Analyticity and scaling in quantum field theory

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Declaration

This thesis is based on work carried out by the author from January 1998 to August 2000 at the Department of Mathematical Sciences of the University of Durham under the supervision of Paul Mansfield.

Chapters two and three serve as introduction and no claim is made for originality. Chapters four, five, six, seven and eight contain mainly original work. Non-original work is acknowledged. Chapters four, five and six are based on work published in the papers Physics Letters **B 462** (1999) 103 and Journal of High Energy Physics **03** (2000) 008 written jointly with Paul Mansfield [1, 2]. The main part of all original calculations in these chapters is done by the author. Chapter seven is based on the publication Journal of High Energy Physics **07** (2000) 053 [3], written solely by the author, and chapter eight on the work [4] also solely by the author.

Abstract

The theory describing the scaling properties of quantum field theory is introduced. The symmetry principles behind scale and conformal transformations are reviewed together with the renormalisation group. A method for improving perturbative calculations of physical quantities in the infra-red limit is developed using general analyticity properties valid for all unitary quantum field theories. The infra-red limit of a physical quantity is shown to equal the limiting value of the Borel transform in a complex scale parameter, where the order of the Borel transform is related to the domain of analyticity. It is shown how this general result can be used to improve perturbative calculations in the infra-red limit. First, the infra-red central charge of a perturbed conformal field theory is considered, and for the unitary minimal models perturbed by $\Phi_{(1,3)}$ the developed approximation is shown to be very close to the exact results by improving only a one loop perturbation. The other example is the infra-red limit of the critical exponents of φ^4 theory in three dimensions, where our approximation is within the limits of other approximations. The exact renormalisation group equation is studied for a theory with exponential interactions and a background charge. It is shown how to incorporate the background charge, and using the operator product expansion together with the equivalence between the quantum group restricted sine-Gordon model and the unitary minimal models perturbed by $\Phi_{(1,3)}$, the equation obtained is argued to describe the flow between unitary minimal models. Finally, a semi-classical approximation of the low energy limit of a bosonic membrane is studied where the action is taken to be the world-volume together with an Einstein–Hilbert term. A solution to the linearized equations of motion is determined describing a membrane oscillating around a flat torus.

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Contents

1

Introduction

Quantum field theory arose from the efforts of unifying quantum mechanics and special relativity. In its present formulation it is the most successful theory in physics describing phenomena from sub-atomic to cosmological scales, and with an agreement between theory and experiment of up to 10^{-10} in quantum electrodynamics. It provides the framework for the standard model of particle physics that describes the electro-magnetic, weak and strong nuclear interactions, and it finds important applications in nuclear, atomic and condensed matter physics together with cosmology.

The formulation of the physical theories describing the fundamental interactions, at the currently accessible energy scale, shows the important role that symmetry principles play in nature. The standard model has a gauge symmetry with gauge group $U(1) \otimes SU(2) \otimes SU(3)$, and Einstein's classical field theory of gravity, general relativity, has four dimensional space-time diffeomorphisms as its gauge symmetry.

Another important consequence of quantum field theory is that the dynamics are scale dependent, and this corresponds to what is observed in nature namely that phenomena observed at one scale are very different from those observed at a much larger scale, i.e. they decouple. Local field theories have ultra-violet divergences which have to be renormalised by comparing the observables in a theory with measured quantities at a certain length scale. The theory will then make unambiguous

predictions, but the couplings will now depend on the scale, this scale dependence is described by the renormalisation group.

A quantum field theory is specified by its coupling constants, and the renormalisation group therefore describes a flow in the coupling space under scaling where the fixed points of the flow are scale invariant theories. The renormalisation group thereby specifies how different quantum field theories are organized by scale in the coupling, or theory, space.

Despite the successes of quantum field theory the understanding of it and the fundamental laws of nature is still very limited. It has not been possible to incorporate gravity into a local quantum field theory, and the emergence of the renormalisation group has meant that quantum field theory is now viewed as an effective theory valid below some cut-off scale [5, 6, 7]. A quantum theory of gravity relevant at the Planck scale 10^{-33}cm (and perhaps even at lower scales) therefore has to be described by a more fundamental theory, and the only present candidate includes extended objects like strings and membranes.

In contrast to the many different applications of quantum field theory the actual calculational methods are few, and the main one is perturbation theory around the gaussian fixed point describing the free theory. This means that for a general quantum field theory in the vast infinite dimensional theory space we have only got information about the neighborhood of a single point, the gaussian fixed point.

To get a better understanding of quantum field theory it is therefore important to develop new approximation methods that extend the region in parameter space where calculations can be performed. In this thesis, we have derived a method of improving perturbative calculations of infra-red quantities using general analyticity properties of correlation functions valid in any unitary quantum field theory.

Another way of extending the knowledge about theory space, and the renormalisation group flows that determine its structure, is to study specific models in dimensions, or with symmetries, that constrain the dynamics so non-perturbative results can be derived.

Two dimensional theories are an important example, here the fixed points are two

dimensional conformal field theories with an infinite dimensional symmetry group, and perturbation theory can be defined around non-trivial fixed points. Also, in two dimensions the c-theorem states that the flows of unitary theories are irreversible and have an infra-red fixed point. The sub-space of renormalisable theories is larger in two than in higher dimensions, and a classification of all conformal field theories will therefore also constrain the fixed point structure in higher dimensions.

The more fundamental high energy theories may also prove helpful in getting a better understanding of the theory space of quantum field theory, even if the physical reality of such theories remains unclear because of a lack of experimental evidence. For example, dualities in string theory have lead to conjectured dualities in supersymmetric quantum field theories by dimensional reduction, and a relation between the strong coupling limit and supergravity; it might be possible to find similar arguments related to more realistic field theories in four dimensions.

1.1 Outline of the thesis

In chapter two and three the theory describing the scaling of quantum field theories is briefly reviewed. In chapter two Nöther's theorem and the Ward identities are introduced, and scale and conformal transformations are defined together with the energy-momentum tensor. In chapter three renormalisation is described and the Callan–Symanzik equation is derived. The operator product expansion and the linearization of the renormalisation group are also reviewed.

In chapter four a method that improves perturbative calculations of physical quantities in the infra-red limit is developed. The method uses the analyticity of correlation functions in a scale parameter to express the infra-red physics as an integral in the ultra-violet region. This expression can be written as the Borel transform of a certain order, and the infra-red limit is then given as the limiting value of the Borel transform. First, the Källen–Lehmann spectral representation is reviewed together with the perturbative series, and then our method is derived.

This method is applied to calculate the infra-red limit of the central charge in

chapter five and the critical exponents of φ^4 theory in three dimensions in chapter six. In chapter five some of the important properties of a two dimensional conformal field theory are first introduced, and Zamolodchikov's c-theorem is then shown. Based on the results from chapter four an approximation method is developed for calculating the infra-red central charge of a perturbed conformal field theory. This is applied to the free bosonic and fermionic theories perturbed by a mass term, and the unitary minimal models perturbed by $\Phi_{(1,3)}$. For the minimal models we obtain very good results by maximizing the domain of analyticity, with approximation values remarkably close to the exact ones by improving only a one loop perturbative calculation.

In chapter six the critical exponents of φ^4 theory are first defined and then an approximation method for calculating ν and η is developed based on the results in chapter four and a conformal mapping and Padé approximation. The results for ν and η are within the limits of other results obtained with different calculational methods.

Chapter seven describes a new application of the exact renormalisation group equation to a theory with a background charge. The motivation for considering this case is the equivalence between the perturbed unitary minimal models $\mathcal{M}_m$, studied in chapter five, and the quantum group restricted sine-Gordon model. First the Coulomb gas representation of minimal models is introduced together with the quantum group restricted sine-Gordon model. Then it is shown how the background charge can be incorporated into the exact renormalisation group equation, and using the operator product expansion this gives an equation describing the renormalisation group flow for all the perturbed unitary minimal models. The higher order terms in the coupling appear in this equation in the off-critical structure constants, and in the perturbative limit $m \to \infty$, the well known perturbative renormalisation group equation is reproduced.

In the last chapter a semi-classical approximation of the bosonic membrane in eleven dimensions with an Einstein–Hilbert term is considered. The action is then a sum of the membrane world-volume and an integral over the scalar curvature,

and can be viewed as an approximation of the low energy effective action for the membrane. The equations of motion are non-linear, but can be approximated by linearizing around a classical solution that is here taken to be a flat torus. A new solution of the linearized equations is obtained that describes a membrane oscillating around the flat torus. First the bosonic membrane is introduced and the solution is then presented.

2

Symmetries and scaling

Physics deals with the understanding of the dynamics in the physical world, and is expressed in terms of physical laws. These can mathematically be expressed in many equivalent ways, and this invariance or symmetry in our description of the dynamics in nature has been one of the main guidelines when formulating new physical laws.

If a theory is described by an action principle, where the classical path is given by the stationary value of the action, then Nöther's theorem states that for every continuous symmetry transformation of the action, there is a classically conserved current and the associated conserved charge is the generator of the transformation.

The conserved quantities in a physical process thereby determine the symmetries of the interaction and these symmetries can then be used to build an action for the theory that describes its dynamical properties (for a quantum field theory it is the partition function which has this role, i.e. here the path integral measure also needs to be specified). One of the successes of physics has been the isolation and identification of conserved quantities, and their very existence shows the regularity and symmetry in nature that has allowed the formulation of the physical laws.

First Nöther's theorem and the Ward identities are shown, the energy-momentum tensor is then introduced and finally scale and conformal transformations are discussed. The section about scale transformations follows Callan, Coleman and Jackiw [8] and [9]. In this chapter the Minkowski signature $(+,-,-,\cdots,-)$ is used. Gen-

eral discussions about symmetries in field theory can be found in [5, 10, 11, 12, 13] and for gauge theories in [14, 15].

2.1 Nöther's theorem and Ward identities

Nöther's theorem [16] shows how the symmetries of the action correspond to conserved quantities. In a local field theory depending on the fields φ^i the action functional can generally[1] be written as

$$S[\varphi] = \int d^d x \; \mathcal{L}(\varphi^i, \partial_\mu \varphi^i) \tag{2.1}$$

and we will consider the effect of the infinitesimal transformation

$$\begin{aligned} x'^\mu &= x^\mu + \delta x^\mu = x^\mu + \delta w^a(x) X_a^\mu(x), \\ \varphi'^i(x') &= \varphi^i(x) + \widetilde{\delta\varphi}^i(x) = \varphi^i(x) + \delta w^a(x) \Phi_a^i(x), \end{aligned} \tag{2.2}$$

where $\delta w^a(x)$ is infinitesimal and has compact support. A symmetry is defined as a transformation that leaves the action invariant. Any infinitesimal transformation will by definition leave the action invariant at the classical path, for it to be a symmetry transformation this invariance must be preserved also away from the classical path, i.e. when the equations of motion are not satisfied. (2.2) is a rigid (or global) symmetry of the action when $\delta S = 0$ for δw^a constant, i.e. $\partial_\mu \delta w^a(x) = 0$. Consider now a transformation where X_a^μ and Φ_a^i are the same as for a rigid symmetry but $\delta w^a(x)$ is not constant. δS can then be written as

$$\delta S = -\int d^d x j_a^\mu(x) \partial_\mu \delta w^a(x), \tag{2.3}$$

for some current $j_a^\mu(x)$, because then $\delta S = 0$ for δw^a constant and δS is the variation of a local quantity and is therefore local [5]. Taking the fields φ^i in δS to satisfy the

[1] Classical equations of motion which are higher than second order differential equations often leads to non-causal effects, so the lagrangian is generally taken to contain at most two derivatives [10].

equations of motion means that (2.3) vanishes for an arbitrary $\delta w^a(x)$ and integrating by parts yields that the current $j_a^\mu(x)$ is conserved $\partial_\mu j_a^\mu(x) = 0$. Assuming that the current vanishes at spatial infinity the Nöther charge

$$Q_a = \int d^{d-1}x \; j_a^0(x) \tag{2.4}$$

is then a constant of the motion $\partial_0 Q_a = 0$, showing that a symmetry leads to a conserved quantity which is Nöther's theorem. If the lagrangian density is invariant under a rigid transformation (2.2) then the form of the current follows by considering the variation of the action. To first order in δw^a the variation in the field at fixed x becomes

$$\begin{aligned} \delta\varphi^i(x) &= \varphi'^i(x) - \varphi^i(x) = \varphi'^i(x') - \delta x^\mu \partial_\mu \varphi^i(x) - \varphi^i(x) \\ &= \delta w^a(x)(\Phi_a^i(x) - X_a^\mu(x)\partial_\mu\varphi^i(x)) \equiv \delta w^a(x)\delta_a\varphi^i(x), \end{aligned} \tag{2.5}$$

and the action changes as

$$\begin{aligned} \delta S = S[\varphi'] - S[\varphi] &= \int d^dx' \mathcal{L}'(x') - \int d^dx \mathcal{L}(x) \\ &= \int \delta(d^dx)\mathcal{L} + \int d^dx \left(\frac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial \varphi^i}\delta\varphi^i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^i}\delta\partial_\mu\varphi^i \right) \\ &= \int d^dx \left[\mathcal{L}\partial_\mu(X_a^\mu \delta w^a) + \partial_\mu(\mathcal{L})\delta w^a X_a^\mu + \left(\frac{\partial \mathcal{L}}{\partial \varphi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^i} \right) \right) \delta\varphi^i \right. \\ &\qquad \left. + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^i}\delta\varphi^i \right) \right] \\ &= \int d^dx \left[\partial_\mu \left(\mathcal{L} X_a^\mu(x)\delta w^a(x) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^i} \left(\Phi_a^i(x) - X_a^\nu(x)\partial_\nu\varphi^i \right) \delta w^a(x) \right) \right. \\ &\qquad \left. + \left(\frac{\partial \mathcal{L}}{\partial \varphi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^i} \right) \right) \delta w^a(x)\delta_a\varphi^i \right]. \end{aligned} \tag{2.6}$$

If now $\mathcal{L}'(x')d^dx' - \mathcal{L}(x)d^dx = 0$ under a rigid transformation (2.2)[2] where δw^a is constant, then the term proportional to $\delta w^a(x)$ in (2.6) vanishes, and it follows from

[2]The boundary conditions of the integral are invariant under (2.2) as $\delta w^a(x)$ has compact support.

(2.6) and (2.3) that $j_a^\mu(x)$ is given by the canonical Nöther current

$$j_a^\mu = X_a^\nu \left(-\mathcal{L}\delta_\nu^\mu + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^i} \partial_\nu \varphi^i \right) - \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^i} \Phi_a^i. \tag{2.7}$$

In the hamiltonian picture where $\pi_i(x) = \frac{\partial \mathcal{L}}{\partial \partial_0 \varphi^i(x)}$ the Nöther charge becomes

$$Q_a = \int d^{d-1}x \left(\pi_i X_a^\nu \partial_\nu \varphi^i - \mathcal{L} X_a^0 - \pi_i \Phi_a^i \right), \tag{2.8}$$

and the Poisson bracket[3] with φ^i is

$$\{Q_a, \varphi^i(x)\}_P = \int d^{d-1}x \frac{\delta Q_a}{\delta \pi_j(x)} \frac{\delta \varphi^i(x)}{\delta \varphi^j(y)} = X_a^\nu(x) \partial_\nu \varphi^i(x) - \Phi_a^i(x) = -\delta_a \varphi^i. \tag{2.9}$$

Q_a is in this sense infinitesimal generator for the transformation (2.2).

Not all conserved charges appear in this way from a continuous symmetry, the other example is topological charges that arise from non-trivial global boundary conditions of the fields. There is a conjectured duality, the Montonen–Olive duality [17], between a quantum theory having gauge particles with Nöther (electric) charge and a theory with solitonic objects with topological (magnetic) charge[4].

[3]The Poisson bracket is here defined as $\{F, G\}_P = \int d^{d-1}x \left(\frac{\delta F}{\delta \pi_i(x)} \frac{\delta G}{\delta \varphi^i(x)} - \frac{\delta G}{\delta \pi_i(x)} \frac{\delta F}{\delta \varphi^i(x)} \right)$.

[4]The motivation for this conjecture is the classical electromagnetic duality in Maxwell's equations (in the absence of sources) under $B \to E$, $E \to -B$, or using differential forms $F \to *F$ ($** = -1$). A more concrete motivation is the duality in the spectrum of $SU(2)$ Yang-Mills-Higgs theory between the massive vector boson with mass $m \propto e$ and charge $q \propto e$ and the BPS monopole with $m \propto 1/e$ and charge $q \propto 1/e$ (when dyons are considered the duality becomes a transformation of the charge lattice under $SL(2, \mathbb{Z})$). The spectrum is therefore invariant under vector boson $\leftrightarrow$ soliton and $e \leftrightarrow 1/e$ exchange, and it is therefore a strong-weak coupling duality. This type of duality is important when trying to understand the non-perturbative strong coupling region of supersymmetric quantum field theory and for theories with extended objects. The strong-weak coupling duality was first seen in two dimensions between the massive Thirring model and the sine-Gordon model [18]. There are a number of obstructions against a Montonen-Olive type of duality, the vector boson has spin one whereas the monopole seems to be a scalar and the electric charge runs, but the topological charge does not. Supersymmetry can potentially cure these as it relates particles with different spin and leads to non-renormalisation theorems, the only present candidate where Montonen–Olive duality seems to be exactly realized is for $SU(2)$ $N = 4$ supersymmetric Yang-Mills-Higgs [19, 20, 21]

2.1.1 Ward Identities

The conservation equation $\partial_\mu j_a^\mu(x) = 0$ was derived for the classical theory, requiring that the equations of motion were satisfied. The quantum mechanical consequences of a symmetry transformation are described by Ward identities that are relations among the correlation functions of the theory. Physical amplitudes are given in terms of the correlation functions that are formally defined by the path integral

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi\; e^{iS[\phi]}\phi(x_1)\cdots\phi(x_n) \quad (2.10)$$

where $Z = \int \mathcal{D}\phi\; e^{iS[\phi]}$ is the partition function. If a transformation (2.2) is a symmetry of the action $S[\phi'] = S[\phi]$ and the path integral measure $\mathcal{D}\phi' = \mathcal{D}\phi$ the correlation function (2.10) then transforms as

$$\langle\phi(x_1')\cdots\phi(x_n')\rangle = \langle\phi'(x_1')\cdots\phi'(x_n')\rangle. \quad (2.11)$$

For translations $\phi'(x+a) = \phi(x)$ this shows the translation invariance of the correlators. The case $\mathcal{D}\phi' \neq \mathcal{D}\phi$ is called an anomaly, here quantum corrections break the symmetry present at the classical level. The Ward identities follow from (2.10) and (2.3) when the path integral measure is invariant, to lowest order

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \frac{1}{Z}\int \mathcal{D}\phi' e^{iS[\phi]+i\int d^dx\delta w^a(x)\partial_\mu j_a^\mu(x)}(\phi(x_1)+\delta\phi(x_1))\cdots(\phi(x_n)+\delta\phi(x_n))$$

and this shows that

$$\int d^dx\delta w^a(x)\frac{\partial}{\partial x^\mu}\langle j_a^\mu(x)\phi(x_1)\cdots\phi(x_n)\rangle = i\sum_{i=1}^n\langle\phi(x_1)\cdots\delta w^a(x_i)\delta_a\phi(x_i)\cdots\phi(x_n)\rangle$$

$$= i\int d^dx\delta w^a(x)\sum_{i=1}^n\delta(x-x_i)\langle\phi(x_1)\cdots\delta_a\phi(x_i)\cdots\phi(x_n)\rangle.$$

Again $\delta w^a(x)$ is arbitrary so that

$$\frac{\partial}{\partial x^\mu}\langle j_a^\mu(x)\phi(x_1)\cdots\phi(x_n)\rangle = i\sum_{i=1}^n\delta(x-x_i)\langle\phi(x_1)\cdots\delta_a\phi(x_i)\cdots\phi(x_n)\rangle, \quad (2.12)$$

this is the Ward identity for the current $j_a^\mu(x)$. When integrated over space this leads for one field to[5] [22]

$$\int d^{d-1}y \frac{\partial}{\partial y^0}\langle 0|T\{j_a^0(y)\phi(x)\}|0\rangle = \langle 0|T\{\dot{Q}_a\phi(x)\}|0\rangle - \delta(x^0-y^0)\langle 0|[\phi(x),Q_a]|0\rangle$$
$$= i\delta(x^0-y^0)\langle 0|\delta_a\phi(x)|0\rangle. \tag{2.13}$$

For $x^0 \neq y^0$ then $\langle 0|[Q_a,\phi(x)]|0\rangle = i\langle 0|\delta_a\phi(x)|0\rangle$, and this argument generalizes to an arbitrary number of fields ϕ [22] so that

$$i[Q_a,\phi(x)] = -\delta_a\phi(x) \tag{2.14}$$

holds as an operator equation, it is the quantum version of (2.9) and it again shows that the charge generates the transformation.

2.2 Scale and conformal transformations

In flat space-time a conformal transformation is defined as a coordinate transformation that leaves the metric invariant up to a scale factor[6]. By definition, Poincaré transformations leave the metric invariant and are therefore conformal, the others are dilatations and special conformal transformations [24]. The infinitesimal generators of the conformal group obey the Lie algebra $so(2,d)$ in Minkowski space and $so(1,d+1)$ in Euclidean space [25].

2.2.1 The energy-momentum tensor

In a Poincaré invariant theory the energy-momentum tensor is defined as the conserved current associated with translations: $x' = x+a$, $\phi'(x') = \phi(x)$. The canonical energy-momentum tensor (2.7) then becomes

$$T_{c\mu\nu}(x) = \frac{\partial\mathcal{L}}{\partial\partial^\mu\phi^i(x)}\partial_\nu\phi^i(x) - \eta_{\mu\nu}\mathcal{L}(x), \tag{2.15}$$

[5] $T\{j_a^0(y)\phi(x)\} = j_a^0(y)\phi(x) + \theta(x^0-y^0)[\phi(x),j_a^0(y)]$.

[6] For a diffeomorphism invariant quantum field theory on a curved space-time the transformation $g_{ij}(x) \to \Omega^2(x)g_{ij}(x)$ of the metric is called a Weyl transformation, see e.g. [23] for a review.

and the conserved charges $P^\mu = \int d^{d-1}x\ T_c^{\mu 0}(x)$ are the energy and momentum $P^\mu = (P^0, P^i)$. The energy-momentum tensor can be made symmetric (put in Belifante form) without changing P^μ and its conservation by adding to $T_c^{\mu\nu}$ total divergences of antisymmetric tensors: $\partial_\rho B^{\rho\mu\nu}$, where $B^{\rho\mu\nu} = -B^{\mu\rho\nu} = -B^{\nu\mu\rho}$ [8]. We will always assume a symmetric energy-momentum tensor.

The presence of gravity breaks the global Poincaré invariance. In a curved space-time the energy-momentum tensor is defined as the source of gravity i.e.[7]

$$T_{\mu\nu}(x) = \frac{2}{\sqrt{-g}} \frac{\delta S[\phi^i, g_{\mu\nu}]}{\delta g^{\mu\nu}(x)} \tag{2.16}$$

which is symmetric. The quantum version becomes

$$\langle T_{\mu\nu}(x) \rangle = \frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g^{\mu\nu}(x)} \tag{2.17}$$

where W is the generating functional for the connected correlators, $Z = e^{iW}$ [26].

2.2.2 Scale transformations

A scale transformation, or dilatation, is defined as

$$\begin{aligned} x' &= sx \\ \phi'(x') &= s^{-\Delta}\phi(x) \end{aligned} \tag{2.18}$$

where Δ is the scaling dimension of $\phi(x)$. The scaling dimension is defined so that the classical action is scale invariant. From the kinetic term in the action it follows that $\Delta = \frac{d-2}{2}$ for a scalar field and $\Delta = \frac{d-1}{2}$ for a Fermi field, and any mass term will therefore break scale invariance. A mass sets a scale in the theory. The mass (or engineering) dimension of a physical quantity is written as $[\mathcal{F}]$, it determines the physical unit in which $\mathcal{F}$ is measured. It is a fundamental principle that all physical quantities are homogeneous with respect to the mass dimension

$$\mathcal{F}(tm) = t^{[\mathcal{F}]}\mathcal{F}, \tag{2.19}$$

[7] The components of the metric tensor satisfy that $g(x) = (-1)^d \det g_{\mu\nu}(x) < 0$, $\delta g = -g\, g_{\mu\nu}\delta g^{\mu\nu}$ and $\delta g^{\mu\rho} = -g^{\alpha\rho}g^{\mu\nu}\delta g_{\nu\alpha}$.

which therefore defines a grading of all physical quantities [27].

In a scale invariant quantum field theory it follows from (2.11) and (2.18) that the correlators transform as

$$\langle\phi_1(sx_1)\cdots\phi_n(sx_n)\rangle = s^{-\Delta_1}\cdots s^{-\Delta_n}\langle\phi_1(x_1)\cdots\phi_n(x_n)\rangle \tag{2.20}$$

under scaling. For the transformation (2.18) the canonical scaling current (2.7) becomes

$$D_c^\mu = x_\nu T_c^{\mu\nu} + \Delta\phi^i \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}. \tag{2.21}$$

In [8] it was shown that this current can always (in $d > 2$) be written as $D^\mu = x_\nu T^{\mu\nu}$ for an improved energy-momentum tensor $T^{\mu\nu}$ by adding divergences of anti-symmetric tensors. Then $T_c^{\mu\nu}$ and $T^{\mu\nu}$ generate the same charges and $\partial_\mu T_c^{\mu\nu} = \partial_\mu T^{\mu\nu} = 0$. This shows that for a classically scale invariant theory, with $\partial_\mu D^\mu = 0$, the trace of the improved energy-momentum tensor vanishes $T_\mu^\mu = 0$. From (2.14) and (2.18) it follows that the scaling charge D is given by

$$i[D,\phi(x)] = -\delta\phi(x) = \delta w(\Delta + x^\mu\partial_\mu)\phi(x), \qquad s = 1+\delta w, \qquad |\delta w| \ll 1. \tag{2.22}$$

Using that $i[P^\mu,\phi(x)] = \partial^\mu\phi(x)$ it follows that $i[D,P^\mu] = P^\mu - \eta^{0\mu}\int d^{d-1}x\partial_\nu D^\nu$. For a scale invariant theory then $P^{2\prime} = e^{i\alpha D}P^2e^{-i\alpha D} = e^{2\alpha}P^2$, and this again shows that masses must vanish $m^2 = 0$ [25, 28].

2.2.3 Conformal transformations

A conformal transformation $x \to x'$ changes the scale of the metric: $g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = g_{\rho\kappa}(x)\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\kappa}{\partial x'^\nu} = \gamma(x)g_{\mu\nu}(x)$, and hence leaves angles $\frac{x\cdot y}{|x||y|}$ invariant. For the infinitesimal transformation $x'^\mu = x^\mu + \epsilon^\mu(x)$ the change in the metric is to first order

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu = \gamma(x)g_{\mu\nu}(x) \quad\Rightarrow\quad \partial_\mu\epsilon_\nu + \partial_\mu\epsilon_\nu = \frac{2}{d}\,\partial_\rho\epsilon^\rho g_{\mu\nu}(x), \tag{2.23}$$

by taking the trace and using that $\partial_\mu g_{\rho\nu}(x) = 0$ in flat space. Conformal invariance completely fixes the form of the two and three point correlations functions in any number of dimensions [24]. For scalar fields that transform as

$$\phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \phi(x), \tag{2.24}$$

where $|\frac{\partial x'}{\partial x}|$ is the Jacobian of the conformal transformation, rotation and translation invariance require that $\langle\phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1 - x_2|)$, and invariance under scale transformations (2.20) shows that $\langle\phi_1(x_1)\phi_2(x_2)\rangle = c_{12}/|x_1 - x_2|^{\Delta_1+\Delta_2}$ for some $c_{12} \in \mathbb{C}$. The Jacobian becomes $|\frac{\partial x'}{\partial x}| = \frac{1}{\sqrt{-g'}} = \gamma(x)^{-d/2}$. The special conformal transformations are given by

$$x \to x' = \frac{x + bx^2}{1 + 2b \cdot x + b^2x^2}, \qquad \left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1 + 2b \cdot x + b^2x^2)^d},$$

and it then directly follows that[8]

$$(x_1' - x_2')^2 = \frac{(x_1 - x_2)^2}{(1 + 2b \cdot x_1 + b^2x_1^2)(1 + 2b \cdot x_2 + b^2x_2^2)}. \tag{2.25}$$

Hence, together with (2.24) invariance under special conformal transformations sets $c_{12} = k\delta_{\Delta_1-\Delta_2,0}$, and the fields are normalized so that $k = 1$, i.e.

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \frac{\delta_{\Delta_1-\Delta_2,0}}{|x_1 - x_2|^{2\Delta_1}}. \tag{2.26}$$

A similar expression is obtained for the three point function.

From (2.16) and (2.23) the variation of the action becomes

$$\delta S = \frac{1}{2}\int d^dx\sqrt{-g}\ \delta g^{\mu\nu}(x)\ T_{\mu\nu}(x) = -\frac{1}{2}\int d^dx\sqrt{-g}\ T^\mu_\mu(x)\ \frac{2}{d}\ \partial_\rho\epsilon^\rho. \tag{2.27}$$

Classical scale invariance $\partial_\mu D^\mu = T^\mu_\mu = 0$ therefore implies conformal invariance $\delta S = 0$ for the improved energy-momentum tensor[9]. There are no general arguments

[8] This is most easily seen writing $(x_1' - x_2')^2 = x_1'^2 + x_2'^2 - 2x_1' \cdot x_2'$.

[9] This argument does not show that $T^\mu_\mu = 0$ for a conformally invariant theory (as sometimes stated, $\partial_\rho\epsilon^\rho$ is not an arbitrary function), but it is shown in [29] that conformal invariance is equivalent to the existence of a traceless energy-momentum tensor and this is the one we will use when considering conformal field theories. In a curved space-time Weyl invariance does imply that $T^\mu_\mu = 0$.

saying that this also holds at the quantum level, but no counter examples exists for unitary theories in flat space (see [30] for a recent example in curved space), and it is often stated that a vanishing beta function leads to conformal invariance [29, 27]. In two dimensions scale invariance can be proven to imply conformal invariance also at the quantum level, this is shown in chapter 5 using the spectral representation.

Quantum corrections change the scaling of a theory. A local field theory has an infinite number of degrees of freedom that give rise to ultra-violet divergences which have to be renormalised, the scaling behaviour of renormalised quantum field theory is described by the renormalisation group.

3

The renormalisation group and universality

The renormalisation group describes the behaviour of a quantum field theory under scaling. It also describes scaling in statistical mechanics and it provides an important link between relativistic quantum field theory, statistical mechanics and critical phenomena. The structure of the space of all quantum field theories is also determined by the renormalisation group.

The relation between field theory and statistical mechanics has led to an understanding in statistical mechanics of critical phenomena and universality, and calculations of critical exponents using quantum field theory methods. On the other hand, the idea of effective field theories used in statistical mechanics is now also used for quantum field theory which is seen as an effective field theory only valid below some fundamental scale Λ_0.

The requirement of renormalisability was originally used as a selection criterion for physical theories, but an effective theory must include all terms obeying the symmetries of the theory, also the non-renormalisable terms. However, these terms are irrelevant in the renormalisation group sense and are therefore suppressed at energies $\Lambda \ll \Lambda_0$. The perturbative calculations in the wilsonian effective theory are equivalent to normal perturbative calculations, where the running coupling constant

can be seen as the effective coupling. Non-renormalisable interactions are allowed with a very small coupling, and gravity has been speculated [6] to be one such example[1].

A quantum field theory is determined by its symmetries and the coupling constants of the interaction. The space of quantum field theories is therefore spanned by the infinite dimensional coupling constant space. The renormalisation group describes a one parameter flow under scaling in the coupling constant space, where fixed points correspond to scale invariant theories. The coupling constants describe the microscopic interactions, but different quantum field theories have the same long range physics if they flow to the same fixed point under the renormalisation group. Quantum field theories can therefore be grouped into different equivalence classes according to which fixed point they flow to. The renormalisation group in this way determines the structure of the coupling constant space. To get a better understanding of the space of quantum field theories it is therefore important to study the infra-red and ultra-violet limits of quantum field theories and their renormalisation group flows.

The analogy between statistical mechanics and quantum field theory, which is an exact equivalence when the continuum limit of the statistical mechanical system is considered, stems from the path integral formulation of the latter. Under a Wick rotation $it = t_E$, $(x,x)_M = -(x,x)_E$ and $iS = -S_E$: $Z[J] = \int \mathcal{D}\phi e^{iS+\int d^dx J(x)\phi(x)} \rightarrow \int \mathcal{D}\phi e^{-S_E+\int d^dx J(x)\phi(x)}$ which is analogous to the partition function of a statistical mechanical system in d space dimensions where S_E is the classical hamiltonian. The n-point Green functions become euclidean Green functions, or Schwinger functions, that are equivalent to correlation functions in statistical mechanics, and this is the term we will use throughout the thesis[2].

[1] To explain why gravity is irrelevant and the other interactions are relevant one needs a fundamental finite theory.

[2] The usual convention in physics is applied where euclidean signature means a riemannian manifold $(\mathcal{M}, g_{ij})$ where g_{ij} has signature $(+,\ldots,+)$ and Minkowski signature means a pseudo-riemannian manifold $(\mathcal{M}, \eta_{ij})$ of signature $(+,-,\ldots,-)$.

3.1 Renormalisation

Loop diagrams in quantum field theory contain ultra-violet divergences (infra-red divergences are discussed in chapter 6). For example in euclidean φ^4 theory the tadpole diagram

$$= \frac{\lambda}{(2\pi)^d} \int d^d q \, (q^2 + m^2)^{-1} > k_1 \int_{k_2}^{\infty} dr \, r^{(d-2)-1},$$

for some $k_1, k_2 \in \mathbb{R}_+$, and the diagram is therefore divergent for $d \geqslant 2$. A theory is renormalised by first regularizing it[3] introducing a convergent expression for the loop integrals depending on some cut-off parameter ϵ, so that the original expression is obtained in the limit where the cut-off is removed $\epsilon \to 0$. The divergences are then absorbed into bare couplings by redefinitions of the coupling constants of the theory so that the correlation functions, and thus physical quantities, are finite when $\epsilon \to 0$, but the bare couplings diverge. Here a wave function renormalisation is used, $\varphi_0 = Z_1^{1/2}(\mu)\varphi_R$, $\varphi_0^2 = Z_2(\mu)(\varphi^2)_R$, where φ_R, $(\varphi^2)_R$ and λ are the renormalised and φ_0, φ_0^2, λ_0 are the bare quantities,

$$\mathcal{L}_R(\varphi_R, \lambda) = \mathcal{L}(\varphi_R, \lambda) + \mathcal{L}_{c.t.}(\varphi_R, \lambda) = \mathcal{L}_0(\varphi_0, \lambda_0). \tag{3.1}$$

A theory is called renormalisable if it is possible to absorb the infinities into a finite number of couplings at every order in the perturbation. Studying the loop diagrams the requirement of renormalisability of a scalar field interaction $\lambda\varphi(x)^n$ becomes

$$[\lambda] \geqslant 0. \tag{3.2}$$

The interactions of the theory (like the self-interaction above) are incorporated into the renormalised couplings, and it is these couplings that have to be compared with measured quantities. The bare couplings are the ultra-violet approximation of the

[3]There is an operator formulation of perturbative quantum field theory where no regularization and renormalisation is performed. In the (Boguliubov-Shirkov-Epstein-Glaser) Causal approach (see e.g. [31]) ultra-violet divergences are removed by proper definitions of products of operator valued tempered distributions in position space.

renormalised couplings in the sense that no interactions are here taken into account[4].

