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Analysis of a reaction-diffusion system of $\lambda - \omega$ type

Marcus Roland GARVIE

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A thesis presented for the degree of
Doctor of Philosophy

10 NOV 2003

Numerical Analysis Group
Department of Mathematical Sciences
University of Durham
UK

August 2003

Dedicated to

my father, whose enthusiasm for mathematics is unbounded.

Analysis of a reaction-diffusion system of $\lambda - \omega$ type

Marcus Roland GARVIE

Submitted for the degree of Doctor of Philosophy
August 2003

Abstract

The author studies two coupled reaction-diffusion equations of '$\lambda - \omega$' type, on an open, bounded, convex domain $\Omega \subset \mathbb{R}^d$ ($d \leq 3$), with a boundary of class C^2, and homogeneous Neumann boundary conditions. The equations are close to a supercritical Hopf bifurcation in the reaction kinetics, and are model equations for oscillatory reaction-diffusion equations. Global existence, uniqueness and continuous dependence on initial data of strong and weak solutions are proved using the classical Faedo-Galerkin method of Lions and compactness arguments. The work provides a complete case study for the application of this method to systems of nonlinear reaction-diffusion equations. The author also undertook the numerical analysis of the reaction-diffusion system. Results are presented for a fully-practical piecewise linear finite element method by mimicking results in the continuous case. Semi-discrete and fully-discrete error estimates are proved after establishing *a priori* bounds for various norms of the approximate solutions. Finally, the theoretical results are illustrated and verified via the numerical simulation of periodic plane waves in one space dimension, and preliminary results representing target patterns and spiral solutions presented in two space dimensions.

Declaration

The work in this thesis is based on research carried out at the Numerical Analysis Group, Department of Mathematical Sciences, University of Durham, UK. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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Chapter 1

Introduction

In Section 1.1 we introduce the system of partial differential equations studied in this thesis and in Section 1.2 the derivation of this system is briefly sketched out. Section 1.3 deals with previous work and explains the relationship of the theme of work to the wider field of knowledge, and Section 1.4 outlines the specific research objectives and methodology undertaken.

1.1 Problem statement

We consider a reaction-diffusion system of '$\lambda - \omega$' type [45], with the following general form:
Find $\{u(\boldsymbol{x},t), v(\boldsymbol{x},t)\}$ such that

$$\frac{\partial u}{\partial t} = \Delta u + \lambda(r)\, u - \omega(r)\, v \qquad \text{in} \quad \Omega_T, \tag{1.1.1a}$$

$$\frac{\partial v}{\partial t} = \Delta v + \omega(r)\, u + \lambda(r)\, v \qquad \text{in} \quad \Omega_T, \tag{1.1.1b}$$

where $\Omega_T := \Omega \times (0,T)$, $T > 0$ and the 'amplitude' is given by

$$r := \sqrt{u^2 + v^2}, \tag{1.1.1c}$$

with initial and boundary conditions

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \qquad v(\boldsymbol{x},0) = v_0(\boldsymbol{x}), \tag{1.1.1d}$$

$$\frac{\partial u}{\partial \boldsymbol{\nu}} = \frac{\partial v}{\partial \boldsymbol{\nu}} = 0 \quad \text{on} \quad \Sigma := \partial\Omega \times (0,T), \tag{1.1.1e}$$

where $\boldsymbol{\nu}$ denotes the outward normal to $\partial\Omega$. Throughout Δ denotes $\sum_{i=1}^{d} \partial^2/\partial x_i^2$ and $\Omega \subset \mathbb{R}^d$ $(d \leq 3)$ is an open, bounded, convex domain. We make the further assumption that the boundary is of class[1] C^2. The Lebesgue measure is denoted $dx \equiv dx_1\, dx_2 \ldots dx_d$. The specific class of λ and ω functions we study are

$$\lambda(r) := \lambda_0 - \lambda_1 r^\rho, \qquad \omega(r) := \omega_0 + \omega_1 r^\rho, \tag{1.1.1f}$$

where λ_0, λ_1, $\rho > 0$ and ω_0, ω_1 are non-zero numbers; all parameters are assumed real and finite. We consider a more general class of nonlinear functions than was originally proposed in [45] with an arbitrary power ρ of the amplitude instead of a quadratic power (cf. [74], [75], [78], [43]). From an applications point of view (discussed in Section 1.3) the most important case is $\rho = 2$. Note that there are no restrictions on T, which is an arbitrary real positive parameter.

The role of a convex domain is two-fold. Firstly, a bounded, open, convex domain has a Lipschitz boundary ([32], Corollary 1.2.2.3) and secondly, is amenable to the application of an elliptic regularity result. A Lipschitz continuous boundary is important as we undertake the numerical analysis of the $\lambda - \omega$ system. For the same reason, a *bounded* domain is used. The requirement that the boundary be of class C^2 for the proof of the well-posedness of the $\lambda - \omega$ system is due to another elliptic regularity result needed in Chapter 3.

In Chapter 2 we rewrite the $\lambda - \omega$ system in complex form and a vector form for later analysis. A key feature of this system is that when the $\lambda - \omega$ system is written in matrix form the coefficient matrix of the reaction term has the 'real canonical form'

$$\begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix}, \qquad \lambda, \omega \in \mathbb{R} \tag{1.1.1g}$$

which is isomorphic to $\lambda + i\omega \in \mathbb{C}$. Furthermore, if λ is positive then this matrix is positive definite. This structure of the nonlinearity has important implications for later energy estimates and also leads to complex numerical methods. We also

[1] I.e., the boundary $\partial\Omega$ can be locally represented as a graph of a C^2 function. For a more precise definition see [70], p.129.

comment that the $\lambda - \omega$ system is invariant under the transformation

$$\{\omega_0, \omega_1, u, v\} \to \{-\omega_0, -\omega_1, v, u\},$$

(Sherratt, J. (2001) *Pers. comm.*), thus existence, uniqueness and stability of solutions will be independent of the change of sign of both ω_0 and ω_1.

1.2 Derivation of the $\lambda - \omega$ system

We justify that close to a supercritical Hopf bifurcation[2], any two coupled ODEs will have the $\lambda - \omega$ form, where the functions $\lambda(r)$, $\omega(r)$ are defined as in (1.1.1f), with $\rho = 2$ and the same conditions on the parameters $\lambda_0, \lambda_1, \omega_0$ and ω_1. When we say that a system is *close* to a Hopf bifurcation we mean that the bifurcation parameter is approximately equal to the bifurcation point. We also mention the relevance to the corresponding reaction-diffusion system with equal diffusion coefficients. These results are stated in [78].

We assume the standard conditions in the Hopf Bifurcation Theorem (e.g., see [29], p.227, [93], p.270, [88], p.203). Consider the system of 2 ODEs

$$\dot{x} = f(x, y; \mu), \tag{1.2.1a}$$

$$\dot{y} = g(x, y; \mu), \tag{1.2.1b}$$

where μ is the bifurcation parameter. We assume with no loss in generality that the equilibrium solution is at the origin for μ near zero (the bifurcation point) [29], p.226. Furthermore, assume that the Jacobian matrix associated with this system evaluated at $(0,0)$, denoted $A(\mu)$, has eigenvalues $\alpha(\mu) \pm i\omega(\mu)$ with $\alpha(0) = 0$, $\omega(0) = \omega \neq 0$, and $d\alpha(0)/d\mu \neq 0$. We interpret this to mean that as μ varies the eigenvalues cross the imaginary axis with finite speed[3]. With these assumptions on the ODE system we explain how to derive the $\lambda - \omega$ system.

[2]The transition of a stable equilibrium solution into a stable periodic orbit containing an unstable fixed point, as the bifurcation parameter is varied.

[3]It is stated in [88], p.203 that this last condition $d\alpha(0)/d\mu \neq 0$ is unnecessary.

The Taylor expansion of (1.2.1a) - (1.2.1b) about the origin is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A(\mu) \begin{pmatrix} x \\ y \end{pmatrix} + \dots \tag{1.2.2}$$

We write this system in Jordan canonical form via the linear change of variable

$$\begin{pmatrix} x \\ y \end{pmatrix} := P \begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix}, \quad P := [\mathrm{Im}(z), \mathrm{Re}(z)],$$

where z is the complex eigenvector of $A(\mu)$ associated with the eigenvalue $\alpha(\mu) + i\omega(\mu)$ (see, e.g., [29], p.59). So after noting $A(\mu)P = PD$ where the canonical form

$$D = \begin{pmatrix} \alpha(\mu) & -\omega(\mu) \\ \omega(\mu) & \alpha(\mu) \end{pmatrix},$$

we can write (1.2.2) as

$$\begin{pmatrix} \dot{\widehat{x}} \\ \dot{\widehat{y}} \end{pmatrix} = D \begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} + \begin{pmatrix} F(\widehat{x}, \widehat{y}) \\ G(\widehat{x}, \widehat{y}) \end{pmatrix},$$

for some nonlinear functions F and G.

The nonlinear functions are then expanded and transformed to a standard form via a lengthy manipulation process called 'normalisation' using 'near identity transformations' (see, e.g., [89], [93]) that removes even order terms (details omitted). Using x and y for $\widehat{x}$ and $\widehat{y}$ respectively, the normalised system can be written as

$$\dot{x} = \alpha(\mu)x - \omega(\mu)y + [a(\mu)x - b(\mu)y]\,(x^2 + y^2) + \mathcal{O}(|x|^5, |y|^5),$$
$$\dot{y} = \omega(\mu)x + \alpha(\mu)y + [b(\mu)x + a(\mu)y]\,(x^2 + y^2) + \mathcal{O}(|x|^5, |y|^5).$$

After Taylor expanding this system about $\mu = 0$ and taking $d := \dot{\alpha}(0)$, $a := a(0)$, $c := \dot{\omega}(0)$, and $b := b(0)$ yields

$$\dot{x} = d\mu x - (\omega + c\mu)y + (x^2 + y^2)(ax - by) + \dots, \tag{1.2.3a}$$

$$\dot{y} = (\omega + c\mu)x + d\mu y + (x^2 + y^2)(bx + ay) + \dots, \tag{1.2.3b}$$

([93], p.271, [89], p.190, [33], p.151).

The Hopf Bifurcation Theorem then tells us that the qualitative properties of (1.2.3a) and (1.2.3b) near the origin remain unchanged if we neglect the higher order terms [33], p.151. So setting $\lambda_0 := d\mu$, $\omega_0 := \omega + c\mu$, $\lambda_1 := -a$, $\omega_1 := b$, and $\lambda(r) := \lambda_0 - \lambda_1 r^2$, $\omega(r) := \omega_0 + \omega_1 r^2$ ($r^2 := x^2 + y^2$) we can write (1.2.3a) and (1.2.3b) in the $\lambda - \omega$ form

$$\dot{x} = \lambda(r)x - \omega(r)y, \tag{1.2.4a}$$

$$\dot{y} = \omega(r)x + \lambda(r)y, \tag{1.2.4b}$$

where we have neglected higher order terms. To facilitate analysis we write this system in polar coordinates. If we multiply (1.2.4a) by x and (1.2.4b) by y and then add, this gives

$$\dot{r} = \lambda(r)r = (\lambda_0 - \lambda_1 r^2)r \equiv d\mu r + ar^3. \tag{1.2.5a}$$

However, if we multiply (1.2.4a) by y and (1.2.4b) by x and then subtract the second equation from the first we get

$$\dot{\theta} = \omega(r) = \omega_0 + \omega_1 r^2 \equiv \omega + c\mu + br^2, \quad \theta := \arctan\left(\frac{y}{x}\right). \tag{1.2.5b}$$

The analysis in [93], pp.272-275, reveals that for $r_0 := (\lambda_0/\lambda_1)^{1/2}$ and μ sufficiently small

$$\begin{aligned}(r, \theta(t)) &= (r_0, \omega(r_0)t + \theta_0) \\ &\equiv \left(\sqrt{\frac{-\mu d}{a}}, \left[\omega + \left(c - \frac{bd}{a}\right)\mu\right]t + \theta_0\right)\end{aligned} \tag{1.2.6}$$

is an asymptotically stable periodic orbit[4] for (1.2.4a), (1.2.4b) provided

$$a < 0 \quad \text{and} \quad \frac{\mu d}{a} < 0. \tag{1.2.7}$$

From (1.2.5b) for a periodic solution we require $\dot{\theta} \neq 0$. To show this first note that ω_0 and ω_1 are both $\mathcal{O}(1)$ and r_0^2 is $\mathcal{O}(\mu)$. Thus near and inside the limit cycle (i.e., near the bifurcation point $\mu = 0$) we have $\omega(r_0) = \omega_0 + \omega_1 r_0^2 \approx \omega_0$. That is, $\omega(r_0)$ is non-zero as ω_0 is non-zero. We also remark that there does not appear to be any reason why we cannot take $\omega_1 = 0$, but it seems traditional to assume this parameter is non-zero (e.g. [78], [43]). We restrict ourselves to the case where as μ increases past the bifurcation point a stable equilibrium solution bifurcates into

[4] I.e., the periodic orbit attracts neighbouring points.

a stable periodic orbit (and not vice-versa). The analysis in [93], p.274, shows this corresponds to the case where $d > 0$. So from (1.2.7) we must have $\mu > 0$ and thus by definition $\lambda_0 > 0$. To summarise, for the $\lambda - \omega$ system (1.2.4a)-(1.2.4b) to possess a supercritical Hopf bifurcation (in the sense described above) we impose

(i) λ_0, $\lambda_1 > 0$,

(ii) ω_0, ω_1 of either sign (not zero),

(e.g., [78]). Furthermore, the limit cycle in the x-y plane has radius $(\lambda_0/\lambda_1)^{1/2}$, with anti-clockwise orbits if $\omega_0 > 0$ and clockwise orbits if $\omega_0 < 0$.

The derivation above is relevant to general reaction-diffusion equations provided the diffusion coefficients are equal, thus they can be scaled to unity [45]. Furthermore, if the amplitude r tends to the zero of $\lambda(r)$, i.e., $(\lambda_0/\lambda_1)^{1/2}$, then the solution of the PDE becomes spatially homogeneous (i.e., reduces to the ODE situation), with a limit cycle in the reaction kinetics.

1.3 Previous work and relevance to the wider field of knowledge

Systems of reaction-diffusion equations have the following general form, with appropriate boundary and initial conditions:

$$\frac{\partial u}{\partial t} = D\,\Delta u + f(u), \qquad x \in \Omega \subset \mathbb{R}^d,\ t > 0, \tag{1.3.1}$$

where $u \in \mathbb{R}^n$, D is the (diagonal) diffusion matrix, and f is the reaction term. The diffusion term tends to 'smooth' the solution u, while the nonlinear term $f(u)$ can produce solutions that grow rapidly. Thus there is the possibility of threshold phenomena. However, the combined effect of reaction and diffusion can produce new mathematical features of the solution, distinct from either mechanism alone. Depending on the nature of the nonlinearity and the initial data it is possible that solutions 'blow-up' in finite time, i.e., we only have local existence of solutions. For further details see [81], [11].

Reaction-diffusion equations can be interpreted in the context of interacting biological or chemical species. The introduction of space into these models is relatively new, and traditional mathematical models involve the spatially homogeneous situation, that is, systems of ODEs without the diffusion term present (e.g., [58], [57]). A classic example is the (deterministic) predator-prey model of Lotka [55] and Volterra [90]. Another classic equation, and the simplest case of a nonlinear reaction-diffusion equation, is the Fisher equation ([27] cited in [62], p.277). This equation arises in the study of population genetics and has the form

$$\frac{\partial u}{\partial t} = ku(1-u) + D\frac{\partial^2 u}{\partial x^2} \tag{1.3.2}$$

where k and D are positive parameters. The Fisher equation is relevant to our study as it possesses travelling wave solutions, in common with systems of $\lambda - \omega$ type.

Reaction-diffusion equations of $\lambda - \omega$ type were first studied almost thirty years ago by Kopell and Howard [45], who were motivated by an attempt to describe the formation of patterns in the Belousov-Zhabotinskii reaction. The system is important and interesting for a number of reasons. Firstly, systems of $\lambda - \omega$ type display features in common with many real biological and chemical models, although they are independent of any specific physical problem. Furthermore, the system is a model for general reaction-diffusion equations with a limit cycle in the reaction kinetics ('oscillatory' reaction diffusion equations). As shown in Section 1.2, from normal form theory and the Hopf Bifurcation Theorem, any system of two ODEs near a supercritical Hopf bifurcation will have the general $\lambda - \omega$ form (1.1.1a)-(1.1.1b), with the λ and ω functions defined as in (1.1.1f) with $\rho = 2$. This is relevant to general reaction-diffusion systems provided the diffusion coefficients are equal [45]. For a study of reaction-diffusion equations close to a subcritical[5] Hopf bifurcation see [25].

For a review of the $\lambda - \omega$ system and early work see [62]. Recent interest in the $\lambda-\omega$ system is due to a series of papers by Sherratt [74], [75], [76], [77], [78], [43] who used a combination of analytical and numerical methods to investigate the dynamics

[5]The transition of an unstable equilibrium solution into an unstable periodic orbit containing a stable fixed point, as the bifurcation parameter is varied.

of solutions in one space dimension, with for example, locally exponentially decaying initial conditions.

In spite of the importance of this system, there are few existence, uniqueness and stability results in the literature and these concern specific (ansatz) solutions. Kopell and Howard [45] proved the $\lambda - \omega$ system on the real line has a simple one-parameter family of 2π-periodic (in space and time) travelling wave solutions (see (6.1.1)), called 'periodic plane waves', which are linearly stable under certain known conditions (see (6.1.2), (6.1.3)). The nonlinear stability of travelling wave solutions in one space dimension has been investigated [42], where it was proved that solutions are stable in a 'polynomially weighted L^∞ space'. In two space dimensions the periodic plane waves correspond to spiral waves or concentric ring waves ('target patterns') (see [62], pp.343-356 and the references therein). Cohen, Neu and Rosales [16] were the first to prove the existence of rotating spiral wave solutions and at about the same time Greenberg [30] proved the existence of target pattern solutions. Soon after these works, Kopell and Howard [46] generalised their earlier one-dimensional results by proving the existence of homogeneous target patterns and spiral solutions to systems of $\lambda - \omega$ type in two and three space dimensions.

We comment on the similarity of the $\lambda - \omega$ system to the Complex Ginzburg-Landau (CGL) equation (see, e.g., [85], p.226 and the references therein). The $\lambda - \omega$ system written in complex form with $\rho = 2$ is

$$c_t = \Delta c + (\lambda_0 + i\,\omega_0)c + (-\lambda_1 + i\,\omega_1)|c|^2 c,$$

where $c := u + i\,v$ and $r := |c| \equiv \sqrt{u^2 + v^2}$. If we rotate the solution vector in the u-v plane by $\omega_0 t$ via the transformation $c \mapsto c\exp(i\,\omega_0\,t)$ (effectively removing ω_0), followed by the rescaling of dependent and independent variables $t \mapsto (1/\lambda_0)t$, $x_i \mapsto (1/\lambda_0^{1/2})x_i$, $c \mapsto (\lambda_0/\lambda_1)^{1/2}\,c$, then[6] we obtain the CGL equation

$$c_t = \Delta c + (1 + i\,\alpha)c - (1 + i\,\beta)|c|^2 c, \tag{1.3.3}$$

where $\alpha = 0$ and $\beta = -\omega_1/\lambda_1$. In general for the CGL equation α is non-zero and the Laplacian has a complex coefficient, so if we split this equation into real and

[6] Thus $\frac{\partial}{\partial t} \mapsto \lambda_0 \frac{\partial}{\partial t}$, $\frac{\partial^2}{\partial x_i^2} \mapsto \lambda_0 \frac{\partial^2}{\partial x_i^2}$ so $\Delta \mapsto \lambda_0 \Delta$.

imaginary parts then the diffusion matrix is antisymmetric (in the $\lambda - \omega$ case the off-diagonal terms in this matrix are zero). Thus our $\lambda - \omega$ model, with $\rho = 2$, is a special case of the CGL equation. In the CGL equation it is necessary that the α term be non-zero for the existence of unstable spatially homogeneous oscillations, a feature not possible in the $\lambda - \omega$ system [49], p.21, p.140. There is an extensive literature on the regularity of the CGL equations (and a generalised CGL equation) and two key papers relevant to bounded domains are [19], [51].

Of particular relevance is a theorem in ([85], p.228) for a CGL equation in a form covering (1.3.3) with $\alpha = 0$ and therefore also applicable to the $\lambda - \omega$ system with $\rho = 2$. From this theorem we have the following results that are consistent with the results proved in this thesis. Assume that $d = 1$ or 2, $\rho = 2$ and c, c_0 are complex valued functions. Given initial data $c_0 \in L^2(\Omega)$, then there exists a unique weak solution of (1.1.1a)-(1.1.1f) where

$$c \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)) \qquad \forall\, T < \infty,$$

and the solution depends continuously on the initial data in $L^2(\Omega)$. Given initial data $c_0 \in H^1(\Omega)$, then a unique strong solution exists where

$$c \in C([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)) \qquad \forall\, T < \infty.$$

Finally, we mention the connection between the $\lambda - \omega$ system and the Allen-Cahn equation arising from phase transitions in materials science (see [26] and the references therein). In the $\lambda - \omega$ system if we take $\rho = 2$, $u = v$, $\lambda_0 = 1/\varepsilon^2$ and $\lambda_1 = 1/2\varepsilon^2$, where ε is a small parameter, then adding (1.1.1a) to (1.1.1b) and simplifying we obtain

$$u_t = \Delta u + \frac{1}{\varepsilon^2} u(1 - u^2). \tag{1.3.4}$$

With $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ and appropriate initial and boundary conditions this is a typical Allen-Cahn equation.

With regard to the numerical analysis of the $\lambda - \omega$ system, to our knowledge, since Kopell and Howard [45] introduced the $\lambda - \omega$ system, there have been no studies except a short paper comparing two numerical methods for a specific example in one space dimension [77].

1.4 Research objectives and methodology

There were three main objectives of this work:

(1) classical analysis of the $\lambda - \omega$ system,

(2) numerical analysis of the $\lambda - \omega$ system,

(3) scientific computing and simulations.

The thesis focuses mainly on the first two objectives, however, all objectives were successfully completed and their description and methodology are given below.

The first objective involved the rigorous proof of the existence, uniqueness and regularity of weak and strong solutions to the $\lambda - \omega$ system, in $d \leq 3$ space dimensions [9]. Continuous dependence of these solutions on the initial data was also shown. This was achieved using the Faedo-Galerkin method of Lions [53] and compactness arguments from Functional Analysis (see, e.g., [18]). Our work also collects together a number of results that are often used implicitly in the literature. These methods belong to the area of Infinite Dimensional Dynamical Systems and two important sources for our work have been [85] and [70]. The basic idea behind the Faedo-Galerkin method is first to reduce the infinite dimensional dynamical system to a finite dimensional one via the introduction of a truncated Galerkin expansion. Then standard results from the theory of ordinary differential equations can be applied. A crucial step in the analysis is the derivation of *a priori* estimates for bounding energy functionals, i.e., the so called 'energy method'. A rigorous treatment requires the formalism of Banach Spaces and Sobolev Spaces (e.g., [1]) to precisely characterise the regularity of solutions. Some standard technical tools we used in this process include various Young's inequalities, Hölder's inequality, a Gagliardo-Nirenberg inequality, Sobolev Embedding results and a Grönwall lemma.

The continuous results provide the foundation for a numerical analysis of the $\lambda - \omega$ system. The second objective was achieved with the finite element method (e.g., [14]) using the canonical piecewise linear basis functions and a non-uniform mesh to discretise the $\lambda - \omega$ system in space, giving a semi-discrete approximation. *A priori* bounds for various norms of the semi-discrete solutions led to a semi-discrete error bound. These calculations then provided the basis for obtaining a semi-implicit (in time), fully-discrete approximation for each time step and the derivation of fully-discrete error estimates. Furthermore, it was proved that these approximate solutions exist and are unique [28]. A crucial approach was to seek approximations that mimicked the properties of the continuous solutions. For example, when deriving stability estimates for the approximations it was important to look carefully at the steps taken in the basic *a priori* estimate for the continuous solutions. Bearing in mind that the Allen-Cahn equation (1.3.4) given at the end of the last section is obtained from a simplification of the $\lambda - \omega$ system, an approach that often proved successful was to initially analyse the Allen-Cahn equation (with $\varepsilon = 1$) and then generalise the calculations to the full $\lambda - \omega$ system. This was particularly true in the error analysis.

A technique that helped simplify the practical calculations was to use 'lumped mass integration', where the mass matrix is approximated with the aid of the 'vertex quadrature rule', i.e., a simple numerical integration rule that is exact for piecewise linear functions. We comment that there are various numerical integration schemes that can be used in the finite dimensional weak forms, depending on the structure of the nonlinearity. For example, consider the semi-discrete weak form corresponding to the Allen-Cahn equation (1.3.4) (with $\varepsilon = 1$): Find $u^h \in S^h$ with an appropriate initial approximation $u^h(\cdot, 0)$ such that

$$\left(\frac{\partial u^h}{\partial t}, \chi\right) + (\nabla u^h, \nabla \chi) = (u^h - (u^h)^3, \chi), \quad \forall \chi \in S^h,$$

where S^h is the standard finite element space of continuous piecewise linear functions. Noting that the integrand of the nonlinear term is a piecewise polynomial of degree ≤ 4 we could choose to evaluate this term exactly with the 3-point Gauss-rule, which is exact for polynomials of degree ≤ 5.

Another technique that simplified some of the analysis was to rewrite the $\lambda - \omega$ system in complex form, giving a single (complex) equation that led to complex numerical methods. Some standard technical tools used in the estimates were the discrete analogues of the corresponding continuous ones, for example, discrete embedding results, a discrete Hölder's inequality and a discrete Grönwall lemma.

We comment on the importance of undertaking a rigorous numerical analysis of the $\lambda - \omega$ system before applying a practical numerical method. It is well-known that the dynamics of numerical discretisations of nonlinear differential equations (DEs) can differ significantly from that of the original DEs themselves (see [94] and the references therein). For example, when investigating simulated spatiotemporal chaos, it is of crucial importance to know whether this behaviour is a feature of the continuous model, or represents the onset of numerical instability.

We comment on the advantages of the finite element method over the finite difference method, when undertaking rigorous analysis. Consider the problem of proving that a numerical scheme is stable. Expressing the equations in weak form, either at the continuous, semi-discrete, or fully-discrete stage, facilitates different approaches for analysing stability, depending on the test function chosen. The (fully-discrete) finite element method expressed in weak form has the added advantage of suppressing the nodal indices, thus simplifying the notation. Let us contrast this with the finite difference approach for solving the one-dimensional heat equation $u_t = u_{xx}$, with appropriate initial and boundary conditions. A popular finite difference method is the Crank-Nicolson scheme:

$$\frac{U_j^n - U_j^{n-1}}{\Delta t} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{2h^2} + \frac{U_{j+1}^{n-1} - 2U_j^{n-1} + U_{j-1}^{n-1}}{2h^2},$$

where h is the space step and Δt is the time step and U_j^n approximates the exact solution u at time level $n\Delta t$ and node x_j. This scheme leads to a linear system of the form

$$A\boldsymbol{U}^n = B\boldsymbol{U}^{n-1}, \quad \{\boldsymbol{U}^n\}_j := U_j^n,$$

where A and B are tri-diagonal matrices. For stability of this method we need the eigenvalues of the stability matrix $A^{-1}B$ to lie within the unit circle in the complex plane. An analytical expression for these eigenvalues is given in [80], p.65, showing unconditional stability of the scheme. Alternatively, we could apply Fourier analysis (see, e.g., [60], p.26) to obtain the same result. Now suppose we try and go through the same procedure for a simple scalar reaction diffusion equation $u_t = u_{xx} + f(u)$, for some nonlinear function f, with appropriate initial and boundary conditions. This situation is much more complicated and the previous techniques for proving stability no longer work. A popular approach for nonlinear problems is the 'energy method', which roughly speaking is a technique for showing that some positive function of the approximate solution values is bounded in some norm (usually by the initial data) and hence the approximation is bounded. This approach becomes very cumbersome with finite differences due to the nodal indices (e.g., see [60], pp.149-156) and is much more straightforward to apply using the finite element method. Similar comments apply for proving the existence and uniqueness of numerical solutions and in error analysis.

For the final objective, programming in Fortran 77 and Matlab led to numerical simulations in one space dimension, and preliminary results in the two-dimensional case. Numerical simulations allowed the expected theoretical behaviour to be verified and the qualitative features of solutions investigated. As the numerical methods are semi-implicit (in time) the approximations lead to a set of (complex) linear equations, which must be solved for at each time step.

In addition to the Faedo-Galerkin method for proving existence results for the $\lambda - \omega$ system, the 'invariant region' method of Smoller ([81], [13]) is also applicable. Although this approach is not central to the work in this thesis, for completeness we briefly review the relevant application here and refer the reader to the above mentioned references for further details.

Assume the reaction-diffusion system (1.3.1) is supplemented with appropriate boundary conditions of either the Dirichlet or Neumann type, and the solution u satisfies the initial condition $u(x,0) = u_0(x)$ for all $x \in \Omega$. Now define a 'bounded invariant' region $\mathbb{D}$ to be a subset of the phase space $\mathbb{R}^n$ with the property that if $u_0 \in \mathbb{D}$ and the boundary lies entirely in $\mathbb{D}$ then $u(x,t) \in \mathbb{D}$ for all $x \in \Omega$ and all $t > 0$. Now from Corollary 14.8(b) in [81], if the diffusion matrix D is the identity matrix, then any convex region Σ, in which f points into Σ on $\partial\Sigma$ is invariant for (1.3.1). Thus from Corollary 14.9 in [81], the solution exists for all $t > 0$.

To find an invariant region we first transform the ODE corresponding to the $\lambda - \omega$ system to polar form in phase space (cf. (1.2.5a)-(1.2.5b)) to give:

$$r_t = (\lambda_0 - \lambda_1 r^\rho)r, \quad r := \sqrt{u^2+v^2} \tag{1.4.1a}$$

$$\theta_t = \omega(r), \quad \theta := \arctan\left(\frac{v}{u}\right). \tag{1.4.1b}$$

From (1.4.1a) it is seen that

$$r_t > 0 \quad \text{for } 0 < r < (\lambda_0/\lambda_1)^{1/\rho} \text{ and } \quad r_t < 0 \quad \text{for } r > (\lambda_0/\lambda_1)^{1/\rho}.$$

Thus orbits that start inside or outside the circle

$$u^2 + v^2 = \left(\frac{\lambda_0}{\lambda_1}\right)^{2/\rho} \equiv r_0^2,$$

tend to $r = r_0$ and the limit cycle solution is given by

$$r = r_0, \quad \theta(t) = \omega(r_0)t + \theta_0.$$

We argue as in Example 3, p.210 in [81] and Example D in [13]. Let B be any convex region containing the disk $u^2 + v^2 = r_0^2$, then it is easy to see that the vector field corresponding to the reaction terms of the $\lambda - \omega$ system points into B and thus by Corollary 14.8(b) in [81] we deduce that B is an invariant region for the $\lambda - \omega$ system. In particular, the region

$$B_\delta := \{(u,v) \mid u^2 + v^2 \leq (\lambda_0/\lambda_1)^{2/\rho} + \delta\}, \quad \delta > 0,$$

is invariant. We claim that the ball B_0 is invariant for the $\lambda - \omega$ system, however we cannot apply Corollary 14.8(b) in [81] directly, since the vector field vanishes identically on ∂B_0. However, if $(u_0(x), v_0(x)) \in B_0$ for all x, then $(u_0(x), v_0(x)) \in B_\delta$ for all x and every $\delta > 0$. Thus the solution $(u(x,t), v(x,t)) \in B_\delta$ for all x and every $\delta > 0$ which implies $(u(x,t), v(x,t)) \in B_0$ for all x and all $t > 0$. Thus provided we have $L^\infty(\Omega)$ initial data then the solution asymptotically lies in B_0. That is, we have global existence in time of solutions to the full reaction-diffusion system.

In this thesis we are mainly interested in investigating solutions of the $\lambda - \omega$ system (and their approximations) evolving from less regular initial data, e.g., data in $L^2(\Omega)$ or $H^1(\Omega)$. There is also the question of uniqueness of solutions to consider.

The modern theory of differential equations (and hence the finite element method) relies on a considerable body of knowledge from distribution theory, measure theory and the Lebesgue integral. It would take us too far from our intended area of study to discuss in detail these topics. Instead we refer the reader to standard works, e.g., see [73] for the theory of generalised function (distribution) theory and for measure theory and the Lebesgue integral see [61] or [37]. For a more accessible introduction to the Lebesgue integral and measure theory see [91]. We shall recall results from functional analysis and the Sobolev spaces when needed. For further details (and more advanced treatments) of the Sobolev spaces see [1], [54] and [70]. For background in functional analysis see [95], [59] and [17], and for a modern grounding in the theory of differential equations see [69]. The two main texts we relied on for background theory of the finite element method were [86] and [14].

The thesis is organised as follows. Chapter 2 deals with existence, uniqueness and regularity of the weak solutions. In Chapter 3 these results are extended to cover the existence, regularity and continuous dependence of strong solutions on the initial data. The overall approach via a 'composite' Galerkin approximation is a generalisation of that in [70] applied to a model reaction-diffusion equation with a polynomial nonlinearity. In Chapter 4 we use the finite element method with piecewise linear basis functions to obtain a semi-discrete approximation, *a priori* bounds of the semi-discrete solutions and then a semi-discrete error bound. In Chapter 5 these calculations provide the basis for obtaining a semi-implicit, fully-discrete approximation, *a priori* bounds for various norms of the fully-discrete solutions and the derivation of fully-discrete error estimates. In Chapter 6 some numerical experiments are performed in one space dimension and the fully-discrete error bound verified numerically. We also present some preliminary results in two space dimensions. Finally, in Chapter 7 we summarise our results and discuss possible further developments.

Note that mathematical analysis estimates are numbered separately from the numerical analysis estimates.

Chapter 2

Weak solutions

In Section 2.1 the basic notation is laid out and abstract Sobolev and Banach spaces are reviewed. In Section 2.2 we rewrite the $\lambda - \omega$ system in two different ways, which then leads to two equivalent weak formulations and the statement of the main theorem of this chapter. Local existence (Section 2.3) and global existence (Section 2.4) of the weak solutions are discussed and the most theoretical part of this thesis, passage to the limit of the Galerkin approximations, is achieved in Section 2.5, followed by a uniqueness proof in Section 2.6. There are a large number of theoretical results needed in this chapter and so to improve the flow of the arguments many of the auxiliary results are put in the appendices. If a result cited in the literature does not provide a proof then we give a proof with the quoted result.

2.1 Notation and preliminaries

For our purposes it will be sufficient to note that integration is defined in the Lebesgue sense and all partial derivatives are to be understood in the context of distribution theory, i.e., as 'weak' derivatives. Two measurable functions are identified as equal in an $L^p(\Omega)$ space if they are equal 'almost everywhere' (a.e.) on a domain[1] Ω. Thus any function is identified with an equivalence class of functions that differ on a set of measure zero (see Definition A.0.1).

[1] Ω may be different to that in Section 1.1.

We denote D^α to be the standard multi-index notation for the mixed partial derivative of order $|\alpha|$ (e.g., see [67]):

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n, \quad D^\alpha \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad |\alpha| = \sum_{i=1}^{n} \alpha_i,$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

In this study the dual space of a Banach space X is written X'. We use the usual Sobolev spaces $W^{m,p}(\Omega)$, $m \in \mathbb{N}$, $p \in [1, \infty]$, with associated norms and semi-norms given by

$$\|u\|_{m,p} := \left(\sum_{0 \le |\alpha| \le m} \|D^\alpha u\|_{0,p}^p \right)^{1/p}, \qquad |u|_{m,p} := \left(\sum_{|\alpha| = m} \|D^\alpha u\|_{0,p}^p \right)^{1/p},$$

respectively. Another standard Banach space we use is $L^\infty(\Omega)$, with associated essential supremum norm[2]

$$\|u\|_{0,\infty} \equiv \|u\|_{L^\infty(\Omega)} := \inf\{M \,:\, |u(x)| \le M \quad \text{a.e. on } \Omega\}.$$

Some standard properties of the Sobolev spaces we assume are collected together as Theorem A.0.2. $W^{m,2}(\Omega)$ will be denoted by $H^m(\Omega)$ with norm $\|\cdot\|_m$ and semi-norm $|\cdot|_m$ and if additionally $m = 0$, $W^{0,2}(\Omega) \equiv L^2(\Omega)$. The usual $L^2(\Omega)$ inner product over Ω with norm $\|\cdot\|_0$ is denoted by $(\cdot,\cdot)$, except in the complex weak formulation of Section 2.2 where it is understood that

$$(z, w) := \int_\Omega z(x)\overline{w(x)}\, dx.$$

Furthermore, $\langle\cdot,\cdot\rangle$ will represent the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$. In general we shall write $\langle\cdot,\cdot\rangle_{X',X}$ for the duality pairing between a Banach space X and its dual X' (see Definition A.0.3 for a brief review of duality pairings and their properties). We denote the Euclidean norm by $|\cdot|$. To simplify notation, we define $H := L^2(\Omega)$ and $V := H^1(\Omega)$ so $V' = [H^1(\Omega)]'$ and the inner product on V is denoted $(\cdot,\cdot)_V$.

[2]The use of the essential supremum norm instead of the maximum norm takes into account a possible set of points at infinity with measure zero. However, if we know $u \in C(\overline{\Omega})$ (for example, if $u \in H^2(\Omega)$ for $d \le 3$) then this norm reduces to $\|u\|_{L^\infty(\Omega)} = \max_{x \in \overline{\Omega}} |u(x)|$ [48], p.83.

It will be useful in the work that follows to note the easily proven 'equivalence of semi-norms' result for $1 \le p < \infty$:

$$|u|_{2,p} \le \left(\int_\Omega \sum_{i,j=1}^{d} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^p dx \right)^{1/p} \le 2^{1/p} |u|_{2,p}. \tag{2.1.1}$$

We define function spaces depending on space and time (e.g., [85], p.45). Let X be a Banach space and $p \in [1, \infty]$. Denote $L^p(0, T; X)$ to be the Banach space of all measurable functions $u : (0, T) \mapsto X$ such that $t \mapsto \|u(t)\|_X$ is in $L^p(0, T)$, with norm

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p dt \right)^{1/p} \quad \text{for} \quad 1 \le p < \infty, \tag{2.1.2}$$

$$\|u\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{t \in (0,T)} \|u(t)\|_X \quad \text{if} \quad p = \infty. \tag{2.1.3}$$

In addition we write $L^p(\Omega_T) \equiv L^p(0, T; L^p(\Omega))$. We assume some well-known properties of these time-dependent Sobolev spaces and collect them together in the appendix (Theorem A.0.4).

We also need to define $C([0, T]; X)$, the space of continuous functions from $[0, T]$ into[3] X, which is dense in $L^p(0, T; X)$ with respect to the norm $\|\cdot\|_{L^p(0,T;X)}$ ([69], p.378), but is a Banach space when equipped with the norm

$$\sup_{t \in [0,T]} \|u(t)\|_X$$

[85], p.45. The space $C^1([0, T]; X)$ consists of those functions and their first derivatives (in $[0, T]$) belonging to $C([0, T]; X)$ and so is a Banach space for the norm

$$\|u\|_{C^1([0,T];X)} := \|u\|_{C([0,T];X)} + \left\| \frac{du}{dt} \right\|_{C([0,T];X)}.$$

We shall also need to use $C_0^\infty(0, T; X)$, the space of infinitely differentiable functions[4] from $(0, T)$ into X with compact support in $(0, T)$; this space is dense in $L^p(0, T; X)$ with respect to the norm $\|\cdot\|_{L^p(0,T;X)}$.

---

[3] $u(t)$ is continuous at t_0 in $X([0, T])$ means given any $\varepsilon > 0$ $\exists \delta > 0$ s.t. $|t - t_0| < \delta \implies \|u(t) - u(t_0)\|_X < \varepsilon$ for all $t, t_0 \in [0, T]$.

[4] If $u \in C_0^\infty(0, T; X)$ this implies $d^k u/dt^k \in C_0^\infty(0, T; X)$ for all $k \in \mathbb{N}$.

We assume the standard Hilbert space setup (see Lemma A.0.5)

$$V \overset{c}{\hookrightarrow} H \equiv H' \hookrightarrow V', \tag{2.1.4}$$

where each space is dense in the previous one; '$\overset{c}{\hookrightarrow}$' denotes compact injection, '$\hookrightarrow$' denotes continuous injection, and $\equiv$ indicates that we explicitly identify the elements in the two spaces (see Definition A.0.10).

For later purposes we recall a Gagliardo-Nirenberg inequality, which is a Sobolev interpolation result (e.g., see [2]): let $s \in [1, \infty]$, $m \geq 1$ and assume $v \in W^{m,s}(\Omega)$. Then there are constants C and $\mu = \frac{d}{m}\left(\frac{1}{s} - \frac{1}{r}\right)$ such that the inequality

$$\|v\|_{0,r} \leq C\|v\|_{0,s}^{1-\mu}\|v\|_{m,s}^{\mu} \quad \text{holds for} \quad r \in \begin{cases} [s, \infty] & \text{if } m - \frac{d}{s} > 0, \\ [s, \infty) & \text{if } m - \frac{d}{s} = 0, \\ [s, -\frac{d}{m-(d/s)}] & \text{if } m - \frac{d}{s} < 0. \end{cases} \tag{2.1.5}$$

We shall make frequent use of the following well-known version of the Sobolev embedding theorem (e.g., [70], p.142):

$$V \hookrightarrow L^r(\Omega) \quad \text{holds for} \quad r \in \begin{cases} [1, \infty] & \text{if } d = 1, \\ [1, \infty) & \text{if } d = 2, \\ [1, 6] & \text{if } d = 3. \end{cases}$$

We also require the following Grönwall lemma in differential form (see the more general result in [23], Proposition 2.2): let $E(t) \in W^{1,1}(0,T)$ and $Q(t), P(t), R(t) \in L^1(0,T)$, where all functions are non-negative. Then,

$$\frac{dE}{dt} + P(t) \leq R(t)E(t) + Q(t) \quad \text{a.e. in } [0,T]$$

implies

$$E(T) + \int_0^T P(\tau)\, d\tau \leq e^{\Lambda(T)}E(0) + e^{\Lambda(T)} \int_0^T Q(\tau)\, d\tau, \tag{2.1.6}$$

where $\Lambda(s) := \int_0^s R(\tau) d\tau$.

For completeness we also mention some additional well-known inequalities that we need on a regular basis. Hölder's inequality states: for $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, if $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and ([10], p.22)

$$\|fg\|_{0,1} \leq \|f\|_{0,p}\, \|g\|_{0,q}.$$

Obvious extensions apply, for example, applying the Hölder inequality twice gives

$$\|fgh\|_{0,1} \leq \|f\|_{0,p} \, \|g\|_{0,q} \, \|h\|_{0,n},$$

where $\frac{1}{p} + \frac{1}{q} + \frac{1}{n} = 1$. For later use we also recall the simple Young's inequality

$$cab \leq Ca^2 + \frac{1}{2\varepsilon}b^2, \quad \forall a, b, c, \varepsilon > 0, \tag{2.1.7}$$

for some positive constant C. We shall also need Young's inequality in the form

$$ab \leq \varepsilon^{m/n}\frac{a^m}{m} + \frac{1}{\varepsilon}\frac{b^n}{n}, \qquad \frac{1}{m} + \frac{1}{n} = 1, \tag{2.1.8}$$

valid for any $\varepsilon > 0$, $a, b \geq 0$ and $m, n > 1$ (e.g., [70], p.23).

Another elementary, but useful result, is the following, valid for any $a, b \geq 0$:

$$(i) \quad a^p + b^p \leq (a+b)^p \leq 2^{p-1}(a^p + b^p) \quad \text{if } p \geq 1,$$

$$(ii) \quad 2^{p-1}(a^p + b^p) \leq (a+b)^p \leq a^p + b^p \quad \text{if } p < 1.$$

The argument for a weaker result can be found in [1], pp.34-35. The proof for the above results involves consideration of the turning points of the function $f(x) := (1+x)^p/(1+x^p)$ with x set equal to b/a.

Throughout this thesis C represents a generic bounded positive constant, possibly depending on T, N, Ω, u_0 and v_0, which may change from expression to expression.

2.2 Alternative formulations of the $\lambda - \omega$ system, key lemmata and main result

We rewrite the $\lambda - \omega$ system (1.1.1a) - (1.1.1f) in the following vector form:

$$\boldsymbol{u}_t = \Delta \boldsymbol{u} + B\boldsymbol{u} + |\boldsymbol{u}|^p A \boldsymbol{u} \quad \text{in} \quad \Omega_T, \tag{2.2.1a}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} = \boldsymbol{0} \quad \text{on} \quad \Sigma, \tag{2.2.1b}$$

$$\text{where} \quad \boldsymbol{u} = (u, v)^T, \quad B = \begin{bmatrix} \lambda_0 & -\omega_0 \\ \omega_0 & \lambda_0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -\lambda_1 & -\omega_1 \\ \omega_1 & -\lambda_1 \end{bmatrix}. \tag{2.2.1c}$$

We note for later use the following identities, which hold for all $\boldsymbol{x} \in \mathbb{R}^2$:

$$(B\boldsymbol{x}) \cdot \boldsymbol{x} = \lambda_0 |\boldsymbol{x}|^2, \qquad (A\boldsymbol{x}) \cdot \boldsymbol{x} = -\lambda_1 |\boldsymbol{x}|^2 \tag{2.2.2a}$$

$$|B\boldsymbol{x}| = \sqrt{\lambda_0^2 + \omega_0^2}\, |\boldsymbol{x}|, \qquad |A\boldsymbol{x}| = \sqrt{\lambda_1^2 + \omega_1^2}\, |\boldsymbol{x}|. \tag{2.2.2b}$$

The equalities (2.2.2a) express the fact that B and $-A$ are positive definite matrices, which leads to an energy property exploited in the form of an *a priori* estimate in Section 2.4. It will also be convenient to express the $\lambda - \omega$ system in the equivalent complex form:

$$c_t = \Delta c + [\lambda(|c|) + i\,\omega(|c|)]c \quad \text{in} \quad \Omega_T, \tag{2.2.3a}$$

$$c(x,0) = c_0(x), \quad \frac{\partial c}{\partial \boldsymbol{\nu}} = 0 \quad \text{on} \quad \Sigma, \tag{2.2.3b}$$

$$\text{where} \quad c := u + i\,v, \quad r := |c| \equiv \sqrt{u^2 + v^2}. \tag{2.2.3c}$$

The following lemmata concern two key properties of the $\lambda - \omega$ system that are required for later estimates.

Lemma 2.2.1 The nonlinearity in (2.2.1a) is locally Lipschitz, namely for all $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R}^2$

$$\Big| |\boldsymbol{u}_1|^\rho A\boldsymbol{u}_1 - |\boldsymbol{u}_2|^\rho A\boldsymbol{u}_2 \Big| \leq (\rho+1)\sqrt{\lambda_1^2 + \omega_1^2}\, \left(|\boldsymbol{u}_1|^\rho + |\boldsymbol{u}_2|^\rho\right) |\boldsymbol{u}_1 - \boldsymbol{u}_2|. \tag{2.2.4}$$

As the term $B\boldsymbol{u}$ in (2.2.1a) is linear we have the corollary that the reaction terms in the $\lambda - \omega$ system are locally Lipschitz.

Proof. The proof is based on Lemma A.0.11. Let $r_1 = |\boldsymbol{u}_1|$ and $r_2 = |\boldsymbol{u}_2|$, then

$$|r_1^\rho A\boldsymbol{u}_1 - r_2^\rho A\boldsymbol{u}_2| = |A(r_1^\rho \boldsymbol{u}_1 - r_2^\rho \boldsymbol{u}_2)| \leq ||A||_2 |r_1^\rho \boldsymbol{u}_1 - r_2^\rho \boldsymbol{u}_2|,$$

where $||A||_2$ is the spectral norm of A. Now

$$A^T A = \begin{bmatrix} \lambda_1^2 + \omega_1^2 & 0 \\ 0 & \lambda_1^2 + \omega_1^2 \end{bmatrix}, \quad \text{so } ||A||_2 \equiv \sqrt{\varrho(A^T A)} = \sqrt{\lambda_1^2 + \omega_1^2},$$

where $\varrho(\cdot)$ denotes the spectral radius. After a simple calculation we obtain that the Jacobian of $r^\rho \boldsymbol{u}$ is given by:

$$J = \begin{bmatrix} r^{\rho-2}(r^2 + \rho u^2) & \rho u v r^{\rho-2} \\ \rho u v r^{\rho-2} & r^{\rho-2}(r^2 + \rho v^2) \end{bmatrix}.$$

As J is symmetric $\|J\|_2 = \varrho(J)$. Solving $|J - \lambda I| = 0$, where λ is an eigenvalue of J, leads to $\lambda_{1,2} = r^\rho,\ r^\rho(\rho+1)$. Let $L(\boldsymbol{u}_1, \boldsymbol{u}_2)$ denote the maximum value of J evaluated on the line joining $\boldsymbol{u}_1$ to $\boldsymbol{u}_2$. Then $L(\boldsymbol{u}_1, \boldsymbol{u}_2) = (\rho+1)\max_{0\le s\le 1} |\boldsymbol{z}|^\rho$, where $\boldsymbol{z} = s\boldsymbol{u}_1 + (1-s)\boldsymbol{u}_2$, $0 \le s \le 1$ and so $|\boldsymbol{z}| \le s|\boldsymbol{u}_1| + (1-s)|\boldsymbol{u}_2| \le \max\{|\boldsymbol{u}_1|, |\boldsymbol{u}_2|\}$, which yields the desired result. ■

We also prove a monotonicity property relevant to the nonlinearity in (2.2.1a).

