Mean Curvature Flow with a Neumann Boundary Condition in Flat Spaces

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Abstract

In this thesis I study mean curvature flow in both Euclidean and Minkowski space with a Neumann boundary condition.

In Minkowski space I show that for a convex timelike cone boundary condition, with compatible spacelike initial data, mean curvature flow with a perpendicular Neumann boundary condition exists for all time. Furthermore, by a blowdown argument I show convergence as $t \to \infty$ to a homothetically expanding hyperbolic hyperplane.

I also study the case of graphs over convex domains in Minkowski space. I obtain long time existence for spacelike initial graphs which are taken by mean curvature flow with a Neumann boundary condition to a constant function as $t \to \infty$.

In Euclidean space I consider boundary manifolds that are rotational tori where I write $\mathbf{t}$ for the unit vector field in the direction of the rotation. If the initial manifold M_0 is compatible with the boundary condition, and at no point has $\mathbf{t}$ as a tangent vector, then mean curvature flow with a perpendicular Neumann boundary condition exists for all time and converges to a flat cross-section of the boundary torus. I also discuss other constant angle boundary conditions.

Declaration

The work in this thesis is based on research carried out in the Pure Mathematics Group at the Department of Mathematical Sciences, Durham University. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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I would also like to thank all of my family and in particular my Mum and Dad, without whose support and encouragement I would never have started this PhD, let alone finished it. I would also like to thank all of my non mathematical friends, they know who they are.

This thesis is dedicated to my grandfathers, Arthur Lambert and George Layer.

Contents

Chapter 0

Introduction

Geometric flows have been of great interest in recent years, successfully proving many new results in differential geometry and topology. Mean Curvature Flow (MCF) is one such flow, and I begin this introduction with a qualitative description of the simplest incarnation of this, namely the *curve shortening flow.* In the curve shortening flow, we start with a smooth embedded curve in the plane $\gamma_0 : S^1 \to \mathbb{R}^2$, and look for a 1–parameter family of curves $\gamma : S^1 \times [0, T] \to \mathbb{R}^2$ such that

$$\begin{cases} \gamma(\theta, 0) = \gamma_0(\theta) & \forall \theta \in S^1 \\ \frac{\partial \gamma}{\partial t} = -\kappa\nu & \forall (\theta, t) \in S^1 \times [0, T] \end{cases}$$

where ν is a normal to the embedded curve and κ is the curvature on the curve with respect to ν. We refer to the interval $[0, T]$ as the time interval, and imagine that the curve is deformed over time.

For example, suppose that γ_0 is a circle of radius r_0. Then under the effect of curve shortening flow, easy calculations (see Section 1.2.2 for similar calculations) show that this circle will remain a circle but its radius will become smaller and smaller until at time $T = \frac{r_0^2}{2}$ the circle will become a point. Indeed, it turns out that this behaviour is true of all smooth initial curves bounding convex regions: If we flow such a curve then the flow exists for some finite T, and at time T the solution will have shrunk down to a single point $\mathbf{p}$.

So far so good, but we want to understand exactly *how* the curve becomes a point. We will do this by blowing up the solution by dilations around the point $\mathbf{p}$.

At each time t we dilate about the point $\mathbf{p}$ by some factor $\lambda(t)$ so that the area enclosed by the dilated curve is equal to 1. We also define a new time variable, $s = -\frac{1}{2}\log(\frac{1}{\lambda(t)})$. This new time interval is increasing with respect to t and has the property that when $t \to T$ then $s \to \infty$. We write the dilated curve as $\widetilde{\gamma}(\theta, s)$. It was shown by Gage and Hamilton [7] that for convex initial data this renormalised flow $\widehat{\gamma}$ converges to a round circle as $s \to \infty$. A vital element of geometric flows is analysing the singularities that occur.

We describe the surprising work of Grayson [10]. He showed that *for any initial embedded curve* γ_0 the renormalised curve shortening flow would become convex *before* it became singular. This implies that any embedded initial curve acting under renormalised curve shortening flow will converge to a circle, regardless of its initial convexity.

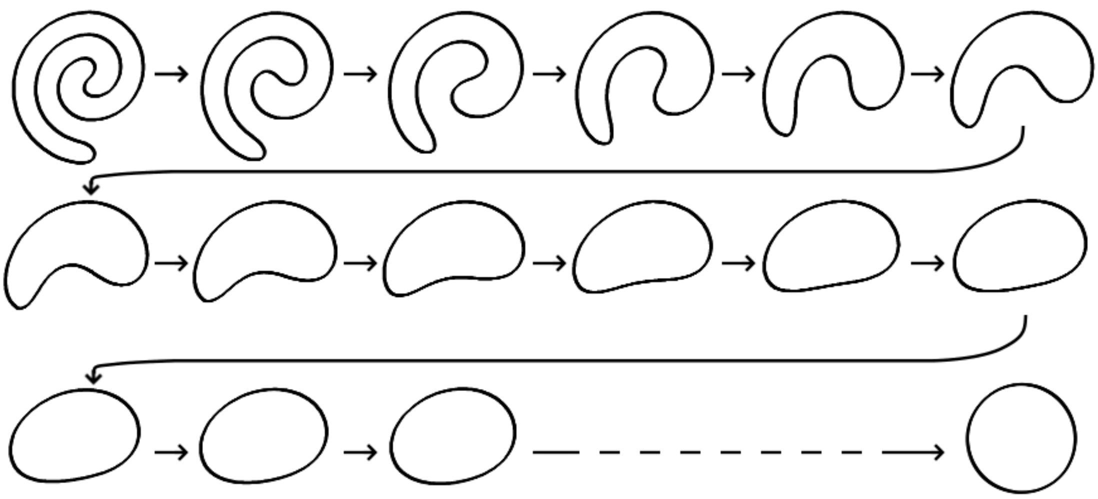


Figure 0.1: Curve shortening flow makes *any* initial closed embedded curve convex after finite time and then converges to a circle as $s \to \infty$

This beautiful result clearly demonstrates that flows can have very desirable properties. A very general initial curve is taken to a very special one via analysis of singularities.

I now briefly digress, to mention a related flow proposed by Hamilton [12]: The incredibly successful *Ricci flow.* In this the metric on a Riemannian manifold (M^n, g)

is deformed by the equation

$$\frac{\partial}{\partial t} g_{ij} = -2\mathrm{Ric}_{ij} \quad .$$

Study of this evolution has been an extremely productive area of mathematics, and this introduction cannot do it justice. I mention only the most famous result: Perelman's [23][24] careful analysis of possible singularities of the Ricci flow in dimension $n = 3$, which completed Hamilton's program proving the Poincaré conjecture and Thurston's geometrisation conjecture.

Returning to the main theme, we apply the heat flow equation to the position vector (see Section 1.2) to get *mean curvature flow*. Suppose we have an initial manifold parametrised by $\mathbf{F}_0 : M^n \to \mathbb{R}^{n+1}$ in $(n+1)$-dimensional Euclidean space, then this flow is defined by the one parameter family of embeddings $\mathbf{F} : M^n \times [0, T) \to \mathbb{R}^{n+1}$ with the properties

$$\begin{cases} \mathbf{F}(\theta, 0) = \mathbf{F}_0(\theta) & \forall \theta \in S^1 \\ \frac{\partial \mathbf{F}}{\partial t} = -H\nu & \forall (\theta, t) \in S^1 \times [0, T] \end{cases}$$

where H is the induced mean curvature and ν is the normal. In 1984 Huisken [13] showed that any initially convex hypersurface in $(n+1)$–dimensional Euclidean space will shrink to a point, and if we parabolically renormalise by dilatations about the singularity (similarly to the curve shortening flow) to hold the area of the flowing manifold constant then the renormalised flow will converge to a sphere. We remark here that Grayson's result does not hold in higher dimensions – much more complicated singularities may occur. A beautiful theory of such singularities has been built up, still with many open questions, I mention here the work of Sinestrari and Huisken [16] [17] in which 2–convex initial manifolds are topologically classified.

I will not give a proper overview of this fascinating subject now, but mention a recent global application of mean curvature flow. This is Guilfoyle and Klingenberg's [11] cunning proof of the Carathéodory conjecture. This conjecture states that any C^2 orientable closed strictly convex sphere M^2 in $\mathbb{R}^3$ must have at least two umbilic points, that is two points at which the second fundamental form has equal eigenvalues. Though this conjecture is simple to state it evaded proof or contradiction for many years. Guilfoyle and Klingenberg's method was to move the problem into the space of oriented geodesics in $\mathbb{R}^3$, that is the semi–Riemannian manifold $(TS^2, \mathbb{G})$, by identifying $\mathbf{p} \in M^2$ with the geodesic going through $\mathbf{p}$ in the direction of the normal to M^2. They then proved

- For *any* M^2 which has only one umbilic, there exists a small deformation of this manifold such that the deformed manifold will *not* admit a holomorphic curve in TS^2 with M^2 as a boundary condition
- *Any* convex M^2 is the boundary condition for a holomorphic curve in TS^2.

and so the existence of a closed convex manifold with only one umbilic leads to a contradiction. The proof of the second part of this is achieved by flowing a 2-dimensional disc with boundary in TS^2 by mean curvature flow. $(TS^2, \mathbb{G})$ is a 4-dimensional manifold which has signature $(2, 2)$, and so at the boundary 2 boundary conditions are required. This gives sufficient flexibility that, by choosing carefully one Neumann and one Dirichlet boundary condition, it is possible to impose asymptotic holomorphicity on the flowing manifold. The flow exists for all time, and in place of analysing a singularity, the question becomes one of what happens as time $t \to \infty$ (a recurring theme in ambient spaces of nonpositive metric). Guilfoyle and Klingenberg showed that there exists a sequence $t_i \to \infty$ such that on this subsequence of times the flowing disc converges to a holomorphic curve with the right properties, giving the second bullet point.