The counter terms $\mathcal{L}_{c.t.}$ are only specified up to finite terms in the limit $\epsilon \to 0$. This means that the theory does not have any predictive power until the parameters of the lagrangian, i.e. the couplings and the constant in the kinetic term, have been fixed by comparing with experimental values measured at a certain length scale. Choosing a different renormalisation condition by fixing the parameters at a different scale then corresponds to a different finite renormalisation of the theory.

The regularization introduces an additional mass parameter μ into the theory, e.g. the lattice spacing $1/\mu$, or in Pauli-Villars: $\frac{1}{q^2+m^2} \to \frac{1}{q^2+m^2} - \frac{1}{q^2+\mu^2}$, and in dimensional regularization $g = \mu^{-\epsilon}\lambda$, where $[g] = 0$. Changing μ corresponds to a finite renormalisation and can be seen as choosing a different renormalisation condition. Physical quantities must be independent of the choice of renormalisation condition and this invariance in the choice of μ leads to the Callan–Symanzik equation [32, 33] which shows how the system changes under scaling. Some references on renormalisation in quantum field theory and some of its history are [22, 11, 5, 34, 35, 36, 37, 38, 39, 40, 41], in which more detailed references can be found.

3.1.1 The Callan–Symanzik equation

The simplest case with one coupling is considered, the equation directly generalizes to more couplings. From the wave-function renormalisation the bare and renormalised correlation functions are related as

$$G_n(p_i, g(\mu), \mu) = \langle \varphi_R(p_1) \cdots \varphi_R(p_n) \rangle = Z_1^{-n/2}(\mu) G_{0n}(p_i, g_0, \epsilon), \tag{3.3}$$

[4]The ultra-violet divergences of the bare couplings then indicate that the perturbative theory is only an effective theory valid below some fundamental cut-off Λ_0, and the renormalisation is then a way of expressing the effective physical degrees of freedom at a length scale much below Λ_0, this is exactly the philosophy in the wilsonian view of the renormalisation group.

where the overall momentum conservation has been extracted so that $G_n(p_i, g, \mu)$ is defined for $\sum_{i=1}^n p_i = 0$ [5]. It then follows that

$$0 = \mu\frac{d}{d\mu}G_{0n}(p_i, g_0, \epsilon) = Z_1^{n/2}\left(n\frac{\mu}{2Z_1}\frac{dZ_1}{d\mu} + \mu\frac{\partial}{\partial\mu} + \mu\frac{\partial g}{\partial\mu}\frac{\partial}{\partial g}\right)G_n(p_i, g, \mu). \tag{3.4}$$

$G_n(p_i, g, \mu)$ is homogeneous with respect to the mass dimension[6] (2.19)

$$G_n(tp_i, g, t\mu) = t^{[G_n(p_i)]}G_n(p_i, g, \mu), \tag{3.5}$$

so that

$$\left(t\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial\mu} - [G_n(p_i)]\right)G_n(tp_i, g, \mu) = 0, \tag{3.6}$$

inserting into (3.4) leads to

$$\left(-t\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g} + n\gamma + [G_n(p_i)]\right)G_n(tp_i, g, \mu) = 0, \tag{3.7}$$

with

$$\beta(g) = \mu\frac{dg}{d\mu}|_{g_0,\epsilon}, \qquad \gamma(g) = \frac{\mu}{2Z_1}\frac{dZ_1}{d\mu}\,|_{g_0,\epsilon}. \tag{3.8}$$

β is the beta function and γ the anomalous field dimension. We call (3.7) the Callan-Symanzik equation and (3.8) the renormalisation group equation[7].

In the Callan–Symanzik equation the invariance of the physics (the S-matrix) in the choice of renormalisation condition is transformed into describing the scaling behaviour of the correlation functions. The correlation functions change as $(n\gamma + [G_n(p_i)])G_n(tp_i, g, \mu)$ under a simultaneous change in t and g in (3.7) and this suggests that the solution has the form $G_n(tp_i, g, \mu) = h(t)G_n(p_i, \bar{g}(t, g), \mu)$ for some

[5] Using translation invariance we can then write the correlator as $\tilde{G}_n(p_1, \ldots, p_n) = \int d^dx_1 e^{ix_1(\sum_{i=1}^n p_i)} \int d^dx_2 \cdots \int d^dx_n e^{i(\sum_{k=2}^n p_k x_k)} G_n(0, x_2, \ldots, x_n) = (2\pi)^d\delta(\sum_{i=1}^n p_i)G_n(p_i)$.

[6] For a scalar field $[G_n(p_i)] = n[\varphi(x)] - nd + d = d - \frac{n}{2}(d+2)$.

[7] (3.7) is also called the renormalisation group equation, and sometimes the word Callan–Symanzik equation is reserved for the case with composite operator insertions, these are considered in chapter 6.

function $h(t)$, where the running coupling constant $\bar{g}(t,g)$ satisfies that $\bar{g}(1,g) = g$ and $t\frac{\partial \bar{g}}{\partial t} = \beta(\bar{g})$ [8]. Inserting this into (3.7) then gives

$$G_n(tp_i, g, \mu) = t^{[G_n(p_i)]} e^{n\int_1^t \frac{\gamma(\bar{g}(t'))}{t'}dt'} G_n(p_i, \bar{g}(t,g), \mu). \tag{3.9}$$

This shows that the scaled correlators with coupling g are the same up to a factor as the unscaled correlators with the running coupling $\bar{g}(t,g)$ at the scale t.

The renormalisation group in this way specifies a one parameter flow in the coupling constant space that determines how the renormalised couplings must change, when the length scale changes, to leave the physics invariant. The cut-off ϵ is kept fixed along the renormalisation group flow; an equivalent view of the renormalisation group is to keep the renormalised couplings fixed by comparing them to measured quantities at a certain scale, and then observe how the bare couplings must be varied in the limit when the cut-off vanishes so that the physical quantities stay unchanged.

The renormalisation group shows how the effective microscopic dynamics, expressed in terms of $\bar{g}(t)$, depends on the length scale where the theory is probed. This dynamical scale dependence is one of the most important consequences of the renormalisation group because it matches what we see in nature, namely that the behaviour of physical systems depends on the scale where they are studied [36]. Physics on different length scales decouples, that is, the dynamics of a system and the relevant parameters needed for its description depend on the scale where it is studied. The value of a physical charge for example depends on the length scale where it is probed because of screening, and the parameters needed to describe a thermo-dynamical system differ from the ones needed to describe its microscopic nuclear properties.

The β-function can be calculated perturbatively and the correlators satisfying (3.7) are called renormalisation group improved, as (3.7) is an additional condition on the perturbative calculation. The renormalisation group equation shows that the

[8] Hence integrating this equation and differentiating with respect to g shows that $\frac{\partial \bar{g}}{\partial g}|_t = \frac{\beta(\bar{g})}{\beta(g)}$, and it also follows that $\int_1^t dt' \frac{\gamma(\bar{g}(t'))}{t'} = \int_g^{\bar{g}} dg' \frac{\gamma(g')}{\beta(g')}$.

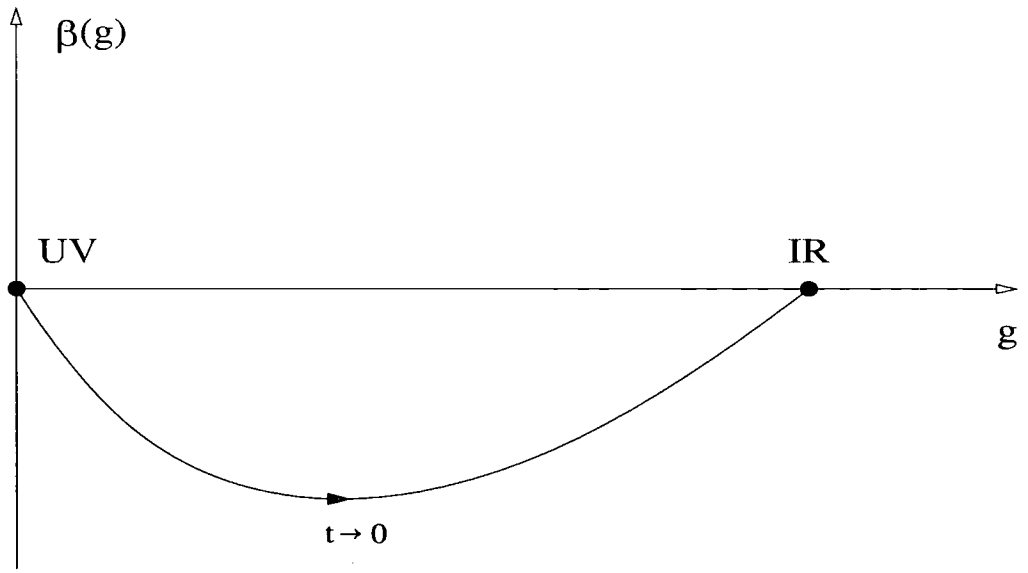


Figure 3.1: β-function with an ultra-violet and an infra-red fixed point.

scaling behaviour is completely determined by the β-function; fixed points of the renormalisation group correspond to scale invariant theories because the effective couplings are the same at all length scales.

We are interested in theories with a β-function with an ultra-violet fixed point where the length scale vanishes, and an infra-red fixed point where the length scale diverges, see figure 3.1. The renormalisation group then flows from the ultra-violet to the infra-red fixed point in the coupling constant space.

If the coupling constant space, or theory space, has the coordinates g^i then the renormalisation group flow is a one parameter transformation along the vector field $\beta = \beta^i \frac{\partial}{\partial g^i}$. In [42, 43] the correlation functions in a theory were argued to be tensor fields in coupling constant space and the renormalisation group flow can then be written as the Lie derivative [44] with respect to the vector field β. It then seems natural and important to try to further develop the geometrical understanding of the renormalisation group by introducing a metric, connection and curvature in theory space, also because geometrical principles have proved very important

when formulating general relativity and quantum gauge theories. In [45, 46, 43, 42] the geometrical properties of the renormalisation group have been studied, but no general conclusions have yet been reached about the geometry. The hope is that geometrical considerations will lead to global non-perturbative constraints on the renormalisation group flow and the structure of theory space. This is the case in two dimensions for unitary theories where Zamolodchikov's c-theorem, which we will prove in chapter 5, shows the irreversible nature of the renormalisation group flow. Another example where the renormalisation group flow has been studied non-perturbatively is the Seiberg–Witten solution to $N = 2$ supersymmetric Yang-Mills, where the exact β-function has been determined together with a metric on coupling constant space [47, 48, 49].

3.1.2 Composite operators and the operator product expansion

In a renormalisable theory correlation functions $\langle \mathcal{O}(x)\phi(y_1)\cdots\phi(y_n)\rangle$ with insertions of local composite operators $\mathcal{O}(x) = \partial_1\cdots\partial_l\phi(x)^k$ will have divergences. These correlators can be calculated by adding a term to the lagrangian $\mathcal{L} \to \mathcal{L} + t(x)\mathcal{O}(x)$ and then use the normal Feynman rules to evaluate $Z^{-1}\frac{\delta}{\delta t(x)}\frac{\delta^n}{\delta J(y_1)\cdots\delta J(y_n)}Z[J,t]|_{J=t=0}$. This corresponds to summing over the usual Feynman diagrams, but with one additional vertex corresponding to $\mathcal{O}(x)$, this vertex has no self-contractions [27]. Some of these diagrams will be divergent and the composite operator is renormalised by adding counter terms $t(x)\mathcal{O}(x) \to t(x)(\mathcal{O}(x) + \sum_i \tilde{Z}_{\mathcal{O}i}\mathcal{O}_i)$. For the theory to be renormalisable the constraint (3.2) shows that[9] $[\tilde{Z}_{\mathcal{O}i}] \geqslant 0$ so that the operator $\mathcal{O}(x)$ only mixes under renormalisation with operators of lower or equal mass dimension.

[9] This argument is for an operator without any derivatives, a general argument is given in [22]; if G is a 1PI graph for a counter term $\tilde{Z}_{\mathcal{O}i}\mathcal{O}_i$ with D_i derivatives then (3.2) implies that the degree of divergence of G: $\delta(G) = [G] - [couplings] \geqslant [G]$, and $[G] = D_i + [\tilde{Z}_{\mathcal{O}i}]$. For $\tilde{Z}_{\mathcal{O}i}\mathcal{O}_i$ to be a counter term then $\delta(G) \geqslant D_i$, and together with the above relation this again shows that $[\tilde{Z}_{\mathcal{O}i}] \geqslant 0$.

The renormalisable composite operator can then be written as

$$\mathcal{O}(x)_R = \sum_i Z_{\mathcal{O}i}\mathcal{O}_i, \quad [\mathcal{O}_i] \leqslant [\mathcal{O}]. \tag{3.10}$$

The correlation functions $\langle \mathcal{O}_1(x_1)_R \cdots \mathcal{O}_i(x_i)_R \mathcal{O}_j(x_j)_R \cdots \mathcal{O}_n(x_n)_R \rangle$ are not defined on the diagonal $x_i = x_j$. We will be interested in the short distance ultra-violet limit where $x_i \to x_j$, this situation is described by the operator product expansion (or short distance expansion). To simplify the notation we will here write the renormalised composite operator as simply $\mathcal{O}(x)$.

The operator product expansion was first conjectured by Wilson in [50] (and independently by Kadanoff in statistical mechanics [51]), it replaces a product of operators by a linear combination of local operators, it has been proved to hold in perturbation theory (see e.g. [22, 52]) and also argued to be true non-perturbatively using the path integral [52]. In the classical theory the product $\mathcal{O}_i(x_i)\mathcal{O}_j(x_j)$ can be calculated for $x_i \to x_j$ using the Taylor series $\mathcal{O}_i(x_i)\mathcal{O}_j(x_j) = \mathcal{O}_i(x_j)\mathcal{O}_j(x_j) + (x_i - x_j)^\mu(\partial_\mu \mathcal{O}_i(x_j))\mathcal{O}_j(x_j) + \cdots$. In quantum theory $\mathcal{O}_i(x_j)\mathcal{O}_j(x_j)$ is singular, it follows for example from (2.20) that $\langle \phi(sx)\phi(0) \rangle \sim s^{-2\Delta}$ for $s \to 0$, and the Taylor series must be replaced by an asymptotic series for $x_i \to x_j$:

$$\mathcal{O}_i(x_i)\mathcal{O}_j(x_j) \sim \sum_k C_{ijk}(x_i - x_j)\mathcal{O}_k(x_j). \tag{3.11}$$

From the scaling behaviour (2.20) it follows that $C_{ijk}(s(x_i - x_j)) \sim s^{-\Delta_i - \Delta_j + \Delta_k}$ for $s \to 0$, and since there is only a finite number of operators[10] $\mathcal{O}_k$ with $\Delta_k < \Delta_i + \Delta_j$ there will only be a finite number of singular terms. The dominant contribution in (3.11) for $x_i \to x_j$ is therefore given by the operators with lowest scale dimension. For scalar fields we will write (3.11) as

$$\mathcal{O}_i(x_i)\mathcal{O}_j(x_j) \sim \sum_k |x_i - x_j|^{-\Delta_i - \Delta_j + \Delta_k} \tilde{C}_{ijk}\mathcal{O}_k(x_j), \tag{3.12}$$

where $\tilde{C}_{ijk}$ is the operator product expansion coefficients.

[10] This is the case for all known theories and it can be taken as one of the defining properties of quantum field theory [53].

3.2 Linearizing the renormalisation group and universality

The structure of the coupling constant space is determined by the renormalisation group flow, the qualitative behaviour of which is given by the stability of the fixed points (in the absence of limit cycles and strange attractors). The stability of a critical point, i.e. if it is attractive or repulsive, can be found by linearizing the renormalisation group near the fixed point.

It is useful to adopt the wilsonian viewpoint where the renormalisation group is implemented by integrating out the high energy degrees of freedom not relevant at lower energy scales. This procedure can also be used in statistical mechanics and the renormalisation group has made an important connection between relativistic quantum field theory and critical phenomena, which has led to a better understanding of divergences in quantum field theory and universality in critical phenomena.

In statistical mechanics a renormalisation group transformation is given as a coarse graining of the system focusing on the long range effects that are the important ones in critical phenomena. For a spin system on a (fixed) lattice of spacing a this can be done by Kadanoff blocking [11]. The blocking, or renormalisation transformation, will then generate new couplings between the blocked spins. This is similar to the renormalisation described by (3.8) where the cut-off is held fixed and the renormalisation group describes the dependence on the renormalisation scale. Equivalently, the lattice can be used as a regulator by adjusting the (bare) couplings so that physical quantities, e.g. the correlation length, are held fixed while the lattice spacing a goes to zero. This is again analogous to the quantum field theory case where $\epsilon \to 0$. The continuum theory defined in this limit then gives a field theory with a non-perturbative regulator, namely the lattice, and non-perturbative effects can in principle be studied in this way.

A continuum formulation is also obtained with a momentum space cut-off, either

[11] The lattice is divided into blocks of size s^d and the spin field in the block is defined as the average value, length are given in terms of the lattice spacing so they must be rescaled $x' = x/s$.

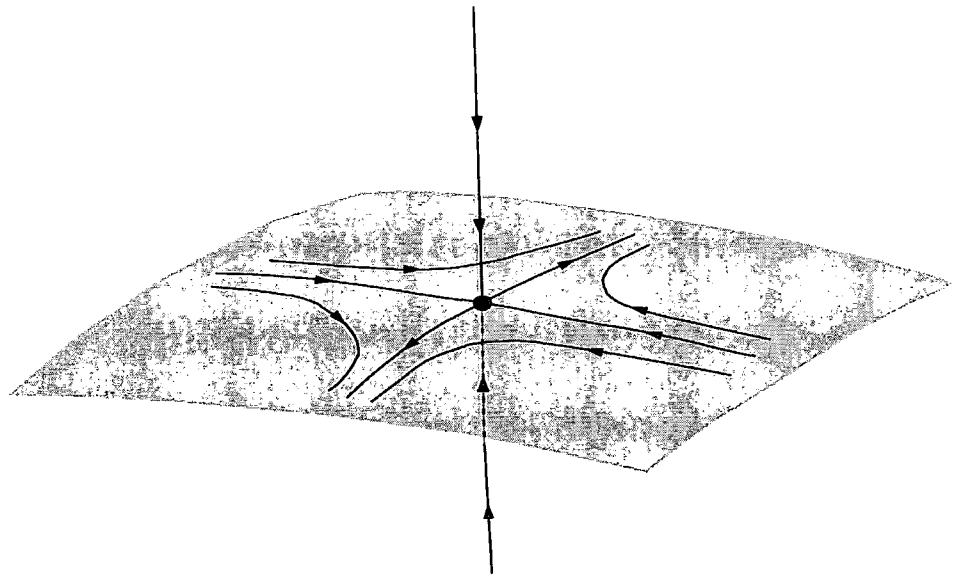


Figure 3.2: A renormalisation group fixed point in coupling constant space with one relevant and two irrelevant operators.

sharp or smooth. One prescription is to include only momentum modes $\phi(k)$ with $|k| < \Lambda$ in the effective partition function $Z = \int \mathcal{D}\phi(k)|_{|k|<\Lambda} e^{-S[\phi]}$. The equivalent of the blocking transformation in this formulation is to integrate out the momentum modes with $\Lambda/s < |k| < \Lambda$ followed by a rescaling $k' = sk$, $\phi'(k') = s^{\Delta-d}\phi(k)$ (which is the momentum space version of (2.18): $\phi'(x/s) = s^{\Delta}\phi(x)$). This transformation again corresponds to a flow in the effective coupling constant space $\lambda \to \lambda'(s)$ and the β-function is then defined as before $\beta(\lambda) = s\frac{d\lambda}{ds}$.

3.2.1 Linearizing the renormalisation group

Assume that there is a fixed point of the renormalisation group, i.e. a scale invariant theory in the coupling space $S_*[\lambda_*]$. In the vicinity of the fixed point the action can be written as $S[\lambda(s)] = S_* + \Delta S[\lambda(s)]$ where s is the scale parameter. Linearizing

the renormalisation group transformation around the fixed point gives

$$s\frac{d}{ds}\Delta S = \mathsf{L}\Delta S \tag{3.13}$$

where L is a linear operator depending on S_* [38]. If L has a discrete spectrum ΔS can be expanded in terms of the eigenoperators with eigenvalue y_i: $\Delta S = \sum_i \lambda_i(s)\mathcal{O}_i$ so that to lowest order

$$s\frac{d}{ds}\lambda_i(s) = y_i\lambda_i(s), \quad \lambda_i(s) = s^{y_i}\lambda_i(1). \tag{3.14}$$

y_i is the renormalisation group eigenvalue for the operator $\mathcal{O}_i$ at the fixed point S_*. $\mathcal{O}_i$ is called relevant for $\mathrm{Re}(y_i) > 0$ where the coupling grows away from the fixed point, irrelevant when $\mathrm{Re}(y_i) < 0$ where the coupling vanishes along the renormalisation group flow, and marginal for $y_i = 0$ where higher order corrections need to be calculated, see figure 3.2. The critical surface for a fixed point is defined as all points in the coupling constant space which flows to the fixed point under the renormalisation group flow, and is therefore locally spanned by the irrelevant operators. The co-dimension of the critical surface equals the number of relevant operators for the fixed point, and the corresponding couplings are the ones that need to be adjusted to reach the critical surface.

The massless free theory is a fixed point of the renormalisation group, the gaussian fixed point. This fixed point is very important because perturbative quantum field theory is defined as a perturbation around it. The free theory is finite and can be calculated exactly. It is a fixed point because the massless free action is scale invariant, and for a quadratic action there is no mixing of different momentum modes, so that the high momentum modes decouple without changing the quadratic coupling. For example, for a scalar field with scaling dimension $\Delta = \frac{d-2}{2}$:

$$\begin{aligned} Z &= \int \mathcal{D}\phi(q)|_{|q|<\Lambda} e^{-\frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\phi(q)q^2\phi(-q)} = k_1 \int \mathcal{D}\phi(q)|_{|q|<\Lambda/s} e^{-\frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\phi(q)q^2\phi(-q)} \\ &= k_2 \int \mathcal{D}\phi'(q')|_{|q'|<\Lambda} e^{-s^{d-2\Delta-2}\frac{1}{2}\int \frac{d^dq'}{(2\pi)^d}\phi'(q')q'^2\phi'(-q')}, \end{aligned}$$

for some $k_1, k_2 \in \mathbb{R}$, and it follows that the massless free action is invariant under a renormalisation group transformation. In this gaussian case the change in the

couplings under the renormalisation group comes solely from the rescaling of the action. For a non-quadratic interaction term $\lambda \int d^d x \mathcal{O}(x)$ the momentum modes mix, and the couplings will consequently change when integrating out the high momentum modes. This effect however must be of order $O(\lambda)$ as it vanishes for the gaussian theory, so the lowest order change in λ is again given by rescaling the action $x \to x' = x/s$, $\mathcal{O}(x) \to \mathcal{O}'(x')$: $\lambda \int d^d x \mathcal{O}(x) = \lambda s^{d-\Delta} \int d^d x' \mathcal{O}'(x')$, where $\mathcal{O}(x)$ has scaling dimension Δ. The renormalisation group eigenvalue for the gaussian fixed point is therefore $y = d - \Delta$ which is equal to $[\lambda]$. The non-renormalisable interactions with $[\lambda] < 0$ are thus damped by $s^{-|y|}$ on the renormalisation group flow. On scales $\Lambda \ll \Lambda_0$ the only non-zero parameters are therefore the finite number (in $d > 2$) of couplings in front of the renormalisable interactions, irrespective of the nature of the underlying fundamental theory at the scale Λ_0, and the theory therefore has a renormalisable form. A similar argument was given by Polchinski in [54] (see also [5]) directly using dimensional analysis and an effective lagrangian[12] [13].

From the solution to the Callan–Symanzik equation (3.9) it follows that the infra-red limit of a correlation function is proportional to the same correlator evaluated at the infra-red fixed point. This shows that all theories flowing to the same infra-red renormalisation group fixed point have the same long distance behaviour irrespective of the microscopic dynamics, this is called universality. Fixed points are determined by the symmetries of the action and the space-time dimension and are characterized by quantities like the renormalisation group eigenvalues from which such universal properties as the critical exponents, that are discussed in chapter 6, can be derived. Quantum field theories can then be grouped into different universality classes, with

[12] In [55, 11] the renormalisation group is seen as an example of functional self-similarity, where a solution to a physical problem, imposed in terms of a differential equation, has the same functional form under a variation of the boundary conditions. If for example the transformation $R_t : \{x \to x' = x/t, g \to g' = \bar{g}(t,g)\}$ satisfy the group composition $R_t R_{t'} = R_{tt'}$ then $\bar{g}(x,g) = \bar{g}(x/t, \bar{g}(t,g))$, and this shows that $x\frac{\partial \bar{g}(x,g)}{\partial x} = \beta(\bar{g}(x,g))$ where $\beta(\bar{g}(x,g)) = \frac{\partial}{\partial t}\bar{g}(t,g)|_{t=1}$, which are similar to the expressions above.

[13] A correspondence between Feynman diagrams, knots, Hopf algebras and non-commutative geometry has been discussed in [56, 57]. This seems to be an important development which may lead to a more fundamental understanding of the perturbative series and the renormalisation group.

the same critical properties, according to which fixed point they flow to under the renormalisation group, this has been verified experimentally for a number of different materials and fluids. If all theories have an infra-red fixed point (as would be the case if the renormalisation group flow could be shown to be a gradient flow with the interpretation as in the c-theorem) the space of quantum field theories would then be classified by determining all fixed points, i.e. all scale invariant theories.

4

Calculating the infra-red limit using analyticity

In this thesis we are interested in quantum field theories that have a β-function as in figure 3.1 with an ultra-violet and an infra-red fixed point of the renormalisation group.

Perturbation theory is the main calculational tool in obtaining physical quantities from quantum field theory. Here the action is divided into a free gaussian part and an interaction, and using that the path integral is well defined for gaussian integration the full interacting theory can then be written as a formal power series in the coupling constants that define the interaction[1]. This power series in the couplings will often at most be asymptotic, hence the finite number of terms obtained in a perturbative calculation only describe the exact physics correctly in a small region around the gaussian fixed point $g^i = 0$, this is called the perturbative region.

The ability to calculate anything beyond this perturbative region is scarce and limited, and it is therefore important to develop new calculational tools that can reach beyond the perturbative region towards the infra-red.

Conventionally the infra-red limit is studied by extrapolating perturbation theory

[1] A perturbative expansion around a non-gaussian fixed point is possible if the correlation functions in the critical theory are known, we will see an example of this in chapter 5.

from the ultra-violet region using the renormalisation group. In this chapter we will show how to improve this approach with information derived from the analyticity properties of the correlation functions in the theory. The analyticity follows from the Källen–Lehmann spectral representation that is valid for all quantum field theories. We are considering Lorentz invariant and unitary theories in an arbitrary number of dimensions.

Using the analyticity in a complex scale parameter the infra-red limit of the correlation functions is written as an integral in the ultra-violet region. This integral over the complex scale parameter is shown to be the Borel transform of order k of the correlation function, where the order is related to the analytical continuation chosen for the scale parameter. In chapters 5 and 6 we will use this method to derive the infra-red limit of different physical quantities in two and three dimensions.

First the Källen–Lehmann spectral representation is derived, and this is then used to determine the analyticity domain of correlation functions. The perturbative expansion in the coupling is then discussed together with asymptotic series and the Borel transformation, and we then show how to obtain the infra-red limit as an integral in the ultra-violet region.

4.1 Spectral representation of correlation functions

The Källen–Lehmann spectral representation [58, 59] states that the two point correlation function in an interacting theory can be written as a sum over intermediate free states of different mass. It follows from general principles of quantum field theory and can be derived by inserting a resolution of the identity in terms of eigenstates of the momentum P^μ [60]. Let us show it for for a scalar field $\phi(x)$, there is a similar representation for correlators of higher spin fields and we will show it for the spin two energy-momentum tensor in chapter 5. The unitary irreducible representations of the Poincaré group are characterized by the invariants of the group (the Casimirs) which are the mass μ and the spin s. We consider the $s = 0$ case and the identity

can therefore be written as a sum over the different masses [60]

$$\mathbf{1} = \int_0^\infty d\mu^2 \, \mathcal{P}_{P^2=\mu^2} = \int_0^\infty d\mu^2 \int d^d p \; \delta(p^2 - \mu^2) \; \theta(p^0) \; \mathcal{P}_{P^\nu = p^\nu} \tag{4.1}$$

where $\mathcal{P}$ is a projection operator onto eigenstates of P^2 and P^μ. Inserting this into the two point function for $x^0 < 0$ gives, using Cauchy's theorem[2]

$$\begin{aligned}
\langle \phi(x)\phi(0) \rangle &= \langle 0|T\{\phi(x)\phi(0)\}|0\rangle = \langle 0|\phi(0)\phi(x)|0\rangle = \langle 0|\phi(0)e^{iPx}\phi(0)|0\rangle \\
&= \int_0^\infty d\mu^2 \int d^d p \; \delta(p^2 - \mu^2)\theta(p^0) e^{ipx} \; \langle 0|\phi(0)\mathcal{P}_{P^\nu=p^\nu}\phi(0)|0\rangle \\
&= \int_0^\infty d\mu^2 \sigma(\mu^2) \int \frac{d^d p}{(2\pi)^{d-1}} \delta(p^2 - \mu^2) \; \theta(p^0) e^{ipx} \\
&= \int_0^\infty d\mu^2 \sigma(\mu^2) \int \frac{d^{d-1} p}{(2\pi)^{d-1}} \frac{1}{2E_p} e^{i(E_p x^0 - \bar{x}\bar{p})} \\
&= \int_0^\infty d\mu^2 \sigma(\mu^2) \int \frac{d^{d-1} p}{(2\pi)^{d-1}} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{dp^0}{2\pi i} \frac{(-1)}{p^2 - \mu^2} \; e^{ipx} \; \theta(p^0) \\
&= \int_0^\infty d\mu^2 \sigma(\mu^2) \int \frac{d^d p}{(2\pi)^d} \; \frac{i}{p^2 - \mu^2 + i\epsilon} \; e^{ipx} = \int_0^\infty d\mu^2 \sigma(\mu^2) \; G_0(x,\mu).
\end{aligned} \tag{4.2}$$

Here $d\mu^2\sigma(\mu^2) = d\mu^2(2\pi)^{d-1}\langle 0|\phi(0)\mathcal{P}_{P^\nu=p^\nu}\phi(0)|0\rangle|_{p^2=\mu^2}$ is the spectral density at the scale μ, and $G_0(x,\mu)$ is the free Feynman propagator which in momentum space is given by the distribution $G_0(p,\mu) = \frac{i}{p^2-\mu^2+i\epsilon}$ (i.e. $\epsilon \to 0_+$ is taken in integrals). For $x^0 > 0$ the same result is obtained, and the full interacting propagator can be written as: $G(x) = \int_0^\infty d\mu^2\sigma(\mu^2)G_0(x,\mu)$, which is the Källen–Lehmann spectral representation. If the projection operator is written $\mathcal{P}_{P^\nu=p^\nu} = |\bar{p},\mu\rangle\langle\mu,\bar{p}|$ it follows that for a unitary theory the spectral density is non-negative $\sigma(\mu^2) = (2\pi)^{d-1}|\langle 0|\phi(0)|\bar{p},\mu\rangle|^2 \geq 0$ [3]. The calculation with euclidean signature is equivalent and in momentum space

[2] Here $E_p = \sqrt{|\bar{p}|^2 + \mu^2}$ so that $p^2 - \mu^2 = (p^0)^2 - E_p^2$. From the identity $\delta(f(x) - f(a)) = \frac{1}{|f'(a)|}\delta(x-a)$ it follows that $\int d^d p \delta(p^2-\mu^2)\theta(p^0) = \int d^{d-1}p \frac{1}{2E_p}$, which also shows that $\frac{d^{d-1}p}{2E_p}$ is the Lorentz invariant measure on the upper sheet $p^0 > 0$ of the hyperboloid $p^2 = \mu^2$.

[3] The Källen–Lehmann spectral representation is in [61, 5] derived by explicitly constructing the projection operators, and in [62] in terms of actiomatic field theory.

it becomes

$$\langle\phi(p)\phi(0)\rangle_E = \int_0^\infty d\mu^2 \sigma(\mu^2) \frac{1}{p_E^2 + \mu^2}. \tag{4.3}$$

From now on we will use euclidean signature unless otherwise stated and write $\langle\phi(p)\phi(0)\rangle_E = \langle\phi(p)\phi(0)\rangle$, $p_E^2 = p^2$ and $S_E = S$.

4.1.1 Analyticity of correlations functions

The spectral representation (4.3) shows that the correlation functions have an analytical continuation. Introducing the complex scale factor $s \in \mathbb{C}$

$$\langle\phi(sp)\phi(0)\rangle = \int_0^\infty d\mu^2 \, \sigma(\mu^2) \, \frac{1}{s^2p^2 + \mu^2}$$

then shows that $\langle\phi(sp)\phi(0)\rangle$ is analytic in the positive half-plane $\mathrm{Re}(s) > 0$. Of course, a different continuation can be chosen and generally physical quantities will not be analytic in the positive half-plane, but in some sector $S(\alpha)$ of the complex plane defined as

$$S(\alpha) = \{z = re^{i\phi} \mid 0 < r < \infty, \; -\tfrac{\alpha}{2} < \phi < \tfrac{\alpha}{2}\}, \tag{4.4}$$

see figure 4.1. Note that $0 \notin S(\alpha)$. We will consider functions which are bounded at the origin, i.e. $\exists C > 0 : \lim_{z\to 0} |f(z)| \leq C$ for $z \in S(\alpha)$.

It is very difficult to go beyond the perturbative approximation. One way is to use the lattice as a non-perturbative regulator as discussed in chapter 3, and then do numerical calculations of the path integral e.g. using Monte Carlo methods. Another has been to study particular theories, or models, with features that have allowed considerations beyond perturbation theory. Examples are theories with large symmetries, like infinite dimensional affine Lie algebras and supersymmetry, that have non-perturbative consequences; or use duality transformations between specific theories, that maps one theory into another weakly coupled theory where perturbation theory is applicable. Non-perturbative results are also obtained by studying dominant contributions to path integrals in form of instantons and solitons that are not given by free theories.

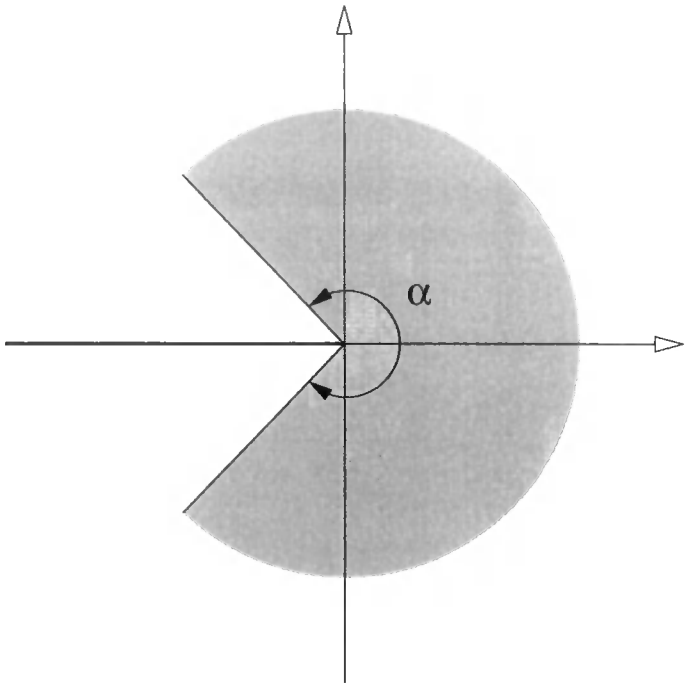


Figure 4.1: The sector $S(\alpha)$ in the complex plane.

