Hopf hypersurfaces

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by

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ABSTRACT.

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This thesis is concerned with Hopf hypersurfaces of Kähler and nearly Kähler manifolds and gives special emphasis to the cases of hypersurfaces of complex projective spaces and of the 6-sphere endowed with its nearly Kähler almost complex structure. Although there is already a wealth of investigations done in the case of complex space forms and the 6-sphere, a full classification of these hypersurfaces in the former spaces was done under assumption of constancy of the rank of its focal map. Here, the classification is revisited and this assumption is removed although a complete classification is still not obtained. The characterization of the Hopf hypersurfaces of the 6-sphere as tubular hypersurfaces around almost complex curves is used to determine among these hypersurfaces special examples which have constant mean curvature or are Einstein hypersurfaces. The invariants needed to decide when a pair of hypersurfaces of S^6 and $\mathbb{C}P^n$ are respectively G_2-congruent and holomorphically congruent are determined and this result is applied to characterize the hypersurfaces of these spaces whose Hopf vector fields are also Killing field. Finally, the linearly full almost complex 2-spheres of S^6 with at most two singularities are determined up to $G_2^{\mathbb{C}}$-congruence of their directrix curves and this is used to determine the space of linearly full almost complex 2-spheres of S^6 with suitably small induced area.

Declaration

This work is the result of research carried out between October 1995 and December 1998. The work presented in this thesis has not been submitted in fulfilment of any other degree or professional qualification.

Throughout this work all non-original material, either in form of statement or proof, is referred to its corresponding original source.

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Along the Journey of the Hero

He who evolves

Spiritualises all matter

But he who is descending to unconciousness

sees only matter everywhere

Knowledge, feelings and thoughts

Are all still matter

Though subtler

How could a carp understand a shark ?

The dolphin shall certainly go far

But he shall only master the Journey

When ignorance and passion and goodness

Are all left behind

Thereafter, there shall only be

Real Love

Real Wisdom

Truth.

(by Ahky)

This Journey
Is dedicated to
Nivarlina and Zambar

Contents

Introduction.

The study of hypersurfaces or curves of Riemannian manifolds has always been facilitated by the fact that these are the submanifolds of lowest co-dimension or dimension. However, we are far from achieving any kind of unified treatment in the investigation of these particular submanifolds despite of the impressive progress made in the 20th century. Nevertherless, mathematicians have obtained enumerable results in this area by considering special ambient spaces or by making further assumptions to be satisfied by these submanifolds.

In this piece of work we shall consider real hypersurfaces of special ambient spaces such as the 6-sphere and the complex projective spaces and also hypersurfaces of more general spaces like Kähler and nearly Kähler manifolds. The 6-sphere is a special example in this category since it is endowed with a non-Kähler nearly Kähler almost complex structure. Nevertherless, an almost complex structure of the ambient space yields in each of its oriented hypersurfaces a special tangent vector field which is obtained by applying the almost complex structure of the ambient space to a unit normal vector field of the hypersurface. Henceforth, we shall name this special vector field as the Hopf vector field of the hypersurface.

A hypersurface of a nearly Kähler manifold is said to be a Hopf hypersurface when the foliation given by its Hopf vector field is geodesic, in other words, when the integral curves of its Hopf vector field are geodesics of the hypersurface.

Some beautiful and elegant studies of these hypersurfaces have already been carried out throughout the last twenty years and to the best of my knowledge it was Yoshiaki Maeda [31] who in 1976 published the first results concerned with these hypersurfaces for the case of the complex projective spaces. In 1982, Cecil and Ryan [18], assuming the constancy of the rank of the focal map of the hypersurface, characterized the Hopf hypersurfaces of the complex projective spaces as open subsets of tubes around complex submanifolds. In 1986, Kimura [29] used the result of Cecil and Ryan and the work of Takagi [39] on homogeneous hypersurfaces of the complex projective spaces to characterize the Hopf hypersurfaces of constant principal curvatures as tubes around some special complex submanifolds of this complex space form. In 1989, Berndt [2] has obtained similar characterizations for the Hopf hypersurfaces of complex hyperbolic space forms. Finally, in 1995 Berndt, Bolton and Woodward [5] gave a complete characterization of the Hopf hypersurfaces of the 6-sphere as tubular hypersurfaces around almost complex curves, these curves being fully classified in [9].

It is worth mentioning here that most of the content of each chapter of this thesis is made up of original research and we have opted for distributing among the sections any basic background material accordingly to the needs of each particular chapter. In the sequel, we give a brief layout of this work.

In chapter 1, we state some of the main results already known about the characterization of Hopf hypersurfaces of complex space forms and of the 6-sphere. We give particular attention to the Hopf hypersurfaces of this sphere, determining special examples of hypersurfaces with constant principal curvatures, constant mean curvature and also those which are Einstein hypersurfaces.

In chapter 2, we use the transitive action of the excepcional Lie group G_2 on the 6-sphere to obtain a special type of rigidity for hypersurfaces of this sphere. Namely, given an isometric immersion $f : M \to S^6$ of a non totally umbilic hypersurface M of the 6-sphere whose second fundamental form has rank greater or equal to 3, we prove that this immersion is extendable to an element of G_2 if and only if its derivative maps the Hopf vector field of M to the Hopf vector field of $f(M)$. This result is firstly obtained for the case of a Hopf hypersurface in section (2.2) and then generalized to any hypersurface in section (2.3). Carrying on with congruence of hypersurfaces, we obtain a new proof for a similar theorem on rigidity of hypersurfaces in $\mathbb{C}P^n$ obtained by Suh-Takagi in [38].

The last two sections of chapter 2 are dedicated to giving an application of these rigidity results above to determine the hypersurfaces of the 6-sphere and of the complex projective spaces whose Hopf vector fields are Killing fields.

In chapter 3, we give the first steps on the way to investigate Hopf hypersurfaces in more general Riemannian manifolds. We do this in section (3.2) whereby we start by characterizing the complex space forms as the Kähler manifolds all of whose geodesic hyperspheres are Hopf hypersurfaces. Still in this section, we consider the reflection map and the push maps induced by a hypersurface of a Kähler manifold and then we determine necessary and sufficient conditions to be satisfied by these maps in order that the hypersurface be a Hopf hypersurface.

In section (3.3), we obtain some geometrical properties of Hopf hypersurfaces of $\mathbb{C}P^n$ which as well as being relevant on their own also suggest that the assumption used by Cecil-Ryan to characterize Hopf hypersurfaces as tubes can actually be proved. This is exactly what is done in section (3.4), that is, we use all the geometrical understanding achieved about Hopf hypersurfaces of this complex space form in order to prove that if we assume that every continuous principal curvature function defined on the hypersurface admits a corresponding continuous principal vector field then the rank of the focal map of a Hopf hypersurface in this space is indeed constant. We prove this by means of a special construction of vector fields along geodesics normal to the hypersurface. Therefore, our approach to this problem is to deal with the Hopf hypersurface from a quite extrinsic geometrical viewpoint.

Chapter 3 is closed with the important fact that the lift of Hopf hypersurfaces under a holomorphic Riemannian submersion $\pi : \widetilde{W} \to W$ are also Hopf hypersurfaces. This can provide us with a means to obtain examples of Hopf hypersurfaces in more general Kähler manifolds which could possibly be non tubular hypersurfaces.

As we have mentioned above the Hopf hypersurfaces of the 6-sphere are characterized as open subsets of tubes around almost complex curves. Thus we are also motivated to obtain explicit examples of such curves. For this reason, in chapter four, we are mainly interested in finding all the linearly full almost complex 2-spheres of S^6 with at most two singularities. In order to do this, we construct in section (4.4) an example of such a curve for each given sigularity type and then we prove that any other such a curve has directrix curve $G_2^{\mathbb{C}}$-equivalent to that one given in our example. In the last section of this chapter we determine the moduli spaces of these curves with suitably small area.

Chapter 1

Special Hypersurfaces of S^6.

1.1 Introduction.

Some authors have investigated Hopf hypersurfaces in complex space forms, obtaining a good wealth of results which essentially characterize these hypersurfaces as tubular hypersurfaces around complex submanifolds. Although the 6-sphere is not a Kähler manifold, it can be endowed with a nearly Kähler almost complex structure and a complete characterization of the Hopf hypersurfaces of S^6 as open subsets of tubes around almost complex curves is known. We state the results concerned with these characterizations in section (1.3) and in section (1.4) we determine those Hopf hypersurfaces of the 6-sphere with particular geometrical properties.

Let M be a submanifold of a Riemannian manifold $\overline{M}$. We shall use, throughout this thesis, the notation $\langle , \rangle$, $\overline{\nabla}$, $\overline{R}$ to denote the metric, Riemannian connection and curvature tensor respectively of $\overline{M}$ whilst all the corresponding induced objects on M shall be denoted simply by $\langle , \rangle$, ∇, R.

The normal bundle $\perp M$ of M in $\overline{M}$ is a manifold. In fact, it can be seen as a vector bundle which is a subbundle of the restriction to M of the tangent bundle of $\overline{M}$.

We shall name the restriction of the exponential map of $\overline{M}$ to the normal bundle of

M, given by $G(p,v) := \exp_p(v)$ for $(p,v) \in \perp M$, as the **normal exponential map** of M. It is possible to prove that if $\overline{M}$ is complete then G is defined for all (p,v) and if M is compact then G maps diffeomorphically a neighbourhood $\mathcal{O}_{\mathrm{M}}$ of $M \subset \perp M$ onto a neighbourhood of $M \subset \overline{M}$. In order to simplify our notation we shall consider the map $\Phi_r : \perp^1 M \longrightarrow \overline{M}$ obtained from G as

$$\Phi_r(p,\eta) = G(p, r\eta), \tag{1.1.1}$$

where $\perp^1 M$ denotes the **unit normal bundle** of M.

Definition 1.1.1 *For each $r > 0$ we define the* **tube** *of radius r around M as the image set $\Phi_r(M)$. In particular, if M is a hypersurface of $\overline{M}$, we say that any open subset $M_s \subset \Phi_s(\perp^1 M)$ is a* **level hypersurface** *associated to M.*

If M is compact then for sufficiently small r, M_r is a submanifold of $\overline{M}$ and the restriction of G can give us a diffeomorphism from $\left\{(p,v) \in \perp M \text{ with } |v| = r\right\}$ onto $M_r \subset G(\mathcal{O}_{\mathrm{M}})$.

We shall use extensively throughout this thesis the notation and abreviation $\gamma = \gamma_{(p,\eta)}$ to denote a geodesic of $\overline{M}$ parametrized by the archlength and satisfying the initial conditions $\gamma(0) = p \in M$ and $\dot\gamma(0) = \eta \in \perp^1_p M$. In terms of the normal exponential map G we have

$$\gamma_{(p,\eta)}(r) = G(p, r\eta). \tag{1.1.2}$$

Definition 1.1.2 *Let π denote the canonical projection of the normal bundle of M onto M. We say that a point $q \in \overline{M}$ is a* **focal point** *of multiplicity $\nu > 0$ of M if there exists a point $(p,\eta) \in \perp M$ such that $q = G(p,\eta)$, and the Jacobian of G has nullity $\nu \neq 0$ at (p,η).*

Remark 1.1.1 *When M is a hypersurface of a Riemannian manifold $\overline{M}$ and ξ is a unit local normal vector field defined on M, we shall consider throughout this thesis the* **focal map** *of M with respect to the field ξ defined as follows. Let W denote the* *set of all the points $q \in M$ such that the geodesic $\gamma_{(q,\xi)}$ contains at least one focal point of M. Then the focal map takes each point $q \in W$ to the first focal point of M situated in the corresponding geodesic $\gamma_{(q,\xi)}$. As we shall see further ahead in this chapter, it turns out that if M is a Hopf hypersurface of a real or complex space form then the focal map is defined on M and its focal points occur at the same distance of the hypersurface, thus in this case the focal map is just Φ_r for a convenient real value r.*

For the next definition and elsewhere in this thesis we shall only be considering orthogonal (almost) complex structures of the manifolds involved.

Definition 1.1.3 *Let $\overline{M}$ be a Riemannian manifold endowed with an orthogonal almost complex structure J. Let M be a hypersurface of $\overline{M}$. Let ξ be a local unit normal vector field on M. The tangential vector field $U := J\xi \in \mathfrak{X}(M)$ will be called the* **Hopf vector field** *of M and we shall say that M is a* **Hopf hypersurface** *of $\overline{M}$ if the integral curves of U are geodesics of M, that is*

$$\nabla_U U = 0. \tag{1.1.3}$$

We shall make an extensive use of Jacobi fields throughout this work. These have been a powerful tool employed by differential geometers to approach a large range of mathematical issues. It is very easy to find a good wealth of the basic theory about these fields in the literature, however, we shall use quite often in this thesis the characterization of a Jacobi field as a variational vector field defined by a geodesic variation and we shall also make some use of the following property.

Lemma 1.1.1 *Given $p \in M$, let η denote a local normal vector field defined on M and let A_η be the shape operator of M with respect η. Then a Jacobi vector field $W(s)$ defined along a geodesic $\gamma = \gamma_{(p,\eta)}(s)$ of $\overline{M}$, shall satisfy the conditions*

$$W(0) \in T_pM \qquad \text{and} \qquad \dot{W}(0) + A_\eta(W(0)) \in \perp_p M, \tag{1.1.4}$$

if and only if $W(s)$ is the variational vector field corresponding to a geodesic variation $f : (-\epsilon, \epsilon) \times [0, r] \longrightarrow \overline{M}$ of γ complying with the following conditions

$$f(t, 0) \in M \textit{ for each } t \in (-\epsilon, \epsilon) \quad \textit{and} \quad \frac{\partial f}{\partial s}(t, 0) \in \perp_{f(t,0)} M. \tag{1.1.5}$$

In this case, we shall say that W is a **M-Jacobi field** *of $\overline{M}$.*

In addition, it is important to highlight here the following facts. Since the M-Jacobi field W is orthogonal to γ, the variation given in the lemma yields locally a surface of $\overline{M}$, which implies $\frac{D}{\partial s}\frac{\partial f}{\partial t} = \frac{D}{\partial t}\frac{\partial f}{\partial s}$ and consequently $[\dot{\gamma}, W] = 0$. Therefore, if ξ denotes a local unit normal field on the tubular hypersurface M_r around M, then the corresponding shape operator A_ξ of M_r satisfies:

$$(A_\xi W)(r) = -(\overline{\nabla}_{\dot{\gamma}} W)(r). \tag{1.1.6}$$

A proof for Lemma (1.1.1) can be found for instance in [21].

We finish this section giving a general illustration of a tube or a tubular hypersurface in order to set up the typical geometrical frame that we have in mind for most of the results to be obtained in this thesis.

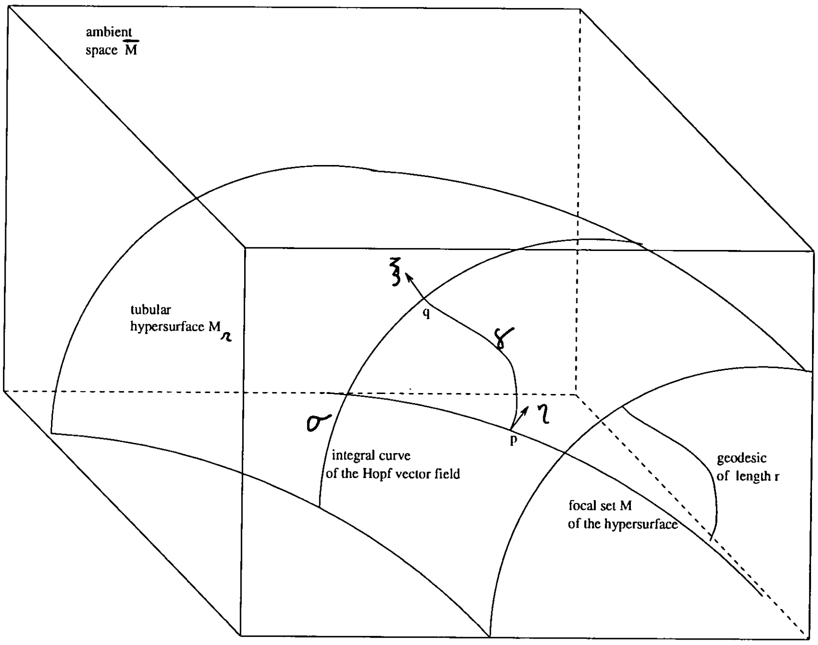

Figure 1.1: A tube M_r around the submanifold M of $\overline{M}$.

1.2 Nearly Kähler hypersurfaces of $\mathbb{R}^7$.

In this section we shall first recall how any hypersurface of the Euclidean space $\mathbb{R}^7$ can be given an almost complex structure J. This is due to the fact that $\mathbb{R}^7$ inherits a special structure (cross product as defined below) when viewed as the imaginary part of the Cayley numbers. Secondly, using this almost complex structure, we shall see that the 6-sphere and $\mathbb{R}^6$ are the only hypersurfaces of $\mathbb{R}^7$ which are nearly Kähler, that is, the almost complex structure J satisfies the condition (1.2.5) below.

On the other hand, as will be made clear in Lemma (1.3.2), the study of a Hopf hypersurface of a nearly Kähler almost complex manifold is facilitated in the sense that, in this case, the integral curves of the Hopf vector field are geodesics of the hypersurface if and only if this vector field is principal. Therefore, it is a natural question to argue which hypersurfaces of $(\mathbb{R}^7, \times)$ shall be nearly Kähler.

Let us recall how to define a cross product on $\mathbb{R}^7$. The Cayley numbers (also called octonians) are defined in terms of the quaternion numbers $\mathbb{H}$ by $\mathcal{O} = \mathbb{H} \oplus \mathbb{H}$. This set can be endowed with a normed algebra structure by defining the multiplication of their elements as

$$(a, b).(c, d) := (ac - \bar{d}b, da + b\bar{c}). \tag{1.2.1}$$

This multiplication induces a vector cross product on $\mathbb{R}^7 = \Im\mathcal{O}$ (viewed as the imaginary part of $\mathcal{O}$) as follows. We first define a cross product $\times : \mathcal{O} \times \mathcal{O} \to \Im\mathcal{O}$ by

$$x \times y = \Im(\bar{y}x) = \frac{1}{2}(\bar{y}x - \bar{x}y) = -\frac{1}{2}(\bar{x}y - \bar{y}x) = -\Im(\bar{x}y). \tag{1.2.2}$$

This map is clearly bilinear and alternating ($x \times x = 0$). Moreover, when restricted to $\mathbb{R}^7 \times \mathbb{R}^7$ it yields a vector cross product $\times$ on $\mathbb{R}^7$ which is related with the ordinary Euclidean inner product $(,)$ by the following elementary relation.

$$u \times (v \times w) + (u \times v) \times w = 2(u, w)v - (u, v)w - (w, v)u. \tag{1.2.3}$$

Remark 1.2.2 *Calabi (c.f. [17]) has shown that the* **triple scalar product** $(u \times v, w)$ *is skew-symmetric. He actually considers this property as part of his axiomatic definition of an abstract vector cross product.*

It follows from this remark that, given a unit vector $u \in \mathbb{R}^7$, the linear operator $J_u : \mathbb{R}^7 \to \mathbb{R}^7$ defined by $J_u(v) = u \times v$ is skew-adjoint. Moreover, using (1.2.3) we see that $J_u \mid_{u^\perp}$ is anti-involutive, that is, $J_u^2 \mid_{u^\perp} = -I$. Therefore, $J_u : u^\perp \to u^\perp$ is an isometry and J_u has kernel spanned by u.

Let M be a hypersurface of $\mathbb{R}^7$ and we consider M endowed with the induced metric and connection $\langle , \rangle$ and $\overline{\nabla}$ respectively. Given $q \in M$, let ξ denote a local unit normal vector field on M defined around q. Then we can define an isomorphism J_q of T_qM by

$$J_q(v) := \xi \times v. \tag{1.2.4}$$

Thus, we get a tensor J which is an orthogonal almost complex structure on M. In general, this tensor is not parallel, that is, $\overline{\nabla} J \neq 0$. However, in the following proposition, we shall determine for which hypersurfaces of $\mathbb{R}^7$, the almost complex structure obtained in this way satisfies the nearly Kähler condition

$$(\overline{\nabla}_X J)X \neq 0. \tag{1.2.5}$$

$\mathbb{R}^6$ is clearly one such hypersurface since $(\mathbb{R}^6, J)$ is isomorphic to $\mathbb{C}^3$.

Another example is the 6-sphere. Indeed, this follows from the fact that $(\overline{\nabla}_X J)Y$ is the component of $X \times Y$ tangent to S^6, which is a straightforward consequence of the definition of J. Hence S^6 with the standard metric $<,>$ and the corresponding Riemannian connection $\overline{\nabla}$ is a nearly Kähler manifold. Henceforth, we shall be always considering S^6 endowed with these structures just defined.

It turns out that these two examples are the only ones, as we prove next.

Proposition 1.2.1 *The only nearly Kähler hypersurfaces of* $\mathbb{R}^7$ *are the open subsets of* S^6 *or* $\mathbb{R}^6$.

Proof:

Let (M, J) be a nearly Kähler hypersurface of $(\mathbb{R}^7, \times)$, where J is defined as above. Let $\widetilde{\nabla}$ and $\overline{\nabla}$ denote the Riemannian connections of $\mathbb{R}^7$ and M (induced) respectively.

Let $q \in M$ and let ξ be a unit local normal vector field on M. Let $A = A_\xi$ denote the shape operator of M. Then for any smooth unit vector field $X \in \mathfrak{X}(M)$, we have

$$\begin{aligned} \widetilde{\nabla}_X(JX) &= (\widetilde{\nabla}_X\xi) \times X + \xi \times (\widetilde{\nabla}_X X) \\ &= X \times AX + \xi \times (\overline{\nabla}_X X). \end{aligned} \tag{1.2.6}$$

On the other hand, using the nearly Kähler condition we get

$$\begin{aligned} \widetilde{\nabla}_X(JX) &= \overline{\nabla}_X(JX) + \langle AX, JX\rangle\xi \\ &= \xi \times (\overline{\nabla}_X X) + \langle AX, JX\rangle\xi. \end{aligned} \tag{1.2.7}$$

Thus, it follows from (1.2.6) and (1.2.7) that

$$\langle AX, JX\rangle\xi = X \times AX. \tag{1.2.8}$$

Taking the cross product of (1.2.8) with X and using (1.2.3) we obtain

$$AX = \alpha X + \beta JX.$$

In particular, if X is a principal vector of A, say $AX = \lambda X$ then so is JX because

$$\beta = \langle JX, AX\rangle = \lambda\langle JX, X\rangle = 0.$$

Therefore, given a point $q \in M$, we can find an orthonormal basis $\{e_1, e_2, e_3, Je_1, Je_2, Je_3\}$ of T_qM with each vector being an eigenvector of A.

Assume $Ae_j = \lambda_j e_j$ for $j = 1, 2, 3$. Then

$$A(e_i + e_j) = \lambda_i e_i + \lambda_j e_j = \alpha(e_i + e_j) + \beta(Je_i + Je_j),$$

which implies that the eigenvalues λ_j are the same, say λ. Similarly, we can verify that the eigenvectors Je_1, Je_2, Je_3 correspond also to a same eigenvalue, say μ.

Finally, choosing $i \neq j$ we get from the following equation that $\lambda = \mu$.

$$A(e_i + Je_j) = \lambda e_i + \mu Je_j = \alpha(e_i + Je_j) + \beta(Je_i - e_j).$$

Therefore, the hypersurface M is totally umbilic and consequently it is an open subset of either a hypersphere or a hyperplane. $\odot$

The exceptional Lie group G_2 is defined as the group of automorphisms of the Cayley algebra $(\mathcal{O}, .)$, that is, $G_2 = \{g \in GL_8(\mathbb{R}) \ / \ g(x.y) = g(x).g(y) \ \forall \ x, y \in \mathcal{O}\}$ but using elementary properties of $\mathcal{O}$ and G_2, we can actually think of this group as the subgroup of $SO(7)$ which preserves the vector cross product of $\mathbb{R}^7$, that is

$$G_2 = \{g \in SO(7)/g(a \times b) = g(a) \times g(b) \ \forall \ a, b \in \mathbb{R}^7\}. \tag{1.2.9}$$

Definition 1.2.4 *A G_2-basis of $\mathbb{R}^7$ is an orthonormal basis $\{e_1, \dots, e_7\}$ for this space satisfying the relations*

$$e_1 \times e_2 = e_3, \ \ e_1 \times e_4 = e_5, \ \ e_2 \times e_4 = e_6, \ \ e_3 \times e_4 = e_7. \tag{1.2.10}$$

Hence, if e_1, e_2, e_4 are orthonormal vectors of $\mathbb{R}^7$ such that $e_4 \perp e_1 \times e_2$ then e_1, e_2, e_4 determine a unique G_2-basis for $\mathbb{R}^7$. Furthermore, an element of $SO(7)$ lies in G_2 if and only if it maps any (and hence every) G_2-basis to a G_2-basis.

It follows from equation (1.2.3) and the relations (1.2.10) that the elements of a G_2-basis for $\mathbb{R}^7$ satisfy the following multiplication table

$e_i \times e_j =$ (1.2.11)

$i \backslash j$	1	2	3	4	5	6	7
1	0	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
2	$-e_3$	0	e_1	e_6	e_7	$-e_4$	$-e_5$
3	e_2	$-e_1$	0	e_7	$-e_6$	e_5	$-e_4$
4	$-e_5$	$-e_6$	$-e_7$	0	e_1	e_2	e_3
5	e_4	$-e_7$	e_6	$-e_1$	0	$-e_3$	e_2
6	e_7	e_4	$-e_5$	$-e_2$	e_3	0	$-e_1$
7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	0

1.3 Classification theorems.

In this section we state the main results known on Hopf hypersurfaces of S^6 and complex space forms. The next two lemmas are proved in [5] and are important basic facts when dealing with Hopf hypersurfaces.

Lemma 1.3.2 *M is a Hopf hypersurface of a nearly Kähler manifold $\widetilde{M}$ if and only if the Hopf vector field is a principal curvature vector field, that is, $A_\xi(U) = \alpha U$ where A_ξ is the shape operator of M with respect to ξ and α is a function on M which we will call* **the Hopf principal curvature**.

Lemma 1.3.3 *The Hopf principal curvature of a Hopf hypersurface of S^6 is constant.*

Proof:

By the Codazzi equation we obtain $grad(\alpha) = (U\alpha)U$. Consider the 4-dimensional orthogonal complementary distribution to U on M: $\mathcal{D} = \perp U \subset TM$. Recall that the gradient vector field satisfies $< grad(\alpha), Z >= Z(\alpha)$ for every smooth section $Z \in \mathfrak{X}(M)$ of the tangent bundle TM. Thus for any $Z \in \mathcal{D}$ we have $Z\alpha = (U\alpha) < Z, U >= 0$ and in order to prove the lemma we just need to show that $U\alpha = 0$ since $TM = D \oplus U$. But, in particular, for X and Y in $\mathcal{D} \cap \mathfrak{X}(M)$ we have $[X,Y]\alpha = X(Y\alpha) - Y(X\alpha) = 0$, so that if $U\alpha \neq 0$ on an open subset of M then the distribution $\mathcal{D}$ would be integrable there and since it is J-invariant this would give us a 4-dim almost complex submanifold of S^6 which does not exist according to the proof given in [26]. Hence, $U\alpha = 0$ and $\alpha = constant$. $\odot$

Remark 1.3.3 *In the case of Hopf hypersurfaces of non-Euclidean complex space forms Maeda [31] has also proved that the Hopf principal curvature is locally constant.*

Example 1.3.1 *The totally geodesic almost complex curves of S^6 are exactly the 2-spheres obtained as the intersection of S^6 with the 3-dimensional vector subspace* *of $\mathbb{R}^7$ which are closed under the vector-cross product $\times$ of $\mathbb{R}^7$ (associative 3-planes). Furthermore, the open subsets of a tube around such curves are Hopf hypersurfaces all of whose principal curvatures are constant.*

Indeed, in order to see the first part of our example let us consider a totally geodesic almost complex curve S of S^6. Since S is totally geodesic, there exists a 3-dimensional subspace V of $\mathbb{R}^7$ such that $S = V \cap S^6$. Given $p \in S$, we can choose an orthonormal basis $\{X_1, X_2\}$ of T_pS such that $X_2 = J(X_1)$, for S is an almost complex curve.

Now, since $\{p, X_1, X_2\}$ is a basis of V, in order to prove that V is an associative 3-plane we just need to check the $\times$-invariance of these basic vectors which is a consequence of the following equations.

$$\begin{aligned} p \times X_j &= J(X_j) \in T_pS \subset V \\ X_1 \times X_2 &= X_1 \times J(X_1) \\ &= X_1 \times (p \times X_1) \\ &= -X_1 \times (X_1 \times p) \\ &= p \in V. \end{aligned}$$

Conversely, let V denote an associative 3-plane and consider the totally geodesic 2-sphere $S^2 = V \cap S^6$. Without loss of generality we can assume that V is generated by the vectors $\{e_1, e_2, e_3\}$ of the standard basis of $\mathbb{R}^7$ because the group G_2 of automorphisms of $\mathbb{R}^7$ which preserve the $\times$-product also takes canonical basis to canonical basis and preserve the almost complex structure of S^6. Thus for $p = e_3$ we have

$$\begin{aligned} J_{e_3}(T_{e_3}(S^2)) &= span\{J_{e_3}(e_1), J_{e_3}(e_2)\} \\ &= span\{e_2, -e_1\} \\ &= T_{e_3}S^2. \end{aligned}$$

Thus, if $p \in V$ and g is an element of G_2 such that $g(V) \subset V$ and $ge_3 = p$, then

$$\begin{aligned} J_p(T_p(S^2)) &= g(J_{e_3}(T_{e_3}(S^2))) \\ &= g(T_{e_3}S^2) \\ &= T_pS^2. \end{aligned}$$

Therefore this S^2 is a totally geodesic almost complex curve of S^6.

To prove the second part of our example, we consider an open subset M_r of a tube $\Phi_r(\perp^1 S)$ around $S = V \cap S^6$ where V is generated by the vectors $\{e_1, e_2, e_3\}$. Let us fix a point $q \in M_r$. By the definition of tube $q = \gamma_{(p,\eta)}(r)$ for some point $(p, \eta) \in \perp^1 S$.

Let $\{X_1 = p, X_2, X_3\}$ be an orthonormal basis for V such that $X_3 = X_1 \times X_2$ and $X_4 = \eta$. Then there exists a unique element $g \in G_2$ such that $g(X_i) = e_i$, so that we can assume without loss of generality that $p = e_3$ and $\eta = e_4$. Thus,

$$\gamma(s) = (\cos s)e_3 + (\sin s)e_4 \tag{1.3.1}$$

$$\dot{\gamma}(s) = -(\sin s)e_3 + (\cos s)e_4 \tag{1.3.2}$$

$$J(\dot{\gamma}(s)) = \gamma(s) \times \dot{\gamma}(s) = e_7. \tag{1.3.3}$$

As a special case of the proof given for Theorem (1.3.3), we can define along γ the vector fields $T_j = (\cos s)e_j$ for $j = 1, 2$ and $T_j = (\sin s)e_j$ for $j = 5, 6, 7$ in such a way that they are Jacobi fields satisfying the conditions (1.1.4). Moreover, using (1.1.6) we have

$$\begin{aligned} A(e_j) &= (\tan r)e_j \text{ for } j = 1, 2 \\ A(e_j) &= -(\cot r)e_j \text{ for } j = 5, 6, 7. \end{aligned}$$

Since the point q is arbitrary, these equations prove that the principal curvatures of the tubular M_r hypersurface around S are $\tan r$ and $-\cot r$. Furthermore, since

$$A(e_7) = -(\cot r)e_7 \quad \text{and} \quad U(q) = J(\dot{\gamma}_\eta(r)) = e_7,$$

we have $A(U) = -(\cot r)U$, in other words, the Hopf vector field U of M_r is a principal field and hence by Lemma (1.3.2) M_r is a Hopf hypersurface.