Lemma 2.2.2 Let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^n$, $n \in \mathbb{N}$, $p \ge 0$, then

$$|\boldsymbol{v}_1|^p \boldsymbol{v}_1 \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2) \ge \frac{|\boldsymbol{v}_1|^{p+2} - |\boldsymbol{v}_2|^{p+2}}{p+2}.$$

Proof. Set $s := |\boldsymbol{v}_2|$, $r := |\boldsymbol{v}_1|$ and $f(t) := t^{p+2}$, $t \in \mathbb{R}$. Then using Taylor series to expand f about r gives

$$f(s) = f(r) + f'(r)(s-r) + f''(\xi)\frac{(s-r)^2}{2},$$

for some ξ between r and s, leading to

$$\frac{|\boldsymbol{v}_2|^{p+2} - |\boldsymbol{v}_1|^{p+2}}{p+2} = |\boldsymbol{v}_1|^{p+1}(|\boldsymbol{v}_2| - |\boldsymbol{v}_1|) + (p+1)\xi^p \frac{(|\boldsymbol{v}_2| - |\boldsymbol{v}_1|)^2}{2},$$

where ξ^p is between $|\boldsymbol{v}_1|^p$ and $|\boldsymbol{v}_2|^p$. Thus

$$\frac{|\boldsymbol{v}_2|^{p+2} - |\boldsymbol{v}_1|^{p+2}}{p+2} \ge |\boldsymbol{v}_1|^{p+1}(|\boldsymbol{v}_2| - |\boldsymbol{v}_1|),$$

or

$$\begin{aligned}
\frac{|\boldsymbol{v}_1|^{p+2} - |\boldsymbol{v}_2|^{p+2}}{p+2} &\le |\boldsymbol{v}_1|^{p+1}(|\boldsymbol{v}_1| - |\boldsymbol{v}_2|) \\
&= \frac{1}{2}|\boldsymbol{v}_1|^p \left(\Big| |\boldsymbol{v}_2| - |\boldsymbol{v}_1| \Big|^2 + |\boldsymbol{v}_1|^2 - |\boldsymbol{v}_2|^2 \right) \\
&\le \frac{1}{2}|\boldsymbol{v}_1|^p \left(|\boldsymbol{v}_1 - \boldsymbol{v}_2|^2 + |\boldsymbol{v}_1|^2 - |\boldsymbol{v}_2|^2 \right) \\
&= |\boldsymbol{v}_1|^p \boldsymbol{v}_1 \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2). \quad ■
\end{aligned}$$

A corollary to this lemma is that $|\boldsymbol{u}|^p\boldsymbol{u}$ is a monotonic function of $\boldsymbol{u}$, namely

$$(|\boldsymbol{v}_1|^p\boldsymbol{v}_1 - |\boldsymbol{v}_2|^p\boldsymbol{v}_2) \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2) \ge 0, \quad \forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^n,\ n \in \mathbb{N},\ p \ge 0. \tag{2.2.5}$$

After application of Green's identity (Lemma A.0.12) and recalling the homogeneous Neumann boundary conditions we rewrite the $\lambda - \omega$ system (1.1.1a) - (1.1.1f) in weak form to give:

($\mathbf{P_0}$) Find $u(\cdot,t), v(\cdot,t) \in V$ such that $u(\cdot,0) = u_0(\cdot)$, $v(\cdot,0) = v_0(\cdot)$ and for almost every $t \in (0,T)$

$$\langle \tfrac{\partial u}{\partial t}, \eta \rangle + (\nabla u, \nabla \eta) = (\lambda(r)\, u, \eta) - (\omega(r)\, v, \eta) \qquad \forall \eta \in V, \tag{2.2.6a}$$

$$\langle \tfrac{\partial v}{\partial t}, \eta \rangle + (\nabla v, \nabla \eta) = (\omega(r)\, u, \eta) + (\lambda(r)\, v, \eta) \qquad \forall \eta \in V. \tag{2.2.6b}$$

Two equivalent weak formulations, based on (2.2.1a)-(2.2.1c) and (2.2.3a)-(2.2.3c) respectively, are

($\mathbf{P_1}$) Find $\boldsymbol{u}(\cdot,t) \in \{V\}^2$ such that $\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0(\cdot)$ and for almost every $t \in (0,T)$

$$\langle \boldsymbol{u}_t, \boldsymbol{\eta} \rangle + (\nabla \boldsymbol{u}, \nabla \boldsymbol{\eta}) = (B\boldsymbol{u}, \boldsymbol{\eta}) + (|\boldsymbol{u}|^\rho A\boldsymbol{u}, \boldsymbol{\eta}) \qquad \forall \boldsymbol{\eta} \in \{V\}^2, \tag{2.2.7}$$

and

($\mathbf{P_2}$) Find $c(\cdot,t) \in \mathbb{H}^1(\Omega)$ such that $c_0(\cdot) := u_0(\cdot) + iv_0(\cdot)$ and for almost every $t \in (0,T)$

$$\langle c_t, \eta \rangle + (\nabla c, \nabla \eta) = (\lambda(|c|)c, \eta) + i(\omega(|c|)c, \eta) \qquad \forall \eta \in \mathbb{H}^1(\Omega), \tag{2.2.8}$$

where $\mathbb{H}^1(\Omega)$ is the 'complexified' space of V, i.e., if $u = u_1 + i\,u_2 \in \mathbb{H}^1(\Omega)$ then $u_j \in V$, $j = 1,2$. The above weak formulations and their finite dimensional equivalents will be the starting points for proving the main technical results of this thesis.

To prove existence of weak solutions we assume:

(A1) ρ is any finite, positive number if $d = 1,2$ and

$\rho \leq 4$ if $d = 3$.

To prove uniqueness of solutions and strong solution results we assume:

(A2)
$$\rho \leq \begin{cases} 4 & \text{if } d = 1, \\ 2 & \text{if } d = 2, \\ 4/3 & \text{if } d = 3. \end{cases}$$

Note by the Sobolev embedding theorem that assumption (A1) is sufficient for V to have continuous injection into $L^{\rho+2}(\Omega)$, while assumption (A2) is sufficient for V to have continuous injection into $L^{2\rho+2}(\Omega)$.

We now state the main theorem of this chapter.

Theorem 2.2.3 Assume $\Omega \subset \mathbb{R}^d$ $(d \leq 3)$ is an open, bounded, convex domain.[5] Let (A1) hold and assume that u_0, $v_0 \in H$, then the $\lambda - \omega$ system (1.1.1a)–(1.1.1f) possesses at least one weak solution $\{u, v\}$ satisfying[6]

$$u, v \in L^2(0, T; V) \cap L^{\rho+2}(\Omega_T) \cap C([0, T]; H),$$

and the equations (1.1.1a) and (1.1.1b) hold as equalities in $L^{\frac{\rho+2}{\rho+1}}(0, T; V')$. Furthermore, with assumption (A2) the weak solution is unique and the map

$$(u_0(\cdot), v_0(\cdot)) \mapsto (u(\cdot, t; u_0, v_0), v(\cdot, t; u_0, v_0)),$$

is continuous in H.

Proof. We prove this theorem using the Faedo-Galerkin method of Lions [53] and classical compactness arguments. An overview of the main steps in the Faedo-Galerkin method is given in Appendix B. We separate the proof into four parts showing: local existence of the Galerkin approximations, global existence of the Galerkin approximations, passage to the limit, and uniqueness. For notational convenience in the proof we define the conjugate exponents by

$$p := \rho + 2, \qquad q := \frac{\rho + 2}{\rho + 1} \in (1, 2), \tag{2.2.9}$$

unless stated to the contrary.

2.3 Local existence of the Galerkin approximations

With $\mathcal{L}_I := -\Delta + I$, domain$(\mathcal{L}_I)$:=$\{u \in V \mid \partial u/\partial \nu = 0 \text{ on } \partial\Omega\}$, $\mathcal{L}_I^{-1}$ is a symmetric, bounded, compact operator from H to H and thus the Hilbert-Schmidt theorem applies (Theorem A.0.13). Consequently, from the spectral theory of such operators we introduce $\{z_i\}_{i=1}^{\infty}$ to be an orthogonal basis for V and an orthonormal

[5]Recall that an open, bounded, convex domain has a Lipschitz continuous boundary [32], Corollary 1.2.2.3.

[6]As $C([0, T]; H) \hookrightarrow L^{\infty}(0, T; H)$ we also have $u, v \in L^{\infty}(0, T; H)$.

basis for H, consisting of eigenfunctions for

$$-\Delta z_i + z_i = \mu_i z_i \quad \text{in } \Omega, \quad \frac{\partial z_i}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \partial\Omega, \tag{2.3.1}$$

where

$$1 \le \mu_1 \le \mu_2 \le \cdots \le \mu_k \le \cdots \quad \text{with } \lim_{i\to\infty} \mu_i = \infty$$

is an infinite set of corresponding eigenvalues (e.g., [22], [8]), (see Theorem A.0.14). Note $(z_i, z_j) = \delta_{ij}$ and the weak form of the eigenvalue problem leads to

$$(z_i, z_j)_V = \mu_i \delta_{ij}, \quad \text{or equivalently} \quad (\nabla z_i, \nabla z_j) = (\mu_i - 1)\delta_{ij}. \tag{2.3.2}$$

Set $V^k := \text{span}\{z_i\}_{i=1}^k \subset V$ and seek a finite dimensional weak form corresponding to (P_0):

Find $u^k(\cdot,t), v^k(\cdot,t) \in V^k$ such that $u^k(\cdot,0) = u_0^k(\cdot)$, $v^k(\cdot,0) = v_0^k(\cdot)$ and for almost every $t \in (0,T)$

$$\left(\tfrac{\partial u^k}{\partial t}, \chi^k\right) + \left(\nabla u^k, \nabla \chi^k\right) = \left(\lambda(r^k)\, u^k, \chi^k\right) - \left(\omega(r^k)\, v^k, \chi^k\right) \qquad \forall\, \chi^k \in V^k, \tag{2.3.3a}$$

$$\left(\tfrac{\partial v^k}{\partial t}, \chi^k\right) + \left(\nabla v^k, \nabla \chi^k\right) = \left(\omega(r^k)\, u^k, \chi^k\right) + \left(\lambda(r^k)\, v^k, \chi^k\right) \qquad \forall\, \chi^k \in V^k, \tag{2.3.3b}$$

$$\text{where} \quad r^k := \sqrt{(u^k)^2 + (v^k)^2}.$$

To derive later estimates we note the finite dimensional weak form corresponding to (P_1) is:

Find $\boldsymbol{u}^k(\cdot,t) \in \{V^k\}^2$ such that $\boldsymbol{u}^k(\cdot,0) = \boldsymbol{u}_0^k(\cdot)$ and for almost every $t \in (0,T)$

$$(\boldsymbol{u}_t^k, \boldsymbol{\chi}^k) + (\nabla \boldsymbol{u}^k, \nabla \boldsymbol{\chi}^k) = (B\boldsymbol{u}^k, \boldsymbol{\chi}^k) + (|\boldsymbol{u}^k|^\rho A\boldsymbol{u}^k, \boldsymbol{\chi}^k) \quad \forall \boldsymbol{\chi}^k \in \{V^k\}^2. \tag{2.3.4}$$

Let $P^k : H \mapsto V^k$ be the orthogonal projection from H onto V^k that satisfies

$$(P^k \eta, \chi^k) = (\eta, \chi^k), \quad \forall \chi^k \in V^k.$$

We assume where necessary additional well-known properties of orthogonal projection operators acting on a Hilbert space, collected together as Theorem A.0.18. For the work that follows we need the fact that the gradient operator satisfies the following symmetry condition:

Lemma 2.3.1 For any $v \in V$ we have

$$(\nabla(P^k v), \nabla \chi^k) = (\nabla v, \nabla \chi^k), \quad \forall \chi \in V^k. \tag{2.3.5}$$

Proof. Let $\chi^k := \sum_{i=1}^{k} c_i z_i \in V^k$. From the weak form of the eigenvalue problem we have that

$$(\nabla z_i, \nabla v) = (\mu_i - 1)(z_i, v), \quad \forall v \in V \tag{2.3.6}$$

and as v is also in H we have

$$P^k v = \sum_{j=1}^{k} (v, z_j) z_j \quad \text{so} \quad \nabla(P^k v) = \sum_{j=1}^{k} (v, z_j) \nabla z_j. \tag{2.3.7}$$

Thus from (2.3.7), (2.3.2) and (2.3.6)

$$\begin{aligned}(\nabla(P^k v), \nabla z_i) &= \sum_{j=1}^{k} (v, z_j)(\nabla z_i, \nabla z_j) \\ &= \sum_{j=1}^{k} (v, z_j)(\mu_j - 1)\delta_{ij} \\ &= (v, z_i)(\mu_i - 1) \\ &= (\nabla z_i, \nabla v).\end{aligned}$$

Multiplying both sides by c_i and summing over $i = 1, \dots, k$ leads to (2.3.5). ■

The following lemmata are corollaries to this result:

Lemma 2.3.2

$$(P^k v, \chi^k)_V = (v, \chi^k)_V, \quad \forall \chi^k \in V^k, \ v \in V. \tag{2.3.8}$$

Proof. Recalling that $(P^k v, \chi^k) = (v, \chi^k)$ for all $\chi^k \in V^k$, $v \in V$, combined with (2.3.5) gives the result. ■

Lemma 2.3.3

$$\|\nabla(P^k v)\|_0 \leq \|\nabla v\|_0, \quad \forall v \in V. \tag{2.3.9}$$

Proof. In (2.3.5) take $\chi^h = P^k v$ leading to

$$\|\nabla(P^k v)\|_0^2 = (\nabla v, \nabla(P^k v)) \leq \|\nabla v\|_0 \|\nabla(P^k v)\|_0,$$

which gives the desired result after dividing both sides by $\|\nabla(P^k v)\|_0$. ■

We use '$\rightarrow$' to denote strong convergence (Definition A.0.19) and require the following lemma:

Lemma 2.3.4

$$P^k v \to v \quad \text{in } V, \qquad \forall v \in V. \tag{2.3.10}$$

Proof. Observe for any $v \in V$ and all $\chi^k \in V^k$ that

$$\begin{aligned} \|P^k v - v\|_1^2 &= (P^k v - v, P^k v - v)_V \\ &= (P^k v - v, \chi^k - v)_V + (P^k v - v, P^k v - \chi^k)_V \\ &= (P^k v - v, \chi^k - v)_V \\ &\leq \|P^k v - v\|_1 \|\chi^k - v\|_1, \end{aligned}$$

after noting (2.3.8) and applying the Cauchy-Schwarz inequality. Thus dividing both sides of this inequality by $\|P^k v - v\|_1$ gives

$$\|P^k v - v\|_1 \leq \|\chi^k - v\|_1, \quad \forall \chi^k \in V^k, \ v \in V.$$

Now as V^k is dense in V we can take any sequence χ^k that converges strongly to v in V and the result follows. ■

Write u^k and v^k as

$$u^k(\cdot, t) = \sum_{i=1}^{k} a_{ik}(t) z_i(\cdot), \qquad v^k(\cdot, t) = \sum_{i=1}^{k} b_{ik}(t) z_i(\cdot) \tag{2.3.11}$$

and set $\chi^k = z_j$ for $j = 1, \ldots, k$ in (2.3.3a) - (2.3.3b), where the a_{ik} and b_{ik} are to be determined. For the initial approximations we take

$$u_0^k := P^k u_0(\cdot), \qquad v_0^k := P^k v_0(\cdot). \tag{2.3.12}$$

Note that we have the strong convergence in H of the initial approximations to the initial data, that is

$$\{u_0^k, v_0^k\} \to \{u_0, v_0\} \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad k \to \infty, \tag{2.3.13}$$

which is easy to see if we recall that $\|P^k u_0 - u_0\|_0 \leq \|\chi^k - u_0\|_0$, for all $\chi^k \in V^k$ and note the density of V^k in H.

With the above setup the substitution of $\{u^k(t), v^k(t)\}$ into the finite dimensional weak form (2.3.3a)-(2.3.3b) leads to

$$\sum_{i=1}^{k} \frac{da_{ik}}{dt}(z_i, z_j) + \sum_{i=1}^{k} a_{ik}(\nabla z_i, \nabla z_j) = \int_\Omega z_j f(u^k, v^k)\, dx,$$
$$\sum_{i=1}^{k} \frac{db_{ik}}{dt}(z_i, z_j) + \sum_{i=1}^{k} b_{ik}(\nabla z_i, \nabla z_j) = \int_\Omega z_j g(u^k, v^k)\, dx,$$

for $j = 1, \ldots, k$, where

$$f(u,v) := \lambda(r)\, u - \omega(r)\, v, \qquad g(u,v) := \omega(r)\, u + \lambda(r)\, v.$$

Using (2.3.2) we obtain an initial value problem for a system of $2k$ ODEs in a_{jk}, b_{jk}

$$\frac{da_{jk}}{dt} + a_{jk}(\mu_j - 1) = \int_\Omega z_j f(u^k, v^k)\, dx,$$
$$\frac{db_{jk}}{dt} + b_{jk}(\mu_j - 1) = \int_\Omega z_j g(u^k, v^k)\, dx,$$

where $a_{jk}(0) = (u_0, z_j)$ and $b_{jk}(0) = (v_0, z_j)$, $j = 1, \ldots, k$. A 'composite' form of this ODE system is obtained by multiplying the system by z_j, summing from $j = 1, \ldots, k$, and using that

$$-\Delta u^k = \sum_{j=1}^{k} (\mu_j - 1) z_j a_{jk}, \quad -\Delta v^k = \sum_{j=1}^{k} (\mu_j - 1) z_j b_{jk},$$

after recalling (2.3.1), yielding the Galerkin approximation

$$\frac{du^k}{dt} = \Delta u^k + P^k f(u^k, v^k), \quad u^k(\cdot, 0) = P^k u_0(\cdot), \tag{2.3.14a}$$
$$\frac{dv^k}{dt} = \Delta v^k + P^k g(u^k, v^k), \quad v^k(\cdot, 0) = P^k v_0(\cdot). \tag{2.3.14b}$$

As in [70], p.222, from standard existence theory (Theorem B.0.33) for systems of ODEs, since f and g are locally Lipschitz, the system has a unique solution $\{u^k, v^k\}$ on some finite time interval $(0, t_k)$, $t_k > 0$.

We show that the 'composite' ODE system (2.3.14a) - (2.3.14b) is equivalent to the finite dimensional weak forms (i.e., we can obtain the finite dimensional weak forms from the ODE system). Initially multiply both composite ODE equations by z_j and integrate over Ω. We integrate by parts the terms involving the Laplacian after noting that $\partial z_i/\partial \boldsymbol{\nu} = 0$ implies $\partial u^k/\partial \boldsymbol{\nu} = \partial v^k/\partial \boldsymbol{\nu} = 0$ on $\partial\Omega$. To deal with the right hand side terms notice

$$(P^k f(u^k, v^k), z_j) = \left(\sum_{i=1}^{k} (f(u^k, v^k), z_i) z_i, z_j \right) = \sum_{i=1}^{k} (f(u^k, v^k), z_i)\delta_{ij} = (f(u^k, v^k), z_j),$$

and similarly $(P^k g(u^k, v^k), z_j) = (g(u^k, v^k), z_j)$. We did not assume the usual symmetry condition for the projection operator as $f(u^k, v^k)$ and $g(u^k, v^k)$ do not necessarily lie in H, and it is through this process that we rigorously attach a meaning to $P^k f(u^k, v^k)$ and $P^k g(u^k, v^k)$. Thus we obtain

$$\left(\frac{\partial u^k}{\partial t}, z_j\right) + \left(\nabla u^k, \nabla z_j\right) = \left(f(u^k, v^k), z_j\right), \quad j = 1, \ldots, k, \tag{2.3.15a}$$

$$\left(\frac{\partial v^k}{\partial t}, z_j\right) + \left(\nabla v^k, \nabla z_j\right) = \left(g(u^k, v^k), z_j\right), \quad j = 1, \ldots, k, \tag{2.3.15b}$$

with $u^k(0) = P^k u_0$, $v^k(0) = P^k v_0$.

2.4 Global existence of the Galerkin approximations and Estimate I

To prove global existence of the Galerkin approximations we derive an *a priori* estimate bounding u^k and v^k (independently of k) in various Banach spaces. From the uniform bounds we conclude $t_k = T$ (independent of k), thus giving global existence of the Galerkin approximations.

Estimate I: Set $\chi^k = z_j$ in the finite dimensional weak form (2.3.3a)-(2.3.3b). Then after recalling (2.3.11) we multiply (2.3.3a) by a_{jk} and (2.3.3b) by b_{jk} and sum both equations from $j = 1, \ldots, k$. This is equivalent to taking $\boldsymbol{\chi}^k = \boldsymbol{u}^k$ in the vectorised finite dimensional weak form (2.3.4), yielding

$$(\boldsymbol{u}_t^k, \boldsymbol{u}^k) + (\nabla \boldsymbol{u}^k, \nabla \boldsymbol{u}^k) = (B\boldsymbol{u}^k, \boldsymbol{u}^k) + (|\boldsymbol{u}^k|^\rho A\boldsymbol{u}^k, \boldsymbol{u}^k).$$

With (2.2.2a) this simplifies to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{u}^k|^2 dx + \int_\Omega |\nabla \boldsymbol{u}^k|^2 dx + \lambda_1 \int_\Omega |\boldsymbol{u}^k|^{\rho+2} dx = \lambda_0 \int_\Omega |\boldsymbol{u}^k|^2 dx,$$

and the application of the Grönwall lemma yields

$$\|\boldsymbol{u}^k\|_0^2 + 2\int_0^T \left(|\boldsymbol{u}^k|_1^2 + \lambda_1 \|\boldsymbol{u}^k\|_{0,\rho+2}^{\rho+2}\right) dt \leq \|\boldsymbol{u}^k(0)\|_0^2 \exp(2\lambda_0 T). \tag{2.4.1}$$

Recalling u_0, $v_0 \in H$ so $\|\boldsymbol{u}^k(0)\|_0 \equiv \|P^k \boldsymbol{u}_0\|_0 \leq \|\boldsymbol{u}_0\|_0 \leq C$ we have

$$u^k,\ v^k \quad \text{are uniformly bounded in } L^\infty(0,T;H) \cap L^{\rho+2}(\Omega_T), \tag{2.4.2}$$

and noting the injection $L^\infty \hookrightarrow L^2$ and the semi-norm bound for V we have

$$u^k,\ v^k \quad \text{are uniformly bounded in } L^2(0,T;V). \tag{2.4.3}$$

We make some comments about our attempts to generalise this estimate and hence obtain additional regularity of the weak solutions. Formally at least, if we take $\boldsymbol{\chi}^k = |\boldsymbol{u}^k|^\rho \boldsymbol{u}^k$ in the weak form (2.3.4), then with the aid of (2.2.2a) and the Grönwall lemma we obtain a generalised version of inequality (2.4.1). With the assumptions $u_0, v_0 \in V$ and (A1) this leads to, for example, that solutions are uniformly bounded in $L^{2\rho+2}(\Omega_T)$. However, this is invalid as $|\boldsymbol{u}^k|^\rho \boldsymbol{u}^k \notin \{V^k\}^2$, so we might attempt to take $\boldsymbol{\chi}^k = P^k\{|\boldsymbol{u}^k|^\rho \boldsymbol{u}^k\}$ and exploit that $(P^k\eta, \chi^k) = (\eta, \chi^k)$ for all $\chi^k \in V^k$. But this approach fails with the nonlinear term as $|\boldsymbol{u}^k|^\rho A\boldsymbol{u}^k \notin \{V^k\}^2$. These ideas are not wasted, as a similar approach can be successfully implemented in the discrete settings (see Estimate I of Section 4.3 and Estimate III of Section 5.3).

2.5 Passage to the limit

Recall that $L^1(0,T;H)$ is a separable Banach space, but not reflexive, while the Banach spaces $L^2(0,T;V)$ and $L^p(\Omega_T)$ are reflexive (Theorems A.0.2 and A.0.4). Thus from classical compactness arguments (Theorems B.0.34, B.0.35), from the uniformly bounded sequences of functions $\{u^k\}_{k=1}^\infty$ and $\{v^k\}_{k=1}^\infty$ in (2.4.2) and (2.4.3) we can extract convergent subsequences, still denoted $\{u^k\}$, $\{v^k\}$, such that

$$\{u^k, v^k\} \rightharpoonup \{u, v\} \quad \text{in} \quad L^p(\Omega_T) \cap L^2(0,T;V) \quad \text{as} \quad k \to \infty, \tag{2.5.1}$$

$$\{u^k, v^k\} \rightharpoonup^* \{u, v\} \quad \text{in} \quad L^\infty(0,T;H) \quad \text{as} \quad k \to \infty, \tag{2.5.2}$$

where '$\rightharpoonup$' and '$\rightharpoonup^*$' represent weak and weak* ('weak-star') convergence respectively (see Definitions A.0.20, A.0.21).

We assumed when extracting an arbitrary number of weak* and weakly convergent subsequences using weak compactness arguments, there is no loss in generality in denoting the convergent subsequences by $\{u^k, v^k\}$ and their limits by the same $\{u, v\}$. This result is implicitly used in the literature (see, e.g., [53], [85], [70]), but not proved in the works we reviewed, thus we provide a proof (see Lemma B.0.36). The proof relies on well-known results concerning weak and weak* convergence, which we collect together as Theorem A.0.22.

We show passage to the limit of the terms in the first composite Galerkin approximation (2.3.14a). The arguments will apply in a similar way to (2.3.14b). Consider first the term $P^k f(u^k, v^k)$. It is easy to show

$$|f(u^k, v^k)| \leq C\left(|u^k| + |v^k| + |u^k|^{\rho+1} + |v^k|^{\rho+1}\right)$$

and thus

$$\begin{aligned}\int_0^T \int_\Omega |f(u^k, v^k)|^{\frac{\rho+2}{\rho+1}}\, dx\, dt &\leq C \int_0^T \int_\Omega \left(|u^k| + |v^k| + |u^k|^{\rho+1} + |v^k|^{\rho+1}\right)^{\frac{\rho+2}{\rho+1}}\, dx\, dt \\ &\leq C \int_0^T \int_\Omega \left(|u^k|^{\frac{\rho+2}{\rho+1}} + |v^k|^{\frac{\rho+2}{\rho+1}} + |u^k|^{\rho+2} + |v^k|^{\rho+2}\right)\, dx\, dt.\end{aligned}$$

That is we have

$$\|f(u^k, v^k)\|^q_{L^p(\Omega_T)} \leq C\left(\|u^k\|^q_{L^q(\Omega_T)} + \|v^k\|^q_{L^q(\Omega_T)} + \|u^k\|^p_{L^p(\Omega_T)} + \|v^k\|^p_{L^p(\Omega_T)}\right) \leq C,$$

after recalling that $u^k, v^k \in L^p(\Omega_T)$ and $L^p(\Omega_T) \hookrightarrow L^q(\Omega_T)$. Thus $f(u^k, v^k)$ is uniformly bounded in $L^q(\Omega_T)$ and so from weak compactness arguments there exists some $\chi \in L^q(\Omega_T)$ such that

$$f(u^k, v^k) \rightharpoonup \chi \quad \text{in } L^q(\Omega_T) \text{ as } k \to \infty. \tag{2.5.3}$$

We show that $P^k f$ also tends weakly to χ in $L^q(\Omega_T)$. Define $Q^k := I - P^k$, the projection orthogonal to P^k. From (2.3.10) we know that $P^k u \to u$ in V, $\forall u \in V$, i.e., $Q^k u \to 0$ in V as $k \to \infty$. By assumption (A1) $V \hookrightarrow L^p(\Omega)$, thus $Q^k u \to 0$ in $L^p(\Omega)$, $\forall\, u \in L^p(\Omega)$. Let $\phi \in L^p(\Omega_T)$ be arbitrary, then using Hölder's inequality and the orthogonality of Q^k

$$\begin{aligned}\left|\int_0^T \left(P^k f(u^k, v^k) - \chi, \phi\right) dt\right| &\leq \left|\int_0^T \left(f(u^k, v^k) - \chi, \phi\right) dt\right| + \left|\int_0^T \left(f(u^k, v^k), Q^k\phi\right) dt\right| \\ &\leq \left|\int_0^T \left(f(u^k, v^k) - \chi, \phi\right) dt\right| + \int_0^T \|f(u^k, v^k)\|_{0,q}\|Q^k\phi\|_{0,p}\, dt \quad \to 0 \quad \text{as } k \to \infty,\end{aligned}$$

on noting the strong convergence of $Q^k\phi$ to 0 in $L^p(\Omega)$, (2.5.3) and the Dominated Convergence Theorem (e.g., [70], Theorem 1.7). See Definition A.0.23 for an explanation of the form of weak convergence we have used. Thus we have

$$P^k f(u^k, v^k) \rightharpoonup \chi \quad \text{in } L^q(\Omega_T) \quad \text{as } k \to \infty. \tag{2.5.4}$$

From Lemma A.0.24 we have $\Delta u^k \in L^2(0,T;V')$. Furthermore, we have from (2.5.4) and part (v) of Theorem A.0.22 that $P^k f(u^k, v^k) \in L^q(\Omega_T)$. Thus it follows from (2.3.14a) that du^k/dt is uniformly bounded in $L^2(0,T;V') + L^q(\Omega_T)$ (see part (ii) of Lemma A.0.25). Moreover, it follows from weak compactness arguments that du^k/dt tends weakly to some $\dot{\eta}$ in this space. We adapt an argument in [70], p.203, to give $\dot{\eta} = du/dt$, i.e.

$$\frac{du^k}{dt} \rightharpoonup \frac{du}{dt} \quad \text{in } L^2(0,T;V') + L^q(\Omega_T) \quad \text{as } k \to \infty.$$

First recall from (2.5.1) that $u^k \rightharpoonup u$ in the space $L^2(0,T;V) \cap L^p(\Omega_T)$, with dual space $L^2(0,T;V') + L^q(\Omega_T)$ (part (i) of Lemma A.0.25). Furthermore, from the Sobolev embedding theorem and the fact that V is dense in H, we have the dense inclusion $V \hookrightarrow L^p(\Omega)$, thus from Lemma A.0.6 $L^q(\Omega) \hookrightarrow V'$ and so $L^2(0,T;V') + L^q(\Omega_T) \subset L^q(0,T;V')$. Now consider an arbitrary $\phi(t) \in C_0^\infty(0,T;V) \subset L^p(0,T;V)$. Integrating by parts, noting that functions in $C_0^\infty(0,T;V)$ have compact support in $(0,T)$ and using the weak convergence of u^k to u in $L^2(0,T;V') + L^q(\Omega_T)$ and hence in $L^q(0,T;V')$ yields

$$\int_0^T \left(\frac{du^k}{dt}, \phi\right) dt = -\int_0^T \left(u^k, \frac{d\phi}{dt}\right) dt \to -\int_0^T \left(u, \frac{d\phi}{dt}\right) dt = \int_0^T \left(\frac{du}{dt}, \phi\right) dt,$$

where we note that $d\phi/dt \in C_0^\infty(0,T;V)$, due to the smoothness of functions in this space. From the weak convergence of du^k/dt to $\dot{\eta}$ in $L^q(0,T;V')$ we also have

$$\int_0^T \left(\frac{du^k}{dt}, \phi\right) dt \to \int_0^T (\dot{\eta}, \phi) dt \quad \text{as } k \to \infty,$$

and so by the uniqueness of weak limits (see Theorem A.0.22) $\dot{\eta} = du/dt$ as required. Due to the density of $C_0^\infty(0,T;V)$ in $L^p(0,T;V)$ the convergence results that hold for functions in $C_0^\infty(0,T;V)$ also hold by extension for functions in $L^p(0,T;V)$.[7]

We claim that as $u^k \rightharpoonup u$ in $L^2(0,T;V)$ we have (cf. [70], p.204)

$$\Delta u^k \rightharpoonup \Delta u \quad \text{in } L^2(0,T;V') \text{ as } k \to \infty.$$

[7]The advantage of this approach is that a proof involving a continuous function space may be less technical than the corresponding one involving the associated Sobolev space.

Take $\mathcal{L} := -\Delta$, domain$(\mathcal{L})$:=$\{u \in V \mid \partial u/\partial \nu = 0 \quad \text{on} \quad \partial\Omega\}$. From Lemma A.0.24 we see that $\mathcal{L}$ is associated with the continuous (in V) bilinear form $a(\chi, \eta) := \int_\Omega \nabla\chi \cdot \nabla\eta \, dx = \langle \mathcal{L}\chi, \eta\rangle_{V',V}$, and symmetry follows from

$$\langle \mathcal{L}\chi, \eta\rangle_{V',V} = a(\chi, \eta) = a(\eta, \chi) = \langle \mathcal{L}\eta, \chi\rangle_{V',V}.$$

Using part (ii) of Lemma A.0.5 allows us to write this condition in the equivalent form

$$(\mathcal{L}\chi, \eta) = (\mathcal{L}\eta, \chi) \quad \forall \chi, \eta \in V, \ \mathcal{L}\chi, \mathcal{L}\eta \in H.$$

Now take $\phi \in L^2(0, T; V)$ and consider

$$\int_0^T (\mathcal{L}u^k, \phi) \, dt = \int_0^T (\mathcal{L}\phi, u^k) \, dt \to \int_0^T (\mathcal{L}\phi, u) \, dt \quad \text{as } k \to \infty, \tag{2.5.5}$$

due to the weak convergence of u^k to u in $L^2(0, T; V)$ (recall $H \hookrightarrow V'$ so $\mathcal{L}\phi \in L^2(0, T; V')$). However,

$$\int_0^T (\mathcal{L}\phi, u) \, dt = \int_0^T (\mathcal{L}u, \phi) \, dt,$$

using the symmetry of $\mathcal{L}$ again, so from (2.5.5) we deduce $\mathcal{L}u^k \rightharpoonup \mathcal{L}u$ in $L^2(0, T; V')$. As $L^2(0, T; V') \subset L^q(0, T; V')$ we have the required passage to the limit of all terms in $L^q(0, T; V')$.

To show $\chi \equiv f(u, v)$ in (2.5.3) we apply some classical theorems. From an application of the Lions-Aubin theorem (Theorem B.0.37) with $E_0 := V$, $E := H$, $E_1 := V'$ and

$$W := \left\{\eta \mid \eta \in L^2(0, T; V), \quad \frac{d\eta}{dt} \in L^q(0, T; V')\right\},$$

we have $W \overset{c}{\hookrightarrow} L^2(\Omega_T)$. As $u^k \in W$ and the injection into $L^2(\Omega_T)$ is compact we extract a subsequence (still denoted u^k), such that $u^k \to u$ in $L^2(\Omega_T)$ (similarly for v^k).[8] From Lemma A.0.26

$$u^k \to u \quad (\text{'pointwise'}) \text{ a.e. in } \Omega_T,$$

[8]Recall that in a metric space compactness is equivalent to sequential compactness.

(similarly, $u^k \to u$ ('pointwise') a.e. in Ω_T). As f is locally Lipschitz in Ω_T this implies by continuity that

$$f(u^k, v^k) \to f(u,v) \quad \text{('pointwise') a.e. in } \Omega_T.$$

The application of a classical lemma of Lions (Lemma B.0.38) gives

$$f(u^k, v^k) \rightharpoonup f(u,v) \quad \text{in } L^q(\Omega_T)$$

and due to the uniqueness of weak limits we deduce $\chi \equiv f(u,v)$, as required.

To summarise, we have shown that

$$\frac{du^k}{dt} - \Delta u^k - P^k f(u^k, v^k) \rightharpoonup \frac{du}{dt} - \Delta u - f(u,v) \quad \text{in } L^q(0,T;V'),$$

which means for all $\eta \in L^p(0,T;V)$

$$\int_0^T \left(\frac{du^k}{dt} - \Delta u^k - P^k f(u^k, v^k), \eta \right) dt \to \int_0^T \left(\frac{du}{dt} - \Delta u - f(u,v), \eta \right) dt,$$

as $k \to \infty$. That is, we have shown passage to the limit of the Galerkin approximation (2.3.14a) to the differential equation (1.1.1a) in $L^q(0,T;V')$ (and using the same arguments we can show passage to the limit of (2.3.14b) to (1.1.1b)). Thus we infer existence of the weak solutions for all $T > 0$. We still have to deal with the initial approximations and show $u(0) = u_0$ (and $v(0) = v_0$).

We show initially that the equality $u(0) = u_0$ requires a proof. We found above that $u^k(t) \to u(t)$ ('pointwise') a.e. in Ω_T. Furthermore, from (2.3.13) we have $P^k u_0 \to u_0$ in $L^2(\Omega)$ and so $P^k u_0 \to u_0$ ('pointwise') a.e. in Ω. Thus if $t = 0$ belongs to the null-set of the almost everywhere statement for the pointwise convergence of $u^k(t)$ to $u(t)$ then possibly $u(0) \neq u_0$ (see [70], p.205, for an example). The arguments used to show $u(\cdot, 0) = u_0(\cdot)$ are well-known (see [70], p.225, or [69], p.381), but for completeness we include them here.

Consider an arbitrary $\phi \in C^1([0,T];V)$ with the property that $\phi(T) = 0$ (recall from Section 2.1 the properties of functions in $C([0,T];V)$, or $C^1([0,T];V)$). We can write the weak form corresponding to the differential equation (1.1.1a) as:

$$\int_0^T \left(\frac{du}{dt}, \eta \right) dt + \int_0^T (\nabla u, \nabla \eta)\, dt = \int_0^T (f(u,v), \eta)\, dt, \quad \forall \eta \in L^p(0,T;V). \tag{2.5.6}$$

The corresponding Galerkin approximation can be written as:

$$\int_0^T \left(\frac{du^k}{dt}, \eta\right) dt + \int_0^T (\nabla u^k, \nabla \eta)\, dt = \int_0^T \left(P^k f(u^k, v^k), \eta\right) dt, \quad \forall \eta \in L^p(0,T;V). \tag{2.5.7}$$

In (2.5.6) and (2.5.7) take $\eta = \phi$ and integrate the first term by parts with respect to time to give:

$$-(u(0), \phi(0)) - \int_0^T \left(u, \frac{d\phi}{dt}\right) dt + \int_0^T (\nabla u, \nabla \phi)\, dt = \int_0^T (f(u,v), \phi)\, dt, \tag{2.5.8a}$$

$$-\left(u^k(0), \phi(0)\right) - \int_0^T \left(u^k, \frac{d\phi}{dt}\right) dt + \int_0^T (\nabla u^k, \nabla \phi)\, dt = \int_0^T \left(f(u^k, v^k), \phi\right) dt. \tag{2.5.8b}$$

From (2.3.13) $P^k u_0 \equiv u^k(0) \to u_0$ in H and as a strongly convergent sequence of functions is also weakly convergent we have $(u^k(0), \eta) \to (u_0, \eta)$ for all $\eta \in H$. Taking limits in (2.5.8b) and comparing with (2.5.8a) leads to $(u(0), \phi(0)) = (u_0, \phi(0))$, or equivalently $\langle u(0), \phi(0)\rangle = \langle u_0, \phi(0)\rangle$, where $\phi(0) \in V \hookrightarrow H$. Thus from the application of Proposition A.0.1 we deduce $u(0) = u_0$ as required (and a similar argument gives $v(0) = v_0$).

To obtain $u \in C([0,T];H)$ we apply a modified version of a classical result (Lemma B.0.39), after noting $u \in L^2(0,T;V) \cap L^p(\Omega_T)$ and $du/dt \in L^2(0,T;V') + L^q(\Omega_T)$.

This completes the existence part of the proof.

We make some comments regarding an equivalent way of expressing passage to the limit. We justified in Section 2.3 the equivalence of the composite ODE system (2.3.14a)-(2.3.14b) and the Galerkin approximation[9] (cf. (2.3.15a) - (2.3.15b)):

$$\int_0^T \left(\frac{du^k}{dt}, \chi^k\right) dt + \int_0^T (\nabla u^k, \nabla \chi^k)\, dt = \int_0^T \left(f(u^k, v^k), \chi^k\right) dt, \quad \forall \chi^k \in L^p(0,T;V^k), \tag{2.5.9a}$$

[9] We now have more information about the spaces that the terms lie in so the Galerkin approximation takes on a more concrete form.

$$\int_0^T \left(\frac{dv^k}{dt}, \chi^k\right) dt + \int_0^T (\nabla v^k, \nabla \chi^k)\, dt = \int_0^T \left(g(u^k, v^k), \chi^k\right) dt, \quad \forall \chi^k \in L^p(0,T;V^k). \tag{2.5.9b}$$

Thus passage to the limit of the ODE system implies passage to the limit of the Galerkin approximation (and vice-versa). To illustrate this we also show passage to the limit of the Galerkin approximation (2.5.9a) (arguments will apply in a similar way to (2.5.9b)).

Multiply the composite ODE (2.3.14a) by $\chi^k \in L^p(0,T;V^k)$ and integrate over Ω_T to give

$$\int_0^T \left(\frac{du^k}{dt}, \chi^k\right) dt - \int_0^T \left(\Delta u^k, \chi^k\right) dt = \int_0^T \left(P^k f(u^k, v^k), \chi^k\right) dt.$$

Integrating by parts (in space) the second term and recalling that $(P^k f, \chi^k) = (f, \chi^k)$ leads to (2.5.9a). Define $\eta \in L^p(0,T;V)$ such that $\chi^k := P^k\eta \in L^p(0,T;V^k)$.[10] Then as $du^k(\cdot,t)/dt \in V^k$ for a.e. t we have

$$\begin{aligned}\int_0^T \left(\frac{du^k}{dt}, \chi^k\right) dt &= \int_0^T \left(\frac{du^k}{dt}, P^k\eta\right) dt \\ &= \int_0^T \left(\frac{du^k}{dt}, \eta\right) dt \\ &\to \int_0^T \left(\frac{du}{dt}, \eta\right) dt,\end{aligned} \tag{2.5.10}$$

after recalling the weak convergence of du^k/dt to du/dt in $L^q(0,T;V')$. Furthermore, applying the projection property (2.3.5) and integrating by parts (in space) gives

$$\int_0^T (\nabla u^k, \nabla \chi^k)\, dt = \int_0^T \left(\nabla u^k, \nabla (P^k\eta)\right) dt = \int_0^T (\nabla u^k, \nabla \eta)\, dt = -\int_0^T \left(\Delta u^k, \eta\right) dt \tag{2.5.11a}$$

[10]This is valid as $V \hookrightarrow H$.

and recalling the weak convergence of Δu^k to Δu in $L^q(0,T;V')$ and integrating by parts again gives

$$-\int_0^T (\Delta u^k, \eta)\, dt \to -\int_0^T (\Delta u, \eta)\, dt = \int_0^T (\nabla u, \nabla \eta)\, dt. \tag{2.5.11b}$$

For the final term we have after recalling the weak convergence of $P^k f(u^k, v^k)$ to $f(u,v)$ in $L^q(0,T;V')$

$$\begin{aligned}\int_0^T \left(f(u^k,v^k), \chi^k\right) dt &= \int_0^T \left(f(u^k,v^k), P^k\eta\right) dt \\ &= \int_0^T \left(P^k f(u^k,v^k), \eta\right) dt \\ &\to \int_0^T (f(u,v), \eta)\, dt. \end{aligned} \tag{2.5.12}$$

Thus equations (2.5.10) - (2.5.12) show passage to the limit of the Galerkin approximation (2.5.9a) to the corresponding weak form (2.5.6), as expected.

2.6 Uniqueness

To prove uniqueness of weak solutions we need the following lemma:

Lemma 2.6.1 Assume ρ and ε are non-negative real numbers, ρ satisfies assumption (A1) and C_1 is an arbitrary positive constant. Let $\eta \in L^{\rho+2}(\Omega)$ and $\psi \in V$ be functions defined on a bounded domain $\Omega \subset \mathbb{R}^d$ $(d \leq 3)$, then there are positive constants $C_2(\varepsilon)$ and $\mu = d\left(\frac{1}{2} - \frac{1}{\rho+2}\right)$, such that

$$C_1 \int_\Omega |\eta|^\rho |\psi|^2 dx \leq \left(\frac{\mu}{\varepsilon} + C_2(\varepsilon)\|\eta\|_{0,\rho+2}^{\frac{\rho}{1-\mu}}\right) \|\psi\|_0^2 + \frac{\mu}{\varepsilon}|\psi|_1^2, \quad \text{where } 0 < \mu < 1. \tag{2.6.1}$$

Proof. Observe using Hölder's inequality, followed by inequality (2.1.5) with $s = 2$, $m = 1$ and $r = \rho + 2$ that

$$\begin{aligned} C_1 \int_\Omega |\eta|^\rho \cdot |\psi|^2 dx &\leq C_1 \left(\int_\Omega |\eta|^{\rho+2}\, dx\right)^{\frac{\rho}{\rho+2}} \left(\int_\Omega |\psi|^{\rho+2}\, dx\right)^{\frac{2}{\rho+2}} \\ &\equiv C_1 \|\eta\|_{0,\rho+2}^\rho \|\psi\|_{0,\rho+2}^2 \leq ab, \end{aligned}$$

where $a := C\|\eta\|_{0,\rho+2}^{\rho}\|\psi\|_0^{2(1-\mu)}$, $b := \|\psi\|_1^{2\mu}$. An application of the Young's inequality (2.1.8) with $m := (1-\mu)^{-1}$, $n := \mu^{-1}$ ($\mu \neq 0, 1$) gives inequality (2.6.1). ■

To prove uniqueness we assume there are two solutions $\boldsymbol{u} := (u_1, u_2)^T$ and $\boldsymbol{v} := (v_1, v_2)^T$ of the weak form (P_1), with initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and $\boldsymbol{v}(0) = \boldsymbol{v}_0$ respectively. Setting $\boldsymbol{\eta} = \boldsymbol{w} := \boldsymbol{u} - \boldsymbol{v}$, subtracting weak forms, using (2.2.2a), and applying Lemma B.0.41 leads to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{w}|^2 dx + \int_\Omega |\nabla \boldsymbol{w}|^2 dx = \lambda_0 \int_\Omega |\boldsymbol{w}|^2 dx + \int_\Omega (|\boldsymbol{u}|^\rho A\boldsymbol{u} - |\boldsymbol{v}|^\rho A\boldsymbol{v}) \cdot \boldsymbol{w}\, dx. \quad (2.6.2)$$

Using (2.2.4), followed by Lemma 2.6.1 with $\varepsilon = 2$, $\eta \in \{\boldsymbol{u}, \boldsymbol{v}\}$, $\psi = \boldsymbol{w}$, we bound the last term in (2.6.2) via

$$\begin{aligned}\int_\Omega (|\boldsymbol{u}|^\rho A\boldsymbol{u} - |\boldsymbol{v}|^\rho A\boldsymbol{v}) \cdot \boldsymbol{w}\, dx &\leq (\rho+1)\sqrt{\lambda_1^2 + \omega_1^2}\int_\Omega (|\boldsymbol{u}|^\rho + |\boldsymbol{v}|^\rho)\, |\boldsymbol{w}|^2 dx, \\ &\leq C\left[1 + \left(\|\boldsymbol{u}\|_{0,\rho+2}^{\nu} + \|\boldsymbol{v}\|_{0,\rho+2}^{\nu}\right)\right] \|\boldsymbol{w}\|_0^2 + \mu|\boldsymbol{w}|_1^2,\end{aligned} \quad (2.6.3)$$

where

$$\nu := \frac{\rho}{(1-\mu)} = \begin{cases} \frac{2\rho(\rho+2)}{\rho+4} & \text{if } d = 1, \\ \frac{\rho(\rho+2)}{2} & \text{if } d = 2, \\ \frac{2\rho(\rho+2)}{4-\rho} & \text{if } d = 3. \end{cases} \quad (2.6.4)$$

Noting $\mu < 1$, from (2.6.2) and (2.6.3) we have after kickback of $\mu|\boldsymbol{w}|_1^2$

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|_0^2 \leq C\left[1 + \left(\|\boldsymbol{u}\|_{0,\rho+2}^{\nu} + \|\boldsymbol{v}\|_{0,\rho+2}^{\nu}\right)\right] \|\boldsymbol{w}\|_0^2.$$

Multiplying through by 2 and applying a Grönwall lemma yields

$$\|\boldsymbol{w}\|_0^2 \leq \|\boldsymbol{w}(0)\|_0^2 \exp\left(2CT + 2C\int_0^T \left(\|\boldsymbol{u}\|_{0,\rho+2}^{\nu} + \|\boldsymbol{v}\|_{0,\rho+2}^{\nu}\right) dt\right). \quad (2.6.5)$$

To use the regularity of solutions in $L^{\rho+2}(\Omega_T)$ we apply Hölder's inequality in time to the right hand side of (2.6.5) giving

$$\begin{aligned}\|\boldsymbol{w}\|_0^2 &\leq \|\boldsymbol{w}(0)\|_0^2 \exp\left(2CT + 2C\int_0^T \left(\|\boldsymbol{u}\|_{0,\rho+2}^{\rho+2} + \|\boldsymbol{v}\|_{0,\rho+2}^{\rho+2}\right) dt\right) \\ &\equiv \|\boldsymbol{w}(0)\|_0^2 \exp\left(2CT + 2C\left(\|\boldsymbol{u}\|_{L^{\rho+2}(\Omega_T)}^{\rho+2} + \|\boldsymbol{v}\|_{L^{\rho+2}(\Omega_T)}^{\rho+2}\right)\right).\end{aligned}$$

For this to be valid we require $\nu \leq \rho + 2$, which is easy to check from (2.6.4) and assumption (A2). Thus we have $\|\boldsymbol{u} - \boldsymbol{v}\|_0^2 \leq C\|\boldsymbol{u}(0) - \boldsymbol{v}(0)\|_0^2$. If $\boldsymbol{u}_0 = \boldsymbol{v}_0$ we deduce uniqueness and if $\boldsymbol{u}_0 \neq \boldsymbol{v}_0$ we have continuous dependence in H.[11] This completes the proof of Theorem 2.2.3. ■

We had to satisfy the condition $\nu \leq \rho + 2$, which forced us to impose assumption (A2). The difficulty in the uniqueness proof is due to the nonlinear term $|\boldsymbol{u}|^\rho A\boldsymbol{u}$. An alternative approach would be to initially split this term via

$$|\boldsymbol{u}|^\rho A\boldsymbol{u} = |\boldsymbol{u}|^\rho \begin{pmatrix} -\lambda_1 & -\omega_1 \\ \omega_1 & -\lambda_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = -\lambda_1 |\boldsymbol{u}|^\rho \boldsymbol{u} + \omega_1 |\boldsymbol{u}|^\rho \begin{pmatrix} -v \\ u \end{pmatrix}, \qquad (2.6.6)$$

and thus in the uniqueness proof as the term $|\boldsymbol{u}|^\rho \boldsymbol{u}$ is monotonic (recall (2.2.5)) the first term on the right hand side of (2.6.6) can be 'discarded'. The second term leads to the following unhelpful expression in the uniqueness proof, after multiplying out the brackets and simplifying:

$$\omega_1 \int_\Omega \left[|\boldsymbol{u}|^\rho \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} - |\boldsymbol{v}|^\rho \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} \right] \cdot \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \end{pmatrix} dx$$
$$= \omega_1 \int_\Omega \left(|\boldsymbol{u}|^\rho - |\boldsymbol{v}|^\rho \right) (u_2 v_1 - u_1 v_2)\, dx.$$

Alternatively,

$$\begin{aligned} &\omega_1 \int_\Omega \left[|\boldsymbol{u}|^\rho \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} - |\boldsymbol{v}|^\rho \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} \right] \cdot \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \end{pmatrix} dx \\ &\leq |\omega_1| \int_\Omega \left| |\boldsymbol{u}|^\rho \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} - |\boldsymbol{v}|^\rho \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} \right| |\boldsymbol{w}|\, dx \\ &= |\omega_1| \int_\Omega \left[|\boldsymbol{u}|^{2\rho+2} + |\boldsymbol{v}|^{2\rho+2} - 2|\boldsymbol{u}|^\rho |\boldsymbol{v}|^\rho (\boldsymbol{u} \cdot \boldsymbol{v}) \right]^{1/2} |\boldsymbol{w}|\, dx \\ &= |\omega_1| \int_\Omega \left| |\boldsymbol{u}|^\rho \boldsymbol{u} - |\boldsymbol{v}|^\rho \boldsymbol{v} \right| |\boldsymbol{w}|\, dx. \end{aligned}$$

From the proof of Lemma 2.2.1 we have

$$|\omega_1| \int_\Omega \left| |\boldsymbol{u}|^\rho \boldsymbol{u} - |\boldsymbol{v}|^\rho \boldsymbol{v} \right| |\boldsymbol{w}|\, dx \leq |\omega_1| (\rho + 2) \int_\Omega \left(|\boldsymbol{u}|^\rho + |\boldsymbol{v}|^\rho \right) |\boldsymbol{w}|^2\, dx,$$

leading to effectively the same expression[12] as the right hand side of (2.6.3).

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[11] I.e., $\boldsymbol{u}_0 \to \boldsymbol{v}_0$ in $H \implies \boldsymbol{u}(t; \boldsymbol{u}_0, \boldsymbol{v}_0) \to \boldsymbol{v}(t; \boldsymbol{u}_0, \boldsymbol{v}_0)$ in H ($\boldsymbol{u}_0 \neq \boldsymbol{v}_0$).

[12] Differing only by a constant.

