-adic Fourier analysis

M. G. T. Scanlon

p-adic Fourier Analysis

M. G. T. Scanlon

A thesis submitted for the
degree of Doctor of Philosophy
at the University of Durham

September 2003

Abstract

Let $\mathfrak{O}_K$ be the ring of integers of a finite extension of $\mathbf{Q}_p$, and let $h \in \mathbf{Q}_{\geqslant 0}$ be in its value group. This thesis considers the space of locally analytic functions of order h on $\mathfrak{O}_K$ with values in $\mathbf{C}_p$: that is, functions that are defined on each disc of radius p^{-h} by a convergent power series. A necessary and sufficient condition for a sequence of polynomials, with coefficient in $\mathbf{C}_p$, to be orthogonal in this space is given, generalising a result of Amice [1]. This condition is used to prove that a particular sequence of polynomials defined in Schneider Teitelbaum [19] is not orthogonal.

p-adic Fourier Analysis

M. G. T. Scanlon

A thesis submitted for the
degree of Doctor of Philosophy
at the University of Durham

September 2003

Michael Gerard Thomas Scanlon
Geometry and Arithmetic Group
Department of Mathematical Sciences
University of Durham

To my parents

Contents

Declaration

I declare that no part of this thesis has been previously submitted by me for a degree or other qualification in this or any other university. It is the result of my own work and includes nothing that is the outcome of work done in collaboration. All material derived from the work of others has been suitably indicated.

M. G. T. Scanlon
Cambridge
September 2003

Acknowledgements

First and foremost, I would like to thank my supervisor, Prof. Tony Scholl. Over the last four years, he has consistently provided me with sound advice, while still allowing me to decide what I wanted to do. He has always made himself available for consultation, even when he was especially busy as head of department in Durham. He was most careful to look after my interests upon his move to Cambridge, for which I am very grateful. Undoubtedly, I could not have done it without him. Thank you, Tony.

In relation to the production of this thesis, I greatly indebted to Tony, Chris Pask, Mum and Dad for proofreading. Particular credit goes to the three non-mathematicians in this list: that they managed to plough through eighty-odd pages without the comfort of understanding, correcting more than a few errors in the process, says much about their commitment, and, perhaps, more than a little about the vagaries of my grammar. Thanks also to Julian Hill, for computer support, and Dr. Steve Wilson, who looked after the administrative details in Durham.

In life outside mathematics, I would like to thank my parents, who, as ever, have provided a safe haven in any storm, and all my other family and friends, for helping to keep me sane. A comprehensive list would be too long — and certainly too dangerous — to contemplate, but I would like to mention my colleagues in both Durham and Cambridge, the residents of Fonteyn Court, and the residents, sometime residents, and unofficial residents of Thoday Street, with whom it has been a pleasure to spend time.

Finally, I gratefully acknowledge the financial support of the Engineering and Physical Sciences Research Council and the Kuwait Foundation for the Advancement of Sciences.

Introduction

Let p be a prime. For $m \in \mathbf{Z}_{\geqslant 0}$, let $\binom{X}{m} = \frac{1}{m!}X(X-1)\dots(X-(m-1))$ be the binomial polynomial of degree m. Mahler [15] proved that every continuous function f on $\mathbf{Z}_p$ with values in $\mathbf{Q}_p$ can be written uniquely in the form

$$f = \sum_{m=0}^{+\infty} a_m \binom{X}{m},$$

where $(a_m)_{m\in\mathbf{Z}_{\geqslant 0}}$ is a null sequence in $\mathbf{Q}_p$. Moreover, we have

$$\|f\|_\infty = \max\left\{|a_m|_p \,|m \in \mathbf{Z}_{\geqslant 0}\right\},$$

where $|\quad|_p$ is the p-adic norm on $\mathbf{Q}_p$ and $\|\quad\|_\infty$ is the supremum norm.

Amice [1] refined this by characterising the Mahler expansions of locally analytic functions. We say that a function $f : \mathbf{Z}_p \to \mathbf{Q}_p$ is locally analytic if everywhere locally it is defined by a convergent power series with coefficients in $\mathbf{Q}_p$; we denote the space of all such functions by $\mathrm{LA}(\mathbf{Z}_p, \mathbf{Q}_p)$. For $h \in \mathbf{Z}_{\geqslant 0}$, we say a locally analytic function is of order h if on each disc of radius p^{-h} it is defined by a convergent power series; we denote the space of all such functions by $\mathrm{LA}_h(\mathbf{Z}_p, \mathbf{Q}_p)$. We can equip $\mathrm{LA}_h(\mathbf{Z}_p, \mathbf{Q}_p)$ with a norm $\|\quad\|_{LA_h}$. Amice proved that every locally analytic function $f \in \mathrm{LA}(\mathbf{Z}_p, \mathbf{Q}_p)$ can be written uniquely in the form

$$f = \sum_{m=0}^{+\infty} a_m \binom{X}{m},$$

where the sequence $(a_m)_{m\in\mathbf{Z}_{\geqslant 0}} \subset \mathbf{Q}_p$ is such that there exists $r \in \mathbf{Q}$, $r > 0$ satisfying $p^{mr}|a_m|_p \to 0$ as $m \to +\infty$. Moreover, if $f \in \mathrm{LA}_h(\mathbf{Z}_p, \mathbf{Q}_p)$ for some $h \in \mathbf{Z}_{\geqslant 0}$, she proved that

$$\|f\|_{LA_h} = \max \left\{ \left\| a_m \binom{X}{m} \right\|_{LA_h} \middle| \, m \in \mathbf{Z}_{\geqslant 0} \right\}.$$

This statement is equivalent to the fact that the set of binomial functions $\left\{ \binom{X}{m} \middle| \, m \in \mathbf{Z}_{\geqslant 0} \right\}$ is orthogonal in $\mathrm{LA}_h(\mathbf{Z}_p, \mathbf{Q}_p)$.

Using this work, Amice and Vélu [2] studied the continuous dual of $\mathrm{LA}(\mathbf{Z}_p, \mathbf{Q}_p)$: that is, the space of locally analytic distributions on $\mathbf{Z}_p$ with values in $\mathbf{Q}_p$. To each locally analytic distribution μ they associated the power series

$$\mathscr{A}(\mu)(T) := \sum_{m=0}^{+\infty} \mu\left(\binom{X}{m}\right) T^m,$$

which is known as the Amice transform of μ. The map $\mu \mapsto \mathscr{A}(\mu)(T)$ is injective, and its image is the space of all power series in $\mathbf{Q}_p[[T]]$ that converge on the maximal ideal of $\mathbf{C}_p$. The Amice transform is closely connected to the group of continuous characters on $\mathbf{Z}_p$ with values in ${\mathbf{Q}_p}^{\times}$. For $z \in p\mathbf{Z}_p$, we define the character

$$\begin{aligned} \kappa_z : \mathbf{Z}_p &\to {\mathbf{Q}_p}^{\times} \\ \alpha &\mapsto \sum_{m=0}^{+\infty} z^m \binom{\alpha}{m}; \end{aligned}$$

the map $z \mapsto \kappa_z$ parameterises the group of continuous characters. We see that κ_z is, in fact, locally analytic, and $\mu(\kappa_z) = \mathscr{A}(\mu)(z)$. A locally analytic distribution is said to be a measure if it can be extended to a continuous $\mathbf{Q}_p$-linear map from $\mathscr{C}(\mathbf{Z}_p, \mathbf{Q}_p)$ into $\mathbf{Q}_p$, where $\mathscr{C}(\mathbf{Z}_p, \mathbf{Q}_p)$ denotes the space of all continuous functions from $\mathbf{Z}_p$ into $\mathbf{Q}_p$. Using the orthogonality of the set of binomial functions, Amice and Vélu characterised the Amice transforms of measures: a locally analytic distribution on $\mathbf{Z}_p$ with values in $\mathbf{Q}_p$ is a measure if and only if its Amice transform is a power series with bounded coefficients.

The Amice transform can be used to help construct p-adic L-functions as follows. Let K be a finite field extension of $\mathbf{Q}_p$, and denote its ring of integers by $\mathfrak{O}_K$. Coleman [7] associated a unique power series in $\mathfrak{O}_K[[T]]$ to each system of norm compatible units in the tower of fields generated by the division points of a Lubin Tate formal group over $\mathfrak{O}_K$. If $K = \mathbf{Q}_p$ then the inverse of the Amice transform can be used to obtain a measure on $\mathbf{Z}_p$ from this power series. By choosing appropriate systems of norm compatible units, it is possible to produce measures that interpolate the values of classical L-functions. This approach was introduced by Coates and Wiles [6], and developed by various authors. As an example, consider an imaginary quadratic field F in which the prime p is split. De Shalit [9] explains how to use Robert's elliptic units to construct a measure that interpolates certain Hecke L-series associated to F. The assumption that p splits in F ensures that the completion of F at a prime ideal above p is isomorphic to $\mathbf{Q}_p$; this is needed in order to apply the Amice transform, but the other ingredients in this construction — elliptic units and Coleman power series — work just as well for inert primes. The p-adic L-function described here was originally constructed by Manin and Višik [16] and Katz [10] using different methods.

More recently, Schneider and Teitelbaum [19] extended much of Amice's theory to the case of a finite extension K of $\mathbf{Q}_p$. Denote the ring of integers of K by $\mathfrak{O}_K$. A function $f : \mathfrak{O}_K \to \mathbf{C}_p$ is said to be locally analytic if everywhere locally it is defined by a convergent power series in $\mathbf{C}_p[[X]]$; we denote the space of all such functions by $\mathrm{LA}(\mathfrak{O}_K, \mathbf{C}_p)$. Properly, these should be referred to as locally K-analytic functions, since it is also possible to define locally $\mathbf{Q}_p$-analytic functions on $\mathfrak{O}_K$ as functions that are defined locally by power series in $[K : \mathbf{Q}_p]$ variables. However, the latter type of function will not be considered in this thesis, so "locally analytic" will always mean "locally K-analytic". Just as in the case of $K = \mathbf{Q}_p$, a locally analytic function is said to be of order h if on each disc of radius p^{-h} it is defined by a convergent power series; we denote the space of all such functions by $\mathrm{LA}_h(\mathfrak{O}_K, \mathbf{C}_p)$. After choosing any Lubin Tate formal group l over $\mathfrak{O}_K$, Schneider and Teitelbaum defined the sequence of polynomials $(P_{l,m}(X))_{m \in \mathbf{Z}_{\geqslant 0}} \subset \mathbf{C}_p[X]$ by the identity

$$\sum_{m=0}^{+\infty} P_{l,m}(X)\, Y^m = \exp\left(\Omega_l X \lambda_l(Y)\right),$$

where $\lambda_l(Y)$ is the logarithm of the Lubin Tate formal group l, and Ω_l is its period: a constant in $\mathbf{C}_p$. I will refer to them as the Schneider Teitelbaum polynomials. They generalise the binomial polynomials. Schneider and Teitelbaum proved that every locally analytic function $f \in \mathrm{LA}(\mathfrak{O}_K, \mathbf{C}_p)$ can be written uniquely in the form

$$f = \sum_{m=0}^{+\infty} a_m P_{l,m},$$

where the sequence $(a_m)_{m \in \mathbf{Z}_{\geqslant 0}} \subset \mathbf{C}_p$ is such that there exists $r \in \mathbf{Q}$, $r > 0$ satisfying $p^{mr} |a_m|_p \to 0$ as $m \to +\infty$. To each locally analytic distribution μ on $\mathfrak{O}_K$ with values in $\mathbf{C}_p$ they associated the power series

$$\mathscr{A}_l(\mu)(T) = \sum_{m=0}^{+\infty} \mu(P_{l,m}(X)) T^m,$$

which I will refer to as the Schneider Teitelbaum transform. The map $\mu \mapsto \mathscr{A}_l(\mu)(T)$ is injective, and its image is the space of all power series in $\mathbf{C}_p[[T]]$ that converge on the maximal ideal $\mathfrak{p}_{\mathbf{C}_p}$ of $\mathbf{C}_p$. For each $z \in \mathfrak{p}_{\mathbf{C}_p}$, they defined a locally analytic character $\kappa_{l,z} : \mathfrak{O}_K \to {\mathbf{C}_p}^{\times}$, and they proved that the Schneider Teitelbaum transform satisfies $\mu(\kappa_{l,z}) = \mathscr{A}_l(\mu)(z)$.

Let F be an imaginary quadratic field in which the prime p is inert. Schneider and Teitelbaum also explained how to extend De Shalit's construction of the p-adic L-function of F to this case, using the Schneider Teitelbaum transform. The p-adic L-function obtained is essentially the same as the one studied by Katz [11] and Boxall [3]. However, we know only that this p-adic L-function is a locally analytic distribution, not that it is a measure. The problem is that we cannot deduce that a distribution is, in fact, a measure even if we know that its Schneider Teitelbaum transform is a power series with bounded coefficients. It would, therefore, be of great interest to characterise the Schneider Teitelbaum transforms of measures, in order to be able to make this sort of deduction.

This thesis demonstrates a difficulty in pursuing such a program. We will approach the theory of Schneider and Teitelbaum from the point of view taken in Amice [1]. The characterisation of the Amice transforms of measures on $\mathbf{Q}_p$ depends on the orthogonality of the set of binomial functions.

We will give a necessary and sufficient condition for a sequence of polynomials $(P_m(X))_{m\in\mathbf{Z}_{\geqslant 0}} \subset \mathbf{C}_p[X]$ to be orthogonal in $\mathrm{LA}_h(\mathfrak{O}_K, \mathbf{C}_p)$, generalising the work of Amice (see proposition II.1.13, p. 38). We will then use this condition to prove that the Schneider Teitelbaum polynomials are not orthogonal in $\mathrm{LA}(\mathfrak{O}_K, \mathbf{C}_p)$ (see corollary V.2.5, p. 83). It is, therefore, impossible to apply the method of Amice to characterise the Schneider Teitelbaum transforms of measures on $\mathfrak{O}_K$. It should, perhaps, be emphasised that Schneider and Teitelbaum made no prediction about the orthogonality of the set of polynomials that they defined.

Here is an outline of the contents of this thesis. In chapter I we will cover some background material on locally analytic functions, orthogonality, and convergent power series. In chapter II the necessary and sufficient condition for a sequence of polynomials $(P_m(X))_{m\in\mathbf{Z}_{\geqslant 0}}$, in which $P_m(X)$ has degree m, to be orthogonal in $\mathrm{LA}_h(\mathfrak{O}_K, \mathbf{C}_p)$ is proved, and we will discuss whether such a sequence forms a Banach basis. In chapter III we will recall, with full proofs, the results of Amice [1] and Amice Vélu [2] on the binomial polynomials. In particular, we will emphasise the importance of the orthogonality of the set of binomial functions in their work. After recalling some Lubin Tate theory, in chapter IV we will define the Schneider Teitelbaum polynomials and state some of their properties, drawing parallels with the binomial functions. In chapter V we will prove that the set of Schneider Teitelbaum polynomials is not orthogonal.

The results of chapters II and V are my own work and, as far as I know, are original. The results of chapters I, III, and IV are not original; they are drawn from various sources as indicated in the text.

Notation

We will use the following notations throughout this thesis.

We denote the ring of integers by $\mathbf{Z}$, the field of rational numbers by $\mathbf{Q}$, and the field of real numbers by $\mathbf{R}$. We set $\mathbf{Z}_{\geqslant 0} := \{n \in \mathbf{Z} \,|\, n \geqslant 0\}$, and use other similar notations. If $r \in \mathbf{R}$, we define $\lfloor r \rfloor$ to be r rounded down to the nearest integer, and $\lceil r \rceil$ to be r rounded up to the nearest integer. If S is a finite set, then $\#S$ will denote its cardinality.

If R is any ring, we write $R[T]$ for the ring of polynomials with coefficients in R, and $R[[T]]$ for the ring of power series with coefficients in R. For $P(T) \in R[T]$, we write $\deg(P(T))$ for the degree of $P(T)$. We denote the group of invertible elements of R by $R^{\times}$.

The letter p will always be a fixed odd prime. We denote the ring of p-adic integers by $\mathbf{Z}_p$, its field of fractions by $\mathbf{Q}_p$, and the completion of the algebraic closure of $\mathbf{Q}_p$ by $\mathbf{C}_p$. We denote the valuation ring of $\mathbf{C}_p$ by $\mathfrak{O}_{\mathbf{C}_p}$, and its maximal ideal by $\mathfrak{p}_{\mathbf{C}_p}$. We write $\mathrm{ord}_p : \mathbf{C}_p \to \mathbf{Q} \cup \{+\infty\}$ for the additive valuation on $\mathbf{C}_p$, normalised such that $\mathrm{ord}_p(p) = 1$.

The letter K will always denote a finite field extension of $\mathbf{Q}_p$ contained in $\mathbf{C}_p$. We denote its valuation ring by $\mathfrak{O}_K$, its maximal ideal by $\mathfrak{p}_K$, its residue class field by k_K, its ramification index by e_K, and its residue class field degree by f_K. We write q_K for the number of elements in the residue class field of K, so that $q_K = p^{f_K}$. The symbol π_K will always denote some prime element of $\mathfrak{O}_K$; that is, $\pi_K \in \mathfrak{O}_K$ such that $\mathrm{ord}_p(\pi_K) = 1/e_K$. The symbol $l(X)$ will always denote a power series in $\mathfrak{O}_K[[X]]$ such that $l(X) \equiv \pi_K X \bmod X^2\mathfrak{O}_K[[X]]$ and $l(X) \equiv X^{q_K} \bmod \pi_K\mathfrak{O}_K[[X]]$.

The letter L will always denote some complete valued subfield of $\mathbf{C}_p$ containing K. We denote its valuation ring by $\mathfrak{O}_L$, its maximal ideal by $\mathfrak{p}_L$, and its residue class field by k_L. We have, therefore, the following system of inclusions:

$$\mathbf{Q}_p \subseteq K \subseteq L \subseteq \mathbf{C}_p.$$

Chapter I

Preliminaries

In this chapter we will introduce the background material that will be needed. It will serve both to fix notations and to provide a firm foundation for the rest of the thesis.

None of the material in this chapter is original; it draws on various sources as noted in the text.

I.1 Locally analytic functions

In this section we will define the space of locally analytic functions on $\mathfrak{O}_K$ with values in L. These functions will be the principal objects of study throughout this thesis.

In the main, I have followed the notation of Colmez [8], ch. I, pp. 495–502.

Definition I.1.1 An L-Banach space (E, ord_E) is an L-vector space E equipped with a valuation function

$$\mathrm{ord}_E : E \to \mathbf{Q} \cup \{+\infty\}$$

that satisfies

i. $\mathrm{ord}_E(e) = +\infty \iff e = 0$,

ii. $\mathrm{ord}_E(ae) = \mathrm{ord}_p(a) + \mathrm{ord}_E(e) \qquad \forall a \in L,\ e \in E$,

iii. $\mathrm{ord}_E(e_1+e_2) \geqslant \min\{\mathrm{ord}_E(e_1), \mathrm{ord}_E(e_2)\} \qquad \forall e_1, e_2 \in E$,

iv. $\mathrm{ord}_E(E) = \mathrm{ord}_p(L)$,

and is such that E is complete with respect to the metric induced by ord_E.

Remark I.1.2 It is more usual to work with a norm on E, defined by $\|e\|_E := p^{-\mathrm{ord}_E(e)}$. However, since Newton polygons will play an important role in this thesis, I have decided to work exclusively with additive valuations.

Notation I.1.3 Let X be a subset of $\mathfrak{O}_K$. We write $\mathbf{1}_X : \mathfrak{O}_K \to \{0,1\}$ for the characteristic function of X; that is, $\mathbf{1}_X(\alpha) = 1$ if $\alpha \in X$ and $\mathbf{1}_X(\alpha) = 0$ if $\alpha \notin X$.

Definition I.1.4 Let $h \in \mathbf{Z}_{\geqslant 0}$ and let $n \in \mathbf{Z}_{\geqslant 0}$. Let $R_{h/e_K} \subset \mathfrak{O}_K$ be a set of representatives of $\mathfrak{O}_K/\pi_K^h\mathfrak{O}_K$. We say that a function $f : \mathfrak{O}_K \to L$ is locally polynomial, of order h/e_K and degree at most n, on $\mathfrak{O}_K$ with values in L if we can write

$$f(\alpha) = \sum_{\beta \in R_{h/e_K}} \left(\mathbf{1}_{\beta+\pi_K^h\mathfrak{O}_K}(\alpha) \left(\sum_{i=0}^{n} a_{\beta,i} \left(\frac{\alpha-\beta}{\pi_K^h} \right)^i \right) \right) \qquad \forall \alpha \in \mathfrak{O}_K,$$

for some $a_{\beta,i} \in L$. We write $\mathrm{LP}_{h/e_K}^{[0,n]}(\mathfrak{O}_K, L)$ for the L-vector space of all such functions. Note that $\mathrm{LP}_{h/e_K}^{[0,n]}(\mathfrak{O}_K, L)$ is independent of our choice of R_{h/e_K}.

Definition I.1.5 Let $h \in \mathbf{Z}_{\geqslant 0}$. Let $R_{h/e_K} \subset \mathfrak{O}_K$ be a set of representatives of $\mathfrak{O}_K/\pi_K^h\mathfrak{O}_K$. We say that a function $f : \mathfrak{O}_K \to L$ is locally analytic of order h/e_K on $\mathfrak{O}_K$ with values in L if we can write

$$f(\alpha) = \sum_{\beta \in R_{h/e_K}} \left(\mathbf{1}_{\beta+\pi_K^h\mathfrak{O}_K}(\alpha) \left(\sum_{i=0}^{+\infty} a_{\beta,i} \left(\frac{\alpha-\beta}{\pi_K^h} \right)^i \right) \right) \qquad \forall \alpha \in \mathfrak{O}_K,$$

for some $a_{\beta,i} \in L$ such that $\mathrm{ord}_p(a_{\beta,i}) \to +\infty$ as $i \to +\infty$ for all $\beta \in R_{h/e_K}$. We write $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$ for the L-vector space of all such functions. Note that $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$ is independent of our choice of R_{h/e_K}.

For all $n \in \mathbf{Z}_{\geqslant 0}$ we have $\mathrm{LP}_{h/e_K}^{[0,n]}(\mathfrak{O}_K, L) \subset \mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$, and if $h_1 \leqslant h_2$ then we have $\mathrm{LA}_{h_1/e_K}(\mathfrak{O}_K, L) \subseteq \mathrm{LA}_{h_2/e_K}(\mathfrak{O}_K, L)$.

Definition I.1.6 Let $h \in \mathbf{Z}_{\geqslant 0}$ and let $\beta \in \mathfrak{O}_K$. We define

$$\begin{aligned} \mathrm{ord}_{\mathrm{LA}_{h/e_K,\beta}} : \mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L) &\to \mathbf{Q} \cup \{+\infty\} \\ f &\mapsto \min\left\{\mathrm{ord}_p(f(z)) \,\middle|\, z \in \beta + \pi_K^h \mathfrak{O}_{\mathbf{C}_p}\right\}. \end{aligned}$$

Note that the domain of f is $\mathfrak{O}_K$, but we can use the power series expansion

$$f(\alpha) = \sum_{i=0}^{+\infty} a_{\beta,i} \left(\frac{\alpha - \beta}{\pi_K^h}\right)^i \qquad \forall \alpha \in \beta + \pi_K^h \mathfrak{O}_K$$

to define $f(z)$ for all $z \in \beta + \pi_K^h \mathfrak{O}_{\mathbf{C}_p}$. For $\beta_1 \equiv \beta_2 \bmod \pi_K^h \mathfrak{O}_K$, we have $\mathrm{ord}_{\mathrm{LA}_{h/e_K,\beta_1}} = \mathrm{ord}_{\mathrm{LA}_{h/e_K,\beta_2}}$.

Let $R_{h/e_K} \subset \mathfrak{O}_K$ be a set of representatives for $\mathfrak{O}_K / \pi_K^h \mathfrak{O}_K$. We define

$$\begin{aligned} \mathrm{ord}_{\mathrm{LA}_{h/e_K}} : \mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L) &\to \mathbf{Q} \cup \{+\infty\} \\ f &\mapsto \min\left\{\mathrm{ord}_{\mathrm{LA}_{h/e_K,\beta}}(f) \,\middle|\, \beta \in R_{h/e_K}\right\}. \end{aligned}$$

Note that $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}$ is independent of our choice of R_{h/e_K}.