The next four theorems show some results already known about the classification of Hopf hypersurfaces in complex space forms. The first one, is just a consequence of the results on homogeneous hypersurfaces of a complex projective space obtained by Kimura [29] and Takagi [39].

Theorem 1.3.1 (Hopf Hypersurfaces in Complex Projective Spaces) *Let M be a connected Hopf hypersurface of $\mathbb{C}P^n$ $(n \geq 2)$ with constant principal curvatures. Then M is holomorphic congruent to an open part of one of the following real hypersurfaces of $\mathbb{C}P^n$:*

(i) *a tube of radius $r \in (0, \frac{\pi}{2})$ around the canonically (totally geodesic) embedded $\mathbb{C}P^k$ for some $k \in \{0, 1, \dots, n-1\}$,*

(ii) *a tube of radius $r \in (0, \frac{\pi}{4})$ around the canonically embedded complex quadric*
$Q^{n-1} = \frac{SO(n+1)}{SO(2)\times SO(n-1)}$,

(iii) *a tube of radius $r \in (0, \frac{\pi}{4})$ around the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^k$ in $\mathbb{C}P^n$, where $n = 2k+1$,*

(iv) *a tube of radius $r \in (0, \frac{\pi}{4})$ around the Plücker embedding of the complex Grassmann manifold $\mathbb{C}G_{2,3}$ in $\mathbb{C}P^9$,*

(v) *a tube of radius $r \in (0, \frac{\pi}{4})$ around the canonical embedding of the Hermitian symmetric space $\frac{SO(10)}{U(5)}$ in $\mathbb{C}P^{15}$.*

Theorem 1.3.2 (Hopf Hypersurfaces in Complex Projective Spaces, [18]) *Let M be a connected orientable Hopf hypersurface of $\mathbb{C}P^n$ with Hopf principal curvature $\mu = -2\cot(2r)$. Assume that the focal map Φ_r of M has constant rank k on M. Then k is even and each point $q \in M$ has a neighbourhood V such that $\Phi_r(\perp^1 V)$ is a complex submanifold of $\mathbb{C}P^n$ and V lies on the tube of radius r over $\Phi_r(\perp^1 V)$. Furthermore, if M is compact then its focal set $N = \Phi_r(\perp^1 M)$ is a complex submanifold of $\mathbb{C}P^n$ and M lies on the tube of radius r around N. Conversely, every open subset of a tube of constant radius over a complex submanifold of $\mathbb{C}P^n$ is a Hopf hypersurface.*

Theorem 1.3.3 (Hopf Hypersurfaces in S^6, [5]) *Let M be a connected hypersurface of S^6. Then M is a Hopf hypersurface of S^6 if and only if M is an open subset of*

(i) *a geodesic hypersphere,*

(ii) *a tube around an almost complex curve of S^6.*

Theorem 1.3.4 (Hopf Hypersurfaces in Complex Hyperbolic Spaces, [2]) *Let M be a connected hypersurface of $\mathbb{C}H^n$. Then M is a Hopf hypersurface with constant principal curvatures if and only if M is an open subset of*

(i) *a tube of radius $r \in \mathbb{R}^+$ around $\mathbb{C}H^k$, for $0 \leq k \leq (n-1)$,*

(ii) *a tube of radius $r \in \mathbb{R}^+$ around $\mathbb{R}H^n$,*

(iii) *a horosphere in $\mathbb{C}H^n$.*

We are compelled to give here a concise proof for at least the first part of Theorem (1.3.3) because most of the ideas and constructions involved in this proof shall be referred to when proving some new results later on in this work. Although we shall not write here the proof for the converse of this theorem, it is worth mentioning that the main idea used in [5] to prove it is to show that the rank of the focal map of a connected Hopf hypersurface of S^6 is constant.

Proof of Theorem (1.3.3):

($\Leftarrow$) Open subsets of Tubes are Hopf Hypersurfaces.

Let M_r be an open subset of the tube $\Phi_r(\perp^1 S)$ around an almost complex curve S of S^6.

If S is degenerate to a single point, that is, M_r is a geodesic hypersphere, then M_r is totally umbilic and in particular $J\xi$ is an eigenvector of the shape operator of M_r. Thus, by Lemma (1.3.2), M_r is a Hopf hypersurface.

Now, assume S is non-degenerate. Let $p \in S$ and $\eta \in \perp_p^1 S$. The geodesic $\gamma = \gamma_{(p,\eta)}$ of S^6 can be written as

$$\gamma(s) = \cos(s)p + \sin(s)\eta.$$

Thus, the unit vector $\xi := \dot{\gamma}(r)$ is normal to the hypersurface M_r at the point $q := \gamma(r)$.

Given $X \in T_pS^6 \cap \{\mathbb{R}\eta\}^\perp$, we can define a Jacobi field W_X along γ complying with the conditions

$W_X(0) = X^T$ (orthogonal projection of T_pS^6 onto T_pS)

$\dot{W}_X(0) = X^\perp - A_\eta X^T$ (orthogonal projection of T_pS^6 onto $\perp_p S$),

where A_η denotes the shape operator of S with respect to η.

Let us denote by $B_v(s)$ the parallel transport of a vector $v \in T_pS^6$ along γ. Then the Jacobi field W_X can be written as

$$W_X(s) = \cos(s)B_{X^T}(s) + \sin(s)B_{X^\perp - A_\eta X^T}(s). \tag{1.3.4}$$

Thus, we can distinguish two particular cases. The first being when X is an eigenvector of A_η, say $A_\eta X = \lambda X$. This implies

$$W_X(s) = (\cos s - \lambda \sin s)B_X(s).$$

The second case is when X lies in $(\perp_p S) \cap (\mathbb{R}\eta)^\perp$, for which we have

$$W_X(s) = (\sin s)B_X(s).$$

By applying (1.1.6) to these equations and writting the principal curvature λ as $\lambda = \tan(\theta)$, we conclude that $B_X(r)$ is a principal vector of A_ξ with eigenvalues $\tan(r \pm \theta)$ and $-\cot(r)$ corresponding to the first and second cases respectively.

Since S is an almost complex curve we have $J(\perp_p S) \subset \perp_p S$, which implies that $J\eta$ lies in $(\perp_p S) \cap (\mathbb{R}\eta)^\perp$ and hence it follows from the nearly Kähler condition that $J\xi$ is the parallel transport of $J\eta$ along γ. Therefore, using the second case above and (1.1.6), we see that $J\xi$ is an eigenvector of A_ξ. $\odot$

1.4 Special Hopf hypersurfaces of S^6.

In this section we use the characterization of the Hopf hypersurfaces of the 6-sphere given in the previous section, to determine those which have constant mean curvature, or constant principal curvatures and also those which are Einstein spaces.

Carrying on with the procedure and notations above for calculating the principal curvatures on an open subset of a tube $M \subset \Phi_r(\perp^1 S)$, let us now identify the Hopf hypersurfaces which have constant mean curvature and also those ones which have constant principal curvatures.

Proposition 1.4.2 *Let S be an almost complex curve of S^6 and let $\pm\lambda(p,\eta) = \pm tan(\theta)$ denote the principal curvature functions corresponding to the shape operator A_η of S where (p,η) varies on the unit normal bundle of S in S^6. Then the function λ (equivalently θ) is constant if and only if S is totally geodesic in S^6.*

Proof:

Let $\{X_1, X_2 = JX_1\}$ denote an orthonormal frame of tangent vectors of S and let h be the second fundamental form of S in S^6. First, we will prove that h satisfies

$$h(X_1, X_2) = J(h(X_1, X_1)) =: v_2. \tag{1.4.1}$$

$$h(X_1, X_1) = -h(X_2, X_2) =: v_1 \tag{1.4.2}$$

Using that J is a nearly Kahler structure we have

$$0 = (\overline{\nabla}_X J)(X) = (\nabla_X J)(X) + h(X, JX) - Jh(X, X). \tag{1.4.3}$$

Equation (1.4.1) then will follow from the fact that the normal component of the right hand side of equation (1.4.3) must be zero, that is, $h(X, JX) = Jh(X, X)$.

Putting $X = JY$ in this equation we get $h(JX, JX) = +Jh(X, JX)$, and so

$$h(X, X) = -Jh(X, JX) = -h(JX, JX),$$

from which equation (1.4.2) follows.

We note in passing that (1.4.2) is also an immediate consequence of the assumption that S is an almost complex curve because it is well known that, in a more general situation, every almost complex submanifold of a nearly Kähler manifold is minimal.

Observe that (1.4.2) and (1.4.1) are just saying that the image of the second fundamental form, when non-trivial, is the 2-dim subspace generated by the unit normal vectors

$$\eta_1 := \frac{v_1}{|v_1|} \quad \text{and} \quad \eta_2 := \frac{v_2}{|v_2|}.$$

Moreover, if we define $\lambda_1 := \lambda(p, \eta_1) := |v_1| = |v_2|$, then it follows from (1.4.2) and (1.4.1) that:

$$\begin{cases} A_{\eta_1}(X_1) = \lambda_1 X_1 \\ A_{\eta_1}(X_2) = -\lambda_1 X_2. \end{cases} \tag{1.4.4}$$

Now, suppose that S is not totally geodesic, then we can choose a point $(p, \eta) \in \perp^1 S$ such that $h \neq 0$ { that is, $v_1 \neq 0 \neq v_2$} on a neighbourhood W of (p, η) in $\perp^1 S$. Thus, the unit normal fields η_1 and η_2 are defined in a neighbourhood V of p in S.

Let us consider vector fields ξ_1 and ξ_2 on S such that $\{\eta_1, \eta_2, \xi_1, \xi_2\}$ is an orthonormal frame of normal fields on V. Then, we can define a curve $\zeta = (p, \zeta(t))$ in the unit normal bundle $\perp^1 S$ by

$$\zeta(t) := \frac{\eta_1 + t\xi_1}{|\eta_1 + t\xi_1|}.$$

If we denote by $A_\zeta := A_{\zeta(t)}$ the shape operators of S with respect to the family of unit normal vector fields $\zeta(t)$, then (1.4.1), (1.4.2) and (1.4.4) yield

$$\langle A_\zeta(X_i), X_j\rangle = \langle h(X_i, X_j), \zeta\rangle = (-1)^{(i+1)}\delta_{i,j}\frac{\lambda_1}{|\eta_1 + t\xi_1|}$$

where $i, j \in \{1, 2\}$. In other words, X_1 and X_2 are eigenvectors of A_ζ and

$$\begin{cases} A_\zeta(X_1) = \frac{\lambda_1}{|\eta_1+t\xi_1|}X_1 \\ A_\zeta(X_2) = \frac{-\lambda_1}{|\eta_1+t\xi_1|}X_2. \end{cases}$$

Thus,

$$\lambda(p, \zeta(t)) = \pm\frac{\lambda_1}{|\eta_1 + t\xi_1|}, \tag{1.4.5}$$

and hence the principal curvature function λ is non-constant along the curve $(p, \zeta(t))$ of the unit normal bundle of S. $\odot$

Remark 1.4.4 *Looking at the proof of Proposition* (1.4.2) *we see that the result stated there does not require the function* λ *to be constant on* $\perp S$ *but it only demands that* λ *be at each base point* $p \in S$ *independent of the choice of normal vector* $\eta \in \perp_p S$. *Indeed, this is clear from* (1.4.5) *since that equation shows that* $\lambda(p, \zeta(t))$ *is not constant even for a fixed base point p. In other words, we can restate that Proposition saying that the function* $\lambda(p, \eta)$ *depends only on the point p if and only if* S *is totally geodesic in* S^6.

For the sake of completeness we must point out that the results in Proposition (1.4.2) and Remark (1.4.4) are valid for the more general situation of any submanifold N^n of any Riemannian manifold $\overline{M}^m$ as far as $m > \frac{n(n+3)}{2}$. This is so because the main part of the proof given above depends solely on the existence of a normal vector orthogonal to the first normal space which has dimension less than or equal to $\frac{n(n+1)}{2}$. Nevertheless, it was convenient having written down the proof for an almost complex curve of the 6-sphere since this is the case we are most interested in this chapter.

In Example (1.3.1) we saw that the tubular hypersurfaces of S^6 around totally geodesic almost complex curves have constant principal curvatures, now using Remark (1.4.4) we give a converse for this fact.

Corollary 1.4.1 *The Hopf hypersurfaces of constant principal curvatures of* S^6 *are the open subsets of either the geodesic hyperspheres or the tubes over the totally geodesic almost complex curves.*

Proof:

Indeed, according to the proof of Theorem (1.3.3) the principal curvatures of a tubular hypersurface M_r around an almost complex curve S are of type $-\cot(r)$ and $\cot(r + \theta)$ respectively, where θ is a function defined on the curve S in such a way that $\pm\tan(\theta)$ are the principal curvature functions of S. Therefore, M_r has constant principal curvatures if and only if θ is constant and hence the Corollary follows from Proposition (1.4.2). $\odot$

As a consequence also of Proposition (1.4.2) we are going now to identify which Hopf hypersurfaces are Einstein (that is, the Ricci tensor is a multiple by a constant of the metric tensor) and which ones have constant mean curvature.

Corollary 1.4.2 *The Einstein Hopf hypersurfaces of S^6 are the open subsets of*

(i) *a geodesic hypersphere.*

(ii) *a tube of radius $r = arctan(\sqrt{2})$ over a totally geodesic almost complex curve of S^6.*

Proof:

The geodesic hyperspheres of S^6 are Einstein spaces because we have more in general that geodesic spheres in S^n have constant curvature.

Let M be a Hopf hypersurface of S^6, say M is an open subset of a tube of radius r around an almost complex curve S. Then the Hopf principal curvature of M is $\alpha = -\cot(r)$.

Given $p \in S$, let η be a local unit normal vector field on S. Let us choose a local orthonormal frame $\{\eta, X_3, X_4, X_5\}$ of normal vector fields on S and also a local orthonormal frame $\{X_1, X_2\}$ of principal vector fields on S with respect to η, say

$$A_\eta X_1 = tan(\theta)X_1 \quad \text{and} \quad A_\eta X_2 = -tan(\theta)X_2.$$

If $A = A_\xi$ is the shape operator of M with respect to a unit normal field ξ on M, and if for each $i \in \{1, \ldots, 5\}$, $B_i = B_i(r)$ denotes the parallel transport of X_i along the geodesic $\gamma = \gamma_{(p,\eta)}(s)$ from $\gamma(0) = p$ to $\gamma(r) = q \in M$, then according to the proof of the first part of Theorem (1.3.3), we have

$$\begin{cases} AB_1 = tan(r+\theta)B_1 \\ AB_2 = tan(r-\theta)B_2 \\ AB_i = \quad -\cot(r)B_i \text{ (i=3,4,5).} \end{cases} \tag{1.4.6}$$

Let R and S denote the curvature and Ricci tensors of M, respectively. Let us denote also by $<,>$, the induced metric on M. As S^6 has constant curvature 1, it follows that the Gauss equation reduces to:

$$\begin{aligned} < R(X,Y)Z, W > = < X, W >< Y, Z > - < X, Z >< Y, W > + \\ + < h(X,W), h(Y,Z) > - < h(X,Z), h(Y,W) >, \end{aligned} \tag{1.4.7}$$

where $X, Y, Z, W \in \mathfrak{X}(M)$ and h is the second fundamental form of M. Then we can rewrite the Ricci tensor as

$$S(X,Y) = 4 < X, Y > + < AX, Y > trA - < AX, AY > . \tag{1.4.8}$$

Using equations (1.4.7) and (1.4.8) we can explicitly calculate the Ricci tensor on the basis $\{B_1, ..., B_5\}$ of T_qM as follows

$$\begin{cases} S(B_1,B_1) = 4 + [\tan(r+\theta) + \tan(r-\theta) - 3\cot(r)].\tan(r+\theta) - \tan^2(r+\theta) \\ S(B_2,B_2) = 4 + [\tan(r+\theta) + \tan(r-\theta) - 3\cot(r)].\tan(r-\theta) - \tan^2(r-\theta) \\ S(B_i,B_i) = 4 - [\tan(r+\theta) + \tan(r-\theta) - 3\cot(r)].\cot(r) - \cot^2(r) \text{ (i=3,4,5)} \end{cases} \tag{1.4.9}$$

The equations (1.4.9) tell us that if M is an Einstein space then $tan(r+\theta)$ and $tan(r-\theta)$ are both equal to either $-\cot r$ or $2\cot r$. Thus, in any case, the function $\theta(p,\eta)$ is constant and hence by Proposition (1.4.2), S is a totally geodesic curve, that is, $\theta \equiv 0$. Moreover, when M is Einstein, equations in (1.4.9) also give us a unique radius for the tube, namely, $r = arctan(\sqrt{2})$. ⊙

Hypersurfaces of constant sectional curvature of real space forms have been studied by several authors (Fialkow [23], Struik [37], Burstin [15]) and are essencially characterized as geodesic hyperspheres and developable hypersurfaces. However, if we consider the additional structure of being Hopf hypersurfaces then we can do the following remark.

Remark 1.4.5 *In the second type of Einstein Hopf hypersurfaces in the Corollary above, the sectional curvatures with respect to the planes spanned by the vectors* $\{B_1, B_2\}$ *and* $\{B_1, B_3\}$ *are respectively equal to* 3 *and* 0. *Thus these hypersurfaces do not have constant curvature and hence, using that Riemannian manifolds of constant curvature are Einstein, we see that the geodesic hyperspheres are the only Hopf hypersurfaces of* S^6 *which have constant sectional curvature.*

Definition 1.4.5 *Let* M *be a hypersurface of an almost complex Riemannian manifold* $(\overline{M}, \langle\,,\,\rangle J)$ *and let* ξ *be a unit normal field on* M. *We say that* M *is* **pseudo-Einstein** *if there exist functions* $a, b : M \to \mathbb{R}$ *such that the Ricci tensor* S *of* M *satisfies*

$$S(X,Y) = a\langle X,Y\rangle + b\langle X,J\xi\rangle\langle Y,J\xi\rangle.$$

The Einstein condition on a Hopf hypersurface is too severe, as we can see from the corollaries above. Nevertheless, we can argue if we could obtain some stronger result by imposing the less stringent condition of being pseudo-Einstein. However, unlike the situation in $\mathbb{C}P^n$, investigated by Cecil-Ryan ([18]), this extra condition does not add anything new to our previous results. Indeed, if M is a pseudo-Einstein Hopf hypersurface of S^6 contained in a tube over an almost complex curve S, it follows from (1.4.9) that $b \equiv 0$. Thus, M is actually an Einstein hypersurface.

Corollary 1.4.3 *The Hopf hypersurfaces with constant mean curvature* $c \neq 0$ *are the open subsets of either a geodesic hypersphere with distance (polar)* $\arctan(\frac{1}{c})$ *or a tube of radius* $r = \arctan(\frac{5c+\sqrt{25c^2+24}}{4})$ *over a totally geodesic almost complex curve.*

Corollary 1.4.4 *The minimal Hopf hypersurfaces of* S^6 *are the open subsets of either the totally geodesic hyperspheres or the tubes of radius* $r = \arctan(\frac{\sqrt{6}}{2})$ *over the totally geodesic almost complex curves.*

Indeed, these corollaries follow from the fact that their extra assumptions on the hypersurface imply that the function $\lambda(p,\eta)$ is constant since the principal curvature functions at a point $q = \gamma(r)$ are given by

$$\alpha = -\cot(r), \quad \beta = \tan(r+\theta), \quad \gamma = \tan(r-\theta).$$

Furthermore, the specific values for the radii in corollaries (1.4.3) and (1.4.4) are calculated directly from the condition on the mean curvature and the constant values, $-\cot(r)$ and $\tan(r)$, for the principal curvatures. ⊙

It is worth remarking in passing that Miquel ([32]) has obtained a particular version of the last two corollaries above for the case of complex space forms. He has proved that the compact Hopf hypersurfaces of constant mean curvature in a non-Euclidean complex space form are geodesic hyperspheres.

Chapter 2

Congruence of Hypersurfaces.

2.1 Introduction.

We start this chapter investigating a special type of rigidity for the hypersurfaces of the 6-sphere, namely we determine when are two hypersurfaces of the 6-sphere G_2-congruent in the sense explained below. Our inspiration to tackle this problem came from a series of three papers [39],[38] and [19] where holomorphic congruence of hypersurfaces of complex projective spaces is studied.

In order to make our approach to rigidity or congruence of hypersurfaces as clear as possible, we give below what this means for us.

Definition 2.1.1 *Let $\overline{M}$ be a Riemannian manifold and let M be a Riemannian submanifold of $\overline{M}$ (with the induced metric). Let G denote a group of isometries of $\overline{M}$. We say that M is* **rigid** *in $\overline{M}$ with respect to G if every isometric immersion $f : M \to \overline{M}$ is extendable to an isometry of the ambient space $\overline{M}$, in other words, there exists an isometry $\overline{f} \in G$ such that $f = \overline{f}\mid_M$. In this case we shall say that the manifolds M and $f(M)$ are* **G-congruent**.

In real space forms, the rigidity of hypersurfaces whose second fundamental forms have rank greater or equal to 3 at every point is a well known classical result that can be found for instance in volume V of the Spivak's work [36].

The papers of Takagi mentioned above give a similar version of this classical rigidity for the case of complex space forms. We shall give later in this chapter a different rather classic approach to this problem in order to produce an alternative proof of Takagi's result.

We dedicate the two last sections of this chapter to do some applications of these rigidity results. There, we determine the hypersurfaces of the 6-sphere and of the complex projective spaces whose Hopf vector fields are Killing fields.

2.2 G_2-Congruence of Hopf hypersurfaces in S^6.

Considering that G_2 is the group of the isometries of S^6 which preserve the almost complex structure, we can naturally be curious to know what are the G_2-rigid hypersurfaces of the 6-sphere.

Definition 2.2.2 *Let $g : M \longrightarrow \overline{M}$ be an isometric immersion of a Riemannian manifold M^{2n-1} into a nearly Kahler manifold $(\overline{M}^{2n}, J)$ and let $\tilde{\xi}$ be a normal vector field on the hypersurface $\widetilde{M} = g(M)$ of $\overline{M}$. Then we can define on M a* **structure vector field** *$\hat{U}$ and tensors $\hat{\phi}$ and $\hat{A}$ of type $(1,1)$, as follows*

$$\hat{U}_q = g_*^{-1}(J\tilde{\xi})$$
$$\hat{\phi}(X) = g_*^{-1}(Jg_*X - < Jg_*X, \tilde{\xi} > \tilde{\xi})$$
$$\hat{A}(X) = -g_*^{-1}(\overline{\nabla}_X\tilde{\xi}).$$

Remark 2.2.1 *It is clear that the structure vector field $\hat{U}$ corresponds to the Hopf vector field $g_*\hat{U}$ of $g(M)$ with respect to the normal field $\tilde{\xi}$.*

When M is a submanifold of $\overline{M}$ and g is taken as the inclusion map then we shall denote these induced structures on the hypersurface M by ξ, U, ϕ, A. In this case they are more simply expressed by

$$U = J\xi, \tag{2.2.1}$$
$$\phi(X) = JX + < X, U > \xi. \tag{2.2.2}$$

We shall make extensive use later on of some basic properties, listed below, of these induced structures. All of them are easy to be checked and essentially they are just consequences of the definitions above and the properties of the almost complex structure J. We should also point out that they are also valid for the induced structures $(\hat{U}, \hat{\phi})$.

$$\phi^2 X = -X + \langle X, U\rangle U, \tag{2.2.3}$$

$$\langle \phi X, \phi Y \rangle = \langle X, Y \rangle - \langle X, U \rangle \langle Y, U \rangle, \tag{2.2.4}$$

$$\phi \text{ is skew symmetric}, \tag{2.2.5}$$

$$Ker(\phi) = span\{U\}, \tag{2.2.6}$$

$$\phi : U^{\perp} \longrightarrow U^{\perp} \text{ is a linear isometry}, \tag{2.2.7}$$

$$(\nabla_X \phi) Y = \langle AX, Y \rangle U - \langle Y, U \rangle AX. \tag{2.2.8}$$

Remark 2.2.2 *Note that up to a choice of the normal vector field for the hypersurface, the tensor ϕ determines and is determined by the vector field U.*

When $\overline{M} = S^6$ and g is the restriction of a linear map (lying on SO(7)) to a hypersurface M of S^6, as will be the situation in most of the results ahead, then we will rather use the explicit definition of J in terms of the vector cross product of $\mathbb{R}^7$ and these structures can be rewritten as

$$g\hat{U}_q = gq \times g\xi, \tag{2.2.9}$$

$$g\hat{\phi}(X) = gq \times gX + \langle X, \hat{U} \rangle g\xi. \tag{2.2.10}$$

In this case the rate of change of the Hopf vector field will play an important role when dealing with the induced structures $(\phi, \hat{\phi})$ because it gives a direct relation between these structures and the corresponding second fundamental forms A and $\hat{A}$. Namely

$$\nabla_X U = -\phi AX + X \times \xi + \langle X, U \rangle q, \tag{2.2.11}$$

$$g\nabla_X \hat{U} = -g\hat{\phi}\hat{A}X + gX \times g\xi + \langle X, \hat{U} \rangle gq. \tag{2.2.12}$$

These equations can be easily obtained by using the Riemannian connections $\widetilde{\nabla}$ and $\overline{\nabla}$ of $\mathbb{R}^7$ and S^6 respectively. For example we obtain (2.2.11) as follows.

$$\begin{aligned}
\nabla_X U &= \overline{\nabla}_X(J\xi) - \langle AX, U \rangle \xi, \\
&= \widetilde{\nabla}_X(q \times \xi) - \langle \widetilde{\nabla}_X(q \times \xi), q \rangle q - \langle AX, U \rangle \xi, \\
&= X \times \xi + q \times \overline{\nabla}_X \xi - \langle X \times \xi, q \rangle q - \langle AX, U \rangle \xi, \\
&= -\phi AX + X \times \xi + \langle X, U \rangle q.
\end{aligned}$$

Proposition 2.2.1 *Let M be a hypersurface of S^6 and $g \in SO(7)$. Consider the tensors $(U, \hat{U}, \phi, \hat{\phi})$ defined on M as above. Then $g \in G_2$ if and only if $\phi = \hat{\phi}$.*

Proof:

$(\Longrightarrow)$

If $g \in G_2$ the conditions will arise naturally from the definitions, indeed for every $q \in M$ and $X \in T_qM$ we have

$$g\hat{U}_q = gq \times g\xi = g(q \times \xi) = gU_q$$

$$\begin{aligned} g\hat{\phi}(X) &= gq \times gX + \langle X, \hat{U} \rangle g\xi \\ &= g(q \times X) + \langle X, U \rangle g\xi \\ &= g\phi(X). \end{aligned}$$

$(\Longleftarrow)$

$A = \hat{A}$ because g is an isometry of S^6. This together with the assumptions gives

$$g(q \times X) = gq \times gX \text{ for every } X \in T_qM \tag{2.2.13}$$

$$g(\xi \times X) = g\xi \times gX \text{ for every } X \in T_qM. \tag{2.2.14}$$

Indeed, the first equation comes from equations (2.2.2) and (2.2.10), and the second one comes from equations (2.2.11) and (2.2.12).

Now, since $U \times X = (q \times \xi) \times X = \xi \times JX = \xi \times \phi X$, we obtain

$$\begin{aligned} g(U \times X) &= g(\xi \times \phi X) = g\xi \times g\phi X && \text{[from (2.2.14)]} \\ &= g\xi \times g(q \times X) = g\xi \times (gq \times gX) && \text{[from (2.2.13)]} \\ &= gU \times gX. && (2.2.15) \end{aligned}$$

If $q \in M$ and $X = X_q \in T_qM$ is a unit tangent vector orthogonal to U_q then elementary calculations using the basic properties of the cross-product of $\mathbb{R}^7$ show that the ordered set

$$\{q, \xi, U, X, q \times X, \xi \times X, U \times X\}$$

is a G_2-basis for $\mathbb{R}^7$. Observe that equations (2.2.13).(2.2.14) and (2.2.15) say that

$$\{gq, g\xi, gU, gX, g(q \times X), g(\xi \times X), g(U \times X)\}$$

is also a G_2-basis and hence $g \in G_2$. $\odot$

Lemma 2.2.1 *Let $g \in SO(7)$ and let M be a hypersurface of S^6 endowed with the induced structures $(\phi, \hat{\phi}, A = \hat{A})$ as described above, then $\phi A - \hat{\phi} A = A\phi - A\hat{\phi}$.*

Proof:

If M is totally umbilic then the lemma holds trivially. Thus assume that M is not totally umbilic. Let a and b be distinct eigenvalues of A and let X and Y be corresponding principal vector fields of M, then from (2.2.11) and (2.2.12) we have

$$-a\langle \phi X, Y\rangle + \langle X \times \xi, Y\rangle = \langle \nabla_X U, Y\rangle = -a\langle \hat{\phi} X, Y\rangle + \langle gX \times g\xi, gY\rangle$$

and

$$-b\langle \phi Y, X\rangle + \langle Y \times \xi, X\rangle = \langle \nabla_Y U, X\rangle = -b\langle \hat{\phi} Y, X\rangle + \langle gY \times g\xi, gX\rangle.$$

As ϕ is skew symmetric, we get

$$\begin{cases} a(\langle \phi X, Y\rangle - \langle \hat{\phi} X, Y\rangle) & = \langle X \times \xi, Y\rangle - \langle gX \times g\xi, gY\rangle, \\ b(\langle \phi X, Y\rangle - \langle \hat{\phi} X, Y\rangle) & = \langle gY \times g\xi, gX\rangle - \langle Y \times \xi, X\rangle. \end{cases}$$

And thus we have

$$(a-b)(\langle \phi X, Y\rangle - \langle \hat{\phi} X, Y\rangle) = 0.$$

Which implies

$$\langle (\phi - \hat{\phi})X, Y\rangle = 0.$$

Therefore, since A is symmetric, we conclude that $(\phi - \hat{\phi})$ leaves all the eigenspaces invariant and consequently the equality $\phi A - \hat{\phi} A = A\phi - A\hat{\phi}$ holds on every eigenspace, and hence everywhere. $\odot$

When M is a Hopf hypersurface of S^6 then in accordance with Theorem (1.3.3) M is an open subset of the tube $\Phi_r(\perp^1 S)$ around an almost complex curve of S^6 which is given in turn as the focal set of the focal map Φ_r of M. In this case, we can give an explicit description of the integral curves of the Hopf vector field U of M. Indeed, given a point q of M, say the end point of the geodesic

$$q = \gamma_{(p,\eta)}(r) = \cos(r)p + \sin(r)\eta. \tag{2.2.16}$$

Consider the curve $\sigma(t)$ of M, passing through q, given by

$$\sigma(t) = \gamma_{(p,\delta(t))}(r) = \cos(r)p + \sin(r)\delta(t), \tag{2.2.17}$$

where $\delta(t) = \cos(\bar{t})\eta + \sin(\bar{t})p \times \eta$ with $\bar{t} = \frac{t}{\sin(r)}$.