Here we will use the analytical properties of the exact correlation functions that are generic for all quantum field theories, to improve the perturbative calculations of the infra-red limit of correlators.

4.2 The perturbative series

The euclidean signature version of the correlators (2.10) can be written as

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \frac{Z[J]}{Z[0]}|_{J=0}, \qquad Z[J] = \int \mathcal{D}\phi e^{-S + \int d^d x J(x)\phi(x)},$$

and they can be evaluated by splitting the action into a quadratic free part and an interaction: $S = S_0 + g\ \mathcal{O}(\phi)$. The free theory has a well defined path integral measure $\mathcal{D}\phi e^{-S_0}$ (the Wiener measure [63]) where correlation functions can be calculated by gaussian integration. The full interacting theory can then be calculated

by a formal power series expansion in the couplings. Wick's theorem

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = e^{\frac{1}{2}\int d^dx d^dy \frac{\delta}{\delta\phi(x)} G_0(x-y) \frac{\delta}{\delta\phi(y)}} \phi(x_1)\cdots\phi(x_n) e^{-g\mathcal{O}(\phi)+\int d^d\tilde{x} J(\tilde{x})\phi(\tilde{x})}|_{\phi=0,J=0}$$
$$= \sum_{n=0}^{\infty} g^n H_n$$

leads to the perturbative expansion in terms of Feynman diagrams. After renormalisation each term in this series is finite, but the series is not necessarily convergent. It was first argued by Dyson in [64] that for QED this perturbative series can at most be asymptotic. If the series is convergent then it is analytic at $g = 0$ and will therefore also converge for some negative coupling $-g$ ($g > 0$), but then like charges will attract and the vacuum will disintegrate by spontaneous polarization.

It is important to understand how well the perturbative series represents the exact physics, and how much information can be extracted from it, since a large part of actual calculations are done using perturbative techniques.

In [65] by counting the number of different Feynman diagrams it was conjectured that the perturbative coefficients grow like $H_n \sim n!$. This was confirmed for large n by Lipatov [66] using a steepest decent approximation of the functional integral with non-trivial saddle points that have a finite euclidean action. For ϕ^4 theory these instanton approximations were calculated in [67][4], and the method has been generalized to theories with fermions, QED and non-abelian gauge theories, see [70] for a review and references.

Let $H(g) = \sum_{n=0}^{\infty} H_n g^n$ be the perturbative series for a physical quantity H. If $H(g) = o(|g|)$ when $|g| \to \infty$ and $H(g^*) = H(g)^*$, where g^* is the complex conjugate, then the once subtracted dispersion relation becomes

$$H(g) = H(0) + \frac{g}{\pi}\int_{-\infty}^{0} dg' \, \frac{\mathrm{Im}(H(g'))}{g'(g'-g)}. \tag{4.5}$$

This follows from Cauchy's theorem by integrating $\frac{gH(g')}{2\pi i g'(g'-g)}$ over a connected contour consisting of a large circle at infinity together with contours running just below

[4] An instanton is a solution to the classical euclidean equations of motion with finite euclidean action. See [68, 69] for an introduction to instantons and solitons.

and above the negative real axis from $-\infty$ to 0 and an infinitesimal circle around 0. Applying Cauchy's theorem again the coefficients with $n > 0$ can be written as

$$H_n = \frac{1}{\pi}\int_{-\infty}^{0} dg\, \frac{\mathrm{Im}(H(g))}{g^{n+1}}. \tag{4.6}$$

For large n the integral is dominated by the small negative g values, and in this region instanton approximations of the functional integral show that [70]

$$\mathrm{Im}(H(g)) \approx \frac{e^{1/ag}}{(-ag)^{b+1}}, \qquad g < 0, \quad a > 0. \tag{4.7}$$

As an example [38] consider the integral $I(g) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx e^{-(x^2+gx^4/2)/2}$. This can be continued to negative values of g if the integration contour is rotated so that $\mathrm{Re}(gx^4)$ stays positive leaving the integral finite, i.e. $\mathrm{Arg}(x) = -\mathrm{Arg}(g)/4$. Hence, there are two contours $C_\pm$ where: $\mathrm{Arg}(x) = \pm\pi/4$ for $g = -|g| \pm i0_+$ respectively. The imaginary part then becomes $\mathrm{Im}(I(g)) = (I(-|g| + i0_+) - I(-|g| - i0_+))/2i \propto (\int_{C_+} - \int_{C_-})dx e^{-(x^2+gx^4/2)/2}$. In a steepest decent approximation the contribution from the saddle point at $x = 0$ vanishes and the contours $C_\pm$ can be split into two contours going through the other saddle points: $x = \pm\sqrt{-1/g}$. For small positive couplings $\mathrm{Re}(I(g)) = I(g)$ is therefore dominated by the saddle point $x = 0$ giving the perturbative expression $\mathrm{Re}(I(g)) = 1 + O(g)$, and for negative g the imaginary part is dominated by the non-trivial saddle points $x = \pm\sqrt{-1/g}$ giving the non-perturbative expression $\mathrm{Im}(I(g)) \propto e^{1/4g}$ [38].

The large order behaviour then follows by inserting (4.7) into (4.6) using that $\int_0^\infty dx\, x^{\nu-1}e^{-x} = \Gamma(\nu)$

$$H_n \approx \int_{-\infty}^{0} dg\, \frac{e^{1/ag}}{g^{n+1}(-ag)^{b+1}} \approx (-a)^n\Gamma(1+n+b). \tag{4.8}$$

Stirling's formula [71] $\Gamma(x) = \sqrt{2\pi x}\, e^{-x}x^{x-1/2}(1+O(1/x))$ shows that $\Gamma(x+a)/\Gamma(x) \sim x^a$ for large x and the large order behaviour can therefore be written as

$$H_n \approx (-a)^n n!\, n^b(1 + O(1/n)), \quad n \to \infty. \tag{4.9}$$

The factorial growth then shows that in these cases the perturbative series is indeed asymptotic.

4.3 Asymptotic series

We shall need a general definition of asymptotic series. Following [72] a formal power series $\hat{f}(z) = \sum_{n=0}^{\infty} f_n z^n$ is asymptotic of (Gevrey) order $k^{-1} > 0$ [5] if $\exists C, K > 0$:

$$|f_n| \leq CK^n \Gamma(1 + n/k), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{4.10}$$

Let $f(z)$ be analytic in the sector $S(\alpha)$. $\hat{f}(z)$ is the asymptotic expansion of $f(z)$ of order $k > 0$ for $z \to 0$ if the function $r_f(z, N) = z^{-N}(f(z) - \sum_{n=0}^{N-1} f_n z^n)$ is bounded at the origin. That is, to every closed subset $\bar{S}' \subset S(\alpha)$ there exists $C, K > 0$:

$$|z|^{-N}|f(z) - \sum_{n=0}^{N-1} f_n z^n| \leq CK^N \Gamma(1 + N/k), \quad \forall N > 0, \quad \forall z \in \bar{S}', \tag{4.11}$$

and this is written as $f(z) \cong_k \hat{f}(z)$ in $S(\alpha)$. It follows that $r_f(z, N+1) = z^{-1}(r_f(z, N) - f_N)$ is bounded at the origin so that $f_N = \lim_{z \to 0} r_f(z, N)$ for $z \in S(\alpha)$. The same argument directly shows that every $f(z)$ analytic in $S(\alpha)$ has a unique asymptotic expansion[6].

The question is, if a given asymptotic series $\hat{f}$ is the asymptotic expansion of only one analytic function (if any), so that the perturbative series uniquely represents the exact physics. This is not generally the case, as the function $e^{-cz^{-k}}$ with $c > 0$ shows. $\mathrm{Re}(cz^{-k}) > 0$ in the sector $S(\pi/k)$ so that $e^{-cz^{-k}} = o(z^N)$ for all $N > 0$ and $z \to 0$ from $S(\pi/k)$, which means that $e^{-cz^{-k}} \cong_k \hat{0}$. An analytic function $f(z) \cong_k \hat{f}(z)$ is therefore only defined modulo $e^{-cz^{-k}}$ (which for small z is minute).

For sectors with larger opening $\alpha > \pi/k$ no such functions exists, and only one function analytic in $S(\alpha)$ satisfies $f(z) \cong_k \hat{f}(z)$. In this case all of the exact physics is for small couplings contained in the perturbative series. The injectivity of the asymptotic expansion for large openings was shown in [73] using an extension of the Phragmen-Lindelöf theorem [71], it can also be shown by explicitly constructing $f(z)$ from $\hat{f}(z)$ by the Borel transform [73, 74].

[5] Then $\hat{f} \in \mathbb{C}[[z]]_{1/k}$. $\mathbb{C}[[z]]_{1/k}$ is a differential algebra [72].

[6] If $f(z)$ is single valued, bounded at the origin and $\alpha = 2\pi$ then $f(z)$ is analytic at $z = 0$, hence $\hat{f}(z)$ converges and equals the Taylor series at the origin.

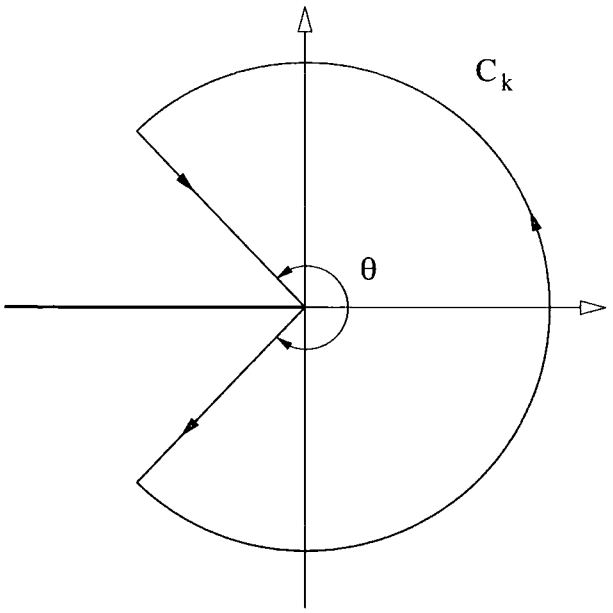


Figure 4.2: The contour C_k where $k > 0$ and $\theta = (\epsilon + \pi)/k$. $\epsilon \in (0, \pi)$ is chosen so that $C_k \subset S(\alpha)$, i.e. $\epsilon < \alpha k - \pi$. The circle has the radius R.

4.3.1 The Borel transform

Let $k > 0$ and $f(z)$ be analytic in $S(\alpha)$ where $\alpha > \pi/k$ and $f(z)$ is bounded at the origin. The Borel transform of $f(z)$ order k is defined by [72]

$$\mathcal{B}_k f(u) = \frac{k}{2\pi i} \int_{C_k} \frac{dz}{z} e^{(u/z)^k} f(z) \tag{4.12}$$

where the contour C_k is given in figure 4.2. If $u \in S(\epsilon/k)$ then $|\mathrm{Arg}((u/z)^k)| > \pi/2$ when z is on the two rays in C_k. The integrand is therefore bounded on C_k and the integral is absolutely convergent and defines an analytic function. From Cauchy's theorem the integral is independent of the choice of radius R and ϵ when $\epsilon < \alpha k - \pi$, and $\mathcal{B}_k f(u)$ is therefore analytic in $S(\alpha - \pi/k)$.

The Borel transform of a formal series $\hat{f}(z) = \sum_{n=0}^{\infty} f_n z^n$ follows from Hankel's integral representation [71]:

$$\frac{1}{\Gamma(1+z)} = \frac{1}{2\pi i} \int_{C_1} \frac{dw}{w} e^{1/w} w^z, \tag{4.13}$$

so that

$$\mathcal{B}_k\hat{f}(u) = \sum_{n=0}^{\infty} \frac{f_n u^n}{\Gamma(1+n/k)}. \tag{4.14}$$

The Borel transform of an asymptotic series of order k^{-1}, i.e. $|f_n| \leq CK^n\Gamma(1+n/k)$, then converges within some positive convergence radius. The inverse of the Borel transform is given by the Laplace transform: $\mathcal{L}_k f(z) = k\int_0^\infty \frac{du}{u} u^k e^{-(u/z)^k} f(u)$. If there exists an analytic function such that $f(z) \cong_k \hat{f}(z)$, then $f(z)$ is uniquely given by $f(z) = \mathcal{L}_k\mathcal{B}_k\hat{f}(z)$ in an appropriate sector and we say that $\hat{f}(z)$ is Borel summable [72]. To perform the integral in the Laplace transform $\mathcal{B}_k\hat{f}(u)$ must be analytically continued to all of the positive real axis, this can be done e.g. by a conformal mapping which we will use in chapter 6.

However, this continuation is not always possible, if e.g. $\mathcal{B}_k\hat{f}(u)$ has a pole on the positive real axis, then the equation $f(z) = \mathcal{L}_k\mathcal{B}_k\hat{f}(z)$ is not satisfied by any $f(z)$. QCD is an example of a theory that has poles on the positive real axis known as renormalons, and it is therefore not Borel summable in this sense (it is possible to distort the contour avoiding the singularities, but this introduces ambiguities [75, 52]). If a theory is not Borel summable in the coupling it still might be Borel summable in the scale parameter s, but the asymptotic expansion in s then has to be obtained by non-perturbative means to describe the full theory because the perturbative series in this case is incomplete. For the instanton approximations in (4.8) the singularity closest to the origin is at the negative axis $g = -a < 0$, and no such problems arise[7]. The φ^4 theory which is considered in chapter 6 has been proved rigorously to be Borel summable for $d < 4$ [77, 78, 79]. Our interest here is not the Borel transform in the coupling, but in the scale parameter s.

[7] A formal way to see that instantons determine the singularities of the Borel transform (of order one) is to formally rewrite the partition function [76]. First rescale the fields so that the action is written as $S[\phi]/g$ (with $g > 0$) and then: $\langle\phi\rangle = Z^{-1}\int \mathcal{D}\phi e^{-S/g}\phi = \int_0^\infty dt \int \mathcal{D}\phi\delta(t-S)e^{-S/g}\phi = \int_0^\infty dt\mathcal{B}_1(t)e^{-t/g}$ where the Borel transform is given by $\mathcal{B}_1(t) = \int \mathcal{D}\phi\delta(t-S[\phi])\phi$. Now using the functional analogue of $\delta(f(x)-f(a)) = \delta(x-a)/|f'(a)|$, shows that $\mathcal{B}_1(t)$ is singular when $\frac{\delta S}{\delta\phi} = 0$, that is, for instantons [76].

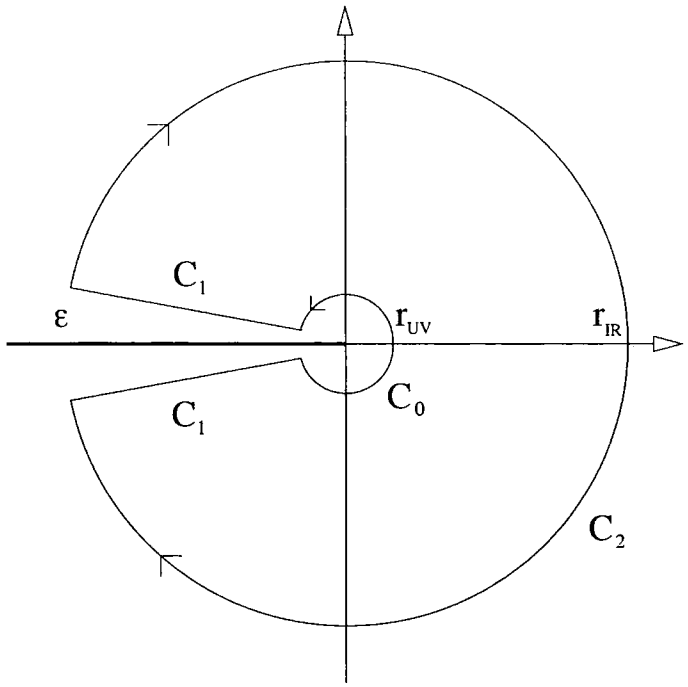


Figure 4.3: Integration contour C in the cut complex plane.

4.4 Relating the ultra-violet and infra-red limits using analyticity

The perturbative calculation of the infra-red limit of physical quantities can be improved utilizing their analytical properties, not in the coupling as above, but in the scale parameter. The analytic behaviour in the scale parameter follows, as shown in section one, from the general characteristics of a quantum field theory and are valid for the exact correlators.

We want to calculate the infra-red limit F_{IR} of a physical quantity $f(x)$, where $F_{IR} = \lim_{|x|\to\infty} f(x)$, and we will denote the ultra-violet limit $F_{UV} = \lim_{|x|\to 0} f(x)$. We are considering functions with well defined limits in $S(\alpha)$, i.e. $\lim_{|s|\to\infty} F(s) = F_{IR}$ and $\lim_{s\to 0} F(s) = F_{UV}$ for $s \in S(\alpha)$.

In the two examples we will consider in chapter 5 and 6, the physical quantities $f(x)$ are given as functionals of a two point correlator resulting in analytic functions

$F(s)$. A large class of physical quantities will in this way preserve the analytic structure of the correlation functions in the theory, which was described in section 4.1.1.

The dimensionless scale parameter s can, as mentioned, be introduced into the physical quantity in a number of ways. We will define by $\tilde{F}(s)$ the analytical continuation of $f(x)$ where s is introduced as $\tilde{F}(s) \equiv f(sx)|_{x=1}$ so that $\tilde{F}(s)$ has the expansion $\tilde{F}(s) = \sum \tilde{F}_n s^n$ around the origin, and $\tilde{F}(s)$ is analytic in $S(\alpha)$. Choosing another positive power $f(s^\gamma x)|_{x=1}$, $\gamma > 0$, gives the same ultra-violet and infra-red limits, but different intermediate behaviour. Let us now introduce the scale parameter so that the opening of the analytic sector is $2(\pi-\epsilon')$ with $\epsilon' \ll 1$. $F(s) = \tilde{F}(s^a)$ is analytic in $S \equiv S(2(\pi - \epsilon'))$ provided that $a = \frac{\alpha}{2(\pi-\epsilon')}$. We will now express the infra-red limit as an integral entirely in the ultra-violet region using analyticity.

Using Cauchy's theorem the analyticity of $F(s)$ implies that the contour integral

$$\frac{1}{2(\pi-\epsilon)i}\int_C ds\,\frac{e^{\rho/s}}{s}F(s) = \frac{1}{2(\pi-\epsilon)i}\left(\int_{C_0} ds + \int_{C_1} ds + \int_{C_2} ds\right)\frac{e^{\rho/s}}{s}F(s) = 0, \tag{4.15}$$

vanishes in the sector S, where $\rho \in \mathbb{R}_+$ and the contour C is given in figure 4.3; $\epsilon > \epsilon'$ so that $C \subset S$. The contribution from the contour C_2 in (4.15), see figure 4.3, in the limit where $r_{IR} \to \infty$ becomes

$$\lim_{r_{IR}\to\infty}\frac{1}{2(\pi-\epsilon)}\int_{\pi-\epsilon}^{-\pi+\epsilon} d\theta\, e^{\rho e^{-i\theta}/r_{IR}}F(r_{IR}e^{i\theta}) = -\lim_{|s|\to\infty} F(s) = -F_{IR}. \tag{4.16}$$

The angular integral and the limit $r_{IR} \to \infty$ can be interchanged as the integrand in (4.15) is bounded by a constant in the limit $r_{IR} \to \infty$. We then get the integral representation of F_{IR}

$$F_{IR} = \frac{1}{2(\pi-\epsilon)i}\left(\int_{C_0} ds\frac{e^{\rho/s}}{s}F(s) + \int_{C_1} ds\frac{e^{\rho/s}}{s}F(s)\right). \tag{4.17}$$

Analogously by considering

$$\frac{1}{2(\pi-\epsilon)i}\int_C ds\,\frac{e^{\rho s}}{s}F(s) = \frac{1}{2(\pi-\epsilon)i}\left(\int_{C_0} ds + \int_{C_1} ds + \int_{C_2} ds\right)\frac{e^{\rho s}}{s}F(s) = 0,$$

we get that

$$F_{UV} = \frac{-1}{2(\pi-\epsilon)i}\left(\int_{C_2} ds \frac{e^{\rho s}}{s} F(s) + \int_{C_1} ds \frac{e^{\rho s}}{s} F(s)\right). \tag{4.18}$$

We denote the integral along the contour C_1 (close to the cut) by

$$cut(\rho) = \frac{1}{2(\pi-\epsilon)i}\int_{C_1} ds \frac{e^{\rho/s}}{s} F(s). \tag{4.19}$$

The integrand in this integral is damped by the factor $e^{-\rho/|s|}$, and we will show that $\lim_{\rho\to\infty} cut(\rho) = 0$ by showing that it is bounded by a finite integral for all ρ allowing us to take the limit $\rho \to \infty$ in the integrand. We substitute $s = re^{\pm i(\pi-\epsilon)}$ for points on C_1 in the upper and lower half-plane, the integral becomes

$$\begin{aligned}
cut(\rho) &= \frac{-1}{2(\pi-\epsilon)i}\left(\int_{r_{IR}}^{r_{UV}} dr \frac{e^{\rho e^{-i(\pi-\epsilon)}/r}}{r} F(re^{i(\pi-\epsilon)}) + \int_{r_{UV}}^{r_{IR}} dr \frac{e^{\rho e^{i(\pi-\epsilon)}/r}}{r} F(re^{-i(\pi-\epsilon)})\right) \\
&= \frac{-1}{(\pi-\epsilon)}\int_{r_{UV}}^{r_{IR}} dr \frac{e^{\rho\cos(\pi-\epsilon)/r}}{r} \mathrm{Im}[e^{i\rho\sin(\pi-\epsilon)/r} F(re^{-i(\pi-\epsilon)})] \\
&= \frac{-1}{(\pi-\epsilon)}\int_{r_{UV}}^{r_{IR}} dr \frac{e^{\rho\cos(\pi-\epsilon)/r}}{r} \big(\cos(\rho\sin(\pi-\epsilon)/r)\mathrm{Im}[F(re^{-i(\pi-\epsilon)})] \\
&\quad + \mathrm{Re}[F(re^{-i(\pi-\epsilon)})]\sin(\rho\sin(\pi-\epsilon)/r)\big).
\end{aligned}$$

We divide the r interval into $(r_{UV}, 1)$ and $(1, r_{IR})$ and write $cut(\rho) = cut_{UV} + cut_{IR}$. The function $F(s)$ is finite in S hence there exists a constant $q > 0$ so that in the limit $r_{UV} \to 0$

$$|cut_{UV}| < q\int_0^1 dr \frac{e^{-\rho/r}}{r} \to 0 \quad \text{for} \quad \rho \to \infty. \tag{4.20}$$

We know that $F(s) \to F_{IR} \in \mathbb{R}$ for $|s| \to \infty$ from S, hence $\mathrm{Im}[F(re^{-i(\pi-\epsilon)})] \to 0$ for $r \to \infty$. If $\mathrm{Im}[F]$ falls off like $r^{-\delta}$ for some $\delta > 0$ then in the limit $r_{IR} \to \infty$, $\exists\ k_1, k_2, k_3 > 0$:

$$|cut_{IR}| < k_1\int_1^\infty dr \frac{\big(\sin(k_2/r) + k_3 r^{-\delta}\big)}{r} < \infty \text{ for all } \rho \in \mathbb{R}_+. \tag{4.21}$$

For a general $F(s)$ where the fall off might be slower we keep a finite r_{IR} then $|cut_{IR}|$ is again finite for all ρ and a finite r_{IR} introduces a $O(\frac{1}{r_{IR}})$ term in (4.17) which is

negligible for r_{IR} large. As $\epsilon \ll 1$ we can replace $\frac{1}{2(\pi-\epsilon)i}$ with $\frac{1}{2\pi i}$ given a term $O(\epsilon)$ on the right hand side in (4.17) which is again negligible for small ϵ. We can then define

$$I_{IR}(\rho) = \lim_{r_{UV}\to 0} \frac{1}{2\pi i}\int_{C_0} ds \frac{e^{\rho/s}}{s} F(s) \tag{4.22}$$

so that $F_{IR} = \lim_{\rho\to\infty} I_{IR}(\rho)$ and we have succeeded in writing the infra-red limit as the limiting value of a contour integral in the ultra-violet region.

We will now use Cauchy's theorem again to rewrite $I_{IR}(\rho)$ and this will show that $I_{IR}(\rho)$ is the Borel transform of $\tilde{F}(s)$. Instead of integrating over C_0 we will integrate over $\tilde{C}$ given as the path from the origin along the ray $\mathrm{Arg}(s) = -\pi + \epsilon$ and then anti-clockwise along $|s| = \tilde{x} > 0$ until $\mathrm{Arg}(s) = \pi - \epsilon$ and then back to the origin. It follows that this contour integral is independent of the choice of $\tilde{x}$ and ϵ as long as $\epsilon' < \epsilon < \frac{\pi}{2}$. The upper limit ensures that the rays stay in the negative half-plane (where $e^{\rho/s}$ is a damping factor), and the lower limit that $F(s)$ is analytic on the contour. Using this contour the ultra-violet limit $r_{UV} \to 0$ can be taken explicitly by extending the rays to the origin.

Using that $F(s) = \tilde{F}(s^a)$ for $a = \frac{\alpha}{2(\pi-\epsilon')}$ then amounts to

$$I_{IR}(\rho) = \frac{1}{2\pi i}\int_{\tilde{C}} \frac{ds}{s} e^{\rho/s} F(s) = \frac{1}{2\pi i}\int_{\tilde{C}} \frac{ds}{s} e^{\rho/s}\tilde{F}(s^a) = \frac{k}{2\pi i}\int_{\tilde{C}'} \frac{ds}{s} e^{(\frac{\tilde{\rho}}{s})^k}\tilde{F}(s), \tag{4.23}$$

where $k = \frac{1}{a}$, $\tilde{\rho} = \rho^{1/k}$ and $\tilde{C}'$ is the contour where the rays satisfy $|\mathrm{Arg}(s)| = |(\pi-\epsilon'')/k|$ for some $\epsilon'' \in (\epsilon', \frac{\pi}{2})$. We will write $\epsilon'' = (\pi-\tilde{\epsilon})/2$ for some $\tilde{\epsilon} \in (0, \pi-2\epsilon')$, then $|\mathrm{Arg}((\frac{\tilde{\rho}}{s})^k)| > \pi/2$ and the integrand in (4.23) is again damped on the rays. Also, on the rays $|\mathrm{Arg}(s)| = (\pi+\tilde{\epsilon})/2k = \frac{\alpha}{4}\frac{\pi+\tilde{\epsilon}}{\pi-\epsilon'} < \alpha/2$ so that $\tilde{F}(s)$ is analytic on the contour, $\tilde{C}' \subset S(\alpha)$, and the contour integral is therefore well defined.

The contour integral in (4.23) is independent of the contour $\tilde{C}'$, i.e. in the choice of $\tilde{\epsilon}$ for $\tilde{\epsilon} \in (0, \pi - 2\epsilon')$ where $\tilde{C}' \subset S(\alpha)$. (4.23) then shows that $I_{IR}(\tilde{\rho})$ is the Borel transform of $\tilde{F}(s)$ of order k. Equation (4.23) holds for all k with $F(s) \equiv \tilde{F}(s^{1/k})$ and $\alpha > \pi/k$, because in this case we can always find an $\tilde{\epsilon} < \alpha k - \pi$, so the contour integral is well defined.

We have then shown the general result:

The infra-red limit of a physical quantity, $F_{IR} = \lim_{|s|\to\infty} \tilde{F}(s)$, is the limiting value of the Borel transform of $\tilde{F}(s)$ in the scale parameter s, and changing the way in which the scale parameter is introduced amounts to changing the order of the Borel transform. The order k has to satisfy the bound $\alpha > \pi/k$ where α determines the analytic sector of the physical quantity $\tilde{F}(s)$.

One way of calculating the contour integral in (4.23) is to insert a series expansion of $\tilde{F}(s)$ around the origin, but because $\tilde{F}(s)$ is only analytic in a sector this series can only be an asymptotic expansion of $\tilde{F}(s)$. For $\alpha > \pi/k$ the asymptotic series will, as explained above, contain all information about the exact function around the origin.

An asymptotic series of $\tilde{F}(s)$ in the scale parameter can be obtained doing the perturbative expansion in the coupling and then inserting the running coupling constant. Assume that $\tilde{F}$ is given as a formal perturbative series, which we write as $\tilde{F}(s,g) = \sum_n \hat{F}_n(s) g^n$. The Callan–Symanzik equation (3.7) states that a theory is invariant, i.e. the correlators are invariant, under a scale transformation if the couplings change according to the renormalisation group. A scaled quantum field theory can therefore equivalently be described by a theory on the same scale, but with couplings changed according to the renormalisation group. In this way scale dependence of a theory can be moved into the running coupling $\bar{g}(s)$. Moving all scale dependence into the running coupling we get an asymptotic series in the scale $\tilde{F}(s=1,\bar{g}(s)) = \sum \tilde{F}_n s^n$. If $\tilde{F}(s)$ is asymptotic of order k'^{-1} ($\tilde{F}(s) \in \mathbb{C}[[s]]_{1/k'}$) we will choose $k = k'$, and then the Borel transform of $\tilde{F}$ has a non-zero convergence radius and the analytically continued value at infinity uniquely determines F_{IR}.

The examples we consider below have $\alpha = \pi - \epsilon'$, where $\epsilon' \ll 1$, hence from the constraint $\alpha > \frac{\pi}{k}$: $k = 1 + \delta$ for some $\delta > 0$. We will generally try to minimize the order k ($k \to 1$), thus maximizing the analytic sector and the convergence of the Borel transform.

5

Scaling in two dimensions

The method developed in chapter 4 is independent of the choice of dimension as it is based on analytical properties present in any dimension. In this chapter we will use this method to calculate the infra-red central charge in two dimensions.

There are many reasons for studying two dimensional theories. Important examples include the two dimensional spin systems in statistical mechanics, the Ising model being one of them, these models describe phenomena in condensed matter physics. Even simple models can capture some of the long range effects in real systems because of universality which, as described in chapter 3, states that the infra-red behaviour only depends on the nature of the fixed point and not the microscopic dynamics. It is often also easier to implement new ideas in a low number of dimensions, and two dimensional theories have therefore served as an important testing ground for new theories with behaviour thought to be generic in higher dimensions[1]. Two dimensions have also received a lot of attention due to the efforts to understand the superstring.

In two dimensions conformal invariance follows from scale invariance and the renormalisation group fixed points are thus conformal field theories. A classification of all conformal field theories would therefore determine all possible fixed points,

[1]However, the dynamics of physical theories often have a strong dependency on the space-time dimension thereby limiting this kind of approach.

and is therefore important in the search for a full understanding of the space of two dimensional field theories. A two dimensional conformal field theory is characterized by its field content, the operator product expansion coefficients and the Virasoro central charge. For unitary theories, the renormalisation group flow is constrained by Zamolodchikov's c-theorem which says that it is an irreversible flow, and it therefore rules out the existence of limit cycles and strange attractors in the coupling constant space. It also states that the infra-red central charge is never larger than the ultra-violet central charge, and it shows that the renormalisation group has an infra-red fixed point.

First the notation used for a conformal field theory is defined, then Zamolodchikov's c-theorem is discussed and $\tilde{F}(s)$ from chapter 4 is determined so that F_{IR} equals the infra-red central charge c_{IR}. Using the method in chapter 4, c_{IR} is then calculated for the free boson and fermion and the unitary minimal models perturbed by $\Phi_{(1,3)}$.

5.1 Conformal field theory

To define the notation we will use for a two dimensional conformal field theory [80] let us describe a few of its fundamental characteristics. There are many good reviews on this subject, see [24, 81, 82, 83, 84, 85, 86].

