproduction. Donnachie and Landshoff [53] represented the photon- ρ -meson-pomeron vertex by a quark loop using an *on-shell approximation* and therefore assume the quarks can be treated as free particles. They found that in order to reproduce experiment an additional quark-pomeron form factor had to be introduced. However, Pichowsky and Lee [51] found that in their approach such a form factor is unnecessary, suggesting it to be an artefact of the *on-shell approximation*.

We conclude that quark confinement has to be taken into account to obtain a reasonable description of the pomeron in terms of gluon exchange. Consequently, an infrared enhanced gluon propagator is not at variance with the pomeron, but is in fact in accord with quark confinement.

4.6 Summary and Conclusion

We have studied the SDE of the gluon propagator to determine analytically the possible infrared solutions for the gluon renormalisation function $G(p^2)$. In both the axial and Landau gauges, one can find a self-consistent solution, which behaves as $1/p^2$ for $p^2 \rightarrow 0$ and hence a propagator which is as singular as $1/p^4$ for $p^2 \rightarrow 0$. This form of the gluon

propagator is consistent with area law behaviour of the Wilson loop, which is regarded as a signal for confinement ². Numerical studies [28], [36] have shown that a gluon propagator with such an enhanced behaviour in the infrared region and connecting to the perturbative regime at a finite momentum (as indicated by experiment) can indeed be found as a self-consistent solution to the gluon SDE. Such a behaviour of the boson propagator has been shown to give quark propagators with no physical poles [54]. Furthermore, extending these non-perturbative methods to hadron physics, it has been found that a regularised, infrared singular gluon propagator together with the SDE for the quark self-energy, gives rise to a good description of dynamical chiral symmetry breaking. For instance, one obtains values for quantities such as the pion decay constant that agree with experimental results [37].

A gluon propagator which is less singular than $1/p^2$ for $p^2 \rightarrow 0$, and hence describes *confined* gluons, cannot be found in either the axial or the Landau gauge. Solutions of this type have only been found using approximations to the gluon SDE with an incorrect sign.

Even softer gluons resulting from the dynamical generation of a gluon mass, though often claimed, only arise if multi-gluon vertices have massless particle singularities that stop the zero momentum limit of the Slavnov-Taylor identity being smooth. Such singularities, though they occur in the vertices of Stingl et al. [38], should not be present in QCD. Furthermore, as shown by Hawes et al. [39] and independently by Alkofer and Bender [40] an infrared vanishing gluon propagator cannot confine quarks and hence the gluon in QCD cannot have this behaviour.

To summarise:

At first sight there appears to be a distinction between a *confining* and a *confined* gluon. A *confining* gluon is one whose interactions lead to quark confinement. $\Delta(p^2) \sim 1/p^4$ behaviour is of this *confining* type. In contrast, it is sometimes argued that $\Delta(p^2)$ must be less singular than $1/p^2$ to ensure that gluons themselves do not propagate over large

²Though as remarked at the end of Sect. 4, axial gauge studies are seriously marred by the simplifying assumption that $H \equiv 0$ in Eq. (4.2).

distances. However, whether gluons are *confining* or *confined* are not real alternatives. Gluons must be both. They confine quarks by having very strong long range interactions. They themselves are confined by not having a Källén-Lehmann representation that any physical asymptotic state must have.

While infrared singular gluons satisfy both criteria, softened gluons though *confined*, do not generate quark confinement or dynamical chiral symmetry breaking, which are features of our world. Remarkably, a study of the field equations of QCD reveals this theory naturally exhibits these aspects with an infrared enhanced gluon propagator.

Chapter 5

Infrared regularisation

As we have illustrated in detail in the previous chapter, studies of the truncated SDE for the gluon propagator show that it is enhanced at low momenta — indeed at all momenta smaller than the scale Λ_{QCD} . This $1/p^4$ -behaviour of the gluon leads to an infrared divergence in the SDEs for both the gluon and the quark. It is therefore only defined up to some regularisation procedure.

Different infrared regularisations have been followed in previous work. The infrared regularisation of the SDE for the gluon was first considered by Mandelstam [34] and incorporated in the imposition of a massless gluon — something gauge invariance requires if gluon vertices have no zero momentum singularities. However, in the more complicated gluon equation studied by Brown and Pennington (Eq. (4.18)) mass renormalisation is not enough to make the SD-integrals infrared safe. As discussed in chapter 4.2, Brown and Pennington [36] chose to treat the potentially infrared divergent integrals by using the plus-prescription in the definition of the gluon renormalisation function. Of course, this prescription is not determined by the theory but put in by hand. Furthermore, it is not the only possible regularisation procedure. Replacing the $1/p^4$ -behaviour by a $\delta^4(p)$ [55], a distribution which is integrable on any domain containing the origin, is another regularisation possibility.

Moreover, in order to study quark confinement and dynamical chiral symmetry breaking (DCSB), which are two crucial features of QCD, one must solve the SDE for the quark

propagator which is given diagrammatically in Fig. (5.1).



Figure 5.1: The Schwinger-Dyson equation for the quark propagator.

In order to solve the quark SDE, knowledge of the behaviour of the gluon propagator at all momenta is required. However, the infrared enhanced, $1/p^4$ -behaviour leads to divergences in the quark equation and needs to be suitably regularised. Thus, in most studies of the quark SDE, a phenomenological (model) form of the infrared behaviour of the gluon is introduced. Again, different regularisation procedures have been followed.

In a study by von Smekal et al. [56] the infrared behaviour of the full quark propagator was investigated modelling the gluon propagator by:

$$G(p^2) = \frac{C}{\ln \left(1 + (p^2 + \mu^2)/\Lambda_{QCD}^2 \right)} + f(p^2) \quad ,$$

where C is a dimensionless parameter, μ^2 has been introduced to regulate the infrared singular part of $G(p^2)$ and $f(p^2)$ is finite as $p \rightarrow 0$. Clearly, for $\mu^2 = 0$ this form of the gluon renormalisation function vanishes at least as fast as $1/p^2$ for $p^2 \rightarrow 0$ and is consistent with the perturbative result for large momenta. If μ^2 is chosen sufficiently small the regularisation procedure employed here, i.e. substitution of $p^2 \rightarrow p^2 + \mu^2$ does not qualitatively alter the behaviour of the gluon. In numerical calculations performed by the authors, μ^2 was varied over several orders of magnitude and a value of $\mu = 10^{-4} \Lambda_{QCD}$ was found to keep the error due to this regularisation procedure below 1%. Making the simplifying approximation $f(p^2) = 0$ this model was shown to lead to an infrared vanishing quark propagator, in accord with confinement, and manifest DCSB.

A qualitatively similar study was carried out by Williams et al. [57] using the following model form for $G(p^2)$:

$$G(p^2) = C \Lambda_{QCD}^2 p^2 \delta^4(p) + \frac{C'}{\ln \left(\tau + p^2/\Lambda_{QCD}^2 \right)} \quad .$$

Here the first term models the dominant infrared behaviour $1/p^4$ in the gluon propagator which has been replaced by the integrable singularity $\delta^4(p)$. The second term reproduces the one-loop perturbative result, and τ has been introduced to regulate its $p^2 \rightarrow 0$ behaviour. Usually τ is chosen so that $\ln \tau = 1$.

Gogokhia et al. [58], in what is called the “zero mode enhancement model of the QCD vacuum”, used the plus prescription as a regularisation procedure. Quark confinement, DCSB and some chiral QCD parameters are successfully described with this model.

More recently, Frank and Roberts [59] employed a model gluon propagator in a calculation of π - and ρ -meson observables using the SD and Bethe-Salpeter equations. In this study, the strong infrared enhancement of the gluon propagator is modelled by $\delta^4(p)$, similar to the form used by Williams et al.. However, Frank and Roberts introduce one extra parameter, m_t , into their model gluon propagator:

$$G(p^2) = C m_t^2 p^2 \delta^4(p) + C' \left(1 - \exp \left\{ -\frac{p^2}{4m_t^2} \right\} \right) .$$

The parameter m_t is the mass scale that marks the transition from the perturbative to the non-perturbative region in this model. It is varied in the numerical calculations to provide a best fit to a range of π -observables and is found to be $m_t = 0.69$ GeV. The second term ensures that $G(p^2)$ has the right ultraviolet behaviour of QCD: $G(p^2) \rightarrow 1$ for large p^2 , logarithmic corrections not being included. With this model gluon, Frank and Roberts [59] calculated the quark propagator from the fermion SDE and found it not to have singularities on the real p^2 -axis, indicating confinement. Furthermore, good agreement with experimental results was obtained for the hadronic observables calculated.

All these studies illustrate that, once regulated, the enhanced gluon propagator leads to a quark propagator without the poles of coloured asymptotic states, but with colour-singlet bound states with properties in accord with experiment. However, the many possible choices of regularisation procedures seem somewhat arbitrary and unsatisfactory.

A slightly different approach to dealing with the infrared divergences in the quark SDE which arise from the enhanced term in the gluon renormalisation has been proposed by Brown and Pennington [36]. Instead of introducing a regulated model gluon, they use

an infrared cutoff λ to make the integrals in the quark SDE finite. The dependence of the quark renormalisation function on this cutoff is studied and λ is found to be fixed entirely by the structure of the quark SDE. Remarkably, λ is a constant of a few MeV, which does not differ very much from the masses of the light quarks.

The introduction of an infrared regulator seems more satisfactory than the regulated model forms of the infrared enhanced gluon, which are somewhat arbitrary. Using an infrared regulator λ , there is the hope of performing the infrared regularisation completely within the context of the SDEs studied, since the non-linearity of the equations may then determine this scale λ .

Here we study a truncated SDE for the gluon in quenched QCD and consider how it depends on the infrared regulator λ , i.e. we study the equation for the gluon renormalisation function over the momentum range $p^2 \in [\lambda^2, \kappa^2]$, not down to $p^2 = 0$. The aim is to eliminate the infrared divergences in the SDEs in a self-consistent way, entirely within the context of the calculational scheme.

In section 5.1 the Mandelstam approximation to the gluon SDE is reviewed. We discuss the ultraviolet renormalisation, introduce the infrared regulator λ to make Mandelstam's equation finite and examine how the masslessness of the gluon is enforced, when $\lambda \neq 0$. We investigate the asymptotic behaviour of $G_R(p^2)$ in the infrared region in section 5.2. We then go on in section 5.3 to set up the equation for numerical analysis. We detail the techniques used and illustrate numerically that with $\lambda \neq 0$, the value of the infrared regulator is bounded by Λ_{QCD} . This is explicitly demonstrated in section 5.4 in a simple model SDE, which can be solved analytically. In section 5.5 we discuss the implications of these results.

5.1 The Mandelstam Approximation

Throughout this chapter we study the gluon SDE in a simpler approximation first proposed by Mandelstam [34] which has been shown to have, qualitatively at least, the same features as the more complicated Brown-Pennington equation [36] (Eq. (4.18)), we studied

in chapter 4.2.

Mandelstam considered the Landau gauge SDE for the gluon in quenched QCD. Ghost contributions and the diagrams involving 4-gluon vertices are neglected in this approximation. Furthermore, Mandelstam [34] replaced the full 3-gluon vertex in the truncated SDE of Fig. (4.3) by its bare value, and simultaneously only used the full gluon propagator for one of the internal lines in the gluon loop. This is justified on the grounds that the STI for the 3-gluon vertex in terms of the gluon renormalisation function, $G(p^2)$, Eq. (4.7), shows that there are cancellations between the corrections to the second internal gluon propagator and those to the vertex function. The longitudinal part of the vertex, which is determined by the STI, Eq (4.8), always involves terms proportional to $1/G(p^2)$. The two gluon propagators in the gluon loop give a contribution of $G(k^2)G(q^2)$ and therefore cancel some of the $1/G(p^2)$ terms. Mandelstam assumed these cancellations to be complete. In addition, he argued that if the full gluon propagator behaves like $1/p^4$ then the full 3-gluon vertex behaves like p^2 . Thus replacing the full by the bare vertex should be matched by the softening of the $1/p^4$ -behaviour of the full propagator to the $1/p^2$ -behaviour of the bare one.

The resulting equation for the gluon vacuum polarisation is:

$$\Pi_{\mu\nu} = \Pi_{\mu\nu}^{(0)} + \frac{C_A g_0^2}{32\pi^4} \int d^4 k \Gamma_{\mu\alpha\delta}^{(0)}(-p, k, q) \Delta^{\alpha\beta}(k) \Delta_{(0)}^{\gamma\delta}(q) \Gamma_{\beta\gamma\nu}^{(0)}(-k, p, -q) \quad (5.1)$$

and is shown diagrammatically in Fig. (5.2).

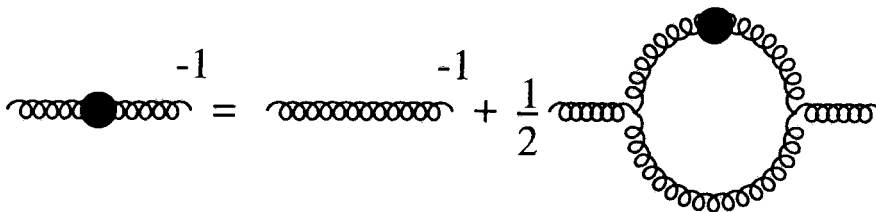


Figure 5.2: The gluon Schwinger-Dyson equation in the Mandelstam approximation

Mandelstam's equation has been studied extensively by Atkinson et al. [61]. These authors give an existence proof of the infrared enhanced, $1/p^4$ -behaviour of the gluon and

a discussion of the singularity structure of the solution. However, as remarked by Brown and Pennington [36], both Mandelstam's original equation and these studies by Atkinson et al. do not ensure that the corrections to the bare 2-point Green's function are transverse as gauge invariance requires. Consequently, their equations have a quadratic ultraviolet divergence and not the logarithmic divergence of an asymptotically free gauge theory. These unphysical quadratic divergences can only occur in the part of the gluon propagator proportional to $g_{\mu\nu}$, the term proportional to $p_\mu p_\nu$ allows logarithmic divergences only. Therefore, Brown and Pennington [36] proposed the use of the projector $P_{\mu\nu}$, which just picks out the $p_\mu p_\nu$ term in the propagator. ($P_{\mu\nu}$ is defined in Eq. (4.17).)