Proposition I.1.7 *Let $h \in \mathbf{Z}_{\geqslant 0}$ and let $R_{h/e_K} \subset \mathfrak{O}_K$ be a set of representatives of $\mathfrak{O}_K / \pi_K^h \mathfrak{O}_K$. Let $f \in \mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$ and write*

$$f(\alpha) = \sum_{\beta \in R_{h/e_K}} \left(\mathbf{1}_{\beta + \pi_K^h \mathfrak{O}_K}(\alpha) \left(\sum_{i=0}^{+\infty} a_{\beta,i} \left(\frac{\alpha - \beta}{\pi_K^h}\right)^i \right) \right) \qquad \forall \alpha \in \mathfrak{O}_K.$$

Then for $\beta \in R_{h/e_K}$ we have

$$\mathrm{ord}_{\mathrm{LA}_{h/e_K,\beta}}(f) = \min\left\{\mathrm{ord}_p(a_{\beta,i}) \,\middle|\, i \in \mathbf{Z}_{\geqslant 0}\right\},$$

and

$$\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(f) = \min\left\{\mathrm{ord}_p(a_{\beta,i}) \,\middle|\, \beta \in R_{h/e_K},\ i \in \mathbf{Z}_{\geqslant 0}\right\}.$$

Proof:
This is an easy consequence of the 'maximum' principle (see proposition I.4.8, p. 24). $\square$

Proposition I.1.8 *Let $h \in \mathbf{Z}_{\geqslant 0}$ and let $n \in \mathbf{Z}_{\geqslant 0}$.*

i. We have that $\left(\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L), \mathrm{ord}_{\mathrm{LA}_{h/e_K}}\right)$ *is an L-Banach space.*

ii. We have that $\left(\mathrm{LP}^{[0,n]}_{h/e_K}(\mathfrak{O}_K, L), \mathrm{ord}_{\mathrm{LA}_{h/e_K}}\right)$ *is a finite dimensional L-Banach space.*

Proof:
We must check that $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}$ satisfies the conditions of definition I.1.1, p. 15. Conditions (i), (ii), and (iii) are obvious. Condition (iv) follows from proposition I.1.7 above, and so does the fact that $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$ is complete. □

Remark I.1.9 Let $h_1, h_2 \in \mathbf{Z}_{\geqslant 0}$ with $h_1 \leqslant h_2$. If $f \in \mathrm{LA}_{h_1/e_K}(\mathfrak{O}_K, L)$, note that $\mathrm{ord}_{\mathrm{LA}_{h_1/e_K}}(f) \leqslant \mathrm{ord}_{\mathrm{LA}_{h_2/e_K}}(f)$. It follows that the natural inclusion $\mathrm{LA}_{h_1/e_K}(\mathfrak{O}_K, L) \hookrightarrow \mathrm{LA}_{h_2/e_K}(\mathfrak{O}_K, L)$ is continuous.

Definition I.1.10 We define the L-vector space $\mathrm{LA}(\mathfrak{O}_K, L)$ of locally analytic functions on $\mathfrak{O}_K$ with values in L to be

$$\mathrm{LA}(\mathfrak{O}_K, L) := \bigcup_{h \in \mathbf{Z}_{\geqslant 0}} \mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L).$$

We give $\mathrm{LA}(\mathfrak{O}_K, L)$ the inductive limit topology; that is, $X \subseteq \mathrm{LA}(\mathfrak{O}_K, L)$ is open if and only if $X \cap \mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$ is open in $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$ for all $h \in \mathbf{Z}_{\geqslant 0}$.

Definition I.1.11 We say that a function $f : \mathfrak{O}_K \to L^\times$ is a locally analytic character on $\mathfrak{O}_K$ with values in L if $f \in \mathrm{LA}(\mathfrak{O}_K, L)$, $f(0) = 1$, and

$$f(\alpha_1 + \alpha_2) = f(\alpha_1) f(\alpha_2) \qquad \forall \alpha_1, \alpha_2 \in \mathfrak{O}_K.$$

We write $\mathrm{Hom}_{\mathrm{LA}}(\mathfrak{O}_K, L^\times)$ for the group of all such characters.

For much of this thesis, we will study the properties of polynomials in $L[X]$, considered as elements of $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$.

Notation I.1.12 Let $P(X) \in L[X]$. Then $P(X)$ induces a function:

$$\begin{aligned} \mathfrak{O}_K &\to L \\ \alpha &\mapsto P(\alpha), \end{aligned}$$

which we will denote simply by P. Clearly $P \in \mathrm{LA}_0(\mathfrak{O}_K, L)$.

I.2 Locally analytic distributions

In this section we will define the space of locally analytic distributions: the continuous dual of the space of locally analytic functions. As before, this section is based on Colmez [8], ch. I, pp. 495–502.

Definition I.2.1 Let $\mu : \mathrm{LA}(\mathfrak{O}_K, L) \to L$ be an L-linear map, and let $h \in \mathbf{Z}_{\geqslant 0}$. We define

$$\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(\mu) := \inf\left\{\mathrm{ord}_p(\mu(f)) - \mathrm{ord}_{\mathrm{LA}_{h/e_K}}(f) \,|\, f \in \mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L) - \{0\}\right\}.$$

We have $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(\mu) \in \mathbf{R} \cup \{\pm\infty\}$. Note that if $h_1, h_2 \in \mathbf{Z}_{\geqslant 0}$, $h_1 \leqslant h_2$ then $\mathrm{ord}_{\mathrm{LA}_{h_1/e_K}}(\mu) \geqslant \mathrm{ord}_{\mathrm{LA}_{h_2/e_K}}(\mu)$.

Proposition I.2.2 *Let $\mu : \mathrm{LA}(\mathfrak{O}_K, L) \to L$ be an L-linear map. Then μ is continuous if and only if $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(\mu) > -\infty$ for all $h \in \mathbf{Z}_{\geqslant 0}$.*

Proof:
From the definition of the topology on $\mathrm{LA}(\mathfrak{O}_K, L)$ (see definition I.1.10, p. 18), we know that μ is continuous if and only if the restriction of μ to $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$ is continuous for all $h \in \mathbf{Z}_{\geqslant 0}$. This reduces the statement to the well known result regarding the continuity of linear maps on normed vector spaces. □

Definition I.2.3 We say that a continuous linear map $\mu : \mathrm{LA}(\mathfrak{O}_K, L) \to L$ is a locally analytic distribution on $\mathfrak{O}_K$ with values in L. We write $\mathscr{D}_{\mathrm{LA}}(\mathfrak{O}_K, L)$ for the L-vector space of all such distributions.

We give $\mathscr{D}_{\mathrm{LA}}(\mathfrak{O}_K, L)$ the least upper bound topology of the topologies induced by $\left\{\mathrm{ord}_{\mathrm{LA}_{h/e_K}} \,|\, h \in \mathbf{Z}_{\geqslant 0}\right\}$; that is, the sets

$$\left\{\mu \in \mathscr{D}_{\mathrm{LA}}(\mathfrak{O}_K, L) \,\middle|\, \mathrm{ord}_{\mathrm{LA}_{h/e_K}}(\mu) \geqslant s\right\},$$

for all $h \in \mathbf{Z}_{\geqslant 0}$ and all $s \in \mathbf{R}$, form a fundamental system of open neighbourhoods of zero in $\mathscr{D}_{\mathrm{LA}}(\mathfrak{O}_K, L)$.

Notation I.2.4 We define $\overline{\mathbf{R}} := \mathbf{R} \cup \{r^- \,| r \in \mathbf{R}\}$. We equip $\overline{\mathbf{R}}$ with the total ordering $<$ that coincides with the usual ordering on $\mathbf{R}$ and is such that $r_1 < r_2^- < r_2$ for all $r_1, r_2 \in \mathbf{R}$, $r_1 < r_2$. If $f : \mathbf{R} \to \mathbf{R}$ is a continuous function and $r \in \mathbf{R}$, we set $f(r^-) := f(r)$. We define

$$\begin{aligned} \eta : \overline{\mathbf{R}} &\to \{\pm 1\} \\ r &\mapsto +1 \qquad r \in \mathbf{R}, \\ r^- &\mapsto -1 \qquad r \in \mathbf{R}. \end{aligned}$$

Definition I.2.5 Let $\eta \in \{\pm 1\}$. We say that a sequence $(a_h)_{h \in \mathbf{Z}_{\geqslant 0}} \subset \mathbf{R}$ is η-bounded below if:

- $\eta = 1$ and $\{a_h \,| h \in \mathbf{Z}_{\geqslant 0}\}$ is bounded below, or
- $\eta = -1$ and $a_h \to +\infty$ as $h \to +\infty$.

Definition I.2.6 Let $r \in \overline{\mathbf{R}}$. We say that $\mu \in \mathscr{D}_{\mathrm{LA}}(\mathfrak{O}_K, L)$ is temperate of order r if the sequence $\left(hr/e_K + \mathrm{ord}_{\mathrm{LA}_{h/e_K}}(\mu)\right)_{h \in \mathbf{Z}_{\geqslant 0}}$ is $\eta(r)$-bounded below. We write $\mathscr{D}_r(\mathfrak{O}_K, L)$ for the L-vector space of all such distributions. Note that if $r < 0$ then $\mathscr{D}_r(\mathfrak{O}_K, L) = \{0\}$, and if $r_1, r_2 \in \overline{\mathbf{R}}$, $r_1 < r_2$ then $\mathscr{D}_{r_1}(\mathfrak{O}_K, L) \subset \mathscr{D}_{r_2}(\mathfrak{O}_K, L)$. We say that a distribution μ is temperate if there exists $r \in \overline{\mathbf{R}}$ such that $\mu \in \mathscr{D}_r(\mathfrak{O}_K, L)$; we write $\mathscr{D}_{\mathrm{temp}}(\mathfrak{O}_K, L)$ for the L-vector space of all such distributions.

I.3 Orthogonality

In this section we will study the notion of orthogonality in L-Banach spaces. This thesis will study sets of polynomials that are orthogonal in $\mathrm{LA}(\mathfrak{O}_K, L)$. None of the work in this section is original; it is based mainly on Schikhof [18], §50, pp. 145–149.

Definition I.3.1 Let (E, ord_E) be an L-Banach space. Let $x, y \in E$. We say that x is orthogonal to y, and write $x \perp y$, if

$$\mathrm{ord}_E(x - ay) \leqslant \mathrm{ord}_E(x) \qquad \forall a \in L.$$

Note that $x \perp y$ if and only if $y \perp x$ (see Schikhof [18], proposition 50.2, p. 146).

Let $D \subset E$. We write $x \perp D$ if $x \perp d$ for all $d \in D$. Note that if D is an L-linear subspace of e then this is equivalent to

$$\operatorname{ord}_E(x-d) \leqslant \operatorname{ord}_E(x) \qquad \forall d \in D.$$

We say that $X \subset E$ is an orthogonal set if

$$x_0 \perp \operatorname{span}_L \{x \in X \,|x \neq x_0\} \qquad \forall x_0 \in X,$$

where $\operatorname{span}_L Y$ denotes the L-linear span of a set $Y \subset E$. If, in addition, we have $\operatorname{ord}_E(x) = 0$ for all $x \in X$, then we say that X is an orthonormal set.

We say that X is a maximal orthogonal set if it is orthogonal and, for any orthogonal set $X_1 \subset E$ such that $X \subset X_1$, we have $X = X_1$; we similarly define a maximal orthonormal set.

Proposition I.3.2 *Let $(E, \operatorname{ord}_E)$ be an L-Banach space.*

i. A set $X \subset E$ is orthogonal if and only if every finite subset of X is orthogonal.

ii. A set $\{x_0, \dots, x_m\} \subset E$ is orthogonal if and only if

$$\operatorname{ord}_E\left(\sum_{i=0}^{m} a_i x_i\right) = \min\{\operatorname{ord}_E(a_i x_i) \,|i \in \{0, \dots, m\}\} \ \forall a_0, \dots, a_n \in L.$$

Proof:
See Schikhof [18], proposition 50.4, p. 146. □

Definition I.3.3 Let $(E, \operatorname{ord}_E)$ be an L-Banach space, and let X be a subset of E. We say that X spans E as an L-Banach space if every element $e \in E$ can be written in the form $e = \sum_{x \in X} a_x x$, where, for every $r \in \mathbf{Q}$, there are only finitely many $x \in X$ such that $\operatorname{ord}_p(a_x x) < r$. If, in addition, this expression is unique then we say that X is an L-Banach basis of E. If X is orthogonal, then we say that it is an orthogonal L-Banach basis of E, and similarly if X is orthonormal.

Definition I.3.4 Let $h \in \mathbf{Z}_{\geqslant 0}$, let $\beta \in \mathfrak{O}_K$, and let $i \in \mathbf{Z}_{\geqslant 0}$. We define:

$$\begin{aligned} \chi_{\beta,i} : \mathfrak{O}_K &\to L \\ \alpha &\mapsto \begin{cases} \left(\frac{\alpha-\beta}{\pi_K^h}\right)^i & \alpha \in \beta + \pi_K^h \mathfrak{O}_K \\ 0 & \alpha \notin \beta + \pi_K^h \mathfrak{O}_K \end{cases} \end{aligned}$$

Note that $\chi_{\beta,i} \in \mathrm{LP}_{h/e_K}^{[0,i]}(\mathfrak{O}_K, L) \subset \mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$.

Proposition I.3.5 *Let $h \in \mathbf{Z}_{\geqslant 0}$, let $n \in \mathbf{Z}_{\geqslant 0}$, and let $R_{h/e_K} \subset \mathfrak{O}_K$ be a set of representatives of $\mathfrak{O}_K / \pi_K^h \mathfrak{O}_K$.*

i. The set $\left\{\chi_{\beta,i} \,\middle|\, \beta \in R_{h/e_K}, i \in \mathbf{Z}_{\geqslant 0}\right\}$ is an orthonormal L-Banach basis of $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$.

ii. The set $\left\{\chi_{\beta,i} \,\middle|\, \beta \in R_{h/e_K}, i \in \{0, \ldots, n\}\right\}$ is an orthonormal L-Banach basis of $\mathrm{LP}_{h/e_K}^{[0,n]}(\mathfrak{O}_K, L)$; in particular, $\mathrm{LP}_{h/e_K}^{[0,n]}(\mathfrak{O}_K, L)$ has dimension $q_K^h(n+1)$ as an L-vector space.

Proof:
Immediate from the definitions and proposition I.1.7, p. 17 (cf. Amice [1], lemma 6, p. 151). □

I.4 Convergent Power Series

Locally analytic functions are defined locally by convergent power series. In this section we will consider some of the properties of such power series.

None of the material in this section is original; it is based mainly on Schikhof [18], §40–42, pp. 117–125.

Definition I.4.1 Let $F(T) = \sum_{i=0}^{+\infty} a_i T^i \in L[[T]]$. We define the order of convergence $r_F \in \mathbf{R} \cup \{\pm\infty\}$ of $F(T)$ to be

$$r_F := \inf\left\{r \in \mathbf{Q} \,\middle|\, ir + \mathrm{ord}_p(a_i) \to +\infty \text{ as } i \to +\infty\right\}.$$

Remark I.4.2 Note that p^{-r_F} is equal to the well known definition of the radius of convergence of a power series $F(T) \in L[[T]]$.

Proposition I.4.3 *Let $F(T) \in L[[T]]$ be a power series and let $r_F \in \mathbf{R} \cup \{\pm\infty\}$ be its order of convergence. Then:*

- *$F(T)$ converges on $\{z \in \mathbf{C}_p \,|\mathrm{ord}_p(z) > r_F\}$, and*
- *$F(T)$ diverges on $\{z \in \mathbf{C}_p \,|\mathrm{ord}_p(z) < r_F\}$.*

Proof:
Follows easily from the definition of r_F. □

We will frequently need to refer to various discs in $\mathbf{C}_p$, so we introduce the following notation.

Notation I.4.4 Let $r \in \mathbf{R}$. Then we define:

$$p^r\mathfrak{p}_{\mathbf{C}_p} := \{z \in \mathbf{C}_p \,|\mathrm{ord}_p(z) > r\},$$

$$p^r\mathfrak{O}_{\mathbf{C}_p} := \{z \in \mathbf{C}_p \,|\mathrm{ord}_p(z) \geqslant r\}.$$

If $r \in \mathbf{Z}$ then $p^r\mathfrak{p}_{\mathbf{C}_p} = \left\{p^r z \,\middle|\, z \in \mathfrak{p}_{\mathbf{C}_p}\right\}$ and $p^r\mathfrak{O}_{\mathbf{C}_p} = \left\{p^r z \,\middle|\, z \in \mathfrak{O}_{\mathbf{C}_p}\right\}$, so there is no clash in our notation.

Definition I.4.5 We say that $F(T) \in L[[T]]$ is a convergent power series if its order of convergence r_F satisfies $r_F < +\infty$.

The following proposition shows that the composition of two convergent power series is again convergent.

Proposition I.4.6 *Let $F(T), G(T) \in L[[T]]$ be two convergent power series with $G(0) = 0$; write $F \circ G(T) \in L[[T]]$ for their formal composition. Then there exists $r \in \mathbf{R}$ such that $r > r_G$, $r > r_{F\circ G}$, $G\left(p^r\mathfrak{O}_{\mathbf{C}_p}\right) \subseteq p^{r_F}\mathfrak{p}_{\mathbf{C}_p}$, and*

$$(F \circ G)(z) = F(G(z)) \qquad \forall z \in p^r\mathfrak{O}_{\mathbf{C}_p}.$$

In particular, $F \circ G(T) \in L[[T]]$ is a convergent power series.

Proof:
See Robert [17], ch. 6, §1.5, theorem, p. 294. □

Definition I.4.7 Let $r \in \mathbf{R}$. We define the function

$$\mathrm{ord}_r : L[[T]] \to \mathbf{R} \cup \{\pm\infty\}$$

as follows:

- if $F(T) \in L[[T]]$ converges on $p^r \mathfrak{p}_{\mathbf{C}_p}$ then we set

$$\mathrm{ord}_r(F(T)) := \inf\left\{\mathrm{ord}_p(F(z)) \,\middle|\, z \in p^r \mathfrak{p}_{\mathbf{C}_p}\right\},$$

- if $F(T) \in L[[T]]$ does not converge on $p^r \mathfrak{p}_{\mathbf{C}_p}$, then we set

$$\mathrm{ord}_r(F(T)) := -\infty.$$

Note that if $r_1, r_2 \in \mathbf{R}$, $r_1 \leqslant r_2$ then $\mathrm{ord}_{r_1}(F(T)) \leqslant \mathrm{ord}_{r_2}(F(T))$ for all $F(T) \in L[[T]]$.

Proposition I.4.8 ('maximum' principle)
Let $F(T) = \sum_{i=0}^{+\infty} a_i T^i \in L[[T]]$, and let $r \in \mathbf{R}$.

i. We have

$$\mathrm{ord}_r(F(T)) = \inf\left\{ir + \mathrm{ord}_p(a_i) \,|\, i \in \mathbf{Z}_{\geqslant 0}\right\}.$$

ii. In addition, if $r \in \mathbf{Q}$ and $F(T)$ converges on $p^r \mathfrak{O}_{\mathbf{C}_p}$, then

$$\mathrm{ord}_r(F(T)) = \inf\left\{\mathrm{ord}_p(F(z)) \,\middle|\, z \in p^r \mathfrak{O}_{\mathbf{C}_p}\right\}.$$

Proof:

i. If $F(T)$ converges on $p^r \mathfrak{p}_{\mathbf{C}_p}$ then see Schikhof [18], theorem 42.3(i), p. 124. If $F(T)$ does not converge on $p^r \mathfrak{p}_{\mathbf{C}_p}$ then $ir + \mathrm{ord}_p(a_i) \to -\infty$ as $i \to +\infty$, as required.

ii. See Schikhof [18], theorem 42.2(i), p. 122. □

Remark I.4.9 Proposition I.4.8 above is an elementary translation of the well known maximum principle into the language of the additive valuation ord_p. Perhaps it would be better nomenclature to refer to this translation as the "minimum principle", but I will not do this.

Definition I.4.10 We write $L\langle\langle T\rangle\rangle \subset L[[T]]$ for the ring of all power series with order of convergence less than or equal to zero; that is, the power series that converge on $\mathfrak{p}_{\mathbf{C}_p}$.

We give $L\langle\langle T\rangle\rangle$ the least upper bound topology of the topologies induced by $\{\mathrm{ord}_r \,| r \in \mathbf{R}_{>0}\}$; that is, the sets

$$\{F(T) \in L\langle\langle T\rangle\rangle \,|\mathrm{ord}_r(F(T)) \geqslant s\},$$

for all $r \in \mathbf{R}_{>0}$ and all $s \in \mathbf{R}$, form a fundamental system of open neighbourhoods of zero in $L\langle\langle T\rangle\rangle$.

I.5 Newton polygons

In this section we will study the Newton polygon of a power series. This construction gives us information about the distribution of the zeros of a power series, a topic that will prove to be very important in this thesis.

None of the material in this section is original; it is based mainly on Koblitz [12], ch. IV, §3–4, pp. 97–107.

Proposition I.5.1 *Let $r \in \mathbf{R}$ and let $F(T) = \sum_{i=0}^{+\infty} a_i T^i \in \mathbf{C}_p[[T]]$ be a power series that converges on $p^r\mathfrak{O}_{\mathbf{C}_p}$. Then the formal derivative $F'(T) := \sum_{i=1}^{+\infty} i a_i T^{i-1}$ also converges on $p^r\mathfrak{O}_{\mathbf{C}_p}$*

Proof:
This is clear since $\mathrm{ord}_p(ia_i) \geqslant \mathrm{ord}_p(a_i)$ for all $i \in \mathbf{Z}_{\geqslant 0}$. □

Definition I.5.2 Let $r \in \mathbf{R}$ and let $F(T) \in \mathbf{C}_p[[T]] - \{0\}$ be a power series that converges on $p^r\mathfrak{O}_{\mathbf{C}_p}$. Let $z \in p^r\mathfrak{O}_{\mathbf{C}_p}$ such that $F(z) = 0$. We define the multiplicity of the zero z of $F(T)$ to be

$$\max\left\{n \in \mathbf{Z}_{\geqslant 1} \,\middle|\, F^{(i)}(z) = 0 \quad \forall i \in \{0, \ldots, n-1\}\right\},$$

where $F^{(i)}(T)$ denotes the i-th formal derivative of $F(T)$. We say that z is a simple zero if it has multiplicity equal to one.

Definition I.5.3 Let $r \in \mathbf{R}$ and let $F(T) \in \mathbf{C}_p[[T]] - \{0\}$ be a power series that is convergent on $p^r\mathfrak{O}_{\mathbf{C}_p}$. Then we define $\mathrm{Z}(r; F(T))$ to be the number of zeros, counting multiplicities, of $F(T)$ in $p^r\mathfrak{O}_{\mathbf{C}_p}$.

Definition I.5.4 (Newton polygon)
Let $F(T) = \sum_{i=0}^{+\infty} a_i T^i \in \mathbf{C}_p[[T]] - \{0\}$ be a power series. We define the Newton polygon of $F(T)$ to be the boundary of the sup convex envelope of the points $(i, \mathrm{ord}_p(a_i))$ in the (x, y)-plane; for details, see Robert [17], ch. 6, §1.6, definition, p. 299. For $n \in \mathbf{Z}_{\geqslant 1}$, we define $\mu(n; F(T))$ to be the slope of the Newton polygon of $F(T)$ between the x co-ordinates $n-1$ and n. If $a_i = 0$ for all $i \in \{0, \ldots, n-1\}$, then we adopt the convention that $\mu(n; F(T)) := -\infty$; if $a_i = 0$ for all $\{i \in \mathbf{Z}_{\geqslant 0} \,|\, i \geqslant n\}$, then we set $\mu(n; F(T)) := +\infty$.

Proposition I.5.5 *Let $F(T) \in \mathbf{C}_p[[T]] - \{0\}$. Then*

$$\sup\{\mu(i; F(T)) \,|\, i \in \mathbf{Z}_{\geqslant 1}\} = -r_F,$$

where r_F is the order of convergence of $F(T)$.

Proof:
This is just a translation of Koblitz [12], ch. IV, §4, lemma 5, p. 101 from the language of the norm $|\ \ |_p$ into that of the valuation ord_p. □

Theorem I.5.6 *Let $F(T) \in \mathbf{C}_p[[T]] - \{0\}$, and let $r \in \mathbf{Q}$ such that $F(T)$ converges on $p^r\mathfrak{O}_{\mathbf{C}_p}$. Then*

$$\mathrm{Z}(r; F(T)) = \#\{i \in \mathbf{Z}_{\geqslant 1} \,|\, \mu(i; F(T)) \leqslant -r\}.$$

Proof:
This is an easy consequence of Koblitz [12], ch. IV, §4, pp. 98–107. Without loss of generality we may assume that $F(0) = 1$. Since $-r < -r_F$, by proposition I.5.5 above we have $N := \#\{i \in \mathbf{Z}_{\geqslant 1} \,|\, \mu(i; F(T)) \leqslant -r\}$ is finite. By ibid., ch. 4, §4, theorem 14, p. 105 (noting that λ there corresponds to $-r$ here), there exists a polynomial $H(T) \in \mathbf{C}_p[T]$ of degree N and a power series $G(T) \in \mathbf{C}_p[[T]]$ that converges and is non-zero on

$p^r\mathfrak{O}_{\mathbf{C}_p}$ such that $H(T) = F(T)G(T)$. Moreover, $\mu(i; H(T)) = \mu(i; F(T))$ for all $i \in \{1, \ldots, N\}$. It follows from ibid., ch. IV, §3, lemma 4, p. 97 that $\mathrm{Z}(r; H(T)) = \#\{i \in \mathbf{Z}_{\geqslant 1} \,|\, \mu(i; F(T)) \leqslant -r\}$, which is equal to N by definition. From $H(T) = F(T)G(T)$ we see that $\mathrm{Z}(r; H(T)) = \mathrm{Z}(r; F(T))$, which completes the proof. □

Definition I.5.7 Let $P(T) \in \mathbf{C}_p[T]$ be a polynomial, let $\alpha \in \mathfrak{O}_K$, and let $r \in \mathbf{Q}$. Then we define $\mathrm{Z}(\alpha, r; P(T))$ to be the number of zeros, counting multiplicities, of $P(T)$ in $\alpha + p^r\mathfrak{O}_{\mathbf{C}_p}$.

Remark I.5.8

i. Note that $\mathrm{Z}(\alpha, r; P(T))$ is equal to the degree of $P(T)$ for all $r \ll 0$, and $\mathrm{Z}(\alpha, r; P(T))$ is integer valued and decreasing as a function of $r \in \mathbf{Q}$.

ii. For $\alpha_1, \alpha_2 \in \mathfrak{O}_K$ and $k \in \mathbf{Z}_{\geqslant 0}$ such that $\alpha_1 \equiv \alpha_2 \bmod \pi_K^k\mathfrak{O}_K$, we have $\mathrm{Z}(\alpha_1, r; P(T)) = \mathrm{Z}(\alpha_2, r; P(T))$ for all $r \in \mathbf{Q}$, $r \leqslant k/e$.

iii. As a function of $r \in \mathbf{Q}$, we note that $\mathrm{Z}(\alpha, r; P(T))$ is a finite $\mathbf{Z}$-linear combination of the unit constant function and the characteristic functions of half open intervals of the form $\{r \in \mathbf{Q} \,|\, a < r \leqslant b\}$, with $a, b \in \{-\infty\} \cup \mathbf{Q}$.