In the following equations we use **dot** and **prime** to denote derivatives with respect to the variables s and t respectively. Now, by elementary calculations we obtain

$$\begin{aligned} \sigma' &= \sin(r)\delta' \\ &= -\sin(\bar{t})\eta + \cos(\bar{t})p \times \eta = p \times \delta. \qquad (2.2.18) \\ U(\sigma) &= \sigma \times \xi(\sigma) \\ &= \sigma \times \dot{\gamma}_{(p,\delta)}(r) \\ &= (\cos(r)p + \sin(r)\delta) \times (-\sin(r)p + \cos(r)\delta) = p \times \delta. \qquad (2.2.19) \end{aligned}$$

Thus σ is the integral curve of U through q and this proves the

Lemma 2.2.2 *The flow $\mathcal{F}_t$ of the Hopf vector field of a Hopf hypersurface $M \subset \Phi_r(\perp^1 S)$ is given by*

$$\mathcal{F}_t(\gamma_{(p,\eta)}(r)) = \gamma_{(p,\delta(t))}(r) = \cos(r)p + \sin(r)\delta(t). \tag{2.2.20}$$

In particular, we note in passing that the integral curve of the Hopf vector field starting at the point $\gamma_{(p,\eta)(r)}$ is geometrically originated from the rotation of the complex 2-plane at p spanned by the vectors $\{\eta, J\eta\}$.

Definition 2.2.3 *We shall name as generic Hopf hypersurfaces of S^6 those ones which are neither the geodesic hyperspheres nor subsets of a tube around totally geodesic almost complex curves.*

Proposition 2.2.2 *Let g be an isometry of S^6 ($g \in SO(7)$) and let M be a generic Hopf hypersurface of S^6, then the following conditions are equivalent*

(i) *$\widetilde{M} = g(M)$ is a Hopf hypersurface,*

(ii) *$g(p \times X) = gp \times gX$ for every $p \in S$ and $X \in T_pS^6$,*

(iii) *g maps the Hopf vector field U of M to the Hopf vector field $\widetilde{U}$ of $\widetilde{M}$ ($U = \hat{U}$), that is $g(q \times \xi) = gq \times g\xi$.*

Proof:

$(iii) \Longrightarrow (i)$

The isometry $f = g\,|_M \colon M \to \widetilde{M}$ maps the geodesics which are integral curves of U to geodesics which are integral curves of $\widetilde{U}$ and hence by definition $\widetilde{M}$ is a Hopf hypersurface.

$(ii) \Longleftrightarrow (iii)$

As an isometry of S^6, g maps the geodesic $\gamma_{(p,\delta)}(r)$ to the geodesic $\widetilde{\gamma}_{(gp,g\delta)}(r)$ thus from equations (2.2.18) and (2.2.19) we obtain respectively

$$g\sigma' = \sin(\bar{t})g\eta + \cos(\bar{t})g(p \times \eta). \tag{2.2.21}$$

$$\begin{aligned} \widetilde{U}_{g\sigma} &= g\sigma \times \dot{\widetilde{\gamma}}_{(gp,g\delta)}(r) \\ &= g\sigma \times g\dot{\gamma}_{(p,\delta)}(r) \\ &= g\sigma \times g\xi \\ &= gp \times g\delta \\ &= \cos(\bar{t})(gp \times g\eta) - \sin(\bar{t})[gp \times g(p \times \eta)]. \end{aligned} \tag{2.2.22}$$

Thus if we use that $g \in SO(7)$ and the vectors $\{g\eta, gp \times g(p \times \eta)\}$ are orthogonal to the vectors $\{g(p \times \eta), gp \times g\eta\}$, then the equivalence $(ii) \Longleftrightarrow (iii)$ follows from equations (2.2.21) and (2.2.22). Note that under the assumption of either condition (i) or (iii), condition (ii) is trivially satisfied for every $X \in T_pS$, because in both cases the isometry g will map the almost complex curve S into the almost complex curve $g(S)$.

$(i) \Longrightarrow (ii)$

Since g is an isometry, it takes the focal set S of M into the focal set $\widetilde{S}$ of $\widetilde{M}$, moreover in accordance with Theorem 1.3.3, the hypersurfaces M and $\widetilde{M}$ lie on tubes around the almost complex curves given by these focal sets.

In order to prove that g and J commute along S we first recall that it follows from equations (1.4.2) and (1.4.1) in Proposition 1.4.2 that the image of the second fundamental form h $(\widetilde{h})$ of S $(\widetilde{S})$ spans a 2-dimensional J-invariant subspace V_1 $(\widetilde{V}_1)$ of the normal space $\perp_p S$ $(\perp_{gp}\widetilde{S})$.

Secondly, as S $(\widetilde{S})$ is an almost complex curve we also have that this normal space is J-invariant, thus it can be decomposed as a direct sum of two J-invariant subspaces

$$\perp_p S = V_1 \oplus V_2$$

$$\perp_{gp}\widetilde{S} = \widetilde{V}_1 \oplus \widetilde{V}_2.$$

However, since g is an isometry of S^6, mapping S to $\widetilde{S}$, we have

$$g(T_pS) = T_{gp}\widetilde{S} \qquad g(\perp_p S) = \perp_{gp}\widetilde{S} \qquad g(V_1) = \widetilde{V}_1 \qquad g(V_2) = \widetilde{V}_2$$

thus using that $g \in SO(7)$ plus the orthogonal properties of J we see that these maps commute on the subspace T_pS and on each subspace V_j and hence they commute on T_pS^6 for every $p \in S$. $\odot$

It is worthwhile observing that the condition (ii) in the proposition above gives a way to improve this result by proving that actually an element $g \in SO(7)$ satisfying those conditions lies in fact in G_2. Although the following examples show that this would not be true for every Hopf hypersurface of S^6, we will prove below that it is true for any generic Hopf hypersurfaces.

Example 2.2.1 *Let M be a geodesic hypersphere of S^6 centred at the point e_4. Consider the element F of $SO(7)$ defined by $F(e_j) = e_j$ for $j \neq 3, 7$, $F(e_3) = e_7$, $F(e_7) = -e_3$, then F is the unique extension of the isometry $f = F|_M \colon M \to M$ and obviously F is not an element of G_2. Moreover, F maps the Hopf vector field to itself.*

Example 2.2.2 *Let M be a Hopf hypersurface contained in a tube around the almost complex curve $S = V^3 \cap S^6$ where $V^3 = span\{e_3, e_4, e_7\}$. Consider the map $F \in$*

$SO(7)$ given by $F(e_j) = e_j$ for $j = 3, 4, 7$, $F(e_1) = e_2$, $F(e_2) = e_1$, $F(e_5) = e_6$, $F(e_6) = e_5$, then $F \notin G_2$ and F is the unique extension of the isometry $f = F|_M$: $M \to M$. Furthermore, F does not map the Hopf vector field U of M to the Hopf vector field of $F(M)$, that is $U \neq \hat{U}$.

In order to see the later part of each example above we just remark that as F is a linear map then from (2.2.19) we have that at each point $q = \gamma_{(p,\eta)} \in M$ and $F(q)$ the Hopf vectors are given respectively by

$$U_q = p \times \eta \qquad \text{and} \qquad F(\hat{U}_q) = Fp \times F\eta. \tag{2.2.23}$$

Therefore, $U_q = \hat{U}_q$ if and only if

$$F(p \times \eta) = Fp \times F\eta,$$

from which the properties stated in the examples follow.

Proposition 2.2.3 *Let M be a generic Hopf hypersurface of the 6-sphere. Let $g \in SO(7)$, then $\widetilde{M} = g(M)$ is a Hopf hypersurface if and only if $g \in G_2$.*

Proof:

($\Longleftarrow$)

If $g \in G_2$ then g maps the Hopf vector field U of M to the Hopf vector field $\widetilde{U}$ of $\widetilde{M}$ so that from Proposition 2.2.2, $\widetilde{M}$ is a Hopf hypersurface.

($\Longrightarrow$)

If $\widetilde{M}$ is a Hopf hypersurface, then from Proposition 2.2.2 we know that $U = \hat{U}$, that is

$$g(q \times \xi) = gq \times g\xi \text{ for every } q \in M \tag{2.2.24}$$

As we have just noted in the proof of Proposition 2.2.1, in order to prove that $g \in G_2$ it suffices to find a unit vector $X = X_q \in T_qM$ orthogonal to U_q and satisfying the following equations

$$g(q \times X) = g(q) \times g(X) \tag{2.2.25}$$

$$g(\xi \times X) = g(\xi) \times g(X) \tag{2.2.26}$$

$$g(U \times X) = g(U) \times g(X) \tag{2.2.27}$$

Consider M as an open subset of the tube $\Phi_r(\perp^1 S)$ of radius $r \in (0, \frac{\pi}{2})$ around an almost complex curve S. We have seen in Theorem 1.3.3, that the orthonormal eigenvectors $\{B_i\}$ $(i = 3, ..., 7)$ of the shape operator A of M at a point $q = \gamma_{p,\eta}(r) \in M$ are just the parallel transport $B_i(t)$ along γ_η of orthonormal vectors $\{X_i \in T_pS^6\}$, where

$$\{X_1 = p, X_2 = \eta, X_3 = p \times \eta, X_4, X_5 = p \times X_4, X_6, X_7 = p \times X_6\}$$

is a G_2-basis for $\mathbb{R}^7$, such that $\{X_6, X_7 = p \times X_6\}$ is a basis for T_pS and $\{X_2 = \eta, X_3 = p \times \eta, X_4, X_5 = p \times X_4\}$ is a basis for $\perp^1_p S$.

We will show now that $X = B_6 = B_6(r)$ satisfies the equations (2.2.25) (2.2.26) and (2.2.27) and therefore g is an element of G_2. Consider the vector field

$$L(s) = g(\gamma_\eta \times B_6) - g\gamma_\eta \times gB_6,$$

then L is a Jacobi field along γ_η. Indeed, $\ddot{\gamma_\eta} = -\gamma_\eta$ and so $\ddot{L} = -L$. Moreover L also satisfies

$L(0) = g(p \times X_6) - gp \times gX_6,$

$L(r) = g(q \times B_6) - gq \times gB_6,$

$\dot{L}(0) = g(\eta \times X_6) - g\eta \times gX_6.$

It follows from Proposition (2.2.2-ii) that for every curve σ in S and every vector field $Z \in \mathfrak{X}(S^6)$ along σ, we have

$$g(\sigma \times Z) = g\sigma \times gZ. \tag{2.2.28}$$

Considering Z as parallel vector field along σ and differentiating this last equation, we see that for each vector $X \in T_pS$ and $Z \in T_pS^6$,

$$g(X \times Z) = gX \times gZ. \tag{2.2.29}$$

Thus it follows from (2.2.28) and (2.2.29) that $L(0) = 0$ and $\dot{L}(0) = 0$ respectively. Therefore, the Jacobi field L vanishes identically. In particular $L(r) = 0$ which proves that B_6 satisfies the equation 2.2.25.

We can similarly prove that B_6 satisfies the equations (2.2.26) and (2.2.27) by using respectively the following Jacobi vector fields

$L(s) = g(\dot{\gamma}_\eta \times B_6) - g\dot{\gamma}_\eta \times gB_6,$

$L(s) = g(B_6 \times (\gamma_\eta \times \dot{\gamma}_\eta)) - gB_6 \times g(\gamma_\eta \times \dot{\gamma}_\eta).$ ⊙

Corollary 2.2.1 *Given a non-totally umbilic Hopf hypersurface M of S^6, that is M is not a geodesic hypersphere, and $g \in SO(7)$ then $U = \hat{U}$ if and only if $g \in G_2$.*

Proof:

If M is a generic Hopf hypersurface this is just a consequence of Proposition 2.2.2 and Proposition 2.2.3. Therefore, we just need to prove the Corollary for the case when M is an open subset of a tube around a totally geodesic almost complex curve S. We can assume without loss of generality that S is the intersection of S^6 with the subspace of $\mathbb{R}^7$ spanned by the cannonical vectors $\{e_1, e_2, e_3\}$.

Now, we know from (2.2.19) that $U = \hat{U}$ if and only if for every $p \in S^6 \cap span\{e_1, e_2, e_3\}$ and $\eta \in span\{e_4, e_5, e_6, e_7\}$ we have

$$g(p \times \eta) = gp \times g\eta,$$

which implies that g maps the cannonical G_2-basis of $\mathbb{R}^7$ to another G_2-basis and hence $g \in G_2$. ⊙

2.3 G_2-Congruence for hypersurfaces in S^6.

Theorem 2.3.1 *Let M be a non totally umbilic hypersurface of S^6 and $g \in SO(7)$, then g maps the Hopf vector field of M to the Hopf vector field of $g(M)$, that is $U = \hat{U}$, if and only if g is an element of G_2.*

Proof:

The converse of the theorem is trivial.

If M is a Hopf hypersurface then the Theorem has already been proved by Corollary (2.2.1), thus we may assume that U and AU are linearly independent vector fields.

Now, let us assume that $U = \hat{U}$. Looking at Proposition (2.2.1), we see that it is only necessary to prove that $\phi = \hat{\phi}$. Moreover, if $U = \hat{U}$ then (2.2.11) and (2.2.12) yield

$$g\phi AX - g\hat{\phi}AX = g(X \times \xi) - gX \times g\xi. \tag{2.3.1}$$

In particular, for $X = U$ we have

$$\phi AU = \hat{\phi}AU. \tag{2.3.2}$$

Using this and the fact that $\phi^2 = \hat{\phi}^2$ when $U = \hat{U}$, then we get

$$\phi(\phi AU) = \phi^2(AU) = \hat{\phi}^2(AU) = \hat{\phi}(\phi AU). \tag{2.3.3}$$

Hence $\phi = \hat{\phi}$ on the space $V = span\{U, AU, \phi AU\}$. Note that this space has always dimension three because (2.2.5) and (2.2.6) imply that $\phi AU \neq 0$ and ϕAU is orthogonal to U and AU. Thus we have

$$T_qM = V_q \oplus W_q$$

where W_q is the 2-dimensional orthogonal complement $V^{\perp}$.

By using those properties, (2.2.3) and (2.2.5), of ϕ and $\hat{\phi}$, we can also see that W is invariant under these maps and it follows particularly from (2.2.4) that $\phi, \hat{\phi} : W \longrightarrow W$ are isometries.

Now, since by their definitions ϕ and $\hat{\phi}$ realize $\frac{\pi}{2}$-rotations and $dim(W) = 2$ then $\phi = \hat{\phi}$ or $\phi = -\hat{\phi}$ on W.

Therefore we just need now to prove that $\phi = -\hat{\phi}$ on W leads us to a contradiction. Henceforth let us assume $\phi = -\hat{\phi}$ on W. First we observe that W is invariant under the tensor A. Indeed, given any vector $X \in W$, we have $\phi X \in W$ and so

$$\langle A\phi X, U\rangle = \langle \phi X, AU\rangle = 0.$$

Using Lemma 2.2.1 together with (2.2.5) and (2.3.2) we get

$$\begin{aligned}\langle A\phi X, AU\rangle &= \frac{1}{2}\langle \phi AX - \hat{\phi}AX, AU\rangle \\ &= -\frac{1}{2}\langle AX, \phi AU\rangle + \frac{1}{2}\langle AX, \hat{\phi}AU\rangle \\ &= 0.\end{aligned}$$

And from Lemma 2.2.1 together with (2.3.2) and (2.2.4) we obtain

$$\begin{aligned}\langle A\phi X, \phi AU\rangle &= \frac{1}{2}\langle \phi AX - \hat{\phi}AX, \phi AU\rangle \\ &= \frac{1}{2}\langle \phi AX, \phi AU\rangle - \frac{1}{2}\langle \hat{\phi}AX, \hat{\phi}AU\rangle \\ &= \frac{1}{2}(\langle AX, AU\rangle - \langle AX, AU\rangle) = 0.\end{aligned}$$

Thus $A\phi X \in W$ for every $X \in W$ and consequently $A(W) \subset W$.

The invariance of W under A together with Lemma 2.2.1 imply that A and ϕ commute on W and hence for each $X \in W$ we have

$$\langle A\phi X, \phi X\rangle = \langle \phi AX, \phi X\rangle = \langle AX, X\rangle,$$

and

$$\langle AX, \phi X\rangle = -\langle \phi AX, X\rangle = -\langle A\phi X, X\rangle = -\langle AX, \phi X\rangle,$$

which implies

$$\langle AX, \phi X\rangle = 0.$$

However, $\{X, \phi X\}$ is an orthonormal basis for W, so $AX = kX$ for every $X \in W$. Considering this in (2.3.1) we have

$$2kg\phi X = g(X \times \xi) - gX \times g\xi,$$

and since

$$g\phi X = -g\hat{\phi} X = -gq \times gX,$$

we deduce that $k = 0$. Therefore the second fundamental form A vanishes on W and consequently (2.2.11) and (2.2.12) are reduced to

$$\nabla_X U = X \times \xi \tag{2.3.4}$$

$$g\nabla_X U = gX \times g\xi. \tag{2.3.5}$$

Substituting X by ϕX in these equations and using (2.2.2), (2.2.9) and (2.2.10), we have

$$g(X \times U) = gU \times gX, \tag{2.3.6}$$

and hence

$$\begin{aligned} g(q \times \nabla_X U) = & \quad g(U \times X) && \text{from (2.3.4)} \\ = & \quad gX \times gU && \text{from (2.3.6)} \\ = & \quad g\nabla_X U \times gq. && \text{from (2.3.5)} \end{aligned} \tag{2.3.7}$$

Therefore, we have proved that

$$\phi(\nabla_X U) = -\hat{\phi}(\nabla_X U).$$

However, $\nabla_X U \in V$ since in accordance with (2.3.4) this vector is orthogonal to W. This contradicts the fact that ϕ and $\hat{\phi}$ coincide on V. ⊙

Corollary 2.3.2 *Let M be a non totally umbilic hypersurface of the 6-sphere whose second fundamental form has rank greater or equal to 3. Let $f : M \to S^6$ be an isometric immersion of M. Then f maps the Hopf vector field of M to the Hopf vector field of $f(M)$ if and only if there exists an element $g \in G_2$ such that f is the restriction of g to the hypersurface M.*

Proof:

Indeed, from the classical rigidity for hypersurfaces of real space forms mentioned in the introduction of this chapter we have that the map f can be extended to an isometry of S^6 and hence the corollary follows from the previous theorem. ⊙

2.4 Congruence for hypersurfaces in $\mathbb{C}P^n$.

In 1973, Takagi ([39]) gave a rigidity theorem for hypersurfaces of the complex projective spaces which was equivalent to that well-known rigidity theorem for hypersurfaces of a real space form, namely, he proved:

Theorem 2.4.2 *Let M be a hypersurface of $\mathbb{C}P^n$ whose second fundamental form A has rank at least 3 everywhere. Let f denote an isometric immersion of M into $\mathbb{C}P^n$ $(n \geq 3)$. Then,*

- **(i)** $\phi = \hat{\phi}$ *if and only if* $A = \hat{A}$,
- **(ii)** *If* $A = \hat{A}$ *then there exists a holomorphic isometry F of $\mathbb{C}P^n$ such that* $F\mid_M = f$.

We call attention here to the fact that the second part of the theorem above, as observed by Takagi, can be proved by following the same method used to deal with rigidity of hypersurfaces in real space forms.

Recently, Takagi et al ([19]) have improved this result by showing that the rigidity of hypersurfaces in $\mathbb{C}P^n$ depends in general only on the invariance of the Hopf vector field, that is $U = \hat{U}$. More precisely they have shown:

Theorem 2.4.3 *Let M be a hypersurface of $\mathbb{C}P^n$ whose second fundamental form A has rank at least 3 everywhere. Let f denote an isometric immersion of M into $\mathbb{C}P^n$ $(n \geq 3)$. If $U = \hat{U}$ then f is a restriction of a holomorphic isometry of $\mathbb{C}P^n$.*

In this section, we shall give a new proof for this result, using the same method as in the case of hypersurfaces of S^6. It turns out that the approach we give here makes the proof clearer, simpler and more geometrical.

Consider the complex projective space $(\mathbb{C}P^n, J, <>, \overline{\nabla}, \overline{R})$ endowed with the Fubini-Study metric of constant holomorphic sectional curvature 4. Then its curvature tensor $\overline{R}$ is given by

$$\overline{R}(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y - \langle Y,JZ\rangle JX + \langle X,JZ\rangle JY + 2\langle X,JY\rangle JZ. \tag{2.4.1}$$

Let M be a hypersurface of $\mathbb{C}P^n$ with second fundamental form h and induced structures $<>, \nabla, R, etc.$ Let ξ be a unit normal vector field on M. The Gauss and Codazzi equations for M are respectively:

$$\langle \overline{R}(X,Y)Z,W\rangle = \langle R(X,Y)Z,W\rangle + \langle AX,Z\rangle\langle AY,W\rangle - \langle AX,W\rangle\langle AY,Z\rangle \tag{2.4.2}$$

$$\langle \overline{R}(X,Y)Z,\xi\rangle = \langle (\tilde{\nabla}_X h)(Y,Z),\xi\rangle - \langle (\tilde{\nabla}_Y h)(X,Z),\xi\rangle, \tag{2.4.3}$$

where the covariant derivative of the tensor h is given by

$$(\tilde{\nabla}_X h)(Y,Z) := \overline{\nabla}_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z).$$

In terms of the shape operator A of M, we can also write (2.4.3) as

$$\begin{aligned}\langle \overline{R}(X,Y)Z,\xi\rangle =& \langle AX,\nabla_Y Z\rangle - \langle AY,\nabla_X Z\rangle + \langle AZ,\nabla_Y X\rangle - \\ &- \langle AZ,\nabla_X Y\rangle + X\langle AY,Z\rangle - Y\langle AX,Z\rangle.\end{aligned} \tag{2.4.4}$$

Thus, using (2.4.1), we have that for every hypersurface M of $\mathbb{C}P^n$, the Gauss and Codazzi equations are simplified to

$$\begin{aligned}R(X,Y)Z &= \langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle \phi Y,Z\rangle\phi X - \langle \phi X,Z\rangle\phi Y \\ &\quad -2\langle \phi X,Y\rangle\phi Z + \langle AY,Z\rangle AX - \langle AX,Z\rangle AY, \end{aligned}\tag{2.4.5}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = 2\langle \phi X,Y\rangle U + \langle Y,U\rangle\phi X - \langle X,U\rangle\phi Y. \tag{2.4.6}$$

The rate of change of the induced vector fields U and $\hat{U}$ (if we are considering an isometric immersion $g : M \longrightarrow \mathbb{C}P^n$) is given by

$$\nabla_X U = -\phi AX \tag{2.4.7}$$

$$\nabla_X \hat{U} = -\hat{\phi}\hat{A}X \tag{2.4.8}$$

This follows immediately from the Kähler condition $\overline{\nabla}_X(JY) = J(\overline{\nabla}_X Y)$ on $\mathbb{C}P^n$.

Theorem 2.4.4 *Let M be a hypersurface of $\mathbb{C}P^n$ whose second fundamental form has rank at least 3 everywhere and let g be an isometric immersion of M into $\mathbb{C}P^n$. If g maps the Hopf vector field of M to the Hopf vector field of $g(M)$, that is $U = \hat{U}$, then g is the restriction of a holomorphic isometry of $\mathbb{C}P^n$.*

Proof:

Since $U = \hat{U}$, it follows from (2.4.7) and (2.4.8) that:

$$\phi AX = \hat{\phi}\hat{A}X \text{ for every } X \in \mathfrak{X}(M). \tag{2.4.9}$$

As g preserves the curvature, that is $R = \hat{R}$, we obtain from (2.4.1) and (2.4.4) that

$$\begin{gathered}\langle X, \phi Z\rangle\phi Y + 2\langle X, \phi Y\rangle\phi Z - \langle Y, \phi Z\rangle\phi X - \langle AX, Z\rangle AY + \langle AY, Z\rangle AX = \\ \langle X, \hat{\phi}Z\rangle\hat{\phi}Y + 2\langle X, \hat{\phi}Y\rangle\hat{\phi}Z - \langle Y, \hat{\phi}Z\rangle\hat{\phi}X - \langle \hat{A}X, Z\rangle\hat{A}Y + \langle \hat{A}Y, Z\rangle\hat{A}X.\end{gathered} \tag{2.4.10}$$

Specializing this equation for $Z = U$ we get:

$$\langle X, AU\rangle AY - \langle Y, AU\rangle AX = \langle X, \hat{A}U\rangle\hat{A}Y - \langle Y, \hat{A}U\rangle\hat{A}X. \tag{2.4.11}$$

Specializing again this equation for $Y = U$ we have for every $X \in \mathfrak{X}(M)$:

$$\langle X, AU\rangle AU - \langle U, AU\rangle AX = \langle X, \hat{A}U\rangle\hat{A}U - \langle U, \hat{A}U\rangle\hat{A}X. \tag{2.4.12}$$

If W denotes the orthogonal complement of the vector space $span\{AU, \hat{A}U\}$ then for any $Y \in \mathfrak{X}(M)$ and $X \in W$, the equations (2.4.11) and (2.4.12) give respectively:

$$\langle Y, AU\rangle AX = \langle Y, \hat{A}U\rangle\hat{A}X, \tag{2.4.13}$$

$$\langle U, AU\rangle AX = \langle U, \hat{A}U\rangle\hat{A}X. \tag{2.4.14}$$

Taking $Y = AU$ and $Y = \hat{A}U$ in (2.4.13) we have for every $X \in W$ respectively:

$$|AU|^2 AX = \langle \hat{A}U, AU\rangle\hat{A}X, \tag{2.4.15}$$

$$\langle \hat{A}U, AU\rangle AX = |\hat{A}U|^2\hat{A}X. \tag{2.4.16}$$

Now we shall split our proof into two cases.

<u>Case 1</u>: $AU \neq 0$.

Since rank$A$ is at least 3, there exists a vector $X \in W$ such that $AX \neq 0$, so from (2.4.15) and (2.4.16) we have $\hat{A}X \neq 0$, $\hat{A}U \neq 0$ and $\langle \hat{A}U, AU\rangle \neq 0$, moreover by taking the quotient between those equations we get $|\langle \hat{A}U, AU\rangle| = |\hat{A}U||AU|$ and hence

$$\hat{A}U = \delta AU,$$

where $\delta = \pm\frac{|\hat{A}U|}{|AU|}$. Using this in (2.4.15), it follows that $AX = \delta\hat{A}X$ for every $X \in W$. However, from (2.4.9) and (2.2.4) we also have $|AX| = |\hat{A}X|$ for every $X \in W$ and so $\delta = \pm 1$. Choosing if necessary, the oposite normal vector field on $g(M)$, we can assume $\delta = 1$. Thus,

$$AX = \hat{A}X, \tag{2.4.17}$$

$$AU = \hat{A}U. \tag{2.4.18}$$

If $\langle AU, U\rangle \neq 0$ then substituting (2.4.18) in (2.4.12) we obtain $A = \hat{A}$.

If $\langle AU, U\rangle = 0$ then from (2.2.7) we can choose a vector $X \in U^{\perp}$such that $\phi X = AU$ and so

$$X = -\phi AU = -\hat{\phi} AU,$$

$$A(AU) = A(\phi X) = \hat{A}\hat{\phi}X = -\hat{A}\hat{\phi}^2(AU) = \hat{A}(AU).$$

This together with (2.4.17) implies $A = \hat{A}$, which reduces (2.4.10) to

$$\langle X, \phi Y\rangle \phi Y = \langle X, \hat{\phi} Y\rangle \hat{\phi} Y.$$

therefore, for every $X \in U^{\perp}$ we have $\phi X = \pm\hat{\phi}X$.

Because $Ker(\phi) = Ker(\hat{\phi}) = span\{U\}$, we must have $\phi = \pm\hat{\phi}$. But we know that $\phi AX = \hat{\phi}AX$ and hence $\phi = \hat{\phi}$.

Case 2: $AU = 0$.

In this case the Codazzi equation (2.4.4) for the hypersurfaces M and $g(M)$ are written respectively as:

$$\langle \overline{R}(X, Y)U, \xi\rangle = 2\langle \phi AX, AY\rangle \tag{2.4.19}$$

$$\langle \overline{R}(g_*X, g_*Y)g_*U, \tilde{\xi}\rangle = 2\langle \hat{\phi}\hat{A}X, \hat{A}Y\rangle \tag{2.4.20}$$

On the other hand, using the curvature tensor of $\mathbb{C}P^n$ as given in (2.4.1), we have for every $X, Y \in U^\perp$:

$$\overline{R}(X,Y)U = 2\langle \phi X, Y\rangle \xi$$
$$\overline{R}(g_*X, g_*Y)g_*U = 2\langle \hat{\phi} X, Y\rangle \tilde{\xi}.$$

thus

$$\langle \phi X, Y\rangle = \langle \phi AX, AY\rangle$$
$$\langle \hat{\phi} X, Y\rangle = \langle \hat{\phi}\hat{A}X, \hat{A}Y\rangle. \tag{2.4.21}$$

In other words, recalling that $\phi A = \hat{\phi}\hat{A}$, we have

$$A\phi A = \phi \tag{2.4.22}$$
$$A\phi\hat{A} = \hat{\phi} \tag{2.4.23}$$

Now, taking $Z = Y$ in (2.4.10) we have

$$\begin{aligned} &3\langle X, \phi Y\rangle \phi Y - \langle AX, Y\rangle AY + \langle AY, Y\rangle AX = \\ &3\langle X, \hat{\phi} Y\rangle \hat{\phi} Y - \langle \hat{A}X, Y\rangle \hat{A}Y + \langle \hat{A}Y, Y\rangle \hat{A}X \end{aligned} \tag{2.4.24}$$

Putting $Y = -\phi AX$ in this equation and using that

$$\phi Y = AX \qquad \text{and} \qquad AY = -A\phi AX = -\phi X,$$

we obtain for every $X \in U^\perp$:

$$< X, AX > AX =< X, \hat{A}X > \hat{A}X.$$

However,

$$|AX| = |\phi AX| = |\hat{\phi}\hat{A}X| = |\hat{A}X|$$

and hence $AX = \pm\hat{A}X$. From (2.4.22) we see that the restriction $A : U^\perp \longrightarrow U^\perp$ is non-singular so that $Ker(A) = span\{U\}$ and hence $A = \pm\hat{A}$ on $U^\perp$. Choosing an apropriate normal vector field, if necessary, we can assume $A = \hat{A}$. Therefore, from (2.4.22) and (2.4.23) we have $\phi = \hat{\phi}$.

Therefore, the proof of the theorem follows from Theorem (2.4.2). $\odot$

2.5 H-K vector fields in S^6.

The Hopf vector field of a hypersurface in a nearly Kähler manifold will be called **H-K vector field** when it is also a Killing field. In this section we intend to characterize those hypersurfaces of S^6 whose Hopf vector fields are H-K vector fields.

For hypersurfaces in $\mathbb{C}P^n$, Berndt (see [4] for details) has proved that the Hopf foliation of a hypersurface is a Riemannian foliation if and only if the Hopf vector field is an H-K vector field.

In investigating in depth the Hopf vector field of any hypersurface (not necessarily Hopf) we shall actually prove here (example (2.5.3) and Theorem (2.5.5)) that the only hypersurfaces of the 6-sphere whose Hopf vector fields are H-K vector fields are the geodesic hyperspheres. This is a surprising fact about the nearly Kahler S^6 since we shall see in Theorem (2.6.6) that for complex projective spaces any Hopf hypersurface around a totally geodesic complex submanifold satisfies this condition on the Hopf vector field.