Chapter 3

Strong solutions

In this chapter we deduce further regularity of solutions to the $\lambda - \omega$ system from additional *a priori* estimates, which leads to results for the 'strong solutions' (for the definition of a strong solution in the context of second order elliptic PDEs see [69], pp.287-288, [70], p.160, [87], pp.42-43). The uniqueness of strong solutions follows from the uniqueness of weak solutions, as a strong solution is also a weak solution (see Section 2.6).

After some preliminaries in Section 3.1, existence of the strong solutions is covered in Section 3.2 and continuous dependence of solutions proved in Section 3.3.

3.1 Notation, preliminaries and main result

Let the assumptions, notation and results of Chapter 2 apply. In particular, note assumption (A2) (Section 2.1) on ρ and the eigenvalue problem (2.3.1) in Section 2.3 with the associated set of orthogonal eigenfunctions $\{z_i\}$ and the finite dimensional weak form corresponding to (P_1) given in (2.3.4).

Theorem 3.1.1 Assume $\Omega \subset \mathbb{R}^d$ $(d \leq 3)$ is an open, bounded, convex domain with a boundary $\partial\Omega$ of class C^2. Let (A2) hold and assume u_0, $v_0 \in V$, then the $\lambda - \omega$ system (1.1.1a)–(1.1.1f) possesses a unique, strong solution $\{u, v\}$ satisfying[1]

$$u, v \in L^2(0,T; H^2(\Omega)) \cap C([0,T]; V),$$

[1]As $C([0,T]; V) \hookrightarrow L^\infty(0,T;V)$ we also have $u, v \in L^\infty(0,T;V)$.

and the equations (1.1.1a) and (1.1.1b) hold as equalities in $L^2(\Omega_T)$. Furthermore, the map

$$(u_0(\cdot), v_0(\cdot)) \mapsto (u(\cdot, t; u_0, v_0), v(\cdot, t; u_0, v_0))$$

is continuous in V.

Proof. In order to obtain existence and uniqueness of strong solutions we require further regularity, which we obtain from additional *a priori* estimates.

3.2 Existence, Estimates II and III

To prove strong solution results we need the following lemma (cf. Lemma 2.6.1):

Lemma 3.2.1 Assume ρ and ε are non-negative real numbers, ρ satisfies assumption (A1) and C_1 is an arbitrary positive constant. Let $\eta \in L^{\rho+2}(\Omega)$ and $\phi \in H^2(\Omega)$ be functions defined on a bounded domain $\Omega \subset \mathbb{R}^d$ ($d \leq 3$), then there are positive constants $C_2(\varepsilon)$ and $\mu = d\left(\frac{1}{2} - \frac{1}{\rho+2}\right)$, such that

$$C_1 \int_\Omega |\eta|^\rho |\nabla\phi|^2 dx \leq \left(\frac{\mu}{\varepsilon} + C_2(\varepsilon)\|\eta\|_{0,\rho+2}^{\frac{\rho}{1-\mu}}\right) |\phi|_1^2 + \frac{2\mu}{\varepsilon}|\phi|_2^2, \quad \text{where } 0 < \mu < 1. \tag{3.2.1}$$

Proof. As $|\nabla\phi|^2 = \sum_{i=1}^d |\partial\phi/\partial x_i|^2$ we have

$$\int_\Omega |\eta|^\rho |\nabla\phi|^2 \, dx = \sum_{i=1}^d \int_\Omega |\eta|^\rho \left|\frac{\partial\phi}{\partial x_i}\right|^2 dx. \tag{3.2.2}$$

Applying Lemma 2.6.1 with $\psi = \partial\phi/\partial x_i$ leads to

$$\int_\Omega |\eta|^\rho \left|\frac{\partial\phi}{\partial x_i}\right|^2 dx \leq \left(\frac{\mu}{\varepsilon} + C_2(\varepsilon)\|\eta\|_{0,\rho+2}^\nu\right) \left\|\frac{\partial\phi}{\partial x_i}\right\|_0^2 + \frac{\mu}{\varepsilon}\sum_{j=1}^d \int_\Omega \left|\frac{\partial^2\phi}{\partial x_i \partial x_j}\right|^2 dx, \tag{3.2.3}$$

where $\nu := \rho/(1-\mu)$. Summing (3.2.3) over $i = 1, \dots, d$, using (3.2.2) and noting (2.1.1) gives

$$\begin{aligned} C_1 \int_\Omega |\eta|^\rho |\nabla\phi|^2 \, dx &\leq \left(\frac{\mu}{\varepsilon} + C_2(\varepsilon)\|\eta\|_{0,\rho+2}^\nu\right) \sum_{i=1}^d \left\|\frac{\partial\phi}{\partial x_i}\right\|_0^2 + \frac{\mu}{\varepsilon}\sum_{i,j=1}^d \int_\Omega \left|\frac{\partial^2\phi}{\partial x_i \partial x_j}\right|^2 dx \\ &\leq \left(\frac{\mu}{\varepsilon} + C_2(\varepsilon)\|\eta\|_{0,\rho+2}^\nu\right) \|\nabla\phi\|_0^2 + \frac{2\mu}{\varepsilon}\sum_{|\alpha|=2} \int_\Omega |D^\alpha\phi|^2 \, dx \\ &\equiv \left(\frac{\mu}{\varepsilon} + C_2(\varepsilon)\|\eta\|_{0,\rho+2}^\nu\right) |\phi|_1^2 + \frac{2\mu}{\varepsilon}|\phi|_2^2, \end{aligned}$$

as required. ■

Estimate II: Set $\chi^k = z_j$ in the finite dimensional weak form (2.3.3a)-(2.3.3b). Then after recalling (2.3.11) we multiply (2.3.3a) by $a_{jk}(\mu_j - 1)$ and (2.3.3b) by $b_{jk}(\mu_j - 1)$ and sum both equations from $j = 1, \ldots, k$. From the eigenvalue problem $-\Delta z_j = (\mu_j - 1)z_j$ and so this procedure is equivalent to taking $\boldsymbol{\chi}^k = -\Delta \boldsymbol{u}^k$ in the vectorised finite dimensional weak form (2.3.4), which leads to after integrating by parts the second term:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla \boldsymbol{u}^k|^2 dx + \int_\Omega |\Delta \boldsymbol{u}^k|^2 dx = -\int_\Omega (B\boldsymbol{u}^k)\cdot \Delta \boldsymbol{u}^k dx - \int_\Omega (|\boldsymbol{u}^k|^\rho A\boldsymbol{u}^k)\cdot \Delta \boldsymbol{u}^k dx. \tag{3.2.4}$$

We deal with the last two terms in this equation separately. Integrating by parts and recalling (2.2.2a) yields

$$-\int_\Omega (B\boldsymbol{u}^k)\cdot \Delta \boldsymbol{u}^k dx = \int_\Omega (B\nabla \boldsymbol{u}^k)\cdot \nabla \boldsymbol{u}^k dx = \lambda_0 \int_\Omega |\nabla \boldsymbol{u}^k|^2 dx. \tag{3.2.5}$$

To control the remaining term first note the identity

$$\nabla\left(|\boldsymbol{u}^k|^\rho A\boldsymbol{u}^k\right) = |\boldsymbol{u}^k|^\rho \left(A\nabla \boldsymbol{u}^k\right) + A\boldsymbol{u}^k\left(\rho|\boldsymbol{u}^k|^{\rho-2}\nabla \boldsymbol{u}^k\cdot \boldsymbol{u}^k\right). \tag{3.2.6}$$

Then integration by parts, use of (3.2.6) and (2.2.2a) again yields

$$\begin{aligned}
&-\int_\Omega (|\boldsymbol{u}^k|^\rho A\boldsymbol{u}^k)\cdot \Delta \boldsymbol{u}^k dx \\
&= \int_\Omega |\boldsymbol{u}^k|^\rho\left(A\nabla \boldsymbol{u}^k\cdot \nabla \boldsymbol{u}^k\right)dx + \rho\int_\Omega |\boldsymbol{u}^k|^{\rho-2}\left(\nabla \boldsymbol{u}^k\cdot \boldsymbol{u}^k\right)\left(A\boldsymbol{u}^k\cdot \nabla \boldsymbol{u}^k\right)dx \\
&\le -\lambda_1\int_\Omega |\boldsymbol{u}^k|^\rho|\nabla \boldsymbol{u}^k|^2 dx - \lambda_1\rho\int_\Omega |\boldsymbol{u}^k|^{\rho-2}(\boldsymbol{u}^k\cdot\nabla \boldsymbol{u}^k)^2 dx + \rho|\omega_1|\int_\Omega |\boldsymbol{u}^k|^\rho\cdot|\nabla \boldsymbol{u}^k|^2 dx.
\end{aligned} \tag{3.2.7}$$

We apply Lemma 3.2.1 to the last term in (3.2.7) to give

$$\rho|\omega_1|\int_\Omega |\boldsymbol{u}^k|^\rho|\nabla \boldsymbol{u}^k|^2 dx \le C\left(1 + \|\boldsymbol{u}^k\|_{0,\rho+2}^\nu\right)|\boldsymbol{u}^k|_1^2 + \frac{2\mu}{\varepsilon}|\boldsymbol{u}^k|_2^2, \tag{3.2.8}$$

where $\nu = \rho/(1-\mu)$. We apply some well-known elliptic regularity results for bounded, convex, open domains with a boundary of class C^2. From an elliptic regularity result (see Definition A.0.27 and Theorem A.0.28) the eigenvalue problem (2.3.1) has $z_i \in H^2(\Omega)$, $1 \le i \le k$ (k fixed and finite). Let C_k denote a positive (bounded) constant depending on k. Now $u^k(\cdot,t) = \sum_{i=1}^{k} a_{ik}(t)\, z_i(\cdot)$ so

$$\|u^k(\cdot,t)\|_2 \leq \max_{1\leq i\leq k} |a_{ik}(t)| \sum_{i=1}^{k} \|z_i(\cdot)\|_2 \leq C_k \quad \text{a.e. } t \in (0,T).$$

Thus for fixed k, $u^k(\cdot,t) \in H^2(\Omega)$ for a.e. t (and a similar argument applies bounding $v^k(\cdot,t) \in H^2(\Omega)$ for a.e. t). Thus from Theorem A.0.29 we have $\|\boldsymbol{u}^k\|_2 \leq \widehat{C}\|\Delta \boldsymbol{u}^k\|_0$ for some positive constant $\widehat{C}$ and a.e. t, and choosing $\varepsilon = 4\mu\widehat{C}$ in (3.2.8) leads to

$$\rho|\omega_1| \int_\Omega |\boldsymbol{u}^k|^\rho |\nabla \boldsymbol{u}^k|^2 dx \leq C\left(1 + \|\boldsymbol{u}^k\|_{0,\rho+2}^\nu\right) |\boldsymbol{u}^k|_1^2 + \frac{1}{2}\|\Delta \boldsymbol{u}^k\|_0^2. \tag{3.2.9}$$

From (3.2.4), (3.2.5), (3.2.7), (3.2.9), a kickback of $\frac{1}{2}\|\Delta \boldsymbol{u}^k\|_0^2$, multiplying through by 2, and applying the Grönwall lemma leads to

$$\begin{aligned} |\boldsymbol{u}^k|_1^2 + \int_0^T \Bigg(\|\Delta \boldsymbol{u}^k\|_0^2 &+ 2\lambda_1 \int_\Omega |\boldsymbol{u}^k|^\rho |\nabla \boldsymbol{u}^k|^2 dx \\ &+ 2\lambda_1 \rho \int_\Omega |\boldsymbol{u}^k|^{\rho-2} (\boldsymbol{u}^k \cdot \nabla \boldsymbol{u}^k)^2 dx \Bigg) dt \\ &\leq |\boldsymbol{u}_0^k|_1^2 \exp\left(2CT + 2C \int_0^T \|\boldsymbol{u}^k\|_{0,\rho+2}^\nu dt \right). \end{aligned} \tag{3.2.10}$$

To use that solutions lie in $L^{\rho+2}(\Omega_T)$ we apply Hölder's inequality in time on the right hand side of (3.2.10). As in the uniqueness proof this requires $\nu \leq \rho+2$, which holds due to assumption (A2). The boundedness of the term $|\boldsymbol{u}_0^k|_1^2 \equiv |P^k \boldsymbol{u}_0|_1^2$ follows from the projection property (2.3.9) and the assumption that the initial data is in V. Then noting bound (2.4.2) we deduce

$$u^k,\ v^k \quad \text{are uniformly bounded in } L^\infty(0,T;V). \tag{3.2.11}$$

Furthermore, as $\boldsymbol{u}^k(\cdot,t), \Delta \boldsymbol{u}^k(\cdot,t) \in \{L^2(\Omega)\}^2$ for a.e. $t \in (0,T)$, elliptic regularity theory (Theorem A.0.28) gives $\boldsymbol{u}^k(\cdot,t) \in \{H^2(\Omega)\}^2$ for a.e. $t \in (0,T)$, thus

$$u^k,\ v^k \quad \text{are uniformly bounded in } L^2(0,T;H^2(\Omega)). \tag{3.2.12}$$

In (3.2.7) if $\rho \leq \lambda_1/|\omega_1|$, then we can apply the Grönwall lemma to deduce the above uniform bounds without assumption (A2) (but for the existence of solutions we still need assumption (A1)). We make a further estimate on $d\boldsymbol{u}/dt$.

Estimate III: Set $\chi^k = z_j$ in the finite dimensional weak form (2.3.3a)-(2.3.3b). Then after recalling (2.3.11) we multiply (2.3.3a) by da_{jk}/dt and (2.3.3b) by db_{jk}/dt and sum both equations from $j = 1, \ldots, k$. This is equivalent to taking $\boldsymbol{\chi}^k = \boldsymbol{u}_t^k$ in the vectorised finite dimensional weak form (2.3.4), which leads to after direct calculation

$$\begin{aligned}
&\int_\Omega |\boldsymbol{u}_t^k|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla \boldsymbol{u}^k|^2\,dx \\
&= \int_\Omega (B\boldsymbol{u}^k)\cdot \boldsymbol{u}_t^k\,dx + \int_\Omega (|\boldsymbol{u}^k|^\rho A\boldsymbol{u}^k)\cdot \boldsymbol{u}_t^k\,dx \\
&= \frac{\lambda_0}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{u}^k|^2\,dx - \frac{\lambda_1}{2}\int_\Omega |\boldsymbol{u}^k|^\rho \frac{\partial |\boldsymbol{u}^k|^2}{\partial t}dx + \int_\Omega (\omega_0 + \omega_1|\boldsymbol{u}^k|^\rho)(u^k v_t^k - v^k u_t^k)\,dx.
\end{aligned} \tag{3.2.13}$$

We apply a simple Young's inequality to the last term in (3.2.13) to give

$$\begin{aligned}
\int_\Omega (\omega_0 + \omega_1|\boldsymbol{u}^k|^\rho)(u^k v_t^k - v^k u_t^k)dx &\equiv \int_\Omega (\omega_0 + \omega_1|\boldsymbol{u}^k|^\rho)\,\boldsymbol{u}^k \cdot \begin{pmatrix} v_t^k \\ -u_t^k \end{pmatrix} dx \\
&\le \int_\Omega (|\omega_0| + |\omega_1||\boldsymbol{u}^k|^\rho)|\boldsymbol{u}^k||\boldsymbol{u}_t^k|dx \\
&\le \omega_0^2||\boldsymbol{u}^k||_0^2 + \omega_1^2||\boldsymbol{u}^k||_{0,2\rho+2}^{2\rho+2} + \tfrac{1}{2}||\boldsymbol{u}_t^k||_0^2.
\end{aligned} \tag{3.2.14}$$

Then after noting

$$|\boldsymbol{u}^k|^\rho \frac{\partial}{\partial t}|\boldsymbol{u}^k|^2 = \frac{2}{(\rho+2)}\frac{\partial}{\partial t}|\boldsymbol{u}^k|^{\rho+2},$$

on combining (3.2.13) and (3.2.14) we have after kickback of $\frac{1}{2}||\boldsymbol{u}_t^k||_0^2$ and multiplying through by 2

$$||\boldsymbol{u}_t^k||_0^2 + \frac{d}{dt}||\nabla \boldsymbol{u}^k||_0^2 + \frac{2\lambda_1}{(\rho+2)}\frac{d}{dt}||\boldsymbol{u}^k||_{0,\rho+2}^{\rho+2} \le \lambda_0 \frac{d}{dt}||\boldsymbol{u}^k||_0^2 + 2\omega_0^2||\boldsymbol{u}^k||_0^2 + 2\omega_1^2||\boldsymbol{u}^k||_{0,2\rho+2}^{2\rho+2}. \tag{3.2.15}$$

Integrating both sides of (3.2.15) over $t \in (0, T)$ yields

$$\begin{aligned}
&\int_0^T ||\boldsymbol{u}_t^k||_0^2 dt + |\boldsymbol{u}^k(T)|_1^2 + \frac{2\lambda_1}{(\rho+2)}||\boldsymbol{u}^k(T)||_{0,\rho+2}^{\rho+2} + \lambda_0||\boldsymbol{u}_0^k||_0^2 \\
&\le \lambda_0||\boldsymbol{u}^k(T)||_0^2 + 2\omega_0^2\int_0^T ||\boldsymbol{u}^k||_0^2 dt + 2\omega_1^2\int_0^T ||\boldsymbol{u}^k||_{0,2\rho+2}^{2\rho+2}dt \\
&+ |\boldsymbol{u}_0^k|_1^2 + \frac{2\lambda_1}{(\rho+2)}||\boldsymbol{u}_0^k||_{0,\rho+2}^{\rho+2}.
\end{aligned} \tag{3.2.16}$$

From the uniform bounds in Estimates I and II, assumption (A2), the continuous injections $V \hookrightarrow L^{\rho+2}(\Omega)$, $L^\infty(0,T;V) \hookrightarrow L^{2\rho+2}(\Omega_T)$ and the projection property (2.3.9), we deduce that the right hand side of (3.2.16) is bounded by a positive constant. Thus we have

$$\frac{\partial u^k}{\partial t},\ \frac{\partial v^k}{\partial t} \quad \text{are uniformly bounded in } L^2(\Omega_T). \tag{3.2.17}$$

By extracting the appropriate subsequences from (3.2.11), (3.2.12) and (3.2.17) we deduce

$$u,\ v \in L^\infty(0,T;V), \quad u,\ v \in L^2(0,T;H^2(\Omega)), \quad \frac{\partial u}{\partial t},\ \frac{\partial v}{\partial t} \in L^2(\Omega_T). \tag{3.2.18}$$

Application of Lemma B.0.40 gives

$$u,\ v \in C([0,T];V).$$

To show the differential equation holds as an equality in $L^2(\Omega_T)$ we can repeat the passage to the limit argument with $p = q = 2$, without any additional complications. Some key points in this process are:

- $f(u^k, v^k)$, $g(u^k, v^k)$ are uniformly bounded in $L^2(\Omega_T)$.
- $P^k u \to u$ in $H \implies Q^k u \to 0$ in H.
- Application of the Lions-Aubin theorem (Theorem B.0.37) leads to $f(u^k, v^k) \rightharpoonup f(u, v)$ in $L^2(\Omega_T)$ (and similarly $g(u^k, v^k) \rightharpoonup g(u, v)$ in $L^2(\Omega_T)$).

We could have obtained the above results more directly by extracting the kick-back term $\frac{1}{2}\|\boldsymbol{u}_t^k\|_0^2$ from $(B\boldsymbol{u}^k + |\boldsymbol{u}^k|^\rho A\boldsymbol{u}^k, \boldsymbol{u}_t^k)$ with the aid of a simple Young's inequality, but the slightly longer approach shows there is nothing to be gained in keeping additional positive terms.

To complete the proof we still need to show continuous dependence of the strong solutions on the initial data in V.

3.3 Continuous dependence

Assume $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfy the weak form (P_1), with initial conditions $\boldsymbol{u}_0 = \boldsymbol{u}(\cdot, 0)$ and $\boldsymbol{v}_0 = \boldsymbol{v}(\cdot, 0)$ respectively and $\boldsymbol{u}_0 \neq \boldsymbol{v}_0$. Setting $\boldsymbol{\eta} = -\Delta \boldsymbol{w} + \boldsymbol{w}$, $\boldsymbol{w} := \boldsymbol{u} - \boldsymbol{v}$ and subtracting weak forms leads to after integrating by parts and noting (2.2.2a):

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(|\boldsymbol{w}|^2 + |\nabla \boldsymbol{w}|^2\right) dx + \int_\Omega |\Delta \boldsymbol{w}|^2 dx + \int_\Omega |\nabla \boldsymbol{w}|^2 dx$$

$$= \lambda_0 \int_\Omega \left(|\boldsymbol{w}|^2 + |\nabla \boldsymbol{w}|^2\right) dx + \int_\Omega \left(|\boldsymbol{u}|^\rho A\boldsymbol{u} - |\boldsymbol{v}|^\rho A\boldsymbol{v}\right)\cdot\left(-\Delta \boldsymbol{w} + \boldsymbol{w}\right) dx. \qquad (3.3.1)$$

We split the last term in (3.3.1) and consider each term separately.

Noting (2.2.4), Hölder's inequality and the continuous injection of V into $L^{\rho+2}(\Omega)$ yields

$$\begin{aligned}\int_\Omega \left(|\boldsymbol{u}|^\rho A\boldsymbol{u} - |\boldsymbol{v}|^\rho A\boldsymbol{v}\right)\cdot \boldsymbol{w}\, dx &\leq (\rho+1)\sqrt{\lambda_1^2+\omega_1^2}\int_\Omega \left(|\boldsymbol{u}|^\rho + |\boldsymbol{v}|^\rho\right)|\boldsymbol{w}|^2 dx \\ &\leq C\left(\|\boldsymbol{u}\|^\rho_{0,\rho+2} + \|\boldsymbol{v}\|^\rho_{0,\rho+2}\right)\|\boldsymbol{w}\|^2_{0,\rho+2} \\ &\leq C\left(\|\boldsymbol{u}\|^\rho_{0,\rho+2} + \|\boldsymbol{v}\|^\rho_{0,\rho+2}\right)\|\boldsymbol{w}\|^2_1. \end{aligned} \qquad (3.3.2)$$

In addition, from (2.2.4) and a simple Young's inequality

$$\begin{aligned}\int_\Omega \left(|\boldsymbol{u}|^\rho A\boldsymbol{u} - |\boldsymbol{v}|^\rho A\boldsymbol{v}\right)\cdot \Delta\boldsymbol{w}\, dx &\leq (\rho+1)\sqrt{\lambda_1^2+\omega_1^2}\int_\Omega \left(|\boldsymbol{u}|^\rho + |\boldsymbol{v}|^\rho\right)|\boldsymbol{w}||\Delta\boldsymbol{w}| dx \\ &\leq C\int_\Omega \left(|\boldsymbol{u}|^{2\rho} + |\boldsymbol{v}|^{2\rho}\right)|\boldsymbol{w}|^2 dx + \frac{1}{2}\int_\Omega |\Delta \boldsymbol{w}|^2 dx. \end{aligned} \qquad (3.3.3)$$

From Hölder's inequality and the continuous injection of V into $L^{2\rho+2}(\Omega)$ we have

$$\begin{aligned}\int_\Omega \left(|\boldsymbol{u}|^{2\rho} + |\boldsymbol{v}|^{2\rho}\right)|\boldsymbol{w}|^2 dx &\leq \left(\|\boldsymbol{u}\|^{2\rho}_{0,2\rho+2} + \|\boldsymbol{v}\|^{2\rho}_{0,2\rho+2}\right)\|\boldsymbol{w}\|^2_{0,2\rho+2} \\ &\leq C\left(\|\boldsymbol{u}\|^{2\rho}_{0,2\rho+2} + \|\boldsymbol{v}\|^{2\rho}_{0,2\rho+2}\right)\|\boldsymbol{w}\|^2_1. \end{aligned} \qquad (3.3.4)$$

Combining (3.3.1) - (3.3.4) leads to after kickback of $\frac{1}{2}\int_\Omega |\Delta \boldsymbol{w}|^2 dx$

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|_1^2 \leq C\left(1 + \|\boldsymbol{u}\|^\rho_{0,\rho+2} + \|\boldsymbol{v}\|^\rho_{0,\rho+2} + \|\boldsymbol{u}\|^{2\rho}_{0,2\rho+2} + \|\boldsymbol{v}\|^{2\rho}_{0,2\rho+2}\right)\|\boldsymbol{w}\|^2_1. \qquad (3.3.5)$$

Multiplying through by 2 and using the Grönwall lemma gives

$$\|\boldsymbol{w}(T)\|_1^2 \leq \|\boldsymbol{w}(0)\|_1^2 \exp\Big(2CT + 2C\int_0^T (\|\boldsymbol{u}\|^\rho_{0,\rho+2} + \|\boldsymbol{v}\|^\rho_{0,\rho+2} + \|\boldsymbol{u}\|^{2\rho}_{0,2\rho+2} + \|\boldsymbol{v}\|^{2\rho}_{0,2\rho+2})\, dt\Big).$$

Using Hölder's inequality in time and recalling that solutions belong to $L^{2\rho+2}(\Omega_T)$ leads to $\|\boldsymbol{u}-\boldsymbol{v}\|_1^2 \leq C\|\boldsymbol{u}_0 - \boldsymbol{v}_0\|_1^2$ (compare with the proof . As $\boldsymbol{u}_0 \neq \boldsymbol{v}_0$ this gives continuous dependence in $\{V\}^2$. This completes the proof of Theorem 3.1.1. ■

Chapter 4

A semi-discrete approximation

In this chapter we discretise the $\lambda-\omega$ system in space using a finite element method, which leads to the proof of an error estimate for the semi-discrete approximations. This is organised in the following way.

In Section 4.1 we recall the relevant continuous results and make some definitions. In Section 4.2 we define the standard piecewise linear finite element method to obtain a semi-discrete approximation and state the necessary assumptions on the partitioning of Ω. We also define some mesh dependent norms, discuss the associated properties and spaces, and prove some technical lemmata. In Section 4.3 we prove the local existence of the semi-discrete approximations and then the global existence of these solutions via *a priori* bounds of the semi-discrete solutions. Finally in Section 4.5 we prove a semi-discrete error bound. The basic approach to obtain the estimates is to mimic the continuous estimates in Chapters 2 and 3.

4.1 Notation and preliminaries

To prove error estimates in the semi-discrete and fully-discrete cases we make the following additional restriction on ρ (cf. (A2)):

(A3)
$$\rho \in \begin{cases} [7/6, 4] & \text{if } d = 1, \\ [7/6, 2] & \text{if } d = 2, \\ [7/6, 4/3] & \text{if } d = 3. \end{cases}$$

Note that assumption (A3) implies assumption (A2).

The results of Chapter 3 require that the domain be of class C^2. However, for the numerical analysis we require a polygonal or polyhedral domain, which does not possess a boundary of class C^2. Thus we assume for ease of exposition that the results of Theorem 3.1.1 also hold in the polygonal/polyhedral setting, that is:

Theorem 4.1.1 Assume $\Omega \subset \mathbb{R}^d$ $(d \leq 3)$ is an open, bounded, convex domain, which is polygonal in $d = 2$ and polyhedral in $d = 3$. Let (A2) hold and assume u_0, $v_0 \in H^1(\Omega)$, then there exists a unique strong solution $\{u, v\}$ to the $\lambda - \omega$ system (1.1.1a)-(1.1.1f) such that[1]

$$u,\, v \in L^2(0,T;H^2(\Omega)) \cap C([0,T];H^1(\Omega)),$$
$$\frac{\partial u}{\partial t},\, \frac{\partial v}{\partial t} \in L^2(\Omega_T).$$

We recall that a simplex τ (plural simplices) is an interval if $d = 1$, a triangle if $d = 2$ and a tetrahedron if $d = 3$. Now define $h_\tau := \operatorname{diam} \tau$ to be the length of τ if $d = 1$, the longest side of τ if $d = 2$ and the longest edge of τ if $d = 3$. Also, we let $\varrho_\tau :=$ the diameter of the sphere inscribed in τ. Then, a partitioning of the domain Ω is said to be quasi-uniform (alternatively, we have a 'regular' family of finite elements)([14], p.132, [68]) if there exists a positive constant σ such that

$$\frac{h_\tau}{\varrho_\tau} \leq \sigma, \quad \forall \tau.$$

We remark that the quasi-uniform condition ensures when $d = 2$ that the angles of triangles do not become too small (see [41], Figure 4.1) and when d is arbitrary it is a natural generalisation of this angle property. Another less well-known condition on the mesh is the 'weak acuteness' property (e.g. [21], [64], p.49), that is in the case $d = 2$, for any pair of adjacent triangles the sum of the opposite angles relative to the common side does not exceed π, and in the case $d = 3$, the angles made by any two faces of the same tetrahedron does not exceed $\pi/2$. We also recall that a finite element method is said to be 'conforming' if the finite element space is a subset of the corresponding continuous space. This is relevant as some of the standard results we use implicitly assume this condition (for further details see [14]).

[1]Recall that $C([0,T];H^1(\Omega)) \hookrightarrow L^\infty(0,T;H^1(\Omega))$ so we also have $u, v \in L^\infty(0,T;H^1(\Omega))$.

4.2 Finite element spaces and associated results

We consider the finite element approximation of the $\lambda-\omega$ system under the following assumptions on the mesh:

($\mathbf{A}^h$) Let $\Omega \subset \mathbb{R}^d$, $d \leq 3$, be a convex polygonal domain in $d = 2$ and a convex polyhedral domain in $d = 3$. Let $\mathcal{T}^h$ be a quasi-uniform partitioning of Ω into disjoint open simplices $\{\tau\}$ with $h_\tau := \operatorname{diam} \tau$ and $h := \max_{\tau \in \mathcal{T}^h} h_\tau$, so that $\overline{\Omega} = \bigcup_{\tau \in \mathcal{T}^h} \overline{\tau}$. Additionally, we assume $\mathcal{T}^h$ is weakly acute.

We introduce the standard finite element space of continuous piecewise linear basis functions:

$$S^h := \{v \in C(\overline{\Omega}) \ : \ v|_\tau \quad \text{is linear } \forall \tau \in \mathcal{T}^h\} \subset H^1(\Omega).$$

Let $\{\varphi_j\}_{j=0}^J$ be the canonical basis associated with S^h, satisfying $\varphi_j(x_i) = \delta_{ij}$, where $\{x_i\}_{i=0}^J$ is the set of nodes of $\mathcal{T}^h$. Let $\pi^h : C(\overline{\Omega}) \mapsto S^h$ be the Lagrange interpolation operator (alternatively, piecewise linear interpolant) such that $\pi^h v(x_j) = v(x_j)$ for all $j = 0, \ldots, J$. We define a discrete L^2 inner product on $C(\overline{\Omega})$ via

$$(u, v)^h := \int_\Omega \pi^h(u(x)v(x))\, dx \equiv \sum_{j=0}^J \widehat{M}_{jj} u(x_j) v(x_j), \tag{4.2.1}$$

where $\widehat{M}_{jj} := (1, \varphi_j) \equiv (\varphi_j, \varphi_j)^h > 0$. It is easy to verify that

$$(\pi^h \eta, \chi)^h = (\eta, \chi)^h \quad \forall \eta, \chi \in C(\overline{\Omega}). \tag{4.2.2}$$

The discrete inner product (4.2.1) approximates the continuous L^2 inner product using the vertex quadrature rule[2] ([14], p.182) and is exact for all piecewise polynomials uv of degree less than or equal to one. For future reference we also define

$$M_{ij} := (\varphi_i, \varphi_j), \quad K_{ij} := (\nabla \varphi_i, \nabla \varphi_j), \quad \widehat{M}_{ij} := (\varphi_i, \varphi_j)^h, \tag{4.2.3}$$

corresponding to the the mass matrix M, stiffness matrix K and lumped mass matrix $\widehat{M}$ respectively. Note that $\widehat{M}$ is a diagonal matrix. Notice that

$$\widehat{M}_{ii} = \sum_{j=0}^J M_{ij}, \quad i = 0, \ldots, J,$$

[2]The composite Trapezium rule in one dimension.

i.e., the elements of the lumped mass matrix $\widehat{M}$ are obtained by adding the off diagonal elements of M in any row to the diagonal element of that row. This is easily proved via

$$\sum_{j=0}^{J} M_{ij} = \sum_{j=0}^{J} \int_\Omega \varphi_i \varphi_j \, dx = \int_\Omega \varphi_i \sum_{j=0}^{J} \varphi_j \, dx = (1, \varphi_i) \equiv \widehat{M}_{ii},$$

using that $\sum_{j=0}^{J} \varphi_j = 1$ (i.e., $\pi^h 1 = 1$). The use of the discrete inner product to approximate the mass matrix is often called 'lumped mass integration' (e.g., [82], p.118). One advantage of mass lumping is that the (diagonal) mass matrix is trivially inverted.

From the above mentioned references, as the partitioning $\mathcal{T}^h$ is weakly acute we have

$$\text{(i)} \sum_{j=0}^{J} K_{ij} \geq 0 \quad \forall i, \quad \text{(ii)} \, K_{ij} \leq 0 \quad i \neq j. \tag{4.2.4}$$

In fact, it follows directly for our method that

$$\sum_{j=0}^{J} K_{ij} = 0,$$

as

$$\sum_{j=0}^{J} K_{ij} = \int_\Omega \sum_{j=0}^{J} \nabla \varphi_i \cdot \nabla \varphi_j \, dx = \int_\Omega \nabla \varphi_i \cdot \nabla \sum_{j=0}^{J} \varphi_j \, dx = 0,$$

using the fact again that $\sum_{j=0}^{J} \varphi_j = 1$.

The following lemma will be important in deriving later stability estimates and is a consequence of the weak acuteness property (see Lemma 6.2 in [65], alt. Section 2.4.2 in [64], which are similar results).

Lemma 4.2.1 Assume the partitioning $\mathcal{T}^h$ is weakly acute and $\boldsymbol{U}(\boldsymbol{\chi}^h) \in \{S^h\}^n$ is a monotonic function for all $\boldsymbol{\chi}^h \in \{S^h\}^n$, $n \in \mathbb{N}$. Then

$$(\nabla \boldsymbol{\chi}^h, \nabla \pi^h \boldsymbol{U}(\boldsymbol{\chi}^h)) \geq 0.$$

Proof. Recall the weak acuteness properties (4.2.4) and the fact that $K_{ii} > 0$. Set $\boldsymbol{\chi}^h = \sum_{j=0}^{J} \boldsymbol{\chi}_j \varphi_j$ where $\boldsymbol{\chi}_j := \boldsymbol{\chi}(x_j)$ and note that $\pi^h \boldsymbol{U}(\boldsymbol{\chi}^h) = \sum_{i=0}^{J} \boldsymbol{U}(\boldsymbol{\chi}_i)\varphi_i$, thus

$$
\begin{aligned}
(\nabla \boldsymbol{\chi}^h, \nabla \pi^h \boldsymbol{U}(\boldsymbol{\chi}^h)) &= \sum_{i=0}^{J} \sum_{j=0}^{J} \boldsymbol{\chi}_j \cdot \boldsymbol{U}(\boldsymbol{\chi}_i) K_{ij} \\
&= \sum_{i=0}^{J} \left(\sum_{\substack{j=0 \\ j \neq i}}^{J} \boldsymbol{\chi}_j \cdot \boldsymbol{U}(\boldsymbol{\chi}_i) K_{ij} + \boldsymbol{\chi}_i \cdot \boldsymbol{U}(\boldsymbol{\chi}_i) K_{ii} \right) \\
&\geq \sum_{i=0}^{J} \sum_{\substack{j=0 \\ j \neq i}}^{J} K_{ij} \boldsymbol{U}(\boldsymbol{\chi}_i) \cdot (\boldsymbol{\chi}_j - \boldsymbol{\chi}_i).
\end{aligned} \tag{4.2.5}
$$

Additionally,

$$
\begin{aligned}
\sum_{i=0}^{J} \sum_{\substack{j=0 \\ j \neq i}}^{J} K_{ij} \boldsymbol{U}(\boldsymbol{\chi}_i) \cdot (\boldsymbol{\chi}_j - \boldsymbol{\chi}_i) &= \sum_{j=0}^{J} \sum_{\substack{i=0 \\ i \neq j}}^{J} K_{ij} \boldsymbol{U}(\boldsymbol{\chi}_i) \cdot (\boldsymbol{\chi}_j - \boldsymbol{\chi}_i) \\
&= \sum_{i=0}^{J} \sum_{\substack{j=0 \\ j \neq i}}^{J} K_{ij} \boldsymbol{U}(\boldsymbol{\chi}_j) \cdot (\boldsymbol{\chi}_i - \boldsymbol{\chi}_j),
\end{aligned}
$$

as $\sum_{i=0}^{J} \sum_{\substack{j=0 \\ j \neq i}}^{J} (\cdot) = \sum_{j=0}^{J} \sum_{\substack{i=0 \\ i \neq j}}^{J} (\cdot)$, $K_{ij} = K_{ji}$ and swapping the indices i and j. Thus from (4.2.5) and the monotonicity of $\boldsymbol{U}(\boldsymbol{\chi}^h)$

$$
(\nabla \boldsymbol{\chi}^h, \nabla \pi^h \boldsymbol{U}(\boldsymbol{\chi}^h)) = \frac{1}{2} \sum_{i=0}^{J} \sum_{\substack{j=0 \\ j \neq i}}^{J} (-K_{ij}) \left(\boldsymbol{U}(\boldsymbol{\chi}_i) - \boldsymbol{U}(\boldsymbol{\chi}_j) \right) \cdot (\boldsymbol{\chi}_i - \boldsymbol{\chi}_j) \geq 0. \quad \blacksquare
$$

We state below results concerning finite dimensional spaces on S^h and the associated norms. It is well-known that the discrete inner product (4.2.1) induces a norm on $S^h \subset C(\overline{\Omega})$ via

$$
|\chi^h|_h := \sqrt{(\chi^h, \chi^h)^h}, \quad \forall \chi^h \in S^h \tag{4.2.6}
$$

and there is an equivalence of norms result between $\|\cdot\|_0$ and $|\cdot|_h$, that is,

$$
c\|\chi^h\|_0 \leq |\chi^h|_h \leq C\|\chi^h\|_0, \quad \forall \chi^h \in S^h, \tag{4.2.7}
$$

(e.g. [68], [66]) where c and C are independent of h. We implicitly assume this result without always referring to it. It will be convenient to generalise (4.2.6) in the following fashion, for all $\chi^h \in S^h$:

$$|\chi^h|_{h,p} := \left(\int_\Omega \pi^h\{|\chi^h(x)|^p\}\,dx\right)^{1/p} \equiv \left(\sum_{j=0}^{J} \widehat{M}_{jj}|\chi^h(x_j)|^p\right)^{1/p} \quad \text{for } 1 \le p < \infty, \tag{4.2.8a}$$

$$|\chi^h|_{h,\infty} := \max_{0\le j\le J}|\chi^h(x_j)| \quad \text{if } p = \infty. \tag{4.2.8b}$$

Let $\boldsymbol{u} := (u_1, \ldots, u_n)^T$, $\boldsymbol{v} := (v_1, \ldots, v_n)^T$ be n-tuples and define the equivalent norms $\|\boldsymbol{u}\|_p := (\sum_{i=1}^n |u_i|^p)^{1/p}$, $1 \le p \le \infty$, where $\|\boldsymbol{u}\|_\infty := \max_i |u_i|$. The spaces l_p of n-tuples with the norm $\|\boldsymbol{u}\|_p$ are discussed in [59], p.158. We thus attach a meaning to $|u|_{h,p} = \left(\sum_{j=0}^J |(\widehat{M}_{jj})^{1/p}u(x_j)|^p\right)^{1/p}$ in the $p = \infty$ case. Using the discrete Hölder inequality

$$\sum_{i=1}^n |u_i v_i| \le \|\boldsymbol{u}\|_p \|\boldsymbol{v}\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 \le p, q, \le \infty \tag{4.2.9}$$

([59], p.272, [17], p.67) and the discrete Minkowski inequalities

$$\left(\sum_{i=1}^n |u_i + v_i|^p\right)^{1/p} \le \|\boldsymbol{u}\|_p + \|\boldsymbol{v}\|_p, \quad 1 \le p < \infty \tag{4.2.10}$$

([59], p.273, [17], p.68), we easily prove the following discrete Minkowski and Hölder inequalities respectively for elements in $C(\overline{\Omega})$ (and hence also for elements in S^h):

Lemma 4.2.2

$$|u + v|_{h,p} \le |u|_{h,p} + |v|_{h,p}, \quad 1 \le p < \infty, \quad \forall u, v \in C(\overline{\Omega}). \tag{4.2.11}$$

Proof.

$$\begin{aligned}
|u + v|_{h,p} &= \left(\sum_{j=0}^J \widehat{M}_{jj}|u(x_j) + v(x_j)|^p\right)^{1/p} \\
&= \left(\sum_{j=0}^J |(\widehat{M}_{jj})^{1/p}u(x_j) + (\widehat{M}_{jj})^{1/p}v(x_j)|^p\right)^{1/p} \\
&\le \|u\|_{h,p} + \|v\|_{h,p},
\end{aligned}$$

using (4.2.10). ■

Lemma 4.2.3

$$|(u,v)^h| \leq |u|_{h,p}|v|_{h,q}, \quad \frac{1}{p}+\frac{1}{q}=1, \quad 1 \leq p,q \leq \infty, \quad \forall u,v \in C(\overline{\Omega}). \qquad (4.2.12)$$

Proof.

$$\begin{aligned} |(u,v)^h| &= \left|\sum_{j=0}^{J} \widehat{M}_{jj} u(x_j) v(x_j)\right| \\ &\leq \sum_{j=0}^{J} |(\widehat{M}_{jj})^{1/p} u(x_j)||(\widehat{M}_{jj})^{1/q} v(x_j)| \\ &\leq |u|_{h,p}|v|_{h,q}, \end{aligned}$$

using (4.2.9). ■

Lemma 4.2.4 (4.2.8a)-(4.2.8b) are norms on S^h (*not* $C(\overline{\Omega})$) and the following space is Banach:

$$L^{h,p}(\Omega) := \{\chi^h \in S^h \; : \; |\chi^h|_{h,p} < \infty, \quad 1 \leq p \leq \infty\}.$$

Proof. We first verify the three axioms of a norm. Recall that S^h is a subset of $C(\overline{\Omega})$. Clearly $|\chi^h|_{h,p} \geq 0$ for all $\chi^h \in S^h$, and $\chi^h = 0$ implies $|\chi^h|_{h,p} = 0$. If $|\chi^h|_{h,p} = 0$ then as $\widehat{M}_{jj} > 0$ for all j this implies $\chi^h(x_j) = 0$ for all j and as $\chi^h \in S^h$ this means $\chi^h \equiv 0$. This is not necessarily true for any continuous function with roots at the nodes x_j, thus the requirement that functions be in S^h. Furthermore, it is clearly true that $|k\chi^h|_{h,p} = |k||\chi^h|_{h,p}$ and the triangle inequality follows from (4.2.11), so $|\cdot|_{h,p}$ is a norm for $1 \leq p < \infty$. That $|\cdot|_{h,\infty}$ is a norm follows from the fact that l_∞ is a norm (see above). Finally, remembering that all finite dimensional normed spaces are complete we conclude that $L^{h,p}(\Omega)$ is Banach. ■

As one would expect, the space $L^{h,p}(\Omega)$ possesses discrete analogues of the properties of $L^p(\Omega)$. In addition to the discrete Hölder and Minkowski inequalities given above, we have a simple injection result for elements in $C(\overline{\Omega})$ (and hence also for elements in S^h):

Lemma 4.2.5

$$|u|_{h,q} \leq C|u|_{h,p}, \quad C := |\Omega|^{1/q-1/p}, \quad 1 \leq q \leq p \leq \infty, \quad \forall u \in L^{h,p}(\Omega). \qquad (4.2.13)$$

Proof. From (4.2.12) we have

$$
\begin{aligned}
|u|_{h,q}^q &= \sum_{j=0}^{J} \widehat{M}_{jj} |u(x_j)|^q \\
&= \sum_{j=0}^{J} \left\{ (\widehat{M}_{jj})^{q/p} |u(x_j)|^q \right\} \left\{ (\widehat{M}_{jj})^{1-q/p} \right\} \\
&\leq \left(\sum_{j=0}^{J} \widehat{M}_{jj} |u(x_j)|^p \right)^{q/p} \left(\sum_{j=0}^{J} \widehat{M}_{jj} \right)^{1-\frac{q}{p}} \\
&= |u|_{h,p}^q |\Omega|^{1-\frac{q}{p}},
\end{aligned}
$$

after noting $\sum_i \varphi_i = 1$. Raising both sides to the power of $1/q$ gives the desired result. ■

Additional embedding results are given below. We extend these finite dimensional spaces to time-dependent ones $L^{h,p}(\Omega_T)$, with norm

$$
|\chi^h|_{h,p,\Omega_T} := \left(\int_0^T |\chi^h(\cdot,t)|_{h,p}^p \, dt \right)^{1/p}, \quad 1 \leq p \leq \infty, \tag{4.2.14}
$$

$\forall \chi^h \in S^h$, which is the discrete analogue of $L^p(\Omega_T) \equiv L^p(0,T;L^p(\Omega))$, with the following injection result:

Lemma 4.2.6

$$
|u|_{h,q,\Omega_T} \leq C|u|_{h,p,\Omega_T}, \quad C := (|\Omega|T)^{1/q-1/p}, \quad 1 \leq q \leq p \leq \infty, \tag{4.2.15}
$$

for all $u \in L^{h,p}(\Omega_T)$.

Proof. We have

$$
\begin{aligned}
|u|_{h,q,\Omega_T}^q &= \int_0^T |u(\cdot,t)|_{h,q}^q \, dt \\
&\leq |\Omega|^{1-q/p} \int_0^T |u(\cdot,t)|_{h,p}^q \, dt \\
&\leq (|\Omega|T)^{1-q/p} \left(\int_0^T |u(\cdot,t)|_{h,p}^p \, dt \right)^{q/p},
\end{aligned}
$$

after applying the discrete injection result (4.2.13) and the standard Hölder inequality. The result then follows after raising both sides to the power of $1/q$. ■

We require the following well-known interpolation error estimates, which follow from Theorem 5 in [15] and the quasi-uniform condition of our triangulation $\mathcal{T}^h$:

$$\|(I-\pi^h)\chi\|_0 + h|(I-\pi^h)\chi|_1 \leq Ch^2|\chi|_2, \quad \forall \chi \in H^2(\Omega), \quad d \leq 3, \tag{4.2.16}$$

$$\|(I-\pi^h)\chi\|_{0,1} \leq Ch^2|\chi|_{2,1}, \quad \forall \chi \in W^{2,1}(\Omega), \quad d \leq 3. \tag{4.2.17}$$

Due to the quasi-uniform condition we have for all $\chi^h \in S^h$ the following inverse estimates (cf. Theorem 3.2.6. in [14]):

$$|\chi^h|_1 \leq \frac{C}{h}|\chi^h|_h, \tag{4.2.18a}$$

$$\|\chi^h\|_{0,q} \leq Ch^{d(1/q-1/r)}\|\chi^h\|_{0,r}, \quad 1 \leq r \leq q \leq \infty, \tag{4.2.18b}$$

$$|\chi^h|_{1,q} \leq Ch^{d(1/q-1/r)}|\chi^h|_{1,r}, \quad 1 \leq r \leq q \leq \infty. \tag{4.2.18c}$$

In order to estimate the error due to numerical integration we need the following estimates:

Lemma 4.2.7 For all $\chi^h, \eta^h \in S^h$

$$|(\chi^h,\eta^h) - (\chi^h,\eta^h)^h| \leq Ch^2|\chi^h|_1|\eta^h|_1, \tag{4.2.19}$$

$$|(\chi^h,\eta^h) - (\chi^h,\eta^h)^h| \leq Ch\|\chi^h\|_0|\eta^h|_1. \tag{4.2.20}$$

As these are slightly sharper than the ones usually quoted in the literature we provide a proof:

Proof. From (4.2.17) and (2.1.1) we have

$$\begin{aligned}|(\chi^h,\eta^h) - (\chi^h,\eta^h)^h| &\equiv \|(I-\pi^h)\chi^h\eta^h\|_{0,1} \\ &\leq Ch^2\int_\Omega \sum_{|\alpha|=2} |D^\alpha(\chi^h\eta^h)|\,dx \\ &\leq Ch^2\int_\Omega \sum_{i,j=1}^{d} \left|\frac{\partial^2(\chi^h\eta^h)}{\partial x_i \partial x_j}\right|\,dx.\end{aligned}$$

Remembering that χ^h and η^h are piecewise linear and applying the Cauchy-Schwarz inequality yields

$$\begin{aligned} Ch^2 \int_\Omega \sum_{i,j=1}^{d} \left| \frac{\partial^2 (\chi^h \eta^h)}{\partial x_i \partial x_j} \right| dx &= Ch^2 \int_\Omega \sum_{i,j=1}^{d} \left| \frac{\partial \eta^h}{\partial x_i} \frac{\partial \chi^h}{\partial x_j} + \frac{\partial \chi^h}{\partial x_i} \frac{\partial \eta^h}{\partial x_j} \right| dx \\ &\leq Ch^2 \sum_{i,j=1}^{d} \left\{ \left\| \frac{\partial \eta^h}{\partial x_i} \right\|_0 \left\| \frac{\partial \chi^h}{\partial x_j} \right\|_0 + \left\| \frac{\partial \chi^h}{\partial x_i} \right\|_0 \left\| \frac{\partial \eta^h}{\partial x_j} \right\|_0 \right\} \\ &\leq Ch^2 |\chi^h|_1 |\eta^h|_1, \end{aligned}$$

which proves (4.2.19). Now applying the inverse inequality (4.2.18a) and using the equivalence of norms result (4.2.7) leads to

$$Ch^2 |\chi^h|_1 |\eta^h|_1 \leq Ch^2 \left(\frac{1}{h} |\chi^h|_h \right) |\eta^h|_1 \leq Ch \|\chi^h\|_0 |\eta^h|_1,$$

proving (4.2.20). ■

In the above proof we have ignored the fact that taking the second derivative of a piecewise linear function leads to contributions due to the resulting Dirac delta functions. To get around this problem we can either show that the contributions lead to an additional factor of h^2, or we can repeat the above calculation for a single simplex τ with associated step size h_τ, and then add the contribution from all the simplices to arrive at the same result (using the fact that $h_\tau \leq h$ and that the partition of Ω is exhaustive).

