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# Black Holes in the Gravity/Gauge Theory Correspondence

James Paul Gregory

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A thesis presented for the degree of  
Doctor of Philosophy

17 SEP 2002



Centre for Particle Theory  
Department of Mathematical Sciences  
University of Durham  
England

June 2002



*For Mum & Dad*

*“Ex antiquis et novissimis, optima”*

— Motto of Fitzwilliam College, Cambridge.

# Black Holes in the Gravity/Gauge Theory Correspondence

James Paul Gregory

Submitted for the degree of Doctor of Philosophy  
June 2002

## Abstract

The AdS/CFT correspondence provides a microscopic description of black hole thermodynamics. In this thesis, I study the relation between the classical physics of black holes and this microscopic description. I first consider the gauge theory's holographic encoding of non-trivial global causal structure, by studying various probes of the black hole. I study the charged black hole, so that the thermal scale is separated from the horizon scale, to demonstrate which relates to the field theory scale size. I find that, when probing the horizon, both Wilson loops and the duals of static supergravity probes have a scale size determined by the horizon, but the field theory scale size is divergent for a time-dependent probe. I also use the bulk black hole geometry to study the physics of the boundary theory. If we consider a dynamical boundary, a braneworld cosmology is induced from the bulk. However, the presence of matter on the brane introduces unconventional quadratic terms in the FRW equations of this braneworld. I find that bulk black holes induce identical unconventional terms on a matterless brane, therefore providing an alternative description of the same cosmology. A new conjecture relating classical and thermodynamic stability of black branes has emerged from the AdS/CFT correspondence. I make progress in proving this for the case of Schwarzschild black holes in a finite cavity. I also extend the conjecture to the supergravity backgrounds of the direct product form Schwarzschild-AdS  $\times$  Sphere, which are relevant to my study of the AdS/CFT correspondence.

# Declaration

The work in this thesis is based on research carried out between October 1999 and May 2002 at the Centre for Particle Theory, Department of Mathematical Sciences, Durham University, England.

No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in this declaration or in the text. Chapter 2 consists largely of a necessary review of earlier work that acts as a preliminary to the new results contained in the subsequent chapters. The work in chapters 3 and 5 was performed in collaboration with Simon Ross [1, 2], and many of the results have been published in the journal *Physical Review D*. The work in chapter 4 was performed in collaboration with Antonio Padilla [3], and the results are to be published in the journal *Classical and Quantum Gravity*. Chapter 6 is my own work which remains unpublished.

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# Chapter 1

## Black Holes in Quantum Gravity

Even if the open windows of science at first make us shiver after the cosy indoor warmth of traditional humanizing myths, in the end the fresh air brings vigour, and the great spaces have a splendour of their own.

— Bertrand Russell, *What I Believe* (1925).

Einstein's General Theory of Relativity [4] adopted a radical new approach to the description of the gravitational force – matter or energy warps the fabric of spacetime itself, and so gravitational attraction is caused by the natural roll of objects in a curved background. With outstanding precision, macroscopic physics is intricately modelled by the equation which describes the geometry of spacetime

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = -8\pi GT_{\mu\nu}, \quad (1.1)$$

in which  $g_{\mu\nu}$  is the metric of the background with Riemann curvature  $\mathcal{R}_{\mu\nu\rho\sigma}$ . The class of allowed geometries is restricted by the nature of the source term in the Einstein equation,  $T_{\mu\nu}$ , the Energy-Momentum tensor.

In modelling the infinitely many degrees of freedom which form the dynamical gravitational field, the elegance of the Einstein equation becomes apparent. Of great interest to us will be the “*black hole*” solutions, first encountered by Schwarzschild [5]. A black hole is a physically viable solution to (1.1) containing a singularity at which spacetime curvature becomes infinite. Obeying the Cosmic Censorship Conjecture, the singularity is cloaked by a one-way surface, the event



horizon. This is a significant part of the global causal structure of the black hole as it represents the boundary of the causal past of future null infinity – anything crossing the event horizon cannot escape to infinity, since within it, all timelike geodesics converge at the singularity. This non-trivial causal structure of black holes provides a large element of our investigation in this thesis.

Following the identification of the importance of the global structure of black hole spacetimes, and the development of methods to study it, the understanding of the differential geometry of black hole solutions has led to a description of their physics in terms of several basic laws of black hole mechanics [6, 7]. Firstly, Hawking proved that, in a classical process, the area of a black hole’s event horizon cannot decrease with time [8]. The analogy between this result and the second law of thermodynamics was observed by Bekenstein [9, 10], when he suggested that we should identify area with the entropy of a black hole (up to some suitable multiplicative factor). This led him to propose a generalised second law of thermodynamics – that in any physical process the total entropy of a system, defined by the entropy of ordinary matter plus black hole entropy, can never decrease. Subsequently, Bardeen *et al* [7], constructed three more laws of black hole mechanics (at least for stationary black holes) based on their classical geometry. These could also be identified with laws of thermodynamics if the black hole surface gravity,  $\kappa$ , was identified as being analogous to temperature. The analogy between the two sets of laws is summarised in table 1.1.

|                   | <b>Black Hole Mechanics</b>                                     | <b>Thermodynamics</b>                                |
|-------------------|---|--|
| <i>Zeroth Law</i> | Surface gravity, $\kappa$ , is constant over the event horizon. | T is constant throughout a body in equilibrium.      |
| <i>First Law</i>  | $dM = \frac{\kappa}{8\pi}dA + \Omega_H dJ + \Phi_H dQ$          | $dE = TdS + \text{work terms}$                       |
| <i>Second Law</i> | $\delta A \geq 0$   | $\delta S \geq 0$                                    |
| <i>Third Law</i>  | Impossible to achieve $\kappa = 0$ by a physical process.       | Impossible to achieve $T = 0$ by a physical process. |

Table 1.1: The four laws of black hole mechanics

This analogy appeared striking because the nature of the two sets of laws could

not be more different – thermodynamics is based on likelihood when one derives macroscopic results from a large set of microscopic systems, but the laws of black hole mechanics are rigorous results of differential geometry derived from the global structure of the spacetime. Considering just the entropy-area relationship, the event horizon is an artefact of the global causal structure and is not detectable by a local observer. Why then should this be related to entropy, which we usually think of as a measure of disorder in a physical system? Despite these conceptual challenges, a concrete realisation of part of the connection between black hole mechanics and thermodynamics was found when Hawking considered the quantum effects on these classical bodies [11].

Quantum Field Theory arose to describe the electromagnetic force (and now describes the weak and strong nuclear forces too). It is the framework within which Special Relativity and Quantum Mechanics are successfully unified. One of the key consequences of the quantum theory, is that it allows for the pair production of particles from the vacuum. Studying this process in the vicinity of a black hole event horizon led Hawking to discover that black holes are not truly “black” – they decay by emitting thermal radiation at a temperature proportional to their surface gravity. This result allowed Hawking to identify the multiplicative factor in Bekenstein’s entropy-area relationship by precisely matching the first laws of black hole mechanics and thermodynamics [12]. The analogy for the zeroth and third laws is also realised given Hawking’s identification of surface gravity with temperature (see table 1.1). The desire to understand this analogy has contributed to the development of the gravity/gauge theory correspondence, but before discussing this, we will comment on another issue in black hole physics arising from the discovery of Hawking radiation.

Quantum field theory involves only unitary time evolution, but it is not understood how to describe black hole formation and evaporation in a unitary manner. Following the formation of a black hole, material crossing the event horizon will fall towards the singularity, but as the black hole decays, it is not known whether the emitted radiation can carry the vast amounts of information that the black hole has previously hidden from observation. Even if the information is encoded in correlations among the particles of the Hawking radiation, this would violate the principles

of causality and locality since the classical view suggested that this information was held at the singularity. These issues are described as the *Information Loss Paradox* [13] – a satisfactory resolution to the problem has not been uncovered, but some of the new ideas developed over the last few years are leading towards a new understanding of the physics of black holes which may unlock their secrets. What we require is a radical shift in our understanding of spacetime causal structure!

We have seen that treating particles quantum mechanically in a classical background has led to useful results in the study of black hole physics, but the problems associated with these results may indeed arise from the shortcomings of this semi-classical approach. A quantum theory of gravity is required if we are to achieve a unified description of fundamental physics. Naïve attempts to directly quantise the gravitational field fail due to non-renormalisability. An entirely new theory of Quantum Gravity was sought and *Superstring Theory* [14,15] is emerging as a viable candidate for such a theory. The basic premise of superstring theory is that vibrating strings, of length  $\sim 10^{-34}\text{m}$ , replace point particles as the building blocks of both fermionic matter and gauge bosons. However, the non-perturbative definition of string theory also contains higher dimensional objects including the *D-branes*. In a low energy limit, these D-branes describe generalisations of black holes in various dimensions – the black  $p$ -branes of supergravity. In Strominger and Vafa’s study of near-extremal D-branes in the weakly coupled theory, an explicit counting of states was made for particular black holes constructed from D-branes [16]. Given this microscopic description of these black holes, they calculated entropy in a statistical mechanical manner and found it agrees with the Bekenstein-Hawking result. As a quantum theory of gravity, string theory is thus beginning to provide feasible solutions to why the laws of black hole mechanics resemble laws of thermodynamics – a microscopic derivation of black hole entropy suggests that these laws of black holes *are* thermodynamic laws. Since the laws arise from studying the global structure of black hole spacetimes, this new interpretation of them may lead to a fuller understanding of this structure from this new perspective.

Strominger and Vafa’s study considered the coincident D1-D5 brane system, which has the near horizon geometry  $\text{AdS}_3 \times S^3$ . The entropy was calculated in

terms of the 1+1 dimensional conformal field theory living on these D-branes. The agreements between the field theory and gravity provided early indication of a gravity/gauge theory correspondence that would follow. This was to be Maldacena's celebrated "*AdS/CFT correspondence*" [17]. Under the auspices of a unification of quantum field theory and general relativity into one overlying theory, string theory has now provided correspondences which have interweaved these two pillars of theoretical physics in an unimaginably fruitful way. In particular, we can now begin to study the problems associated with black holes in quantum gravity from a new vantage point – the gravity/gauge theory correspondences are *holographic*, they match gravity theories to gauge theories in fewer dimensions. Holographic theories state that the amount of information contained in a bulk region is limited by the size of that region's boundary (we will discuss this further in section 2.3) – this requirement was inspired by the black hole entropy bound [18, 19].

Understanding how gravitational spacetime causal structure is encoded in a gauge theory is of great interest, and this motivates our study in chapter 3. In particular we use the AdS/CFT correspondence to study the non-trivial causal structure of black holes. In the next chapter of this thesis we will give the dual gauge theory description of AdS black hole gravity solutions, which provides the test bed for our investigation. At this stage, resolving the black hole information paradox is too great a task, but we undertake a study of the encoding of the black hole event horizon in the field theory, since this is one of the key features of the black hole's non-trivial causal structure. Given that objects in the black hole spacetime must move on a trajectory which lies within their future lightcone, we should be able to understand how this restriction arises from the gauge theory. We extend a proposal of a gauge theory mechanism for this to the case of black holes. We find that this suggests a connection between the physics of probes of the bulk spacetime and their dual sources in the gauge theory as the bulk probe nears the event horizon – this is a possible re-interpretation of UV/IR relations for black hole spacetimes, which connect the scale size of field theory operators to the radial position of a bulk probe. We will review how gauge theory operators act as sources for bulk gravitational fields in section 2.2. We are then able to test this generalised relation by studying a

variety of probes, both static and time-dependent, of the black hole event horizon. Some of the probes we study will illustrate how bulk locality may be encoded in the field theory despite the holographic nature of the correspondence.

One of the appealing features of the gravity/gauge theory correspondence is the fact that it describes a purely gravitational theory in terms of a gauge theory alone and vice versa. However, the gravitational description of a gauge theory coupled to gravity would also be useful, since it may be able to provide a solely gravitational description of the cosmology of our universe. Extensions of the gravity/gauge theory correspondence in this direction have provided what we need to make this a possibility. They relied on bringing together two of the most well studied concepts in recent years – the gravity/gauge theory correspondences and *braneworlds*. The notion of the braneworld [20–22] is that our universe could be a four-dimensional hypersurface living in a higher-dimensional spacetime. The bulk physics of the higher-dimensional spacetime can induce the correct physics on the braneworld to govern the cosmology of our universe. In so doing, braneworlds can be used to propose viable solutions to many of the problems of cosmology. We will discuss these concepts in more detail in section 2.5. For our study, we would like to understand what effect a black hole in the gravity theory has on the dual braneworld cosmology. It was earlier found that when matter was placed in the brane universe, the presence of the extra dimensions induced non-conventional terms in the braneworld cosmology which are not present in the standard cosmology of a four-dimensional universe [23, 24]. In chapter 4, we will show that these non-conventional terms can be found to arise in a completely different way – they can be induced by the black hole on a matterless brane universe. This example illustrates how the black holes can be used to understand their dual theory, rather than the opposite way which we explore in chapter 3. In this context, a complete understanding of the global structure of black hole spacetimes could be used to probe the cosmology of our universe.

One of the remarkable results of studying black holes in the gravity/gauge theory correspondence is the new relations which have been identified between the classical dynamical evolution of black brane solutions and their thermodynamics. It has been proposed that classical black branes are dynamically stable if and only if they are

locally thermodynamically stable [25, 26]. These are new examples of relationships between the local dynamics of black holes and their thermodynamics. The laws of black hole mechanics all relate to the global structure of black holes and we have seen how these are related to thermodynamics. Despite the indication that, within a quantum theory of gravity, there is a microscopic derivation of the black hole thermodynamics, the laws which result are global concepts since they refer to the general macroscopic behaviour of a large number of microscopic systems. It is therefore surprising that we can now identify correspondences between two notions of “local” stability at the classical level, but this is indeed what we have following Gubser and Mitra’s conjecture.

We will discuss the new conjectured correspondence in section 2.6. Whilst this correspondence between dynamic and thermodynamic stability is only a conjecture, an excellent argument for its general validity was given in [27]. However, this still falls short of a rigorous proof – in chapter 5 we will give a more complete argument connecting classical and thermodynamic stability for the specific case of an uncharged black hole in a finite cavity. This is a simpler case than that studied in [27], which allows us to be more rigorous in connecting the two notions of stability. Our argument will rely on explicitly demonstrating the equivalence of the onset of thermodynamic instability with the presence of the Euclidean negative mode associated with classical instability.

Given that, in the context of black branes, dynamic and thermodynamic stability have been identified by this new conjecture, it is natural that we would want to extend this connection to a wider class of backgrounds of the gravitational theory. Stability of a gravitational solution indicates that the solution is physically relevant and thus questions regarding the stability of any background are necessarily asked. We undertake the task of extending the conjecture in chapter 6, where in particular we study backgrounds which are the direct products of black holes and spheres. These are the backgrounds which arise naturally in the construction of the gravity/gauge theory correspondences, hence to understand their stability is certainly a well-motivated problem. In this situation, by the nature of the gravity/gauge theory correspondence, an instability of the gravitational background should be interpreted

as some interesting phenomenon in the dual gauge theory, such as a phase transition. Dual to the black hole spacetime is the finite temperature field theory in which this transition is realised.

We conclude this thesis in chapter 7 with a summary of the findings of our study. Furthermore, we give an outlook on future research directions in this field which naturally follow on from those which have been discussed.

# Chapter 2

## Preliminaries

Mathematical discoveries, small or great... are never born of spontaneous generation. They always presuppose a soil seeded with preliminary knowledge and well prepared labour, both conscious and subconscious.

— Jules Henri Poincaré (1854-1912),  
*in* Alan L. Mackay, *A Dictionary of Scientific Quotations* (1991).

### 2.1 String theory

In its four decades of history, string theory has developed into a broad arena of study for a large field of theoretical physicists. Superstring theories in particular, are interesting for two reasons: they fix traditional ultraviolet problems associated with quantum field theories, and they provide candidate normalisable quantum theories of gravity. An excellent account of the theory, including a historical perspective, is given in [14]. More modern aspects of the theory are described in detail in [15], and in the self-contained introduction [28]. In what follows, we will give a brief review of the basic principles of String theory that will be relevant to this thesis. This will aid us significantly in understanding the gravity/gauge theory dualities which will form the basic premise of our study.

### 2.1.1 A menagerie of strings

We will study the theory of supersymmetric strings propagating in 10 dimensions. One can consider both open and closed strings; we will be mostly interested in the type II closed theories which have 32 supersymmetries. At the perturbative level, the theory of closed strings is a useful tool since, amongst its oscillations, the closed string contains a spin two particle - the graviton - and hence closed string theory is, at least in part, a theory of gravity. Our closed superstrings arise from two copies of the open string spectrum with left- and right-moving levels of the wave-like oscillations matched. In these theories we achieve spacetime supersymmetry by projecting out certain states of the string theory spectrum, but there is a sign choice in the projections that are undertaken. If the same sign is adopted for the left- and right-movers we uncover type IIB superstrings, otherwise we have type IIA superstrings. Furthermore, we must impose boundary conditions on the left- and right-moving fermionic modes of the closed string. We can restrict them to be periodic “Ramond” (R) conditions, or antiperiodic “Neveu-Schwarz” (NS) conditions. These conditions can be applied independently on the left- and right-movers, so closed string theory contains a variety of fields. Of particular importance to us are the Ramond-Ramond fields, since the objects which are charged under these fields are the *D-branes*. We will shortly see how these are responsible for deriving the gravity/gauge theory correspondences that we study.

There are also several other fundamental string theories, the type I open string theory, and the two heterotic string theories. Whilst it was initially thought that each of these five string theories were independent theories, a web of dualities has been uncovered which links each of them into an overlying 11-dimensional theory, *M-theory* (for a review of these ideas, see [29]). This could be the unified theory that is sought, and in studying string theory we are uncovering various aspects of M-theory in differing regimes.

### 2.1.2 D-branes

The discovery of the dualities which link the individual string theories sparked a revolution in our understanding of string theory. It was noticed that to complete the matching of states in these dualities, string theory requires additional objects which are charged under the Ramond-Ramond gauge fields. However, these objects are non-perturbative and hence could never arise in the spectrum of states of the fundamental string – they are the D-branes<sup>1</sup> [32–34].

The role of such objects in string theory can be understood by considering the constraints that must be satisfied by the open string. When the string action is extremized, one discovers a string worldsheet surface term which is usually set to zero by the requirement that momentum does not flow off the ends of the string – this is the traditional Neumann boundary condition. However, this surface term can also be set to zero in a more unconventional way, by requiring that the string’s ends be fixed to hyperplanes – these are the Dirichlet boundary conditions, and the hyperplanes on which the string’s ends are fixed are the D-branes. Imposing these conditions allows open strings to live in an otherwise closed string theory. A closed string theory has a symmetry called T-duality which states that string theory on a compact dimension of radius  $R$ , is exactly the same as string theory on a compact dimension of radius  $\frac{\alpha'}{R}$  ( $\alpha' = l_s^2$  sets the tension of the string to be  $T = \frac{1}{2\pi\alpha'}$ ). The matching of states between the T-dual theories is clear for closed strings which wrap the compact dimension, but open strings only obey this intrinsically stringy duality if we impose the Dirichlet boundary conditions which restrict their endpoints to lie on a fixed plane in the compact dimension. T-duality interchanges Neumann and Dirichlet boundary conditions, realising D-branes of various dimensions. In general, one speaks of  $Dp$ -branes which are  $(p + 1)$ -dimensional.

The elegance of non-perturbative string theory is unveiled when we realise that these D-branes are dynamical solitonic objects in string theory. We can add non-dynamical degrees of freedom to the endpoints of open strings, Chan-Paton factors,

---

<sup>1</sup>An excellent review of D-branes and their applications can be found in [30] and the forthcoming textbook [31].

which run from 1 to  $N$ , and these introduce a  $U(N)$  gauge symmetry into our string theory. These govern the interactions of open strings, and are not ruled out by the restrictions of Poincaré invariance of our spacetime, but how do they modify the D-brane picture? When Chan-Paton factors are taken into consideration strings can end on any of  $N$  coincident D-branes. We therefore find that the open string spectrum on the worldvolume of our D-branes contains a  $U(N)$  gauge theory. The precise details of this theory depend on the particular string theory that is being studied.

## 2.2 The AdS/CFT correspondence

Maldacena conjectured a correspondence between superstring theory and gauge theory, the AdS/CFT correspondence [17], which provides us with a concrete example of the holographic encoding of bulk gravity theories in terms of gauge theories. It identifies the string theory and gauge theory descriptions of D-branes. We have seen that D-branes are non-perturbative objects upon which open strings can end – the low energy dynamics of these strings describes the gauge theory. Furthermore, at low energies there is a dual description of the same physical system: the propagation of closed strings in the near horizon region of the supergravity metric describing the D-branes. Since this description realises a theory of gravity, the field theory in the AdS/CFT correspondence provides us with a non-perturbative theory of quantum gravity. We can therefore use the correspondence to study the physics of bulk black holes via their holographic encodings in dual gauge theories. In the remainder of this section we will review the reasoning which led to Maldacena’s celebrated conjecture.

The starting point is to consider type IIB superstring theory in a 10-dimensional Minkowski background containing  $N$  parallel D3-branes each separated by the vector  $\vec{r}$  in the 6 transverse coordinates. The precise definition of Maldacena’s low energy limit is then to take  $\alpha' \rightarrow 0$  and  $r \rightarrow 0$  (thus making the D-branes coincident) whilst  $U = r/\alpha'$  is held fixed – this holds fixed the energy of strings which stretch between the D-branes. There are now two viewpoints from which one can consider this system and we look at these in turn.

Firstly, at energies below the string scale, we restrict ourselves to the massless modes of the string theory spectrum in the D3-brane background – this system is described perturbatively by open strings ending on the branes and closed strings propagating in the bulk. Consider first the closed string propagation – the low energy effective action describing this consists of a sum of the IIB supergravity action for the free massless modes, and higher derivative corrections to this action proportional to  $\sqrt{G_N} \sim g_s \alpha'^2$ . These corrections vanish when we take Maldacena's  $\alpha' \rightarrow 0$  low energy limit, thus reducing us to the 10-dimensional supergravity theory. The massless modes of the open string sector describe a gauge theory in the 3 + 1-dimensional Minkowski worldvolume of the branes – from our discussion in section 2.1.2, we have some evidence that this should be a  $U(N)$  gauge theory. In fact, strings ending on single D-branes describe a  $U(1)$  theory; the fact that strings ending on  $N$  coincident D-branes describe an enhanced non-abelian  $U(N)$  gauge theory, rather than the naïve product of gauge theories,  $U(1)^N$ , is a highly non-trivial fact [35]. The precise gauge theory which results is  $\mathcal{N} = 4$  supersymmetric  $U(N)$  gauge theory with coupling constant  $g_{YM}^2 = 4\pi g_s$ . This field theory is known to be a conformal field theory – its operators obey the superconformal algebra. There are higher derivative corrections to the action describing strings ending on D-branes, but these too vanish in Maldacena's low energy limit. Finally, in the spectrum of massless modes of the string theory spectrum, one can consider the interaction of the closed and open string sectors, representing the ability of D-branes to emit and absorb closed strings. The leading terms in the interaction Lagrangian are found in [33], and these too are proportional to positive powers of  $\sqrt{G_N}$  and therefore drop out as  $\alpha'$  is taken to zero. From this point of view, we are thus left with 2 decoupled systems.

Our second point of view arises when we consider the fact that D-branes are massive objects which thus warp our 10-dimensional supergravity background. In the case of D3-branes, they also source the Ramond-Ramond self-dual five form field strength,  $F_5$ . The supergravity description of  $N$  coincident D-branes is thus a solution of the low energy effective string action

$$S = \frac{1}{16\pi G} \int d^D x \sqrt{g} \left[ \mathcal{R} - \frac{1}{2} (\nabla\phi)^2 - \frac{1}{2 \cdot 5!} F_5^2 \right]. \quad (2.1)$$

(We must impose the condition of self-duality of  $F_5$  by hand.)

The 3-brane solution is then given by the metric

$$ds^2 = H(r)^{-1/2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H(r)^{1/2} (dr^2 + r^2 d\Omega_5^2), \quad (2.2)$$

$$F_5 = (1 + *)dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge d(H(r)^{-1}), \quad (2.3)$$

$$e^{2\phi} = g_s^2, \quad (2.4)$$

where

$$H(r) = 1 + \frac{R^4}{r^4} \quad \text{and} \quad R^4 \equiv 4\pi g_s \alpha'^2 N. \quad (2.5)$$

So our second point of view amounts to considering string theory on this curved background. The metric of this solution has a horizon at  $r = 0$ . In the decoupling limit ( $\alpha' \rightarrow 0$  and  $r \rightarrow 0$  whilst  $U = r/\alpha'$  is held fixed), the string theory on this background again reduces to two decoupled systems. Firstly, there is the system representing the spectrum of massless string states propagating in flat 10-dimensional Minkowski space (i.e. the bulk region, away from  $r = 0$ ), which once again is the 10-dimensional free supergravity theory. However, in the near horizon region,  $r \rightarrow 0$ , there is an infinite redshift of the spectrum of string states and so the full spectrum remains. From this point of view, the fact that the two systems decouple is more intuitive – excitations in the near horizon region cannot climb the infinite gravitational potential well in the 3-brane solution, and so do not interact with the states in the asymptotic region. The near horizon region of this geometry in the decoupling limit reduces to

$$\frac{ds^2}{\alpha'} = \frac{U^2}{\sqrt{4\pi g_s N}} \left( -dt^2 + \sum_{i=1}^3 dx_i^2 \right) + \sqrt{4\pi g_s N} \frac{dU^2}{U^2} + \sqrt{4\pi g_s N} d\Omega_5^2, \quad (2.6)$$

$$\Rightarrow ds^2 = \frac{U^2}{R^2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2 \frac{dU^2}{U^2} + R^2 d\Omega_5^2, \quad (2.7)$$

where in the second equation, we have rescaled, the  $t$  and  $x_i$  coordinates to absorb a factor of  $\alpha'$ . This is the metric of  $\text{AdS}_5 \times \text{S}^5$ , in which both the anti-de Sitter (AdS) spaces and the sphere have radius of curvature,  $R$ .  $\text{AdS}_5$  and  $\text{S}^5$  are the maximally symmetric solutions of the 5-dimensional Einstein equations with constant negative curvature and positive curvature respectively.

We see that, from both points of view, one of our decoupled systems is supergravity in a flat 10-dimensional Minkowski spacetime. It therefore seems natural

to pair off the other decoupled systems since they both describe the same physical system. What remains is Maldacena's "AdS/CFT Conjecture":

“ $\mathcal{N} = 4$   $U(N)$  Super-Yang Mills theory in  $3 + 1$  dimensions is dual to type IIB superstring theory on  $AdS_5 \times S^5$ .”

We should consider the regimes in which this conjecture will provide us with useful results, it will also exhibit why we have stated that the two theories are *dual*. We first recall the relationships between the various constants in the correspondence

$$\frac{R^4}{l_s^4} = 4\pi g_s N = g_{YM}^2 N = \lambda, \quad (2.8)$$

where  $\lambda = g_{YM}^2 N$  is the 't Hooft coupling of the gauge theory. This is the effective coupling which governs large  $N$  Yang Mills theory – it was observed, long before the advent of the AdS/CFT correspondence, that large  $N$  gauge theory at fixed 't Hooft coupling should be described by a string theory [36], and the AdS/CFT correspondence finally realises this precisely. We thus turn to studying AdS/CFT in this large  $N$ , fixed  $\lambda$  limit. In this limit,  $g_s \rightarrow 0$  and we restrict ourselves to tree level, *classical* string theory. However, in a curved background, our understanding of string theory at this level is still very limited and whilst we await further progress in this area, we can make great use of the AdS/CFT correspondence by taking a further limit. If we consider the limit of fixed, but large, 't Hooft coupling whilst fixing the radius of curvature of the sphere and AdS spaces, then we see that this corresponds to  $\alpha' \rightarrow 0$  ( $l_s \rightarrow 0$ ). We are then considering the classical supergravity limit of our string theory. Furthermore, supergravity calculations on  $AdS_5 \times S^5$  are reliable since the radius of curvature of our spacetimes is large compared to the string length.

On the opposite side of the correspondence, we understand the perturbative regime of the Yang Mills theory. However, this regime holds when the effective coupling of the theory, which is the 't Hooft coupling, is small i.e.

$$1 \gg \lambda = g_{YM}^2 N. \quad (2.9)$$

It is now clear why the correspondence is so powerful. We understand each side of the correspondence in different regimes - it is a "duality" - that is, when one of the

theories is weakly coupled, the dual theory is strongly coupled. Classical supergravity can teach us about strongly coupled gauge theory. A concrete statement which one can make via the correspondence is that the large  $N$ , strong 't Hooft coupling limit of the gauge theory is completely described by supergravity, a result which lies at the heart of the AdS/CFT correspondence. Similarly, perturbative gauge theory defines quantum string theory – since the latter is our candidate theory of quantum gravity, our understanding of the gauge theory can help us to study the problems outlined in chapter 1 from a new perspective. The AdS/CFT correspondence has brought together two fundamental physical theories which were born to solve completely different problems in physics – gauge theory and gravity!

Maldacena's conjecture was given a very firm calculational basis in subsequent works [37, 38] which identified the partition function of the string theory with the partition function of the gauge theory via the relation

$$\left\langle e^{\int d^4x \phi_0(\vec{x}) \mathcal{O}(\vec{x})} \right\rangle_{CFT} = \mathcal{Z}_{string} \left[ \phi(\vec{x}, U) \Big|_{\phi(\vec{x}, 0) = \phi_0(\vec{x})} \right]. \quad (2.10)$$

This relation is interpreted to mean that the boundary value of each bulk field acts as a source for each CFT operator (It is partly due to this reason that we often consider the dual gauge theory to live on the boundary of our bulk spacetime – the two dual theories are thus often referred to as the bulk and boundary theories). Furthermore, the mass of the field  $\phi$  is related to the scaling dimension of the operator. From the left hand side of this equation we can calculate CFT correlation functions by taking functional derivatives with respect to  $\phi_0$ , and thus given the string theory partition function we are able to understand our dual field theory. In the supergravity limit of the AdS/CFT correspondence, this relation provides us with greater calculational power, since we can approximate the string partition function by  $e^{-I_{SUGRA}}$  where  $I_{SUGRA}$  is the supergravity action evaluated on the bulk spacetime. Furthermore, we can consistently truncate the 10-dimensional supergravity theory to a gauged supergravity theory on  $AdS_5$  alone, realising the correspondence between a 5-dimensional bulk theory and a 4-dimensional boundary theory. We will use these aspects of the correspondence in chapter 3.