As stated in chapter 2 is the rigid (or global) euclidean conformal group given by $SO(3,1)$ ($\approx SL(2,\mathbb{C})/\mathbb{Z}_2$). In Cartesian coordinates where $g_{\mu\nu} = \delta_{\mu\nu}$ an infinitesimal conformal transformation (2.23) satisfies for $d = 2$

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \delta_{\mu\nu} \partial_\rho \epsilon^\rho \quad \Rightarrow \quad \partial_0 \epsilon_1 = -\partial_1 \epsilon_0, \ \partial_0 \epsilon_0 = \partial_1 \epsilon_1, \tag{5.1}$$

which are the Cauchy-Riemann differential equations. The local conformal algebra is therefore infinite dimensional as any holomorphic function defines a transformation. It is then natural to introduce complex coordinates $(z, \bar{z}) = (x^0 + ix^1, x^0 - ix^1)$ and

in these coordinates the components of the metric become

$$g_{ij} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \tag{5.2}$$

and we will write the measure as $d^2x = dx^0 \wedge dx^1 = \frac{i}{2} dz \wedge d\bar{z} \equiv d^2z$. It is useful to treat $z, \bar{z}$ as independent variables and then the Cauchy-Riemann differential equations become: $\partial_{\bar{z}} f(z, \bar{z}) = 0$. The physical variables $(x^0, x^1) \in \mathbb{R}^2$ are recovered by the condition $\bar{z} = z^*$. Primary fields are defined to transform similarly as (2.24) under the local conformal transformation $z \to w(z)$, $\bar{z} \to \bar{w}(\bar{z})$:

$$\phi'(w, \bar{w}) = \left(\frac{\partial w}{\partial z}\right)^{-h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) \tag{5.3}$$

where $(h, \bar{h})$ are the conformal scaling dimensions. The infinitesimal version is $w(z) = z + \epsilon(z)$, $\bar{w}(\bar{z}) = \bar{z} + \bar{\epsilon}(\bar{z})$:

$$\begin{aligned} \delta_{\epsilon,\bar{\epsilon}} \phi &= \phi'(z, \bar{z}) - \phi(z, \bar{z}) = \phi'(w, \bar{w}) - \epsilon \partial_z \phi - \bar{\epsilon} \partial_{\bar{z}} \phi - \phi \\ &= -(h \partial_z \epsilon + \bar{h} \partial_{\bar{z}} \bar{\epsilon} + \epsilon \partial_z + \bar{\epsilon} \partial_{\bar{z}}) \phi. \end{aligned} \tag{5.4}$$

Under an infinitesimal scaling $z \to (1 + a)z$, $\bar{z} \to (1 + a)\bar{z}$: $\delta_a \phi = -a(h + \bar{h} + x^\mu \partial_\mu)\phi$ and it follows from (2.22) that the scaling dimension is given by $\Delta = h + \bar{h}$. Analogously for a rotation (Lorentz transformation in Minkowski space): $z \to (1 + ia)z$, $\bar{z} \to (1 - ia)\bar{z}$, the spin becomes $s = h - \bar{h}$. The energy-momentum tensor is defined as in (2.16)

$$T_{\mu\nu}(x) = \frac{2 \cdot 2\pi}{\sqrt{g(x)}} \frac{\delta S}{\delta g^{\mu\nu}}, \tag{5.5}$$

with a different normalization making expressions simpler in two dimensions. In complex coordinates: $T_{zz} = T = \frac{1}{4}(T_{00} - 2iT_{01} - T_{11})$, $T_{\bar{z}\bar{z}} = \bar{T} = \frac{1}{4}(T_{01} + 2iT_{10} - T_{11})$ and $T_{z\bar{z}} = \frac{1}{4} T^\mu_\mu = \frac{1}{4}\Theta$. The conservation law $\partial^\mu T_{\mu\nu} = 0$ together with conformal invariance $\Theta = 0$ shows that

$$\partial_{\bar{z}} T_{zz} + \partial_z T_{z\bar{z}} = 0 \Rightarrow \partial_{\bar{z}} T(z, \bar{z}) = 0. \tag{5.6}$$

The Ward identity (2.12) for translation invariance becomes in complex coordinates

$$\partial_{\bar{z}}\langle T(z)\phi(x_1)\cdots\phi(x_n)\rangle + \frac{1}{4}\partial_z\langle\Theta(z)\phi(x_1)\cdots\phi(x_n)\rangle = \pi\sum_{i=1}^{n}(\delta(z-x_i)\partial_{x_i} + \delta(z-x_i)h_i\partial_{x_i})\langle\phi(x_1)\cdots\phi(x_n)\rangle. \quad (5.7)$$

At the fixed point $\Theta = 0$, so this equation shows that away from the diagonal $\partial_{\bar{z}}T = 0$ in all correlators in the quantum theory, hence $T(z)$ is holomorphic. In chapter 2 it was shown that the currents for Lorentz transformations and scaling can be written in terms of $T_{\mu\nu}$, in the coordinates $(z,\bar{z})$ the corresponding Ward identities can together be written as [84][2]

$$\delta_\epsilon\langle\phi(z_1,\bar{z}_1)\cdots\phi(z_n,\bar{z}_n)\rangle = \frac{-1}{2\pi i}\oint_C dz\ \epsilon(z)\langle T(z)\phi(z_1,\bar{z}_1)\cdots\phi(z_n,\bar{z}_n)\rangle, \quad (5.8)$$

where C includes all $(z_i,\bar{z}_i)$. Using (5.4) the conformal Ward identity becomes

$$\langle T(z)\phi(z_1)\cdots\phi(z_n)\rangle = \sum_{i=1}^{n}\left(\frac{h_i}{(z-z_i)^2} + \frac{1}{(z-z_i)}\frac{\partial}{\partial z_i}\right)\langle\phi(z_1)\cdots\phi(z_n)\rangle. \quad (5.9)$$

$T(z)$ is a density, and because the theory is conformal z is the only dimensionful parameter in the theory, this means that the two point function can be written as

$$\langle T(z)T(0)\rangle = \frac{c}{2z^4}, \quad c\in\mathbb{R}. \quad (5.10)$$

Here c is characteristic for the particular theory and is called the Virasoro central charge. $T(z)$ is analytic hence it can be written as a Laurent series: $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$, and the modes L_n can be shown to satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \quad (5.11)$$

explaining the name for c. (5.11) follows by considering successive conformal transformations, and it implies that if $c < 0$ then

$$\langle h|L_nL_{-n}|h\rangle = (2nh + \tfrac{1}{12}cn(n^2-1))\langle h|h\rangle < 0 \quad (5.12)$$

for sufficiently large $n > 0$, where $L_0|h\rangle = h|h\rangle$, $L_n^\dagger = L_{-n}$ and $L_n|h\rangle = 0$. This shows that $c \geqslant 0$ for unitary theories.

[2]Taking $\bar{\epsilon} \neq 0$ there is an analogous term with $\bar{T}$. Useful identities for manipulating these expressions are: $\pi\delta^2(x) = \partial_{\bar{z}}\frac{1}{z}$, and the residue for an nth order pole $\frac{f(z)}{(z-a)^n}$ is: $\frac{f^{(n-1)}(a)}{(n-1)!}$.

5.2 Zamolodchikov's c-theorem

Zamolodchikov's c-theorem states that for a unitary and renormalisable quantum field theory in two dimensions there exists a function which is monotonically decreasing along the renormalisation group flow, and which is stationary only for conformally invariant theories where it takes the value of the Virasoro central charge. The c-theorem implies that the infra-red limit, where the scale goes to infinity is a fixed point of the renormalisation group.

The central charge is a measure for the number of massless degrees of freedom in the theory. The theorem thus shows the loss of massless degrees of freedom under the renormalisation group flow. At the fixed point there is scale invariance and all degrees of freedom are massless, by following the renormalisation group some of these degrees of freedom become massive with a mass m. These decouple at the length scale $R \gg \frac{1}{m}$, and at the infra-red fixed point all remaining degrees of freedom will again be massless. The c-theorem then shows the irreversible nature of the coarse graining that takes place under the renormalisation group.

The c-theorem describes the local flow of the renormalisation group, but it also says something about the global topological properties of the space of two dimensional quantum field theories. In [87, 88] the c-function was argued to be a Morse function and the Euler and Betti number of the coupling constant space were calculated.

There have been proposals for c-theorems in higher dimensions[3], in theories with supersymmetry and in connection with the AdS/CFT correspondence and the holographic renormalisation group[4].

The proof of the c-theorem [93, 94, 81] follows from Lorentz invariance and unitarity of the quantum field theory away form the fixed point, in euclidean space

[3]See [89, 90] for a recent discussion and references. The most promising candidate for a c-function in higher dimensions is the coefficient a in front of the Euler density in the trace anomaly (first suggested by Cardy) [90].

[4]See e.g. [91, 92]. In [92] a c-theorem is proved for $a = c$ in the AdS/CFT context using the Einstein equations and the weak energy condition.

this becomes rotation and translation invariance and reflection positivity. We saw above that $T(z)$ has spin 2 and $\Theta = T^{\mu}_{\mu}$ is a scalar so it has spin 0. The general form of the two point functions then becomes

$$\langle T(z,\bar{z})T(0,0)\rangle = \frac{F(z\bar{z}\Lambda^2)}{z^4}, \quad \langle T(z,\bar{z})\Theta(0,0)\rangle = \frac{G(z\bar{z}\Lambda^2)}{z^3\bar{z}},$$
$$\langle \Theta(z,\bar{z})\Theta(0,0)\rangle = \frac{H(z\bar{z}\Lambda^2)}{z^2\bar{z}^2}$$

for some functions F, G, H and Λ is a mass scale of the theory. From the conservation equation (5.6) it follows that

$$\begin{aligned} \langle(\partial_{\bar{z}}T(z,\bar{z}) + \frac{1}{4}\partial_z\Theta(z,\bar{z}))T(0,0)\rangle = 0 \quad &\Rightarrow \dot{F} + \frac{1}{4}(\dot{G} - 3G) = 0, \\ \langle(\partial_{\bar{z}}T(z,\bar{z}) + \frac{1}{4}\partial_z\Theta(z,\bar{z}))\Theta(0,0)\rangle = 0 \quad &\Rightarrow \dot{G} - G + \frac{1}{4}(\dot{H} - 2H) = 0, \end{aligned} \tag{5.13}$$

where $\dot{F} = \bar{z}z\frac{\partial F(\bar{z}z)}{\partial(\bar{z}z)}$. Defining $C = 2(F - \frac{1}{2}G - \frac{3}{16}H)$, and substituting into (5.13) gives

$$\dot{C} = -\frac{3}{4}H \leqslant 0. \tag{5.14}$$

The inequality follows from reflection positivity[5] and because Θ is a scalar which means that $H \geqslant 0$. This shows that $C(z\bar{z})$ is a monotonic function only stationary when $H = 0$, here $\Theta = 0$ and the theory has conformal symmetry with $C = 2F = c$ using (5.10). The Callan–Symanzik equation (3.7) can be used to move the scaling dependence into the running coupling constant. By explicitly writing the dependence of the couplings $C(R, g^i)$, with $R^2 = \bar{z}z$, and using that $C(R, g^i)$ is dimensionless and finite, it becomes

$$\left(\frac{\partial}{\partial R} + \sum_i \beta_i \frac{\partial}{\partial g^i}\right) C(R, g^i) = 0. \tag{5.15}$$

$C(R, \bar{g}^i(t))|_{R=1}$ therefore describes a flow in the coupling constant space with the properties in the theorem.

[5] $\langle A(x,\tau)A(x,-\tau)\rangle \geqslant 0$, see [63].

It follows from the above proof that we can obtain an expression for the total change in the central charge when going from short to large distances, as shown by Cardy in [81]

$$\Delta c = c(\infty) - c(0) = \int_0^\infty \frac{d(R^2)}{R^2} \dot{C}(R) = -\frac{3}{4} \int_0^\infty R^2 \langle \Theta(R)\Theta(0) \rangle d(R^2). \tag{5.16}$$

One can thereby calculate Δc knowing the correlator $\langle \Theta(R)\Theta(0) \rangle$ in the entire scaling region, but $\langle \Theta(R)\Theta(0) \rangle$ is normally only known perturbatively and therefore only at scales where the coupling is small.

5.3 Defining $\tilde{F}(s)$ for the central charge

We will now define a function $\tilde{F}(s)$ as in chapter 4, which has the infra-red central charge c_{IR} as its infra-red limit.

It follows from (5.10) that the the correlator $\langle T_{zz}(z) T_{zz}(0) \rangle$ gives the ultra-violet and infra-red central charges in the limits $z \to 0$ and $z \to \infty$ respectively. Using the Källen-Lehmann spectral representation (4.3) $\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle$ was in [60] written as

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = \frac{\pi}{3 \cdot 16} \int_0^\infty d\mu^2 \, \tilde{c}(\mu^2) \int \frac{d^2p}{(2\pi)^2} e^{ipx} \frac{(g_{\mu\nu}p^2 - p_\mu p_\nu)(g_{\rho\sigma}p^2 - p_\rho p_\sigma)}{p^2 + \mu^2}, \tag{5.17}$$

in cartesian coordinates. This is the most general form with the correct mass dimension that respects: $T_{\mu\nu} = T_{\nu\mu}$ and $\partial_\mu T_{\mu\nu} = 0$. Writing this in the complex coordinates $(z, \bar{z})$ using (5.2)

$$\langle T_{zz}(z,\bar{z}) T_{zz}(0,0) \rangle = \frac{\pi}{3 \cdot 16} \int_0^\infty d\mu^2 \, \tilde{c}(\mu^2) \int \frac{d^2p}{(2\pi)^2} \frac{e^{\frac{i}{2}(p\bar{z} + \bar{p}z)}}{p\bar{p} + \mu^2} \bar{p}^4,$$

where $\tilde{c}(\mu^2) d\mu^2$ is the spectral density which represents the density in degrees of freedom of the quantum field theory at the mass μ. If we scale $z, \bar{z}$ by a positive real dimensionless parameter s ($s > 0$) we get

$$\langle T_{zz}(sz, s\bar{z}) T_{zz}(0,0) \rangle = \frac{\pi}{3 \cdot 16} \int_0^\infty d\mu^2 \, \tilde{c}(\mu^2) \int \frac{d^2q}{(2\pi)^2} \frac{e^{\frac{i}{2}(q\bar{z} + \bar{q}z)}}{q\bar{q} + s^2\mu^2} \frac{\bar{q}^4}{s^4}. \tag{5.18}$$

In the ultra-violet limit where $s \to 0$ then $\frac{\bar{q}^4}{q\bar{q}+s^2\mu^2} \to \frac{\bar{q}^4}{q\bar{q}}$ and using that the Fourier transform of $(\pi/24)\bar{p}^4/\bar{p}p$ is $1/z^4$, then (5.18) becomes

$$\langle T_{zz}(sz, s\bar{z})T_{zz}(0,0)\rangle \to \frac{1}{2s^4z^4}\int_0^\infty d\mu^2\,\tilde{c}(\mu^2) = \frac{c_{UV}}{2s^4z^4} \quad \text{for } s \to 0. \tag{5.19}$$

The ultra-violet central charge is therefore $c_{UV} = \int_0^\infty d\mu^2\,\tilde{c}(\mu^2)$ [60]. To calculate the infra-red limit we first note that

$$\int \frac{d^2q}{(2\pi)^2}\frac{e^{\frac{i}{2}(q\bar{z}+\bar{q}z)}}{q\bar{q}+s^2\mu^2}\,\bar{q}^4 = 2^4\left(\frac{\partial}{\partial z}\right)^4 G_0(z,\bar{z},s\mu), \tag{5.20}$$

where $G_0(z,\bar{z},\mu)$ is equal to the free Feynman propagator at mass μ. $G_0(z,\bar{z},\mu)$ can be written in terms of a modified Bessel function [84]

$$G(x,\mu) = \int \frac{d^2p}{(2\pi)^2}\frac{e^{ix\cdot p}}{p^2+\mu^2} = \frac{K_0(|x|\mu)}{2\pi}. \tag{5.21}$$

Performing the differentiation, using the identities $K_n' = -\frac{1}{2}(K_{n-1}+K_{n+1})$, $K_2(x) = K_0(x) + \frac{2}{x}K_1(x)$ and $K_{-1}(x) = K_1(x)$, we can then write (5.18) as

$$\begin{aligned}\langle T_{zz}(sz, s\bar{z})T_{zz}(0,0)\rangle = \frac{1}{2\cdot 48\, s^4z^4}\int_0^\infty d\mu^2\;\tilde{c}(\mu^2)\mu s|z|\,\big((\mu^3s^3|z|^3 \\ +24\mu s|z|)K_0(\mu s|z|) + (8\mu^2s^2|z|^2+48)K_1(\mu s|z|)\big)\,.\end{aligned} \tag{5.22}$$

In the infra-red limit where $s \to \infty$, $K_0(\mu|z|s)$ and $K_1(\mu|z|s)$ have the asymptotic behaviour [95] $e^{-\mu s|z|}$ and the only contribution to (5.22) comes from the massless limit where $\mu \to 0$, hence

$$\langle T_{zz}(sz, s\bar{z})T_{zz}(0,0)\rangle \to \lim_{\epsilon\to 0}\frac{1}{2s^4z^4}\int_0^\epsilon d\mu^2\,\tilde{c}(\mu^2) = \frac{c_{IR}}{2s^4z^4} \quad \text{for } s \to \infty, \tag{5.23}$$

so that $c_{IR} = \lim_{\epsilon\to 0}\int_0^\epsilon d\mu^2\,\tilde{c}(\mu^2)$. This shows that $c_{UV} \geqslant c_{IR}$ as the spectral density is positive for a unitary theory, and this is another way of showing the c-theorem [60]. This representation of the central charge using the spectral representation also shows that the central charge is a measure for the number of (massless) degrees of freedom of the conformal field theory. If the theory is scale invariant dimensional arguments say that the spectral density must be proportional to a delta function as

there is no mass in the theory. The representation (5.17) for the trace becomes in this case

$$\langle\Theta(x)\Theta(0)\rangle = \frac{\pi}{3}\int_0^\infty d\mu^2\,\tilde{c}(\mu^2)\int\frac{d^2p}{(2\pi)^2}e^{ipx}\frac{(p^2)^2}{p^2+\mu^2} \propto \partial_x^2\delta^2(x), \tag{5.24}$$

which vanishes for $|x| \neq 0$ so that $\Theta = 0$, and this shows that scale invariance does imply conformal invariance at the quantum level in two dimensions [60].

We define the function

$$\tilde{F}(s) = 2z^4s^4\langle T_{zz}(sz, s\bar{z})T_{zz}(0,0)\rangle\Big|_{z=\bar{z}=1} \tag{5.25}$$

here s and $\tilde{F}(s)$ are dimensionless and $s \in \mathbb{R}_+$. When we set $|z| = 1$ then μ becomes dimensionless in (5.22), we also denote this dimensionless quantity by μ. This function then satisfies

$$\tilde{F}(s) \to \begin{cases} c_{UV} & \text{for } s \to 0_+, \\ c_{IR} & \text{for } s \to \infty. \end{cases} \tag{5.26}$$

5.3.1 Analyticity of $\tilde{F}(s)$

We will show that $\tilde{F}(s)$ is an analytic continuation of $\tilde{F}(x)$, $x \in \mathbb{R}_+$, for $s \in S = S(\pi - \epsilon')$ (with $\epsilon' \ll 1$); to show this write $\tilde{F}(s) = \int_0^\infty d\mu^2 f(s,\mu^2)$. For $\tilde{F}$ to be holomorphic in S then $f(s,\mu^2)$ must be holomorphic in S for all $\mu^2 \in [0,\infty)$, and both f and $\frac{df}{ds}$ must be integrable over the set $[0,\infty)$. Using (5.22) above we can write $\tilde{F}(s)$ as

$$\tilde{F}(s) = \frac{1}{48}\int_0^\infty d\mu^2\,\tilde{c}(\mu^2)\mu s\left((\mu^3s^3+24\mu s)K_0(\mu s)+(8\mu^2s^2+48)K_1(\mu s)\right). \tag{5.27}$$

$K_\nu(z)$ is holomorphic in S so $f(s,\mu^2)$ is clearly holomorphic in S. The modified Bessel functions also satisfy that $|K_\nu(z)|$ is bounded for $|z| \geq \epsilon$ for any $\epsilon \in \mathbb{R}_+$ and $|\mathrm{Arg}(z)| < \frac{\pi}{2}$, which is the case for $z = \mu s$ when $s \in S$. For large values of $|s|$ integrability is ensured by $\int_0^\infty d\mu^2\,\tilde{c}(\mu^2) = c_{UV} < \infty$ and the asymptotic behaviour $K_\nu(z) = \sqrt{\frac{\pi}{2z}}e^{-z}(1 + O(z^{-1}))$. Around the origin $K_0(z)z \to 0$ for $z \to 0$ and $K_1(z)z \to 1$ for $z \to 0$, hence $\tilde{F}(s)$ is integrable. $\frac{d\tilde{F}(s)}{ds}$ is shown to be integrable in a similar way and the analyticity is shown.

The form of $\tilde{F}(s)$ in (5.27) shows that $\tilde{F}(s)$ has a limit value $\tilde{F}(s) \to k_1$ for $s \to 0$ with $s \in S$, from (5.26) it follows that $k_1 = c_{UV}$, and we define $\tilde{F}(0) = c_{UV}$. It also follows from (5.27) that $\tilde{F}(s)$ has a limit value for $|s| \to \infty$ from S: $\tilde{F}(s) \to k_2$ and (5.26) again sets $k_2 = c_{IR}$. The analytic opening is therefore $\alpha = \pi - \epsilon'$ where $\epsilon' \ll 1$.

5.3.2 The approximation for c_{IR}

We have the following representation of c_{IR} and c_{UV}

$$c_{IR} = \frac{1}{2\pi i}\left(\int_{C_0} ds \frac{e^{\rho/s}}{s} F(s) + \int_{C_1} ds \frac{e^{\rho/s}}{s} F(s)\right) = I_{IR}(\rho) + cut(\rho) \tag{5.28}$$

$$c_{UV} = \frac{-1}{2\pi i}\left(\int_{C_2} ds \frac{e^{\rho s}}{s} F(s) + \int_{C_1} ds \frac{e^{\rho s}}{s} F(s)\right) = I_{UV}(\rho) + cut(\rho), \tag{5.29}$$

where in both cases the contribution from the cut is rapidly decreasing in ρ. Note that in these relations the integral is performed in the opposite scaling limit of the quantity we calculate. We will here concentrate on (5.28) as perturbation theory can be applied to $F(s)$ when $s \in C_0$, we will call $I_{IR}(\rho) = I(\rho)$.

We can choose coordinates in the coupling constant space so that the ultra-violet fixed point corresponds to $\bar{g}(s) = 0$. In the ultra-violet limit where $s \to 0$ we may describe $F(s)$ by perturbation theory as $\bar{g}(s) \to 0$. The nth order perturbative approximation of $F(s)$ is denoted by $F_n(s)$ and the corresponding integral by $I_n(\rho)$. In the limit of large ρ the contribution from the cut vanishes and we get

$$c_{IR} \approx \lim_{\rho\to\infty} I_n(\rho) = \lim_{s\to\infty} F_n(s) \tag{5.30}$$

where the last equality follows setting $s' = s/\rho$ in (5.28) and then taking the limit $\rho \to \infty$ in the integrand valid for all $r_{UV} > 0$. Moving all scale dependence into the running coupling constant $\bar{g}(s)$, as in the proof of the c-theorem, we can write $F_n(s) = \Phi_n(\bar{g}(s))$. Let g^*_{IR} denote the first non-trivial zero of the perturbative β-function, i.e. $\bar{g}(s) \to g^*_{IR}$ for $s \to \infty$ (and $*$ is not complex conjugation). Then equation (5.30) becomes

$$\lim_{s\to\infty} F_n(s) = \lim_{s\to\infty} \Phi_n(\bar{g}(s)) = \Phi_n(g^*_{IR}) = c^*_{IR} \tag{5.31}$$

which is the perturbative estimate of c_{IR} we want to improve.

$F_n(s)$ is the ultra-violet perturbative approximation to $F(s)$ and the integration range in $I(\rho)$ is compact so $\lim_{\rho\to 0} I_n(\rho) = \lim_{\rho\to 0} I(\rho) = c_{UV}$. $I_n(\rho)$ therefore provides a good approximation to $I(\rho)$, for small enough ρ, since the power series expansion of $I(\rho)$ is controlled by the small s expansion of $F(s)$ for which perturbation theory applies. This is illustrated in (5.71) below for the minimal models. For larger values of ρ higher order terms in the expansion of $I_n(\rho)$ become important and the coefficients of the expansion of $I_n(\rho)$ and $F_n(\rho)$ part company. If $c_{IR} < c^*_{IR}$ and if the region where $I_n(\rho)$ is a good approximation to $I(\rho)$ is large enough, then $I_n(\rho)$ will have a minimum before approaching its limiting value of c^*_{IR}. Since this minimum occurs at the largest value of ρ for which $I_n(\rho)$ is a reasonable approximation to $I(\rho)$ and the true value of c_{IR} is given by $I(\infty)$, it is this minimum of $I_n(\rho)$ that we will use to provide a better estimate of c_{IR}. The approximation then becomes

$$c_{IR} = I_n(\rho_m) \tag{5.32}$$

where ρ_m is the value where $I_n(\rho)$ attains its minimum. Below we consider the quantity $\Delta c_{exact} = c_{UV} - c_{IR}$ and denote the approximation to it $\Delta c_{approx} = c_{UV} - I_n(\rho_m)$, we call the perturbative value $\Delta c_{pert} = c_{UV} - \lim_{\rho\to\infty} I_n(\rho) = c_{UV} - c^*_{IR}$. This approximation rests upon the assumption that the exact function is monotonically decreasing from c_{UV} to c_{IR}, or at least that its minimum value is close to c_{IR}. Below we will show that this is indeed the case for $k = 4$, at least to a very good approximation.

5.3.3 Exact bound on Δc

As discussed above can the exact value $I(\rho)$ possibly be smaller than the asymptotic value c_{IR}. We will denote by ρ'_m the value where $I(\rho)$ attains its minimum and $\Delta c_{est} = c_{UV} - I(\rho'_m)$, then $\Delta c_{exact} - \Delta c_{est} \leqslant 0$ measures the undershoot of the exact function $I(\rho)$ compared with its asymptotic value c_{IR}, see figure 5.1.

From the spectral representation of $F(s)$ in (5.27) we can obtain a rigorous lower

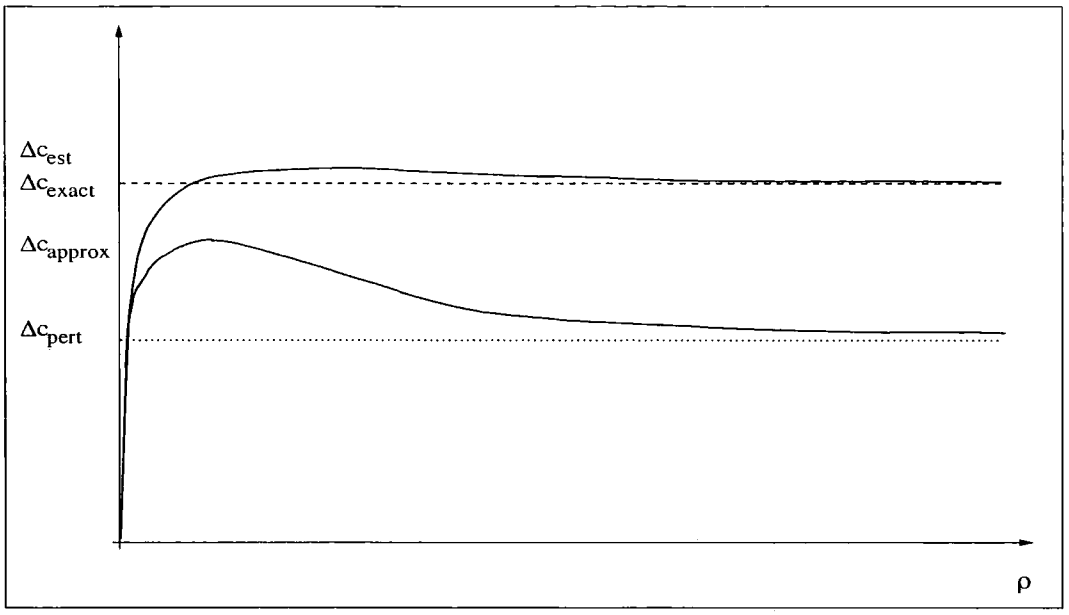


Figure 5.1: The expected behaviour of $c_{UV} - I_n(\rho)$ and $c_{UV} - I(\rho)$.

bound on this exact undershoot. As noted above can the spectral density be written as $\tilde{c}(\mu^2) = k_1\delta(\mu^2)$ for a scale invariant theory, in a general theory an additional term is needed [60]: $\tilde{c}(\mu^2) = k_1\delta(\mu^2) + \hat{c}(\mu^2)$, where $\hat{c}(\mu^2)$ is supported away from $\mu^2 = 0$ and (5.23) then sets $k_1 = c_{IR}$. We showed in (5.19) that $\int_0^\infty d\mu^2\tilde{c}(\mu^2) = c_{UV}$ hence $\int_0^\infty d\mu^2\hat{c}(\mu^2) = \Delta c_{exact}$. Using this in (5.27) we get

$$\begin{aligned}\Delta c_{est} &= c_{UV} - I(\rho'_m) = c_{UV} - \frac{1}{2\pi i}\int_{\tilde{C}}\frac{ds}{s}e^{\rho'_m/s}F(s)\\ &= \Delta c_{exact} - \int_0^\infty d\mu^2\ \hat{c}(\mu^2)\Upsilon(\mu^2,\rho'_m)\end{aligned}$$

where

$$\begin{aligned}\Upsilon(\mu^2,\rho'_m) = \frac{1}{2\pi i}\int_{\tilde{C}}\frac{ds}{s}e^{\rho'_m/s}\frac{\mu s^{1/4}}{48}\big(((\mu s^{1/4})^3\\ +24\mu s^{1/4})K_0(\mu s^{1/4}) + (8(\mu s^{1/4})^2+48)K_1(\mu s^{1/4})\big)\end{aligned} \tag{5.33}$$

and $F(s) = \tilde{F}(s^{1/k})$ with $k = 4$. Rescaling s allows us to move all μ dependence into $\rho'_m(\mu) = \rho'_m\mu^4$, so that we can write $\Upsilon(\mu^2,\rho'_m) = \Upsilon(\rho'_m(\mu))$. Unitarity ensures

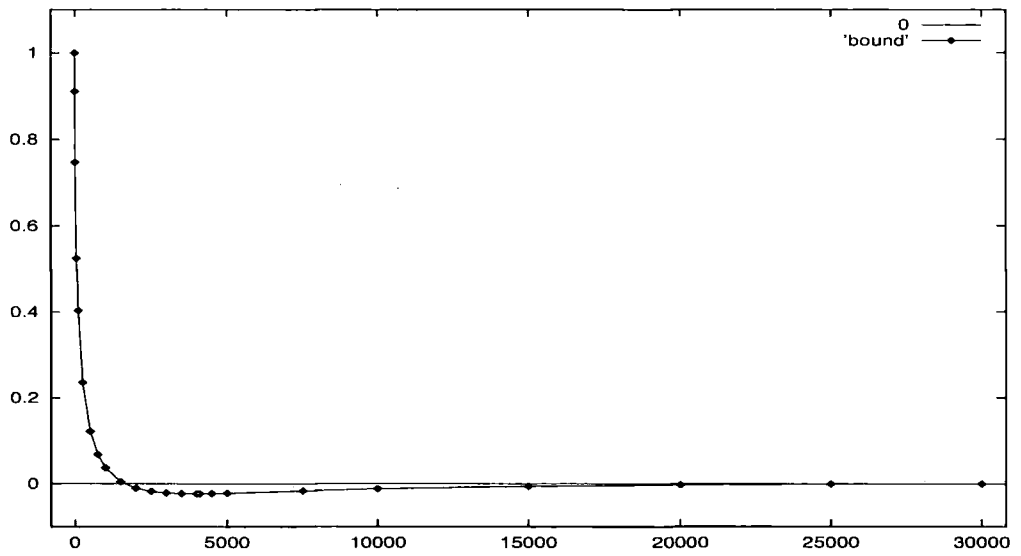


Figure 5.2: Numerical integration of $\Upsilon(\rho)$ given in (5.33).

that $\hat{c}(\mu^2) \geqslant 0$ hence

$$\Delta c_{exact} - \Delta c_{est} \geqslant \min_{\mu} \Upsilon(\rho'_m(\mu)) \int_0^\infty d\tilde{\mu}^2 \hat{c}(\tilde{\mu}^2), \tag{5.34}$$

$\Delta c_{est} > 0$ and we are considering the case where $\Delta c_{exact} \geqslant 0$ so that

$$0 \geqslant \frac{\Delta c_{exact} - \Delta c_{est}}{\Delta c_{exact}} \geqslant \min_{\rho>0} \Upsilon(\rho). \tag{5.35}$$

The lower bound in the relative undershoot of $I(\rho'_m)$ therefore equals $\min_\rho \Upsilon(\rho)$. In figure 5.2 we plot $\Upsilon(\rho)$ for $k = 4$. With this value we get that $\min_\rho \Upsilon(\rho) = -0.0232$ so the relative overshoot in Δc_{est} compared with Δc_{exact} is maximally 2.3%. It follows from (5.34) that the bound in (5.35) is only saturated for free theories (where $\hat{c}(\mu^2) \propto \delta(\mu^2 - m^2)$), for general interacting theories the relative overshoot will be smaller than $|\min_\rho \Upsilon(\rho)|$. We show in figure 5.1 the type of behaviour that we expect and which is confirmed for the minimal models below.

The actual choice of $k = 4$ then is a compromise between maximizing the range of analyticity (small k) and minimizing the bound of the undershoot in $I(\rho'_m)$ (large k).

5.4 Application to different two dimensional models

In this section we calculate the central charge for the free bosonic and fermionic theory perturbed by a mass term. These theories are gaussian and we can calculate $F(s)$ exactly, the central charge can then be written as $\lim_{\rho\to\infty} I(\rho)$ as the contribution from the cut then vanishes. The infra-red limit of these theories is trivial ($c_{IR} = 0$) as all the degrees of freedom are massive, and therefore decouple when approaching the infra-red fixed point where the scale goes to infinity. The free theories are none the less important to consider as we can here obtain the exact function $I(\rho)$.

In 5.4.3 we consider the unitary minimal models perturbed by the relevant operator $\Phi_{(1,3)}$, this theory has a non-trivial infra-red fixed point. We obtain the renormalisation group improved perturbative calculation of c_{IR} and compare with the approximation (5.32).

5.4.1 The free boson

We take the action for the free bosonic theory in two dimensions with a mass m to be

$$S = \int d^2x \left(\frac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x) + \frac{1}{2}m^2\varphi^2(x)\right), \tag{5.36}$$

where the perturbation away from the conformal field theory is given by the mass term $\frac{1}{2}m^2\varphi^2$. The theory is still a free theory off criticality and the correlator $\langle TT\rangle$ can be calculated exactly in the whole scaling region from the ultra-violet to the infra-red. With this normalization the energy-momentum tensor becomes

$$T(z,\bar{z}) = T_{zz}(z,\bar{z}) = -2\pi : \partial\varphi(z,\bar{z})\partial\varphi(z,\bar{z}) : \tag{5.37}$$

with the correlator

$$\begin{aligned}\langle T(z,\bar{z})T(w,\bar{w})\rangle &= (2\pi)^2\langle : \partial\varphi(z,\bar{z})\partial\varphi(z,\bar{z}) : : \partial\varphi(w,\bar{w})\partial\varphi(w,\bar{w}) :\rangle \\ &= 2(2\pi)^2\langle\partial\varphi(z,\bar{z})\partial\varphi(w,\bar{w})\rangle^2,\end{aligned} \tag{5.38}$$

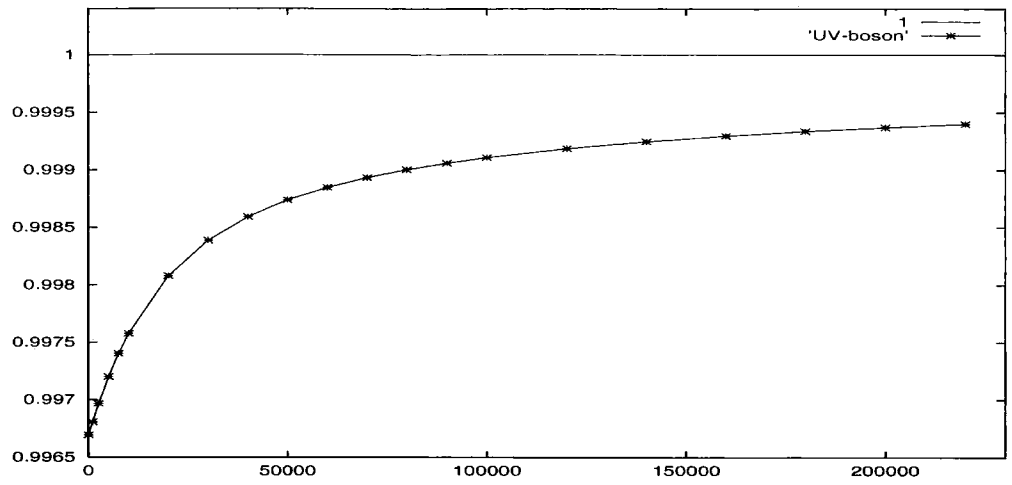


Figure 5.3: Numerical integration of $I_{UV}(\rho)$ given by (5.40) compared with the exact value $c_{UV} = 1$.

as only the double contractions survive. Using the form (5.21) of the free propagator then (5.38) is $2(\partial_z\partial_z K_0(m|z|))^2$ where we have set $w = 0$ using translation invariance. (5.38) then becomes

$$\langle T(z,\bar{z})T(0,0)\rangle = \frac{m^2|z|^2}{8z^4}\left(4K_1^2(m|z|)+m^2|z|^2K_0^2(m|z|)+4m|z|K_1(m|z|)K_0(m|z|)\right),$$

hence $F(s)$ with $k = 4$ is

$$F(s) = \left(\frac{ms^{1/2}}{2}\right)^2\left(\frac{4}{s^{1/2}}K_1^2(ms^{1/4}) + m^2K_0^2(ms^{1/4}) + \frac{4m}{s^{1/4}}K_0(ms^{1/4})K_1(ms^{1/4})\right). \tag{5.39}$$

Knowing $F(s)$ exactly the central charges follow from (5.26) directly, but let us use the relations (5.29), (5.28) in the limit of large ρ where the cut vanishes. For c_{UV} we substitute $s' = ms^{1/4}$, $\rho' = \frac{\rho}{m^4}$ in the contour C_2. From (5.29) we then get that (for $r_{IR} = \frac{1}{m^4}$)

$$I_{UV}(\rho) = \frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\, e^{4i\theta}e^{\rho' e^{2i\theta}}\left(4e^{-2i\theta}K_1^2(e^{i\theta}) + K_0^2(e^{i\theta}) + 4e^{-i\theta}K_0(e^{i\theta})K_1(e^{i\theta})\right). \tag{5.40}$$

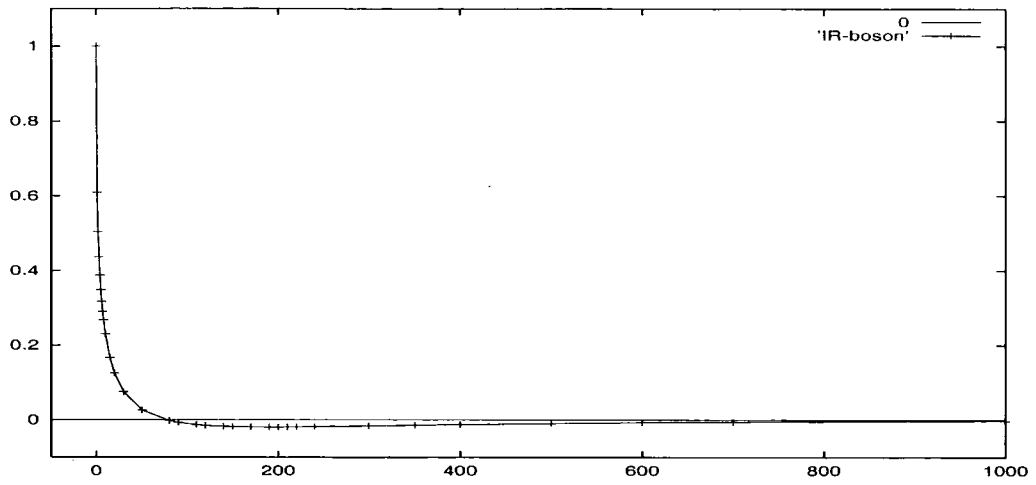


Figure 5.4: Numerical integration of $I_{IR}(\rho)$.