We present several theorems that are the discrete analogues of continuous theorems. For example, in order to obtain error estimates in later sections we need a discrete Sobolev embedding result:

Lemma 4.2.8 Let $v \in S^h$, $r \in \mathbb{R}$, $h \leq 1$ and assume the triangulation $\mathcal{T}^h$ is quasi-uniform, then there exists a positive constant C (independent of h) such that

$$|v|_{h,r} \leq C\|v\|_1 \quad \text{holds for} \quad r \in \begin{cases} [2, \infty] \text{ if } d = 1, \\ [2, \infty) \text{ if } d = 2, \\ [2, 6] \text{ if } d = 3. \end{cases} \tag{4.2.21}$$

Proof. Take $\eta = v^r(x)$ in (4.2.17), note (2.1.1), and apply the Cauchy-Schwarz inequality, yielding (see the comment after the proof of Lemma 4.2.7):

$$\begin{aligned}\int_\Omega |(I-\pi^h)v^r|\,dx &\le Ch^2 \sum_{|\alpha|=2}\int_\Omega |D^\alpha v^r|\,dx \\ &\le Ch^2 \sum_{i,j=1}^{d}\int_\Omega \left|\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} r(r-1)v^{r-2}\right| dx \\ &\le Ch^2 \|v\|_{0,\infty}^{r-2} \sum_{i,j=1}^{d}\left\|\frac{\partial v}{\partial x_i}\right\|_0 \left\|\frac{\partial v}{\partial x_j}\right\|_0 \\ &\le Ch^2\|v\|_{0,\infty}^{r-2}|v|_1^2. \end{aligned} \tag{4.2.22}$$

In the $d = 1$ case we apply the Sobolev embedding $\|v\|_{0,\infty} \le C\|v\|_1$ to the last term in (4.2.22). In the $d = 2$ case we first apply the inverse inequality (4.2.18b) in the form $\|v\|_{0,\infty} \le Ch^{-2/p}\|v\|_{0,p}$, $1 \le p \le \infty$, to the last term in (4.2.22), followed by the Sobolev embedding $\|v\|_{0,p} \le C\|v\|_1$, $p \in [2,\infty)$. In the $d = 3$ case we apply the inverse inequality (4.2.18b) in the form $\|v\|_{0,\infty} \le Ch^{-1/2}\|v\|_{0,6}$ to the last term in (4.2.22), followed by the Sobolev embedding $\|v\|_{0,6} \le C\|v\|_1$. This leads to:

$$\int_\Omega |(I-\pi^h)v^r|\,dx \le \begin{cases} Ch^2\|v\|_1^r \text{ if } d=1, \\ Ch^{2+\frac{2(2-r)}{p}}\|v\|_1^r \quad \forall p\in[2,\infty) \text{ if } d=2, \\ Ch^{3-\frac{r}{2}}\|v\|_1^r \text{ if } d=3. \end{cases} \tag{4.2.23}$$

Now split $|v|_{h,r}^r$ via

$$\begin{aligned}|v|_{h,r}^r &\le \|(I-\pi^h)v^r\|_{0,1} + \|v\|_{0,r}^r \\ &\le \begin{cases} C(h^2+1)\|v\|_1^r,\ r\in[2,\infty] \text{ if } d=1, \\ C\left(h^{2+\frac{2(2-r)}{p}}+1\right)\|v\|_1^r,\ r,p\in[2,\infty) \text{ if } d=2, \\ C\left(h^{3-\frac{r}{2}}+1\right)\|v\|_1^r,\ r\in[2,6] \text{ if } d=3, \end{cases}\end{aligned}$$

after using the Sobolev embedding theorem for $H^1(\Omega) \hookrightarrow L^r(\Omega)$. To obtain constants independent of h note the simple rule that $h < |\Omega|$ implies $h^\alpha < |\Omega|^\alpha$, for all $\alpha \in \mathbb{R}^+$. Additionally, $r < 2+p < \infty$ in the $d = 2$ case and $r \le 6$ in the $d = 3$ case.

■

We also require a discrete Gagliardo-Nirenberg inequality (cf. (2.1.5)):

Lemma 4.2.9 Let $v \in S^h$, $r \in \mathbb{R}$, $h \leq 1$, $\mu := d\left(\frac{1}{2} - \frac{1}{r}\right)$ and assume the triangulation $\mathcal{T}^h$ is quasi-uniform, then

$$|v|_{h,r} \leq \frac{C}{h}|v|_h^{1-\mu}\|v\|_1^{\mu} \quad \text{holds for} \quad r \in \begin{cases} [2,\infty] \text{ if } d=1, \\ [2,\infty) \text{ if } d=2, \\ [2,6] \text{ if } d=3. \end{cases} \tag{4.2.24}$$

Proof. This result follows from: Lemma 4.2.8, the inverse inequality[3] (4.2.18a) with the equivalence of norms result (4.2.7), the embedding $L^r \hookrightarrow L^2$ $(r \geq 2)$, and the Gagliardo-Nirenberg inequality (2.1.5) with $m = 1$ to give:

$$|v|_{h,r} \leq C\|v\|_1 \leq \frac{C}{h}\|v\|_0 \leq \frac{C}{h}\|v\|_{0,r} \leq \frac{C}{h}\|v\|_0^{1-\mu}\|v\|_1^{\mu} \leq \frac{C}{h}|v|_h^{1-\mu}\|v\|_1^{\mu}. \quad \blacksquare$$

In order to prove uniqueness of the semi-discrete approximation we prove the following result:

Lemma 4.2.10 Assume ρ and ε are non-negative real numbers, ρ satisfies assumption (A1) and $\eta, \psi \in S^h$, $h \leq 1$ and the triangulation $\mathcal{T}^h$ is quasi-uniform. Then there are positive constants $C_h(\varepsilon) := C(\varepsilon)h^{2/(\mu-1)} > 0$, $\mu := d\left(\frac{1}{2} - \frac{1}{\rho+2}\right)$ and C such that

$$C\left(|\eta|^\rho, |\psi|^2\right)^h \leq \left(\frac{\mu}{\varepsilon} + C_h(\varepsilon)|\eta|_{h,\rho+2}^{\frac{\rho}{1-\mu}}\right)\|\psi\|_0^2 + \frac{\mu}{\varepsilon}|\psi|_1^2, \quad \text{where } 0 < \mu < 1. \tag{4.2.25}$$

We provide a proof for completeness, which is the discrete analogue of the proof of Lemma 2.6.1.

Proof. Initially apply the Hölder inequality for S^h (4.2.12) followed by the discrete Gagliardo-Nirenberg inequality (4.2.24) yielding

$$C\int_\Omega \pi^h\left(|\eta|^\rho|\psi|^2\right)\,dx \leq C|\eta|_{h,\rho+2}^\rho|\psi|_{h,\rho+2}^2 \leq ab,$$

where $a := \frac{C}{h^2}|\eta|_{h,\rho+2}^\rho|\psi|_h^{2(1-\mu)}$, $b := \|\psi\|_1^{2\mu}$. Application of the Young's inequality (2.1.8) with $m := (1-\mu)^{-1}$, $n := \mu^{-1}$, $(\mu \neq 0, 1)$ and recalling the equivalence of norms result (4.2.7) leads to inequality (4.2.25). $\blacksquare$

[3]It is easy to show from this inverse inequality that provided $h \leq 1$ we have $\|v\|_1 \leq (C/h)\|v\|_0$.

Finally, (see, e.g., [6], [7]), we recall a result for the L^2 projection operator $P^h : L^2(\Omega) \mapsto S^h$ given by

$$(P^h\eta, \chi^h) = (\eta, \chi^h) \quad \forall \chi^h \in S^h,$$

namely,

$$\|(I - P^h)\eta\|_0 + h|(I - P^h)\eta|_1 \leq Ch^m|\eta|_m, \quad m = 1, 2, \quad \forall \eta \in H^m(\Omega). \quad (4.2.26)$$

We will use this lemma to bound the initial semi-discrete approximations in H^1 via:

Lemma 4.2.11

$$\|P^h\chi\|_1 \leq C\|\chi\|_1, \quad \forall \chi \in H^1(\Omega). \quad (4.2.27)$$

Proof. We deal with the terms on the right hand side of

$$\|P^h\chi\|_1^2 = |P^h\chi|_1^2 + \|P^h\chi\|_0^2.$$

The last term is easily controlled via a standard property of orthogonal projections, namely,

$$\|P^h\chi\|_0 \leq \|\chi\|_0 \leq \|\chi\|_1,$$

as $\chi \in H^1(\Omega) \hookrightarrow L^2(\Omega)$. To control the remaining term use from (4.2.26) that $|(I - P^h)\eta|_1 \leq C|\eta|_1$, for all $\eta \in H^1(\Omega)$, and so

$$\begin{aligned} |P^h\chi|_1 &\equiv |P^h\chi - \chi + \chi|_1 \\ &\leq |(I - P^h)\chi|_1 + |\chi|_1 \\ &\leq C\|\chi\|_1, \end{aligned}$$

which proves the lemma. ■

We give two equivalent semi-discrete approximations of (P_1) and (P_2) respectively (see (2.2.7), (2.2.8)):

($\mathbf{P}_1^h$) Find $\boldsymbol{u}^h(\cdot, t) \in \{S^h\}^2$ such that $\boldsymbol{u}^h(\cdot, 0) = P^h\boldsymbol{u}_0(\cdot)$ and for a.e. $t \in (0, T)$

$$(\boldsymbol{u}_t^h, \boldsymbol{\chi}^h)^h + (\nabla \boldsymbol{u}^h, \nabla \boldsymbol{\chi}^h) = (B\boldsymbol{u}^h, \boldsymbol{\chi}^h)^h + (|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h, \boldsymbol{\chi}^h)^h \quad \forall \boldsymbol{\chi}^h \in \{S^h\}^2, \quad (4.2.28)$$

where $\boldsymbol{u}^h := (u^h, v^h)^T$.

($\mathbf{P}_2^h$) Find $c^h(\cdot,t) \in \mathbb{S}^h$ such that $c^h(\cdot,0) = P^h c_0(\cdot)$ and for a.e. $t \in (0,T)$

$$(c_t^h, \chi^h)^h + (\nabla c^h, \nabla \chi^h) = (\lambda(r^h)c^h, \chi^h)^h + i(\omega(r^h)c^h, \chi^h)^h \quad \forall \chi^h \in \mathbb{S}^h,$$

where $c^h := u^h + iv^h$, $r^h \equiv |c^h|$ and $\mathbb{S}^h$ is the 'complexified' space of S^h, i.e., if $c = c_1 + i\, c_2 \in \mathbb{S}^h$ then $c_j \in S^h$, $j = 1,2$. Before considering error estimates for the semi-discrete approximation we need a lemma.

Lemma 4.2.12 Let the assumptions (A1) and (A^h) hold, $u_0, v_0 \in H^1(\Omega)$ and $h \leq 1$. Then (P_1^h) possesses a unique solution $\{u^h, v^h\}$ such that the following stability bounds hold independent of h:

$$u^h, v^h \in L^\infty(0,T; H^1(\Omega)) \cap L^{h,2\rho+2}(\Omega_T),$$
$$\frac{\partial u^h}{\partial t}, \frac{\partial v^h}{\partial t} \in L^2(\Omega_T).$$

Proof. We separate the proof into four parts showing: local existence of the approximations, global existence of the approximations, uniqueness, and an additional stability estimate.

4.3 Existence and uniqueness of the approximations

4.3.1 Local existence of the approximations

Let $c^h(\cdot,t) = \sum_{i=0}^{J} C_i(t)\varphi_i(\cdot)$ in (P_2^h) where $C_i(t) \approx c(x_i,t)$ and take $\chi^h = \varphi_j$, $j = 0,\ldots,J$ yielding:

$$\sum_{i=0}^{J} \frac{dC_i}{dt}(\varphi_i, \varphi_j)^h + \sum_{i=0}^{J} C_i(\nabla\varphi_i, \nabla\varphi_j) = (f(c^h), \varphi_j)^h,$$
$$\sum_{i=0}^{J} C_i(0)(\varphi_i, \varphi_j) = (c_0, \varphi_j), \quad j = 0,\ldots,J, \tag{4.3.1}$$

where $f(c^h) := [\lambda(r^h) + i\omega(r^h)]c^h$, $r^h \equiv |c^h|$. We have used that the initial approximation $c^h(0) = P^h c_0$ can be expressed in the equivalent form $(c^h(0), \chi^h) = (c_0, \chi^h)$

for all $\chi^h \in \mathbb{S}^h$. To deal with the nonlinearity note that

$$(f(c^h), \varphi_j)^h = \sum_{k=0}^{J} \widehat{M}_{kk} f(C_k)\delta_{jk} = \widehat{M}_{jj} f(C_j). \tag{4.3.2}$$

Thus from (4.3.1), (4.3.2) and noting $(\varphi_i, \varphi_j)^h = \delta_{ij}(1, \varphi_j)$ we have the following system of $(J+1)$ complex ODEs:

$$\widehat{M}\frac{d\boldsymbol{C}}{dt} + K\boldsymbol{C} = \widehat{M}\boldsymbol{f}(\boldsymbol{C}), \quad \boldsymbol{c}^0 := M\boldsymbol{C}(0), \tag{4.3.3}$$

where $\boldsymbol{C} := (C_0, \ldots, C_J)^T$, $\{\boldsymbol{c}^0\}_i := (c_0, \varphi_i)$, $\{\boldsymbol{f}(\boldsymbol{C})\}_i := f(C_i)$.

We simplify this system by writing $f(C_i) = \widehat{f}(C_i)C_i$ where $\widehat{f}(C_i) := \lambda(R_i) + i\omega(R_i)$, $R_i \equiv |C_i|$, so that $\boldsymbol{f}(\boldsymbol{C}) = D\boldsymbol{C}$, $D := \text{diag}\{\widehat{f}(C_0), \ldots, \widehat{f}(C_J)\}$. Now, $\widehat{M}$ is non-singular, so the system (4.3.3) becomes after rearrangement

$$\frac{d\boldsymbol{C}}{dt} = (D - L)\boldsymbol{C}, \quad \boldsymbol{C}(0) = M^{-1}\boldsymbol{c}^0, \tag{4.3.4}$$

where $L := (\widehat{M})^{-1}K$. As f is a locally Lipschitz function (see (2.2.4)), from standard existence theory for systems of ODEs, system (4.3.4) has a unique solution $\boldsymbol{C}$ (and hence (P_2^h) has a unique solution c^h) on some finite time interval $(0, t_h)$, $t_h > 0$.

4.3.2 Global existence of the approximations and Estimate I

To obtain global existence of the semi-discrete approximations we derive an *a priori* estimate bounding u^h, v^h independent of h, thus concluding $t_h = T$ (T independent of h).

Estimate I: The estimate is a discrete analogue of a generalised version of Estimate I in Section 2.4 (see the remark at the end of Section 2.4). Choosing $\boldsymbol{\chi}^h = \pi^h\{|\boldsymbol{u}^h|^m\boldsymbol{u}^h\}$, $m \geq 0$, in (P_1^h) leads to

$$\frac{1}{(m+2)}\frac{d}{dt}|\boldsymbol{u}^h|^{m+2}_{h,m+2} + \lambda_1|\boldsymbol{u}^h|^{\rho+m+2}_{h,\rho+m+2} \leq \lambda_0|\boldsymbol{u}^h|^{m+2}_{h,m+2}, \tag{4.3.5}$$

after noting

$$|\boldsymbol{u}^h|^m\boldsymbol{u}^h \cdot \frac{\partial \boldsymbol{u}^h}{\partial t} = \frac{1}{(m+2)}\frac{\partial}{\partial t}|\boldsymbol{u}^h|^{m+2},$$

(4.2.2), Lemma 4.2.1, and (2.2.2a). Multiplying (4.3.5) through by $(m+2)$ and applying the Grönwall lemma yields

$$|\boldsymbol{u}^h(T)|_{h,m+2}^{m+2} + \lambda_1(m+2)\int_0^T |\boldsymbol{u}^h|_{h,\rho+m+2}^{\rho+m+2}\,dx \le |\boldsymbol{u}^h(0)|_{h,m+2}^{m+2}\,e^{\lambda_0(m+2)T}. \tag{4.3.6}$$

It will be sufficient for existence, uniqueness and stability estimates to choose $m=\rho$ in (4.3.6) giving

$$|\boldsymbol{u}^h(T)|_{h,\rho+2}^{\rho+2} + \lambda_1(\rho+2)\int_0^T |\boldsymbol{u}^h|_{h,2\rho+2}^{2\rho+2}\,dx \le |\boldsymbol{u}^h(0)|_{h,\rho+2}^{\rho+2}\,e^{\lambda_0(\rho+2)T}. \tag{4.3.7}$$

To bound the right hand side note assumption (A1), Lemma 4.2.8, the projection property (4.2.27), and the fact that the initial data is in $H^1(\Omega)$, to give

$$|\boldsymbol{u}^h(0)|_{h,\rho+2}^{\rho+2} \equiv |P^h\boldsymbol{u}_0|_{h,\rho+2}^{\rho+2} \le C\|P^h\boldsymbol{u}_0\|_1^{\rho+2} \le C. \tag{4.3.8}$$

Thus from (4.3.7) we deduce the following bounds hold independent of h:

$$u^h, v^h \in L^{h,2\rho+2}(\Omega_T)\cap L^\infty(0,T;L^{h,\rho+2}(\Omega)). \tag{4.3.9}$$

4.3.3 Uniqueness

The proof is a discrete analogue of the corresponding uniqueness proof in Section 2.6.

Assume there are two semi-discrete solutions $\boldsymbol{u}^h, \boldsymbol{v}^h$ of (P_1^h). Setting $\boldsymbol{\chi}^h = \boldsymbol{w}^h := \boldsymbol{u}^h - \boldsymbol{v}^h$ and subtracting semi-discrete approximations yields

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{w}^h|_h^2 + |\boldsymbol{w}^h|_1^2 = \lambda_0|\boldsymbol{w}^h|_h^2 + \left(|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h - |\boldsymbol{v}^h|^\rho A\boldsymbol{v}^h, \boldsymbol{w}^h\right)^h, \tag{4.3.10}$$

after noting (2.2.2a). We bound the last term in this equation using: (2.2.4), Lemma 4.2.10 with $\eta\in\{\boldsymbol{u}^h,\boldsymbol{v}^h\}$, $\varepsilon=2$, and noting (2.2.2a) again to give

$$\begin{aligned}\left(|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h - |\boldsymbol{v}^h|^\rho A\boldsymbol{v}^h, \boldsymbol{w}^h\right)^h &\le (\rho+1)\sqrt{\lambda_1^2+\omega_1^2}\,(|\boldsymbol{u}^h|^\rho + |\boldsymbol{v}^h|^\rho, |\boldsymbol{w}^h|^2)^h \\ &\le C\left\{1 + C_h\left(|\boldsymbol{u}^h|_{h,\rho+2}^k + |\boldsymbol{v}^h|_{h,\rho+2}^k\right)\right\}\|\boldsymbol{w}^h\|_0^2 + \mu|\boldsymbol{w}^h|_1^2 \\ &\le \widehat{C}_h\|\boldsymbol{w}^h\|_0^2 + \mu|\boldsymbol{w}^h|_1^2,\end{aligned} \tag{4.3.11}$$

where $\widehat{C}_h$ is a positive constant depending on h, after applying a uniform bound in (4.3.9). Note $\mu < 1$ and $k := \rho/(1-\mu)$, so after kickback of $\mu|\boldsymbol{w}^h|_1^2$ we have from (4.3.10) and (4.3.11) that

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{w}^h|_h^2 \leq \widehat{C}_h|\boldsymbol{w}^h|_h^2.$$

Applying the Grönwall lemma yields

$$|\boldsymbol{w}(T)|_h^2 \leq |\boldsymbol{w}(0)|_h^2 \exp(\widehat{C}_h T) \equiv 0,$$

leading to $\boldsymbol{u}^h \equiv \boldsymbol{v}^h$, as required.

4.4 Estimate II

Estimate II: The estimate is a discrete analogue of Estimate III in Section 3.2. Choose $\boldsymbol{\chi}^h = \boldsymbol{u}_t^h$ in (P_1^h), then on noting

$$|\boldsymbol{u}^h|^\rho \frac{\partial}{\partial t}|\boldsymbol{u}^h|^2 = \frac{2}{(\rho+2)}\frac{\partial}{\partial t}|\boldsymbol{u}^h|^{\rho+2},$$

a direct calculation with the aid of (2.2.2a) leads to

$$\begin{aligned}&\left|\frac{\partial \boldsymbol{u}^h}{\partial t}\right|_h^2 + \frac{1}{2}\frac{d}{dt}|\boldsymbol{u}^h|_1^2 + \frac{\lambda_1}{(\rho+2)}\frac{d}{dt}|\boldsymbol{u}^h|_{h,\rho+2}^{\rho+2}\\ &= \frac{\lambda_0}{2}\frac{d}{dt}|\boldsymbol{u}^h|_h^2 + (\omega_0 + \omega_1|\boldsymbol{u}^h|^\rho, u^h v_t^h - v^h u_t^h)^h.\end{aligned} \tag{4.4.1}$$

We apply a simple Young's inequality to the last term in (4.4.1) to give

$$\begin{aligned}(\omega_0 + \omega_1|\boldsymbol{u}^h|^\rho, u^h v_t^h - v^h u_t^h)^h &\equiv \int_\Omega \pi^h\left\{(\omega_0 + \omega_1|\boldsymbol{u}^h|^\rho)\boldsymbol{u}^h \cdot \begin{pmatrix} v_t^h \\ -u_t^h \end{pmatrix}\right\} dx\\ &\leq \int_\Omega \pi^h\Big\{(|\omega_0| + |\omega_1||\boldsymbol{u}^h|^\rho)|\boldsymbol{u}^h||\boldsymbol{u}_t^h|\Big\}\, dx\\ &\leq \omega_0^2|\boldsymbol{u}^h|_h^2 + \omega_1^2|\boldsymbol{u}^h|_{h,2\rho+2}^{2\rho+2} + \frac{1}{2}\left|\frac{\partial \boldsymbol{u}^h}{\partial t}\right|_h^2.\end{aligned} \tag{4.4.2}$$

Combining (4.4.1) and (4.4.2) we have after kickback of $\frac{1}{2}\left|\frac{\partial \boldsymbol{u}^h}{\partial t}\right|_h^2$ and multiplication by 2:

$$\left|\frac{\partial \boldsymbol{u}^h}{\partial t}\right|_h^2 + \frac{d}{dt}|\boldsymbol{u}^h|_1^2 + \frac{2\lambda_1}{(\rho+2)}\frac{d}{dt}|\boldsymbol{u}^h|_{h,\rho+2}^{\rho+2} \leq \lambda_0\frac{d}{dt}|\boldsymbol{u}^h|_h^2 + 2\omega_0^2|\boldsymbol{u}^h|_h^2 + 2\omega_1^2|\boldsymbol{u}^h|_{h,2\rho+2}^{2\rho+2}. \tag{4.4.3}$$

Integrating both sides of (4.4.3) over $t \in (0,T)$ yields

$$\int_0^T \left|\frac{\partial \boldsymbol{u}^h}{\partial t}\right|_h^2 dt + |\boldsymbol{u}^h(T)|_1^2 + \frac{2\lambda_1}{(\rho+2)}|\boldsymbol{u}^h(T)|_{h,\rho+2}^{\rho+2} + \lambda_0|\boldsymbol{u}^h(0)|_h^2$$
$$\leq \lambda_0|\boldsymbol{u}^h(T)|_h^2 + 2\omega_0^2\int_0^T |\boldsymbol{u}^h|_h^2\,dt + 2\omega_1^2\int_0^T |\boldsymbol{u}^h|_{h,2\rho+2}^{2\rho+2}\,dt$$
$$+ |\boldsymbol{u}^h(0)|_1^2 + \frac{2\lambda_1}{(\rho+2)}|\boldsymbol{u}^h(0)|_{h,\rho+2}^{\rho+2}. \tag{4.4.4}$$

Applying the discrete injection results (4.2.13) and (4.2.15), noting (4.3.8) and (4.3.9), we conclude the right hand side of (4.4.4) is bounded uniformly. Thus from the injection $L^{h,\rho+2}(\Omega) \hookrightarrow L^{h,2}(\Omega)$, the H^1 semi-norm bound and (4.3.9), we deduce the following bounds independent of h:

$$u^h, v^h \in L^\infty(0,T;H^1(\Omega)), \qquad \frac{\partial u^h}{\partial t}, \frac{\partial v^h}{\partial t} \in L^2(\Omega_T). \tag{4.4.5}$$

This completes the proof of Lemma 4.2.12. ∎

An important step in the continuous estimates was to introduce the test function $\boldsymbol{\chi}^k = -\Delta \boldsymbol{u}^k$ in the finite dimensional weak form, leading to regularity of the strong solutions. In order for the semi-discrete calculations to mimic this step we would need to introduce a discrete Laplacian operator $\Delta_h : H^1(\Omega) \mapsto S^h$ such that

$$(\Delta_h u, \chi^h)^h = -(\nabla u, \nabla \chi^h) \quad \forall \chi^h \in S^h, \quad u \in H^1(\Omega).$$

It is easy to show that Δ_h is well-defined and is a bounded linear operator from S^h to S^h. The strategy would be to choose $\boldsymbol{\chi}^h = -\Delta_h \boldsymbol{u}^h$ in (P_1^h). However, this would lead to the term $(\nabla\pi^h|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h, \nabla\boldsymbol{u}^h)$ on the right hand side, which is not controlled with the aid of Lemma 4.2.1, as $|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h$ is not a monotonic function of $\boldsymbol{u}^h$. Nevertheless, the results from the previous estimate are sufficient for our later calculations. Taking the test function in Estimate I equal to a monotonic function led to increased 'regularity' of the semi-discrete solutions, allowing us to dispense with a discrete analogue of Estimate II in Section 3.2 (see the comments at the end of Section 2.4).

4.5 An error estimate

We prove an error estimate between the continuous solutions of problem (P_1) and the semi-discrete solutions of (P_1^h), which is optimal in H^1, but sub-optimal in L^2. The classical approach to deriving a semi-discrete error bound is via an elliptic ('Ritz') projection, which can be traced back to Wheeler [92] (see also Thomeé [86]). The use of the Ritz projection for linear problems, or problems with a sufficiently regular nonlinearity, leads to optimal error estimates in L^2 and H^1 [63]. The $\lambda - \omega$ system is highly nonlinear and so instead of the Ritz projection we use the interpolant $\pi^h \boldsymbol{u}$ (e.g., [4], [39]), which facilitated handling of the nonlinearity.

Lemma 4.5.1 Let the results and assumptions of Theorem 4.1.1, Lemma 4.2.12, and (A3) hold. Then the solution $\{u^h, v^h\}$ of (P_1^h) satisfies the following semi-discrete error bound:

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{u} - \boldsymbol{u}^h\|^2_{L^2(0,T;H^1(\Omega))} \leq Ch^2. \tag{4.5.1}$$

Proof. Define

$$\left.\begin{aligned} \boldsymbol{e} &:= \boldsymbol{u} - \boldsymbol{u}^h \\ \boldsymbol{e}^A &:= \boldsymbol{u} - \pi^h \boldsymbol{u} \\ \boldsymbol{e}^h &:= \pi^h \boldsymbol{u} - \boldsymbol{u}^h \end{aligned}\right\} \quad \text{so} \quad \boldsymbol{e} := \boldsymbol{e}^A + \boldsymbol{e}^h. \tag{4.5.2}$$

Note that $\pi^h \boldsymbol{u}$ is well-defined as from the Sobolev embedding theorem $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$, $d \leq 3$ and $u(\cdot,t), v(\cdot,t) \in H^2(\Omega)$ for a.e. $t \in (0,T)$ (see Theorem 4.1.1). To facilitate the proof we note the following inequalities, which are easily proved with the aid of (4.2.16) and (4.5.2):

$$\|\boldsymbol{e}^A\|_0 \leq Ch^2 \|\boldsymbol{u}\|_2, \tag{4.5.3a}$$

$$|\boldsymbol{e}^A|_1 \leq Ch \|\boldsymbol{u}\|_2, \tag{4.5.3b}$$

$$\|\boldsymbol{e}^A\|_1 \leq Ch \|\boldsymbol{u}\|_2, \tag{4.5.3c}$$

$$\|\boldsymbol{e}^h\|_0 \leq \|\boldsymbol{e}\|_0 + Ch^2 \|\boldsymbol{u}\|_2, \tag{4.5.3d}$$

$$|\boldsymbol{e}^h|_1 \leq |\boldsymbol{e}|_1 + Ch \|\boldsymbol{u}\|_2, \tag{4.5.3e}$$

$$\|\boldsymbol{e}^h\|_1 \leq \|\boldsymbol{e}\|_0 + |\boldsymbol{e}|_1 + Ch \|\boldsymbol{u}\|_2. \tag{4.5.3f}$$

We choose $\boldsymbol{\eta} = \boldsymbol{e}^h$ in (P$_1$), $\boldsymbol{\chi}^h = \boldsymbol{e}^h$ in (P$_1^h$) and subtract yielding:

$$(\boldsymbol{u}_t, \boldsymbol{e}^h) - (\boldsymbol{u}_t^h, \boldsymbol{e}^h)^h + (\nabla \boldsymbol{e}, \nabla \boldsymbol{e}^h) = (B\boldsymbol{u}, \boldsymbol{e}^h) - (B\boldsymbol{u}^h, \boldsymbol{e}^h)^h + (|\boldsymbol{u}|^\rho A\boldsymbol{u}, \boldsymbol{e}^h) - (|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h, \boldsymbol{e}^h)^h. \tag{4.5.4}$$

Adding and subtracting each of the terms $(B\boldsymbol{u}^h, \boldsymbol{e}^h)$, $(|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h, \boldsymbol{e}^h)$ and $(\boldsymbol{u}_t^h, \boldsymbol{e}^h)$ to (4.5.4) and rearranging leads to

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}\|_0^2 + |\boldsymbol{e}|_1^2 &= \left\{(\boldsymbol{u}_t^h, \boldsymbol{e}^h)^h - (\boldsymbol{u}_t^h, \boldsymbol{e}^h)\right\} + \left(\frac{\partial \boldsymbol{e}}{\partial t}, \boldsymbol{e}^A\right) + (\nabla \boldsymbol{e}, \nabla \boldsymbol{e}^A) + (B\boldsymbol{e}, \boldsymbol{e}^h) \\
&+ \left\{(B\boldsymbol{u}^h, \boldsymbol{e}^h) - (B\boldsymbol{u}^h, \boldsymbol{e}^h)^h\right\} + (|\boldsymbol{u}|^\rho A\boldsymbol{u} - |\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h, \boldsymbol{e}^h) \\
&+ \left\{(|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h, \boldsymbol{e}^h) - (|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h, \boldsymbol{e}^h)^h\right\} \\
&=: \sum_{i=1}^{7} \mathrm{T}_i.
\end{aligned} \tag{4.5.5}$$

We bound each term on the right hand side of (4.5.5) separately.

Using (4.2.20), the Young's inequality (2.1.7) with $\varepsilon = 8$, and (4.5.3e) yields

$$\begin{aligned}
\mathrm{T}_1 &\equiv (\boldsymbol{u}_t^h, \boldsymbol{e}^h)^h - (\boldsymbol{u}_t^h, \boldsymbol{e}^h) \\
&\leq Ch\|\boldsymbol{u}_t^h\|_0|\boldsymbol{e}^h|_1 \\
&\leq Ch^2\|\boldsymbol{u}_t^h\|_0^2 + \frac{1}{16}|\boldsymbol{e}^h|_1^2 \\
&\leq Ch^2\|\boldsymbol{u}_t^h\|_0^2 + \frac{1}{8}|\boldsymbol{e}|_1^2 + Ch^2\|\boldsymbol{u}\|_2^2.
\end{aligned} \tag{4.5.6}$$

With the aid of the Cauchy-Schwarz inequality and (4.5.3a) we have

$$\mathrm{T}_2 \equiv \left(\frac{\partial \boldsymbol{e}}{\partial t}, \boldsymbol{e}^A\right) \leq \left\|\frac{\partial \boldsymbol{e}}{\partial t}\right\|_0 \|\boldsymbol{e}^A\|_0 \leq Ch^2\|\boldsymbol{u}\|_2 \left\|\frac{\partial \boldsymbol{e}}{\partial t}\right\|_0, \tag{4.5.7}$$

and also noting the Young's inequality (2.1.7) with $\varepsilon = 8$, (4.5.3b), and the Cauchy-Schwarz inequality yields

$$\mathrm{T}_3 \equiv (\nabla \boldsymbol{e}, \nabla \boldsymbol{e}^A) \leq |\boldsymbol{e}|_1|\boldsymbol{e}^A|_1 \leq Ch\|\boldsymbol{u}\|_2|\boldsymbol{e}|_1 \leq Ch^2\|\boldsymbol{u}\|_2^2 + \frac{1}{16}|\boldsymbol{e}|_1^2. \tag{4.5.8}$$

Noting the Cauchy-Schwarz inequality, (2.2.2b), a Young's inequality, and (4.5.3d) leads to

$$\mathrm{T}_4 \equiv (B\boldsymbol{e}, \boldsymbol{e}^h) \leq C\|\boldsymbol{e}\|_0\|\boldsymbol{e}^h\|_0 \leq C\|\boldsymbol{e}\|_0^2 + Ch^4\|\boldsymbol{u}\|_2^2. \tag{4.5.9}$$

To bound the fifth term we use: (4.2.20), (2.2.2b), a stability bound in (4.4.5), the Young's inequality (2.1.7) with $\varepsilon = 8$, and (4.5.3e) to give

$$\begin{aligned}
\mathrm{T}_5 &\leq |(B\boldsymbol{u}^h, \boldsymbol{e}^h) - (B\boldsymbol{u}^h, \boldsymbol{e}^h)^h| \\
&\leq Ch\|B\boldsymbol{u}^h\|_0|\boldsymbol{e}^h|_1 \\
&\leq Ch\|\boldsymbol{u}^h\|_0|\boldsymbol{e}^h|_1 \\
&\leq Ch|\boldsymbol{e}^h|_1 \\
&\leq Ch^2 + \frac{1}{16}|\boldsymbol{e}^h|_1^2 \\
&\leq Ch^2 + \frac{1}{8}|\boldsymbol{e}|_1^2 + Ch^2\|\boldsymbol{u}\|_2^2. \qquad (4.5.10)
\end{aligned}$$

Using (2.2.4), a generalised Hölder inequality (see Section 2.1), the continuous injections $H^1 \hookrightarrow L^{3\rho}$, $H^1 \hookrightarrow L^6$, assumption (A3), Theorem 4.1.1, (4.4.5), the Young's inequality (2.1.7) with $\varepsilon = 8$, and (4.5.3f) leads to

$$\begin{aligned}
\mathrm{T}_6 &\leq \int_\Omega \left||\boldsymbol{u}|^\rho A\boldsymbol{u} - |\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h\right| |\boldsymbol{e}^h| \, dx \\
&\leq C\int_\Omega \left(|\boldsymbol{u}|^\rho + |\boldsymbol{u}^h|^\rho\right) |\boldsymbol{e}||\boldsymbol{e}^h| \, dx \\
&\leq C\left(\|\boldsymbol{u}\|_{0,3\rho}^\rho + \|\boldsymbol{u}^h\|_{0,3\rho}^\rho\right) \|\boldsymbol{e}\|_0\|\boldsymbol{e}^h\|_{0,6} \\
&\leq C\left(\|\boldsymbol{u}\|_1^\rho + \|\boldsymbol{u}^h\|_1^\rho\right) \|\boldsymbol{e}\|_0\|\boldsymbol{e}^h\|_1 \\
&\leq C\|\boldsymbol{e}\|_0\|\boldsymbol{e}^h\|_1 \\
&\leq C\|\boldsymbol{e}\|_0^2 + \frac{1}{16}\|\boldsymbol{e}^h\|_1^2 \\
&\leq C\|\boldsymbol{e}\|_0^2 + \frac{1}{8}|\boldsymbol{e}|_1^2 + Ch^2\|\boldsymbol{u}\|_2^2. \qquad (4.5.11)
\end{aligned}$$

Bounding the final term is more technical than the calculations for the previous terms. First noting (4.2.17) and (2.1.1) we write

$$\mathrm{T}_7 \leq \int_\Omega \left|(I - \pi^h)|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h \cdot \boldsymbol{e}^h\right| \, dx \leq Ch^2 \sum_{i,j=1}^d \int_\Omega \left|\frac{\partial^2(|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h \cdot \boldsymbol{e}^h)}{\partial x_i \partial x_j}\right| \, dx =: \mathrm{T}_{7,1}, \qquad (4.5.12)$$

(see the comment after the proof of Lemma 4.2.7). To expand the right hand side of (4.5.12) we need the following identities, which are easily verified for arbitrary differentiable vector valued functions $\boldsymbol{\psi}$, $\boldsymbol{\phi}$, $\boldsymbol{\eta} \in \mathbb{R}^2$ and all $p \in \mathbb{R}$:

$$\frac{\partial}{\partial x_k}|\boldsymbol{\psi}|^p = p|\boldsymbol{\psi}|^{p-2}\left(\boldsymbol{\psi}\cdot\frac{\partial \boldsymbol{\psi}}{\partial x_k}\right), \tag{4.5.13}$$

$$\frac{\partial}{\partial x_k}\left(|\boldsymbol{\psi}|^p A\boldsymbol{\phi}\cdot\boldsymbol{\eta}\right) = |\boldsymbol{\psi}|^p\left(A\boldsymbol{\phi}\cdot\frac{\partial \boldsymbol{\eta}}{\partial x_k}\right) + |\boldsymbol{\psi}|^p\left(\boldsymbol{\eta}\cdot A\frac{\partial \boldsymbol{\phi}}{\partial x_k}\right) + p|\boldsymbol{\psi}|^{p-2}\left(\boldsymbol{\psi}\cdot\frac{\partial \boldsymbol{\psi}}{\partial x_k}\right)(A\boldsymbol{\phi}\cdot\boldsymbol{\eta}). \tag{4.5.14}$$

Thus

$$\frac{\partial}{\partial x_i}\left(|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h\cdot\boldsymbol{e}^h\right) = |\boldsymbol{u}^h|^\rho\left(A\boldsymbol{u}^h\cdot\frac{\partial \boldsymbol{e}^h}{\partial x_i}\right) + |\boldsymbol{u}^h|^\rho\left(\boldsymbol{e}^h\cdot A\frac{\partial \boldsymbol{u}^h}{\partial x_i}\right) + \rho|\boldsymbol{u}^h|^{\rho-2}\left(\boldsymbol{u}^h\cdot\frac{\partial \boldsymbol{u}^h}{\partial x_i}\right)(A\boldsymbol{u}^h\cdot\boldsymbol{e}^h), \tag{4.5.15}$$

$$\frac{\partial}{\partial x_j}\left(|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h\cdot\frac{\partial \boldsymbol{e}^h}{\partial x_i}\right) = |\boldsymbol{u}^h|^\rho\left(\frac{\partial \boldsymbol{e}^h}{\partial x_i}\cdot A\frac{\partial \boldsymbol{u}^h}{\partial x_j}\right) + \rho|\boldsymbol{u}^h|^{\rho-2}\left(\boldsymbol{u}^h\cdot\frac{\partial \boldsymbol{u}^h}{\partial x_j}\right)\left(A\boldsymbol{u}^h\cdot\frac{\partial \boldsymbol{e}^h}{\partial x_i}\right), \tag{4.5.16}$$

$$\frac{\partial}{\partial x_j}\left(|\boldsymbol{u}^h|^\rho A\frac{\partial \boldsymbol{u}^h}{\partial x_i}\cdot\boldsymbol{e}^h\right) = |\boldsymbol{u}^h|^\rho\left(A\frac{\partial \boldsymbol{u}^h}{\partial x_i}\cdot\frac{\partial \boldsymbol{e}^h}{\partial x_j}\right) + \rho|\boldsymbol{u}^h|^{\rho-2}\left(\boldsymbol{u}^h\cdot\frac{\partial \boldsymbol{u}^h}{\partial x_j}\right)\left(A\frac{\partial \boldsymbol{u}^h}{\partial x_i}\cdot\boldsymbol{e}^h\right). \tag{4.5.17}$$

Using the simple product rule $(abc)' = a'bc + ab'c + abc'$ we have

$$\begin{aligned}
&\frac{\partial}{\partial x_j}\left[\rho|\boldsymbol{u}^h|^{\rho-2}\left(\boldsymbol{u}^h\cdot\frac{\partial \boldsymbol{u}^h}{\partial x_i}\right)(A\boldsymbol{u}^h\cdot\boldsymbol{e}^h)\right]\\
&= \rho(\rho-2)|\boldsymbol{u}^h|^{\rho-4}\left(\boldsymbol{u}^h\cdot\frac{\partial \boldsymbol{u}^h}{\partial x_j}\right)\left(\boldsymbol{u}^h\cdot\frac{\partial \boldsymbol{u}^h}{\partial x_i}\right)(A\boldsymbol{u}^h\cdot\boldsymbol{e}^h)\\
&+ \rho|\boldsymbol{u}^h|^{\rho-2}\left(\frac{\partial \boldsymbol{u}^h}{\partial x_i}\cdot\frac{\partial \boldsymbol{u}^h}{\partial x_j}\right)(A\boldsymbol{u}^h\cdot\boldsymbol{e}^h)\\
&+ \rho|\boldsymbol{u}^h|^{\rho-2}\left(\boldsymbol{u}^h\cdot\frac{\partial \boldsymbol{u}^h}{\partial x_i}\right)\left(A\boldsymbol{u}^h\cdot\frac{\partial \boldsymbol{e}^h}{\partial x_j} + \boldsymbol{e}^h\cdot A\frac{\partial \boldsymbol{u}^h}{\partial x_j}\right).
\end{aligned} \tag{4.5.18}$$

Combining (4.5.15)-(4.5.18) leads to

$$\begin{aligned}
&\frac{\partial^2}{\partial x_i\partial x_j}\left(|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h\cdot\boldsymbol{e}^h\right)\\
&= |\boldsymbol{u}^h|^\rho\left(\frac{\partial \boldsymbol{e}^h}{\partial x_i}\cdot A\frac{\partial \boldsymbol{u}^h}{\partial x_j}\right) + \rho|\boldsymbol{u}^h|^{\rho-2}\left(\boldsymbol{u}^h\cdot\frac{\partial \boldsymbol{u}^h}{\partial x_j}\right)\left(A\boldsymbol{u}^h\cdot\frac{\partial \boldsymbol{e}^h}{\partial x_i}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ |\boldsymbol{u}^h|^\rho \left(A \frac{\partial \boldsymbol{u}^h}{\partial x_i} \cdot \frac{\partial \boldsymbol{e}^h}{\partial x_j} \right) + \rho |\boldsymbol{u}^h|^{\rho-2} \left(\boldsymbol{u}^h \cdot \frac{\partial \boldsymbol{u}^h}{\partial x_j} \right) \left(A \frac{\partial \boldsymbol{u}^h}{\partial x_i} \cdot \boldsymbol{e}^h \right) \\
&+ \rho(\rho-2) |\boldsymbol{u}^h|^{\rho-4} \left(\boldsymbol{u}^h \cdot \frac{\partial \boldsymbol{u}^h}{\partial x_j} \right) \left(\boldsymbol{u}^h \cdot \frac{\partial \boldsymbol{u}^h}{\partial x_i} \right) (A\boldsymbol{u}^h \cdot \boldsymbol{e}^h) \\
&+ \rho |\boldsymbol{u}^h|^{\rho-2} \left(\frac{\partial \boldsymbol{u}^h}{\partial x_i} \cdot \frac{\partial \boldsymbol{u}^h}{\partial x_j} \right) (A\boldsymbol{u}^h \cdot \boldsymbol{e}^h) \\
&+ \rho |\boldsymbol{u}^h|^{\rho-2} \left(\boldsymbol{u}^h \cdot \frac{\partial \boldsymbol{u}^h}{\partial x_i} \right) \left(A\boldsymbol{u}^h \cdot \frac{\partial \boldsymbol{e}^h}{\partial x_j} + \boldsymbol{e}^h \cdot A \frac{\partial \boldsymbol{u}^h}{\partial x_j} \right). \qquad (4.5.19)
\end{aligned}
$$

Thus from (4.5.12), (4.5.19) and (2.2.2b) we obtain

$$
\begin{aligned}
\mathrm{T}_{7,1} \leq Ch^2 \sum_{i,j=1}^{d} \int_\Omega & \left\{ |\boldsymbol{u}^h|^\rho \left| \frac{\partial \boldsymbol{e}^h}{\partial x_i} \right| \left| \frac{\partial \boldsymbol{u}^h}{\partial x_j} \right| + |\boldsymbol{u}^h|^\rho \left| \frac{\partial \boldsymbol{e}^h}{\partial x_j} \right| \left| \frac{\partial \boldsymbol{u}^h}{\partial x_i} \right| \right. \\
& \left. + |\boldsymbol{u}^h|^{\rho-1} \left| \frac{\partial \boldsymbol{u}^h}{\partial x_j} \right| \left| \frac{\partial \boldsymbol{u}^h}{\partial x_i} \right| |\boldsymbol{e}^h| \right\} dx \\
=: \mathrm{T}_{7,2}. & \qquad (4.5.20)
\end{aligned}
$$

To bound the right hand side of (4.5.20) note: a generalised Hölder inequality, $\|\partial\zeta/\partial x_k\|_{0,3} \leq |\zeta|_{1,3}$ for all ζ in $W^{1,3}$, assumption (A3), the injections $H^1 \hookrightarrow L^{3\rho}$ and $H^1 \hookrightarrow L^{6\rho-6}$, the inverse inequality (4.2.18c) in the form $|\chi^h|_{1,3} \leq Ch^{-d/6}|\chi^h|_{1,2}$, an estimate in (4.4.5), (4.5.3f), and the simple Young's inequality (2.1.7) with $\varepsilon = 8$ again to obtain

$$
\begin{aligned}
\mathrm{T}_{7,2} \leq Ch^2 \sum_{i,j=1}^{d} & \left\{ \|\boldsymbol{u}^h\|_{0,3\rho}^\rho \left\| \frac{\partial \boldsymbol{u}^h}{\partial x_j} \right\|_{0,3} \left\| \frac{\partial \boldsymbol{e}^h}{\partial x_i} \right\|_{0,3} + \|\boldsymbol{u}^h\|_{0,3\rho}^\rho \left\| \frac{\partial \boldsymbol{u}^h}{\partial x_i} \right\|_{0,3} \left\| \frac{\partial \boldsymbol{e}^h}{\partial x_j} \right\|_{0,3} \right. \\
& \left. + \|\boldsymbol{u}^h\|_{0,6\rho-6}^{\rho-1} \|\boldsymbol{e}^h\|_{0,6} \left\| \frac{\partial \boldsymbol{u}^h}{\partial x_i} \right\|_{0,3} \left\| \frac{\partial \boldsymbol{u}^h}{\partial x_j} \right\|_{0,3} \right\} \\
& \leq Ch^2 \left\{ \|\boldsymbol{u}^h\|_{0,3\rho}^\rho |\boldsymbol{u}^h|_{1,3} |\boldsymbol{e}^h|_{1,3} + \|\boldsymbol{u}^h\|_{0,6\rho-6}^{\rho-1} \|\boldsymbol{e}^h\|_{0,6} |\boldsymbol{u}^h|_{1,3}^2 \right\} \\
& \leq Ch^{2-\frac{d}{3}} \left\{ \|\boldsymbol{u}^h\|_1^\rho |\boldsymbol{u}^h|_1 |\boldsymbol{e}^h|_1 + \|\boldsymbol{u}^h\|_1^{\rho-1} |\boldsymbol{u}^h|_1^2 \|\boldsymbol{e}^h\|_1 \right\} \\
& \leq Ch^{2-\frac{d}{3}} \|\boldsymbol{e}^h\|_1 \\
& \leq Ch^{2-\frac{d}{3}} (\|\boldsymbol{e}\|_0 + |\boldsymbol{e}|_1 + Ch\|\boldsymbol{u}\|_2) \\
& \leq Ch^{2-\frac{d}{3}} (\|\boldsymbol{e}\|_0 + Ch\|\boldsymbol{u}\|_2) + Ch^{2-\frac{d}{3}} |\boldsymbol{e}|_1 \\
& \leq Ch^{4-\frac{2d}{3}} + \|\boldsymbol{e}\|_0^2 + Ch^2\|\boldsymbol{u}\|_2^2 + \frac{1}{16}|\boldsymbol{e}|_1^2 \\
& =: \mathrm{T}_{7,3} \qquad (4.5.21)
\end{aligned}
$$

Thus from: (4.5.5)-(4.5.12), (4.5.20), (4.5.21), a kickback of $\frac{1}{2}|e|_1^2$, and noting that $h^{4-\frac{2d}{3}} \leq h^2$ as $h \leq 1$, $d \leq 3$, we have after multiplying through by 2

$$\frac{d}{dt}\|e\|_0^2 + |e|_1^2 \leq C\left(\|e\|_0^2 + h^2\|u_t^h\|_0^2 + h^2\|u\|_2^2 + h^2\|u\|_2\left\|\frac{\partial e}{\partial t}\right\|_0 + h^2\right). \quad (4.5.22)$$

Applying the Grönwall lemma and Cauchy-Schwarz to (4.5.22) yields

$$\begin{aligned}\|e(T)\|_0^T + \int_0^T |e|_1^2\,dt &\leq \exp(CT)\|e(0)\|_0^2 \\ &+ Ch^2\exp(CT)\int_0^T\left(1 + \|u_t^h\|_0^2 + \|u\|_2^2 + \|u\|_2\left\|\frac{\partial e}{\partial t}\right\|_0\right)dt \\ &\leq C\|e(0)\|_0^2 + Ch^2\Big(T + \|u_t^h\|_{L^2(\Omega_T)}^2 + \|u\|_{L^2(0,T;H^2(\Omega))}^2 \\ &+ \|u\|_{L^2(0,T;H^2(\Omega))}\left\|\frac{\partial e}{\partial t}\right\|_{L^2(\Omega_T)}\Big). \end{aligned} \quad (4.5.23)$$

To bound the right hand side of (4.5.23) first note from the projection property (4.2.26) that

$$\|e(0)\|_0^2 \equiv \|u_0 - P^h u_0\|_0^2 \leq Ch^2|u_0|_1^2 \leq Ch^2,$$

after recalling that $u_0, v_0 \in H^1(\Omega)$. Furthermore, from Theorem 4.1.1 and Lemma 4.2.12 the *a priori* bound becomes

$$\|e(T)\|_0^2 + \int_0^T |e|_1^2\,dt \leq Ch^2. \quad (4.5.24)$$

So $\|e\|_0 \leq Ch$ for a.e. $t \in (0,T)$, which implies $\|e\|_{L^\infty(0,T;L^2(\Omega))} \leq Ch$. However, $L^\infty(0,T;L^2(\Omega)) \hookrightarrow L^2(\Omega_T)$ and so we also have $\|e\|_{L^2(\Omega_T)} \leq Ch$. Thus with the semi-norm bound in (4.5.24) we deduce $\|e\|_{L^2(0,T;H^1(\Omega))} \leq Ch$, i.e.,

$$\|u - u^h\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|u - u^h\|_{L^2(0,T;H^1(\Omega))}^2 \leq Ch^2. \quad \blacksquare$$

A corollary to the error bound is the convergence of the semi-discrete approximations to the strong solutions:

$$\{u^h, v^h\} \to \{u, v\} \quad \text{in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \text{ as } h \to 0.$$

Chapter 5

A fully-discrete approximation

In this chapter we discretise the $\lambda - \omega$ system in space using a finite element method (see Chapter 4) and discretise in time using finite differences, leading eventually to the proof of a fully-discrete error estimate.

In Section 5.1 we briefly cover the assumptions and results needed for the subsequent analysis and present a fully-discrete, semi-implicit in time, finite element approximation. In Section 5.2 we prove the existence and uniqueness of the fully-discrete approximations, while in Section 5.3 two stability estimates are proved. Finally, in Section 5.5 we prove a fully-discrete error estimate. The basic approach to obtain estimates is to mimic the semi-discrete estimates of Chapter 4.