At this point we should remark that the AdS/CFT correspondence is not restricted to the  $AdS_5/CFT_4$  case – this is just the case which arises from studying

D3-branes. However, in chapter 6 we will study the 11-dimensional backgrounds  $\text{AdS}_4 \times S^7$  and  $\text{AdS}_7 \times S^4$  which arise in the near horizon regions of the M-theory M2-branes and M5-branes respectively. We will also consider  $\text{AdS}_3 \times S^3$  which one can arrive at by considering the near horizon region in the intersecting D1-D5-brane geometry. In this case one considers IIB string theory compactified on an appropriate manifold  $M^4$ , to reduce the theory to 6 dimensions. The dual field theories in each of these cases are again the low energy brane worldvolume theories.

Furthermore, although our D-brane construction of the correspondence arises at the exact  $\text{AdS} \times S$  spacetime, Maldacena's conjecture is not limited to this spacetime. It only requires that the bulk spacetime under consideration is asymptotically  $\text{AdS} \times S$  as the boundary is approached. More non-trivial bulk spacetimes have more complicated dual theories. In particular, we can construct field theory duals to bulk AdS black hole spacetimes and we will demonstrate this in section 2.4. It was demonstrated in [37] that we should consider the partition function in (2.10) to sum over bulk spacetimes which satisfy the appropriate boundary conditions. We expect the partition function to be dominated by the contribution of a particular background, but this prescription of the correspondence allows for us to understand field theory phase transitions via the AdS/CFT correspondence.

## 2.3 UV/IR relations

We have seen in the previous section that a gravity theory in a bulk AdS space is dual to a field theory in fewer dimensions, in particular, a CFT which can be considered to live on the boundary of AdS space. As we have already observed, this is a “holographic” correspondence – the precise nature of holography also requires that the number of degrees of freedom in the bulk theory be bounded by one per unit Planck area of the boundary theory [19]. For AdS/CFT, this is hard to realise as both theories contain an infinite number of degrees of freedom. Nevertheless, we can understand the holographic bound by regularising both theories. Susskind and Witten demonstrated [39] that if we consider a cutoff to our AdS space by considering it to have a finite size boundary, inside the infinite size AdS boundary,

then this *infrared* cutoff provides an *ultraviolet* cutoff to the field theory. This is known as the UV/IR relation. Furthermore, in these cutoff theories, the number of degrees of freedom of each theory can be calculated and the holographic bound is realised exactly.

In the AdS/CFT correspondence, the introduction of a source probe in the bulk is reflected in a change in one-point functions in the field theory [40–42] (higher-point functions are also needed to resolve some probes; see e.g. [43,44]). The UV/IR relation, representing the holographic nature of the correspondence, is then reflected in a relation between the radial position of the probe and the characteristic scale of the one-point function in the field theory [39,45]. In the AdS<sub>5</sub>/CFT<sub>4</sub> case, this can be expressed as a distance/distance relationship

$$\delta x_{\parallel} = \frac{\sqrt{g_{YM}^2 N}}{U} \quad (2.11)$$

(i.e., a source at radius  $U$  in AdS<sub>5</sub> corresponds to perturbing the field theory in a region of size  $\delta x_{\parallel}$ ) [45]. Here,  $U$  is the radial coordinate in a Poincaré coordinate system, such that  $U \rightarrow \infty$  at the boundary of AdS. Hence (2.11) relates large distances in spacetime (the IR) to short distances in the field theory (the UV).

In pure AdS<sub>5</sub>, this relationship follows from the isometry  $x^i \rightarrow \lambda x^i, U \rightarrow \lambda^{-1}U$  in the bulk. The UV/IR relationship has also been studied for more general metrics with Poincaré invariance in the directions parallel to the boundary. It is used to relate non-trivial solutions of this form to renormalization group flows in the dual field theory (a huge industry now; early works are [46–49]). However, the class of spacetimes for which the description of bulk position in field theory terms is understood is still very limited. One of the goals of chapter 3 of this thesis is to extend the understanding of this relation for the simplest examples of spacetimes with a non-trivial causal structure.

## 2.4 AdS/CFT and bulk black holes

We have seen that the AdS/CFT correspondence allows us to consider any bulk spacetime which is asymptotically AdS and thus we can now use the correspondence to study AdS black holes. We will see in what follows that such black holes

naturally arise in the correspondence – they correspond to thermal states of the field theory. These states break both supersymmetry and conformal invariance, and so the identification of the string theory with the gauge theory becomes more complicated. However, at the same time, the gravity theory in the bulk black hole spacetime is far less trivial – the reward for understanding the correspondence in these spacetimes is thus clear. In the examples we will study, a D-brane construction of the correspondence is known and we briefly review this here. For the original example discussed in section 2.2, we considered a stack of  $N$  coincident D-branes. The field theory on the worldvolume of these D-branes was dual to the type IIB superstring theory on the near horizon region of the corresponding supergravity solution to the low energy effective string action. We generalise this matching of theories in the next two sections.

### 2.4.1 Schwarzschild AdS black holes

The D3-branes that we originally considered were extremal branes. That is to say that they were in their groundstate and the worldvolume theory of the open strings ending on the stack of branes was at minimum energy, i.e. zero temperature. We stated that our required supergravity solution was the 3-brane given by equation (2.2). However, this metric only represents the extremal 3-brane. In general, the following metric is a solution to the low energy effective action and represents the near-extremal 3-brane solution

$$ds^2 = H(r)^{-1/2} (-f(r)dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H(r)^{1/2} (f^{-1}(r)dr^2 + r^2d\Omega_5^2), \quad (2.12)$$

where

$$f(r) = 1 - \frac{r_0^4}{r^4} \quad \text{and} \quad H(r) = 1 + \frac{R^4}{r^4}. \quad (2.13)$$

This supergravity solution is interpreted in string theory as representing D-branes in an excited state in which the worldvolume theory on the D-branes is a finite temperature  $U(N)$  Yang Mills theory. The near horizon geometry of the solution is given by

$$ds^2 = \frac{U^2}{R^2} (-V(U)dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2 \frac{dU^2}{U^2V(U)} + R^2 d\Omega_5^2, \quad (2.14)$$

where

$$V(U) = 1 - \frac{U_T^4}{U^4}. \quad (2.15)$$

This is the metric of Schwarzschild-AdS<sub>5</sub> × S<sub>5</sub>, in which the black hole factor in this cross product is the toroidal black hole with an event horizon at  $U = U_T$ , and temperature  $T = \frac{U_T}{\pi R^2}$ . It is thus conjectured that IIB string theory on this background is dual to the finite temperature gauge theory.

Although we have motivated this duality with a D-brane prescription, Witten's prescription for the correspondence suggests that such a duality should hold irrespective of this motivation. In particular the D-brane construction leads us to consider toroidal black holes, but the AdS-Schwarzschild solution can also have a spherical or hyperbolic boundary and these can also be studied in the context of the AdS/CFT correspondence. The metrics for each of the possible  $(n+1)$ -dimensional AdS black holes that we've discussed are given below for completeness.

$$ds^2 = -V_k(r)dt^2 + \frac{dr^2}{V_k(r)} + \frac{r^2}{R^2}d\Sigma_{k,n-1}^2, \quad (2.16)$$

with

$$V_k(r) = k - \frac{m}{r^{n-2}} + \frac{r^2}{R^2}, \quad (2.17)$$

where the  $(n-1)$ -dimensional metric  $d\Sigma_{k,n-1}^2$  is

$$d\Sigma_{k,n-1}^2 = \begin{cases} R^2 d\Omega_{n-1}^2 & \text{for } k = +1 \\ \sum_{i=1}^{n-1} dx_i^2 & \text{for } k = 0 \\ R^2 dH_{n-1}^2 & \text{for } k = -1. \end{cases} \quad (2.18)$$

Here,  $d\Omega_{n-1}^2$  is the unit metric on  $S^{n-1}$ , and  $dH_{n-1}^2$  is the unit metric on the  $(n-1)$ -dimensional hyperbolic space  $H^{n-1}$ .

### 2.4.2 Reissner-Nordström AdS black holes

In much of what follows, charged (Reissner-Nordström) Anti-de Sitter black holes will play a significant role, so we review their main properties here and subsequently discuss how these too, fit into the AdS/CFT correspondence.

The  $n$ -dimensional RNAdS black hole is the most general solution of the action of Einstein-Maxwell gravity

$$I = \frac{1}{16\pi G_n} \int_{\mathcal{M}} d^n x \sqrt{-g} (\mathcal{R} - 2\Lambda_n - F^2), \quad (2.19)$$

where the cosmological constant is

$$\Lambda_n = -\frac{(n-1)(n-2)}{2R^2}. \quad (2.20)$$

The bulk equations of motion which result from this action are given by

$$R_{ab} - \frac{1}{2}Rg_{ab} = -\Lambda_n g_{ab} + 2F_{ac}F_b{}^c - \frac{1}{2}g_{ab}F^2, \quad (2.21)$$

$$\partial_a (\sqrt{g}F^{ab}) = 0. \quad (2.22)$$

These admit the following two parameter family of electrically charged black hole solutions for the bulk metric

$$ds_n^2 = -h(U)dt^2 + \frac{dU^2}{h(U)} + U^2 d\Omega_{n-2}^2, \quad (2.23)$$

in which

$$h(U) = \frac{U^2}{R^2} + 1 - \frac{m}{U^{n-3}} + \frac{q^2}{U^{2n-6}}, \quad (2.24)$$

and the electromagnetic field strength

$$F = dA \quad \text{where} \quad A = \left( -\frac{1}{\kappa} \frac{q}{U^{n-3}} + \Phi \right) dt \quad \text{and} \quad \kappa = \sqrt{\frac{2(n-3)}{n-2}}. \quad (2.25)$$

In this solution, the presence of  $q$  introduces the black hole charge for which  $\Phi$  is an electrostatic potential difference. In this general metric,  $h(U)$  has two zeros, the larger of which,  $U_H$ , represents the event horizon of the black hole.

The charge and mass of these black holes can be evaluated by Gauss' Law and the generalised ADM formalism [50] respectively, and are found to be

$$Q = \frac{(n-2)\kappa\Omega_{n-2}q}{8\pi G_n}, \quad \text{and} \quad M = \frac{(n-2)\Omega_{n-2}m}{16\pi G_n}. \quad (2.26)$$

The above solution describes spherical black holes, since they have boundary  $\mathbb{R} \times S^{n-2}$ . However, as seen in the previous section, black holes in AdS space can also have toroidal (planar) boundaries given by  $\mathbb{R}^{n-1}$ , and we will also study these

solutions. They can be found by taking an infinite volume limit of the spherical black holes. Consider the following scaling by the dimensionless parameter  $\lambda$

$$U \rightarrow \lambda^{\frac{1}{n-1}} U, \quad t \rightarrow \lambda^{-\frac{1}{n-1}} t, \quad m \rightarrow \lambda m, \quad q \rightarrow \lambda^{\frac{n-2}{n-1}} q, \quad R^2 d\Omega_{n-2}^2 \rightarrow \lambda^{-\frac{2}{n-1}} \sum_{i=1}^{n-2} dx_i^2. \quad (2.27)$$

Under this rescaling, in the limit  $\lambda \rightarrow \infty$ , the RNAdS metric tends to the planar solution

$$ds^2 = -V(U)dt^2 + \frac{dU^2}{V(U)} + \frac{U^2}{R^2} \sum_{i=1}^{n-2} dx_i^2, \quad (2.28)$$

where

$$V(U) = \frac{U^2}{R^2} - \frac{m}{U^{n-3}} + \frac{q^2}{U^{2n-6}}. \quad (2.29)$$

These black holes have an event horizon at radius,  $U_T$ , given by  $V(U_T) = 0$ , and their temperature  $T$  is related to the period  $\beta$  in Euclidean time by

$$\beta = \frac{1}{T} = \frac{4\pi}{V'(U_T)} = \frac{4\pi R^2 U_T^{2n-5}}{(n-1)U_T^{2n-4} - (n-3)q^2 R^2}. \quad (2.30)$$

The black hole will be extremal ( $T = 0$ , and the horizons coincide) at  $U_T = U_e$ , where  $U_e^{2n-4} = (n-3)R^2 q^2 / (n-1)$ .

In these solutions for Reissner-Nordström black holes, we regain the solutions for Schwarzschild-AdS black holes by setting the charge parameter,  $q$ , equal to zero.

The AdS<sub>5</sub> charged black hole solutions are derived from a reduction of a 10-dimensional solution to the AdS<sub>5</sub> gauged supergravity theory in five dimensions. This theory has an  $SO(6)$  gauge symmetry associated with the isometries of the S<sup>5</sup>. In particular the charge of these black holes arises from introducing equal non-zero rotation parameters in the  $U(1)^3$  maximal abelian subgroup of this  $SO(6)$  R-charge. In the case when these parameters are zero, we previously observed that the solution arose in the near horizon limit of a 3-brane. However, when the rotation parameters are non-zero, the 3-brane is spinning in the S<sup>5</sup> directions. We view this as the supergravity solution associated with spinning D3-branes [51]. Following the construction of the AdS/CFT correspondence, we then understand the dual field theory to the charged black hole solution to be the field theory on the worldvolume of these branes. The rotation parameters give a nonvanishing expectation value to the dual global  $U(1)$  current of the gauge theory. In this case, the chemical potential

of the gauge theory is non-zero since this is conjugate to the non-zero momentum we have introduced in the  $S^5$  directions.

## 2.5 Braneworlds

### 2.5.1 AdS/CFT and braneworlds

Whilst braneworlds were of importance long before the advent of the AdS/CFT correspondence [20], the wealth of ideas emerging from these two areas of physics were not intertwined until after Randall and Sundrum wrote the seminal paper [52], showing that there could be more than four non-compact dimensions. They proposed an “alternative to compactification” in which gravity is allowed to propagate in non-compact extra dimensions, providing these additional dimensions are curved. It is this curvature which sets the Planck mass, rather than the size of compact extra dimensions in more conventional braneworld scenarios [53]. Furthermore, the curvature of the extra dimensions results in their being a dominant bound state of the higher-dimensional graviton that is localised to the four dimensions of our brane universe. With one extra dimension, this Randall Sundrum II (RS2) model therefore considers a 4-dimensional brane embedded in a 5-dimensional bulk anti-de Sitter spacetime. The original analysis of the model required that we fine-tune the tension of the brane,  $\sigma$ , in relation to the cosmological constant in the bulk,  $\Lambda_5 \equiv -6k_5^2$ , so as to obey the relation

$$\frac{4\pi G_5}{3} \sigma = k_5. \quad (2.31)$$

This results in the braneworld having zero cosmological constant and it is referred to as a critical brane. Considering the linearized gravity in this brane, then allows the effective gravitational potential to be calculated

$$V(r) = \frac{G_4}{r} \left( 1 + \frac{2}{3k_5^2 r^2} \right), \quad (2.32)$$

where the four dimensional Newton’s constant is  $G_4 = G_5 k_5$ . The corrections to Newton’s Law are only significant at and below the AdS length scale,  $1/k_5$ .

In this model, the brane is considered to separate two  $\mathbb{Z}_2$  symmetric AdS spacetimes, but the possibility of reinterpreting the brane as a cutoff on a single AdS spacetime was soon considered by a variety of authors (see [54] and references therein) – this shows the close relation between this setup and the AdS/CFT correspondence. In the latter, we consider gravity in the bulk spacetime to be dual to a CFT on the boundary. However, we have seen in section 2.3 that if we cutoff the bulk spacetime in the IR then this is dual to a UV cutoff in the field theory. We thus interpret the physics of a braneworld boundary of AdS spacetime as being governed by a broken CFT. Furthermore, it is only in the limit of the braneworld being at the boundary of AdS space, that this field theory decouples from a gravity theory on the brane. When the braneworld actually cuts off the bulk spacetime, the 5-dimensional gravity theory induces a 4-dimensional gravitational cosmology on the braneworld. In this sense, the braneworld is a truly dynamical boundary of AdS spacetime and its evolution can be studied from this point of view.

### 2.5.2 Braneworld cosmology

The idea of the cosmology of our universe being driven by a gravitational force in an extra dimension has received much interest, not least because such braneworld scenarios are well-motivated by string theory [55, 56]. Understanding the cosmology which arises from these possibilities could provide an interesting test for extra dimensions and string theory.

Consider a braneworld given by the section  $(\mathbf{x}^\mu, t(\tau), Z(\tau))$  in the bulk AdS<sub>5</sub> space

$$ds_5^2 = -h(Z)dt^2 + \frac{1}{h(Z)}dZ^2 + Z^2d\Omega_3^2, \quad \text{where} \quad h(Z) = 1 + k_5^2 Z^2. \quad (2.33)$$

The brane has a normal vector given by  $n_a = (\mathbf{0}, -\dot{Z}, \dot{t})$  and the condition that this be a unit vector implies

$$-h(Z)\dot{t}^2 + \frac{\dot{Z}^2}{h(Z)} = -1. \quad (2.34)$$

Here, an overdot represents the  $\tau$  derivative. We wish the induced metric on the brane,  $h_{ab}$ , to represent a Friedmann-Robertson-Walker (FRW) universe,

$$ds_4^2 = -d\tau^2 + Z^2(\tau) d\Omega_3^2. \quad (2.35)$$

As in the RS2 scenario, we assume  $\mathbb{Z}_2$  symmetry across the brane. However, this implies that the extrinsic curvature tensor,  $K_{ab} = h_a^c h_b^d \nabla_{(c} n_{d)}$ , of the brane is different when calculated from either side. The jump is given by the Israel junction conditions which imply the relation

$$K_{ab} = \frac{\kappa^2}{2} \left( S_{ab} - \frac{1}{3} S h_{ab} \right), \quad (2.36)$$

in which  $S_{ab}$  is the stress-energy tensor on the brane, and  $\kappa^2 = 8\pi G_5$ . Assuming our universe is made from homogeneous and isotropic matter, we have

$$S_{ab} = (\rho_{brane} + p_{brane}) u_a u_b + p_{brane} h_{ab}. \quad (2.37)$$

From the three equations (2.34), (2.36) and (2.37), the conditions representing the cosmological evolution of the brane in the bulk can be derived

$$\frac{\ddot{Z}}{Z} = -k_5^2 - \frac{\kappa^4}{36} \rho_{brane} (2\rho_{brane} + 3p_{brane}), \quad (2.38)$$

$$\left( \frac{\dot{Z}}{Z} \right)^2 = -k_5^2 - \frac{1}{Z^2} + \frac{\kappa^4}{36} \rho_{brane}^2. \quad (2.39)$$

In order to make braneworld cosmology consistent with nucleosynthesis, it is necessary to split  $\rho_{brane}$  into energy density due to matter,  $\rho$ , and that due to a constant tension of the brane,  $\sigma$ .

$$\rho_{brane} = \rho + \sigma \quad \text{and} \quad p_{brane} = p - \sigma. \quad (2.40)$$

The induced Friedmann equation of the brane universe is then

$$H^2 = \frac{\Lambda_4}{3} - \frac{1}{Z^2} + \frac{8\pi G_4}{3} \rho + \frac{\kappa^4}{36} \rho^2, \quad (2.41)$$

where  $H \equiv \dot{Z}/Z$  is the Hubble parameter and  $\Lambda_4$  and  $G_4$  are the 4-dimensional cosmological constant and Newton's constant respectively. These are given by

$$\Lambda_4 = 3(\sigma_5^2 - k_5^2), \quad G_4 = G_5 \sigma_5 \quad \text{where} \quad \sigma_5 = \frac{4\pi G_5 \sigma}{3}. \quad (2.42)$$

Such relations between the 4-dimensional and 5-dimensional constants are standard in the brane cosmology literature. The first arose in the RS2 model, where we saw that  $\sigma_5$  was fine-tuned to give  $\Lambda_4 = 0$ . However, non-critical braneworlds,

$\Lambda_4 \neq 0$ , are now often studied as recent observations suggest we live in a de Sitter universe [57, 58]. The second relation is a particular case of the more general result,  $G_{n-1} = \frac{n-3}{2}G_n\sigma_n$ , which generalises Randall and Sundrum's original connection between the Newton's constants to non-critical branes [52, 59–63].

The first three terms of this braneworld Friedmann equation represent the standard cosmology. However, the  $\rho^2$  term represents a high energy correction to the standard Friedmann equation, the extra dimension is inducing a *nonconventional cosmology* on the brane [23, 24] – testing the compatibility of this model with current observations is of great interest as it probes the feasibility of the braneworld model.

To study a braneworld just in pure AdS space is not embracing the power of dual theories. As in the AdS/CFT correspondence, we can similarly study asymptotically AdS spaces and the fact that this replacement can be made without disturbing the matter content of the braneworld is demonstrated in [54, 64]. Therefore, bulk black holes play their role in the study of braneworld cosmology. Such work was inspired by Savonije and Verlinde [65, 66] who considered a braneworld in a Schwarzschild AdS background. They demonstrated that the standard cosmology driven by the energy density/pressure of an  $(n-1)$ -dimensional conformal field theory (CFT), has a dual description – it can be regarded as being a braneworld cosmology driven by the bulk black hole. One can similarly study the braneworld in a charged RNAdS black hole background. The cosmology is then viewed as being driven either by the charge of the bulk black hole or by the chemical potential of the dual field theory which acts as stiff matter in the brane universe – this proposal has been widely studied in [67–73].

## 2.6 Black hole thermodynamics and geometry

The gravity/gauge theory correspondence represents an exciting possibility for giving further insight into why black hole mechanics (which arises from the geometrical gravitational theory) should be related to thermodynamics. In fact, we have already seen that Maldacena's correspondence arose following studies of black branes which

were undertaken for the purpose of explaining black hole entropy [16]. In this section, we discuss the more modern developments in the geometry/thermodynamic connections. As our depth of understanding increases, more conjectures in this field are now being made and it is a new conjecture between the dynamics of black branes and their thermodynamics that we study in this thesis.

### 2.6.1 Euclidean quantum gravity and the negative mode

One of the best ways to get a grasp on the connections between classical general relativity and black hole thermodynamics is to study the path integral approach to quantum gravity. Following the construction of a path integral, one then uses this as a partition function from which thermodynamic quantities can be derived. The path integral is constructed by integrating over all possible solutions to the Einstein equations. To make the path integral better behaved, we integrate over Euclidean metrics.

The Euclidean path integral is given by

$$Z = \int D[g] e^{-I_E[g]}, \quad (2.43)$$

where  $I_E[g]$  is the Euclidean Einstein-Hilbert action evaluated for an asymptotically flat ( $\sim S^1 \times \mathbb{R}^{n-1}$ ) Riemannian manifold,  $g$ . The  $S^1$  is the compactified Euclidean time direction and has period  $\beta = 1/T$ . In the “semi-classical approximation”, the Euclidean path integral is evaluated by expanding around solutions of the Euclidean Einstein equations (gravitational instantons),  $\bar{g}_{\mu\nu}$ , which are thus extrema of the action – the metric perturbation,  $h_{\mu\nu}$ , is considered to be a quantum field on a fixed classical background with geometry determined by the instanton

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \quad (2.44)$$

$$\Rightarrow I_E[g] = I_E[\bar{g}] + \int d^4x \sqrt{\bar{g}} h^{\mu\nu} A_{\mu\nu\rho\sigma} h^{\rho\sigma} + \dots \quad (2.45)$$

The quadratic part of this action dominates higher order corrections so we can ignore these in our expansion in  $h$ . The contribution to the partition function from a given instanton,  $\bar{g}$ , goes like  $B \exp(-I_E[\bar{g}])$ , where  $B$  is a functional determinant which arises from path integrals over the metric perturbations – it includes a factor

of  $(\det A)^{-\frac{1}{2}}$ . Therefore, if the operator  $A$  has negative eigenvalues, there will be an imaginary part in the partition function and this represents a pathology in the canonical ensemble for quantum gravity. We refer to an eigenvector with such a negative eigenvalue as a candidate negative mode. Its presence is an indication of a thermodynamic instability of the instanton,  $\bar{g}$ , which we then know to be a saddle point of the action rather than a local minimum. We must however, consider the physical nature of any candidate negative mode,  $h_{\mu\nu}$ . If we decompose the perturbation into a transverse tracefree part,  $h_{\mu\nu}^{TT}$ , a trace and a longitudinal part, then the trace will have a negative quadratic term, but this is just a sign of the usual conformal factor problem (see [74] and references therein), and the resulting imaginary part is cancelled by a corresponding contribution from integrating over ghosts – it does not correspond to a physical instability [75]. The quadratic term for the longitudinal part is positive definite. The physical negative modes in pure gravity therefore arise only from the quadratic term for  $h_{\mu\nu}^{TT}$ , which involves the Euclidean Lichnerowicz operator

$$G_{\mu\nu\rho\sigma} = -\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma}\nabla_\beta\nabla^\beta - 2\bar{R}_{\mu\rho\nu\sigma}. \quad (2.46)$$

The negative modes are therefore given by the eigenvectors of

$$G_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma}^{TT} = \lambda h_{\mu\nu}^{TT} \quad (2.47)$$

with negative eigenvalues.

### 2.6.2 Classical instability of a black brane

Black branes are higher-dimensional generalisations of black holes. We can construct the simplest case of a black brane, a 5d black string, by taking the direct product of a black hole and an extra translationally invariant direction, i.e. 4d-Schwarzschild  $\times \mathbb{R}$ . The entropy of a length  $L$  portion of this string can be written in terms of the mass of that portion,  $\mathcal{M}$ , as

$$S_{\text{string}} \propto \frac{\mathcal{M}^2}{L}. \quad (2.48)$$

However, a linear array of 5-dimensional Schwarzschild black holes, inter-spaced to give the same mass density as the string,  $\mathcal{M}/L$ , has an entropy  $S_{BH} \propto \mathcal{M}^{3/2}$ .

Therefore, for large enough  $L$ , the black string entropy is less than the entropy of the array of black holes. Since the entropy of such black objects is proportional to their horizon area, then the 2nd law of black hole mechanics permits a decay from the black string to the array of black holes. This is a thermodynamic argument, but Gregory and Laflamme [76, 77] searched for a perturbative instability of a black string which may be able to realise such a decay. They found that the black string was unstable to long enough wavelength (hence large  $L$ ) perturbations<sup>2</sup>. Their work was initially carried out for black branes carrying magnetic charge with respect to a Neveu-Schwarz  $n$ -form field strength, but it has been generalised to a variety of cases. In particular, for the simple case of uncharged black branes with metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \delta_{ij} dz^i dz^j, \quad (2.49)$$

where  $g_{\mu\nu}$  is the  $d$ -dimensional Lorentzian Schwarzschild metric and  $z^i$  are  $p$  flat spatial directions. If we consider a metric perturbation

$$h_{\mu\nu} = \exp(i\mu_i z^i) H_{\mu\nu}, \quad h_{\mu i} = h_{ij} = 0, \quad (2.50)$$

where  $H_{\mu\nu}$  is transverse traceless and spherically symmetric, then the equation of motion implies [76, 77]

$$\tilde{G}_{\mu\nu}{}^{\rho\sigma} H_{\rho\sigma} = -\mu^2 H_{\mu\nu}, \quad (2.51)$$

where  $\tilde{G}$  is now the Lorentzian Lichnerowicz operator for the metric  $g_{\mu\nu}$ . The results of a Gregory-Laflamme analysis of these perturbations indicate that there exists a critical value of  $\mu$ ,  $\mu^*$ , such that exponentially growing (unstable) solutions exist if, and only if,  $\mu < \mu^*$ . For  $\mu > \mu^*$ , solutions are oscillatory in time and hence these are stable perturbations. Therefore, a time-independent solution of this equation with non-zero  $\mu$ , represents a threshold unstable mode, separating stable and unstable perturbations.

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<sup>2</sup>More recent developments [78] have found that the GL perturbative instability may not be able to realise the decay to an array of black holes – instead a non-uniform black brane [79] may be the endpoint of such a decay. An excellent review of these ideas can be found in [80]

### 2.6.3 The Gubser-Mitra conjecture

We have seen that Gregory and Laflamme discovered a classical instability of black branes, based on a thought argument inspired by thermodynamic stability. A connection between classical and thermodynamic stability was later conjectured by Gubser and Mitra [25, 26]:

“A black brane with a non-compact translational symmetry is classically stable if, and only if, it is locally thermodynamically stable.”

This conjecture was inspired by the AdS/CFT correspondence. They studied the RNAdS solution in 4 dimensions, which arises in the AdS/CFT correspondence in a manner analogous to that discussed in section 2.4.2 – RNAdS<sub>5</sub>. The RNAdS<sub>4</sub> spacetime is a solution of  $\mathcal{N} = 8$  gauged supergravity which is a consistent truncation of 11d supergravity. Therefore any classical instability of the 4-dimensional space can be lifted to be a classical instability of the 11-dimensional space RNAdS<sub>4</sub> × S<sup>7</sup>, which is the near horizon region of  $N$  coincident rotating M2-branes (for large  $N$ ). Therefore, this 4-dimensional black hole spacetime has a field theory dual – namely, the 2 + 1-dimensional superconformal field theory on its boundary, with global R-symmetry charges, dual to the rotation parameters of the M2-brane. It was argued in [25, 81] that a thermodynamic instability is expected in this field theory – the condensation of bosons which carry the relevant global  $U(1)$  charges. Given that the dual field theory exhibits such an instability it is natural to assume that some fluctuation mode will grow exponentially in time. From the AdS/CFT correspondence we then infer that there should be some normalisable mode in AdS which also grows exponentially with time, i.e. the RNAdS spacetime should exhibit a dynamical instability. Gubser and Mitra varied the charge parameters of the RNAdS<sub>4</sub> solution and demonstrated that dynamic and thermodynamic instabilities occur at the same time for this spacetime. They then lifted this to 11 dimensions to propose their conjecture for black branes.

In subsequent work [27], Reall provided a general argument for the validity of this conjecture, based on establishing a relationship between the classical Gregory-Laflamme instability and the Euclidean negative mode associated with thermody-

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dynamic instability. We will exhibit this argument when we study the Schwarzschild black hole in a finite cavity in chapter 5.

## Chapter 3

# Probing the Causal Structure of Black Holes

In holographic theories of gravity, such as the AdS/CFT correspondence [17, 37, 38], the true, fundamental causal structure of the gravitational theory is the fixed background causal structure of a lower dimensional field theory. The dynamical spacetime description is supposed to emerge from this underlying field theory in some approximation. Since spacetime has more dimensions than the space the field theory lives in, the encoding of information about the dynamical spacetime should be quite subtle. Understanding how the spacetime, especially its causal structure, are encoded in the field theory is one of the main open questions about these models. The aim of this chapter is to see to what extent non-trivial causal structures, such as a black hole horizon, effect the values of simple gauge theory observables.