This is not expressible in terms of elementary functions, but it can be calculated using the analytical properties of the Bessel functions. The integration contour can be collapsed into a contour running along the imaginary axis together with an infinitesimal semi-circle around the origin. For $\rho \to \infty$ only the contribution from the infinitesimal semi-circle C_ϵ survives and here we can insert the asymptotic form of the modified Bessel functions: $K_0(z) \sim -\log z$ and $K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)(\frac{1}{2}z)^{-\nu}$ for $|z| \ll 1$. In this limit only the first term with ${K_1}^2$ then contributes, and taking into account that we only integrate over half a circle we then get the well known result $c_{UV} = 1$, which is an exact result as the contribution from the cut vanishes in this limit. This also directly follows from (5.26) and (5.39) using the asymptotic form for K_0 and K_1 for small s.

To calculate the infra-red central charge we use (5.30): $c_{IR} = \lim_{\rho\to\infty} I_{IR}(\rho) = \lim_{s\to\infty} F(s)$. From the asymptotic form of the modified Bessel functions for large $|z|$: $K_n(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}$, together with (5.39), it follows that $c_{IR} = 0$. Figure 5.3 shows the exact function $I_{UV}(\rho)$ computed by a numerical integration of (5.40) using a NAG Fortran Library integration routine. The figure shows how the contribution from the cut vanishes in the limit of large ρ.

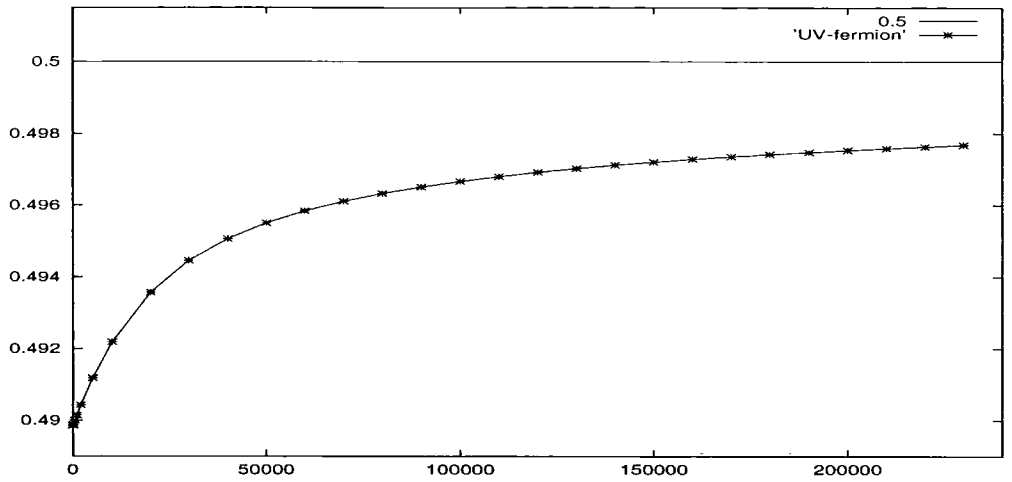


Figure 5.5: Numerical integration of $I_{UV}(\rho')$ using (5.42) compared with the exact value $c_{UV} = \frac{1}{2}$.

The exact infra-red function $I_{IR}(\rho)$ is plotted in figure 5.4, it is important to note that the minimum value of $I_{IR}(\rho)$ is very small, namely -0.019.

5.4.2 The free fermion

The free massive Majorana fermion with the component fields $\Psi = (\psi, \bar{\psi})$ has the action

$$S = \int d^2x \left(\bar{\psi}\partial\bar{\psi} + \psi\bar{\partial}\psi + im\bar{\psi}\psi\right). \tag{5.41}$$

Where the Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$ in two euclidean dimensions is represented by $\gamma^0 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\gamma^1 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$.

A calculation analogous to the Bose case gives, for $k = 4$, that

$$F(s) = \frac{m^4 s}{8}\left(K_1^2(ms^{1/4})\left(1 + \frac{4}{m^2 s^{1/2}}\right) - K_0^2(ms^{1/4})\right). \tag{5.42}$$

Taking the limit $s \to 0$ and again using $K_1(x) \to 1/x$ for $x \to 0$ yields that $c_{UV} = \frac{1}{2}$, and the infra-red contribution $c_{IR} = \lim_{s\to\infty} F(s) = 0$, using that $K_n(x) \sim e^{-x}$ for $x \to \infty$. We have calculated the exact function $I(\rho)$ by a numerical computation.

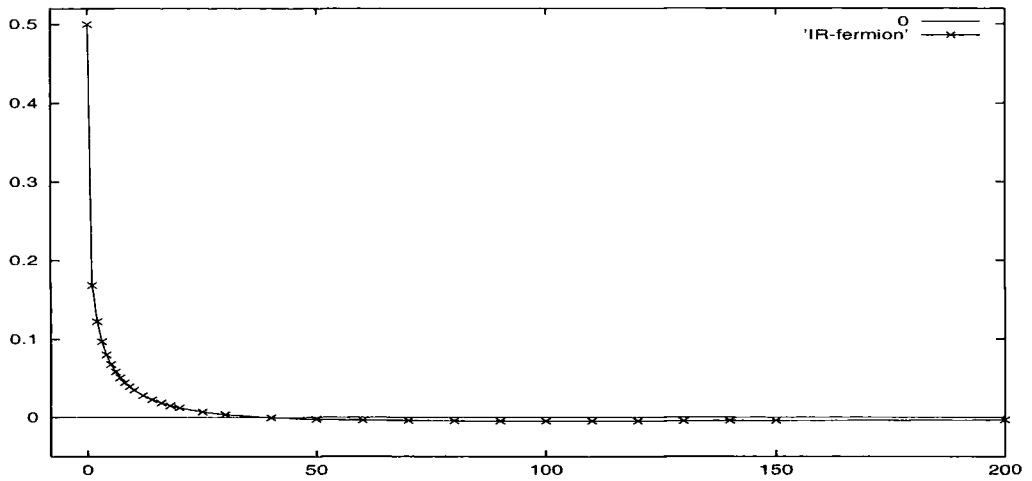


Figure 5.6: Numerical integration of $I_{IR}(\rho)$ and the exact value $c_{IR} = 0$.

In figure 5.5 $I_{UV}(\rho)$ is plotted, and in figure 5.6 $I_{IR}(\rho)$. We get the same behaviour of $I_{IR}(\rho)$ as in the bosonic case, but with the minimum value -0.0048, so this exact function does again not differ much from the infra-red central charge at its minimum value.

5.4.3 The unitary minimal models

The off critical quantum field theory picks out a specific renormalisation group flow from the ultra-violet to the infra-red conformal field theory. If the quantum field theory is in the neighborhood of one of the renormalisation group fixed points λ^* in coupling constant space, we may choose the coordinates so $\lambda^* = 0$ corresponds to the ultra-violet conformal field theory, the action can formally be written as [96]

$$S = S_{CFT} + \sum_{i=1}^{N} \lambda^i \int d^2x \; \Phi_i(x). \tag{5.43}$$

Here the $\Phi_i(x)$'s are scaling fields with scaling dimension Δ_i, the coupling constants λ^i then have mass dimension $[\lambda^i] = 2 - \Delta_i = y_i$. It was shown in chapter 3 that y_i is the renormalisation group eigenvalue of the scaling fields Φ_i. For renormalisable quantum field theories we need $y_i \geqslant 0$ and the scaling operators therefore have to be

relevant ($y > 0$) or marginal ($y = 0$). For relevant operators we will move away from the fixed point when the scale increases and S_{CFT} thus corresponds to an ultra-violet critical point.

We consider quantum field theories that have the unitary minimal models as their scaling limits. The minimal models are conformal field theories with central charge $c < 1$ and a finite number of primary fields. Descendant fields are fields formed by acting with the modes L_{-n} of $T(z)$ on a primary field and an arbitrary field is a linear combination of primary and descendant fields. The correlation functions of descendant fields can be obtained from correlation functions of primary fields acting with differential operators. The two and three point functions are, as shown in chapter 2 for the two point function, fixed by conformal symmetry. Hence, if the central charge, the operator product expansion coefficients, the finite number of primary fields and their conformal dimensions are known then all correlation functions can be calculated in the theory. The minimal models are reducible representations of the Virasoro algebra.

We denote by $\mathcal{M}_m$ the unitary minimal model that has the spin zero primary fields

$$\Phi_{(p,q)}, \quad 1 \leqslant p \leqslant m-1, \quad 1 \leqslant q \leqslant m, \tag{5.44}$$

and a central charge

$$c = 1 - \frac{6}{m(m+1)}, \quad m \geqslant 3. \tag{5.45}$$

The minimal models represent the critical points of well known statistical models, e.g. $\mathcal{M}_3$ describes the fixed point of the Ising model with $c = \frac{1}{2}$, $\mathcal{M}_4$ is the tri-critical Ising model with $c = \frac{7}{10}$ and $\mathcal{M}_5$ is the 3-state Potts model with $c = \frac{4}{5}$. The unitary minimal models $\mathcal{M}_m$ have the same $(m-1)$ critical behaviour as a Landau-Ginzburg theory with a bosonic field and even polynomial interactions up to the power $2(m-1)$ [97, 98].

$\Phi_{(p,q)}$ has the conformal scaling dimension

$$h_{(p,q)} = \bar{h}_{(p,q)} = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}, \tag{5.46}$$

which shows that $\Phi_{(p,q)} = \Phi_{(m-p,m+1-q)}$ leaving only $\frac{m(m-1)}{2}$ different primary fields. The unitary minimal model $\mathcal{M}_m$ perturbed by the relevant operator $\Phi_{(1,3)}$ can formally be written as

$$S = \mathcal{M}_m - \lambda_0 \int d^2z \; \Phi_{(1,3)}(z,\bar{z}), \quad \lambda_0 < 0, \tag{5.47}$$

and the theory is defined by the correlators

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \frac{\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{\lambda_0 \int d^2z \Phi_{(1,3)}} \rangle_{\mathcal{M}_m}}{\langle e^{\lambda_0 \int d^2z \Phi_{(1,3)}} \rangle_{\mathcal{M}_m}} \tag{5.48}$$

where $\langle \cdots \rangle_{\mathcal{M}_m}$ is the correlator in the minimal model $\mathcal{M}_m$ and $\mathcal{O}_i(x)$ are the local scaling fields in the theory. The renormalisation group eigenvalue for $\Phi_{(1,3)}$ is then: $y = 2 - \Delta = \frac{4}{m+1}$ where $\Delta = 2h_{(1,3)}$. $\Phi_{(1,3)}$ is therefore a relevant operator ($y > 0$), and from the form of $h_{(p,q)}$ in (5.46) it directly follows that the other relevant primary fields are

$$\Phi_{(p,p+s)}, \quad -1 \leqslant s \leqslant 2, \quad 1 \leqslant p+s \leqslant m, \quad 1 \leqslant p \leqslant m-1. \tag{5.49}$$

There are therefore $2m-3$ different relevant primary fields including $\Phi_{(1,1)} = \mathbf{1}$. These are the only relevant scalar fields as all the spin zero descendants will have scaling dimension $2h_{(p,q)} + 2n$ with $n \in \mathbb{N}$.

The reasons for using the operator $\Phi_{(1,3)}$ as a perturbation are that: [6]

i) $\Phi_{(1,3)}$ is a relevant field, $h_{(1,3)} = 1 - \frac{2}{m+1} < 1$, which exists in all $\mathcal{M}_m$.

ii) $\Phi_{(m,n)}$ forms an algebra under the operator product expansion. From the fusion rules of minimal models [83] it follows that $\Phi_{(1,r)}\Phi_{(1,s)} = \sum_p C_{rs}^p \Phi_{(1,p)}$ where $p \in \{|r-s|+1, |r-s|+3, \ldots, r+s-1\}$ in steps of two. This shows that $\Phi_{(1,n)}$ constitutes a sub-algebra in which only $\Phi_{(1,1)}$, $\Phi_{(1,2)}$ and $\Phi_{(1,3)}$ are relevant as seen from (5.46). $\Phi_{(1,3)}$ is normalized so that the structure constant $C_{(1,3)(1,3)}^{(1,1)} = 1$ where $\mathbf{1} = \Phi_{(1,1)} = \Phi_{(m-1,m)}$ is the identity, and it has the self coupling $C_{(1,3)(1,3)}^{(1,3)} = b(m) >$

[6] The model (5.47) is integrable and was first studied by Zamolodchikov in [96]. The only other integrable perturbations of the unitary minimal models are with $\Phi_{(1,2)}$ and $\Phi_{(2,1)}$. These three models correspond respectively to the Korteweg-de Vries, Gibbon-Samede-Kotera and Kupersmidth equations [99].

0. It does not couple to $\Phi_{(1,2)}$ so it has no coupling to other relevant operators in the sub-algebra $\Phi_{(1,n)}$ [94, 98, 83]. This means that there is perturbatively a renormalisation group flow connecting the ultra-violet and infra-red fixed points along the direction of $\Phi_{(1,3)}$ so it is consistent to include only the one relevant field $\Phi_{(1,3)}$. This can be seen perturbatively by writing the Zamolodchikov metric [93] in normal coordinates around the ultra-violet fixed point[7] $g = 0$, $G_{ij} = \delta_{ij} + O(g^2)$, the beta-functions become in these coordinates $\beta^i(g) = -y_i g^i - \pi \sum_{j,k} C^i_{jk} g^j g^k + O(g^3)$ (see below). If $g^i = 0$ for $i \neq (1,3)$ and $C^j_{(1,3)(1,3)} = 0$ for $j \neq (1,3)$ then $\beta^j(g) = 0$ for $j \neq (1,3)$ and there is no flow transverse to the $\Phi_{(1,3)}$ direction [98, 100]. It is therefore perturbatively a geodesic renormalisation group trajectory [42].

iii) $\Phi_{(1,3)}$ is the least relevant field, and the perturbation in (5.47) becomes marginal in the limit of $m \to \infty$ where $y = \frac{4}{m+1} \to 0$. In this limit the fixed points are arbitrarily close in coupling constant space, and perturbation theory is viable in the whole region from the ultra-violet to the infra-red.

We want to calculate the difference between the ultra-violet and the infra-red central charge $\Delta c = c_{UV} - c_{IR}$. It has been argued that the infra-red conformal field theory of (5.47) is given by the unitary minimal model $\mathcal{M}_{m-1}$, as Δc in the perturbative limit $y \to 0$ is given by $\Delta c = \frac{3}{16} y^3 + O(y^4)$ [94, 98], and from (5.45) we get that

$$c(m) - c(m-1) = \frac{12}{m(m^2-1)} = \frac{3y^3}{2(2-y)(4-y)} = \frac{3y^3}{16} + O(y^4). \tag{5.50}$$

A general argument for all m has been given by a thermodynamic Bethe ansatz method in [101][8]. We now describe how Δc is calculated using our approximation method, and we will compare this with the exact result $\Delta c_{exact} = c(m) - c(m-1)$.

To construct the term $\langle T(z,\bar{z}) T(0,0) \rangle$ we use the Ward identity (5.7)

$$\begin{aligned} &\partial_{\bar{z}} \langle T(z,\bar{z}) \Phi_1(x_1) \cdots \Phi_n(x_n) \rangle + \frac{1}{4} \partial_z \langle \Theta(z,\bar{z}) \Phi_1(x_1) \cdots \Phi_n(x_n) \rangle \\ &= \pi \sum_{i=1}^{n} (\delta(z-x_i)\partial_{x_i} + \delta(z-x_i) h_i \partial_{x_i}) \langle \Phi_1(x_1) \cdots \Phi_n(x_n) \rangle, \end{aligned} \tag{5.51}$$

[7] $G_{ij} = \langle \Phi_i(1) \Phi_j(0) \rangle$.

[8] In [102, 103] the first fixed points $\mathcal{M}_m$, $m = 3, 4, ..., 12$ were found numerically using the exact renormalisation group.

here Φ_i are primary fields with conformal dimension h_i. For the correlator we are interested in contact terms vanish and we get

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}\langle T(z_1,\bar{z}_1)T(z_2,\bar{z}_2)\rangle = \frac{1}{4^2}\partial_{z_1}\partial_{z_2}\langle\Theta(z_1,\bar{z}_1)\Theta(z_2,\bar{z}_2)\rangle. \tag{5.52}$$

For the perturbed theory scale invariance is broken $\partial_\mu D_\mu = \Theta \neq 0$ and in a renormalisable field theory Θ must belong to the space spanned by the relevant and marginal fields defining the perturbation away from criticality in (5.43)

$$\Theta(x) \equiv 2\pi \sum_{i=1}^{N} \beta^i(g)\Phi_i(x), \tag{5.53}$$

where $\beta^i(g)$ is the beta-function given in terms of the renormalised coupling constants g [94]. (5.52) can thus be written

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}\langle T(z_1,\bar{z}_1)T(z_2,\bar{z}_2)\rangle = \frac{\pi^2}{4}\beta^i(g)\beta^j(g)\partial_{z_1}\partial_{z_2}\langle\Phi_i(z_1,\bar{z}_1)\Phi_j(z_2,\bar{z}_2)\rangle. \tag{5.54}$$

The correlator $\langle\Phi_i\Phi_j\rangle$ can be calculated in perturbative conformal field theory using the operator product expansion in the ultra-violet conformal field theory. Let us write $\Phi_{(1,3)}(x) = \phi(x)$. The bare correlator is in the lowest order in λ_0 given by

$$\begin{aligned}\langle\phi(x)\phi(0)\rangle &= \frac{\langle\phi(x)\phi(0)e^{\lambda_0\int d^2x'\phi(x')}\rangle_{\mathcal{M}_m}}{\langle e^{\lambda_0\int d^2x'\phi(x')}\rangle_{\mathcal{M}_m}} \\ &= \langle\phi(x)\phi(0)\rangle_{\mathcal{M}_m} + \lambda_0\int d^2x'\langle\phi(x)\phi(0)\phi(x')\rangle_{\mathcal{M}_m} + O(\lambda_0^2).\end{aligned} \tag{5.55}$$

$\langle\phi(x)\phi(0)\rangle_{\mathcal{M}_m}$ was derived in (2.26), and

$$\int d^2x'\langle\phi(x)\phi(0)\phi(x')\rangle_{\mathcal{M}_m} = \frac{C^{(1,3)}_{(1,3)(1,3)}}{|x|^{2h}}\int d^2x'\frac{1}{|x'|^{2h}|x-x'|^{2h}}. \tag{5.56}$$

Using the formulas in [104, 105], (5.55) can then be written as

$$\langle\phi(x)\phi(0)\rangle = \frac{1}{|x|^{2(2-y)}}\left(1+\lambda_0\frac{4\pi b(y)A(y)}{y}|x|^y + O(\lambda_0^2)\right), \tag{5.57}$$

where $A(y) = \frac{\Gamma(1-y)\Gamma(1+y/2)^2}{\Gamma(1-y/2)^2\Gamma(1+y)} = 1+O(y^3)$. The operator product expansion coefficient $b(y)$ can be calculated from a Coulomb gas representation of the minimal models

again using the formulas in [104, 105]

$$(C^{(1,3)}_{(1,3)(1,3)})^2 = b(y)^2 = \frac{16}{3}\frac{(1-y)^4}{(1-y/2)^2(1-3y/4)^2}\left(\frac{\Gamma(1+y/2)}{\Gamma(1-y/2)}\right)^4\left(\frac{\Gamma(1-y/4)}{\Gamma(1+y/4)}\right)^3 \times \\ \times \left(\frac{\Gamma(1-y)}{\Gamma(1+y)}\right)^2\left(\frac{\Gamma(1+3y/4)}{\Gamma(1-3y/4)}\right) = \frac{16}{3} + O(y), \tag{5.58}$$

and we use the standard branch $b(y) > 0$. Choosing the renormalisation conditions $\langle\phi(x,g)\phi(0,g)\rangle|_{|x|=\mu^{-1}} \equiv \mu^4$, the renormalised correlator and the β-function becomes [100]

$$\begin{aligned} \langle\phi(x,g)\phi(0,g)\rangle &= \frac{\mu^4}{|\mu x|^{2(2-y)}}\left(1 + \frac{4\pi A(y)b(y)g}{y}(|\mu x|^y - 1) + O(g^2)\right), \\ \beta(g) &= -yg - \pi b(y) g^2 A(y) + O(g^3), \end{aligned} \tag{5.59}$$

where $\phi(x,g)$ is the renormalised field and g is the renormalised coupling. The zeros of the $\beta-$function, the renormalisation group fixed points, are thus $g_{UV} = 0$, $g^*_{IR} = \frac{-y}{\pi A(y)b(y)}$ and therefore $g \in (\frac{-y}{\pi A(y)b(y)}, 0)$ as the theory (5.43) lies between the two scaling limits. Correlators are renormalisation group improved by demanding that they satisfy the Callan–Symanzik equation (3.7). The running coupling constant becomes [100]

$$t\frac{\partial\bar{g}}{\partial t} = -\beta(\bar{g}) \quad \Rightarrow \quad \bar{g}(t) = \frac{t^y g}{1 - \frac{\pi A(y)b(y)g}{y}(t^y - 1)}, \tag{5.60}$$

interpolating between g_{UV} for $t \to 0$ and g^*_{IR} for $t \to \infty$ and satisfying $\bar{g}(1) = g$. Euclidean invariance allows us to write the correlator of the energy-momentum tensor as $\langle T(z,\bar{z})T(0,0)\rangle = \frac{\tilde{F}(\tilde{R})}{2z^4}$ in terms of the dimensionless quantity $\tilde{R} = \mu^2 z\bar{z}$. The differential equation (5.54) then becomes

$$\frac{\partial^2}{\partial\tilde{R}^2}\tilde{F}(\tilde{R}) = \frac{\pi^2\beta^2}{2\mu^4}\tilde{R}^2\frac{\partial^2}{\partial\tilde{R}^2}\langle\phi(\tilde{R})\phi(0)\rangle. \tag{5.61}$$

A solution to this equation is given by

$$\tilde{F}(\tilde{R}) = \frac{\pi^2\beta^2}{2\mu^4}\left(\tilde{R}^2\langle\phi\phi\rangle - 4\tilde{R}\int^{\tilde{R}} d\tilde{R}'\,\langle\phi\phi\rangle + 6\int^{\tilde{R}} d\tilde{R}'\int^{\tilde{R}'} d\tilde{R}''\,\langle\phi\phi\rangle\right) + \alpha_1 + \alpha_2\tilde{R},$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$. These equations directly generalize to the case with more couplings. The differential equation (5.61) is a boundary value problem as $\tilde{F}(\tilde{R})$ is known in the scaling limits

$$\tilde{F}(\tilde{R}) \to \begin{cases} c_{UV} & \text{for } \tilde{R} \to 0, \\ c^*_{IR} & \text{for } \tilde{R} \to \infty. \end{cases} \tag{5.62}$$

In the limit where $\tilde{R} \to 0$ the correlator $\langle\phi\phi\rangle$ scales as in $\mathcal{M}_m$ i.e. $\langle\phi\phi\rangle \sim \frac{1}{\tilde{R}^{2h}}$, hence $\tilde{R}^2\langle\phi\phi\rangle \sim \tilde{R}^y \to 0$, $\tilde{R}\int^{\tilde{R}} d\tilde{R}'\langle\phi\phi\rangle \sim \tilde{R}^y \to 0$, and finally $\int^{\tilde{R}} d\tilde{R}' \int^{\tilde{R}'} d\tilde{R}''\langle\phi\phi\rangle \sim \tilde{R}^y \to 0$ for $\tilde{R} \to 0$. This sets the boundary value $\alpha_1 = c_{UV}$. As $\tilde{F}(\tilde{R})$ attains a finite value for $\tilde{R} \to \infty$ then all linear terms in $F(\tilde{R})$ must cancel and we set $\alpha_2 = 0$. Integrating (5.59) and inserting the boundary conditions gives

$$\begin{aligned} \tilde{F}(\tilde{R}) =& c_{UV} + \frac{\pi^2 g^2 \tilde{R}^y}{2}\left(\frac{y(2-y)(3-y)}{y-1} + 2\pi A(y)b(y)g\left(\frac{(2-y)(3-y)}{1-y}\right.\right. \\ &\left.\left. + \tilde{R}^{\frac{y}{2}}\frac{(3y-4)(3y-6)}{3(\frac{3}{2}y-1)}\right)\right). \end{aligned} \tag{5.63}$$

As in the proof of the c-theorem we can use the Callan–Symanzik equation to move all scale dependence into the running coupling constant, defining $F(s)$ as $F(s) = \tilde{F}(\tilde{R})|_{\tilde{R}=1, g=\bar{g}(s)}$. With $t \to s^{1/4y}$ the running coupling constant becomes

$$\bar{g}(s) = \frac{g s^{\frac{1}{4}}}{1 - \frac{\pi A(y)b(y)g}{y}(s^{\frac{1}{4}} - 1)}, \tag{5.64}$$

and scale-transformations move around in the coupling constant space. The 1 loop renormalisation group improved approximation to $F(s)$ then becomes

$$\begin{aligned} F_1(s) = c_{UV} + \frac{\pi^2}{2}\bar{g}^2(s)\left(\frac{y(2-y)(3-y)}{y-1}\right. \\ \left. + 2\pi A(y)b(y)\bar{g}(s)\left(\frac{(2-y)(3-y)}{1-y} + \frac{(3y-4)(3y-6)}{3(\frac{3}{2}y-1)}\right)\right). \end{aligned} \tag{5.65}$$

Thus $k = 4$ as $F_1(s) = \tilde{F}_1(s^{1/4})$. To obtain the approximation for c_{IR} we then have to calculate the contour integral

$$I_1(\rho) = \frac{1}{2\pi i}\int_{C_0} ds \frac{e^{\rho/s}}{s} F_1(s). \tag{5.66}$$

In the limit $m \to \infty$ the ultra-violet and infra-red fixed points are perturbatively close in coupling constant space as noted above, hence $F_1(s)$ correctly describes $F(s)$ in this limit and we should take $\rho \to \infty$ in (5.28) thus eliminating the contribution from the cut. The approximation then becomes $\lim_{\rho\to\infty} I_1(\rho) = \lim_{s\to\infty} F_1(s) = c^*_{IR}$ using (5.31), and the approximation in this limit thus equals the renormalisation group improved perturbative result which is

$$\begin{aligned}\Delta c_{pert} = c_{UV} - c^*_{IR} = -\frac{\pi^2}{2}(g^*_{IR})^2 \Bigg(& \frac{y(2-y)(3-y)}{y-1} \\ & +2\pi A(y)b(y)g^*_{IR}\left(\frac{(2-y)(3-y)}{1-y} + \frac{(3y-4)(3y-6)}{3(\frac{3}{2}y-1)}\right)\Bigg) \\ = \frac{3y^3}{16} & + O(y^4),\end{aligned} \tag{5.67}$$

this is equal to the asymptotic form of the exact value $c(m) - c(m-1)$ in (5.50). We wish to improve this result using the approximation (5.32). We rewrite the running coupling constant

$$\bar{g}(s) = \frac{g s^{1/4}}{1 - \frac{\pi A(y)b(y)g}{y}(s^{1/4}-1)} = \frac{g^*_{IR}|\tilde{g}|s^{1/4}}{1+|\tilde{g}|s^{1/4}}, \qquad \tilde{g} = \frac{g}{g - g^*_{IR}} \in (-\infty, 0). \tag{5.68}$$

All dependence of the renormalised coupling g is now moved into the parameter ρ setting $s' = s|\tilde{g}|^4$ and $\rho' = \rho|\tilde{g}|^4$

$$I_1(\rho, g) = \frac{1}{2\pi i}\int_{C_0} ds \frac{e^{\rho/s}}{s} F_1(s) = \frac{1}{2\pi i}\int_{C'_0} ds' \frac{e^{\rho'/s'}}{s'} F_1(s') = \tilde{I}_1(\rho') \tag{5.69}$$

and then $\bar{g}(s') = \frac{g^*_{IR} s'^{1/4}}{1+s'^{1/4}}$. This contour integral can be evaluated for example doing a numerical integration or a series expansion in $F_1(s')$, we have done both. A numerical integration of (5.69) with $m = 14$ using the NAG Fortran Library is shown in figure 5.7. In figure 5.7 the dashed line indicates the expected behaviour of the exact function $\tilde{I}(0) - \tilde{I}(\rho')$ which is taken analogous to the curves of the free theories in figure 5.3 and 5.4. Writing $F_1(s')$ as a power series

$$F_1(s') = \Phi_1(\bar{g}(s')) = \sum_{n=0}^{\infty} h_n (s')^{n/4} \tag{5.70}$$

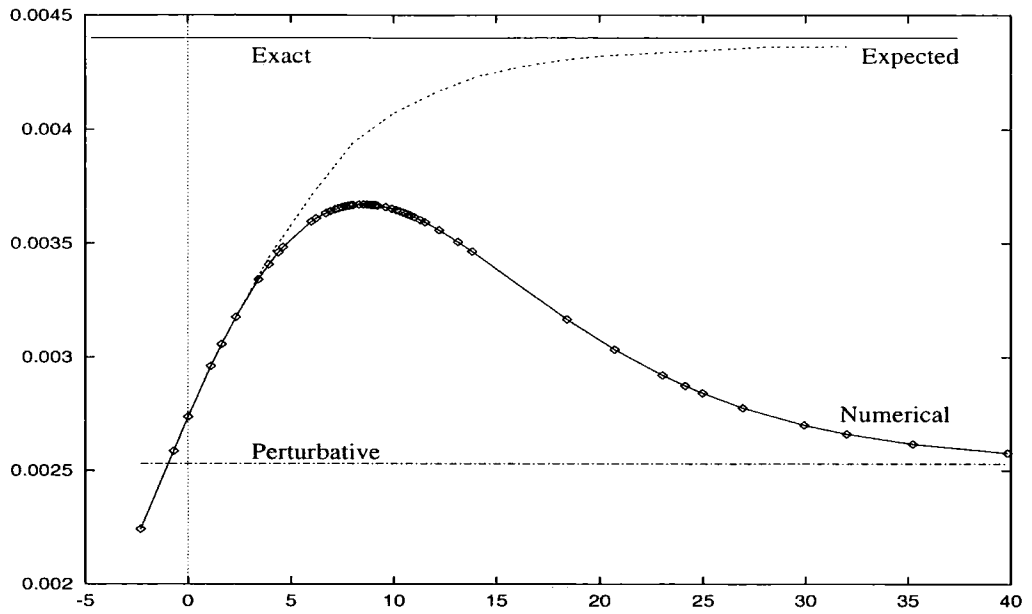


Figure 5.7: The numerical-result $c_{UV} - \tilde{I}_1(\rho')$ against $\log \rho'$ for $m = 14$. Also plotted is the exact and perturbative values Δc_{exact} and Δc_{pert}. The dashed line is the expected behaviour of $\tilde{I}(0) - \tilde{I}(\rho')$.

and substituting this into (5.69), we then obtain using the integral representation (4.13)

$$\tilde{I}_1(\rho') = \sum_{n=0}^{\infty} \frac{h_n (\rho')^{n/4}}{\Gamma(1 + n/4)} \tag{5.71}$$

which is recognized as the Borel transform of order $k = 4$. This expression can be computed numerically (using e.g. Maple) by truncating to a finite n, and the minimal value can be found. In figure 5.8 $c_{UV} - \tilde{I}_1(\rho')$ is plotted for $m = 14$ with 50 terms. The horizontal lines are the exact value and the renormalisation group improved perturbative calculation. The punctured line is the value of the numerical integration which is seen to match the value of the truncated series.

In table 5.1 the obtained values of the numerical integration denoted Δc_{approx} are listed together with the exact results Δc_{exact} and the renormalisation group improved perturbative values Δc_{pert}. For $m < 11$ the perturbative result will break

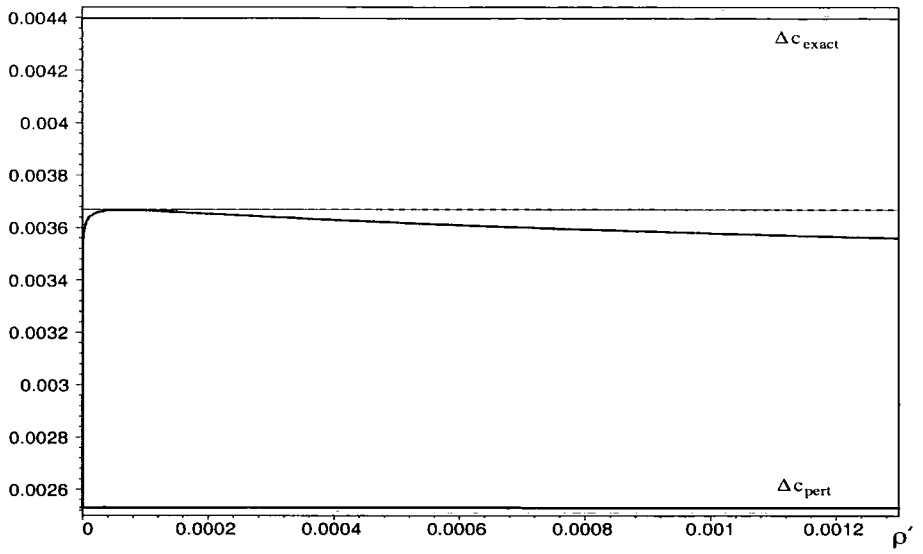


Figure 5.8: A numerical summation of $c_{UV} - \tilde{I}_1(\rho')$ for $m = 14$ with 50 terms. The upper line is the exact value and the lower line is the improved perturbative calculation. The punctured line is the value of the numerical integration.

down as Δc_{pert} becomes negative and thereby violates unitarity. In [1] we used $k = 2/y = (m+1)/2$, but here (and in [2]) we have chosen $k = 4$ as we then get a stricter bound on the exact function as explained in section 5.3.3. The results for $k = 2/y$ and $k = 4$ are similar for all m calculated, except $m = 11, 12$ where they differ slightly. The improvement of the approximation (5.32) over the renormalisation group improved perturbative result is seen to be significant. In figure 5.9 we plot $\Delta c_{exact} - \Delta c_{approx}$ and $\Delta c_{exact} - \Delta c_{pert}$ scaled with $m(m^2 - 1)$ so that all the points can be distinguished. The horizontal axis is then the exact value. The figure shows that the approximation improves the perturbative results with more than a factor two.