5.1 Notation and preliminaries

We let the assumptions and results of Chapter 4 apply. We shall also need the following discrete analogue of the Grönwall lemma:

Lemma 5.1.1 Assume $w_n, \alpha_n, p_n \geq 0$, $0 \leq \beta < 1$, satisfy

$$w_n + p_n \leq \alpha_n + \beta \sum_{k=0}^{n-1} w_{k+1}, \quad \forall n \geq 1, \tag{5.1.1a}$$

$$w_0 + p_0 \leq \alpha_0, \tag{5.1.1b}$$

where $\{\alpha_n\}$ is non-decreasing. Then

$$w_n + \frac{p_n}{1-\beta} \le \left(\frac{\alpha_n - \beta w_0}{1-\beta}\right) \exp\left(\frac{n\beta}{1-\beta}\right).^{1} \tag{5.1.2}$$

Proof. We adapt a proof for a similar result (Lemma 10.5 in [86]). From (5.1.1a) we have

$$w_n + p_n \le \alpha_n + \beta \sum_{k=0}^{n-1} w_k - \beta w_0 + \beta w_n,$$

and as $\beta < 1$ this inequality rearranges to

$$w_n + \widehat{p}_n \le \widehat{\alpha}_n + \widehat{\beta} \sum_{k=0}^{n-1} w_k \quad \forall n \ge 1 \quad \text{where}$$

$$\widehat{p}_n := \frac{p_n}{1-\beta}, \quad \widehat{\alpha}_n := \frac{\alpha_n - \beta w_0}{1-\beta}, \quad \widehat{\beta} := \frac{\beta}{1-\beta}. \tag{5.1.3}$$

Now fix $n \ge 1$ and define

$$u_m := \widehat{\alpha}_n + \widehat{\beta} \sum_{k=0}^{m-1} w_k, \quad 1 \le m \le n, \quad u_0 := \widehat{\alpha}_n. \tag{5.1.4}$$

From (5.1.3) and (5.1.4) we have

$$w_m + \widehat{p}_m \le \widehat{\alpha}_m + \widehat{\beta} \sum_{k=0}^{m-1} w_k \le \widehat{\alpha}_n + \widehat{\beta} \sum_{k=0}^{m-1} w_k \equiv u_m, \tag{5.1.5}$$

as by assumption $\{\alpha_n\}$ and hence $\{\widehat{\alpha}_n\}$ is non-decreasing. Now

$$\begin{aligned} u_n &= u_{n-1} + \widehat{\beta}\, w_{n-1} \\ &\le (1+\widehat{\beta})\, u_{n-1} \\ &\le \exp(\widehat{\beta})\, u_{n-1}, \end{aligned} \tag{5.1.6}$$

after noting from (5.1.5) that $w_{n-1} \le u_{n-1}$,[2] and that $1 + x \le \exp(x)$, $x \ge 0$. It follows inductively from (5.1.6) and noting (5.1.4) that

$$u_n \le u_0 \exp(n\widehat{\beta}) \equiv \widehat{\alpha}_n \exp(n\widehat{\beta}),$$

thus taking $m = n$ in (5.1.5) yields (5.1.2), as required. ∎

Let N be a positive integer and $\Delta t := T/N$ be the time step. We consider the following fully-discrete, semi-implicit in time, finite element approximation of (P_1) (see (2.2.7)):

[1] Taking $n = 0$ in (5.1.2) leads to $w_0 + p_0 \le \alpha_0$, which is true by assumption.

[2] This also holds for $n = 1$, as from (5.1.1b) and (5.1.4) $w_0 + p_0 \le \alpha_0 \le \alpha_n \le \widehat{\alpha}_n \equiv u_0$.

$(\mathbf{P}_1^{h,\Delta t})$ For $n = 1, \dots, N$ find $\boldsymbol{U}^n \in \{S^h\}^2$ such that $\boldsymbol{U}^0 := P^h \boldsymbol{u}_0$ and

$$\left(\frac{\boldsymbol{U}^n - \boldsymbol{U}^{n-1}}{\Delta t}, \boldsymbol{\chi}^h\right)^h + (\nabla \boldsymbol{U}^n, \nabla \boldsymbol{\chi}^h) = (B\boldsymbol{U}^n, \boldsymbol{\chi}^h)^h + (|\boldsymbol{U}^{n-1}|^\rho A\boldsymbol{U}^n, \boldsymbol{\chi}^h)^h, \quad (5.1.7)$$

for all $\boldsymbol{\chi}^h \in \{S^h\}^2$ where $\boldsymbol{U}^n := (U^n, V^n)^T$. Before considering error estimates for the fully-discrete approximation we prove the following stability lemma:

Lemma 5.1.2 Let the results and assumptions of Lemma 4.5.1 hold and $\Delta t < \min\{1, \frac{1}{\lambda_0(2\rho+2)}\}$. Then for all $h \leq 1$ there exists a unique solution to $(\mathrm{P}_1^{h,\Delta t})$ such that

$$\max_{1 \leq n \leq N} \|\boldsymbol{U}^n\|_1 \leq C, \quad (5.1.8)$$

$$\sum_{n=1}^{N} |\boldsymbol{U}^n - \boldsymbol{U}^{n-1}|_h^2 \leq C\Delta t, \quad (5.1.9)$$

$$\sum_{n=1}^{N} |\boldsymbol{U}^n - \boldsymbol{U}^{n-1}|_1^2 \leq C. \quad (5.1.10)$$

Proof. We separate the proof into two parts, showing existence and uniqueness, followed by two stability estimates.

5.2 Existence and uniqueness of the approximations

The linear system $(\mathrm{P}_1^{h,\Delta t})$ can be written as a square matrix system

$$M_{n-1}\boldsymbol{U}^n = \boldsymbol{U}^{n-1}, \quad \boldsymbol{U}^0 := \boldsymbol{U}_0,$$

where M_{n-1} is the coefficient matrix depending on the solution at time $(n-1)\Delta t$, and so existence of the fully-discrete approximation follows from the well-known fact that for a square linear system existence is equivalent to uniqueness.

To prove uniqueness assume there are two fully-discrete solutions $\boldsymbol{U}^n, \boldsymbol{V}^n$ $(n \geq 1)$ of $(\mathrm{P}_1^{h,\Delta t})$. We use proof by induction. Assume uniqueness of the approximation at time $t_{n-1} := (n-1)\Delta t$ and note that we have uniqueness at time t_0. Now setting $\boldsymbol{\chi}^h = \boldsymbol{W}^n := \boldsymbol{U}^n - \boldsymbol{V}^n$ and subtracting the fully-discrete approximations yields on noting (2.2.2a)

$$\frac{1}{\Delta t}|\boldsymbol{W}^n|_h^2 + |\boldsymbol{W}^n|_1^2 + \lambda_1 \left(|\boldsymbol{U}^{n-1}|^\rho, |\boldsymbol{W}^n|^2\right)^h = \lambda_0 |\boldsymbol{W}^n|_h^2. \qquad (5.2.1)$$

By assumption $\lambda_0 < 1/\Delta t$ so

$$C(\Delta t)\,|\boldsymbol{W}^n|_h^2 + |\boldsymbol{W}^n|_1^2 \le 0,$$

where $C(\Delta t)$ is a positive constant depending on Δt. We thus conclude $\boldsymbol{U}^n \equiv \boldsymbol{V}^n$ for all $n \ge 1$ as required.

5.3 Estimate III

Estimate III: The estimate is a fully-discrete analogue of Estimate I in Section 4.3. Choosing $\boldsymbol{\chi}^h = \pi^h\{|\boldsymbol{U}^n|^m \boldsymbol{U}^n\}$, $m \ge 0$, in $(\mathrm{P}_1^{h,\Delta t})$ and noting Lemma 4.2.1, (4.2.2), and (2.2.2a) yields

$$\frac{1}{\Delta t}\left(|\boldsymbol{U}^n|^m, \boldsymbol{U}^n \cdot (\boldsymbol{U}^n - \boldsymbol{U}^{n-1})\right)^h + \lambda_1 \left(|\boldsymbol{U}^{n-1}|^\rho, |\boldsymbol{U}^n|^{m+2}\right)^h \le \lambda_0 |\boldsymbol{U}^n|_{h,m+2}^{m+2}. \qquad (5.3.1)$$

Applying the monotonicity property of Lemma 2.2.2 to the first term in (5.3.1) and multiplying through by $\Delta t(m+2)$ leads to

$$\begin{aligned}&\left[|\boldsymbol{U}^n|_{h,m+2}^{m+2} - |\boldsymbol{U}^{n-1}|_{h,m+2}^{m+2}\right] + \lambda_1 \Delta t(m+2)\left(|\boldsymbol{U}^{n-1}|^\rho, |\boldsymbol{U}^n|^{m+2}\right)^h \\ &\le \lambda_0 \Delta t(m+2)|\boldsymbol{U}^n|_{h,m+2}^{m+2}. \qquad (5.3.2)\end{aligned}$$

Summing both sides of (5.3.2) from $n = 1, \ldots, N$ yields

$$\begin{aligned}&|\boldsymbol{U}^N|_{h,m+2}^{m+2} + \lambda_1 \Delta t(m+2)\sum_{n=1}^{N}\left(|\boldsymbol{U}^{n-1}|^\rho, |\boldsymbol{U}^n|^{m+2}\right)^h \\ &\le |\boldsymbol{U}^0|_{h,m+2}^{m+2} + \lambda_0 \Delta t(m+2)\sum_{n=0}^{N-1}|\boldsymbol{U}^{n+1}|_{h,m+2}^{m+2}. \qquad (5.3.3)\end{aligned}$$

Applying the discrete Grönwall lemma to (5.3.3) for $\Delta t < \frac{1}{\lambda_0(m+2)}$ gives

$$\begin{aligned}&|\boldsymbol{U}^N|_{h,m+2}^{m+2} + \left(\frac{\lambda_1 \Delta t(m+2)}{1 - \lambda_0 \Delta t(m+2)}\right)\sum_{n=1}^{N}\left(|\boldsymbol{U}^{n-1}|^\rho, |\boldsymbol{U}^n|^{m+2}\right)^h \\ &\le |\boldsymbol{U}^0|_{h,m+2}^{m+2}\exp\left(\frac{\lambda_0(m+2)T}{1 - \lambda_0 \Delta t(m+2)}\right), \quad N\Delta t \equiv T. \qquad (5.3.4)\end{aligned}$$

We choose $m = 2\rho$ in (5.3.4). To bound the right hand side of (5.3.4), note assumption (A2), Lemma 4.2.8, and Lemma 4.2.11 to give (cf. (4.3.8))

$$|\boldsymbol{U}^0|_{h,2\rho+2}^{2\rho+2} \equiv |P^h\boldsymbol{u}_0|_{h,2\rho+2}^{2\rho+2} \leq C\|P^h\boldsymbol{u}_0\|_1^{2\rho+2} \leq C.$$

This leads to the following stability bound after noting the assumption on Δt:

$$\max_{1\leq n\leq N} |\boldsymbol{U}^n|_{h,2\rho+2} \leq C. \tag{5.3.5}$$

From the discrete injection property (4.2.13) it follows that $|\boldsymbol{U}^N|_h \leq C|\boldsymbol{U}^N|_{h,2\rho+2}$ and hence we also have

$$\max_{1\leq n\leq N} |\boldsymbol{U}^n|_h \leq C. \tag{5.3.6}$$

5.4 Estimate IV

The estimate is a fully-discrete analogue of Estimate II in Section 4.4. Choosing $\boldsymbol{\chi}^h = (\boldsymbol{U}^n - \boldsymbol{U}^{n-1})/\Delta t$ in $(\mathrm{P}_1^{h,\Delta t})$ leads to

$$\begin{aligned}\left|\frac{\boldsymbol{U}^n - \boldsymbol{U}^{n-1}}{\Delta t}\right|_h^2 &+ \frac{1}{2\Delta t}\left(|\boldsymbol{U}^n - \boldsymbol{U}^{n-1}|_1^2 + |\boldsymbol{U}^n|_1^2 - |\boldsymbol{U}^{n-1}|_1^2\right) \\ &= \left(B\boldsymbol{U}^n + |\boldsymbol{U}^{n-1}|^\rho A\boldsymbol{U}^n, \frac{\boldsymbol{U}^n - \boldsymbol{U}^{n-1}}{\Delta t}\right)^h,\end{aligned} \tag{5.4.1}$$

where we have used the elementary identity

$$2b(b-a) \equiv |b-a|^2 + b^2 - a^2 \quad \forall a, b.$$

We apply a simple Young's inequality to the last term in (5.4.1) after noting (2.2.2b) to give

$$\begin{aligned}\left(B\boldsymbol{U}^n + |\boldsymbol{U}^{n-1}|^\rho A\boldsymbol{U}^n, \frac{\boldsymbol{U}^n - \boldsymbol{U}^{n-1}}{\Delta t}\right)^h &\leq \left(\left|B\boldsymbol{U}^n + |\boldsymbol{U}^{n-1}|^\rho A\boldsymbol{U}^n\right|, \left|\frac{\boldsymbol{U}^n - \boldsymbol{U}^{n-1}}{\Delta t}\right|\right)^h \\ &\leq C|\boldsymbol{U}^n|_h^2 + C\left(|\boldsymbol{U}^{n-1}|^{2\rho}, |\boldsymbol{U}^n|^2\right)^h + \frac{1}{2}\left|\frac{\boldsymbol{U}^n - \boldsymbol{U}^{n-1}}{\Delta t}\right|_h^2.\end{aligned} \tag{5.4.2}$$

Thus from: (5.4.1), (5.4.2), a kickback of $\frac{1}{2}|(\boldsymbol{U}^n - \boldsymbol{U}^{n-1})/\Delta t|_h^2$, and multiplying through by 2, we have

$$\begin{aligned}\left|\frac{\boldsymbol{U}^n-\boldsymbol{U}^{n-1}}{\Delta t}\right|_h^2&+\frac{1}{\Delta t}\left(|\boldsymbol{U}^n-\boldsymbol{U}^{n-1}|_1^2+|\boldsymbol{U}^n|_1^2-|\boldsymbol{U}^{n-1}|_1^2\right)\\&\le C|\boldsymbol{U}^n|_h^2+C\left(|\boldsymbol{U}^{n-1}|^{2\rho},|\boldsymbol{U}^n|^2\right)^h\\&\le C|\boldsymbol{U}^n|_h^2+C|\boldsymbol{U}^{n-1}|_{h,2\rho+2}^{2\rho}|\boldsymbol{U}^n|_{h,2\rho+2}^2\\&\le C,\end{aligned}\tag{5.4.3}$$

where we have used the discrete Hölder inequality for S^h (see (4.2.12)) and noted the stability bounds (5.3.5) and (5.3.6). Multiplying (5.4.3) through by Δt, summing from $n=1,\ldots,N$, rearranging, and noting Lemma 4.2.11 leads to

$$\begin{aligned}\Delta t\sum_{n=1}^{N}\left|\frac{\boldsymbol{U}^n-\boldsymbol{U}^{n-1}}{\Delta t}\right|_h^2&+\sum_{n=1}^{N}|\boldsymbol{U}^n-\boldsymbol{U}^{n-1}|_1^2+|\boldsymbol{U}^N|_1^2\\&\le|\boldsymbol{U}^0|_1^2+\sum_{n=1}^{N}C\Delta t\\&=|P^h\boldsymbol{u}_0|_1^2+CT\\&\le C.\end{aligned}\tag{5.4.4}$$

The bound (5.1.10) follows directly. We also have $|\boldsymbol{U}^N|_1^2\le C$, which implies $\max_{1\le n\le N}|\boldsymbol{U}^n|_1\le C$, and noting (5.3.6) we deduce bound (5.1.8). To show bound (5.1.9) multiply (5.4.4) through by Δt ■

The fully-discrete stability estimates mimic the corresponding semi-discrete estimates, however the match is not perfect. The main reason for differences is due to the semi-implicit time discretisation of the nonlinearity, which effectively breaks the structure of terms in the semi-discrete and continuous cases. This is because some of the terms are at time level $t_n:=n\Delta t$, while others are at $t_{n-1}:=(n-1)\Delta t$. Thus terms that were previously positive may no longer be so. For example, in Estimate III we would have liked to choose $\boldsymbol{\chi}^h=\pi^h\left\{|\boldsymbol{U}^{n-1}|^\rho\boldsymbol{U}^n\right\}$ in $(\mathrm{P}_1^{h,\Delta t})$ leading to control of the term $\sum_{n=1}^N(|\boldsymbol{U}^{n-1}|^{2\rho},|\boldsymbol{U}^n|^2)^h$ in Estimate IV, but this choice would also give the term $(\nabla\boldsymbol{U}^n,\nabla\{\pi^h|\boldsymbol{U}^{n-1}|^\rho\boldsymbol{U}^n\})$ on the left hand side. If we attempt to duplicate the approach used in the proof of Lemma 4.2.1 this leads to,

$$(\nabla \boldsymbol{U}^n, \nabla\{\pi^h|\boldsymbol{U}^{n-1}|^\rho \boldsymbol{U}^n\}) =$$
$$\frac{1}{2}\sum_{i=0}^{J}\sum_{\substack{j=0\\ j\neq i}}^{J}(-K_{ij})\left(|\boldsymbol{U}_i^{n-1}|^\rho \boldsymbol{U}_i^n - |\boldsymbol{U}_j^{n-1}|^\rho \boldsymbol{U}_j^n\right)\cdot(\boldsymbol{U}_i^n - \boldsymbol{U}_j^n),$$

which is not necessarily positive.

5.5 An error estimate

In this section we prove an error estimate between the continuous solutions of (P_1) and the fully-discrete solutions of $(\mathrm{P}_1^{h,\Delta t})$, with no additional assumptions. For notational convenience we extend the fully-discrete solutions via the piecewise linear interpolant, or piecewise constant interpolant in time. This approach can be found, for example, in [4], [6], [7].

We present the main numerical result of this thesis:

Theorem 5.5.1 Let the results and assumptions of Lemma 5.1.2 hold. Then we have

$$\|\boldsymbol{u} - \boldsymbol{U}^+\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{u} - \boldsymbol{U}^+\|^2_{L^2(0,T;H^1(\Omega))} \leq C(\Delta t + h^2), \tag{5.5.1}$$

where

$$\boldsymbol{U}(t) := \left(\frac{t - t_{n-1}}{\Delta t}\right)\boldsymbol{U}^n + \left(\frac{t_n - t}{\Delta t}\right)\boldsymbol{U}^{n-1}, \quad t \in [t_{n-1}, t_n],\ n \geq 1,$$

and

$$\boldsymbol{U}^+(t) := \boldsymbol{U}^n, \quad \boldsymbol{U}^-(t) := \boldsymbol{U}^{n-1}, \quad t \in (t_{n-1}, t_n],\ n \geq 1. \tag{5.5.2}$$

Proof. Note for future reference that

$$\frac{\partial \boldsymbol{U}}{\partial t} = \frac{\boldsymbol{U}^+ - \boldsymbol{U}^-}{\Delta t} = \frac{\boldsymbol{U} - \boldsymbol{U}^-}{t - t_{n-1}} = \frac{\boldsymbol{U}^+ - \boldsymbol{U}}{t_n - t}, \quad t \in (t_{n-1}, t_n],\ n \geq 1. \tag{5.5.3}$$

We restate $(\mathrm{P}_1^{h,\Delta t})$ as follows:

Find $\boldsymbol{U} \in \{H^1(0,T;S^h)\}^2$ such that $\boldsymbol{U}(0) := P^h \boldsymbol{u}_0$ and for a.e. $t \in (0,T)$

$$\left(\frac{\partial \boldsymbol{U}}{\partial t}, \boldsymbol{\chi}^h\right)^h + (\nabla \boldsymbol{U}^+, \nabla \boldsymbol{\chi}^h) = (B\boldsymbol{U}^+, \boldsymbol{\chi}^h)^h + (|\boldsymbol{U}^-|^\rho A \boldsymbol{U}^+, \boldsymbol{\chi}^h)^h \quad \forall \boldsymbol{\chi}^h \in \{S^h\}^2. \tag{5.5.4}$$

Define $\boldsymbol{E}^+ := \boldsymbol{u}^h - \boldsymbol{U}^+ \in \{S^h\}^2$, $\boldsymbol{E} := \boldsymbol{u}^h - \boldsymbol{U} \in \{S^h\}^2$ and $\boldsymbol{E}^- := \boldsymbol{u}^h - \boldsymbol{U}^- \in \{S^h\}^2$, so that $\boldsymbol{E}^+ - \boldsymbol{E} \equiv \boldsymbol{U} - \boldsymbol{U}^+ \equiv (t - t_n)\frac{\partial \boldsymbol{U}}{\partial t}$ and $\boldsymbol{E}^- - \boldsymbol{E} \equiv \boldsymbol{U} - \boldsymbol{U}^- \equiv \Delta t \frac{\partial \boldsymbol{U}}{\partial t}$. Using these definitions we note for later use the following inequalities

$$|\boldsymbol{E}^+|_h \leq |\boldsymbol{E}|_h + |\boldsymbol{U}^+ - \boldsymbol{U}^-|_h, \tag{5.5.5a}$$

$$|\boldsymbol{E}^-|_h \leq |\boldsymbol{E}|_h + |\boldsymbol{U}^+ - \boldsymbol{U}^-|_h, \tag{5.5.5b}$$

$$\|\boldsymbol{E}^+\|_1 \leq C|\boldsymbol{E}|_h + C|\boldsymbol{U}^+ - \boldsymbol{U}^-|_h + |\boldsymbol{E}^+|_1, \tag{5.5.5c}$$

$$|\boldsymbol{E}|_1 \leq |\boldsymbol{E}^+|_1 + |\boldsymbol{U}^+ - \boldsymbol{U}^-|_1, \tag{5.5.5d}$$

which are easily verified with the aid of (5.5.3), and the equivalence of norms result (4.2.7) for (5.5.5c).

We choose $\boldsymbol{\chi}^h = \boldsymbol{E}^+$ in (5.5.4) and (4.2.28), and subtract, which leads to after noting (2.2.2a)

$$\left(\frac{\partial \boldsymbol{E}}{\partial t}, \boldsymbol{E}^+\right)^h + |\boldsymbol{E}^+|_1^2 = \lambda_0 |\boldsymbol{E}^+|_h^2 + (|\boldsymbol{u}^h|^\rho A \boldsymbol{u}^h - |\boldsymbol{U}^-|^\rho A \boldsymbol{U}^+, \boldsymbol{E}^+)^h, \quad \text{where } \boldsymbol{E}(0) = 0.$$

We rewrite this as

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\boldsymbol{E}|_h^2 + |\boldsymbol{E}^+|_1^2 &= \left(\frac{\partial \boldsymbol{E}}{\partial t}, \boldsymbol{U}^+ - \boldsymbol{U}\right)^h + \lambda_0 |\boldsymbol{E}^+|_h^2 + (|\boldsymbol{u}^h|^\rho A \boldsymbol{u}^h - |\boldsymbol{U}^-|^\rho A \boldsymbol{U}^+, \boldsymbol{E}^+)^h \\ &\equiv \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \end{aligned} \tag{5.5.6}$$

We deal with the terms on the right hand side of (5.5.6) separately.

Noting (4.2.12) and (5.5.3) we have

$$\begin{aligned} \mathrm{I}_1 &= \left(\frac{\partial \boldsymbol{E}}{\partial t}, \boldsymbol{U}^+ - \boldsymbol{U}\right)^h \\ &\leq \left|\frac{\partial \boldsymbol{E}}{\partial t}\right|_h |\boldsymbol{U}^+ - \boldsymbol{U}|_h \\ &\leq \left(\left|\frac{\partial \boldsymbol{u}^h}{\partial t}\right|_h + \left|\frac{\partial \boldsymbol{U}}{\partial t}\right|_h\right) |\boldsymbol{U}^+ - \boldsymbol{U}|_h \\ &\leq \left(\left|\frac{\partial \boldsymbol{u}^h}{\partial t}\right|_h + \frac{1}{\Delta t}|\boldsymbol{U}^+ - \boldsymbol{U}^-|_h\right) \frac{|t_n - t|}{\Delta t}|\boldsymbol{U}^+ - \boldsymbol{U}^-|_h \\ &\leq \left|\frac{\partial \boldsymbol{u}^h}{\partial t}\right|_h |\boldsymbol{U}^+ - \boldsymbol{U}^-|_h + \frac{1}{\Delta t}|\boldsymbol{U}^+ - \boldsymbol{U}^-|_h^2. \end{aligned} \tag{5.5.7}$$

With the aid of (5.5.5a) we have

$$\begin{aligned} \mathrm{I}_2 &= \lambda_0 |\boldsymbol{E}^+|_h^2 \\ &\leq C|\boldsymbol{E}|_h^2 + C|\boldsymbol{U}^+ - \boldsymbol{U}^-|_h^2. \end{aligned} \tag{5.5.8}$$

We split the third term via

$$\begin{aligned} \mathrm{I}_3 &= \left(|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h - |\boldsymbol{U}^-|^\rho A\boldsymbol{U}^+, \boldsymbol{E}^+\right)^h \\ &= \left(|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h - |\boldsymbol{U}^-|^\rho A\boldsymbol{U}^-, \boldsymbol{E}^+\right)^h + \left(|\boldsymbol{U}^-|^\rho A\boldsymbol{U}^- - |\boldsymbol{U}^-|^\rho A\boldsymbol{U}^+, \boldsymbol{E}^+\right)^h \\ &\equiv \mathrm{I}_{3,1} + \mathrm{I}_{3,2}. \end{aligned} \tag{5.5.9}$$

It follows from: (2.2.4), a generalised Hölder inequality for S^h (cf. (4.2.12)), assumption (A3), Lemma 4.2.8, a stability bound in (4.4.5), (5.1.8), the Young's inequality (2.1.7) with $\varepsilon = 4$, (5.5.5b), and (5.5.5c) that

$$\begin{aligned} \mathrm{I}_{3,1} &= \left(|\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h - |\boldsymbol{U}^-|^\rho A\boldsymbol{U}^-, \boldsymbol{E}^+\right)^h \\ &\leq \int_\Omega \pi^h \left\{ \left| |\boldsymbol{u}^h|^\rho A\boldsymbol{u}^h - |\boldsymbol{U}^-|^\rho A\boldsymbol{U}^- \right| |\boldsymbol{E}^+| \right\} dx \\ &\leq (\rho+1)\sqrt{\lambda_1^2 + \omega_1^2} \int_\Omega \pi^h \left\{ \left(|\boldsymbol{u}^h|^\rho + |\boldsymbol{U}^-|^\rho\right) |\boldsymbol{E}^-||\boldsymbol{E}^+| \right\} dx \\ &\leq C\left(|\boldsymbol{u}^h|_{h,3\rho}^\rho + |\boldsymbol{U}^-|_{h,3\rho}^\rho\right) |\boldsymbol{E}^-|_h |\boldsymbol{E}^+|_{h,6} \\ &\leq C\left(\|\boldsymbol{u}^h\|_1^\rho + \|\boldsymbol{U}^-\|_1^\rho\right) |\boldsymbol{E}^-|_h \|\boldsymbol{E}^+\|_1 \\ &\leq C|\boldsymbol{E}^-|_h \|\boldsymbol{E}^+\|_1 \\ &\leq C|\boldsymbol{E}^-|_h^2 + \frac{1}{8}\|\boldsymbol{E}^+\|_1^2 \\ &\leq C|\boldsymbol{E}|_h^2 + C|\boldsymbol{U}^+ - \boldsymbol{U}^-|_h^2 + \frac{1}{4}|\boldsymbol{E}^+|_1^2. \end{aligned} \tag{5.5.10}$$

To bound $\mathrm{I}_{3,2}$ note: (2.2.2b), a generalised Hölder inequality for S^h (cf. (4.2.12)), assumption (A3), Lemma 4.2.8, bound (5.1.8), the Young's inequality (2.1.7) with $\varepsilon = 4$, and (5.5.5c) to give

$$\begin{aligned} \mathrm{I}_{3,2} &= \left(|\boldsymbol{U}^-|^\rho A\boldsymbol{U}^- - |\boldsymbol{U}^-|^\rho A\boldsymbol{U}^+, \boldsymbol{E}^+\right)^h \\ &\leq \sqrt{\lambda_1^2 + \omega_1^2} \int_\Omega \pi^h \left\{ |\boldsymbol{U}^-|^\rho |\boldsymbol{U}^+ - \boldsymbol{U}^-||\boldsymbol{E}^+| \right\} dx \\ &\leq C|\boldsymbol{U}^-|_{h,3\rho}^\rho |\boldsymbol{U}^+ - \boldsymbol{U}^-|_h |\boldsymbol{E}^+|_{h,6} \end{aligned}$$

$$\begin{aligned}
&\leq C\|\boldsymbol{U}^-\|_1^\rho|\boldsymbol{U}^+ - \boldsymbol{U}^-|_h\|\boldsymbol{E}^+\|_1 \\
&\leq C|\boldsymbol{U}^+ - \boldsymbol{U}^-|_h\|\boldsymbol{E}^+\|_1 \\
&\leq |\boldsymbol{U}^+ - \boldsymbol{U}^-|_h^2 + \frac{1}{8}\|\boldsymbol{E}^+\|_1^2 \\
&\leq C|\boldsymbol{E}|_h^2 + C|\boldsymbol{U}^+ - \boldsymbol{U}^-|_h^2 + \frac{1}{4}|\boldsymbol{E}^+|_1^2. \qquad (5.5.11)
\end{aligned}$$

Thus from equations (5.5.6) - (5.5.11), a kickback of $\frac{1}{2}|\boldsymbol{E}^+|_1^2$, and multiplying through by 2 we have

$$\frac{d}{dt}|\boldsymbol{E}|_h^2 + |\boldsymbol{E}^+|_1^2 \leq C|\boldsymbol{E}|_h^2 + C\left|\frac{\partial \boldsymbol{u}^h}{\partial t}\right|_h |\boldsymbol{U}^+ - \boldsymbol{U}^-|_h + \frac{C}{\Delta t}|\boldsymbol{U}^+ - \boldsymbol{U}^-|_h^2. \qquad (5.5.12)$$

Using the Grönwall lemma and recalling that $\boldsymbol{E}(0) = 0$ yields

$$|\boldsymbol{E}(T)|_h^2 + \int_0^T |\boldsymbol{E}^+|_1^2\,dt \leq C\exp(CT)\int_0^T \left\{\left|\frac{\partial \boldsymbol{u}^h}{\partial t}\right|_h |\boldsymbol{U}^+ - \boldsymbol{U}^-|_h + \frac{1}{\Delta t}|\boldsymbol{U}^+ - \boldsymbol{U}^-|_h^2\right\} dt. \qquad (5.5.13)$$

To bound the right hand side of (5.5.13), observe using Lemma 5.1.2 that

$$\frac{C}{\Delta t}\int_0^T |\boldsymbol{U}^+ - \boldsymbol{U}^-|_h^2\,dt = \frac{C}{\Delta t}\sum_{n=1}^N \int_{t_{n-1}}^{t_n} |\boldsymbol{U}^n - \boldsymbol{U}^{n-1}|_h^2\,dt \leq C\Delta t,$$

and with the aid of the Cauchy-Schwarz inequality and Lemmata 5.1.2 and 4.2.12 we have

$$\begin{aligned}
\int_0^T \left|\frac{\partial \boldsymbol{u}^h}{\partial t}\right|_h |\boldsymbol{U}^+ - \boldsymbol{U}^-|_h\,dt &\leq \left(\int_0^T \left|\frac{\partial \boldsymbol{u}^h}{\partial t}\right|_h^2 dt\right)^{1/2} \left(\int_0^T |\boldsymbol{U}^+ - \boldsymbol{U}^-|_h^2\,dt\right)^{1/2} \\
&\leq C\left(\sum_{n=1}^N \int_{t_{n-1}}^{t_n} |\boldsymbol{U}^n - \boldsymbol{U}^{n-1}|_h^2\,dt\right)^{1/2} \leq C\Delta t.
\end{aligned}$$

Thus (5.5.13) becomes

$$|\boldsymbol{E}(T)|_h^2 + \int_0^T |\boldsymbol{E}^+|_1^2\,dt \leq C\Delta t. \qquad (5.5.14)$$

Thus we have

$$\|\boldsymbol{E}\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq C\Delta t. \qquad (5.5.15)$$

With the aid of (5.4.3) and the equivalence of norms result (4.2.7) we have

$$\begin{aligned}
\|\boldsymbol{E} - \boldsymbol{E}^+\|^2_{L^\infty(0,T;L^2(\Omega))} &\equiv \|\boldsymbol{U}^+ - \boldsymbol{U}\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\leq \|\boldsymbol{U}^+ - \boldsymbol{U}^-\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&= \max_{1\leq n\leq N} \|\boldsymbol{U}^n - \boldsymbol{U}^{n-1}\|^2_0 \\
&\leq C \max_{1\leq n\leq N} |\boldsymbol{U}^n - \boldsymbol{U}^{n-1}|^2_h \\
&\leq C \sum_{n=1}^{N} |\boldsymbol{U}^n - \boldsymbol{U}^{n-1}|^2_h \\
&\leq C\Delta t, \qquad (5.5.16)
\end{aligned}$$

after noting the stability bound (5.1.9). Thus combining (5.5.15) and (5.5.16) and applying the triangle inequality yields

$$\begin{aligned}
\|\boldsymbol{E}^+\|^2_{L^\infty(0,T;L^2(\Omega))} &\leq 2\|\boldsymbol{E}^+ - \boldsymbol{E}\|^2_{L^\infty(0,T;L^2(\Omega))} + 2\|\boldsymbol{E}\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\leq C\Delta t,
\end{aligned}$$

and so noting (5.5.14) and applying a similar argument to the one given at the end of Section 4.5 gives

$$\|\boldsymbol{E}^+\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{E}^+\|^2_{L^2(0,T;H^1(\Omega))} \leq C\Delta t. \qquad (5.5.17)$$

After recalling the semi-discrete error bound in Lemma 4.5.1 and the splitting $\boldsymbol{u} - \boldsymbol{U}^+ \equiv \boldsymbol{e} + \boldsymbol{E}^+$ we obtain the desired result (5.5.1), after application of the triangle inequality. ■

A corollary to the fully-discrete error bound is the convergence of the fully-discrete approximations to the strong solutions:

$$\boldsymbol{U}^+ \to \boldsymbol{u} \quad \text{in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \quad \text{as} \quad h,\, \Delta t \to 0.$$

Chapter 6

Numerical experiments

In this chapter we present the results of numerical experiments in one space dimension (Section 6.1) that verify the theoretical results of the previous chapter. In Section 6.2 we present preliminary numerical results in two space dimensions, and in Section 6.3 we make some concluding remarks. Figures are collected together in Section 6.4.

6.1 One-dimensional simulations

6.1.1 Preliminaries

In this subsection we recall some facts concerning an explicit solution that facilitates use of time-dependent Neumann boundary conditions, and then develop the tools needed to check numerically the fully-discrete error bound (5.5.1). We also make some comments regarding the applicability of the fully-discrete error bound in the time-dependent boundary condition case, and undertake a linear stability analysis of the $\lambda - \omega$ system about the origin.

The $\lambda - \omega$ system with $\rho = 2$ has on the real line a unique one parameter family of periodic plane wave solutions given by[1]

$$\begin{matrix} u(x,t) \\ v(x,t) \end{matrix} = \widehat{r} \begin{matrix} \cos \\ \sin \end{matrix} \left\{ \omega(\widehat{r})t + [\lambda(\widehat{r})]^{1/2} x \right\}, \qquad (6.1.1)$$

[1]There is also a '$-$' solution moving in the opposite direction, but the '+' solution is sufficient for our purposes.

where $\widehat{r}$ is the constant amplitude [45]. A necessary condition for the $\lambda - \omega$ system to possess periodic plane waves is

$$\widehat{r} < r_{\max} := \left(\frac{\lambda_0}{\lambda_1}\right)^{1/2}. \tag{6.1.2}$$

From a result in [45] (equation (41), p.317) it follows that the travelling wave solutions of the $\lambda - \omega$ system are linearly stable if and only if

$$\widehat{r} \geq r_{\min} := \sqrt{\frac{2\lambda_0(\omega_1^2 + \lambda_1^2)}{\lambda_1(2\omega_1^2 + 3\lambda_1^2)}}. \tag{6.1.3}$$

Thus the condition $r_{\min} \leq \widehat{r} < r_{\max}$ provides a practical range of values for the amplitude to choose from in the explicit calculation of travelling wave solutions. If we rearrange the analytical solution in the form

$$\begin{matrix} u(x,t) \\ v(x,t) \end{matrix} = \widehat{r} \begin{matrix} \cos \\ \sin \end{matrix} \left([\lambda(\widehat{r})]^{1/2} \left\{ x + \frac{\omega(\widehat{r})}{[\lambda(\widehat{r})]^{1/2}} t \right\} \right),$$

we see that the wave speed and wavelength are given by

$$c := -\frac{\omega(\widehat{r})}{[\lambda(\widehat{r})]^{1/2}}, \quad w_l := \frac{2\pi}{[\lambda(\widehat{r})]^{1/2}}, \tag{6.1.4}$$

respectively.

For the purpose of numerically checking the fully-discrete error bound (5.5.1), note that the norms in space can be evaluated exactly, since for all $v^h \in S^h$

$$\begin{aligned} |v^h|_1^2 &= \int_0^L \left| \frac{dv^h}{dx} \right|^2 dx \\ &= \sum_{j=0}^{J-1} \int_{x_j}^{x_{j+1}} \left[\frac{v_{j+1}^h - v_j^h}{h} \right]^2 dx \\ &= \frac{1}{h} \sum_{j=0}^{J-1} (v_{j+1}^h - v_j^h)^2, \end{aligned}$$

where $v_j^h \equiv v^h(x_j)$ and $v^h(x_{j+1}) \equiv v_{j+1}^h$. Furthermore

$$\begin{aligned}
\|v^h\|_0^2 &= \int_0^L |v^h|^2\,dx \\
&= \sum_{j=0}^{J-1} \int_{x_j}^{x_{j+1}} |v^h|^2\,dx \\
&= \sum_{j=0}^{J-1} \left[\frac{(v^h)^3}{3(v^h)'}\right]_{x_j}^{x_{j+1}}, \quad \text{since} \quad (v^h)' = \frac{v_{j+1}^h - v_j^h}{h} = \text{constant} \\
&= \frac{h}{3}\sum_{j=0}^{J-1} \left[\frac{(v_{j+1}^h)^3 - (v_j^h)^3}{v_{j+1}^h - v_j^h}\right], \quad v_j^h \neq v_{j+1}^h \\
&= \frac{h}{3}\sum_{j=0}^{J-1} \left[(v_{j+1}^h)^2 + v_{j+1}^h v_j^h + (v_j^h)^2\right], \quad v_j^h \neq v_{j+1}^h.
\end{aligned}$$

This last expression also holds when $v_j^h = v_{j+1}^h$ as then

$$\int_{x_j}^{x_{j+1}} |v^h|^2\,dx = h(v_j^h)^2 \equiv \frac{h}{3}\left[(v_{j+1}^h)^2 + v_{j+1}^h v_j^h + (v_j^h)^2\right].$$

Recall from Section 5.5 the definition

$$\boldsymbol{U}^+(t) := \boldsymbol{U}^n, \quad t \in (t_{n-1}, t_n],\ n \geq 1,$$

i.e., we extend the finite element solution in time via the piecewise constant interpolant. We make a similar definition for the exact solution $\boldsymbol{u}$ via

$$\boldsymbol{u}^+(t) := \pi^h \boldsymbol{u}(t_n), \quad t \in (t_{n-1}, t_n],\ n \geq 1,$$

that is, we take the piecewise linear interpolant of the exact solution in space and extend this solution in time using the piecewise constant interpolant[2]. We numerically verify the fully-discrete error bound with the aid of the following proposition:

Proposition 6.1.1

$$\|\boldsymbol{u}^+ - \boldsymbol{U}^+\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{u}^+ - \boldsymbol{U}^+\|^2_{L^2(0,T;H^1(\Omega))} \leq C(\Delta t + h^2), \qquad (6.1.5)$$

where $\boldsymbol{u}$ corresponds to the analytical solution (6.1.1).

[2]Note that the exact solution (6.1.1) is in $C^\infty(\Omega_T)$.

Proof. Observe that

$$\|\boldsymbol{u}^{+} - \boldsymbol{U}^{+}\|_{L^2(0,T;H^1(\Omega))} \leq \|\boldsymbol{u} - \boldsymbol{u}^{+}\|_{L^2(0,T;H^1(\Omega))} + \|\boldsymbol{u} - \boldsymbol{U}^{+}\|_{L^2(0,T;H^1(\Omega))}$$
$$=: \mathrm{I} + \mathrm{II}. \tag{6.1.6}$$

From the error bound (5.5.1) we have

$$(\mathrm{II})^2 \leq C(\Delta t + h^2). \tag{6.1.7}$$

Consider the splitting

$$\begin{aligned}(\mathrm{I})^2 &\equiv \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\boldsymbol{u}(t) - \pi^h \boldsymbol{u}(t_n)\|_1^2 \, dt \\ &\leq 2\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\boldsymbol{u}(t) - \boldsymbol{u}(t_n)\|_1^2 \, dt + 2\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\boldsymbol{u}(t_n) - \pi^h \boldsymbol{u}(t_n)\|_1^2 \, dt \\ &=: \mathrm{III} + \mathrm{IV}.\end{aligned} \tag{6.1.8}$$

To deal with IV first note from (6.1.1) that

$$|\boldsymbol{u}(t_n)|_2^2 \equiv \int_0^L \left|\frac{\partial^2 \boldsymbol{u}}{\partial x^2}\right|^2 dx \leq C.$$

Thus from the interpolation error estimate (4.2.16) we have

$$\mathrm{IV} \leq Ch^2 \int_0^T |\boldsymbol{u}(t_n)|_2^2 \, dt \leq Ch^2. \tag{6.1.9}$$

We rewrite the first term on the right hand side of (6.1.8) as

$$\begin{aligned}\mathrm{III} &= 2\int_0^T \left\| \int_t^{t_n} \boldsymbol{u}_t(s)\, ds \right\|_1^2 dt \\ &= 2\int_0^T \left[\left\| \int_t^{t_n} \boldsymbol{u}_t(s)\, ds \right\|_0^2 + \left\| \nabla \int_t^{t_n} \boldsymbol{u}_t(s)\, ds \right\|_0^2 \right] dt, \quad t \in (t_{n-1}, t_n].\end{aligned} \tag{6.1.10}$$

Using the Cauchy-Schwarz inequality and noting (6.1.1) again we have

$$\left| \int_t^{t_n} \boldsymbol{u}_t(s)\, ds \right| \leq \left(\int_t^{t_n} |\boldsymbol{u}_t(s)|^2 \, ds \right)^{1/2} (t_n - t)^{1/2} \leq (\Delta t)^{1/2} \left(\int_{t_{n-1}}^{t_n} |\boldsymbol{u}_t(s)|^2 \, ds \right)^{1/2} \leq C\Delta t.$$

Thus

$$\left\| \int_t^{t_n} \boldsymbol{u}_t(s)\, ds \right\|_0^2 \leq C(\Delta t)^2 \tag{6.1.11}$$

and similarly

$$\left\| \nabla \int_t^{t_n} \boldsymbol{u}_t(s)\, ds \right\|_0^2 \equiv \left\| \int_t^{t_n} \boldsymbol{u}_{xt}(s)\, ds \right\|_0^2 \leq C(\Delta t)^2. \tag{6.1.12}$$

Thus from (6.1.10), (6.1.11), and (6.1.12) we have

$$\mathrm{III} \leq C(\Delta t)^2 \int_0^T dt = C(\Delta t)^2 \leq C\Delta t, \tag{6.1.13}$$

as by assumption $\Delta t \leq 1$. Combining (6.1.6)-(6.1.9) and (6.1.13) leads to

$$\|\boldsymbol{u}^+ - \boldsymbol{U}^+\|^2_{L^2(0,T;H^1(\Omega))} \leq C(\Delta t + h^2).$$

The proof for

$$\|\boldsymbol{u}^+ - \boldsymbol{U}^+\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C(\Delta t + h^2)$$

is similar. ■

Given an analytical solution $\boldsymbol{u}$ we can exactly calculate the left hand side of the error bound (6.1.5) via the quantities

$$\begin{aligned} \xi_0(h, \Delta t) :&= \|\boldsymbol{u}^+ - \boldsymbol{U}^+\|^2_{L^2(0,T;H^1(\Omega))} \\ &\equiv \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\pi^h u(t_n) - U^n\|_1^2\, dt \\ &= \Delta t \sum_{n=1}^{N} \left[\|\pi^h u(t_n) - U^n\|_0^2 + |\pi^h u(t_n) - U^n|_1^2 \right], \end{aligned} \tag{6.1.14}$$

$$\begin{aligned} \xi_\infty(h, \Delta t) :&= \|\boldsymbol{u}^+ - \boldsymbol{U}^+\|^2_{L^\infty(0,T;L^2(\Omega))} \\ &\equiv \max_{1 \leq n \leq N} \|\pi^h u(t_n) - U^n\|_0^2. \end{aligned} \tag{6.1.15}$$

The analytical solution (6.1.1) is given on the unbounded domain $\mathbb{R}$, thus in order to make comparisons with the approximate solution on $\Omega = (0, L)$ (see Section 6.1.3) we use the finite element method with time-dependent Neumann boundary conditions corresponding to this analytical solution. We make some comments regarding the applicability of the fully-discrete error bound (5.5.1) in this situation.

Consider the semi-discrete weak form corresponding to ($\mathrm{P_1}^h$) (see Section 4.2) with time-dependent Neumann boundary conditions:

$$\left(\frac{\partial \boldsymbol{u}^h}{\partial t}, \boldsymbol{\chi}^h\right)^h + \left(\frac{\partial \boldsymbol{u}^h}{\partial x}, \frac{\partial \boldsymbol{\chi}^h}{\partial x}\right) = (\boldsymbol{f}(\boldsymbol{u}^h), \boldsymbol{\chi}^h)^h + \left(\frac{\partial \boldsymbol{u}^h}{\partial \boldsymbol{\nu}}, \boldsymbol{\chi}^h\right)_{\partial\Omega},$$

$$\text{with} \quad \frac{\partial \boldsymbol{u}^h}{\partial \boldsymbol{\nu}} = \boldsymbol{g} \quad \text{on} \quad \partial\Omega,$$

where $\boldsymbol{f}$ is the reaction term, $\boldsymbol{g} := (u_x, v_x)^T$ corresponds to (6.1.1), $(\cdot,\cdot)_{\partial\Omega} := \int_{\partial\Omega} \cdot \, ds$, ($ds$ an element of 'length' on $\partial\Omega$) and $\boldsymbol{\nu}$ is the unit normal to $\partial\Omega$. In one dimension the boundary term is $\boldsymbol{u}_x(L,t) \cdot \boldsymbol{\chi}^h(L) - \boldsymbol{u}_x(0,t) \cdot \boldsymbol{\chi}^h(0)$. We have the same expression on the right hand side of the fully-discrete weak formulation ($\mathrm{P}_1^{h,\Delta t}$) and so in the error estimate proofs for the semi-discrete and the fully-discrete cases these boundary terms cancel. With regard to the stability estimates, note that the exact solutions (6.1.1) on Ω_T are smooth and thus the regularity results of Theorem 4.1.1 automatically apply. We can attempt to control the boundary terms via (and similarly in the fully-discrete case):

$$\begin{aligned}\left(\frac{\partial \boldsymbol{u}^h}{\partial \boldsymbol{\nu}}, \boldsymbol{\chi}^h\right)_{\partial\Omega} &\equiv (\boldsymbol{g}, \boldsymbol{\chi}^h)_{\partial\Omega} \leq \|\boldsymbol{g}\|_{L^2(\partial\Omega)} \|\boldsymbol{\chi}^h\|_{L^2(\partial\Omega)} \\ &\leq C\|\boldsymbol{g}\|_1 \|\boldsymbol{\chi}^h\|_1 \\ &\leq C\|\boldsymbol{g}\|_1^2 + \frac{1}{2\varepsilon}\|\boldsymbol{\chi}^h\|_1^2,\end{aligned}$$

where we have applied the Cauchy-Schwarz inequality, a well-known trace inequality (Theorem A.0.32) and finally the Young's inequality (2.1.7). As the analytical solutions are smooth we can control the term $\|\boldsymbol{g}\|_1^2$ after the application of the Grönwall lemma. Control of the final term depends on the specific choice of test function (see Estimates I and II in Chapter 4 and Estimates III and IV in Chapter 5). So provided the regularity of the semi-discrete and fully-discrete solutions are sufficient to deal with this term, then the error bound (5.5.1) will still apply. Numerical experiments in one space dimension indicate the following. If we are sufficiently far from the boundary to avoid 'pollution' of the solution due to homogeneous Neumann boundary conditions, then there is good qualitative agreement with the approximations using time-dependent Neumann boundary conditions.

To help assess the behaviour of the approximations in Section 6.1.3, we prove that the origin in the u-v phase plane is an unstable fixed point of the linearised

$\lambda - \omega$ system in one space dimension. The linearised $\lambda - \omega$ system is given by

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u \\ v \end{pmatrix}_{xx} + \begin{pmatrix} \lambda_0 & -\omega_0 \\ \omega_0 & \lambda_0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

We perform a linear stability analysis by introducing the Fourier modes

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} e^{\sigma t + ikx},$$

into the linearised PDE system above, for constants μ_1, μ_2, $\sigma \in \mathbb{C}$ and $k \in \mathbb{R}$, yielding

$$\begin{pmatrix} \sigma + k^2 - \lambda_0 & \omega_0 \\ -\omega_0 & \sigma + k^2 - \lambda_0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \mathbf{0}.$$

This matrix system has a non-zero solution if and only if

$$(\sigma + k^2 - \lambda_0)^2 + {\omega_0}^2 = 0.$$

Taking the imaginary part of this equation leads to

$$\mathrm{Re}(\sigma) = \lambda_0 - k^2.$$

For a non-growing solution we require $\mathrm{Re}(\sigma) < 0$ for all real values of the wave number k. When $k = 0$ this condition is never satisfied as $\lambda_0 > 0$. Thus the origin is an unstable critical point of the linearised $\lambda - \omega$ system.

6.1.2 Practical algorithms

We give details of some practical algorithms in one space dimension, written in complex form, with either time-dependent Neumann boundary conditions, or with homogeneous Neumann boundary conditions. A uniform discretisation is used in both space and time.