We undertake this study by considering possible implications of a UV/IR relationship in black hole spacetimes. In the relation (2.11),  $U \rightarrow 0$  is mapped to diverging scale size in the field theory. From the spacetime point of view,  $U = 0$  in Poincaré coordinates is an event horizon, and one can think of the divergence in the scale size as reflecting the one-way nature of the horizon: particles at the event horizon cannot move to larger  $U$ , and an infinite scale excitation can't return to smaller scale. As we will review in section 3.1, at least in pure AdS, the relation (2.11) provides a connection between spacetime and field theory causality throughout spacetime [82].

We would like to know if this connection between the UV/IR relation and causality can be generalised. A simple question to ask is whether the horizon of a black hole is also associated with an infinite scale size in the CFT. We will consider a variety of probes of black hole spacetimes, and find that the characteristic scale in the field theory description of time-independent probes is typically finite, even when they are very close to the horizon. Considering time-dependent probes is more complicated, but we argue that the scale size diverges at late times, although the leading behaviour is not directly related to the black hole structure.

The example that we study is a charged black hole in AdS (see section 2.4.2). Considering charged black holes allows us to have a large separation between the horizon size and the thermal scale. In an uncharged black hole, the standard relation (2.11), and some probe calculations, would assign a scale size which is of the order of the thermal scale when a probe is at the black hole horizon. We would like to investigate if this connection between the horizon and the thermal scale persists when there are other scales in the problem, or if we can see some sign of a divergent scale associated with the horizon. The presence of charge allows us to see which boundary scales are related to the thermal fluctuations in the gauge theory and which depend on the scale set by the black hole horizon.

In [51, 83], the thermodynamic properties of charged Reissner-Nordström AdS black holes was investigated. As observed earlier, these black holes can be spherical or toroidal. From the point of view of thermodynamics, the spherical black holes are more interesting, but to analyse UV/IR relations, we will focus on the simpler case of toroidal black holes. It was shown in [51] that these black holes are thermodynamically stable (in both the canonical and the grand canonical ensembles) for arbitrary values of the mass and charge, so the black hole solutions carry information about the CFT in the corresponding ensemble. For most of this work, we will focus on the case of AdS<sub>5</sub>, as a generic example for our study.

As we discovered in section 2.4.2, the charged AdS<sub>5</sub> black holes are derived from a reduction of a spinning 3-brane metric. If the  $S^5$  spins faster than the speed of light, then the Killing vector,  $k = \frac{\partial}{\partial t}$ , will become spacelike outside the black hole horizon. This allows for the possibility of superradiant modes of scattered waves

to carry away the rotational energy of the black brane. The occurrence of such a process indicates the presence of a classical instability of the metric. However, it was shown in [84] that the spherical black hole solutions in the  $\text{AdS}_5 \times S^5$  context will not have superradiant modes, as the internal  $S^5$  rotates at a speed less than the speed of light everywhere in the spacetime. It is easy to extend this argument to the toroidal black holes. The charged black hole (2.28) is derived from the reduction ansatz

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \sum_{i=1}^3 \left[ d\mu_i^2 + \mu_i^2 \left( d\phi_i + \frac{2}{\sqrt{3}} A_\mu dx^\mu \right)^2 \right], \quad (3.1)$$

where  $g_{\mu\nu}$  is the five-dimensional metric,  $\mu_i$  are the direction cosines and  $\phi_i$  are the rotation angles on the  $S_5$ . Non-zero  $A_t$  gives the electric potential

$$A = (\Phi(U_T) - \Phi(U))dt, \quad \text{where } \Phi(U) = \frac{\sqrt{3}}{2} \frac{q}{U^2}. \quad (3.2)$$

The norm of the Killing vector field  $k = \partial/\partial t$  with respect to our ten dimensional metric is then

$$\begin{aligned} k^2 &= -V(U) + \frac{4}{3} A_t^2 \\ &= - \left( 1 - \frac{U_T^2}{U^2} \right) \left[ U^2 \left( 1 - \frac{U_-^2}{U^2} \right) + \left( 1 - \frac{U_-^2}{U_T^2} \right) (U_T^2 + U_-^2) \right], \end{aligned} \quad (3.3)$$

where  $U_-$  is the inner horizon. We see that  $k^2$  is always negative outside the black hole horizon. Hence,  $k$  is always timelike in this region and thus superradiance cannot occur for the toroidal black hole.

A useful rewriting of the metric (2.28) is

$$ds^2 = \frac{R^2}{U^2} \frac{dU^2}{f(U)} + \frac{U^2}{R^2} \left[ -f(U) dt^2 + \sum_{i=1}^3 dx_i^2 \right], \quad (3.4)$$

$$f(U) = 1 - (1 + \theta) \frac{U_T^4}{U^4} + \theta \frac{U_T^6}{U^6}, \quad (3.5)$$

where we have defined a dimensionless parameter  $\theta = q^2 R^2 / U_T^6$ ; for the uncharged black hole  $\theta = 0$ , while the extremal black hole has  $\theta = 2$ . This rewriting of the metric is particularly useful for our study, as varying  $\theta$  allows us to consider the range of different black holes metrics with a fixed horizon scale. The case  $\theta = U_T = 0$  is pure  $\text{AdS}_5$  in Poincaré coordinates, where (2.11) is valid.

After reviewing the connection between (2.11) and causality in section 3.1, we move on to consider specific probes in this black hole background. We begin with a discussion of Wilson loops in section 3.2. We find that the qualitative behaviour of the loop expectation values is independent of the charge, and the non-trivial physics associated with the presence of a black hole appears at a scale given by (2.11). That is, the characteristic scale for these observables is the inverse horizon size, and is not in general related to the temperature. We then go on to discuss supergravity probes in section 3.3. We find that the scale of the expectation value for time-dependent probes diverges at late times, but the expectation value is primarily determined by the asymptotic metric. We study the small contribution from the near-horizon region in the BTZ metric, and argue that it is spatially constant. For static supergravity probes, there is a finite scale size associated with the horizon in general. The scale size associated with the static propagator diverges like  $\ln(T)$  in the extremal limit  $T \rightarrow 0$ . This behaviour provides the main element of surprise in this chapter, and it would be very interesting to gain a better understanding of this logarithmic dependence from the field theory point of view.

In addition to addressing the implications for our understanding of bulk causality, we will briefly comment on the physical significance of these UV/IR relations from the field theory point of view, and remark on the relation to previous work [85, 86] which studied the Euclidean rotating brane solutions as models for pure gauge theories.

We conclude in section 3.4 with a discussion of the difficulties in identifying the origins of spacetime causal structure in the gauge theory, and some speculations for future directions.

### 3.1 UV/IR relations in black hole spacetimes

To begin our investigation of the UV/IR relation in spacetimes with horizons, we review the connection between the UV/IR relation and causality in pure AdS proposed in [82]. In pure AdS, the condition for a bulk probe to move inside the light

cone in the radial direction is

$$-\frac{U^2}{R^2}dt^2 + \frac{R^2}{U^2}dU^2 \leq 0. \quad (3.6)$$

For a supergravity probe, the UV/IR relation (2.11) implies that (3.6) is equivalent to<sup>1</sup>

$$\left| \frac{d\delta x_{\parallel}}{dt} \right| \leq 1, \quad (3.7)$$

which is just the statement that the field theory excitation's size cannot change faster than the speed of light in the field theory. This thus connects the causality of AdS to the causality of the space the field theory lives in.

It would be interesting to see if a similar relation could exist for black hole spacetimes, through a suitable modification of the UV/IR relation (2.11). In a black hole spacetime, the condition (3.6) is modified to

$$-\frac{U^2}{R^2}f(U)dt^2 + \frac{R^2}{U^2f(U)}dU^2 \leq 0. \quad (3.8)$$

We could derive a candidate UV/IR relation on the black hole spacetime by assuming that this spacetime condition is still equivalent to the kinematical condition (3.7). If we assume a UV/IR relationship of the form  $\delta x_{\parallel} = g(U)$ , then (3.7) would imply

$$\left| \frac{dU}{dt} \right| \leq \left| \frac{dU}{dg(U)} \right|. \quad (3.9)$$

Requiring that this condition is equivalent to (3.8) gives

$$\left| \frac{dg(U)}{dU} \right| = \frac{R^2}{U^2f(U)}. \quad (3.10)$$

This can be solved exactly for the black hole spacetimes we are considering, and the results of this are given below. It should be noted that the behaviour differs according to whether or not the black hole is extremal. For non-extremal black holes ( $\theta \neq 2$ ) we find that

$$\delta x_{\parallel} = \frac{R^2}{2(2-\theta)U_T} \left[ \ln \left( \frac{U+U_T}{U-U_T} \right) + \frac{(\theta - (1-\theta)x_2^2)}{x_2} \ln \left( \frac{U+U_Tx_2}{U-U_Tx_2} \right) + \frac{(\theta - (1-\theta)x_1^2)}{x_1} \left( 2 \tan^{-1} \left( \frac{U}{U_Tx_1} \right) - \pi \right) \right] \quad (3.11)$$

<sup>1</sup>We are working in units where  $l_s = 1$ , so from equation (2.8), we have  $R = (g_{YM}^2 N)^{1/4}$ .

where  $x_1$  and  $x_2$  are the positive and negative roots of  $x^4 + x^2 - \theta = 0$  respectively. For extremal black holes ( $\theta = 0$ ) we find that

$$\delta x_{\parallel} = \frac{R^2}{36U_T} \left[ \frac{6UU_T}{U^2 - U_T^2} + 7 \ln \left( \frac{U + U_T}{U - U_T} \right) - 4\sqrt{2} \left( 2 \tan^{-1} \left( \frac{U}{\sqrt{2}U_T} \right) - \pi \right) \right] \quad (3.12)$$

The important feature of these results is how the boundary scale size behaves for a probe near the black hole horizon. It is clear that in both cases the scale size becomes infinite as the black hole horizon is approached.

$$\theta \neq 2 \Rightarrow \delta x_{\parallel} \sim -\frac{R^2}{2(2-\theta)} \ln(U - U_T), \quad (3.13)$$

$$\theta = 2 \Rightarrow \delta x_{\parallel} \sim \frac{R^2}{12} \frac{1}{U - U_T}. \quad (3.14)$$

Thus, a diverging scale size for probes at finite radial position would be a signature of a horizon, under the assumption that bulk causality arises from a kinematical restriction in the field theory. This is the main prediction we will investigate in our discussion of the probes.

In studies of the uncharged black holes [40, 87, 88], it was found that sources near the black hole horizon produced expectation values with a scale  $\delta x_{\parallel} \sim 1/T$ , where  $T$  is the temperature. Such a connection was found for string worldsheets dual to gauge theory Wilson loops in [87, 88], we will discuss these calculations in the next section. It was further demonstrated for infalling fundamental strings wrapped around a compactified spatial dimension of the AdS boundary in [40]. It was proposed that the physical interpretation of these results is that probes which fall into the horizon become entangled with the thermal bath in the gauge theory. This suggests that the presence of finite-temperature Hawking radiation acts as a barrier to our ability to probe the non-trivial classical causal structure in the region near the horizon. Although this is a very feasible argument to explain these results, the connection between the field theory scale size and the thermal scale is not forced upon us, since in the bulk black hole, there are two non-trivial length scales, one set by the horizon scale and the other set by the thermal scale. For the uncharged black hole these scales are the same and one can talk of the horizon scale and the thermal scale interchangeably. The introduction of charge into the black hole allows us to separate the horizon scale from the thermal scale. In particular we can consider

black holes of arbitrarily low temperature, but fixed horizon scale, allowing us to study, in more depth, whether the field theory scale is governed by the black hole horizon or the thermal scale.

## 3.2 Wilson loop calculations

For each operator in the boundary field theory, we expect to be able to find a dual bulk observable and in [89,90], this was studied for the case of Wilson loop operators of the Yang Mills theory. The Wilson loop operator is defined by

$$W(\mathcal{C}) = \frac{1}{N} \text{Tr} \mathcal{P} \exp \left( \oint_{\mathcal{C}} A \right), \quad (3.15)$$

where  $\mathcal{C}$  is a closed loop, and the trace is taken over the fundamental representation of the gauge group. This Wilson loop is associated with the phase factor of the propagation of a very massive quark.

In order to understand the dual picture, it is necessary to consider how massive quarks would appear in the D-brane picture of the AdS spacetime. We begin with the  $U(N+1)$  gauge group of  $N+1$  coincident D-branes and Higgs this theory to  $U(N) \times U(1)$  by giving an appropriate expectation value to the scalars,  $\vec{\Phi}$ . This is equivalent to separating one of the branes and introducing a fundamental string of mass  $\propto |\vec{\Phi}|$ , which stretches from the  $N$  coincident branes to the single brane. This string is the Higgsing  $W$  boson and since it transforms in the fundamental representation of the gauge group we refer to it as the massive quark of the gauge theory. It is important to note that at energy scales  $\ll |\vec{\Phi}|$ , the  $U(N)$  theory is decoupled from the  $U(1)$ , so in the supergravity limit we can study the massive quarks of the  $SU(N)$  SYM theory dual to  $\text{AdS}_5 \times S^5$  by considering strings which stretch from the boundary ( $U = \infty$ ) of AdS space to the horizon at  $U = 0$ . We consider quarks at the same position on the  $S_5$ .

It is natural to propose, via the AdS/CFT correspondence, that the expectation value of the Wilson loop should be given by

$$\langle W(\mathcal{C}) \rangle \sim \exp(-S), \quad (3.16)$$

where  $S$  is the action of the string worldsheet which terminates on the path of the

Wilson loop,  $\mathcal{C}$ , at the boundary of AdS. We can therefore investigate its leading-order behaviour by finding the minimal-area surface for the string worldsheet. The symmetries of the metric (3.4) lead us to consider rectangular Wilson loops with two long sides, and a separation  $L$  between them. The long sides lie either along the  $t$  direction (timelike Wilson loops) or along one of the  $x^i$  directions (spacelike Wilson loops). Both of these scenarios have been studied in the Schwarzschild-AdS bulk, dual to the finite temperature field theory, and we study them each in turn for the charged AdS bulk below.

### 3.2.1 Timelike Wilson loops

We wish to consider Wilson loop operators in the boundary theory, where the loop  $\mathcal{C}$  forms a rectangle with two long sides extended along the  $t$  direction, of length  $t_0$ , and two sides of length  $L$  along one of the  $x^i$  directions. We consider the limit in which  $t_0 \rightarrow \infty$ .

Consider first the vacuum case (pure AdS) [89, 90]. There are two competing scenarios for the dual string picture in AdS of this gauge theory Wilson loop – the preferred one is the one which is energetically favoured. Firstly, we have the worldsheets of two infinite strings stretching from the AdS boundary to  $U = 0$ , which represents a non-interacting quark and antiquark – if this is the dual picture one would want to understand, in gauge theory terms, why the interaction of the  $q\bar{q}$  pair does not occur. The second possibility is that the string would loop in the bulk from the quark to the antiquark, which more naturally realises the interaction of the  $q\bar{q}$  pair. Since (for large  $N$  and large  $g_{YM}^2 N$ ) the expectation value of the Wilson loop is given by (3.16), in which  $S$  is the Nambu-Goto action for a fundamental string worldsheet, then the shape of this loop would be the minimal length curve derived from the minimum-area principle for the worldsheet action. From the gauge theory we know that in the limit  $t_0 \rightarrow \infty$ , the expectation value of the Wilson loop behaves like  $\langle W(\mathcal{C}) \rangle \sim \exp(-t_0 E)$ , where  $E = V(L)$  is the lowest possible energy of the quark-antiquark configuration. Therefore we can extract this quark-antiquark potential energy from this configuration, by solving the Euler-Lagrange equations to minimize  $S$ . We must also subtract the infinite mass of the W boson to regularise

this supergravity calculation – this contribution is equal to the energy of the two infinite straight strings (We will shortly see the details of this calculation when we study the equivalent calculation in the charged black hole background). For pure AdS, the results of this calculation give  $V \propto -(g_{YM}^2 N)^{1/2}/L$ . The  $1/L$  behaviour is, in fact, dictated by conformal invariance.

This calculation was extended to the field theory at finite temperature by considering a string worldsheet in a Schwarzschild-AdS background in [87, 88]<sup>2</sup>. In this calculation there is an extra subtlety – the subtraction to remove the infinite mass of the  $W$  bosons is not made by subtracting the worldsheet area of strings which stretch from infinity to  $U = 0$ . Instead, the worldsheet of strings stretching from infinity to the black hole horizon,  $U = U_T$ , is subtracted. It has become standard to adopt this subtraction in black hole spacetimes, and there are several reasons for this [87]. An early argument comes from D-brane probes of black hole spacetimes [92]. In such calculations, a coordinate transformation is required to match the coordinates of the supergravity solution and the coordinates of the field theory living on the D-brane. This transformation then indicates that, from the point of view of the field theory living on the brane, the horizon is the origin. Furthermore, the Euclidean (Wick rotated) black hole solution only contains the region outside of the horizon. Brandhuber *et al* [87] also gave a physical argument for why the strings must end at the horizon – they burn near the horizon due to the infinite local Hawking temperature there.

The potential one finds in the Schwarzschild-AdS bulk is illustrated in figure 3.2. There are two branches to the potential curve, the upper branch (with  $E \geq 0$ ) is unphysical. The lower branch gives the energy of the U-shaped string configuration hanging into the bulk that we are studying. The shape of the string worldsheet in this configuration is illustrated in figure 3.1.

From the potential curve, we see that the small- $L$  behaviour is similar to the Coulomb behaviour we observe for pure AdS, but at a scale set by  $L \sim 1/T$ , where  $T$  is the temperature, we encounter  $V = 0$ . Since we have subtracted the mass of

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<sup>2</sup>A review of Wilson loops from the string/gauge correspondence can be found in [91], which includes extensive references.

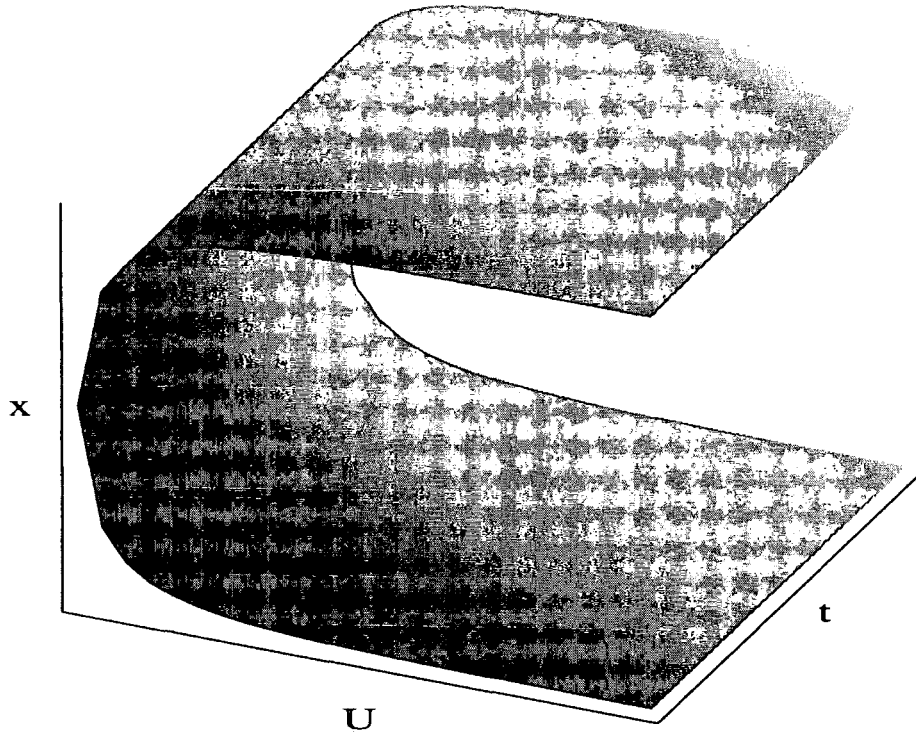


Figure 3.1: Profile of the string worldsheet dual to the interacting  $q\bar{q}$  pair

the W boson which is equal to the energy of the two string configuration, then in the situation for which  $V > 0$  for the single U-shaped string, it is the two string picture which is energetically favourable. This latter solution would have zero potential energy. We should therefore only consider the section of the  $E$  vs.  $L$  curves with negative energy. Where the curve crosses the axis, the potential becomes constant. From the point of view of gauge theory, this corresponds to the screening of the quark charge by the plasma in the field theory which carries the energy of the thermal state. Therefore the quarks become “free” at a great enough separation.

Our objective is to investigate the quark-antiquark potential obtained from the string worldsheet in the charged AdS black hole background. We begin with the metric (3.4), and calculate the string worldsheet area from the Nambu-Goto action

$$S = \int \mathcal{L} dx = \frac{t_0}{2\pi} \int dx \sqrt{(\partial_x U)^2 + \frac{U^4}{R^4} f(U)}. \quad (3.17)$$

We are working in the static gauge  $\tau = t$ ,  $\sigma = x$  for the string worldsheet coordinates, where  $x$  is the position in the boundary direction along which the quarks are separated. This action is independent of  $x$ , so we can calculate  $x$  from the

Euler-Lagrange equations

$$\mathcal{L} - \partial_x U \frac{\partial \mathcal{L}}{\partial (\partial_x U)} = \text{const.} \quad (3.18)$$

The constant in this equation is fixed by imposing that there is a minimum value of the radial coordinate for the string worldsheet,  $U_0$ , at which  $\partial_x U = 0$ . We then find that  $\partial_x U$  is given by

$$\partial_x U = \frac{U^2}{R^2} \sqrt{f(U) \left[ \frac{U^4 f(U)}{U_0^4 f(U_0)} - 1 \right]}, \quad (3.19)$$

and so we can calculate the profile of the string to be

$$x = \frac{R^2}{U_T} \alpha \sqrt{1 - (1 + \theta)\alpha^4 + \theta\alpha^6} \times \int_1^{\frac{U\alpha}{U_T}} \frac{y^2 dy}{\sqrt{(y^2 - 1)(y^2 - \alpha^2)(y^4 + \alpha^2 y^2 - \theta\alpha^4)(y^4 + y^2 - \theta\alpha^6)}}. \quad (3.20)$$

We have introduced a dimensionless parameter  $\alpha = U_T/U_0$ , and we see that  $x = 0$  at  $U = U_0$ , so the string profile is symmetric about  $x = 0$ . Furthermore, the separation of the quarks is given by  $L = 2x(U = \infty)$ . We wish to calculate the energy, which is naively given by  $S/t_0$ , where  $S$  is the action (3.17) integrated over the range  $-L/2 \leq x \leq L/2$ . However, as we've already discussed, this would give an infinite result due to the contribution from the mass of the W boson. We must therefore regularise the expression by only integrating up to  $U = U_{max}$ , and subtracting the regularised mass of the W boson,  $U_{max}/(2\pi)$ . Taking our cutoff to infinity, we then find the solution for the energy

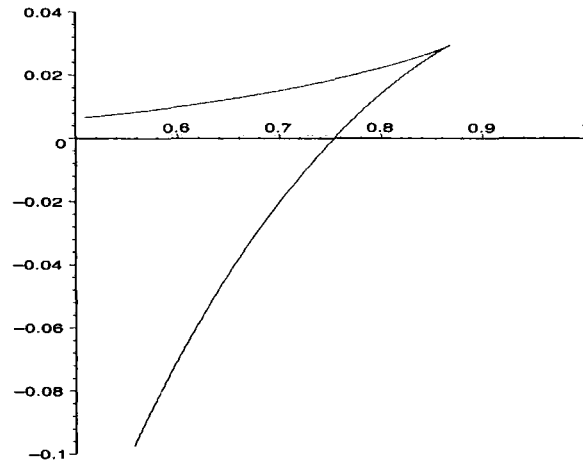
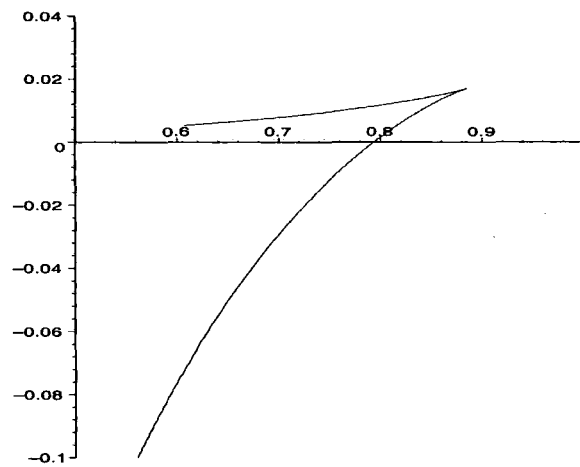
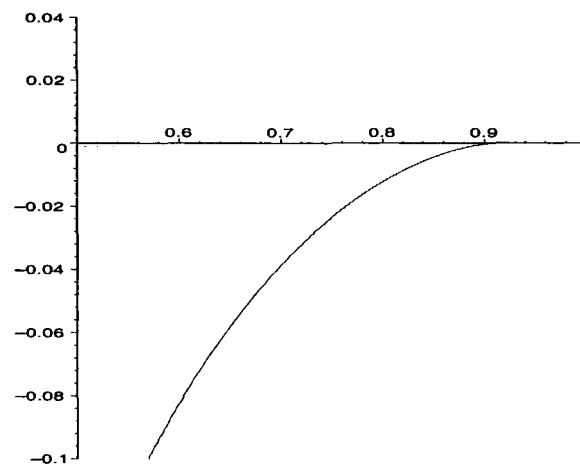
$$\begin{aligned} E &= \lim_{U_{max} \rightarrow \infty} \left[ \frac{1}{2\pi} \int_{-L/2}^{L/2} dx \sqrt{(\partial_x U)^2 + \frac{U^4}{R^4} f(U)} - \frac{U_{max}}{2\pi} \right] \\ &= \lim_{U_{max} \rightarrow \infty} \left[ \frac{1}{\pi} \int_{U_0}^{U_{max}} dU \partial_x U \sqrt{(\partial_x U)^2 + \frac{U^4}{R^4} f(U)} - \frac{U_{max}}{2\pi} \right] \\ &= \frac{U_T}{\pi\alpha} \left[ \int_1^\infty \left( \frac{\sqrt{(y^2 - \alpha^2)(y^4 + \alpha^2 y^2 - \theta\alpha^4)}}{\sqrt{(y^2 - 1)(y^4 + y^2 - \theta\alpha^6)}} - 1 \right) dy - 1 + \alpha \right]. \quad (3.21) \end{aligned}$$

We can only evaluate this integral by numerical methods. If we plot  $E/U_T$  against  $LU_T/R^2$ , the only free variable is  $\theta$ , which specifies the charge on the black hole. We can study the effect of the charge by varying  $\theta$ . The results obtained in the uncharged case have been reproduced here for comparative purposes in figure 3.2.

The typical behaviour of  $E$  vs.  $L$  for a charged black hole (plotted here is the case  $\theta = 1$ ) is given in figure 3.3, and the behaviour for the extremal black hole is given in figure 3.4. It should be noted that in these plots the parameter  $\alpha$  is not plotted over the complete range, as numerical methods to solve the integral break down when  $\alpha$  is close to 0 or 1. However, the behaviour of the  $E$  vs.  $L$  plot in the unplotted regions can be observed from studying  $L$  and  $E$  separately as functions of  $\alpha$ . As  $\alpha \rightarrow 0$ ,  $L \rightarrow 0$  and  $E \rightarrow -\infty$ . With  $\alpha$  increasing from 0, the  $E$  vs.  $L$  plot rises smoothly to the cusp with increasing  $L$  and increasing  $E$ , at which point both  $L$  and  $E$  begin to decrease along the upper branch. This monotonic decreasing behaviour in  $L$  and  $E$  continues until  $\alpha = 1$ , where both  $L$  and  $E$  are zero.

We see that the qualitative behaviour of the potential remains the same as in the uncharged case as the charge of the black hole is increased. The separation at which screening sets in increases slightly, but the overall scale is still set by the horizon radius  $U_T$ . This is reasonable from the point of view of the field theory, since we interpret this screening as due to polarization in the plasma in the field theory which carries the energy density in this state (which corresponds to the black hole mass from the spacetime point of view). In the charged black hole case, this energy density goes like  $U_T$ , and not like the temperature. Even in the extremal,  $T \rightarrow 0$  limit, there is still a finite energy density, which is responsible for the screening behaviour in figure 3.4.

It is interesting to observe that the maximum value of the parameter  $\alpha$  for which  $E$  is negative increases from  $\sim 0.66$  in the uncharged case to 1 in the extremal charged case. That is, as we increase the charge, the string worldsheet probes deeper into the interesting region near the horizon before the cross-over to the disconnected solution. Despite this behaviour, these loops are not a good probe of the bulk causality. In particular, there is no sign of any special behaviour as  $T \rightarrow 0$ . From the field theory point of view, the qualitative screening behaviour is associated with the background energy density, and one seems to find qualitatively similar behaviour independent of the details of the energy distribution. It is also possible to construct examples which display the same screening behaviour without a black hole horizon, for example by considering states on the Coulomb branch of the field

Figure 3.2: Timelike Wilson loop -  $E$  vs.  $L$  plot for uncharged black hole ( $\theta = 0$ )Figure 3.3: Timelike Wilson loop -  $E$  vs.  $L$  plot for charged black hole ( $\theta = 1$ )Figure 3.4: Timelike Wilson loop -  $E$  vs.  $L$  plot for extremal black hole ( $\theta = 2$ )

theory [44, 93, 94]. Thus, while the value of the screening length is dictated by the horizon radius,  $U_T$ , this probe is insensitive to the horizon as a horizon.

### 3.2.2 Spacelike Wilson loops

We can use similar techniques to study spacelike Wilson loops on the boundary of the charged black hole. In earlier studies of the uncharged AdS Schwarzschild black hole, spacelike Wilson loops were considered with the Euclidean black hole metric in the bulk. For the uncharged black hole, the physics of Wilson loops in the Lorentzian and Euclidean black holes is the same, as the two are related by the analytic continuation,  $t \rightarrow i\tau$ . Therefore these studies of spacelike Wilson loops in the Euclidean solution can be directly translated to statements about the Lorentzian solution, which is what we must consider in our study of the causal structure. These spacelike Wilson loops for the finite temperature field theory were considered in [95, 96]. Because of the thermal boundary conditions, the  $t$  direction is compactified on a circle of period  $\beta = 1/T$  in the Euclidean solution. At energies smaller than the compactification scale, this Euclidean bulk solution is dual to a pure gauge theory living in the  $2 + 1$  uncompactified directions, as all the other modes of the original field theory get a mass proportional to the temperature. In this analysis of the Euclidean solution, one of the spatial directions is interpreted as the time direction in the  $2 + 1$  theory; the spacelike Wilson loop with the long side along this direction is interpreted as giving the quark-antiquark potential in this gauge theory. The supergravity calculation indicated that it would display an area-law behaviour for  $L \gg \beta$ , in agreement with the expectation that this pure Yang-Mills theory is confining. Unlike the timelike loops, the string worldsheet always approaches arbitrarily close to the horizon as we increase the separation in the boundary.