We start by recalling that a vector field X in a Riemannian manifold $(M, <>, \nabla)$ is a Killing field when its flow is locally an isometry. We shall also make use of the classical equivalent to this definition given by the so called **Killing equation**:

$$< \nabla_Y X, Z >= - < \nabla_Z X, Y > . \qquad (2.5.1)$$

Lemma 2.5.3 *If M is a Riemannian manifold which admits a Killing field X of constant length, then the integral curves of X are geodesics.*

Proof:

From the Killing equation (2.5.1) we have for every $Y \in \mathfrak{X}(M)$

$$< \nabla_X X, Y >= - < \nabla_Y X, X >= 0.$$

Thus, $\nabla_X X = 0$. ⊙

Lemma 2.5.4 *Let $(S, \langle,\rangle, \nabla)$ be an almost complex curve of a nearly Kähler Riemannian manifold $(M, \langle,\rangle, \overline{\nabla})$. Then S is totally geodesic in M if and only if every unit normal vector field $\eta(t)$ of M, defined along a smooth curve $p(t)$ of S, satisfies*

$$\overline{\nabla}_{\dot{p}(t)}\eta(t) \in \perp_{p(t)} S.$$

Proof:

$(\Longrightarrow)$

If S is a totally geodesic almost complex curve, then the property stated in the lemma follows from the fact that $\{\dot{p}, J\dot{p}\}$ is a basis for T_pS and the following calculation

$$\begin{cases} \langle \overline{\nabla}_{\dot{p}}\eta, \dot{p}\rangle = -\langle \eta, \overline{\nabla}_{\dot{p}}\dot{p}\rangle = 0 \\ \langle \overline{\nabla}_{\dot{p}}\eta, J\dot{p}\rangle = -\langle \eta, \overline{\nabla}_{\dot{p}}J\dot{p}\rangle = \langle J\eta, \overline{\nabla}_{\dot{p}}\dot{p}\rangle = 0. \end{cases} \tag{2.5.2}$$

$(\Longleftarrow)$

Conversely, assume that S is an almost complex curve satisfying the property in the Lemma. Since the vector field $\eta(t)$ is orthogonal to the tangent vector fields $\dot{p}(t)$ and $J\dot{p}(t)$, we have

$$\begin{cases} \langle A_\eta\dot{p}, \dot{p}\rangle = \langle \eta, \overline{\nabla}_{\dot{p}}\dot{p}\rangle = -\langle \overline{\nabla}_{\dot{p}}\eta, \dot{p}\rangle = 0 \\ \langle A_\eta\dot{p}, J\dot{p}\rangle = \langle \eta, \overline{\nabla}_{\dot{p}}J\dot{p}\rangle. = -\langle \overline{\nabla}_{\dot{p}}\eta, J\dot{p}\rangle = 0, \end{cases} \tag{2.5.3}$$

where the last equality in each case follows from the assumption. Therefore, $A_\eta(\dot{p}) = 0$ and hence $A_\eta = 0$ because the curve $p(s)$ of S is given arbitrarily. $\odot$

Example 2.5.3 *The Hopf vector field of a geodesic hypersphere of S^6 is an H-K vector field.*

Let us first consider a great hypersphere. There is no loss of generality if we choose $M = V \cap S^6$ where $V = e_4{}^\perp$ because this hypersphere can be mapped to any other one via an element $g \in G_2$ and this transformation shall certainly map the H-K vector field of M to an H-K vector field of $g(M)$.

In this case, the unit normal vector field $\xi = e_4$ to M is constant and the Hopf vector field at a point $q \in M$ is just $U_q = e_4 \times q$. Thus

$$\langle \nabla_X U, Y \rangle = \langle \nabla_X (e_4 \times q), Y \rangle = \langle e_4 \times X, Y \rangle. \tag{2.5.4}$$

Therefore, using equation (2.5.1) and the fact that the product $\langle X \times Y, Z \rangle$ is skew-symmetric we conclude that U is a Killing field.

Now, let M be the small hypersphere of S^6 centred at the point $p = e_4$. This hypersurface is just a degenerate tube of radius r around the degenerate curve $S = \{p\}$. However, we note that all of our calculations to determine the flow $\mathcal{F}_t$ of the Hopf vector field remain valid in this situation.

In order to prove that U is a Killing field, we start by assuming this to be true and out of that assumption we deduce the natural candidate for the local isometry $\mathcal{F}_t$ which describes the flow of U.

As the rank of the second fundamental form of M is 5, we have by the rigidity of the hypersurfaces of spheres that $\mathcal{F}_t$ can be extended yielding a 1-parameter subgroup of $SO(7)$ which we shall still denote by $\mathcal{F}_t$. From linearity of $\mathcal{F}_t$ and Lemma (2.2.2) we obtain

$$\cos(r)\mathcal{F}_t e_4 + \sin(r)\mathcal{F}_t \eta = \cos(r) e_4 + \sin(r)\delta(t). \tag{2.5.5}$$

Each $\mathcal{F}_t$ must map the focal set of M to itself and since the focal set of M is just $\{e_4\}$, we have $\mathcal{F}_t e_4 = e_4$. Thus (2.5.5) can be simplified to

$$\mathcal{F}_t \eta = \cos(\bar{t})\eta + \sin(\bar{t})(e_4 \times \eta) \text{ for every } \eta \in e_4^{\perp}. \tag{2.5.6}$$

It is immediate to verify that the map $\mathcal{F}_t$ defined as above is indeed an element of $SO(7)$. Moreover, it is worth mentioning that $\mathcal{F}_t$ lies in G_2 only for the values $t = 0$ and $t = \pi \sin(r)$.

In the following, we determine the action of $\mathcal{F}_t$ on an integral curve σ of U in order to check that $\mathcal{F}_t$ corresponds, indeed, to the flow of U.

$$\begin{aligned}\mathcal{F}_t\sigma(0) &= \cos(r)e_4 + \sin(r)\mathcal{F}_t\eta\\ &= \cos(r)e_4 + \sin(r)\delta(t)\\ &= \sigma(t).\end{aligned}$$

Theorem 2.5.5 *The geodesic hyperspheres are the only connected hypersurfaces of S^6 whose Hopf vector fields are H-K vector fields.*

Proof:

Let M be a connected hypersurface of S^6 with unit normal field ξ and H-K vector field U. It follows from Lemma 2.5.3 that M is a Hopf hypersurface, say that M is a subset of a tube $\Phi_r(\perp {}^1S)$ where S is an almost complex curve of S^6. We shall assume that the Hopf principal curvature $\alpha = -\cot(r)$ is not zero, that is, $r \neq \frac{\pi}{2}$. Thus the second fundamental form of M has rank at least 3 everywhere since it is proved in Theorem (1.3.3) that the α-eigenspace of M has dimension at least 3.

From the well known rigidity of hypersurfaces of a real space form [36] we have that under the assumption that the second fundamental form having rank at least 3 everywhere, any isometry between hypersurfaces of a sphere is extendable to an ambient isometry. Therefore, the flow $\mathcal{F}_t$ of the H-K vector field U can be realised as the restriction to the hypersurface of an isometry of S^6, which we shall still name as $\mathcal{F}_t$.

Now, we prove that the almost complex curve is a connected component of the fixed point set of each isometry $\mathcal{F}_t$. Geometrically, this is almost evident for since $\mathcal{F}_t$ is the flow of the Hopf vector field, we can expect that the action of the isometry $\mathcal{F}_t$ on M, and similarly on each tubular hypersurface around S, is just to turn it around the curve S. This idea is based on the fact that we already know from Theorem (1.3.3) that the Hopf hypersurfaces of S^6 are subsets of tubes. However, we call attention to the rather subtle fact that the proof we give in the sequel does rely only upon the formula obtained in Lemma (2.2.2) for the flow of the Hopf vector field and this formula in turn depends only on the fact that the focal map of a Hopf hypersurface is constant along the integral curves of the Hopf vector field.

S is connected because it is the image of the connected tubular hypersurface M under the focal map, which is a continuous map.

Since $\mathcal{F}_t$ is an isometry of S^6 and maps open subsets of M to open subsets of M then $\mathcal{F}_t$ also maps open subsets of the focal set S to open subsets of S.

On the other hand, by construction, the map $\mathcal{F}_t$ maps an integral curve σ of the Hopf vector field of M to itself. Thus it follows from Lemma (2.2.2) that $\mathcal{F}_t$ must fix the point $p \in S$ which corresponds to the integral curve σ. Therefore, $\mathcal{F}_t$ fixes every point of S.

Moreover, it follows from the fact that S is the set fixed by $\mathcal{F}_t$, the linearity of $\mathcal{F}_t$ and Lemma (2.2.2) that for $s \in (0, \frac{\pi}{2})$ we also have $\mathcal{F}_t(\gamma_{(p,\eta)}(s)) = \gamma_{(p,\delta)}(s)$, that is the isometries $\mathcal{F}_t$ perform a non-trivial rotation of each tube of constant radius s around the curve S.

Now, using the property that S is a connected component of the fixed point set of $\mathcal{F}_t$ we shall prove that the curve S is totally geodesic by two different methods.

The first method shall essentially make use of the linearity of $\mathcal{F}_t$ and the formula (2.2.20) for the flow of the Hopf vector field whilst the second one will explore the fact that $\mathcal{F}_t$ is a one-parameter subgroup of $SO(7)$.

Method 1:

Using that $\mathcal{F}_t$ is linear and fixes S, from (2.2.20) we obtain

$$\mathcal{F}_t\eta = \cos(\bar{t})\eta - \sin(\bar{t})(p \times \eta) \text{ for every } (p, \eta) \in \perp^1 S. \tag{2.5.7}$$

Let us consider an arbitrary smooth curve $(p(s), \eta(s))$ in $\perp^1 S$. Then by differentiating (2.5.7) along this curve we have

$$\mathcal{F}_t\dot{\eta} = \cos(\bar{t})\dot{\eta} - \sin(\bar{t})\{(\dot{p} \times \eta) + (p \times \dot{\eta})\}. \tag{2.5.8}$$

Thus, since $\mathcal{F}_t^{transpose}(\dot{p}) = \mathcal{F}_t^{-1}(\dot{p}) = \dot{p}$ and using (2.5.8) we have

$$\begin{aligned}
&\langle \mathcal{F}_t\dot{\eta}, \dot{p}\rangle = \cos(\bar{t})\langle \dot{\eta}, \dot{p}\rangle - \sin(\bar{t})\langle \dot{p}\times\eta, \dot{p}\rangle - \sin(\bar{t})\langle p\times\dot{\eta}, \dot{p}\rangle \\
&\Longrightarrow \langle \dot{\eta}, \dot{p}\rangle(1-\cos(\bar{t})) = \sin(\bar{t})\langle J\dot{p}, \dot{\eta}\rangle \\
&\Longrightarrow \langle \dot{\eta}, J\dot{p}\rangle(1-\cos(\bar{t})) = -\sin(\bar{t})\langle \dot{p}, \dot{\eta}\rangle \\
&\Longrightarrow \langle \dot{\eta}, \dot{p}\rangle(1-\cos(\bar{t}))^2 = -\sin^2(\bar{t})\langle \dot{p}, \dot{\eta}\rangle \\
&\Longrightarrow \langle \dot{\eta}, \dot{p}\rangle(1-\cos(\bar{t})) = 0 \\
&\Longrightarrow \langle \dot{\eta}, \dot{p}\rangle = 0,
\end{aligned}$$

and hence by Lemma 2.5.4 the almost complex curve S is totally geodesic.

Method 2:

We want to prove that the almost complex curve S is totally geodesic and we already know that S is a connected component of the fixed point set of the isometry $\mathcal{F} = \mathcal{F}_t$. In order to do that we shall consider a conjugation $\widetilde{\mathcal{F}} = g^{-1}\mathcal{F}g$ by an element $g \in SO(7)$ such that $\widetilde{\mathcal{F}}$ be an element of the standard maximal torus of $SO(7)$. Then g maps the fixed point set of $\mathcal{F}$ exactly to the fixed point set of $\widetilde{\mathcal{F}}$ and hence $\widetilde{S} = g(S)$ is also a connected component of $\widetilde{\mathcal{F}}$.

Therefore, we can describe $\widetilde{\mathcal{F}}$ by $\widetilde{\mathcal{F}}X = AX$ where A is the matrix

$$A = \begin{pmatrix} R_0 & 0 & 0 & 0 \\ 0 & R_1 & 0 & 0 \\ 0 & 0 & R_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with } R_j = \begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix} \quad \text{and } \theta_j = \theta_j(t)$$

Since $\widetilde{\mathcal{F}}$ fixes any point $p = (p_0, ..., p_6) \in \widetilde{S}$ then our matricial representation for $\widetilde{\mathcal{F}}$ yields for each $j \in \{0, 1, 2\}$ a homogeneous system as follows

$$\begin{cases} \cos(\theta_j)\ p_{2j} + \sin(\theta_j)\ p_{2j+1} = p_{2j} \\ -\sin(\theta_j)\ p_{2j} + \cos(\theta_j)\ p_{2j+1} = p_{2j+1} \end{cases}$$

Which implies

$$\begin{cases} (\cos(\theta_j) - 1)p_{2j} + \sin(\theta_j)p_{2j+1} = 0 \\ -\sin(\theta_j)p_{2j} + (\cos(\theta_j) - 1)p_{2j+1} = 0 \end{cases}$$

This homogeneous system must hold for every real value t and every point $p \in \widetilde{S}$, thus its discriminant $\Delta_j = 2(1 - \cos\theta_j)$ vanishes if and only if the function $\cos\theta_j(t)$ is identically equal to 1 so that for at least one value $j \in \{0, 1, 2\}$, say $j = 0$, we must have $\Delta_0 \neq 0$ otherwise $\widetilde{\mathcal{F}}$ would be the Identity map. Therefore, the two first coordinates of any point of $\widetilde{S}$ vanish.

By using these systems, we can also conclude that $R_j = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ if and only if for some point $p \in \widetilde{S}$ we have $p_{2j} \neq 0$ or $p_{2j+1} \neq 0$. Consequently, there are only three possibilities for our isometries $\widetilde{\mathcal{F}}$ and their fixed point sets V, namely:

case1.

$$A = \begin{pmatrix} R_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$V = S^6 \cap span\{e_3, .., e_7\}$$

case2.

$$A = \begin{pmatrix} R_0 & 0 & 0 & 0 & 0 \\ 0 & R_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$V = S^6 \cap span\{e_5, e_6, e_7\}$$

case3.

$$A = \begin{pmatrix} R_0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & R_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$V = S^6 \cap span\{e_3, e_4, e_7\}$$

Since in all these possibilities the set V would be connected, we should have $\widetilde{S} = V$. However, in the first case $dimV = 4$. Since the other two possibilities give $\widetilde{S}$ totally geodesic and g is an isometry then S is also totally geodesic.

Now, in order to conclude the proof of the theorem we recall the example (1.3.1) where we have showed that a totally geodesic almost complex curve of S^6 is given by $S = V^3 \cap S^6$ where V^3 is spanned by vectors $\{v_1, v_2, v_3\}$ of a G_2 basis $\{v_1, ..., v_7\}$.

This gives us an obvious contradiction in the equation (2.5.7). Indeed, for $\eta = v_4$ and $p \in \{v_1, v_2\}$ we have

$$v_5 = v_1 \times v_4 = v_2 \times v_4 = v_6.$$

$\odot$

2.6 H-K vector fields in $\mathbb{C}P^n$.

In this section, we shall make use of Theorem (2.4.4) on holomorphic congruence for hypersurfaces in $\mathbb{C}P^n$, to prove that the hypersurfaces of $\mathbb{C}P^n$ which have a Killing Hopf vector field are exactly the open subsets of tubes around totally geodesic complex submanifolds. This result has already been proved by Berndt [4] but we give here a simpler and more geometrical proof.

We will think of S^{2n+1} as naturally included in $\mathbb{C}^{n+1}$ so that the Hopf fibration $\pi : S^{2n+1} \to \mathbb{C}P^n$ is a Riemannian submersion with linear isomorphism $\pi_\star : H_z \to T_{\pi(z)}\mathbb{C}P^n$ for each $z \in S^{2n+1}$, where H_z denotes the tangent subspace $\{\mathbb{R}z\}^\perp \cap \{\mathbb{R}iz\}^\perp$ of T_zS^{2n+1}. The natural complex structure on H_z, given by the complex multiplication by i, induces via $\pi_\star$ the standard complex structure J on $\mathbb{C}P^n$.

The Hopf fibration can be used to describe the geodesics of $\mathbb{C}P^n$ as projection of horizontal geodesics of S^{2n+1}, in other words, given $\zeta \in T_p(\mathbb{C}P^n)$, as π is a Riemannian submersion, then the geodesic $\gamma_{(p,\zeta)}(s)$ of $\mathbb{C}P^n$ is the projection of the horizontal geodesic

$$\tilde{\gamma}_{(\tilde{p},\tilde{\zeta})}(s) = \cos(s)\tilde{p} + \sin(s)\tilde{\zeta}, \tag{2.6.1}$$

where the tilde notation is used here to denote corresponding points and horizontal vectors under the maps π and $\pi_\star$ respectively, that is

$$\gamma_{(p,\zeta)}(s) = \pi(\tilde{\gamma}_{(\tilde{p},\tilde{\zeta})}(s)), \qquad \text{with} \qquad \begin{cases} \pi(\tilde{p}) & = p \\ \pi_\star(\tilde{\zeta}) & = \zeta. \end{cases} \tag{2.6.2}$$

Let M be a Hopf hypersurface of $\mathbb{C}P^n$ and as usual let ξ denote a local normal field on M.

By using similar procedure here, as in the case of the 6-sphere, we can also describe the integral curves of the Hopf vector field explicitly.

However, we shall recall first the fundamental calculation done by Cecil-Ryan ([18]) for the derivative of the normal exponential map G of M. They have shown that given $q \in M$ and a vector $X \in T_qM$, if we denote by $\tilde{X}$ its horizontal lift to H_z where z is a point in the fibre $\pi^{-1}(q)$, then

$$G_* \mid_{(q,r\xi)} (X) = d\pi_w\{\cos(r)\tilde{X} - \sin(r)(\tilde{Y} - \langle\tilde{X}, i\tilde{\xi}\rangle iz)\}, \tag{2.6.3}$$

Where $\xi = d\pi_z(\tilde{\xi})$, $w = \cos(r)z + \sin(r)\tilde{\xi} \in \pi^{-1}(F(q, xi))$, $Y = A_\xi X$, and the vector on the right hand side belongs to T_zS^{2n+1} but not necessarily to H_z.

We call attention to the difference between our notation for the points z and w and that of Cecil-Ryan, which unfortunately is swapped.

Using (2.6.3), Cecil-Ryan located the focal points of a Hopf hypersurface of $\mathbb{C}P^n$, as we summarize in the following

Lemma 2.6.5 *Let M be a Hopf hypersurface of $\mathbb{C}P^n$. If $U = J\xi$ denotes the Hopf vector field of M and $\alpha = -2\cot(2r)$ is the Hopf principal curvature of M, then given $q \in M$*

(i) $G_* \mid_{(q,r\xi)} (U) = 0$

(ii) $G_* \mid_{(q,r\xi)} (X) = 0$ *whenever $X \in T_qM$ is a principal vector of (M, ξ) corresponding to the principal value $-\cot(r)$.*

(iii) $G_* \mid_{(q,r\xi)} (X) \neq 0$ *otherwise.*

Now, to determine the integral curve σ of U through a given point $q \in M$, we first note from the lemma above that the focal map of M is constant along the integral curves of the Hopf vector field, that is, $G(\sigma, \xi) \equiv p$.

Next, we consider a geodesic $\gamma = \gamma_{(p,\eta)}$ of $\mathbb{C}P^n$ normal to M at q and connecting the points q and p, where η denotes the tangent vector to γ at the point p. We shall assume γ to be parametrized by the arclength s from p to q and so $\gamma(0) = p$ and $\gamma(r) = q$.

Let $\tilde{\sigma}$ be the curve in S^{2n+1} obtained as the end points of the geodesics $\tilde{\gamma}_{(\tilde{p},\tilde{\delta})}$ where

$$\tilde{\delta} = \tilde{\delta}(t) = \cos(\bar{t})\tilde{\eta} + i\sin(\bar{t})\tilde{\eta} \qquad \text{and} \qquad \bar{t} = \frac{t}{\sin(r)\cos(r)},$$

in other words, $\tilde{\sigma}(t) = \widetilde{\gamma}_{(\tilde{p},\tilde{\delta}(t))}(r)$. Let us define the vector $\delta = \pi_* \tilde{\delta}$ and the curve $\sigma(t) = \widetilde{\gamma}_{(p,\delta(t))}(r)$. Then the following calculations show that σ is indeed the integral curve of U through q.

In equations (2.6.4) and (2.6.5) below we must consider carefully along the curve $\tilde{\sigma}$ only the horizontal components of the vectors A and B, that is, their projections on $H_{\tilde{\sigma}}$.

$$\begin{aligned} \sigma' &= (\pi_\star|_{\tilde{\sigma}})(\tilde{\sigma}') \\ &= (\pi_\star|_{\tilde{\sigma}})(\sin(r)\tilde{\delta}') \\ &= (\pi_\star|_{\tilde{\sigma}})(A) \text{ where } A = \frac{1}{\cos(r)}\{\cos(\bar{t})i\tilde{\eta} - \sin(\bar{t})\tilde{\eta}\}. \end{aligned} \tag{2.6.4}$$

$$\begin{aligned} U(\sigma) &= J\dot{\gamma}_{(p,\delta)}(r) \\ &= J\pi_\star|_{\tilde{\sigma}}\ (\dot{\tilde{\gamma}}_{(\tilde{p},\tilde{\delta})}(r)) \\ &= \pi_\star|_{\tilde{\sigma}}\ (-\sin(r)i\tilde{p} + \cos(r)i\tilde{\delta}) \\ &= \pi_\star|_{\tilde{\sigma}}\ (B) \text{ where } B = (-\sin(r)i\tilde{p} + \cos(r)\cos(\bar{t})i\tilde{\eta} - \cos(r)\sin(\bar{t})\tilde{\eta}). \end{aligned} \tag{2.6.5}$$

Noting that

$$\langle B, \tilde{\sigma}\rangle = \langle B, i\tilde{\sigma}\rangle = 0 = \langle A, \tilde{\sigma}\rangle \quad \text{and} \quad \langle A, i\tilde{\sigma}\rangle = \tan(r),$$

we can see that the projections of the vectors A and B on the space $H_{\tilde{\sigma}}$ are exactly the same and hence $\sigma' = U(\sigma)$. Therefore, using all the notation above we have

Proposition 2.6.4 *The flow of the Hopf vector field of a Hopf hypersurface of* $\mathbb{C}P^n$ *can be described as:*

$$\mathcal{F}_t(\gamma_{(p,\eta)}(r)) = \gamma_{(p,\delta)}(r) = \pi(\cos(r)\tilde{p} + \sin(r)\tilde{\delta}). \tag{2.6.6}$$

As we noted in the introduction of this section there is a contrast between the hypersurfaces with H-K vector fields in S^6 and those ones in $\mathbb{C}P^n$. The later being a broader category than the former. Although Berndt ([4]) has proved the following result, we give here an alternative proof as a nice application of the holomorphic congruence of hypersurfaces in $\mathbb{C}P^n$ discussed in section (2.4).

Theorem 2.6.6 *Let M be a connected real hypersurface of $\mathbb{C}P^n$. Then the Hopf vector field U of M is an H-K vector field if and only if M lies on a tube of constant radius around a totally geodesic complex submanifold.*

Proof:

Let us assume first that U is an H-K vector field. This implies, using Lemma 2.5.3, that M is a Hopf hypersurface. Let $\mathcal{F}_t$ denote the flow of U on an open subset of M. It follows from Theorem (2.4.4) that for each t the map $\mathcal{F}_t$ can be extended to a holomorphic isometry of $\mathbb{C}P^n$ which we shall also name as $\mathcal{F}_t$. Thus, we obtain a 1-parameter subgroup B_t of $SU(n+1)$ such that

$$\mathcal{F}_t(\pi(z)) = \pi(B_t(z)). \tag{2.6.7}$$

As in the case of S^6 (cf. Theorem (2.5.5)), we can also show here that the focal set N of M is a connected component of the fixed point set of $\mathcal{F}_t$ and the proof is exactly the same as in that proposition since, as we mentioned above, Lemma (2.6.5) shows that the focal map is constant along the integral curves of U.

On the other hand, it follows from (2.6.7) that the inverse image $\widetilde{N} = \pi^{-1}(N)$ consists only of points in $\mathbb{C}^{n+1}$ which are eigenvectors for the linear operator B_t. Therefore, $\widetilde{N}$ is a disjoint union of eigenspaces of B_t, say

$$\widetilde{N} = V_{\lambda_1} \cup \ldots \cup V_{\lambda_k}. \tag{2.6.8}$$

However, if $\widetilde{N}$ is not just a single eigenspace V_λ, we would have a contradiction to the connectivity of N since in this case we would write N as a disjoint union of closed sets

$$N = \pi(V_{\lambda_1}) \cup \ldots \cup \pi(V_{\lambda_k}).$$

Therefore, the focal set of M is the totally geodesic complex submanifold of $\mathbb{C}P^n$ given by the projectivisation of the complex linear subspace $\widetilde{N} = V_\lambda$.

Conversely, let M be an open subset of a tube $\Phi_r(N)$ of radius r around a totally geodesic complex submanifold. If we make use here of some properties of the Hopf hypersurfaces of $\mathbb{C}P^n$, then we can give a short proof of the fact that U is an H-K vector field. Indeed, in this situation we have in accordance with Proposition (3.4.4) that the only possible eigenvalues for M are $\lambda_0 = \alpha = -2\cot(2r)$, $\lambda_1 = -\cot(r)$ and $\lambda_2 = \tan(r)$. Using this we can verify that U satisfies (2.5.1) as follows.

Since U has unit length and $\nabla_U U = 0$, we just need to verify (2.5.1) for vectors Y and Z orthogonal to U. Moreover, because of the linearity of $\langle \nabla_Y U, Z\rangle$ with respect to these variables, we just need to prove that equation for any pair of eigenvectors Y and Z. Thus, we have a few cases to consider.

If Y and Z lie in the same eigenspace V_λ then it follows from (2.4.7) and (2.2.5) that

$$\begin{aligned}\langle \nabla_Y U, Z\rangle &= -\langle \phi AY, Z\rangle \\ &= -\lambda\langle \phi Y, Z\rangle \\ &= \lambda\langle Y, \phi Z\rangle \\ &= -\langle Y, \nabla_Z U\rangle.\end{aligned}$$

For the other possibility we need to mention the property proved by Maeda (cf. Remark (3.3.3)) which shows in particular that the eigenspaces $V_1 = V_{\lambda_1}$ and $V_2 = V_{\lambda_2}$ are invariant under the operator ϕ. Thus, for each $i \in \{0, 1, 2\}$, the space ϕV_i is orthogonal to the spaces $\{V_j\}_{j\neq i}$. Therefore, given $i \in \{1, 2\}$, $Y \in V_i$ and $Z \in V_j$ with $i \neq j$, we have

$$\begin{aligned}\langle \nabla_Y U, Z\rangle &= -\lambda_i\langle \phi Y, Z\rangle \\ &= 0 \\ &= \lambda_j\langle Y, \phi Z\rangle \\ &= -\langle Y, \nabla_Z U\rangle.\end{aligned}$$

$\odot$

Chapter 3

Hopf Hypersurfaces in the Large.

3.1 Introduction.

The main purpose in this chapter is to prove the assumption used by Cecil-Ryan in Theorem (1.3.2) to characterize the Hopf hypersurfaces of the complex projective spaces. However, we shall start the chapter by investigating Hopf hypersurfaces in more general spaces.

In the next section we define the reflection and push maps induced by hypersurfaces of (nearly) Kähler manifolds in order to give an alternative characterization of Hopf hypersurfaces in these spaces.

In section (3.3), we deduce some properties of the principal curvatures of the level hypersurfaces of a Hopf hypersurface. These properties together with the constructions of special vector fields enable us to prove our main result.

In the last section of this chapter, we prove that the lift of a Hopf hypersurface under a holomorphic Riemannian submersion is also a Hopf hypersurface and this provides a way to obtain examples of Hopf hypersurfaces in spaces other than complex space forms.

3.2 Hopf hypersurfaces of Kähler manifolds.

The geodesic hyperspheres of complex space forms are the simplest examples of Hopf hypersurfaces, as we can see from Theorems (1.3.1),(1.3.2) and (1.3.4). Consequently, we can question naturally about what can be said of the geometry of a Kähler manifold all of whose geodesic hyperspheres are Hopf hypersurfaces. It turns out that this fact actually characterizes the complex space forms as we shall prove now.

We shall be considering some common terminology throughout this chapter which we mention next. Given a Kähler manifold $\overline{M}$ with metric $\langle , \rangle$ and complex structure J, let M be a hypersurface of $\overline{M}$ and let ξ denote a unit normal vector field defined on a neighbourhood $\mathcal{O} \subset M$ of a point $q \in M$. We can use the exponential map of $\overline{M}$ to extend ξ to a local unit vector field $\dot{\gamma}_{(p,\xi)}(s)$ on $\overline{M}$, where $p \in \mathcal{O}$ and $\gamma_{(p,\xi)}(s) = exp_p(s\xi_p)$.

When M is a hypersurface of a nearly Kähler manifold $\overline{M}$ then the vector field U_s defined along $\gamma = \gamma_{(p,\xi)}(s)$ by $U_s = J\dot{\gamma}$, is parallel along γ. Indeed,

$$\overline{\nabla}_{\dot{\gamma}} U_s = J(\overline{\nabla}_{\dot{\gamma}}\dot{\gamma}) = 0. \tag{3.2.1}$$

Theorem 3.2.1 *Let $(\overline{M}, J)$ be a Kahler manifold. Then $\overline{M}$ is a complex space form if and only if every geodesic hypersphere of $\overline{M}$ is a Hopf hypersurface.*

Proof:

($\Longrightarrow$)

It is clear from the results on Hopf hypersurfaces of complex space forms stated in the first chapter that every geodesic hypersphere of these spaces is indeed a Hopf hypersurface.

($\Longleftarrow$)

Given $q \in \overline{M}$ and a unit vector $X \in T_q\overline{M}$, let $\gamma(s) = exp_q(sX)$ be the geodesic of $\overline{M}$ starting at q in the direction X. Then, $U_s = J\dot{\gamma}(s)$ is the Hopf vector at $\gamma(s)$

of the geodesic hypersphere $\mathcal{G}_s$ centred at q and radius s. Thus, if A_s denotes the shape operator of the hypersurface $\mathcal{G}_s$, we have

$$A_s(U_s) = \alpha_s U_s, \tag{3.2.2}$$

where α_s is the Hopf principal curvature of $\mathcal{G}_s$.