Inserting the explicit expressions for $\Pi^{\mu\nu}$, $\Delta^{\alpha\beta}$ and $\Gamma_{\mu\alpha\delta}^{(0)}(p, k, q)$ in Eq. (5.1) and contracting with $P_{\mu\nu}$ we find:

$$\frac{1}{G(p^2)} = 1 + \frac{C_A g_0^2}{32\pi^4} \frac{1}{p^2} \int d^4k \left(\frac{G(k^2)}{k^2 q^2} A(k, p) + \frac{G(k^2)}{k^2 q^4} B(k, p) \right) , \quad (5.2)$$

where

$$\begin{aligned} A(k, p) &= -\frac{8}{3}p^2 - \frac{11}{3}k^2 - \frac{34}{3}(k \cdot p) + \frac{44}{3} \frac{(k \cdot p)^2}{p^2} + \frac{14}{3} \frac{(k \cdot p)^2}{k^2} - \frac{8}{3} \frac{(k \cdot p)^3}{p^2 k^2} , \\ B(k, p) &= -\frac{4}{3}p^2 k^2 + \frac{2}{3}p^4 - \frac{1}{3}k^4 - \frac{2}{3}(k \cdot p)^2 + \frac{4}{3} \frac{k^2 (k \cdot p)^2}{p^2} + \frac{1}{3} \frac{p^2 (k \cdot p)^2}{k^2} . \end{aligned}$$

We can now use the results of Appendix A to perform the angular integrals to obtain the following equation:

$$\begin{aligned} \frac{1}{G(p^2)} = 1 + \frac{C_A g_0^2}{16\pi^2} \frac{1}{p^2} \left\{ \int_0^{p^2} dk^2 G(k^2) \left[\frac{7}{6} \frac{k^4}{p^4} - \frac{17}{6} \frac{k^2}{p^2} - \frac{3}{8} \right] \right. \\ \left. + \int_{p^2}^{\kappa^2} dk^2 G(k^2) \left[-\frac{7}{3} \frac{p^2}{k^2} + \frac{7}{24} \frac{p^4}{k^4} \right] \right\} , \quad (5.3) \end{aligned}$$

where we have introduced an ultraviolet cutoff κ^2 making all the integrals in Eq. (5.2) ultraviolet finite. As it stands the equation logarithmically depends on the ultraviolet cutoff as well as containing potential infrared divergences. These must be dealt with to give a renormalised equation for $G(p^2)$.

5.1.1 Renormalisation

Because of the presence of the ultraviolet cutoff κ^2 we have really only defined $G(p^2, \kappa^2)$.

We now define a renormalised gluon function $G_R(p^2)$ by:

$$G_R(p^2)Z\left(\frac{\kappa^2}{\mu^2}\right) = G(p^2, \kappa^2) \quad . \quad (5.4)$$

Mandelstam's equation, Eq. 5.3, is ultraviolet renormalised by first evaluating the integral equation at $p^2 = \mu^2$, where μ^2 is some arbitrary momentum scale, and then subtracting this from Eq. 5.3. Using Eq. 5.4 and defining the renormalised coupling by

$$g^2(\mu^2) = Z\left(\frac{\kappa^2}{\mu^2}\right)g_0^2 \quad ,$$

where as usual $\alpha_S(\mu^2) = g^2(\mu^2)/4\pi$, we obtain the following equation:

$$\begin{aligned} \frac{1}{G_R(p^2)} &= \frac{1}{G_R(\mu^2)} + \\ &C \left\{ \int_0^{p^2} dk^2 G_R(k^2) I_1(k^2, p^2) + \int_{p^2}^{\kappa^2} dk^2 G_R(k^2) I_2(k^2, p^2) \right. \\ &\quad \left. - \int_0^{\mu^2} dk^2 G_R(k^2) I_1(k^2, \mu^2) - \int_{\mu^2}^{\kappa^2} dk^2 G_R(k^2) I_2(k^2, \mu^2) \right\} \quad , \quad (5.5) \end{aligned}$$

where the kernels $I_i(k^2, p^2)$ can be simply read off from Eq. 5.3, to be

$$I_1(k^2, p^2) = \frac{1}{p^2} \left(\frac{7k^4}{6p^4} - \frac{17k^2}{6p^2} - \frac{3}{8} \right) \quad \text{and} \quad I_2(k^2, p^2) = \frac{1}{k^2} \left(-\frac{7}{3} + \frac{7p^2}{24k^2} \right)$$

and

$$C = \frac{C_A \alpha_S(\mu^2)}{4\pi} \quad .$$

In order to avoid numerical problems in evaluating the ultraviolet behaviour, it is useful to perform the ultraviolet renormalisation in Mandelstam's equation under the integral. Consequently, we rewrite:

$$\int_{p^2}^{\kappa^2} = \int_{p^2}^{\mu^2} + \int_{\mu^2}^{\kappa^2}$$

and divide the integral involving the ultraviolet cutoff κ^2 into a convergent and a divergent part:

$$\int_{\mu^2}^{\kappa^2} dk^2 G_R(k^2) I_2(k^2, p^2) = -\frac{7}{3} \int_{\mu^2}^{\kappa^2} dk^2 \frac{G_R(k^2)}{k^2} + \int_{\mu^2}^{\kappa^2} dk^2 G_R(k^2) I_3(k^2, p^2) \quad ,$$

where obviously

$$I_3(k^2, p^2) = \frac{7}{24} \frac{p^2}{k^4} \quad .$$

Note that the ultraviolet divergent term is independent of the external momentum p^2 and thus renormalising the equation as described above this term exactly cancels and Eq. (5.5) becomes:

$$\begin{aligned} \frac{1}{G_R(p^2)} &= \frac{1}{G_R(\mu^2)} + \\ &C \left\{ \int_0^{p^2} dk^2 G_R(k^2) I_1(k^2, p^2) + \int_{p^2}^{\mu^2} dk^2 G_R(k^2) I_2(k^2, p^2) \right. \\ &\quad + \int_{\mu^2}^{\kappa^2} dk^2 G_R(k^2) I_3(k^2, p^2) \\ &\quad \left. - \int_0^{\mu^2} dk^2 G_R(k^2) I_1(k^2, \mu^2) - \int_{\mu^2}^{\kappa^2} dk^2 G_R(k^2) I_3(k^2, \mu^2) \right\} \quad . \quad (5.6) \end{aligned}$$

We can now take the limit $\kappa \rightarrow \infty$ for the cutoff explicitly, but we still obtain contributions violating the masslessness condition of the gluon. Gauge invariance requires

$$\lim_{p^2 \rightarrow 0} \frac{p^2}{G(p^2)} = 0 \quad , \quad (5.7)$$

as we stated earlier (Eq. (4.11)). Mass terms arising from the SD integrals have to be subtracted and therefore a mass renormalised gluon function is defined by:

$$\frac{p^2}{G_{MR}(p^2)} = \frac{p^2}{G_R(p^2)} - \lim_{p^2 \rightarrow 0} \frac{p^2}{G_R(p^2)} \quad . \quad (5.8)$$

We note that gluon mass terms only arise from the infrared enhanced term in the gluon renormalisation function, $G(p^2)$. Thus, Brown and Pennington proposed dealing with these mass terms by writing $G(p^2) = A\mu^2/p^2 + G_1(p^2)$ and subtracting the contributions of the first term. Since the Mandelstam equation is linear in $G(p^2)$, this subtraction means that only $G_1(p^2)$ appears under the integrals.

However, it is important to stress at this point that taking the infrared enhanced term out of the integral is more than just enforcing the masslessness condition, Eq. (5.7). It is an infrared regularisation procedure as well.

In contrast, we will here adopt a different approach. We introduce an infrared regulator λ in the ultraviolet renormalised equation, Eq. (5.6), and study how the gluon renormalisation function depends on this λ . This procedure obviously takes care of the infrared divergences, however, gluon mass terms are still possible and need to be renormalised.

In analogy with Eq. (5.8) we define the mass renormalisation by:

$$\frac{p^2 - \lambda^2}{G_{MR}(p^2)} = \frac{p^2}{G_R(p^2)} - \frac{\lambda^2}{G_R(\lambda^2)} \quad (5.9)$$

which in the limit $\lambda \rightarrow 0$ goes back to the original definition, Eq. (5.8). In this respect $p^2 = \lambda^2$ can be thought of as “the gluon mass-shell”.

The fully renormalised Mandelstam equation is then:

$$\begin{aligned} \frac{1}{G_{MR}(p^2)} &= \frac{1}{G_{MR}(\mu^2)} + C \left[\right. \\ &\frac{1}{p^2 - \lambda^2} \left\{ p^2 \left(\int_{\lambda^2}^{p^2} dk^2 G_{MR}(k^2) I_1(k^2, p^2) + \int_{p^2}^{\mu^2} dk^2 G_{MR}(k^2) I_2(k^2, p^2) \right. \right. \\ &\quad \left. \left. + \int_{\mu^2}^{\kappa^2} dk^2 G_{MR}(k^2) I_3(k^2, p^2) \right) \right. \\ &\quad \left. - \lambda^2 \left(\int_{\lambda^2}^{\mu^2} dk^2 G_{MR}(k^2) I_2(k^2, \lambda^2) + \int_{\mu^2}^{\kappa^2} dk^2 G_{MR}(k^2) I_3(k^2, \lambda^2) \right) \right\} \\ &- \frac{1}{\mu^2 - \lambda^2} \left\{ \mu^2 \left(\int_{\lambda^2}^{\mu^2} dk^2 G_{MR}(k^2) I_1(k^2, \mu^2) + \int_{\mu^2}^{\kappa^2} dk^2 G_{MR}(k^2) I_3(k^2, \mu^2) \right) \right. \\ &\quad \left. - \lambda^2 \left(\int_{\lambda^2}^{\mu^2} dk^2 G_{MR}(k^2) I_2(k^2, \lambda^2) + \int_{\mu^2}^{\kappa^2} dk^2 G_{MR}(k^2) I_3(k^2, \lambda^2) \right) \right\} \left. \right] \quad (5.10) \end{aligned}$$

This is the equation we study throughout this chapter. Having discussed how to deal with ultraviolet and mass renormalised gluon functions, $G_{MR}(p^2)$, we drop the subscript MR henceforth.

5.2 Consistent Infrared Behaviour of $G(p^2)$

The structure of our equation for the gluon renormalisation function $G(p^2)$, Eq. (5.10), does not allow a complete analytic solution and we attempt a numerical study. Before doing that we investigate the possible infrared behaviour which we can then build into our numerical solution. In chapter 4, we determined the possible asymptotic behaviour of the gluon renormalisation function, $G(p^2)$, by choosing trial input functions, $G_{in}(p^2)$, and demanding that they solve the truncated SDE self-consistently. In contrast, we make use of the simpler structure of Mandelstam's equation here. The fact that the integrals only depend linearly on the gluon renormalisation function enables us to derive a differential equation for $G(\lambda^2)$ which can be solved analytically in an approximate form. We demonstrate this in the following.

To derive an equation for $G(\lambda^2)$ from Eq. (5.10), we first rewrite the term proportional to $1/(p^2 - \lambda^2)$ as:

$$\begin{aligned} & \frac{1}{p^2 - \lambda^2} \left\{ p^2 \int_{\lambda^2}^{p^2} dk^2 G(k^2) [I_1(k^2, p^2) - I_2(k^2, p^2)] \right. \\ & \quad + p^2 \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) I_2(k^2, p^2) - \lambda^2 \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) I_2(k^2, \lambda^2) \\ & \quad \left. + p^2 \int_{\mu^2}^{\kappa^2} dk^2 G(k^2) I_3(k^2, p^2) - \lambda^2 \int_{\mu^2}^{\kappa^2} dk^2 G(k^2) I_3(k^2, \lambda^2) \right\} . \quad (5.11) \end{aligned}$$

We now take the limit $p^2 \rightarrow \lambda^2$ of the above term by term:

$$\begin{aligned} & \lim_{p^2 \rightarrow \lambda^2} \left\{ \frac{p^2}{p^2 - \lambda^2} \int_{\lambda^2}^{p^2} dk^2 G(k^2) [I_1(k^2, p^2) - I_2(k^2, p^2)] \right\} \\ & \quad = \frac{\lambda^2}{p^2 - \lambda^2} \frac{p^2 - \lambda^2}{\lambda^2} G(\lambda^2) [I_1(\lambda^2, \lambda^2) - I_2(\lambda^2, \lambda^2)] = 0 \quad , \end{aligned}$$

$$\begin{aligned} & \lim_{p^2 \rightarrow \lambda^2} \left\{ \frac{p^2}{p^2 - \lambda^2} \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) I_2(k^2, p^2) - \frac{\lambda^2}{p^2 - \lambda^2} \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) I_2(k^2, \lambda^2) \right\} \\ & \quad = \lim_{p^2 \rightarrow \lambda^2} \left\{ \frac{p^2}{p^2 - \lambda^2} \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) (I_2(k^2, p^2) - I_2(k^2, \lambda^2)) - \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) I_2(k^2, \lambda^2) \right\} \\ & \quad = \lambda^2 \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) \frac{d}{dp^2} I_2(k^2, p^2) \Big|_{p^2=\lambda^2} + \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) I_2(k^2, \lambda^2) \quad , \end{aligned}$$

and similarly:

$$\begin{aligned} \lim_{p^2 \rightarrow \lambda^2} & \left\{ \frac{p^2}{p^2 - \lambda^2} \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) I_3(k^2, p^2) - \frac{\lambda^2}{p^2 - \lambda^2} \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) I_3(k^2, \lambda^2) \right\} \\ & = \lambda^2 \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) \frac{d}{dp^2} I_3(k^2, p^2) \Big|_{p^2=\lambda^2} + \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) I_3(k^2, \lambda^2) \quad . \end{aligned}$$

Putting all this together and noting that

$$\frac{d}{dp^2} I_2(k^2, p^2) \Big|_{p^2=\lambda^2} = \frac{d}{dp^2} I_3(k^2, p^2) \Big|_{p^2=\lambda^2}$$

we obtain the following equation for $G(\lambda^2)$:

$$\begin{aligned} \frac{1}{G(\lambda^2)} & = \frac{1}{G(\mu^2)} + C \left[\lambda^2 \int_{\lambda^2}^{\kappa^2} dk^2 G(k^2) \frac{d}{dp^2} I_2(k^2, p^2) \Big|_{p^2=\lambda^2} \right. \\ & \quad + \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) I_2(k^2, \lambda^2) + \int_{\mu^2}^{\kappa^2} dk^2 G(k^2) I_3(k^2, \lambda^2) \\ & \quad - \frac{1}{\mu^2 - \lambda^2} \left\{ \mu^2 \left(\int_{\lambda^2}^{\mu^2} dk^2 G(k^2) I_1(k^2, \mu^2) + \int_{\mu^2}^{\kappa^2} dk^2 G(k^2) I_3(k^2, \mu^2) \right) \right. \\ & \quad \left. \left. - \lambda^2 \left(\int_{\lambda^2}^{\mu^2} dk^2 G(k^2) I_2(k^2, \lambda^2) + \int_{\mu^2}^{\kappa^2} dk^2 G(k^2) I_3(k^2, \lambda^2) \right) \right\} \right] \quad .(5.12) \end{aligned}$$

With $\mu^2 \gg \lambda^2$, which should be true since the renormalisation scale μ^2 is usually chosen in the perturbative regime, Eq. (5.12) simplifies to:

$$\begin{aligned} \frac{1}{G(\lambda^2)} & = \frac{1}{G(\mu^2)} + C \left[\lambda^2 \int_{\lambda^2}^{\kappa^2} dk^2 G(k^2) \frac{d}{dp^2} I_2(k^2, p^2) \Big|_{p^2=\lambda^2} \right. \\ & \quad + \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) I_2(k^2, \lambda^2) + \int_{\mu^2}^{\kappa^2} dk^2 G(k^2) I_3(k^2, \lambda^2) \\ & \quad \left. - \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) I_1(k^2, \mu^2) \int_{\mu^2}^{\kappa^2} dk^2 G(k^2) I_3(k^2, \lambda^2) \right] \quad .(5.13) \end{aligned}$$