It is clear that mean curvature flow in semi–Riemannian spaces have applications to other problems, and also is an interesting subject in its own right. In codimension one although the Dirichlet problem and the entire problem has been studied (by Ecker, see [3]), to the authors knowledge until this thesis nothing has been done on the Neumann problem. I give some initial results on this.

At some point in a thesis every person must ask themselves the following:

0.1 What have I done?

Before going into this I will briefly describe what the Neumann boundary condition is (following Stahl [26]), see Section 1.2.3 for a full definition. We let Σ be a hypersurface in the ambient space (which will be Euclidean or Minkowski space), which will be referred to as the boundary manifold. We will be flowing another manifold with boundary, M, by mean curvature flow. At the boundary we impose two conditions: Firstly we require that the boundary of M, that is ∂M is contained within Σ. Secondly we require that the normal to the flowing manifold and the normal to Σ are held at some constant angle, generally $\frac{\pi}{2}$.

There are two main results in this thesis:

The first is concerned with mean curvature flow in Minkowski space $\mathbb{R}^{n+1}_1$. Here I choose a boundary manifold to be a cone of timelike vectors, and flow a disc with boundary inside this cone with a perpendicular Neumann boundary condition. In this setting if we assume the boundary cone to be convex, then a solution to mean curvature flow exists for all time, and moves "upwards" away from the origin. If we renormalise to hold the area of the flowing disc to be constant then in fact the solution converges to a hyperbolic hyperplane solution (see Example 1.2.5), the Minkowski equivalent of the homothetic sphere solution. I therefore describe this as a Minkowski–Neumann equivalent of [13]. This material is contained in Chapter 3.

The second main result is on mean curvature flow with a Neumann boundary condition in Euclidean space $\mathbb{R}^{n+1}$. We take the boundary manifold to be *any* smooth torus of rotation and again flow a disc inside the torus with a perpendicular Neumann boundary condition. We start with any manifold M_0 which satisfies the boundary condition and whose normal ν_0 is nowhere perpendicular to the rotational vector field inside the torus. By modifying the Stampaccia iteration method in [14] I have shown that under mean curvature flow with a Neumann boundary condition any such M_0 will converge to a flat sheet perpendicular to the rotational vector field. This material is contained in Chapter 5, also see Figure 0.2.

Additional results are contained in Chapter 4 where I considered graphical mean curvature flow in $\mathbb{R}^{n+1}_1$, and showed that mean curvature flow inside a convex cylinder with a perpendicular boundary condition exists for all time and converges to a constant solution. Also in this Chapter there is a brief discussion on integral methods in Minkowski space.

Also in Chapter 5 I give a sufficient condition on boundary manifolds to get long time existence of mean curvature flow with constant angle boundary conditions for the boundary angle close to $\frac{\pi}{2}$. This is in the form of a rough gradient estimate and was motivation for looking at the torus problem.

In Chapters 1 and 2 I give some supporting material for these results. In Chapter 1 I first give some background semi–Riemannian geometry before defining mean curvature with a Neumann boundary condition in Minkowski space. I also calculate many of the evolution equations needed in Chapters 3 and 4. In Chapter 2 I give a review of some of the standard quasilinear existence theory needed.

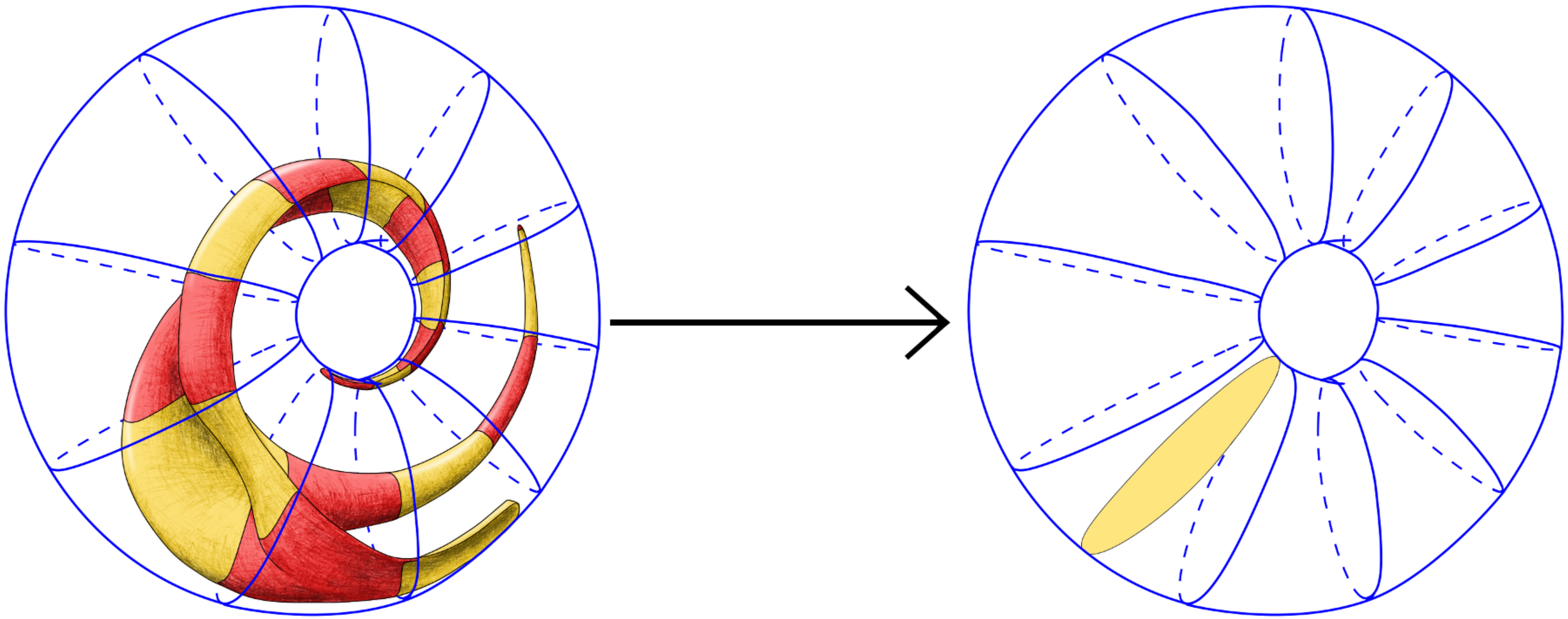


Figure 0.2: Under the effect of mean curvature flow inside a torus, a jesters hat initial manifold, M_0 (which is perpendicular to the boundary torus on ∂M_0), converges to a flat disc as $t \to \infty$

Chapter 1

Semi–Riemannian geometry and mean curvature flow

This Chapter sets out some of the basics of semi-Riemannian geometry, before introducing mean curvature flow with a Neumann boundary condition in Minkowski space.

I include the following brief section on semi–Riemannian geometry to remove any questions regarding signs in any derived equations. For further information see [22].

In Section 1.2 we define mean curvature flow with a Neumann boundary condition, and derive many of the evolution equations necessary for later chapters. Though the evolution equations are stated in [5], few explicit calculations are included. Therefore, I give the full details here.

1.1 Semi–Riemannian geometry

I will be working on manifolds contained in Minkowski space, some spacelike some of indefinite metric. As the signs of the various geometric quantities will be vital in calculations, care must be taken in derivations to ensure that signs arising from the spacelikeness or timelikeness of vectors are correct. Also since no consistent standard is agreed upon in the literature on choices of signs in the definitions of curvature tensors and other geometrical objects, to avoid confusion I will give the definitions of these quantities and briefly derive the main results I will need later. The main reference for the standard semi-Riemannian geometry I will be using here will be [22] although I use different sign conventions to fit in with the conventions adopted in papers on mean curvature flow.

1.1.1 Semi–Riemannian manifolds

Definition 1.1.1. Let (M^n, s) be a semi-Riemannian manifold, which for the purposes of this thesis we will define to be a smooth manifold M^n endowed with a nondegenerate smooth scalar product s (sometimes called a semi-Riemannian metric). For $V \in T_pM^n$ we say

- V is *spacelike* if $s(V, V) > 0$
- V is *lightlike* if $s(V, V) = 0$
- V is *timelike* if $s(V, V) < 0$

Additionally, we will describe the tangent space at a point $\mathbf{p}$ of a manifold as spacelike if $\forall V \in T_{\mathbf{p}}M^n$, V is spacelike. If all tangent spaces of a manifold M are spacelike then we will call M a spacelike manifold. We note that by the nondegeneracy of the metric spacelikeness of a manifold is equivalent to spacelikeness of the tangent space at any one point $\mathbf{p}$. We say that a manifold is indefinite if the tangent space contains both spacelike and timelike vectors.