Now, we show that the rate of change, in a radial direction, of the shape operators of tubular hypersurfaces satisfies a Riccati differential equation, namely:

$$(\overline{\nabla}_{\dot\gamma} A_s)(Z) = A_s^2(Z) + \overline{R}(Z, \dot\gamma)\dot\gamma, \tag{3.2.3}$$

where Z is a vector field, orthogonal to $\dot\gamma$, defined along γ and $(\overline{\nabla}_{\dot\gamma} A_s)(Z) = \overline{\nabla}_{\dot\gamma}(A_s Z) - A_s(\overline{\nabla}_{\dot\gamma} Z)$. Indeed, equation (3.2.3) follows from the definition of the curvature tensor

$$\begin{aligned}\overline{R}(Z, \dot\gamma)\dot\gamma &= \overline{\nabla}_Z \overline{\nabla}_{\dot\gamma}\dot\gamma - \overline{\nabla}_{\dot\gamma}\overline{\nabla}_Z \dot\gamma - \overline{\nabla}_{[Z,\dot\gamma]}\dot\gamma \\ &= \overline{\nabla}_{\dot\gamma}(A_s Z) - \overline{\nabla}_{(-A_s Z - \overline{\nabla}_{\dot\gamma} Z)}\dot\gamma \\ &= \overline{\nabla}_{\dot\gamma}(A_s Z) - A_s^2 Z - A_s(\overline{\nabla}_{\dot\gamma} Z).\end{aligned}$$

By using (3.2.1), (3.2.2) and (3.2.3), we obtain

$$\overline{R}(U_s, \dot\gamma)\dot\gamma = (\dot\alpha_s - \alpha_s^2)U_s. \tag{3.2.4}$$

Given a tangential vector $Y \in T_q\overline{M}$ such that Y is orthogonal to both vectors X and JX, let Y_s denote the parallel transport of Y along γ. Then (3.2.4) implies $\langle \overline{R}(U_s, \dot\gamma)\dot\gamma, Y_s\rangle = 0$ for any $s \neq 0$ and hence by continuity we have

$$\langle \overline{R}(JX, X)X, Y\rangle = 0. \tag{3.2.5}$$

However, it is well known (see for example [33] or [41]) that the condition (3.2.5) on the curvature tensor characterizes the complex space forms. ⊙

It is convenient to point out here that the Riccati equation (3.2.3) for the second fundamental forms of the tubular hypersurfaces around a submanifold P, encompasses essentially the same information as the Jacobi differential equation which defines Jacobi fields on $\overline{M}$. This equation has been useful to study the geometry of tubular hypersurfaces in general. (c.f. [25] and references mentioned there.)

We remark that Vanhecke et al used Jacobi fields (for details see [42]) to show that the complex space forms are characterized by the fact that their geodesic hyperspheres are quasi-umbilical with respect to their Hopf vector field. Thus the result we have proved above improves that of Vanhecke et al in the sense that, being Hopf hypersurfaces, the geodesic hyperspheres of complex space forms satisfy some further geometrical properties.

Remark 3.2.1 *The theorem above can be proved also using Jacobi fields instead of the Riccati equation, however, the proof would be less elegant.*

Definition 3.2.1 *Let M be a hypersurface of a Riemannian manifold $\overline{M}$. Then for some $\epsilon > 0$ and locally on M, we can define the* **reflection map** $\mathfrak{R}$ *on a tubular neighbourhood M^ϵ of an open subset $\mathcal{O}$ of M by putting*

$$\mathfrak{R}(\gamma_{(q,\xi)}(s)) = \gamma_{(q,\xi)}(-s), \tag{3.2.6}$$

where $q \in \mathcal{O}$, $s \in (-\epsilon, \epsilon)$ and ξ is a unit normal vector field on $\mathcal{O}$. For each $s \in (-\epsilon, \epsilon)$, we also define the **push map** $\mathfrak{P}_s$ *by*

$$\mathfrak{P}_s(q) = \gamma_{(q,\xi)}(s). \tag{3.2.7}$$

We will denote the level hypersurfaces of M^ϵ by M_s so that $M^\epsilon = \bigcup_{|s|<\epsilon} M_s$ and the restriction $\mathfrak{R}_s$ of $\mathfrak{R}$ maps M_s into M_{-s}, whilst $\mathfrak{P}_s$ maps M into M_s.

Lemma 3.2.1 *Let $\sigma(t)$ be a smooth curve of a hypersurface M in $\mathbb{C}P^n$. Let ξ denote a unit normal vector field on M. Then the variational vector field $W(s)$ defined along $\gamma_{(\sigma(t),\xi)}(s)$ by $W(s) = \frac{d}{dt}(\gamma_{(\sigma(t),\xi)}(s))$ satisfies*

$$\mathfrak{R}_*(W(s)) = W(-s) \tag{3.2.8}$$

$$\mathfrak{P}_{s*}(\sigma'(t)) = W(s). \tag{3.2.9}$$

Proof:

Indeed, the lemma follows from direct application of (3.2.6) and (3.2.7). ⊙

Remark 3.2.2 *Note that M is the fixed point set of $\mathfrak{R}$ and so if $\mathfrak{R}$ is an isometry then M is a totally geodesic submanifold of $\overline{M}$.*

Indeed, this is just a consequence of the well known fact that a connected component of the fixed point set of an isometry of $\overline{M}$ is a totally geodesic submanifold of $\overline{M}$. But we should note that particularly for the reflection map we can also prove this directly. Although the proof we give below is assuming that M is a hypersurface, it can be similarly applied to submanifolds of higher codimension.

Given $q \in M$, let ξ be a local unit normal vector field on M and let $X \in T_qM$ be an eigenvector of A_ξ. Let us consider a curve σ on M with $\sigma(0) = q$ and $\sigma'(0) = X$. Then the geodesic variation $\gamma_{(\sigma,\xi)}$ of the geodesic $\gamma_{(q,\xi)}$ gives the variational vector field $W(s)$ along $\gamma_{(q,\xi)}(s)$ which is a Jacobi field satisfying conditions (1.1.4) and so the shape operator A_ξ of M satisfies (1.1.6), so that $\dot{W}(0) = -A_\xi(W(0)) = -\lambda W(0)$. Now, if $\mathfrak{R}$ is an isometry then it follows from (3.2.8) that the function $|W(s)|^2$ is even. Thus, its derivative is an odd function which implies $\dot{W}(0) = 0$ and hence $\lambda = 0$.

Theorem 3.2.2 *If M is a hypersurface of a nearly Kähler manifold $\overline{M}$ satisfying condition $(\star)$ below, then M is a Hopf hypersurface.*

$(\star)$: *for each $s \in (-\epsilon, \epsilon)$, $\mathfrak{R}$ maps the Hopf vector field of M_s to a scalar multiple of the Hopf vector field of M_{-s}.*

Proof:

Let M be a hypersurface of $\overline{M}$ satisfying the condition in the Theorem, then in order to prove that M is a Hopf hypersurface we will just verify that the Hopf vector field U of M is a principal vector field.

Given $q \in M$, consider a local unit normal vector field ξ of M defined around q. Let

$A = A_\xi$ denote the second fundamental form of M. It follows from (3.2.6) that

$$\mathfrak{R}_*|_{\gamma_{(q,\xi)}(s)}\ (\dot\gamma_{(q,\xi)}(s)) = -\dot\gamma_{(q,\xi)}(-s). \tag{3.2.10}$$

By assumption there exists a smooth function $g(s) = g(q,s)$ such that

$$\mathfrak{R}_*|_{\gamma_{(q,\xi)}(s)}\ (U_s) = g(s)U_{-s}. \tag{3.2.11}$$

We can fix the point q because in what follows we shall be considering the rate of change of the function g only in the radial direction. Since $\mathfrak{R}$ is a smooth map, using (3.2.1), we have

$$\begin{aligned}
&\mathfrak{R}_*[\dot\gamma, U_s] = [\mathfrak{R}_*\dot\gamma, \mathfrak{R}_*U_s] \\
\Longrightarrow &\mathfrak{R}_*(\overline\nabla_{\dot\gamma}U_s) - \mathfrak{R}_*(\overline\nabla_{U_s}\dot\gamma) = \overline\nabla_{\mathfrak{R}_*\dot\gamma}\mathfrak{R}_*U_s - \overline\nabla_{\mathfrak{R}_*U_s}\mathfrak{R}_*\dot\gamma \\
\Longrightarrow &\overline\nabla_{\mathfrak{R}_*U_s}\mathfrak{R}_*\dot\gamma = \overline\nabla_{\mathfrak{R}_*\dot\gamma}\mathfrak{R}_*U_s + \mathfrak{R}_*(\overline\nabla_{U_s}\dot\gamma).
\end{aligned} \tag{3.2.12}$$

Now, let A_s denote the shape operator of the level hypersurface M_s with respect to the normal field $\dot\gamma(s)$, then if we substitute (3.2.10) and (3.2.11) in (3.2.12) we obtain

$$g(s)A_{-s}U_{-s} = -\dot g(s)U_{-s} - \mathfrak{R}_*(A_sU_s), \tag{3.2.13}$$

so that by taking the limit when s goes to zero and recalling that the reflection restricts to the identity map on M, we finally have

$$AU(q) = -\frac{1}{2}\dot g(0)U(q). \tag{3.2.14}$$

And hence M is a Hopf hypersurface. $\odot$

Proposition 3.2.1 *A hypersurface of S^6 satisfies the condition $(\star)$ if and only if it is a Hopf hypersurface.*

Proof:

Let M be a Hopf hypersurface of S^6. Then by the characterization given in Theorem (1.3.3) we may assume $M \subset \Phi_r(\perp^1 S)$ for some almost complex curve S in S^6. Let us define $q^{\pm} = \gamma_{(p,\eta)}(r \pm s) \in M_{\pm s}$. According to the characterization of integral curves of the Hopf vector field given by Lemma (2.2.2), we have that the integral curve $\sigma_s(t)$ of the Hopf vector field U_s of M_s passing through q^+ is $\gamma_{(p,\delta_s(t))}(r+s)$ where

$$\delta_s(t) = \cos(\frac{t}{\sin(r+s)})\eta + \sin(\frac{t}{\sin(r+s)})p \times \eta. \tag{3.2.15}$$

On the other hand the integral curve $\sigma_{-s}(t)$ of U_{-s} passing through $q^- = \mathfrak{R}(q^+)$ is $\gamma_{(p,\delta_{-s}(t))}(r-s)$ where

$$\delta_{-s}(t) = \cos(\frac{t}{\sin(r-s)})\eta + \sin(\frac{t}{\sin(r-s)})p \times \eta. \tag{3.2.16}$$

From the definition of the reflection map we have

$$\mathfrak{R}(\gamma_{(p,\delta_s(t))}(r+s)) = \gamma_{(p,\delta_s(t))}(r-s). \tag{3.2.17}$$

By observing that

$$\delta_s(t) = \delta_{-s}(kt), \tag{3.2.18}$$

where $k = \frac{\sin(r-s)}{\sin(r+s)}$, we see that (3.2.17) is reduced to $\mathfrak{R}(\sigma_s(t)) = \sigma_{-s}(kt)$ and hence

$$\mathfrak{R}_*(U_s(q^+)) = \mathfrak{R}_*(\dot{\sigma}_s(0)) = k\dot{\sigma}_{-s}(0) = kU_{-s}(q^-). \tag{3.2.19}$$

$\odot$

In the next section, we prove a similar converse to Theorem (3.2.2) for hypersurfaces of $\mathbb{C}P^n$.

Theorem 3.2.3 *If a hypersurface M of a nearly Kahler manifold $\overline{M}$ satisfies the condition $(\star\star)$ below, then each level hypersurface M_s is a Hopf hypersurface.*

$(\star\star)$: *for each $s \in (-\epsilon, \epsilon)$, $\mathfrak{P}_s$ maps the Hopf vector field of M to a scalar multiple of the Hopf vector field of M_s.*

Proof:

Let us use the same notation and terminology as in the proof of Theorem (3.2.2).

Here we can give a simpler proof since by using the assumption we see that the push map $\mathfrak{P}_s$ will map the integral curve σ of U to the integral curve, possibly reparametrised, σ_s of U_s. This fact implies that there exists a smooth function $f(s)$ such that the Jacobi field $V(s)$ along $\gamma_{(q,\xi)}(s)$ defined by $V(s) = \frac{d}{dt}\gamma_{(\sigma,\xi)}(s)$ can be expressed by:

$$V(s) = f(s)U_s. \tag{3.2.20}$$

Using (3.2.1), we obtain $\dot{V} = \dot{f}U_s$. Now, observing that V satisfies the conditions (1.1.4), that is, V is a M_s-Jacobi field, then we have from (1.1.6) that

$$A_sU_s = A_sV(s) = -\dot{V} = -\dot{f}U_s, \tag{3.2.21}$$

and hence M_s is a Hopf hypersurface. $\odot$

In the next section, we give a similar converse to Theorem (3.2.3) for hypersurfaces of $\mathbb{C}P^n$.

The following result is proved in a manner similar to that of Proposition (3.2.1).

Proposition 3.2.2 *A hypersurface of S^6 satisfies the condition ($\star\star$) if and only if it is a Hopf hypersurface.*

3.3 Properties of the Hopf hypersurfaces of $\mathbb{C}P^n$.

We give in this section some further geometrical properties of Hopf hypersurfaces in $\mathbb{C}P^n$ which not only point out more evidence that they are indeed tubular but also highlight some special features of the geometry of such hypersurfaces.

Let M be a Hopf hypersurface of $\mathbb{C}P^n$. Let ξ and $U = J\xi$ denote a unit local normal field and the corresponding Hopf vector field respectively. As we mentioned in Remark (1.3.3), the Hopf principal curvature α of M is locally constant, thus we may consider

$$\alpha = -2\cot(2r),$$

for some constant $r \in (0, \frac{\pi}{4}]$. Moreover, using Gauss and Codazzi equations, Maeda ([31]) has shown the following main result known about the geometry of a Hopf hypersurface of $\mathbb{C}P^n$.

Theorem 3.3.4 *Let M be a Hopf hypersurface of $\mathbb{C}P^n$ with Hopf principal curvature α. Let A denote the shape operator of M with respect to a unit normal vector field ξ on M and let ϕ be the tensor on M induced by the complex structure of $\mathbb{C}P^n$ as defined by (2.2.2). Then these tensors satisfy the following relation*

$$A\phi A = \phi + \frac{\alpha}{2}(A\phi + \phi A). \tag{3.3.1}$$

Remark 3.3.3 *As Maeda observed, the result above shows in particular that if X is a principal vector field of M with principal curvature λ, orthogonal to the Hopf vector field U, then ϕX is also a principal vector field with corresponding principal curvature $\tilde{\lambda}$, where*

$$\tilde{\lambda} = \frac{\alpha\lambda + 2}{2\lambda - \alpha}. \tag{3.3.2}$$

Equivalently, if we consider the principal curvature λ given in terms of a new function $\theta : M \to \mathbb{R}$ by $\lambda = -\cot(r + \theta)$ then $\tilde{\lambda} = -\cot(r - \theta)$.

The local constancy of the Hopf principal curvature α of M is not an isolated fact in the sense that using the Codazzi equation (2.4.6) for M we can actually prove the following

Proposition 3.3.3 *Let X be a unit smooth principal vector field of a Hopf hypersurface $M \subset \mathbb{C}P^n$ corresponding to a principal curvature function λ. Then λ is constant along any integral curve of the Hopf vector field U.*

Proof:

Using that the Hopf principal curvature α is constant, it follows from (2.4.6) that

$$U(\lambda)X = \alpha\nabla_X U - A(\nabla_X U) - \lambda\nabla_U X + A(\nabla_U X) - \phi X.$$

Thus, using (2.4.7) and (3.3.2) we get

$$U(\lambda)X = A(\nabla_U X) - \lambda\nabla_U X - (1 + \alpha\lambda - \lambda\tilde{\lambda})\phi X. \tag{3.3.3}$$

Consequently, the inner product of this equation with X yields

$$U(\lambda) = 0.$$

⊙

Remark 3.3.4 *If X is a unit smooth principal vector field with corresponding principal curvature λ, then it follows also from* (3.3.3) *that $\nabla_U X$ is a principal vector field corresponding to the same principal curvature λ if and only if $\lambda = \tilde{\lambda}$ and hence, using* (3.3.2), *if and only if $\lambda = -\cot(r)$ or $\lambda = \tan(r)$.*

We shall see now that, in general, almost every principal vector of a level hypersurface of a Hopf hypersurface M in $\mathbb{C}P^n$ is obtained simply by parallel transporting principal vectors of M along normal geodesics, more precisely we have

Theorem 3.3.5 *Let M be a Hopf hypersurface of $\mathbb{C}P^n$. Let X be a unit principal vector at $q \in M$ corresponding to an eigenvalue λ such that X is either equal or* *orthogonal to the Hopf eigenvector U at q. Then for each s, the parallel transport $X(s)$ of X along the normal geodesic $\gamma = \gamma_{(q,\xi)}(s)$ (that is, $\xi = \dot{\gamma}(0) \in \perp_q M$) starting at q, is a principal vector of the level hypersurface M_s.*

Proof:

Let us consider a curve σ of M such that $\sigma(0) = q$ and $\sigma'(0) = X$. Then the 2-parameter family of geodesics in $\mathbb{C}P^n$

$$F(s,t) = \gamma_{(\sigma,\xi)}(s)$$

yields the Jacobi field

$$W(s) = \frac{\partial F}{\partial t}(s,0),$$

defined along the geodesic $\gamma(s)$. Then W satisfies the initial conditions $W(0) = X$ and $\dot{W}(0) = -A_0 X = -\lambda X$.

On the other hand, let $f : \mathbb{R} \to \mathbb{R}$ denote either the solution of the differential equation $\ddot{f} = -f$ if X is orthogonal to U, or the solution of $\ddot{f} = -4f$ if $X = U$, satisfying the initial conditions $f(0) = 1$ and $\dot{f}(0) = -\lambda$. Then, using the curvature tensor $\overline{R}$ for $\mathbb{C}P^n$ as given by (2.4.1) we obtain

$$\overline{R}(fX(s), \dot{\gamma})\dot{\gamma} = -\ddot{f}X(s)s.$$

Therefore, $fX(s)$ is also a Jacobi field along γ having the same initial conditions as W and hence $W = fX(s)$. However, by construction, W is a M_s-Jacobi field for each s and so $A_s(W(s)) = -\dot{W}(s)$, which implies

$$A_s X(s) = -\frac{\dot{f}}{f}X(s).$$

If we denote the Hopf principal curvature by $\alpha = -2\cot(2r)$ and if we write the principal curvature as $\lambda = -\cot(r+\theta)$ then for the case when X is orthogonal to U, the function f is given by

$$f(s) = \frac{\sin(r+\theta+s)}{\sin(r+\theta)}. \tag{3.3.4}$$

⊙

Note that in particular, applying this result to the Hopf eigenvector, we have also proved the following

Corollary 3.3.1 *The level hypersurfaces of a Hopf hypersurface in $\mathbb{C}P^n$ are also Hopf hypersurfaces.*

Theorem 3.3.6 *A hypersurface of $\mathbb{C}P^n$ satisfies either the condition $(\star)$ or $(\star\star)$ if and only if it is a Hopf hypersurface.*

Proof:

One direction has already been proved in accordance with Theorems (3.2.2) and (3.2.3). So if M is a Hopf hypersurface, let us show firstly that the push maps $\mathfrak{P}_s$ satisfy $(\star\star)$.

Indeed, given $q \in M$, let σ be the integral curve of the Hopf vector field U which passes through q, say $\sigma(0) = q$. Then it follows from (3.2.9) that

$$\begin{aligned} d\mathfrak{P}_s \mid_q (U_q) &= \frac{d}{dt} \mid_{t=0} \mathfrak{P}_s(\sigma(t)) \\ &= \frac{dF}{dt}(s,0) \\ &= f(s)U_s, \end{aligned} \tag{3.3.5}$$

From which $(\star\star)$ follows.

Now, for the reflection map we set up $W(s) = fX(s)$ in (3.2.8), which yields

$$d\mathfrak{R}(U_s) = \frac{f(-s)}{f(s)}U_{-s}. \tag{3.3.6}$$

Therefore, M also satisfies $(\star)$. $\odot$

3.4 Tubular hypersurfaces of $\mathbb{C}P^n$.

We recall here Theorem (1.3.2), where Cecil-Ryan, under the assumption of constancy of the rank of the focal map of a hypersurface, characterize the Hopf hypersurfaces of $\mathbb{C}P^n$ as open subsets of tubes around complex submanifolds. Now, looking at the calculation for the derivative of the focal map, Lemma (2.6.5), we can see that this assumption is equivalent to the constancy of the rank of the principal curvature $-\cot(r)$, wherever this value be a principal curvature for the hypersurface.

We shall investigate hereafter what is the behaviour of the principal curvatures of a Hopf hypersurface of $\mathbb{C}P^n$ from the extrinsic viewpoint, that is, considering the hypersurface as part of a family of level hypersurfaces. Our intention in doing that shall be to collect as much information as possible about the principal curvatures in order to determine the rank of the principal value $-\cot(r)$.

It is worthwhile highlighting here that from the intrinsinc point of view the best geometrical property that we know about a Hopf hypersurface in $\mathbb{C}P^n$ is the elegant result of Maeda stated in Theorem (3.3.4). However, this result does not seem to be sufficient to evaluate the behaviour of $-\cot(r)$ as an eigenvalue of the shape operator of the hypersurface.

The main results of this section which are concerned with the characterization of the Hopf hypersurfaces of $\mathbb{C}P^n$, shall use the crucial constructions of tangent vector fields X_t and V_t that we now start to describe.

Let us fix a point q in a Hopf hypersurface M of $\mathbb{C}P^n$. Then, according to Proposition (2.6.4) we can write $q = \gamma(r) = \gamma_{(p,\eta)}(r)$ and the integral curve of the Hopf vector field U of M is described by $\sigma(t) = \gamma_t(r) = \gamma_{(p,\delta_t)}(r)$ where $\delta_t = \cos(\bar{t})\eta + \sin(\bar{t})J\eta$ and $\bar{t} = \frac{2t}{\sin(2r)}$.

In Corollary (3.3.1), we showed that the level hypersurfaces M_s of M are also Hopf hypersurfaces. Thus, we just need to replace r by s in the description above in order to describe the integral curve σ_s of the Hopf vector field U_s of the level hypersurface

M_s starting at the point $\gamma(s)$.

Definition 3.4.2 *Given a vector $X_0 \in T_qM$ orthogonal to the Hopf vector $U(q)$, let $X_0(s)$ denote the parallel transport of X_0 along $\gamma(s)$. Then we can construct a smooth vector field X_t along $\sigma(t)$ in two different manners which we shall name hereafter as*

Case I.

The vector field $X_t(r)$ is defined as the parallel transport of $X_0(0)$ along $\gamma_t(s)$ from the point p to the point $\sigma(t)$.

Case II.

The vector field $X_t(r)$ is defined as the parallel transport along $\gamma_t(s)$ from the point p to the point $\sigma(t)$ of the vector

$$X_t(0) = \cos(\frac{\bar{t}}{2})X_0(0) + \sin(\frac{\bar{t}}{2})JX_0(0). \tag{3.4.1}$$

Remark 3.4.5 *The vectors $\{X_0(0), JX_0(0)\}$ are orthogonal to the vectors $\{\eta, J\eta\}$ because $X_0(r)$ and $JX_0(r)$ are both also orthogonal to the vectors $\{\dot\gamma_0(r), U_0(q)\}$. Therefore, in both constructions above $X_t(0)$ is orthogonal to δ_t for every t. Thus, by elementary properties of parallel vector fields we have that $X_t(s)$ is orthogonal to $\dot\gamma_t(s)$ for each value of t and s. In particular, this makes it clear that $X_t(s)$ is indeed a tangent vector field defined along $\sigma_s(t)$ on the level hypersurface M_s.*

Definition 3.4.3 *Let us denote the induced Riemannian connection of each hypersurface M_s by the same symbol ∇. Then, for each construction of $X_t(s)$ as given in Definition* (3.4.2), *we associate the following vector field*

$$V_t(s) = \nabla_{U_s} X_t(s) + \frac{\alpha_s}{2}\phi X_t(s). \tag{3.4.2}$$

The constructions of X_t and V_t may appear artificial at first. However, as we shall see in Theorem (3.4.7), they arise quite naturally when considering the case of tubular hypersurfaces.

In the sequel, we shall need to recall some basic facts about Jacobi fields of $\mathbb{C}P^n$ in order to prove our next proposition.

The Hopf fibration $\pi : S^{2n+1} \to \mathbb{C}P^n$ can be used to write down Jacobi fields of $\mathbb{C}P^n$ in terms of Jacobi fields of the sphere. Indeed, let $\gamma = \gamma_{(q,\xi)}(s)$ be a geodesic of $\mathbb{C}P^n$ which is lifted to a horizontal geodesic $\tilde{\gamma} = \tilde{\gamma}_{(\tilde{q},\tilde{\xi})}(s)$ in the sphere. Then the Jacobi fields along γ are given as follows.

Lemma 3.4.2 *The Jacobi field $W(s)$ along $\gamma(s)$ satisfying the initial conditions $W(0) = X$ and $\dot{W}(0) = Y$ is determined by*

$$W(s) = \cos(s)B_X(s) + \sin(s)B_Y(s), \tag{3.4.3}$$

where $B_Z(s)$ denotes the image under π_* of the parallel transport $\tilde{B}_{\tilde{Z}}(s)$ of $\tilde{Z}$ along $\tilde{\gamma}(s)$.

Proof:

Indeed, we first need to observe that the variation of $\tilde{\gamma}$ given by

$$\tilde{F}(s,t) = \cos(s)\big(\cos(t)\tilde{q} + \sin(t)\tilde{X}\big) + \sin(s)\big(\cos(t)\tilde{\xi} + \sin(t)\tilde{Y}\big),$$

consists of horizontal geodesics since the initial tangent vector of each of these geodesics are horizontal. Consequently, the projection $F(s,t) = \pi(\tilde{F}(s,t))$ is also a geodesic variation. Secondly, we note that the former variation corresponds to the Jacobi field

$$\tilde{W} = \cos(s)\tilde{B}_{\tilde{X}}(s) + \sin(s)\tilde{B}_{\tilde{Y}}(s).$$

$\odot$

Let us first have a close look at tubes around complex submanifolds of $\mathbb{C}P^n$. We shall do this in order to get a good picture of the geometrical relation between the principal curvatures of a tubular hypersurface around a complex submanifold and the principal curvatures of this core.

Proposition 3.4.4 *Let M be an open subset of a tube $\Phi_r(\perp^1 N)$ of radius r around a complex submanifold N^m of $\mathbb{C}P^n$. The principal vectors of M at a point $q = \gamma(r) = \gamma_{(p,\eta)}(r)$ are obtained according to the following cases*

(i) $A_\xi B = -2\cot(2r)B$,
where $B(s) = \pi_*(\tilde{B}_{i\tilde{\eta}})$ and $W(s) = \sin(s)B(s)$ is the Jacobi field along γ satisfying the initial conditions $W(0) = 0$ and $\dot{W}(0) = J\eta$. Note that $B = J\xi = U$.

(ii) $A_\xi B = -\cot(r)B$,
where $B(s) = \pi_*(\tilde{B}_{\tilde{X}})$ and $W(s) = \sin(s)B(s)$ is the Jacobi field along γ satisfying the initial conditions $W(0) = 0$ and $\dot{W}(0) = X \in (\perp_p N) \cap \{\eta\}^\perp$

(iii) $A_\xi B = -\cot(r+\theta)B$,
where X is a principal vector of the shape operator A_η of N corresponding to the principal value $\cot(\theta)$, $B(s) = \pi_*(\tilde{B}_{\tilde{X}})$ and $W(s) = \big(\cos(s) - \cot(\theta)\sin(s)\big)B(s)$ is the Jacobi field along γ satisfying the initial conditions $W(0) = X \in T_pN$ and $\dot{W}(0) = -A_\eta X = -\cot(\theta)X$.

Proof:

The proposition follows immediately from Lemma (3.4.2) and the fact that the Jacobi field W satisfies conditions (1.1.4) and hence satisfies (1.1.6), that is, $\dot{W}(0) = -A_\xi W(0)$. ⊙

Remark 3.4.6 *In the Proposition above we can highlight some useful facts. The first, being that* **(i)** *shows that every tube around a complex submanifold is indeed a Hopf hypersurface. Secondly, it follows from* **(ii)** *that the multiplicity of the eigenvalue $-\cot(r)$ is exactly $2(n-m)$ at each point of the hypersurface M.*

The next theorem points out the geometrical relevance of the vector field V_t for the study of the principal curvatures in the case of a tubular hypersurface.

Theorem 3.4.7 *Let M be an open subset of the tube $\Phi_r(\perp^1 N)$ of radius r around a complex submanifold N of $\mathbb{C}P^n$. Let $q = \gamma_{(p,\eta)}(r) \in M$ and let $X_0 \in T_qM$ be a vector orthogonal to $U(q)$. Then the vector fields X_t and V_t, as given in Definitions* (3.4.2) *and* (3.4.3) *satisfy the following properties*

(i) *If X_0 is an eigenvector of M corresponding to the eigenvalue $-\cot(r)$ (respectively, $-\cot(r+\theta)$). Then, for every $s \in (0,r]$, the vector field $X_t(s)$ constructed in case I (case II) is a principal field along σ_s corresponding to the eigenvalue $-\cot(s)$ (respectively, $-\cot(s+\theta)$).*

(ii) *$V_t(s) \neq 0$ for every $s \in (0,r]$, in Case I.*

(iii) *$V_t(s) \equiv 0$, in case II and consequently $\nabla_{U_s} X_t = -\frac{\alpha_s}{2}\phi X_t$.*

Proof:

Item (i), for Case I, follows imediately from item (ii) of Proposition (3.4.4). To prove (i) for Case II, we note that from item (iii) of Proposition (3.4.4), we have $\overline{\nabla}_{X_0(0)}\eta = -\cot(\theta)X_0(0)$, which implies

$$\overline{\nabla}_{X_t(0)}\delta_t = -\cot(\theta)X_t(0). \tag{3.4.4}$$

And hence, using again item (iii) of that proposition, (i) follows.

Now, in order to prove (ii) and (iii), we need to consider the geodesic variation $F(s,t) = \gamma_{(p,\delta_t)}(s)$ with its corresponding variational Jacobi field $W_t(s) = \frac{\partial F}{\partial t}(s,t)$. Then for each t, W_t is a M_s-Jacobi field since it satisfies condition (1.1.4) and hence $A_sW_t = -\dot{W}_t$, where A_s denotes the shape operator of the level hypersurface M_s.

Using Proposition (2.6.4), we see that W_t satisfies the initial conditions

$$W_t(r) = U(\sigma(t)) \quad \text{and} \quad \dot{W}_t(r) = 2\cot(2r)U(\sigma(t)).$$

Therefore, setting $h(s) = \frac{\sin(2s)}{\sin(2r)}$, we have $W_t(s) = h(s)U_s$ because $h(s)U_s$ is also a Jacobi field along γ_t satisfying the same initial conditions.

It follows from (2.4.1) that

$$\overline{R}(\dot{\gamma}_t, W_t)X_t = -2h\phi X_t. \tag{3.4.5}$$

On the other hand, using the definition of the curvature tensor and the following fact

$$[\dot{\gamma}_t, W_t] = \overline{\nabla}_{\dot{\gamma}_t}(hU_s) - \overline{\nabla}_{hU_s}(\dot{\gamma}_t) = (\alpha_s h + \dot{h})U = 0, \tag{3.4.6}$$

we obtain

$$\begin{aligned} \overline{\nabla}_{\dot{\gamma}_t}\overline{\nabla}_{W_t}X_t &= -2h\phi X_t \\ &= -\frac{2\sin(2s)}{\sin(2r)}\phi X_t \\ &= \overline{\nabla}_{\dot{\gamma}_t}\left(\frac{\cos(2s)}{\sin(2r)}\phi X_t\right) \end{aligned} \tag{3.4.7}$$

and hence

$$\begin{aligned} \overline{\nabla}_{\dot{\gamma}_t}(hV_t) &= \overline{\nabla}_{\dot{\gamma}_t}(\overline{\nabla}_W X_t + \frac{\alpha_s h}{2}\phi X_t) \\ &= \overline{\nabla}_{\dot{\gamma}_t}(\overline{\nabla}_W X_t - \frac{\cos(2s)}{\sin(2r)}\phi X_t) \\ &= 0. \end{aligned}$$

In other words, the vector field $h(s)V_t(s)$ is parallel along the geodesic $\gamma_t(s)$. Therefore, using this parallelism, (ii) follows from the limit

$$\lim_{s\to 0}(hV_t) = -\frac{1}{\sin(2r)}JX_t = 0,$$

and (iii) follows from (3.4.1) and the limit

$$\lim_{s\to 0}(hV_t) = \frac{dX_t}{dt} - \frac{1}{\sin(2r)}JX_t = 0.$$

$\odot$

Remark 3.4.7 *The parallelism of the vector field hV_t is equivalent to the property $\overline{\nabla}_{\dot{\gamma}_t}V_t = \alpha_s V_t$ because of $h(s)$ satisfying $\dot{h}(s) = -\alpha_s h(s)$.*

In each construction given by Definition (3.4.2), the vector field $X_t(s)$ satisfies the basic property below which shall be used extensively to prove some oncoming results in this section.