Chapter II

Orthogonal bases consisting of polynomials

In this chapter we consider the question: when does a sequence of polynomials $(P_m(X))_{m\in\mathbf{Z}_{\geqslant 0}} \subset L[X]$, with $\deg(P_m(X)) = m$, form an orthogonal L-Banach basis of $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$? In §II.1 we prove a necessary and sufficient condition for such a sequence to be an orthogonal set; in §II.2 we consider whether it is also a basis.

This generalises, in two different respects, the situation studied in Amice [1], §9, pp. 150–158. Firstly, in our case the coefficient field L is not necessarily discretely valued. Secondly, Amice has $P_m(X)$ dividing $P_{m+1}(X)$, but we make no such assumption. We will need this extra generality in order to apply our results to the Schneider Teitelbaum polynomials.

II.1 Orthogonal sets

In this section we consider which polynomials $P_m(X) \in L[X]$ of degree m are orthogonal to $\mathrm{span}_L\{X^i \,|i \in \{0, \ldots, m-1\}\}$ in $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$. First we prove a formula that computes $\mathrm{ord}_{\mathrm{LA}_{h/e_K,\alpha}}(P_m)$ in terms of the distribution of the the zeros of $P_m(X)$. Through a series of estimates, we then produce an upper bound for $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_m)$, where $P_m(X)$ is any monic polynomial of given degree m; and we show that this bound is achieved. It is then easy to prove that $P_m(X)$ is orthogonal to $\operatorname{span}_L\{X_i \mid i \in \{0,\dots,m-1\}\}$ in $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$ if and only if $\operatorname{ord}_{\mathrm{LA}_{h/e_K}}(P_m)$ achieves this bound.

I have followed the ideas of Amice [1], §9, pp. 150–158 closely throughout this section. The main difference is that, since L need not be discretely valued, what appears in Amice [1], loc. cit. as finite sums appears here as integrals.

Proposition II.1.1 *Let $P_m(X) \in L[X]$ be a polynomial of degree m with leading coefficient a_m. Let $\alpha \in \mathfrak{O}_K$ and let $h \in \mathbf{Z}_{\geqslant 0}$. Then*

$$\operatorname{ord}_{\mathrm{LA}_{h/e_K,\alpha}}(P_m) = \operatorname{ord}_p(a_m) + \frac{hm}{e_K} - \int_{-\infty}^{h/e_K} (m - \mathrm{Z}(\alpha, r; P_m(X)))\,\mathrm{d}r.$$

Remark II.1.2 Recall, from remark I.5.8, p. 27, that $\mathrm{Z}(\alpha, r; P_m(X))$ is a step function in r, and for all $r \ll 0$ we have $\mathrm{Z}(\alpha, r; P_m(X)) = m$. Hence the integral in the proposition above is just a finite sum.

Proof of proposition II.1.1:
By definition I.1.6, p. 17 we have

$$\operatorname{ord}_{\mathrm{LA}_{h/e_K,\alpha}}(P_m) = \inf\left\{\operatorname{ord}_p(P_m(z)) \,\middle|\, z \in \alpha + \pi_K^h \mathfrak{O}_{\mathbf{C}_p}\right\}.$$

Write $P_m(X) = a_m \prod_{j=1}^m (X - u_j)$, with $u_1, \dots, u_m \in \mathbf{C}_p$. For $j \in \{1, \dots, m\}$, set $s_j := \operatorname{ord}_p(u_j - \alpha)$, and order the u_j such that $s_1 \leqslant s_2 \leqslant \dots \leqslant s_m$. Set $k := m - \mathrm{Z}(\alpha, h/e_K; P_m(X))$, so that $s_j < h/e_K$ if $j \leqslant k$ and $s_j \geqslant h/e_K$ if $j \geqslant k+1$. Note that

$$\operatorname{ord}_{\mathrm{LA}_{h/e_K,\alpha}}(P_m) = \operatorname{ord}_p(a_m) + \inf\left\{\sum_{j=1}^m \operatorname{ord}_p((z - u_j)) \,\middle|\, z \in \alpha + \pi_K^h \mathfrak{O}_{\mathbf{C}_p}\right\}. \tag{1}$$

If $\operatorname{ord}_p(u_j - \alpha) < h/e_K$, then by the strong triangle inequality we have $\operatorname{ord}_p(z - u_j) = \operatorname{ord}_p(u_j - \alpha)$ for all $z \in \alpha + \pi_K^h \mathfrak{O}_{\mathbf{C}_p}$. Hence

$$\sum_{j=1}^k \operatorname{ord}_p(z - u_j) = \sum_{j=1}^k s_j \qquad \forall z \in \alpha + \pi_K^h \mathfrak{O}_{\mathbf{C}_p}. \tag{2}$$

We now consider

$$\inf\left\{\sum_{j=k+1}^{m}\operatorname{ord}_p(z-u_j)\,\middle|\, z\in\alpha+\pi_K^h\mathfrak{O}_{\mathbf{C}_p}\right\}.$$

Since $\operatorname{ord}_p(z-u_j)\geqslant h/e_K$ for all $z\in\alpha+\pi_K^h\mathfrak{O}_{\mathbf{C}_p}$ and all $j\in\{k+1,\dots,m\}$, this infimum is certainly greater than or equal to $\frac{h}{e_K}(m-k)$. We will show that equality holds. For all $j\in\{k+1,\dots,m\}$, we have $\frac{u_j-\alpha}{\pi_K^h}\in\mathfrak{O}_{\mathbf{C}_p}$. Since the residue class field of $\mathfrak{O}_{\mathbf{C}_p}$ is infinite, we can choose $y\in\mathfrak{O}_{\mathbf{C}_p}$ whose residue class is not equal to the residue class of $\frac{u_j-\alpha}{\pi_K^h}$ for all $j\in\{k+1,\dots,m\}$; that is,

$$\begin{aligned} & y-\frac{u_j-\alpha}{\pi_K^h}\in{\mathfrak{O}_{\mathbf{C}_p}}^{\times},\\ \Rightarrow\qquad & \left(\alpha+\pi_K^h y\right)-u_j\in\pi_K^h{\mathfrak{O}_{\mathbf{C}_p}}^{\times}.\end{aligned}$$

Let $z=\alpha+\pi_K^h y$. Then $\operatorname{ord}_p(z-u_j)=h/e_K$ for all $j\in\{k+1,\dots,m\}$, and so

$$\sum_{j=k+1}^{m}\operatorname{ord}_p(z-u_j)=\frac{h}{e_K}(m-k).$$

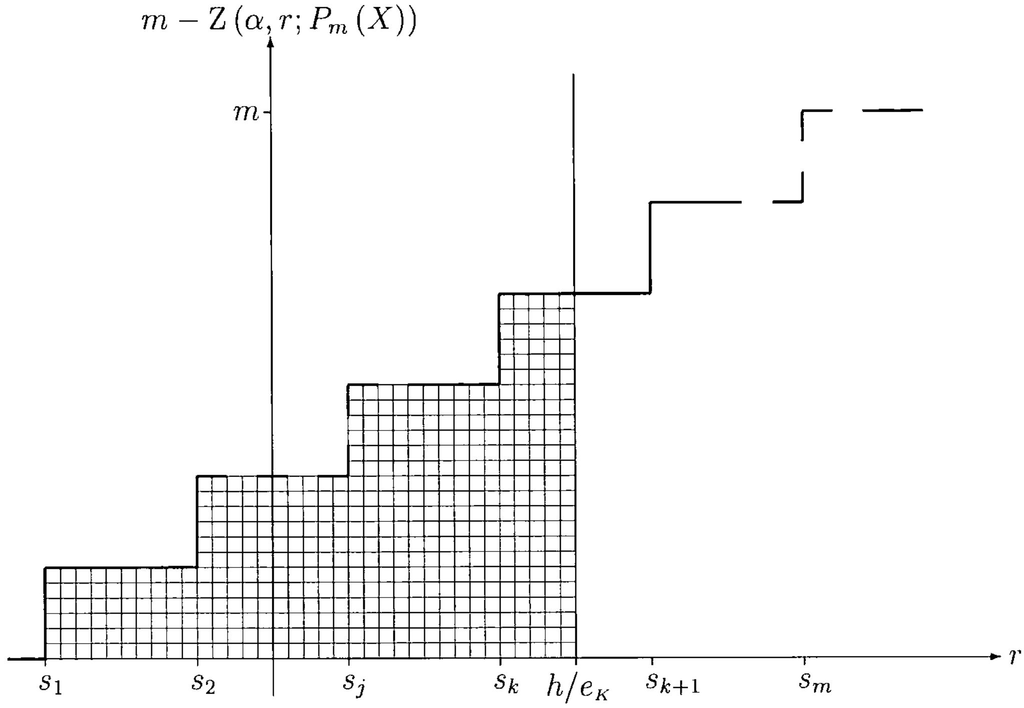
Therefore

$$\inf\left\{\sum_{j=k+1}^{m}\operatorname{ord}_p(z-u_j)\,\middle|\, z\in\alpha+\pi_K^h\mathfrak{O}_{\mathbf{C}_p}\right\}=\frac{h}{e_K}(m-k). \tag{3}$$

Combining equations (1), (2) and (3) we have

$$\operatorname{ord}_{\mathrm{LA}_{h/e_K,\alpha}}(P_m)=\operatorname{ord}_p(a_m)+\sum_{j=1}^{k}s_j+\frac{hm}{e_K}-\frac{hk}{e_K}. \tag{4}$$

To complete the proof, we calculate the area hatched below in two different ways.



We have

$$\begin{aligned}\int_{-\infty}^{h/e_K} m - \mathrm{Z}\left(\alpha, r; P_m\left(X\right)\right) \mathrm{d}r &= \sum_{j=1}^{k} \left(h/e_K - s_j\right) \\ &= \frac{hk}{e_K} - \sum_{j=1}^{k} s_j.\end{aligned}$$

Comparing this with equation 4 gives the result. □

In order to apply proposition II.1.1, we will need to calculate integrals of the form $\int_{-\infty}^{h/e_K} m - \psi(r)\mathrm{d}r$, where ψ is a step function such that $\psi(r) = m$ for all $r \ll 0$. We will start by defining an important example of such a function, and calculating the associated integral.

Definition II.1.3 Let $m \in \mathbf{Z}_{\geqslant 0}$. We define the step function

$$\begin{aligned}\psi_m : \mathbf{Q} &\to \mathbf{Z} \\ r &\mapsto \begin{cases} m & r \leqslant 0 \\ \lfloor m/q_K^k \rfloor & (k-1)/e_K < r \leqslant k/e_K \qquad \forall k \in \mathbf{Z}_{\geqslant 1} \end{cases}.\end{aligned}$$

Recall that $\lfloor m/q_K^k \rfloor$ denotes m/q_K^k rounded down to the nearest integer.

Remark II.1.4 Note that ψ_m is a finite $\mathbf{Z}$-linear combination of characteristic functions of half open intervals of the form $\{r \in \mathbf{Q} \,|a < r \leqslant b\}$, with $a, b \in \{-\infty\} \cup \mathbf{Q}$.

Lemma II.1.5 *Let $m \in \mathbf{Z}_{\geqslant 0}$ and let $h \in \mathbf{Z}_{\geqslant 0}$. Then*

$$\int_{-\infty}^{h/e_K} m - \psi_m (r) \, \mathrm{d}r = \frac{hm}{e_K} - \frac{1}{e_K} \sum_{k=1}^{h} \left\lfloor \frac{m}{q_K^k} \right\rfloor .$$

Proof:
We have

$$\begin{aligned} \int_{-\infty}^{h/e_K} m - \psi_m (r) \, \mathrm{d}r &= \int_{0}^{h/e_K} m - \psi_m (r) \, \mathrm{d}r \\ &= \frac{hm}{e_K} - \sum_{k=1}^{h} \frac{1}{e_K} \left\lfloor \frac{m}{q_K^k} \right\rfloor . \end{aligned}$$

□

In the following lemma, by using proposition II.1.1 and lemma II.1.5, we will see how inequality relationships between $\psi_m(r)$ and $\mathrm{Z}(\alpha, r; P_m(X))$ lead to bounds on $\mathrm{ord}_{\mathrm{LA}_{h/e_K, \alpha}}(P_m)$. These bounds will eventually give us our upper bound on $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_m)$, where $P_m(X) \in L[X]$ is any monic polynomial of given degree m.

Lemma II.1.6 *Let $P_m(X) \in L[X]$ be a polynomial of degree m with leading coefficient a_m. Let $\alpha \in \mathfrak{O}_K$ and let $h \in \mathbf{Z}_{\geqslant 0}$.*

i. If $\mathrm{Z}(\alpha, r; P_m(X)) \leqslant \psi_m(r)$ for all $r < h/e_K$, then

$$\mathrm{ord}_{\mathrm{LA}_{h/e_K, \alpha}}(P_m) \leqslant \mathrm{ord}_p(a_m) + \frac{1}{e_K} \sum_{k=1}^{h} \left\lfloor \frac{m}{q_K^k} \right\rfloor .$$

ii. If $\mathrm{Z}(\alpha, r; P_m(X)) \geqslant \psi_m(r)$ for all $r < h/e_K$, then

$$\mathrm{ord}_{\mathrm{LA}_{h/e_K, \alpha}}(P_m) \geqslant \mathrm{ord}_p(a_m) + \frac{1}{e_K} \sum_{k=1}^{h} \left\lfloor \frac{m}{q_K^k} \right\rfloor .$$

iii. *If* $\mathrm{Z}(\alpha, r; P_m(X)) \leqslant \psi_m(r)$ *for all* $r < h/e_K$*, and there exist* $a, b \in \mathbf{Q}$*,* $a < b \leqslant h/e_K$ *such that* $\mathrm{Z}(\alpha, r; P_m(X)) < \psi_m(r)$ *for all* $a < r \leqslant b$*, then*

$$\mathrm{ord}_{\mathrm{LA}_{h/e_K,\alpha}}(P_m) < \mathrm{ord}_p(a_m) + \frac{1}{e_K} \sum_{k=1}^{h} \left\lfloor \frac{m}{q_K^k} \right\rfloor .$$

Proof:

i. We have

$$m - \mathrm{Z}(\alpha, r; P_m(X)) \geqslant m - \psi_m(r) \qquad \forall r < h/e_K,$$

so, by integrating with respect to r and using lemma II.1.5 above, we obtain

$$\int_{-\infty}^{h/e_K} m - \mathrm{Z}(\alpha, r; P_m(X))\,\mathrm{d}r \geqslant \frac{hm}{e_K} - \frac{1}{e_K} \sum_{k=1}^{h} \left\lfloor \frac{m}{q_K^k} \right\rfloor .$$

Substituting this into proposition II.1.1, p. 29 we have

$$\begin{aligned} \mathrm{ord}_{\mathrm{LA}_{h/e_K,\alpha}}(P_m) &\leqslant \mathrm{ord}_p(a_m) + \frac{hm}{e_K} - \left(\frac{hm}{e_K} - \frac{1}{e_K} \sum_{k=1} h \left\lfloor \frac{m}{q_K^k} \right\rfloor \right) \\ &= \mathrm{ord}_p(a_m) + \frac{1}{e_K} \sum_{k=1}^{h} \left\lfloor \frac{m}{q_K^k} \right\rfloor , \end{aligned}$$

as required.

ii. As for part (i), with the inequality signs reversed.

iii. Note that

$$\int_{-\infty}^{h/e_K} m - \mathrm{Z}(\alpha, r; P_m(X))\,\mathrm{d}r > \frac{hm}{e_K} - \frac{1}{e_K} \sum_{k=1}^{h} \left\lfloor \frac{m}{q_K^k} \right\rfloor .$$

The proof then follows as in part (i), with inequalities replaced by strict inequalities. □

In order to use the lemma above, we will need to prove inequalities of the form $\mathrm{Z}(\alpha, r; P_m(X)) \leqslant \psi_m(r)$ for all $r < h/e_K$. We will use the following technical lemma to do this.

Lemma II.1.7 *Let $P(X) \in L[X]$, let $\alpha \in \mathfrak{O}_K$, and let $k_0 \in \mathbf{Z}_{\geqslant 0}$. Let $B \in \mathbf{Z}_{\geqslant 0}$ such that $\mathrm{Z}(\alpha, k_0/e_K; P(X)) \leqslant B$. Then for all $h \in \mathbf{Z}$, $h \geqslant k_0$, there exists $\alpha_h \in \mathfrak{O}_K$ such that $\alpha_h \equiv \alpha \bmod \pi_K^{k_0}\mathfrak{O}_K$ and, for all $k \in \{k_0+1, \dots, h\}$, we have*

$$\mathrm{Z}(\alpha_h, r; P(X)) \leqslant \left\lfloor \frac{B}{q_K^{k-k_0}} \right\rfloor \qquad \forall r > \frac{k-1}{e_K}.$$

Proof:

We will prove this by induction on h.

For $h = k_0$, the choice $\alpha_h := \alpha$ trivially satisfies the required conditions.

Now let $h \geqslant k_0 + 1$. By induction, there exists $\alpha_{h-1} \in \mathfrak{O}_K$ such that $\alpha_{h-1} \equiv \alpha \bmod \pi_K^{k_0}\mathfrak{O}_K$ and, for all $k \in \{k_0 + 1, \dots, h-1\}$, we have

$$\mathrm{Z}(\alpha_{h-1}, r; P(X)) \leqslant \left\lfloor \frac{B}{q_K^{k-k_0}} \right\rfloor \qquad \forall r > \frac{k-1}{e_K}.$$

Set $m := \mathrm{Z}\left(\alpha_{h-1}, \frac{h-1}{e_K}; P(X)\right)$. If $h \geqslant k_0 + 2$, then for $k = h-1$ we have $k_0 < k \leqslant h-1$, and so:

$$\begin{aligned} \mathrm{Z}(\alpha_{h-1}, r; P(X)) &\leqslant \left\lfloor \frac{B}{q_K^{h-1-k_0}} \right\rfloor \qquad \forall r > \frac{(h-1)-1}{e_K} \\ \Rightarrow \qquad m &\leqslant \left\lfloor \frac{B}{q_K^{h-1-k_0}} \right\rfloor. \end{aligned}$$

If $h = k_0 + 1$, then

$$\begin{aligned} m &\leqslant B \\ &= \left\lfloor \frac{B}{q_K^{h-1-k_0}} \right\rfloor. \end{aligned}$$

Hence in both cases we have proved

$$m \leqslant \left\lfloor \frac{B}{q_K^{h-1-k_0}} \right\rfloor.$$

We will now find a suitable choice of α_h. Let $u_1, \dots, u_m$ be the zeros of $P(X)$ contained in $\alpha_{h-1} + \pi_K^{h-1}\mathfrak{O}_{\mathbf{C}_p}$. Choose $\beta_1, \dots, \beta_{q_K} \in \mathfrak{O}_K$ such that

$$\alpha_{h-1} + \pi_K^{h-1}\mathfrak{O}_K = \coprod_{i=1}^{q_K} \beta_i + \pi_K^h \mathfrak{O}_K.$$

For $i \in \{1, \ldots, q_K\}$, set $m_i := \#\{j \in \{1, \ldots, m\} \,|u_j \in \beta_i + \pi_K^{h-1}\mathfrak{p}_{\mathbf{C}_p}\}$. Note that the sets $\beta_i + \pi_K^{h-1}\mathfrak{p}_{\mathbf{C}_p}$, for $i \in \{1, \ldots q_K\}$, are pairwise disjoint and contained in $\alpha_{h-1} + \pi_K^{h-1}\mathfrak{O}_{\mathbf{C}_p}$; hence

$$\sum_{i=1}^{q_K} m_i \leqslant m.$$

It follows that there exists $i_0 \in \{1, \ldots, q_K\}$ such that $m_{i_0} \leqslant m/q_K$. Set $\alpha_h := \beta_{i_0}$.

We will now prove that α_h satisfies the required properties. We have $\alpha_h = \beta_{i_0} \equiv \alpha_{h-1} \bmod \pi_K^{h-1}\mathfrak{O}_K$. This implies that $\alpha_h \equiv \alpha \bmod \pi_K^{k_0}\mathfrak{O}_K$ and, from remark I.5.8(ii), p. 27, that $\mathrm{Z}\left(\alpha_h, r; P\left(X\right)\right) = \mathrm{Z}\left(\alpha_{h-1}, r; P\left(X\right)\right)$ for all $r \leqslant \frac{h-1}{e_K}$. Hence by the induction hypothesis, for all $k \in \{k_0+1, \ldots, h-1\}$, we have

$$\mathrm{Z}\left(\alpha_h, r; P\left(X\right)\right) \leqslant \left\lfloor \frac{B}{q_K^{k-k_0}} \right\rfloor \qquad \forall r > \frac{k-1}{e_K}.$$

In addition, for $k = h$ and for all $r > \frac{h-1}{e_K}$, we have

$$\begin{aligned} \mathrm{Z}\left(\alpha_h, r; P\left(X\right)\right) &\leqslant m_{i_0} \\ &\leqslant m/q_K \\ &\leqslant \frac{1}{q_K}\frac{B}{q_K^{h-1-k_0}} \\ &= \frac{B}{q_K^{h-k_0}}, \\ \Rightarrow \qquad \mathrm{Z}\left(\alpha_h, r; P\left(X\right)\right) &\leqslant \left\lfloor \frac{B}{q_K^{h-k_0}} \right\rfloor \qquad \forall r > \frac{h-1}{e_K}. \end{aligned}$$

□

We are now in a position to obtain our upper bound on $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}\left(P_m\right)$.

Proposition II.1.8 *Let $P_m(X) \in L[X]$ be a polynomial of degree m with leading coefficient a_m. Let $h \in \mathbf{Z}_{\geqslant 0}$. Then*

$$\mathrm{ord}_{\mathrm{LA}_{h/e_K}}\left(P_m\right) \leqslant \mathrm{ord}_p\left(a_m\right) + \frac{1}{e_K}\sum_{k=1}^{h}\left\lfloor \frac{m}{q_K^k} \right\rfloor.$$

Proof:
For any $\alpha \in \mathfrak{O}_K$, we certainly have $\mathrm{Z}(\alpha, 0; P_m(X)) \leqslant m$. Now, by applying lemma II.1.7 above with $k_0 := 0$ and $B := m$, we have $\alpha_h \in \mathfrak{O}_K$ such that, for all $k \in \{1, \ldots, h\}$, we have

$$\mathrm{Z}(\alpha, r; P_m(X)) \leqslant \left\lfloor \frac{m}{q_K^k} \right\rfloor \qquad \forall r > \frac{k-1}{e_K}.$$

Hence

$$\mathrm{Z}(\alpha, r; P_m(X)) \leqslant \psi_m(r) \qquad \forall r \leqslant h/e_K,$$

so by lemma II.1.6(i), p. 32 we have

$$\mathrm{ord}_{\mathrm{LA}_{h/e_K, \alpha_h}}(P_m) \leqslant \mathrm{ord}_p(a_m) + \frac{1}{e_K} \sum_{k=1}^{h} \left\lfloor \frac{m}{q_K^k} \right\rfloor.$$

The result now follows from the definition of $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}$ (see definition I.1.6, p. 17). $\square$

Our next task is to prove that this upper bound is achieved. We define below a set of polynomials that do so.

Definition II.1.9 Let $P_m(X) \in L[X]$ be a polynomial of degree m and let $h \in \mathbf{Z}_{\geqslant 0}$. We say that $P_m(X)$ is evenly distributed of order h if, for all $\alpha \in \mathfrak{O}_K$ and all $k \in \{0, \ldots, h\}$, we have

$$\mathrm{Z}(\alpha, k/e_K; P_m(X)) \geqslant \lfloor m/q_K^k \rfloor.$$

We say that $P_m(X)$ is very evenly distributed if it is evenly distributed of order h for all $h \in \mathbf{Z}_{\geqslant 0}$.

Remark II.1.10

i. A polynomial $P(X) \in L[X]$ is evenly distributed of order 0 if and only if all the zeros of $P(X)$ lie in $\mathfrak{O}_{\mathbf{C}_p}$.

ii. If $h_1 \leqslant h_2$ and $P(X)$ is evenly distributed of order h_2, then $P(X)$ is also evenly distributed of order h_1.

Before we prove that evenly distributed polynomials achieve our upper bound for $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_m)$, we will first check that they actually exist.

Proposition II.1.11 *For all $m \in \mathbf{Z}_{\geqslant 0}$ and all $a_m \in L$, there exists a polynomial of degree m with leading coefficient a_m that is very evenly distributed.*

Proof:
We will explicitly define such a polynomial. Write $m = \sum_{i=0}^{n} s_i q_K^i$, with $s_i \in \{0, \ldots, q_K - 1\}$ for all $i \in \{0, \ldots, n\}$. For each $i \in \{0, \ldots, n\}$, let $R_{i/e_K} \subset \mathfrak{O}_K$ be a set of representatives of $\mathfrak{O}_K/\pi_K^i \mathfrak{O}_K$; note that $\# R_{i/e_K} = q_K^i$. Set

$$P(X) := a_m \prod_{i=0}^{n} \left(\left(\prod_{\beta \in R_{i/e_K}} (X - \beta) \right)^{s_i} \right).$$

Then $P(X) \in L[X]$ has degree m and leading coefficient a_m. Let $\alpha \in \mathfrak{O}_K$ and let $k \in \mathbf{Z}_{\geqslant 0}$. Then

$$\begin{aligned}
\mathrm{Z}(\alpha, k/e_K; P(X)) &= \sum_{i=0}^{n} s_i \, \#\{\beta \in R_{i/e_K} \,|\, \alpha \equiv \beta \bmod \pi_K^k \mathfrak{O}_K\} \\
&\geqslant \sum_{i=k}^{n} s_i \, \#\{\beta \in R_{i/e_K} \,|\, \alpha \equiv \beta \bmod \pi_K^k \mathfrak{O}_K\} \\
&= \sum_{i=k}^{n} s_i q_K^{i-k} \\
&= \frac{1}{q_K^k} \sum_{i=k}^{n} s_i q_K^i \\
&= \left\lfloor \frac{m}{q_K^k} \right\rfloor.
\end{aligned}$$

Therefore $P(X)$ is very evenly distributed. $\square$

As previously promised, we will now show that evenly distributed polynomials achieve the upper bound for $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_m)$.