We want to consider spacelike Wilson loops in the Lorentzian charged black hole metric (3.4). Once we introduce charge, the physics of a real Euclidean solution is not the same as that of the Lorentzian solution we want to consider. The analytic continuation from (3.4) to a real Euclidean solution involves continuing both  $t \rightarrow i\tau$  and  $q \rightarrow iq'$ . (This is easy to see if we remember that charge comes from rotation in

the higher-dimensional solution; that is, it is an angular momentum parameter.) The analytic continuation  $q \rightarrow iq'$  in (3.4) drastically changes the physics – for example, the analytically continued metric no longer has an extremal limit. Thus, while the results obtained in the Euclidean metric would have an interpretation in terms of a (2+1)-dimensional gauge theory, as in the uncharged case, the spacelike Wilson loops in the Lorentzian metric (3.4) we will consider, do not have the same interpretation. In our solution,  $q$  is not analytically continued, and there is no connection between the behaviour of the Wilson loops in this metric and any (2+1)-dimensional theory. Our motivation for studying the spacelike Wilson loops is thus simply that they are an interesting probe of the state in the (3+1)-dimensional supersymmetric field theory corresponding to our charged black holes.

The action for the string worldsheet spanning a loop along two spatial dimensions is

$$S = \frac{Y}{2\pi} \int dx \sqrt{\frac{(\partial_x U)^2}{f(U)} + \frac{U^4}{R^4}}, \quad (3.22)$$

where  $Y$  is the length of the long side of the loop. Repeating the method of calculation of  $L$  and  $E$  used in the calculations for the timelike Wilson loops, we find

$$L = \frac{2R^2\alpha}{U_T} \int_1^\infty \frac{y dy}{\sqrt{(y^4 - 1)(y^2 - \alpha^2)(y^4 + \alpha^2 y^2 - \alpha^4 \theta)}}, \quad (3.23)$$

$$E = \frac{U_T}{\pi\alpha} \left[ \int_1^\infty \frac{y^5}{\sqrt{(y^4 - 1)(y^2 - \alpha^2)(y^4 + \alpha^2 y^2 - \alpha^4 \theta)}} - 1 dy - 1 + \alpha \right] \quad (3.24)$$

We would like to find the behaviour of the theory at large  $L$ .  $L$  is increasing as a function of  $\alpha$ , so this requires us to consider the behaviour for  $\alpha \rightarrow 1$  (i.e., we consider strings which hang close to the horizon). For any value of the parameter  $\theta$ , both of the integrals are then dominated by the region  $y = 1$ . So for  $\alpha \rightarrow 1$ , the integrals in  $L$  and  $E$  become the same and we uncover the same area law as in the uncharged case:

$$E = \mathcal{T}L, \quad (3.25)$$

where the tension  $\mathcal{T} = \frac{U_T^2}{2\pi R^2}$ . The  $E$  vs.  $L$  plot for the extremal black hole is given in figure 3.5, to illustrate the similarity with the uncharged case (As before, we plot  $E/U_T$  against  $LU_T/R^2$ ). The scale at which the linear behaviour sets in is once

again determined primarily by the horizon radius  $U_T$ ; the effect of  $\theta$  is just some multiplicative factor of order unity.

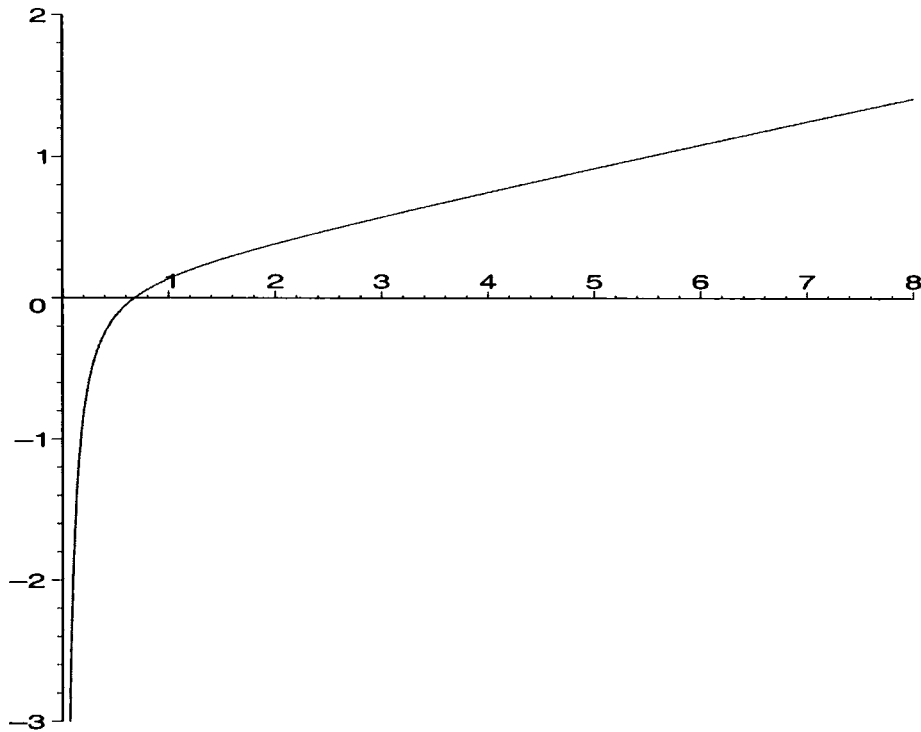


Figure 3.5: Spacelike Wilson loop -  $E$  vs.  $L$  plot for extremal black hole

These probes see the horizon basically as a boundary, providing a lower bound on  $g_{x^i x^i}$ , and hence enforcing an area law behaviour at large distances. Since they don't probe the  $g_{tt}$  part of the metric, it is not surprising that they're not good probes of the causal structure, or particularly sensitive to the temperature.

### 3.3 Supergravity probes

We now consider the propagators for supergravity probes of the charged black hole background. These can be used to calculate the expectation value of a field theory operator that is dual to a source coupled to supergravity fields near the horizon. One might hope that these supergravity sources will better probe the causal structure, as unlike the string worldsheets considered above, these sources can have compact support in the radial direction. However, the fact that the one-point functions are determined by the asymptotic fields will still complicate the story.

### 3.3.1 Retarded propagator

We consider first the retarded propagator, defined as the solution to the wave equation

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu G(x, x')) = \delta(x - x') \quad (3.26)$$

subject to the boundary condition  $G(x, x') = 0$  for  $t < t'$ . Here,  $x'$  is the position of the source, and  $x$  is the position where the measurement is made. It is natural to assume that sources in the bulk will follow geodesics, and in the charged black hole spacetimes, this implies that they will fall into the black hole. To study the effects of such source probes, we must calculate boundary expectation values using the retarded propagator; to learn about the causal structure, we must respect it. Unfortunately, explicit calculations are extremely difficult, except in  $2 + 1$  dimensions, where the black hole is locally AdS. We will remark on the qualitative features of geodesic probes, assuming the solution has similar properties to the solution in pure AdS.

The retarded propagator in pure AdS was previously investigated in [97], where an explicit solution was constructed. In  $\text{AdS}_{2n+1}$ , the retarded propagator is non-zero only in the part of the forward light cone that can be reached by a causal geodesic. In  $\text{AdS}_{2n}$ , it is non-zero only on the forward light cone. In either case, the retarded propagator from a point in the bulk to a point on the boundary is non-zero only where the forward light cone of the point in the bulk intersects the boundary.

The expectation values dual to probes following geodesics in pure AdS were constructed in [97]. There is an isometry which maps any geodesic in pure AdS to any other, which acts on the boundary as a conformal transformation. This can be used to obtain the expectation value dual to a boosted probe. These geodesics start from large radius in the Poincaré coordinates and fall towards the horizon. As the boost is increased, the initial position is moved to larger radius, bringing the probe closer to the boundary. It was found that as the boost is increased, the expectation value became concentrated in a ‘bubble’ around the light cone of the point where the source makes its closest approach to the boundary (see [97] for details). An exact metric describing a lightlike particle falling from a point on the boundary of AdS was found in [98]. In that case, the dual expectation value was a delta-function along

the light cone of the point on the boundary that the particle fell from, in agreement with the general arguments for test particles. Thus, the expectation value spreads to infinite scale size as the particle falls towards the horizon. (Note that for a time-dependent probe, we have to wait for an infinitely long time to see this infinite scale size, which makes this a little inconvenient as a probe of causality.)

Thus, most of the expectation value is along the light cone of the initial point when the source starts near the boundary. By causality, only the part of the source trajectory near the boundary can be contributing to this part of the expectation value; the region where the expectation value is large is outside of the light cone of all but the initial part of the probe's trajectory. Thus, in pure AdS, the expectation value lies mostly in an expanding bubble around the initial point, and is determined by the part of the source trajectory at large radial distances.

This has several lessons for the black hole spacetimes. In black hole spacetimes, the metric at large distances is approximately AdS, so we would expect propagation in this region to be well-approximated by the propagation in pure AdS. Thus, for a source starting at large radial distance in the black hole geometry, the expectation value will have a contribution which produces a delta-function along the light cone. This part will spread out to infinite scale size along the light cone, just as it did in AdS. In an uncharged black hole background, there are thermal fluctuations around this average value, so it was argued in [40] that in practice, we will see the bubble expand until it reaches the thermal scale, where it becomes confused with the thermal fluctuations. In the charged black hole, we can suppress these thermal fluctuations, so we should be able to see the bubble expanding to larger and larger scales, just as in pure AdS.

Although it is tempting to see this as a sign that the horizon is associated with infinite scale size, we should note that this expansion of the bubble is not connected with the horizon in the interior; it is determined by the behaviour of the probe far from the horizon. To see the effects of the non-trivial causal structure, we should consider the contribution to the expectation value from the region near the horizon. Because of the non-trivial causal structure, there can be a contribution from the region near the horizon only at very late times. Furthermore, the discussion in pure

AdS suggests this contribution will be very small.

If we consider the BTZ black hole in  $\text{AdS}_3$ , we can study the propagator from the near-horizon region, since the spacetime is locally pure AdS. The metric is [99]

$$ds^2 = -\frac{(r^2 - r_+^2)}{\ell^2} dt^2 + \frac{\ell^2 dr^2}{(r^2 - r_+^2)} + r^2 d\phi^2, \quad (3.27)$$

where  $\ell$  (the analogue of  $R$  in our higher-dimensional discussion) is the cosmological length scale. If  $\phi$  ranges over all values, this is a peculiar coordinate system for  $\text{AdS}_3$ . If  $\phi$  is periodically identified with period  $2\pi$ , this is a black hole with a horizon at  $r = r_+$ . A lightlike geodesic starting at the boundary point  $t = 0, \phi = 0$  is described by

$$\phi = 0, r = r_+ \coth \frac{tr_+}{\ell^2}. \quad (3.28)$$

The propagator from a point on this trajectory to the boundary at  $r = \infty$  will be non-zero only at the intersection of the light-cone of the point with the boundary (since the space is locally AdS, we can use the propagator obtained in [97]). For a point at  $r = r_+(1 + \epsilon)$ , the light cone meets the boundary at

$$t \approx \frac{\ell^2}{r_+} \ln \left( \frac{2 \cosh(r_+ \phi / \ell)}{\epsilon} \right). \quad (3.29)$$

As expected,  $t$  here goes to infinity as the source approaches the horizon. More importantly, the point contributes at later times as we increase  $\phi$ . We consider the contribution to the expectation value at some fixed late time from the source worldline near the horizon. If we take the source sufficiently close to the horizon, we need only consider the contribution from the source at  $\phi = 0$ , and not that of the images under the identification at  $\phi = \pm 2\pi n$ . That is, we can disregard the compactification of  $\phi$  for this calculation.

The expectation value for a geodesic source in these coordinates with  $\phi \in (-\infty, \infty)$  is [97]

$$\langle \mathcal{O} \rangle = \frac{(ar_+)^{\Delta}}{(a^2 + (1 + a^2) \sinh^2[r_+(t + \phi)/2])^{\Delta/2} (a^2 + (1 + a^2) \sinh^2[r_+(t - \phi)/2])^{\Delta/2}} \quad (3.30)$$

for an operator of conformal dimension  $\Delta$ , where  $a$  is the boost parameter (the source is lightlike for  $a \rightarrow 0$ ). This expectation value is approximately independent of  $\phi$  for  $\phi \in (-\pi, \pi)$  at large  $t$ . Thus, the contribution to the expectation value from

the region near the black hole horizon is  $\phi$ -independent, which is as close as we can come to infinite scale in the present context of a compact spatial direction on the boundary. We reiterate that this is just the contribution from the region near the horizon; the main contribution to the expectation value is at  $t = \pm\phi$  as discussed in [43], and comes from the part of the worldline near the boundary.

### 3.3.2 Static propagator

We will next turn to static sources. The advantage of considering static sources is that since the source is always near the horizon, the expectation value will be affected by the near-horizon structure. However, the static propagator is independent of  $g_{tt}$ , so this is not guaranteed to produce a result which reflects the causal structure near the black hole, and in fact the answer we obtain does not have a straightforward relation to statements about the causality. However, it does appear to encode information about the near-horizon region in a non-trivial way.

The calculation of the appropriate propagator in the uncharged black hole background was discussed in [97]. The static propagator for a massless scalar field, such as the dilaton, is defined as a solution to the equation

$$\partial_i (\sqrt{g} g^{ij} \partial_j G(x, x')) = \sqrt{g_{tt}} \delta(x - x'). \quad (3.31)$$

Here  $x'$  is the position of the source, while  $x$  is the point at which the field is measured. We take the metric (3.4) and rescale the coordinates by  $U \rightarrow U_T u$ ,  $t \rightarrow tR^2/U_T$  and  $x \rightarrow xR^2/U_T$ . The equation for the static propagator is then

$$\begin{aligned} (u^7 - (1 + \theta)u^3 + \theta u) \partial_u^2 \tilde{G} + (5u^6 - (1 + \theta)u^2 - \theta) \partial_u \tilde{G} - u^3 k^2 \tilde{G} \\ = \frac{\sqrt{u^6 - (1 + \theta)u^2 + \theta}}{R^7} \delta(u - u'). \end{aligned} \quad (3.32)$$

Here we have Fourier transformed the propagator equation with respect to  $x_i$ , so

$$\tilde{G}(u, u', k_i) = \int d^3k e^{i\vec{k}\cdot\vec{x}} G(u, u', x_i). \quad (3.33)$$

For  $u < u'$  and  $u > u'$  the Green's function is given by the solutions of the homogeneous equation

$$(u^7 - (1 + \theta)u^3 + \theta u) y''(u) + (5u^6 - (1 + \theta)u^2 - \theta) y'(u) - u^3 k^2 y(u) = 0. \quad (3.34)$$

We must now consider the indicial equations which arise for solutions near the horizon  $u = 1$  and the boundary  $u = \infty$ . For  $u > u'$  this is  $\sigma^2 + 4\sigma = 0$ , and the presence of electric charge makes no difference to the boundary behaviour of the Green's function. In order for the solution to vanish at infinity, as required, we therefore have

$$\tilde{G}(u > u', k) = Ay_1(u, k) \text{ where } y_1(u, k) \sim \frac{1}{u^4}, u \rightarrow \infty. \quad (3.35)$$

For the behaviour near  $u = 1$ , we have the indicial equation  $(4 - 2\theta)(\sigma(\sigma - 1) + \sigma) = 0$ . So long as  $\theta \neq 2$  (i.e. for a nonextremal black hole) this is also the same as in the uncharged case, and regularity at the horizon requires

$$\tilde{G}(u < u', k) = By_2(u, k) \text{ where } y_2(u, k) \sim 1, u \rightarrow 1. \quad (3.36)$$

We will return to the extremal case later in this section. The constants  $A$  and  $B$  are calculated by continuity in the Green's function at  $u = u'$  and the correct discontinuity in its derivative, giving

$$A = \frac{y_2}{W(y_1, y_2)} \frac{1}{R^7 u' \sqrt{u'^6 - (1 + \theta)u'^2 + \theta}}, \quad (3.37)$$

$$B = \frac{y_1}{W(y_1, y_2)} \frac{1}{R^7 u' \sqrt{u'^6 - (1 + \theta)u'^2 + \theta}}. \quad (3.38)$$

From (3.34) we can establish that the Wronskian,  $W(y_1, y_2) \equiv y_1 y_2' - y_2 y_1'$ , satisfies the differential equation

$$\frac{dW}{W} = \frac{d(u^5 - (1 + \theta)u + \frac{\theta}{u})}{u^5 - (1 + \theta)u + \frac{\theta}{u}}. \quad (3.39)$$

It is therefore given by

$$W(y_1, y_2) = \frac{w(k)}{u^5 - (1 + \theta)u + \frac{\theta}{u}}, \quad (3.40)$$

where the constant of integration,  $w(k)$ , incorporates the  $k$ -dependence of (3.34).

As in [97], we observe that a charge in the boundary theory is dual to a string hanging from the boundary to the horizon – the minimum action configuration is that of a string extended in the  $U$  and  $t$  directions only. Such a string sources the bulk dilaton field,  $\phi$ , through the coupling

$$S_{\text{int}} = \frac{1}{4\pi} \int dU dt \sqrt{g_{tt} g_{UU}} \phi = \frac{1}{4\pi} \int du dt \sqrt{g_{tt} g_{uu}} \frac{u^2}{\sqrt{u^6 - (1 + \theta)u^2 + \theta}} \phi, \quad (3.41)$$

where we have expanded the string worldsheet action to linear order in the dilaton field.

We compute the dilaton field by noting that the position-space propagator may be written as the Fourier transform of our momentum-space propagator, and the integrals in this Fourier transform can be expressed as a sum over the poles of  $\tilde{G}(u, u', k_i)$ .

$$\phi(u, \vec{x}) = \frac{1}{4\pi} \int du' \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \frac{u'^2 R}{\sqrt{u'^6 - (1+\theta)u'^2 + \theta}} \tilde{G}(u, u', \vec{k}). \quad (3.42)$$

The boundary behaviour of the dilaton field is then found to be

$$\phi(U, x) \stackrel{U \rightarrow \infty}{\approx} \frac{U_T^4}{8\pi R^6 U^4} \sum_{n=1}^{\infty} \frac{e^{-m_n r}}{m_n r w'(m_n)} \int du' y_2(u', m_n) du'. \quad (3.43)$$

Here,  $m_n$  are the zeros of  $w(k)$ , which give poles in the propagator. These correspond to the special values of  $k$  for which we can construct solutions regular at both the horizon and infinity.  $y_2(u', m_n)$  are the corresponding solutions of equation (3.34). Thus, the propagator has an exponential suppression for  $r > 1/m_n$ , and these poles hence provide a maximum length scale for the expectation value dual to sources anywhere in the bulk. We proceed to determine this maximum scale by finding the poles  $m_n$ .

In the case of the uncharged black hole, this problem of determining the zeros of the Wronskian physically corresponded to finding the glueball mass spectrum for the 2 + 1 dimensional pure Yang-Mills theory, and was first described in [95], and both numerical methods and analytic approximations have since been used to calculate the spectrum [97, 100, 101]. As was emphasised in the discussion of spacelike Wilson loops, the Lorentzian metric (3.4) is not directly related to the 2 + 1 field theory obtained from a Euclidean rotating brane metric. Thus, the zeros,  $m_n$ , found here will not be simply related to the glueball mass spectra obtained from studies of the rotating brane metrics in [85, 86].

We follow the approach of [101] in calculating  $m_n$ , as the change of coordinates employed there makes the interpretation in the extremal limit clear. Returning to equation (3.34), with the change of variables  $x = u^2$  ( $k = i\kappa$ ), we find

$$\partial_x [(x^3 - (1+\theta)x + \theta)\partial_x y] + \kappa^2 y = 0. \quad (3.44)$$

In order to use WKB methods on a second order linear differential equation, it is necessary to redefine the dependent variable so that it satisfies a differential equation with no first derivative term. The WKB analysis is greatly simplified with the change of variables  $x = 1 + e^z$ . Defining

$$\psi = \sqrt{\frac{x^3 - (1 + \theta)x + \theta}{x - 1}} y = \sqrt{f(z)} y, \quad (3.45)$$

where

$$f(z) = e^{2z} + 3e^z + (2 - \theta), \quad (3.46)$$

we obtain a differential equation which is completely analogous to the uncharged case,

$$\psi'' + V(z)\psi = 0, \quad (3.47)$$

where

$$V(z) = \frac{\kappa^2}{f} e^z - \frac{f''}{2f} + \left(\frac{f'}{2f}\right)^2. \quad (3.48)$$

The only change in this equation is that  $f(z)$  is altered by the  $\theta$  term. To perform the WKB analysis we need to find the points where the potential in this equation is zero, as these are the turning points of the WKB approximation. In the limits of large  $|z|$ , for  $\theta \neq 2$ , we have

$$V(z) \approx \left[ \frac{\kappa^2}{2 - \theta} - \frac{3}{2(2 - \theta)} \right] e^z \text{ for } z \ll 0, \quad (3.49)$$

$$V(z) \approx \kappa^2 e^{-z} - 1 \text{ for } z \gg 0. \quad (3.50)$$

For  $\kappa$  sufficiently large there are thus turning points at  $z = -\infty$  and  $z = z_0 \approx 2 \ln(\kappa)$ . The WKB approximation therefore gives

$$\left(n + \frac{1}{2}\right) \pi = \int_{-\infty}^{z_0} dz \sqrt{V(z)}. \quad (3.51)$$

To leading order in  $\kappa$ , we can approximate the integral

$$\left(n + \frac{1}{2}\right) \pi = \int_{-\infty}^{\infty} \kappa \sqrt{\frac{e^z}{f(z)}} dz = \kappa \int_1^{\infty} \frac{dx}{\sqrt{x^3 - (1 + \theta)x + \theta}} \equiv \kappa \alpha, \quad (3.52)$$

where the last equality defines  $\alpha$ . The zeros,  $m_n$ , of the Wronskian are thus approximately given by

$$m_n = \frac{\pi}{\alpha} \left(n + \frac{1}{2}\right), \quad (3.53)$$

where  $n$  is a positive integer<sup>3</sup>.

In the uncharged case  $\alpha$  can be evaluated exactly. For the charged case, we evaluate  $\alpha$  numerically, and see that as the charge of the black hole is increased,  $\alpha$  increases and thus each  $m_n$  decreases. In figure 3.6, we plot the value of the lowest zero  $m_1$  as a function of the parameter  $\theta$  determining black hole charge. As  $\theta \rightarrow 2$ ,  $\alpha$  diverges like  $\ln(T) \propto \ln(2 - \theta)$ , where  $T$  is the black hole temperature. Since we are obtaining a divergent answer, we should consider the validity of the approximation more carefully.

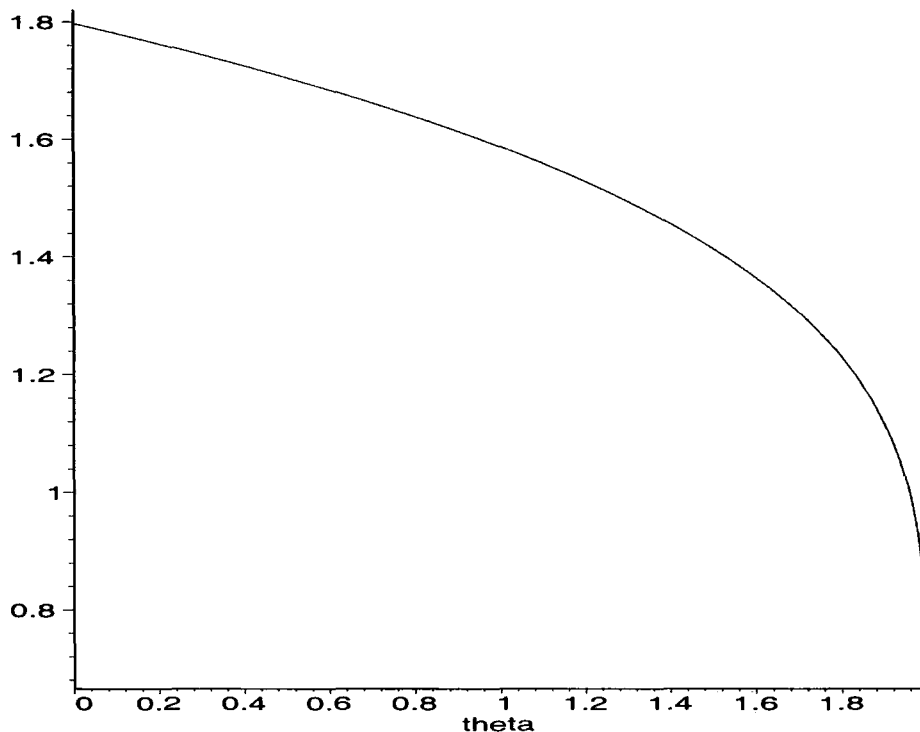


Figure 3.6: The lowest zero of the Wronskian,  $m_1$  vs.  $\theta$

The divergence found in the WKB approximation in the extremal case can be explained by considering the potential. For non-extremal black holes the behaviour of  $V(z)$  for  $z \ll 0$  was as given in (3.49). This is the case for any value of  $\theta$  other than 2, but for  $\theta = 2$  it is the next term in  $f(z)$  which contributes to  $V(z)$  for  $z \ll 0$

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<sup>3</sup>It is argued in [97] that the extra zero for  $n = 0$  does not contribute and this claim is substantiated by comparison to numerical results for calculation of the zeros in the uncharged black hole

and we find that the leading behaviour in  $V(z)$  is given by

$$V(z) \approx \frac{\kappa^2}{3} - \frac{1}{4} \text{ for } z \ll 0. \quad (3.54)$$

We see that the potential now no longer decays exponentially for  $z \ll 0$ : instead it becomes constant, and the potential never reaches zero in this region. Therefore there is no second turning point of the equation and the bound state problem has no solutions. The shape of the potential is illustrated in figure 3.7. The lower curve is that for an uncharged black hole, the middle curve illustrates the shape for a generic charged black hole, whilst the upper curve illustrates the special behaviour in the extremal case.

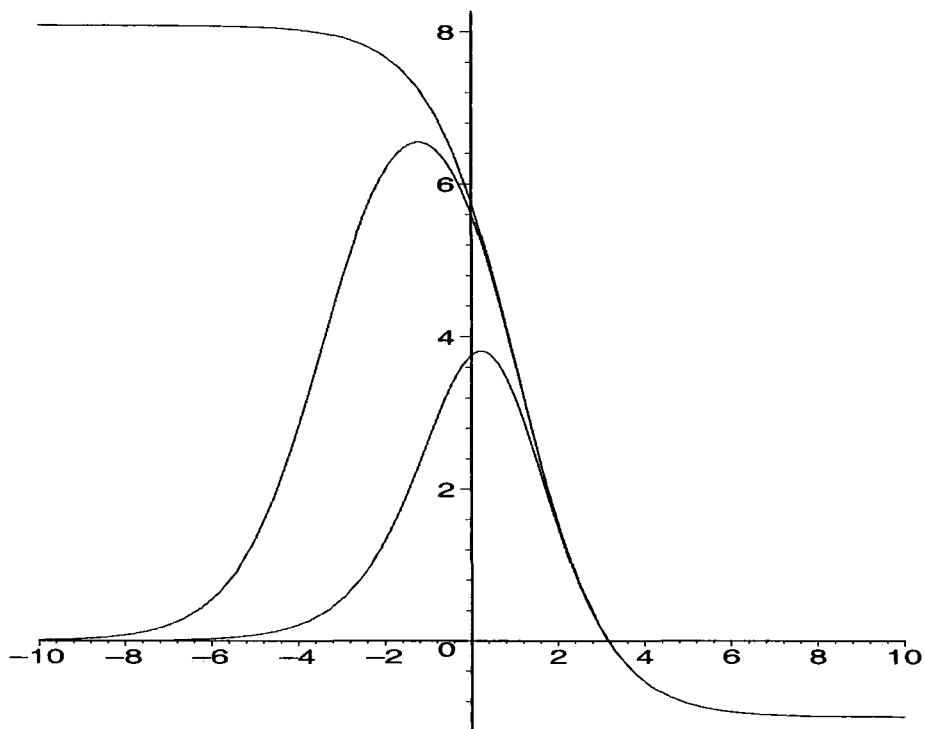


Figure 3.7: Plot of the potential,  $V(z)$  vs.  $z$

The different nature of the problem in the extremal case was discovered earlier, when we found for equation (3.34), that the indicial equation for solutions near the black hole horizon is given by  $(4 - 2\theta)(\sigma(\sigma - 1) + \sigma) = 0$ . This is true for  $\theta \neq 2$ , but for  $\theta = 2$  the dominant terms in the solution near the horizon are those of a lower exponent and they lead to the indicial equation

$$12\sigma(\sigma - 1) + 24\sigma - k^2 = 0. \quad (3.55)$$

For solutions to be well behaved near the horizon this requires  $k^2 \geq 0$ . This is problematic since the zeros are given by  $m^2 = -k^2$ . In fact, in the extremal case, (3.34) can be solved exactly, enabling us to see how it differs from the non-extremal case. Making the substitution  $x = u^2$  reduces the homogeneous equation for  $\theta = 2$  to

$$(x-1)^2(x+2)\partial_x^2 y + 3(x-1)(x+1)\partial_x y - \frac{k^2}{4}y = 0. \quad (3.56)$$

Whereas in both the uncharged case and the nonextremal charged case our homogeneous differential equation was linear second order with four regular singularities, this equation only has three regular singularities, at  $x = -2$ ,  $x = 1$  and  $x = \infty$ . We recognise this as the hypergeometric equation and reduce it to the standard form by the transformation  $z = 3/(1-x)$ . Then

$$z(1-z)\partial_z^2 y - \partial_z y + \frac{k^2}{12}y = 0. \quad (3.57)$$

We will look for solutions of (3.57) satisfying our boundary conditions without restricting the sign of  $k^2$ , to see if any such solutions exist. We need the solution to be normalisable, so  $y \approx u^{-4}$  as  $u \rightarrow \infty$ ; after coordinate transformations this condition becomes  $y \approx z^2$  as  $z \rightarrow 0$ . We also require the solution to be well-behaved at  $u = 1$ , i.e., at  $z = \infty$ . The hypergeometric equation has one solution with the correct behaviour at  $z = 0$ ,

$$y(z) = z^2 F\left(\frac{3}{2} + \lambda, \frac{3}{2} - \lambda, 3; z\right), \quad \lambda = \frac{1}{6}\sqrt{9 + 3k^2} \geq 0. \quad (3.58)$$

To examine the behaviour near  $z = \infty$ , we use the asymptotic expansion in terms of hypergeometric functions in  $1/z$  to give

$$\begin{aligned} y(z) &= \frac{\Gamma(3)\Gamma(-2\lambda)}{\Gamma(3/2 - \lambda)^2} (-1)^{\frac{3}{2} + \lambda} z^{\frac{1}{2} - \lambda} F\left(\frac{3}{2} + \lambda, -\frac{1}{2} + \lambda, 1 + 2\lambda; \frac{1}{z}\right) \\ &+ \frac{\Gamma(3)\Gamma(2\lambda)}{\Gamma(3/2 + \lambda)^2} (-1)^{\frac{3}{2} - \lambda} z^{\frac{1}{2} + \lambda} F\left(\frac{3}{2} - \lambda, -\frac{1}{2} - \lambda, 1 - 2\lambda; \frac{1}{z}\right). \end{aligned} \quad (3.59)$$

Since the hypergeometric function as a function of  $1/z$  is asymptotic to 1 at  $z = \infty$ , for our function to be well-behaved at  $z = \infty$  we require that  $z$  does not appear outside the hypergeometric function with a positive exponent. In the second term of this expression for  $y(z)$  this cannot be achieved, so the gamma function in the denominator must diverge to set this term to zero. However, this would only happen

if  $3/2 + \lambda = -n$  for  $n$  a non-negative integer, and  $\lambda$  is positive. Thus, in the extremal case there are no solutions of the time-independent wave equation satisfying the boundary conditions at both the horizon and the boundary. The breakdown in the WKB analysis near extremality is therefore physical.