Lemma 3.4.3 *The vector field $X_t(s)$ is orthogonal to the Hopf vector field U_s.*

Proof:

Let $W_t(s)$ be the M_s-Jacobi field defined along γ_t as in Theorem (3.4.7). Then, using that $W_t(s) = h(s)U_s$ we have

$$h(s)\langle X_t, U_s\rangle = \langle X_t, W_t\rangle,$$

which by differentiation with respect to s yields

$$\begin{aligned} \dot{h}(s)\langle X_t, U_s\rangle \quad &= \langle X_t, \overline{\nabla}_{\dot{\gamma}} W_t\rangle \quad \text{Using that } X_t \text{ and } U_s \text{ are parallel along } \gamma_t \\ &= \langle X_t, \overline{\nabla}_{W_t}\dot{\gamma}\rangle \quad \text{Using (3.4.6).} \end{aligned} \tag{3.4.8}$$

Thus, calculating the limit of (3.4.8) when s goes to zero we obtain

$$\begin{aligned} \lim_{s\to 0}\langle X_t, U_s\rangle \quad &= \lim_{s\to 0}\frac{1}{\dot{h}(s)}\langle X_t, \overline{\nabla}_{W_t}\dot{\gamma}\rangle \\ &= \frac{\sin(2r)}{2}\langle X_0(0), \frac{d\delta_t}{dt}\rangle \\ &= 0. \end{aligned}$$

Therefore, using again that X_t and U_s are parallel along γ_t, we have $\langle X_t, U_s\rangle \equiv 0$ which proves the Lemma. ⊙

Inspired by the geometrical properties of the tubular hypersurfaces of $\mathbb{C}P^n$ described above, we can show now that these properties hold in general for any Hopf hypersurface of this space form. Thus, we shall be henceforth considering M as an arbitrary Hopf hypersurface of $\mathbb{C}P^n$.

We shall prove next that in the case of our generic Hopf hypersurface M, the vector fields X_t and V_t also satisfy properties similar to those obtained in Theorem (3.4.7) for tubular Hopf hypersurfaces.

Theorem 3.4.8 *Let M be a Hopf hypersurface of $\mathbb{C}P^n$. Let $q = \gamma_{(p,\eta)}(r) \in M$ and let $X_0 \in T_qM$ be a vector orthogonal to $U(q)$. Then the vector fields X_t and V_t, as given in Definitions* (3.4.2) *and* (3.4.3) *satisfy the following properties*

(i) $V_t(s) \neq 0$ *for every* $s \in (0, r]$, *in Case I.*

(ii) $V_t(s) \equiv 0$, *in case II and consequently* $\nabla_{U_s} X_t = -\frac{\alpha_s}{2}\phi X_t$.

(iii) *If in addition we assume that* M *is analytic and* X_0 *is an eigenvector of* M *corresponding to the eigenvalue* $-\cot(r)$ *(respectively,* $-\cot(r+\theta)$*), then for every* $s \in (0, r]$, *the vector field* $X_t(s)$ *constructed in case I (case II) is a principal field along* σ_s *corresponding to the eigenvalue* $-\cot(s)$ *(respectively,* $-\cot(s+\theta)$*).*

Proof:

The proof of item (i) and (ii) can be carried out in the same manner as that given in Theorem (3.4.7) for tubular hypersurfaces as far as we can show that the vector field $h(s)V_t$ here is also parallel. According to the Remark (3.4.7), this parallelism is equivalent to $\overline{\nabla}_{\dot{\gamma}_t} V_t = \alpha_s V_t$, which can be proved as follows.

The vector field X_t is orthogonal to $\dot{\gamma}_t$ by Remark (3.4.5) and is also orthogonal to U_s by Lemma (3.4.3). Thus, we have

$$\overline{\nabla}_{U_s} X_t = \nabla_{U_s} X_t. \tag{3.4.9}$$

We have proved in Theorem (3.3.1) that M_s is a Hopf hypersurface and so

$$\begin{aligned} [\,\dot{\gamma}_t, U_s\,] &= \overline{\nabla}_{\dot{\gamma}_t} U_s - \overline{\nabla}_{U_s} \dot{\gamma}_t \\ &= \alpha_s U_s. \end{aligned} \tag{3.4.10}$$

Thus, using the results above and (2.4.1), we obtain

$$\begin{aligned} \overline{\nabla}_{\dot{\gamma}_t} \overline{\nabla}_{U_s} X_t &= \overline{\nabla}_{U_s} \overline{\nabla}_{\dot{\gamma}_t} X_t + \overline{\nabla}_{[\,\dot{\gamma}_t, U_s\,]} X_t + \overline{R}(\dot{\gamma}_t, U_s) X_t \\ &= -2\phi X_t + \alpha_s \nabla_{U_s} X_t. \end{aligned} \tag{3.4.11}$$

Now, applying (3.4.9),(3.4.10) and (3.4.11) to (3.4.2), we have

$$\begin{aligned} \overline{\nabla}_{\dot{\gamma}_t} V_t &= \overline{\nabla}_{\dot{\gamma}_t} (\nabla_{U_s} X_t) + \frac{\dot{\alpha}_s}{2}\phi X_t \\ &= (\frac{\dot{\alpha}_s}{2} - 2)\phi X_t + \alpha_s \nabla_{U_s} X_t \\ &= \alpha_s V_t \quad \text{since } \dot{\alpha}_s = 4 + \alpha_s^2. \end{aligned} \tag{3.4.12}$$

The last part of the theorem shall be proved by showing that the following vector field $Z_t(s)$ defined along $\sigma_s(t)$ is identically zero.

$$Z_t = A_s X_t - \lambda X_t, \tag{3.4.13}$$

where A_s denotes the shape operator of the level hypersurface M_s.

First, we notice that the analyticity of the ambient space $\mathbb{C}P^n$ and of the hypersurface M imply that we can construct a local analytical unit normal field on M. Thus, the field Z_t is also analytic and it is identically zero if and only if all the derivatives of Z_t with respect to t vanish at $t = 0$.

In order to simplify our notation we shall omit any subscript s since it is clear that we are considering all the geometrical objects involved as defined on each level hypersurface M_s.

It follows from the Codazzi equation (2.4.6) together with (2.4.7) and the fact that M_s is a Hopf hypersurface that

$$\nabla_U(AX_t) = \nabla_{X_t}(AU) - A(\nabla_{X_t}U) + A(\nabla_U X_t) - \phi X_t. \tag{3.4.14}$$

We can also differentiate ϕAX_t using (2.2.8) as follows.

$$\nabla_U(\phi AX_t) = \phi\nabla_U(AX_t) \tag{3.4.15}$$

We shall first consider the situation when $AX_0 = -\cot(r+\theta)X_0$ and X_t is constructed as in Case II. Then, recalling that the Hopf principal curvature α is constant, we have from (3.4.14)

$$\begin{aligned} \nabla_U(AX_t) &= -\alpha\phi AX_t + A\phi AX_t - \frac{\alpha}{2}A\phi X_t - \phi X_t && \text{Using item (ii)} \\ &= -\frac{\alpha}{2}\phi AX_t && \text{Using (3.3.1),} \end{aligned} \tag{3.4.16}$$

and hence (3.4.15) can be simplified to

$$\nabla_U(\phi AX_t) = \frac{\alpha}{2}AX_t. \tag{3.4.17}$$

Thus, it follows from (3.4.16) and (3.4.17) that the n-th derivative of AX_t is given by

$$\nabla^n_U(AX_t) = \begin{cases} (-1)^{m+1}(\frac{\alpha}{2})^n \phi AX_t & \text{if } n = 2m+1. \\ (-1)^m(\frac{\alpha}{2})^n AX_t & \text{if } n = 2m. \end{cases} \tag{3.4.18}$$

On the other hand, it follows from item (ii) and (2.2.8) that the n-th derivative of X_t is given by

$$\nabla^n_U(X_t) = \begin{cases} (-1)^{m+1}(\frac{\alpha}{2})^n \phi X_t & \text{if } n = 2m+1. \\ (-1)^m(\frac{\alpha}{2})^n X_t & \text{if } n = 2m. \end{cases} \tag{3.4.19}$$

Therefore, it follows from (3.4.18), (3.4.19) and the assumption $AX_0 = \lambda X_0$, that all the derivatives of Z_t at $t = 0$ vanish.

The proof for the situation when $AX_0 = -\cot(r)X_0$ and X_t is constructed as in Case I, is now just a consequence of the previous case. Indeed, in accordance with Theorem (3.3.4), if Y_0 is any eigenvector at q corresponding to a principal curvature $\cot(r+\theta)$ with $\theta \neq 0$ then X_0 is orthogonal to both vectors Y_0 and JY_0 since the eigenvalues are all distinct. Consequently, the parallel transport X_t along γ_t remains orthogonal to the parallel transport Y_t of the rotated vector $Y_t(0) = \cos(\frac{t}{2})Y_0(0) + \sin(\frac{t}{2})JY_0(0)$. Thus, the vector $X_t(s)$ must lie in the eigenspace $V_{-\cot(r)}$. ⊙

Theorem 3.4.9 *Let M be a connected Hopf hypersurface of $\mathbb{C}P^n$ with Hopf principal curvature being $-2\cot(2r)$. Let X be a continuous principal vector field on M corresponding to a continuous principal curvature function $\lambda : M \to \mathbb{R}$. If λ assumes the value $-\cot(r)$ at a particular point $q_0 \in M$ then λ is constant.*

Proof:

The set of points where the function λ assumes the value $-\cot(r)$ is certainly closed and so because of M being connected we can obtain our theorem by proving that this set is also open. Next, we shall prove this by contradiction.

Let us assume the existence of a sequence of points $q_n \in M$ converging to $q_0 \in M$ such that at each of these points we have $\lambda_n = \lambda(q_n) \neq -\cot(r)$. Thus, if we define

λ in terms of a new function θ by putting $\lambda = -\cot(r+\theta)$, our assumption can be read as $\lim \theta_n = 0$.

For each $n \in \{0, 1, \ldots\}$ we shall denote the integral curve of the Hopf vector field starting at the point q_n by $\sigma^n(t)$. Along each of these curves we can apply the construction given in Definition (3.4.2) to obtain a vector field X_t^n satisfying the initial condition $X_t^n(r) = X(q_n)$.

Thus, using Definition (3.4.3), we have vector fields X_t^n and V_t^n satisfying the properties stated in Theorem (3.4.8), that is, for each t we have $V_t^0 \neq 0$ and for each $n \neq 0$ we have $V_t^n \equiv 0$.

Now, since the vector field V_t^n depends continuously on the initial condition given for the vector field X_t^n, we must have

$$\lim_{n\to\infty} V_t^n(q_n) = V_t^0(q),$$

which contradicts the properties satisfied by these vector fields that we have just mentioned. ⊙

Corollary 3.4.2 *Let M be a connected Hopf hypersurface of $\mathbb{C}P^n$ such that every continuous principal curvature function on M corresponds to a continuous principal vector field. If $-\cot(r)$ is an eigenvalue at a point of M then it will be an eigenvalue at any point of M with the same multiplicity.*

Proof:

If we order the principal curvatures of M at each point as

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2n-1},$$

then each λ_j is a continuous principal curvature function and using the theorem above we see that if λ_j assumes the value $-\cot(r)$ at some point then it must be constant and hence $-\cot(r)$ must have constant multiplicity. ⊙

Corollary 3.4.3 *Let* M *be a connected Hopf hypersurface of* $\mathbb{C}P^n$ *such that to every continuous principal curvature function there corresponds a continuous principal vector field. Then* M *lies in a tube around a complex submanifold of* $\mathbb{C}P^n$.

Proof:

The result follows from Corollary (3.4.2) and Theorem (1.3.2). $\odot$

3.5 Liftings under Riemannian submersions.

In this section we point out a natural way of getting Hopf hypersurfaces in Kähler manifolds other than complex space forms. In order to accomplish this we shall see in the next theorem that a holomorphic Riemannian submersion lifts Hopf hypersurfaces to Hopf hypersurfaces and so the following theorem can be useful to establish a relation between the Hopf hypersurfaces of two (Nearly) Kähler manifolds. In particular, we could use this result to understand the geometrical behaviour of the Hopf hypersurfaces in one ambient space in terms of the known behaviour of their corresponding projections in another ambient space. Furthermore, the next theorem also gives new examples of Hopf hypersurfaces in Kahler manifolds not of constant holomorphic sectional curvature.

Theorem 3.5.10 *Let $(\tilde{W}, \tilde{J})$ and (W, J) be Kähler manifolds. Let $\pi : \tilde{W} \to W$ be a holomorphic Riemannian submersion. If $M \subset W$ is a Hopf hypersurface of W then $\widetilde{M} := \pi^{-1}(M)$ is also a Hopf hypersurface of $\tilde{W}$.*

Proof:

Given $q \in M$, let ξ be a local unit normal vector field on M and let $\tilde{\xi}$ be the corresponding horizontal local vector field defined along $\widetilde{M}$, that is, $d\pi(\tilde{\xi}) = \xi$. It follows that $\tilde{\xi}$ is also a unit normal vector field on $\widetilde{M}$. Indeed, given a tangent vector $\tilde{v} \in T_{\tilde{q}}(\widetilde{M})$, let us consider its decomposition into horizontal and vertical components $\tilde{v} = \tilde{v}^H + \tilde{v}^V$. Then

$$\langle \tilde{\xi}, \tilde{v} \rangle = \langle \tilde{\xi}, \tilde{v}^H \rangle = \langle d\pi(\tilde{\xi}), d\pi(\tilde{v}^H) \rangle = \langle \xi, v \rangle = 0,$$

where in the last equality we have used that $d\pi$ preserves the length of horizontal vectors. Thus, $\tilde{U} = \tilde{J}\tilde{\xi}$ is the Hopf vector field of $\tilde{M}$ with respect to $\tilde{\xi}$, moreover, because of π being holomorphic, we have $d\pi(\tilde{U}) = U$.

The vertical space $V = Ker(d\pi)$ and horizontal space $H = V^{\perp}$ are invariant under $\tilde{J}$, since $d\pi(\tilde{J}X) = Jd\pi(X) = 0$ for every $X \in V$ and $\tilde{J}$ is an isometry of $V \oplus H$. Consequently, $\tilde{U}$ is the horizontal lift of U.

The connection $\widetilde{\nabla}$ induced on $\tilde{W}$ by the Riemannian submersion is related to the Riemannian connection ∇ of W by the O'Neil's formula (cf. [24])

$$\widetilde{\nabla}_{\tilde{X}}\tilde{Y} = \widetilde{\nabla_X Y} + \frac{1}{2}[\tilde{X}, \tilde{Y}]^v,$$

where $\tilde{X}$ and $\tilde{Y}$ are horizontal lifts of X and Y respectively. In particular, for $X = Y = U$ we have

$$\widetilde{\nabla}_{\tilde{U}}\tilde{U} = \widetilde{\nabla_U U} = -\alpha\tilde{\xi},$$

which implies that

$$\nabla^{\tilde{M}}_{\tilde{U}}\tilde{U} = 0.$$

Therefore, it follows from Lemma (1.3.2) that $\widetilde{M}$ is a Hopf hypersurface. ⊙

One more point we would like to remark here is that tubular hypersurfaces of a Riemannian manifold are lifted under Riemannian submersions to tubular hypersurfaces. More precisely, we have

Proposition 3.5.5 *Let $\widetilde{W}$ and W be Riemannian manifolds. Let $\pi : \widetilde{W} \to W$ be a Riemannian submersion. If N is a Riemannian submanifold of W then for r sufficiently small, the tube $T = \Phi_r(\perp^1 N)$ of radius r around N is lifted to the tube $\tilde{T} = \Phi_r(\perp^1 \tilde{N})$ of radius r around the submanifold $\tilde{N} = \pi^{-1}(N)$, that is, $\tilde{T} = \pi^{-1}(T)$.*

Proof:

Given $q \in T$, let $\tilde{q} \in \pi^{-1}(q)$. By assumption, there exists a geodesic $\gamma_{(p,\eta)}$ in W, of length r, connecting the point $q = \gamma(r)$ to a point $p = \gamma(0) \in N$ such that $\eta = \dot{\gamma}(0) \in \perp_p N$. The Riemannian submersion π lifts γ to a horizontal geodesic $\tilde{\gamma}$ of $\widetilde{W}$ with $\tilde{\gamma}(0) = \tilde{p} \in \tilde{N}$ and $\tilde{\gamma}(r) = \tilde{q}$. Moreover, this geodesic lift meets $\tilde{T}$ orthogonally as we now show.

Given any vector $\tilde{v} \in T_{\tilde{p}}\tilde{N}$, we can decompose it into horizontal and vertical components, say $\tilde{v} = \tilde{v}^H + \tilde{v}^V$. Then,

$$\langle\dot{\tilde{\gamma}}(0), \tilde{v}\rangle = \langle\dot{\tilde{\gamma}}(0), \tilde{v}^H\rangle = \langle\pi_*\dot{\tilde{\gamma}}(0), \pi_*\tilde{v}^H\rangle = \langle\eta, \pi_*\tilde{v}^H\rangle = 0,$$

and hence $\tilde{q}$ lies on the tube of radius r around $\widetilde{N}$.

On the other hand, every vector normal to $\tilde{N}$ is horizontal and consequently, using some basic properties of Riemannian submersions (see for instance [24] page 97.), every geodesic $\tilde{\gamma}$ starting at a point of $\tilde{N}$ and orthogonal to $\tilde{N}$ is horizontal. Then, for small r, it follows by uniqueness of geodesics that $\tilde{\gamma}$ is the lift of a geodesic γ of same length and normal to N. Therefore, $\Phi_r(\perp^1 \tilde{N}) = \pi^{-1}(\Phi_r(\perp^1 N))$. ⊙

Theorem (3.5.10) can be applied to flag manifolds, providing a way to find Hopf hypersurfaces in spaces other than complex space forms. In order to do such example we need first a few definitions and preparation.

Definition 3.5.4 *A* **flag manifold** *is a homogeneous space G/H where G is a compact Lie group and H is the centralizer of a torus in G.*

If G/H is a flag manifold then the subgroup H contains a maximal torus T of G. Our example of Riemannian submersion will come out of the fact that we can build up G-invariant metrics and complex structures on the flag manifolds G/T and G/H in such a way that the projection $\pi : G/T \to G/H$ is a holomorphic Riemannian submersion. In order to do so, let us first look at flag manifolds from the infinitesimal viewpoint.

We refer to Burstall and Rawnsley [14] for a detailed treatment of flag manifolds.

Definition 3.5.5 *Let g be a semisimple Lie algebra and let $g^{\mathbb{C}}$ denote its complexification. A subalgebra of $g^{\mathbb{C}}$ is a* **Borel subalgebra** *if it is a maximal solvable subalgebra of $g^{\mathbb{C}}$. A subalgebra of $g^{\mathbb{C}}$ is a* **parabolic subalgebra** *if it contains a Borel subalgebra. If $G^{\mathbb{C}}$ is a Lie group with Lie algebra $g^{\mathbb{C}}$ then a subgroup P of $G^{\mathbb{C}}$ is a* **parabolic subgroup** *if its Lie algebra is parabolic.*

A flag manifold G/H can also be realised as a homogeneous space $G^{\mathbb{C}}/P$ where P is a parabolic subgroup of $G^{\mathbb{C}}$ such that $H = G \cap P$.

Let us denote by $\mathbf{g}, \mathbf{h}, \mathbf{t}$ and $\mathbf{p}$ the Lie algebras of G, H, T and P respectively. It is well known that by choosing a set of simple roots $\Delta^+ = \{\alpha_1, \ldots, \alpha_l\}$ with respect to the Cartan subalgebra $\mathbf{t}^{\mathbb{C}}$, we can decompose the complex semisimple Lie algebra $\mathbf{g}^{\mathbb{C}}$ into root eigenspaces $\{\mathbf{g}^{\alpha_k}\}$ as follows

$$\mathbf{g}^{\mathbb{C}} = \mathbf{t}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta^+} \mathbf{g}^{\alpha} \oplus \sum_{\alpha \in \Delta^+} \mathbf{g}^{-\alpha}.$$

Furthermore, the parabolic subalgebras of $\mathbf{g}^{\mathbb{C}}$ are determined (c.f. [28]), up to conjugation, by the subsets of Δ^+. More precisely, given a subset $S \subset \Delta^+$ and if we denote by $T(S)$ the set of all positive roots which are linear combination of roots in S, then we have a further decomposition of $\mathbf{g}^{\mathbb{C}}$

$$\mathbf{g}^{\mathbb{C}} = \mathbf{t}^{\mathbb{C}} \oplus \sum_{\alpha \in T(S)} \mathbf{g}^{\alpha} \oplus \sum_{\alpha \in T(S)} \mathbf{g}^{-\alpha} \oplus \sum_{\beta \in \Delta^+ \setminus T(S)} \mathbf{g}^{\beta} \oplus \sum_{\beta \in \Delta^+ \setminus T(S)} \mathbf{g}^{-\beta},$$

where the complexification of the Lie algebra $\mathbf{h}$ and the parabolic subalgebra $\mathbf{p}$ are given respectively by

$$\mathbf{h}^{\mathbb{C}} = \mathbf{t}^{\mathbb{C}} \oplus \sum_{\alpha \in T(S)} \mathbf{g}^{\alpha} \oplus \sum_{\alpha \in T(S)} \mathbf{g}^{-\alpha}$$

and

$$\mathbf{p} = \mathbf{t}^{\mathbb{C}} \oplus \sum_{\alpha \in T(S)} \mathbf{g}^{\alpha} \oplus \sum_{\alpha \in T(S)} \mathbf{g}^{-\alpha} \oplus \sum_{\beta \in \Delta^+ \setminus T(S)} \mathbf{g}^{\beta}.$$

Thus, we have in particular, that the Lie algebra $\mathbf{g}$ admits the following decompositions into direct sums

$$\mathbf{g} = \mathbf{t} \oplus \mathbf{n} \quad \text{and} \quad \mathbf{g} = \mathbf{h} \oplus \mathbf{m},$$

where

$$\mathbf{n} = \sum_{\alpha \in \Delta^+} \left(\mathbf{g} \cap (\mathbf{g}^{\alpha} \oplus \mathbf{g}^{-\alpha})\right) \quad \text{and} \quad \mathbf{m} = \sum_{\beta \in \Delta^+ \setminus T(S)} \left(\mathbf{g} \cap (\mathbf{g}^{\beta} \oplus \mathbf{g}^{-\beta})\right).$$

Now, as G is semisimple and compact, the Killing form B of $\mathbf{g}$ can be used to define an $Ad(G)$-invariant inner product $\langle X, Y \rangle = -B(X, Y)$ on $\mathbf{g}$. Therefore, the restrictions $\langle , \rangle|_{\mathbf{n}}$ and $\langle , \rangle|_{\mathbf{m}}$ yield $Ad(T)$ and $Ad(H)$ invariant inner products defined on these subspaces respectively.

Since the metric $\langle , \rangle$ is $Ad(G)$-invariant, we have in particular that for any $t \in T$ and $X, Y \in \mathbf{n}$

$$\langle Ad(t)X, Ad(t)Y \rangle = \langle X, Y \rangle, \tag{3.5.1}$$

Thus, giving $Z \in \mathbf{t}$ and setting $t = \exp(sZ)$, we obtain from the differentiation of (3.5.1) that

$$\langle ad_Z X, Y \rangle + \langle ad_Z Y, X \rangle = 0.$$

Therefore, the subspaces $\mathbf{g} \cap (\mathbf{g}^{\alpha} \oplus \mathbf{g}^{-\alpha})$ and $\mathbf{g} \cap (\mathbf{g}^{\beta} \oplus \mathbf{g}^{-\beta})$ are orthogonal whenever $\alpha \neq \pm\beta$, and hence $\mathbf{n} = \mathbf{t}^{\perp}$ and $\mathbf{m} = \mathbf{h}^{\perp}$.

This implies that $\mathbf{m}$ is the horizontal subspace $H_{\overline{e}} \subset T_{\overline{e}}(G/T)$ of the projection π at the identity $\overline{e}$, where here we are identifying the tangent spaces $T_{\overline{e}}(G/T)$ and $T_{\overline{e}}(G/H)$ with the subspaces $\mathbf{n}$ and $\mathbf{m}$ respectively, via the derivative at $\overline{e}$ of the natural projections $G \to G/T$ and $G \to G/H$. Consequently, we have that the identity map $d\pi_{\overline{e}} : H_{\overline{e}} \to \mathbf{m}$ is an isometry.

Using the left translations $L_g : G/T \to G/T$ and $\tilde{L}_g : G/H \to G/H$, we can induce G-invariant metrics on these homogeneous spaces by requiring that for each $g \in G$ these translations are isometries. Then the projection π is a Riemannian submersion because we can write down its differential at the point $\overline{g} \in G/T$ as

$$d\pi_{\overline{g}} = d\tilde{L}_g \circ d\pi_{\overline{e}} \circ dL_{g^{-1}}. \tag{3.5.2}$$

The equation above follows from the fact that for each $g \in G$ we have $\pi_{\overline{g}} \circ L_g = \tilde{L}_g \circ \pi_{\overline{e}}$.

A comprehensive study of almost complex structures on homogeneous spaces has been done by Borel-Hirzebruch in [12] and for the special case of flag manifolds Wang [43] has characterized these homogeneous spaces exactly as the ones which admit a complex structure. We shall not go into the details of their work but in order to set up complex structures on our flag manifolds we need to mention the following main result

Theorem 3.5.11 *The G-invariant complex structures on a flag manifold G/H are in one-to-one correspondence with the splittings of $T^{\mathbb{C}}_{\overline{e}}(G/H)$ into $Ad(H)$-invariant* *subspaces*

$$T^{1,0}_{\bar{e}}(G/H) = \sum_{\beta_k \in (\Delta^+ \setminus T(S))} \mathbf{g}^{\epsilon_k \beta_k} \quad \textit{and} \quad T^{0,1}_{\bar{e}}(G/H) = \sum_{\beta_k \in (\Delta^+ \setminus T(S))} \mathbf{g}^{-\epsilon_k \beta_k},$$

where $\epsilon_k = \pm 1$.

Let us give a G-invariant complex structure to G/H by fixing a splitting for $T^{\mathbb{C}}_{\bar{e}}(G/H)$ as in the theorem above. Then, we can obtain a complex structure of G/T by considering a splitting of $T^{\mathbb{C}}_{\bar{e}}(G/T)$ as follows

$$T^{1,0}_{\bar{e}}(G/T) = \sum_{\beta_k \in T(S)} (\mathbf{g}^{\delta_k \beta_k}) \oplus T^{1,0}_{\bar{e}}(G/H) \quad \text{and} \quad T^{0,1}_{\bar{e}}(G/T) = \sum_{\beta_k \in T(S)} (\mathbf{g}^{-\delta_k \beta_k}) \oplus T^{0,1}_{\bar{e}}(G/H),$$

where $\delta_k = \pm 1$.

Indeed, this splitting is $Ad(T)$-invariant because of the very definition of the root subspaces $\{\mathbf{g}^{\beta}\}$.

Therefore, using (3.5.2) and the fact that the root spaces $\mathbf{g}^{\alpha}$ and $\mathbf{g}^{-\alpha}$ are complex conjugate for every $\alpha \in \Delta^+$, we can see that π is a holomorphic map with respect to the complex structures given to G/T and G/H.

We conclude this section ilustrating the theory above with the specific example of complex projective spaces.

Example 3.5.1 *Let T be a maximal torus for the Lie group $SU(n+1)$. Then every Hopf hypersurface of $\mathbb{C}P^n = SU(n+1)/S(U(1) \times U(n))$ is lifted by the projection $\pi : SU(n+1)/T \to SU(n+1)/S(U(1) \times U(n))$ to a Hopf hypersurface of the flag manifold $SU(n+1)/T$. The metrics and complex structures considered in these flag manifolds are constructed as above.*

Chapter 4

Superminimal Surfaces of S^6.

4.1 Introduction.

Although the Hopf hypersurfaces of the 6-sphere are tubular hypersurfaces around almost complex curves, we cannot find explicit examples of these hypersurfaces for the simple reason that in general it is not so easy to get an explicit description of these curves.

In this chapter, we use the harmonic sequence associated to a weakly conformal harmonic map $f : S \to S^6$ in order to determine explicitly all the linearly full almost complex 2-spheres of S^6 with at most two singularities. The use of harmonic sequence is a very well known technique since, in recent times, it has been used by several authors ([20],[22],[6]) and we shall give here the same treatment as in [6], which was suitably specialized for the case of the spheres. Thus, our next section is entirely dedicated to establish this background and to state the main results already known about almost complex curves of the 6-sphere that we shall need in this chapter.

We shall see that the singularity type of the particular almost complex 2-sphere mentioned above has an extra symmetry and this shall permit us to determine the moduli space of such curves with suitably small area.

4.2 Harmonic sequences and harmonic maps.

A map $\psi : S \to W$ between Riemannian manifolds is **harmonic** if it satisfies the **Euler-Lagrange equation**

$$\mathrm{tr}(\nabla d\psi) = 0.$$

Throughout this chapter, we shall use S to denote a Riemann surface and $z = x + iy$ shall denote a local complex coordinate z on S. In this case, the harmonicity condition of ψ is simplified to

$$(\nabla_{\frac{\partial}{\partial \bar{z}}} d\psi)(\frac{\partial}{\partial z}) = 0. \tag{4.2.1}$$

Let $V \to S$ be a complex vector bundle over the Riemann surface S and assume that $\widetilde{\nabla}$ is a connection on V. By the Koszul-Malgrange Theorem, V admits the structure of a holomorphic vector bundle. Here a section s is a **holomorphic section** if and only if

$$\widetilde{\nabla}_{\frac{\partial}{\partial \bar{z}}} s = 0. \tag{4.2.2}$$

Given a harmonic map $\psi_0 := \psi : S \to \mathbb{C}P^n$, several authors ([44],[22],[20],[6]) have dealt with the sequence of harmonic maps $\psi_k : S \to \mathbb{C}P^n$ obtained from ψ via an inductive construction of a sequence of complex line bundles over S. In the sequel, we outline this construction and give some of the main features of this sequence.

Let $\mathfrak{L}$ be the tautological line bundle over $\mathbb{C}P^n$. Let L_0 and $L_0^\perp$ be the pullbacks via ψ_0 of $\mathfrak{L}$ and $\mathfrak{L}^\perp$ respectively. L_0 and $L_0^\perp$ are endowed with naturally induced connections for they are vector subbundles of the trivial $\mathbb{C}^{n+1}$-bundle over S.