Remark 1.1.2. A submanifold of a semi-Riemannian manifold is not necessarily semi-Riemannian. If s is nondegenerate when restricted to the tangent space of a submanifold $\widetilde{M} \subset M$ then we say that $\widetilde{M}$ is a *semi-Riemannian submanifold* of M. The notions of spacelikeness and indefiniteness, now apply to semi-Riemannian submanifolds.

Remark 1.1.3. A spacelike submanifold is also a Riemannian manifold .

We will say that $V \in T_{\mathbf{p}}M$ is a *unit vector* if $|s(V, V)| = 1$. Note that this allows timelike unit vectors of length -1 .

An example of such structures is the ambient space I will be using, Minkowski space.

Example 1.1.4. We define n-dimensional *Minkowski space* to be $\mathbb{R}^{n+1}_1$, that is $\mathbb{R}^{n+1}$ equipped with the indefinite metric $\langle\cdot,\cdot\rangle$ where

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + \ldots + x_ny_n - x_{n+1}y_{n+1} \quad .$$

Identically to Riemannian manifolds, we may define the Levi-Civita connection as the unique connection which is torsion free and compatible with the metric. We may also use this to define an induced connection on semi-Riemannian submanifolds, which again will be the Levi-Civita connection on the submanifold with respect to the induced metric. As is usual in a coordinate system we define the Christoffel symbols of the connection by writing

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k} \tag{1.1}$$

and we may calculate

$$\Gamma^k_{ij} = g^{kr} \left(\frac{\partial g_{ri}}{\partial x^j} + \frac{\partial g_{rj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) \tag{1.2}$$

where $g_{ij} = s\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ and g^{ij} is the inverse of this matrix.

We define $\mathfrak{X}(M)$ to be the set of smooth vector fields on a manifold M, $\mathfrak{X}^*(M)$ to be the set of smooth covector fields and $\mathfrak{F}(M) = \{f : M \to \mathbb{R} | f \text{ smooth}\}$ to be the set of smooth real valued functions on M.

1.1.2 Tensor fields

Let

$$A : \mathfrak{X}^*(M)^r \times \mathfrak{X}(M)^s \to \mathfrak{F}(M)$$

then A is a tensor field of type (r, s) if it is $\mathfrak{F}(M)$-multilinear. We will denote the set of all tensors of type (r, s) to be $\mathfrak{T}^r_s(M)$ and we will use the convention that $\mathfrak{T}^0_0(M) = \mathfrak{F}(M)$. Although we will not go through all the properties of tensors here, we will mention a few definitions. All that is stated here is standard, and may be found in more detail in [22, Chapter 2].

Often tensors will be written in coordinate form. For $A \in \mathfrak{T}^r_s(M)$ then in the basis $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ and corresponding cobasis $\{dx^1, \ldots, dx^n\}$ we will write

$$A^{j_1,\ldots,j_r}_{i_1,\ldots,i_s} = A(dx^{j_1}, \ldots, dx^{j_r}, \frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_s}}) \quad .$$

We can define the contraction of A over a, b locally by

$$C_b^a A_{i_1,\dots,i_s}^{j_1,\dots,j_r} = A_{i_1,\dots,i_{b-1},k,i_{b+1},\dots,i_s}^{j_1,\dots,j_{a-1},k,j_{a+1},\dots,j_r}$$

where for the rest of this section we are using summation convention on repeated indices. This contraction is again tensorial. We may change the type of tensors using the metric – for a $(r, s+1)$–tensor we may define a $(r+1, s)$–tensor by

$$A_{i_1,\dots,i_s}^{j_1,\dots,j_r,k} = A_{i_1,\dots,i_s,p}^{j_1,\dots,j_r} g^{pk} \quad .$$

We will define tensor derivatives firstly for covariant tensors (that is tensors of type $(0, s)$):

$$\begin{aligned}
&(\nabla_Z T)(X_1, X_2, \dots, X_s) \\
&\quad := Z(T(X_1, X_2, \dots, X_s)) - T(\nabla_Z X_1, X_2, \dots, X_s) \\
&\qquad - T(X_1, \nabla_Z X_2, \dots, X_s) - \dots - T(X_1, X_2, \dots, \nabla_Z X_s) \quad .
\end{aligned}$$

Remark 1.1.5. This may be considered a $(0, s+1)$-tensor, since this is also tensorial in Y.

By considering elements of $\mathfrak{X}^*(M)$ as a $(0,1)$ tensor then we see we have defined a covariant derivative on covector fields. Specifically for $X = X_i dx^i \in \mathfrak{X}^*(M)$ then for any $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^q} \in \mathfrak{X}(M)$

$$\begin{aligned}
(\nabla_{\frac{\partial}{\partial x^q}} X)(\frac{\partial}{\partial x^i}) &= \frac{\partial X_i}{\partial x^q} - X(\nabla_{\frac{\partial}{\partial x^q}} \frac{\partial}{\partial x^i}) \\
&= \frac{\partial X_i}{\partial x^q} - X_k \Gamma_{qi}^k \quad .
\end{aligned}$$

We hence have a covariant derivative on $\mathfrak{X}^*(M)$ defined by $\nabla_{\frac{\partial}{\partial x^i}} dx^j = -\Gamma_{qi}^j dx^q$ with the usual multiplication formula: For $f \in \mathfrak{F}(M)$, $X \in \mathfrak{X}^*(M)$, $Y \in \mathfrak{X}(M)$ then $\nabla_Y f X = Y(f)X + f\nabla_Y X$.

We may therefore extend the definition of tensor derivatives to tensors of type (r, s) using the same familiar formula (above) to get a $(r, s+1)$–tensor.

Lemma 1.1.6 (Tensor Product Rule)**.** *For $A \in \mathfrak{T}_0^2(M)$ and $A \in \mathfrak{T}_2^0(M)$ then let $C \in \mathfrak{T}_1^1$ be defined by $C_j^i = A^{ik} B_{kj}$ then*

$$(\nabla_p C)_j^i = (\nabla_p A)^{ik} B_{kj} + A^{ir} (\nabla_p B)_{rj}$$

Proof. This is just a question of doing the calculation.

$$
\begin{aligned}
(\nabla_p C)^j_i &= \frac{\partial}{\partial x^p}(A^{ik}B_{kj}) + \Gamma^i_{pq}A^{qr}B_{rj} - \Gamma^s_{jp}A^{iu}B_{us} \\
&= \left(\frac{\partial A^{ir}}{\partial x^p} + \Gamma^i_{pq}A^{qr}\right)B_{rj} + A^{ik}\left(\frac{\partial B_{kj}}{\partial x^p} - \Gamma^s_{jp}B_{ks}\right) \\
&= (\nabla_p A)^{ir}B_{rj} - \Gamma^v_{pw}A^{wi}B_{vj} + A^{ik}(\nabla_p B)_{kj} + A^{il}\Gamma^r_{pl}B_{rj} \\
&= (\nabla_p A)^{ir}B_{rj} + A^{ik}(\nabla_p B)_{kj} \quad .
\end{aligned}
$$

□

An analogous statements can be made for tensors $C \in \mathfrak{T}^{r+q-1}_{q+s}(M)$ made from $A \in \mathfrak{T}^r_s(M)$ and $B \in \mathfrak{T}^p_q(M)$. The proof is identical.

If we are on a *spacelike* manifold we may turn $\mathfrak{T}^r_s$ into an inner product space in a natural way by extending the metric. For $A, B \in \mathfrak{T}^r_S$ we define

$$
\left\langle A^{i_1,\dots,i_r}_{j_1,\dots j_s}, B^{i_1,\dots,i_r}_{j_1,\dots j_s}\right\rangle = A^{i_1,\dots,i_r}_{j_1,\dots j_s} B^{k_1,\dots,k_r}_{l_1,\dots l_s} g_{i_1,k_1}\dots g_{i_r,k_r} g^{j_1,l_1}\dots g^{j_n,l_n} \quad .
$$

Note that we have the Cauchy–Schwarz inequality for this inner product. This is often used to estimate otherwise complicated expressions. As a simple example let $A \in \mathfrak{T}^0_2(M)$ and $X, Y \in \mathfrak{X}(M)$:

$$
\begin{aligned}
A(X,Y) &= A_{ij}X^iY^j \\
&= A_{ij}g_{ac}g_{bd}X^aY^bg^{ci}g^{dj} \\
&= \left\langle A_{ij}, g_{ai}g_{bj}X^aY^b\right\rangle \\
&\le |A|\sqrt{g_{ai}g_{bj}X^aY^bg_{el}g_{fh}X^eY^fg^{li}g^{hj}} \\
&= |A|\sqrt{|X|^2|Y|^2} \\
&= |A||X||Y| \quad .
\end{aligned}
$$

If we are working on a indefinite manifold although we may define a *scalar* product as above this is not generally useful since in applications we will usually need the Cauchy–Schwarz inequality.

1.1.3 Curvature

Now we will briefly deal with both intrinsic and extrinsic curvatures. Firstly, for intrinsic curvature: Let M be a semi-Riemannian manifold with scalar product $\langle\cdot,\cdot\rangle$

and Levi-Civita connection ∇.