To derive a differential equation for $G(\lambda^2)$, we take the derivative of Eq. (5.13) with respect to λ^2 twice. Noting that the second derivative of both $I_2(k^2, p^2)$ and $I_3(k^2, p^2)$ with respect to p^2 vanishes, we find:

$$\begin{aligned} \frac{d^2}{d(\lambda^2)^2} \frac{1}{G(\lambda^2)} &= C \left\{ G(\lambda^2) \left[-2 \frac{d}{d\lambda^2} I_2(k^2 \lambda^2) \Big|_{k^2=\lambda^2} \right. \right. \\ &\quad \left. \left. + \frac{d}{d\lambda^2} \left(I_1(\lambda^2, \mu^2) - I_2(\lambda^2, \lambda^2) - \frac{d}{dp^2} I_2(\lambda^2, p^2) \Big|_{p^2=\lambda^2} \right) \right] \right. \\ &\quad \left. + \frac{d}{d\lambda^2} G(\lambda^2) \left(I_1(\lambda^2, \mu^2) - I_2(\lambda^2, \lambda^2) - \frac{d}{dp^2} I_2(\lambda^2, p^2) \Big|_{p^2=\lambda^2} \right) \right\}. \end{aligned} \quad (5.14)$$

Inserting the specific forms of the functions $I(k^2, p^2)$, as given in Eq. (5.5), and again using the fact that $\mu^2 \gg \lambda^2$ the differential equation can be written as:

$$\frac{d^2}{d(\lambda^2)^2} \frac{1}{G(\lambda^2)} = C \left\{ -\frac{7}{3} \frac{G(\lambda^2)}{\lambda^4} + \frac{7}{4} \frac{1}{\lambda^2} \frac{d}{d\lambda^2} G(\lambda^2) \right\}.$$

Substituting

$$\frac{1}{G(\lambda^2)} = H(x) \quad , \quad \frac{d}{d\lambda^2} \frac{1}{G(\lambda^2)} = H'(x) \quad \text{and} \quad \frac{d^2}{d(\lambda^2)^2} \frac{1}{G(\lambda^2)} = H''(x) \quad ,$$

where $x = \lambda^2$. This has the form:

$$H(x)H''(x) = C \left(-\frac{7}{3} \frac{1}{x^2} - \frac{7}{4} \frac{H'(x)}{xH(x)} \right). \quad (5.15)$$

With the ansatz $H(x) = A [\ln x]^n$ the left hand side of Eq. (5.15) becomes

$$H(x)H''(x) = -\frac{nA^2}{x^2} [\ln x]^{2n-1} (1 - (n-1) [\ln x]^{-1})$$

which has to be consistent with the right hand side

$$C \left(-\frac{7}{3} \frac{1}{x^2} - \frac{7}{4} \frac{H'(x)}{xH(x)} \right) = -\frac{7C}{3x^2} \left(1 + \frac{3n}{4} [\ln x]^{-1} \right).$$

Only matching leading log-terms as is appropriate for $\lambda^2 \ll \mu^2$, we can easily see that n has to be equal to 1/2 and the solution to the differential equation for $G(\lambda^2)$ is:

$$G(\lambda^2) = \frac{1}{A} \left[\ln \left(\frac{\lambda^2}{M^2} \right) \right]^{-\frac{1}{2}}, \quad \text{where} \quad A = \sqrt{\frac{14}{3}} C \quad (5.16)$$

and M^2 is some arbitrary mass scale, introduced to keep the argument of the logarithm dimensionless. We will use this result in chapter 5.4, where we study a simplified equation, which can be transformed into a differential equation. This makes numerical analysis easier and enables us to determine the scale M^2 which, as we shall show, is fixed by Λ_{QCD} , the scale of our theory. However, before doing this we study the more realistic gluon equation, Eq. (5.10), numerically.

5.3 Numerical Analysis

In this section we will set up the fully renormalised Mandelstam equation, Eq. (5.10), for numerical analysis.

We evaluate the integrals in the approximate SDE using Simpson's rule at N integration points, x_i . Because of the expected behaviour of the gluon renormalisation function these integration points are best chosen on a logarithmic scale in momentum squared. We define,

$$x_i = \log_{10}(\lambda^2) + \frac{i}{N-1} \left\{ \log_{10}(\kappa^2) - \log_{10}(\lambda^2) \right\} \quad , i = 1, 2, \dots, N \quad , \quad (5.17)$$

where λ^2 and κ^2 are the infrared and ultraviolet cutoff respectively defined in arbitrary units so that the argument of the logarithms are dimensionless. After renormalisation our integral equation should not depend on the value of the ultraviolet cutoff κ^2 , and in general $\kappa^2 \rightarrow \infty$ is understood implicitly. However, for numerical purposes we choose a value for κ^2 and increase it until the result of the integrations is stable to within 0.1%. Furthermore, the number of integration points N is increased until a similar numerical accuracy is reached.

We change variables in Eq. (5.10) to

$$v = \log_{10}(p^2) \quad , \quad w = \log_{10}(k^2) \quad , \quad dw = \frac{dk^2}{k^2 \ln 10} \quad .$$

Substituting this into Eq. (5.10) we obtain:

$$\begin{aligned}
\frac{1}{G(p^2)} &= \frac{1}{G(\mu^2)} + \ln 10 C \left[\right. \\
&\frac{1}{p^2 - \lambda^2} \left\{ p^2 \left(\int_a^v dw G(k^2) \tilde{I}_1(k^2, p^2) + \int_v^b dw G(k^2) \tilde{I}_2(k^2, p^2) \right. \right. \\
&\quad \left. \left. + \int_b^c dw G(k^2) \tilde{I}_3(k^2, p^2) \right) \right. \\
&\quad \left. - \lambda^2 \left(\int_a^b dw G(k^2) \tilde{I}_2(k^2, \lambda^2) + \int_b^c dw G(k^2) \tilde{I}_3(k^2, \lambda^2) \right) \right\} \\
&- \frac{1}{\mu^2 - \lambda^2} \left\{ \mu^2 \left(\int_a^b dw G(k^2) \tilde{I}_1(k^2, \mu^2) + \int_b^c dw G(k^2) \tilde{I}_3(k^2, \mu^2) \right) \right. \\
&\quad \left. - \lambda^2 \left(\int_a^b dw G(k^2) \tilde{I}_2(k^2, \lambda^2) + \int_b^c dw G(k^2) \tilde{I}_3(k^2, \lambda^2) \right) \right\} \left. \right] , \quad (5.18)
\end{aligned}$$

where

$$a = \log_{10}(\lambda^2) \quad , \quad b = \log_{10}(\mu^2) \quad , \quad c = \log_{10}(\kappa^2) \quad \text{and} \quad k^2 = 10^w$$

and where we have redefined the functions I_i of Eq. (5.5) to absorb the extra factor of k^2 from the integration measure, so that $\tilde{I}_i = k^2 I_i$, i.e.

$$\begin{aligned}
\tilde{I}_1(k^2, p^2) &= \frac{7 k^6}{6 p^6} - \frac{17 k^4}{6 p^4} - \frac{3 k^2}{8 p^2} \quad , \\
\tilde{I}_2(k^2, p^2) &= -\frac{7}{3} + \frac{7 p^2}{24 k^2} \quad , \\
\tilde{I}_3(k^2, p^2) &= \frac{7 p^2}{24 k^2} \quad .
\end{aligned}$$

To enable a numerical solution of Eq. (5.18) we have to approximate the unknown gluon renormalisation function $G(p^2)$. Here we choose to use an expansion in Chebyshev polynomials.

5.3.1 Chebyshev Expansion

Chebyshev polynomials define a polynomial approximation of a given function $f(x)$ over the interval $[-1, 1]$.

$$f(x) = \sum_{j=0}^{\infty} \gamma_j T_j(x) \quad , \quad (5.19)$$

where the Chebyshev polynomial of degree n is denoted $T_n(x)$ and is given by the formula [62]

$$T_n(x) = \cos(n \arccos x) \quad .$$

Using trigonometric identities we find the following polynomial forms:

$$\begin{aligned} T_0(x) &= 1 \quad , \\ T_1(x) &= x \quad , \\ T_2(x) &= 2x^2 - 1 \quad , \\ T_3(x) &= 4x^3 - 3x \quad , \\ T_4(x) &= 8x^4 - 8x^2 + 1 \quad , \\ &\vdots \\ T_n(x) &= 2x T_{n-1}(x) - T_{n-2}(x) \quad . \end{aligned}$$

The first 4 polynomials are plotted in Fig. (5.3). Approximating a function by a Chebyshev expansion, the series of Eq. (5.19) is truncated at some N .

5.3.2 Chebyshev Approximation for $G(p^2)$

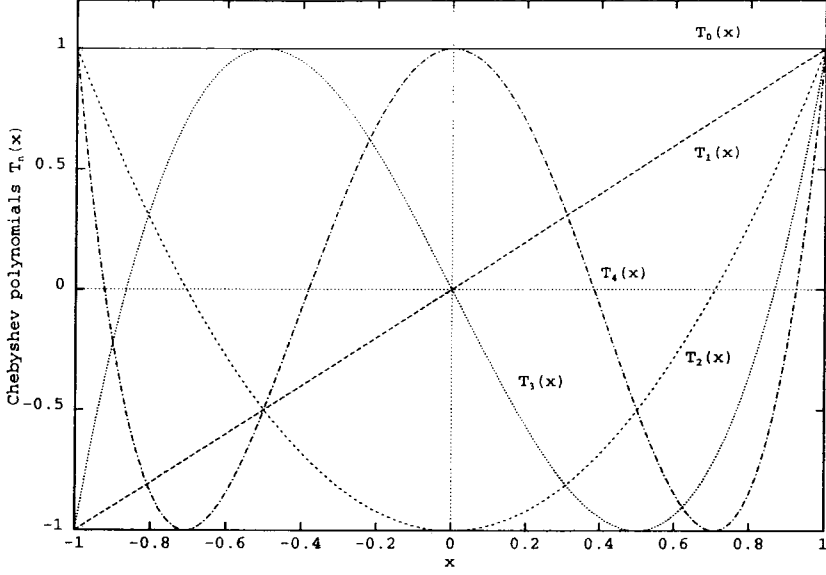
In order to approximate $G(p^2)$ by a Chebyshev expansion we introduce a mapping of variables onto the interval $[-1, 1]$ over which the Chebyshev polynomials are defined. As we discussed before,

$$v = \log_{10}(p^2)$$

is a convenient variable to perform the numerical integrations. We now map the variable $v \in [a, c] \mapsto x \in [-1, 1]$ by defining:

$$x = \frac{v - \frac{1}{2}(c + a)}{\frac{1}{2}(c - a)}$$

Performing the necessary change of variables, Eq. (5.18) becomes:

Figure 5.3: Chebyshev polynomial $T_0(x)$ through $T_4(x)$

$$\begin{aligned}
\frac{1}{G(p^2(x))} &= \frac{1}{G(\mu^2)} + \ln 10 C \frac{(c-a)}{2} \left[\right. \\
&\frac{1}{p^2 - \lambda^2} \left\{ p^2 \left(\int_{-1}^x dy G(k^2(y)) \tilde{I}_1(k^2, p^2) + \int_x^{x(\mu)} dy G(k^2(y)) \tilde{I}_2(k^2, p^2) \right. \right. \\
&\quad \left. \left. + \int_{x(\mu)}^1 dy G(k^2(y)) \tilde{I}_3(k^2, p^2) \right) \right. \\
&\quad \left. - \lambda^2 \left(\int_{-1}^{x(\mu)} dy G(k^2(y)) \tilde{I}_2(k^2, \lambda^2) + \int_{x(\mu)}^1 dy G(k^2(y)) \tilde{I}_3(k^2, \lambda^2) \right) \right\} \\
&- \frac{1}{\mu^2 - \lambda^2} \left\{ \mu^2 \left(\int_{-1}^{x(\mu)} dy G(k^2(y)) \tilde{I}_1(k^2, \mu^2) + \int_{x(\mu)}^1 dy G(k^2(y)) \tilde{I}_3(k^2, \mu^2) \right) \right. \\
&\quad \left. - \lambda^2 \left(\int_{-1}^{x(\mu)} dy G(k^2(y)) \tilde{I}_2(k^2, \lambda^2) + \int_{x(\mu)}^1 dy G(k^2(y)) \tilde{I}_3(k^2, \lambda^2) \right) \right\} \left. \right] , \quad (5.20)
\end{aligned}$$

where

$$p^2(x) = 10^v \quad \text{with} \quad v = \frac{1}{2}[x(c-a) + c + a] \quad ,$$

$$k^2(y) = 10^w \quad \text{with} \quad w = \frac{1}{2}[y(c-a) + c + a] \quad \text{and}$$

$$x(\mu) = \frac{\log_{10}(\mu^2) - \frac{1}{2}(c+a)}{\frac{1}{2}(c-a)}$$

and both $x \in [-1, 1]$ and $y \in [-1, 1]$.

We now parametrise the gluon renormalisation function in terms of Chebyshev polynomials in the following way:

$$G(p^2(x)) = \sum_{j=0}^N \gamma_j T_j(x) \quad , \quad (5.21)$$

where the Chebyshev coefficients γ_j are to be determined by self-consistency of Eq. (5.20).

Substituting our parametrisation for $G(p^2(x))$, Eq. (5.21), into Eq. (5.20), we find:

$$\begin{aligned} \left(\sum_{j=0}^N \gamma_j T_j(x) \right)^{-1} &= \frac{1}{G(\mu^2)} + \ln 10 \, C \frac{(c-a)}{2} \sum_{j=0}^N \gamma_j \left[\right. \\ &\quad \frac{1}{p^2 - \lambda^2} \left\{ p^2 \left(\int_{-1}^x dy \, T_j(y) \tilde{I}_1(k^2, p^2) + \int_x^{x(\mu)} dy \, T_j(y) \tilde{I}_2(k^2, p^2) \right. \right. \\ &\quad \left. \left. + \int_{x(\mu)}^1 dy \, T_j(y) \tilde{I}_3(k^2, p^2) \right) \right. \\ &\quad \left. - \lambda^2 \left(\int_{-1}^{x(\mu)} dy \, T_j(y) \tilde{I}_2(k^2, \lambda^2) + \int_{x(\mu)}^1 dy \, T_j(y) \tilde{I}_3(k^2, \lambda^2) \right) \right\} \\ &\quad - \frac{1}{\mu^2 - \lambda^2} \left\{ \mu^2 \left(\int_{-1}^{x(\mu)} dy \, T_j(y) \tilde{I}_1(k^2, \mu^2) + \int_{x(\mu)}^1 dy \, T_j(y) \tilde{I}_3(k^2, \mu^2) \right) \right. \\ &\quad \left. - \lambda^2 \left(\int_{-1}^{x(\mu)} dy \, T_j(y) \tilde{I}_2(k^2, \lambda^2) + \int_{x(\mu)}^1 dy \, T_j(y) \tilde{I}_3(k^2, \lambda^2) \right) \right\} \left. \right] , \quad (5.22) \end{aligned}$$

To solve this equation numerically we choose a starting set of parameters and perform the integrals numerically. We then vary the parameters until good agreement is obtained over a range of values x between the left and the right hand side of Eq. (5.22).