We present the following fully-discrete, semi-implicit in time, finite element approximation, which is the complex equivalent of $(\mathrm{P}_1^{h,\Delta t})$ (see (5.1.7)), but with Neumann boundary data corresponding to (6.1.1):

$(\mathbf{P}_2^{h,\Delta t})^*$ For $n = 1, \ldots, N$ find $C^n \in \mathbb{S}^h$ such that $C^0 := P^h c_0$ and

$$\left(\frac{C^n - C^{n-1}}{\Delta t}, \chi^h\right)^h + (\nabla C^n, \nabla \chi^h) = \left(\widehat{f}(C^{n-1})C^n, \chi^h\right)^h +$$
$$c_x(L,t)\chi^h(x_J) - c_x(0,t)\chi^h(x_0), \quad \forall \chi^h \in \mathbb{S}^h$$
$$\text{where} \quad \widehat{f}(C) := \lambda(R) + i\omega(R), \quad R \equiv |C|, \quad C := U + iV,$$

and from the Neumann boundary data we have

$$c_x(0,t) := u_x(0,t) + i\, v_x(0,t) = i\,\widehat{r}[\lambda(\widehat{r})]^{1/2} \exp\{i\,\omega(\widehat{r})t\},$$
$$c_x(L,t) := u_x(L,t) + i\, v_x(L,t) = i\,\widehat{r}[\lambda(\widehat{r})]^{1/2} \exp\left\{i\,\left(\omega(\widehat{r})t + [\lambda(\widehat{r})]^{1/2}L\right)\right\}.$$

Choosing $C^n = \sum_{j=0}^J C_j^n \varphi_j$, $\chi^h = \varphi_i$, $i = 0, \ldots, J$ where $C_j^n \approx c(jh, n\Delta t)$ leads to

$$\frac{1}{\Delta t}\sum_{j=0}^J (C_j^n - C_j^{n-1})(\varphi_i, \varphi_j)^h + \sum_{j=0}^J C_j^n (\nabla\varphi_i, \nabla\varphi_j) = (\widehat{f}(C^{n-1})C^n, \varphi_i)^h$$
$$+ c_x(x_J,t)\delta_{iJ} - c_x(x_0,t)\delta_{i0}, \quad i = 0, \ldots, J. \quad (6.1.16)$$

To deal with the nonlinearity note

$$(\widehat{f}(C^{n-1})C^n, \varphi_i)^h = \sum_{j=0}^J \widehat{M}_{jj}\widehat{f}(C_j^{n-1})C_j^n\delta_{ij} = \widehat{M}_{ii}\widehat{f}(C_i^{n-1})C_i^n, \quad (6.1.17)$$

(recall (4.2.1), (4.2.3)). Thus multiplying (6.1.16) through by Δt and noting $(\varphi_i, \varphi_j)^h = \widehat{M}_{jj}\delta_{ij}$ and (6.1.17) we have

$$(C_i^n - C_i^{n-1})\widehat{M}_{ii} + \Delta t \sum_{j=0}^J C_j^n K_{ij} = \Delta t\, \widehat{M}_{ii}\widehat{f}(C_i^{n-1})C_i^n$$
$$+ \Delta t\, c_x(x_J,t)\delta_{iJ} - \Delta t\, c_x(x_0,t)\delta_{i0}, \quad i = 0, \ldots, J. \quad (6.1.18)$$

Thus

$$\widehat{M}(\boldsymbol{C}^n - \boldsymbol{C}^{n-1}) + \Delta t\, K\boldsymbol{C}^n = \Delta t\, \widehat{M}\, \mathrm{diag}\{\widehat{f}(C_0^{n-1}), \ldots, \widehat{f}(C_J^{n-1})\}\boldsymbol{C}^n + \Delta t\, \widehat{\boldsymbol{b}}(t),$$
$$M\boldsymbol{C}^0 = \boldsymbol{C}_0,$$

where $\widehat{\boldsymbol{b}}(t) := (-c_x(x_0,t), 0, \ldots, 0, c_x(x_J,t))^T$, $\boldsymbol{C}^n := (C_0^n, \ldots, C_J^n)^T$,
$C_j^n := U_j^n + i\, V_j^n$, $\quad R_j^n \equiv |C_j^n|$, and $\{\boldsymbol{C}_0\}_j := (c_0, \varphi_j)$. This leads to the following

tri-diagonal system of $(J+1)$ linear equations, with complex coefficients:

$$A_{n-1}\boldsymbol{C}^n = \boldsymbol{C}^{n-1} + \boldsymbol{b}(t), \quad \boldsymbol{C}^0 := M^{-1}\boldsymbol{C}_0, \tag{6.1.19}$$

$$\text{where} \quad A_{n-1} := I - \Delta t \, \text{diag}\{\widehat{f}(C_0^{n-1}), \ldots, \widehat{f}(C_J^{n-1})\} + \Delta t \, (\widehat{M})^{-1}K,$$

$$\boldsymbol{C}^n := (C_0^n, \ldots, C_J^n)^T, \quad C_j^n := U_j^n + i\, V_j^n, \quad R_j^n \equiv |C_j^n|, \quad \{\boldsymbol{C}_0\}_j := (c_0, \varphi_j),$$

$$\text{and} \quad \{\boldsymbol{b}(t)\}_j := \frac{2\Delta t}{h} \begin{cases} -c_x(0,t) & \text{for } j = 0, \\ 0 & \text{for } 1 \le j \le J-1, \\ c_x(L,t) & \text{for } j = J, \end{cases}$$

(see Appendix C). For concreteness we chose the initial approximations to correspond to the interpolant of the analytical solutions at $t = 0$.

For the purposes of numerical comparison, we also present the following fully-discrete, semi-implicit in time, finite element approximation, which is the complex equivalent of $(\mathrm{P}_1^{h,\Delta t})$ (see (5.1.7)):[3]

$(\mathbf{P}_2^{h,\Delta t})$ For $n = 1, \ldots, N$ find $C^n \in \mathbb{S}^h$ such that $C^0 := P^h c_0$ and

$$\left(\frac{C^n - C^{n-1}}{\Delta t}, \chi^h\right)^h + (\nabla C^n, \nabla \chi^h) = \left(\widehat{f}(C^{n-1})C^n, \chi^h\right)^h \quad \forall \chi^h \in \mathbb{S}^h,$$

$$\text{where} \quad \widehat{f}(C) := \lambda(R) + i\omega(R), \quad R \equiv |C|, \quad C := U + iV. \tag{6.1.20}$$

Following the same steps as in the derivation of (6.1.19), but with the boundary terms set equal to zero yields:

$$A_{n-1}\boldsymbol{C}^n = \boldsymbol{C}^{n-1}, \quad \boldsymbol{C}^0 := M^{-1}\boldsymbol{C}_0. \tag{6.1.21}$$

Alternatively, and more directly, we could approximate the system of ODEs (4.3.4) by

$$\frac{(\boldsymbol{C}^n - \boldsymbol{C}^{n-1})}{\Delta t} = (D_{n-1} - L)\boldsymbol{C}^n,$$

where $D_{n-1} := \text{diag}\{\widehat{f}(C_0^{n-1}), \ldots, \widehat{f}(C_J^{n-1})\}$ and $L := (\widehat{M})^{-1}K$, leading again to (6.1.21). We employed a direct linear system solver to compute the solutions of (6.1.19) and (6.1.21). For Δt sufficiently small the coefficient matrix A_{n-1} is strictly diagonally dominant and thus no partial pivoting is required (Theorem A.0.30).

[3] I.e., the same scheme as $(\mathrm{P}_2^{h,\Delta t})^*$, but with homogeneous Neumann boundary conditions.

We remark that the finite element method $(\mathrm{P}_2^{h,\Delta t})$ is equivalent to the following semi-implicit finite difference scheme:

$(\mathbf{Q}_2^{h,\Delta t})$ For $n = 1, \ldots, N$ and $j = 0, \ldots, J$ find C_j^n such that

$$\frac{C_j^n - C_j^{n-1}}{\Delta t} = \frac{C_{j+1}^n - 2C_j^n + C_{j-1}^n}{h^2} + \widehat{f}(C_j^{n-1})C_j^n,$$

$$\text{with} \quad C_j^0 := c(jh, 0) \quad \text{(initial approximation)}$$

$$\text{and} \quad C_{-1}^n := C_1^n, \quad C_{J+1}^n := C_{J-1}^n \quad \text{('reflective' boundary conditions).} \qquad (6.1.22)$$

The 'reflective' boundary conditions arise from the use of fictitious nodes x_{-1} and x_{J+1} to approximate the homogeneous Neumann boundary conditions via:

$$\frac{(C_1^n - C_{-1}^n)}{2h} = 0 = \frac{(C_{J+1}^n - C_{J-1}^n)}{2h}.$$

To see that $(\mathrm{Q}_2^{h,\Delta t})$ and $(\mathrm{P}_2^{h,\Delta t})$ are equivalent, observe that the set of linear expressions

$$\frac{1}{h^2}(-C_{j+1}^n + 2C_j^n - C_{j-1}^n), \quad j = 0, \ldots, J$$

can be written in matrix form as $(\widehat{M})^{-1}K\mathbf{C}^n$ (see Appendix C), and thus after multiplying (6.1.22) through by Δt we obtain, after simplification, the linear system (6.1.21). One reason for presenting the finite difference scheme is that applied mathematicians and scientists are often more familiar with finite differences than they are with finite elements. Moreover, due to the equivalence of these two methods the theoretical results of previous chapters will also apply to the finite difference scheme.

For the purposes of comparing/reproducing results in the literature, we also present the following fully-discrete, semi-implicit finite element approximation:

$(\mathbf{P}_3^{h,\Delta t})$ For $n = 1, \ldots, N$ find $C^n \in \mathbb{S}^h$ such that $C^0 := P^h c_0$ and

$$\left(\frac{C^n - C^{n-1}}{\Delta t}, \chi^h\right)^h + \frac{1}{2}(\nabla C^n + \nabla C^{n-1}, \nabla \chi^h) = \left(f(C^{n-1}), \chi^h\right)^h \quad \forall \chi^h \in \mathbb{S}^h,$$

$$\text{where} \quad f(C) := \widehat{f}(C)C, \quad C := U + iV. \qquad (6.1.23)$$

It is easy to prove[4] the existence and uniqueness of solutions to this scheme for arbitrary d, but we were unable to prove this is a stable numerical method, however, all the numerical simulations we have performed behaved in a stable manner.

[4]Results not provided.

Following a similar procedure to the derivation of (6.1.21), leads to the following complex linear system:

$$\begin{aligned} B\boldsymbol{C}^n &= E_{n-1}\boldsymbol{C}^{n-1}, \quad \boldsymbol{C}^0 := M^{-1}\boldsymbol{C}_0, \qquad (6.1.24)\\ \text{where} \quad B :&= I + \frac{\Delta t}{2}(\widehat{M})^{-1}K, \\ E_{n-1} :&= I - \frac{\Delta t}{2}(\widehat{M})^{-1}K + \Delta t \ \mathrm{diag}\{\widehat{f}(C_0^{n-1}),\ldots,\widehat{f}(C_J^{n-1})\}. \end{aligned}$$

Alternatively, and more directly, we could approximate the system of ODEs (4.3.4) by

$$\frac{(\boldsymbol{C}^n - \boldsymbol{C}^{n-1})}{\Delta t} = D_{n-1}\boldsymbol{C}^{n-1} - \frac{1}{2}L(\boldsymbol{C}^n + \boldsymbol{C}^{n-1}),$$

where $D_{n-1} := \mathrm{diag}\{\widehat{f}(C_0^{n-1}),\ldots,\widehat{f}(C_J^{n-1})\}$ and $L := (\widehat{M})^{-1}K$, leading again to (6.1.24). Note that B is (unconditionally) strictly diagonally dominant and thus in order to solve (6.1.24) for $n = 1,\ldots,N$ we perform the LU factorisation of B once, followed by repeated forward and backward substitutions and updating of the right hand side of the linear system.

To make comparisons with a numerical method in the literature we present a finite difference approximation equivalent to the finite element method $(\mathrm{P}_3^{h,\Delta t})$. This 'semi-implicit Crank-Nicolson' type scheme is similar to the usual Crank-Nicolson scheme, except that the reaction term is kept entirely at the previous time level.

$(\mathbf{Q}_3^{h,\Delta t})$ For $n = 1,\ldots,N$ and $j = 0,\ldots,J$ find C_j^n such that

$$\frac{C_j^n - C_j^{n-1}}{\Delta t} = \frac{C_{j+1}^n - 2C_j^n + C_{j-1}^n}{2h^2} + \frac{C_{j+1}^{n-1} - 2C_j^{n-1} + C_{j-1}^{n-1}}{2h^2} + f(C_j^{n-1}),$$

$$\text{with} \quad C_j^0 := c(jh, 0) \quad \text{(initial approximation)}$$

$$\text{and} \quad C_{-1}^n := C_1^n, \quad C_{J+1}^n := C_{J-1}^n \quad \text{('reflective' boundary conditions).} \qquad (6.1.25)$$

To see that $(\mathrm{Q}_3^{h,\Delta t})$ and $(\mathrm{P}_3^{h,\Delta t})$ are equivalent, observe that the set of linear expressions

$$\frac{1}{h^2}\left[(-C_{j+1}^n + 2C_j^n - C_{j-1}^n) + (-C_{j+1}^{n-1} + 2C_j^{n-1} - C_{j-1}^{n-1})\right], \quad j = 0,\ldots,J$$

can be written in matrix form as $\widehat{M}^{-1}K(\boldsymbol{C}^n + \boldsymbol{C}^{n-1})$ (see Appendix C), and thus after multiplying (6.1.25) through by Δt we obtain, after simplification, the linear system (6.1.24).

Implementation of the real finite element method $(\mathrm{P}_1^{h,\Delta t})$ (see (5.1.7)) leads to the following block-matrix form of $(2J+2)$ linear equations, with real coefficients

$$\begin{pmatrix} D_{n-1} & -B_{n-1} \\ B_{n-1} & D_{n-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{U}^n \\ \boldsymbol{V}^n \end{pmatrix} = \begin{pmatrix} \boldsymbol{U}^{n-1} \\ \boldsymbol{V}^{n-1} \end{pmatrix},$$

where D_{n-1} is tri-diagonal, and B_{n-1} is a diagonal matrix (both depending on the solution at time level t_{n-1}). Thus an advantage of a complex numerical method is it leads to a simpler linear system to solve (i.e., reduced size and bandwidth).

6.1.3 Results

Numerical results are presented in one dimension on a uniform partition of $\Omega = (0, L)$, for $0 \le t \le T$, with mesh points $x_j = jh$, $j = 0, \ldots, J$, where $h := L/J$. We undertake some experiments with $(\mathrm{P}_2^{h,\Delta t})^*$ and then make some comparisons with results in the literature using $(\mathrm{P}_2^{h,\Delta t})$ and $(\mathrm{P}_3^{h,\Delta t})$. Programs were run on a Linux PC and written in Fortran 77 and Matlab.

To test the error bound (6.1.5) we chose the following data for $(\mathrm{P}_2^{h,\Delta t})^*$: $L = 60$, $T = 1/6$, $\rho = 2$, $\lambda_0 = 3$, $\lambda_1 = 2$, $\omega_0 = -5$, $\omega_1 = 1$. The amplitude was set at $\hat{r} = (r_{\max} + r_{\min})/2 \approx 1.1299$. We computed the ratios (see (6.1.14), (6.1.15))

$$R_i^h := \frac{\xi_i(h,\Delta t) - \xi_i(h/2,\Delta t)}{\xi_i(h/2,\Delta t) - \xi_i(h/4,\Delta t)}, \quad R_i^{\Delta t} := \frac{\xi_i(h,\Delta t) - \xi_i(h,\Delta t/2)}{\xi_i(h,\Delta t/2) - \xi_i(h,\Delta t/4)} \quad i = 0, \infty, \tag{6.1.26}$$

which led to the results in Table 6.1 and Table 6.2 for discretisation in space and time respectively. If we assume the quantities $\xi_0(h,\Delta t)$, $\xi_\infty(h,\Delta t)$ can be written in the form

$$ah^p + A(\Delta t)^q, \quad p, q \in \mathbb{N}, \quad a, A \in \mathbb{R},$$

then $R_i^h = 2^p$ and $R_i^{\Delta t} = 2^q$ $(i = 0, \infty)$. From the tabulated results we conclude

$$p = q = 2, \quad |a| \ll |A|.$$

This suggests it may be possible to improve the theoretical result of first order in the time step for the error bound. Furthermore, the condition $|a| \ll |A|$ implies the contribution to the error from space discretisation is much less than the contribution

h	$\xi_0(h, 1/80)$	$\xi_\infty(h, 1/80)$	R_0^h (3 s.f.)	R_∞^h (3 s.f.)
1/2	0.0198878034	0.0891816059	3.92	3.92
1/4	0.020155689	0.0909897348	3.91	3.89
1/8	0.0202240452	0.0914510397	4.00	3.81
1/16	0.020241534	0.0915695369	–	–
1/32	0.0202459105	0.0916006106	–	–

Table 6.1: Numerical results from $(\mathrm{P}_2^{h,\Delta t})^*$ used to test the error bound in Theorem 5.5.1. $\Delta t = 1/80$ and the space step is successively halved.

Δt	$\xi_0(1/4, \Delta t)$	$\xi_\infty(1/4, \Delta t)$	$R_0^{\Delta t}$ (3 s.f.)	$R_\infty^{\Delta t}$ (3 s.f.)
1/80	0.0201556965	0.0909897574	3.74	3.92
1/160	0.00527169516	0.0230973836	4.12	4.01
1/320	0.0012926067	0.00575748705	3.95	4.01
1/640	0.000325840155	0.00143159815	–	–
1/1280	8.08132988E-05	0.000352038359	–	–

Table 6.2: Numerical results from $(\mathrm{P}_2^{h,\Delta t})^*$ used to test the error bound in Theorem 5.5.1. $h = 1/4$ and the time step is successively halved.

to the error from time discretisation. Thus it is possible to have $\frac{\xi_i(h,\Delta t)}{\xi_i(h/2,\Delta t)} \approx 1$, $(i = 0, \infty)$, even when Δt is much smaller than h. This is reflected in the observation that provided the space step is small compared to the wavelength of the travelling wave solutions (see below), then the qualitative features of the solution appear independent of refinements of the mesh in space.

In plots (a) - (d) of Figure 6.2, the numerical solution U^n of $(\mathrm{P}_2^{h,\Delta t})^*$ and the exact solution $u(x,t)$ of (6.1.1) are plotted together at time intervals of 5 units, with initial data corresponding to (6.1.1). The mesh is refined by reducing Δt with h fixed at 1/8. As Δt is increased beyond the critical value of $1/[\lambda_0(2\rho + 2)] = 1/18$ (see Lemma 5.1.2) the amplitude reduces to zero over time, corresponding to a stable fixed point at (0,0) of the numerical scheme. As the origin in the u-v plane is an unstable fixed point of the linearised $\lambda - \omega$ system this behaviour illustrates a spurious solution of the numerical scheme for large Δt. At the critical value of

$\Delta t = 1/18$ the amplitude of the numerical solution matches the amplitude of the exact solution well, but the phase is poorly reproduced. As Δt is reduced from $1/18$ the poorly represented phase recovers until at $\Delta t = 1/320$ there is good qualitative agreement between the approximate and the exact solution, except at $x = L$. This may be due to the fact that the travelling waves are moving in the positive x-direction ($c \approx 5.57$), resulting in the propagation of errors to the right hand side of the domain. Numerical experiments reveal the reverse effect when the wave speed is negative (for example, reversing the signs of ω_0 and ω_1 gives $c \approx -5.57$). As Δt is further reduced we observe convergence of the approximate solutions to the continuous ones.

Similar results were obtained for a wide variety of other parameter values.

As discussed in Section 6.1.1, the family of travelling waves (6.1.1) is posed on the unbounded domain $\mathbb{R}$, thus we would expect the scheme $(\mathrm{P}_2^{h,\Delta t})$ to represent poorly the analytical solution near the boundary. Furthermore, as the solution evolves we would expect the discrepancy between these two solutions to increase due to 'pollution' of the solution near the boundary. In Figure 6.3 we plot the numerical solution U^n of $(\mathrm{P}_2^{h,\Delta t})$ at equally spaced times t, starting from initial data corresponding to (6.1.1). Results of experiments indicate ω_1 is a key parameter in the 'pollution' of the periodic plane waves. When ω_1 is greater than zero the plane waves are affected mostly on the left hand side of the domain (Figure 6.3(a)) and the reverse situation occurs for ω_1 less than zero (Figure 6.3(b)). If we let ω_1 approach zero, the periodic plane waves are eroded from both ends of the boundary at an approximately equal rate as the solution evolves (Figure 6.3(c) and Figure 6.3(d))[5]. The 'pollution' of the approximate solutions does not disappear with additional refinements of the time step.

An interesting feature occurs in our numerical solutions if we take both ω_0 and ω_1 equal to zero[6]. From (6.1.4) this implies the wave speed is zero, i.e., we have

[5]Strictly speaking, from the definition of the $\lambda - \omega$ system (see (1.1.1f)) we must have $\omega_1 \neq 0$, however from the analysis in Section 1.2 there does not appear to be any reason why we cannot take $\omega_1 = 0$.

[6]From the analysis in Section 1.2 recall that $\dot{\theta} = \omega(r_0)$, thus taking both ω_0 and ω_1 equal to

standing waves. This situation is illustrated in Figure 6.4 where we compare the numerical solution U^n of $(\mathrm{P}_2^{h,\Delta t})$ with the corresponding standing wave solution $u(x,t)$ in (6.1.1). We chose the domain so the analytical solution has zero flux on the boundary. The approximations are symmetric about the line $x = 3w_l$ (w_l is the wavelength). A consideration of the behaviour of the approximations in the central portion of the domain was the aim of the next experiment.

In plots (a) - (d) of Figure 6.5, we compare the numerical solutions U^n of $(\mathrm{P}_2^{h,\Delta t})$, U^n of $(\mathrm{P}_3^{h,\Delta t})$, and the exact travelling wave solution $u(x,t)$ of (6.1.1). Solutions are plotted together at regular time intervals with initial data corresponding to (6.1.1). The mesh is refined by reducing Δt with h fixed at $1/2$. The solutions are calculated with $L = 40$, but displayed on the interval $(10, 30)$ to investigate behaviour of the approximations away from the 'polluted' solutions discussed above. As Δt is reduced we observe convergence of the approximate solutions to the corresponding exact solutions in the central region of the domain. Furthermore, the solutions of $(\mathrm{P}_2^{h,\Delta t})$ and $(\mathrm{P}_3^{h,\Delta t})$ perform approximately equally well. Similar results were obtained for a wide range of other parameter values.

In the next set of experiments we compare/reproduce numerical results in the literature.

In Figure 6.6 we illustrate the numerical solution U^n of $(\mathrm{P}_2^{h,\Delta t})$ at successive times t, where the vertical separation of solutions is proportional to Δt. We chose exponentially decaying initial data and parameter values to make comparisons with the corresponding results in [75], which were obtained using Gear's method with the Method of Lines. Gear's method is a variable order, variable step-size scheme for stiff ODE systems, utilising a parameter ε to bound the estimated local error at each time step. In our plots the periodic plane waves are clearly visible and there is good qualitative agreement with the corresponding plots in [75] (e.g., speed and direction of travelling waves and the speed and direction of the decaying wavefronts). We would have liked to reproduce results using Gear's method, but the corresponding NAG routine is now obsolete.

zero would imply a non-periodic solution. However, this is invalid for systems of $\lambda - \omega$ type as in the derivation of the system it is an assumption that ω_0 is non-zero.

A similar experiment is illustrated in Figures 6.7(a) and 6.7(b) where we have plotted U^n of $(P_3^{h,\Delta t})$ and U^n of $(P_2^{h,\Delta t})$ respectively, evolving from an initial 'pulse' at the origin, namely, U^0, $V^0 \in S^h$ such that

$$U^0(x_j) = V^0(x_j) = \begin{cases} 0.01, & \text{if} \quad j = 0, \\ 0.0, & \text{if} \quad j \neq 0. \end{cases}$$

Note that each value of h leads to a different initial approximation. In Figure 6.7(a) we reproduce a plot presented in [77], using the same parameter values.[7] This paper uses the finite difference method $(Q_3^{h,\Delta t})$ (see (6.1.25)), which is equivalent to the finite element method $(P_3^{h,\Delta t})$. The paper in [77] employs homogeneous Dirichlet boundary conditions at $x = L$, but as explained in this paper, provided the domain is sufficiently large compared to the evolving wavefront, then the solution is independent of the specific boundary condition at $x = L$. The qualitative features of the approximations in Figure 6.7(a) match well the corresponding plot in [77]. For example, both plots possess similar regions of irregular oscillations, periodic plane waves and decaying wavefronts. The main difference is in the region of irregular oscillations, but evidence is presented in [76] that this behaviour is temporally chaotic, so these differences may be due to small differences in the data. In Figure 6.7(b) we have used $(P_2^{h,\Delta t})$ to generate the corresponding numerical solution U^n. Results are similar to those in Figure 6.7(a), but the region of periodic plane waves is significantly smaller. As Δt is well below the critical value of $1/6$ (see Lemma 5.1.2) we know the irregular oscillations cannot be the result of numerical instability. Evidence is given in [77] suggesting that at $t = 76$ the transition point between the irregular oscillations and the periodic plane waves occur approximately at $x = 70$. However, additional refinements of the approximations using $(P_2^{h,\Delta t})$ (see Figures 6.8(a) and 6.8(b)), indicate that this transition point is considerably smaller and the solutions presented in Sherratt's paper do not yet represent convergence. A puzzling phenomena is that there is a corresponding shrinkage of the region containing the irregular oscillations. Additional experiments are needed to understand this phenomena, however with $\Delta t = 1 \times 10^{-7}$ the computations took many hours to

[7] However, the space and time steps are not given in this paper.

complete on a Linux PC.[8]

For our final experiment we compare more closely the numerical solutions U^n of $(\mathrm{P}_2^{h,\Delta t})$ with the corresponding solutions U^n of $(\mathrm{P}_3^{h,\Delta t})$, without the added complication of the irregular oscillations in the last set of experiments. Approximations are plotted at time intervals of 10 units, evolving from an initial 'pulse' at the origin (see Figure 6.9). As Δt is reduced we observe the solutions converge to each other over the whole domain. As in previous experiments, the match between the two solutions gets worse as time progresses.

6.2 Two-dimensional simulations

We take $\Omega := (-L, L) \times (-L, L)$, a square uniform mesh with vertices $(x_i, y_j) = (ih - L, jh - L)$ where $i, j = 0, \ldots, J$ (see Figure 6.1). Note $h = 2L/J$, i.e., we

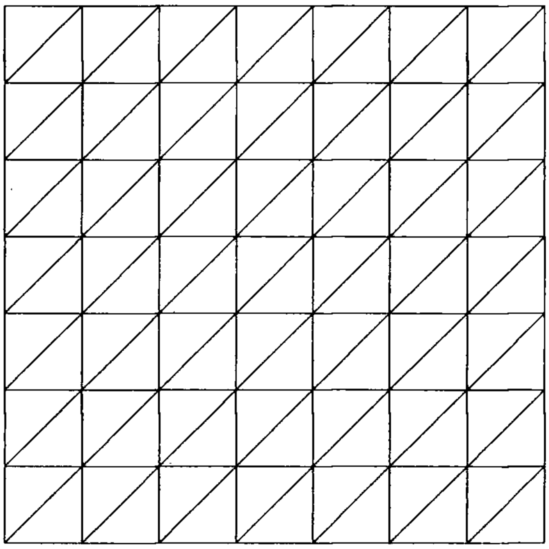


Figure 6.1: Mesh for two-dimensional finite element approximations.

used the same space step in both the x and y directions. We employ a 'right-angled' triangulation where each square is bisected by a diagonal running from the top-right corner to the bottom-left corner. Nodes are ordered in the 'natural way', that is, we number the nodes consecutively left to right starting with the bottom row. We implemented the fully-discrete, semi-implicit in time, finite element approximation

[8]With 512 Mb RAM and a 2 GHz processor

($P_2^{h,\Delta t}$) presented in Section 6.1.2, except now we have $2(J+1)^2$ unknowns and the resulting complex linear system (see (6.1.21)) has a block matrix structure (see Appendix C). Programs were run on a Linux PC and written in Matlab. The resulting linear systems were solved directly with sparse matrix facilities in Matlab. As in the one dimensional case, the linear system is strictly diagonally dominant for Δt sufficiently small and so no partial pivoting is required (see Theorem A.0.31).

6.2.1 Preliminary results

In Figures 6.10 - 6.13 we illustrate the numerical solution V^n of ($P_2^{h,\Delta t}$) at times $t = 10, 17, 24$, and 31 respectively, evolving from Gaussian initial data at the origin, namely, $0.1\exp\{-0.8(x^2+y^2)\}$. Numerical results represent radially symmetric ring waves ('target patterns'), centred at the origin, with a rapid decay to zero beyond the wavefront. As discussed in Section 1.3, it is known that target patterns exist as solutions to reaction-diffusion equations of the $\lambda - \omega$ type [30], [46]. We remark that the two-dimensional results correspond to the one-dimensional results illustrated in Figure 6.6.[9] The semi-infinite spatial domain in the one-dimensional case (the positive x-axis) corresponds to a radial component of an expanding (circular) wavefront in the two-dimensional case. The crests (or troughs) of the travelling waves behind the front at some fixed time in the one-dimensional case correspond to the rings of the target patterns. If the wave speed in the one-dimensional case is positive, then this corresponds to an expansion of the concentric rings of the target patterns in the two-dimensional case (with the reverse situation if the wave speed is negative).[10]

For the next set of experiments we were interested in numerically simulating spiral wave solutions, which have been proved to exist as solutions of $\lambda - \omega$ systems [16]. Numerous authors have investigated spiral solutions of $\lambda - \omega$ systems, for example [20], [31], [50], [35], [44]. A rotating spiral wave has the form in polar

[9]We also obtain qualitatively similar results to those in Figure 6.6 with the Gaussian initial data $0.1\exp\{-0.8x^2\}$.

[10]However, this is distinct from the speed of the advancing front, which is always positive.

coordinates[11] (R, ϕ)

$$\begin{matrix} u(R,\phi) \\ v(R,\phi) \end{matrix} = r(R) \begin{matrix} \cos \\ \sin \end{matrix} \{\Theta t \pm m\phi + S(R)\},$$

where Θ is the frequency of rotation, m is the number of arms on the spiral and $S(R)$ is a function that determines the type of spiral, e.g., Archimedian if $S(R) = aR$, or logarithmic if $S(R) = a\ln(R)$, for some constant a. The $\pm$ in the $m\phi$ term determines whether rotation is counter-clockwise or clockwise spatially.[12] For behaviour near the 'core' (i.e., the centre of the spiral) it can be shown that spiral solutions of the $\lambda - \omega$ system have $r(R) \propto R^m$ [44], [62], p.352. This suggests we might be able to generate rotating Archimedian spirals from the initial data $u_0 = c_1 R^m \cos\{m\phi\}$, $v_0 = c_1 R^m \sin\{m\phi\}$ for some constants c_1 and c_2. However, we have seen no analytical conditions for the stability of spiral waves of $\lambda - \omega$ systems.

In Figures 6.14 - 6.17 we illustrate the numerical solution U^n of $(\mathrm{P}_2^{h,\Delta t})$ at times $t = 100, 200, 300$, and 400 respectively, evolving from initial data $C^0 = R\exp\{i\phi\}$. As expected, a rotating 1-armed spiral develops and persists. Furthermore, when we changed the sign of ϕ this reversed the sense of rotation of the spiral in space. Moreover, changing m to be 2 or 3 (and the initial data appropriately) resulted in the production of 2-armed or 3-armed spirals respectively.

We have also undertaken some simulations in two dimensions using $(\mathrm{P}_3^{h,\Delta t})$ with qualitatively similar result to those from $(\mathrm{P}_2^{h,\Delta t})$.[13]

6.3 Concluding remarks

Scientific computing has an important part to play in the investigation of oscillatory reaction-diffusion equations. This is due to the fact that systems are nonlinear and analytical solutions are only known in a few specific cases. We undertook various experiments and investigated some of the qualitative features of our solutions.

[11] We use this notation for the polar coordinates to distinguish it from the polar coordinates in phase space (r, θ) of Section 1.4.

[12] The spiral wave also rotate counter-clockwise or clockwise temporally.

[13] Results not presented.

The $\lambda - \omega$ system possesses a range of solution dynamics depending on the specific model parameters and the data. The advantage of using a numerical method with known stability and accuracy properties cannot be overstated. For example, our theoretical results prove the irregular behaviour in Figure 6.7(b) cannot be due to numerical instability. It is therefore tempting to conclude that the irregular oscillations represent chaos in the underlying continuous solutions, but this behaviour disappears with additional refinements of the time step (see Figure 6.8(b)). Further investigations are needed to understand this behaviour.

The fully-discrete finite element method was tested in various ways. We checked the convergence of our approximations and verified the fully-discrete error bound with the aid of a family of analytical solution on $\mathbb{R}$. As these analytical solutions are posed on the unbounded domain we employed two different strategies to make comparisons meaningful. In the first approach, we used time-dependent Neumann boundary conditions corresponding to this family of solutions. In the second approach, we looked at the numerical solutions for $(\mathrm{P}_2^{h,\Delta t})$ sufficiently far from the 'pollution' effects due to the homogeneous Neumann boundary conditions. In both cases we illustrated convergence of the approximate solutions to the analytical ones. These studies also highlight the problems associated with truncating problems naturally posed on an unbounded domain. Most studies of nonlinear parabolic equations seem to ignore this issue; a notable exception is a paper by Hagstrom and Keller [36].

We compared our approximations with the corresponding approximations in the literature obtained from Gear's method and a semi-implicit Crank-Nicolson method $(\mathrm{P}_3^{h,\Delta t})$. Results were qualitatively similar. We also coded the semi-implicit Crank-Nicolson method. This allowed us to reproduce numerical results in the literature and to illustrate the convergence of the methods $(\mathrm{P}_2^{h,\Delta t})$ and $(\mathrm{P}_3^{h,\Delta t})$ to each other. Note however that we have no underlying convergence theory for $(\mathrm{P}_3^{h,\Delta t})$.

The preliminary two-dimensional results are consistent with what is known about the specific 'ansatz' solutions of the $\lambda - \omega$ system. However, the studies we looked at did not state all the conditions needed to reproduce numerical results (notably, the initial data), thus a detailed comparison was not possible.

6.4 Figures

In this section we present the figures resulting from the numerical experiments discussed in Sections 6.1.3 and 6.2.1.

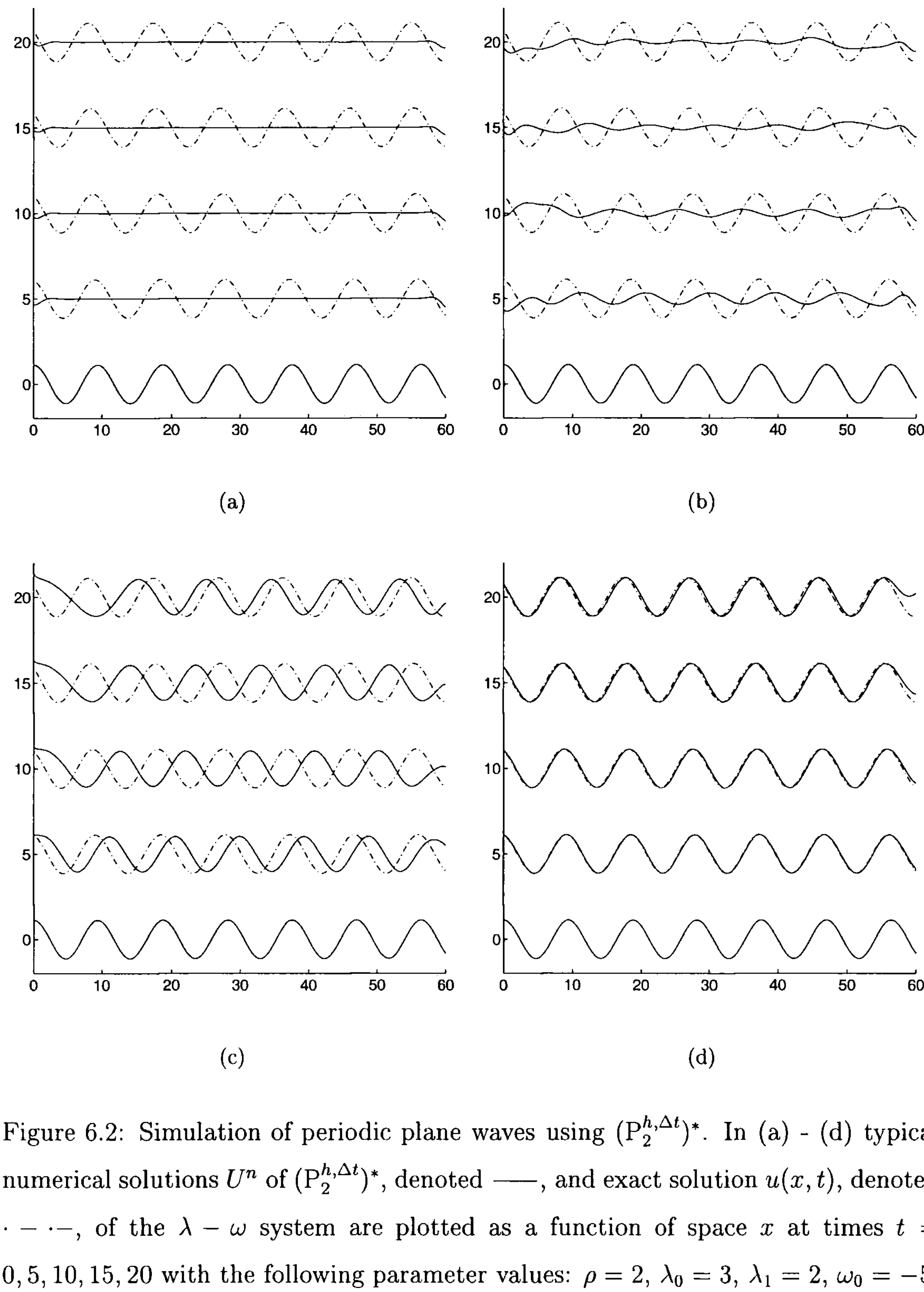


Figure 6.2: Simulation of periodic plane waves using $(\mathrm{P}_2^{h,\Delta t})^*$. In (a) - (d) typical numerical solutions U^n of $(\mathrm{P}_2^{h,\Delta t})^*$, denoted ——, and exact solution $u(x,t)$, denoted $\cdot - \cdot -$, of the $\lambda - \omega$ system are plotted as a function of space x at times $t = 0, 5, 10, 15, 20$ with the following parameter values: $\rho = 2$, $\lambda_0 = 3$, $\lambda_1 = 2$, $\omega_0 = -5$, $\omega_1 = 1$, $\widehat{r} \approx 1.1299$. Plots show successive refinement of Δt with h fixed at 1/8: (a) $\Delta t = 1/3$, (b) $\Delta t = 1/6$, (c) $\Delta t = 1/18$, (d) $\Delta t = 1/320$. The initial approximations correspond to (6.1.1).

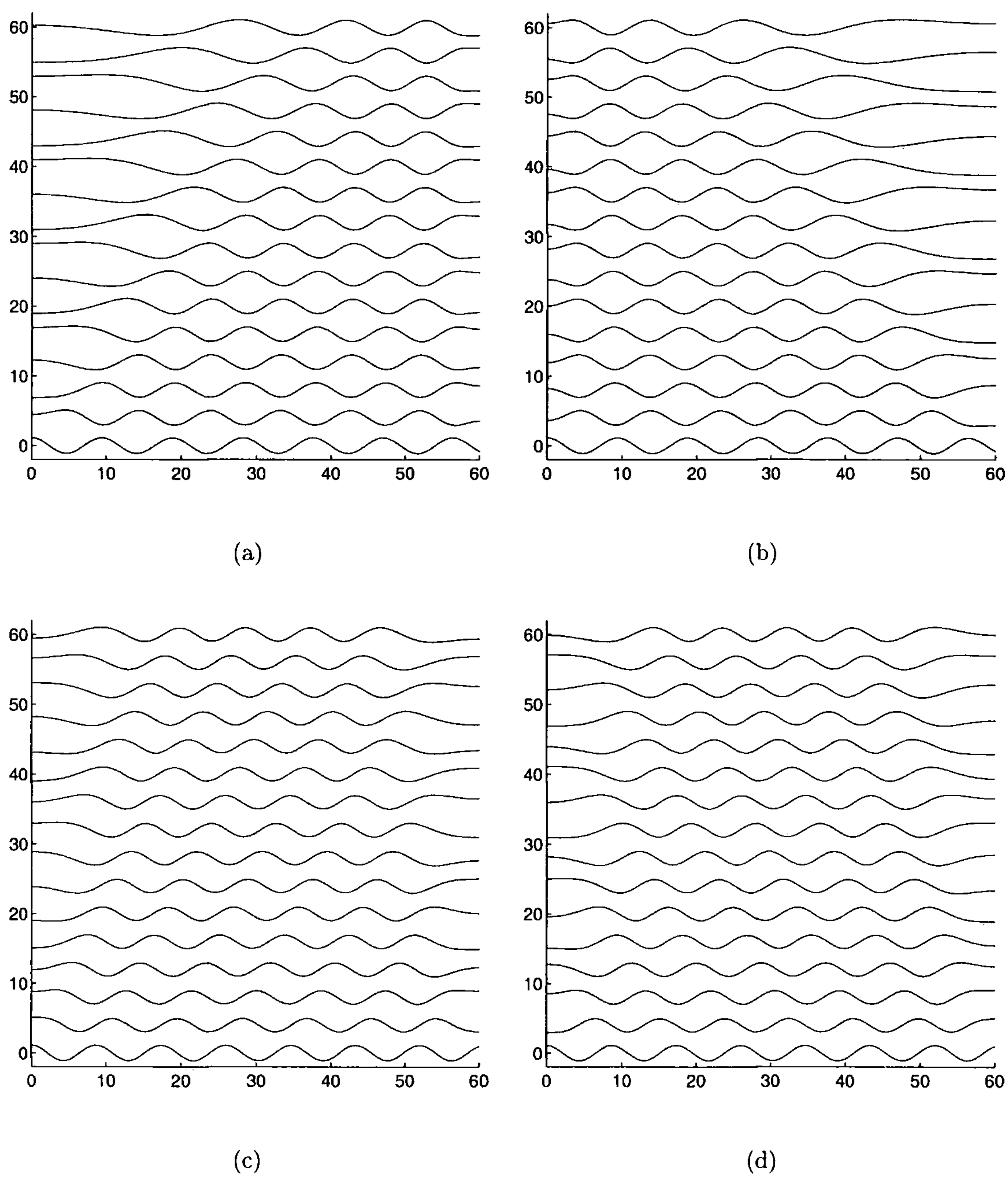


Figure 6.3: Simulation of periodic plane waves using $(P_2^{h,\Delta t})$. Numerical solutions U^n of $(P_2^{h,\Delta t})$ plotted as a function of space x at equally spaced times t with $h = 0.5$, $\Delta t = 0.05$ and the following parameter values: (a) $\rho = 2$, $\lambda_0 = 3$, $\lambda_1 = 2$, $\omega_0 = -5$, $\omega_1 = 1$, $\widehat{r} \approx 1.1299$, $c \approx 5.57$. (b) $\rho = 2$, $\lambda_0 = 3$, $\lambda_1 = 2$, $\omega_0 = 5$, $\omega_1 = -1$, $\widehat{r} \approx 1.1299$, $c \approx -5.57$. (c) $\rho = 2$, $\lambda_0 = 3$, $\lambda_1 = 2$, $\omega_0 = -5$, $\omega_1 = 0$, $\widehat{r} \approx 1.1124$, $c \approx 6.90$. (d) $\rho = 2$, $\lambda_0 = 3$, $\lambda_1 = 2$, $\omega_0 = 5$, $\omega_1 = 0$, $\widehat{r} \approx 1.1124$, $c \approx -6.90$. The initial approximations correspond to (6.1.1).

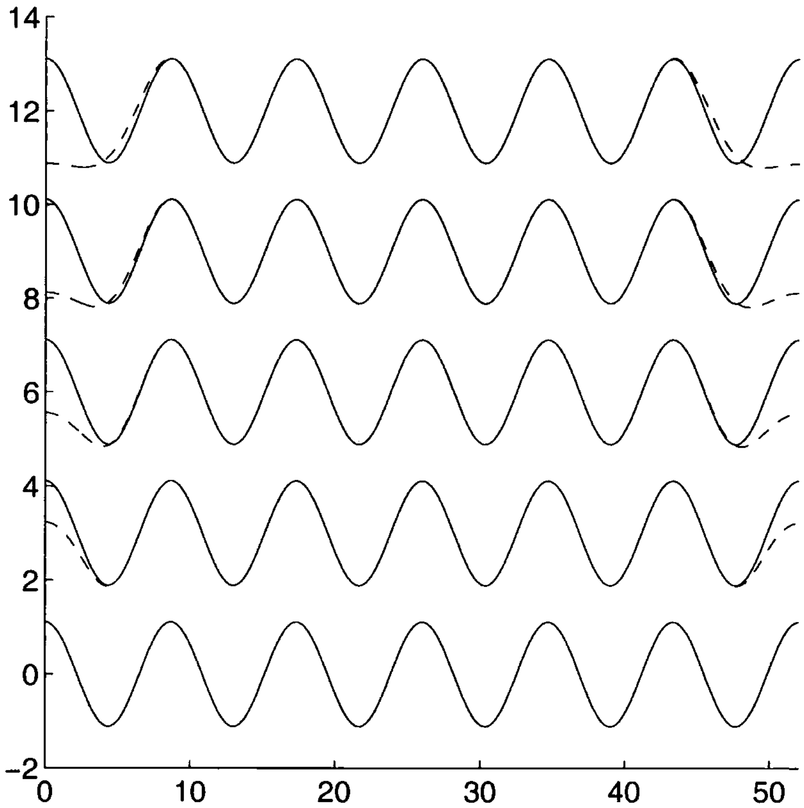


Figure 6.4: Simulation of standing waves using $(\mathrm{P}_2^{h,\Delta t})$. Numerical solutions U^n of $(\mathrm{P}_2^{h,\Delta t})$, denoted – – –, and exact solution $u(x,t)$, denoted ——, plotted as a function of space x at times $t = 0, 3, 6, 9, 12$. The parameter values are: $\rho = 2$, $\lambda_0 = 3$, $\lambda_1 = 2$, $\omega_0 = 0$, $\omega_1 = 0$, $h = 1/10$, $\Delta t = 1/20$, $\widehat{r} \approx 1.1124$, $L = 6w_l$, $c = 0$. The initial approximations correspond to (6.1.1).

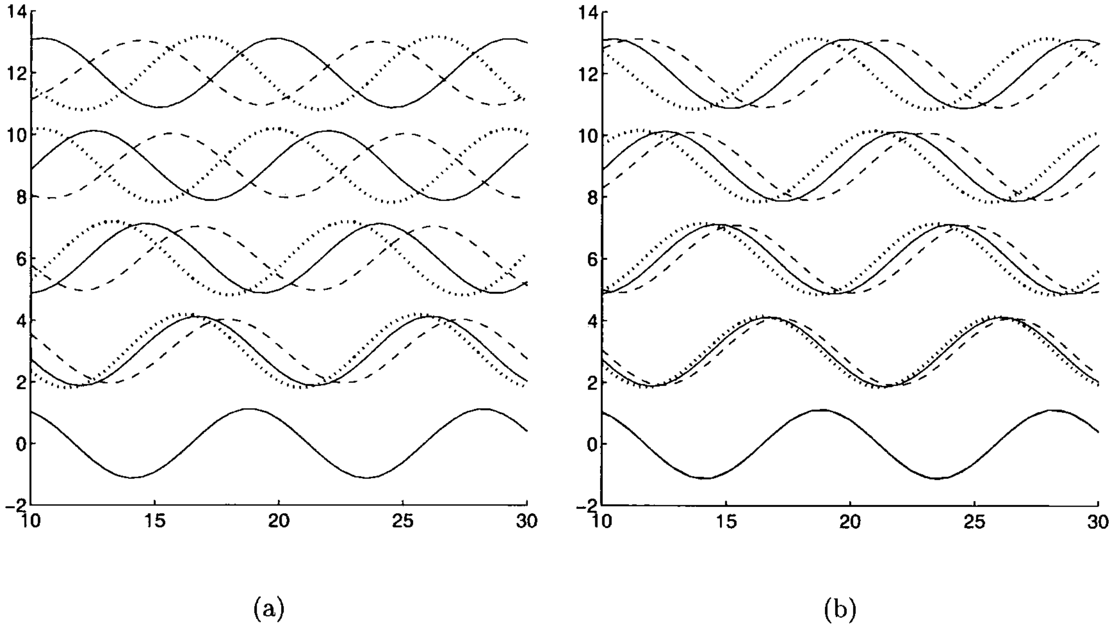


Figure 6.5: A comparison of numerical methods for simulating periodic plane waves. In (a) - (d) typical numerical solutions U^n of $(\mathrm{P}_2^{h,\Delta t})$ denoted $- - -$, U^n of $(\mathrm{P}_3^{h,\Delta t})$ denoted $\cdots\cdots$, and exact solution $u(x,t)$ denoted ——, plotted at times $t = 0, 3, 6, 9, 12$ with the following parameter values: $\rho = 2$, $\lambda_0 = 3$, $\lambda_1 = 2$, $\omega_0 = -5$, $\omega_1 = 1$, $T = 12$, $L = 40$ (solution near boundary not shown). Plots show successive refinement of Δt with h fixed at $1/2$: (a) $\Delta t = 1/20$, (b) $\Delta t = 1/50$, (c) $\Delta t = 1/80$, (d) $\Delta t = 1/160$. The initial approximations correspond to (6.1.1)

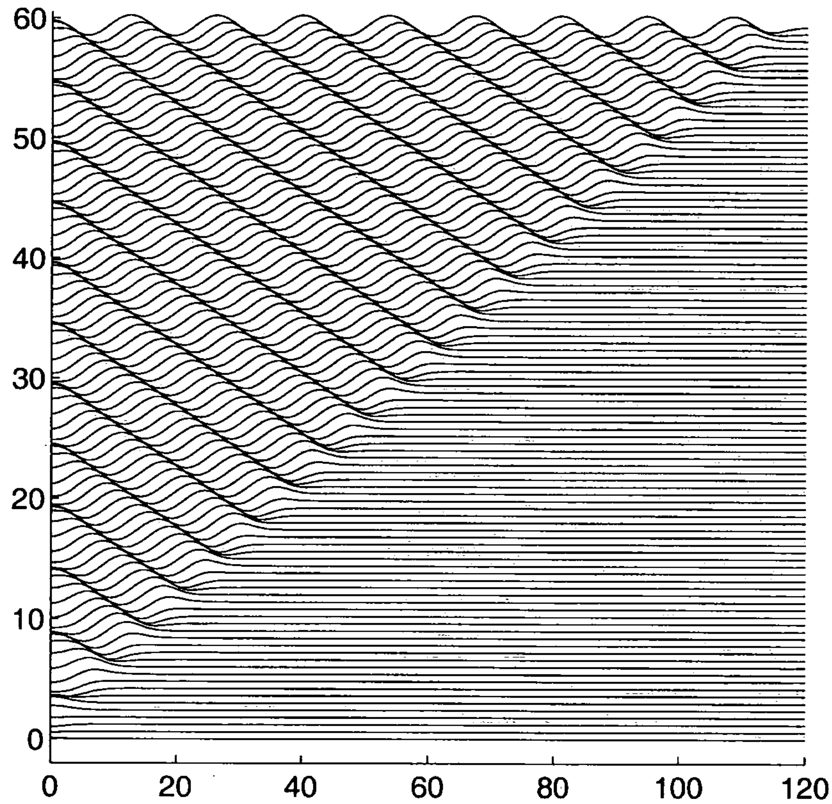
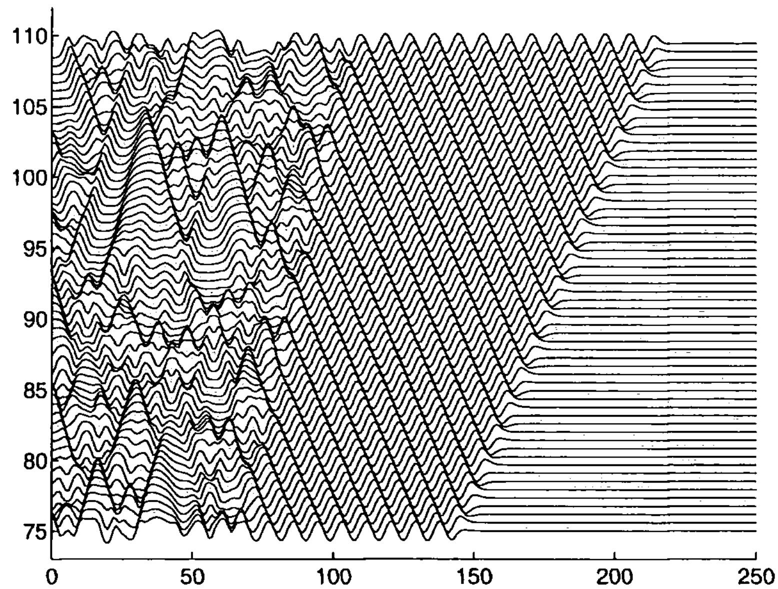
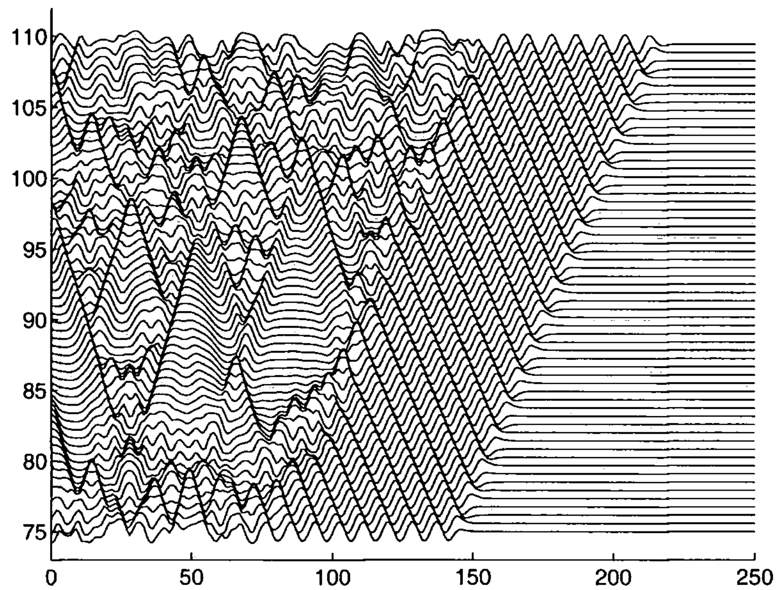


Figure 6.6: Typical numerical solution evolving from locally exponentially decaying initial data. Numerical solutions U^n of $(\mathrm{P}_2^{h,\Delta t})$ plotted as a function of space x at equally spaced times t with initial approximation $U^0 = V^0 = \pi^h(0.1\exp\{-0.8x\})$. The parameter values are: $\rho = 1.8$, $\lambda_0 = 1$, $\lambda_1 = 1$, $\omega_0 = 2$, $\omega_1 = -1$, $h = 1/4$, $\Delta t = 1/20$.