We saw above that the analytical opening $\alpha = \pi - \epsilon'$, $\epsilon' \ll 1$. $k > \pi/\alpha$ so the best approximation is expected to be in the limit $k \to 1$. For $k = 1$ we get results

m	Δc_{exact}	Δc_{pert}	Δc_{approx}	$\Delta c_{approx2}$
11	0.00909	0	0.00642	0.00970
12	0.00699	0.00180	0.00533	0.00721
13	0.00549	0.00248	0.00437	0.00556
14	0.00440	0.00253	0.00368	0.00440
15	0.00357	0.00237	0.00310	0.00357
16	0.00294	0.00215	0.00262	0.00293
17	0.00245	0.00191	0.00222	0.00244
18	0.00206	0.00169	0.00190	0.00205
19	0.00175	0.00149	0.00163	0.00175
20	0.00150	0.00131	0.00140	0.00150
21	0.00130	0.00116	0.00122	0.00130
22	0.00113	0.00103	0.00109	0.00113
23	0.000988	0.000911	0.000956	0.000989
24	0.000870	0.000811	0.000846	0.000871
25	0.000769	0.000725	0.000752	0.000771
26	0.000684	0.000650	0.000671	0.000686
27	0.000611	0.000583	0.000601	0.000613

Table 5.1: The exact, perturbative and approximate values for Δc. Δc_{approx} is with $k = 4$ and $\Delta c_{approx2}$ is with $k = 1$, which is the minimal value of k allowed by the bound $k > \pi/\alpha$.

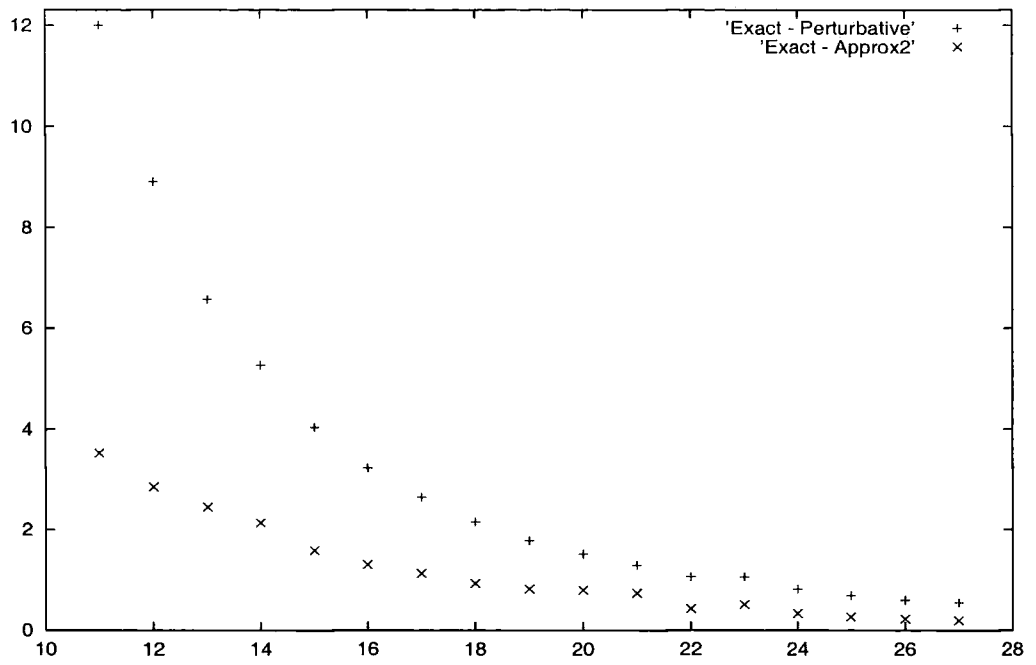


Figure 5.9: $(\Delta c_{exact} - \Delta c_{approx})m(m^2-1)$ and $(\Delta c_{exact} - \Delta c_{pert})m(m^2-1)$ against m.

which are almost identical to the exact values, we denote these by $\Delta c_{approx2}$, and plot in figure 5.10 Δc_{exact} and $\Delta c_{approx2}$ against m, the numbers are also listed in table 5.1. In this case though we do not have a strict bound on the exact function $I(\rho)$ as for the case with $k = 4$ described in section 5.3.3. The exact function might still have a very small undershoot if for example the smallest non-zero mass of a one particle state is small compared with ρ'_m, i.e. $\Delta c_{est} \sim \Delta c_{exact}$. Figure 5.10 shows the very good correspondence between Δc_{exact} and $\Delta c_{approx2}$, this is remarkable because the approximation is based on only a one loop calculation.

5.5 Conclusions

We have in this chapter, using the method derived in chapter 4 and the exact bound obtained in section 5.5.3, attained an approximation method for calculating the

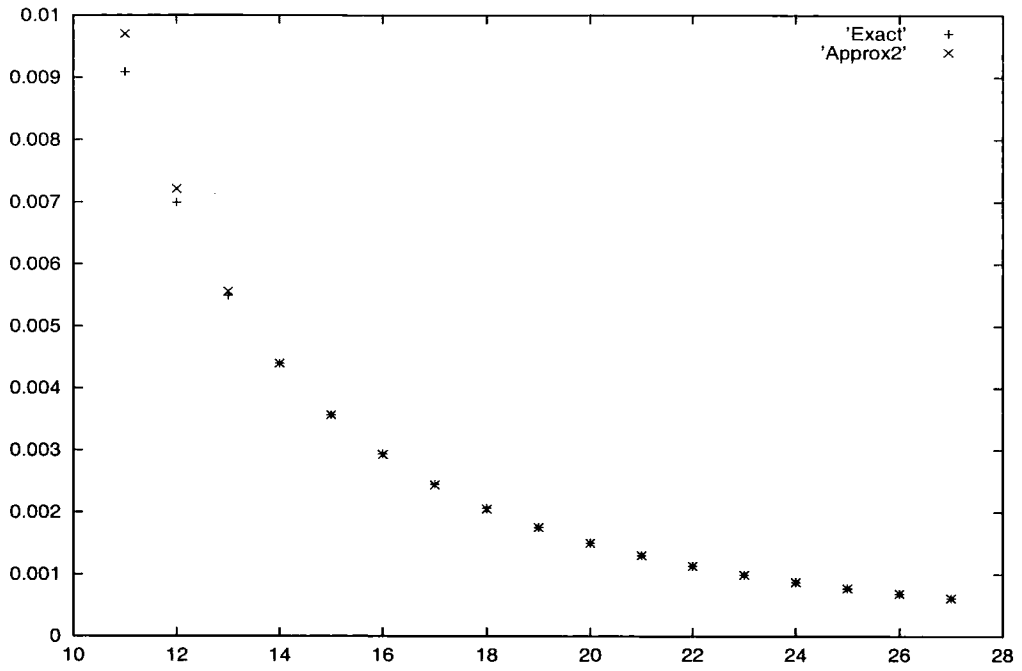


Figure 5.10: Δc_{exact} and $\Delta c_{approx2}$ against m.

infra-red central charge of a two dimensional theory. This method is applied to the free bosonic and fermionic theory, and although these theories have a trivial infra-red limit they are important as the functions used in the approximation $\tilde{F}(s)$ and $I(\rho)$ can be calculated exactly. The infra-red central charge for the minimal models perturbed by $\Phi_{(1,3)}$ is calculated using the approximation method on a renormalisation group improved one loop perturbative result, where the Callan–Symanzik equation is used to move the scale dependence into the running coupling constant. Optimizing the order of the Borel transform a remarkably good approximation for the infra-red central charge is obtained already at the one loop perturbative level.

6

Scaling in three dimensions: φ^4 theory

The other example where we will use the method developed in chapter 4 is the calculation of the critical exponents of φ^4 theory in three dimensions with $O(N)$ symmetry. This model is one of the most studied models in critical phenomena and it is important because it shares its infra-red fixed point with a number of physical models, such as: polymers ($N = 0$), the Ising model ($N = 1$), super-fluid Bose-liquid ($N = 2$) and the Heisenberg ferro-magnet ($N = 3$) [106]. We will study the Ising $N = 1$ case in detail here, but the method applies for a general N.

Quantum field theories in dimensions less than four have generic infra-red divergences. For φ^4 theory this can be seen by studying the 1PI four point function. Diagrams with n 'bubbles' are divergent when $(4-d)n \geqslant d$[1]. A way of regulating these infra-red divergences is to either do an ϵ expansion in $\epsilon = 4 - d$, or work with a massive theory. We will here do the latter, and work in $d = 3$ following Parisi [106]. We introduce the renormalised field, mass and coupling as in chapter 3: $\varphi = (Z_1)^{1/2}\varphi_R$, $\varphi^2 = Z_2(\varphi^2)_R$ and m, g denote the renormalised mass and coupling, and Z_1, Z_2 are the renormalisation constants. The infra-red divergences are now

[1]This follows as the diagram generally behaves as $p^{(d-4)n}$ in external momentum, which has a Fourier transform proportional to $\Gamma(\frac{1}{2}(d - n(d-4)))$ divergent for $n(4-d) \geqslant d$ [76, 106].

removed and only show up in the bare correlators as non-analytic dependence in the bare coupling [106]. We will calculate two of the infra-red critical exponents ν and η, all other exponents follow from scaling relations.

The calculation of the critical exponents in φ^4 theory are among the most precise calculations in quantum field theory, and they have been calculated using both the ϵ expansion, exact renormalisation group arguments, perturbative quantum field theory, high temperature expansions, strong coupling expansions and Monte Carlo simulations [107]. These exponents describe the scaling behaviour of the theory in the scaling region and are universal, as described in chapter 3, they are uniquely determined by the infra-red fixed point and are therefore identical for all quantum field theories flowing to this point under the renormalisation group, irrespective of the underlying microscopic dynamics.

6.1 Critical exponents for φ^4 theory

We are considering a theory with $O(1)$ symmetry, i.e. the $\mathbb{Z}_2$ symmetry $\varphi \to -\varphi$, in three dimensions with the lagrangian[2]

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi(x)\partial_\mu\varphi(x) + \frac{1}{2}m_0^2\varphi(x)^2 + \frac{\lambda_0}{4!}\varphi(x)^4, \quad \lambda_0 > 0. \tag{6.1}$$

In Landau theory the free energy $\mathcal{F}$ (from which all thermodynamic quantities can be derived) is written as an expansion in some order parameter [108]. For the Ising model with $\mathbb{Z}_2$ symmetry: $\mathcal{F} = \mathcal{F}_0 + a_2M^2 + a_4M^4 + O(M^6)$ where M is the magnetization. For $a_2 < 0$ there is a spontaneous symmetry breaking where $\mathcal{F} - \mathcal{F}_0$ has minima with non-zero magnetization $M_0 = \pm\sqrt{\frac{-a_2}{2a_4}}$ (for $a_4 > 0$) see figure 6.1 (b). The system is said to be in a ferro-magnetic ordered phase. For $a_2 > 0$ the minimum is at $M_0 = 0$ and the system is in a high temperature symmetric (para-magnetic) phase, figure 6.1 (a). It easily follows that $\frac{\partial M_0}{\partial a_2}$ diverges when $a_2 \to 0_-$ so the phase transition is second order with power law behaviour and critical exponents.

[2] $\varphi(x)^6$ is marginal so the critical properties are dominated by the φ^4 term, higher orders with φ^{2n} are irrelevant as $[g_n] = 3 - n$.

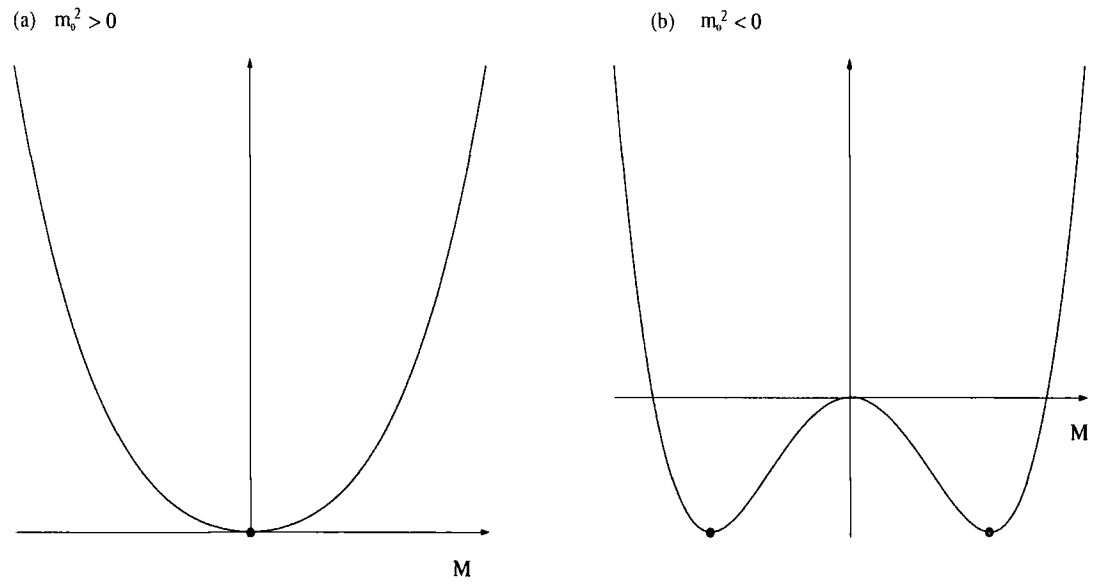


Figure 6.1: (a) The high temperature symmetric phase with $m_0^2 > 0$. (b) When $m_0^2 < 0$ the system is in a different phase with a spontaneous magnetization.

We can then write $a_2 = \tilde{a}_2(T - T_c)$ for some $\tilde{a}_2 > 0$ so the phase transition occurs at the critical temperature T_c [108]. Comparing with (6.1) it follows that the bare mass squared is proportional to the reduced temperature $m_0^2 \propto \theta = \frac{T-T_c}{T_c}$. We are considering the high temperature symmetric phase with $m_0^2 > 0$. For the free theory with $\lambda_0 = 0$ the two point correlator, or propagator, is given as

$$G_2(p, m) = \int d^d x \; e^{ipx} \langle \varphi(x)\varphi(0) \rangle = \frac{1}{p^2 + m^2}, \tag{6.2}$$

hence the free propagator can be written as [106]

$$G_2(p, \theta) = \theta^{-\gamma} h(\tfrac{p}{\theta^\nu}) \tag{6.3}$$

where h is regular at the origin and $\gamma = 1$ and $\nu = \frac{1}{2}$. This scaling behaviour is assumed to hold also for the interacting theory, where $\lambda_0 > 0$, with critical exponents ν and γ, and one of the main problems in critical phenomena is to determine these exponents. From the scaling relations described below all critical exponents can be

obtained from the propagator. In real space the free propagator becomes [60, 11]

$$G_2(x,m) = \int \frac{d^dp}{(2\pi)^d}\frac{e^{ipx}}{p^2+m^2} = \frac{1}{2\pi}\left(\frac{m}{2\pi|x|}\right)^{\frac{d-2}{2}} K_{\frac{d-2}{2}}(m|x|). \tag{6.4}$$

The asymptotic form of the modified Bessel function (see section 5.4.1) then shows that away from the fixed point where $\frac{1}{m} \ll |x|$ the propagator is exponentially damped, and close to the fixed point where $|x| \ll \frac{1}{m}$ it has a power law behaviour: $G_2(\tilde{t}|x|,m) \sim \tilde{t}^{2-d}$. For the interacting theory this is generalized to $G_2(\tilde{t}|x|,m) \sim \tilde{t}^{2-d-\eta}$ for the critical exponent η, in momentum space it becomes

$$G_2(tp) \sim \frac{1}{t^{2-\eta}}, \tag{6.5}$$

and the gaussian fixed point therefore has $\eta = 0, \gamma = 1$ and $\nu = \frac{1}{2}$.

6.2 $\tilde{F}(s)$ for η and ν

We will now define the functions $\tilde{F}(s)$ from chapter four for respectively ν and η, and show that they are analytic in some sector with the correct scaling limits. We denote the infra-red fixed point coupling g_c so that $\beta(g_c) = 0$, where g is the dimensionless renormalised coupling $g = f(\lambda_0/m) = \frac{\lambda_0}{m} + O((\frac{\lambda_0}{m})^2)$. In the infra-red limit $t \to 0$ the Callan–Symanzik equation (3.7) becomes

$$\left(-t\frac{\partial}{\partial t} + \gamma_1(g_c) - 2\right) G_2^R(tp) = 0, \tag{6.6}$$

where[3] $\gamma_1 = \frac{\partial \ln Z_1}{\partial \ln m}$ so that for small t: $G_2^R(tp) \sim \frac{1}{t^{2-\gamma_1(g_c)}}$ and this shows that $\eta = \gamma_1(g_c)$ (see e.g. [106, 109, 110] for further details). Hence defining

$$\tilde{F}_\eta(t) = t\frac{\partial G_2^R(tp)}{\partial t}/G_2^R(tp) + 2, \quad t \in \mathbb{R}_+, \tag{6.7}$$

would satisfy the infra-red behaviour $\lim_{t\to 0}\tilde{F}_\eta(t) = \eta$. The ultra-violet limit follows from the spectral representation

$$\tilde{F}_\eta(t) = -2\,\frac{\int_0^\infty d\mu^2\ \tilde{c}(\mu^2,m,g)\frac{t^2p^2}{(t^2p^2+\mu^2)^2}}{\int_0^\infty d\mu^2\ \tilde{c}(\mu^2,m,g)\frac{1}{t^2p^2+\mu^2}} + 2 \to 0 \quad \text{for } t \to \infty, \tag{6.8}$$

[3]Note that γ from chapter three is $\gamma = \gamma_1/2$.

or simply using that $G_2^R(p) \to \frac{1}{p^2+m^2}$ in the ultra-violet (the spectral density will only contribute with $\delta(\mu^2 - m^2)$) also showing that $\lim_{t\to\infty} \tilde{F}_\eta(t) = 0$. From this spectral decomposition it also directly follows that $\tilde{F}_\eta(s) = \tilde{F}_\eta(ts)|_{t=1}$ is analytic for $s \in S(\pi - \epsilon')$ for $\epsilon' \ll 1$ [4], and that

$$\tilde{F}_\eta(s) \to \begin{cases} \eta & \text{for } s \to 0_+, \\ 0 & \text{for } s \to \infty. \end{cases} \tag{6.9}$$

Equation (3.6) states that

$$-m\frac{\partial G_2^R(tp,m)}{\partial m} = t\frac{\partial G_2^R(tp,m)}{\partial t} - [G_2]G_2^R(tp,m) = \left(t\frac{\partial}{\partial t} + 2\right) G_2^R(tp,m),$$

hence it follows that

$$\tilde{F}_\eta(t) = -m\frac{\partial G_2^R(tp,m)}{\partial m}/G_2^R(tp,m) = Z_1^{-1} m\frac{\partial Z_1}{\partial m}, \tag{6.10}$$

also showing that the ultra-violet value is zero as the gaussian theory is ultra-violet finite. We also want to compute ν, but let us first comment on the scaling relations. The scaling hypothesis states that the singular power law behaviour near a second order phase transition stems from the divergence of one quantity at the critical point, namely the correlation length $\xi \equiv m^{-1} \sim \theta^{-\nu}$. From the definition of γ_1 it follows that for small m [76]: $Z_1 \sim k_1 m^{\gamma_1(g_c)}(1 + o(1))$ for some $k_1 \in \mathbb{R}$, and $G_2(p) = Z_1 G_2^R(p) = Z_1/(p^2 + m^2 + O(p^4))$, hence together with (6.3) this shows that

$$G_2(0) \sim \theta^{-\gamma} \sim \theta^{-\nu(2-\eta)} \quad \Rightarrow \gamma = \nu(2-\eta), \tag{6.11}$$

which is Fisher's scaling relation, clearly satisfied by the free theory[5].

Similar to γ_1 we define [109] $\gamma_2 = \frac{\partial \ln Z_2}{\partial \ln m}$. In [106] it was shown that $Z_2 \propto \frac{\partial m^2}{\partial m_0^2}$ which means that $Z_2 \sim \frac{\partial m^2}{\partial \theta} \sim \theta^{2\nu-1}$ and from the definition of γ_2 it follows that

[4] Actually it will be analytic in $S(\pi)$ but to get the Borel transform $\tilde{F}_\eta(s)$ needs to be bounded at the origin so that $\tilde{F}_\eta(s)$ has to be bounded for all closed subsets of S which would not be the case with $S(\pi)$.

[5] The others: $\alpha = 2 - \nu d$, $\beta = \nu(d - 2 + \eta)/2$, $\delta = (d + 2 - \eta)/(d - 2 + \eta)$ are shown in [109] using the ϵ expansion.

$Z_2 \sim \theta^{\nu\gamma_2(g_c)}$ in the infra-red limit, hence $\gamma_2(g_c) = 2 - \nu^{-1}$. The Callan–Symanzik equation for correlators $G^R_{n,l}(p)$ with l insertions of $\frac{1}{2}\varphi^2$ reads [109]

$$\left(-t\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g} + [G_{n,l}(p)] + \frac{1}{2}n\gamma_1 + l\gamma_2\right) G^R_{n,l}(tp, tq) = 0 \tag{6.12}$$

where $[G_{n,l}(p,q)] = [G_n(p)] - 2l = d - n(d+2)/2 - 2l$. Hence for $t \to 0$ at the critical point is [109]

$$G^R_{n,l}(tp, tq, m) \sim t^{d-\frac{1}{2}n(d+2-\eta)-\frac{l}{\nu}}, \tag{6.13}$$

and in the limit $t \to 0$ will

$$t\frac{\partial G^R_{2,1}(tp, tq, m)}{\partial t}/G^R_{2,1}(tp, tq, m) = -2 + \eta - \frac{1}{\nu}, \tag{6.14}$$

so we define

$$\tilde{F}_\nu(t) = -\tilde{F}_\eta(t) + 4 + t\frac{\partial G^R_{2,1}(tp, -tp, q=0, m)}{\partial t}/G^R_{2,1}(tp, -tp, q=0, m). \tag{6.15}$$

Again using the homogeneity of $G_{2,1}$ we see that $\tilde{F}_\nu(t)$ can be rewritten as

$$\tilde{F}_\nu(t) = -\frac{m\frac{\partial}{\partial m}(G^R_{2,1}(tp, -tp, q=0, m)/G^R_2(tp, m)}{(G^R_{2,1}(tp, -tp, q=0, m)/G^R_2(tp, m))} = {Z_2}^{-1} m\frac{\partial Z_2}{\partial m}, \tag{6.16}$$

which shows that $\lim_{t\to\infty} \tilde{F}_\nu(t) = 0$, again because Z_2 is a constant in this limit. The equation (6.15) shows that $\tilde{F}_\nu(s) = \tilde{F}_\nu(ts)|_{t=1}$ is analytic in $S(\pi - \epsilon')$ for all $\epsilon' \in (0, \pi)$. This follows from writing $G^R_{2,1} = G_{2,1}Z_1^{-1}Z_2^{-1}$ and using that $G_{2,1}(p, -p, q=0) = -\frac{\partial}{\partial m_0^2}G_2(p)$ [109] and then rewriting in terms of renormalised quantities and doing the spectral representation. $\tilde{F}_\nu(s)$ then satisfies

$$\tilde{F}_\nu(s) \to \begin{cases} 2 - \frac{1}{\nu} & \text{for } s \to 0_+, \\ 0 & \text{for } s \to \infty. \end{cases} \tag{6.17}$$

6.3 The infra-red limit

The functions $\tilde{F}_\eta(g)$ and $\tilde{F}_\nu(g)$ have been calculated up to an amazing 7 loops in the coupling constant g. We will follow the convention of [111] where $\tilde{g} = \frac{3}{16\pi}g$ and

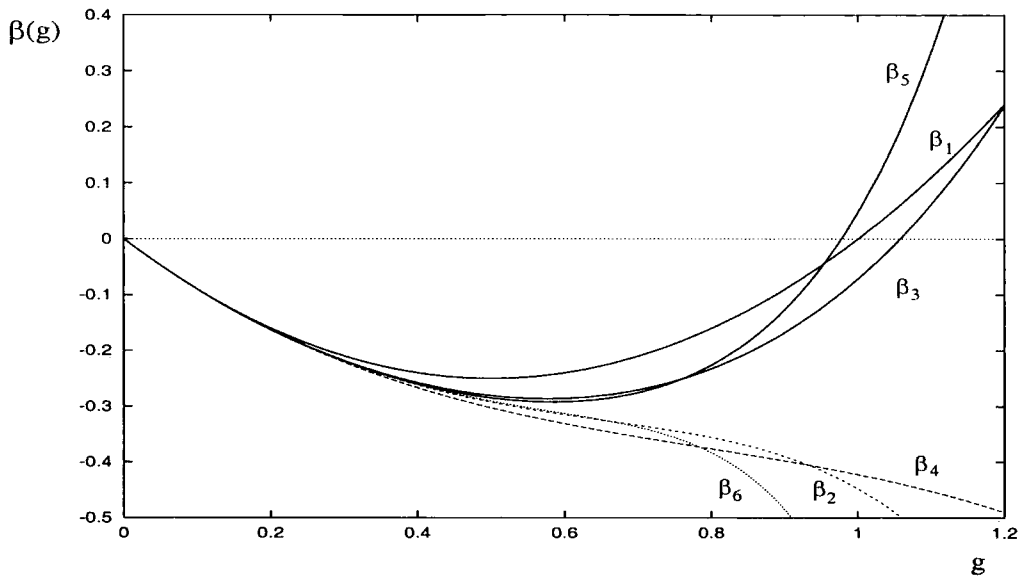


Figure 6.2: The first 6 orders of the perturbative β-function, the plot shows the dependence of the critical coupling on the order of perturbation theory.

$\tilde{\beta}(\tilde{g}) = \frac{3}{16\pi}\beta(g)$ giving ± 1 as the first two coefficients in the $\tilde{\beta}$ function; we will use β, g to denote $\tilde{\beta}, \tilde{g}$ below to simplify the notation. The results are that [111]

$$\begin{aligned}
\tilde{F}_\eta(g) &= 0.0109739369g^2 + 0.0009142223g^3 + 0.0017962229g^4 \\
&\quad - 0.0006536980g^5 + 0.0013878101g^6 - 0.001697694g^7, \\
\tilde{F}_\nu(g) &= \tfrac{1}{3}g - 0.0631001372g^2 + 0.0452244754g^3 - 0.0377233459g^4 \\
&\quad + 0.0437466494g^5 - 0.0589756313g^6 + 0.09155179g^7, \\
\beta(g) &= -g + g^2 - 0.4224965707g^3 + 0.3510695978g^4 \\
&\quad - 0.3765268283g^5 + 0.49554751g^6 - 0.74968893g^7,
\end{aligned} \tag{6.18}$$

where the beta-function has been calculated up to 6-loops [111, 112]. In figure 6.2 we plot the nth order perturbative β-function for $n = 1, \ldots, 6$: $\beta_n(g) = -g + \sum_{k=2}^{n+1} g^k \tilde{\beta}_k$.

The perturbative series for φ^4 theory was in [77, 78, 79] rigorously shown to be Borel summable in the coupling of order $k = 1$, so we set $F(s) = \tilde{F}(s^1)$. As discussed

in chapter 4 (see (4.9)), the high order coefficients of the perturbative correlation functions $f(g) = \sum_n f_n g^n$ for large n are given as

$$f_n = c(-\tilde{b})^n \Gamma(1 + b_l + n)(1 + O(1/n)) \tag{6.19}$$

where $\tilde{b}, b_l, c$ were calculated in [67, 113]. $\tilde{b} = 0.14777422$ and it follows that the convergence radius of the Borel transform of $f(g)$ is given by $b_s = 1/\tilde{b}$ with a singularity at $-1/\tilde{b}$. The critical exponents can now be obtained by first Borel transforming the series (6.18) and then analytically continuing them to $\mathbb{R}_+$ by either a conformal mapping or a Padé approximation. The Laplace transform can then be performed, e.g. numerically, and resummed series are thereby obtained. The first non-trivial zero of the resummed β-function on the real axis determines the critical coupling g_c, which substituted into the resummed $\tilde{F}_\eta(g)$, $\tilde{F}_\nu(g)$ gives an approximation for η and $2 - 1/\nu$. This procedure has some deficiencies, it is very sensitive to variations of g_c and the continuation with Padé approximation might introduce new poles at or near the real axis that will lead to artificially large contributions in the Laplace integral. Continuation with a conformal mapping might also lead to poles close to the positive real axis if there are sub-leading instanton contributions giving poles away from the negative real axis.

Here we will use the approximation developed in chapter 4 and move the scale dependence in $\tilde{F}_\eta(s)$, $\tilde{F}_\nu(s)$ into the running coupling constant and then get the infra-red limit as the limiting value of the Borel transform. An analytical continuation still has to be performed and we will use a conformal mapping that maps $s \to \infty$ to $s \to 1$. A final approximation is then to do a Padé approximation at $s = 1$. The advantage of this approach is that g_c does not have to be estimated and the Laplace integral is not calculated.

When $b_l > 0$ we want to perform a Borel-Leroy transform of order k, which replaces a formal series $\hat{f} = \sum_n f_n z^n$ with $\sum_n (f_n/\Gamma(1 + b_l + n/k) z^n$. Instead of (4.23) we then define

$$I(\rho) = \frac{\Gamma(1 + b_l)}{\rho^{kb_l}} \frac{k}{2\pi i} \int_{\tilde{C}} \frac{ds}{s} s^{kb_l} e^{(\rho/s)^k} F(s), \tag{6.20}$$

this function will then have the limits $\lim_{\rho\to\infty} I(\rho) = F_{IR}$ and $\lim_{\rho\to 0} I(\rho) = F_{UV}$ as before.

We introduce the scale parameter s via the exact running coupling $\bar{g}(s)$ ($s \to 0$ in the ultra-violet) then we know that $F_\eta(\bar{g}(s))$ and $F_\nu(\bar{g}(s))$ are asymptotic series in $\bar{g}(s)$. We now approximate the exact running coupling $\bar{g}(s)$ with the solution to

$$s\frac{d\bar{g}_n(s)}{ds} = -\beta_n(\bar{g}_n) \tag{6.21}$$

with the boundary condition $\bar{g}_n'(0) = 1$, where $\beta_n(g)$ is the perturbative β function to the nth order. $F_\eta(\bar{g}_n(s))$, $F_\nu(\bar{g}_n(s))$ are then asymptotic series in s, which we will write as $F(s) = \sum_m F_m s^m$. The Borel-Leroy transform given by (6.20) becomes

$$I(\rho, b_l, k) = \Gamma(1+b_l)\cdot\sum_{m=1}^{n+1}\frac{F_m\ \rho^m}{\Gamma(1+b_l+m/k)} = \sum_{m=1}^{n+1} I_m\rho^m. \tag{6.22}$$

As we are only working with a truncated series we will set $k = 1$ (only infinitesimally different from $k = 1+\delta$, $\delta \ll 1$), note we are summing up to $n+1$ as the nth order β function has $n+1$ terms, and $F(s)$ are known to the $(n+1)$th order.

6.4 Conformal mapping and Padé approximation

Let us describe how we do the analytical continuation of the Borel transform to get its infra-red limit.

The Borel transform is analytic in a sector $S(\epsilon'')$ where $\epsilon'' \ll 1$. The series given by (6.22) will have a pole at the negative real axis for some value $\rho = -r_c$ ($r_c > 0$) determining its convergence radius.

6.4.1 Conformal mapping

We will now analytically continue $I(\rho)$ given by (6.22) doing the conformal transformation (as in [76])

$$t = \frac{\rho}{r_c+\rho},\ \rho \notin (-\infty, -r_c], \quad (\rho = \frac{r_c t}{1-t}), \tag{6.23}$$

so that $\tilde{I}(t, b_l) = I(\frac{r_c t}{1-t}, b_l)$, and the limit $\rho \to \infty$ is replaced by $t \to 1$. Then $\tilde{I}(t) = I(\frac{r_c t}{1-t}) = \sum_{m=1}^{n+1} \tilde{I}_m t^m$. Using that $(1-y)^{-k} = (\sum_{m=0}^{\infty} y^m)^k = \sum_{m=0}^{\infty} \frac{(k+m-1)!}{(k-1)!(m)!} y^m$ it follows that $\tilde{I}_m = \sum_{j=1}^{m} (r_c)^j I_j \frac{(m-j+j-1)!}{(m-j)!(j-1)!}$ hence

$$\tilde{I}(t) = \sum_{m=1}^{n+1} \tilde{I}_m \, t^m, \qquad \tilde{I}_m = \sum_{j=1}^{m} I_j r_c^j \binom{m-1}{j-1}. \tag{6.24}$$

If all poles of $I(\rho)$ lies in $(-\infty, -r_c]$ then $\tilde{I}(t)$ given by this series is convergent for $t \in [0, 1)$. In [114] it was argued that this is indeed the case for the φ^4 theory since all known instanton contributions have negative action leading to a Borel transform analytic in a cut plane with the singularity closest to the origin being given by the large order estimates. We want to evaluate this expression at the convergence radius $t = 1$ (where $\tilde{I}(t)$ is regular), to do this we will use Padé approximants [115, 116].