In these static propagator calculations, we have found that there is a finite screening length associated with most of the black hole spacetimes. From the calculations in the uncharged black hole, where the screening length is the thermal scale, one might have suspected that this is associated with the thermal fluctuations, which are concealing a divergence in the true behaviour. However, as we increase the charge, the screening length grows only logarithmically in the temperature, and soon falls below the thermal scale. There is thus really some limit on the characteristic scale for probes near the horizon, and we see no sign of a divergent scale size associated with the horizon in this calculation. It also would be interesting to understand the origin of this behaviour in the field theory: since the scale is not simply fixed by the horizon radius  $U_T$ , there may be some interesting physics here. From this point of view, it should be stressed that  $1/m_n$  only provides an upper bound on the possible scale size; there may be power-law suppression at a smaller scale that this calculation is not sensitive to.

### 3.4 Conclusions

We only found signs of the expected divergence in the scale size of dual expectation values for probes near the horizon in the discussion of time-dependent probes. The failure to find such a relationship for Wilson loops is perhaps understandable, since the extended nature of the worldsheet implies that the part of the worldsheet that changes as we vary the asymptotic separation is probing a range of radii near the horizon, and not a specific value. Thus, the Wilson loops do depend non-trivially on the structure of the metric near the horizon, but don't really see the horizon as a horizon. This fact has also been observed recently in [102], where Wilson loops in spacetimes with naked singularities were studied.

It is more surprising that static supergravity probes produce dual expectation

values with a finite scale size. From the spacetime point of view, this statement means that a point charge close to the horizon produces an asymptotic field which still depends non-trivially on the transverse coordinates  $x_i$ . This is quite different from the case of Schwarzschild black holes in flat space, where the field of a charged particle close to the horizon becomes completely spherically symmetric (see [103] and references therein). Note that this is not just the usual difference between the asymptotic behaviours of flat space and AdS: in the Schwarzschild case, the field measured at some finite radius is becoming spherically symmetric as the source approaches the horizon. One might expect that we are violating a possible *no scalar hair* theorem for these black hole solutions. However, a closer inspection of the solution for our scalar propagator (equations (3.35)–(3.40)) reveals that this is not the case. As a scalar point charge is lowered towards the horizon ( $u' \rightarrow 1$ ) then the field outside of it is determined by (3.35). The constant,  $A$ , in this equation tends to zero in this limit – i.e. although the asymptotic field continues to depend non-trivially on the transverse coordinates  $x_i$ , there is an overall factor which reduces the magnitude of the field to zero when the test particle producing the field hits the horizon.

The static propagator also exhibits a mysterious logarithmic dependence on the temperature for small temperatures. It should be noted that this only provides an upper limit for the behaviour of the scale size of excitations for static probes near the horizon, and the actual scale could be constant. Nevertheless, it would be interesting to try to understand this behaviour from the field theory point of view. While no analogue of this behaviour was seen in the glueball mass calculations in rotating backgrounds [85,86], this should not cause concern. As previously emphasised, these calculations address the physically different Euclidean solution obtained by  $t \rightarrow i\tau$ ,  $q \rightarrow iq'$ . In this Euclidean solution, it is not possible to take the temperature to zero; in fact, the minimum value of the temperature is achieved when  $q' = 0$ .

One interesting example of a ‘static’ source that is not covered by the foregoing analysis is a D-instanton. To consider a D-instanton, we need to pass to the Euclidean solution, so we cannot really think of it as a probe of the causal structure. However, the expectation value dual to a D-instanton is sensitive to the presence of

the horizon in a dramatic way. For non-extremal black holes, the horizon in the Euclidean solution is simply a point where the proper length of the periodic  $\tau$  direction goes to zero. If we place a D-instanton at this point, the translational symmetry in  $\tau$  is preserved, so the dual expectation value must be  $\tau$ -independent. That is, for D-instantons near the horizon, the scale of the dual field theory instanton goes to infinity in the  $\tau$  direction (and from the above comments on static propagators, in the  $\tau$  direction only). This may be a useful test for a horizon in the Euclidean solution, but it does not help us to understand the causality of the Lorentzian solution.

The key to a more satisfactory representation of the bulk causal structure may be to develop a relation between the bulk theory and the boundary which does not require us to propagate effects out to the boundary in spacetime (such issues are studied in [104–106]). This is difficult to achieve with the present correspondence, as the relation between spacetime and field theory is phrased in terms of boundary conditions on the gravity side. It is worth stressing that this problem is distinct from the problem of studying local physics on scales smaller than the AdS scale; the black holes here can be as large as one wants. It is clear that resolving this apparently simple question requires much further study in the field.

## Chapter 4

# Braneworld Cosmology From Bulk Black Holes

In this chapter we study the holographic description of braneworld cosmology. This is undertaken by embedding a braneworld in a bulk AdS black hole spacetime – all of the details of the cosmology can be captured by the parameters defining the background spacetime. What will interest us here is a brane embedded in a background of bulk Reissner-Nordström AdS black holes. We choose to study the RNAdS background as it exhibits the full generality of many of the results discussed in earlier studies of braneworld holography.

In braneworld holography, we normally think of the bulk gravity theory as being dual to a CFT on the brane itself. However, the CFT has an ultraviolet cut off, so it is actually a broken CFT [54, 107]. In most studies of dual theories on the brane, the precise nature of the dual theory is unknown. We regard it as an abstract field theory, some of whose properties we can derive. We should also note that early studies of branes [52] had fine tuned tensions that ensured the cosmological constant on the brane vanished (critical branes). If we avoid this fine tuning we are able to induce a non-zero cosmological constant on the brane [108] (non-critical branes). In particular, recent observations indicating that our universe has a small positive cosmological constant [57, 58] suggest it is important to consider de Sitter braneworlds. These inflationary braneworlds are naturally induced by quantum effects of a field theory on the brane [109–111].

When the bulk spacetime has a non-vanishing Weyl tensor<sup>1</sup>, Shiromizu *et al* [59] pointed out that the electric part of this tensor appears in the Einstein's equations on the brane. This contribution is thought of as being holographic in that it can also be interpreted as coming from a dual theory on the brane. By considering a braneworld observer, we can derive properties of the dual theory and get a better understanding of the Weyl tensor contribution. However, in all the previous literature, it has been assumed that the brane is at large radial distance from the centre of AdS space, i.e. near the boundary. This allows us to assume that the energy density of the braneworld universe is small, and the true holographic description of an  $(n - 1)$ -dimensional braneworld in an  $n$ -dimensional bulk can be understood. However, these results are all approximations in the sense that for a general brane evolution it is not necessary for the brane to remain close to the boundary. In this chapter, it is our objective to undertake a new study of the braneworld in which we calculate the energy of the braneworld field theory exactly, regardless of the brane's position in the bulk. We also allow the brane tension to be arbitrary, thereby including both critical and non-critical branes.

We find that this exact approach introduces quadratic terms in the Friedmann Robertson Walker (FRW) equations of the braneworld universe. Remarkably these turn out to have exactly the same form as the quadratic terms found in the cosmology of branes embedded in a pure AdS background, when extra matter is placed on the brane (see section 2.5.2). This suggests a new, enlarged duality that includes even the nonconventional quadratic terms in the FRW equation. We may regard the holographic dualities seen earlier for braneworlds as being an approximation of more general "unconventional" dualities. Furthermore, unlike in section 2.5.2, there is no ambiguity in splitting the braneworld energy momentum tensor into tension and additional matter. This is because we do not have any additional matter, only tension and a bulk Weyl tensor contribution that we have interpreted on the brane.

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<sup>1</sup>This can occur naturally for a hot critical braneworld due to the emission of radiation into the bulk [64].

## 4.1 Equations of motion

We will consider an  $(n - 1)$ -dimensional brane of tension  $\sigma$  sandwiched in between two  $n$ -dimensional black holes. In order to show the generality of our work we are allowing the black holes to be charged, although our brane will be uncharged. Since this means that lines of flux must not converge to or diverge from the brane, we must have black holes of equal but opposite charge. In this case, the flux lines will pass through the brane since one black hole will act as a source for the charge whilst the other acts as a sink. It should be noted that although we do not have  $\mathbb{Z}_2$  symmetry across the brane for the electromagnetic field, the geometry is  $\mathbb{Z}_2$  symmetric.

We denote our two spacetimes by  $\mathcal{M}^+$  and  $\mathcal{M}^-$  for the positively and negatively charged black holes respectively. Their boundaries,  $\partial\mathcal{M}^+$  and  $\partial\mathcal{M}^-$ , both coincide with the brane. Our braneworld scenario is then described by the following action:

$$S = \frac{1}{16\pi G_n} \int_{\mathcal{M}^+ + \mathcal{M}^-} d^n x \sqrt{g} (R - 2\Lambda_n - F^2) + \frac{1}{8\pi G_n} \int_{\partial\mathcal{M}^+ + \partial\mathcal{M}^-} d^{n-1} x \sqrt{h} K + \frac{1}{4\pi G_n} \int_{\partial\mathcal{M}^+ + \partial\mathcal{M}^-} d^{n-1} x \sqrt{h} F^{ab} n_a A_b + \sigma \int_{brane} d^{n-1} x \sqrt{h}, \quad (4.1)$$

where  $g_{ab}$  is the bulk metric and  $h_{ab}$  is the induced metric on the brane.  $K$  is the trace of the extrinsic curvature of the brane, and  $n_a$  is the unit normal to the brane pointing from  $\mathcal{M}^+$  to  $\mathcal{M}^-$ . Notice the presence of the Hawking-Ross term in the action (4.1) which is necessary for black holes with a fixed charge [112].

The bulk equations of motion which result from this action are given in section 2.4.2. There, we saw that they admitted the following 2 parameter family of electrically charged black hole solutions for the bulk metric

$$ds_n^2 = -h(Z)dt^2 + \frac{dZ^2}{h(Z)} + Z^2 d\Omega_{n-2}, \quad (4.2)$$

in which

$$h(Z) = k_n^2 Z^2 + 1 - \frac{c}{Z^{n-3}} + \frac{q^2}{Z^{2n-6}}, \quad (4.3)$$

and the electromagnetic field strength

$$F = dA \quad \text{where} \quad A = \left( -\frac{1}{\kappa} \frac{q}{Z^{n-3}} + \Phi \right) dt \quad \text{and} \quad \kappa = \sqrt{\frac{2(n-3)}{n-2}}. \quad (4.4)$$

Note that we have relabelled the radial coordinate as  $Z$ , and the AdS radius is now  $\frac{1}{k_n}$  so that  $k_n$  is related to the bulk cosmological constant by  $\Lambda_n = -\frac{1}{2}(n-1)(n-2)k_n^2$ ,

whereas  $c$  and  $q$  are constants of integration. If  $q$  is set to zero in this solution, we regain the AdS-Schwarzschild solution in which  $c$  introduces a black hole mass. The presence of  $q$  introduces black hole charge for which  $\Phi$  is an electrostatic potential difference. Furthermore,  $h(Z)$  has two zeros, the larger of which,  $Z_H$ , represents the event horizon of the black hole.

Since charge is a localised quantity, it can be evaluated from a surface integral on any closed shell wrapping the black hole (Gauss' Law). In  $\mathcal{M}^\pm$  the total charge is

$$Q = \pm \frac{(n-2)\kappa\Omega_{n-2}}{8\pi G_n} q, \quad (4.5)$$

where  $\Omega_{n-2}$  is the volume of a unit  $(n-2)$  sphere. As in section 2.4.2, the ADM mass [50] of each black hole is given by

$$M = \frac{(n-2)\Omega_{n-2}c}{16\pi G_n}. \quad (4.6)$$

Let us now consider the dynamics of our brane embedded in this background of charged black holes. This argument was outlined for a brane in pure AdS space in section 2.5.2. We use  $\tau$  to parametrise the brane so that it is given by the section  $(\mathbf{x}^\mu, t(\tau), Z(\tau))$  of the bulk metric. The Israel equations for the jump in extrinsic curvature across the brane give the brane's equations of motion. One might suspect that the presence of the Hawking-Ross term in the action will affect the form of these equations. However, since the charge on the black holes is fixed, the flux across the brane does not vary and the Israel equations take their usual form which we write here as

$$2K_{ab} - 2Kh_{ab} = -8\pi G_n \sigma h_{ab}, \quad (4.7)$$

where  $K_{ab}$  and  $n_a$  are as given in 2.5.2. This equation is the same as (2.36), but here the energy-momentum tensor of the brane only contains the brane tension – our braneworld does not contain the additional matter which we assumed in section 2.5.2. Importantly, we again have the condition defined by the fact that  $n_a$  is a unit normal (2.34), but now  $h(Z)$  is the potential in our black hole metric (4.3).

The resulting equations of motion for the brane are:

$$\dot{Z}^2 = \sigma_n^2 Z^2 - h(Z) = aZ^2 - 1 + \frac{c}{Z^{n-3}} - \frac{q^2}{Z^{2n-6}}, \quad (4.8a)$$

$$\ddot{Z} = \sigma_n^2 Z - \frac{1}{2}h'(Z) = aZ - \left(\frac{n-3}{2}\right)\frac{c}{Z^{n-2}} + (n-3)\frac{q^2}{Z^{2n-5}}, \quad (4.8b)$$

$$\dot{t} = \frac{\sigma_n Z}{h(Z)}, \quad (4.8c)$$

where  $a = \sigma_n^2 - k_n^2$  and  $\sigma_n = \frac{4\pi G_n \sigma}{n-2}$ . This analysis has been presented in more detail, at least for uncharged black holes, in [113, 114].

We shall now examine the cosmology of our braneworld in this background. The induced metric is an FRW universe given by the following

$$ds_{n-1}^2 = -d\tau^2 + Z(\tau)^2 d\Omega_{n-2}. \quad (4.9)$$

We see that we may interpret  $Z(\tau)$  as corresponding to the scale factor in an expanding/contracting universe and that equations (4.8a) and (4.8b) should be regarded as giving rise to the Friedmann equations of our braneworld. Indeed, if we define the Hubble parameter as  $H = \frac{\dot{Z}}{Z}$ , we arrive at the following equations for the cosmological evolution of the brane

$$H^2 = a - \frac{1}{Z^2} + \frac{c}{Z^{n-1}} - \frac{q^2}{Z^{2n-4}}, \quad (4.10a)$$

$$\dot{H} = \frac{1}{Z^2} - \left(\frac{n-1}{2}\right)\frac{c}{Z^{n-1}} + (n-2)\frac{q^2}{Z^{2n-4}}. \quad (4.10b)$$

Let us examine these equations in more detail. Equation (4.10a) contains the cosmological constant term  $a$ . For  $a = 0$  we have a critical wall with vanishing cosmological constant. For  $a > 0/a < 0$  we have super/subcritical walls that correspond to asymptotically de Sitter/anti-de Sitter spacetimes. Note that for subcritical and critical walls,  $Z$  has a maximum and minimum value. For supercritical walls, we have two possibilities: either  $Z$  is bounded above and below or it is only bounded below and may stretch out to infinity.

However, our real interest in equations (4.10a) and (4.10b), lies in understanding the  $c$  and  $q^2$  terms. As discussed in [66, 115], we can interpret the contribution of the  $c$  term to the cosmological evolution in two different ways. On the one hand, the evolution is driven, in part, by the masses of the bulk black holes, as is

evident in equations (4.10a) and (4.10b). On the other hand, we can ignore the bulk and describe the evolution as being driven by the energy density and pressure of radiation in a dual field theory living on the brane. At least for critical walls, it was showed by Biswas and Mukherji [68] that we could interpret the  $q^2$  terms as corresponding to stiff matter in the dual theory. These discussions were motivated by “holography” in the sense that the duality related an  $n$ -dimensional bulk theory to an  $(n - 1)$ -dimensional field theory on the brane. In the spirit of AdS/CFT, the brane was close to the AdS boundary. In the next section we will not make that assumption. We will nevertheless discover an interesting dual picture, although it will be “unconventional”, at least in terms of cosmology!

## 4.2 Energy on the brane

Consider an observer living on the brane. He measures time using the braneworld coordinate,  $\tau$ , rather than the bulk time coordinate,  $t$ . This of course affects his measurement of the energy density. In previous literature, the energy on the brane has been obtained by scaling the energy of the bulk accordingly. For example, in [115], a detailed calculation using Euclidean gravity techniques yielded a bulk energy,  $E_{bulk} \approx 2M \left( \frac{k_n^2}{\sigma_n^2} \right)$ , where  $M$  is the ADM mass of the black holes. This was then scaled using  $\dot{t}$  to give the energy on the brane. In the limit of large  $Z(\tau)$ ,  $\dot{t} \approx \frac{\sigma_n}{k_n^2 Z}$  so that a braneworld observer measures the energy to be  $E \approx \frac{2M}{\sigma_n Z}$ . This analysis enabled us to have the dual picture described in the previous section, even for non-critical branes.

While the results obtained from these methods are interesting, they are approximations. Indeed, the Euclidean analysis of [115] is only valid in certain limits, imposed because we require time translational symmetry to Wick rotate to Euclidean signature. We should note that the deviation from the expected bulk energy,  $2M$ , is not so startling as we might originally think. The limits of the analysis impose that as the brane approaches the AdS boundary,  $\sigma_n \rightarrow k_n$ . Therefore, if the brane actually strikes the AdS boundary, it has to be critical and we recover what we might have naïvely expected: the bulk energy is given by the sum of the ADM masses.

In this chapter, we will bypass all of these approximations and limitations by ignoring the bulk energy and calculating the energy on the brane directly. We will use the techniques of [116] to evaluate the gravitational energy using  $\tau$  as our chosen time coordinate. Happily we will not need to Wick rotate to Euclidean signature, enabling us to exactly calculate the energy density on the brane, even at smaller values of  $Z(\tau)$ .

We begin by focusing on the contribution from the positively charged black hole spacetime,  $\mathcal{M}^+$  and its boundary,  $\partial\mathcal{M}^+$ . This boundary of course coincides with the brane. Consider the timelike vector field defined on  $\partial\mathcal{M}^+$

$$\tau^a = (\mathbf{0}, \dot{t}, \dot{Z}). \quad (4.11)$$

This maps the boundary/brane onto itself, and satisfies  $\tau^a \nabla_a \tau = 1$ . In principle we can extend the definition of  $\tau^a$  into the bulk, stating only that it approaches the form given by equation (4.11) as it nears the brane. We now introduce a family of spacelike surfaces,  $\Sigma_\tau$ , labelled by  $\tau$  that are always normal to  $\tau^a$ . This family provide a slicing of the spacetime,  $\mathcal{M}^+$  and each slice meets the brane orthogonally. As usual we decompose  $\tau^a$  into the lapse function and shift vector,  $\tau^a = Nr^a + N^a$ , where  $r^a$  is the unit normal to  $\Sigma_\tau$ . However, when we lie on the brane,  $\tau^a$  is the unit normal to  $\Sigma_\tau$ , because there we have the condition (2.34). Therefore, on  $\partial\mathcal{M}^+$ , the lapse function,  $N = 1$  and the shift vector,  $N^a = 0$ . Before we consider whether or not we need to subtract off a background energy, let us first state that the relevant part of the action at this stage of our analysis is the following:

$$I^+ = \frac{1}{16\pi G_n} \int_{\mathcal{M}^+} R - 2\Lambda_n - F^2 + \frac{1}{8\pi G_n} \int_{\partial\mathcal{M}^+} K + \frac{1}{4\pi G_n} \int_{\partial\mathcal{M}^+} F_{ab} n^a A^b. \quad (4.12)$$

As stated earlier, we do not include any contribution from  $\mathcal{M}^-$  or  $\partial\mathcal{M}^-$ , nor do we include the term involving the brane tension. This is because we want to calculate the gravitational Hamiltonian, without the extra contribution of a source. The brane tension has already been included in the analysis as a cosmological constant term, and it would be wrong to double count.

Given the slicing  $\Sigma_\tau$ , the Hamiltonian that we derive from  $I^+$  is given by

$$H^+ = \frac{1}{8\pi G_n} \int_{\Sigma_\tau} N\mathcal{H} + N^a\mathcal{H}_a - 2NA_\tau\nabla_a E^a - \frac{1}{8\pi G_n} \int_{S_\tau} N\Theta + N^a p_{ab}n^b - 2NA_\tau n_a E^a + 2NF^{ab}n_a A_b, \quad (4.13)$$

where  $\mathcal{H}$  and  $\mathcal{H}_a$  are the Hamiltonian and the momentum constraints respectively.  $p^{ab}$  is the canonical momentum conjugate to the induced metric on  $\Sigma_\tau$  and  $E^a$  is the momentum conjugate to  $A_a$ . The surface  $S_\tau$  is the intersection of  $\Sigma_\tau$  and the brane, while  $\Theta$  is the trace of the extrinsic curvature of  $S_\tau$  in  $\Sigma_\tau$ .

Note that the momentum  $E^a = F^{a\tau}$ . In particular,  $E^\tau = 0$  and we regard  $A_\tau$  as an ignorable coordinate. We will now evaluate this Hamiltonian for the RNAdS spacetime described by equations (4.2), (4.3) and (4.4). Each of the constraints vanish because this is a solution to the equations of motion.

$$\mathcal{H} = \mathcal{H}_a = \nabla_a E^a = 0. \quad (4.14)$$

The last constraint is of course Gauss' Law. When evaluated on the surface  $S_\tau$ , the potential,  $A = \left(-\frac{1}{\kappa} \frac{q}{Z(\tau)^{n-3}} + \Phi\right) \dot{t} d\tau$ . The important thing here is that it only has components in the  $\tau$  direction. This ensures that the last two terms in the Hamiltonian cancel one another. Since  $N = 1$  and  $N^a = 0$  on  $S_\tau$ , it only remains to evaluate the extrinsic curvature  $\Theta$ . If  $\gamma_{ab}$  is the induced metric on  $S_\tau$ , it is easy to show that

$$\Theta = \Theta_{ab}\gamma^{ab} = K_{ab}\gamma^{ab} = (n-2)\frac{h(Z)\dot{t}}{Z}. \quad (4.15)$$

The energy is then evaluated as

$$\mathcal{E} = -\frac{1}{8\pi G_n} \int_{S_\tau} (n-2)\frac{h(Z)\dot{t}}{Z}. \quad (4.16)$$

We will now address the issue of background energy. This is usually necessary to cancel divergences in the Hamiltonian. In our case, the brane cuts off the spacetime. If the brane does not stretch to the AdS boundary there will not be any divergences that need to be cancelled. However it is important to define a zero energy solution. In this work we will choose pure AdS space. This is because the FRW equations for a brane embedded in pure AdS space would include all but the holographic terms

that appear in equations (4.10a) and (4.10b). These are the terms we are trying to interpret with this analysis.

We will denote the background spacetime by  $\mathcal{M}_0$ . We have chosen this to be pure AdS space cut off at a surface  $\partial\mathcal{M}_0$  whose geometry is the same as our brane. As is described in [115] this means we have the bulk metric given by

$$ds_n^2 = -h_{AdS}(Z)dT^2 + \frac{dZ^2}{h_{AdS}(Z)} + Z^2 d\Omega_{n-2}, \quad (4.17)$$

in which

$$h_{AdS}(Z) = k_n^2 Z^2 + 1. \quad (4.18)$$

There is of course no electromagnetic field. The surface  $\partial\mathcal{M}_0$  is described by the section  $(\mathbf{x}^\mu, T(\tau), Z(\tau))$  of the bulk spacetime. In order that this surface has the same geometry as our brane we impose the condition

$$-h_{AdS}(Z)\dot{T}^2 + \frac{\dot{Z}^2}{h_{AdS}(Z)} = -1, \quad (4.19)$$

which is analogous to the condition given in equation (2.34).

We now repeat the above evaluation of the Hamiltonian for the background spacetime. This gives the following value for the background energy

$$\mathcal{E}_0 = -\frac{1}{8\pi G_n} \int_{S_\tau} (n-2) \frac{h_{AdS}(Z)\dot{T}}{Z}. \quad (4.20)$$

Making use of equations (4.8a), (4.8c) and (4.19) we find that the energy of  $\mathcal{M}^+$  above the background  $\mathcal{M}_0$  is given by

$$\begin{aligned} E_+ = \mathcal{E} - \mathcal{E}_0 &= \frac{(n-2)}{8\pi G_n} \int_{S_\tau} \frac{1}{Z} \left[ \sqrt{\dot{Z}^2 + h_{AdS}(Z)} - \sqrt{\dot{Z}^2 + h(Z)} \right] \\ &= \frac{(n-2)}{8\pi G_n} \int_{S_\tau} \sqrt{\sigma_n^2 - \frac{\Delta h}{Z^2}} - \sigma_n, \end{aligned} \quad (4.21)$$

where

$$\Delta h = h(Z) - h_{AdS}(Z) = -\frac{c}{Z^{n-3}} + \frac{q^2}{Z^{2n-6}}. \quad (4.22)$$

In this relation  $\Delta h$  is negative everywhere outside of the black hole horizon and so it is clear that  $E_+$  is positive. We now turn our attention to the contribution to the energy from  $\mathcal{M}^-$ . Since the derivation of  $E_+$  saw the cancellation of the last two terms in the Hamiltonian (4.13), we note that the result is purely geometrical.

Even though  $\mathcal{M}^+$  and  $\mathcal{M}^-$  have opposite charge, they have the same geometry and so  $E_+ = E_-$ . We deduce then that the total energy

$$E = E_+ + E_- = \frac{(n-2)}{4\pi G_n} \int_{S_r} \sqrt{\sigma_n^2 - \frac{\Delta h}{Z^2}} - \sigma_n. \quad (4.23)$$

Since the spatial volume of the braneworld  $V = \int_{S_r} = \Omega_{n-2} Z^{n-2}$ , we arrive at the following expression for the energy density measured by an observer living on the brane

$$\rho = \frac{(n-2)\sigma_n}{4\pi G_n} \left( \sqrt{1 - \frac{\Delta h}{\sigma_n^2 Z^2}} - 1 \right), \quad (4.24)$$

where we have pulled out a factor of  $\sigma_n$ . If we insert this expression back into the first Friedmann equation (4.10a) we find that

$$H^2 = a - \frac{1}{Z^2} + \frac{8\pi G_n \sigma_n}{n-2} \rho + \left( \frac{4\pi G_n}{n-2} \right)^2 \rho^2. \quad (4.25)$$

Although we will leave a more detailed analysis of the pressure,  $p$ , until section 4.3, we can make use of the Conservation of Energy equation

$$\dot{\rho} = -(n-2)H(\rho + p), \quad (4.26)$$

to derive the second Friedmann equation

$$\dot{H} = \frac{1}{Z^2} - 4\pi G_n \sigma_n (\rho + p) - (n-2) \left( \frac{4\pi G_n}{n-2} \right)^2 \rho (\rho + p). \quad (4.27)$$

These are clearly not the standard Friedmann equations of an  $(n-1)$ -dimensional universe with energy density  $\rho$  and pressure  $p$ . However, we should not expect them to be. We have not made any approximations in arriving at these results so it is possible that we would see non-linear terms. What is exciting is that the quadratic terms we see here have exactly the same form as the unconventional terms that were reviewed in section 2.5.2, and originally noticed by Binetruy *et al* in the study of brane cosmologies [23, 24]. In that case, they arose from extra matter on the brane. We have no extra matter on the brane here, but by including a bulk black hole, we get exactly the same type of cosmology. Clearly there is a duality relating these two pictures.

We also note that in [23, 24], the energy momentum tensor on the brane is split between tension and additional matter in an arbitrary way, equation (2.40). In our

analysis the tension is the **only** explicit source of energy momentum on the brane so there is no split required. With this in mind we are able to interpret each term in the FRW equations more confidently, in particular, the cosmological constant term. Furthermore, we have not yet made any assumptions on the form of the braneworld Newton's constant.

Finally, we see that for small  $\rho$  and  $p$ , we can neglect the  $\rho^2$  and  $\rho p$  terms and recover the standard Friedmann equations for an  $(n - 1)$  dimensional universe

$$H^2 = a - \frac{1}{Z^2} + \frac{16\pi G_{n-1}}{(n-2)(n-3)}\rho, \quad (4.28)$$

$$\dot{H} = \frac{1}{Z^2} - \frac{8\pi G_{n-1}}{(n-3)}(\rho + p). \quad (4.29)$$

Here we have taken the  $(n - 1)$  dimensional Newton's Constant to be

$$G_{n-1} = \frac{(n-3)}{2}G_n\sigma_n, \quad (4.30)$$

as is suggested by [52, 59–63]. We see, then, how the dualities noticed in previous studies of braneworld holography arise as approximations to the duality described here.