Explicitly, if s is a section of a subbundle L of the trivial bundle $S \times \mathbb{C}^{n+1}$, then s may be regarded as a map $S \to \mathbb{C}^{n+1}$. Given $X \in T_xS$, we can define a connection $\nabla_X s$ by the orthogonal projection $(Xs)^L$ of Xs onto L. Similarly, we also define a connection in $L^\perp$. Thus the line bundles L_0 and $L_0^\perp$ have structures of holomorphic vector bundles over S.

The map ψ_0 determines a bundle map $\partial_0 : L_0 \to L_0^\perp$. Indeed, if we consider a holomorphic local section $f_0 : S \to \mathbb{C}^{n+1} \setminus \{0\}$ of L_0, then we define $\partial_0 f_0 = (\frac{\partial f_0}{\partial z})^{L_0^\perp}$, and $\overline{\partial}_0 f_0 = (\frac{\partial f_0}{\partial \bar{z}})^{L_0^\perp}$.

It follows from (4.2.1) and (4.2.2) that ∂_0 ($\overline{\partial}_0$) is a holomorphic (anti-holomorphic) bundle map if and only if $\psi_0 = [f_0]$ is a harmonic map. Therefore if $\tilde{f}_1 := \partial_0 f_0$ is not identically zero then it is a holomorphic section of the bundle $L_0^\perp$ and hence its zeros (if any) are isolated. Let z_0 be such a zero, then for some holomorphic local section f_1 we have $\tilde{f}_1(z) = (z - z_0)^r f_1(z)$ with $f_1(z_0) \neq 0$. This latter map will then yield a well-defined map $\psi_1(z) := [f_1(z)]$ from S into $\mathbb{C}P^n$ and f_1 is a meromorphic local section for a complex line bundle $L_1 \subset L_0^\perp$.

By defining ∂_1 in a similar way, and verifying that $\tilde{f}_2 := \partial_1 f_1$ is a holomorphic local section of $L_1^\perp$, we can prove that ψ_1 is also harmonic.

Therefore, as long as the bundle section f_k is not identically zero, that is, ψ_{k-1} is not anti-holomorphic (or ψ_{k+1} is not holomorphic, when considering the descending sequence given by $\overline{\partial}$), we can carry on with this process, defining a sequence $\psi_k = [f_k(z)]$ of harmonic maps such that the local sections f_k are characterized by the following properties:

$$\begin{aligned}
\frac{\partial f_p}{\partial z} &= f_{p+1} + \frac{1}{|f_p|^2} < \frac{\partial}{\partial z} f_p, f_p > f_p \\
&= f_{p+1} + \partial_z (log|f_p|^2) f_p \\
&= f_{p+1} + \alpha_p f_p \quad \text{where} \quad \alpha_p := \partial_z (log|f_p|^2) \qquad (4.2.3) \\
\frac{\partial f_{p+1}}{\partial \bar{z}} &= -\gamma_p f_p \qquad \text{where} \quad \gamma_p := \frac{|f_{p+1}|^2}{|f_p|^2}. \qquad (4.2.4)
\end{aligned}$$

It is known [22] that the harmonic sequence terminates at one end if and only if it terminates at both ends. If this happens, we say that each element of the sequence is **superminimal** and it is customary to consider the range for the indices starting at the holomorphic element of the sequence, that is, $(\psi_j)_{j=0}^n$ denotes the harmonic sequence of ψ. This holomorphic map is usually named in the literature as the **directrix curve** associated to ψ referring to the terminology adopted when dealing with harmonic 2-spheres in S^{2m}(cf. [1]).

If $\psi_m = [f_m]$ for some harmonic map $f_m : S \to S^n$, then we can consider f_m as a nowhere vanishing global holomorphic section of L_0 so that the sequence of meromorphic sections f_j will also satisfy the condition:

$$\overline{f}_{m+k} = (-1)^k |f_{m+k}|^2 f_{m-k}. \tag{4.2.5}$$

In particular,

$$|f_{m+k}||f_{m-k}| \equiv 1. \tag{4.2.6}$$

Note that in this situation, the element ψ_0 will necessarily be in the middle of the sequence, that is $n = 2m$, because $f_{m+k} \equiv 0$ if and only if $f_{m-k} \equiv 0$. Moreover, (4.2.3) and $|f_m| \equiv 1$ implies

$$\partial_z f_m = f_{m+1}. \quad (Notation\ \partial_z := \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})] \tag{4.2.7}$$

Definition 4.2.1 *We say that a map from a Riemannian manifold N into $\mathbb{C}P^n$ is* **linearly full**, *when its image is not contained in any complex space form $\mathbb{C}P^k$ for $k < n$.*

If the Riemann surface is homeomorphic to the sphere S^2 then Wolfson [44] shows that the corresponding complex line bundles are mutually orthogonal and consequently the harmonic sequence terminates, that is, all the harmonic 2-spheres of $\mathbb{C}P^n$ are superminimal. Moreover, in this case, the length of the sequence achieves its maximum, $n + 1$, if and only if ψ is linearly full.

A detailed discussion of the holomorphic curves of a complex projective space can be found in [27] (pages 263-268), but here we describe some material on this topic to be used in this chapter.

Let $\psi(z) = [f(z)] : S \to \mathbb{C}P^n$ be a holomorphic curve from the Riemann surface S into $\mathbb{C}P^n$, where $f : S \to \mathbb{C}^{n+1} \setminus \{0\}$ is a local holomorphic lift of ψ. Then the $\boldsymbol{j^{th}}$**-osculating curve** of ψ is the holomorphic curve $\sigma_j : S \to \mathbb{C}P^{n_j}$ (where $n_j := \left[\binom{n+1}{j+1}\right] - 1$) defined by

$$\sigma_j(z) = [f \wedge \ldots \wedge f^{(j)}](z),$$

where $f^{(j)} = \frac{\partial^j f}{\partial z^j}$ and $j = 0, ..., n-1$.

A **higher order singularity** of ψ is a point $p \in S$ which is a singularity for some j^{th}-osculating curve $(j = 0, 1, .., n-1)$. The **ramification index** of σ_j at a point p is the order r_{j+1} of this point as a zero of the derivative of the curve σ_j.

Thus, the holomorphic curve ψ is said to have **singularity type** $(r_1, r_2, , .., r_n)$ at the point p.

If S is a compact Riemann surface, then the curve ψ has a finite set $\mathcal{Z}_\psi = \{p_1, .., p_k\}$ of higher order singularities. We shall denote by R_{j+1} the sum of the ramification indices of σ_j at each singularity, that is,

$$R_{j+1} = \sum_{i=1}^{k} r_{j+1}(p_i) \tag{4.2.8}$$

and we shall refer to R_{j+1} just as the ramification index of σ_j. Moreover, we can define the **total ramification index** of ψ as the sum $\sum_{j=1}^{n} R_j$.

Then, we shall say that the holomorphic curve ψ or any element of its corresponding harmonic sequence has **singularity type** $(r_1(p), \ldots, r_n(p))$ at the point p and has **total singularity type** $(R_1, \ldots, R_n)$.

The curve ψ is **totally unramified** when its total ramification index is zero. Otherwise, ψ is said to be ***k*-point ramified** if the set $\mathcal{Z}_\psi$ has cardinality k.

In terms of a local complex coordinate z for S centred on p, that is $z(p) = 0$, it is possible to determine a basis $\{v_0, .., v_n\}$ for $\mathbb{C}^{n+1}$ in such a way that the holomorphic map f can be written in the normal form

$$f(z) = \sum_{i=0}^{n} z^{k_0 + \ldots + k_i} h_i(z) v_i, \tag{4.2.9}$$

where $k_0 = 0$, $k_i = r_i(p) + 1$ $(j = 1, \ldots, n)$ and $h_i(z)$ denotes a holomorphic function satisfying $h_i(0) \neq 0$.

If $\psi : S^2 \to \mathbb{C}P^n$ is 2-point ramified, then we can find a local complex coordinate for S^2 so that the higher order singularities of ψ (if any) occur at $z = 0$ and

$z = \infty$. Indeed, this follows from the fact that the Möbius group of conformal transformations of the 2-sphere acts triply transitively on S^2.

We observe that ψ is a holomorphic map between algebraic varieties, since $S^2 = \mathbb{C}P^1$, and so ψ is an algebraic map (c.f. [27]), that is f is a rational function. Without loss of generality, we may assume f to be a $\mathbb{C}^{n+1}$-valued polynomial function.

Definition 4.2.2 *We say that a harmonic map $\psi : S^2 \to \mathbb{C}P^n$ has* ***S^1-symmetry*** *if there exists non-trivial S^1-actions on S^2 and $\mathbb{C}P^n$, where the action on $\mathbb{C}P^n$ is by holomorphic isometries, such that for all $z \in S^2$,*

$$\psi(e^{i\theta}z) = e^{i\theta}\psi(z).$$

Remark 4.2.1 *It is shown in [7] (see Theorem* (4.2.1) *below) that such a map ψ is either 0-point or 2-point ramified, where in the later case, the singularities of ψ occur exactly at the two points of S^2 fixed by the S^1-action.*

We are now able to state the main results ([7],[10]) to be used in the following sections, which are concerned with the characterization of the k-point ramified harmonic 2-spheres of $\mathbb{C}P^n$ for $k \leq 2$.

Theorem 4.2.1 *Let $\psi : S^2 \to \mathbb{C}P^n$ be a linearly full harmonic map with S^1-symmetry. Then there exists a holomorphic coordinate z on S^2 such that the directrix curve $\psi_0 = [f_0]$ of ψ can be expressed up to holomorphic isometries of $\mathbb{C}P^n$ by*

$$f_0(z) = \sum_{p=0}^{n} z^{k_1+\ldots+k_p} v_p,$$

where $\{v_0, \ldots, v_n\}$ is an orthogonal basis of $\mathbb{C}^{n+1}$ and the scalars k_j are positive integers. Furthermore, ψ is either totally unramified or 2-point ramified. In the latter case ψ has singularities at $z = 0$ and $z = \infty$ with corresponding singularity type at $z = 0$ and $z = \infty$ given respectively by $(k_1 - 1, \ldots, k_n - 1)$ and $(k_n - 1, \ldots, k_1 - 1)$.

Theorem 4.2.2 *Let $\psi : S^2 \to \mathbb{C}P^n$ be a linearly full harmonic map which is k-point ramified for $k \leq 2$. Let z be a complex coordinate on S^2, then*

(i) *The higher order singularities of* ψ *(if any) occur at* $z = 0$ *and* $z = \infty$ *if and only if its directrix curve* $\psi_0(z) = [f_0(z)]$ *can be written in the form*

$$f_0(z) = \sum_{p=0}^{n} z^{k_1+\ldots+k_p} v_p, \tag{4.2.10}$$

where the scalars k_j *are positive integers and the vectors* v_j *constitutes a basis of* $\mathbb{C}^{n+1}$*. Furthermore,* ψ *has* S^1*-symmetry (with fixed points of the* S^1*-action at* $z = 0$ *and* $z = \infty$*) if and only if the basis* $\{v_0, \ldots, v_n\}$ *is orthogonal.*

(ii) $\psi(S^2) \subset \mathbb{R}P^n$ *and one of the conditions (and so both) in the first equivalence stated in* **(i)** *occurs if and only if* $n = 2m$ *for some integer* m *and the following properties are satisfied.*

$$\begin{aligned} k_j &= k_{n-j+1} \quad \textit{for } i, j \in \{1, \ldots, n\}, \\ \langle v_j, \overline{v}_i \rangle &= (-1)^j \delta_{(j,n-i)} \mu \lambda_j, \quad \textit{for } i, j \in \{0, \ldots, n\}, \end{aligned}$$

where μ *is a constant and* $\lambda_j := \frac{\prod_{1 \le r \le s \le n}(k_r + \ldots + k_s)}{\prod_{r=1}^{j}(k_r+\ldots+k_j)\prod_{r=1}^{n-j}(k_{j+1}+\ldots+k_{n-r+1})}$.

The constant μ in the theorem can be chosen to be 1 by rescaling the homogeneous coordinates of $\mathbb{C}P^n$. However, we will avoid this in order to facilitate our calculations later on when determining examples of superminimal almost complex curves.

Remark 4.2.2 *The theorem above shows in particular that there does not exist a 1-point ramified linearly full harmonic 2-sphere in* $\mathbb{C}P^n$.

Definition 4.2.3 *Two maps* $\psi, \widetilde{\psi} : S \to \mathbb{C}P^n$ *are said to be projectively equivalent if there exists* $[A] \in PGL(n+1, \mathbb{C})$ *so that* $\widetilde{\psi} = [A](\psi)$.

Corollary 4.2.1 *Any two k-point ramified (*$k \le 2$*) linearly full harmonic maps* $\psi, \widetilde{\psi} : S^2 \to \mathbb{C}P^n$ *with the same singularity type are projectively equivalent.*

Proof:

According to Theorem (4.2.2) these curves are uniquely determined by their singularity type and a choice of basis for $\mathbb{C}^{n+1}$. Thus, item (i) of that theorem shows that the corresponding directrix curves differ by an element of $GL(n+1, \mathbb{C})$. ⊙

4.3 The twistor fibration $\pi : Q^5 \to S^6$.

Let Q^5 denote the quadric of $\mathbb{C}P^n$, which is the Kähler submanifold defined

$$Q^5 = \{[x] \in \mathbb{C}P^n \text{ such that } (x, x) = 0\},$$

where $(,)$ denotes the Euclidean inner product of $\mathbb{C}^{n+1}$.

The twistor fibration $\pi : Q^5 \to S^6$ is defined by the map

$$\pi([x]) = \frac{i}{|x|^2}\overline{x} \times x.$$

We shall see in the next section that the superminimal almost complex curves of the 6-sphere can be characterized as the projections of a special type of holomorphic curves of this quadric. This led us to investigate what is the group of holomorphic transformations of the quadric which preserves the superhorizontal distribution to be defined ahead.

Although π is not a Riemannian submersion, it is quite close to that as we shall see below. Moreover, the fact that π can be easily expressed in terms of the cross product $\times$ on $\mathbb{R}^7$ (extended $\mathbb{C}$-linearly to $\mathbb{C}^7$) yields some good methods to investigate properties of any lifting.

The exceptional Lie group G_2 acts transitively on the manifolds Q^5 and S^6 in such a way that these manifolds can be realized also as the homogeneous spaces $G_2/U(2)$ and $G_2/SU(3)$ respectively (c.f. [35] for details). By considering these homogeneous spaces, it is possible to show ([35]) that the twistor fibration just defined is nothing but the canonical projection of the first space onto the second one.

We can write any element of Q^5, without loss of generality, as $[x] = [a - ib]$ where a and b are orthonormal vectors of $\mathbb{R}^7$. In this case π reduces to

$$\pi[x] = a \times b. \tag{4.3.1}$$

Remark 4.3.3 *It follows from the characterization of G_2 as the group of automorphisms of $(\mathbb{R}^7, \times)$ that the map π is G_2-equivariant, that is, $\pi[gx] = g(\pi[x])$ for every $g \in G_2$.*

Using the Hopf fibration we know that the tangent space of $\mathbb{C}P^n$ at a point $[x] \in \mathbb{C}P^n$ is $T_{[x]}\mathbb{C}P^n = \{v \in \mathbb{C}^{n+1} \text{ such that } (v, \overline{x}) = 0\}$. This implies that the tangent space of Q^5 at a point $[x] \in Q^5$ is given by

$$T_{[x]}Q^5 = \{v \in \mathbb{C}^{n+1} \text{ such that } (v, \overline{x}) = 0 \text{ and } (v, x) = 0.\}$$

Using the definition of π we have

$$\pi_*|_{[x]}(v) = \frac{i}{|x|^2}.(\overline{x} \times v - x \times \overline{v}). \tag{4.3.2}$$

Thus, the vertical distribution on Q^5 defined as the kernel of π_* is given by the set of those tangent vectors $v \in T_{[x]}Q^5$ such that the imaginary part of $x \times \overline{v}$ is zero. However, writting $v = c + id$, using (1.2.3) and the notation $\Im$ and $\Re$ to denote the imaginary and real part of a vector, we have

$$\begin{aligned} \Im(x \times \overline{v}) \times d &= a - (b \times c) \times d &= -\Re(x \times \overline{v}) \times c \\ \Im(x \times \overline{v}) \times c &= b - (a \times d) \times c &= \Re(x \times \overline{v}) \times d. \end{aligned}$$

Thus, $\Im(x \times \overline{v}) = 0$ if and only if $\Re(x \times \overline{v}) = 0$ which implies that the vertical distribution is characterized by

$$V_{[x]} = Ker\pi_* = \{v \in T_{[x]}Q^5 \text{ such that } x \times \overline{v} = 0\}. \tag{4.3.3}$$

We should also note that $V_{[x]}$ is an isotropic subspace of $T_{[x]}Q^5$ since using (1.2.3) we have for any $v \in V_{[x]}$

$$0 = \overline{v} \times (x \times \overline{v}) = 2(\overline{v}, \overline{v})x.$$

This yields a distribution of isotropic subspaces $H' := \overline{V}$ of the horizontal spaces $H = V^\perp$, which we henceforth will name as the **superhorizontal distribution**. It follows then from (4.3.3) that this vector space is characterized at $[x]$ by:

$$H' = \{v \in T_{[x]}Q^5 \text{ such that } x \times v = 0\}. \tag{4.3.4}$$

Incidentally, looking at the point $v_0 = \pi[x] \in S^6$ as a real vector of $\mathbb{C}^7$, it is clear that $v_0 \in T_{[x]}Q^5$. Moreover, v_0 is a horizontal vector since using Remark (1.2.2) we have for any vertical vector $v \in V$

$$\langle \pi[x], v\rangle = (\overline{x} \times x, \overline{v}) = -(\overline{v} \times x, \overline{x}) = 0. \quad \forall\ v \in V.$$

Furthermore, the equation above also shows that v_0 is orthogonal to H'. Thus, using the 1-dimensional complex space D spanned by v_0, we can split the horizontal distribution as follows.

$$H = D \oplus H'. \tag{4.3.5}$$

We shall investigate now how far the map π is prevented from being a holomorphic Riemannian submersion, by looking at its behaviour concerning to length-preservation and $\mathbb{C}$-linearity of its differential. We split these properties into two lemmas. But we must notice firstly that as G_2 acts transitively on Q^5 and π is G_2-equivariant, it suffices to verify these properties at any convenient point of Q^5, say $x_0 = e_1 - ie_5$.

Lemma 4.3.1 *π_* is length-preserving in H' and it reduces the length by a $\sqrt{2}$-factor in D.*

Proof:

At this point x_0, we have $v_0 = \pi[x_0] = -e_4$, $V = span_{\mathbb{C}}(v_1 = e_2 - ie_6, v_2 = e_3 - ie_7)$ and $H' = span_{\mathbb{C}}(\overline{v}_1, \overline{v}_2)$. Now, using (4.3.2) we get

$$\begin{aligned}
\pi_*|_{[x_0]}(v_0) &= -e_1 \\
\pi_*|_{[x_0]}(\overline{v}_1) &= 2e_7 \\
\pi_*|_{[x_0]}(\overline{v}_2) &= -2e_6.
\end{aligned}$$

Thus, the lemma follows from the fact that with respect to the Fubini-Study metric in $\mathbb{C}P^6$, these vectors have lengths $|v_0| = \sqrt{2}$ and $|\overline{v}_1| = |\overline{v}_2| = 2$. ⊙

Lemma 4.3.2 *π_* is $\mathbb{C}$-linear in H' and it is $\mathbb{C}$-anti-linear in D.*

Proof:

This lemma also follows easily from (4.3.2) and the definition of the almost complex structure on S^6 since in this case we have

$$\begin{aligned}
\pi_*|_{[x_0]}(iv_0) &= e_5 &&= -\pi[x_0] \times \pi_*(v_0) &&= -J_{\pi[x_0]}(\pi_*(v_0)) \\
\pi_*|_{[x_0]}(i\overline{v}_1) &= -2e_3 &&= \pi[x_0] \times \pi_*(\overline{v}_1) &&= J_{\pi[x_0]}(\pi_*(\overline{v}_1)) \\
\pi_*|_{[x_0]}(i\overline{v}_2) &= 2e_2 &&= \pi[x_0] \times \pi_*(\overline{v}_2) &&= J_{\pi[x_0]}(\pi_*(\overline{v}_2)).
\end{aligned}$$

⊙

These lemmas show that although π is not a Riemannian submersion it is not so far from this. Consequently, bearing in mind the main result of section (3.5), we could argue if it is possible that the lift $\widetilde{M} = \pi^{-1}(M)$ of a Hopf hypersurface M of S^6 would still be a Hopf hypersurface in Q^5. However, it is not hard to see that the horizontal lift of a normal vector field on M cannot lie either in the distribution D or in H'. This fact makes clear that $\widetilde{M}$ cannot be a tubular hypersurface. Furthermore, the decomposition (4.3.5) of the horizontal distribution makes it rather complicated to work with the Riemannian connection of Q^5.

We shall see later on in this chapter that the superminimal almost complex curves of S^6 are in 1-1 correspondence with the holomorphic curves of Q^5 which are tangential to the superhorizontal distribution. This motivates us to determine what is the group of holomorphic transformations of Q^5 which preserves the superhorizontal distribution.

Let us consider the Lie group $H_1 = \{\lambda I \in GL(n+1, \mathbb{C}) : \lambda \in \mathbb{C}^*\}$ and its Lie subgroup $H_2 = \{\lambda I \in SO(n+1, \mathbb{C}) : \lambda \in \mathbb{C}$ and $\lambda^{n+1} = 1\}$. It is well known (for instance, [27] page 65) that $PGL(n+1, \mathbb{C}) = GL(n+1, \mathbb{C})/H_1$ is the group of holomorphic transformations of $\mathbb{C}P^n$.

Lemma 4.3.3 *$SO(n+1, \mathbb{C})/H_2$ is the group of holomorphic transformations of the quadric Q^{n-1}.*

Proof:

Every holomorphic transformation $\widetilde{T}$ of the quadric Q^{n-1} is extendable to a holomorphic transformation of $\mathbb{C}P^n$, see for instance page 178 of [27], thus we can assume without loss of generalization that $\widetilde{T} \in PGL(n+1, \mathbb{C})$, say $\widetilde{T} = [X]$ where $X \in GL(n+1, \mathbb{C})$.

Let $q = [a - ib] \in Q^{n-1}$, where a and b are orthonormal vectors of $\mathbb{R}^{n+1}$. We can decompose the matrix X as $X = A + iB$, with A and B being real matrices. Then

$X(q) \in Q^{n-1}$ if and only if

$$\begin{aligned} |Aa - Bb| &= |Ab + Ba| \\ (Aa - Bb) &\perp (Ab + Ba). \end{aligned}$$

From which, by taking $a = e_i$ and $b = e_j$, and using the notation $A_k := Ae_k$ and $B_k := Be_k$, we obtain for $i, j \in \{1, 2, .., n+1\}$:

$$|A_i|^2 + |B_j|^2 - 2(A_i, B_j) = |A_j|^2 + |B_i|^2 + 2(A_j, B_i) \tag{4.3.6}$$

$$(A_i, A_j) - (B_i, B_j) = (B_j, A_j) - (B_i, A_i). \tag{4.3.7}$$

If we consider twice (4.3.6) interchanging i and j, we obtain

$$\begin{aligned} (A_i, B_j) &= -(A_j, B_i) \qquad \text{for every} \quad i \neq j \\ |A_i|^2 - |B_i|^2 &= |A_j|^2 - |B_j|^2 \qquad \text{for every} \quad i, j. \end{aligned}$$

From the second equation above we can define the scalar $\alpha = |A_i|^2 - |B_i|^2$.

Similarly, we can use (4.3.7) twice to get

$$\begin{aligned} (A_i, A_j) &= (B_i, B_j) \qquad \text{for every} \quad i \neq j \\ (B_i, A_i) &= (B_j, A_j) \qquad \text{for every} \quad i, j, \end{aligned}$$

and from the second equation above we can also define the scalar $\beta = (B_i, A_i)$.

Therefore, it follows at once from these properties that

$$X^t X = (A^t A - B^t B) + i(A^t B + B^t A) = (\alpha + 2i\beta) I$$

and hence if we define $\lambda := \alpha + 2i\beta$ then $\lambda \neq 0$ because $X \in GL(n+1, \mathbb{C})$ and we have $\frac{1}{\sqrt{\lambda}} X \in SO(n+1, \mathbb{C})$. ⊙

In the next proposition we will need the following elementary properties of the distributions D, H' and V.

$$\begin{aligned} D_{[x]} &= D_{[\overline{x}]} \\ H'_{[x]} &= V_{[\overline{x}]} \\ V_{[x]} &= H'_{[\overline{x}]}. \end{aligned}$$

Remark 4.3.4 *Let $G_2^{\mathbb{C}}$ be the group of automorphisms of $(\mathbb{C}^7, \times)$. It is clear from the definition of G_2 and the characterization of the superhorizontal distribution given in (4.3.4), that $G_2^{\mathbb{C}}$ is a Lie subgroup of $SO(7,\mathbb{C})$ and it preserves the superhorizontal distribution.*

Proposition 4.3.1 *The group $\widetilde{G}$ of holomorphic transformations of Q^5 which preserves the superhorizontal distribution is $G_2^{\mathbb{C}}$.*

Proof:

In accordance with the remark (4.3.4) above, we have $G_2^{\mathbb{C}} \subset \widetilde{G}$. Let $[g]$ be an arbitrary element of $\widetilde{G}$, that is, $[g] \in SO(7,\mathbb{C})/H_2$. Thus $+g$ or $-g$ lies in $SO(7,\mathbb{C})$. By assumption we have for every $[x] \in Q^5$ and $v \in H'_{[x]}$ that

$$gx \times gv = 0. \tag{4.3.8}$$

We shall first observe that $\overline{g}$ also lies in $\widetilde{G}$. Indeed, given $v \in H'_{[x]}$ we have $\overline{v} \in V_{[x]} = H'_{\overline{x}}$ and hence

$$\overline{g}x \times \overline{g}v = \overline{g\overline{x} \times g\overline{v}} = 0. \tag{4.3.9}$$

The superhorizontal subspaces at the points $[x_0] = [e_1 - ie_5]$ and $[x_1] = [e_1 - ie_4]$ are $H'_{[x_0]} = span_{\mathbb{C}}(e_2 + ie_6, e_3 + ie_7)$ and $H'_{[x_1]} = span_{\mathbb{C}}(e_7 + ie_2, e_3 + ie_6)$ respectively. If we then apply (4.3.8) and (4.3.9) to these vectors at their corresponding points, we obtain

$$ge_1 \times ge_2 = ge_4 \times ge_7 = ge_6 \times ge_5 \tag{4.3.10}$$

$$ge_1 \times ge_3 = ge_6 \times ge_4 = ge_7 \times ge_5 \tag{4.3.11}$$

$$ge_1 \times ge_6 = ge_5 \times ge_2 = ge_4 \times ge_3 \tag{4.3.12}$$

$$ge_1 \times ge_7 = ge_5 \times ge_3 = ge_2 \times ge_4 \tag{4.3.13}$$

The vectors $\{ge_1, .., ge_7\}$ are orthonormal with respect to the Euclidean product $(,)$ in $\mathbb{C}^7$ since

$$(ge_i, ge_j) = \langle ge_i, \overline{g}e_j \rangle = \langle g^t ge_i, e_j \rangle = \delta_{ij}.$$

Recalling that $(* \times *, *)$ is skew-symetric, we see that

$$(ge_i \times ge_j, ge_j) = 0.$$

Thus it follows from (4.3.10) that $ge_1 \times ge_2 = \pm ge_3$.

If $ge_1 \times ge_2 = ge_3$ then we can use directly (4.3.11), (4.3.12) and (4.3.13) to show that $\{ge_1, ..ge_7\}$ is a G_2-basis for $\mathbb{C}^7$ and hence $g \in G_2^{\mathbb{C}}$.

Similarly, if $ge_1 \times ge_2 = -ge_3$ then we can repeat the same process above to deduce that $-g \in G_2^{\mathbb{C}}$. $\odot$

4.4 Almost complex 2-spheres of S^6.

Definition 4.4.4 *Let S be a Riemann surface. We say that a smooth map $f : S \to S^6$ is an* **almost complex curve** *of the nearly Kähler S^6 if f_* is complex linear.*

Therefore, using a local complex coordinate $z = x + iy$ for S we can characterize these curves by

$$\partial_y f = f \times \partial_x f. \tag{4.4.1}$$

It follows that almost complex curves of S^6 are weakly conformal maps which are also harmonic because if we differentiate (4.4.1) again, we obtain $f \times (\partial_{xx} f + \partial_{yy} f) = 0$.

Therefore an almost complex curve $f : S \to S^6$ determines a harmonic sequence of maps $\psi_k : S \to \mathbb{C}P^6$ so that $\psi_0 = [f]$. Using this sequence and some invariants associated to their elements, a full classification of these curves was obtained in [9] according to the following four types:

(I) Linearly full in S^6 and superminimal,

(II) Linearly full in S^6 but not superminimal,

(III) Linearly full in a totally geodesic S^5 in S^6,

(IV) Totally geodesic.

A result of Bryant [13] highlights the importance of the Type-I almost complex curves of S^6. Bryant has shown that every compact Riemann surface of any genus can be realised as such an almost complex curve of the 6-sphere.

In this section we shall obtain explicitly all the 0-point and 2-point ramified linearly full almost complex 2-spheres of the 6-sphere. This is done by using the normal form for such surfaces as given by Theorem (4.2.2).

In particular, we will also prove that these surfaces are uniquely determined by their singularity type up to $G_2^{\mathbb{C}}$-equivalence of their directrix curves. It is worthwhile mentioning here that a similar result in the more general situation of harmonic 2-

spheres of S^n and $\mathbb{C}P^n$ has been obtained in [8] but replacing, of course, the group $G_2^{\mathbb{C}}$ by $SO(n+1,\mathbb{C})$.

Part of the multiplication table given in the following proposition was obtained in [9] by using slightly different calculations.