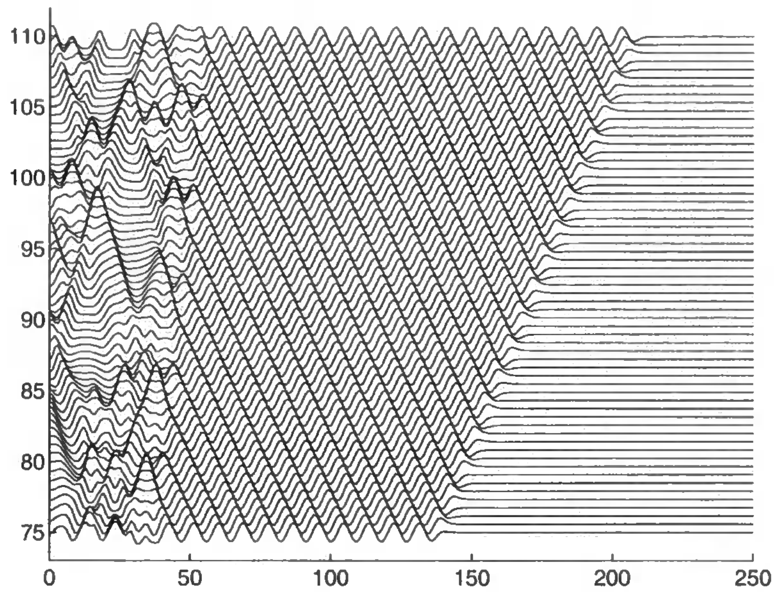


(a)

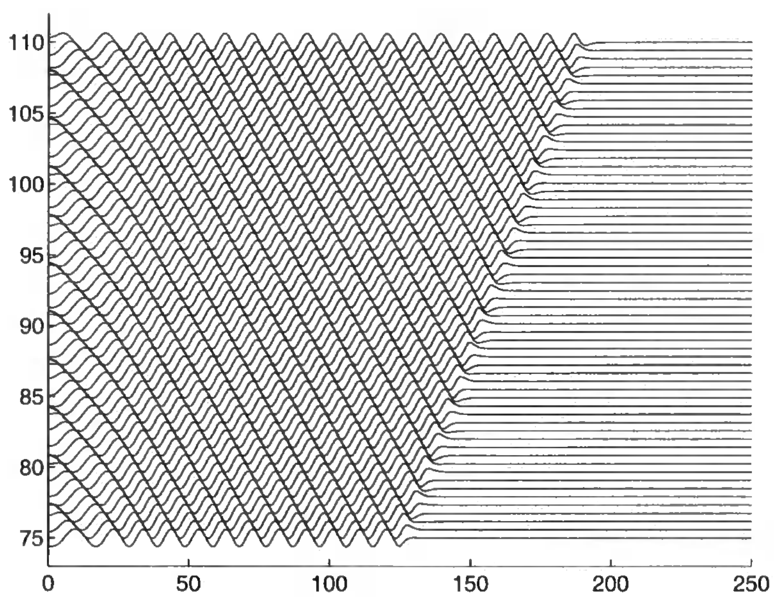


(b)

Figure 6.7: A comparison of numerical solutions illustrating spatiotemporal chaos. (a) Numerical solutions U^n of $(\mathrm{P}_3^{h,\Delta t})$ plotted as a function of space x at equally spaced times t. (b) Numerical solutions U^n of $(\mathrm{P}_2^{h,\Delta t})$ plotted as a function of space x at equally spaced times t. For the initial approximations we take $U^0, V^0 \in S^h$ s.t. $U^0(x_0) = V^0(x_0) = 0.01$, $U^0(x_j) = V^0(x_j) = 0$ if $j \neq 0$. The parameter values are: $\rho = 2$, $\lambda_0 = 1$, $\lambda_1 = 1$, $\omega_0 = 3$, $\omega_1 = -3$, $h = 1.0$, $\Delta t = 5 \times 10^{-4}$.



(a)



(b)

Figure 6.8: Additional refinements of Δt in Figure 6.7(b). Numerical solutions U^n of $(\mathrm{P}_2^{h,\Delta t})$ plotted as a function of space x at equally spaced times t. For the initial approximations we take $U^0, V^0 \in S^h$ s.t. $U^0(x_0) = V^0(x_0) = 0.01$, $U^0(x_j) = V^0(x_j) = 0$ if $j \neq 0$. The parameter values are: $\rho = 2$, $\lambda_0 = 1$, $\lambda_1 = 1$, $\omega_0 = 3$, $\omega_1 = -3$. Plots show successive refinement of Δt with h fixed at 1.0: (a) $\Delta t = 1 \times 10^{-6}$, (b) $\Delta t = 1 \times 10^{-7}$.

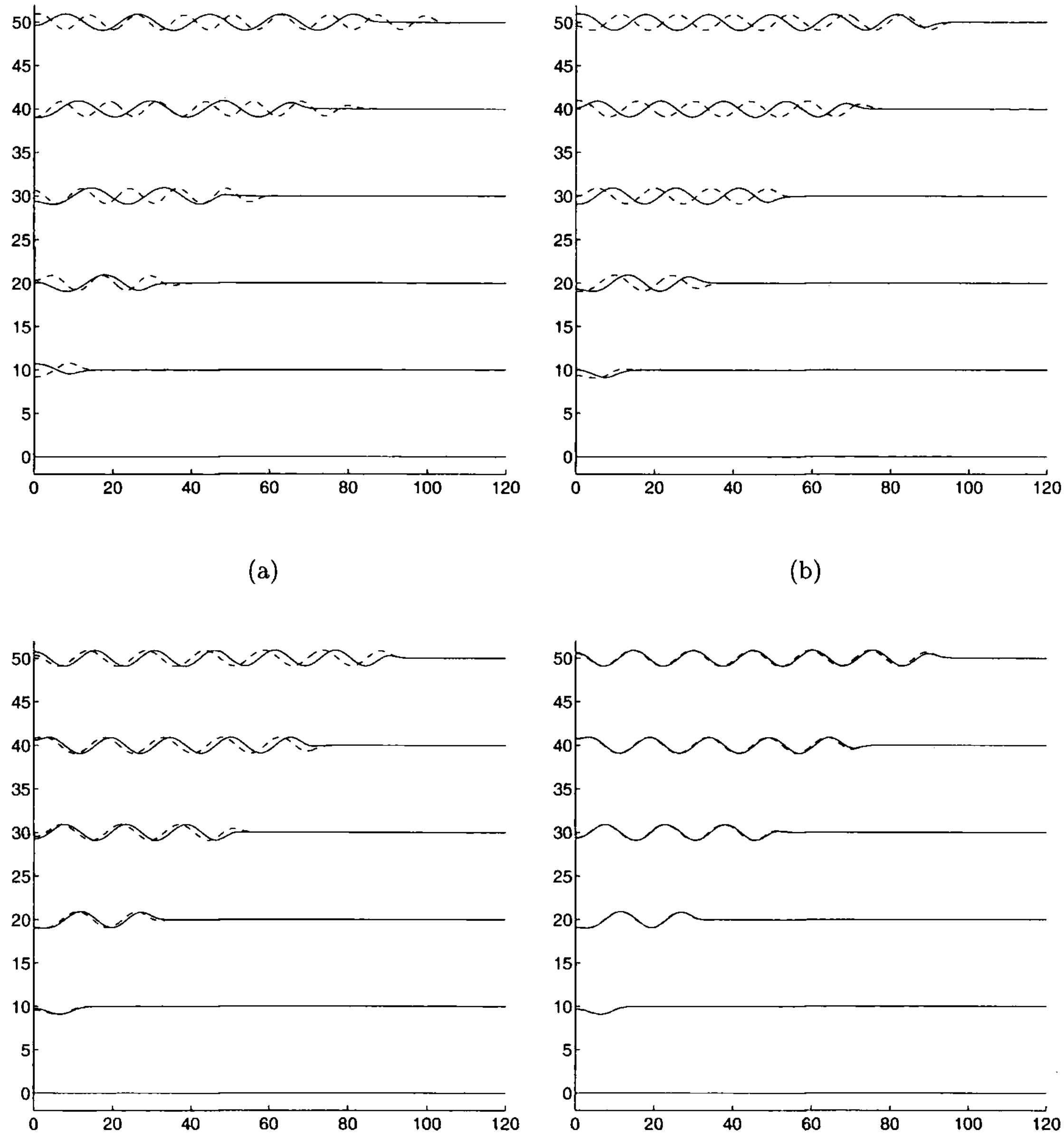


Figure 6.9: A comparison of numerical solutions evolving from an initial pulse. In (a) - (d) numerical solutions U^n of $(\mathrm{P}_2^{h,\Delta t})$, denoted $-\ -\ -$, and numerical solutions U^n of $(\mathrm{P}_3^{h,\Delta t})$, denoted ——, are plotted as a function of space x at times $t = 0, 10, 20, 30, 40, 50$ with the following parameter values: $\rho = 2$, $\lambda_0 = 1$, $\lambda_1 = 1$, $\omega_0 = 1$, $\omega_1 = -1$. Plots show successive refinement of Δt with h fixed at $1/2$: (a) $\Delta t = 1/5$, (b) $\Delta t = 1/20$, (c) $\Delta t = 1/80$, (d) $\Delta t = 1/320$. For the initial approximations we take $U^0, V^0 \in S^h$ s.t. $U^0(x_0) = V^0(x_0) = 0.01$, $U^0(x_j) = V^0(x_j) = 0$ if $j \neq 0$.

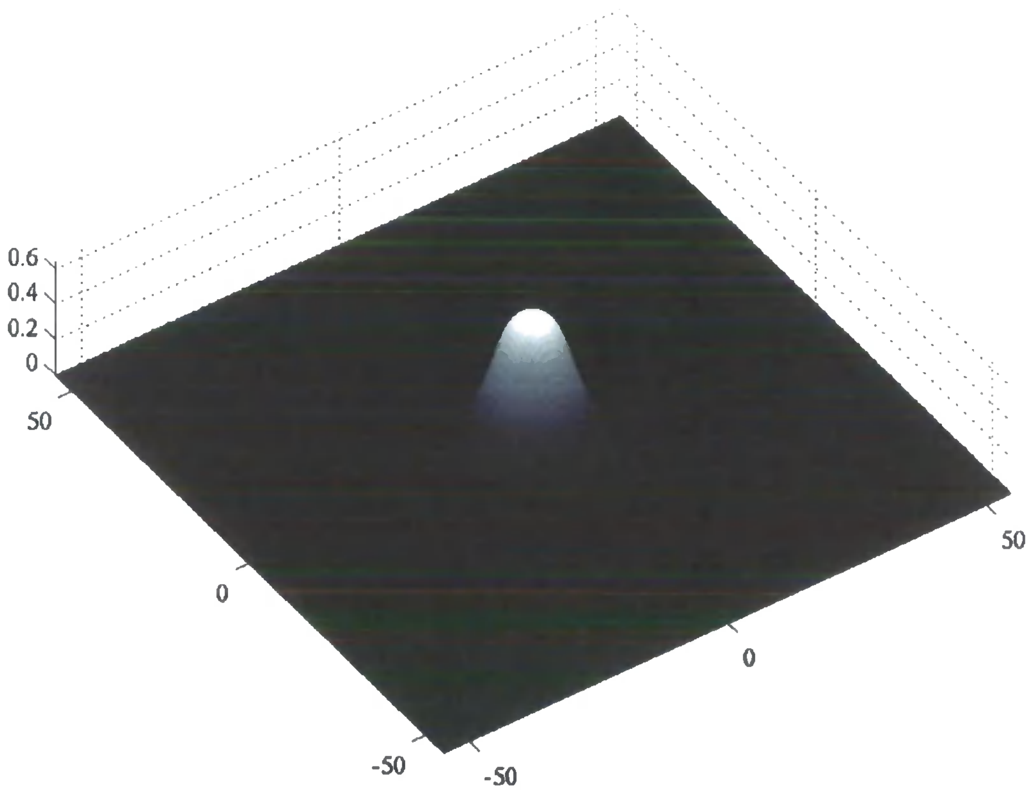


Figure 6.10: Simulation of concentric ring waves using $(\mathrm{P}_2^{h,\Delta t})$. Numerical solution V^n of $(\mathrm{P}_2^{h,\Delta t})$ plotted at time $t = 10$ with initial approximations $U^0, V^0 = 0.1\exp\{-0.8(x^2+y^2)\}$. Parameter values: $\rho = 1.5$, $\lambda_0 = 0.5$, $\lambda_1 = 0.1$, $\omega_0 = 0.1$, $\omega_1 = -1$, $\Omega = (-55,55) \times (-55,55)$, $h = 0.6$, $\Delta t = 0.5$.

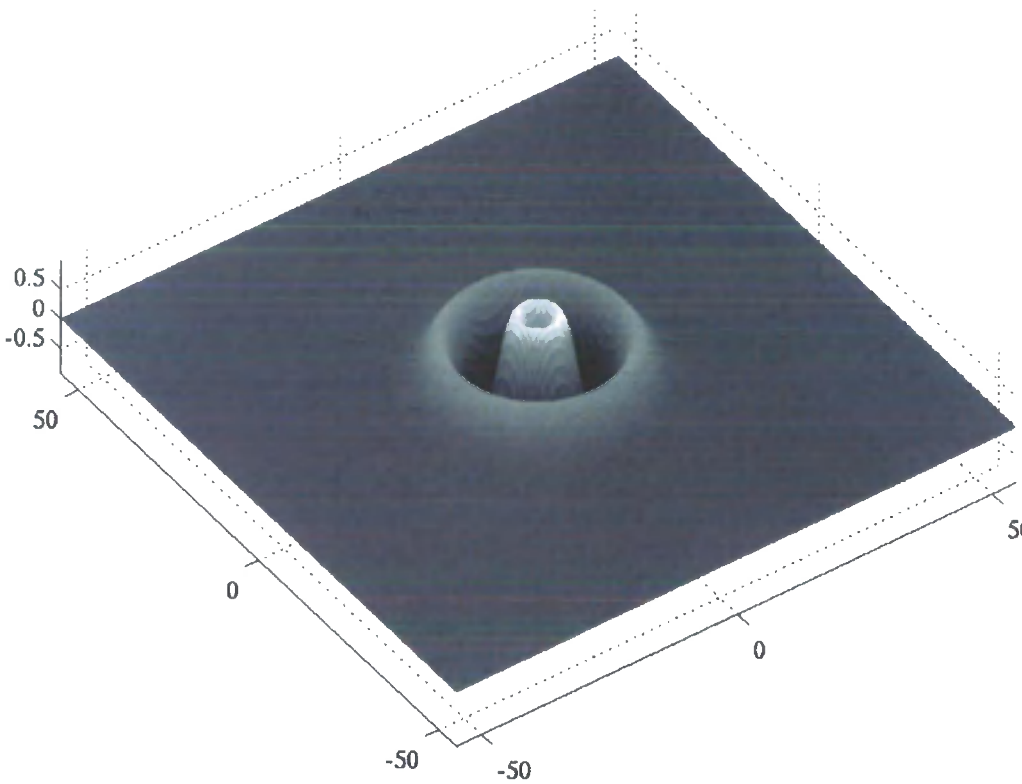


Figure 6.11: Simulation of concentric ring waves using $(\mathrm{P}_2^{h,\Delta t})$. Numerical solution V^n of $(\mathrm{P}_2^{h,\Delta t})$ plotted at time $t = 17$. Parameter values and initial data are given in Figure 6.10.

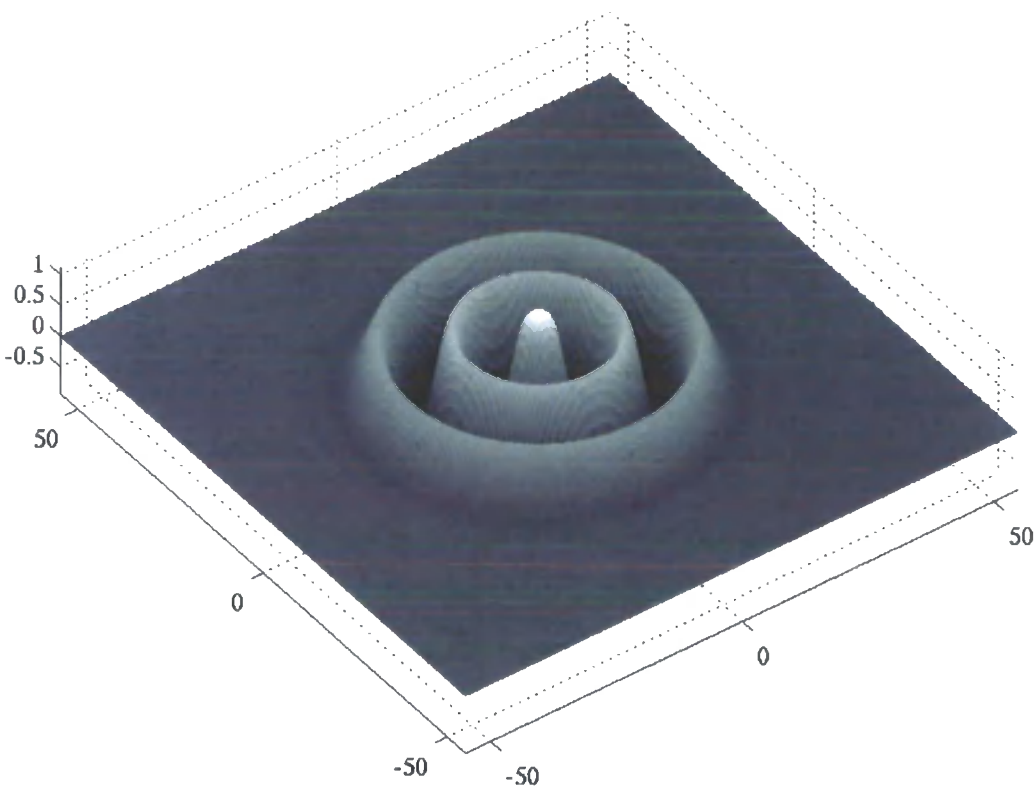


Figure 6.12: Simulation of concentric ring waves using $(\mathrm{P}_2^{h,\Delta t})$. Numerical solution V^n of $(\mathrm{P}_2^{h,\Delta t})$ plotted at time $t = 24$. Parameter values and initial data are given in Figure 6.10.

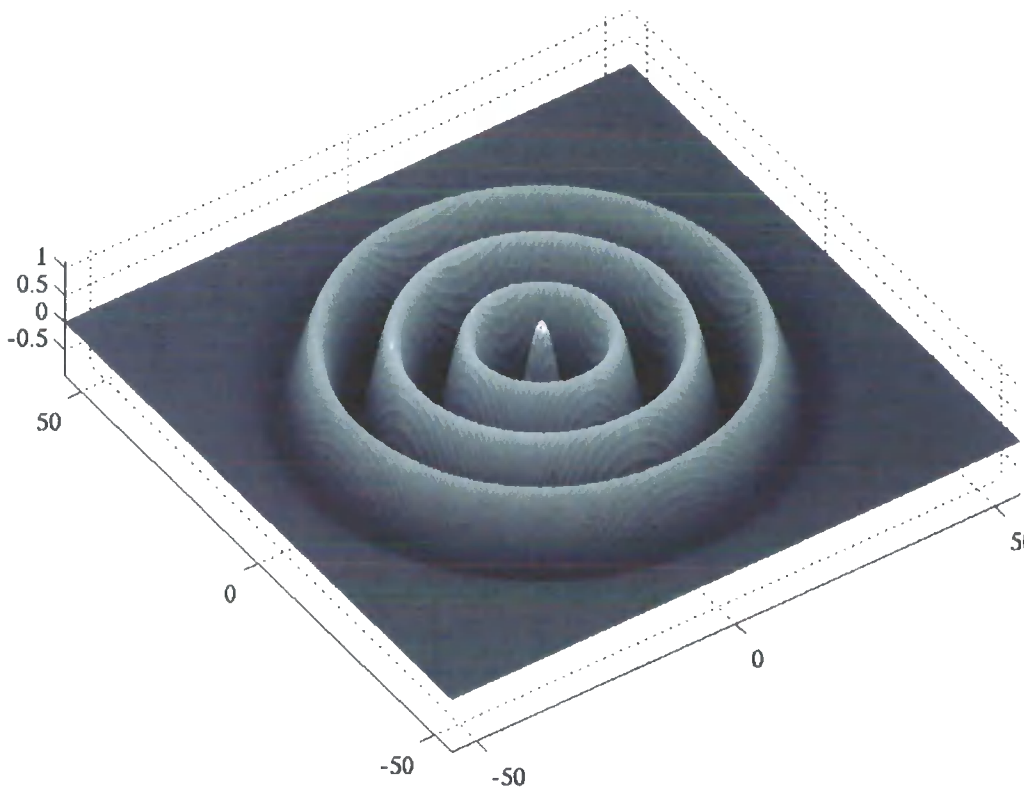


Figure 6.13: Simulation of concentric ring waves using $(P_2^{h,\Delta t})$. Numerical solution V^n of $(P_2^{h,\Delta t})$ plotted at time $t = 31$. Parameter values and initial data are given in Figure 6.10.

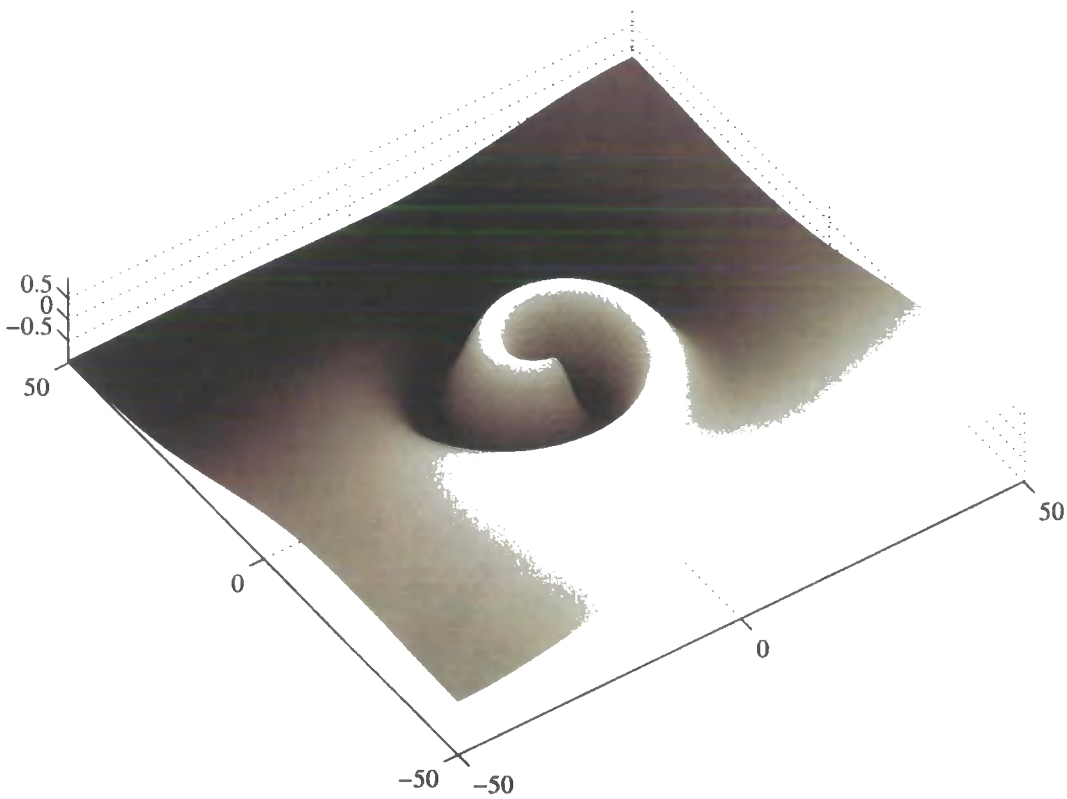


Figure 6.14: Simulation of spiral waves using $(\mathrm{P}_2^{h,\Delta t})$. Numerical solution U^n of $(\mathrm{P}_2^{h,\Delta t})$ plotted at time $t = 100$ with initial approximation $C^0 := R\exp\{i\phi\}$, where $R := \sqrt{x^2+y^2}$ and $\phi := \arctan(y/x)$. Parameter values: $\rho = 2$, $\lambda_0 = 1$, $\lambda_1 = 1$, $\omega_0 = 1$, $\omega_1 = -1$, $\Omega = (-50,50) \times (-50,50)$, $h = 0.6$, $\Delta t = 0.5$.

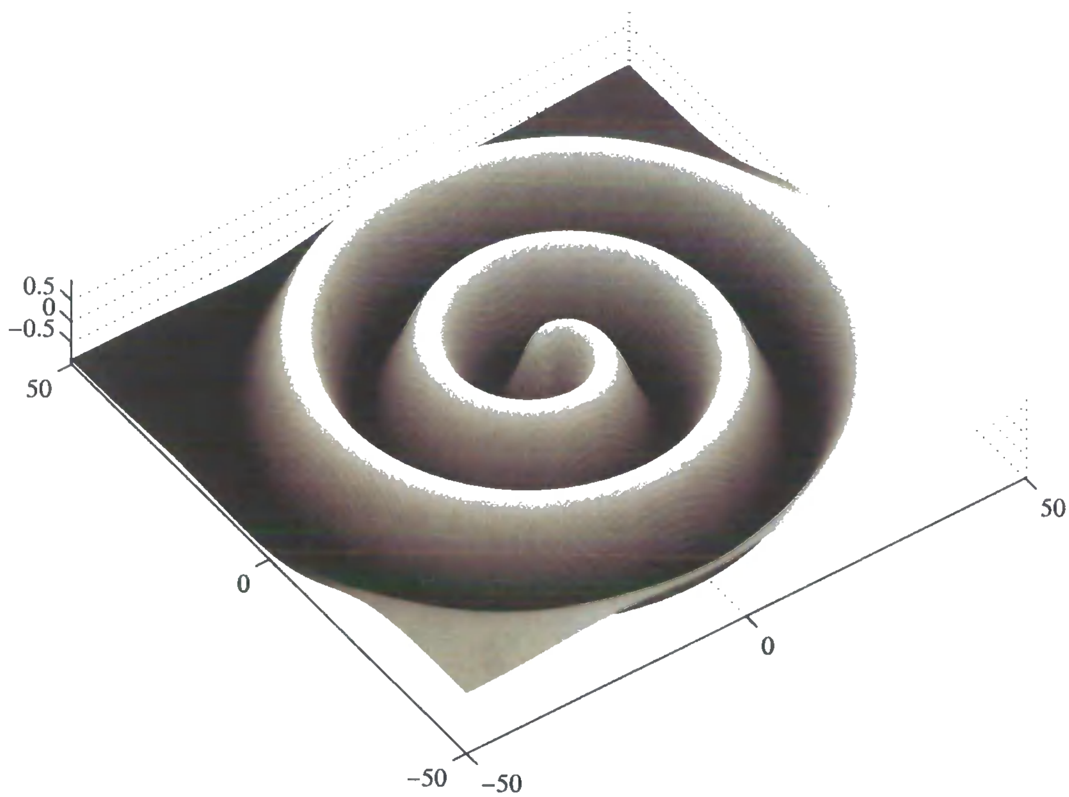


Figure 6.15: Simulation of spiral waves using $(P_2^{h,\Delta t})$. Numerical solution U^n of $(P_2^{h,\Delta t})$ plotted at time $t = 200$. Parameter values and initial data are given in Figure 6.14.

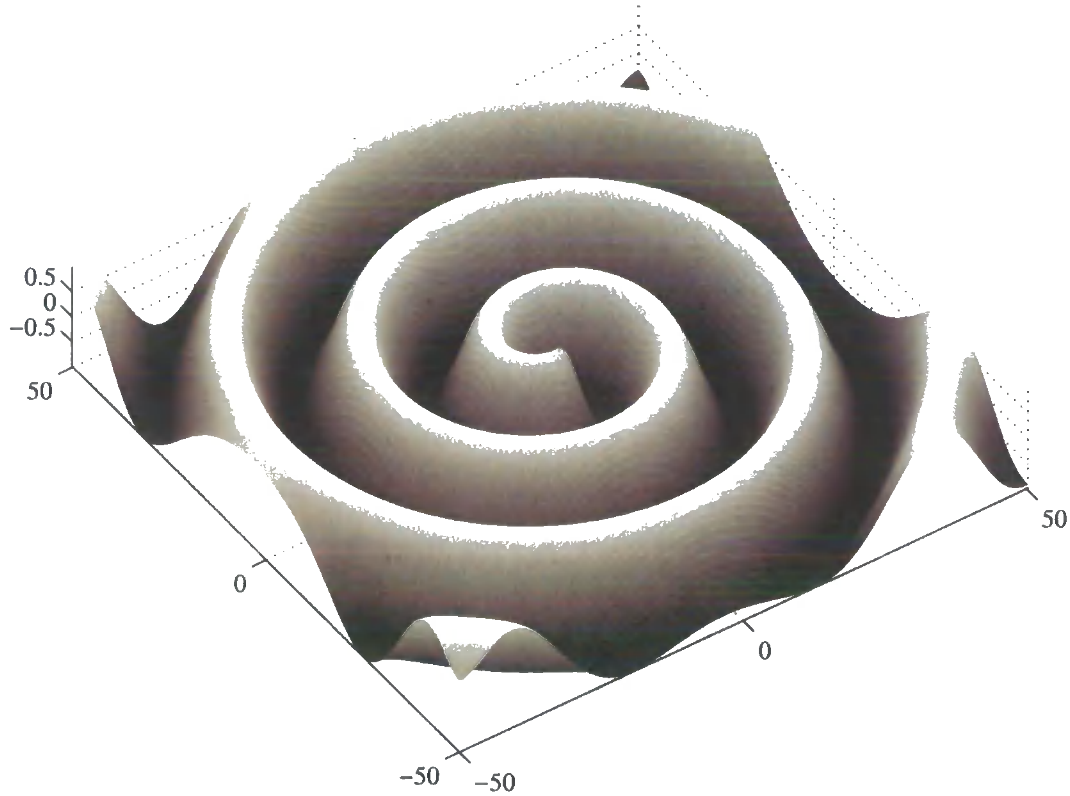


Figure 6.16: Simulation of spiral waves using $(\mathrm{P}_2^{h,\Delta t})$. Numerical solution U^n of $(\mathrm{P}_2^{h,\Delta t})$ plotted at time $t = 300$. Parameter values and initial data are given in Figure 6.14.

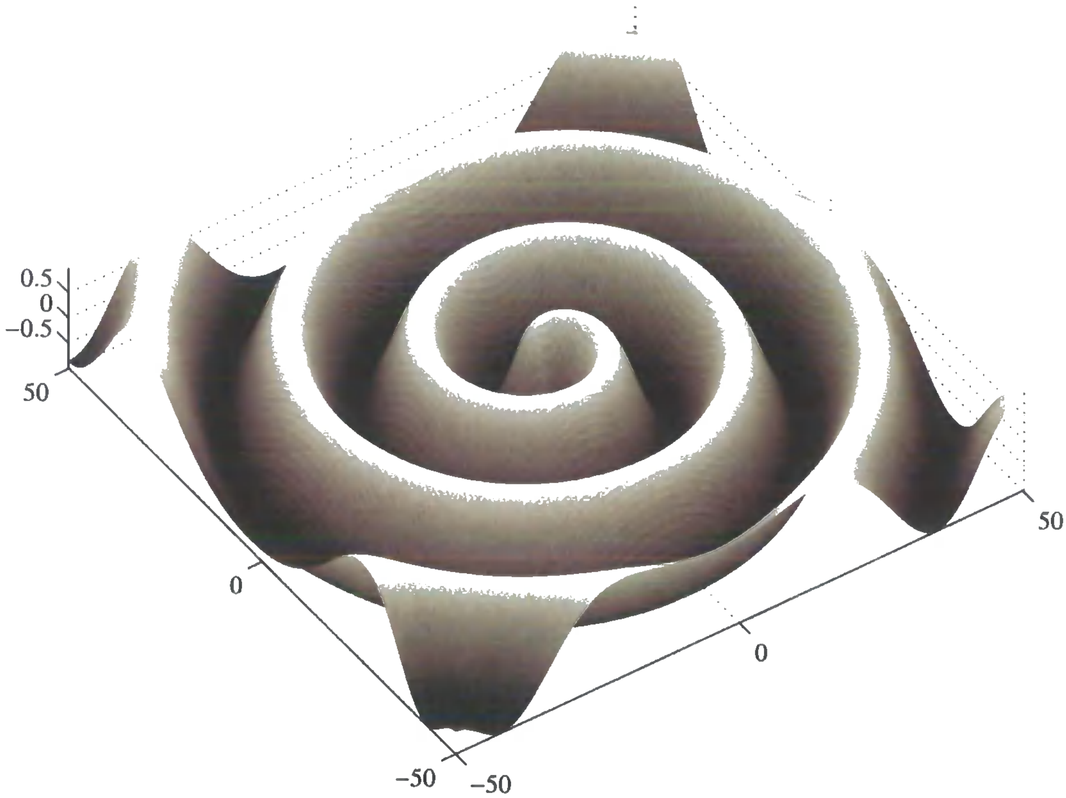


Figure 6.17: Simulation of spiral waves using $(\mathrm{P}_2^{h,\Delta t})$. Numerical solution U^n of $(\mathrm{P}_2^{h,\Delta t})$ plotted at time $t = 400$. Parameter values and initial data are given in Figure 6.14.

Chapter 7

Summary and discussion

We studied the weak and strong solutions of a generalised $\lambda - \omega$ reaction-diffusion system in $d \leq 3$ space dimensions. With minor adjustments the proofs are applicable to the homogeneous Dirichlet and periodic boundary conditions as well. Provided the initial data is square integrable, we proved global existence, uniqueness and continuous dependence on initial data of the weak solution, subject to restrictions on the parameter ρ. Furthermore, if the initial data is in $H^1(\Omega)$, then there is a unique global strong solution depending continuously on the initial data, subject to additional restrictions on the parameter ρ. When $\rho = 2$, $d = 3$, we were unable to prove uniqueness of weak solutions, or global regularity results, except in the special case (see Estimate II of Chapter 3) $\rho \leq \lambda_1/|\omega_1|$. Results in one and two space dimensions cover the important case $\rho = 2$.

The main difficulty in this work was the lack of $L^{2\rho+2}(\Omega_T)$ regularity, which forced us to severely restrict the admissible values of ρ via assumption (A2). Bearing in mind the results obtainable by the invariant region method of Smoller (see Section 1.4), this may be a limitation of the Faedo-Galerkin method and the fact that we took the initial data in $L^2(\Omega)$ or $H^1(\Omega)$.

There is still additional mathematical analysis to be done. For example, we could: extend results to cover the $\rho = 2$, $d = 3$ case; prove the continuous dependence of solutions on the system parameters; and investigate how the solutions depend on the data (initial and boundary conditions). Given more time we would have liked to investigate further the invariant region method of Smoller and explore the connection with the Faedo-Galerkin method of Lions [53]. Alternatively, it may be profitable to apply semigroup methods, or the concepts of absorbing sets and attractors from infinite dimensional dynamical systems [70], [85].

We proved an error bound for a semi-discrete finite element approximation that was optimal in H^1, but sub-optimal in L^2. The advantage of initially analysing the semi-discrete problem is two-fold. Firstly, we could have analysed the error between the continuous solution and the fully-discrete approximation directly, but such an approach can be technically cumbersome. We thus split the error using the semi-discrete approximation to isolate the errors due to discretisations in space and discretisations in time. Secondly, we can investigate various time stepping methods and the semi-discrete results will apply to all of them [4].

We proved an error bound for a fully-practical[1] piecewise linear finite element approximation, using a semi-implicit time discretion of the $\lambda - \omega$ system. The fully-discrete error bound was proved to be first order in the time step and second order in the space step. All results cover the important case when $\rho = 2$ in one and two space dimensions. We also extended the theoretical framework of the finite element space S^h, by generalising mesh dependent norms and establishing a number of new properties and lemmata. We implemented several complex numerical methods in one space dimension. The error bound was numerically verified with the aid of an explicit solution in one space dimension and results indicate the fully-discrete error bound is second order in the time step. Furthermore, results indicate that contributions to the error from space discretisations are considerably less than contributions to the error from time discretisations.

We were very fortunate in having a family of analytical solutions to verify the fully-discrete error bound. If no analytical solution is available, then there is a simple procedure one can adopt, where the solution on a coarse mesh is compared with the solution on a fine mesh with the aid of the triangle inequality (see [4], [5] for further details).

There is still much numerical work to be done. For example, there is the problem

[1]By fully-practical we mean that the numerical method it is easy to implement on a computer and there are no restrictive conditions on the space or time steps.

of how best to truncate the infinite domain (see the discussion in Section 6.3) and the related problem of developing front tracking procedures. It would be sensible in the case of solutions evolving from locally perturbed initial conditions (see Figure 6.6) to use a fine mesh near the wavefront and a coarse mesh beyond the wavefront where the solution is effectively zero. Furthermore, we have not investigated the use of adaptive meshes (h-refinement), method order variation (p-refinement), or adaptive time-stepping techniques. Recall that we defined the step size via $h := \max_{\tau \in \mathcal{T}^h} h_\tau$ (see Section 4.2), thus our theoretical results also apply to adaptive mesh technique with the step sizes h_τ bounded above by h.[2] The p-refinement technique is likely to be difficult due to the lack of regularity of solutions to the $\lambda-\omega$ system. For example, if we employ a finite element method with continuous piecewise polynomials of degree $k \geq 1$ and the interpolation operator obeys $\pi_h u = u$ for all $u \in P^k$, then the standard interpolation error estimate gives (see Theorem 5 in [14]):

$$\|u - \pi_h u\|_0 + h|u - \pi_h u|_1 \leq Ch^{k+1}|u|_{k+1}.$$

We also ran some preliminary simulations in two space dimensions with the finite element scheme ($\mathrm{P}_2^{h,\Delta t}$) and a uniform triangulation of the square $\Omega = (-L, L) \times (-L, L)$. Numerical results represent radially symmetric ring waves ('target patterns'), or rotating spiral waves. There are many open questions concerning the rich dynamics of solutions in two space dimensions, for example, the persistence and stability of spiral solutions of $\lambda - \omega$ systems and the possibility of turbulence [50]. Numerical experiments are much more expensive than in the one-dimensional case, and so additional efforts are needed to solve the resulting large, sparse, linear system in an efficient manner.

The analysis of the fully-discrete, semi-implicit finite element method ($\mathrm{P}_2^{h,\Delta t}$) can be carried out for several closely related semi-implicit methods. For example, we have proved[3] the existence, uniqueness and stability of solutions to the following methods (cf. $\mathrm{P}_3^{h,\Delta t}$):

[2]Recall from Section 4.2 that we must also assume the quasi-uniform and weak acuteness properties.

[3]Results not provided.

For $n = 1, \ldots, N$ find $C^n \in \mathbb{S}^h$ such that $C^0 := P^h c_0$ and

$$\left(\frac{C^n - C^{n-1}}{\Delta t}, \chi^h\right)^h + \frac{1}{2}(\nabla C^n + \nabla C^{n-1}, \nabla\chi^h) = \left(\widehat{f}(C^{n-1})C^n, \chi^h\right)^h \quad \forall \chi^h \in \mathbb{S}^h,$$

and

For $n = 1, \ldots, N$ find $C^n \in \mathbb{S}^h$ such that $C^0 := P^h c_0$ and

$$\left(\frac{C^n - C^{n-1}}{\Delta t}, \chi^h\right)^h + (\nabla C^n, \nabla\chi^h) = \frac{1}{2}\left(\widehat{f}(C^{n-1})(C^n + C^{n-1}), \chi^h\right)^h \quad \forall \chi^h \in \mathbb{S}^h,$$

where $\widehat{f}(C) := \lambda(R) + i\omega(R)$, $R \equiv |C|$, $C := U + iV$. In the first case we approximate the gradient using a Crank-Nicolson approach, while in the second case we have slightly altered the approximation of the non-linearity. The error analysis for these schemes is along similar lines to that given for $(\mathrm{P}_2^{h,\Delta t})$. We expect these schemes to have slightly different convergence properties to those of $(\mathrm{P}_2^{h,\Delta t})$.

We did not investigate nonlinear schemes as we wished to focus more on numerical analysis and less on scientific computing and implementation issues needed to solve large sets of nonlinear algebraic equations.

The overall approach and techniques developed in this thesis are applicable to general reaction-diffusion systems. The $\lambda - \omega$ system is not derived from any specific physical context and so it would be natural to try and use the methods in this work to undertake the analysis of more realistic problems, for example, in ecology or epidemiology. Alternatively, we could attempt the numerical analysis of complex Ginzburg-Landau equations (see the discussion in Section 1.3), bearing in mind that the $\lambda - \omega$ system with $\rho = 2$ is a special case of these type of equations. Also, much work has been done on the existence, uniqueness and regularity of solutions to a generalised complex Ginzburg-Landau equation with an arbitrary power of the nonlinearity, which might have interesting connections with the $\lambda - \omega$ system for arbitrary ρ. We leave this work and additional numerical experiments in higher space dimensions for future study.

Our results significantly contribute to the mathematical and numerical analysis of reaction-diffusion systems with a supercritical Hopf bifurcation in the reaction kinetics, and pave the way for further study.

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Appendix A

Auxiliary results and definitions

Definition A.0.1 (**almost everywhere (a.e.)**)

A property is said to hold 'almost everywhere' (a.e.) in Ω (or, for almost every (a.e.) x in Ω) if the property is true for all $x \in \Omega \backslash \Gamma$, where Γ is a subset of Ω with (Lebesgue) measure zero ([83], p.10).

Thus two functions $f, g : \Omega \mapsto \mathbb{R}$ are equal a.e. if the set $\{x \in \Omega \mid f(x) \neq g(x)\}$ has measure zero. Also, $f(x) = g(x)$ *a.e.* in Ω if $\|f - g\|_{0,p} = 0$.

Theorem A.0.2 (**Sobolev spaces: - collected results**)

The Sobolev spaces $W^{m,p}(\Omega)$, $m \geq 0$, equipped with the appropriate norms satisfy the following:

(i) For $1 \leq p \leq \infty$, $W^{m,p}(\Omega)$ is a Banach space ([1], p.45),

(ii) $W^{m,p}(\Omega)$ is separable if $1 \leq p < \infty$ ([1], p.47),

(iii) $W^{m,p}(\Omega)$ is reflexive if $1 < p < \infty$ ([1], p.47).

Definition A.0.3 (**duality pairing**)

If E is a Banach Space with norm $\|\cdot\|$, and l is a bounded linear functional $l : E \mapsto \mathbb{R}$, then we denote this functional via the 'duality pairing' between E and E' ([71], p.55):

$$l(v) := \langle l, v \rangle \quad v \in E,\ l \in E'.$$

Thus a duality pairing is in fact a bounded *bilinear* functional from $E' \times E$ into $\mathbb{R}$.

We recall that the smallest constant C satisfying the boundedness requirement of l (i.e., $|l(v)| \leq C\|v\|_E$, $\forall v \in E$) is given by the dual norm

$$\|l\|_{E'} := \sup_{v\neq 0} \frac{|\langle l, v\rangle|}{\|v\|_E} = \sup_{\|v\|=1} |\langle l, v\rangle|.$$

We also note that a duality pairing satisfies a Cauchy-Schwarz type inequality, i.e.:

$$|\langle l, x\rangle| \leq \|l\|_{E'}\|x\|_E, \ \forall l \in E' \text{ and } x \in E.$$

This follows directly from the definition of a bounded linear functional with the smallest possible constant C.

Theorem A.0.4 (evolution spaces: - collected results)

Let X and Y be Banach spaces. The evolution spaces $L^p(0,T;X)$ with appropriate norms satisfy

(i) $L^p(0,T;X)$, $(1 \leq p \leq \infty)$, is a Banach space ([48], pp.114-116),

(ii) $L^p(0,T;X)$, $(1 \leq p < \infty)$, is separable if and only if X is separable ([48], p.118),

(iii) $L^p(0,T;X)$, $(1 < p < \infty)$, is reflexive if X is reflexive ([96], p.40),

(iv) If X is a reflexive (or separable) Banach space and $(1 \leq p < \infty)$ then $[L^p(0,T;X)]' \simeq L^{p'}(0,T;X')$ where $\frac{1}{p} + \frac{1}{p'} = 1$ (the symbol between the spaces means 'isometrically isomorphic') ([96], p.40),

(v) The continuous injection $X \hookrightarrow Y$ implies $L^q(0,T;X) \hookrightarrow L^p(0,T;Y)$ if $1 \leq p \leq q \leq \infty$ ([56], p.34).

Lemma A.0.5 (Hilbert space setup)

Let V and H be Hilbert spaces, where the inner product on H is denoted $(\cdot,\cdot)$, such that

$$V \hookrightarrow H, \quad V \text{ is dense in } H,$$

then

$$\text{(i)} \qquad V \hookrightarrow H \equiv H' \hookrightarrow V', \quad H' \text{ is dense in } V'$$

$$\text{(ii)} \qquad \langle f, v\rangle_{V',V} = (f, v) \quad \forall f \in H,\ v \in V,$$

([85], p.55, [79], p.54, [34], p.133), where we have identified H with its dual H' by the Riesz Representation theorem.

Proof. The continuity of the embedding $H' \subset V'$ follows from Lemma A.0.6. Now from the definition of the dual operator (Definition A.0.7) we have

$$\langle i'f, v\rangle_{V',V} = \langle f, iv\rangle_{H',H} \qquad \forall f \in H',\ v \in V, \tag{A.0.1}$$

where we denote the identity mapping from V to H by i, and the corresponding dual operator from H' to V' by i' (an identity operator from H' to V'). From Corollary A.0.9 we deduce that H' is dense in V'. To prove the second part of this lemma, note as i and i' are identity mappings that we can write

$$\langle f, v\rangle_{V',V} = \langle f, v\rangle_{H',H} \qquad \forall f \in H',\ v \in V,$$

(alternatively, due to the density of V in H this follows directly if we remember that $f \in H'$ is an extension of $f \in V'$). From the Riesz Representation theorem (Theorem A.0.17) and due to the explicit identification of elements in H with those in H' we have

$$\langle f, v\rangle_{V',V} = (f, v) \qquad \forall f \in H,\ v \in V,$$

as required. ∎

By identifying H with H' we obtain the so called 'Gelfand Triple'

$$V \hookrightarrow H \hookrightarrow V',$$

where each space is dense in the previous one, and when working with Sobolev spaces one always chooses $H = L^2(\Omega)$ ([34], pp.133-134).

Examples of Hilbert spaces satisfying this setup are: $V = H^1(\Omega)$, $H = L^2(\Omega)$, and $V' = [H^1(\Omega)]'$, or $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and $V' = H^{-1}(\Omega)$. Now if Ω is a bounded Lipschitz domain then from a Sobolev embedding theorem for bounded domains ('Rellich's theorem') we have the following compact injection results:

(a) $H^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega) \overset{c}{\hookrightarrow} [H^1(\Omega)]'$ *dense inclusions,*

(b) $H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega) \overset{c}{\hookrightarrow} H^{-1}(\Omega)$ *dense inclusions.*

Lemma A.0.6

Let the X and Y be Banach spaces. Then

$$X \hookrightarrow Y \quad \text{densely} \implies Y' \hookrightarrow X',$$

([34], p.131 and cf. Lemma A.0.5).

Definition A.0.7 (dual operator)

Let X, Y be normed vector spaces and X', Y' their duals. Also let $T : X \mapsto Y$ be a bounded linear operator (i.e., $T \in \mathcal{L}(X,Y)$). Then we define the 'dual operator' $T' : Y' \mapsto X'$ via

$$\langle T'y', x\rangle_{X',X} = \langle y', Tx\rangle_{Y',Y} \quad \forall x \in X,\ y' \in Y', \tag{A.0.2}$$

([47], p.232. See also [59], p.173, [79], p.17).

Note that equation (A.0.2) is more usually written as $(T'y')(x) = y'(Tx)$, to indicate the action of the bounded linear functionals, however we prefer the more explicit notation via duality pairings. To be strictly correct we would then define the dual *operator* as $T'y' = y' \circ T$. The dual operator is also called the 'algebraic dual', or the 'adjoint operator' in some sources. The situation is illustrated diagrammatically by:

$$\begin{array}{ccccccc} X & \ni & x & \xrightarrow{T} & Tx & \in & Y \\ & & \| & & \| & & \\ & & \textit{dual} & & \textit{dual} & & \\ & & \| & & \| & & \\ X' & \ni & T'y' & \xleftarrow{T'} & y' & \in & Y'. \end{array}$$

Theorem A.0.8 (dual operator)

Let X, Y be normed vector spaces and X', Y' their duals. If $T \in \mathcal{L}(X,Y)$, and T' is the dual operator, then

(i) $T' \in \mathcal{L}(Y', X')$ ([59], p.173),

(ii) $\|T'\|_{\mathcal{L}(Y',X')} = \|T\|_{\mathcal{L}(X,Y)}$ ([59], p.173),

(iii) T' is injective $\iff$ the range of T is dense in Y, ([52], p.107).

As Hilbert spaces are reflexive we have the following corollary to the last part of this theorem:

Corollary A.0.9 (Hilbert space dual operator)
Let V, W be Hilbert spaces and V', W' their duals. If $T \in \mathcal{L}(V, W)$ and $T' \in \mathcal{L}(W', V')$ is the dual operator, then ([79], p.18):

$$T' \text{ is injective} \iff \text{the range of } T \text{ is dense in } W,$$
$$T \text{ in injective} \iff \text{the range of } T' \text{ is dense in } V'.$$

Definition A.0.10 (continuous/compact injection)
We say that the normed vector space X is 'embedded' in the normed vector space Y, or that X has 'continuous injection' into Y and write $X \hookrightarrow Y$, if ([1], p.9)

(i) X is a vector subspace of Y, and

(ii) the identity operator I defined on X into Y by $Ix = x$ for all $x \in X$ is continuous.