### 4.3 Thermodynamics on the brane

In [65, 66], it was noticed that there was a remarkable relationship between the thermodynamics of a field theory on the brane and its gravitational dynamics in the AdS bulk. One can derive a ‘‘Cardy-Verlinde’’ formula which relates the thermodynamic variables of the field theory, thus generalising Cardy’s formula for the entropy of a  $1 + 1$ -dimensional CFT [117]. At the point in the brane’s evolution when it crosses the event horizon of the bulk black hole, this Cardy-Verlinde formula reproduces the Friedmann equation of the brane universe. Furthermore, the field theory entropy then realises the holographic Hubble entropy bound of the bulk spacetime. Much of the work in this area has been based on the assumption that the brane reached close to the AdS boundary. Our work is not limited by these approximations and we shall now attempt to generalise those results to this exact setting.

We shall assume that our dual field theory is in thermodynamic equilibrium. We would therefore expect it to satisfy the first law of thermodynamics

$$TdS = dE - \Phi dQ + pdV. \quad (4.31)$$

Notice the presence of the chemical potential,  $\Phi$ , which is conjugate to the R charge,  $Q$ , of the field theory. This is typical of theories that are dual to RNAdS in the bulk [118], and was discussed in 2.4.2.

In reality, this first law arises from the contributions of both of our black holes,  $\mathcal{M}^+$  and  $\mathcal{M}^-$ . Written in terms of the field theory quantities which are derived from each of these black holes, the first law should actually take the extended form

$$T_+ dS_+ + T_- dS_- = dE_+ + dE_- - \Phi_+ dQ_+ - \Phi_- dQ_- + (p_+ + p_-) dV. \quad (4.32)$$

We noted in section 4.1, that  $\mathcal{M}^+$  and  $\mathcal{M}^-$  have the same geometry. Field theory quantities that only see this geometry, as opposed to the difference in charge, will therefore be the same whether they are derived from  $\mathcal{M}^+$  or  $\mathcal{M}^-$ . We deduce then that we should derive a single temperature,  $T$ , entropy,  $S$ , energy,  $E$ , and pressure,  $p$ , where

$$\begin{aligned} T &= T_+ = T_-, \\ S &= 2S_+ = 2S_-, \\ E &= 2E_+ = 2E_-, \\ p &= 2p_+ = 2p_-. \end{aligned}$$

However we need to be more careful in discussing the chemical potential and the charge because these are aware of the different sign in the charges in  $\mathcal{M}^+$  and  $\mathcal{M}^-$ . Since the charges are equal and opposite we define

$$\begin{aligned} \Phi &= \Phi_+ = -\Phi_-, \\ Q &= 2Q_+ = -2Q_-. \end{aligned}$$

Given these relations, we do indeed recover a simplified first law of the form (4.31).

To proceed further we will have to assume that the entropy and charge of our field theory is given exactly by the entropy and charge of the black holes. Taking

into account the doubling up from the two black holes we find that

$$S = \frac{\Omega_{n-2} Z_H^{n-2}}{2G_n} \quad \text{and} \quad Q = \frac{(n-2)\kappa\Omega_{n-2}}{4\pi G_n} q. \quad (4.33)$$

Recall that the energy  $E$  is given by

$$E = \frac{(n-2)\sigma_n\Omega_{n-2}Z^{n-2}}{4\pi G_n} \left( \xi(Z) - 1 \right). \quad (4.34)$$

where we have taken out a factor of  $\sigma_n$  in equation (4.23) and used the simplifying notation

$$\xi(Z) = \sqrt{1 - \frac{\Delta h}{\sigma_n^2 Z^2}}. \quad (4.35)$$

The remaining thermodynamic variables,  $p$ ,  $T$  and  $\Phi$ , can now be obtained by making use of the first law (4.31).

$$p = - \left( \frac{\partial E}{\partial V} \right)_{S,Q} = -\rho + \frac{1}{8\pi G_n \sigma_n \xi(Z)} \left[ \frac{(n-1)c}{Z^{n-1}} - \frac{2(n-2)q^2}{Z^{2n-4}} \right], \quad (4.36)$$

$$T = \left( \frac{\partial E}{\partial S} \right)_{Q,V} = \frac{1}{\xi(Z)} \frac{T_{BH}}{\sigma_n Z}, \quad (4.37)$$

$$\Phi = \left( \frac{\partial E}{\partial Q} \right)_{S,V} = \frac{1}{\kappa \sigma_n \xi(Z) Z} \left( \frac{q}{Z_H^{n-3}} - \frac{q}{Z^{n-3}} \right), \quad (4.38)$$

where  $T_{BH} = \frac{h'(Z_H)}{4\pi}$  is the temperature of the black holes. Given the value of  $p$  that we have derived above, it is a useful check of the consistency of these results, to observe that equations (4.27) and (4.10b) are indeed the same. Notice also that the chemical potential,  $\Phi$ , vanishes at the horizon.

In earlier studies of such braneworlds, the nonlinear terms in  $\rho$  were not taken into account due to various approximations which were made during the calculations. It is therefore far from trivial to state that any further results connecting the bulk spacetime physics and the field theory thermodynamics should continue to hold in our scenario. We therefore proceed to consider these connections between the FRW equations and the field theory variables, including all such non-linear terms, to observe how the connection is affected in their presence.

Our first task is to rewrite the first law of thermodynamics in terms of densities and we find that

$$Tds = d\rho - \Phi d\rho_Q - \gamma d\left(\frac{1}{Z^2}\right), \quad (4.39)$$

where

$$\gamma = \frac{(n-2)Z^2}{2}(\rho + p - \Phi\rho_Q - Ts). \quad (4.40)$$

and the densities  $s = \frac{S}{V}$  and  $\rho_Q = \frac{Q}{V}$ . This equation defines  $\gamma$  which is the variation of  $\rho$  with respect to the spatial curvature  $1/Z^2$ . It therefore represents the geometrical Casimir part of the energy density. Using the values we have derived for our field theory thermodynamic variables, we find that

$$\gamma = \frac{n-2}{8\pi G_n \sigma_n \xi(Z)} \frac{Z_H^{n-3}}{Z^{n-3}}. \quad (4.41)$$

We are now ready to present a generalised (local) Cardy-Verlinde formula for the entropy density,  $s$ , in terms of  $\gamma$  and the thermodynamic variables

$$s = \frac{4\pi\sigma_n}{(n-2)k_n} \left[ \left( 1 + \frac{4\pi G_n \rho}{(n-2)\sigma_n} \right)^2 \gamma \left( \rho - \frac{1}{2}\Phi\rho_Q - \frac{\gamma}{Z^2} \right) - \frac{2\pi G_n}{(n-2)\sigma_n} \rho^2 \gamma \left( 1 + \frac{4\pi G_n \rho}{(n-2)\sigma_n} \right) \right]^{1/2}. \quad (4.42)$$

In the limit of small  $\rho$ , we can ignore the quadratic terms and this reduces to the formula found in [66–68, 119]

$$s = \frac{4\pi\sigma_n}{(n-2)k_n} \sqrt{\gamma \left( \rho - \frac{1}{2}\Phi\rho_Q - \frac{\gamma}{Z^2} \right)} \quad (4.43)$$

In [66], it was noticed that, when a critical brane ( $\sigma_n = k_n$ ) crosses the black hole horizon, the Hubble parameter is given by  $H^2 = k_n^2$ . At this instant, the entropy density can be written as

$$s = \frac{H}{2G_n k_n} = \frac{(n-3)H}{4G_{n-1}} \quad (4.44)$$

where we have made use of equation (4.30) when  $\sigma_n = k_n$ . Notice that this corresponds to the Hubble entropy described in [65]. The key observation is that, at the horizon, equation (4.43) then reduces to the linear form of the first Friedmann equation (4.28). Confining ourselves to the case of critical branes, this result generalises to the non-linear case when we evaluate equation (4.42) at the horizon, since we then recover the exact form of the Friedmann equation (4.25).

For non-critical branes ( $\sigma_n \neq k_n$ ) at the horizon, the connection between entropy density and the Hubble parameter is not a linear one. We instead find that  $H^2 =$

$\sigma_n^2 = k_n^2 + a$ , therefore using equation (4.33), the entropy density on non-critical branes should be rewritten as

$$s = \frac{\sqrt{H^2 - a}}{2G_n k_n} = \left(\frac{\sigma_n}{k_n}\right) \frac{(n-3)\sqrt{H^2 - a}}{4G_{n-1}} \quad (4.45)$$

where we have once again made use of equation (4.30). Unlike in the critical case, it is not clear how we should interpret this entropy formula. Nevertheless we note that when equations (4.43) and (4.42) are evaluated at the horizon they reproduce the linear (4.28) and non-linear (4.25) Friedmann equations respectively.

We also present the generalised form of the temperature at the horizon crossing

$$T = -\frac{k_n \dot{H}}{2\pi\sigma_n\sqrt{H^2 - a}} \left(1 + \frac{4\pi G_n \rho}{(n-2)\sigma_n}\right)^{-1}. \quad (4.46)$$

Notice that this reduces to the formula quoted in [66] when we ignore quadratic terms and set  $\sigma_n = k_n$  for the critical brane. Using this generalised form, we see that if we evaluate equation (4.40) at the horizon, we reproduce the second Friedmann equation (4.27), even in the exact, non-linear case.

## 4.4 Conclusions

We have noticed a remarkable new duality for braneworld cosmology that includes the unconventional terms first spotted in [23,24]. Consider the approach to the cosmology of braneworlds discussed in section 2.5.2, where we embedded a brane in pure AdS space and placed additional matter on it. It was observed that the braneworld observer sees a different cosmology to what we might expect – the Friedmann equations include quadratic terms in the energy density and pressure of this additional matter. In this chapter, we have obtained exactly the same unconventional terms when one considers a brane embedded in between two black holes, with no extra matter being placed on the brane. In previous studies of branes in black hole backgrounds, the cosmological evolution has been described in two different ways. As above, it may be regarded as being driven by the bulk black hole or by a dual theory living on the brane. However, these results assume that the brane is close to the AdS boundary. We have used techniques that do not require us to make such

approximations in our calculations. It seems that the “holographic” dualities of [66] are approximations to the “unconventional” duality we have noticed.

Another appealing feature of our analysis is that it is free of ambiguities. In [23, 24] it is not clear why the decomposition of the brane energy momentum tensor into tension and additional matter should be carried out as it is, i.e. as in equation (2.40). In our approach there is no additional matter, only tension, so there is no ambiguous split. We find that our exact interpretation of the bulk Weyl tensor suggests that the FRW equations should indeed take the form given in [23, 24]. However, by comparing our values for  $\rho$  and  $p$  we also see that we have a highly non-trivial equation of state. It only takes a recognisable form as the brane approaches the AdS boundary. This is expected because by allowing the brane to probe deep into the bulk we are inserting a significant cutoff in the dual field theory. The approach of [23, 24] has the benefit that it allows for an arbitrary linear equation of state.

Given our exact work, we have also been able to present a generalised Cardy-Verlinde formula for the field theory on the brane. We should note that we made various assumptions (e.g. entropy on brane is entropy of bulk) that perhaps require further justification – our results are as given but it would be worth taking a closer look at how a braneworld observer should really see entropy in this exact setting. Nevertheless, we find that just as in [66], the Friedmann equations are reproduced when we evaluate the relevant thermodynamic formulas at the horizon, including all non-linear terms. We should note that we do not know how we should interpret the expression for the entropy (4.45) when there is a cosmological constant induced on the brane.

# Chapter 5

## Stability of Schwarzschild in a Finite Cavity

Gubser and Mitra [25, 26] have made a conjecture relating classical and thermodynamic stability of black branes that considers their dynamical evolution and so goes beyond the traditional correspondences between black hole mechanics and thermodynamics.

“A black brane with a non-compact translational symmetry is classically stable if, and only if, it is locally thermodynamically stable.”

Interesting tests of this conjecture are provided by solutions which pass from (classical or thermodynamic) stability to instability as some parameter is varied and we will study a simple example of this in this chapter.

### 5.1 Proving the conjecture

In [27], charged black  $p$ -brane solutions were considered, where in some cases the specific heat becomes positive near extremality, and it was explicitly demonstrated that for suitable gauge choices, the equations satisfied by the Euclidean negative mode and the classical Gregory-Laflamme perturbation are equivalent.

In this chapter, we will consider the simpler case of an uncharged black  $p$ -brane in  $d + p$  dimensions, constructed by taking a direct product of a  $d$ -dimensional

Schwarzschild black hole and  $p$  flat directions. We consider this system in a finite cavity, so we can vary the specific heat by varying the ratio of the cavity size and the black hole's mass. The relation between the classically unstable mode and the Euclidean negative mode for this solution was already considered in infinite volume in [27]. We show that the introduction of the cavity walls does not spoil the equivalence, and furthermore show that the negative mode coincides precisely with the regime of thermodynamic instability.<sup>1</sup>

In [120], it was shown that Schwarzschild black holes with negative specific heat will necessarily have a negative mode. This was extended to the charged black brane solutions in [27]. Thus, thermodynamic instability in the canonical ensemble implies the existence of a negative mode. No general proof of the converse exists, but we have seen in section 2.6.1 that a negative mode suggests there should indeed be a thermodynamic instability since the definition of the canonical ensemble for Quantum Gravity breaks down in the presence of such a mode. In what follows, we will check this explicitly by finding the lowest eigenvalue of (2.47) and showing it is negative if, and only if, the specific heat is negative.

The key remaining step in relating thermodynamic instability and classical instability is relating the negative mode satisfying (2.47) in the black hole solution to the unstable mode in the Gregory-Laflamme analysis of the perturbations about a black brane solution. For the uncharged black branes we are interested in, this connection is very direct [27]. We observed in section 2.6.2 that the black hole factor in the Gregory-Laflamme perturbation,  $H_{\mu\nu}$ , satisfied equation (2.51), and that a time-independent solution of this equation with non-zero eigenvalue given by  $\mu = \mu^*$ , represents a threshold unstable mode, separating stable and unstable perturbations. But for such time-independent perturbations,  $H_{\mu\nu}$  is static, and Wick rotating it just changes the sign of  $H_{tt}$ . Furthermore, upon Wick rotation, the Lorentzian Lichnerowicz operator  $\tilde{G}_{\mu\nu}{}^{\rho\sigma}$  becomes the Euclidean Lichnerowicz operator  $G_{\mu\nu}{}^{\rho\sigma}$ . We therefore see that, on time-independent perturbations, equations (2.47) and (2.51)

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<sup>1</sup>A similar calculation of the negative mode for Euclidean Schwarzschild-anti de Sitter was carried out in [120]. It was also shown in [121] that for Reissner-Nordström black holes in infinite volume, the negative mode coincides with the regime of thermodynamic instability.

are equivalent – the eigenvalue for the Euclidean negative mode equation is given by  $\lambda = -\mu^2$ . Thus the conditions for classical instability and existence of a Euclidean negative mode give rise to the same equation.

The above argument relies on the fact that the boundary conditions satisfied by the Euclidean negative mode in the black hole solution and the classical Gregory-Laflamme instability of the uncharged black brane coincide – in both cases we look for perturbations which are regular at the horizon and decay at infinity. However, the introduction of a finite boundary introduces little modification to the argument. For the case of a black hole in a finite cavity, the appropriate boundary condition in both the Euclidean negative mode and the classical perturbation is that the induced metric on the boundary be unchanged. In what follows, we will show that this Euclidean negative mode (and hence the classical instability) occurs precisely when the specific heat of the black hole is negative, and hence when the black hole is thermodynamically unstable (it is trivial to observe that thermodynamic stability of the black hole and black brane are equivalent, since the specific heats,  $c = \frac{\partial E}{\partial T}$ , of the brane and the black hole are related by a volume factor which does not affect their sign). This simple example of the uncharged  $p$ -brane therefore allows this identification between the Euclidean negative mode and negative specific heat to be made, and so we have a more complete argument demonstrating Gubser and Mitra's conjectured equivalence – it is thus a particularly attractive example of Reall's general approach to demonstrating why the conjecture should be true.

## 5.2 Schwarzschild in a finite cavity

Gravitational thermodynamics in a finite cavity was considered by York [122]. He considered four-dimensional spacetime, with the proper area of the spherical cavity,  $A = 4\pi r_b^2$ , and the local temperature,  $T$ , at the cavity wall fixed. In the Euclidean solution, these boundary conditions amount to fixing the induced metric on the boundary. If the product of cavity radius and temperature was sufficiently high,  $r_b T > \sqrt{27}/8\pi$ , there were two Schwarzschild solutions of mass  $M_1, M_2$  which satisfied the boundary conditions. As  $r_b T$  runs from this minimum value to infinity,

$M_1/r_b$  runs from  $1/3$  to  $0$ , and  $M_2/r_b$  runs from  $1/3$  to  $1/2$  (thus, in the limit, the black hole fills the cavity). The stability in the canonical ensemble was analysed by calculating the specific heat at constant area of the boundary, with the result

$$C_A = 8\pi M^2 \left(1 - \frac{2M}{r_b}\right) \left(\frac{3M}{r_b} - 1\right)^{-1}. \quad (5.1)$$

Thus, the smaller black hole, of mass  $M_1$ , has negative specific heat and is thermodynamically unstable, while the larger black hole, of mass  $M_2$ , has positive specific heat and is thermodynamically stable.

In considering perturbations, it is more convenient to use the black hole's mass,  $M$ , and  $r_b$  as parameters, rather than  $T$  and  $r_b$ , and we will imagine varying  $r_b$ . We can consider any  $r_b > 2M$ . For  $2M < r_b < 3M$ , we have a positive specific heat, while for  $r_b > 3M$ , we have a negative specific heat. We therefore expect to see a negative mode only for  $r_b > 3M$ . Allen analysed the problem of finding the negative mode for black holes in a finite cavity prior to York's study of the thermodynamics [123]. He found that the negative mode exists if  $r_b \gtrsim 2.89M$ . A resolution of this apparent contradiction was presented by York [122], who observed that Allen had fixed the coordinate location of the boundary, which, in his choice of gauge for the perturbation, did not correspond to fixing the area of the boundary.

In the next section, we will discuss the solution of the eigenvalue equation (2.47) with the induced metric on the boundary fixed. First, however, we will briefly discuss the straightforward extension of York's thermodynamic analysis to higher dimensions. The  $d$ -dimensional Schwarzschild black hole is

$$ds^2 = \left(1 - \frac{\omega M}{r^{d-3}}\right) dt^2 + \left(1 - \frac{\omega M}{r^{d-3}}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}, \quad (5.2)$$

where  $\omega = 16\pi/[(d-2)V_{d-2}]$  and  $V_{d-2}$  is the volume of the unit  $S^{d-2}$ . Hence, the relation between the black hole mass,  $M$ , and the temperature of the cavity,  $T_b$ , is

$$T_b \equiv T(r_b) = \frac{d-3}{4\pi(\omega M)^{1/(d-3)}} \left(1 - \frac{\omega M}{r_b^{d-3}}\right)^{-1/2}. \quad (5.3)$$

The entropy of the black hole is

$$S = \frac{1}{4} V_{d-2} (\omega M)^{\frac{d-2}{d-3}}. \quad (5.4)$$

Therefore, we can calculate the specific heat of a black hole in a cavity of radius  $r_b$

$$C_A \equiv T_b \left( \frac{\partial S}{\partial T_b} \right)_A = 4\pi M (\omega M)^{1/(d-3)} \left( 1 - \frac{\omega M}{r_b^{d-3}} \right) \left( \frac{d-1}{2} \frac{\omega M}{r_b^{d-3}} - 1 \right)^{-1}. \quad (5.5)$$

We therefore expect there to be a negative mode in the Euclidean solution only for  $r_b^{d-3} > \frac{d-1}{2} \omega M$ .

## 5.3 Finding the negative mode

We will now discuss the calculation of the negative mode for a  $d$ -dimensional Euclidean Schwarzschild black hole in a finite cavity with the induced metric on the wall of the cavity fixed. By the foregoing discussion, this also gives a threshold unstable mode for the uncharged black brane solution. We will first discuss the analysis for four dimensions in detail and then sketch the extension to higher dimensions.

### 5.3.1 Four dimensions

We wish to consider perturbations of the four-dimensional Euclidean Schwarzschild metric,

$$ds^2 = \left( 1 - \frac{2M}{r} \right) d\tau^2 + \frac{dr^2}{\left( 1 - \frac{2M}{r} \right)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (5.6)$$

We are looking for perturbations which are eigenvectors of (2.47) with negative eigenvalues. This problem was analysed in [75] for the black hole in infinite volume, and it was found that in an expansion in spherical harmonics, the only part which can have a negative eigenvalue is the spherically symmetric static mode. In considering the black hole in a finite box, it will therefore suffice for us to consider a perturbation

$$h^\mu{}_\nu = \text{Diag} \left[ H_0(r), H_1(r), -\frac{1}{2} (H_0(r) + H_1(r)), -\frac{1}{2} (H_0(r) + H_1(r)) \right], \quad (5.7)$$

where the condition of transversality,  $\nabla_\mu h^\mu{}_\nu = 0$ , implies

$$H_0(r) = \left[ -\frac{r(r-2M)}{r-3M} \frac{d}{dr} - \frac{3r-5M}{r-3M} \right] H_1(r). \quad (5.8)$$

The task of finding the negative mode then reduces to finding negative eigenvalues of

$$\left[ -\left( 1 - \frac{2M}{r} \right) \frac{d^2}{dr^2} - \frac{2(r-4M)(2r-3M)}{r^2(r-3M)} \frac{d}{dr} + \frac{8M}{r^2(r-3M)} \right] H_1(r) = \lambda H_1(r). \quad (5.9)$$

This equation has regular singular points at  $r = 0$ ,  $r = 2M$  and  $r = 3M$  and an irregular singular point at  $r = \infty$ , so an explicit solution to it can not be found. As in previous works, we must therefore use numerical methods to find the eigenvalues. In the infinite cavity the required boundary conditions for the solution are regularity at the horizon,  $r = 2M$ , and normalisability at infinity. Imposing these conditions on the solution then yields one negative eigenvalue,  $\lambda \approx -0.19M^{-2}$  [75].

To search for negative modes in a finite cavity, we need to impose the condition that the induced metric on the walls of the cavity is fixed. This is complicated by the fact that the perturbation is generically non-zero in the spherical directions. In the unperturbed metric the area of a spherical cavity of radius  $r = r_b$  is  $A = 4\pi r_b^2$ . The perturbed metric is

$$ds^2 = \left(1 - \frac{2M}{r}\right) (1 + \epsilon H_0(r)) d\tau^2 + \frac{1 + \epsilon H_1(r)}{1 - \frac{2M}{r}} dr^2 + r^2 \left[1 - \frac{1}{2}\epsilon(H_0(r) + H_1(r))\right] d\Omega^2, \quad (5.10)$$

and so after the perturbation the surface  $r = r_b$  has area

$$A = 4\pi r_b^2 \left[1 - \frac{1}{2}\epsilon(H_0(r) + H_1(r))\right]. \quad (5.11)$$

The change of area could be corrected by allowing the boundary of the cavity  $r_b$  to move to  $r'_b$  after the perturbation, where  $r'_b$  is chosen such that the area of the cavity wall in the perturbed metric is the same as that in the unperturbed metric at  $r_b$ . We will instead give a more careful analysis by making a change of coordinates to work in a different gauge for the perturbation, in which the area of the cavity wall at fixed radius is unchanged by the perturbation. The appropriate coordinate transformation is

$$r \rightarrow \rho: \quad \rho^2 = r^2 \left[1 - \frac{1}{2}\epsilon(H_0(r) + H_1(r))\right]. \quad (5.12)$$

Rewriting the perturbed metric in the new coordinates and keeping only the dominant first order terms of the metric perturbation gives

$$ds^2 = \left(1 - \frac{2M}{\rho}\right) (1 + \epsilon F_0(\rho)) d\tau^2 + \left(1 - \frac{2M}{\rho}\right)^{-1} (1 + \epsilon F_1(\rho)) d\rho^2 + \rho^2 d\Omega^2, \quad (5.13)$$

where

$$F_0(\rho) = \frac{(2\rho - 3M)H_0(\rho) + MH_1(\rho)}{2(\rho - 2M)},$$

$$F_1(\rho) = \frac{(\rho - 3M)H_0(\rho) + (3\rho - 7M)H_1(\rho) + \rho(\rho - 2M)(H_0'(\rho) + H_1'(\rho))}{2(\rho - 2M)}.$$

Thus the metric perturbation in this gauge is

$$h^\mu{}_\nu = \text{Diag}[F_0(\rho), F_1(\rho), 0, 0]. \quad (5.14)$$

With this new gauge we see that to hold the surface area fixed when the metric perturbation is switched on is a trivial task: we simply hold  $\rho_b$  fixed.

In section 2.6.1, we saw that physical negative modes are determined by the eigenvalue equation (2.47), which involves only the transverse traceless part of the perturbation. An infinitesimal coordinate transformation changes the metric by  $g_{\mu\nu} \rightarrow g_{\mu\nu} + \epsilon \nabla_{(\mu} \xi_{\nu)}$  for some vector field  $\xi^\mu$ , and thus only changes the longitudinal part. That is, although the metric perturbation in our new gauge choice looks more complicated, the equation we need to solve is unchanged, as the transverse traceless part is not affected by the gauge transformation. The transverse traceless part of (5.14) is just (5.7) with  $r \rightarrow \rho$ , that is,

$$h^{TT\ \mu}{}_\nu = \text{Diag}\left[H_0(\rho), H_1(\rho), -\frac{1}{2}(H_0(\rho) + H_1(\rho)), -\frac{1}{2}(H_0(\rho) + H_1(\rho))\right] \quad (5.15)$$

with the transversality condition

$$H_0(\rho) = \left[-\frac{\rho(\rho - 2M)}{\rho - 3M} \frac{d}{d\rho} - \frac{3\rho - 5M}{\rho - 3M}\right] H_1(\rho). \quad (5.16)$$

We are thus seeking negative eigenvalues of the equation

$$\left[-\left(1 - \frac{2M}{\rho}\right) \frac{d^2}{d\rho^2} - \frac{2(\rho - 4M)(2\rho - 3M)}{\rho^2(\rho - 3M)} \frac{d}{d\rho} + \frac{8M}{\rho^2(\rho - 3M)}\right] H_1(\rho) = \lambda H_1(\rho), \quad (5.17)$$

subject to the boundary conditions of regularity at  $\rho = 2M$  and the isothermal boundary condition at the fixed radial position  $\rho = \rho_b$ . The first boundary condition is imposed by finding a regular series solution to the differential equation about  $\rho = 2M$ . There is only one such solution:

$$H_1(\rho) = \sum_{n=0}^{\infty} a_n (\rho/M - 2)^n, \quad (5.18)$$

where

$$\begin{aligned}
 a_1 &= -(\lambda M^2 + 2)a_0, \\
 a_2 &= \frac{1}{6}(\lambda M^2 + 2)(2\lambda M^2 + 7)a_0, \\
 a_3 &= -\frac{1}{36}(\lambda M^2 + 5)(2\lambda^2 M^4 + 10\lambda M^2 + 15)a_0, \\
 a_4 &= \frac{1}{360}(585 + 520\lambda M^2 + 168\lambda^2 M^4 + 30\lambda^3 M^6 + 2\lambda^4 M^8)a_0.
 \end{aligned}$$

The isothermal boundary condition fixes the proper length around the  $S^1$  in the  $\tau$  direction, which is given by  $\sqrt{g_{\tau\tau}(\rho_b)}\Delta\tau$ . It therefore imposes

$$\begin{aligned}
 \left(1 - \frac{2M}{\rho_b}\right) &= \left(1 - \frac{2M}{\rho_b}\right)(1 + F_0(\rho_b)) \\
 \Rightarrow F_0(\rho_b) &= 0 \\
 \Rightarrow H_0(\rho_b) &= \frac{M}{3M - 2\rho_b}H_1(\rho_b).
 \end{aligned} \tag{5.19}$$

$H_0$  is given in terms of  $H_1$  by (5.16), so the isothermal boundary condition reduces to a mixed boundary condition for  $H_1$ ,

$$\frac{H_1'(\rho_b)}{H_1(\rho_b)} + \frac{6\rho_b^2 - 20M\rho_b + 18M^2}{\rho_b(\rho_b - 2M)(2\rho_b - 3M)} = 0. \tag{5.20}$$

This can be contrasted to the condition derived by Allen by holding  $r_b$  fixed [123],

$$\frac{H_1'(r_b)}{H_1(r_b)} + \frac{3r_b - 5M}{r_b(r_b - 2M)} = 0. \tag{5.21}$$

It should be noted that in deriving these conditions we have multiplied through by  $\rho_b - 3M$  in (5.16) [ $r_b - 3M$  in (5.8)]. This is acceptable so long as  $\rho_b \neq 3M$ , but for  $\rho_b = 3M$ , we need to consider more carefully the true boundary condition (5.19).

The method used in solving the differential equation is to numerically integrate from  $\rho = 2M$  to  $\rho = \rho_b$  using the form of the solution given by (5.18) as the initial data. However, the numerical method breaks down at  $\rho = 3M$  due to the singularity here in the differential equation. To overcome this problem we find a power series solution about  $\rho = 3M$  and find that the solution is in fact well behaved there. To evolve our solution through  $\rho = 3M$ , it is therefore necessary to numerically integrate up to  $3M - \delta$ , for some small  $\delta$ , and use the results of this numerical integration to fit the power series solution at  $3M$  to the data. This power series can

then be evaluated at  $3M + \delta$ , and the value of  $H_1$  and its first derivative can be extracted there to provide new initial conditions for a numerical solution beginning at  $3M + \delta$ , which can then be evolved up to  $\rho_b$ .

Unlike at  $\rho = 2M$ , where only one of the series solutions was well behaved, there are two independent well behaved solutions at  $\rho = 3M$ . One of these is of order 1 whilst the other is  $O((\rho - 3M)^3)$ . Since the numerical solution breaks down in a fairly large region around  $3M$ , we can not approach the singularity with too small a  $\delta$ . It is therefore necessary to go to fifth order in the series to maintain accuracy with a  $\delta$  of order  $10^{-3}$ , and so the particular solution to the differential equation which we require here is a linear combination of both series. We therefore use the values of  $H_1$  and  $H_1'$  at  $3M - \delta$  to find the coefficients  $b_0$  and  $c_0$  in the general form for the solution at  $\rho = 3M$ :

$$H_1(\rho) = \sum_{n=0}^{\infty} (b_n + c_n(\rho/M - 3)^3) (\rho/M - 3)^n, \quad (5.22)$$

where

$$\begin{aligned} b_1 &= -\frac{4}{3}b_0, & c_1 &= -\frac{11}{6}c_0, \\ b_2 &= \left(\frac{4}{3} + \frac{3}{2}\lambda M^2\right)b_0, & c_2 &= \left(\frac{7}{3} - \frac{3}{10}\lambda M^2\right)c_0, \\ b_3 &= -\left(\frac{40}{27} + 5\lambda M^2\right)b_0, \\ b_4 &= \left(\frac{130}{81} + \frac{15}{2}\lambda M^2 - \frac{9}{8}\lambda^2 M^4\right)b_0, \\ b_5 &= -\left(\frac{136}{81} + \frac{79}{9}\lambda M^2 - 3\lambda^2 M^4\right)b_0. \end{aligned}$$

The eigenvalue spectrum is now found by a shooting method. An arbitrary value of  $\lambda$  is input, the solution at  $\rho = \rho_b$  is found and tested to see if it obeys the correct boundary condition. If it doesn't,  $\lambda$  is adjusted appropriately and the process is repeated until  $\lambda$  is found to the required precision. We have repeated this process with varying cavity sizes so that the value of the lowest eigenvalue can be plotted against  $\rho_b$ .