Proposition II.1.12 *Let $h \in \mathbf{Z}_{\geqslant 0}$. Let $P_m(X) \in L[X]$ be a polynomial of degree m with leading coefficient a_m that is evenly distributed of order h. Then*

$$\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_m) = \mathrm{ord}_p(a_m) + \frac{1}{e_K}\sum_{k=1}^{h}\left\lfloor \frac{m}{q_K^k}\right\rfloor .$$

Proof:
Since $P_m(X)$ is evenly distributed of order h, and since, by remark I.5.8(i), p. 27, $\mathrm{Z}(\alpha, r; P_m(X))$ is a decreasing function of r; we see, for all $\alpha \in \mathfrak{O}_K$ and all $k \in \{0, \ldots, h\}$, that

$$\mathrm{Z}(\alpha, r; P_m(X)) \geqslant \lfloor m/q_K^k \rfloor \qquad \forall r \leqslant k/e_K;$$

that is,

$$\mathrm{Z}(\alpha, r; P_m(X)) \geqslant \psi_m(r) \qquad \forall r \leqslant h/e_K.$$

Hence, by lemma II.1.6(ii), p. 32, for all $\alpha \in \mathfrak{O}_K$, we have

$$\mathrm{ord}_{\mathrm{LA}_{h/e_K,\alpha}}(P_m) \geqslant \mathrm{ord}_p(a_m) + \frac{1}{e_K}\sum_{k=1}^{h}\left\lfloor \frac{m}{q_K^k}\right\rfloor .$$

Therefore

$$\begin{aligned}\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_m) &= \inf\left\{\mathrm{ord}_{\mathrm{LA}_{h/e_K,\alpha}}(P_m) \,|\, \alpha \in \mathfrak{O}_K\right\} \\ &\geqslant \mathrm{ord}_p(a_m) + \frac{1}{e_K}\sum_{k=1}^{h}\left\lfloor \frac{m}{q_K^k}\right\rfloor ,\end{aligned}$$

and by proposition II.1.8, p. 35 equality must hold. □

We can now give the connection between achieving our upper bound for $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_m)$ and orthogonality.

Proposition II.1.13 *Let $P_m(X) \in L[X]$ be a polynomial of degree m with leading coefficient a_m. Let $h \in \mathbf{Z}_{\geqslant 0}$. Then the function P_m is orthogonal to $\mathrm{span}_L\{X^j \,|\, j \in \{0, \ldots, m-1\}\}$ in $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$ if and only if*

$$\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_m) = \mathrm{ord}_p(a_m) + \frac{1}{e_K}\sum_{k=1}^{h}\left\lfloor \frac{m}{q_K^k}\right\rfloor .$$

Proof:
First assume that $P_m(X)$ is orthogonal to $\mathrm{span}_L \{X^j \,|j \in \{0, \dots, m-1\}\}$ in $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$. By proposition II.1.11, p. 37 we can find $\lambda_0, \lambda_1, \dots, \lambda_{m-1} \in L$ such that $\sum_{j=0}^{m-1} \lambda_j X^j + P_m(X)$ is very evenly distributed. Then by proposition II.1.12 above we have

$$\mathrm{ord}_{\mathrm{LA}_{h/e_K}}\left(\sum_{j=0}^{m-1} \lambda_j X^j + P_m(X)\right) = \mathrm{ord}_p(a_m) + \frac{1}{e_K}\sum_{k=1}^{h}\left\lfloor \frac{m}{q_K^k}\right\rfloor .$$

Now by definition I.3.1, p. 20 we have

$$\mathrm{ord}_{\mathrm{LA}_{h/e_K}}\left(\sum_{j=0}^{m-1} \lambda_j X^j + P_m(X)\right) \leqslant \mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_m);$$

hence

$$\mathrm{ord}_p(a_m) + \frac{1}{e_K}\sum_{k=1}^{h}\left\lfloor \frac{m}{q_K^k}\right\rfloor \leqslant \mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_m)$$

and by proposition II.1.8, p. 35 equality must hold, as required.

Conversely, assume that $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_m) = \mathrm{ord}_p(a_m) + \frac{1}{e_K}\sum_{k=1}^{h}\left\lfloor \frac{m}{q_K^k}\right\rfloor$. Let $R(X) \in \mathrm{span}_L \{X^j \,|j \in \{0, \dots, m-1\}\}$. By proposition II.1.8, p. 35 we have

$$\begin{aligned}\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_m(X) + R(X)) &\leqslant \mathrm{ord}_p(a_m) + \frac{1}{e_K}\sum_{k=1}^{h}\left\lfloor \frac{m}{q_K^k}\right\rfloor \\ &= \mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_m).\end{aligned}$$

Hence by definition I.3.1, p. 20 we have $P_m(X)$ is orthogonal to the L-linear span of $\{X^j \,|j \in \{0, \dots, m-1\}\}$, as required. □

Corollary II.1.14 *Let $h \in \mathbf{Z}_{\geqslant 0}$ and let $P_m(X) \in L[X]$ be a polynomial of degree m that is evenly distributed of order h. Then P_m is orthogonal to $\mathrm{span}_L \{X^j \,|j \in \{0, \dots, m-1\}\}$ in $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$.*

Proof:
Follows immediately from proposition II.1.12, p 37 and proposition II.1.13 above. □

We will now prove a partial converse to this corollary. In chapter V we will prove, for certain values of $m \in \mathbf{Z}_{\geqslant 0}$, that the Schneider Teitelbaum polynomial $P_{l,m}(X) \in \mathbf{C}_p[X]$ is not very evenly distributed. We will then use this converse to conclude that $\{P_{l,m} \,|m \in \mathbf{Z}_{\geqslant 0}\}$ is not an orthogonal set in $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, \mathbf{C}_p)$ for certain values of $h \in \mathbf{Z}_{\geqslant 0}$.

Proposition II.1.15 *Let $h_0 \in \mathbf{Z}_{\geqslant 0}$, and let $m \in \mathbf{Z}_{\geqslant 1}$ such that $q_K^{h_0}$ divides m. Let $P_m(X) \in L[X]$ be a polynomial of degree m that is not evenly distributed of order h_0. Then, for all $h \in \mathbf{Z}$, $h \geqslant h_0$, we have P_m is not orthogonal to* $\mathrm{span}_L\{X^j \,|j \in \{0, \ldots, m-1\}\}$ *in* $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$.

Proof:
Let $a_m \in L$ be the leading coefficient of $P_m(X)$. By proposition II.1.13, p. 38 it is enough to prove that

$$\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_m) < \mathrm{ord}_p(a_m) + \frac{1}{e_K}\sum_{k=1}^{h}\left\lfloor \frac{m}{q_K^k}\right\rfloor .$$

Hence, by lemma II.1.6(iii), p. 33, it is enough to show that there exists $\alpha \in \mathfrak{O}_K$ such that $\mathrm{Z}(\alpha, r; P_m(X)) \leqslant \psi_m(r)$ for all $r < h/e_K$, and that there exist $a, b \in \mathbf{Q}$, $a < b \leqslant h/e_K$ such that $\mathrm{Z}(\alpha, r; P_m(X)) < \psi_m(r)$ for all $a < r \leqslant b$. We will find such an $\alpha \in \mathfrak{O}_K$.

The idea of the proof is now quite simple, although the details become rather intricate. Since $P_m(X)$ is not evenly distributed of order h_0, there exists $\alpha_1 \in \mathfrak{O}_K$ and $k_1 \in \{0, \ldots, h_0\}$ such that

$$\mathrm{Z}(\alpha_1, k_1/e_K; P_m(X)) < \psi_m(k_1/e_K).$$

Using the geometry of $\mathfrak{O}_K$, we can adjust α_1 to ensure that

$$\mathrm{Z}(\alpha, r; P_m(X)) \leqslant \psi_m(r) \qquad \forall r \in \mathbf{Q},\ k_1/e_K \leqslant r \leqslant h/e_K;$$

we will use lemma II.1.7, p. 34 for this step. However, we also must be careful that

$$\mathrm{Z}(\alpha, r; P_m(X)) \leqslant \psi_m(r) \qquad \forall r \in \mathbf{Q},\ r < k_1/e_K.$$

The following paragraph and subsequent claim will deal with this potential problem; it is here that we will use the hypothesis $q_K^{h_0}$ divides m.

For $\alpha \in \mathfrak{O}_K$, set

$$a_\alpha := \sup \{ r_0 \in \mathbf{Q} \,|\, \mathrm{Z}(\alpha, r; P_m(X)) - \psi_m(r) \geqslant 0 \quad \forall r \leqslant r_0 \}.$$

By remark I.5.8(iii), p. 27 and remark II.1.4, p. 32, we see that $a_\alpha \in \mathbf{Q} \cup \{+\infty\}$ and, if $a_\alpha < +\infty$, that $\mathrm{Z}(\alpha, a_\alpha; P_m(X)) - \psi_m(a_\alpha) \geqslant 0$. Set

$$a := \inf \{ a_\alpha \,|\, \alpha \in \mathfrak{O}_K \}.$$

Note that $a_{\alpha_1} < k_1/e_K$, so $a < h_0/e_K$. Let R_{h_0/e_K} be a set of representatives of $\mathfrak{O}_K/\pi_K^{h_0}\mathfrak{O}_K$. If $\beta \in \mathfrak{O}_K$ with $a_\beta \leqslant h_0/e_K$, then by remark I.5.8(ii), p. 27 there exists $\alpha \in R_{h_0/e_K}$ such that $a_\alpha = a_\beta$. Hence $a = \min \left\{ a_\alpha \,\middle|\, \alpha \in R_{h_0/e_K} \right\}$ and so there exists $\alpha_0 \in \mathfrak{O}_K$ such that $a_{\alpha_0} = a$. We will now consider $r \leqslant a$ and $a < r \leqslant h/e_K$ separately. The following claim deals with the first of these two ranges.

Claim:
For all $\alpha \in \mathfrak{O}_K$, we have

$$\mathrm{Z}(\alpha, r; P_m(X)) = \psi_m(r) \qquad \forall r \leqslant a.$$

Proof of claim:
Fix any $\alpha \in \mathfrak{O}_K$ and any $r \in \mathbf{Q}$, $r \leqslant a$. Set $k := \min \{ i \in \mathbf{Z}_{\geqslant 0} \,|\, r \leqslant i/e_K \}$; note that $k \leqslant h_0$. For all $\beta \in \mathfrak{O}_K$, we have

$$\begin{aligned} \mathrm{Z}(\beta, r; P_m(X)) &\geqslant \psi_m(r) \\ &= \lfloor m/q_K^k \rfloor \\ &= m/q_K^k. \end{aligned}$$

Let $R_{k/e_K} \subset \mathfrak{O}_K$ be a set of representatives of $\mathfrak{O}_K/\pi_K^k\mathfrak{O}_K$ with $\alpha \in R_{k/e_K}$. The sets $\beta + p^r\mathfrak{O}_{\mathbf{C}_p}$, for $\beta \in R_{k/e_K}$, are pairwise disjoint, so

$$\begin{aligned} m &\geqslant \sum_{\beta \in R_{k/e_K}} \mathrm{Z}(\beta, r; P_m(X)) \\ &\geqslant q_K^k \frac{m}{q_K^k} \\ &= m. \end{aligned}$$

Hence equality must hold and $\mathrm{Z}(\beta, r; P_m(X)) = m/q_K^k$ for all $\beta \in R_{k/e_K}$; in particular, $\mathrm{Z}(\alpha, r; P_m(X)) = \psi_m(r)$, as required. This concludes the proof of the claim.

We now turn our attention to the range $r \in \mathbf{Q}$, $a < r \leqslant h/e_K$. Recall that there exists $\alpha_0 \in \mathfrak{O}_K$ such that $a_{\alpha_0} = a$, and that $\mathrm{Z}(\alpha_0, a; P_m(X)) - \psi_m(a) \geqslant 0$. Set $k_0 := \min\{i \in \mathbf{Z}_{\geqslant 0} \,|a < i/e_K\}$, and set $b := k_0/e_K$. Since $\mathrm{Z}(\alpha_0, r; P_m(X))$ is a decreasing function of r, and $\psi_m(r)$ is constant on the range $r \in \mathbf{Q}$, $a < r \leqslant b$, we must have $\mathrm{Z}(\alpha_0, r; P_m(X)) < \psi_m(r)$ for all $r \in \mathbf{Q}$, $a < r \leqslant b$. Now, by applying lemma II.1.7, p. 34 with $\alpha := \alpha_0$ and $B := m/q_K^{k_0}$, we obtain $\alpha_h \in \mathfrak{O}_K$ such that $\alpha_h \equiv \alpha_0 \bmod \pi_K^{k_0}\mathfrak{O}_K$ and, for all $k \in \{k_0 + 1, \ldots, h\}$, we have

$$\mathrm{Z}(\alpha_h, r; P_m(X)) \leqslant \left\lfloor \frac{m/q_K^{k_0}}{q_K^{k-k_0}} \right\rfloor = \left\lfloor \frac{m}{q_K^k} \right\rfloor \qquad \forall r > \frac{k-1}{e_K}.$$

Note that, by remark I.5.8(ii), p. 27, we have

$$\mathrm{Z}(\alpha_h, r; P_m(X)) = \mathrm{Z}(\alpha_0, r; P_m(X)) \qquad \forall r \leqslant b.$$

Combining this information with the results of the claim above, we have shown that $\alpha_h \in \mathfrak{O}_K$ satisfies

$$\mathrm{Z}(\alpha_h, r; P_m(X)) \begin{cases} = & \psi_m(r) & r \leqslant a \\ < & \psi_m(r) & a < r \leqslant b \\ \leqslant & \psi_m(r) & b < r \leqslant h/e_K, \end{cases}$$

which is what we required. □

II.2 Maximal orthogonal sets and bases

In this section we will prove that a sequence of polynomials $(P_m(X))_{m \in \mathbf{Z}_{\geqslant 0}}$, with $\deg(P_m(X)) = m$, that is orthogonal in $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$ is in fact a maximal orthogonal set. We will then note that if L is discretely valued then every maximal orthogonal set is a basis, but that this is not the case if L is not discretely valued.

The proof that such a set of polynomials is maximal orthogonal again follows the ideas of Amice [1], §9, pp. 150–158. My reference for facts regarding orthogonality in L-Banach spaces was Rooij [21], ch. 5, pp. 165–202.

Definition II.2.1 Let (E, ord_E) be an L-Banach space. We define $\mathfrak{O}_E := \{x \in E \,|\, \mathrm{ord}_E(x) \geqslant 0\}$ and $\mathfrak{p}_E := \{x \in E \,|\, \mathrm{ord}_E(x) > 0\}$. The residue class space $\overline{E}$ of E is defined as $\overline{E} := \mathfrak{O}_E/\mathfrak{p}_E$. It naturally has the structure of a vector space over the residue class field k_L of L. For $x \in \mathfrak{O}_E$, we denote the image of x in $\overline{E}$ by $\overline{x}$.

We denote the residue class space of $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$ by $\overline{\mathrm{LA}}_{h/e_K}(\mathfrak{O}_K, L)$, and that of $\mathrm{LP}^{[0,n]}_{h/e_K}(\mathfrak{O}_K, L)$ by $\overline{\mathrm{LP}}^{[0,n]}_{h/e_K}(\mathfrak{O}_K, L)$.

The relevance of the residue class space of an L-Banach space to this section is revealed by the following proposition.

Proposition II.2.2 *Let E be an L-Banach space and let X be a subset of $\mathfrak{O}_E$.*

i. The set X is orthonormal in E if and only if $\{\overline{x} \,|\, x \in X\} \subset \overline{E}$ is k_L-linearly independent in $\overline{E}$.

ii. The set X is maximal orthonormal in E if and only if $\{\overline{x} \,|\, x \in X\} \subset \overline{E}$ is a k_L-basis of $\overline{E}$.

Proof:
See Rooij [21], ch. 5, exercise 5.A, p. 167. □

We have the following elementary results about $\overline{\mathrm{LA}}_{h/e_K}(\mathfrak{O}_K, L)$.

Proposition II.2.3 *Let $h \in \mathbf{Z}_{\geqslant 0}$.*

i. Let $n \in \mathbf{Z}_{\geqslant 0}$, then $\overline{\mathrm{LP}}^{[0,n]}_{h/e_K}(\mathfrak{O}_K, L)$ has dimension $q_K^h(n+1)$ as a k_L-vector space.

ii. We have

$$\overline{\mathrm{LA}}_{h/e_K}(\mathfrak{O}_K, L) = \bigcup_{n \in \mathbf{Z}_{\geqslant 0}} \overline{\mathrm{LP}}^{[0,n]}_{h/e_K}(\mathfrak{O}_K, L)\,.$$

Proof:

i. By proposition I.3.5(i), p. 22 we know that

$$\left\{\chi_{\alpha,i}\,\middle|\,\alpha \in \mathfrak{O}_K/\pi_K^h\mathfrak{O}_K,\ i \in \{0,\dots,n\}\right\}$$

forms an orthonormal L-Banach basis of $\mathrm{LP}_{h/e_K}^{[0,n]}(\mathfrak{O}_K, L)$; hence, by proposition II.2.2(ii) above, its image in $\overline{\mathrm{LP}}_{h/e_K}^{[0,n]}(\mathfrak{O}_K, L)$ is a k_L- basis. Clearly $\#\left\{\chi_{\alpha,i}\,\middle|\,\alpha \in \mathfrak{O}_K/\pi_K^h\mathfrak{O}_K,\ i \in \{0,\dots,n\}\right\} = q_K^h(n+1)$.

ii. Let $f \in \mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$ such that $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(f) \geqslant 0$. We will show that its image $\overline{f} \in \overline{\mathrm{LA}}_{h/e_K}(\mathfrak{O}_K, L)$ lies in $\overline{\mathrm{LP}}_{h/e_K}^{[0,n]}(\mathfrak{O}_K, L)$ for some $n \in \mathbf{Z}_{\geqslant 0}$. We can write

$$f = \sum_{\substack{\alpha \in \mathfrak{O}_K/\pi_K^h\mathfrak{O}_K \\ i \in \mathbf{Z}_{\geqslant 0}}} a_{\alpha,i}\chi_{\alpha,i}$$

with $a_{\alpha,i} \in L$, $\mathrm{ord}_p(a_{\alpha,i}) \to +\infty$ as $i \to +\infty$ for each $\alpha \in \mathfrak{O}_K/\pi_K^h\mathfrak{O}_K$. Choose $n \in \mathbf{Z}_{\geqslant 0}$ large enough such that $\mathrm{ord}_p(a_{\alpha,i}) > 0$ for all $i > n$ and all $\alpha \in \mathfrak{O}_K/\pi_K^h\mathfrak{O}_K$. Then $\overline{f} \in \overline{\mathrm{LP}}_{h/e_K}^{[0,n]}(\mathfrak{O}_K, L)$. □

Recall that in this section we are seeking to prove that an orthonormal sequence of polynomials $(P_m(X) \in L[X])_{m \in \mathbf{Z}_{\geqslant 0}}$, with $\deg(P_m(X)) = m$, is in fact a maximal orthonormal set. Using propositions II.2.2 and II.2.3 above it is possible to reduce this to proving that enough of the images $\overline{P_m} \in \overline{\mathrm{LA}}_{h/e_K}(\mathfrak{O}_K, L)$ lie in each $\overline{\mathrm{LP}}_{h/e_K}^{[0,n]}(\mathfrak{O}_K, L)$; the details of this argument are given in the proof of proposition II.2.8. Given a polynomial $P_m(X)$ of degree m, we are interested, therefore, in determining a small value of $n \in \mathbf{Z}_{\geqslant 0}$ such that $\overline{P_m} \in \overline{\mathrm{LP}}_{h/e_K}^{[0,n]}(\mathfrak{O}_K, L)$. This problem is solved by the next two propositions.

Definition II.2.4 Let $h \in \mathbf{Z}_{\geqslant 0}$, let $f \in \mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$, and let $\alpha \in \mathfrak{O}_K$. We define $f_{\alpha,h} := \mathbf{1}_{\alpha+\pi_K^h\mathfrak{O}_K} f \in \mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$; that is, $f_{\alpha,h}$ is the locally analytic function that is equal to f on $\alpha + \pi_K^h\mathfrak{O}_K$ and identically zero outside this set.

Proposition II.2.5 *Let $h \in \mathbf{Z}_{\geqslant 0}$. Let $P(X) \in L[X]$ be a polynomial such that* $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P) = 0$. *Let $\alpha \in \mathfrak{O}_K$ such that* $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_{\alpha,h}) = 0$. *Then $\overline{P_{\alpha,h}} \in \overline{\mathrm{LP}}^{[0,n]}_{h/e_K}(\mathfrak{O}_K, L)$, where $n = \mathrm{Z}(\alpha, h/e_K; P(X))$.*

Proof:
Set $m := \deg(P(X))$. Write

$$P(X) = \sum_{i=0}^{m} a_i \binom{\frac{X-\alpha}{\pi_K^h}}{i} \qquad a_i \in L.$$

Since $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_{\alpha,h}) = 0$, we have $\min\{\mathrm{ord}_p(a_i) \,|\, i \in \{0, \ldots, m\}\} = 0$. We must prove that $\mathrm{ord}_p(a_i) > 0$ for all $i \in \mathbf{Z}$, $i > \mathrm{Z}(\alpha, h/e_K; P(X))$.

Set

$$Q(Y) := \sum_{i=0}^{m} a_i Y^i.$$

Counting multiplicities, the polynomial $P(X)$ has $\mathrm{Z}(\alpha, h/e_K; P(X))$ zeros in $\alpha + \pi_K^h \mathfrak{O}_K$; hence we see that, counting multiplicities, the polynomial $Q(Y)$ has $\mathrm{Z}(\alpha, h/e_K; P(X))$ zeros in $\mathfrak{O}_{\mathbf{C}_p}$. Hence by theorem I.5.6, p. 26 we have $\mu(i; Q(Y)) > 0$ for all $i > \mathrm{Z}(\alpha, h/e_K; P(X))$. Therefore $\mathrm{ord}_p(a_i) > 0$ for all $i > \mathrm{Z}(\alpha, h/e_K; P(X))$, as required. □

Proposition II.2.6 *Let $h \in \mathbf{Z}_{\geqslant 0}$. Let $P_m(X) \in L[X]$ be a polynomial of degree m. Let $\alpha \in \mathfrak{O}_K$ such that* $\mathrm{ord}_{\mathrm{LA}_{h/e_K,\alpha}}(P_m) = \mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_m)$. *Then*

$$\mathrm{Z}(\alpha, h/e_K; P_m(X)) \leqslant \lfloor m/q_K^h \rfloor.$$

Proof:
Set $s := \mathrm{Z}(\alpha, h/e_K; P_m(X))$. We will prove by induction that

$$\mathrm{Z}\left(\alpha, \frac{h-i}{e_K}; P_m(X)\right) \geqslant q_K^i s \qquad \forall i \in \{0, \ldots, h\}.$$

The case $i = 0$ is trivial.

We will now prove the induction step; assume that the statement is true for all $i \in \{0, \ldots, i_0\}$. For contradiction, assume that

$$\mathrm{Z}\left(\alpha, \frac{h-(i_0+1)}{e_K}; P_m(X)\right) \leqslant q_K^{i_0+1} s - 1.$$

Then, by applying lemma II.1.7, p. 34 with $k_0 = h - (i_0 + 1)$ and $B = q_K^{i_0+1} s - 1$, there exists $\alpha_h \in \mathfrak{O}_K$ such that $\alpha_h \equiv \alpha \bmod \pi_K^{h-(i_0+1)} \mathfrak{O}_K$ and, for all $k \in \mathbf{Z}$, $h - (i_0 + 1) < k \leqslant h$, we have

$$\mathrm{Z}\left(\alpha_h, r; P_m\left(X\right)\right) \leqslant \left\lfloor \frac{q_K^{i_0+1} s - 1}{q_K^{k-(h-(i_0+1))}} \right\rfloor = q_K^{h-k} s - 1 \qquad \forall r > \frac{k-1}{e_K}.$$

Hence we have shown that

$$\mathrm{Z}\left(\alpha_h, r; P_m\left(X\right)\right) = \mathrm{Z}\left(\alpha, r; P_m\left(X\right)\right) \qquad -\infty < r \leqslant \frac{h - (i_0 + 1)}{e_K}$$

and

$$\mathrm{Z}\left(\alpha_h, r; P_m\left(X\right)\right) < \mathrm{Z}\left(\alpha, r; P_m\left(X\right)\right) \qquad \frac{h - (i_0 + 1)}{e_K} < r \leqslant \frac{h}{e_K}.$$

Therefore

$$\int_{-\infty}^{h/e_K} m - \mathrm{Z}\left(\alpha_h, r; P_m\left(X\right)\right) \mathrm{d}r > \int_{-\infty}^{h/e_K} m - \mathrm{Z}\left(\alpha, r; P_m\left(X\right)\right) \mathrm{d}r,$$

and so by proposition II.1.1, p. 29 we have

$$\mathrm{ord}_{\mathrm{LA}_{h/e_K, \alpha_h}}\left(P_m\right) < \mathrm{ord}_{\mathrm{LA}_{h/e_K, \alpha}}\left(P_m\right),$$

which is a contradiction to $\mathrm{ord}_{\mathrm{LA}_{h/e_K, \alpha}}\left(P_m\right) = \mathrm{ord}_{\mathrm{LA}_{h/e_K}}\left(P_m\right)$. This completes the induction.