6.4.2 Padé approximation

In Padé approximation a function $f(z) = \sum c_n z^n$ is approximated by a rational fraction, called the Padé approximant $[L/M]$, with polynomials of degree L and M in the numerator and denominator, and the constant 1 in the denominator

$$[L/M](z) = \frac{a_0 + a_1 z + \cdots a_L z^L}{1 + b_1 z + \cdots + b_M z^M}. \tag{6.25}$$

If the n first terms in a power series expansion of the function are known then these coefficients are matched with the coefficients of the polynomials where $0 < L + N \leq n$, so that[6]

$$f(z) = [L/M](z) + O(z^{L+M+1}). \tag{6.26}$$

Because of the polynomial in the denominator $[L/M]$ can describe functions with poles, and it can therefore serve as an approximation of the analytical continuation of $f = \sum c_n z^n$ outside of the convergence radius. An illustrative example [117] is the function

$$f(z) = \sqrt{\frac{1+2z}{1+z}} = 1 + \frac{1}{2}z - \frac{5}{8}z^2 + O(z^3), \tag{6.27}$$

[6]See [115] for a discussion about uniqueness of Padé approximants and their convergence.

where we are interested in the value $\lim_{x\to\infty} f(x)$, $x \in \mathbb{R}$. The Taylor series diverges for $|z| > \frac{1}{2}$ (as $f'(-1/2)$ diverges) and one way to get the limiting value $x \to \infty$ from the series data is to do an analytical continuation by performing a Taylor expansion around x_0 close to $\frac{1}{2}$ and so on, but it will take a long time to reach infinity. A more feasible way is to make the conformal mapping: $z = \frac{w}{1-2w}$, $w = \frac{z}{1+2z}$ sending $z \to \infty$ into $w \to \frac{1}{2}$. The Taylor expansion of $f(z(w))$ becomes $f(w) = (1-w)^{-1/2} = 1 + \frac{1}{2}w + \frac{3}{8}w^2 + O(w^3)$, which is convergent at $w = \frac{1}{2}$ and the first truncations gives: 1, 1.25, 1.34375. For the Padé approximation the first approximant easily follows from (6.27)

$$[1/1] = \frac{1 + \frac{7}{4}z}{1 + \frac{5}{4}z}, \tag{6.28}$$

giving $\lim_{x\to\infty}[1/1](x) = 7/5 = 1.4$. This is less than 2% off the exact value at infinity $\lim_{x\to\infty} f(x) = \sqrt{2} = 1.4142356237$ (the next two diagonal approximants give 1.414201183 and 1.414213198).

The Padé table is a matrix with entries $[L/M]$. The Padé approximation is based on the conjecture that there is a subsequence of diagonal Padé approximants $[L/L]$ which converge uniformly to the function, and this conjecture [118] has shown to hold in practice. The diagonal Padé approximants are conformally invariant and are therefore independent of r_c above. Generally we should use Padé approximants close to the diagonal. Note that according to the Padé conjecture one should still expect convergence of the Padé approximants even if there are a finite number of poles within $|t| < 1$, which is the case if not all poles of $I(\rho)$ are in $(-\infty, -r_c]$.

We also have to determine the values of r_c and b_l. Let us first note that from the boundary condition $\bar{g}_n'(0) = 1$ we have that $\bar{g}_n(s) = s + O(s^2)$ and in $F(s) = \sum_m F_m s^m$ we thus have that $F_m = \hat{F}_m + \cdots$ where $\hat{F}_m$ is the mth coefficient of g in $F(g)$, this means that the convergence radius of the transformed series cannot be larger than the convergence radius of the series in the coupling g, i.e. $0 < r_c \leq b_s = 1/\tilde{b}$. Also we would suspect that $b_l \sim b'_l$, where b'_l is the Leroy parameter of the series in g. If b_l is chosen too large we will divide by more than the actual asymptotic increase in (6.24) and the approximation value will be too small, if b_l is

chosen too small we expect to get a poor convergence in the Padé table. In the same way we expect the approximate value to be too small if r_c is chosen smaller than the actual convergence radius because the values we are calculating are increasing from zero. This is the behaviour we see in the tables and we estimate r_c and b_l to be respectively the largest and smallest value so that there is convergence in the Padé table.

From the asymptotic form of the β-function (6.19) it follows that it is alternating, and an approximation to a truncated alternating asymptotic series $\sum_{l=1}^{n} f_l$ is to use the series $\sum_{l=1}^{n} \tilde{f}_l$ where $\tilde{f}_l = f_l$ for $l < n$ and $\tilde{f}_n = f_n/2$ [119]. We have also obtained the approximation where the perturbative β-function to the nth order (for $n = 5$ and $n = 6$) is approximated in this way.

Using the β function with $1/2$ times the last coefficient, and the choice of coefficients r_c, b_l mentioned above, i.e. $r_c = b_s$ and $b_l = 2.4$, we get the Padé table (the best convergence is with the 5th order expression)[7]

$$\begin{pmatrix} \cdot & \cdot & 0.407663 & 0.405138 & 0.404239 \\ \cdot & 0.362383^{*} & 0.405248 & 0.403736 & \\ \cdot & 0.395111 & 0.404338 & & \\ \cdot & 0.400335 & & & \\ \cdot & & & & \end{pmatrix}, \tag{6.29}$$

for the function $F_\nu(s)$. Only numbers in the table for which there is convergence should be sampled [118], hence discarding 0.395111 gives $F_{\nu,IR} = 0.404 \pm 0.004$, where the error is the inter-tabular error for the points chosen (which is here larger than the Baker-Hunter error [118, 120]). With the usual β-function we get, with the

[7]The numbers marked with a * has a pole close to or in the interval $(0, 1)$ and are therefore discarded.

same values for r_c and b_l

$$\begin{pmatrix} \cdot & \cdot & 0.407663 & 0.405138 & 0.386709 & 0.396082 \\ \cdot & 0.362383^* & 0.405248 & 0.408042 & 0.392824 & \\ 0.414164 & 0.395111 & 0.392686 & 0.395422 & & \\ 0.377688 & 0.392075 & 0.394118 & & & \\ 0.4843^* & 0.403122 & & & & \\ 0.406194 & & & & & \end{pmatrix} \tag{6.30}$$

for the function $F_\nu(s)$ leading to $F_{\nu,IR} = 0.402 \pm 0.008$. Averaging over these two we get for the critical exponent $\nu = 0.626 \pm 0.003$. At this value of b_l and r_c we have the best convergence in the Padé table, a more conservative estimate of $F_{\nu,IR}$ is obtained by varying r_c and b_s in a region around these values and then calculate the corresponding Padé tables and sample the highest and lowest values for which there is some convergence in the Padé table. From this variation we find that $F_{\nu,IR} = 0.40 \pm 0.01$ leading to

$$\nu = 0.625 \pm 0.004. \tag{6.31}$$

Such a variation also has to be performed when expanding in the coupling as the asymptotic behaviour might not have been reached for the low number of terms available. In [121, 122] it was argued that the results obtained using Borel-Leroy transforms were very stable under variation of b_l and b_s in a wide range around the exact asymptotic values, and we see the same stability here.

For the other exponent η we get approximately the same Padé tables using either the β-function given in (6.18) or with a half times the last term, the convergent Padé table becomes

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0.0291884 & 0.0321865 & 0.0326601 & \\ \cdot & 0.0271003 & 0.0489937^* & 0.0327393 & & \\ 0.0323937 & 0.0310551 & 0.031359 & & & \\ 0.0321239 & 0.0312882 & & & & \\ 0.0312289 & & & & & \end{pmatrix}, \tag{6.32}$$

and we obtain the critical exponent $\eta = 0.0319 \pm 0.0010$, where $r_c = b_s$ and $b_l = 1.8$. Again varying around these values gives the estimate

$$\eta = 0.0315 \pm 0.0020. \tag{6.33}$$

These numbers should be compared with $\nu = 0.6304 \pm 0.0013$ [107], $\nu = 0.6300 \pm 0.0015$ [114], $\nu = 0.6290 \pm 0.0025$ (ϵ expansion [107]), $\nu = 0.6289 \pm 0.0008$ (Monte Carlo) and for the other exponent $\eta = 0.0355 \pm 0.0025$ [107], $\eta = 0.032 \pm 0.003$ [114], $\eta = 0.0360 \pm 0.0050$ (ϵ expansion [107]), $\eta = 0.0374 \pm 0.0014$ (Monte Carlo), $\eta = 0.0347 \pm 0.001$ (strong coupling [123]), most of these numbers are taken from [107].

The error in ν for our result is seen to be larger than the one in η in contrast to the errors from other methods. The reason for this is that we have chosen a conservative estimate where we have averaged over the results with two different β-functions (using its alternating behaviour). The two β-functions differ only in the 7th order term, and here the F_η term is about a factor 100 smaller than the F_ν term resulting in less sensitivity to this averaging. The errors obtained could most likely be lowered e.g. using one of the techniques from [114] or as in [116] using the information about the pole at r_c to obtain an extra order in the Padé table.

6.5 Conclusions

In this chapter the approximation method developed in chapter 4 has been used together with a conformal mapping and a Padé approximation to calculate the critical exponents of φ^4 theory in three dimensions. Our estimates of the critical exponents are within the errors of other, more elaborate approximations. In the usual evaluation one determines the value of the critical coupling g_c and then evaluates the resummed series at this point. The values then become very dependent on the estimate of g_c. One advantage of our method is that we do not have to estimate g_c, likewise we do not have to perform the Laplace integral, but instead the critical point is reached taking the limit of the Borel transform. The disadvantage of

the method is that we do not have specific knowledge of the quantities r_c, b_l governing the asymptotic behaviour for the transformed series in the scale parameter. In the usual case of the perturbative expansion in the coupling these parameters are obtained from estimates of the higher order behaviour of perturbation theory. However, our results showed stability in a variation of these parameters around the values from high order estimates.

7

The exact renormalisation group with exponential interaction

In this chapter the exact renormalisation group equation is used to study the renormalisation group flow of a two dimensional theory.

The exact renormalisation group equation is a functional differential equation which gives a non-perturbative description of how the wilsonian effective action must vary when the cut-off changes so that the physics is invariant. It is derived by integrating out the ultra-violet degrees of freedom in the partition function. The equation is not solvable and has to be approximated.

We will do this by including in the wilsonian effective action only relevant operators together with a background charge at infinity. The motivation for studying this interaction is the renormalisation group flow between the unitary minimal models $\mathcal{M}_m$ perturbed by $\Phi_{(1,3)}$ that we considered in chapter 5. Using the equivalence between the perturbed minimal models and the quantum group restricted sine-Gordon model, we will argue that this interaction describes the renormalisation group flow of the perturbed unitary minimal models.

The exact renormalisation group allows for a study of the renormalisation group flow of the perturbed model $\mathcal{M}_m$ for all m, and not only in the limit $m \to \infty$ where the ultra-violet and infra-red fixed points approach each other in the coupling

constant space as seen in chapter 5.

In the next section the Coulomb gas representation of the perturbed minimal models is described together with its realization as a quantum group restricted sine-Gordon model. Then the exact renormalisation group equation is introduced, and it is approximated by only allowing relevant operators in the effective action, which for the minimal models are given by exponential operators. The wilsonian effective potential does not contain any field derivatives so the approximation is similar to the local potential approximation. The non-linear term in the exact renormalisation group equation is approximated using the operator product expansion.

For the perturbed minimal models a renormalisation group equation is obtained where all higher order terms in the coupling are contained in the off-critical structure constant for the operator product expansion. This renormalisation group equation is valid for all $m > 3$, and in the perturbative limit $m \to \infty$ we show that it is equal to the perturbative renormalisation group equation.

7.1 Coulomb gas representation and quantum group restriction

We use the same notation as in chapter 5 for the minimal model $\mathcal{M}_m$ and its primary fields. Recall from chapter 5 that the perturbed minimal model formally can be written as

$$S = \mathcal{M}_m + \lambda \int d^2z \; \Phi_{(1,3)}(z, \bar{z}), \tag{7.1}$$

with the correlators

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \frac{\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{-\lambda \int d^2z \Phi_{(1,3)}} \rangle_{\mathcal{M}_m}}{\langle e^{-\lambda \int d^2z \Phi_{(1,3)}} \rangle_{\mathcal{M}_m}}, \tag{7.2}$$

and $\Phi_{(1,3)}$ has the renormalisation group eigenvalue $y = 2 - \Delta = \frac{4}{m+1}$.

7.1.1 The Coulomb gas representation

In the Coulomb gas [104, 105, 83] (or Feigin–Fuchs, Dotsenko–Fateev) representation the minimal model $\mathcal{M}_m$ is realized as a lagrangian free field theory with a background charge $-2\alpha_0$ at infinity where $\alpha_0 > 0$. The action can then be written as[1]

$$S = \frac{1}{8\pi} \int d^2x \sqrt{g(x)} (g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) + 2\sqrt{2}i\alpha_0\phi(x)R(x)) \tag{7.3}$$

on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, which is the one point compactification of the complex plane with genus zero, $R(x)$ is the scalar curvature. The primary fields are given by the vertex operators $V_\alpha(x) = e^{i\sqrt{2}\alpha\phi(x)}$. With a background charge the action is complex, but for specific choices of parameters in the theory α_0 and α, the theory becomes unitary and can be identified with the unitary minimal models $\mathcal{M}_m$.

The presence of a background charge changes the energy-momentum tensor from the free value (5.37) where $T = -\frac{1}{2}\partial\phi\partial\phi$. The new energy-momentum tensor directly follows from the definition (2.16) in chapter 2 and the usual formulas for the variation of the determinant of the metric δg and the scalar curvature δR. In complex coordinates the background charge adds a term $i\sqrt{2}\alpha_0\partial^2\phi$ [83], hence the energy-momentum tensor undergoes the transformation

$$T = -\frac{1}{2}\partial\phi\partial\phi \xrightarrow{-2\alpha_0} T = -\frac{1}{2}\partial\phi\partial\phi + i\sqrt{2}\alpha_0\partial^2\phi. \tag{7.4}$$

The corresponding change in the central charge from the free value $c = 1$ then follows from its definition (5.10) using that $-2\alpha_0^2\partial_w\partial_z\langle\partial_z\phi(z)\partial_w\phi(w)\rangle = -\frac{24\alpha_0^2}{2(z-w)^4}$:

$$c = 1 - 24\alpha_0^2. \tag{7.5}$$

To match the value (5.45) for the minimal model then $\alpha_0 = 1/\sqrt{4m(m+1)}$ (for $\alpha_0 > 0$). The background charge changes the energy-momentum tensor and hence also the scaling behaviour of the theory. The conformal dimension of the vertex operator V_α becomes $h = \alpha^2 - 2\alpha\alpha_0$, which follows from the operator product

[1] We are here using the standard normalization used in the Coulomb gas representation which differs from the one in chapter 5 for the free boson.

expansion of TV_α. The background charge therefore breaks the symmetry between the operators V_α and $V_{-\alpha}$, and instead the operators V_α and $V_{2\alpha_0-\alpha}$ now have the same conformal dimension.

The Ward identity associated with the transformation $\phi \to \phi + a$, $a \in \mathbb{R}$ (which is not a symmetry for $\alpha_0 \neq 0$) implies that the overall charge of a non-vanishing correlation function $\langle V_{\alpha_1}(x_1) \cdots V_{\alpha_n}(x_n) \rangle$ must satisfy that [84][2]

$$\sum_{j=1}^{n} \alpha_j = 2\alpha_0, \tag{7.6}$$

so the overall charge vanishes. To get a consistent theory screening charges must be inserted into correlators so that this condition holds, but these charges must have a vanishing conformal dimension so the conformal properties of the correlators are left unchanged. Non-local screening charges[3] can be constructed as $Q_\pm = \int d^2z V_{\alpha_\pm}$ which is conformally invariant if $h_{\alpha_\pm} = 1$, so that $\alpha_-\alpha_+ = -1$ and $\alpha_- + \alpha_+ = 2\alpha_0$. We choose $\alpha_- < 0$, hence from the constraint on α_0 above it follows that $\alpha_- = -\sqrt{\frac{m}{m+1}}$. For the equation (7.6) to be satisfied for the four point function $\langle V_\alpha V_{2\alpha_0-\alpha} V_\alpha V_\alpha \rangle$ it must be replaced with $\langle Q_+^m Q_-^n V_\alpha V_{2\alpha_0-\alpha} V_\alpha V_\alpha \rangle$, for appropriate $m, n \in \mathbb{N}$ so that $2\alpha + m\alpha_+ + n\alpha_- = 0$. This shows that 2α has to be an integer combination of α_+ and α_-, we will write it as $\alpha_{r,s} = \frac{1-r}{2}\alpha_+ + \frac{1-s}{2}\alpha_-$. To correspond with the unitary minimal models it turns out[4] that $1 \leqslant r < m$, $1 \leqslant s < m+1$. It then follows that the vertex operator $V_{\alpha_{r,s}}(x) = e^{i\sqrt{2}\alpha_{r,s}\phi(x)}$ has the conformal dimension

$$h_{r,s} = \bar{h}_{r,s} = \alpha_{r,s}(\alpha_{r,s} - 2\alpha_0) = \frac{1}{4}(r\alpha_+ + s\alpha_-)^2 - \alpha_0^2 = \frac{(r(m+1) - sm)^2 - 1}{4m(m+1)}, \tag{7.7}$$

[2]This can also be shown using that the correlators are independent of the large scale cut-off [83].

[3]The Coulomb gas representation can be given a BRST formulation [124], where the BRST charge is given in terms of the screening operators $Q_\pm$.

[4]This follows from the values of the operator product coefficients of the minimal models because the three point function of vertex operators vanishes when the corresponding operator product coefficient vanishes.

which is recognized as the conformal dimension (5.46) of the primary field $\Phi_{(r,s)}$. Hence, the primary fields $\Phi_{(r,s)}$ are represented by the vertex operators $V_{\alpha_{r,s}}(x) = V_{(r,s)} = e^{i\sqrt{2}\alpha_{r,s}\phi(x)}$, and in particular is $\Phi_{(1,3)}$ represented by $V_{(1,3)} = e^{-i\sqrt{2}\alpha_-\phi}$.

7.1.2 The quantum group restricted sine-Gordon model

In [125, 126, 127, 128, 129, 130] it was observed that the perturbed unitary minimal model (7.1) is equivalent to a quantum group restriction of the sine-Gordon theory. The sine-Gordon theory[5] is given by the euclidean action[6]

$$S_{SG} = \frac{1}{4\pi}\int d^2x \left(\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - 4\pi\tilde{\lambda}\cos(\beta\phi(x)/\sqrt{4\pi})\right). \tag{7.8}$$

In two dimensions all polynomial interactions are super-renormalisable and the degree of divergence for a graph G becomes $\delta(G) = 2 - 2V$ where V is the number of vertices. The tadpole diagrams with only one vertex and k external legs and l loops for $k, l \in \mathbb{N}$ are then divergent as $\delta(G) = 0$. The theory is renormalised by adding counterterms and summing over all tadpoles, and this directly leads, as first shown by Coleman [18][7], to a multiplicative renormalisation of $\tilde{\lambda}$, where $\tilde{\lambda} \to 0$ in the ultra-violet limit that therefore corresponds to a free theory with $c = 1$. β and ϕ are not renormalised.

The sine-Gordon model has the infinite dimensional quantum group symmetry

[5]The sine-Gordon theory has been used to model phenomena in condensed matter physics, e.g. crystal dislocations, magnetic flux in Josephson lines and magnetic crystals [69].

[6]Note that β is here a parameter and not a beta function.

[7]In [18] it was shown that the energy is unbounded from below for $\beta^2 > 8\pi$, and for $\beta^2 < 8\pi$, which is the case we are considering, the theory is equivalent to the massive Thirring model which is a fermionic theory. The case of $\beta^2 > 8\pi$ is treated in [131, 132], here β is also renormalised and this is used to establish a flow between infra-red and ultra-violet theories, but in this case no identification of the Hilbert space of states exists between the restricted sine-Gordon model and the perturbed unitary minimal models. The sine-Gordon model is an integrable system, that has a spectrum consisting of solitons and anti-solitons, with topological charge $\pm n, n \in \mathbb{N}$, and their bound states breathers (or kinks), the number of boundstates is $[\pi/\chi] = [\frac{8\pi}{\beta^2}] - 1$ where $\chi = \frac{\pi\beta^2}{8\pi-\beta^2}$ [68, 127].

$U_q(\widehat{sl_2})$ [129, 126] where q is a root of unity[8] $q = e^{-i8\pi^2/\beta^2}$. In [129, 126] this was shown creating four non-local charges $\tilde{Q}_\pm, \bar{\tilde{Q}}_\pm$ that together with the topological charge $\mathcal{T}$ generate the quantum group $U_q(\widehat{sl_2})$, which has two $U_q(sl_2)$ sub-algebras generated by $\{\tilde{Q}_+, \bar{\tilde{Q}}_-, \mathcal{T}\}$ and $\{\tilde{Q}_-, \bar{\tilde{Q}}_+, \mathcal{T}\}$.

In [126, 130] the quantum group restriction of the sine-Gordon model is performed by adding a background charge $-2\alpha_0$ at infinity. The background charge will again change the energy-momentum tensor as in (7.4) and the two vertex operators in the potential will now scale differently[9]. Two different couplings are needed, we will write this as

$$\tilde{\lambda}\cos(\beta\phi/\sqrt{4\pi}) \xrightarrow{-2\alpha_0} \frac{\tilde{\lambda}}{2}e^{-i\beta\phi/\sqrt{4\pi}} + \frac{\tilde{\lambda}_-}{2}e^{i\beta\phi/\sqrt{4\pi}}. \tag{7.9}$$

Analogous to the Coulomb gas representation for the minimal models the background charge α_0 is determined from the central charge $c = 1 - 24\alpha_0^2 = 1 - \frac{6}{m(m+1)}$. β is chosen from the requirement that $e^{i\beta\phi/\sqrt{4\pi}}$ becomes marginal (i.e. $h = 1$) so that it survives in the ultra-violet limit. $\beta = \sqrt{8\pi}\alpha_- = -\sqrt{8\pi}\sqrt{m/(m+1)}$. The conformal dimension of $e^{-i\beta\phi/\sqrt{4\pi}}$ is then $h = \frac{m}{m+1} - \frac{1}{m+1} = \frac{m-1}{m+1} = h_{(1,3)}$ and $e^{-i\beta\phi/\sqrt{4\pi}}$ represents the perturbing operator $\Phi_{(1,3)}$ [10]. Only relevant operators are included in the effective action, as explained below, so the marginal screening term $e^{i\beta\phi/\sqrt{4\pi}}$ (which is needed when considering correlators) is excluded.

In [126, 129, 130] it was argued that the perturbed unitary minimal model (7.1) is a quantum group restriction of the sine-Gordon theory, i.e. a massive theory, with the coupling taken as $\tilde{\lambda} = -\lambda$ in (7.8). Here we are only interested in the massless

[8] $U_q(\widehat{sl_2})$ has special representations when q is a root of unity [130]. A quantum group is a Hopf algebra with a non-trivial co-multiplication, see [129] for a definition and a review of $U_q(\widehat{sl_2})$.

[9] The quantum group restriction can be described in terms of cohomology in analogy with the BRST description of minimal models [126]. When the background charge is added the quantum group symmetry is reduced by breaking the $\phi \to -\phi$ symmetry in the potential. Choosing the background charge so that $\tilde{Q}_+$ becomes a marginal screening operator the quantum group restriction is then defined by gauging away the $U_q(sl_2)$ sub-algebra containing $\tilde{Q}_+$.

[10] The ultra-violet limit of the restricted sine-Gordon model is a Liouville theory which is equivalent to the unitary minimal models $\mathcal{M}_m$, the perturbed minimal models can therefore also be seen as a perturbed Liouville theory [133].

flow to $\mathcal{M}_{m-1}$ ($\lambda > 0$ in [94]) where the limit $m \to \infty$ is given by perturbation theory. We will therefore take the opposite sign, $\tilde{\lambda} = k\lambda$ for some $k \in \mathbb{R}_+$, and in the exact renormalisation group equation we consider the action

$$S = \frac{1}{4\pi} \int d^2x \sqrt{g(x)} \left(\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - 4\pi \frac{\tilde{\lambda}}{2} e^{-i\beta\phi(x)/\sqrt{4\pi}} + \sqrt{2} i \alpha_0 \phi(x) R(x) \right) \tag{7.10}$$

as a model for the perturbed unitary minimal model (7.1). The operator $\Phi_{(1,3)}$ in (7.1) is normalized as in [134] so that the ultra-violet limit of correlators in the perturbed theory (7.2) equals the corresponding correlators in $\mathcal{M}_m$ where we take $\langle \Phi_{(1,3)}(1)\Phi_{(1,3)}(0) \rangle_{\mathcal{M}_m} = 1$. The scaling dimensions of $V_{(1,3)}$ and $\Phi_{(1,3)}$ are equal so the normalization constant $N_{(1,3)}$ relating them is therefore independent of $\tilde{\lambda}$. $V_{(1,3)} = N_{(1,3)}\Phi_{(1,3)}$ and we therefore take $\tilde{\lambda} = {N_{(1,3)}}^{-1}\lambda$. The normalization constant is given by the formulas in [135, 104, 105]

$${N_{(1,3)}}^2 = 2\pi^2 \frac{\gamma(1-\alpha_-^2)^3 \gamma(3\alpha_-^2 - 1)}{(1-2\alpha_-^2)(2\alpha_-^2 - 1)} \tag{7.11}$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ and we use the normal convention that $N_{(1,3)} > 0$.

In [136] the exact relation between the sine-Gordon coupling $\tilde{\lambda}$ (i.e. in the theory (7.8) without a background charge where the scaling dimensions of $\cos(\beta\phi/\sqrt{4\pi})$ and $\Phi_{(1,3)}$ differ) and λ was established $\lambda = \frac{\pi\tilde{\lambda}^2(\gamma(1-\beta^2/(8\pi))^3\gamma(3\beta^2/(8\pi)))^{1/2}}{4(1-\beta^2/(4\pi))(3\beta^2/(8\pi))}$ using the Coulomb gas representation of minimal models [83]. We do not use this representation here because we want the ultra-violet limit to be $\mathcal{M}_m$, and not as in the sine-Gordon model where it is the free theory with $c = 1$, also we want a linear relation between λ and $\tilde{\lambda}$.

For a non-zero background charge the scaling of the theory is not determined by the usual beta function related to the ultra-violet divergences, but by a generalized beta function as discussed in [131]. (We denote the beta function $\boldsymbol{\beta}$ to avoid confusion with the parameter β in (7.8)). The generalized $\boldsymbol{\beta}$-function takes into account the change in the energy-momentum tensor when the background charge is added.

To lowest order the background charge adds a term [131]

$$\beta(\tilde{\lambda}) = \Lambda\frac{\partial\tilde{\lambda}}{\partial\Lambda} \rightarrow \tilde{\beta}(\tilde{\lambda}) = \beta(\tilde{\lambda}) - 2\cdot 2\alpha_{(1,3)}\alpha_0\tilde{\lambda} \tag{7.12}$$

to the normal perturbative β-function for the interaction term $\tilde{\lambda}e^{i\sqrt{2}\alpha_{(1,3)}\phi(x)}$ in the action. This follows from the definition of the renormalisation group eigenvalue (3.14)[11] which says that $\beta = -y\tilde{\lambda} + O(\tilde{\lambda}^2)$, and when the background charge is added $y \rightarrow y + 2\cdot 2\alpha_{(1,3)}\alpha_0$ because of the change in the scaling dimension of $\Phi_{(1,3)}$ given by (7.7): $\Delta \rightarrow \Delta - 2\cdot 2\alpha_{(1,3)}\alpha_0$. We discuss below how this term appears in the exact renormalisation group.

7.2 The exact renormalisation group equation

The wilsonian effective action at the scale Λ is obtained by 'integrating out' momentum modes between Λ and Λ_0, where Λ_0 is the fundamental wilsonian cut-off. The exact renormalisation group equation[12] then describes how the effective action must change so as to describe the same physics when Λ changes. Following Polchinski [54], the partition function is written as

$$Z(\Lambda) = \int \mathcal{D}\phi\; e^{-S[\phi]} = \int \mathcal{D}\phi\; e^{-\frac{1}{2}\int\frac{d^2p}{(2\pi)^2}\phi(p)\phi(-p)p^2K^{-1}(p^2/\Lambda^2) - V(\phi,\Lambda)} \tag{7.13}$$

in a continuum formulation with the cut-off propagator $\frac{K(p^2/\Lambda^2)}{p^2}$. $K(p^2/\Lambda^2)$ is constant for small $s = p^2/\Lambda^2$ and vanishes faster than any power for large s. The physics must be independent of the choice of effective scale Λ so that $\Lambda\frac{d}{d\Lambda}Z(\Lambda) = 0$, and performing the differentiation in (7.13) it follows that

$$\Lambda\frac{d}{d\Lambda}Z(\Lambda) = \langle(-\frac{1}{2}\int\frac{d^2p}{(2\pi)^2}\phi(p)\phi(-p)p^2\Lambda\frac{\partial K^{-1}(p^2/\Lambda^2)}{\partial\Lambda} - \Lambda\frac{\partial V(\phi,\Lambda)}{\partial\Lambda})\rangle = 0. \tag{7.14}$$

[11] With a sign change as we are here considering changes in the momentum space.

[12] Or the Wilson-Polchinski renormalisation group equation [137, 54, 41].

When the wilsonian effective potential obeys the operator equation

$$\Lambda\frac{\partial}{\partial\Lambda}V(\phi,\Lambda)=\frac{1}{2}\int d^dxd^dy\int\frac{d^dk}{(2\pi)^d}e^{ik(x-y)}\frac{2K'(k^2/\Lambda^2)}{\Lambda^2}\left(\frac{\delta^2V}{\delta\phi(x)\delta\phi(y)}-\frac{\delta V}{\delta\phi(x)}\frac{\delta V}{\delta\phi(y)}\right) \tag{7.15}$$

where $K'(k^2/\Lambda^2)=\frac{dK(s)}{ds}|_{s=k^2/\Lambda^2}$, the partition function then changes as a total derivative up to a field independent term [54], and equation (7.14) is satisfied

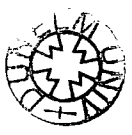
$$\begin{aligned}\Lambda\frac{d}{d\Lambda}Z&=\int d^2p\int D\phi\frac{\delta}{\delta\phi}\Psi=0,\\ \Psi&=\Lambda\frac{\partial K(p^2/\Lambda^2)}{\partial\Lambda}\left(-\phi(p)K^{-1}(p^2/\Lambda^2)+\frac{(2\pi)^d}{2}p^{-2}\frac{\delta S}{\delta\phi(-p)}\right)e^{-S}.\end{aligned} \tag{7.16}$$

(7.15) is the exact renormalisation group equation[13]. It is a non-linear functional equation in $V(\phi,\Lambda)$ which is not directly solvable and approximations must be made, either by truncating the operator space or by performing a derivative expansion [139].

7.2.1 Truncating the operator space

We will make a truncation in the operator space. Firstly, we will only consider the relevant operators, this approximation becomes exact in the infra-red ($\Lambda\ll\Lambda_0$) where the effective theory is determined by the relevant (and marginal) couplings. Secondly, we saw in chapter 5 that the primary fields $\{\Phi_{(1,2p+1)}\}$ with $0\leqslant p\leqslant[\frac{m-1}{2}]$ form a sub-algebra in $\mathcal{M}_m$, and there is therefore a renormalisation group flow from the minimal model $\mathcal{M}_m$ spanned by these operators. When considering perturbations with operators in a sub-algebra, all divergences that arise away from the fixed point will be contained in this sub-algebra, and this shows that there is a renormalisation group flow in the sub-space of the corresponding couplings. The operator structure at the non-trivial ultra-violet fixed point thereby determines the renormalisation group flow.

[13]See [138] for a recent review of the exact renormalisation group equation. A constructive proof was given by Zinn-Justin in [38] by explicitly integrating out the high momentum modes in the partition function and observing how the potential then changes under an infinitesimal change in the cut-off.



In the sub-algebra $\{\Phi_{(1,2p+1)}\}$ only $\Phi_{(1,1)} = \mathbf{1}$ and $\Phi_{(1,3)}$ are relevant fields[14]. The exact renormalisation group is only determined up to a field independent term, so we will only consider perturbations with respect to $\Phi_{(1,3)}$. None of the relevant operators $V_{(1,n)}$ contain any derivatives in ϕ, so the situation is similar to the local potential approximation of the exact renormalisation group equation. The local potential approximation is the lowest order term in a derivative expansion of $V = V(\Lambda, \phi, \partial\phi, \dots)$ [140, 141]. The second order term containing derivatives $f(\Lambda)(\partial\phi)^2$ leads to a renormalisation of ϕ and an anomalous dimension $\eta > 0$. In our approximation there are no derivatives in ϕ because we only keep relevant operators, and ϕ is not renormalised in the sine-Gordon model. In this approximation the first term in (7.15) can be rewritten [140, 141] as

$$\frac{1}{\Lambda^2}\int \frac{d^2k}{(2\pi)^2} K'(k^2/\Lambda^2) \int d^2x \frac{\partial^2 \mathcal{V}}{\partial\phi\partial\phi}(x) = -k_1 \int d^2x \frac{\partial^2 \mathcal{V}}{\partial\phi\partial\phi}(x) \tag{7.17}$$

where $\mathcal{V}(x)$ is the potential density $V = \int d^2x \mathcal{V}(x)$, and $k_1 = -\int \frac{d^2k}{(2\pi)^2} K'(k^2) > 0$ as $K(k^2)$ is decreasing. In the local potential approximation [142, 140, 141, 143] the second non-local term in (7.15)

$$-\frac{1}{\Lambda^2}\int d^2x \int d^2y \int \frac{d^2k}{(2\pi)^2} K'(k^2/\Lambda^2) e^{ik(x-y)} \frac{\partial \mathcal{V}}{\partial\phi(x)} \frac{\partial \mathcal{V}}{\partial\phi(y)} \tag{7.18}$$

is approximated by $k_2 \int d^2x (\frac{\partial \mathcal{V}}{\partial\phi(x)})^2$ where $k_2 \in \mathbb{R}_+$ depends on the cut-off function $K(s)$. This is a good approximation in the ultra-violet limit $\Lambda \to \infty$ [143] where $\int \frac{d^2k}{(2\pi)^2} K'(k^2/\Lambda^2) e^{ik(x-y)} \to K'(0) \int \frac{d^2k}{(2\pi)^2} e^{ik(x-y)} = K'(0)\delta^2(x-y)$. We consider the operator version of (7.15) and a different approximation therefore has to be used for the term (7.18) otherwise divergences will appear in the operator product $\frac{\partial \mathcal{V}}{\partial\phi(x)} \frac{\partial \mathcal{V}}{\partial\phi(x)}$, whereas (7.15) is finite for $\Lambda < \infty$. The correct form of the approximation in the operator case is obtained from the operator product expansion, which for spin zero operators is (3.12)

$$\mathcal{O}_i(x)\mathcal{O}_j(y) \sim \sum_k |x|^{\Delta_k - \Delta_i - \Delta_j} \tilde{C}_{ij}^k(\tilde{\lambda}, \Lambda) \mathcal{O}_k(y), \tag{7.19}$$

[14]The operator $\Phi_{(1,3)}$ does not mix with other relevant operators off criticality [94, 81].

where $\mathcal{O}_k(y)$ is a complete set of local scaling fields. The non-local term (7.18) then takes the form

$$\int d^2x \mathcal{O}(x) \int d^2y \int \tfrac{d^2k}{(2\pi)^2} e^{ik(x-y)} K'(k^2/\Lambda^2) h(|x-y|^2) \tag{7.20}$$

for some $h(|x-y|^2)$ and operator $\mathcal{O}(x)$. $\delta^d(x)$ is the Fourier transformation of 1 which shows that $\delta^2(x) = \frac{1}{\pi}\delta(|x|^2)$, and we use this identity to write the ultra-violet limit $\Lambda \to \infty$ of the k integral as $-\int \frac{d^2k}{(2\pi)^2} e^{ik(x-y)} K'(k^2/\Lambda^2) \to k_2 \delta(|x-y|^2)$ where again $k_2 \in \mathbb{R}_+$ depends on the choice of $K(s)$. For finite Λ this expression is approximately $k_2\delta(|x-y|^2 - a/\Lambda^2)$ where $a \in \mathbb{R}_+$ depends on $K(s)$. The cut-off dependence in a is removed by redefining $\Lambda^2 \to a\Lambda^2$, and the approximation becomes

$$-\int \frac{d^2k}{(2\pi)^2} e^{ik(x-y)} K'(k^2/\Lambda^2) = k_2\ \delta(|x-y|^2 - \Lambda^{-2}) \tag{7.21}$$

in (7.18)[15]. If the effective potential is written as $V = \tilde{\lambda}^i \int d^2x \mathcal{O}_i(x)$, where $[\tilde{\lambda}^i] = y^i > 0$, then (7.15) becomes using (7.17) and the operator product expansion

$$\begin{aligned} \frac{k_1}{k_2} \int d^2x \Lambda \frac{d}{d\Lambda} \mathcal{V}(\phi, \Lambda) = &- \frac{k_1}{k_2} \int d^2x \frac{\partial^2 \mathcal{V}(\phi, \Lambda)}{\partial\phi\partial\phi} \\ &+ \frac{k_1}{k_2} \frac{\pi}{\Lambda^2} \int d^2x \sum_{i,j,k} \tilde{\lambda}^i \tilde{\lambda}^j \Lambda^{\Delta'_i + \Delta'_j - \Delta'_k} \tilde{C}''^k_{ij}(\tilde{\lambda}, \Lambda) \mathcal{O}'_k(x). \end{aligned} \tag{7.22}$$

Here V and ϕ have been rescaled: $V \to \frac{k_1}{k_2} V$, $\phi \to \sqrt{k_1}\phi$ and Δ'_i and $\tilde{C}''^k_{ij}$ are the scaling dimension and structure constant for the field $\mathcal{O}'_i = \frac{\partial \mathcal{O}_i}{\partial \phi}$. If $\mathcal{O}_i$ is a vertex operator then $\Delta' = \Delta$, $\tilde{C}''^k_{ij} \propto \tilde{C}^k_{ij}$ and $\mathcal{O}'_i \propto \mathcal{O}_i$. It follows from (7.22) that the implicit dependence in the choice of cut-off function $K(s)$ contained in k_1 and k_2 drops out [140, 141]. The renormalisation group equation is usually written in terms of dimensionless couplings $\tilde{g}^i = \Lambda^{-y_i}\tilde{\lambda}^i$ so that $V = \tilde{g}^i \Lambda^2 \int d^2x \tilde{\mathcal{O}}_i$ where $[\tilde{\mathcal{O}}_i] = 0$.