For $X, Y \in \mathfrak{X}(M)$ we define the *Lie bracket*, $[X, Y] \in \mathfrak{X}(M)$, at $\mathbf{p} \in M$ by

$$[X, Y](\mathbf{p})f = (XY - YX)(\mathbf{p})f$$

for all $f \in \mathfrak{F}(M)$. As usual we have that this is zero if X and Y are coordinate directions.

We define

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad .$$

This is tensorial, and we now define the *Riemann curvature tensor* on a M is defined by

$$R(X, Y, Z, V) = \langle R(X, Y)Z, V\rangle \quad .$$

Now for extrinsic curvature. For the rest of this section let $(\overline{M}, \langle -, -\rangle)$ be a semi-Riemannian manifold and (M, g) a semi-Riemmannian submanifold. We will denote the Levi-Civita connection on these manifolds by $\overline{\nabla}$ and ∇ respectively. A quantity on M denoted f will be denoted $\overline{f}$ for the equivalent quantity on $\overline{M}$.

We define for $X, Y \in \mathfrak{X}(M)$ the *shape operator* to be

$$\mathrm{I\!I}(X, Y) = \left(\overline{\nabla}_X Y\right)^{\perp} \quad .$$

This is symmetric and tensorial on M. We define the related notion of *second fundamental form* in the direction of some vector field ν normal to M by

$$A_\nu(X, Y) = -\langle \mathrm{I\!I}(X, Y), \nu\rangle = \left\langle \overline{\nabla}_X \nu, Y\right\rangle \tag{1.3}$$

where the equality comes from the compatibility of the Levi-Civita connection and applying X to the identity $\langle Y, \nu\rangle = 0$. In the case of orientable hypersurfaces we will drop the subscript on the second fundamental form since the choice of normal will either be clear or not matter. We will often write

$$h_{ij} = A\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \quad .$$

On hypersurfaces locally parametrised by $\mathbf{F}$ with ν locally defined it will also be useful to derive the *Weingarten relations*. Suppose that ν is a timelike unit normal, then from equations (1.2) and (1.3) we see

$$\frac{\partial^2 \mathbf{F}}{\partial x^i \partial x^j} = \Gamma^k_{ij} \frac{\partial \mathbf{F}}{\partial x^k} + h_{ij}\nu \tag{1.4}$$

whereas for a spacelike unit normal μ we have that

$$\frac{\partial^2 \mathbf{F}}{\partial x^i \partial x^j} = \Gamma^k_{ij} \frac{\partial \mathbf{F}}{\partial x^k} - h_{ij}\mu \quad . \tag{1.5}$$

1.1.4 Curvature identities

We now want relations between intrinsic and extrinsic curvatures. I will state these first in semi-Riemannian generality but also as a corollary I will include the statement I will actually use, that of the special case of a spacelike hypersurface with a timelike unit normal.

Proposition 1.1.7 (Gauss Lemma)**.** *For $X, Y, Z, V \in T_{\mathbf{p}}M$ then*

$$\langle R(X,Y)Z, V\rangle = \left\langle \overline{R}(X,Y)Z, V\right\rangle + \langle \mathrm{I\!I}(Y,Z), \mathrm{I\!I}(X,V)\rangle - \langle \mathrm{I\!I}(X,Z), \mathrm{I\!I}(Y,V)\rangle \quad .$$

Proof. Without loss of generality we consider this locally and take $X = \frac{\partial}{\partial x^X}$, $Y = \frac{\partial}{\partial x^Y}$, ...which implies that all the Lie brackets are zero. From the above we have $\overline{\nabla}_V W = \nabla_V W + \mathrm{I\!I}(V,W)$. Using this we calculate:

$$\begin{aligned}
\left\langle \overline{\nabla}_X \overline{\nabla}_Y Z, V\right\rangle &= \left\langle \overline{\nabla}_X \nabla_Y Z, V\right\rangle + \left\langle \overline{\nabla}_X (\mathrm{I\!I}(Y,Z)), V\right\rangle \\
&= \left\langle \overline{\nabla}_X \nabla_Y Z, V\right\rangle + X\left\langle \mathrm{I\!I}(Y,Z), V\right\rangle - \left\langle \mathrm{I\!I}(Y,Z), \overline{\nabla}_X V\right\rangle \\
&= \left\langle \nabla_X \nabla_Y Z, V\right\rangle - \left\langle \mathrm{I\!I}(Y,Z), \overline{\nabla}_X V\right\rangle \\
&= \left\langle \nabla_X \nabla_Y Z, V\right\rangle - \left\langle \mathrm{I\!I}(Y,Z), \mathrm{I\!I}(X,V)\right\rangle
\end{aligned}$$

where we used the compatibility of the connection, that $\langle \mathrm{I\!I}(Y,Z), V\rangle = 0$ and that $V \in T_{\mathbf{p}}M$. Since $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_Z Z$, we use the difference of the above with itself with Y and X switched to give the Proposition. □

Corollary 1.1.8. *On a hypersurface with a timelike unit normal ν then since $\mathrm{I\!I}(X,Y) = A(X,Y)\nu$ the above may be written*

$$R(X,Y,Z,V) = \overline{R}(X,Y,Z,V) + A(X,Z)A(Y,V) - A(Y,Z)A(X,V) \quad .$$

If $\overline{M}$ is Minkowski space then

$$R_{xyzv} = h_{xz}h_{yv} - h_{yz}h_{xv} \quad .$$

At any point $\mathbf{p} \in M$ we may write $T_{\mathbf{p}}\overline{M} = T_{\mathbf{p}}M \oplus N_{\mathbf{p}}$. In an identical way to the construction of the tangent bundle we may now construct the normal vector bundle NM. A section of this bundle is a normal vector field, and we write the set of normal vector fields on M by $\mathfrak{X}^{\perp}(M)$.

We define the normal connection $\nabla^{\perp} : \mathfrak{X}(M) \times \mathfrak{X}^{\perp}(M) \to \mathfrak{X}^{\perp}(M)$ as follows

$$\nabla^{\perp}_X Y = \left(\overline{\nabla}_X Y\right)^{\perp}$$

where $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}^{\perp}(M)$. As usual for a tensor field (that is a $\mathfrak{F}(M)$-linear mapping) $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}^{\perp}(M)$ we may define tensor derivatives by

$$(\nabla_Z T)(X,Y) := \nabla^{\perp}_Z T(X,Y) - T(\nabla_Z X, Y) - T(X, \nabla_Z Y) \quad .$$

As usual this is also a tensor field.

Proposition 1.1.9 (Codazzi Equation)**.** *If $X, Y, Z \in T_{\mathbf{p}}M$ then*

$$\left(\overline{R}(X,Y)Z\right)^{\perp} = (\nabla_X \mathrm{I\!I})(Y,Z) - (\nabla_Y \mathrm{I\!I})(X,Z)$$

Proof. Similarly to the proof of the Gauss Lemma we again assume $[X,Y] = [Y,Z] = [Z,X] = 0$ and consider

$$\begin{aligned}
\left(\overline{\nabla}_X \overline{\nabla}_Y Z\right)^{\perp} &= \left(\overline{\nabla}_X \mathrm{I\!I}(Y,Z) + \overline{\nabla}_X \nabla_Y Z\right)^{\perp} \\
&= \nabla^{\perp}_X(\mathrm{I\!I}(Y,Z)) + \mathrm{I\!I}(X, \nabla_Y Z) \\
&= (\nabla_X \mathrm{I\!I})(Y,Z) + \mathrm{I\!I}(\nabla_X Y, Z) + \mathrm{I\!I}(Y, \nabla_X Z) + \mathrm{I\!I}(X, \nabla_Y Z) \quad .
\end{aligned}$$

But now as before by using the above identity with switched X and Y we calculate

$$\begin{aligned}
\left(\overline{R}(X,Y)Z\right)^{\perp} &= \left(\overline{\nabla}_X \overline{\nabla}_Y Z\right)^{\perp} - \left(\overline{\nabla}_Y \overline{\nabla}_X Z\right)^{\perp} \\
&= (\nabla_X \mathrm{I\!I})(Y,Z) - (\nabla_Y \mathrm{I\!I})(X,Z) \quad .
\end{aligned}$$

□

If we are in codimension 1 we note that

$$\begin{aligned}
\nabla_X A(Y,Z) &= X(A(Y,Z)) - A(\nabla_X Y, Z) - A(Y, \nabla_X Z) \\
&= -\left\langle \overline{\nabla}_X \mathrm{I\!I}(Y,Z) - \mathrm{I\!I}(\nabla_X Y, Z) - \mathrm{I\!I}(Y, \nabla_X Z), \nu \right\rangle - \left\langle \mathrm{I\!I}(Y,Z), \overline{\nabla}_X \nu \right\rangle \\
&= -\left\langle (\nabla_X \mathrm{I\!I})(Y,Z), \nu \right\rangle
\end{aligned}$$

since $\left\langle \overline{\nabla}_X \nu, \nu \right\rangle = 0$.

Corollary 1.1.10. *If we are in codimension* 1 *with a timelike unit normal* ν *then*

$$\left\langle \overline{R}(X,Y)Z,\nu\right\rangle = (\nabla_Y A)(X,Z) - (\nabla_X A)(Y,Z)$$

and on a hypersurface in Minkowski space we have

$$(\nabla_X A)(Y,Z) = (\nabla_Y A)(X,Z) = (\nabla_Z A)(X,Y) \quad .$$

1.1.5 Derivative interchange and Simon's identity

We will need the following useful identity.