In figure 5.1, we give the results obtained for the boundary condition (5.21), demonstrating Allen's finding that the critical value of the radius is at  $r_b \approx 2.89M$ .

In figure 5.2, we give the results for our boundary condition (5.20), showing clearly the expected result that there is a negative mode only for  $\rho_b > 3M$ .

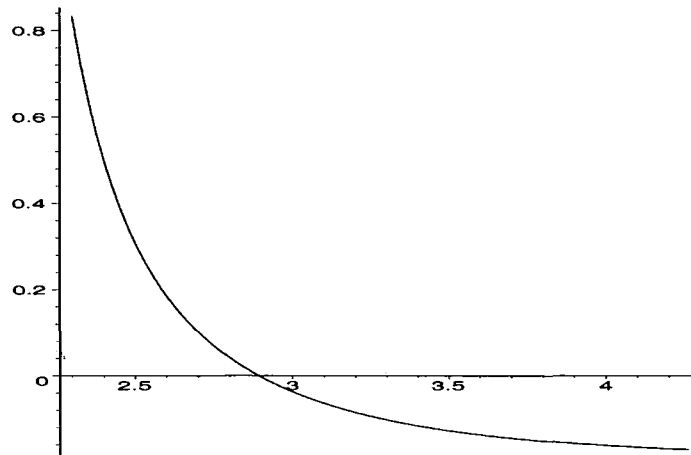


Figure 5.1:  $\lambda M^2$  vs  $\rho_b/M$  for the boundary condition (5.21)

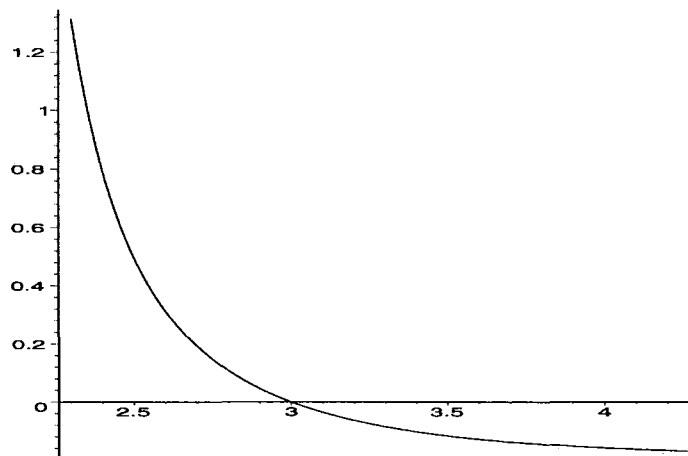


Figure 5.2:  $\lambda M^2$  vs  $\rho_b/M$  for the boundary condition (5.20)

### 5.3.2 Higher dimensions

We will now briefly discuss the extension of the results above to uncharged  $p$ -branes in  $d + p$  dimensions for  $d > 4$ . We are interested in studying the negative modes of the Euclidean black hole geometry (5.2). In  $d > 4$ , the spherically symmetric

transverse tracefree perturbation is

$$h^\mu{}_\nu = \text{Diag} \left[ H_0(r), H_1(r), -\frac{1}{d-2} (H_0(r) + H_1(r)), \dots, -\frac{1}{d-2} (H_0(r) + H_1(r)) \right], \quad (5.23)$$

and the transversality condition  $\nabla_\mu h^\mu{}_\nu = 0$  gives

$$H_0(r) = -\frac{2r(r^{d-3} - \omega M)}{2r^{d-3} - (d-1)\omega M} H_1'(r) - \frac{2(d-1)r^{d-3} - (d+1)\omega M}{2r^{d-3} - (d-1)\omega M} H_1(r). \quad (5.24)$$

The Euclidean negative mode is still given by the eigenvectors of (2.47), which becomes

$$\begin{aligned} -\left(1 - \frac{\omega M}{r^{d-3}}\right) H_1''(r) - \frac{[2dr^{2d-6} - \omega M r^{d-3}(3d^2 - 11d + 18) + d(d-1)\omega^2 M^2]}{r^{d-2}(2r^{d-3} - (d-1)\omega M)} H_1'(r) \\ + \frac{2d(d-3)^2 \omega M}{r^2(2r^{d-3} - (d-1)\omega M)} H_1(r) = \lambda H_1(r). \end{aligned} \quad (5.25)$$

As in the four-dimensional case, it is convenient to make a coordinate transformation before applying the boundary conditions at the shell. Here the appropriate transformation is

$$r \rightarrow \rho: \quad \rho^2 = r^2 \left[ 1 - \frac{1}{d-2} \epsilon (H_0(r) + H_1(r)) \right]. \quad (5.26)$$

The condition that the induced metric on the shell is fixed then implies that the shell is at some fixed  $\rho = \rho_b$ , and that

$$H_0(\rho_b) + \frac{d-3}{2(d-2)} \frac{\omega M}{\rho_b^{d-3} - \omega M} (H_0(\rho_b) + H_1(\rho_b)) = 0. \quad (5.27)$$

Using the transversality condition (5.24), this becomes a mixed boundary condition for  $H_1$ ,

$$\frac{H_1'(\rho_b)}{H_1(\rho_b)} + \frac{2(d-1)(d-2)\rho_b^{2d-6} - 2(d^2 - d - 2)\omega M \rho_b^{d-3} + (d-1)^2 \omega^2 M^2}{\rho_b(\rho_b^{d-3} - \omega M)[2(d-2)\rho_b^{d-3} - (d-1)\omega M]} = 0. \quad (5.28)$$

We analyse this eigenvalue problem using the same numerical methods as previously. The results are displayed in figure 5.3. In this graph the eigenvalue has been plotted against the cavity radius in a scaled and shifted radial coordinate,  $\tilde{\rho}$ , given by

$$\tilde{\rho} = \frac{\rho(\omega M)^{\frac{1}{3-d}} - 1}{\left(\frac{d-1}{2}\right)^{\frac{1}{d-3}} - 1}. \quad (5.29)$$

This choice is made so that in all dimensions we see that the black hole horizon is at  $\tilde{\rho} = 0$  and the specific heat (5.5) is negative only when  $\tilde{\rho}_b > 1$ . It is clear that in all cases this corresponds exactly to the existence of the negative mode.

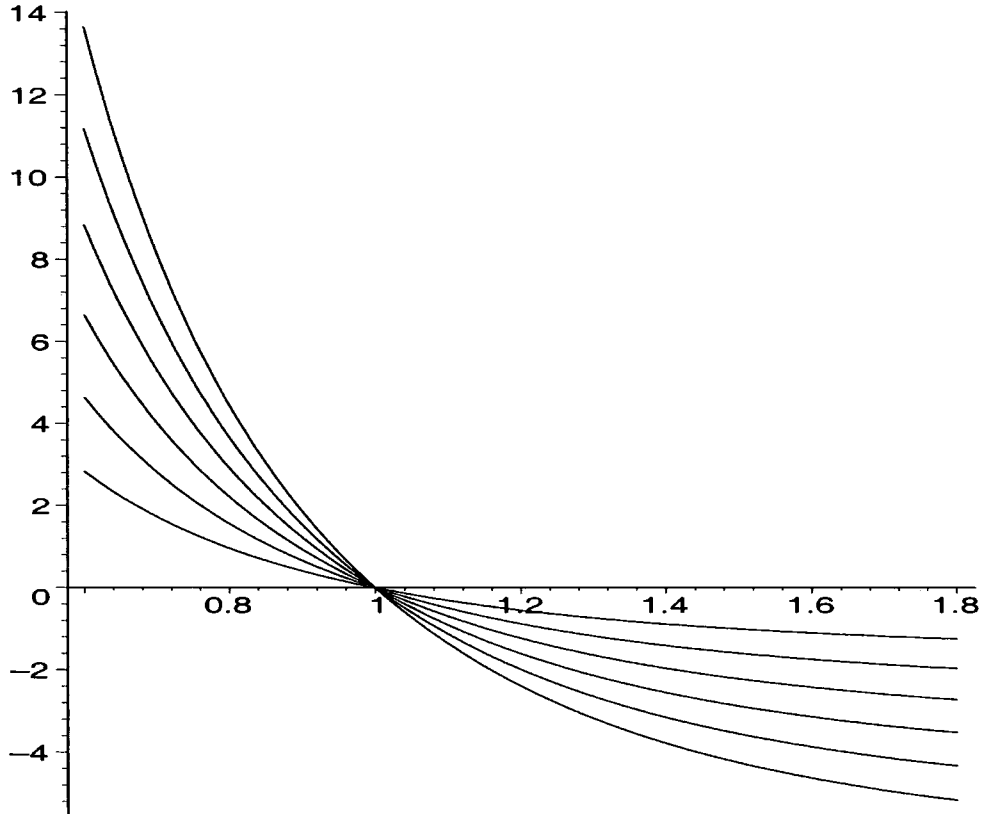


Figure 5.3:  $\lambda(\omega M)^{\frac{2}{d-3}}$  vs  $\tilde{\rho}_b$  for  $d = 5$  (closest to axis) to  $d = 10$  (furthest from axis) dimensions

## 5.4 Conclusions

We have demonstrated that an uncharged black brane in a spherical cavity is classically unstable if, and only if, it is locally thermodynamically unstable. This provides a particularly simple and elegant example of the general connection between thermodynamic and classical instability conjectured in [25, 26]. The key element of the proof was showing that as we vary the volume of the cavity, the classical instability disappears at precisely the point where the specific heat changes sign. To do this, we used the observation that the threshold unstable mode for the classical insta-

bility is the analytic continuation of the Euclidean negative mode [27]. We then provided the first analysis of the Euclidean negative mode for Schwarzschild in a finite cavity using the boundary conditions appropriate to the canonical ensemble. (A similar calculation of the negative mode for the Euclidean Schwarzschild anti-de Sitter solution was carried out in [120]; unfortunately, it is not straightforward to construct a corresponding black brane solution in that case. The issue of stability for a warped black string solution in anti-de Sitter space was considered in [124].)

One might be surprised that the presence of a boundary affects the classical instability at all; after all, the bulk metric is unchanged, so the initial behaviour of a perturbation with compact support near the horizon should be unaffected by the introduction of the boundary. However, in general relativity, the initial data are subject to constraints, so we are not free to specify an arbitrary initial perturbation of compact support, and the boundary conditions can influence the allowed possibilities for initial perturbations. It is hence reasonable that the boundary can turn off the classical instability.

# Chapter 6

## Extending the Gubser-Mitra Conjecture

In the last chapter we studied Gubser and Mitra's conjectured equivalence of classical and thermodynamic stability of black branes. The key restriction of the conjecture is that it only applies to branes which have a translationally-invariant direction. This condition is imposed to exclude black holes from the conjecture – it is well known that the Schwarzschild black hole is classically stable, yet has negative specific heat and is therefore locally thermodynamically unstable. Nevertheless, it would be appealing to be able to extend Gubser and Mitra's conjecture to other product spacetimes in which a black hole is crossed with a more general spacetime, other than the flat directions of the supergravity brane solutions.

A particularly interesting extension in this context is to consider the stability of black holes crossed with spheres, in particular the Schwarzschild  $\text{AdS}_m \times S^n$  backgrounds. These backgrounds are solutions to the supergravity action that is derived from string theory in a low energy limit. Therefore, as we have seen in section 2.4.1 and chapter 3, they play a significant role in the AdS/CFT correspondence – in this context, the stability of the supergravity background has consequences in the dual gauge theory. The need to understand issues of stability of any background upon which we wish to study the correspondence is thus clear. In this chapter, we match the two notions of stability for these spacetime by deriving the explicit connection between the Euclidean negative mode of the black hole (associated with thermody-

dynamic instability), and the onset of a dynamical instability of the spacetime. Further aspects of the proof of this conjecture then follow as in chapter 5.

Similar issues regarding the stability of the Schwarzschild  $\text{AdS}_5 \times S^5$  background have been independently studied by Hubeny and Rangamani in [125]. However, unlike in their paper, we will perform an explicit analysis of the perturbations of all the supergravity fields and demonstrate that it is possible to consider perturbations of the metric field alone. These considerations are similar to those undertaken by Gregory and Laflamme where they explicitly demonstrated that perturbations of the form field were not relevant to their study of the stability of black branes [76, 77].

## 6.1 The equations of motion

When studying the AdS-Schwarzschild black hole, one can consider a variety of metrics due to the freedom of having a spherical, toroidal or hyperbolic boundary. The toroidal case arises from the near horizon region of the  $p$ -brane metric and has been studied earlier. It is very similar to our study in chapter 5, since the translationally-invariant directions in the black hole metric allows the Gubser-Mitra conjecture to be applied as it stands. Far less attention has been received by the hyperbolic black holes in the AdS/CFT correspondence. Their properties in this respect were discussed in [126]; we will not discuss them any further here. In what follows we restrict our study to the spherical case. The  $D(= m + n)$ -dimensional metric then takes the general form

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_{m-2}^2 + \tilde{R}^2 d\Omega_n^2, \quad (6.1)$$

where

$$V(r) = 1 - \frac{\mu}{r^{m-3}} + \frac{r^2}{R^2}. \quad (6.2)$$

This metric represents an  $m$ -dimensional AdS black hole smeared over an  $n$ -sphere. The black hole has an event horizon at  $r = r_+$ , where  $V(r_+) = 0$ , and mass,  $M$ , related to the parameter,  $\mu$ , and the  $m$ -dimensional Newton constant,  $G_m$ , by

$$M = \frac{(m-2)\text{Vol}(S^{m-2})\mu}{16\pi G_m}. \quad (6.3)$$

The AdS and sphere curvatures are related by  $R = \frac{m-1}{n-1} \tilde{R}$ .

The restriction of our study to spacetimes which arise in string/M-theory allows the following possibilities for the pair  $(m, n)$ : (5, 5), (4, 7), (7, 4) and (3, 3). In all that follows we standardise our use of indices:  $\alpha, \beta, \gamma, \dots$  represent indices on the AdS part of the metric whilst  $a, b, c, \dots$  represent indices on the sphere.  $A, B, C, \dots$  run over all possible indices. The coordinate  $x$  refers to the AdS black hole directions and  $y$  refers to the sphere directions.

These spaces arise from the supergravity action

$$I_D = \int d^D z \sqrt{-g} \left[ \mathcal{R} - \frac{1}{2p!} F_p^2 \right] + (\text{Chern-Simons term in D=11}), \quad (6.4)$$

where  $p = m$  (electric ansatz) or  $p = n$  (magnetic ansatz). The Chern-Simons term is a topological term and in considering perturbations of our solution it is safe to ignore this. Self-duality of the 5-form and 3-form field strengths for the cases  $(m, n) = (5, 5)$  and  $(3, 3)$  respectively must be imposed by hand.

The equations of motion which follow from this action are

$$R_{AB} = \frac{1}{2(p-1)!} \left( F_{AC_1 \dots C_{p-1}} F_B{}^{C_1 \dots C_{p-1}} - \frac{(p-1)}{p(D-2)} F_p^2 g_{AB} \right), \quad (6.5)$$

$$0 = \nabla_A F^{AB_1 \dots B_{p-1}}. \quad (6.6)$$

## 6.2 Perturbations

In Reall's study [27] of the equivalence of classical and thermodynamic stability of black branes, he only considered perturbations of the metric. However, it is not trivial that setting the field strength perturbations to zero should be a consistent restriction upon the perturbations of our supergravity solution. We will thus initially consider the most general perturbation equations and then reconsider this question.

The perturbations of the fields involved in the problem are

$$\tilde{g}_{AB} = g_{AB} + h_{AB}, \quad (6.7)$$

$$\Rightarrow \tilde{g}^{AB} = g^{AB} - h^{AB}. \quad (6.8)$$

$$\tilde{F}_{AB_1 \dots B_{p-1}} = F_{AB_1 \dots B_{p-1}} + \delta F_{AB_1 \dots B_{p-1}}, \quad (6.9)$$

$$\begin{aligned} \Rightarrow \tilde{F}^{AB_1 \dots B_{p-1}} &= F^{AB_1 \dots B_{p-1}} + \delta F^{AB_1 \dots B_{p-1}} - h_C^A F^{CB_1 \dots B_{p-1}} \\ &\quad - (p-1) F^{AC[B_2 \dots B_{p-1} h_C^{B_1}],} \end{aligned} \quad (6.10)$$

$$\begin{aligned} \text{and } \tilde{F}_A^{B_1 \dots B_{p-1}} &= F_A^{B_1 \dots B_{p-1}} + \delta F_A^{B_1 \dots B_{p-1}} \\ &\quad - (p-1) F_A^{C[B_2 \dots B_{p-1} h_C^{B_1}].} \end{aligned} \quad (6.11)$$

Perturbing the equations of motion, (6.5) and (6.6), then results in the following equations for our perturbations (to first order)

$$\begin{aligned} \Delta_L h_{AB} + \frac{1}{(p-1)!} \left[ 2\delta F_{(A}{}^{C_1 \dots C_{p-1}} F_{B)C_1 \dots C_{p-1}} - (p-1) F_{ADC_1 \dots C_{p-2}} F_{BE}{}^{C_1 \dots C_{p-2}} h^{DE} \right. \\ \left. - \frac{p-1}{p(D-2)} [F^2 h_{AB} + 2F \cdot \delta F g_{AB} - p F_{DC_1 \dots C_{p-1}} F_E{}^{C_1 \dots C_{p-1}} h^{DE} g_{AB}] \right] = 0 \end{aligned} \quad (6.12)$$

and

$$\begin{aligned} \nabla_A \delta F^{AC_1 \dots C_{p-1}} - (p-1) F^{AB[C_2 \dots C_{p-1} \nabla_A h_B^{C_1}] - h_B^A \nabla_A F^{BC_1 \dots C_{p-1}} \\ - \nabla_A \left( h_B^A - \frac{1}{2} h \delta_B^A \right) F^{BC_1 \dots C_{p-1}} = 0, \end{aligned} \quad (6.13)$$

where  $\Delta_L$  is the  $D$ -dimensional Lichnerowicz operator defined by

$$\Delta_L h_{AB} = \square h_{AB} + 2R_{ACBD} h^{CD} - R_{CA} h_B^C - R_{CB} h_A^C - \nabla_A \nabla_C h_B^C - \nabla_B \nabla_C h_A^C + \nabla_A \nabla_B h_C^C, \quad (6.14)$$

and  $\square = g^{AB} \nabla_A \nabla_B$ .

We would like to simplify the analysis by only considering metric perturbations, i.e. by setting  $\delta F = 0$ . This simplification is undertaken by Gregory and Laflamme in their studies of black brane stability [76, 77], and by Reall in his outline of a proof for the Gubser-Mitra conjecture [27]. It was argued in both these cases that no unstable mode would be present in the perturbation of  $F$ , and it is reasonable to assume that these arguments should extend to the case we are studying here.

However, Reall only demonstrated that  $\delta F = 0$  was consistent in the case of the magnetic field strength. We see that the perturbation equations, (6.12) and (6.13), are coupled. Therefore, if we wish to consider setting  $\delta F = 0$  then we must check that this does not restrict our metric perturbation to be zero. With  $\delta F = 0$  the perturbation equations reduce to

$$\begin{aligned} & \Delta_L h_{AB} - \frac{1}{(p-2)!} F_{ADC_1 \dots C_{p-2}} F_{BE}{}^{C_1 \dots C_{p-2}} h^{DE} \\ & - \frac{(p-1)}{p!(D-2)} [F^2 h_{AB} - p F_{DC_1 \dots C_{p-1}} F_E{}^{C_1 \dots C_{p-1}} h^{DE} g_{AB}] = 0, \end{aligned} \quad (6.15)$$

and

$$\begin{aligned} & (p-1) F^{AB[C_2 \dots C_{p-1}] \nabla_A h_B^{C_1]} + h_B^A \nabla_A F^{BC_1 \dots C_{p-1}} \\ & + \nabla_A \left( h_B^A - \frac{1}{2} h \delta_B^A \right) F^{BC_1 \dots C_{p-1}} = 0. \end{aligned} \quad (6.16)$$

Now we only wish to consider perturbations of the metric that are restricted to the AdS black hole factor, i.e.  $h_{\alpha a} = h_{ab} = 0$ . We also have the freedom to impose a gauge choice on the metric perturbations and we adopt the transverse-traceless gauge,  $\nabla_\beta h_\alpha^\beta = 0$  and  $h_\alpha^\alpha = 0$ . It will soon become clear that this is a requirement for our analysis to hold.

We now consider, in turn, each of the three separate ansätze for the field strength tensor.

### 6.2.1 Magnetic field strength (Schwarzschild-AdS<sub>7</sub> × S<sup>4</sup>)

The magnetic ansatz is one in which the field strength tensor is only non-zero on the sphere and so we consider  $p = n$  in the equations of motion. This solution arises in  $D = 11$  when we set  $n = 4$ , our equations of motion are then the relevant equations for the magnetic M5-brane of M-theory. We choose to leave the dimensionality factors arbitrary in what follows since then it can be seen that in each of the three cases for the field strength that we study, the equations of motion restrict us to the same general equations for Schwarzschild-AdS <sub>$m$</sub>  × S <sup>$n$</sup> .

The only non-zero components of the magnetic field strength are

$$F_{a_1 \dots a_n} = \sqrt{\frac{2(D-2)(n-1)}{m-1}} \frac{1}{\tilde{R}} \epsilon_{a_1 \dots a_n}, \quad (6.17)$$

where  $\epsilon$  is the volume form on the  $n$ -sphere.

With this field strength ansatz, equation (6.16) reduces to the following equation

$$(p-1)\epsilon^{ab[c_2\dots c_{p-1}]\nabla_a h_b^{c_1]} + h_b^a \nabla_a \epsilon^{bc_1\dots c_{p-1}} + \nabla_a \left( h_b^a - \frac{1}{2} h \delta_b^a \right) \epsilon^{bc_1\dots c_{p-1}} + h_b^\alpha \nabla_\alpha \epsilon^{bc_1\dots c_{p-1}} + \nabla_\alpha h_b^\alpha \epsilon^{bc_1\dots c_{p-1}} = 0. \quad (6.18)$$

Given that we have imposed localisation of our metric perturbation to the AdS part of the metric, then the only term in this equation which could possibly be non-zero is  $\nabla h$ , but this too is zero since we have imposed a TT gauge. Therefore it is consistent to set  $\delta F = 0$  when we adopt the magnetic field strength ansatz.

### 6.2.2 Electric field strength (Schwarzschild-AdS<sub>4</sub> × S<sup>7</sup>)

The electric ansatz is one in which the field strength tensor is only non-zero on the AdS black hole and so we consider  $p = m$  in the equations of motion. This solution arises in  $D = 11$  when we set  $m = 4$ , our equations of motion are then the relevant equations for the electric M2-brane of M-theory.

The only non-zero components of the electric field strength are

$$F_{\alpha_1\dots\alpha_m} = \sqrt{\frac{2(D-2)(m-1)}{n-1}} \frac{1}{R} \epsilon_{\alpha_1\dots\alpha_m}, \quad (6.19)$$

where  $\epsilon$  is the volume form on the Schwarzschild-AdS <sub>$m$</sub> .

As in the previous section we now consider the consistency of setting  $\delta F = 0$  by looking at equation (6.16) with this field strength ansatz. It reduces to the following equation

$$(p-1)\epsilon^{\alpha\beta[\gamma_2\dots\gamma_{p-1}]\nabla_\alpha h_\beta^{\gamma_1]} + h_\beta^\alpha \nabla_\alpha \epsilon^{\beta\gamma_1\dots\gamma_{p-1}} + \nabla_\alpha \left( h_\beta^\alpha - \frac{1}{2} h \delta_\beta^\alpha \right) \epsilon^{\beta\gamma_1\dots\gamma_{p-1}} + h_\beta^\alpha \nabla_\alpha \epsilon^{\beta\gamma_1\dots\gamma_{p-1}} + \nabla_\alpha h_\beta^\alpha \epsilon^{\beta\gamma_1\dots\gamma_{p-1}} = 0. \quad (6.20)$$

The final two terms of this equation are zero by the localisation of our metric perturbation. Furthermore, our choice of gauge ensures that the third term is also zero, and the second term is zero since the volume form is covariantly constant. The only non-trivial part of ensuring that we can set  $\delta F = 0$  in this electric case is demonstrating that the first term of this equation is zero. However, it can be shown that

$$\epsilon^{\alpha\beta[\gamma_2\dots\gamma_{p-1}]\nabla_\alpha h_\beta^{\gamma_1]} \propto \nabla_\alpha h_{\gamma_p}^\alpha - \nabla_{\gamma_p} h, \quad (6.21)$$

where  $\gamma_p$  is the remaining coordinate which does not appear in  $\gamma_1 \dots \gamma_{p-1}$ . It is then clear that this term is also equal to zero as a result of the gauge choice for our metric perturbations. Thus even in the electric case, which was not considered by Reall [27] in his study of charged black branes, we can once again simplify the perturbative analysis by assuming that unstable perturbations only appear in the metric perturbations.

### 6.2.3 Self-dual field strength (Schwarzschild-AdS<sub>5/3</sub> × S<sup>5/3</sup>)

Equation (6.16) is linear in  $F$ . Therefore, when we consider a self-dual field strength which is a sum of a magnetic field strength and an electric field strength, we can see that, if it is possible to set  $\delta F = 0$  in the magnetic and electric cases alone, then it is also possible to set  $\delta F = 0$  in the self-dual case. We need to assume that any field strength perturbation is also self-dual, but this seems a reasonable assumption as we would want to maintain self-duality even for the perturbed field strength.

The self-dual field strength ansatz is adopted in  $D = 10$ , when we set  $n = 5$ . Our equations then become the relevant equations for the D3-brane of type IIB string theory. Furthermore, we can also adopt this ansatz with  $n = 3$  in  $D = 6$ . This arises for intersecting D1-branes and D5-branes in type IIB string theory compactified on a 4-dimensional manifold.

The ansatz is that

$$F_n = G_n + *G_n, \quad (6.22)$$

where

$$G_{\alpha_1 \dots \alpha_n} = \sqrt{D-2} \frac{1}{R} \epsilon_{\alpha_1 \dots \alpha_n}. \quad (6.23)$$

### 6.3 The threshold unstable mode and the negative mode

Substituting any of the above field strengths into the equations of motion results in (6.6) being satisfied trivially, whilst (6.5) reduces to

$$R_{\alpha\beta} = -\frac{m-1}{R^2}g_{\alpha\beta} \quad (6.24)$$

$$R_{ab} = \frac{n-1}{\tilde{R}^2}g_{ab} \quad (6.25)$$

$$R_{\alpha a} = 0 \quad (6.26)$$

We now know that it is consistent to only consider metric perturbations and equation (6.12) reduces to the following

$$\Delta_L h_{\alpha\beta} - \frac{2(n-1)^2}{(m-1)\tilde{R}^2}(h_{\alpha\beta} - h_c^c g_{\alpha\beta}) = 0 \quad (6.27)$$

$$\Delta_L h_{\alpha a} - \frac{2(n-1)^2}{(m-1)\tilde{R}^2}h_{\alpha a} = 0 \quad (6.28)$$

$$\Delta_L h_{ab} + \frac{2(n-1)}{\tilde{R}^2}(h_{ab} - h_c^c g_{ab}) = 0. \quad (6.29)$$

As already stated we are only considering perturbations that are localised on the AdS black hole, i.e.  $h_{\alpha a} = h_{ab} = 0$ . It is necessary to check that we can still solve the metric perturbation equations without this ansatz enforcing only trivial perturbations of the black hole. The restrictions which arise on  $h_{\alpha\beta}$  are

$$(6.28) \Rightarrow (\nabla_\beta h_\alpha^\beta)_{,a} = 0, \quad (6.30)$$

$$(6.29) \Rightarrow h_{\alpha,ab}^\alpha - \Gamma_{ab}^c h_{\alpha,c}^\alpha = 0. \quad (6.31)$$

Since we are working in the transverse-traceless gauge for our perturbations,  $\nabla_\beta h_\alpha^\beta = 0$  and  $h_\alpha^\alpha = 0$ , then it is clear that these equations are obeyed and it only remains to find perturbations satisfying equation (6.27) which reduces to

$$\Delta_L h_{\alpha\beta} = \frac{2(n-1)^2}{(m-1)\tilde{R}^2}h_{\alpha\beta}. \quad (6.32)$$

The term on the right hand side of this equation is introduced by the fact that our black hole is smeared over a sphere and not a planar space. We return to the importance of this term in this derivation later.

Since our AdS black hole is smeared over a sphere, we can decompose our perturbation into spherical harmonics,  $Y^k(y)$ , in the following manner

$$h_{\alpha\beta}(x, y) = \sum_k H_{\alpha\beta}^k(x) Y^k(y). \quad (6.33)$$

Using this decomposition of  $h$ , we see that we can rewrite this in terms of the Lichnerowicz operator for the AdS black hole  $\Delta_L^x$ .

$$(\Delta_L^x H_{\alpha\beta}^k) Y^k + H_{\alpha\beta}^k \square_y Y^k = \frac{2(n-1)^2}{(m-1)\tilde{R}^2} H_{\alpha\beta}^k Y^k, \quad (6.34)$$

where the  $m$ -dimensional Lichnerowicz operator is

$$\Delta_L^x H_{\alpha\beta} = \square_x H_{\alpha\beta} + 2R_{\alpha\gamma\beta\delta} H^{\gamma\delta} + \frac{2(n-1)^2}{(m-1)\tilde{R}^2} H_{\alpha\beta}. \quad (6.35)$$

$Y^k$  is an eigenvector of  $\square_y$  satisfying the equation

$$\square_y Y^k = -\frac{\lambda^k}{\tilde{R}^2} Y^k, \quad (6.36)$$

where

$$\lambda^k = k(k+n-1) \quad \text{for } k \geq 0, \quad (6.37)$$

and, given the uniqueness of decomposition into spherical harmonics, the equation for  $H^k$  is true individually for each  $k$

$$\left( -\Delta_L^x + \frac{2(n-1)^2}{(m-1)\tilde{R}^2} \right) H_{\alpha\beta}^k = -\frac{\lambda^k}{\tilde{R}^2} H_{\alpha\beta}^k. \quad (6.38)$$

A time-independent solution to this equation is the threshold unstable mode representing the boundary between unstable and stable modes. The existence of the threshold unstable mode therefore ensures the presence of a classical instability. We do not consider the zero mode ( $k=0$ ) to represent a physical threshold unstable mode. Therefore the eigenvalues we consider are strictly positive so we can rewrite them as  $\frac{\lambda^k}{\tilde{R}^2} = \mu_k^2$ . Given a threshold unstable mode we can Wick rotate equation (6.38) to find

$$\left( -\Delta_L^{Eucl} + \frac{2(n-1)^2}{(m-1)\tilde{R}^2} \right) H_{\alpha\beta}^k = -\mu_k^2 H_{\alpha\beta}^k. \quad (6.39)$$

The action of the operator  $\Delta_L^{Eucl}$  is again given by equation (6.35), only now we implicitly understand that we are in Euclidean signature.