The advantage of using a polynomial expansion to parametrise the gluon renormalisation function is that all the integrals in our truncated SDE, Eq. (5.22), depend linearly on the expansion coefficients and hence they can be taken out as an overall factor, enabling

us to perform the numerical integrals once and store the output. This enormously shortens computing time. The Chebyshev expansion is used because the error generated by replacing the function $G(p^2)$ by the expansion is smeared out over the complete interval over which the function is defined. Note that if we had built in the behaviour of the gluon in the infrared which we determined in chapter 5.2 (Eq. (5.16)), we would have gained an extra coefficient in our parametrisation of the gluon function, namely the mass scale M^2 . However this is at the cost of extremely lengthening the computing time needed, since the dependence of our equation on M^2 is nonlinear. Furthermore, we should point out that Eq. (5.22) does not only depend on the Chebyshev coefficients γ_j , but also on the infrared cutoff λ^2 . However, for the same reason which made us decide not to explicitly build the infrared behaviour of Eq. (5.16) into our parametrisation, we vary λ^2 *by hand* instead of making it a parameter. That is, for a fixed value of the infrared cutoff, we now allow the Chebyshev coefficients γ_j to vary within the CERN numerical minimisation program MINUIT [63]. We then repeat the calculation for different values of the cutoff λ^2 .

5.3.3 Results

We choose the renormalisation scale $\mu^2 = 10 \text{ GeV}^2$, a scale we know from experiment to be in the perturbative region and thus we expect the gluon renormalisation function $G(\mu^2) \approx 1$. However, we find that fixing $G(\mu^2)$ as well as the infrared cutoff λ^2 in the calculation, we are unable to match the right and left hand side of Eq. (5.22) to good numerical accuracy. This suggests that by fixing both these values we overconstrain the problem, indicating that the integral equation does indeed determine the scale λ^2 as hoped. Since for numerical purposes it is easier to keep λ^2 fixed and make $G(\mu^2)$ a parameter to be determined by the minimisation program, we will follow this route. For each chosen value of the infrared cutoff λ^2 we carry out the numerical calculation for different values of the renormalised strong coupling constant $\alpha_S(\mu^2)$. Recalling the form of $\alpha_S(\mu^2)$ Eq. (2.27):

$$\alpha_S(\mu^2) = \frac{4\pi}{\beta'_0 \ln(\mu^2/\Lambda_{QCD}^2)} \quad .$$

this is equivalent to choosing different values of the QCD-scale Λ_{QCD} , once μ^2 is fixed and β'_0 is known. It must be pointed out here that, due to the approximations made to derive Mandelstam's equation, β'_0 is not equal to the familiar $\beta_0 = \frac{11}{3}C_A$ from perturbation theory. However, the value for β'_0 is easily determined by studying the asymptotic behaviour of Mandelstam's equation for large momenta. By expanding $G(p^2) = 1 + O(\alpha_S(\mu^2))$ and only working to $O(\alpha_S(\mu^2))$, we obtain:

$$\frac{1}{G(p^2)} = \frac{1}{G(\mu^2)} + \frac{C_A \alpha_S(\mu^2)}{4\pi} \frac{7}{3} \ln\left(\frac{p^2}{\mu^2}\right) \quad ,$$

so that

$$\frac{G(\mu^2)}{G(p^2)} = 1 + \frac{C_A \alpha_S(\mu^2)}{4\pi} \frac{7}{3} \ln\left(\frac{p^2}{\mu^2}\right) + O(\alpha_S^2(\mu^2)) \quad . \quad (5.23)$$

Since renormalisation must be scale invariant we have the identity:

$$\frac{G(p^2)}{G(\mu^2)} = \left(\frac{\alpha_S(p^2)}{\alpha_S(\mu^2)}\right)^{\frac{1}{2}} \quad .$$

Inverting this, inserting Eq. (5.23) and again expanding in powers of $\alpha_S(\mu^2)$ we obtain:

$$\frac{1}{\alpha_S(p^2)} = \frac{1}{\alpha_S(\mu^2)} + \frac{\beta'_0}{4\pi} \ln\left(\frac{p^2}{\mu^2}\right) \quad , \quad (5.24)$$

where $\beta'_0 = \frac{14}{3}C_A$.

For each value of the infrared cutoff λ^2 for which we carry out the minimisation we choose different values for the renormalised coupling constant.

Parametrising the gluon renormalisation function by a Chebyshev expansion, which we truncate at the 10th order, we find that the right and left hand side of Eq. (5.22) are matched to impressive numerical accuracy. However, in order to obtain the required $G(\mu^2) \approx 1$ the infrared cutoff has to be bigger than Λ_{QCD} .

We detail the solutions for $\lambda^2 = 10^{-4} \text{ GeV}^2$, 10^{-3} GeV^2 and 10^{-2} GeV^2 . The results are plotted in Fig. (5.4-5.6) and the values found for the parameter $G(\mu^2)$ are given in Tab. (5.1).

We find that the solution of the truncated SDE of the gluon propagator, using Mandelstam's approximation, does indeed determine the infrared regulator λ , which depends

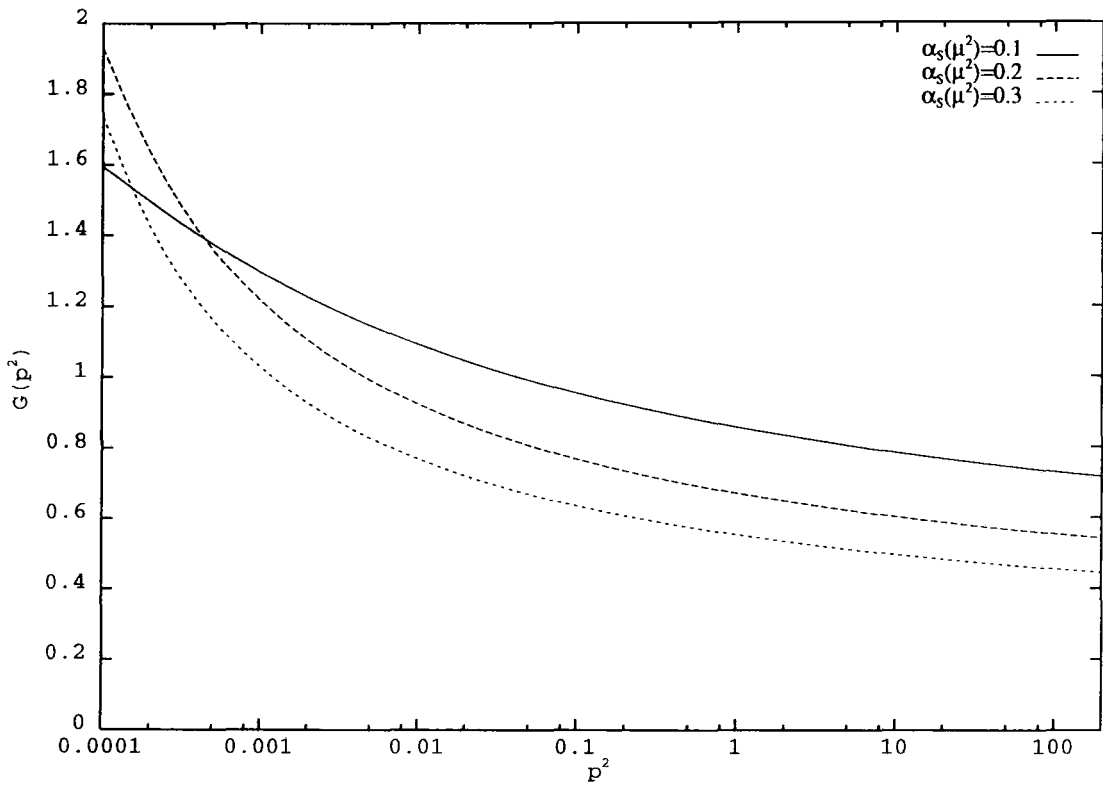
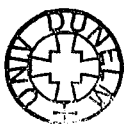


Figure 5.4: The gluon renormalisation function $G(p^2)$ as a function of p^2 with the infrared cutoff $\lambda^2 = 10^{-4} \text{ GeV}^2$.



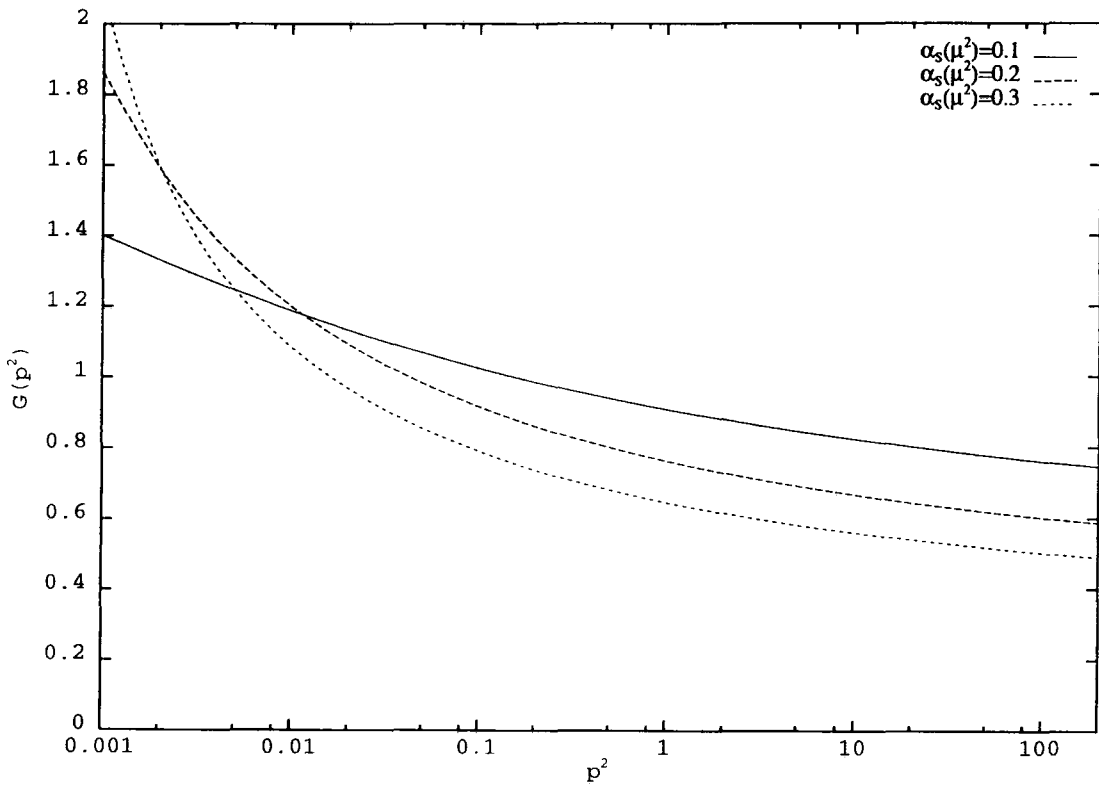


Figure 5.5: The gluon renormalisation function $G(p^2)$ as a function of p^2 with the infrared cutoff $\lambda^2 = 10^{-3} \text{ GeV}^2$.

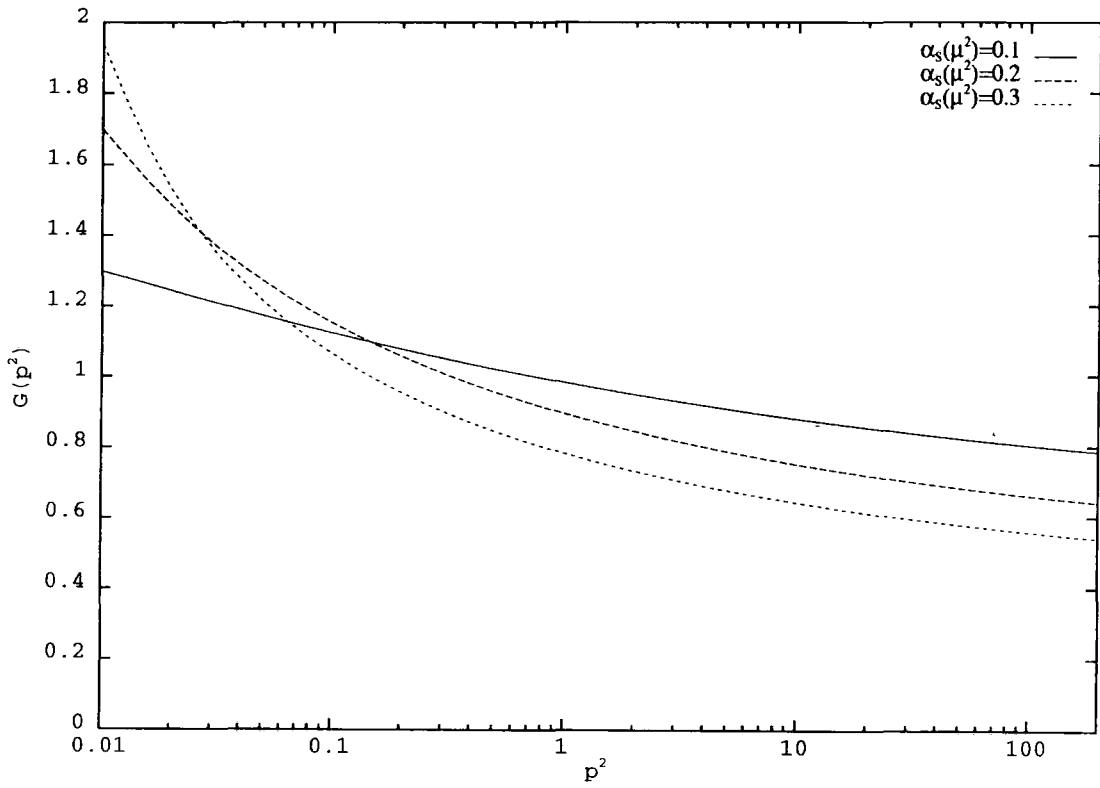


Figure 5.6: The gluon renormalisation function $G(p^2)$ as a function of p^2 with the infrared cutoff $\lambda^2 = 10^{-2} \text{ GeV}^2$.

| | | $\lambda^2 = 10^{-4} \text{ GeV}^2$ | $\lambda^2 = 10^{-3} \text{ GeV}^2$ | $\lambda^2 = 10^{-2} \text{ GeV}^2$ |
|-------------------|--------------------------------|-------------------------------------|-------------------------------------|-------------------------------------|
| $\alpha_s(\mu^2)$ | $\Lambda_{QCD}^2/\text{GeV}^2$ | $G(\mu^2)$ | $G(\mu^2)$ | $G(\mu^2)$ |
| 0.3 | 0.502 | 0.5523 | 0.6463 | 0.7827 |
| 0.25 | 0.276 | 0.6097 | 0.6976 | 0.8396 |
| 0.2 | 0.112 | 0.6700 | 0.7630 | 0.8953 |
| 0.15 | 0.025 | 0.7526 | 0.8382 | 0.9450 |
| 0.1 | $1.264 \cdot 10^{-3}$ | 0.8572 | 0.9084 | 0.9828 |
| 0.08 | $1.340 \cdot 10^{-4}$ | 0.8878 | 0.9304 | 1.1995 |

Table 5.1: Parameters for $G(\mu^2)$ for solution shown in Fig. (5.3–5.5).

on the scale Λ_{QCD} . To investigate the relation between Λ_{QCD} and the infrared regulator further and, in particular, check whether the results obtained here are a specific feature of the Mandelstam approximation or a more general property of the gluon equation, we turn to a simple model *SDE* which as we will show does indeed possess the same qualitative solutions as those obtained here.