Proposition 1.1.11 (Derivative Interchange)**.** *Let* $T : \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{F}(M)$, *then*

$$(\nabla_X\nabla_Y T)(A,B) - (\nabla_Y\nabla_X T)(A,B) = -T(R(X,Y)A,B) - T(A,R(X,Y)B) \quad .$$

Proof. As usual without loss of generality assume that $0=[X,Y]=[X,A]=\dots$ and calculate

$$\begin{aligned}
&(\nabla_X\nabla_Y T)(A,B)\\
&\quad= X\left((\nabla_Y T)(A,B)\right) - (\nabla_{\nabla_X Y}T)(A,B) - (\nabla_Y T)(\nabla_X A,B)\\
&\qquad\qquad - (\nabla_Y T)(A,\nabla_X B)\\
&\quad= XY(T(A,B)) - X(T(\nabla_Y A,B) - X(T(A,\nabla_Y B))\\
&\qquad - (\nabla_{\nabla_X Y}T)(A,B) - (\nabla_Y T)(\nabla_X A,B) - (\nabla_Y T)(A,\nabla_X B)\\
&\quad= XY(T(A,B)) - (\nabla_X T)(\nabla_Y A,B) - T(\nabla_X\nabla_Y A,B) - T(\nabla_Y A,\nabla_X B)\\
&\qquad - (\nabla_X T)(A,\nabla_Y B) - T(\nabla_X A,\nabla_Y B) - T(A,\nabla_X\nabla_Y B)\\
&\qquad - (\nabla_{\nabla_X Y}T)(A,B) - (\nabla_Y T)(\nabla_X A,B) - (\nabla_Y T)(A,\nabla_X B) \quad .
\end{aligned}$$

Hence, using that $0=[X,Y]=\nabla_X Y - \nabla_Y X$ we get the formula

$$\begin{aligned}
&(\nabla_X\nabla_Y T)(A,B) - (\nabla_Y\nabla_X T)(A,B)\\
&\quad= [X,Y](T(A,B)) - T(\nabla_X\nabla_Y A - \nabla_Y\nabla_X A,B)\\
&\qquad - T(A,\nabla_X\nabla_Y B - \nabla_Y\nabla_X B) - \nabla_{\nabla_X Y-\nabla_Y X}T(A,B)\\
&\quad= -T(R(X,Y)A,B) - T(A,R(X,Y)B) \quad .
\end{aligned}$$

□

Remark 1.1.12. The above proof holds for arbitrary covariant tensors: Let $T : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{n \text{ times}} \to \mathfrak{F}(M)$ then

$$\begin{aligned}
&\nabla_X \nabla_Y T(A_1, \ldots, A_n) - \nabla_Y \nabla_X T(A_1, \ldots, A_n) \\
&\qquad = -T(R(X,Y)A_1, A_2, \ldots, A_n) - \ldots - T(A_1, \ldots, A_{n-1}, R(X,Y)A_n) \quad .
\end{aligned}$$

Corollary 1.1.13. *For a spacelike hypersurface in Minkoski space we have*

$$\begin{aligned}
&\nabla_x \nabla_y T_{ab} - \nabla_y \nabla_x T_{ab} \\
&\qquad = -R_{xyai} g^{ij} T_{jb} - R_{xybk} g^{kl} T_{la} \\
&\qquad = h_{ya} h_{xi} g^{ij} T_{jb} + h_{yb} h_{xk} g^{kl} T_{la} - h_{xa} h_{yp} g^{pq} T_{qb} - h_{xb} h_{ys} g^{st} T_{ta}
\end{aligned}$$

The following will be useful when deriving evolution equations

Proposition 1.1.14 (Simon's Identity)**.** *For a spacelike hypersurface of Minkowski space then*

$$\Delta h_{ab} = \nabla_a \nabla_b H + h_{ab} \|A\|^2 - H h_{ak} g^{kl} h_{lb} \quad .$$

Proof. We see that

$$\begin{aligned}
\nabla_i \nabla_j h_{ab} &= \nabla_i \nabla_a h_{jb} \\
&= \nabla_a \nabla_i h_{jb} + h_{aj} h_{ic} g^{cd} h_{db} + h_{ab} h_{ie} g^{ef} h_{fj} \\
&\qquad - h_{ij} h_{ak} g^{kl} h_{lb} - h_{ib} h_{ap} g^{pq} h_{qj} \\
&= \nabla_a \nabla_b h_{ij} + h_{aj} h_{ic} g^{cd} h_{db} + h_{ab} h_{ie} g^{ef} h_{fj} \\
&\qquad - h_{ij} h_{ak} g^{kl} h_{lb} - h_{ib} h_{ap} g^{pq} h_{qj}
\end{aligned}$$

where we used the Codazzi equation on the first and third lines, and the interchange formula on the second. Taking a metric contraction over i and j we have

$$\begin{aligned}
\Delta h_{ab} &= g^{ij} \nabla_a \nabla_b h_{ij} + h_{ab} \|A\|^2 - H h_{ak} g^{kl} h_{lb} \\
&= \nabla_a \nabla_b H + h_{ab} \|A\|^2 - H h_{ak} g^{kl} h_{lb}
\end{aligned}$$

since tensor derivatives of the metric are zero. □

1.2 Mean curvature flow

In this section I will define mean curvature flow, and the Neumann boundary conditions I will be using.

1.2.1 Mean curvature flow without boundary

Let M^n be a smooth n-dimensional orientable manifold and for $m > n$ let $(N, \overline{g})$ be a m-dimensional manifold with a (semi-)Riemannian metric $\overline{g}$. N will be referred to as the *ambient space*. Suppose we are given an immersion $\mathbf{F}_0 : M^n \to N$. Then a family of immersions $\mathbf{F} : M^n \times [0, T) \to N$ satisfies *mean curvature flow* if

$$\begin{cases} \mathbf{F}(x,0) = \mathbf{F}_0(x) & \forall x \in M^n \\ \frac{d\mathbf{F}}{dt}(x,t) = \Delta_g \mathbf{F}(x,t) & \forall (x,t) \in M^n \times [0,T) \end{cases} \tag{1.6}$$

where the Δ_g is the trace of the second derivative with respect to g, the metric induced on M^n by $\overline{g}$ at time t. We define $\mathbf{F}_t(\cdot) = \mathbf{F}(\cdot, t)$ and M_t to be the immersion of M^n at time t defined by $\mathbf{F}_t$.

Remark 1.2.1. A useful property of this flow is that it is invariant under isometries of the ambient space N in that if $\mathbf{G}_0 = P(\mathbf{F}_0)$ where P is some isometry of A, then flowing both by equation (1.6) we have $\mathbf{G}_t = P(\mathbf{F}_t)$.

The above is the most general definition, indicating relations between the heat flow equation and mean curvature flow. In the introduction we tacitly used that in $(n+1)$-dimensional Euclidean space $\Delta_g \mathbf{F} = -H\nu$. For most of this thesis we will be considering spacelike hypersurfaces in Minkowski space and we will need a similar identity. In this case recalling the Weingarten relations (equation (1.4)) we calculate:

$$\begin{aligned} \Delta_g \mathbf{F} &= g^{ij}\left(\frac{\partial^2 \mathbf{F}}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial \mathbf{F}}{\partial x^k}\right) \\ &= g^{ij} h_{ij} \nu \\ &= H\nu \end{aligned}$$

where H and ν are the mean curvature and unit normal of M_t respectively.

Remark 1.2.2. The vector $H\nu$ in Minkowski space ($-H\nu$ in Euclidean space) is called the mean curvature vector and is invariant under choice of ν: Changing the sign of the ν changes the sign of the second fundamental form and hence H.

Also note that the signature of the metric on the flowing manifold is important. If we want parabolicity of equation (1.6) – a desirable property, allowing the application of the existence theory in Chapter 2 – we will require spacelikeness of the initial manifold $\mathbf{F}_0$. For further details of this see Section 3.1 and Remark 4.0.5. If not, we will get a hyperbolic equation or worse, and even short time existence of a solution is not guaranteed.

1.2.2 Special solutions in Minkowski space

Although we are very rarely going to be able to solve the above system explicitly, it is useful to get an idea of what's going on by considering special solutions which are solvable by assuming some kind of symmetry.

For example we may choose to consider rotationally symmetric spacelike manifolds. All such manifolds in Minkowski space may be written as graphs, and so we may write the mean curvature flow equations as a graph $u : \mathbb{R}^n \times [0, T) \to \mathbb{R}$. We will show in Appendix B that for a graph we have

$$\frac{du}{dt} = H\sqrt{1 - |Du|^2} = D_{ij}u\left(\delta_{ij} + \frac{D_iuD_ju}{1 - |Du|^2}\right) \quad .$$

Specifying that $u = u(r)$ is a function of $r = \sqrt{x_1^2 + \dots x_n^2}$ then we have

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}, \quad \frac{\partial^2 r}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}$$
$$D_iu = u'\frac{\partial r}{\partial x_i}, \quad D_{ij}u = u''\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j} + u'\frac{\partial^2 r}{\partial x_i \partial x_j} \quad .$$

So for such a radial function

$$\begin{aligned} Hv &= \left[u''\frac{x_ix_j}{r^2} + \frac{u'}{r}\left(\delta_{ij} - \frac{x_ix_j}{r^2}\right)\right]\left(\delta_{ij} + \frac{(u')^2x_ix_j}{r^2(1-(u')^2)}\right) \\ &= u''\left(1 + \frac{(u')^2}{1-(u')^2}\right) + \frac{(n-1)u'}{r} \\ &= \frac{u''}{1-(u')^2} + \frac{(n-1)u'}{r} \quad . \end{aligned}$$

Now we may construct various examples.