Now for $i = h$ we have shown that $\mathrm{Z}\left(\alpha, 0; P_m\left(X\right)\right) \geqslant q_K^h s$, and clearly $m \geqslant \mathrm{Z}\left(\alpha, 0; P_m\left(X\right)\right)$, so $m/q_K^h \geqslant s$, as required. □

Corollary II.2.7 *Let $h \in \mathbf{Z}_{\geqslant 0}$. Let $P(X) \in L[X]$ be a polynomial of degree m such that* $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}\left(P\right) \geqslant 0$. *Then* $\overline{P} \in \overline{\mathrm{LP}}_{h/e_K}^{[0, \lfloor m/q_K^h \rfloor]}\left(\mathfrak{O}_K, L\right)$.

Proof:
Let R_{h/e_K} be a set of representatives of $\mathfrak{O}_K/\pi_K^h \mathfrak{O}_K$, so $P = \sum_{\alpha \in R_{h/e_K}} P_{\alpha,h}$. If $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}\left(P_{\alpha,h}\right) > 0$, then $\overline{P_{\alpha,h}} = 0$. If $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}\left(P_{\alpha,h}\right) = 0$, then by proposition II.2.6 above we have $\mathrm{Z}\left(\alpha, h/e_K; P\left(X\right)\right) \leqslant \lfloor m/q_K^h \rfloor$, and then by proposition II.2.5, p. 45 we have $\overline{P_{\alpha,h}} \in \overline{\mathrm{LP}}_{h/e_K}^{[0, \lfloor m/q_K^h \rfloor]}\left(\mathfrak{O}_K, L\right)$. □

We can now prove the main result of this section.

Proposition II.2.8 *Let $h \in \mathbf{Z}_{\geqslant 0}$. Let $(P_m(X))_{m \in \mathbf{Z}_{\geqslant 0}} \subset L[X]$ be a sequence of polynomials that are orthogonal in* $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$ *and such that* $\deg(P_m(X)) = m$. *Then $\{P_m \,|\, m \in \mathbf{Z}_{\geqslant 0}\}$ is a maximal orthogonal set in* $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$.

Proof:
For $m \in \mathbf{Z}_{\geqslant 0}$, set $v_m := \mathrm{ord}_{\mathrm{LA}_{h/e_K}}(P_m)$ and set $R_m := \pi_K^{-e_K v_m} P_m$; so we have $\mathrm{ord}_{\mathrm{LA}_{h/e_K}}(R_m) = 0$. We must prove that $\{R_m \,|\, m \in \mathbf{Z}_{\geqslant 0}\}$ is a maximal orthonormal set in $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, L)$. By proposition II.2.2(ii), p. 43 it is enough to prove that $\{\overline{R_m} \,|\, m \in \mathbf{Z}_{\geqslant 0}\}$ is a k_L-basis of $\overline{\mathrm{LA}}_{h/e_K}(\mathfrak{O}_K, L)$.

Let $n \in \mathbf{Z}_{\geqslant 0}$. We will show that $\{\overline{R_m} \,|\, m \in \{0, 1, \ldots, q_K^h(n+1) - 1\}\}$ forms a k_L-basis of $\overline{\mathrm{LP}}^{[0,n]}_{h/e_K}(\mathfrak{O}_K, L)$. If $m \in \{0, \ldots, q_K^h(n+1) - 1\}$, then $\lfloor m/q_K^h \rfloor \leqslant n$; hence by corollary II.2.7 above we have $\overline{R_m} \in \overline{\mathrm{LP}}^{[0,n]}_{h/e_K}(\mathfrak{O}_K, L)$. By proposition II.2.2(i), p. 43 we see that $\{\overline{R_m} \,|\, m \in \{0, \ldots, q_K^h(n+1) - 1\}\}$ is k_L-linearly independent in $\overline{\mathrm{LP}}^{[0,n]}_{h/e_K}(\mathfrak{O}_K, L)$. By proposition II.2.3(i), p. 43 we know that $\overline{\mathrm{LP}}^{[0,n]}_{h/e_K}(\mathfrak{O}_K, L)$ has dimension $q_K^h(n+1)$ as a k_L-vector space; hence $\{\overline{R_m} \,|\, m \in \{0, \ldots, q_K^h(n+1) - 1\}\}$ forms a k_L-basis of $\overline{\mathrm{LP}}^{[0,n]}_{h/e_K}(\mathfrak{O}_K, L)$.

Now by proposition II.2.3(ii), p. 43 it follows that $\{\overline{R_m} \,|\, m \in \mathbf{Z}_{\geqslant 0}\}$ is a k_L-basis of $\overline{\mathrm{LA}}_{h/e_K}(\mathfrak{O}_K, L)$, as required. □

We conclude this section by stating a theorem on the relationship between maximal orthogonal sets and orthogonal bases.

Theorem II.2.9 *Let E be an infinite dimensional L-Banach space.*

i. If L is discretely valued, then every maximal orthogonal set in E is an orthogonal L-Banach basis.

ii. If L is not discretely valued, then there exists a maximal orthogonal set in E that is not an L-Banach basis.

Proof:
See Rooij [21], theorem 5.13(α), p. 177 and theorem 5.16(ι), p. 179. □

Chapter III

Binomial functions

The main purpose of this chapter is to illustrate how the orthogonality of the binomial functions is essential in characterising the Amice transforms of measures. In section III.1 we will prove that the set of binomial functions is an orthogonal Banach basis of $\mathrm{LA}_h(\mathbf{Z}_p, L)$, for all $h \in \mathbf{Z}_{\geqslant 0}$. This fact is used in section III.2 to study the dual of $\mathrm{LA}_h(\mathbf{Z}_p, L)$ and the Amice transform.

None of the material in this chapter is original. The results are due to Amice [1] and Amice and Vélu [2]. My presentation also owes much to Colmez [8], ch. I, pp. 495–502.

III.1 Locally analytic functions on $\mathbf{Z}_p$

In this section we recall the definition of the binomial polynomials $\binom{X}{m} \in \mathbf{Q}[X]$ and, for each $h \in \mathbf{Z}_{\geqslant 0}$, prove that they form an orthogonal L-Banach basis of $\mathrm{LA}_h(\mathbf{Z}_p, L)$. The results of this section are due to Amice [1], with the minor difference that here we do not assume that L is discretely valued.

Definition III.1.1 Let $m \in \mathbf{Z}_{\geqslant 0}$. We define the binomial polynomial

$$\binom{X}{m} := \frac{1}{m!}\prod_{i=0}^{m-1}(X-i) \in \mathbf{Q}[X].$$

When we consider $\binom{X}{m}$ as an element of $\mathrm{LA}_0(\mathbf{Z}_p, L) \subset \mathrm{LA}(\mathbf{Z}_p, L)$, we will call it a binomial function.

Proposition III.1.2 *i. Let $m \in \mathbf{Z}_{\geqslant 0}$. In $\mathbf{Q}[X_1, X_2]$ we have*

$$\binom{X_1 + X_2}{m} = \sum_{\substack{i_1, i_2 \in \mathbf{Z}_{\geqslant 0} \\ i_1 + i_2 = m}} \binom{X_1}{i_1}\binom{X_2}{i_2}.$$

ii. Let $m \in \mathbf{Z}_{\geqslant 0}$ and let $\alpha \in \mathbf{Z}_p$. Then $\operatorname{ord}_p\left(\binom{\alpha}{m}\right) \geqslant 0$.

Proof:
See Schikhof [18], proposition 47.2(iii,v), p. 138. □

In order to allow for the case of L not discretely valued, we will need the following technical lemma.

Lemma III.1.3 *Let E be a K-Banach space, and let F be an L-Banach space such that $E \subset F$. Assume that there exists a set $\{b_i \mid i \in \mathbf{Z}_{\geqslant 0}\} \subset E$ that is simultaneously an orthonormal K-Banach basis for E and an orthonormal L-Banach basis for F.*

Then any orthogonal K-Banach basis for E spans F as an L-Banach space.

Proof:
Let $\{c_j \mid j \in \mathbf{Z}_{\geqslant 0}\} \subset E$ be an orthogonal K-Banach basis for E. Let $x \in F$. We must show that x can be written as the limit of a series of elements in the L-linear span of $\{c_j \mid j \in \mathbf{Z}_{\geqslant 0}\}$.

Since $\{b_i \mid i \in \mathbf{Z}_{\geqslant 0}\}$ is an orthonormal L-Banach basis of F, we can write

$$x = \sum_{i=0}^{+\infty} \beta_i b_i,$$

where $(\beta_i)_{i \in \mathbf{Z}_{\geqslant 0}} \subset L$ is such that $\operatorname{ord}_p(\beta_i) \to +\infty$ as $i \to +\infty$. By orthonormality, we have that $\operatorname{ord}_p(\beta_i) \geqslant \operatorname{ord}_F(x)$ for all $i \in \mathbf{Z}_{\geqslant 0}$.

Since $\{c_j \mid j \in \mathbf{Z}_{\geqslant 0}\}$ is an orthogonal K-Banach basis for E, for each $i \in \mathbf{Z}_{\geqslant 0}$ we can write

$$b_i = \sum_{j=0}^{+\infty} \gamma_{ij} c_j,$$

where $(\gamma_{ij})_{j\in\mathbf{Z}_{\geqslant 0}} \subset K$ is such that $\mathrm{ord}_E(\gamma_{ij}c_j) \to +\infty$ as $j \to +\infty$. By orthogonality, we have $\mathrm{ord}_E(\gamma_{ij}c_j) \geqslant 0$ for all $(i,j) \in \mathbf{Z}_{\geqslant 0} \times \mathbf{Z}_{\geqslant 0}$.

We have, therefore,

$$x = \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} \beta_i\gamma_{ij}c_j.$$

Set $a_{ij} := \beta_i\gamma_{ij}c_j \in F$. For any $s \in \mathbf{Q}$, there are only finitely many $(i,j) \in \mathbf{Z}_{\geqslant 0} \times \mathbf{Z}_{\geqslant 0}$ such that $\mathrm{ord}_F(a_{ij}) < s$. Hence we have

$$\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} a_{ij} = \sum_{j=0}^{+\infty}\sum_{i=0}^{+\infty} a_{ij}$$

(see, for example, Robert [17], ch. 2, §1.2, corollary, p. 76). Therefore

$$x = \sum_{j=0}^{+\infty}\left(\sum_{i=0}^{+\infty}\beta_i\gamma_{ij}\right)c_j$$

and $\mathrm{ord}_F\left(\left(\sum_{i=0}^{+\infty}\beta_i\gamma_{ij}\right)c_j\right) \to +\infty$ as $j \to +\infty$, as required. □

We can now prove the main result of this section.

Theorem III.1.4 (Amice) *The set of binomial functions* $\left\{\binom{X}{m} \,|m \in \mathbf{Z}_{\geqslant 0}\right\}$ *forms an orthogonal L-Banach basis of* $\mathrm{LA}_h(\mathbf{Z}_p, L)$ *for all* $h \in \mathbf{Z}_{\geqslant 0}$. *We have*

$$\mathrm{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right) = -\mathrm{ord}_p(m!) + \sum_{k=1}^{h}\left\lfloor\frac{m}{p^k}\right\rfloor.$$

Proof:
First we will prove that the polynomial $\binom{X}{m}$ is very evenly distributed, for all $m \in \mathbf{Z}_{\geqslant 0}$. Let $\alpha \in \mathbf{Z}_p$ and let $k \in \mathbf{Z}_{\geqslant 0}$. We must show that

$$\mathrm{Z}\left(\alpha, k; \binom{X}{m}\right) \geqslant \left\lfloor\frac{m}{p^k}\right\rfloor.$$

Since $\mathbf{Z}$ is dense in $\mathbf{Z}_p$ we can choose $n \in \mathbf{Z} \cap \left(\alpha + p^k\mathbf{Z}_p\right)$. The set of zeros of $\binom{X}{m}$ is $\{0, 1, \ldots, m-1\}$, and $\#\left(n + p^k\mathbf{Z}_p\right) \cap \{0, 1, \ldots, m-1\} \geqslant \left\lfloor\frac{m}{p^k}\right\rfloor$, as required.

Fix $h \in \mathbf{Z}_{\geqslant 0}$. By corollary II.1.14, p. 39 we have that the binomial functions are orthogonal in both $\mathrm{LA}_h(\mathbf{Z}_p, \mathbf{Q}_p)$ and $\mathrm{LA}_h(\mathbf{Z}_p, L)$. It follows by proposition II.1.13, p. 38 that

$$\mathrm{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right) = -\mathrm{ord}_p(m!) + \sum_{k=1}^{h} \left\lfloor \frac{m}{p^k} \right\rfloor .$$

It remains to prove that the binomial functions span $\mathrm{LA}_h(\mathbf{Z}_p, L)$ as an L-Banach space. By proposition II.2.8, p. 47 we have that the binomial functions $\left\{\binom{X}{m} \middle| m \in \mathbf{Z}_{\geqslant 0}\right\}$ form a maximal orthogonal set in $\mathrm{LA}_h(\mathbf{Z}_p, \mathbf{Q}_p)$, so by theorem II.2.9(i), p. 47 they form an orthogonal $\mathbf{Q}_p$-Banach basis of $\mathrm{LA}_h(\mathbf{Z}_p, \mathbf{Q}_p)$. Now, by proposition I.3.5, p. 22 the set

$$\left\{\chi_{\alpha,i} \middle| \alpha \in \mathbf{Z}_p/p^h\mathbf{Z}_p,\ i \in \mathbf{Z}_{\geqslant 0}\right\}$$

is simultaneously an orthonormal $\mathbf{Q}_p$-Banach basis of $\mathrm{LA}_h(\mathbf{Z}_p, \mathbf{Q}_p)$ and an orthonormal L-Banach basis of $\mathrm{LA}_h(\mathbf{Z}_p, L)$. Hence, by lemma III.1.3 above, the binomial functions $\left\{\binom{X}{m} \middle| m \in \mathbf{Z}_{\geqslant 0}\right\}$ span $\mathrm{LA}_h(\mathbf{Z}_p, L)$ as an L-Banach space, as required. □

Next we will simplify the expression for $\mathrm{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right)$. To do so, we will need the following useful fact.

Lemma III.1.5 *Let $m \in \mathbf{Z}_{\geqslant 0}$ and write m in its base p expansion: $m = \sum_{i=0}^{n} s_i p^i$, with $s_i \in \mathbf{Z}$, $0 \leqslant s_i \leqslant p-1$ for all $i \in \{0, \dots, n\}$. Then*

$$\mathrm{ord}_p(m!) = \frac{1}{p-1}\left(m - \sum_{i=0}^{n} s_i\right).$$

Proof:
See Schikhof [18], lemma 22.5, p. 70. □

Proposition III.1.6 *Let $h \in \mathbf{Z}_{\geqslant 0}$ and let $m \in \mathbf{Z}_{\geqslant 0}$. Then*

$$\mathrm{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right) = -\mathrm{ord}_p\left(\lfloor m/p^h \rfloor !\right).$$

Proof:
Write $m = \sum_{i=0}^{n} s_i p^i$ with $s_i \in \mathbf{Z}$, $0 \leqslant s_i \leqslant p-1$ for all $i \in \{0, \ldots, n\}$. If $n < h$, then set $s_i := 0$ for all $i \in \{n+1, \ldots, h\}$. We have

$$\begin{aligned}
\sum_{k=1}^{h} \left\lfloor \frac{m}{p^k} \right\rfloor &= \sum_{k=1}^{h} \left(\sum_{i=k}^{n} s_i p^{i-k} \right) \\
&= \sum_{i=1}^{n} \left(\sum_{k=1}^{\min\{i,h\}} s_i p^{i-k} \right) \\
&= \sum_{i=1}^{h-1} \left(\sum_{k=1}^{i} s_i p^{i-k} \right) + \sum_{i=h}^{n} \left(\sum_{k=1}^{h} s_i p^{i-k} \right) \\
&= \frac{1}{p-1} \left(\sum_{i=1}^{h-1} s_i \left(p^i - 1\right) + \sum_{i=h}^{n} s_i \left(p^i - p^{i-h}\right) \right) \\
&= \frac{1}{p-1} \left(\sum_{i=1}^{n} s_i \left(p^i - 1\right) - \sum_{i=h}^{n} s_i \left(p^{i-h} - 1\right) \right) \\
&= \frac{1}{p-1} \left(\left(m - \sum_{i=1}^{n} s_i \right) - \left(\left\lfloor \frac{m}{p^h} \right\rfloor - \sum_{i=h}^{n} s_i \right) \right) \\
&= \mathrm{ord}_p\,(m!) - \mathrm{ord}_p\left(\left\lfloor m/p^h \right\rfloor ! \right),
\end{aligned}$$

where we have used lemma III.1.5 above to obtain the last equality. The result now follows by theorem III.1.4, p. 50. □

Using this formula, we will now give simple upper and lower bounds for $\mathrm{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right)$. These bounds will allow us to determine which expressions of the form $\sum_{m=0}^{+\infty} a_m \binom{X}{m}$, with $(a_m)_{m \in \mathbf{Z}_{\geqslant 0}} \subset L$, converge to some element of $\mathrm{LA}\,(\mathbf{Z}_p, L)$.

Definition III.1.7 For $h \in \mathbf{Z}_{\geqslant 0}$, we define

$$r_h := \frac{1}{(p-1)p^h}.$$

Lemma III.1.8 *Let $m \in \mathbf{Z}_{\geqslant 0}$ and let $h \in \mathbf{Z}_{\geqslant 0}$. We have*

$$m r_{h+1} - 1 \leqslant -\mathrm{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right) \leqslant m r_h.$$

Proof:
Write $m = \sum_{i=0}^{n} s_i p^i$ with $s_i \in \mathbf{Z}$, $0 \leqslant s_i \leqslant p-1$ for all $i \in \{0, \ldots, n\}$. From proposition III.1.6, above, and lemma III.1.5, p. 51, we have

$$\begin{aligned}
-\mathrm{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right) &= \mathrm{ord}_p\left(\left\lfloor \frac{m}{p^h} \right\rfloor !\right) \\
&= \mathrm{ord}_p\left(\left(\sum_{i=h}^{n} s_i p^{i-h}\right)!\right) \\
&= \frac{1}{p-1}\left(\sum_{i=h}^{n} s_i p^{i-h} - \sum_{i=h}^{n} s_i\right) \\
&= \frac{1}{p-1}\sum_{i=h+1}^{n} s_i\left(p^{i-h}-1\right).
\end{aligned}$$

Now we have

$$\begin{aligned}
mr_h &= \frac{1}{p-1}\sum_{i=0}^{n} s_i p^{i-h} \\
&\geqslant \frac{1}{p-1}\sum_{i=h}^{n} s_i p^{i-h} \\
&\geqslant \frac{1}{p-1}\sum_{i=h}^{n} s_i\left(p^{i-h}-1\right) \\
&= -\mathrm{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right),
\end{aligned}$$

as required. Also

$$\begin{aligned}
mr_{h+1} - 1 &= \left(r_{h+1}\sum_{i=0}^{h} s_i p^i\right) - 1 + r_{h+1}\sum_{i=h+1}^{n} s_i p^i \\
&\leqslant r_{h+1}\left(\sum_{i=h+1}^{n} s_i p^i\right) \\
&= \frac{1}{p-1}\sum_{i=h+1}^{n} s_i \frac{p^{i-h}}{p}.
\end{aligned}$$

If $j \in \mathbf{Z}_{\geqslant 1}$ then $p^j \leqslant p^{j+1} - p$, hence $\frac{p^j}{p} \leqslant p^j - 1$. Therefore

$$\begin{aligned} mr_{h+1} - 1 &\leqslant \frac{1}{p-1} \sum_{i=h+1}^{n} s_i \left(p^{i-h} - 1\right) \\ &= -\mathrm{ord}_{\mathrm{LA}_h} \left(\binom{X}{m} \right), \end{aligned}$$

as required. □

Proposition III.1.9 (Amice)
Every locally analytic function $f \in \mathrm{LA}(\mathbf{Z}_p, L)$ can be written uniquely in the form

$$f = \sum_{m=0}^{+\infty} a_m \binom{X}{m},$$

with $(a_m)_{m \in \mathbf{Z}_{\geqslant 0}} \subset L$ such that there exists $r \in \mathbf{Q}_{>0}$ satisfying $\mathrm{ord}_p(a_m) - mr \to +\infty$ as $m \to +\infty$.

Moreover, if a sequence $(a_m)_{m \in \mathbf{Z}_{\geqslant 0}} \subset L$ satisfies this condition, then $\sum_{m=0}^{+\infty} a_m \binom{X}{m}$ converges to an element of $\mathrm{LA}(\mathbf{Z}_p, L)$.

Proof:
By theorem III.1.4, p. 50, it is enough to prove that a sequence $(a_m)_{m \in \mathbf{Z}_{\geqslant 0}} \subset L$ satisfies this condition if and only if there exists $h \in \mathbf{Z}_{\geqslant 0}$ such that $\mathrm{ord}_{\mathrm{LA}_h}\left(a_m \binom{X}{m}\right) \to +\infty$ as $m \to +\infty$.

Assume that there exists $r \in \mathbf{Q}_{>0}$ satisfying $\mathrm{ord}_p(a_m) - mr \to +\infty$ as $m \to +\infty$. Choose $h \in \mathbf{Z}_{\geqslant 0}$ sufficiently large that $r_h \leqslant r$. By lemma III.1.8, p. 52 we have

$$\begin{aligned} & \mathrm{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right) \geqslant -mr_h, \\ \Rightarrow \quad & \mathrm{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right) \geqslant -mr, \\ \Rightarrow \quad & \mathrm{ord}_{\mathrm{LA}_h}\left(a_m\binom{X}{m}\right) \geqslant \mathrm{ord}_p(a_m) - mr. \end{aligned}$$

Hence $\mathrm{ord}_{\mathrm{LA}_h}\left(a_m \binom{X}{m}\right) \to +\infty$ as $m \to +\infty$, as required.

Conversely, assume that there exists $h \in \mathbf{Z}_{\geqslant 0}$ such that $\mathrm{ord}_{\mathrm{LA}_h}\left(a_m\binom{X}{m}\right) \to +\infty$ as $m \to +\infty$. By lemma III.1.8, p. 52, we have

$$
\begin{aligned}
& & 1 - mr_{h+1} &\geqslant \mathrm{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right), \\
\Rightarrow & & \mathrm{ord}_p\left(a_m\right) - mr_{h+1} &\geqslant \mathrm{ord}_{\mathrm{LA}_h}\left(a_m\binom{X}{m}\right) - 1.
\end{aligned}
$$

Hence $\mathrm{ord}_p\left(a_m\right) - mr_{h+1} \to +\infty$ as $m \to +\infty$, so the sequence $\left(a_m\right)_{m\in\mathbf{Z}_{\geqslant 0}} \subset L$ satisfies the required condition with $r := r_{h+1}$. $\square$

III.2 The Amice transform

In this section we will use the binomial functions to study the space of locally analytic distributions on $\mathbf{Z}_p$ with values in L. For $\mu \in \mathscr{D}_{\mathrm{LA}}\left(\mathbf{Z}_p, L\right)$, we recall the definition of the Amice transform $\mathscr{A}\left(\mu\right)\left(T\right) \in L\left[\left[T\right]\right]$. The orthogonality of the binomial functions will allow us to characterise the Amice transforms of measures and of temperate distributions. We will also examine the relationship between locally analytic characters and the Amice transform. The results in this section are due to Amice and Vélu [2].

Definition III.2.1 Let $\mu \in \mathscr{D}_{\mathrm{LA}}\left(\mathbf{Z}_p, L\right)$. We define the Amice transform $\mathscr{A}\left(\mu\right)\left(T\right) \in L\left[\left[T\right]\right]$ of μ to be the power series

$$
\mathscr{A}\left(\mu\right)\left(T\right) := \sum_{m=0}^{+\infty} \mu\left(\binom{X}{m}\right) T^m.
$$

Recall, from definition I.4.10, p. 25, that $L\left\langle\left\langle T\right\rangle\right\rangle$ denotes the ring of all power series with coefficients in L that have order of convergence less than or equal to zero.

Proposition III.2.2 *Let $\mu \in \mathscr{D}_{\mathrm{LA}}\left(\mathbf{Z}_p, L\right)$. Then $\mathscr{A}\left(\mu\right)\left(T\right) \in L\left\langle\left\langle T\right\rangle\right\rangle$.*

Proof:
For all $m \in \mathbf{Z}_{\geqslant 0}$, set $c_m := \mu\left(\binom{X}{m}\right)$. By the definition of the order of convergence (see definition I.4.1, p. 22), it is enough to show that $mr +$

$\text{ord}_p(c_m) \to +\infty$ as $m \to +\infty$ for all $r \in \mathbf{Q}_{>0}$. Fix $r \in \mathbf{Q}_{>0}$. Choose $h \in \mathbf{Z}_{\geqslant 0}$ sufficiently large that $r_h < r$. From the definition of $\text{ord}_{\text{LA}_h}(\mu)$ we have

$$\text{ord}_p(c_m) \geqslant \text{ord}_{\text{LA}_h}(\mu) + \text{ord}_{\text{LA}_h}\left(\binom{X}{m}\right) \qquad \forall m \in \mathbf{Z}_{\geqslant 0}.$$

By lemma III.1.8, p. 52 we have

$$\text{ord}_{\text{LA}_h}\left(\binom{X}{m}\right) \geqslant -mr_h \qquad \forall m \in \mathbf{Z}_{\geqslant 0}.$$

Hence

$$\begin{aligned} & \text{ord}_p(c_m) \geqslant \text{ord}_{\text{LA}_h}(\mu) - mr_h && \forall m \in \mathbf{Z}_{\geqslant 0}, \\ \Rightarrow \quad & mr + \text{ord}_p(c_m) \geqslant \text{ord}_{\text{LA}_h}(\mu) + m(r - r_h) && \forall m \in \mathbf{Z}_{\geqslant 0}. \end{aligned}$$

Therefore $mr + \text{ord}_p(c_m) \to +\infty$ as $m \to +\infty$, as required. □

Given a power series $F(T) \in L\langle\langle T\rangle\rangle$, we will now construct a distribution $\mu_F \in \mathscr{D}_{\text{LA}}(\mathbf{Z}_p, L)$ such that $\mathscr{A}(\mu_F)(T) = F(T)$.