5.4 Simple Model *SDE*

In this section we “derive” a simple model *SDE* from Mandelstam’s approximation. This is constructed to allow the integral equation to be turned into a differential equation and solved using the Runge-Kutta method, which greatly simplifies the numerical analysis. Furthermore, studying the simple model equation has the advantage that we can explicitly

build in the infrared behaviour of the gluon we derived in chapter 5.2. For Mandelstam's equation, recalling Eq. (5.16), this has the form:

$$G(\lambda^2) = \frac{1}{A} \left[\ln \left(\frac{\lambda^2}{M^2} \right) \right]^{-\frac{1}{2}}, \quad \text{where } A = \sqrt{\frac{14}{3}} C$$

and depends on the mass scale M^2 . Here we demonstrate that this mass scale is fixed by Λ_{QCD}^2 and, at least for the simplified equation, makes it impossible to find a solution with the infrared regulator $\lambda^2 < \Lambda_{QCD}^2$.

We “derive” the simple model *SDE* by approximating the integrands in Mandelstam's original equation, Eq. (5.3), by their leading terms and furthermore “adjusting” the numerical factors so that the integrands match at $k^2 = p^2$. That is, the simplified “toy” equation takes the form:

$$\frac{1}{G(p^2)} = 1 + \frac{C_A g_0^2}{16\pi^2} \frac{1}{p^2} \left\{ \int_0^{p^2} dk^2 G(k^2) \left[-\frac{7}{3} \frac{1}{p^2} \right] + \int_{p^2}^{\kappa^2} dk^2 G(k^2) \left[-\frac{7}{3} \frac{1}{k^2} \right] \right\}. \quad (5.25)$$

This equation is now renormalised in the way we described in chapter 5.1.1 for Mandelstam's original equation. Inserting the appropriate functions I_i for our simplified equation, given by:

$$I_1 = -\frac{7}{3} \frac{1}{p^2}, \quad I_2 = -\frac{7}{3} \frac{1}{k^2} \quad \text{and} \quad I_3 = 0, \quad (5.26)$$

into the fully renormalised Eq. (5.10) we obtain:

$$\begin{aligned} \frac{1}{G_{MR}(p^2)} = & \frac{1}{G_{MR}(\mu^2)} + C \left[\right. \\ & \frac{1}{p^2 - \lambda^2} \left\{ p^2 \left(\int_{\lambda^2}^{p^2} dk^2 G_{MR}(k^2) I_1(k^2, p^2) + \int_{p^2}^{\mu^2} dk^2 G_{MR}(k^2) I_2(k^2, p^2) \right) \right. \\ & \left. \left. - \lambda^2 \int_{\lambda^2}^{\mu^2} dk^2 G_{MR}(k^2) I_2(k^2, \lambda^2) \right\} \right. \\ & \left. - \frac{1}{\mu^2 - \lambda^2} \left\{ \mu^2 \int_{\lambda^2}^{\mu^2} dk^2 G_{MR}(k^2) I_1(k^2, \mu^2) - \lambda^2 \int_{\lambda^2}^{\mu^2} dk^2 G_{MR}(k^2) I_2(k^2, \lambda^2) \right\} \right] \end{aligned} \quad (5.27)$$

Because of the simple forms of the integrands, Eq. (5.26), we can transform Eq.(5.27) into a differential equation which we then solve numerically.

Taking one derivative of Eq. (5.27) with respect to the external momentum p^2 we obtain:

$$\begin{aligned} \frac{d}{dp^2} \frac{1}{G(p^2)} = C \left[\frac{1}{(p^2 - \lambda^2)^2} \left\{ \lambda^2 \left(\int_{\lambda^2}^{\mu^2} dk^2 G(k^2) I_2(k^2, \lambda^2) \right. \right. \right. \\ \left. \left. \left. - \int_{\lambda^2}^{p^2} dk^2 G(k^2) I_1(k^2, p^2) - \int_{p^2}^{\mu^2} dk^2 G(k^2) I_2(k^2, p^2) \right) \right. \right. \\ \left. \left. p^2 (p^2 - \lambda^2) \int_{\lambda^2}^{p^2} dk^2 G(k^2) \frac{d}{dp^2} I_1(k^2, p^2) \right\} \right] , \end{aligned} \quad (5.28)$$

where the subscript MR , indicating that we are dealing with both ultraviolet and mass renormalised quantities, has again been drop for notational convenience. Noting that

$$p^2 \frac{d}{dp^2} I_1(k^2, p^2) = -I_1(k^2, p^2) \quad ,$$

Eq. (5.28) can be simplified to:

$$\begin{aligned} \frac{d}{dp^2} \frac{1}{G(p^2)} = \\ C \left[\frac{1}{(p^2 - \lambda^2)^2} \left\{ \lambda^2 \int_{p^2}^{\mu^2} dk^2 G(k^2) I_2(k^2, p^2) - p^2 \int_{\lambda^2}^{p^2} dk^2 G(k^2) I_1(k^2, p^2) \right\} \right] . \end{aligned} \quad (5.29)$$

Multiplying both sides of the above equation by $(p^2 - \lambda^2)^2$ and taking one further derivative with respect to p^2 gives:

$$\frac{d}{dp^2} \left[(p^2 - \lambda^2)^2 \frac{d}{dp^2} \frac{1}{G(p^2)} \right] = C \left\{ \lambda^2 \frac{G(p^2)}{p^2} I_2(p^2, p^2) - p^2 \frac{G(p^2)}{p^2} I_1(p^2, p^2) \right\}$$

which, inserting the specific forms of the functions I_i , Eq.(5.26), becomes:

$$\frac{d}{dp^2} \left[(p^2 - \lambda^2)^2 \frac{d}{dp^2} \frac{1}{G(p^2)} \right] = \frac{7}{3} C \frac{(p^2 - \lambda^2)}{p^2} G(p^2) \quad ,$$

or

$$\frac{d^2}{dp^4} \frac{1}{G(p^2)} = \left(\frac{7}{3} C \frac{G(p^2)}{p^2} - 2 \frac{d}{dp^2} \frac{1}{G(p^2)} \right) \frac{1}{p^2 - \lambda^2} \quad . \quad (5.30)$$

We now determine the initial conditions for this differential equation, these being the value of $G(p^2)$ and its first derivative at $p^2 = \lambda^2$. The gluon renormalisation function at $p^2 = \lambda^2$ is obtained by simply inserting Eq. (5.26) into Eq. (5.13) giving

$$\frac{1}{G(\lambda^2)} = \frac{1}{G(\mu^2)} + C \left\{ \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) \left[-\frac{7}{3} \frac{1}{k^2} \right] + \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) \left[-\frac{7}{3} \frac{1}{\mu^2} \right] \right\} \quad . \quad (5.31)$$

By taking the derivative of Eq. (5.31) with respect to λ^2 and using $\mu^2 \gg \lambda^2$ we find the following differential equation for $G(\lambda^2)$:

$$\frac{d}{d\lambda^2} \frac{1}{G(\lambda^2)} = \frac{7}{3} C \frac{G(\lambda^2)}{\lambda^2}$$

which is solved exactly by

$$G(\lambda^2) = \sqrt{\frac{3}{14C}} \left(\ln \left(\frac{\lambda^2}{M^2} \right) \right)^{-\frac{1}{2}} . \quad (5.32)$$

We note that this is equal to the approximate form of $G(\lambda^2)$ we found for Mandelstam's equation, Eq. (5.16).

The second initial condition is easily obtained by taking the limit $p^2 \rightarrow \lambda^2$ of Eq. (5.29) giving:

$$\left. \frac{d}{dp^2} \frac{1}{G(p^2)} \right|_{p^2=\lambda^2} = \frac{7}{6} C \frac{G(\lambda^2)}{\lambda^2} . \quad (5.33)$$

The boundary conditions of the differential equation can be obtained by inserting

$$G(p^2) = \frac{3p^2}{7C} \left((p^2 - \lambda^2) \frac{d^2}{dp^4} \frac{1}{G(p^2)} + 2 \frac{d}{dp^2} \frac{1}{G(p^2)} \right)$$

from Eq. (5.30) into the right hand side of our integral equation, Eq. (5.27). We find:

$$\begin{aligned} \frac{1}{G(p^2)} = \frac{1}{G(\mu^2)} - \left[\right. \\ & \frac{1}{p^2 - \lambda^2} \left\{ p^2 \left(\int_{\lambda^2}^{p^2} dk^2 \left((k^2 - \lambda^2) \frac{d^2}{dk^4} \frac{1}{G(k^2)} + 2 \frac{d}{dk^2} \frac{1}{G(k^2)} \right) \frac{k^2}{p^2} \right. \right. \\ & \quad \left. \left. + \int_{p^2}^{\mu^2} dk^2 \left((k^2 - \lambda^2) \frac{d^2}{dk^4} \frac{1}{G(k^2)} + 2 \frac{d}{dk^2} \frac{1}{G(k^2)} \right) \right) \right. \\ & \quad \left. - \lambda^2 \int_{\lambda^2}^{\mu^2} dk^2 \left((k^2 - \lambda^2) \frac{d^2}{dk^4} \frac{1}{G(k^2)} + 2 \frac{d}{dk^2} \frac{1}{G(k^2)} \right) \right\} \\ & - \frac{1}{\mu^2 - \lambda^2} \left\{ \mu^2 \int_{\lambda^2}^{\mu^2} dk^2 \left((k^2 - \lambda^2) \frac{d^2}{dk^4} \frac{1}{G(k^2)} + 2 \frac{d}{dk^2} \frac{1}{G(k^2)} \right) \frac{k^2}{\mu^2} \right. \\ & \quad \left. - \lambda^2 \int_{\lambda^2}^{\mu^2} dk^2 \left((k^2 - \lambda^2) \frac{d^2}{dk^4} \frac{1}{G(k^2)} + 2 \frac{d}{dk^2} \frac{1}{G(k^2)} \right) \right\} \left. \right] . \end{aligned}$$

Integrating the above by parts gives the trivial result:

$$\frac{1}{G(p^2)} = \frac{1}{G(p^2)} \quad .$$

Thus the differential equation is independent of μ^2 and hence does not know anything of our choice of renormalisation scale and the requirement $G(\mu^2) \approx 1$. Of all the solutions we find by numerically solving the differential equation, Eq. (5.27), only those which satisfy $G(\mu^2) \approx 1$ are physical meaningful solution to our simple model *SDE*.

5.4.1 Numerical Solution

Eq. (5.30), which we want to solve numerically, is a second order differential equation.

We rewrite it as:

$$K''(p^2) = \left[\frac{C}{p^2 K(p^2)} - 2K'(p^2) \right] \frac{1}{p^2 - \lambda^2} \quad , \quad (5.34)$$

where we have defined

$$\frac{1}{G(p^2)} \equiv K(p^2) \quad .$$

The initial conditions, Eq. (5.32) and Eq. (5.33) then take the form:

$$K(\lambda^2) = \sqrt{\frac{14C}{3}} \sqrt{\ln\left(\frac{\lambda^2}{M^2}\right)} \quad , \quad \text{and} \quad K'(\lambda^2) = \frac{7C}{6} \frac{1}{K(\lambda^2)\lambda^2} \quad .$$

In general, second order differential equations can always be reduced to coupled sets of first order differential equations. This is most commonly done by introducing the function

$$L(p^2) = K'(p^2) \quad .$$

Doing this we obtain the following two coupled equations from equation Eq. (5.34):

$$\begin{aligned} L'(p^2) &= \left[\frac{C}{p^2 K(p^2)} - L(p^2) \right] \frac{1}{p^2 - \lambda^2} \quad , \\ K'(p^2) &= L(p^2) \quad . \end{aligned}$$

However, for our problem this is not a good choice of the auxiliary function $L(p^2)$, since now $L'(\lambda^2)$ is infinite and a numerical solution of the system of equations as they stand

is not possible. Instead we define

$$L(p^2) = \lambda^2 [(p^2 - \lambda^2)K'(p^2) - K(p^2)] \quad ,$$

so that

$$L'(p^2) = \lambda^2(p^2 - \lambda^2)K''(p^2) \quad .$$

Now

$$L'(\lambda^2) = C \left[\frac{1}{K(\lambda^2)} - K(\lambda^2) \right]$$

which is perfectly well behaved.

The coupled set of first order differential equations we have to solve is then:

$$\begin{aligned} L'(p^2) &= \lambda^2 \left[\frac{C}{p^2 K(p^2)} - 2 \left(\frac{L(p^2)}{\lambda^2} + K(p^2) \right) \frac{1}{p^2 - \lambda^2} \right] \quad , \\ K'(p^2) &= \left[\frac{L(p^2)}{\lambda^2} + K(p^2) \right] \frac{1}{p^2 - \lambda^2} \quad , \end{aligned} \quad (5.35)$$

with the initial conditions being:

$$\begin{aligned} K(\lambda^2) &= \sqrt{\frac{14C}{3}} \sqrt{\ln \left(\frac{\lambda^2}{M^2} \right)} \quad , \\ L(\lambda^2) &= -\lambda^2 K(\lambda^2) \quad . \end{aligned}$$

Again we discretise the problem, defining a grid of N points x_i on a logarithmic scale in momentum squared. Recall Eq. (5.17):

$$x_i = \log_{10}(\lambda^2) + \frac{i}{N-1} \left\{ \log_{10}(\kappa^2) - \log_{10}(\lambda^2) \right\} \quad .$$

Changing variables in Eq. (5.35) to $x = \log_{10}(p^2)$ this becomes:

$$\begin{aligned} \frac{d}{dx} L(p^2) &= \lambda^2 p^2 \ln 10 \left[\frac{C}{p^2 K(p^2)} - 2 \left(\frac{L(p^2)}{\lambda^2} + K(p^2) \right) \frac{1}{p^2 - \lambda^2} \right] \quad , \\ \frac{d}{dx} K(p^2) &= p^2 \ln 10 \left[\frac{L(p^2)}{\lambda^2} + K(p^2) \right] \frac{1}{p^2 - \lambda^2} \quad , \end{aligned} \quad (5.36)$$

where $p^2 = 10^x$.