Prestidge studied the stability of AdS black holes [120] and demonstrated that the Euclidean negative mode is a solution to equation (2.47) with negative eigenvalue, say  $\lambda = -\mu^2$ . The fact that we are in asymptotically anti-de Sitter space does not alter the operator which defines the Euclidean negative modes. We rewrite equation (2.47) here in a manner which is more easily comparable to the equation for the classical instability.

$$-\square_x h_{\alpha\beta} - 2R_{\alpha\gamma\beta\delta} h^{\gamma\delta} = -\mu^2 h_{\alpha\beta} \quad (6.40)$$

Using (6.35), it is clear that (6.39) and (6.40) are identical, and hence the Euclidean negative mode and the threshold unstable mode satisfy the same equation.

In Reall's proof of the Gubser-Mitra conjecture [27], the connection between the equations for the Lorentzian threshold unstable mode of the black brane and the Euclidean negative mode of the dimensionally-reduced black hole was more trivial since the operator governing negative modes, was identical to the Lichnerowicz operator of the black hole. It is less obvious that we should expect the connection between the two equations which we demonstrate here. As noted earlier, the smearing of the black hole over a sphere, instead of a flat space, introduced an extra term on the right hand side of equation (6.32). This term is evident in the equation (6.39) which defines the threshold unstable mode and hence the presence of a classical instability. However, it is also present in the Lichnerowicz operator (6.35). The cancellation between the two appearances of this term results in the exact matching of the equation for a classical instability (6.39), and the equation for the Euclidean negative mode (6.40).

## 6.4 Conclusions

In this chapter we have demonstrated the equivalence of classical and thermodynamic stability of spacetimes of the form, Schwarzschild-AdS<sub>m</sub> × S<sup>n</sup>. This is a non-trivial extension of the case of branes with translationally invariant worldspace directions – the Lichnerowicz operator and the Euclidean mode operator in this case are not identical. The sphere induces an extra term in the Lichnerowicz operator, which is then cancelled by the extra term which appears in the equation defining

metric perturbations (6.32). One should note that when the radius of curvature of the sphere is taken to infinity, this extra term tends to zero as we are returning to the case of translational invariance described by the Gubser-Mitra conjecture.

The calculations in this chapter thus support a generalisation of the Gubser-Mitra conjecture to include additional directions with different symmetries rather than translational invariance. It would be interesting to study further generalisations of the conjecture, but this work is indication that the connection between classical and thermodynamic stability of a supergravity background is a general phenomenon.

# Chapter 7

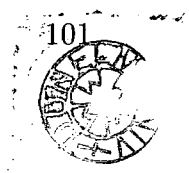
## Conclusions and Discussion

The history of the living world can be summarised as the elaboration of ever more perfect eyes within a cosmos in which there is always something more to be seen.

— Pierre Teilhard de Chardin, *The Phenomenon of Man* (1959).

We have used the AdS/CFT correspondence in a variety of ways to begin to investigate the questions regarding black hole physics that were posed at the beginning of this thesis. In this chapter, we will review the findings of our studies, and discuss prospects for further work in this field.

In chapter 3, we considered how the AdS/CFT correspondence could encode the non-trivial global causal structure of black hole spacetimes. Our study demonstrated the difficulties associated with unravelling the holographic encoding of a bulk spacetime – such difficulties are the hallmark of the major conceptual leap which was required to arrive at the AdS/CFT correspondence. We successfully demonstrated that UV/IR relations in black hole spacetimes did not follow directly from a causal propagation argument in the field theory. In fact, the study of bulk probes could not give a unique generalisation of the UV/IR relation of pure AdS space, since the field theory scale sizes associated with the probes were dependent on the probe being studied. We believe that the AdS/CFT correspondence should provide a representation of the global causal structure of spacetime in terms of a field theory, but to unveil the precise nature of this representation requires further investigation.



The reward for achieving this understanding is undoubtedly exciting – following an understanding of the structure of spacetime, we could further study the dynamical evolution of black holes to answer more of the questions associated with black hole physics.

In chapter 4, our perspective on the correspondence was reversed. We studied the cosmology of a braneworld induced by a higher dimensional bulk AdS black hole spacetime. Studying matter filled braneworlds in AdS space led Binetruy *et al* to discover an unconventional cosmology on the brane [23, 24]. We have discovered a new way of arriving at this same cosmology without inserting matter onto the braneworld – all the physics of our brane universe is induced by the bulk black hole and in this model there is no need for an arbitrary splitting of the source term into brane tension and matter. This study extends the concept of correspondences between gravitational theories in bulk spacetimes and boundary theories. Although the beauty of Maldacena’s decoupling of the gravity and gauge theory modes in the D-brane construction of the correspondence is lost in this braneworld model, the fact that the braneworld correspondence can produce realistic cosmologies will undoubtedly continue to be of great interest.

The four laws of black hole physics, which we reviewed in chapter 1, relate the global structure of black hole spacetimes to thermodynamics. The AdS/CFT correspondence was born out of an effort to give a quantum realisation of this correspondence. In chapters 5 and 6, we studied a new resulting connection between the classical dynamics of black branes and their thermodynamics, which emerged from the AdS/CFT correspondence. In chapter 5, we successfully demonstrated that the onset of dynamic and thermodynamic instabilities do indeed occur at the same time for black holes in a finite cavity – this is a crucial step in proving Gubser and Mitra’s conjectured equivalence of the two notions of stability for black branes. Their correspondence was conjectured for branes which have translationally invariant directions, but we extended this conjecture to additional supergravity backgrounds in chapter 6. This work is substantial support for the idea that thermodynamics and classical dynamics are connected in a general manner and not just for specific examples.

The examples of the use of the AdS/CFT correspondence that we have seen, demonstrate the powerful potential of further application of gravity/gauge theory correspondences in the study of black hole physics. Our study of the field theory encoding of bulk causality leads to further open questions. We found difficulty in uncovering the field theory representation of the event horizon, because of the nature of the correspondence – events in the bulk spacetime are only recognised in the gauge theory following their propagation to the boundary. To propagate from the event horizon to the boundary takes infinite time and thus probing the horizon and beyond it requires a new understanding of the holographic encoding. Gauge theory operators which do not require causal propagation to the boundary are required, but early studies of these “precursors” [104–106] have not yet revealed the guise of these operators in the gauge theory. If we could achieve this understanding a study of the issue of *black hole complementarity* [127, 128] could be undertaken – this proposes that freely falling observers who cross the horizon observe the same set of degrees of freedom as external observers who remain outside the horizon. Precursors would potentially allow us to realise this principle in the gauge theory.

Our new understanding of the braneworld cosmology induced from bulk AdS black holes could also have further applications. Verlinde noticed a remarkable connection between the FRW equations of the brane universe and the generalised Cardy formula for the dual description of the field theory on the brane [65, 66]. It was argued that this relation sheds light on the meaning of the holographic principle in a cosmological setting [129] – the braneworld field theory incorporates the holographic entropy bounds in the cosmological evolution of the brane universe. The new relationship we have proposed includes non-linear terms in both the Cardy-Verlinde formula and the FRW equations. We have demonstrated equivalence of these two results in this non-linear case, but it would be of great interest to be able to re-interpret our generalised Cardy-Verlinde formula in terms of holographic entropy bounds. This would hopefully lead to a greater understanding of holography and cosmology. The interpretation of the effect of a non-zero braneworld cosmological constant is still yet to be made in both the linear and non-linear cases.

Gubser and Mitra’s conjectured equivalence of classical and thermodynamic sta-

bility of black branes answers questions about the dynamical evolution of the branes – this has been further studied in [78–80]. It is now proposed that the endpoint of the evolution of a brane exhibiting a classical Gregory-Laflamme instability, is a non-uniform black brane. The explicit forms of these solutions are currently unknown, what we know is that these higher-dimensional analogues to black holes exhibit a rich structure of possible forms, and it would be interesting to try to understand this structure. Furthermore, the Gubser-Mitra conjecture again interlinks the classical and quantum properties of black holes, since in quantum gravity we believe the thermodynamics to be derived in a traditional statistical mechanical manner. Arising from a quantum theory of gravity, the gravity/gauge theory correspondence is the new synthesis of these connections between the quantum and classical physics of black holes, and it clearly provides much hope for developing a complete understanding of black holes in quantum gravity.

# Bibliography

- [1] J. P. Gregory and S. F. Ross, “Looking for event horizons using UV/IR relations,” *Phys. Rev.* **D63** (2001) 104023, [hep-th/0012135](#).
- [2] J. P. Gregory and S. F. Ross, “Stability and the negative mode for Schwarzschild in a finite cavity,” *Phys. Rev.* **D64** (2001) 124006, [hep-th/0106220](#).
- [3] J. P. Gregory and A. Padilla, “Exact braneworld cosmology induced from bulk black holes,” [hep-th/0204218](#).
- [4] A. Einstein, “The foundation of the general theory of relativity,” *Annalen Phys.* **49** (1916) 769–822.
- [5] K. Schwarzschild, “On the gravitational field of a mass point according to Einstein’s theory,” *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys. )* **1916** (1916) 189–196, [physics/9905030](#).
- [6] S. W. Hawking, “Black holes in general relativity,” *Commun. Math. Phys.* **25** (1972) 152–166.
- [7] J. M. Bardeen, B. Carter, and S. W. Hawking, “The four laws of black hole mechanics,” *Commun. Math. Phys.* **31** (1973) 161–170.
- [8] S. W. Hawking, “Gravitational radiation from colliding black holes,” *Phys. Rev. Lett.* **26** (1971) 1344–1346.
- [9] J. D. Bekenstein, “Black holes and entropy,” *Phys. Rev.* **D7** (1973) 2333–2346.

- [10] J. D. Bekenstein, "Generalized second law of thermodynamics in black hole physics," *Phys. Rev.* **D9** (1974) 3292–3300.
- [11] S. W. Hawking, "Particle creation by black holes," *Commun. Math. Phys.* **43** (1975) 199–220.
- [12] S. W. Hawking, "Black holes and thermodynamics," *Phys. Rev.* **D13** (1976) 191–197.
- [13] S. W. Hawking, "Breakdown of predictability in gravitational collapse," *Phys. Rev.* **D14** (1976) 2460–2473.
- [14] M. B. Green, J. H. Schwarz, and E. Witten, "Superstring theory. Vols. 1 and 2," . Cambridge, UK: Univ. Pr. (1987) (Cambridge Monographs On Mathematical Physics).
- [15] J. Polchinski, "String theory. Vols. 1 and 2," . Cambridge, UK: Univ. Pr. (1998) (Cambridge Monographs On Mathematical Physics).
- [16] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," *Phys. Lett.* **B379** (1996) 99–104, [hep-th/9601029](#).
- [17] J. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200](#).
- [18] G. 't Hooft, "Dimensional reduction in quantum gravity," [gr-qc/9310026](#).
- [19] L. Susskind, "The world as a hologram," *J. Math. Phys.* **36** (1995) 6377–6396, [hep-th/9409089](#).
- [20] V. A. Rubakov and M. E. Shaposhnikov, "Do we live inside a domain wall?," *Phys. Lett.* **B125** (1983) 136–138.
- [21] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, "The hierarchy problem and new dimensions at a millimeter," *Phys. Lett.* **B429** (1998) 263–272, [hep-ph/9803315](#).

- [22] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, “New dimensions at a millimeter to a Fermi and superstrings at a TeV,” *Phys. Lett.* **B436** (1998) 257–263, [hep-ph/9804398](#).
- [23] P. Binetruy, C. Deffayet, and D. Langlois, “Non-conventional cosmology from a brane-universe,” *Nucl. Phys.* **B565** (2000) 269–287, [hep-th/9905012](#).
- [24] P. Binetruy, C. Deffayet, U. Ellwanger, and D. Langlois, “Brane cosmological evolution in a bulk with cosmological constant,” *Phys. Lett.* **B477** (2000) 285–291, [hep-th/9910219](#).
- [25] S. S. Gubser and I. Mitra, “Instability of charged black holes in anti-de Sitter space,” [hep-th/0009126](#).
- [26] S. S. Gubser and I. Mitra, “The evolution of unstable black holes in anti-de Sitter space,” *JHEP* **08** (2001) 018, [hep-th/0011127](#).
- [27] H. S. Reall, “Classical and thermodynamic stability of black branes,” *Phys. Rev.* **D64** (2001) 044005, [hep-th/0104071](#).
- [28] U. Danielsson, “Introduction to string theory,” *Rept. Prog. Phys.* **64** (2001) 51–96.
- [29] A. Sen, “An introduction to non-perturbative string theory,” [hep-th/9802051](#).
- [30] C. V. Johnson, “D-brane primer,” [hep-th/0007170](#).
- [31] C. V. Johnson, “D-branes,” Cambridge, UK: Univ. Pr. (2002) 544 p. (Cambridge Monographs On Mathematical Physics).
- [32] J. Dai, R. G. Leigh, and J. Polchinski, “New connections between string theories,” *Mod. Phys. Lett.* **A4** (1989) 2073–2083.
- [33] R. G. Leigh, “Dirac-Born-Infeld action from Dirichlet sigma model,” *Mod. Phys. Lett.* **A4** (1989) 2767.

- [34] J. Polchinski, “Dirichlet-branes and Ramond-Ramond charges,” *Phys. Rev. Lett.* **75** (1995) 4724–4727, [hep-th/9510017](#).
- [35] E. Witten, “Bound states of strings and p-branes,” *Nucl. Phys.* **B460** (1996) 335–350, [hep-th/9510135](#).
- [36] G. ’t Hooft, “A planar diagram theory for strong interactions,” *Nucl. Phys.* **B72** (1974) 461.
- [37] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150](#).
- [38] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” *Phys. Lett.* **B428** (1998) 105, [hep-th/9802109](#).
- [39] L. Susskind and E. Witten, “The holographic bound in anti-de Sitter space,” [hep-th/9805114](#).
- [40] T. Banks, M. R. Douglas, G. T. Horowitz, and E. Martinec, “AdS dynamics from conformal field theory,” [hep-th/9808016](#).
- [41] V. Balasubramanian, P. Kraus, and A. Lawrence, “Bulk vs. boundary dynamics in anti-de Sitter spacetime,” *Phys. Rev. D* **59** (1999) 046003, [hep-th/9805171](#).
- [42] V. Balasubramanian, P. Kraus, A. Lawrence, and S. P. Trivedi, “Holographic probes of anti-de Sitter space-times,” *Phys. Rev. D* **59** (1999) 104021, [hep-th/9808017](#).
- [43] V. Balasubramanian and S. F. Ross, “Holographic particle detection,” *Phys. Rev. D* **61** (2000) 044007, [hep-th/9906226](#).
- [44] S. B. Giddings and S. F. Ross, “D3-brane shells to black branes on the Coulomb branch,” *Phys. Rev. D* **61** (2000) 024036, [hep-th/9907204](#).
- [45] A. W. Peet and J. Polchinski, “UV/IR relations in AdS dynamics,” *Phys. Rev. D* **59** (1999) 065011, [hep-th/9809022](#).

- [46] J. Distler and F. Zamora, “Non-supersymmetric conformal field theories from stable anti-de Sitter spaces,” *Adv. Theor. Math. Phys.* **2** (1999) 1405, [hep-th/9810206](#).
- [47] L. Girardello, M. Petrini, M. Porrati, and A. Zaffaroni, “Novel local CFT and exact results on perturbations of  $N = 4$  super Yang-Mills from AdS dynamics,” *JHEP* **12** (1998) 022, [hep-th/9810126](#).
- [48] D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner, “Renormalization group flows from holography supersymmetry and a c-theorem,” *Adv. Theor. Math. Phys.* **3** (1999) 363–417, [hep-th/9904017](#).
- [49] I. R. Klebanov and E. Witten, “AdS/CFT correspondence and symmetry breaking,” *Nucl. Phys.* **B556** (1999) 89, [hep-th/9905104](#).
- [50] S. W. Hawking and D. N. Page, “Thermodynamics of black holes in anti-de Sitter space,” *Commun. Math. Phys.* **87** (1983) 577.
- [51] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, “Charged AdS black holes and catastrophic holography,” *Phys. Rev. D* **60** (1999) 064018, [hep-th/9902170](#).
- [52] L. Randall and R. Sundrum, “An alternative to compactification,” *Phys. Rev. Lett.* **83** (1999) 4690–4693, [hep-th/9906064](#).
- [53] L. Randall and R. Sundrum, “A large mass hierarchy from a small extra dimension,” *Phys. Rev. Lett.* **83** (1999) 3370–3373, [hep-ph/9905221](#).
- [54] S. S. Gubser, “AdS/CFT and gravity,” *Phys. Rev.* **D63** (2001) 084017, [hep-th/9912001](#).
- [55] P. Horava and E. Witten, “Heterotic and type I string dynamics from eleven dimensions,” *Nucl. Phys.* **B460** (1996) 506–524, [hep-th/9510209](#).
- [56] P. Horava and E. Witten, “Eleven-dimensional supergravity on a manifold with boundary,” *Nucl. Phys.* **B475** (1996) 94–114, [hep-th/9603142](#).

- [57] **Supernova Search Team** Collaboration, A. G. Riess *et. al.*, “Observational evidence from supernovae for an accelerating universe and a cosmological constant,” *Astron. J.* **116** (1998) 1009–1038, [astro-ph/9805201](#).
- [58] **Supernova Cosmology Project** Collaboration, S. Perlmutter *et. al.*, “Measurements of omega and lambda from 42 high-redshift supernovae,” *Astrophys. J.* **517** (1999) 565–586, [astro-ph/9812133](#).
- [59] T. Shiromizu, K. Maeda, and M. Sasaki, “The Einstein equations on the 3-brane world,” *Phys. Rev.* **D62** (2000) 024012, [gr-qc/9910076](#).
- [60] G. Uchida and M. Sasaki, “Radion on the de Sitter brane,” *Prog. Theor. Phys.* **105** (2001) 591–606.
- [61] R. Gregory and A. Padilla, “Nested braneworlds and strong brane gravity,” *Phys. Rev.* **D65** (2002) 084013, [hep-th/0104262](#).
- [62] R. Gregory and A. Padilla, “Braneworld instantons,” *Class. Quant. Grav.* **19** (2002) 279–302, [hep-th/0107108](#).
- [63] P. Singh and N. Dadhich, “Localization of gravity in brane world cosmologies,” [hep-th/0204190](#).
- [64] A. Hebecker and J. March-Russell, “Randall-Sundrum II cosmology, AdS/CFT, and the bulk black hole,” *Nucl. Phys.* **B608** (2001) 375–393, [hep-ph/0103214](#).
- [65] E. Verlinde, “On the holographic principle in a radiation dominated universe,” [hep-th/0008140](#).
- [66] I. Savonije and E. Verlinde, “CFT and entropy on the brane,” *Phys. Lett.* **B507** (2001) 305–311, [hep-th/0102042](#).
- [67] R.-G. Cai, “The Cardy-Verlinde formula and AdS black holes,” *Phys. Rev.* **D63** (2001) 124018, [hep-th/0102113](#).
- [68] A. K. Biswas and S. Mukherji, “Holography and stiff-matter on the brane,” *JHEP* **03** (2001) 046, [hep-th/0102138](#).

- [69] Y. S. Myung, "Standard cosmology from the brane cosmology with a localized matter," hep-th/0103241.
- [70] R.-G. Cai, Y. S. Myung, and N. Ohta, "Bekenstein bound, holography and brane cosmology in charged black hole background," *Class. Quant. Grav.* **18** (2001) 5429–5440, hep-th/0105070.
- [71] D. Youm, "The Cardy-Verlinde formula and charged topological AdS black holes," *Mod. Phys. Lett.* **A16** (2001) 1327–1334, hep-th/0105249.
- [72] B. Wang, E. Abdalla, and R.-K. Su, "Friedmann equation and Cardy formula correspondence in brane universes," *Mod. Phys. Lett.* **A17** (2002) 23–30, hep-th/0106086.
- [73] P. Creminelli, "Holography of asymmetrically warped space-times," *Phys. Lett.* **B532** (2002) 284–290, hep-th/0111107.
- [74] S. W. Hawking, "The path-integral approach to quantum gravity," in *General Relativity. An Einstein Centenary Survey*, S. W. Hawking and W. Israel, eds., p. 919. Cambridge, UK: Univ. Pr. (1979).
- [75] D. J. Gross, M. J. Perry, and L. G. Yaffe, "Instability of flat space at finite temperature," *Phys. Rev. D* **25** (1982) 330–355.
- [76] R. Gregory and R. Laflamme, "Black strings and p-branes are unstable," *Phys. Rev. Lett.* **70** (1993) 2837, hep-th/9301052.
- [77] R. Gregory and R. Laflamme, "The instability of charged black strings and p-branes," *Nucl. Phys.* **B428** (1994) 399–434, hep-th/9404071.
- [78] G. T. Horowitz and K. Maeda, "Fate of the black string instability," *Phys. Rev. Lett.* **87** (2001) 131301, hep-th/0105111.
- [79] S. S. Gubser, "On non-uniform black branes," hep-th/0110193.
- [80] G. T. Horowitz, "Playing with black strings," hep-th/0205069.

- [81] S. S. Gubser, "Thermodynamics of spinning D3-branes," Nucl. Phys. **B551** (1999) 667–684, [hep-th/9810225](#).
- [82] D. Kabat and G. Lifschytz, "Gauge theory origins of supergravity causal structure," JHEP **05** (1999) 005, [hep-th/9902073](#).
- [83] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, "Holography, thermodynamics and fluctuations of charged AdS black holes," Phys. Rev. D **60** (1999) 104026, [hep-th/9904197](#).
- [84] S. W. Hawking and H. S. Reall, "Charged and rotating AdS black holes and their CFT duals," Phys. Rev. D **61** (2000) 024014, [hep-th/9908109](#).
- [85] C. Csaki, Y. Oz, J. Russo, and J. Terning, "Large N QCD from rotating branes," Phys. Rev. D **59** (1999) 065012, [hep-th/9810186](#).
- [86] J. G. Russo and K. Sfetsos, "Rotating D3 branes and QCD in three dimensions," Adv. Theor. Math. Phys. **3** (1999) 131, [hep-th/9901056](#).
- [87] A. Brandhuber, N. Itzhaki, J. Sonnenschein, and S. Yankielowicz, "Wilson loops in the large N limit at finite temperature," Phys. Lett. **B434** (1998) 36–40, [hep-th/9803137](#).
- [88] S.-J. Rey, S. Theisen, and J.-T. Yee, "Wilson-Polyakov loop at finite temperature in large N gauge theory and anti-de Sitter supergravity," Nucl. Phys. **B527** (1998) 171, [hep-th/9803135](#).
- [89] S.-J. Rey and J. Yee, "Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity," Eur. Phys. J. **C22** (2001) 379–394, [hep-th/9803001](#).
- [90] J. Maldacena, "Wilson loops in large N field theories," Phys. Rev. Lett. **80** (1998) 4859–4862, [hep-th/9803002](#).
- [91] J. Sonnenschein, "What does the string / gauge correspondence teach us about Wilson loops?," [hep-th/0003032](#).

- [92] J. Maldacena, "Probing near extremal black holes with D-branes," *Phys. Rev. D* **57** (1998) 3736–3741, [hep-th/9705053](#).
- [93] D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner, "Continuous distributions of D3-branes and gauged supergravity," *JHEP* **07** (2000) 038, [hep-th/9906194](#).
- [94] A. Brandhuber and K. Sfetsos, "Wilson loops from multicentre and rotating branes, mass gaps and phase structure in gauge theories," *Adv. Theor. Math. Phys.* **3** (1999) 851–887, [hep-th/9906201](#).
- [95] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," *Adv. Theor. Math. Phys.* **2** (1998) 505, [hep-th/9803131](#).
- [96] A. Brandhuber, N. Itzhaki, J. Sonnenschein, and S. Yankielowicz, "Wilson loops, confinement, and phase transitions in large N gauge theories from supergravity," *JHEP* **06** (1998) 001, [hep-th/9803263](#).
- [97] U. H. Danielsson, E. Keski-Vakkuri, and M. Kruczenski, "Vacua, propagators, and holographic probes in AdS/CFT," *JHEP* **01** (1999) 002, [hep-th/9812007](#).
- [98] G. T. Horowitz and N. Itzhaki, "Black holes, shock waves, and causality in the AdS/CFT correspondence," *JHEP* **02** (1999) 010, [hep-th/9901012](#).
- [99] M. Banados, C. Teitelboim, and J. Zanelli, "The black hole in three-dimensional space-time," *Phys. Rev. Lett.* **69** (1992) 1849–1851, [hep-th/9204099](#).
- [100] C. Csaki, H. Ooguri, Y. Oz, and J. Terning, "Glueball mass spectrum from supergravity," *JHEP* **01** (1999) 017, [hep-th/9806021](#).
- [101] J. A. Minahan, "Glueball mass spectra and other issues for supergravity duals of QCD models," *JHEP* **01** (1999) 020, [hep-th/9811156](#).
- [102] V. Husain, "Naked singularities and the Wilson loop," [hep-th/0204180](#).

- [103] D. A. Macdonald, R. H. Price, W.-M. Suen, and K. S. Thorne, “Nonrotating and slowly rotating black holes,” in *Black holes: the membrane paradigm*, R. H. P. Kip S. Thorne and D. A. Macdonald, eds., p. 13. Yale University Press, New Haven, USA, 1986.
- [104] J. Polchinski, L. Susskind, and N. Toumbas, “Negative energy, superluminality and holography,” *Phys. Rev.* **D60** (1999) 084006, [hep-th/9903228](#).
- [105] L. Susskind and N. Toumbas, “Wilson loops as precursors,” *Phys. Rev.* **D61** (2000) 044001, [hep-th/9909013](#).
- [106] S. B. Giddings and M. Lippert, “Precursors, black holes, and a locality bound,” *Phys. Rev.* **D65** (2002) 024006, [hep-th/0103231](#).
- [107] P. Kraus, “Dynamics of anti-de Sitter domain walls,” *JHEP* **12** (1999) 011, [hep-th/9910149](#).
- [108] A. Karch and L. Randall, “Locally localized gravity,” *JHEP* **05** (2001) 008, [hep-th/0011156](#).
- [109] S. W. Hawking, T. Hertog, and H. S. Reall, “Brane new world,” *Phys. Rev.* **D62** (2000) 043501, [hep-th/0003052](#).
- [110] S. Nojiri and S. D. Odintsov, “Brane world inflation induced by quantum effects,” *Phys. Lett.* **B484** (2000) 119–123, [hep-th/0004097](#).
- [111] S. Nojiri and S. D. Odintsov, “AdS/CFT and quantum-corrected brane entropy,” *Class. Quant. Grav.* **18** (2001) 5227–5238, [hep-th/0103078](#).
- [112] S. W. Hawking and S. F. Ross, “Duality between electric and magnetic black holes,” *Phys. Rev.* **D52** (1995) 5865–5876, [hep-th/9504019](#).
- [113] P. Bowcock, C. Charmousis, and R. Gregory, “General brane cosmologies and their global spacetime structure,” *Class. Quant. Grav.* **17** (2000) 4745–4764, [hep-th/0007177](#).

- [114] A. C. Petkou and G. Siopsis, “dS/CFT correspondence on a brane,” JHEP **02** (2002) 045, hep-th/0111085.
- [115] A. Padilla, “CFTs on non-critical braneworlds,” Phys. Lett. **B528** (2002) 274–282, hep-th/0111247.
- [116] S. W. Hawking and G. T. Horowitz, “The gravitational hamiltonian, action, entropy and surface terms,” Class. Quant. Grav. **13** (1996) 1487–1498, gr-qc/9501014.
- [117] J. L. Cardy, “Operator content of two-dimensional conformally invariant theories,” Nucl. Phys. **B270** (1986) 186–204.
- [118] M. Cvetič and S. S. Gubser, “Phases of R-charged black holes, spinning branes and strongly coupled gauge theories,” JHEP **04** (1999) 024, hep-th/9902195.
- [119] A. J. M. Medved, “A note on holography on a curved brane,” hep-th/0112009.
- [120] T. Prestidge, “Dynamic and thermodynamic stability and negative modes in Schwarzschild-anti-de Sitter,” Phys. Rev. D **61** (2000) 084002, hep-th/9907163.
- [121] T. Prestidge, “Making sense of the information loss paradox,” University of Cambridge, Ph.D. Thesis (2000).
- [122] J. W. York, Jr., “Black hole thermodynamics and the Euclidean Einstein action,” Phys. Rev. D **33** (1986) 2092–2099.
- [123] B. Allen, “Euclidean Schwarzschild negative mode,” Phys. Rev. D **30** (1984) 1153–1157.
- [124] T. Hirayama and G. Kang, “Stable black strings in anti-de Sitter space,” Phys. Rev. **D64** (2001) 064010, hep-th/0104213.
- [125] V. E. Hubeny and M. Rangamani, “Unstable horizons,” JHEP **05** (2002) 027, hep-th/0202189.

- [126] R. Emparan, “AdS/CFT duals of topological black holes and the entropy of zero-energy states,” *JHEP* **06** (1999) 036, [hep-th/9906040](#).
- [127] L. Susskind, L. Thorlacius, and J. Uglum, “The stretched horizon and black hole complementarity,” *Phys. Rev.* **D48** (1993) 3743–3761, [hep-th/9306069](#).
- [128] L. Susskind and L. Thorlacius, “Gedanken experiments involving black holes,” *Phys. Rev.* **D49** (1994) 966–974, [hep-th/9308100](#).
- [129] W. Fischler and L. Susskind, “Holography and cosmology,” [hep-th/9806039](#).