Lemma III.2.3 *Let $F(T) = \sum_{m=0}^{+\infty} c_m T^m \in L\langle\langle T\rangle\rangle$. We define*

$$\begin{aligned} \mu_F : \text{LA}(\mathbf{Z}_p, L) &\to L \\ f &\mapsto \sum_{m=0}^{+\infty} a_m c_m, \end{aligned}$$

where $f = \sum_{m=0}^{+\infty} a_m \binom{X}{m}$ is the unique expression for f given by proposition III.1.9, p. 54.

Then μ_F is a well defined element of $\mathscr{D}_{\text{LA}}(\mathbf{Z}_p, L)$ and, for all $h \in \mathbf{Z}_{\geqslant 0}$, we have

$$\text{ord}_{\text{LA}_h}(\mu_F) = \inf\left\{ \text{ord}_p(c_m) - \text{ord}_{\text{LA}_h}\left(\binom{X}{m}\right) \,\middle|\, m \in \mathbf{Z}_{\geqslant 0} \right\}.$$

Proof:
We first check that the series $\sum_{m=0}^{+\infty} a_m c_m$ converges, so that $\mu_F(f)$ is well defined. Fix $f = \sum_{m=0}^{+\infty} a_m \binom{X}{m} \in \text{LA}(\mathbf{Z}_p, L)$. From proposition III.1.9, p. 54 there exists $r \in \mathbf{Q}_{>0}$ such that $\text{ord}_p(a_m) - mr \to +\infty$ as $m \to +\infty$.

Since $F(T)$ has order of convergence less than or equal to zero, we have $mr + \mathrm{ord}_p(c_m) \to +\infty$ as $m \to +\infty$. Hence

$$\mathrm{ord}_p(a_m c_m) = (\mathrm{ord}_p(a_m) - mr) + (mr + \mathrm{ord}_p(c_m)) \to +\infty \quad \text{as } m \to +\infty.$$

Therefore $\sum_{m=0}^{+\infty} a_m c_m$ converges, as required.

It is clear that μ_F is L-linear, so $\mathrm{ord}_{\mathrm{LA}_h}(\mu_F) \in \mathbf{R} \cup \{\pm\infty\}$ is defined for all $h \in \mathbf{Z}_{\geqslant 0}$.

Claim:
Let $h \in \mathbf{Z}_{\geqslant 0}$. Then

$$\mathrm{ord}_{\mathrm{LA}_h}(\mu_F) = \inf\left\{ \mathrm{ord}_p(c_m) - \mathrm{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right) \,\middle|\, m \in \mathbf{Z}_{\geqslant 0} \right\}.$$

Proof of claim:
From the definition of $\mathrm{ord}_{\mathrm{LA}_h}$, it is clear that

$$\mathrm{ord}_{\mathrm{LA}_h}(\mu_F) \leqslant \inf\left\{ \mathrm{ord}_p(c_m) - \mathrm{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right) \,\middle|\, m \in \mathbf{Z}_{\geqslant 0} \right\}.$$

We will prove the opposite inequality. Fix $f = \sum_{m=0}^{+\infty} a_m \binom{X}{m} \in \mathrm{LA}_h(\mathbf{Z}_p, L)$. From the definition of μ_F, we have

$$\mathrm{ord}_p(\mu_F(f)) \geqslant \min\{\mathrm{ord}_p(a_m c_m) \,|\, m \in \mathbf{Z}_{\geqslant 0}\}.$$

From the orthogonality of the binomial functions, by proposition I.3.2(ii), p. 21, we have

$$\mathrm{ord}_{\mathrm{LA}_h}(f) = \min\left\{ \mathrm{ord}_{\mathrm{LA}_h}\left(a_m \binom{X}{m}\right) \,\middle|\, m \in \mathbf{Z}_{\geqslant 0} \right\}.$$

Hence

$$\begin{aligned}
&\mathrm{ord}_p(\mu_F(f)) - \mathrm{ord}_{\mathrm{LA}_h}(f) \geqslant \\
&\qquad \min\{\mathrm{ord}_p(a_m c_m) \,|\, m \in \mathbf{Z}_{\geqslant 0}\} - \min\left\{\mathrm{ord}_{\mathrm{LA}_h}\left(a_m \tbinom{X}{m}\right) \,\middle|\, m \in \mathbf{Z}_{\geqslant 0}\right\} \\
&\quad \geqslant \min\left\{\mathrm{ord}_p(a_m c_m) - \mathrm{ord}_{\mathrm{LA}_h}\left(a_m \tbinom{X}{m}\right) \,\middle|\, m \in \mathbf{Z}_{\geqslant 0}\right\} \\
&\quad = \min\left\{\mathrm{ord}_p(c_m) - \mathrm{ord}_{\mathrm{LA}_h}\left(\tbinom{X}{m}\right) \,\middle|\, m \in \mathbf{Z}_{\geqslant 0}\right\}.
\end{aligned}$$

Therefore

$$\operatorname{ord}_{\mathrm{LA}_h}(\mu_F) \geqslant \inf\left\{ \operatorname{ord}_p(c_m) - \operatorname{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right) \middle| \, m \in \mathbf{Z}_{\geqslant 0} \right\},$$

as required. This concludes the proof of the claim.

It remains to prove that μ_F is continuous. By proposition I.2.2, p. 19 it is enough to show that $\operatorname{ord}_{\mathrm{LA}_h}(\mu_F) > -\infty$ for all $h \in \mathbf{Z}_{\geqslant 0}$, and we have proved that $\operatorname{ord}_{\mathrm{LA}_h}(\mu_F) = \inf\left\{\operatorname{ord}_p(c_m) - \operatorname{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right) \middle| \, m \in \mathbf{Z}_{\geqslant 0}\right\}$. For all $m \in \mathbf{Z}_{\geqslant 0}$, by lemma III.1.8, p. 52 we have

$$-\operatorname{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right) \geqslant m r_{h+1} - 1,$$

hence

$$\operatorname{ord}_p(c_m) - \operatorname{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right) \geqslant \operatorname{ord}_p(c_m) + m r_{h+1} - 1.$$

Now $\operatorname{ord}_p(c_m) + m r_{h+1} \to +\infty$ as $m \to +\infty$, so

$$\inf\left\{ \operatorname{ord}_p(c_m) - \operatorname{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right) \middle| \, m \in \mathbf{Z}_{\geqslant 0} \right\} > -\infty,$$

as required. □

This lemma has two important consequences.

Corollary III.2.4 *The Amice transform*

$$\begin{aligned} \mathscr{A} : \mathscr{D}_{\mathrm{LA}}(\mathbf{Z}_p, L) &\to L\langle\langle T \rangle\rangle \\ \mu &\mapsto \mathscr{A}(\mu)(T) \end{aligned}$$

is an L-linear isomorphism.

Proof:
It is clearly L-linear; it is injective by proposition III.1.9, p. 54; it is surjective by lemma III.2.3 above. □

Corollary III.2.5 *Let $\mu \in \mathscr{D}_{\mathrm{LA}}(\mathbf{Z}_p, L)$ and let $h \in \mathbf{Z}_{\geqslant 0}$. Then*

$$\operatorname{ord}_{\mathrm{LA}_h}(\mu) = \inf\left\{ \operatorname{ord}_p\left(\mu\left(\binom{X}{m}\right)\right) - \operatorname{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right) \middle| m \in \mathbf{Z}_{\geqslant 0}\right\}.$$

Proof:
Follows immediately from lemma III.2.3 and the injectivity of the Amice transform. □

Remark III.2.6 Corollary III.2.5 allows us to read off information about the continuity properties of a distribution $\mu \in \mathscr{D}_{\mathrm{LA}}(\mathbf{Z}_p, L)$ from its Amice transform $\mathscr{A}(\mu)(T)$; it is the key result that allows us to characterise the Amice transforms of temperate distributions and measures. Notice that the proof of this corollary depends on the claim made in the proof of lemma III.2.3, in which the orthogonality of the binomial functions played an essential role.

Lemma III.2.7 *Let $\mu \in \mathscr{D}_{\mathrm{LA}}(\mathbf{Z}_p, L)$ and let $h \in \mathbf{Z}_{\geqslant 0}$. We have*

$$\operatorname{ord}_{r_{h+1}}(\mathscr{A}(\mu)(T)) - 1 \leqslant \operatorname{ord}_{\mathrm{LA}_h}(\mu) \leqslant \operatorname{ord}_{r_h}(\mathscr{A}(\mu)(T)).$$

Proof:
From lemma III.1.8, p. 52 we have

$$mr_{h+1} - 1 \leqslant -\operatorname{ord}_{\mathrm{LA}_h}\left(\binom{X}{m}\right) \leqslant mr_h,$$

$$\begin{aligned}\Rightarrow \qquad & \operatorname{ord}_p\left(\mu\left(\tbinom{X}{m}\right)\right) + mr_{h+1} - 1 \\ & \qquad \leqslant \operatorname{ord}_p\left(\mu\left(\tbinom{X}{m}\right)\right) - \operatorname{ord}_{\mathrm{LA}_h}\left(\tbinom{X}{m}\right) \\ & \qquad\qquad \leqslant \operatorname{ord}_p\left(\mu\left(\tbinom{X}{m}\right)\right) + mr_h,\end{aligned}$$

$$\begin{aligned}\Rightarrow \qquad & \inf\left\{\operatorname{ord}_p\left(\mu\left(\tbinom{X}{m}\right)\right) + mr_{h+1} - 1 \middle| m \in \mathbf{Z}_{\geqslant 0}\right\} \\ & \qquad \leqslant \inf\left\{\operatorname{ord}_p\left(\mu\left(\tbinom{X}{m}\right)\right) - \operatorname{ord}_{\mathrm{LA}_h}\left(\tbinom{X}{m}\right) \middle| m \in \mathbf{Z}_{\geqslant 0}\right\} \\ & \qquad\qquad \leqslant \inf\left\{\operatorname{ord}_p\left(\mu\left(\tbinom{X}{m}\right)\right) + mr_h \middle| m \in \mathbf{Z}_{\geqslant 0}\right\}.\end{aligned}$$

The result now follows from the 'maximum principle' (proposition I.4.8, p. 24) and from corollary III.2.5 above. □

The bounds in lemma III.2.7 above immediately allow us to prove that the Amice transform is bi-continuous.

Proposition III.2.8 *The Amice transform*

$$\begin{aligned} \mathscr{A} : \mathscr{D}_{\mathrm{LA}}\left(\mathbf{Z}_p, L\right) &\rightarrow L\left\langle\langle T\rangle\right\rangle \\ \mu &\mapsto \mathscr{A}\left(\mu\right)\left(T\right) \end{aligned}$$

is an L-linear homeomorphism.

Proof:
By corollary III.2.4, p. 58, it is enough to show that $\mathscr{A}$ is bi-continuous. Recall that we have defined fundamental systems of open neighbourhoods of zero in $\mathscr{D}_{\mathrm{LA}}\left(\mathbf{Z}_p, L\right)$ and $L\left\langle\langle T\rangle\right\rangle$, see definition I.2.3, p. 19 and definition I.4.10, p. 25 respectively.

Let $r \in \mathbf{R}_{>0}$, let $s \in \mathbf{R}$, and set

$$U := \left\{F(T) \in L\left\langle\langle T\rangle\right\rangle \middle| \mathrm{ord}_r\left(F\left(T\right)\right) \geqslant s\right\},$$

so U is a general element of the fundamental system of open neighbourhoods of zero in $L\left\langle\langle T\rangle\right\rangle$. Choose $h \in \mathbf{Z}_{\geqslant 0}$ sufficiently large that $r_h \leqslant r$. Then, by lemma III.2.7 above, we have

$$\left\{\mu \in \mathscr{D}_{\mathrm{LA}}\left(\mathbf{Z}_p, L\right) \middle| \mathrm{ord}_{\mathrm{LA}_h}\left(\mu\right) \geqslant s\right\} \subset \mathscr{A}^{-1}(U),$$

so $\mathscr{A}$ is continuous.

Conversely, let $h \in \mathbf{Z}_{\geqslant 0}$, let $s \in \mathbf{R}$, and set

$$V := \left\{\mu \in \mathscr{D}_{\mathrm{LA}}\left(\mathbf{Z}_p, L\right) \middle| \mathrm{ord}_{\mathrm{LA}_h}\left(\mu\right) \geqslant s\right\},$$

so V is a general element of the fundamental system of open neighbourhoods of zero in $\mathscr{D}_{\mathrm{LA}}\left(\mathbf{Z}_p, L\right)$. Then, by lemma III.2.7 again, we have

$$\left\{F(T) \in L\left\langle\langle T\rangle\right\rangle \middle| \mathrm{ord}_{r_{h+1}}\left(F\left(T\right)\right) \geqslant s+1\right\} \subset \mathscr{A}(V),$$

so $\mathscr{A}^{-1}$ is continuous. □

In fact, the bounds in lemma III.2.7 are far stronger than would be necessary just to prove continuity; they also allow us to prove the stronger statements that follow. Recall the definition of temperate distributions from definition I.2.6, p. 20.

Proposition III.2.9 *Let* $\mu \in \mathscr{D}_{\mathrm{LA}}(\mathbf{Z}_p, L)$ *and let* $r \in \overline{\mathbf{R}}_{\geqslant 0}$. *Then* μ *is temperate of order* r *if and only if the sequence* $(hr + \mathrm{ord}_{r_h}(\mathscr{A}(\mu)(T)))_{h \in \mathbf{Z}_{\geqslant 0}}$ *is* $\eta(r)$*-bounded below.*

Proof:
Assume that μ is temperate of order r. From definition I.2.6, p. 20 the sequence $(hr + \mathrm{ord}_{\mathrm{LA}_h}(\mu))_{h \in \mathbf{Z}_{\geqslant 0}}$ is $\eta(r)$-bounded below. Now by lemma III.2.7, p. 59, for all $h \in \mathbf{Z}_{\geqslant 0}$, we have

$$\begin{aligned} & \mathrm{ord}_{\mathrm{LA}_h}(\mu) \leqslant \mathrm{ord}_{r_h}(\mathscr{A}(\mu)(T)), \\ \Rightarrow \qquad & hr + \mathrm{ord}_{\mathrm{LA}_h}(\mu) \leqslant hr + \mathrm{ord}_{r_h}(\mathscr{A}(\mu)(T)). \end{aligned}$$

Hence $(hr + \mathrm{ord}_{r_h}(\mathscr{A}(\mu)(T)))_{h \in \mathbf{Z}_{\geqslant 0}}$ is $\eta(r)$-bounded below, as required.

Conversely, assume that the sequence $(hr + \mathrm{ord}_{r_h}(\mathscr{A}(\mu)(T)))_{h \in \mathbf{Z}_{\geqslant 0}}$ is $\eta(r)$-bounded below. By lemma III.2.7, for all $h \in \mathbf{Z}_{\geqslant 0}$, we have

$$\begin{aligned} & \mathrm{ord}_{r_{h+1}}(\mathscr{A}(\mu)(T)) - 1 \leqslant \mathrm{ord}_{\mathrm{LA}_h}(\mu), \\ \Rightarrow \qquad & hr + \mathrm{ord}_{r_{h+1}}(\mathscr{A}(\mu)(T)) - 1 \leqslant hr + \mathrm{ord}_{\mathrm{LA}_h}(\mu). \end{aligned}$$

Hence the sequence $(hr + \mathrm{ord}_{\mathrm{LA}_h}(\mu))_{h \in \mathbf{Z}_{\geqslant 0}}$ is $\eta(r)$-bounded below. □

Remark III.2.10 Recall that a distribution $\mu \in \mathscr{D}_{\mathrm{LA}}(\mathbf{Z}_p, L)$ is a measure if it can be extended to a continuous L-linear map $\mu : \mathscr{C}(\mathbf{Z}_p, L) \to L$, where $\mathscr{C}(\mathbf{Z}_p, L)$ denotes the space of all continuous functions on $\mathbf{Z}_p$ with values in L. It is not hard to see that μ is a measure if and only if it is temperate of order 0 (cf. Colmez [8], remark I.4.3(ii), p. 499). The following corollary, therefore, characterises the Amice transforms of measures.

Corollary III.2.11 *Let* $\mu \in \mathscr{D}_{\mathrm{LA}}(\mathbf{Z}_p, L)$*; write* $\mathscr{A}(\mu)(T) = \sum_{m=0}^{+\infty} c_m T^m$. *Then* μ *is temperate of order* 0 *if and only if the sequence* $(\mathrm{ord}_p(c_m))_{m \in \mathbf{Z}_{\geqslant 0}}$ *is bounded below.*

Proof:

By proposition III.2.9 above, it is enough to show that $(\mathrm{ord}_p(c_m))_{m\in\mathbf{Z}_{\geqslant 0}}$ is bounded below if and only if $(\mathrm{ord}_{r_h}(\mathscr{A}(\mu)(T)))_{h\in\mathbf{Z}_{\geqslant 0}}$ is bounded below. By the 'maximum' principle (proposition I.4.8, p. 24) we have

$$\mathrm{ord}_{r_h}(\mathscr{A}(\mu)(T)) = \inf\{mr_h + \mathrm{ord}_p(c_m)\,|m \in \mathbf{Z}_{\geqslant 0}\}.$$

Assume that there exists $B \in \mathbf{Q}$ such that $\mathrm{ord}_p(c_m) \geqslant B$ for all $m \in \mathbf{Z}_{\geqslant 0}$. Then, for all $h \in \mathbf{Z}_{\geqslant 0}$ and all $m \in \mathbf{Z}_{\geqslant 0}$, we have $mr_h + \mathrm{ord}_p(c_m) \geqslant B$; hence $\mathrm{ord}_{r_h}(\mathscr{A}(\mu)(T)) \geqslant B$ for all $h \in \mathbf{Z}_{\geqslant 0}$, as required.

Conversely, assume that there exist $B \in \mathbf{Q}$ such that $\mathrm{ord}_{r_h}(\mathscr{A}(\mu)(T)) \geqslant B$ for all $h \in \mathbf{Z}_{\geqslant 0}$. Then we claim that $(\mathrm{ord}_p(c_m))_{m\in\mathbf{Z}_{\geqslant 0}}$ is bounded below by $B-1$. Fix $m_0 \in \mathbf{Z}_{\geqslant 0}$, and choose $h_0 \in \mathbf{Z}_{\geqslant 0}$ sufficiently large that $m_0 r_{h_0} \leqslant 1$. We have

$$\begin{aligned} B &\leqslant \mathrm{ord}_{r_{h_0}}(\mathscr{A}(\mu)(T)) \\ &= \inf\{mr_{h_0} + \mathrm{ord}_p(c_m)\,|m \in \mathbf{Z}_{\geqslant 0}\} \\ &\leqslant m_0 r_{h_0} + \mathrm{ord}_p(c_{m_0}) \\ &\leqslant 1 + \mathrm{ord}_p(c_{m_0}). \end{aligned}$$

Hence $\mathrm{ord}_p(c_{m_0}) \geqslant B-1$, as claimed. □

We will now turn our attention to the relationship between locally analytic characters and the Amice transform. We start by associating a locally analytic character to each element $z \in \mathfrak{p}_L$.

Definition III.2.12 Let $z \in \mathfrak{p}_L$. We define

$$\begin{aligned} \kappa_z : \mathbf{Z}_p &\to L \\ \alpha &\mapsto \sum_{m=0}^{+\infty} z^m \binom{\alpha}{m}. \end{aligned}$$

Note that by proposition III.1.2(ii), p. 49 we have $\mathrm{ord}_p\left(\binom{\alpha}{m}\right) \geqslant 0$ for all $m \in \mathbf{Z}_{\geqslant 0}$ and all $\alpha \in \mathbf{Z}_p$, so the series $\sum_{m=0}^{+\infty} z^m \binom{\alpha}{m}$ is convergent and $\kappa_z(\alpha)$ is well defined.

Proposition III.2.13 *Let $z \in \mathfrak{p}_L$. Then $\kappa_z \in \mathrm{Hom}_{\mathrm{LA}}(\mathbf{Z}_p, L^\times)$.*

Proof:
By proposition III.1.9, p. 54 we see that $\kappa_z \in \mathrm{LA}(\mathbf{Z}_p, L)$. Clearly $\kappa_z(0) = 1$. Let $\alpha_1, \alpha_2 \in \mathbf{Z}_p$. Using proposition III.1.2(i), p. 49 we have

$$\begin{aligned}
\kappa_z(\alpha_1)\kappa_z(\alpha_2) &= \left(\sum_{i_1=0}^{+\infty} z^{i_1}\binom{\alpha_1}{i_1}\right)\left(\sum_{i_2=0}^{+\infty} z^{i_2}\binom{\alpha_2}{i_2}\right) \\
&= \sum_{m=0}^{+\infty} z^m \left(\sum_{\substack{i_1,i_2\in\mathbf{Z}_{\geqslant 0}\\ i_1+i_2=m}} \binom{\alpha_1}{i_1}\binom{\alpha_2}{i_2}\right) \\
&= \sum_{m=0}^{+\infty} z^m \binom{\alpha_1+\alpha_2}{m} \\
&= \kappa_z(\alpha_1+\alpha_2),
\end{aligned}$$

as required. □

Proposition III.2.14 *Let $\mu \in \mathscr{D}_{\mathrm{LA}}(\mathbf{Z}_p, L)$ and let $z \in \mathfrak{p}_L$. Then $\mu(\kappa_z) = \mathscr{A}(\mu)(z)$.*

Proof:
By the linearity and continuity of μ, we have

$$\begin{aligned}
\mu(\kappa_z) &= \mu\left(\sum_{m=0}^{+\infty} z^m \binom{X}{m}\right) \\
&= \sum_{m=0}^{+\infty} z^m \mu\left(\binom{X}{m}\right) \\
&= \mathscr{A}(\mu)(z).
\end{aligned}$$

□

Chapter IV

Schneider Teitelbaum functions

In this chapter we will give the definition of the Schneider Teitelbaum polynomials and state some of their properties, drawing parallels with the binomial polynomials. This shall be done in section IV.2, after recalling some Lubin Tate theory in section IV.1.

None of the material in this chapter is original. The results of section IV.1 are taken from various sources as noted in the text; those of section IV.2 are from Schneider Teitelbaum [19].

IV.1 Lubin Tate formal groups

We will use Lubin Tate formal groups in order to generalise the ideas of the previous chapter; they are essential for the definition of the Schneider Teitelbaum functions. This section summarises the parts of Lubin Tate theory that will be required in the remainder of this thesis. This class of formal group was introduced in Lubin Tate [14]. I have taken material from this paper and several other sources, as noted in the text.

Notation IV.1.1 Let R be any ring and let $F, G \in R[[X_1, \dots, X_n]]$, with $n \in \mathbf{Z}_{\geqslant 1}$. We will write $F \equiv G \mod \deg 2$ if $F - G$ is a power series of the form

$$\sum_{i_1,\dots,i_n \in \mathbf{Z}_{\geqslant 0}} a_{i_1,\dots,i_n} X_1^{i_1} \dots X_n^{i_n}$$

with $a_{i_1,\dots,i_n} = 0$ for all $\{i_1, \dots, i_n \in \mathbf{Z}_{\geqslant 0} \,|i_1 + \dots + i_n \leqslant 1\}$.

Definition IV.1.2 A power series $l(X) \in \mathfrak{O}_K[[X]]$ such that:

- $l(X) \equiv \pi_K X \mod \deg 2$, and
- $l(X) \equiv X^{q_K} \mod \pi_K \mathfrak{O}_K[[X]]$

is called a Frobenius power series.

Notation IV.1.3 In the rest of this thesis $l(X)$, or just l, will denote a Frobenius power series as above.

Proposition IV.1.4 *Let $l(X)$ be a Frobenius power series. Then there exists a unique power series in $\mathfrak{O}_K[[X_1, X_2]]$, which we will denote by $X_1[+]_l X_2$, such that:*

i. $X_1[+]_l X_2 \equiv X_1 + X_2 \mod \deg 2$,

ii. $(X_1[+]_l X_2)[+]_l X_3 = X_1[+]_l(X_2[+]_l X_3)$,

iii. $X_1[+]_l X_2 = X_2[+]_l X_1$,

iv. $l(X_1[+]_l X_2) = l(X_1)[+]_l l(X_2)$.

We call $X_1[+]_l X_2$ the Lubin Tate formal group associated to $l(x)$.

For all $\alpha \in \mathfrak{O}_K$, there exists a unique power series $[\alpha]_l(X) \in \mathfrak{O}_K[[X]]$ such that:

v. $[\alpha]_l(X) \equiv \alpha X \mod \deg 2$,

vi. $l([\alpha]_l(X)) = [\alpha]_l(l(X))$.

We have $l(x) = [\pi_K]_l(X)$.