[15] Adding a function $f(|x-y|^2, 1/\Lambda^2)$ satisfying $f \to 0$ for $\Lambda \to \infty$ corresponds to a higher order derivative expansion in ϕ [143] and it is therefore neglected in the approximation considered here where there are no derivatives in ϕ.

(7.22) then becomes

$$\Lambda^2 \int d^2x \Lambda \frac{\partial \tilde{g}^i(\Lambda)}{\partial \Lambda} \tilde{\mathcal{O}}_i = -2\Lambda^2 \int d^2x \tilde{g}^i \tilde{\mathcal{O}}_i - \Lambda^2 \int d^2x \tilde{g}^i \frac{\partial \tilde{\mathcal{O}}_i}{\partial \phi \partial \phi} + \pi \Lambda^2 \int d^2x \sum_{i,j,k} \tilde{g}^i \tilde{g}^j \tilde{C}_{ij}^{\prime k} \tilde{\mathcal{O}}'_k. \tag{7.23}$$

For the perturbed minimal models we will use the action (7.10), and for the moment neglect the curvature term. (7.23) then becomes

$$\int d^2x e^{-i\beta\phi/\sqrt{4\pi}} \left(\Lambda \frac{\partial \tilde{g}}{\partial \Lambda} + 2\tilde{g}(\Lambda) + \tilde{g}(\Lambda) (\frac{-i\beta}{\sqrt{4\pi}})^2 + \pi \frac{\tilde{g}(\Lambda)^2}{2} \tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(\Lambda), \Lambda) (\frac{-i\beta}{\sqrt{4\pi}})^2 \right) = 0,$$

up to field independent and irrelevant terms. Introducing the dimensionless variable $t = \ln \frac{\Lambda_0}{\Lambda}$ (and $\tilde{x} = \Lambda x$) the renormalisation group equation for $\tilde{g}$ can then be written as

$$\frac{\partial \tilde{g}(t)}{\partial t} = \dot{\tilde{g}} = 2\tilde{g}(t) - \frac{2m}{m+1} \tilde{g}(t) - \pi \frac{m}{m+1} \tilde{g}(t)^2 \tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t). \tag{7.24}$$

Adding the additional term (7.12) from the background charge $2 \cdot 2\alpha_{(1,3)}\alpha_0 \tilde{g}(t) = \frac{2}{m+1}\tilde{g}(t)$ gives

$$\dot{\tilde{g}}(t) = \frac{4}{m+1} \tilde{g}(t) - \pi \frac{m}{m+1} \tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t) \tilde{g}(t)^2 = y\tilde{g}(t) - \pi \tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t) \tilde{g}(t)^2 \frac{m}{m+1}. \tag{7.25}$$

The contribution from the background charge to the scaling behaviour can be incorporated directly into the exact renormalisation group equation if the curvature term in (7.10) is taken into account.

7.2.2 Incorporating the background charge

The action (7.10) is defined on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ where all the curvature is situated at infinity so that the topological invariant $\int d^2\tilde{x} \sqrt{g(\tilde{x})} R(\tilde{x}) = 8\pi$ is satisfied. The Riemann surface $\mathbb{C} \cup \{\infty\}$ cannot be covered by a single coordinate chart, one chart is needed for the flat space and another in the neighborhood of infinity. We take polar coordinates in flat space $x = (r \sin\theta, r \cos\theta)$ and define the contribution

from infinity as in [144] by a limit value. We consider the theory on a disk Γ with radius $r \to \infty$ so that all curvature is at the boundary $\partial\Gamma$. From the Gauss-Bonnet theorem $\lim_{r\to\infty} \int_{\partial\Gamma} d\theta\ rR(r) = 8\pi$ so that $R(r) = \frac{4}{r}$ on $\partial\Gamma$. We define as in [144] the contribution from infinity to be $\phi_\infty = \lim_{r\to\infty} \frac{1}{2\pi r} \int_{\partial\Gamma} dl\ \phi = \lim_{r\to\infty} \frac{1}{2\pi} \int_{\partial\Gamma} d\theta\ \phi$ and similarly for the vertex operators. The effect of the background term is seen by evaluating the exact renormalisation group equation at infinity where the curvature is non-vanishing. Hence we replace $\int d^2x$ in (7.23) by $\lim_{r\to\infty} \frac{1}{2\pi} \int_{\partial\Gamma} d\theta$. Only the term $\frac{\partial V}{\partial\phi}\frac{\partial V}{\partial\phi}$ in the exact renormalisation group equation (7.15) will give an additional term proportional to $e^{-i\beta\phi/\sqrt{4\pi}}$ when the background charge is added. This term becomes

$$\begin{aligned}
&\lim_{r\to\infty} \frac{-1}{2\pi} \int_{\partial\Gamma} d\theta \int d^2y \delta(|x-y|^2 - \Lambda^{-2}) \frac{2^2(\sqrt{2})^2(i)^2\alpha_0\alpha_{(1,3)}}{8\pi} rR(r) e^{i\sqrt{2}\alpha_{(1,3)}\phi(y)} \tilde{g}(t) \\
&= (2)^3 \alpha_0\alpha_{(1,3)}\tilde{g}(t) \lim_{r\to\infty} \int_{\partial\Gamma} \frac{d\theta}{2\pi} \int \frac{d\theta'}{2\pi} \int \frac{d(r')^2}{2} \delta((r')^2 - \Lambda^{-2}) e^{i\sqrt{2}\alpha_{(1,3)}\phi(x-y')} \\
&\approx 2^2\alpha_0\alpha_{(1,3)}\tilde{g}(t) \lim_{r\to\infty} \int_{\partial\Gamma} \frac{d\theta}{2\pi} e^{i\sqrt{2}\alpha_{(1,3)}\phi(x)} = 2^2\alpha_0\alpha_{(1,3)}\tilde{g}(t) V_{(1,3)}(\infty).
\end{aligned}$$

The second to last equation holds in the limit $r \to \infty$ where $r \gg \Lambda^{-1}$. When the coefficient of $e^{-i\beta\phi/\sqrt{4\pi}}$ from this term is added to (7.24) then (7.25) is again obtained, but it now holds to all orders in the coupling $\tilde{g}$.

This shows how the change in scaling behaviour due to the background charge is seen in the exact renormalisation group equation when it is evaluated at a point with non-zero curvature. Equation (7.25) is then valid to all orders in perturbation theory for the chosen truncation of the operator space. Hence, in

$$\dot{\tilde{g}}(t) = y\tilde{g}(t) - \pi \tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t)\tilde{g}(t)^2 \tfrac{m}{m+1} \tag{7.26}$$

higher order terms appear via the off-critical structure constant $\tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t)$. The structure constant $\tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t)$ is regular in the coupling $\tilde{g}$ [145, 146, 147]. The wilsonian effective action (7.10) is for zero coupling equal to the Coulomb gas representation (7.3) of the minimal model $\mathcal{M}_m$. The structure constant therefore has the following expansion $\tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t) = \tilde{C}^{\Phi}_{\Phi\Phi} + O(\tilde{g}(t))$, where $\tilde{C}^{\Phi}_{\Phi\Phi}$ is the structure constant for the vertex operators $V_{(1,3)}$ in the minimal model $\mathcal{M}_m$. $V_{(1,3)}(1)V_{(1,3)}(0) =$

$\mathbf{1} + \tilde{C}^{\Phi}_{\Phi\Phi} V_{(1,3)}(0) + \cdots$ and the structure constant for $\Phi_{(1,3)}$ is thus $C^{\Phi}_{\Phi\Phi} = N^{-1}_{(1,3)} \tilde{C}^{\Phi}_{\Phi\Phi}$ [135, 104], which is given in (5.58). $\tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t)$ can in principle be calculated in conformal perturbation theory [147, 145, 146] for strictly relevant perturbations ($y > 0$), and for small t it is given by [147]

$$\tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}) = \tilde{C}^{\Phi}_{\Phi\Phi} - \tilde{g}\pi \int' d^2 z \langle V_{(1,3)}(\infty) V_{(1,3)}(z) V_{(1,3)}(1) V_{(1,3)}(0) \rangle_{\mathcal{M}_m} + \mathcal{O}(\tilde{g}^3). \quad (7.27)$$

The four-point function is the first non-trivial correlator and it can be evaluated by pairing the fields and inserting the short distance expansion. There is then a crossing symmetry and this leads to the bootstrap equation for the structure constants which expresses the associativity of the operator algebra. In the Coulomb gas representation monodromy invariance of physical correlators is used to get the form of the four-point function which is written in terms of structure constants and conformal blocks. Unfortunately no closed form has been found for the conformal blocks with two screening charges ($\langle V_{(1,3)} V_{(1,3)} V_{(1,3)} V_{(1,3)} \rangle$ requires the screening Q_-^2 for the overall charge to be $2\alpha_0$). When only one screening Q_- is needed a closed expression of the conformal blocks can be found in terms of hyper-geometric functions using Euler's integral representation [84]. In the marginal case $y = 0$ the four-point function can be written in a closed form as the conformal blocks become meromorphic functions of z [42], but conformal perturbation theory is only defined for strictly relevant perturbations with $y > 0$.

It follows from (7.26) by inserting $\tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t) = \tilde{C}^{\Phi}_{\Phi\Phi} + O(\tilde{g}(t))$ that the infra-red fixed point coupling vanishes in the limit $m \to \infty$, and this limit can therefore be compared with the perturbative renormalisation group equation. For large m (7.26) becomes $\dot{g} = yg - \pi C^{\Phi}_{\Phi\Phi} g^2 + O(g^3)$ where $g = N_{(1,3)}\tilde{g}$, and this has the infra-red fixed point $g_{IR} = \frac{y}{\pi C^{\Phi}_{\Phi\Phi}}$ as obtained in [94, 98], and chapter five.

7.3 Conclusions

We studied in this chapter the exact renormalisation group equation for a two dimensional quantum field theory. It was approximated by only including relevant

operators of exponential form together with a background charge at infinity in the wilsonian effective action. The effective action does not contain any derivatives in the field and the approximation is therefore similar to the local potential approximation. The non-linear term in the exact renormalisation group equation is approximated by the operator product expansion. We showed that the effect of the background charge can be incorporated into the exact renormalisation group equation by evaluating it at a point of non-zero curvature.

Using the equivalence between the unitary minimal models perturbed by $\Phi_{(1,3)}$ and the quantum group restricted sine-Gordon model, the obtained renormalisation group equation was argued to describe the renormalisation group flow for the perturbed unitary minimal models from $\mathcal{M}_m$ to $\mathcal{M}_{m-1}$. The resulting renormalisation group equation is valid to all orders in the coupling for our truncation of the operator space, and for all $m > 3$. The higher order terms in the coupling appear in the off-critical structure constant. In the limit of large m, where the ultra-violet and infra-red fixed points approach each other, the renormalisation group equation agrees with the well known perturbative result.

8

The self-gravitating bosonic membrane

In this chapter we will consider the field theory of an extended object, namely the bosonic membrane with an Einstein–Hilbert term.

Extended objects play a role in different areas of physics e.g. condensed matter physics and hydro-dynamics (see e.g. [148]), and have also showed to be important in the theories describing the fundamental interactions. The only present candidate for quantum gravity includes extended objects like strings and membranes[1]. The membrane was first described by Dirac in [149] where a spherical charged membrane was used as a model for the electron. In [150, 151, 152] the supermembrane in 11 dimensions was described and in [153] it was shown that the IIA superstring can be obtained from it by a dimensional reduction [154]. In [155] the supermembrane was shown to be a solution of the field equations of 11 dimensional supergravity[2].

[1]The efforts to unify general relativity and the three interactions in the standard model in a local quantum field theory have failed for different reasons. The main one being the non-renormalisability of the Einstein–Hilbert term which shows that such a local theory can only be an effective one that has to be replaced by a more fundamental theory at high energies, such as in the early universe and for black holes where quantum gravity is relevant. Another problem is the understanding of quantum field theory when the space-time background is dynamical.

[2]For these reasons it is believed to play an important role in $\mathcal{M}$-theory [156]. $\mathcal{M}$-theory is a conjectured theory in 11 dimensions that has the five 10 dimensional superstring theories as

In this chapter we will consider a membrane with its dynamics determined by a minimization of the world-volume together with the world-volume scalar curvature. The action can then be written as a sum over the world-volume Dirac–Nambu–Goto action and an Einstein–Hilbert term: $S = \text{vol} + \int R$. In chapter three we saw that all terms respecting the symmetries of a theory should be included in the action. The motivation for studying the model given by S is that the Einstein–Hilbert term is the simplest world-volume diffeomorphism invariant and target-space Poincaré invariant term that can be added to the Dirac–Nambu–Goto action, and S can therefore be seen as an approximation to the low energy effective action for the bosonic membrane.

The solution of a semi-classical approximation of the model is determined, where the field equations are linearized around a toroidal background. We follow the paper [159] closely, there a semi-classical quantisation around this background is described for the supermembrane.

8.1 The bosonic membrane

The Dirac–Nambu–Goto [149, 160, 161] action for a p dimensional extended object is given by its $p+1$ dimensional world-volume

$$S = T_{p+1} \int d^{p+1}\xi \sqrt{-\det(\eta_{\mu\nu}\partial_i X^\mu(\xi)\partial_j X^\nu(\xi))} \tag{8.1}$$

where ξ^i, $i = 1, \ldots, p+1$ are the world-volume coordinates and $X^\mu(\xi)$, $\mu = 1, \ldots, d$ describe the embedding of the p dimensional object into d dimensional Minkowski space-time (target space) with the metric $\eta_{\mu\nu}$. T_{p+1} is the p brane tension. In this chapter we will take $\eta_{\mu\nu} = (-, +, \ldots, +)$ which is the standard convention in theories with extended objects. The equations of motion that are derived from (8.1)

different limits and 11 dimensional supergravity as its low energy limit. The supermembrane is non-renormalisable, but can be regularized by rewriting it as the large N limit of $SU(N)$ Yang-Mills theory, and this is one way of getting the matrix-model proposal for $\mathcal{M}$-theory [156, 154, 157, 158].

can equivalently be obtained from the action[3] [162, 163, 164, 165]

$$S = T_{p+1} \int d^{p+1}\xi \, \sqrt{-g} \left(-\frac{1}{2} g^{ij} \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} + \frac{p-1}{2} \right) \tag{8.2}$$

where $g_{ij}(\xi)$ is an auxiliary variable. (8.2) is invariant under Poincaré transformations of X^μ and diffeomorphisms in ξ^i, and for the string $p = 1$ there is also invariance under the Weyl transformations $g_{ij} \to \Omega(\xi)^2 g_{ij}$. We are here considering the membrane, i.e. $p = 2$, in 11 dimensions and the equations of motion become

$$\frac{\delta S}{\delta g_{ij}} = 0 \Rightarrow \sqrt{-g} \, (\frac{1}{2} g^{ij} g^{kl} \partial_k X^\mu \partial_l X^\nu \eta_{\mu\nu} - g^{ik} g^{jl} \partial_k X^\mu \partial_l X^\nu \eta_{\mu\nu}) = \frac{1}{2} \sqrt{-g} g^{ij}, \tag{8.3}$$

thus taking the trace gives $g^{ij} \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} = 3$ and g_{ij} is therefore the induced metric on the membrane $g_{ij} = \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}$. (2.16) then shows that in this case the energy-momentum tensor vanishes $T_{ij} = 0$. The other field equations are

$$\frac{\delta S}{\delta X^\mu} = 0 \Rightarrow \partial_i (\sqrt{-g} g^{ij} \partial_j X^\nu \eta_{\mu\nu}) = 0. \tag{8.4}$$

We will use the light-cone-Hoppe gauge as in [159] and it is useful to write the metric in the ADM (Arnowitt-Deser-Misner [166, 167]) decomposition:

$$g_{ij} = \begin{pmatrix} -N^2 + \gamma_{ab} N^a N^b & N^a \gamma_{ab} \\ N^a \gamma_{ab} & \gamma_{ab} \end{pmatrix}, \quad a = 1, 2. \tag{8.5}$$

$\xi^i = (\tau, \sigma, \rho)$ and $\gamma_{ab} = g_{ab}$ is the metric on the membrane, i.e. on the surface determined by $X^\mu(\tau, \sigma, \rho)$ for fixed τ. N and N^a are the shift and the lapse functions that determine the foliation of the membrane world-volume into constant τ surfaces, $\gamma^{ab}\gamma_{bc} = \delta^a_c$ and $\sqrt{-g} = N\sqrt{\gamma}$. (8.2) can then be written as

$$S = T_3 \int d^3\xi \, N\sqrt{\gamma} \left(-\frac{1}{2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + N^{-2} (\dot{X} - N^a \partial_a X)^2 + \frac{1}{2} \right). \tag{8.6}$$

This action is independent in the choice of world-volume foliation hence the number of dynamical degrees of freedom become $11 - 3 = 8$. The light cone coordinates are

[3] Which is often called the Polyakov action.

$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{10})$ so that $X^2 = X^\mu X_\mu = -X^+X^- - X^-X^+ + X^I X^I$, summing over $I = 3, \ldots, 9$, and the light-cone-Hoppe gauge is

$$X^+ = p^+\tau, \quad g_{0a} = 0, \quad g_{00} = -\gamma. \tag{8.7}$$

The last condition holds for all τ as the equation of motion for X^+ in this gauge shows that $\partial_\tau(\frac{-\gamma}{g_{00}})^{1/2} = 0$. $g_{0a} = \dot{X}^\mu \partial_a X_\mu = 0$ then fixes X^- in terms of X^I

$$\partial_a X^- = \frac{1}{p^+}(\partial_a X^I)\dot{X}^I. \tag{8.8}$$

One gauge degree of freedom is left unfixed until after the field equations are linearized as in [159].

8.2 The self-gravitating bosonic membrane

Let us now consider the action

$$S = T_3 \int d^3\xi \sqrt{-g}\left(-\frac{1}{2}g^{ii}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} + \frac{1}{2} + GR\right), \quad G \in \mathbb{R}_+ \tag{8.9}$$

where R is the scalar curvature on the membrane world-volume, it contains derivatives in γ_{ab} which is therefore now a dynamical field. The number of degrees of freedom becomes $11 - 3 + 3 = 11$, and viewing (8.9) as $2 + 1$ dimensional gravity with 11 minimally coupled scalar fields this counting simply becomes $0 + 11$ [4]. Due to the non-linearity of (8.9) we will make a semi-classical approximation of the field equations where they are linearized around a classical solution that is taken to be a flat torus following [159]. Taking space-time to be $S^1 \times S^1 \times \mathbb{R}^9$ a classical solution

[4] Pure $2 + 1$ dimensional gravity is specified by the three components of the spatial metric γ_{ab} together with its embedding given by the three components of the exterior curvature K_{ab}, in addition there are three constraints from diffeomorphism invariance and three initial constraints for the Einstein equations [44]. There are thus no local degrees of freedom, and pure $2+1$ dimensional gravity can be seen as a topological field theory [168, 169].

is obtained by wrapping the membrane around $S^1 \times S^1$ with a flat metric

$$\begin{aligned} X^1_{cl} &= l_1 R_1\, \sigma, \quad X^2_{cl} = l_2 R_2\, \rho, \quad X^I_{cl} = 0, \quad I = 3, \dots, 9, \\ g_{cl} &= \begin{pmatrix} -k_0^2 & 0 & 0 \\ 0 & k_1^2 & 0 \\ 0 & 0 & k_2^2 \end{pmatrix}, \quad k_0, k_1, k_2 \in \mathbb{R}. \end{aligned} \tag{8.10}$$

$\sigma, \rho \in [0, 2\pi]$ and R_1, R_2 are the radii of the torus and $l_1, l_2 \in \mathbb{Z}$ are the winding numbers. (8.10) is clearly a solution to $\partial_i(\sqrt{-g}g^{ij}\partial_j X^\mu) = 0$ and the Einstein equations that become $T_{ij} = 0$ (because the curvature vanishes), where

$$T_{ij} = T_3 \left(\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} - g_{ij} \left(\frac{1}{2} g^{kl} \partial_k X^\mu \partial_l X^\nu \eta_{\mu\nu} - \frac{1}{2} \right) \right). \tag{8.11}$$

$(g_{cl})_{ij}$ is therefore the induced metric and (8.10) together with the gauge condition (8.7) shows that $k_1 = l_1 R_1$, $k_2 = l_2 R_2$ and $k_0 = k_1 k_2$, (8.8) sets $X^- = \frac{1}{2p_+}(l_1 R_1 l_2 R_2)^2 \tau$. The perturbation around the classical solution is written as

$$\begin{aligned} X^i &= X^i_{cl} + Z^i, \quad i = 1, \dots, 9, \\ g_{ij} &= \begin{pmatrix} -(l_1 R_1 l_2 R_2)^2 & 0 & 0 \\ 0 & (l_1 R_1)^2 & 0 \\ 0 & 0 & (l_2 R_2)^2 \end{pmatrix} + \begin{pmatrix} \tilde{\gamma}_{00} & 0 & 0 \\ 0 & \tilde{\gamma}_{11} & \tilde{\gamma}_{12} \\ 0 & \tilde{\gamma}_{12} & \tilde{\gamma}_{22} \end{pmatrix} = \begin{pmatrix} -\gamma & 0 & 0 \\ 0 & \gamma_{11} & \gamma_{12} \\ 0 & \gamma_{12} & \gamma_{22} \end{pmatrix}, \end{aligned}$$

where $\gamma = \gamma_{11}\gamma_{22} - \gamma_{12}^2$. X^i and g_{ij} are then inserted into the field equations keeping terms up to $O((Z^i)^2, (\tilde{\gamma}_{ab})^2, Z^i \tilde{\gamma}_{ab})$. (8.8) no longer holds and we set $X^- = X^-_{cl} = \frac{1}{2p^+}(l_1 R_1 l_2 R_2)^2 \tau$ which is always a solution because of $g_{0a} = 0$ and $g_{00} = -\gamma$. The solution to the field equations that we consider below requires that $(l_1 R_1)^2 = (l_2 R_2)^2 = 1$ so that $R_1 = \frac{1}{|l_1|}$, $R_2 = \frac{1}{|l_2|}$ and we set $l_1 R_1 = l_2 R_2 = 1$ from now on.

8.2.1 Solution to the semi-classical approximation

Using that $\sqrt{-g} = \gamma = 1 + \tilde{\gamma}_{11} + \tilde{\gamma}_{22}$ and

$$\gamma^{ab} = \begin{pmatrix} 1 - \tilde{\gamma}_{11} & -\tilde{\gamma}_{12} \\ -\tilde{\gamma}_{12} & 1 - \tilde{\gamma}_{22} \end{pmatrix}, \tag{8.12}$$

the field equations for X^μ become

$$\begin{aligned}\ddot{Z}^I &= (\partial_\sigma^2 + \partial_\rho^2)Z^I, \quad I = 3, \dots, 9,\\ \ddot{Z}^1 &= (\partial_\sigma^2 + \partial_\rho^2)Z^1 + \partial_\sigma\tilde{\gamma}_{22} - \partial_\rho\tilde{\gamma}_{12},\\ \ddot{Z}^2 &= (\partial_\sigma^2 + \partial_\rho^2)Z^2 + \partial_\rho\tilde{\gamma}_{11} - \partial_\sigma\tilde{\gamma}_{12}.\end{aligned} \tag{8.13}$$

The field equations for $\tilde{\gamma}_{ab}$ are the Einstein equations

$$2GT_3\left(R_{ab} - \frac{1}{2}g_{ab}R\right) = 2GT_3E_{ab} = T_{ab}. \tag{8.14}$$

Γ^k_{ij} is already first order in $\tilde{\gamma}_{ab}$, hence to first order $R_{ab} = \partial_c\Gamma^c_{ab} - \partial_b\Gamma^c_{ac}$ and

$$R = g^{ij}R_{ij} = \ddot{\tilde{\gamma}}_{11} + \ddot{\tilde{\gamma}}_{22} - (\partial_\sigma^2 + \partial_\rho^2)(\tilde{\gamma}_{11} + \tilde{\gamma}_{22}) - \partial_\sigma^2\tilde{\gamma}_{22} - \partial_\rho^2\tilde{\gamma}_{11} + 2\partial_\sigma\partial_\rho\tilde{\gamma}_{12}. \tag{8.15}$$

The components of the Einstein tensor are

$$\begin{pmatrix}E_{11}\\ E_{22}\\ E_{12}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\partial_\rho^2 & -\partial_\tau^2 + \partial_\rho^2 & 0\\ \partial_\sigma^2 & -\partial_\tau^2 + \partial_\sigma^2 & 0\\ -\partial_\rho\partial_\sigma & -\partial_\rho\partial_\sigma & \partial_\tau^2\end{pmatrix}\begin{pmatrix}\tilde{\gamma}_{11}\\ \tilde{\gamma}_{22}\\ \tilde{\gamma}_{12}\end{pmatrix} \tag{8.16}$$

and

$$\begin{pmatrix}E_{01}\\ E_{02}\\ E_{00}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0 & -\partial_\tau\partial_\sigma & \partial_\tau\partial_\rho\\ -\partial_\tau\partial_\rho & 0 & \partial_\tau\partial_\sigma\\ -\partial_\rho^2 & -\partial_\sigma^2 & 2\partial_\rho\partial_\sigma\end{pmatrix}\begin{pmatrix}\tilde{\gamma}_{11}\\ \tilde{\gamma}_{22}\\ \tilde{\gamma}_{12}\end{pmatrix}. \tag{8.17}$$

The $(0,1)$ and $(0,2)$ components of the Einstein equations (8.14) then become

$$\begin{aligned}\dot{Z}^1 &= G(\partial_\rho\dot{\tilde{\gamma}}_{12} - \partial_\sigma\dot{\tilde{\gamma}}_{22}),\\ \dot{Z}^2 &= G(\partial_\sigma\dot{\tilde{\gamma}}_{12} - \partial_\rho\dot{\tilde{\gamma}}_{11}).\end{aligned} \tag{8.18}$$

Hence, $Z^1 = G(\partial_\rho\tilde{\gamma}_{12} - \partial_\sigma\tilde{\gamma}_{22}) + h^1(\rho,\sigma)$ for some function h^1 and we can use the remaining gauge freedom to set $h^1(\rho,\sigma) = 0$ [159]. The field equations are symmetric under $\rho \leftrightarrow \sigma$, $l_1R_1 \leftrightarrow l_2R_2$, $Z^1 \leftrightarrow Z^2$ and we demand the solution to have the same symmetry so that

$$\begin{aligned}Z^1 &= G(\partial_\rho\tilde{\gamma}_{12} - \partial_\sigma\tilde{\gamma}_{22}),\\ Z^2 &= G(\partial_\sigma\tilde{\gamma}_{12} - \partial_\rho\tilde{\gamma}_{11}).\end{aligned} \tag{8.19}$$

Substituting (8.19) into (8.13) for $G \neq 0$:

$$\begin{aligned} \ddot{Z}^1 &= (\partial_\sigma^2 + \partial_\rho^2) Z^1 - \frac{1}{G} Z^1, \\ \ddot{Z}^2 &= (\partial_\sigma^2 + \partial_\rho^2) Z^2 - \frac{1}{G} Z^2, \end{aligned} \tag{8.20}$$

and the field equations for Z^1, Z^2 and Z^I decouple[5]. The $(0,0)$ component of (8.14) is seen to be automatically satisfied using (8.19), and the $(1,1)$, $(2,2)$ and $(1,2)$ components give

$$\ddot{\tilde{\gamma}}_{ab} = (\partial_\sigma^2 + \partial_\rho^2)\tilde{\gamma}_{ab} - \frac{1}{G}\tilde{\gamma}_{ab}, \quad a, b \in \{1, 2\} \tag{8.21}$$

again using (8.19). The dynamical equations therefore all decouple and have the same form. The diagonal Einstein equations require that $(l_1 R_1)^2 = (l_2 R_2)^2 = 1$ to get the oscillatory behaviour in (8.21) otherwise a constant term is added. From the periodicity in σ and ρ the solutions are given by a Fourier expansion and using that Z^i and $\tilde{\gamma}_{ab}$ are real they become

$$\begin{aligned} Z^j &= \sum_{m,n \in \mathbb{Z}} e^{i(m\sigma + n\rho)} \left((\alpha^j_{m,n})^\dagger e^{i\omega_{mn}\tau} + \alpha^j_{-m,-n} e^{-i\omega_{mn}\tau} \right), \quad j = 1, 2, \\ \tilde{\gamma}_{ab} &= \sum_{m,n \in \mathbb{Z}} e^{i(m\sigma + n\rho)} \left((\beta^{ab}_{m,n})^\dagger e^{i\omega_{mn}\tau} + \beta^{ab}_{-m,-n} e^{-i\omega_{mn}\tau} \right), \quad a, b \in \{1, 2\}, \end{aligned} \tag{8.22}$$

where $\omega_{mn} = \sqrt{m^2 + n^2 + 1/G}$, and

$$Z^I = z_0^I + p^I \tau + \sum_{m^2+n^2 \neq 0} e^{i(m\sigma + n\rho)} \left((\alpha^I_{m,n})^\dagger e^{i\omega' {}_{mn}\tau} + \alpha^I_{-m,-n} e^{-i\omega' {}_{mn}\tau} \right), \quad I = 3, \ldots, 9, \tag{8.23}$$

with $\omega' = \sqrt{m^2 + n^2}$. The solution to the semi-classical approximation is therefore a sum of oscillating toroidal membranes with a metric that oscillates around the flat metric. Taking the two constraints (8.19) into account the number of degrees of freedom becomes $9 + 3 - 2 + 1 = 11$ as X^- is not constrained.

[5]Note that taking the limit $G \to 0$ does not lead to the equations for the bosonic sector of the supermembrane. For the membrane the equations $G = 0 \Rightarrow T_{ij} = 0$ are solved exactly before linearizing taking g_{ij} to be the induced metric.

8.3 Discussion and conclusions

In [159] a semi-classical quantisation of the supermembrane is performed by a Dirac quantisation where $\alpha^\dagger$ and α are interpreted as creation and annihilation operators, where states are created acting with $\alpha^\dagger$ on a vacuum that is annihilated by α. Supersymmetry seems to be essential for a consistent quantisation of the membrane, the vacuum-energy contributions for example cancel out between the bosonic and fermionic sectors [159]. It would be interesting to try and extend the above semi-classical solution to include either space-time or world-volume supersymmetry, and if the analogy with the supermembrane carries through the solutions of the linearized fermionic equations will be similar to (8.22) and (8.23). It would also be interesting to consider a generalization with a cosmological constant in (8.9) thus linearizing around de Sitter or anti de Sitter space.

To conclude, in this chapter we have determined a solution to the dynamical equations of a membrane action with an Einstein–Hilbert term linearized around a flat torus. The solution is an oscillation around the toroidal membrane with the metric oscillating around the flat metric.

9

Conclusions

In this thesis we have studied the scaling behaviour of different quantum field theories given by the renormalisation group.

We have shown how perturbative calculations of the infra-red limit can be improved using general analyticity properties valid for all unitary quantum field theories. It was also shown how the exact renormalisation group equation together with the operator product expansion can be used to get equations describing the scaling of theories with a background charge. Finally, it was shown how new solutions to a semi-classical approximation of the bosonic membrane can be found studying the low energy effective action.

In chapter four the infra-red limit of a physical quantity is shown to equal the limiting value of the Borel transform in the scale parameter, where the order of the Borel transform is related to the analyticity domain of the physical quantity in a complex scale parameter. In chapter five this was used to develop an approximation method for the infra-red central charge of a perturbed conformal field theory. Improving a one loop perturbative calculation, for the unitary minimal models perturbed by $\Phi_{(1,3)}$, the approximation is very close to the exact results when maximizing the domain of analyticity.

In chapter six the critical exponents of φ^4 theory in three dimensions are approximated using the method from chapter four together with a conformal mapping

and a Padé approximation. The results are within the limits of other calculations. It would be interesting to use the result in chapter four to calculate the infra-red limit of other physical quantities in theories with a flow between an ultra-violet and an infra-red fixed point. Examples could be the theories discussed in [170] where there is a renormalisation group flow between two different WZW models in two dimensions, or the theory in [171] where there is a flow between different three dimensional conformal field theories constructed so that the ultra-violet and infra-red fixed points are S-dual.

In chapter seven using the equivalence between the quantum group restricted sine-Gordon model and the perturbed unitary minimal models together with the exact renormalisation group equation and the operator product expansion, an equation is obtained describing the flow between unitary minimal models, also when they are not infinitesimally close in coupling space. By combining the operator product expansion and the exact renormalisation group equation higher order corrections in the coupling are moved into the off-critical structure constants (operator product expansion coefficients). This seems to be a useful way of studying the renormalisation group in theories where higher order corrections for the structure constants can be calculated, e.g. using the methods developed in [147, 146, 145] or solving to lowest order the renormalisation group equations for the structure constants.

As already mentioned at the end of chapter eight, it would also be interesting to try to extend the solutions of the linearized equations of motion for the bosonic membrane to a theory with space-time supersymmetry. If a solution can be found the spectrum from a semi-classical quantisation can then be compared with the spectrum for the supermembrane in [159], and it can be checked if such a self-gravitating supermembrane is still a solution to 11 dimensional supergravity. Finally, additional terms can be considered in the lagrangian like a cosmological term and the equation of motions can be linearized around different backgrounds.

Quantum field theories are organized according to their scaling behaviour, and the study of scaling is therefore essential when trying to understand the structure of quantum field theory.

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