Example 1.2.3 (The Hyperplane)**.** Any non-degenerate hyperplane (i.e. the metric restricted to the hyperplane is non degenerate) has $H = 0$, and hence we have that this solution will remain stationary under mean curvature flow.

This is an example of a *maximal surface*, the Minkoski equivalent of a minimal surface defined by $H = 0$. We may also write down a non-planar rotational maximal surface.

Example 1.2.4 (A Maximal Surface in $\mathbb{R}^3_1$)**.** The hypersurface defined by $u = \sinh^{-1}(r)$ is a maximal surface in $\mathbb{R}^3_1$ for $r > 0$. It is easy to verify this by simply substituting u into the equation above. Note that at $\mathbf{0}$ the surface is tangent to the light cone.

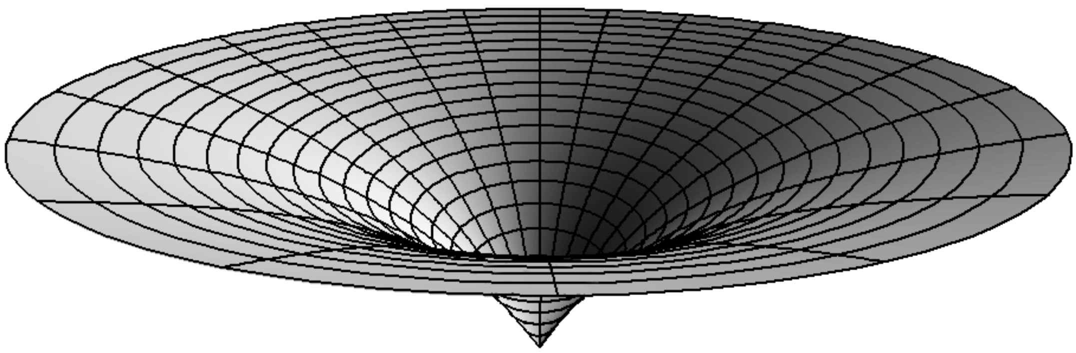


Figure 1.1: A maximal surface of revolution in $\mathbb{R}^3_1$

The next is a non-stationary example.

Example 1.2.5 (The Hyperbolic Hyperplane)**.** Let

$$Y_R = \{x \in \mathbb{R}^{n+1}_1 | \, \langle x, x\rangle = -R^2, x^{n+1} > 0\} \quad .$$

Then Y_R is a spacelike hypersurface, which is isometric to hyperbolic space of constant negative gauss curvature $K = \frac{-1}{R^n}$. This is an analogy of the sphere in Euclidean space, and has similar properties:

- Y_R is invariant (as a set) under isometries of $\mathbb{R}^{n+1}_1$ that preserve the half of the light cone with $x^n > 0$: Isometries preserve $\langle \cdot, \cdot \rangle$ and therefore Y_R is mapped

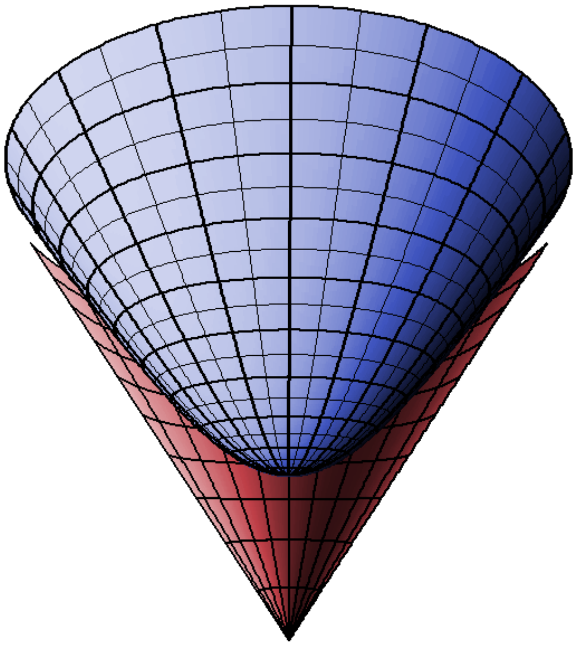


Figure 1.2: The Hyperbolic plane (blue) sits inside the light cone (red)

to Y_R. Indeed if M is a hypersurface contained in the upper half space defined by $\{x \in \mathbb{R}^{n+1}_1 : x^{n+1} > 0\}$ that is invariant under these isometries then it is a union of such Y_R.

- The position vector is normal to the surface: Suppose $\mathbf{F}$ is a parametrisation of Y_R. Then by definition $\langle \mathbf{F}, \mathbf{F} \rangle = -R^2$ and so by differentiating in any tangent direction we see $\left\langle \frac{\partial \mathbf{F}}{\partial x^i}, \mathbf{F} \right\rangle = 0$.

- Y_R is totally umbilic: Since the normal $\nu = \frac{\mathbf{F}}{R}$ then for $X, Y \in T_{\mathbf{F}} Y_R$ we have $A(X,Y) = \frac{1}{R} \left\langle \overline{\nabla}_X \mathbf{F}, Y \right\rangle = \frac{1}{R} \langle X, Y \rangle$.

Using Remark 1.2.1, if mean curvature flow is initially invariant under a set of isometries then it remains so. Therefore if we flow Y_{R_0} we know that the flow must remain a hyperbolic hyperplane but with varying "radii". Using that $H = \frac{n}{R}$ we get the differential equation

$$\frac{dR}{dt} = \frac{n}{R}, \qquad R(0) = R_0 \quad .$$

Hence $\frac{dR^2}{dt} = 2n$ and $R(t) = \sqrt{R_0^2 + 2nt}$ and $Y_{R(t)}$ is a solution to (1.6) for $N = \mathbb{R}^{n+1}_1$.

This shows that a hyperbolic hyperplane starting "near" the light cone moves away in the x_{n+1} direction towards infinity, existing for all time. This is the Minkowski equivalent of the sphere solution in Euclidean space.

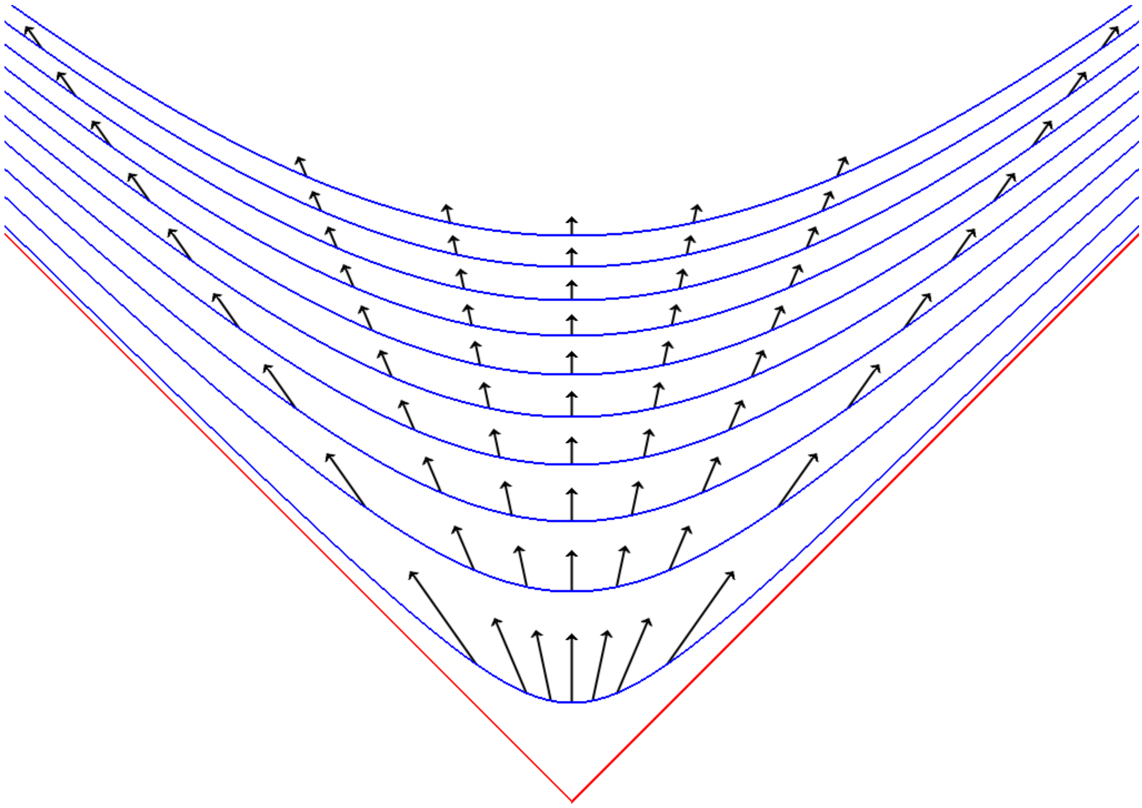


Figure 1.3: Under the flow the Hyperbolic hyperplane exists for all time, expanding by dilations.