Proposition 4.4.2 *Let $f : S \to S^6$ be a linearly full superminimal almost complex curve. If $\left(\psi_j = [f_j]\right)_{j=0}^{6}$ is the harmonic sequence corresponding to the harmonic map $\psi_3 = [f]$ then the meromorphic local sections $f_k : \mathbb{C} \to \mathbb{C}^7$ have the following multiplication table for $f_i \times f_j$, where the cross product $\times$ is extended $\mathbb{C}$-linearly to $\mathbb{C}^7$:*

$i\backslash j$	0	1	2	3	4	5	6
0	0	0	0	$-if_0$	$-2if_1$	$-2if_2$	$-if_3$
1	0	0	if_0	if_1	0	$-if_3$	$-if_4$
2	0	$-if_0$	0	if_2	if_3	0	$-if_5$
3	if_0	$-if_1$	$-if_2$	0	if_4	if_5	$-if_6$
4	$2if_1$	0	$-if_3$	$-if_4$	0	$2if_6$	0
5	$2if_2$	if_3	0	$-if_5$	$-2if_6$	0	0
6	if_3	if_4	if_5	if_6	0	0	0

(4.4.2)

Furthemore, the following relation holds

$$|f_4|^2|f_5|^2 = 2|f_6|^2. \tag{4.4.3}$$

Proof:

It follows immediately from (4.4.1) and (4.2.7) that $f_3 = f$ is an almost complex curve if and only if

$$f_3 \times f_4 = if_4, \tag{4.4.4}$$

which by differentiation with respect to z and $\overline{z}$ (using (4.2.4) in the latter differentiation) yields respectively:

$$f_3 \times f_5 = if_5 \tag{4.4.5}$$

$$f_2 \times f_4 = if_3. \tag{4.4.6}$$

Using (4.2.5) and (4.4.4) we get

$$f_2 \times f_3 = if_2, \tag{4.4.7}$$

which together with (4.4.5) gives

$$f_2 \times f_5 = -if_2 \times (f_3 \times f_5) = i(f_2 \times f_3) \times f_5 = -f_2 \times f_5,$$

Thus,

$$f_2 \times f_5 = 0. \tag{4.4.8}$$

Now, by differentiating (4.4.8) with respect to $\overline{z}$ and using (4.2.4) we obtain

$$f_1 \times f_5 = -if_3. \tag{4.4.9}$$

Differentiating (4.4.5) with respect to z and using (4.2.3):

$$f_3 \times f_6 + f_4 \times f_5 = if_6. \tag{4.4.10}$$

Differentiating this equation once more and using now the superminimality condition $f_7 \equiv 0$ (that is, $\partial_z f_6 = \alpha_6 f_6$) we finally have

$$f_4 \times f_6 = \frac{\lambda}{2} f_4 \times f_5 \quad \text{where} \quad \lambda := \alpha_6 - \alpha_5 - \alpha_4 \tag{4.4.11}$$

It follows straightforwardly from (4.2.5), (4.2.6) and the orthogonality of the bundle sections f_k that

$$(f_i, f_j) = (-1)^{(i+1)} \delta_{(i,6-j)}. \tag{4.4.12}$$

Thus if we take the cross product of (4.4.11) with f_2 and if we use (1.2.3), we get

$$f_3 \times f_6 = -if_6 + i\lambda f_5. \tag{4.4.13}$$

Substituting (4.4.13) in (4.4.10) yields

$$f_4 \times f_5 = 2if_6 - i\lambda f_5. \quad (4.4.14)$$

The Euclidean product of this equation with f_1 gives

$$\begin{aligned} -i\lambda &= (f_4 \times f_5, f_1) \\ &= -(f_1 \times f_5, f_4) \\ &= i(f_3, f_4) \\ &= 0. \end{aligned}$$

Using (1.2.3) and (4.4.12), we have

$$\begin{aligned} |f_4 \times f_5|^2 &= (f_4 \times f_5, \overline{f}_4 \times \overline{f}_5) \\ &= -|f_4|^2|f_5|^2(f_4 \times f_5, f_2 \times f_1) \\ &= |f_4|^2|f_5|^2(f_1, f_2 \times (f_4 \times f_5)) \\ &= |f_4|^2|f_5|^2(f_1, -(f_2 \times f_4) \times f_5 - (f_2, f_4)f_5) \\ &= |f_4|^2|f_5|^2(f_1, -if_3 \times f_5 + f_5) \\ &= 2|f_4|^2|f_5|^2. \end{aligned}$$

This, together with (4.4.14), proves (4.4.3). The remaining cross products in the multiplication table can now be easily verified by using the methods and equations obtained so far. ⊙

Remark 4.4.5 *It is worth mentioning that the condition* (4.4.3) *characterizes the linearly full superminimal almost complex curves of the 6-sphere (cf. [9]) in the sense that a weakly conformal harmonic map $f : S \to S^6$ is $O(7)$-congruent to a linearly full superminimal almost complex curve if and only if* (4.4.3) *holds.*

We say that a map ψ from a Riemann surface S into Q^5 is superhorizontal if at each point of S, ψ_* takes values in the superhorizontal distribution. We shall recall now the 1-1 correspondence (cf.[34]) between superminimal almost complex curves in S^6 and holomorphic superhorizontal curves in Q^5. We intend to make use of this correspondence later on in this chapter in order to work out explicit examples of superminimal 2-spheres of S^6.

By using (4.3.4), the superhorizontal condition of $\psi = [f]$ can be described analitically as follows.

$$\begin{aligned} \psi \text{ is superhorizontal } &\iff f_*|_p(T_pS) \subseteq H'_{\psi(p)} = \{v \in T_{[f(p)]}Q^5 : f \times v = 0\} \\ &\iff f \times f_*(a\frac{\partial}{\partial z} + b\frac{\partial}{\partial \bar{z}}) = 0 \quad \forall\ a, b \in \mathbb{R} \\ &\iff f \times f_z = 0 \quad \text{and} \quad f \times f_{\bar{z}} = 0. \end{aligned}$$

thus a holomorphic map $\psi : S \to Q^5$ is superhorizontal if and only if

$$f \times \frac{df}{dz} = 0. \tag{4.4.15}$$

The caracterization given above fits nicely with the following theorem due essentially to Bryant (see [30] and references in there).

Theorem 4.4.3 *A map $g : S \to Q^5$ is holomorphic and superhorizontal if and only if $\psi = \pi(g) : S \to S^6$ is a superminimal almost complex curve in S^6 with directrix curve g where π denotes the twistor map from Q^5 onto S^6.*

Theorem 4.4.4 *Let f and $\tilde{f}$ be linearly full almost complex 2-spheres of S^6. Then their directrix curves are projectively equivalent if and only if they are also $G_2^{\mathbb{C}}$-equivalent.*

Proof:

$(\Longleftarrow)$

The converse in the theorem is obvious since $G_2^{\mathbb{C}}$ is a subgroup of $SO(7, \mathbb{C})$.

$(\Longrightarrow)$

Let $(\psi_j = [f_j])_{j=0}^6$ denote the harmonic sequence corresponding to the harmonic map $[f]$ and let $\widetilde{\psi}_0$ denote the directrix curve of the harmonic map $[\tilde{f}]$. By assumption there exists an element $[A] \in PGL(7, \mathbb{C})$ such that $\widetilde{\psi}_0 = [Af_0]$.

It is shown in [8] (Theorem 3.3) that two linearly full harmonic 2-spheres of S^{2m} are projectively equivalent if and only if they are $SO(2m+1,\mathbb{C})$-equivalent. Thus we can assume in our particular case here that A lies in $SO(7,\mathbb{C})$.

According to theorem (4.4.3) the map $Af_0 : S^2 \to \mathbb{C}^7$ is holomorphic and superhorizontal and hence

$$Af_0 \times Af_0' = 0. \quad (f_0' = \tfrac{df_0}{dz}) \tag{4.4.16}$$

We shall make use in the sequel of the following properties satisfied by the functions $(\alpha_j)_{j=0}^{6}$ defined by equation (4.2.3).

$$\begin{aligned} \alpha_3 &= 0. && \text{Follows from (4.2.7),} \\ \alpha_j &= -\alpha_{6-j}. && \text{Follows from (4.2.6),} && (4.4.17) \\ \alpha_6 &= \alpha_5 + \alpha_4. && \text{Follows from (4.4.3).} && (4.4.18) \end{aligned}$$

We differentiate (4.4.16) with respect to z and use (4.2.3), obtaining in this way the cross product between different vectors Af_j. By repeating this process we can derive some relations among the cross product of the vectors Af_j, namely

$$\begin{aligned} Af_0 \times Af_1 &= 0 && (4.4.19) \\ Af_0 \times Af_2 &= 0 && (4.4.20) \\ Af_0 \times Af_3 &= -Af_1 \times Af_2 && (4.4.21) \\ Af_0 \times Af_4 &= -2Af_1 \times Af_3 && (4.4.22) \\ Af_0 \times Af_5 &= -2Af_2 \times Af_3 - 3Af_1 \times Af_4 && (4.4.23) \\ Af_0 \times Af_6 &= -5Af_2 \times Af_4 - 4Af_1 \times Af_5 + 3\alpha_5 Af_1 \times Af_4. && (4.4.24) \end{aligned}$$

As $A \in SO(7,\mathbb{C})$, it follows from (4.2.5) that

$$\begin{aligned} (Af_i, Af_j) &= \langle Af_i, \overline{Af_j} \rangle \\ &= \langle A^t A f_i, \overline{f}_j \rangle \\ &= \langle f_i, \overline{f}_j \rangle \\ &= (-1)^i \delta_{i,6-j}. \end{aligned} \tag{4.4.25}$$

The vectors $\{Af_0, \ldots, Af_6\}$ form a basis for $\mathbb{C}^7$ since $A \in SO(7, \mathbb{C})$. Thus, from (4.4.21) we see immediately that $Af_0 \times Af_3$ can be written as the linear combination:

$$Af_0 \times Af_3 = aAf_0 + bAf_1 + cAf_2.$$

But if we take the cross product of this equation with Af_2 and Af_1, then it follows that $b = c = 0$. Indeed,

$$\begin{aligned} 0 &= Af_2 \times (Af_0 \times Af_3) && \text{use (1.2.3) and (4.4.25)} \\ &= bAf_2 \times Af_1 && \text{use (4.4.20)} \\ &= b(Af_0 \times Af_3), && \text{use (4.4.21)} \end{aligned}$$

and

$$\begin{aligned} 0 &= Af_1 \times (Af_0 \times Af_3) && \text{use (4.4.25) and (1.2.3)} \\ &= cAf_1 \times Af_2 && \text{use (4.4.19)} \\ &= -c(Af_0 \times Af_3). && \text{use (4.4.21)} \end{aligned}$$

On the other hand, if we also take the cross product with Af_6, we see that $a = \pm i$. Indeed,

$$\begin{aligned} a(Af_0 \times Af_6) &= -Af_6 \times (Af_0 \times Af_3) \\ &= (Af_6 \times Af_0) \times Af_3 - Af_3, \end{aligned}$$

and the Euclidean product of this with Af_0 gives:

$$\begin{aligned} -1 &= a(Af_0 \times Af_6, Af_3) \\ &= -a(Af_0 \times Af_3, Af_6) \\ &= -a^2(Af_0, Af_6) \\ &= a^2. \end{aligned}$$

Let us first assume the case $Af_0 \times Af_3 = -iAf_0$. Then (4.4.22) yields $Af_1 \times Af_3 = iAf_1$, indeed

$$\begin{aligned} -2Af_1 \times Af_3 &= Af_0 \times Af_4 \\ &= i(Af_0 \times Af_3) \times Af_4 \\ &= -i(Af_0 \times Af_4) \times Af_3 \\ &= 2i(Af_1 \times Af_3) \times Af_3 \\ &= -2iAf_1. \end{aligned}$$

Thus from (4.4.22) we get

$$Af_0 \times Af_4 = -2iAf_1.$$

And this yields:

$$Af_1 \times Af_4 = \frac{i}{2}(Af_0 \times Af_4) \times Af_4 = 0. \quad \text{(Using (1.2.3) and (4.4.1)).}$$

Now, by using the equations (4.4.16),...,(4.4.25), we can carry on with this process to determine all the cross products of the vectors $\{Af_0, .., Af_6\}$ and to verify that they satisfy the multiplication table (4.4.2) in the following sense

$$A(f_i \times f_j) = Af_i \times Af_j.$$

Therefore, A is an element of $G_2^{\mathbb{C}}$ since $\{f_0, .., f_6\}$ is a basis for $\mathbb{C}^7$.

In the other case to be considered, that is, when $Af_0 \times Af_3 = iAf_0$, we can use the same procedure as above to prove that $-A \in G_2^{\mathbb{C}}$. However, this contradicts our assumption that $A \in SO(7, \mathbb{C})$. ⊙

Let $f : S^2 \to S^6$ be a k-point ramified ($k \leq 2$) linearly full almost complex curve and let $\psi_j = [g_j]$ ($j = 0, \ldots, 6$) denote the harmonic sequence corresponding to the harmonic map $\psi_3 = [f]$. Then according to Theorem (4.2.2), we can find a local complex coordinate z for S^2 and a basis $\{v_0, .., v_6\}$ for $\mathbb{C}^7$, so that the directrix curve $\psi_0 = [g_0]$ can be expressed by:

$$\begin{aligned} g_0 = & v_0 + z^{k_1}v_1 + z^{k_1+k_2}v_1 + z^{k_1+k_2+k_3}v_3 + \\ & + z^{k_1+k_2+2k_3}v_4 + z^{k_1+2k_2+2k_3}v_5 + z^{2k_1+2k_2+2k_3}v_6. \end{aligned} \tag{4.4.26}$$

Using the meromorphic sections g_j we can choose a particular orthonormal basis $\{u_0, \ldots, u_6\}$ for $\mathbb{C}^7$ so that each vector u_j spans the same complex line bundle as g_j. Indeed, as for $z \neq 0$ the function $\frac{g_j}{|g_j|}$ takes values in the sphere S^{13} then by compactness there exists a sequence $z_k \mapsto 0$ so that for each $j \in \{0, \ldots, 6\}$ we have

$$\lim_{k \mapsto \infty} \frac{g_j}{|g_j|}(z_k) = u_j \tag{4.4.27}$$

It follows immediately from (4.4.26) that the vectors u_j and v_j are related in a triangular way, that is

$$v_0 = a_{(0,0)}u_0$$

$$v_1 = a_{(1,0)}u_0 + a_{(1,1)}u_1$$

$$v_2 = a_{(2,0)}u_0 + a_{(2,1)}u_1 + a_{(2,2)}u_2$$

$$v_3 = a_{(3,0)}u_0 + a_{(3,1)}u_1 + a_{(3,2)}u_2 + a_{(3,3)}u_3$$

$$v_4 = a_{(4,0)}u_0 + a_{(4,1)}u_1 + a_{(4,2)}u_2 + a_{(4,3)}u_3 + a_{(4,4)}u_4$$

$$v_5 = a_{(5,0)}u_0 + a_{(5,1)}u_1 + a_{(5,2)}u_2 + a_{(5,3)}u_3 + a_{(5,4)}u_4 + a_{(5,5)}u_5$$

$$v_6 = a_{(6,0)}u_0 + a_{(6,1)}u_1 + a_{(6,2)}u_2 + a_{(6,3)}u_3 + a_{(6,4)}u_4 + a_{(6,5)}u_5 + a_{(6,6)}u_6$$

where the scalars $(a_{(i,j)})$ appearing in the linear combinations are complex numbers.

From (4.4.27) and Proposition (4.4.2) we can easily determine the cross product of the vectors u_j and consequently also of the vectors v_j. Namely, the vectors u_j have the following multiplication table for $u_i \times u_j$:

$i\backslash j$	0	1	2	3	4	5	6
0	0	0	0	$-iu_0$	$-i\sqrt{2}u_1$	$-i\sqrt{2}u_2$	$-iu_3$
1	0	0	$i\sqrt{2}u_0$	iu_1	0	$-iu_3$	$-i\sqrt{2}u_4$
2	0	$-i\sqrt{2}u_0$	0	iu_2	iu_3	0	$-i\sqrt{2}u_5$
3	iu_0	$-iu_1$	$-iu_2$	0	iu_4	iu_5	$-iu_6$
4	$i\sqrt{2}u_1$	0	$-iu_3$	$-iu_4$	0	$i\sqrt{2}u_6$	0
5	$i\sqrt{2}u_2$	iu_3	0	$-iu_5$	$-i\sqrt{2}u_6$	0	0
6	iu_3	$i\sqrt{2}u_4$	$i\sqrt{2}u_5$	iu_6	0	0	0

(4.4.28)

Now, we notice that the coefficient $a_{(6,6)}$ must be non-zero since f is linearly full and also f_0 is a holomorphic superhorizontal curve because of the characterization given in Theorem (4.4.3). These facts together with equation (4.4.15), give us a cumbersome but straigthforward calculation to determine the following example.

Example 4.4.1 *Let $\{e_1, \ldots, e_7\}$ denote a G_2-basis for $\mathbb{R}^7$. Let k_1, k_2 denote positive integers. Then the holomorphic map $\psi_0 = [g_0] : S^2 \to \mathbb{C}P^6$, determined by the polynomial $g_0(z) = \sum_{j=1}^{7} a_j(z)e_j$ where the $a_j(z)$ are given by*

$$a_1(z) = -\frac{3\sqrt{30}k_2(k_1+k_2)}{(3k_1+k_2)(2k_1+k_2)}z^{k_1+k_2} + \frac{\sqrt{30}}{2}z^{3k_1+k_2},$$

$$a_2(z) = \frac{15\sqrt{3}k_1k_2}{(3k_1+2k_2)(2k_1+k_2)}z^{k_1} + \sqrt{3}z^{3k_1+2k_2},$$

$a_3(z) = \frac{i45\sqrt{2}k_1k_2^2(k_1+k_2)}{(3k_1+2k_2)(3k_1+k_2)(2k_1+k_2)^2} + i\frac{\sqrt{2}}{2}z^{4k_1+2k_2},$
$a_4(z) = \frac{6\sqrt{5}k_2}{2k_1+k_2}z^{2k_1+k_2},$
$a_5(z) = i\frac{3\sqrt{30}k_2(k_1+k_2)}{(3k_1+k_2)(2k_1+k_2)}z^{k_1+k_2} + i\frac{\sqrt{30}}{2}z^{3k_1+k_2},$
$a_6(z) = -i\frac{15\sqrt{3}k_1k_2}{(3k_1+2k_2)(2k_1+k_2)}z^{k_1} + i\sqrt{3}z^{3k_1+2k_2},$
$a_7(z) = -\frac{45\sqrt{2}k_1k_2^2(k_1+k_2)}{(3k_1+2k_2)(3k_1+k_2)(2k_1+k_2)^2} + \frac{\sqrt{2}}{2}z^{4k_1+2k_2},$

is the directrix curve of a linearly full S^1-symmetric almost complex 2-sphere in S^6 with the same singularity type $(k_1-1, k_2-1, k_1-1, k_1-1, k_2-1, k_1-1)$ at $z=0$ and $z=\infty$.

Theorem 4.4.5 *Let $f : S^2 \to S^6$ be a k-point ramified ($k \le 2$) linearly full almost complex curve. Then for a suitable choice of complex coordinate on S^2, the harmonic map $[f(z)] : S^2 \to \mathbb{C}P^6$ has the same singularity type $(k_1-1, k_2-1, k_1-1, k_1-1, k_2-1, k_1-1)$ at $z=0$ and $z=\infty$. Moreover, the directrix curve of f is $G_2^{\mathbb{C}}$-equivalent to the S^1-symmetric curve given in the Example* (4.4.1).

Proof:

Let $\psi_0 = [g_0(z)]$ denote the directrix curve of the map $[f(z)]$. The first part of the statement follows from item (ii) of Theorem (4.2.2) and the following observation.

By comparing the exponents of the variable z appearing in the polynomial $g \times g_z = 0$, we obtain the symmetry $k_3 = k_1$ in the singularity type.

We can use (4.4.15) and Proposition (4.4.2) in order to determine the vectors (v_j) in the simplest way so that they comply with the properties stated in Theorem (4.2.2).

This would envolve some cumbersome calculation if the computer program MAPLE did not help us to execute these boring algebraic manipulations. Therefore, out of this little work, we can write down the vectors (v_j) in terms of complex parameters $\{r_1, \dots, r_8\}$ as follows.

$v_0 = \frac{k_1k_2^2(k_1+k_2)r_1^2r_8^2}{(3k_1+2k_2)(3k_1+k_2)(2k_1+k_2)^2}u_0,$
$v_1 = \frac{k_1k_2r_1^2r_8}{(3k_1+2k_2)(2k_1+k_2)}(r_5u_0 + u_1),$
$v_2 = \frac{k_2(k_1+k_2)r_1r_8}{(2k_1+k_2)(3k_1+k_2)}((r_2r_5 - r_4r_8)u_0 + r_2u_1 + r_8u_2),$

$v_3 = \frac{k_2 r_1 r_8}{(2k_1+k_2)}(\sqrt{2}r_3u_0 + 2r_4u_1 + 2r_5u_2 + \sqrt{2}u_3),$
$v_4 = \frac{r_1}{2}((\sqrt{2}r_3r_5 - 2r_6)u_0 + (2r_4r_5 - \sqrt{2}r_3)u_1 + 2r_5^2u_2 + 2\sqrt{2}r_5u_3 + 2u_4),$
$v_5 = (\frac{\sqrt{2}}{2}r_2r_3r_5 - \frac{\sqrt{2}}{2}r_3r_4r_8 + r_7r_8 - r_2r_6)u_0 + (r_2r_4r_5 - \frac{\sqrt{2}}{2}r_2r_3 - r_4^2r_8)u_1,$
$\quad + (r_2r_5^2 - \frac{\sqrt{2}}{2}r_3r_8 - r_4r_5r_8)u_2 + \sqrt{2}(r_2r_5 - r_4r_8)u_3 + r_2u_4 + r_8u_5,$
$v_6 = (r_5r_7 + \frac{1}{2}r_3^2 - r_4r_6)u_0 + r_7u_1 + r_6u_2 + r_3u_3 + r_4u_4 + r_5u_5 + u_6.$

By Corollary (4.2.1) we can assume f to be S^1-symmetric. Theorem (4.2.2) shows that the S^1-symmetric linearly full almost complex 2-spheres are characterized by the orthogonality of the vectors (v_j) and hence according to the formulae above we must have $r_2 = \ldots = r_7 = 0$. Thus, the directrix curve is described by the 2-parameter family

$$\begin{aligned} g_0(z) = & \frac{k_1k_2^2(k_1+k_2)r_1^2r_8^2}{(3k_1+2k_2)(3k_1+k_2)(2k_1+k_2)^2}u_0 + \frac{k_1k_2r_1^2r_8}{(3k_1+2k_2)(2k_1+k_2)}z^{k_1}u_1 \\ & + \frac{k_2(k_1+k_2)r_1r_8^2}{(2k_1+k_2)(3k_1+k_2)}z^{k_1+k_2}u_2 + \sqrt{2}\frac{k_2r_1r_8}{(2k_1+k_2)}z^{2k_1+k_2}u_3 \\ & + r_1z^{3k_1+k_2}u_4 + r_8z^{3k_1+2k_2}u_5 + z^{4k_1+2k_2}u_6. \end{aligned}$$

Now, we shall apply a suitable conformal transformation to the domain and also apply an appropriate element of G_2 to the co-domain in order to prove that f is indeed equivalent to the curve given in the example above.

Let r be a complex root for the equation

$$r^{2k_1+k_2}r_1r_8 = \sqrt{90}. \tag{4.4.29}$$

Then we shall consider the conformal transformation $z \mapsto rz$, and the element $A \in G_2$ defined by

$$\begin{aligned} &Au_0 := (\tfrac{90}{r_1^2r_8^2})u_0, && Au_1 := (\tfrac{15\sqrt{6}}{r^{k_1}r_1^2r_8})u_1, && Au_2 := (\tfrac{6\sqrt{15}}{r^{(k_1+k_2)}r_1r_8^2})u_2, \\ &Au_3 := (\tfrac{\sqrt{90}}{r^{(2k_1+k_2)}r_1r_8})u_3, && Au_4 := (\tfrac{\sqrt{15}}{r^{(3k_1+k_2)}r_1})u_4, && Au_5 := (\tfrac{\sqrt{6}}{r^{(3k_1+2k_2)}r_8})u_5, \\ &Au_6 := (\tfrac{1}{r^{(4k_1+2k_2)}})u_6. \end{aligned}$$

Using (4.4.29) and the multiplication table (4.4.28) for the vectors u_j we deduce that

$$A(u_i \times u_j) = Au_i \times Au_j,$$

which implies that $A \in G_2$. Thus, the holomorphic curve $g(z)$ is reduced to

$$
\begin{aligned}
g_0(z) =& \frac{90k_1k_2^2(k_1+k_2)}{(3k_1+2k_2)(3k_1+k_2)(2k_1+k_2)^2}u_0 + \frac{15\sqrt{6}k_1k_2}{(3k_1+2k_2)(2k_1+k_2)}z^{k_1}u_1 \\
&+ \frac{6\sqrt{15}k_2(k_1+k_2)}{(2k_1+k_2)(3k_1+k_2)}z^{k_1+k_2}u_2 + \sqrt{2}\frac{6\sqrt{5}k_2}{(2k_1+k_2)}z^{2k_1+k_2}u_3 \\
&+ \sqrt{15}z^{3k_1+k_2}u_4 + \sqrt{6}z^{3k_1+2k_2}u_5 + z^{4k_1+2k_2}u_6.
\end{aligned}
$$

Using again that multiplication table we can also deduce by straigthforward calculations that the vectors $e_j \in \mathbb{R}^7$ $(j = 1, \ldots, 7)$ defined by

$u_0 = \frac{1}{\sqrt{2}}(-e_7 + ie_3)$,

$u_1 = \frac{1}{\sqrt{2}}(e_2 - ie_6)$

$u_2 = \frac{1}{\sqrt{2}}(-e_1 + ie_5)$,

$u_3 = e_4$,

$u_4 = \frac{1}{\sqrt{2}}(e_1 + ie_5)$,

$u_5 = \frac{1}{\sqrt{2}}(e_2 + ie_6)$,

$u_6 = \frac{1}{\sqrt{2}}(e_7 + ie_3)$,

form a G_2-basis for $\mathbb{R}^7$ and the holomorphic curve $g_0(z)$ is written in terms of this basis exactly as the one we gave in the example. $\odot$

Corollary 4.4.2 *If $H^{0,0}$ is the space of linearly full totally unramified almost complex maps of S^2 into S^6 then*

$$\mathcal{H}^{0,0} = G_2^{\mathbb{C}}.$$

Proof:

Indeed, this follows from the theorem above and the fact that the harmonic sequence corresponding to a harmonic map $[f]$, where $f \in \mathcal{H}^{0,0}$, is uniquely determined by its directrix curve. Some care is required here since the composition of f with the antipodal map of S^6 gives also a harmonic map with that same directrix curve. However, the map $-f$ is fortunately an almost anticomplex curve as we can see from (4.4.1). $\odot$

Let M denote the quotient set of the manifold $N = \{(p,q) \in S^2 \times S^2 / p \neq q\}$ by the equivalence relation: $(p,q) \cong (a,b)$ if and only if $(p,q) = (a,b)$ or $(p,q) = (b,a)$.

Corollary 4.4.3 *Let H^{r_1,r_2} denote the space of linearly full almost complex maps of S^2 into S^6 with 2 higher singularities each of type $(r_1, r_2, r_1, r_1, r_2, r_1)$. Then*

$$\mathcal{H}^{r_1,r_2} = M \times G_2^{\mathbb{C}},$$

4.5 Superminimal surfaces of low area.

Let $\psi_0, \ldots, \psi_n : S^2 \to \mathbb{C}P^n$ be a harmonic sequence with corresponding local lifts $f_0, \ldots, f_n : S^2 \backslash W \to \mathbb{C}P^n$ given in accordance with (4.2.3) and (4.2.4), where W is the set of all singularities of the harmonic maps ψ_p. Bolton et al have proved in [11] that when ψ_p is an immersion, the area $A(\psi_p)$ of S^2 with the metric induced by ψ_p is given by

$$A(\psi_p) = \pi(\delta_{p-1} + \delta_p), \qquad (4.5.1)$$

where $\delta_{-1} = 0$ and δ_p is the degree of the $(p-1)$-st osculating curve σ_{p-1}. Moreover, they calculate this degree in terms of the γ_p invariants. Namely,

$$\delta_p = \frac{1}{2\pi i} \int_{S^2} \gamma_p d\bar{z} \wedge dz. \qquad (4.5.2)$$

Bolton et al carry on working out the following global Plücker formula, relating the ramification indices R_p of the curves σ_{p-1} to the degrees δ_p by

$$R_p = -2 - \delta_{p-2} + 2\delta_{p-1} - \delta_p, \qquad \text{where } p = 1, \ldots, n. \qquad (4.5.3)$$

Finally, they also write down the degrees δ_p in terms of the the ramification indices R_p as follows

$$\delta_p = (p+1)(n-p) + \frac{n-p}{n+1} \sum_{k=0}^{p-1} (k+1) R_k + \frac{p+1}{n+1} \sum_{k=p}^{n-1} (n-k) R_k. \qquad (4.5.4)$$

Using these results for the case $n = 6$ we can now produce the following consequence

Lemma 4.5.4 *Let $f : S^2 \to S^6$ be a linearly full almost complex curve. Let $(\psi_p)_{p=0}^6$ be the harmonic sequence determined by f. Then the ramification indices R_p of the associated osculating curves σ_{p-1} satisfy*

(i) $R_j = R_{7-j}$ *for* $j = 1, \ldots, 6$.

(ii) $R_3 = R_2$.

Proof:

Considering that f is superminimal (see paragraph after Definition (4.2.1)) item (i) follows from direct application of (4.5.3), (4.5.2),(4.2.6) and (4.4.3), while item (ii) follows from (4.5.3), (4.5.2) and (4.4.3). ⊙

Proposition 4.5.3 *Let $f : S^2 \to S^6$ be a linearly full almost complex curve with total singularity type $(R_1, \ldots, R_6)$. Then the area $A(\psi)$ of the harmonic map $\psi = [f] : S^2 \to \mathbb{C}P^6$ is given by*

$$A(\psi) = 4\pi(6 + 2R_1 + R_2). \tag{4.5.5}$$

Proof:

Using the lemma above and (4.5.4) we have

$$\delta_3 = 12 + 4R_1 + 2R_2 = \delta_4.$$

Thus, the Corollary follows from (4.5.1). ⊙

Theorem 4.5.6 *Let $\mathcal{H}^d$ be the space of linearly full almost complex maps of S^2 into S^6 of area $4\pi d$. Then $d \geq 6$ and*

(i) $\mathcal{H}^6 = \mathcal{H}^{0,0} = G_2^{\mathbb{C}}$,

(ii) $\mathcal{H}^7$ *is empty,*

(iii) $\mathcal{H}^8 = \mathcal{H}^{(0,1)} = M \times G_2^{\mathbb{C}}$.

Furthermore, every element of $\mathcal{H}^8$ has directrix curve $G_2^{\mathbb{C}}$-equivalent to the following S^1-symmetric case

$$\begin{aligned} g(z) \quad &= (70\sqrt{15}z^5 - 126\sqrt{15}z^3)e_1 + (70\sqrt{6}z^7 + 75\sqrt{6}z)e_2 \\ &+(135i + 70iz^8)e_3 + 210\sqrt{10}z^4 e_4 + (70i\sqrt{15}z^5 + 126i\sqrt{15}z^3)e_5 \\ &+(70i\sqrt{6}z^7 - 75i\sqrt{6}z)e_6 + (-135 + 70z^8)e_7. \end{aligned} \tag{4.5.6}$$

Proof:

It follows from (4.5.5) that the scalar d is given in terms of the total ramification indices, which are non negative integers, by

$$d = 6 + 2R_1 + R_2. \tag{4.5.7}$$

From which we see that $d = 6$ if and only if $R_1 = 0 = R_2$, in other words the space $\mathcal{H}^6$ is made up of the totally unramified curves. Thus item (i) follows from Corollary (4.4.2).

If $d = 7$ then (4.5.7) implies $R_1 = 0$ and $R_2 = 1$ which means that an element of $\mathcal{H}^7$ would have only one singularity. But this is not possible as we have noticed in Remark (4.2.2).

If $d = 8$ then we have two possibilities for the total ramification indices of an element of $\mathcal{H}^8$. The first one being $R_1 = 1$ and $R_2 = 0$ which cannot occur for the same reason above. Thus every element of this moduli space has total ramification indices $R_1 = 0$ and $R_2 = 2$. In particular, an element of $\mathcal{H}^8$ must be 2-point ramified with singularity type at each point given by $(0, 1, 0, 0, 1, 0)$ and hence (iii) and (4.5.6) follow from Theorem (4.4.5).

It is worth mentioning here that an element of $\mathcal{H}^8$ is always an immersion since it occupies the middle position in the corresponding harmonic sequence and its singularities occur only in the second and fifth element of this sequence as we have just shown. $\odot$

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