This coupled set of equations is then solved numerically using the Runge-Kutta method [62]. Again we choose different values for the infrared cutoff λ^2 and then change the unknown mass scale M^2 in our initial conditions. As mentioned before, we furthermore demand that the gluon renormalisation function is $G(\mu^2) \approx 1$ at the renormalisation scale μ^2 , which again we choose to be $\mu^2 = 10 \text{ GeV}^2$.

We find that, in order to fulfill this requirement, the mass scale M^2 in the gluon renormalisation function has to be equal to Λ_{QCD}^2 . With $M^2 \neq \Lambda_{QCD}^2$ it is not possible to find a value of the infrared cutoff λ^2 so that $G(\mu^2) = 1$. This can be compared with the results presented in chapter 5.3.3 for Mandelstam's equation which showed that λ^2 has to be bigger than Λ_{QCD}^2 in order to find a physically meaningful result (i.e. $G(\mu^2) \approx 1$), see Fig. (5.4-5.6). Here we have the extra parameter M^2 which, as we shall demonstrate acts, as a lower limit on the infrared cutoff λ^2 . Thus our simplified "toy" model *SDE* does in fact have the same qualitative behaviour as Mandelstam's equation.

In Tab. (5.2) we give the values for λ^2 and $G(\mu^2)$ for a fixed $M^2 = 10^{-4} \text{ GeV}^2$ and three different values of the renormalised coupling $\alpha_S(\mu^2) = 0.25$, $\alpha_S(\mu^2) = 0.1$ and $\alpha_S(\mu^2) = 0.08$, i.e. $\Lambda_{QCD}^2 = 0.276$, $\Lambda_{QCD}^2 = 1.264 \cdot 10^{-3}$ and $\Lambda_{QCD}^2 = 1.340 \cdot 10^{-4} \text{ GeV}^2$.

In Fig. (5.7) and Fig. (5.8) we plot our results for different values of M^2 for two different value of $\alpha_S(\mu^2)$. For illustrative purposes, we use an infrared cutoff close to the scale Λ_{QCD}^2 , since this gives the maximal infrared enhancement of the gluon renormalisation function $G(p^2)$. However, as we now discuss the infrared cutoff is only constrained by a lower limit, which is given by M^2 .

It is important to note here that $G(\lambda^2)$, Eq. (5.32), is given by

$$G(\lambda^2) \propto \left[\sqrt{\ln \left(\frac{\lambda^2}{M^2} \right)} \right]^{-\frac{1}{2}}$$

and a solution of Eq. (5.31) of the form:

$$G(\lambda^2) \propto \left[\sqrt{\ln \left(\frac{M^2}{\lambda^2} \right)} \right]^{-\frac{1}{2}}$$

is not possible. Hence the simple model *SDE*, we are studying, does not allow values of

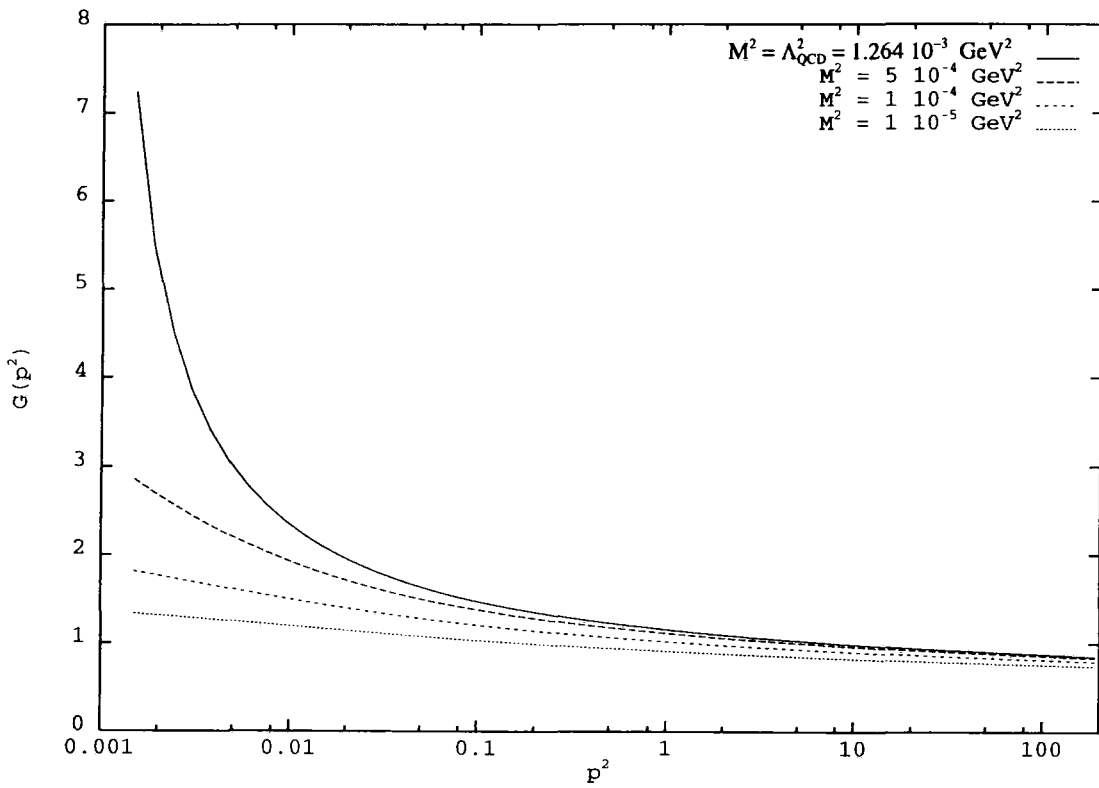


Figure 5.7: The gluon renormalisation function $G(p^2)$ as a function of p^2 with the renormalised coupling $\alpha_S(\mu^2) = 0.1$ and the infrared cutoff $\lambda^2 = 1.5 \cdot 10^{-3} \text{ GeV}^2$.

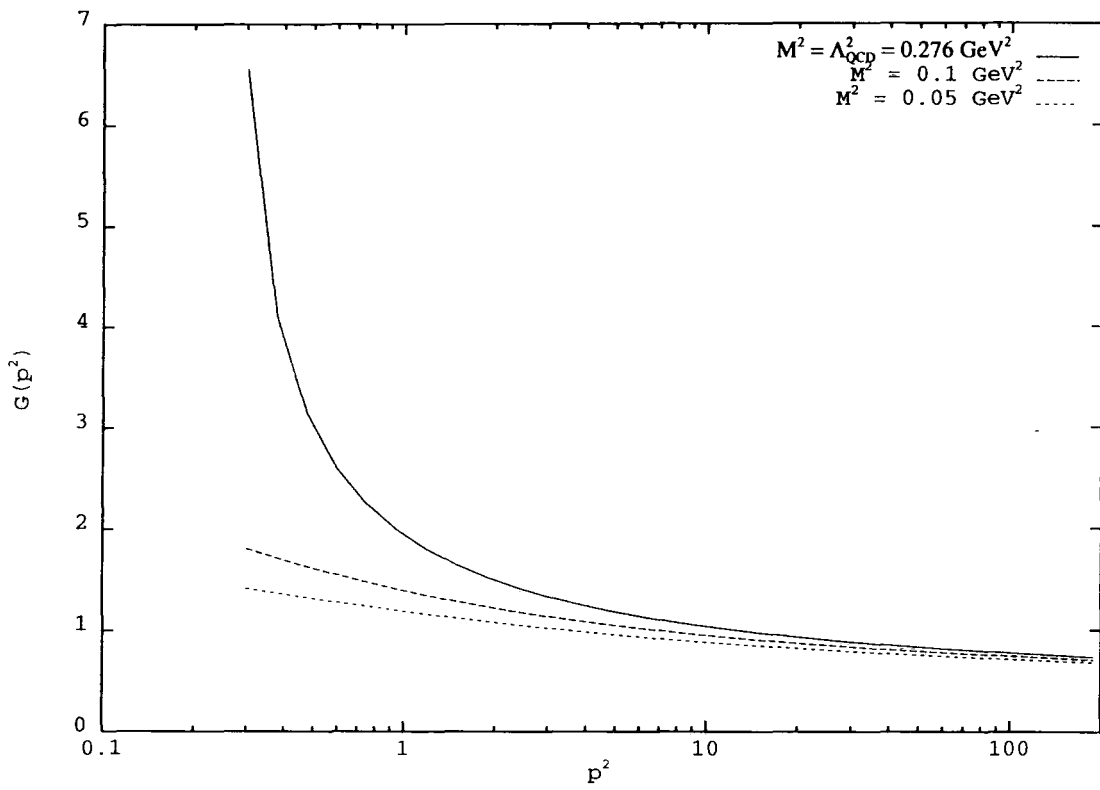


Figure 5.8: The gluon renormalisation function $G(p^2)$ as a function of p^2 with the renormalised coupling $\alpha_S(\mu^2) = 0.25$ and the infrared cutoff $\lambda^2 = 0.3 \text{ GeV}^2$.

| | $\alpha_S(\mu^2) = 0.25$ | $\alpha_S(\mu^2) = 0.1$ | $\alpha_S(\mu^2) = 0.08$ |
|--------------------------|--------------------------|-------------------------|--------------------------|
| λ^2/GeV^2 | $G(\mu^2)$ | $G(\mu^2)$ | $G(\mu^2)$ |
| $5 \cdot 10^{-4}$ | 0.560 | 0.892 | 0.996 |
| $1 \cdot 10^{-3}$ | 0.568 | 0.898 | 1.002 |
| $5 \cdot 10^{-3}$ | 0.571 | 0.908 | 1.010 |
| $1 \cdot 10^{-2}$ | 0.576 | 0.910 | 1.017 |
| $5 \cdot 10^{-2}$ | 0.578 | 0.915 | 1.020 |
| $1 \cdot 10^{-1}$ | 0.580 | 0.917 | 1.025 |

Table 5.2: Parameters for $G(\mu^2)$ for three different values of α_S , i.e. Λ_{QCD} for a gluon renormalisation function with $M^2 = 1 \cdot 10^{-4} \text{ GeV}^2$

$\lambda^2 < M^2$ which, with $M^2 = \Lambda_{QCD}^2$, means that we can only find solutions with $p^2 > \Lambda_{QCD}^2$. This is exactly the momentum region covered by perturbation theory. However, we should stress that our result is non-perturbative in nature in that it does not require the strong coupling constant to be small. The choice of λ^2 is therefore constrained to be greater than the QCD scale Λ_{QCD}^2 . This relation between the infrared cutoff we introduce to make the SDE infrared safe, and the scale Λ_{QCD} is explicitly demonstrated in the next section.

5.4.2 Analytically Relating λ^2 to Λ_{QCD}^2

Recalling the form of Eq. (5.31) for our “toy” model:

$$\frac{1}{G(\lambda^2)} = \frac{1}{G(\mu^2)} - \frac{7}{3}C \left\{ \int_{\lambda^2}^{\mu^2} dk^2 \frac{G(k^2)}{k^2} + \frac{1}{\mu^2} \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) \right\} ,$$

and making use of the fact $\mu^2 \gg \lambda^2$ we rewrite the above as:

$$\frac{1}{G(\lambda^2)} - \frac{1}{G(\mu^2)} = -\frac{7}{3}C \int_{\lambda^2}^{\mu^2} dk^2 G(k^2) \frac{1}{k^2} . \quad (5.37)$$

This equation can be solved analytically without any further approximation. Making the ansatz

$$G(k^2) = B \left(\ln \left(\frac{k^2}{M^2} \right) \right)^{-\frac{1}{2}} \quad (5.38)$$

we analytically calculate the integral on the right hand side of Eq. (5.37) giving:

$$-\frac{7}{3}C B \int_{\lambda^2}^{\mu^2} dk^2 \frac{(\ln(k^2/M^2))^{-\frac{1}{2}}}{k^2} = -\frac{14}{3}C B \left[\left(\ln \left(\frac{\mu^2}{M^2} \right) \right)^{\frac{1}{2}} - \left(\ln \left(\frac{\lambda^2}{M^2} \right) \right)^{\frac{1}{2}} \right] , \quad (5.39)$$

and inserting our ansatz, Eq. (5.38), into the left hand side of Eq. (5.37) we obtain:

$$\frac{1}{G(\lambda^2)} - \frac{1}{G(\mu^2)} = -\frac{1}{B} \left[\left(\ln \left(\frac{\mu^2}{M^2} \right) \right)^{\frac{1}{2}} - \left(\ln \left(\frac{\lambda^2}{M^2} \right) \right)^{\frac{1}{2}} \right] . \quad (5.40)$$

Therefore we find:

$$\frac{1}{B^2} = \frac{14}{3}C .$$

Note that, recalling our discussion of the value of β'_0 for Mandelstam's equation (chapter 5.3.3):

$$\frac{14}{3}C_A \frac{\alpha_S(\mu^2)}{4\pi} = \beta'_0 \frac{\alpha_S(\mu^2)}{4\pi} = \frac{1}{\ln(\mu^2/\Lambda_{QCD}^2)} \quad (5.41)$$

and hence

$$B = \left(\ln \left(\mu^2/\Lambda_{QCD}^2 \right) \right)^{\frac{1}{2}}$$

and our ansatz $G(k^2)$, Eq. (5.38), becomes:

$$G(k^2) = \left(\frac{\ln \left(\mu^2/\Lambda_{QCD}^2 \right)}{\ln \left(k^2/M^2 \right)} \right)^{\frac{1}{2}} . \quad (5.42)$$

Requiring $G(\mu^2) = 1$ we finally get $M^2 = \Lambda_{QCD}^2$.

In the limit of $p^2 \rightarrow \lambda^2$ the solution of our simplified model *SDE*, Eq. (5.27), matches the renormalisation group improved perturbative result and choosing $\lambda^2 < \Lambda_{QCD}^2$ is not

allowed. However, we stress again, that since in the derivation of the equation no perturbative expansion has been made, i.e. we are not requiring that the strong coupling constant is small, our results are non-perturbative in nature. Though the relation of $\alpha_S(k^2)$ to Λ_{QCD} we have used is that of one loop perturbation theory (see Eqs. (5.24, 5.41)), this has only been used for $k^2 = \mu^2$ where we assume perturbation theory applies.

5.5 Conclusion

We have studied the truncated SDE for the gluon propagator in quenched QCD and its dependence on the infrared regulator λ . Using the Mandelstam approximation [34] λ then turns out to be determined by the equation, giving us the possibility of performing the infrared regularisation entirely within the context of the SDEs. We have managed to show that the value of the infrared regulator is bounded by the QCD-scale Λ_{QCD} .

It is not surprising that the regulator λ we introduced to make the SD-integrals infrared safe is fixed by Λ_{QCD} , since the behaviour of the gluon propagator at all momenta does in fact depend on this scale when $\lambda \equiv 0$, as noted by Brown and Pennington [36]. However, the fact that the infrared cutoff has to be bigger than Λ_{QCD} , as we explicitly demonstrated in a simple analytically solvable model *SDE* is unexpected.