By the 'compact injection' $X \overset{c}{\hookrightarrow} Y$ we mean that the identity operator I is compact, i.e., I maps bounded sets in X to precompact (and hence compact [17], pp.189-191) sets in Y.

We recall from the elementary properties of linear operators that boundedness is equivalent to continuity ([69], p.197), thus (ii) is equivalent to the existence of a constant C such that $\|x\|_Y \leq C\|x\|_X$. This is useful for example, in bounding terms in numerical analysis. We also note from the definition of a compact operator that if $X \overset{c}{\hookrightarrow} Y$, then any bounded sequence in X contains a subsequence that converges (strongly) in Y.

Lemma A.0.11 (existence of a Lipschitz condition)
Let $\boldsymbol{f} : U \mapsto \mathbb{R}^n$ be a continuously differentiable mapping of the Euclidean space $\mathbb{R}^m$ into the Euclidean space $\mathbb{R}^n$. Then the mapping $\boldsymbol{f}$ satisfies a Lipschitz condition on each convex compact subset V of the domain U with Lipschitz constant L equal to the supremum of the derivative $\boldsymbol{f}$ on V:

$$L = \sup_{\boldsymbol{x} \in V} \left\| \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \right\|, \quad \text{where} \quad \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \text{ is the Jacobian matrix of } \boldsymbol{f},$$

([3], pp.272-273).

Proof. The proof is simple and instructive, so we give it here. Consider the line arc in V given by $\boldsymbol{z} := \boldsymbol{v}s + (1-s)\boldsymbol{u} = s(\boldsymbol{v}-\boldsymbol{u}) + \boldsymbol{u}$, $0 \le s \le 1$. Then

$$\boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{v}) = \int_0^1 \frac{d}{ds}\boldsymbol{f}(\boldsymbol{z}(\tau))\,d\tau = \int_0^1 \frac{d\boldsymbol{f}}{d\boldsymbol{z}}\frac{d\boldsymbol{z}}{ds}\,d\tau,$$
$$\text{so} \quad \|\boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{v})\|_f \le \sup_{\boldsymbol{z}\in V}\left\|\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{z}}\right\|_J \|\boldsymbol{u}-\boldsymbol{v}\|_z,$$

where the matrix norm $\|\cdot\|_J$ is compatible with the vector norm $\|\cdot\|_z$. ■

$\boldsymbol{f} \in C^1$, consequently the supremum of the norm of the Jacobian matrix is attained on the compact set V.

Lemma A.0.12 (Green's identity)

Let Ω be a bounded Lipschitz domain, $\phi \in H^2(\Omega)$ and $\psi \in H^1(\Omega)$, then ([10], p.124)

$$\int_\Omega \psi\Delta\phi\,dx = \int_{\partial\Omega} \psi\frac{\partial\phi}{\partial\nu}\,ds - \int_\Omega \nabla\phi\cdot\nabla\psi\,dx,$$

where ν is the outward unit normal to $\partial\Omega$, and ds is the element of arc length if $d=2$, or surface area if $d=3$.

It is understood that it is the traces of the functions that occur in the boundary terms. Formally, the derivation of Green's identity follows from 'integration by parts', i.e., integrating over Ω after applying the product rule: $\nabla\cdot(\psi\nabla\phi) = \psi(\Delta\phi) + \nabla\phi\cdot\nabla\psi$, followed by application of the Divergence (Gauss') theorem $\int_\Omega \nabla\cdot F\,dx = \int_{\partial\Omega} F\cdot\nu\,ds$.

Theorem A.0.13 (Hilbert-Schmidt theorem)

Let H be an infinite dimensional Hilbert space and let $L \in \mathcal{L}(H)$ be a compact, self-adjoint operator. Then there is a sequence of non-zero, real eigenvalues $\{\mu_i\}_{i=1}^\infty$ of L, s.t.

$$\lim_{i\to\infty}\mu_i = 0, \qquad \cdots \le |\mu_{i+1}| \le |\mu_i| \le \cdots \le |\mu_1|,$$

where each eigenvalue is repeated in the sequence according to its multiplicity. Furthermore, there exists an orthonormal set $\{z_i\}_{i=1}^\infty$ of corresponding eigenfunctions, i.e.,

$$Lz_i = \mu_i z_i.$$

Moreover, $\{z_i\}_{i=1}^{\infty}$ is an orthonormal basis for the range of L, ([69], p.267, [70], p.75).

Theorem A.0.14 (spectral theorem)
Consider the elliptic eigenvalue problem

$$A z_i = \mu_i z_i \quad \text{in } \Omega, \quad (z_i \neq 0)$$
$$\gamma_n z_i = 0,$$

where the operator γ_n is defined by $\gamma_n u := \gamma_0 \frac{\partial^n u}{\partial \nu^n}$, γ_0 is the usual trace operator restricting functions to $\partial\Omega$, and ν is the outward unit normal to $\partial\Omega$. Then for the cases

(a) $A := -\Delta$, $n = 0$, $V := H_0^1(\Omega)$, $H := L^2(\Omega)$,

(b) $A := -\Delta + I$, $n = 1$, $V := H^1(\Omega)$, $H := L^2(\Omega)$,

the following facts hold for the associated eigenvalues and eigenfunctions:

(i) For case (a) we have

$$0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_i \leq \text{with } \lim_{i\to\infty} \mu_i = \infty,$$

while for case (b) we have

$$1 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_i \leq \text{with } \lim_{i\to\infty} \mu_i = \infty,$$

(ii) $\{z_i\}_{i=1}^{\infty}$ is an orthogonal basis for V, with $(z_i, z_j)_V = \mu_i \delta_{ij}$,

(iii) $\{z_i\}_{i=1}^{\infty}$ (after normalisation) is an orthonormal basis for H, i.e., $(z_i, z_j) = \delta_{ij}$.

Proof. The argument here is partly based on the one given in [70], pp.162-163, for case (a) above. The basic idea is to show that A^{-1} is a symmetric, bounded, compact operator from H to H and thus the Hilbert-Schmidt theorem applies.

Consider the elliptic boundary value problem

$$Au = f \quad \text{in } \Omega, \quad f \in V', \tag{A.0.3}$$

where A is either the 'Dirichlet' or 'Neumann' Laplacian defined in (a) and (b) respectively. The weak form of (A.0.3) is

$$\text{Find } u \in V \text{ s.t.} \quad a(u,v) \equiv (u,v)_V = \langle f, v\rangle_{V',V} \quad \forall v \in V, \tag{A.0.4}$$

where we assume the results of Lemma A.0.5.

Now by the Lax-Milgram theorem (Theorem A.0.16) we know that equation (A.0.4) has a unique solution in V, thus the operator A is invertible, i.e.

$$u = A^{-1} f. \tag{A.0.5}$$

We note that A^{-1} is linear[1].

Using Lemma A.0.24 we can write the inner product as $a(u,v) \equiv (u,v)_V = \langle Au, v\rangle_{V',V}$ where $A \in \mathcal{L}(V, V')$ and so by the Riesz Representation theorem (Theorem A.0.17) we have

$$\|f\|_{V'} = \|u\|_V. \tag{A.0.6}$$

Recall from Lemma A.0.5 that $H \hookrightarrow V'$ and so

$$\|f\|_{V'} \leq C\|f\|_H, \tag{A.0.7}$$

for some constant C. Thus (A.0.6) and (A.0.7) together give

$$\|u\|_V \leq C\|f\|_H \qquad \text{or} \qquad \|A^{-1}f\|_V \leq C\|f\|_H,$$

i.e., A^{-1} is a bounded operator from H to V. But from Lemma A.0.5 we have $V \overset{c}{\hookrightarrow} H$, so A^{-1} is a bounded compact operator from H to H.

To show that A^{-1} is a self-adjoint operator notice first that A is in general an unbounded operator[2] which is symmetric due to $\langle Au, v\rangle_{V',V} = a(u,v) = a(v,u) = \langle Av, u\rangle_{V',V}$. Now we would like to use the fact from Lemma A.0.5 that $(f,v) = \langle f, v\rangle_{V',V}$ for all $v \in V$, $f \in H$ and thus the symmetry condition for A becomes

$$(Au, v) = (u, Av) \quad \forall u, v \in V,\ Au, Av \in H, \tag{A.0.8}$$

[1] Let $Au := f_1$, $Av := f_2$, so $u = A^{-1}f_1$, $v = A^{-1}f_2$. Then $A(\alpha u + \beta v) := f \implies \alpha u + \beta v = A^{-1}f \implies \alpha A^{-1}f_1 + \beta A^{-1}f_2 = A^{-1}(\alpha f_1 + \beta f_2)$, using the linearity of A.

[2] E.g., see the example in [70], p.79, with the operator $\frac{d}{dx}$ replaced by $\Delta := -\frac{d^2}{dx^2}$.

where we denote the inner product on H by $(\cdot, \cdot)$. We have to be a little careful as A is an unbounded operator and the domain of an unbounded operator becomes an integral part of the definition of the operator ([69], p.253). However, domain$(A) = V$ which is dense in H and so equation (A.0.8) is valid (see [59], p.222, [69], p.253). Now let $Au = x$, $Av = y$ for all $x, y \in H$, then

$$(x, A^{-1}y) = (A^{-1}x, y) \qquad \forall x, y \in H.$$

Thus A^{-1} is self-adjoint. We now apply the Hilbert-Schmidt theorem (Theorem A.0.13) with $L := A^{-1}$, noting that

$$Az_i = \mu_i z_i \iff A^{-1}z_i = \mu_i^{-1} z_i, \tag{A.0.9}$$

thus the μ_i^{-1} are real and we have the infinite sequence

$$\lim_{i\to\infty} \mu_i^{-1} = 0, \qquad \cdots \leq |\mu_{i+1}^{-1}| \leq |\mu_i^{-1}| \leq \cdots \leq |\mu_1^{-1}|, \tag{A.0.10}$$

where the eigenfunctions z_i form an orthonormal basis (after normalisation) for range$(A^{-1}) = V \subset H$. In fact, we now show that the z_i form an orthonormal basis for the *whole* of H and an orthogonal basis for V. To show this we recall the easily proven result that if H is a Hilbert space then M is a dense subspace of H if and only $M^\perp = \{0\}$ (see Corollary 6.27 in [69]), i.e., the only element in H that is orthogonal to the elements in M is the zero vector. Now take $M := \text{Span}\{z_i\}_{i=1}^\infty \subset V \subset H$ and as V is dense in H we have $V^\perp = \{0\}$, which implies $M^\perp = \{0\}$ (with respect to H), which implies M is dense in H, i.e., by definition $\{z_i\}_{i=1}^\infty$ is a basis for the whole of H. Also, from the weak form of the eigenvalue problem we have

$$(z_i, z_j)_V = \mu_i \delta_{ij},$$

that is the z_i are an orthogonal basis for V. As $\|\cdot\|_V$ is a norm we have

$$a(z_i, z_i) \equiv \|z_i\|_V^2 = \mu_i > 0 \qquad (z_i \neq 0),$$

thus the eigenvalues are strictly positive. Finally, note that for case (b), again using the weak form of the eigenvalue problem, we have

$$\|\nabla z_i\|_0^2 = \mu_i - 1 \geq 0,$$

so $\mu_i \geq 1$. Clearly $\mu_1 = 1 \iff z_1 = c$, where c is a constant. In fact from $||z_1||_0^2 = 1$ we deduce that $c = \pm 1/|\Omega|^{1/2}$. ∎

Case (a) is relevant to partial differential equations with the homogeneous Dirichlet boundary conditions, while case (b) (see for example [22]) is relevant to partial differential equations with the homogeneous Neumann boundary conditions. We use the orthonormal bases in this theorem to construct a Galerkin approximation in the Faedo-Galerkin method (see Appendix B).

Definition A.0.15 (continuous, coercive bilinear form)

A bilinear form $a(\cdot,\cdot)$ on a normed vector space V is said to be 'continuous' (or 'bounded') if there exists $C < \infty$ such that ([10], p.55, [69], p.290)

$$|a(u,v)| \leq C||v||_V||v||_V \quad \forall u,v \in V$$

and 'coercive' (or 'V-elliptic') on $W \subset V$ if there exists $\alpha > 0$ such that

$$a(v,v) \geq \alpha||v||_V^2 \quad \forall v \in W.$$

The constant α is sometimes called the 'coercivity' constant. Notice that the bilinear form is coercive on a subspace W of V, thus in general
$\sqrt{a(v,v)} =: ||v||_a \neq ||v||_V$.

Theorem A.0.16 (Lax-Milgram)

Given a real Hilbert space $(V, (\cdot,\cdot))$, a continuous, coercive ('V-elliptic') bilinear functional $a(\cdot,\cdot)$ on $V \times V$ and a continuous linear functional $F \in V'$, then there exists a unique $u \in V$ such that ([14], p.8):

$$a(u,v) = \langle F, v\rangle_{V',V} \quad \forall v \in V,$$

furthermore,

$$||v||_V \leq \frac{1}{\alpha}||F||_{V'} \quad \text{where } \alpha \text{ is the coercivity constant.}$$

The proof of this theorem is based on a generalisation of the Riesz Representation theorem to bilinear forms that are not necessarily symmetric and utilises a contraction mapping theorem for Banach spaces (see any book on functional analysis). The last part of the theorem expresses the continuous dependence of the solution on the data. Identifying f and F through the Riesz Representation theorem, so $||f|| = ||F||$, then we see that if f is small, then so is v. Thus the Lax-Milgram theorem asserts that the variational problem is well-posed.

Theorem A.0.17 (Riesz Representation theorem for Hilbert spaces)
Let H be a Hilbert space and H' be the corresponding dual space. Then there exists an isometric isomorphism between H and H', so each bounded linear functional $l \in H'$ acting on H is identified with a unique element $u \in H$ via ([69], p.199):

$$l_u(v) \equiv \langle l, v \rangle = (u, v) \quad \forall v \in H, \quad \text{with} \quad ||l||_{H'} = ||u||_H.$$

Thus the Riesz Representation theorem identifies a linear bijective correspondence between the elements of H' and H

$$H' \ni l_u \longleftrightarrow u \in H,$$

that preserves distance (in norm). Implicit in this definition is the fact that $(\cdot, \cdot)$ is an inner product on H (i.e., a positive definite, symmetric bilinear form). One consequence of this theorem is that weak convergence in a Hilbert space takes on a concrete form (see Definition A.0.23).

Theorem A.0.18 (orthogonal projections:- collected results)
Let $(H, (\cdot, \cdot))$ be a Hilbert space with the norm $||v|| := \sqrt{(v, v)}$ and a closed subspace M with $v \in H \backslash M$. Define the projection operators $P_M : H \mapsto M$, $P_{M^\perp} : H \mapsto M^\perp$ $(P_{M^\perp} = I - P_M)$ and denote by $M^\perp$ the orthogonal complement[3] of M, then ([10], [69], [47]):

(i) $H = P_M \oplus P_{M^\perp}$, i.e., each $v \in H$ has the unique decomposition $v = x + y$ for $x \in M$ and $y \in M^\perp$.

(ii) $(v - P_M v, w) = 0 \quad \forall w \in M \quad (v - P_M v \in M^\perp)$, or equivalently, P_M is self-adjoint (or symmetric), i.e., $(v, P_M w) = (P_M v, w) \quad \forall w \in M$. Thus P_M is an orthogonal projection.

[3]The elements in H orthogonal to M.

(iii) $\|v - P_M v\| = \inf \|v - w\| \quad \forall w \in M$. $P_M v$ is the unique 'best approximation' to v out of M. This property is a consequence of Cauchy-Schwarz and (ii).

(iv) $P_M \in \mathcal{L}(H, M)$; $\|P_M v\| \leq \|v\| \quad \forall v \in H$. This property is also a consequence of (ii).

(v) If $M \equiv V^k = \text{Span}\{z_i\}_{i=1}^k$, where $\{z_i\}_{i=1}^\infty$ is an orthonormal set in H, i.e. $(z_i, z_j) = \delta_{ij}$ and $P^k : H \mapsto V^k$, then for every $v^k \in V^k$ we have $P^k v := v^k = \sum_{j=1}^k (v, z_j) z_j$. This property is also a consequence of (ii). Furthermore, $\|v\|^2 \geq \sum_{j=1}^\infty |(v, z_j)|^2$ ('Bessel's inequality').

(vi) The orthonormal set in (v) is a basis for H (i.e., complete , alt. 'maximal', in the sense that $\text{Span}\{z_i\}_{i=1}^\infty$ is dense in H), or equivalently $v = \sum_{j=1}^\infty (v, z_j) z_j$ $\forall v \in H$, if and only if $\|v\|^2 = \sum_{j=1}^\infty |(v, z_j)|^2$ ('Parseval's relation').

Definition A.0.19 (strong convergence)

Let V be a normed vector space. Then $x_n \to x$ ('strongly') in V means

$$\|x_n - x\|_V \to 0 \quad \text{as } n \to \infty.$$

This type of convergence is also called 'convergence with respect to the norm', or just 'norm convergence'.

Definition A.0.20 (weak convergence (in E))

Let E be a Banach space. Then $x_n \rightharpoonup x$ ('weakly') in E means ([69], p.203, [71], p.55)

$$\langle l, x_n \rangle \to \langle l, x \rangle \quad \text{as } n \to \infty, \quad \forall l \in E',$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between E' and E.

Definition A.0.21 (weak* convergence (in E'))

Let E be a Banach space. Then $l_n \rightharpoonup^* l$ ('weak*') in E' means ([69], p.203, [71], p.56)

$$\langle l_n, x \rangle \to \langle l, x \rangle \quad \text{as } n \to \infty, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between E' and E.

There is a natural relationship between weak* and weak convergence. If we consider the definition of weak convergence in E', and recall that in general $E \subset E''$, then we can write: $x_n \rightharpoonup x$ in E' means

$$\langle l, x_n \rangle_{E'',E'} \to \langle l, x \rangle_{E'',E'} \quad \text{as } n \to \infty, \quad \forall l \in E'' \supset E.$$

Thus we see that weak* convergence in E' is in fact weak convergence in E' where we have restricted the admissible functionals l to the subset $E \subset E''$ ([70], p.105, [59], p.171).

Theorem A.0.22 (weak and weak* convergence:- collected results) Let E be a Banach space and E' its dual. Then

(i) If E is reflexive, then weak convergence in E' is equivalent to weak* convergence in E' (by definition). But E is reflexive if and only if E' is reflexive, thus weak convergence in E is equivalent to weak* convergence in E.

(ii) A weakly convergent sequence in E' also converges weak* in E'. This follows by definition and the fact that $E \subset E''$([70], p.105, [59], p.171).

(iii) Strong ('norm') convergence implies weak convergence (follows directly from the definitions and a Cauchy-Schwarz type inequality for the duality pairing).

(iv) Weak and weak* limits are unique ([69], p.203). This follows directly from the definition in the weak* convergence case. For the weak convergence case we need Proposition A.0.1 to prove uniqueness.

(v) It follows from a uniform boundedness principle that weak and weak* limits are bounded ([69], p.203). Additionally, if u_n converges weakly to u in E, then $\|u_n\|_E$ is bounded and $\|u\|_E \leq \liminf \|u_n\|_E$ ([71], p.55).

(vi) Strong and weak convergence are equivalent in finite dimensions ([69], p.203).

(vii) If a sequence $\{u_n\}$ converges weakly (resp. weak *), then every subsequence converges weakly (resp. weak*) to the same limit. The proof is a direct analogy of the corresponding elementary result in real analysis (e.g., see [72], p.51).

Proposition A.0.1

If x and y lie in some Banach space X and $\langle f, x\rangle_{X',X} = \langle f, y\rangle_{X',X}$ for all $f \in X'$, then $x = y$ ([70], p.92).

Definition A.0.23 (weak & weak* convergence:-concrete forms)

Let H be a Hilbert space, $1 < p, q < \infty$ (p and q conjugate indices) and $(\cdot,\cdot)$ be the usual L^2 inner product. Then

$$\left.\begin{array}{l} u^k \rightharpoonup^* u \text{ in } L^\infty(\Omega), \\ u^k \rightharpoonup u \text{ (or } u^k \rightharpoonup^* u) \text{ in } L^p(\Omega), \\ u^k \rightharpoonup u \text{ (or } u^k \rightharpoonup^* u) \text{ in } H, \end{array}\right\} \quad \text{means}$$

$$(u^k, v) \to (u, v) \quad \text{as } k \to \infty,$$

$$\left\{\begin{array}{l} \forall v \in L^1(\Omega), \\ \forall v \in L^q(\Omega), \\ \forall v \in H, \end{array}\right.$$

respectively. Some analogous examples for evolution spaces are:

$$\left.\begin{array}{l} u^k \rightharpoonup^* u \text{ in } L^\infty(0,T;H), \\ u^k \rightharpoonup u \text{ (or } u^k \rightharpoonup^* u) \text{ in } L^p(\Omega_T), \\ u^k \rightharpoonup u \text{ (or } u^k \rightharpoonup^* u) \text{ in } L^2(0,T;H), \end{array}\right\} \quad \text{means}$$

$$\int_0^T \left(u^k(\cdot,t), v(\cdot,t)\right) dt \to \int_0^T \left(u(\cdot,t), v(\cdot,t)\right) dt \quad \text{as } k \to \infty,$$

$$\left\{\begin{array}{l} \forall v \in L^1(0,T;H), \\ \forall v \in L^q(\Omega_T), \\ \forall v \in L^2(0,T;H), \end{array}\right.$$

respectively.

These results follow from the abstract definitions of weak and weak* convergence (Definitions A.0.20 and A.0.21) after noting Riesz Representation theorems for Hilbert spaces [69], p.199, L^p spaces [48], pp.79-85 and time-dependent spaces of the form $L^p(0,T,X)$ [96], p.40. In common with other authors we treat these concrete forms as definitions when required.

Lemma A.0.24 (bilinear form result)

Let a bilinear form $a(\cdot,\cdot)$ be continuous on a Hilbert space V, then one can assign a unique operator $A \in \mathcal{L}(V, V')$ s.t.

(i) $a(u,v) = \langle Au, v\rangle_{V',V} \equiv Au(v) \qquad \forall u, v \in V.$

(ii) $||A||_{\mathcal{L}(V',V)} \leq C_s$. C_s corresponds to $|a(u,v)| \leq C_s||u||_V||v||_V \quad \forall\, u, v \in V.$

([34], p.138, [10], p.60, [85], p.54).

For example, take $V := H^1(\Omega)$, $a(u,v) := \int_\Omega \nabla u \cdot \nabla v\, dx$ and $\partial u/\partial \nu = 0$ on $\partial\Omega$. Then $a(u,v) \leq C||\nabla u||_0||\nabla v||_0 \leq C||u||_1||v||_1$ and so $a(\cdot,\cdot)$ is continuous on V. Thus result (i) corresponds to the application Green's Formula giving

$$\int_\Omega \nabla u \cdot \nabla v\; dx = -\int_\Omega v\,\Delta u\; dx, \quad \text{and} \quad A = -\Delta \in \mathcal{L}(V, V').$$

For time-dependent problems if $u \in L^2(0,T;V)$ then $-\Delta u \in L^2(0,T;V')$.

Lemma A.0.25 Let X and Y be Banach spaces with dual spaces X' and Y' respectively. Then

(i) The space $Z := X \cap Y$ with the norm $||u||_Z = ||u||_X + ||u||_Y$ is Banach and the dual space of Z is given by $Z' = X' + Y'$.

(ii) The space $W := X + Y = \{x + y \mid x \in X,\ y \in Y\}$ with the norm $||w||_W := ||x||_X + ||y||_Y$, where $w := x + y$ is Banach.

Proof. (i) Clearly for a function u bounded in X and Y we have u bounded in Z. It is straightforward to check that $||\cdot||_Z$ is a norm. To show completeness in Z, recall the completeness of X and Y and consider a sequence $\{u_n\}$ that converges to u in both X and Y. Thus

$$||u_n - u||_Z = ||u_n - u||_X + ||u_n - u||_Y \to 0 \quad \text{as } n \to \infty,$$

i.e., $u_n \to u$ in Z and so $\{u_n\}$ converges in Z. We have shown that Cauchy sequences converge in Z (convergent sequences are Cauchy sequences) and so by definition Z is a Banach Space.

The proof for $Z' = X' + Y'$ is as follows (Robinson, J. (2003) *Pers. comm.*): assume X and Y are both continuously embedded in another Banach Z, given $f \in (X \cap Y)'$, consider the subspace D of $X \times Y$, given by

$$D = \{(u, u) \; : \; u \in X \times Y\}.$$

Then D is a linear subspace of $X \times Y$, and f induces a bounded linear functional g on D via the definition

$$g(u, u) = f(u).$$

The linear functional g can then be extended to a bounded linear functional F on $X \times Y$. Now define linear functionals $f_1 \in X'$ and $f_2 \in Y'$ by

$$f_1(x) = F(x, 0) \quad \text{and} \quad f_2(y) = F(0, y),$$

and then for $u \in X \cap Y$,

$$f(u) = g(u, u) = g(u, 0) + g(0, u) = f_1(u) + f_2(u),$$

as required.

(ii) Clearly for x bounded in X and y bounded in Y we have $w := x + y$ bounded in W. It is straightforward to check that $\|\cdot\|_W$ is a norm. To show completeness in W (again recalling the completeness of X and Y) consider a sequence $\{x_n\}$ that converges to x in X and another sequence $\{y_n\}$ that converges to y in Y. Then with $w_n := x_n + y_n$ and $w := x + y$ we have

$$\|w_n - w\|_W = \|(x_n - x) + (y_n - y)\|_Z = \|x_n - x\|_X + \|y_n - y\|_Y \to 0 \quad \text{as } n \to \infty,$$

i.e., $w_n \to w$ in W and so $\{w_n\}$ converges in W and so as above we deduce that W is a Banach space. ■

Lemma A.0.26

If a sequence $u_n \to u$ in L^p $(1 \leq p < \infty)$, then there is a subsequence, still denoted u_n, such that

$$u_n \to u \quad \text{('pointwise') a.e. in } \Omega,$$

as $n \to \infty$ ([70], p.27, [71], p.59).

Definition A.0.27 (strongly elliptic operators)

Define an operator A by

$$Au = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial u}{\partial x_j}\right), \quad a_{i,j} = a_{j,i} \in C^{0,1}(\overline{\Omega}),$$

where $C^{0,1}(\overline{\Omega})$ is the space of Lipschitz continuous functions. We say that $-A$ is strongly elliptic ([32], p.142), if there exists an $\alpha > 0$ such that

$$\sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\xi_j \le -\alpha|\xi|^2,$$

for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^d$.

For example, if we take $[a_{i,j}] = -I$, that is $a_{i,j} = -\delta_{ij}$, then $Au = -\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} \equiv -\Delta u$, i.e., $A = -\Delta$. Notice that $-\Delta$ is strongly elliptic as

$$\sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\xi_j = -\sum_{i,j}^{d} \delta_{ij}\xi_i\xi_j = -|\xi|^2, \quad (\alpha = 1).$$

Theorem A.0.28 (elliptic regularity property 1)

Let A be a strongly elliptic operator (see Definition A.0.27) and Ω be a convex, bounded and open subset of $\mathbb{R}^d$. Then for each $f \in L^2(\Omega)$ and each $\lambda > 0$ there exists a unique $u \in H^2(\Omega)$ that is the solution of

$$\begin{aligned} Au + \lambda u &= f \quad \text{in } \Omega, \\ \sum_{i,j=1}^{d} \nu_i\, \gamma(a_{i,j}\frac{\partial u}{\partial x_j}) &= 0 \quad \text{a.e. on } \partial\Omega, \end{aligned}$$

where γ is the trace operator and ν_i is the ith component of the outward unit normal $\boldsymbol{\nu} := (\nu_1, \ldots, \nu_d)^T$ to $\partial\Omega$ (Theorem 3.2.1.3 in [32]).

With $a_{i,j} = -\delta_{ij}$ (corresponding to $A = -\Delta$) the boundary condition becomes $-\sum_{i=1}^{d} \nu_i \frac{\partial u}{\partial x_i} = -\boldsymbol{\nu} \cdot \nabla u = 0$, i.e., we have the homogeneous Neumann boundary condition $\partial u/\partial \boldsymbol{\nu} = 0$ on $\partial\Omega$. For a similar elliptic regularity result applicable to the homogeneous Dirichlet boundary condition case see Theorem 3.2.1.2 in [32].

This theorem is useful for deducing regularity via an *a priori* estimate. Suppose we know that $u \in L^2(\Omega)$ and we deduce from an *a priori* estimate the uniform bound $\|\Delta u\|_0 \le C$, i.e., $-\Delta u + \lambda u \in L^2(\Omega)$, then $u \in H^2(\Omega)$.

Theorem A.0.29 (elliptic regularity property 2)

Let A be a strongly elliptic operator (see Definition A.0.27) and Ω be a convex, bounded and open subset of $\mathbb{R}^d$ with a boundary of class C^2. Then there exists a constant C such that

$$\|u\|_2 \leq C\|Au + \lambda u\|_0,$$

for all $u \in H^2(\Omega)$ such that $\sum_{i,j=1}^{d} \nu_i \ \gamma(a_{i,j} \frac{\partial u}{\partial x_j}) = 0$ a.e. on $\partial\Omega$ and for all $\lambda > 0$.

See the comments for Theorem A.0.28.

Theorem A.0.30 (diagonally dominant tri-diagonal systems)

Consider the following (real) coefficient matrix of a tri-diagonal system

$$\begin{pmatrix} a_1 & c_1 & & & & \\ b_2 & a_2 & c_2 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & b_{n-1} & a_{n-1} & c_{n-1} \\ & & & & b_n & a_n \end{pmatrix},$$

where the elements satisfy:

(i) $|a_1| > |c_1| > 0$,

(ii) $|a_i| \geq |b_i| + |c_i|, \quad b_i c_i \neq 0, \quad i = 2, \ldots, n-1$,

(iii) $|a_n| > |b_n| > 0$,

then LU factorisation is guaranteed to give a unique bounded solution, without the need for partial pivoting [40], p.56. The proof of this theorem is also valid for complex systems.

This theorem can be generalised to give:

Theorem A.0.31 (strictly diagonally dominant systems)

A strictly diagonally dominant matrix A is nonsingular. Moreover, in this case, Gaussian elimination can be performed on any linear system of the form $Ax = b$ to obtain its unique solution without row or column interchanges and the computations are stable with respect to the growth of roundoff errors ([12], p.404).

Theorem A.0.32 (trace inequality)

Let Ω be a bounded domain with a sufficiently smooth boundary (e.g., Lipschitzian), then there exists a constant $C = C(\Omega)$ s.t.

$$\|\gamma u\|_{L^2(\partial\Omega)} \leq C\|u\|_1 \quad \forall u \in H^1(\Omega),$$

where the continuous linear 'trace operator' γ restricts functions to the boundary. The norm $\|\gamma u\|$ is frequently written $\|u\|$ (see e.g., [14], p.13).

Appendix B

The Faedo-Galerkin method and associated results

We give an overview of the Faedo-Galerkin method used to prove the well-posedness of weak solutions to second order, linear and nonlinear PDEs. Some associated theorems and lemmata are also given.

(a) Assume we have a set $\{z_i\}_{i=1}^{\infty}$ of linearly independent elements of $H^1(\Omega)$ (or $H_0^1(\Omega)$) such that the linear span of the z_i is dense in $H^1(\Omega)$ (or $H_0^1(\Omega)$). A Galerkin approximation $u^k(\cdot, t) = \sum_{i=1}^{k} c_{ik}(t) z_i(\cdot)$ is substituted into the finite dimensional weak form of the PDE to give a system of k ODEs (an IVP) for $c_{ik}(t)$. Standard ODE theory then gives local existence (and uniqueness) of the $c_{ik}(t)$ and hence of the approximate solution u^k on the finite time interval $(0, t_k)$, $t_k > 0$. This relies on the (local) Lipschitz continuity of the nonlinearities on the right hand side of the system of ODEs.

(b) We deduce that the functions u^k are uniformly bounded with respect to some norm, i.e., $\|u^k\| \leq C$. This bound is called an '*a priori* estimate'. Then $t_k = T$ is independent of k, that is we have global existence of u^k.

(c) We use 'weak compactness' arguments to extract a convergent subsequence (in some sense) from the uniformly bounded sequence of functions. This process is called 'passage to the limit'. We must also show passage to the limit of each finite dimensional term in the ODE (or, each term in the finite dimensional weak form). It is typically the nonlinear term that gives the most difficulty in this process. This leads to global existence of the weak solution u.

(d) To obtain uniqueness of the weak solutions assume there are two weak solutions u_1 and u_2 with the *same* initial data. Subtract the weak form for u_2 from the weak form for u_1, let the test function $\eta = u_1 - u_2 =: w$ and bound w in terms of the initial data. The aim is to deduce that $w = 0$, i.e., there is one and only one weak solution. If the initial data of the weak solutions u_1 and u_2 are assumed different, then this process leads to continuous dependence of the weak solution on the initial data.

Theorem B.0.33 (Picard's existence/uniqueness theorem for ODEs)
Let $\boldsymbol{y}, \boldsymbol{f} \in \mathbb{R}^d$; $\boldsymbol{f}(t, \boldsymbol{y})$ be continuous on a parallelepiped $R : t_0 \leq t \leq t_0 + a$, $\|\boldsymbol{y} - \boldsymbol{y}_0\| \leq b$ and uniformly Lipschitz continuous with respect to $\boldsymbol{y}$. Let L be a bound for $\|\boldsymbol{f}(t, \boldsymbol{y})\|$ on R; $\alpha = \min\{a, b/L\}$. Then ([38], p.9)

$$y' = \boldsymbol{f}(t, \boldsymbol{y}), \quad \boldsymbol{y}(t_0) = \boldsymbol{y}_0$$

has a unique solution on $[t_0, t_0 + \alpha]$.

Note that $\|\boldsymbol{y}\|$ can be any norm on $\mathbb{R}^d$ (as all norms are equivalent in finite dimensions).

Theorem B.0.34 (weak sequential compactness)
A Banach space E is reflexive if and only if every infinite (strongly[1] and uniformly[2]) bounded sequence in E contains a subsequence that converges weakly to an element of E ([18], p.289).

Theorem B.0.35 (weak* sequential compactness)
Let E be a separable Banach space. Then every infinite (strongly[1] and uniformly[2]) bounded sequence in E' contains a subsequence that is weak* convergent in E' ([18], p.291).

[1]Bounded in norm.

[2]The bound on the sequence $\{u^k\}$ is independent of k.

We give an example frequently encountered in applications. Let H be a Hilbert space and assume a uniformly bounded sequence is in $L^\infty(\Omega)$ (resp. $L^\infty(0,T;H)$). Now by a Riesz Representation type theorem ([96], p.40), we can identify this space with $[L^1(\Omega)]'$ (resp. $[L^1(0,T;H)]'$). If we recall that the space $L^1(\Omega)$ (resp. $L^1(0,T;H)$) is separable, but not reflexive, then using the 'weak* sequential compactness' theorem we can extract a weak* convergent subsequence in $L^\infty(\Omega)$ (resp. $L^\infty(0,T;H)$).

Lemma B.0.36 (convergence lemma)

Suppose a sequence of functions $\{u^k\}$ is uniformly bounded in the Banach spaces E_i $(1 \leq i \leq n)$, which are reflexive for $j+1 \leq i \leq n$ and whose pre-duals[3] are separable for $1 \leq i \leq j$. Assume $E := \bigcap_{1\leq i\leq n} E_i$ is non-empty. Then there exists a function $u \in E$ and a subsequence $\{u^{\tilde{k}}\}$ s.t.

$$\begin{aligned} u^{\tilde{k}} &\rightharpoonup^* u \quad \text{in } E_i \quad \text{for} \quad 1 \leq i \leq j, \\ u^{\tilde{k}} &\rightharpoonup u \quad \text{in } E_i \quad \text{for} \quad j+1 \leq i \leq n. \end{aligned}$$

Proof. From the uniform bound in E_1 and the weak* sequential compactness theorem (B.0.35) we deduce the existence of a subsequence $\{u^{k_1}\}$ of $\{u^k\}$ converging weak* to some u_1 in E_1, i.e.,

$$u^{k_1} \rightharpoonup^* \quad u_1 \quad \text{in } E_1.$$

Now $\{u^{k_1}\}$ is uniformly bounded in E_2, so again using the weak* sequential compactness theorem we extract another subsequence $\{u^{k_2}\}$ of $\{u^{k_1}\}$ converging weak* to some u_2 in E_2, i.e.,

$$u^{k_2} \rightharpoonup^* \quad u_2 \quad \text{in } E_2.$$

We continue in this fashion, repeatedly extracting weak* convergent subsequences until we have

$$u^{k_j} \rightharpoonup^* \quad u_j \quad \text{in } E_j.$$

[3]Given Banach spaces X and Y such that $X' = Y$, then the pre-dual of Y is X.

Now $\{u^{k_j}\}$ is uniformly bounded in E_{j+1}, and so using the 'weak compactness theorem' (B.0.34) we can extract a subsequence $\{u^{k_{j+1}}\}$ of $\{u^{k_j}\}$ converging weakly to some u_{j+1} in E_{j+1}, i.e.,

$$u^{k_{j+1}} \rightharpoonup \quad u_{j+1} \quad \text{in } E_{j+1}.$$

We repeatedly extract convergent subsequences as before, the only difference being that the subsequences converge weakly, instead of weak*, until we have

$$u^{k_n} \rightharpoonup \quad u_n \quad \text{in } E_n.$$

Now in a reflexive Banach space weak convergence is equivalent to weak* convergence (see Theorem A.0.22), thus we can write

$$u^{k_i} \rightharpoonup^* \quad u_i \quad \text{in } E_i, \qquad 1 \le i \le n.$$

Recall that $\{u^{k_{i+1}}\}$ is a subsequence of $\{u^{k_i}\}$ and subsequences of weak* convergent subsequences converge weak* to the same limit (see Theorem A.0.22). Thus we have

$$u^{k_n} \rightharpoonup^* \quad u_i \quad \text{in } E_i, \qquad 1 \le i \le n.$$

Finally, as weak* limits are unique (see Theorem A.0.22) we deduce that

$$u_1 = u_2 = \cdots = u_i = \cdots = u_n =: u \in E,$$

and after setting $u^{\tilde{k}} := u^{k_n}$ the proof is complete. ∎

Theorem B.0.37 (Lions-Aubin compactness lemma)
Let E_0, E and E' be three Banach spaces such that

$$E_0 \overset{c}{\hookrightarrow} E \hookrightarrow E_1,$$

E_0, E_1 reflexive. Let T be finite and $1 < p_0, p_1 < \infty$, then the space

$$W = \left\{ v \quad | \quad v \in L^{p_0}(0,T;E_0), \quad v_t = \frac{dv}{dt} \in L^{p_1}(0,T;E_1) \right\},$$

with the norm

$$\|v\|_W := \|v\|_{L^{p_0}(0,T;E_0)} + \|v_t\|_{L^{p_1}(0,T;E_1)},$$

is a Banach space and the injection of W into $L^{p_0}(0,T;E)$ is compact ([84], p.271, [53], p.58).

Lemma B.0.38 (lemma of Lions)

Let $\Omega_T := \Omega \times (0,T)$ be a bounded open set in $\mathbb{R}^d \times \mathbb{R}$ and $\{g^k\}$, $g \in L^q(\Omega_T) \equiv L^q(0,T;L^q(\Omega))$, where $\{g^k\}$ is a sequence of functions s.t.

$$\|g^k\|_{L^q(\Omega_T)} \leq C, \quad g^k \to \quad g \quad \text{('pointwise')} \quad \text{a.e. in } \Omega_T,$$

then $g^k \rightharpoonup g$ in $L^q(\Omega_T)$ ([53], p.12).

This lemma is also proved in [70], p.218 in a form involving $\Omega \subset \mathbb{R}^d$, instead of $\Omega_T := \Omega \times (0,T)$.

Lemma B.0.39 (continuous in time property 1)

Let $u \in L^2(0,T;H^1(\Omega)) \cap L^p(\Omega_T)$ and $\frac{du}{dt} \in L^2(0,T;[H^1(\Omega)]') + L^q(\Omega_T)$ where $p := \rho+2$, $q := \frac{\rho+2}{\rho+1} \in (1,2)$, $\rho > 0$. Then u is almost everywhere equal to a function continuous from $[0,T]$ into $L^2(\Omega)$, i.e., $u \in C([0,T];L^2(\Omega))$ a.e.

Proof. The proof is an adapted (and expanded) version of a proof for a similar result ([70], Theorem 7.2) in the supplementary booklet 'Solutions to Exercises', Exercise 8.2, available at http://www.cup.org/titles/0521635640.html. See also [85], p.71, [84], p.260 for proofs of the more standard (simpler) result (Lemma B.0.41).

Initially we define some function spaces on the general time interval $I := (t^*, t)$, $0 \leq t^* < t \leq T$, where $V := H^1(\Omega)$ and $V' := [H^1(\Omega)]'$:

$$\begin{aligned} X(I) &:= L^2(I;V) \cap L^p(\Omega \times I), \\ X'(I) &:= L^2(I;V') + L^q(\Omega \times I), \\ H(I) &:= L^2(\Omega \times I), \\ H'(I) &:= [H(I)]'. \end{aligned}$$

It follows from Lemma A.0.25 that $X'(I)$ is the dual space of $X(I)$ and by considering the associated Banach norms it is easy to verify the following chain of injections

$$X(I) \hookrightarrow H(I) \hookrightarrow X'(I). \tag{B.0.1}$$

Consequently, by an analogous argument to the proof of part (ii) of Lemma A.0.5, and the application of a Riesz Representation theorem for $L^2(I;L^2(\Omega))$ [96], p.40, which allows us to explicitly identify the elements of this space with those in the corresponding dual space, we have

$$\langle f, x\rangle_{X'(I),X(I)} = \langle f, x\rangle_{H'(I),H(I)} = \int_{t^*}^{t} (f, x)\, dt, \quad \forall f \in H(I),\ x \in X(I). \qquad \text{(B.0.2)}$$

Now as in [70], p.191, we extend u outside $[0, T]$ by zero and set $u_n(t) := (u(t))_{1/n}$, a mollified version of u (see [70], p.19, for the precise definitions of a general mollified function). With this setup we approximate u by $u_n \in C^1([0,T]; V)$ such that

$$u_n \to u \quad \text{in } X(0,T), \quad \frac{du_n}{dt} \to \frac{du}{dt} \quad \text{in } X'(0,T).$$

From (B.0.2) we have

$$\begin{aligned}
2\left\langle \frac{du_n}{dt}, u_n \right\rangle_{X'(I),X(I)} &= 2\int_{t^*}^{t} \left(\frac{du_n}{ds}, u_n\right) ds, \\
&= \int_{t^*}^{t} \frac{d}{ds}\|u_n\|_0^2\, ds, \\
&= \|u_n(t)\|_0^2 - \|u_n(t^*)\|_0^2.
\end{aligned}$$

We now choose t^* so that $\|u_n(t^*)\|_0^2$ is the mean value of $\|u_n(t)\|_0^2$ over $[0, T]$, i.e.,

$$\|u_n(t^*)\|_0^2 = \frac{1}{T}\int_0^T \|u_n(t)\|_0^2\, dt.$$

Thus with the application of Cauchy-Schwarz, a simple Young's inequality and noting the injections (B.0.1) we have

$$\begin{aligned}
\|u_n(t)\|_0^2 &= \frac{1}{T}\|u_n\|_{L^2(\Omega_T)}^2 + 2\left\langle \frac{du_n}{dt}, u_n \right\rangle_{X'(I),X(I)} \\
&\leq C\|u_n\|_{X(0,T)}^2 + 2\left\|\frac{du_n}{dt}\right\|_{X'(0,T)} \|u_n\|_{X(0,T)} \\
&\leq \left(C + \frac{1}{2}\right)\|u_n\|_{X(0,T)}^2 + \left\|\frac{du_n}{dt}\right\|_{X'(0,T)}^2.
\end{aligned}$$

Thus

$$\sup_{t\in[0,T]} \|u_n(t)\|_0 \leq C\left(\|u_n\|_{X(0,T)} + \left\|\frac{du_n}{dt}\right\|_{X'(0,T)}\right)$$

and as u_n and du_n/dt are Cauchy sequences in $X(0,T)$ and $X'(0,T)$ respectively, this implies u_n is a Cauchy sequence in $C([0,T]; L^2(\Omega))$ and hence $u \in C([0,T]; L^2(\Omega))$ as claimed. ∎

Lemma B.0.40 (continuous in time property 2)

Let $u \in L^2(0,T;H^2(\Omega))$ and $\frac{du}{dt} \in L^2(\Omega_T)$. Then u is almost everywhere equal to a function continuous from $[0,T]$ into $H^1(\Omega)$, i.e., $u \in C([0,T];H^1(\Omega))$ a.e.

This Lemma is proved in the first part of the proof of Corollary 7.3 in [70].

Lemma B.0.41 (continuous in time property 3)

Let V, H, V' be three Hilbert spaces, each space included and dense in the following one, V' being the dual of V. If $u \in L^2(0,T;V)$ and $u_t \in L^2(0,T;V')$ then

(i) u is almost everywhere equal to a function continuous from $[0,T]$ into H, i.e., $u \in C([0,T];H)$ a.e.,

(ii) and the following result holds in the scaler distribution sense on $(0,T)$:

$$\frac{1}{2}\frac{d}{dt}\|u\|_0^2 = \langle u_t, u\rangle_{V',V},$$

([85], p.71 [84], p.260).

Appendix C

Matrices associated with the finite element discretisation

For completeness we give some matrices needed for the implementation of the fully discrete finite element methods in Chapter 6. We assume the homogeneous Neumann boundary conditions. Let

$$M_{ij} := (\varphi_i, \varphi_j), \quad K_{ij} := (\nabla\varphi_i, \nabla\varphi_j),$$

where M and K are the mass and stiffness matrices respectively. Associated with these matrices are

$$\widehat{M}_{jj} := (1, \varphi_j), \quad L := (\widehat{M})^{-1}K,$$

where $\widehat{M}$ is the 'lumped' mass matrix, effectively obtained by adding the off-diagonal elements of M to the diagonal elements (see [82], p.118 and Section 4.2). The lumped mass matrix arises during the derivation of the linear algebraic systems and the relation

$$(\varphi_j, \varphi_i)^h = \widehat{M}_{jj}\delta_{ij}.$$

See also the notes in [24], pp.359-367, for calculating the stiffness matrix in the two dimensional case.

C.1 One-dimensional case

The matrices below are of dimension $(J+1) \times (J+1)$. It is easy to calculate the following (global) matrices (see [82], pp.24-39):

$$M = \frac{h}{6}\begin{bmatrix} 2 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 4 & 1 & \\ & & & & 1 & 4 & 1 \\ & & & & & 1 & 2 \end{bmatrix}, \quad K = \frac{1}{h}\begin{bmatrix} 1 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{bmatrix}.$$

We also have the following (global) lumped mass matrix and the matrix L:

$$\widehat{M} = \frac{h}{2}\,\mathrm{diag}\{1,2,2,\ldots,2,2,1\}, \quad L = \frac{1}{h^2}\begin{bmatrix} 2 & -2 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -2 & 2 \end{bmatrix}.$$

C.2 Two-dimensional case

The matrices below are of dimension $(J+1)^2 \times (J+1)^2$, while the blocks are of dimension $(J+1) \times (J+1)$. We have the following (global) stiffness matrix:

$$K = \begin{bmatrix} A & B & & & & & \\ B & C & B & & & & \\ & B & C & B & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & B & C & B & \\ & & & & B & C & B \\ & & & & & B & A \end{bmatrix},$$

with associated blocks

$$A = \begin{bmatrix} 1 & -1/2 & & & & & \\ -1/2 & 2 & -1/2 & & & & \\ & -1/2 & 2 & -1/2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1/2 & 2 & -1/2 & \\ & & & & -1/2 & 2 & -1/2 \\ & & & & & -1/2 & 1 \end{bmatrix},$$

$$B = \operatorname{diag}\{-1/2, -1, -1, \ldots, -1, -1, -1/2\},$$

$$C = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 4 & -1 & & & & \\ & -1 & 4 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 4 & -1 & \\ & & & & -1 & 4 & -1 \\ & & & & & -1 & 2 \end{bmatrix}.$$

We also have the following (global) mass matrix:

$$M = \frac{h^2}{12} \begin{bmatrix} D & E & & & & & \\ E^T & F & E & & & & \\ & E^T & F & E & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & E^T & F & E & \\ & & & & E^T & F & E \\ & & & & & E^T & G \end{bmatrix},$$

with associated blocks

$$D = \begin{bmatrix} 2 & 1/2 & & & & & \\ 1/2 & 3 & 1/2 & & & & \\ & 1/2 & 3 & 1/2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1/2 & 3 & 1/2 & \\ & & & & 1/2 & 3 & 1/2 \\ & & & & & 1/2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1/2 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & 1 & 1 & & & \\ & & & \ddots & \ddots & & \\ & & & & 1 & 1 & \\ & & & & & 1 & 1 \\ & & & & & & 1/2 \end{bmatrix},$$

$$F = \begin{bmatrix} 3 & 1 & & & & & \\ 1 & 6 & 1 & & & & \\ & 1 & 6 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 6 & 1 & \\ & & & & 1 & 6 & 1 \\ & & & & & 1 & 3 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 1/2 & & & & & \\ 1/2 & 3 & 1/2 & & & & \\ & 1/2 & 3 & 1/2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1/2 & 3 & 1/2 & \\ & & & & 1/2 & 3 & 1/2 \\ & & & & & 1/2 & 2 \end{bmatrix}.$$

The lumped mass matrix is:

$$\widehat{M} = \frac{h^2}{6} \begin{bmatrix} H & & & & & & \\ & P & & & & & \\ & & P & & & & \\ & & & \ddots & & & \\ & & & & P & & \\ & & & & & P & \\ & & & & & & Q \end{bmatrix},$$

with associated blocks

$$H = \mathrm{diag}\{2, 3, 3, \ldots, 3, 3, 1\},$$

$$P = \mathrm{diag}\{3, 6, 6, \ldots, 6, 6, 3\},$$

$$Q = \mathrm{diag}\{1, 3, 3, \ldots, 3, 3, 2\}.$$

We use the above matrices to calculate

$$L = \frac{1}{h^2} \begin{bmatrix} S & T & & & & & \\ W & X & W & & & & \\ & W & X & W & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & W & X & W & \\ & & & & W & X & W \\ & & & & & Y & Z \end{bmatrix},$$

with associated blocks

$$\begin{aligned} T &= \mathrm{diag}\{-3/2, -2, -2, \ldots, -2, -2, -3\}, \\ W &= -I, \\ Y &= \mathrm{diag}\{-3, -2, -2, \ldots, -2, -2, -3/2\}, \end{aligned}$$

$$S = \begin{bmatrix} 3 & -3/2 & & & & & \\ -1 & 4 & -1 & & & & \\ & -1 & 4 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 4 & -1 & \\ & & & & -1 & 4 & -1 \\ & & & & & -3 & 6 \end{bmatrix},$$

$$X = \begin{bmatrix} 4 & -2 & & & & & \\ -1 & 4 & -1 & & & & \\ & -1 & 4 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 4 & -1 & \\ & & & & -1 & 4 & -1 \\ & & & & & -2 & 4 \end{bmatrix}, \quad Z = \begin{bmatrix} 6 & -3 & & & & & \\ -1 & 4 & -1 & & & & \\ & -1 & 4 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 4 & -1 & \\ & & & & -1 & 4 & -1 \\ & & & & & -3/2 & 3 \end{bmatrix}.$$