This is an example of a *homothetic* solution to mean curvature flow, that is, a solution that remains the same up to dilations. For spacelike homothetic solutions of codimension 1 we have

$$\mathbf{F}(x,t) = \lambda(t)\mathbf{F}_0(\tau(x,t))$$

where $\tau : M^n \times [0,T) \to M^n$ is at each time some diffeomorphism. We can immediately see that

$$H(x,t) = \frac{H_0(\tau(x,t))}{\lambda(t)}\ , \qquad \nu(x,t) = \nu_0(\tau(x,t))\quad .$$

Hence

$$\frac{d\mathbf{F}}{dt} = \frac{H_0(\tau(x,t))\nu_0(\tau(x,t))}{\lambda(t)} = \lambda'(t)\mathbf{F}_0(\tau(x,t)) + \lambda(t)\frac{\partial \mathbf{F}_0}{\partial x^i}\bigg|_{\tau(x,t)} \frac{d\tau^i}{dt}\bigg|_{(x,t)} \quad .$$

Taking inner products with $\nu_0(\tau(x,t))$ then

$$-H_0 = \lambda'(t)\lambda(t)\left\langle \mathbf{F}_0, \nu_0 \right\rangle \quad .$$

Since only λ has time dependence, then we see $\lambda(t) = \sqrt{C_1 + C_2 t}$. We therefore see that manifolds which move homothetically under mean curvature flow may be characterised by one of

$$H = \left\langle \mathbf{F}, \nu \right\rangle \tag{1.7}$$

$$H = -\left\langle \mathbf{F}, \nu \right\rangle \quad . \tag{1.8}$$

where solutions to equation (1.7) will shrink towards the origin while solutions to equation (1.8) will expand away, as in the above. In Euclidean space such solutions are extremely important. Huisken [15] used his monotonicity formula to show that any Type I singularity of mean curvature flow in $\mathbb{R}^{n+1}$ under renormalisation will converge to a homothetic solution. The monotonicity formula has been considered in Minkowski space by Thorpe [28].

Example 1.2.6 (A Translating Solution in $\mathbb{R}^2_1$)**.** We may find the equivalent of the "grim reaper" solution in the plane. Choosing $n = 1$ we look for a solution that remains the same but moves upwards with speed 1 and has mirror symmetry. Hence such a solution must satisfy

$$1 = Hv = \frac{u''}{1-(u')^2} = \left(\tanh^{-1}(u')\right)' \quad .$$

Choosing $u'(0) = 0$ and solving explicitly we have $u'(r) = \tanh(r)$ and so

$$u(r,t) = \log(\cosh(r)) + t \quad .$$

We therefore have a translating solution tangent to the light cone at infinity and with mean curvature that increases exponentially as $r \to \infty$.

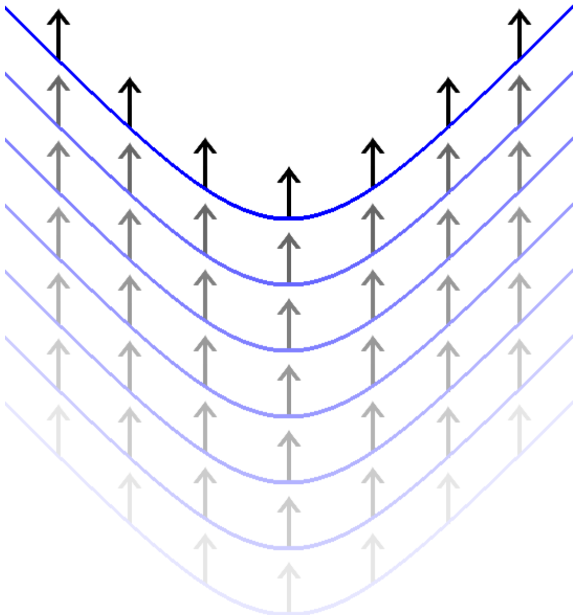


Figure 1.4: The $\mathfrak{Grim\ Reaper}$ solution in $\mathbb{R}^2_1$

Example 1.2.7 (General Rotationally Symmetric Translating Solutions)**.** In [19] Huai-Yu Jian has studied the equation

$$1 = \frac{u''}{1-(u')^2} + \frac{(n-1)u'}{r}$$

to give that there exists exactly one such rotationally symmetric solution in all dimensions. This solution is smooth, spacelike and convex.

1.2.3 Mean curvature flow with a Neumann boundary condition

For a graph, a Neumann boundary condition is control of the derivative of the graph in some outwards direction at the boundary. The graph only has this derivative specified, but the height of the graph at the boundary is allowed to be any function.

We wish to get a similar notion but for some smooth n–dimensional topological manifold M^n with boundary ∂M^n. Let Σ be a smooth embedded manifold in $\mathbb{R}^{n+1}_1$ with an outward pointing unit normal, μ. This will be called the *boundary manifold*. The Neumann condition corresponding to the graphical case now becomes that the boundary of M^n must be contained within Σ and free to move within Σ, but $\langle \nu, \mu \rangle$ is specified.

More precisely, suppose we are given an initial spacelike embedding $\mathbf{F}_0 : M^n \to \mathbb{R}^{n+1}_1$ where we specify that $\mathbf{F}_0(\partial M) \subset \Sigma$ with the extra compatibility condition $\langle \nu_0, \mu \rangle (x) = 0$, where as usual ν_0 is the normal to $M_0 = \mathbf{F}_0(M^n)$.

Definition 1.2.8. Let $\mathbf{F} : M^n \times [0, T] \to \mathbb{R}^{n+1}_1$ be such that

$$\begin{cases} \frac{d\mathbf{F}}{dt} = \mathbf{H} = H\nu & \forall (x,t) \in M^n \times [0,T] \\ \mathbf{F}(x,0) = \mathbf{F}_0(x) & \forall x \in M^n \\ \mathbf{F}(x,t) \subset \Sigma & \forall (x,t) \in \partial M^n \times [0,T] \\ \langle \nu, \mu \rangle (x,t) = 0 & \forall (x,t) \in \partial M^n \times [0,T] \end{cases} \tag{1.9}$$

then $\mathbf{F}$ moves by *Mean Curvature Flow with a perpendicular Neumann boundary condition* Σ (here $\nu(x,t)$ is the normal to $\mathbf{F}$ at time t.)

Remark 1.2.9. When in Minkowski space with this boundary condition Σ *must* be an indefinite manifold, since the flowing manifold is spacelike the boundary condition implies that μ has positive length.

1.2.4 Notation

It is clear that geometric properties on several different manifolds will be required. The following notation will be used throughout: A bar will imply quantities on the ambient space $\mathbb{R}^{n+1}_1$, for example $\overline{\Delta}, \overline{\nabla}, \ldots$ and so on; no extra markings $\Delta, \nabla, \ldots$ will be geometric quantities on M_t the flowing surface at time t and for any other manifold Z $\Delta^Z, \nabla^Z, \ldots$ will be the Laplacian, covariant derivatives, $\ldots$ on Z.

We will adopt summation convention on repeated indices, and the summation will always be from 1 to n unless otherwise specified.

1.2.5 Evolution of curvature and metric

I end this chapter by deriving the evolution equations for curvature on the interior of the flowing manifolds. Many of the results of this section are written down in [5] and [3], although few explicit calculations are given. I write them here and remark only that they are unremarkable, and very similar to the calculations in [13], for example.

Proposition 1.2.10. *On the interior of the flowing manifold we have*

$$\frac{d\nu}{dt} = \nabla H \quad .$$

Proof. This is almost exactly the same as the Euclidean case proved in [13]:

$$\begin{aligned}
\left\langle \frac{d\nu}{dt}, \frac{\partial \mathbf{F}}{\partial x_i} \right\rangle &= - \left\langle \nu, \frac{\partial}{\partial x_i} \frac{d\mathbf{F}}{dt} \right\rangle \\
&= - \left\langle \nu, \frac{\partial}{\partial x_i} (H\nu) \right\rangle \\
&= - \left\langle \nu, \frac{\partial H}{\partial x_i} \nu \right\rangle \\
&= \frac{\partial H}{\partial x_i} \quad .
\end{aligned}$$

□

Proposition 1.2.11. *On the interior of M^n we have*

$$\frac{dg_{ij}}{dt} = 2Hh_{ij} \quad .$$

Proof. We see

$$\begin{aligned}
\frac{dg_{ij}}{dt} &= \frac{d}{dt} \left\langle \frac{\partial \mathbf{F}}{\partial x^i}, \frac{\partial \mathbf{F}}{\partial x^j} \right\rangle \\
&= \left\langle \frac{\partial (H\nu)}{\partial x^i}, \frac{\partial \mathbf{F}}{\partial x^j} \right\rangle + \left\langle \frac{\partial \mathbf{F}}{\partial x^i}, \frac{\partial (H\nu)}{\partial x^j} \right\rangle \\
&= H \left(\left\langle \frac{\partial \nu}{\partial x^i}, \frac{\partial \mathbf{F}}{\partial x^j} \right\rangle + \left\langle \frac{\partial \mathbf{F}}{\partial x^i}, \frac{\partial \nu}{\partial x^j} \right\rangle \right) \\
&= 2Hh_{ij} \quad .
\end{aligned}$$

□

Corollary 1.2.12. *For the inverse of the metric we have*

$$\frac{dg^{ij}}{dt} = -2Hh^{ij} \quad .$$

Proof. Since $g^{ik}g_{kj} = \delta^i_j$ then $\frac{dg^{ik}}{dt}g_{kj} + g^{il}\frac{dg_{lj}}{dt} = 0$ and so

$$\frac{dg^{ip}}{dt} = -g^{il}\frac{dg_{lj}}{dt}g^{jp} \quad .$$

Substituting Proposition 1.2.11 into this gives the corollary. □

We will need to make estimates on the curvature, and so we calculate the following.