Apart from showing that for $\lambda > \Lambda_{QCD}$ the truncated SDE is correctly solved by perturbation theory, we expect that it should be possible to find a different, non-perturbative solution with $\lambda < \Lambda_{QCD}$. Since the truncated SDE is non-linear, it may accommodate several solutions. However, we have not been able to find a second solution of the infrared regularised Mandelstam equation, Eq. (5.22).

In analogy with Mandelstam's original approach [34] to solving the integral equation and infrared regularise it by enforcing the masslessness condition, which has also been followed by Brown and Pennington [36], we have looked for a gluon renormalisation function $G(p^2)$ of the form

$$G(p^2) = G_{pole}(p^2) + G_1(p^2) \quad ,$$

where $G_{pole}(p^2)$ is the infrared dominant part, giving only a mass term contribution under

the integral. Then, since the integrals in Mandelstam's equation are linear in $G(p^2)$ mass renormalisation implies that only $G_1(p^2)$ appears under the integrals. This is to be compared with the ansatz $G(p^2) = A\mu^2/p^2 + G_1(p^2)$ by Brown and Pennington [36] which we discussed in section 5.1.

Recalling the masslessness condition, Eq. (5.9):

$$\frac{p^2 - \lambda^2}{G_{MR}(p^2)} = \frac{p^2}{G_R(p^2)} - \frac{\lambda^2}{G_R(\lambda^2)}$$

we derive the following constraint for $G_{pole}(p^2)$ from Eq. (5.22):

$$\begin{aligned} p^2 \left(\int_{\lambda^2}^{p^2} dk^2 G_{pole}(k^2) I_1(k^2, p^2) + \int_{p^2}^{\mu^2} dk^2 G_{pole}(k^2) I_2(k^2, p^2) + \int_{\mu^2}^{\kappa^2} dk^2 G_{pole}(k^2) I_3(k^2, p^2) \right) \\ = \lambda^2 \left(\int_{\lambda^2}^{\mu^2} dk^2 G_{pole}(k^2) I_2(k^2, \lambda^2) + \int_{\mu^2}^{\kappa^2} dk^2 G_{pole}(k^2) I_3(k^2, \lambda^2) \right) \end{aligned}$$

However, the only solution seems to be $G_{pole}(p^2) = 0$ and again $\lambda < \Lambda_{QCD}$ is not allowed. This indicates that the infrared regulator λ plays an important role here. Indeed, the limit $\lambda \rightarrow 0$ is not analytic, so this is not the same as setting $\lambda \equiv 0$.

One problem with our approach to solving the truncated SDE could be that by imposing the masslessness condition, Eq. (5.9), we assume that the gluon renormalisation function under the integrals of Eq. (5.22) is automatically mass renormalised as well. This assumption is usually made in SDE studies of the gluon propagator ([28]–[36]). However there is no renormalisation group equation which ensures that this is true. In general the mass renormalised gluon function could have the form

$$G_{MR}(p^2) = f(p^2, \lambda^2) G(p^2)$$

with $f(p^2, \lambda^2 = 0) = 1$. Clearly this introduces a certain arbitrariness into our equations, since there are infinitely many choices for $f(p^2, \lambda^2)$.

We have repeated our analysis of the simple model SDE which we presented in section 5.4 with one choice for $f(p^2, \lambda^2)$, namely:

$$G_{MR}(p^2) = \frac{p^2}{p^2 + \lambda^2} G(p^2) \quad .$$

Inserting this into the right hand side of our simplified equation, Eq. (5.27), the differential equation (Eq. (5.30)) changes slightly, giving:

$$\frac{d^2}{dp^4} \frac{1}{G_{MR}(p^2)} = \left(\frac{7}{3} C \frac{p^2 + \lambda^2}{p^4} G_{MR}(p^2) - 2 \frac{d}{dp^2} \frac{1}{G_{MR}(p^2)} \right) \frac{1}{p^2 - \lambda^2} . \quad (5.43)$$

However, the initial conditions remain the same as before, see Eq. (5.32) and Eq. (5.33), and again it seems impossible to find a solution to Eq. (5.43) with $\lambda < \Lambda_{QCD}$.

To summarise:

Studying the dependence of the truncated gluon SDE on an infrared regulator λ we find that, although derived in a non-perturbative calculation including contributions from all orders in the strong coupling constant, α_S , the gluon renormalisation function $G(p^2)$ matches the renormalisation group improved perturbative result for $\lambda^2 > \Lambda_{QCD}^2$. Furthermore we find that choosing $\lambda^2 < \Lambda_{QCD}^2$ is not allowed. There is no known physical reason why the truncated SDE should not have a second solution, enabling us to explore the behaviour of the gluon propagator at much lower momenta. However, we have not been able to find such a solution for the infrared regularised Mandelstam equation, Eq. (5.22).

As explained in section 5.1, an infrared regularised form of the enhanced gluon propagator is needed to study quark confinement and DCSB. However, some important questions remain open: What are the implications of our results, presented in section 5.3.3 and 5.4.1, for confinement? Is the enhancement of the gluon propagator, once regulated in the way we described here, enough to confine quarks? To answer this question one could just insert our result for $G(p^2)$ into the quark SDE and cut off the integrals at the same value of λ determined from the gluon equation. However, without carrying out the calculations explicitly it is not clear what this would mean for the quark propagator.

Perhaps more importantly, one should study the coupled set of gluon and quark SDEs. In fact, doing this another mass scale, the mass of the light quarks, is introduced into the problem and there is the possibility that this second scale then fixes our infrared regulator λ , allowing a different, “non-perturbative” solution.

Chapter 6

Summary and Conclusions

In this chapter we summarise the results and conclusions of the work presented in this thesis. Furthermore, we discuss the application of a model gluon propagator which has a regularised infrared enhanced behaviour to SDE studies of quark confinement, DCSB and finally to hadron phenomenology. This is important because it highlights that although the infinite tower of SDEs has to be truncated to make any progress in their study and hence an uncertainty regarding their solution is introduced, there is a relation to physical observables which allows us to test the reliability of our solution. In particular, the application of SDE studies to hadron phenomenology means that experiment actually does probe the infrared behaviour of the gluon propagator. As we will highlight in section 6.2 the results found with model forms of the gluon propagator are very promising, however the aim remains to use $\Delta_{\mu\nu}$ obtained directly from QCD, i.e. from the gluon SDE, to calculate hadron observables.

6.1 “Confining” Gluons

In the preceding chapters we have investigated the infrared behaviour of the gluon propagator using the Schwinger-Dyson equation approach to study QCD non-perturbatively. Extensive work had been done studying the behaviour of the gluon and three different solutions had been reported in the literature. However, as discussed in chapter 4.4, an

infrared vanishing gluon had been ruled out by the fact that this behaviour does not lead to quark confinement. Therefore we have concentrated on the other two suggested solutions: the infrared enhanced, *confining* solution and the *confined* solution which has a singularity softer than a pole. Our analytic study of the possible infrared behaviour of the gluon propagator in both the Landau and the axial gauge reveals that the SDE of the gluon in QCD does not support an infrared softened behaviour, despite claims to the contrary. This *confined* solution had only been found in the axial gauge due to an incorrect sign in the approximate SDE. Only an infrared enhanced gluon propagator as singular as $1/p^4$ as $p^2 \rightarrow 0$ is a self-consistent solution of the truncated SDE.

We stress again, that this enhanced infrared behaviour of the gluon is an indication of confinement. This can be demonstrated in a gauge invariant way by the fact that the Wilson loop operator obeys an area law. Hence, could we show, without approximating or truncating the SDE, that the exact, full gluon propagator possesses this behaviour, we would have proven that QCD can indeed confine quarks. Furthermore, as we discuss in more detail in the next section, the $1/p^4$ -behaviour of the gluon, when employed in studies of the quark SDE, yields a quark propagator without a pole at timelike momenta, corresponding to quarks being unable to propagate on mass-shell and thus being confined. For this reason we call the infrared enhanced gluon “*confining*”. However, importantly, gluons with this behaviour are also *confined* themselves by not having a Källen-Lehmann spectral representation that any physical asymptotic state must have.

We then proceeded to investigate the consequences of the infrared behaviour of the gluon for the modelling of the pomeron in terms of dressed gluon exchange. The Landshoff-Nachtmann pomeron model requires an infrared softened gluon propagator which cannot be obtained as a self-consistent solution of the SDE. Is the behaviour of the gluon found in this study at variance with the pomeron? We argue it is not, since Landshoff and Nachtmann’s belief in an infrared softened, rather than enhanced, gluon results from a perturbative treatment of the quarks inside the hadrons to which the pomeron couples. However, since pomeron exchange is a soft process, perturbation theory is not valid. On the contrary, quark confinement should be taken into account in the modelling of the pomeron. An enhanced gluon propagator, as singular as $1/p^4$ as $p^2 \rightarrow 0$ is not at variance

with the pomeron but in accord with quark confinement.

However, as well as having all these desirable properties the enhanced, *confining* gluon propagator has one major drawback: it introduces infrared divergences in the SDE that need to be regulated. So far different regularisation procedures had been followed in the literature which we have described in chapter 5.1. This arbitrariness in regularising the infrared enhanced gluon causes one to question how stable and qualitatively reliable these procedures are.

We therefore continued our study of the infrared behaviour of the gluon by addressing the problem of regularising the $1/p^4$ -behaviour. In chapter 5, we studied Mandelstam's approximation to the gluon SDE and how it depends on an infrared regulator, λ . We have shown that the value of the infrared regulator is fixed by the QCD-scale Λ_{QCD} , which in quenched QCD is the only scale in the theory. However, although our results are derived from the SDE which is non-perturbative in nature, we find that $\lambda^2 > \Lambda_{QCD}^2$ is required and that our results match the renormalisation group improved perturbative ones. Choosing $\lambda^2 < \Lambda_{QCD}^2$ is not allowed.

The fact that an infrared regulator turns out to be determined by the non-linearity of the gluon equation holds out the possibility of performing the infrared regularisation entirely within the context of the SDEs. However, studying Mandelstam's approximation in quenched QCD we have not been able to find a second solution allowing us to investigate the properties of QCD at momenta smaller than Λ_{QCD} . It could be necessary to include quarks into the theory which would provide us with another scale and the hope that λ would then be fixed by the mass of the light quarks.

Obviously, our study of the infrared regularisation of the SDE is far from being complete and evidently open problems remain, requiring further research. We detailed these in section 5.5.

6.2 From Gluon Propagator to Hadron Phenomenology

There are numerous physical consequences connected with the behaviour of the gluon propagator in QCD. As we have mentioned a number of times throughout this thesis, the gluon propagator is an important element in the SDE of the quark propagator, Fig. (5.1). Studying the quark SDE we can determine whether the behaviour of the gluon does support DCSB and quark confinement, two important properties of our world which are responsible for the nature of the hadron spectrum and hadronic observables.

The fermion SDE is the easiest of all the SDEs to write down. It constitutes a relationship between the full quark propagator, the full gluon propagator and the full quark gluon vertex function (see Fig. (5.1)) and has the form:

$$S^{-1}(p^2) = S_{(0)}^{-1}(p^2) - \frac{g_0^2 C_F}{16\pi^4} \int d^4 k \gamma^\mu S(k) \Gamma^\mu(k, p) \Delta_{\mu\nu}(q^2) \quad . \quad (6.1)$$

Here $S(p)$ is the full quark propagator defined by:

$$S(p) = -i \frac{Z(p^2)}{\gamma_\mu p^\mu - M(p^2)} \equiv -i \gamma_\mu p^\mu \sigma_V(p^2) + \sigma_S(p^2) \quad , \quad (6.2)$$

where $Z(p^2)$ is the fermion renormalisation function,

$M(p^2)$ is the dynamical mass function of the quark and

$$\sigma_V(p^2) = \frac{Z(p^2)}{p^2 - M^2(p^2)} \quad \text{and} \quad \sigma_S(p^2) = M(p^2) \sigma_V(p^2) \quad .$$

The bare quark propagator, $S_{(0)}(p^2)$, is obtained from Eq. (6.2) by substituting $Z(p^2) = 1$ and $M(p^2) = m^{bare}$, where m^{bare} is the bare quark mass. C_F is the appropriate colour factor, $C_F = (N_C^2 - 1)/2N_C$, and $\Gamma^\mu(k, p)$ is the full quark-gluon vertex.

Again, in a way similar to the approach followed in studies of the gluon SDE, one has to make an ansatz for the full vertex function which, of course, satisfies yet another SDE involving the complete quark-antiquark scattering (Bethe-Salpeter) kernel. However, so far no attempts have been made to solve the vertex SDE. Instead an ansatz is made,

constructed in a way not to violate any of the physical constraints which this SDE would satisfy. The vertex ansatz should:

- satisfy the Slavnov-Taylor identity.
- be free of kinematic singularities.
- have the same C, P, T transformation properties as the bare vertex.
- reduce to the bare vertex in the free field limit.
- ensure multiplicative renormalisability of the SDEs.
- be gauge covariant.

Extensive work has been carried out in QED trying to find a suitable vertex ansatz which satisfies the above criteria, see Ref. [29] and [64, 65]. Neglecting the effects of ghosts in QCD one can make use of what has been learnt in QED and use the vertex ansätze obtained there.

Choosing an ansatz for the full quark gluon vertex and using the full gluon propagator found in studies of the gluon SDE one can derive a pair of coupled, nonlinear integral equations for the two unknown quark functions, $Z(p^2)$ and $M(p^2)$, or equivalently $\sigma_V(p^2)$ and $\sigma_S(p^2)$. These equations are then solved numerically.

The analytic structure of the quark propagator obtained from the SDE has some important implications: Should the quark propagator have no singularities at timelike momenta, i.e. should it not have a mass-pole so that the dynamical mass function of the quark, Eq. (6.2), has the property $M^2(p^2) \neq p^2$ for any $p^2 \geq 0$, then a sufficient condition for confinement is satisfied.

In addition, the quark propagator determines whether our model supports DCSB. The quark condensate, $\langle \bar{q}q \rangle$, is an order parameter for DCSB and is easily related to the trace of the quark propagator:

$$\langle \bar{q}q \rangle = - \lim_{x \rightarrow 0^+} \text{tr} S(x, 0) = -12i \int \frac{d^4 p}{(2\pi)^4} \frac{Z(p^2) M(p^2)}{p^2 - M^2(p^2)} ,$$

where $S(x, y)$ is the coordinate space quark propagator and where the trace over spinor and colour indices give a factor 12. A nonzero value of the quark condensate indicates DCSB.

As discussed in chapter 5.1, the infrared enhanced, *confining* gluon propagator, once regularised, leads to a quark propagator with no physical poles, in accord with confinement, and manifest DCSB (see Ref. [56]-[59]).

Furthermore, once the behaviour of both gluon and quark propagator is known one can develop a phenomenology of hadrons based on the SD and the Bethe-Salpeter equations (BSE). The BSE relates the proper meson-quark vertex function Γ_{meson} to the full quark propagator and the quark-antiquark scattering kernel K and is illustrated in Fig. (6.1).