Proof:
See Lubin Tate [14], §1, equations (4, 5) and theorem 1, p. 382. □

Proposition IV.1.5 *The maps*

$$\begin{aligned} \mathfrak{p}_L \times \mathfrak{p}_L &\to \mathfrak{p}_L \\ (z_1, z_2) &\mapsto z_1[+]_l z_2 \end{aligned}$$

and

$$\begin{aligned} \mathfrak{O}_K \times \mathfrak{p}_L &\to \mathfrak{p}_L \\ (\alpha, z) &\mapsto [\alpha]_l (z) \end{aligned}$$

equip $\mathfrak{p}_L$ with the structure of a $\mathfrak{O}_K$-module. We denote this $\mathfrak{O}_K$-module by $\mathrm{A}_l(\mathfrak{p}_L)$.

Proof:
See Lubin Tate [14], §1, equations (12, 13), p. 383. $\square$

Proposition IV.1.6 *Let $z \in \mathfrak{p}_{\mathbf{C}_p}$. Then:*

i. $\mathrm{ord}_p\left([\pi_K^n]_l(z)\right) \to +\infty$ *as* $n \to +\infty$,

ii. $\mathrm{ord}_p\left([p^n]_l(z)\right) \to +\infty$ *as* $n \to +\infty$.

Proof:
Since $[\pi_K]_l(X) = l(X)$ satisfies $[\pi_K]_l(X) \equiv \pi_K X \bmod \deg 2$ and $[\pi_K]_l(X) \equiv X^{q_K} \bmod \pi_K \mathfrak{O}_K[[X]]$, we have

$$\mathrm{ord}_p\left([\pi_K]_l(z)\right) \geqslant \min\left\{1/e_K + \mathrm{ord}_p(z), q_K \mathrm{ord}_p(z)\right\}.$$

The result follows. $\square$

Definition IV.1.7 For $n \in \mathbf{Z}_{\geqslant 0}$, we define

$$\Lambda_{l,n/e_K} := \left\{ z \in \mathrm{A}_l\left(\mathfrak{p}_{\mathbf{C}_p}\right) \middle| \, [\pi_K^n]_l(z) = 0 \right\} \subset \mathrm{A}_l\left(\mathfrak{p}_{\mathbf{C}_p}\right).$$

We also write

$$\Lambda_{l,+\infty} := \bigcup_{n \in \mathbf{Z}_{\geqslant 0}} \Lambda_{l,n/e_K} \subset \mathrm{A}_l\left(\mathfrak{p}_{\mathbf{C}_p}\right).$$

Proposition IV.1.8 *Let $n \in \mathbf{Z}_{\geqslant 1}$ and let $\gamma_n \in \Lambda_{l,\frac{n}{e_K}} - \Lambda_{l,\frac{n-1}{e_K}}$.*

i. The map

$$\begin{aligned} \mathfrak{O}_K / \pi_K^n \mathfrak{O}_K &\to \Lambda_{l,n/e_K} \\ \alpha &\mapsto [\alpha]_l(\gamma_n) \end{aligned}$$

is an isomorphism of $\mathfrak{O}_K$-modules. In particular, $\#\Lambda_{l,n/e_K} = q_K^n$.

ii. We have $\mathrm{ord}_p(\gamma_n) = \frac{1}{e_K\left(q_K^n - q_K^{n-1}\right)}$.

iii. All the zeros $\gamma \in \Lambda_{l,n/e_K}$ *of* $[\pi_K^n]_l(X)$ *are simple.*

Proof:

i. See Lubin Tate [14], §1, theorem 2(b), p. 383.

ii. By Lang [13], ch. 8, theorem 2.1(ii), p. 197, the extension $K(\gamma_n)/K$ is totally ramified of degree $q_K^n - q_K^{n-1}$, and the result follows.

iii. We have $[\pi_K^n]_l(X) \equiv X^{q_K^n} \bmod \pi_K \mathfrak{O}_K[[X]]$. By theorem I.5.6, p 26, for any $r \in \mathbf{Q}_{>0}$, we have

$$\mathrm{Z}\left(r; [\pi_K^n]_l(X)\right) \leqslant q_K^n.$$

But by part (i), there are q_K^n distinct zeros of $[\pi_K^n]_l(X)$ in $\mathfrak{p}_{\mathbf{C}_p}$; hence they must all be simple zeros. □

We will now review the logarithm of a Lubin Tate formal group. My main reference for this material was Lang [13].

Proposition IV.1.9 *There exists a unique power series* $\lambda_l(X) \in K[[X]]$ *such that:*

- $\lambda_l(X) \equiv X \bmod \deg 2$, *and*
- $\lambda_l(X_1 [+]_l X_2) = \lambda_l(X_1) + \lambda_l(X_2)$.

We call $\lambda_l(X)$ *the logarithm of the Lubin Tate formal group associated to* $l(X)$.

Proof:
See Lang [13], ch. 8, §6, lemma 1, p. 212. □

Proposition IV.1.10 *i. For* $\alpha \in \mathfrak{O}_K$, *we have* $\lambda_l([\alpha]_l(X)) = \alpha\lambda_l(X)$.

ii. We have $\lambda_l'(X) \in \mathfrak{O}_K[[X]]$, *where* $\lambda_l'(X)$ *denotes the formal derivative of the power series* $\lambda_l(X)$.

iii. The logarithm $\lambda_l(X)$ *converges on* $\mathfrak{p}_{\mathbf{C}_p}$.

iv. Let $z \in \mathfrak{p}_{\mathbf{C}_p}$. *Then* $z \in \Lambda_{l,+\infty}$ *if and only if* $\lambda_l(z) = 0$.

Proof:

i. See Lang [13], ch. 8, §6, lemma 2, p. 213.

ii. See Lang [13], ch. 8, §6, lemma 3(i), p. 214.

iii. See Lang [13], ch. 8, §6, lemma 3(ii), p. 214.

iv. See Lang [13], ch. 8, §6, lemma 5, p. 217. □

Proposition IV.1.11 *Write* $\lambda_l(X) = \sum_{i=0}^{+\infty} c_i X^i$. *Let* $n \in \mathbf{Z}_{\geqslant 0}$. *Then*

$$\mathrm{ord}_p(c_i) \geqslant -n/e_K \qquad \forall i \in \mathbf{Z}_{\geqslant 0},\ q_K^n \leqslant i < q_K^{n+1}$$

and

$$\mathrm{ord}_p\left(c_{q_K^n}\right) = -n/e_K.$$

Proof:
By Cartier [5], §8, p. 282 there exists a Frobenius power series $l'(X)$ such that

$$\lambda_{l'}(X) = \sum_{i=0}^{+\infty} \frac{X^{q_K^i}}{\pi_K^i}.$$

Now if $[1]_{l',l}(X) \in \mathfrak{O}_K[[X]]$ is the formal isomorphism from $X_1[+]_l X_2$ into $X_1[+]_{l'} X_2$ (see Lubin Tate [14], §1, equation (5), p. 382), then we have $\lambda_l(X) = \lambda_{l'}([1]_{l',l}(X))$, and the result follows. □

In the final part of this section, we will study the formal group homomorphisms from a Lubin Tate formal group into the multiplicative group. This study depends on an important result of Tate [20]. I have followed the presentation of Boxall [4].

Notation IV.1.12 For the remainder of this thesis, we will assume that K is not equal to $\mathbf{Q}_p$.

Definition IV.1.13 We define

$$\mathscr{H}_l := \left\{ t(Y) \in \mathfrak{O}_{\mathbf{C}_p}[[Y]] \,\middle|\, t(0) = 1,\ t\left(Y_1[+]_l Y_2\right) = t(Y_1)t(Y_2) \right\}.$$

Note that $\mathscr{H}_l$ is a group under multiplication. The map

$$\begin{aligned} \mathfrak{O}_K \times \mathscr{H}_l &\to \mathscr{H}_l \\ (\alpha, t(Y)) &\mapsto t\left([\alpha]_l(Y)\right) \end{aligned}$$

equips $\mathscr{H}_l$ with the structure of an $\mathfrak{O}_K$-module.

Proposition IV.1.14 *The $\mathfrak{O}_K$-module $\mathscr{H}_l$ is free of rank 1.*

Proof:
See Boxall [4], p. 6. □

Definition IV.1.15 Once and for all, we choose an $\mathfrak{O}_K$-module generator of $\mathscr{H}_l$ and denote it by $t_l^1(Y)$. For $\alpha \in \mathfrak{O}_K$, we set $t_l^\alpha(Y) := t_l^1\left([\alpha]_l(Y)\right) \in \mathscr{H}_l$. We define the constant $\Omega_l \in \mathfrak{O}_{\mathbf{C}_p}$ to be the coefficient of Y in the power series $t_l^1(Y)$. Note that Ω_l is independent, up to an element of $\mathfrak{O}_K{}^\times$, of our choice of generator $t_l^1(Y)$ of $\mathscr{H}_l$.

Proposition IV.1.16 *Let $\alpha, \beta \in \mathfrak{O}_K$.*

i. We have $t_l^{\alpha+\beta}(Y) = t_l^\alpha(Y)\, t_l^\beta(Y)$ and $t_l^0(Y) = 1$.

ii. We have $t_l^{\alpha\beta}(Y) = t_l^\alpha\left([\beta]_l(Y)\right)$.

iii. If $\beta \in \mathbf{Z}_p$, then we have $t_l^{\alpha\beta}(Y) = t_l^\alpha(Y)^\beta$, where we define $t(Y)^\beta := \sum_{i=0}^{+\infty} \left(t(Y) - 1\right)^i \binom{\beta}{i}$.

Proof:
See Boxall [4], §1, lemma 1(iv), p. 7. □

Definition IV.1.17 Let $n \in \mathbf{Z}_{\geqslant 1}$. We define μ_n to be the group of n-th roots of unity; that is, $\mu_n := \left\{ z \in \mathbf{C}_p \,\middle|\, z^n = 1 \right\}$.

Proposition IV.1.18 *The map*

$$\begin{aligned}\Lambda_{l,1/e_K} &\rightarrow \mu_p \\ \gamma &\mapsto t_l^1(\gamma)\end{aligned}$$

is a surjective group homomorphism.

Proof:
From the definition of $\mathscr{H}_l$, this map is clearly a group homomorphism. Let $\eta \in \Lambda_{l,1/e_K} - \{0\}$. By Boxall [4], §1, fact 2, p. 6 there exists $\alpha \in \mathfrak{O}_K$ such that $t_l^\alpha(\eta)$ is a primitive p-th root of unity. Now, by the definition of $t_l^\alpha(Y)$, we have $t_l^1([\alpha]_l(\eta)) = t_l^\alpha(\eta)$, so $\gamma \mapsto t_l^1(\gamma)$ is surjective. □

Proposition IV.1.19 *Let* $\alpha \in \mathfrak{O}_K$. *We have*

$$t_l^\alpha(Y) = \exp(\alpha\Omega_l\lambda_l(Y)),$$

where $\exp(Y) = \sum_{i=0}^{+\infty} \frac{Y^i}{i!}$.

Proof:
Let $\log(1+X) := \sum_{i=1}^{+\infty}(-1)^{i-1}X^i/i$. Note that $\frac{1}{\alpha\Omega_l}\log(t_l^\alpha(Y))$ satisfies the defining properties of the logarithm of the formal group l (see proposition IV.1.9, p. 67), so, by uniqueness, it must be equal to $\lambda_l(Y)$. The result follows. □

Proposition IV.1.20 *We have*

$$\mathrm{ord}_p(\Omega_l) = \frac{1}{p-1} - \frac{1}{e_K(q_K-1)}.$$

Proof:
See Schneider Teitelbaum [19], appendix, theorem (c), p. 33. □

IV.2 Locally analytic functions on $\mathfrak{O}_K$

In this section we will give the definition of the Schneider Teitelbaum functions and state some of their properties. All the results in this section are from Schneider Teitelbaum [19].

We continue to assume that $K \neq \mathbf{Q}_p$.

Definition IV.2.1 For all $m \in \mathbf{Z}_{\geqslant 0}$, we define the polynomials $P_{l,m}(X) \in \mathbf{C}_p[X]$ by the identity

$$\sum_{m=0}^{+\infty} P_{l,m}(X) Y^m = \exp\left(\Omega_l X \lambda_l(Y)\right),$$

where $\exp(Y) = \sum_{i=0}^{+\infty} \frac{Y^i}{i!}$.

We will call the polynomials $P_{l,m}(X)$ the Schneider Teitelbaum polynomials or, when we consider them as elements of $\mathrm{LA}_0(\mathfrak{O}_K, \mathbf{C}_p) \subset \mathrm{LA}(\mathfrak{O}_K, \mathbf{C}_p)$, the Schneider Teitelbaum functions.

Remark IV.2.2

i. Note that this definition is slightly different from the one in Schneider Teitelbaum [19], which does not include the factor Ω_l.

ii. Note that

$$\sum_{m=0}^{+\infty} \binom{X}{m} Y^m = \exp\left(X \log(1+Y)\right),$$

where $\log(1+Y) = \sum_{i=1}^{+\infty}(-1)^{i-1}Y^i/i$; so the Schneider Teitelbaum polynomials are a direct generalisation of the binomial polynomials.

Proposition IV.2.3 *i. We have $P_{l,0}(X) = 1$ and $P_{l,m}(0) = 0$ for all $m \in \mathbf{Z}_{\geqslant 1}$. For all $m \in \mathbf{Z}_{\geqslant 0}$, we have $P_{l,m}(X)$ is a polynomial of degree exactly m, with leading coefficient $\Omega_l^m/m!$.*

ii. Let $\alpha \in \mathfrak{O}_K$. We have

$$t_l^{\alpha}(Y) = \sum_{m=0}^{+\infty} P_{l,m}(\alpha) Y^m.$$

iii. For all $m \in \mathbf{Z}_{\geqslant 0}$ and all $\alpha \in \mathfrak{O}_K$, we have $P_{l,m}(\alpha) \in \mathfrak{O}_{\mathbf{C}_p}$.

iv. Let $m \in \mathbf{Z}_{\geqslant 0}$. Write $\lambda_l(X) = \sum_{i=1}^{+\infty} c_i Y^i \in K[[X]]$. Then the formal derivative $P'_{l,m}(X)$ of $P_{l,m}(X)$ satisfies

$$P'_{l,m}(X) = \Omega_l \sum_{i=1}^{m} c_i P_{l,m-i}(X).$$

Proof:

i. Immediate from the definition.

ii. By definition

$$\sum_{m=0}^{+\infty} P_{l,m}(X)\,Y^m = \exp\left(\Omega_l X \lambda_l(Y)\right).$$

Substituting $X := \alpha$ we obtain

$$\sum_{m=0}^{+\infty} P_{l,m}(\alpha)\,Y^m = \exp\left(\alpha \Omega_l \lambda_l(Y)\right),$$

and the result now follows from proposition IV.1.19, p. 70.

iii. Recall that $t_l^{\alpha}(Y) \in \mathfrak{O}_{\mathbf{C}_p}[[Y]]$ and use part (ii).

iv. By definition we have

$$\sum_{m=0}^{+\infty} P_{l,m}(X)\,Y^m = \exp\left(\Omega_l X \lambda_l(Y)\right).$$

By differentiating with respect to X we obtain

$$\begin{aligned}
\sum_{m=0}^{+\infty} P'_{l,m}(X) Y^m &= \Omega_l \lambda_l(Y) \exp\left(\Omega_l X \lambda_l(Y)\right) \\
&= \Omega_l \left(\sum_{i=0}^{+\infty} c_i Y^i\right) \left(\sum_{j=0}^{+\infty} P_{l,j}(X)\,Y^j\right) \\
&= \sum_{m=0}^{+\infty} \left(\Omega_l \sum_{i=1}^{m} c_i P_{l,m-i}(X)\right) Y^m.
\end{aligned}$$

Now equate coefficients of Y^m.

□

The following theorem shows how the Schneider Teitelbaum functions generalise the work of §III.1.

Theorem IV.2.4 (Schneider Teitelbaum)
Every locally analytic function $f \in \mathrm{LA}(\mathfrak{O}_K, \mathbf{C}_p)$ *can be written uniquely in the form*

$$f = \sum_{m=0}^{+\infty} a_m P_{l,m},$$

with $(a_m)_{m \in \mathbf{Z}_{\geqslant 0}} \subset \mathbf{C}_p$ *such that there exists* $r \in \mathbf{Q}_{>0}$ *satisfying* $\mathrm{ord}_p(a_m) - mr \to +\infty$ *as* $m \to +\infty$.

Moreover, if a sequence $(a_m)_{m \in \mathbf{Z}_{\geqslant 0}} \subset \mathbf{C}_p$ *satisfies this condition, then* $\sum_{m=0}^{+\infty} a_m P_{l,m}$ *converges to an element of* $\mathrm{LA}(\mathfrak{O}_K, \mathbf{C}_p)$.

Proof:
See Schneider Teitelbaum [19], theorem 4.7, p. 26 and proposition 4.5, p. 24. □

Remark IV.2.5 This is a direct generalisation of proposition III.1.9, p. 54.

We will now consider how the Schneider Teitelbaum functions can be used to generalise the results of §III.2.

Definition IV.2.6 Let $\mu \in \mathscr{D}_{\mathrm{LA}}(\mathfrak{O}_K, \mathbf{C}_p)$. We define the Schneider Teitelbaum transform $\mathscr{A}_l(\mu)(T) \in \mathbf{C}_p[[T]]$ of μ to be the power series

$$\mathscr{A}_l(\mu)(T) := \sum_{m=0}^{+\infty} \mu(P_{l,m}) T^m.$$

Proposition IV.2.7 (Schneider Teitelbaum)
The Schneider Teitelbaum transform

$$\begin{aligned} \mathscr{A}_l : \mathscr{D}_{\mathrm{LA}}(\mathfrak{O}_K, \mathbf{C}_p) &\to \mathbf{C}_p\langle\langle T \rangle\rangle \\ \mu &\mapsto \mathscr{A}_l(\mu)(T) \end{aligned}$$

is a $\mathbf{C}_p$*-linear homeomorphism.*

Proof:
See Schneider Teitelbaum [19], comments before lemma 4.6, p. 24. □

Remark IV.2.8 This is a direct generalisation of proposition III.2.8, p. 60.

We will conclude this section by considering the relationship between locally analytic characters and the Schneider Teitelbaum transform. We start by associating a locally analytic character to each element $z \in \mathfrak{p}_{\mathbf{C}_p}$.

Definition IV.2.9 Let $z \in \mathfrak{p}_{\mathbf{C}_p}$. We define

$$\begin{aligned} \kappa_{l,z} : \mathfrak{O}_K &\to \mathbf{C}_p \\ \alpha &\mapsto t_l^{\alpha}(z) \end{aligned}$$

Proposition IV.2.10 *Let $z \in \mathfrak{p}_{\mathbf{C}_p}$. Then $\kappa_{l,z} \in \mathrm{Hom}_{\mathrm{LA}}\left(\mathfrak{O}_K, \mathbf{C}_p{}^{\times}\right)$.*

Proof:
By proposition IV.2.3(ii), p. 71 we can write $\kappa_{l,z} = \sum_{m=0}^{+\infty} z^m P_{l,m}$, so by theorem IV.2.4, p. 73 we have $\kappa_{l,z} \in \mathrm{LA}\left(\mathfrak{O}_K, \mathbf{C}_p\right)$. Now by proposition IV.1.16(i), p. 69 we have $\kappa_{l,z}(0) = 1$ and $\kappa_{l,z}(\alpha_1 + \alpha_2) = \kappa_{l,z}(\alpha_1)\,\kappa_{l,z}(\alpha_2)$, as required. $\square$

Proposition IV.2.11 (Schneider Teitelbaum)
Let $\mu \in \mathscr{D}_{\mathrm{LA}}\left(\mathfrak{O}_K, \mathbf{C}_p\right)$ and let $z \in \mathfrak{p}_{\mathbf{C}_p}$. Then

$$\mu\left(\kappa_{l,z}\right) = \mathscr{A}_l(\mu)(z).$$

Proof:
By proposition IV.2.3(ii), p. 71 we have $\kappa_{l,z} = \sum_{m=0}^{+\infty} z^m P_{l,m}$, so by the linearity and continuity of μ we have

$$\begin{aligned} \mu\left(\kappa_{l,z}\right) &= \mu\left(\sum_{m=0}^{+\infty} z^m P_{l,m}\right) \\ &= \sum_{m=0}^{+\infty} z^m \mu\left(P_{l,m}\right) \\ &= \mathscr{A}_l(\mu)(z). \end{aligned}$$

$\square$

Remark IV.2.12 This is a direct generalisation of proposition III.2.14, p. 63.

Chapter V

Non-orthogonality of the Schneider Teitelbaum functions

In this chapter we will prove that the set of Schneider Teitelbaum functions $\{P_{l,m} \,|m \in \mathbf{Z}_{\geqslant 0}\}$ is not orthogonal in $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, \mathbf{C}_p)$ for all $h \geqslant \max\{1, [K : \mathbf{Q}_p] - e_K\}$. The strategy of the proof is as follows. In §V.1 we will study the zeros of the power series $t_l^\alpha(Y) - 1$. Using the Newton polygon, this gives us information about the coefficients of $t_l^\alpha(Y) - 1$, and we know that the coefficient of Y^m in $t_l^\alpha(Y)$ is equal to $P_{l,m}(\alpha)$. In §V.2 we will study the polynomial $P_{l,m}(\alpha + X)$. We know its leading coefficient, and the work of §V.1 tells us about its constant coefficient $P_{l,m}(\alpha)$. We can, therefore, use Newton polygons again; this time to derive information about the zeros of $P_{l,m}(\alpha + X)$ from our knowledge of its coefficients. In particular, for certain values of $m \in \mathbf{Z}_{\geqslant 0}$, we can prove that $P_{l,q_K^m}(X)$ is not evenly distributed of order m. The result then follows from proposition II.1.15, p. 40.

Throughout this chapter we continue to assume that $K \neq \mathbf{Q}_p$.

V.1 The Newton polygon of $t_l^\alpha(Y) - 1$

In this section we will determine the Newton polygon of $t_l^\alpha(Y) - 1$. The following three propositions will: show that all the zeros of $t_l^\alpha(Y) - 1$ lie in $\Lambda_{l,+\infty}$, count $\#\{\gamma \in \Lambda_{l,n/e_K} \,|t_l^\alpha(\gamma) - 1 = 0\}$, and prove that all these zeros are simple. This is enough to give us complete information about the Newton polygon.

Proposition V.1.1 *Let* $\alpha \in \mathfrak{O}_K - \{0\}$, *and let* $z \in \mathfrak{p}_{\mathbf{C}_p}$ *such that* $t_l^\alpha(z) - 1 = 0$. *Then* $z \in \Lambda_{l,+\infty}$.

Proof:
Let $F(Y) = \exp(Y)$ and let $G(Y) = \alpha \Omega_l \lambda_l(Y)$. By proposition I.4.6, p. 23 there exists $r \in \mathbf{Q}$ such that

$$(F \circ G)(x) = F(G(x)) \qquad \forall x \in p^r \mathfrak{O}_{\mathbf{C}_p}.$$

By proposition IV.1.6(ii), p. 66 we can choose $n \in \mathbf{Z}_{\geqslant 0}$ sufficiently large that $\mathrm{ord}_p([p^n]_l(z)) \geqslant r$. We have

$$\begin{array}{rrcll} & t_l^\alpha(z) & = & 1, & \\ \Rightarrow & t_l^\alpha(z)^{p^n} & = & 1, & \\ \Rightarrow & t_l^\alpha([p^n]_l(z)) & = & 1 & \text{(by proposition IV.1.16, p. 69)}, \\ \Rightarrow & F \circ G([p^n]_l(z)) & = & 1, & \\ \Rightarrow & F(G([p^n]_l(z))) & = & 1. & \end{array}$$

Now $\exp(Y)$ is injective where it converges (see, for example, Schikhof [18], proposition 44.1, p. 128), so we must have

$$\begin{array}{rrcll} & G([p^n]_l(z)) & = & 0, & \\ \Rightarrow & \lambda_l([p^n]_l(z)) & = & 0, & \\ \Rightarrow & p^n \lambda_l(z) & = & 0 & \text{(by proposition IV.1.10(i), p. 67)}, \\ \Rightarrow & \lambda_l(z) & = & 0. & \end{array}$$

The result now follows by proposition IV.1.10(iv), p. 68. $\square$

Proposition V.1.2 *Let* $\alpha \in \mathfrak{O}_K - \{0\}$. *Let* $n \in \mathbf{Z}$, $n/e_K \geqslant \mathrm{ord}_p(\alpha)$ *and set* $i := \lceil n/e_K - \mathrm{ord}_p(\alpha) \rceil \in \mathbf{Z}_{\geqslant 0}$, *where* $\lceil r \rceil$ *denotes* r *rounded up to the nearest integer. Then the map*

$$\begin{aligned} t_l^\alpha : \Lambda_{l,n/e_K} &\to \mu_{p^i} \\ z &\mapsto t_l^\alpha(z) \end{aligned}$$

is a surjective group homomorphism.

Proof:
First we check that $t_l^\alpha(z) \in \mu_{p^i}$ for all $z \in \Lambda_{l,n/e_K}$. We have

$$\begin{array}{rrcl} & [\pi_K^n]_l(z) & = & 0, \\ \Rightarrow & [p^i\alpha]_l(z) & = & 0, \\ \Rightarrow & t_l^1([p^i\alpha]_l(z)) & = & 1, \\ \Rightarrow & t_l^\alpha(z)^{p^i} & = & 1 \quad \text{(by proposition IV.1.16, p. 69).} \end{array}$$

Hence $t_l^\alpha(z) \in \mu_{p^i}$, as required.

Since $t_l^\alpha \in \mathscr{H}_l$, it is clear that the map is a group homomorphism.