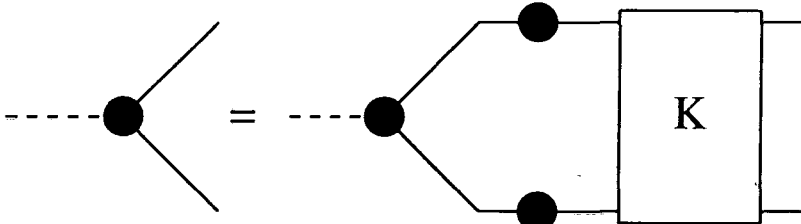


Figure 6.1: The Bethe-Salpeter equation for mesons. Here the dashed line represents a meson.

The meson BSE has the form

$$\Gamma_{meson}(p; P) = i \int \frac{d^4k}{(2\pi)^4} \Gamma_{meson}(k; P) S(k - \frac{1}{2}P) S(k + \frac{1}{2}P) K \quad ,$$

where P is the centre-of-mass momentum of the bound state.

The quark-antiquark scattering kernel is usually approximated by one gluon exchange and so involves an effective single gluon propagator and two bare quark-gluon vertices, i.e.

$$K \approx g^2 \gamma_\mu \Delta_{ab}^{\mu\nu}(p) \gamma_\nu \quad .$$

Using a gluon propagator whose infrared behaviour is enhanced (but regulated) and whose ultraviolet behaviour matches the known perturbative result, and a quark propagator obtained from the fermion SDE with the same gluon propagator, a number of hadronic

observables have been calculated from the BSE and found to be in good agreement with experimental data. (For an extensive review see Ref. [37].) For illustrative purposes we detail some of the more recent results found by Frank and Roberts [59] with a *confining*, one parameter model gluon propagator in Tab. (6.1). We see the agreement is most encouraging.

| | Calculated | Experiment |
|-------------------------|------------|---------------------|
| m_π | 138.7 MeV | 138.3 ± 0.5 MeV |
| f_π | 92.3 MeV | 92.4 ± 0.3 MeV |
| r_π | 0.24 fm | 0.31 ± 0.004 fm |
| $g_{\pi^0\gamma\gamma}$ | 0.45 | 0.50 ± 0.02 |

Table 6.1: Some hadron observables calculated in Ref. [59] compared to the corresponding experimental data.

6.3 Final Conclusions

This thesis has been concerned with the investigation of the infrared behaviour of the gluon propagator and its implications for confinement. We have found that the solution of the gluon SDE incorporates both quark and gluon confinement. Furthermore, the behaviour of the gluon found as the only self-consistent solution of the truncated SDE once infrared regulated gives good agreement for hadron observables calculated from the BSE. We then attempted to perform the infrared regularisation of the SDEs consistently within the calculational framework. This is possible with the introduction of an infrared regulator. However, further research is required here to allow a non-perturbative study of

the gluon and quark SDEs and their application to hadron phenomenology at momenta below the QCD-scale Λ_{QCD} .

Appendix A

Angular Integrals

This appendix will give a derivation of the angular integrals we have used in the calculations of chapter 4 and 5, as given in [66].

As mentioned before our calculations are all performed in Euclidian space. We choose our coordinate system so that the external momentum p is defined:

$$p^\mu = (p, 0, 0, 0) \quad (\text{A.1})$$

and the loop momentum is:

$$k^\mu = (k \cos \psi, k \sin \psi \sin \theta \cos \phi, k \sin \psi \sin \theta \sin \phi, k \sin \psi \cos \theta) \quad , \quad (\text{A.2})$$

where $\psi \in [0, \pi]$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. The four dimensional integrals over the loop momentum k can then be written as:

$$\int d^4k = \int \frac{k^2 dk^2}{2} \sin^2 \psi d\psi \sin \theta d\theta d\phi \quad . \quad (\text{A.3})$$

All functions in the integrand depend only on p^2 , k^2 and $k \cdot p$ and thus from Eq. (A.1) and (A.2) are independent of θ and ϕ . Thus the integration over these two angles can be performed trivially giving a factor of 4π . The general form of the angular integrals (over ψ) we are left to calculate is then:

$$\int_0^\pi \sin^2 \psi d\psi \frac{(k \cdot p)^n}{(q^2)^m} \quad ,$$

where $q = p - k$. Using $k \cdot p = |k| |p| \cos \psi$ and $q^2 = a - b \cos \psi$, where we defined $a \equiv k^2 + p^2$ and $b \equiv 2 |k| |p|$ we can rewrite the above in the following way:

$$I_{n,m} = \left(\frac{b}{2}\right)^n \int_0^\pi \sin^2 \psi \, d\psi \frac{(\cos \psi)^n}{(a - b \cos \psi)^m} \quad . \quad (\text{A.4})$$

We start by considering the simplest of these integrals, $I_{0,1}$:

$$I_{0,1} = \int_0^\pi d\psi \frac{\sin^2 \psi}{(a - b \cos \psi)} \quad . \quad (\text{A.5})$$

Changing variable to $z = \cos \psi$, giving

$$I_{0,1} = \int_{-1}^1 dz \frac{\sqrt{1-z^2}}{a-bz} \quad ,$$

and then to $y = a - bz$:

$$I_{0,1} = \frac{1}{b^2} \int_{a-b}^{a+b} dy \frac{\sqrt{b^2 - (a-y)^2}}{y} \quad .$$

We now substitute

$$R = b^2 - (a-y)^2 = b^2 - a^2 + 2ay - y^2 = A + By + Cy^2$$

and where we define

$$\begin{aligned} A &= (b^2 - a^2) \quad , \\ B &= 2a \quad , \\ C &= 1 \quad \text{and} \\ \Delta &= 4AC - B^2 = -4b^2 \end{aligned}$$

The integral we have to solve is then:

$$I_{0,1} = \frac{1}{b^2} \int_{a-b}^{a+b} dy \frac{\sqrt{R}}{y} \quad (\text{A.6})$$

and can be calculated explicitly [67], giving:

$$I_{0,1} = \frac{1}{b^2} \left\{ \sqrt{R} \Big|_{a-b}^{a+b} + A \int_{a-b}^{a+b} \frac{dy}{y\sqrt{R}} + \frac{B}{2} \int_{a-b}^{a+b} \frac{dy}{\sqrt{R}} \right\} \quad , \quad (\text{A.7})$$

where the first term vanishes by symmetry. Since $A < 0$ and $\Delta < 0$ the integral of the second term of Eq. (A.7) is:

$$\begin{aligned}
 \int_{a-b}^{a+b} \frac{dy}{y\sqrt{R}} &= \frac{1}{\sqrt{-A}} \left[\arcsin \left(\frac{2A + By}{y\sqrt{-\Delta}} \right) \right]_{a-b}^{a+b} \\
 &= \frac{1}{\sqrt{a^2 - b^2}} \left[\arcsin \left(\frac{2b^2 - 2a^2 + 2a(a+b)}{(a+b)2b} \right) \right. \\
 &\quad \left. - \arcsin \left(\frac{2b^2 - 2a^2 + 2a(a-b)}{(a-b)2b} \right) \right] \\
 &= \frac{\pi}{\sqrt{a^2 - b^2}} .
 \end{aligned} \tag{A.8}$$

Finally the integral of the last term of Eq. (A.7), since $C = -1 < 0$, is:

$$\begin{aligned}
 \int_{a-b}^{a+b} \frac{dy}{\sqrt{R}} &= -\frac{1}{\sqrt{-C}} \left[\arcsin \left(\frac{2Cy + B}{\sqrt{-\Delta}} \right) \right]_{a-b}^{a+b} \\
 &= - \left[\arcsin \left(\frac{-2(a+b) + 2a}{2b} \right) - \arcsin \left(\frac{-2(a-b) + 2a}{2b} \right) \right] \\
 &= \pi .
 \end{aligned} \tag{A.9}$$

Inserting Eq. (A.8) and Eq. (A.9) into Eq. (A.7) gives:

$$I_{0,1} = \frac{\pi}{b^2} \left(a - \sqrt{a^2 - b^2} \right) \tag{A.10}$$

Once we have calculated this integral all the other integrals $I_{n,m}$ are easily obtained from Eq. (A.10) by differentiation. However, first we need to know the form of $I_{r,0}$. With:

$$I_{r,0} = \int_0^\pi d\psi \sin^2\psi \cos^r\psi$$

Making the substitutions $z = \cos\psi$ and then $w = z^2$ we find:

$$I_{r,0} = \int_{-1}^1 dz z^r \sqrt{1-z^2} = \int_0^1 dw w^{\frac{r-1}{2}} \sqrt{1-w} = B\left(\frac{r+1}{2}, \frac{3}{2}\right) ,$$

where $B(a, b)$ is the well known Beta function which can be written in terms of Gamma functions, which we define in Appendix B, as:

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} .$$

Using the property of the Gamma function $\Gamma(n) = (n-1)\Gamma(n-1)$, we finally obtain:

$$I_{r,0} = \frac{\Gamma(\frac{r+1}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{r+4}{2})} = \left(\frac{r-1}{r+2}\right) I_{r-2,0} \quad (\text{A.11})$$

For r odd $I_{r,0}$ vanishes by symmetry and furthermore

$$I_{0,0} = \frac{\pi}{2}$$

so that using Eq. (A.11) all integrals of the form $I_{r,0}$ are known.

The integrals $I_{n,m}$ can now be calculated using the relations:

$$I_{n,1} = -\frac{1}{b}I_{n-1,0} + \frac{a}{b}I_{n-1,1} \quad (\text{A.12})$$

and

$$\frac{\partial}{\partial a} I_{n,1} = -I_{n,2} \quad (\text{A.13})$$

Using Eq. (A.12) together with Eq. (A.10) and Eq. (A.11) we find:

$$\begin{aligned} I_{0,1} &= \frac{\pi}{b^2} (a - \sqrt{a^2 - b^2}) \quad , \\ I_{1,1} &= \frac{\pi a}{2b^2} (a - \sqrt{a^2 - b^2}) - \frac{\pi}{4} \quad , \\ I_{2,1} &= \frac{\pi a^2}{4b^2} (a - \sqrt{a^2 - b^2}) - \frac{\pi a}{8} \quad , \\ I_{3,1} &= \frac{\pi a^3}{8b^2} (a - \sqrt{a^2 - b^2}) - \frac{\pi a^2}{16} - \frac{\pi b^2}{64} \quad . \end{aligned}$$

With these and the relation Eq. (A.13) we compute:

$$\begin{aligned} I_{0,2} &= \frac{\pi}{b^2} \left(\frac{a}{\sqrt{a^2 - b^2}} - 1 \right) \quad , \\ I_{1,2} &= \frac{\pi}{2b^2} \left(\frac{a^2}{\sqrt{a^2 - b^2}} - 2a + \sqrt{a^2 - b^2} \right) \quad , \\ I_{2,2} &= \frac{\pi}{4b^2} \left(\frac{a^3}{\sqrt{a^2 - b^2}} - 3a^2 + \frac{1}{2}b^2 + 2a\sqrt{a^2 - b^2} \right) \quad , \\ I_{3,2} &= \frac{\pi}{8b^2} \left(\frac{a^4}{\sqrt{a^2 - b^2}} - 4a^3 + ab^2 + 3a^2\sqrt{a^2 - b^2} \right) \quad . \end{aligned}$$

These are all the integrals needed for the calculation of this study. We have $a = p^2 + k^2$ and $b = 2pk$, noting that all the integrals contain the quantity:

$$\sqrt{a^2 - b^2} = \left((p^2 + k^2)^2 - 4p^2k^2 \right)^{\frac{1}{2}} = (p^4 + k^4 - 2p^2k^2)^{\frac{1}{2}} = |p^2 - k^2|$$

We introduce the following function $h(x)$:

$$h(x) = \frac{1}{2}(1 + x - |1 - x|) = \begin{cases} x & \text{for } x < 1 \\ 1 & \text{for } x \geq 1 \end{cases} .$$

Inserting the quantities for a and b and making use of the function $h(x)$ we can now write our integrals in the following way:

$$\begin{aligned} I_{0,1} &= \frac{\pi}{4k^2p^2} \left((p^2 + k^2) - |p^2 - k^2| \right) \\ &= \frac{\pi}{4k^2} \left(1 + \frac{k^2}{p^2} - \left| 1 - \frac{k^2}{p^2} \right| \right) \\ &= \frac{\pi}{2k^2} h \left(\frac{k^2}{p^2} \right) \quad , \\ I_{0,2} &= \frac{\pi}{4k^2p^2} \left(\frac{(p^2 + k^2)}{|p^2 - k^2|} - 1 \right) \\ &= \frac{\pi}{4k^2p^2} \frac{1}{|p^2 - k^2|} \left((p^2 + k^2) - |p^2 - k^2| \right) \\ &= \frac{\pi}{2k^2} \frac{1}{|p^2 - k^2|} h \left(\frac{k^2}{p^2} \right) \quad . \end{aligned}$$

Equivalently:

$$\begin{aligned} I_{1,1} &= \frac{\pi p^2}{4k^2} h \left(\frac{k^4}{p^4} \right) \quad , \\ I_{2,1} &= \frac{\pi p^2}{8k^2} (p^2 + k^2) h \left(\frac{k^4}{p^4} \right) \quad , \\ I_{3,1} &= \frac{\pi p^6}{16k^2} h \left(\frac{k^8}{p^8} \right) + \frac{\pi p^4}{8} h \left(\frac{k^4}{p^4} \right) \quad , \\ I_{1,2} &= \frac{\pi p^2}{2k^2} \frac{1}{|p^2 - k^2|} h \left(\frac{k^4}{p^4} \right) \quad , \end{aligned}$$

$$I_{2,2} = \frac{3\pi p^4}{8k^2} \frac{1}{|p^2 - k^2|} h\left(\frac{k^6}{p^6}\right) + \frac{\pi p^2}{8} \frac{1}{|p^2 - k^2|} h\left(\frac{k^2}{p^2}\right) ,$$
$$I_{3,2} = \frac{\pi p^6}{4k^2} \frac{1}{|p^2 - k^2|} h\left(\frac{k^8}{p^8}\right) + \frac{\pi p^4}{4} \frac{1}{|p^2 - k^2|} h\left(\frac{k^4}{p^4}\right) .$$

Appendix B

Special Functions

In this Appendix we will give the definition of the Γ -function and its logarithmic derivative, the Ψ -function. These two special functions and some of their properties, which we also give here, have been used in the calculations of chapter 4.

The function $\Gamma(x)$ can be defined by :

$$\Gamma(x) = \int_0^1 dy e^{-y} y^{x-1}, \quad (x > 0). \quad (\text{B.1})$$

It has the following properties :

$$\Gamma(x + 1) = x \Gamma(x)$$

and

$$\Gamma(1) = 1 \quad , \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad .$$

The ψ -function is the logarithmic derivative of the Γ -function which is defined by:

$$\psi(z) \equiv \frac{d}{dz} [\ln \Gamma(z)] \quad (\text{B.2})$$

and has the property:

$$\psi(z + 1) = \psi(z) - \frac{1}{z} \quad . \quad (\text{B.3})$$

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