It remains to prove that the map is surjective. Since any primitive p^i-th root of unity generates μ_{p^i}, it is enough to find $\gamma \in \Lambda_{l,n/e_K}$ such that $t_l^\alpha(\gamma)^{p^{i-1}} \neq 1$. By proposition IV.1.18, p. 70, the map $t_l^1 : \Lambda_{l,1/e_K} \to \mu_p$ is surjective. The map $[\pi_K^{n-1}]_l : \Lambda_{l,n/e_K} \to \Lambda_{l,1/e_K}$ is certainly surjective, so we can find $\gamma_1 \in \Lambda_{l,n/e_K}$ such that $t_l^1([\pi_K^{n-1}]_l(\gamma_1)) \neq 1$. Set $\beta := \pi_K^{n-1}/\alpha p^{i-1}$. We have

$$\begin{aligned} \operatorname{ord}_p(\beta) &= \frac{n-1}{e_K} - (\operatorname{ord}_p(\alpha) + i - 1) \\ &= \frac{n}{e_K} - \operatorname{ord}_p(\alpha) - \left\lceil \frac{n}{e_K} - \operatorname{ord}_p(\alpha) \right\rceil + 1 - \frac{1}{e_K} \\ &\geqslant 0, \end{aligned}$$

so $\beta \in \mathfrak{O}_K$. Set $\gamma := [\beta]_l(\gamma_1)$. Now, using proposition IV.1.16, p. 69 again, we have

$$\begin{aligned} t_l^\alpha(\gamma)^{p^{i-1}} &= t_l^1\left(\left[p^{i-1}\alpha\right]_l(\gamma)\right) \\ &= t_l^1\left(\left[p^{i-1}\alpha\beta\right]_l(\gamma_1)\right) \\ &= t_l^1\left(\left[\pi_K^{n-1}\right]_l(\gamma_1)\right) \\ &\neq 1, \end{aligned}$$

as required. □

Corollary V.1.3 *Let* $\alpha \in \mathfrak{O}_K - \{0\}$.

i. Let $n \in \mathbf{Z}_{\geqslant 0}$, $n/e_K \leqslant \operatorname{ord}_p(\alpha)$. *Then*

$$\#\left\{\gamma \in \Lambda_{l,n/e_K} \,\middle|\, t_l^\alpha(\gamma) - 1 = 0\right\} = q_K^n.$$

ii. Let $n \in \mathbf{Z}, n/e_K \geqslant \mathrm{ord}_p(\alpha)$ and set $i := \lceil n/e_K - \mathrm{ord}_p(\alpha) \rceil \in \mathbf{Z}_{\geqslant 0}$. Then

$$\#\left\{\gamma \in \Lambda_{l,n/e_K} \left| t_l^\alpha(\gamma) - 1 = 0\right.\right\} = q_K^n/p^i.$$

Proof:

i. For all $\gamma \in \Lambda_{l,n/e_K}$, we have $t_l^\alpha(\gamma) = t_l^1([\alpha]_l(\gamma)) = t_l^1(0) = 1$, and $\#\Lambda_{l,n/e_K} = q_K^n$ by proposition IV.1.8(i), p. 66.

ii. By proposition V.1.3 above we have that $t_l^\alpha : \Lambda_{l,n/e_K} \to \mu_{p^i}$ is a surjective group homomorphism; hence

$$\begin{aligned}\#\left\{\gamma \in \Lambda_{l,n/e_K} \left| t_l^\alpha(\gamma) - 1 = 0\right.\right\} &= \frac{\#\Lambda_{l,n/e_K}}{\#\mu_{p^i}} \\ &= q_K^n/p^i.\end{aligned}$$

□

Proposition V.1.4 *Let $\alpha \in \mathfrak{O}_K - \{0\}$. Then the power series $t_l^\alpha(Y) - 1$ has only simple zeros in $\mathfrak{p}_{\mathbf{C}_p}$.*

Proof:
Let $z \in \mathfrak{p}_{\mathbf{C}_p}$ such that $t_l^\alpha(z) - 1 = 0$. We must show that $(t_l^\alpha)'(z) \neq 0$, where $(t_l^\alpha)'(Y)$ denotes the formal derivative of $t_l^\alpha(Y)$.

As a formal power series we have

$$\begin{aligned} t_l^\alpha(Y) &= \exp(\alpha\Omega_l\lambda_l(Y)), \\ \Rightarrow \qquad (t_l^\alpha)'(Y) &= \alpha\Omega_l\lambda_l'(Y)\exp(\alpha\Omega_l\lambda_l(Y)) \\ &= \alpha\Omega_l\lambda_l'(Y)t_l^\alpha(Y). \end{aligned}$$

Now $t_l^\alpha(Y) \in \mathfrak{O}_{\mathbf{C}_p}[[Y]]$ and, by proposition IV.1.10(ii), p. 67, we have $\lambda_l'(Y) \in \mathfrak{O}_K[[Y]]$, so both these power series converge on $\mathfrak{p}_{\mathbf{C}_p}$. It follows that $(t_l^\alpha)'(z) = 0$ if and only if either $\lambda_l'(z) = 0$ or $t_l^\alpha(z) = 0$. But $t_l^\alpha(z) = 1$, so it is enough to prove that $\lambda_l'(z) \neq 0$.

By proposition V.1.1, p. 76 we have $z \in \Lambda_{l,+\infty}$; choose $n \in \mathbf{Z}_{\geqslant 0}$ sufficiently large that $[\pi_K^n]_l(z) = 0$. By proposition IV.1.8(iii), p. 67, we have $[\pi_K^n]_l'(z) \neq$

0. By proposition IV.1.10(i), p. 67 we have

$$\begin{aligned} \lambda_l\left(\left[\pi_K^n\right]_l(Y)\right) &= \pi_K^n \lambda_l(Y), \\ \Rightarrow \quad \left[\pi_K^n\right]_l'(Y)\lambda_l'\left(\left[\pi_K^n\right]_l(Y)\right) &= \pi_K^n \lambda_l'(Y), \\ \Rightarrow \quad \left[\pi_K^n\right]_l'(z)\lambda_l'(0) &= \pi_K^n \lambda_l'(z), \\ \Rightarrow \quad \left[\pi_K^n\right]_l'(z) &= \pi_K^n \lambda_l'(z). \end{aligned}$$

Hence $\lambda_l'(z) \neq 0$, as required. □

We summarise our results about the zeros of $t_l^\alpha(Y) - 1$ in the following proposition.

Proposition V.1.5 *Let $\alpha \in \mathfrak{O}_K - \{0\}$.*

i. *We have*

$$\mathrm{Z}\left(r; t_l^\alpha(Y) - 1\right) = 1 \qquad \forall r \in \mathbf{Q},\ \frac{1}{e_K(q_K - 1)} < r \leqslant +\infty.$$

ii. *Let $n \in \mathbf{Z}_{\geqslant 1}$, $n/e_K \leqslant \mathrm{ord}_p(\alpha)$. Then*

$$\mathrm{Z}\left(r; t_l^\alpha(Y) - 1\right) = q_K^n \qquad \frac{1}{e_K\left(q_K^{n+1} - q_K^n\right)} < r \leqslant \frac{1}{e_K\left(q_K^n - q_K^{n-1}\right)}.$$

iii. *Let $n \in \mathbf{Z}_{\geqslant 1}$, $n/e_K \geqslant \mathrm{ord}_p(\alpha)$ and set $i := \lceil n/e_K - \mathrm{ord}_p(\alpha)\rceil \in \mathbf{Z}_{\geqslant 0}$. Then*

$$\mathrm{Z}\left(r; t_l^\alpha(Y) - 1\right) = q_K^n/p^i \qquad \frac{1}{e_K\left(q_K^{n+1} - q_K^n\right)} < r \leqslant \frac{1}{e_K\left(q_K^n - q_K^{n-1}\right)}.$$

Proof:
Let $n \in \mathbf{Z}_{\geqslant 1}$. Let $\gamma_n \in \Lambda_{l,\frac{n}{e_K}} - \Lambda_{l,\frac{n-1}{e_K}}$. By proposition IV.1.8(ii), p. 67 we have

$$\mathrm{ord}_p(\gamma_n) = \frac{1}{e_K\left(q_K^n - q_K^{n-1}\right)}.$$

By proposition V.1.1, p. 76 and proposition V.1.4 above, part (i) now follows and, for all $n \in \mathbf{Z}_{\geqslant 1}$, we have

$$\begin{aligned} \mathrm{Z}\left(r; t_l^\alpha(Y) - 1\right) &= \#\left\{\gamma \in \Lambda_{l,n/e_K} \,\middle|\, t_l^\alpha(\gamma) - 1 = 0\right\} \\ &\qquad \forall r \in \mathbf{Q},\ \tfrac{1}{e_K\left(q_K^{n+1} - q_K^n\right)} < r \leqslant \tfrac{1}{e_K\left(q_K^n - q_K^{n-1}\right)}. \end{aligned}$$

The result now follows by corollary V.1.3, p. 77. □

Remark V.1.6 It is a simple task to use proposition V.1.5 above to build the Newton polygon of $t_l^{\alpha}(Y) - 1$ for any $\alpha \in \mathfrak{O}_K - \{0\}$. Here I will just extract the information we will need in the following section.

Proposition V.1.7 *Let $\alpha \in \mathfrak{O}_K{}^{\times}$. Let $n \in \mathbf{Z}_{\geqslant 1}$, and if $K/\mathbf{Q}_p$ is totally ramified assume that $n \geqslant 2$. Set $i := \lceil n/e_K \rceil$. Then*

$$\operatorname{ord}_p\left(P_{l,q_K^n/p^i}(\alpha)\right) < \operatorname{ord}_p(\Omega_l).$$

Proof:
The coefficient of Y in $t_l^{\alpha}(Y) - 1$ is $\alpha\Omega_l$, and $\operatorname{ord}_p(\alpha\Omega_l) = \operatorname{ord}_p(\Omega_l)$. By theorem I.5.6, p. 26 and proposition V.1.5 above, the slopes of the Newton polygon $\mu(j; t_l^{\alpha}(Y) - 1) < 0$ for all $j \in \mathbf{Z}_{\geqslant 0}$. It follows that all the vertices of the Newton polygon of $t_l^{\alpha}(Y) - 1$ lie below $\operatorname{ord}_p(\Omega_l)$. By proposition IV.2.3(ii), p 71 we know that $P_{l,q_K^n/p^i}(\alpha)$ is the coefficient of $Y^{q_K^n/p^i}$ in $t_l^{\alpha}(Y)$, and using proposition V.1.5 again we see that $\left(q_K^n/p^i, \operatorname{ord}_p\left(P_{l,q_K^n/p^i}(\alpha)\right)\right)$ is a vertex of the Newton polygon of $t_l^{\alpha}(Y) - 1$. The result follows. □

V.2 The zeros of $P_{l,q_K^m}(\alpha + X)$

In this section, for certain values of $m \in \mathbf{Z}_{\geqslant 0}$, we will prove that the Schneider Teitelbaum polynomial $P_{l,q_K^m}(X)$ is not evenly distributed of order m. It then follows from proposition II.1.15, p. 40 that, for all $h \in \mathbf{Z}$, $h \geqslant m$, the function P_{l,q_K^m} is not orthogonal to $\operatorname{span}_{\mathbf{C}_p}\{X^j \,|\, j \in \{0, 1, \ldots, q_K^m - 1\}\}$ in $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K, \mathbf{C}_p)$.

For $\alpha \in \mathfrak{O}_K{}^{\times}$, we will study the Newton polygon of the polynomial $P_{l,q_K^m}(\alpha + X)$. The valuation of its constant term was studied in §V.1, and we know that its leading coefficient is $\Omega_l^{q_K^m}/q_K^m!$. The assumption $P_{l,q_K^m}(X)$ is evenly distributed allows us to estimate the slopes of the Newton polygon of $P_{l,q_K^m}(\alpha + X)$, and leads to a contradiction.

In this section we will work exclusively with Schneider Teitelbaum polynomials whose degree is a power of q_K. The following proposition obtains the information we will require about $\operatorname{ord}_p\left(P_{l,q_K^m}(\alpha)\right)$ from the work of the previous section.

Proposition V.2.1 *Let $m \in \mathbf{Z}_{\geqslant 1}$ such that we can write $m = n - \lceil n/e_K \rceil / f_K$ for some $n \in \mathbf{Z}_{\geqslant 2}$. Let $\alpha \in \mathfrak{O}_K{}^{\times}$. Then*

$$\operatorname{ord}_p\left(P_{l,q_K^m}(\alpha)\right) < \operatorname{ord}_p(\Omega_l).$$

Proof:
Set $i := \lceil n/e_K \rceil$, so $q_K^n/p^i = q_K^{n - \lceil n/e_K \rceil / f_K} = q_K^m$. Hence, by proposition V.1.7, p. 80, we have $\operatorname{ord}_p\left(P_{l,q_K^m}(\alpha)\right) < \operatorname{ord}_p(\Omega_l)$. □

Remark V.2.2 There are infinitely many $m \in \mathbf{Z}_{\geqslant 1}$ satisfying the condition of proposition V.2.1, the smallest of which is $m = [K : \mathbf{Q}_p] - e_K$ if $K/\mathbf{Q}_p$ is not totally ramified, or $m = 1$ if $K/\mathbf{Q}_p$ is totally ramified.

We will now estimate the slopes of the Newton polygon of $P(\alpha + X)$, where $P(X) \in \mathbf{C}_p[X]$ is an evenly distributed polynomial of degree q_K^m.

Proposition V.2.3 *Let $m \in \mathbf{Z}_{\geqslant 0}$ and let $P(X) \in \mathbf{C}_p[X]$ be a polynomial of degree q_K^m that is evenly distributed of order m. Let $\alpha \in \mathfrak{O}_K$. Then*

$$\sum_{i=1}^{q_K^m} \mu(i; P(\alpha + X)) \leqslant -\frac{q_K^m - 1}{e_K(q_K - 1)}.$$

Proof:
Since $P(X)$ is evenly distributed of order m, from definition II.1.9, p. 36, for all $k \in \{0, \ldots, m\}$, we have

$$\mathrm{Z}(\alpha, k/e_K; P(X)) \geqslant q_K^{m-k}.$$

Hence, by theorem I.5.6, p. 26, we have

$$\mu(i; P(\alpha + X)) \leqslant -k/e_K \qquad \forall i \in \{1, \ldots, q_K^{m-k}\}.$$

Therefore

$$
\begin{aligned}
\sum_{i=1}^{q_K^m} \mu\left(i; P\left(\alpha+X\right)\right) &= \mu\left(1; P\left(\alpha+X\right)\right) + \sum_{j=1}^{m} \sum_{i=q_K^{j-1}+1}^{q_K^j} \mu\left(i; P\left(\alpha+X\right)\right) \\
&\leqslant -\frac{m}{e_K} + \sum_{j=1}^{m} \left(q_K^j - q_K^{j-1}\right)\left(-\frac{m-j}{e_K}\right) \\
&= -\frac{1}{e_K}\left(m + \sum_{j=1}^{m} q_K^j (m-j) - \sum_{j=0}^{m-1} q_K^j (m-(j+1))\right) \\
&= -\frac{1}{e_K}\left(m + \left(\sum_{j=1}^{m-1} q_K^j\right) - (m-1)\right) \\
&= -\frac{1}{e_K}\sum_{j=0}^{m-1} q_K^j \\
&= -\frac{q_K^m - 1}{e_K\left(q_K - 1\right)}.
\end{aligned}
$$

□

We are now ready to prove the main result of this section.

Proposition V.2.4 *Let $m \in \mathbf{Z}_{\geqslant 1}$ such that we can write $m = n - \lceil n/e_K \rceil / f_K$ for some $n \in \mathbf{Z}_{\geqslant 2}$. Then the Schneider Teitelbaum polynomial $P_{l,q_K^m}(X)$ is not evenly distributed of order m.*

Proof:
Let $\alpha \in \mathfrak{O}_K^{\times}$. By proposition IV.2.3(i), p. 71, we know that $P_{l,q_K^m}(\alpha+X)$ has degree exactly m and leading coefficient $\Omega_l^{q_K^m}/q_K^m!$. By considering its Newton polygon we see that

$$
\sum_{i=1}^{q_K^m} \mu\left(i; P_{l,q_K^m}\left(\alpha+X\right)\right) = \mathrm{ord}_p\left(\frac{\Omega_l^{q_K^m}}{q_K^m!}\right) - \mathrm{ord}_p\left(P_{l,q_K^m}\left(\alpha\right)\right).
$$

Now, by propositions V.2.1, p. 81 and IV.1.20, p. 70, and by lemma III.1.5,

p. 51, we have

$$
\begin{aligned}
\operatorname{ord}_p\left(\frac{\Omega_l^{q_K^m}}{q_K^m!}\right)-\operatorname{ord}_p\left(P_{l,q_K^m}(\alpha)\right) &> \operatorname{ord}_p\left(\frac{\Omega_l^{q_K^m}}{q_K^m!}\right)-\operatorname{ord}_p\left(\Omega_l\right)\\
&=\left(q_K^m-1\right)\left(\frac{1}{p-1}-\frac{1}{e_K\left(q_K-1\right)}\right)-\frac{q_K^m-1}{p-1}\\
&=-\frac{q_K^m-1}{e_K\left(q_K-1\right)}.
\end{aligned}
$$

Therefore

$$
\sum_{i=1}^{q_K^m}\mu\left(i;P_{l,q_K^m}(\alpha+X)\right)>-\frac{q_K^m-1}{e_K\left(q_K-1\right)},
$$

so, by proposition V.2.3 above, the polynomial $P_{l,q_K^m}(X)$ is not evenly distributed of order m. $\square$

Corollary V.2.5 *For all $h\in\mathbf{Z}$, $h\geqslant\max\{1,[K:\mathbf{Q}_p]-e_K\}$, the set of Schneider Teitelbaum functions $\{P_{l,m}\,|m\in\mathbf{Z}_{\geqslant 0}\}$ is not orthogonal in the $\mathbf{C}_p$-Banach space* $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K,\mathbf{C}_p)$.

Proof:
If $K/\mathbf{Q}_p$ is totally ramified, set $n:=2$, so $\lceil n/e_K\rceil=1$. If $K/\mathbf{Q}_p$ is not totally ramified, set $n:=[K:\mathbf{Q}_p]-(e_K-1)$, so $\lceil n/e_K\rceil=f_K$. In both cases, set $m:=n-\lceil n/e_K\rceil/f_K=n-1$. Note that $m=\max\{1,[K:\mathbf{Q}_p]-e_K\}$. By proposition V.2.4 above the polynomial $P_{l,q_K^m}(X)$ is not evenly distributed of order m. Hence by proposition II.1.15, p. 40, for all $h\in\mathbf{Z}$, $h\geqslant m$, the function P_{l,q_K^m} is not orthogonal to $\operatorname{span}_{\mathbf{C}_p}\{X^j\,|j\in\{0,\ldots,q_K^m-1\}\}$ in $\mathrm{LA}_{h/e_K}(\mathfrak{O}_K,\mathbf{C}_p)$. $\square$

Remark V.2.6 In proposition V.2.4, p. 82 we have shown that there are infinitely many $m\in\mathbf{Z}_{\geqslant 1}$ such that $P_{l,q_K^m}(X)$ is not very evenly distributed. However, for fixed $h\in\mathbf{Z}, h\geqslant\max\{1,[K:\mathbf{Q}_p]-e_K\}$, we have only exhibited finitely many $m\in\mathbf{Z}_{\geqslant 0}$ such that $P_{l,m}(X)$ is not evenly distributed of order h.

I will conclude this thesis with a small result that is, perhaps, a little more encouraging than corollary V.2.5.

Proposition V.2.7 *Assume that $e_K \leqslant p-1$. For $m \in \mathbf{Z}_{\geqslant 1}$, the polynomial $P_{l,q_K^m}(X)$ is evenly distributed of order $m-1$.*

Proof:
Let $\alpha \in \mathfrak{O}_K$ and let $k \in \{0, \dots, m-1\}$. We must show that

$$\mathrm{Z}\left(\alpha, k/e_K; P_{l,q_K^m}(X)\right) \geqslant q_K^{m-k}.$$

Choose a set $R \subset \mathfrak{O}_K$ of representatives of $\alpha + \pi_K^k \mathfrak{O}_K$ in $\mathfrak{O}_K/\pi_K^m \mathfrak{O}_K$; that is, such that

$$\alpha + \pi_K^k \mathfrak{O}_K = \coprod_{\beta \in R} \beta + \pi_K^m \mathfrak{O}_K.$$

Note that $\#R = q_K^{m-k}$. By proposition IV.1.20, p. 70 we have $\mathrm{ord}_p(\Omega_l) < \frac{1}{p-1}$, so we have $\mathrm{ord}_p(\Omega_l) < 1/e_K$ since $e_K \leqslant p-1$. Set $r := m/e_K - \mathrm{ord}_p(\Omega_l)$; we have $r > \frac{m-1}{e_K}$. It follows that the sets $\beta + p^r \mathfrak{O}_{\mathbf{C}_p}$, for $\beta \in R$, are pairwise disjoint and contained in $\alpha + \pi_K^k \mathfrak{O}_{\mathbf{C}_p}$. Hence it is enough to prove that $\mathrm{Z}\left(\beta, r; P_{l,q_K^m}(X)\right) \geqslant 1$ for all $\beta \in R$.

Fix $\beta \in R$. We will consider the Newton polygon of $P_{l,q_K^m}(\beta + X)$. By proposition IV.2.3(iii), p. 71 we know that $\mathrm{ord}_p\left(P_{l,q_K^m}(\beta)\right) \geqslant 0$. We wish to estimate $\mathrm{ord}_p\left(P'_{l,q_K^m}(\beta)\right)$. Write $\lambda_l(Y) = \sum_{i=1}^{+\infty} c_i Y^i \in K[[Y]]$. By proposition IV.2.3(iv), p. 71 we have

$$P'_{l,q_K^m}(\beta) = \Omega_l \sum_{i=1}^{q_K^m} c_i P_{l,q_K^m - i}(\beta).$$

Hence, by proposition IV.1.11, p. 68, we have $\mathrm{ord}_p\left(P'_{l,q_K^m}(\beta)\right) = \mathrm{ord}_p(\Omega_l) - m/e_K = -r$. It follows that

$$\begin{aligned} \mu\left(1; P_{l,q_K^m}(\beta + X)\right) &\leqslant \mathrm{ord}_p\left(P'_{l,q_K^m}(\beta)\right) - \mathrm{ord}_p\left(P_{l,q_K^m}(\beta)\right) \\ &\leqslant -r. \end{aligned}$$

Hence, by theorem I.5.6, p. 26, we have $\mathrm{Z}\left(\beta, r; P_{l,q_K^m}(X)\right) \geqslant 1$, as required.
□

Bibliography

[1] Y. Amice. Interpolation p-adique. *Bull. Soc. Math. France*, 92:117–180, 1964.

[2] Y. Amice and J. Vélu. Distributions p-adiques associées aux séries de Hecke. In *Journées Arithmétiques de Bordeaux (Conf., Univ. Bordeaux, Bordeaux, 1974)*, pages 119–131. Astérisque, Nos. 24–25. Soc. Math. France, Paris, 1975.

[3] J. L. Boxall. A new construction of p-adic L-functions attached to certain elliptic curves with complex multiplication. *Ann. Inst. Fourier (Grenoble)*, 36(4):31–68, 1986.

[4] J. L. Boxall. p-adic interpolation of logarithmic derivatives associated to certain Lubin-Tate formal groups. *Ann. Inst. Fourier (Grenoble)*, 36(3):1–27, 1986.

[5] P. Cartier. Groupes de Lubin-Tate généralisés. *Invent. Math.*, 35:273–284, 1976.

[6] J. Coates and A. Wiles. On p-adic L-functions and elliptic units. *J. Austral. Math. Soc. Ser. A*, 26(1):1–25, 1978.

[7] R. F. Coleman. Division values in local fields. *Invent. Math.*, 53(2):91–116, 1979.

[8] P. Colmez. Théorie d'Iwasawa des représentations de de Rham d'un corps local. *Ann. of Math. (2)*, 148(2):485–571, 1998.

[9] E. de Shalit. *Iwasawa theory of elliptic curves with complex multiplication*, volume 3 of *Perspectives in Mathematics.* Academic Press Inc., Boston, MA, 1987.

[10] N. M. Katz. p-adic interpolation of real analytic Eisenstein series. *Ann. of Math. (2)*, 104(3):459–571, 1976.

[11] N. M. Katz. Formal groups and p-adic interpolation. In *Journées Arithmétiques de Caen (Univ. Caen, Caen, 1976)*, pages 55–65. Astérisque No. 41–42. Soc. Math. France, Paris, 1977.

[12] N. Koblitz. *p-adic numbers, p-adic analysis, and zeta-functions*, volume 58 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1984.

[13] S. Lang. *Cyclotomic fields I and II*, volume 121 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1990.

[14] J. Lubin and J. Tate. Formal complex multiplication in local fields. *Ann. of Math. (2)*, 81:380–387, 1965.

[15] K. Mahler. An interpolation series for continuous functions of a p-adic variable. *J. Reine Angew. Math.*, 199:23–34, 1958.

[16] J. I. Manin and M. M. Višik. p-adic Hecke series of imaginary quadratic fields. *Math. USSR-Sb.*, 24(3):345–371, 1974.

[17] A. M. Robert. *A course in p-adic analysis*, volume 198 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2000.

[18] W. H. Schikhof. *Ultrametric calculus*, volume 4 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1984.

[19] P. Schneider and J. Teitelbaum. p-adic Fourier theory. *Doc. Math.*, 6:447–481 (electronic), 2001.

[20] J. T. Tate. p-divisible groups. In *Proc. Conf. Local Fields (Driebergen, 1966)*, pages 158–183. Springer, Berlin, 1967.

[21] A. C. M. van Rooij. *Non-Archimedean functional analysis*, volume 51 of *Monographs and Textbooks in Pure and Applied Math.* Marcel Dekker Inc., New York, 1978.