Proposition 1.2.13. *On the interior of the flowing manifold*

$$\left(\frac{d}{dt} - \Delta\right) h_{ij} = 2Hh_{jl}g^{lm}h_{mi} - h_{ij}|A|^2 \quad .$$

Proof. Using equation (1.4) and Proposition 1.2.10 and the definition of the second fundamental form (equation (1.3)) we see

$$\begin{aligned}
\frac{dh_{ij}}{dt} &= -\frac{d}{dt}\left\langle \frac{\partial^2 \mathbf{F}}{\partial x^i \partial x^j}, \nu \right\rangle \\
&= -\left\langle \frac{\partial^2 (H\nu)}{\partial x^i \partial x^j}, \nu \right\rangle - \left\langle \frac{\partial^2 \mathbf{F}}{\partial x^i \partial x^j}, \nabla H \right\rangle \\
&= -\left\langle \frac{\partial}{\partial x^i}\left(\frac{\partial H}{\partial x^j}\nu + Hh_{jl}g^{lm}\frac{\partial \mathbf{F}}{\partial x^m}\right), \nu \right\rangle - \Gamma^k_{ij}\frac{\partial H}{\partial x^k} \\
&= -\left\langle \left(\frac{\partial^2 H}{\partial x^j \partial x^i}\nu + Hh_{jl}g^{lm}\frac{\partial^2 \mathbf{F}}{\partial x^m \partial x^i}\right), \nu \right\rangle - \Gamma^k_{ij}\frac{\partial H}{\partial x^k} \\
&= \frac{\partial^2 H}{\partial x^j \partial x^i} - \Gamma^k_{ij}\frac{\partial H}{\partial x^k} + Hh_{jl}g^{lm}h_{mi} \\
&= \nabla_i \nabla_j H + Hh_{jl}g^{lm}h_{mi} \quad .
\end{aligned}$$

We may now use Simon's identity, that is Proposition 1.1.14, to get

$$\frac{dh_{ij}}{dt} = \Delta h_{ij} + 2Hh_{jl}g^{lm}h_{mi} - h_{ij}|A|^2$$

completing the proof. □

Corollary 1.2.14. *On the interior of M^n we have*

$$\left(\frac{d}{dt} - \Delta\right) H = -H|A|^2 \tag{1.10}$$

$$\left(\frac{d}{dt} - \Delta\right) |A|^2 = -2|A|^4 - 2|\nabla A|^2 \quad . \tag{1.11}$$

Proof. For the first, we see using the proof of Proposition 1.2.13 and Corollary 1.2.12 that

$$\begin{aligned}\frac{dH}{dt} &= \frac{dg^{ij}h_{ij}}{dt}\\ &= g^{ij}\left(\nabla_i\nabla_j H + Hh_{jl}g^{lm}h_{mi}\right) - 2Hh^{ij}h_{ij}\\ &= \Delta H - H|A|^2 \quad .\end{aligned}$$

For the second

$$\begin{aligned}\frac{d|A|^2}{dt} &= \frac{dg^{ij}h_{jk}g^{kl}h_{li}}{dt}\\ &= 2g^{ij}\left(\Delta h_{jk} + 2Hh_{kl}g^{lm}h_{mj} - h_{jk}|A|^2\right)g^{kl}h_{li} - 4Hh^{ij}h_{jk}g^{kl}h_{li}\\ &= 2h^{kj}\Delta h_{jk} - 2|A|^4 \quad .\end{aligned}$$

Because the compatibility of the metric implies $\nabla g = 0$ we now use the Tensor Product Rule (Lemma 1.1.6) to conclude

$$\begin{aligned}\Delta|A|^2 &= g^{ij}\nabla^2_{ij}\left(g^{ab}h_{bc}g^{cd}h_{da}\right)\\ &= 2g^{ij}g^{ab}g^{cd}(h_{da}(\nabla^2_{ij}h)_{bc} + (\nabla_j h)_{bc}(\nabla_i h)_{da})\\ &= 2h^{ab}(\Delta h)_{ab} + 2|\nabla A|^2 \quad .\end{aligned}$$

Substituting gives equation (1.11). □

We now wish to derive corresponding results for higher derivatives of A. Generally the precise form of the equations are not needed so we will use the convention adopted by Huisken in [13]: For any two tensors S and T we write $S \star T$ in place of any linear combination of S and T, possibly with metric contractions. This will be used as a shorthand for terms of lower order in an evolution equation. We will use $\nabla^m A$ to be the m^{th} tensorial derivative of the second fundamental form where we will use the convention $\nabla^0 A = A$.

Lemma 1.2.15. *The Christoffel symbols evolve by*

$$\frac{d\Gamma^k_{ij}}{dt} = g^{kr}\left[(\nabla_{\frac{\partial}{\partial x^i}}Hh)_{rj} + (\nabla_{\frac{\partial}{\partial x^j}}Hh)_{ri} - (\nabla_{\frac{\partial}{\partial x^r}}Hh)_{ij}\right] \quad .$$

Proof. This is simply an exercise in cancellation: Using Proposition 1.2.11 and Corollary 1.2.12 then

$$\begin{aligned}\frac{d\Gamma^k_{ij}}{dt} &= \frac{d}{dt}\left(\frac{g^{kr}}{2}\left(\frac{\partial}{\partial x^i}g_{rj} + \frac{\partial}{\partial x^j}g_{rj} - \frac{\partial}{\partial x^r}g_{ij}\right)\right)\\ &= g^{kr}\left[\frac{\partial}{\partial x^i}(Hh_{rj}) + \frac{\partial}{\partial x^j}(Hh_{ri}) - \frac{\partial}{\partial x^r}(Hh_{ij})\right] - 2Hh^{kr}g_{rp}\Gamma^p_{ij}\\ &= g^{kr}\big[(\nabla_{\frac{\partial}{\partial x^i}}Hh)_{rj} + \Gamma^p_{ir}Hh_{pj} + \Gamma^q_{ij}Hh_{qr} + (\nabla_{\frac{\partial}{\partial x^j}}Hh)_{ri} + \Gamma^s_{jr}Hh_{si}\\ &\qquad + \Gamma^u_{ji}Hh_{ur} - (\nabla_{\frac{\partial}{\partial x^r}}Hh)_{ij} - \Gamma^v_{ri}Hh_{vj} - \Gamma^w_{rj}Hh_{wi}\big] - 2Hh^{kr}g_{rp}\Gamma^p_{ij}\\ &= g^{kr}\left[(\nabla_{\frac{\partial}{\partial x^i}}Hh)_{rj} + (\nabla_{\frac{\partial}{\partial x^j}}Hh)_{ri} - (\nabla_{\frac{\partial}{\partial x^r}}Hh)_{ij}\right] \quad .\end{aligned}$$

□

Thus using tensor product rule $\frac{d\Gamma^k_{ij}}{dt} = A \star \nabla A$.

Proposition 1.2.16. *For $m \geq 1$ then on the interior of M the following holds*

$$\left(\frac{d}{dt} - \Delta\right)|\nabla^m A|^2 = -2|\nabla^{m+1}A|^2 + \sum_{\substack{a,b,c\geq 0\\ a+b+c=m}} \nabla^a A \star \nabla^b A \star \nabla^c A \star \nabla^m A \quad .$$

Proof. Throughout this proof we will write ∇_i in place of $\nabla_{\frac{\partial}{\partial x^i}}$, and all covariant derivatives of tensors are tensor derivatives as defined in Section 1.1.2, that is, $\nabla_i T_{jk} = (\nabla_{\frac{\partial}{\partial x^i}}T)(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k})$. Firstly, we will make some observations. For a tensor T of any type (although here we will use a $(0, r)$ tensor) then using that $\frac{dg_{ij}}{dt} = A \star A$ (Proposition 1.2.11) then

$$\begin{aligned}\frac{d}{dt}|T|^2 &= \frac{d}{dt}\left(T_{i_1 i_2,\dots i_r}T_{j_1 j_2,\dots j_r}g^{i_1 j_1}\dots g^{i_r j_r}\right)\\ &= 2\frac{dT_{i_1 i_2,\dots i_r}}{dt}T_{j_1 j_2,\dots j_r}g^{i_1 j_1}\dots g^{i_r j_r} + A\star A\star T\star T\\ &= 2\left\langle T_{i_1 i_2,\dots i_r}, \frac{dT_{i_1 i_2,\dots i_r}}{dt}\right\rangle + A\star A\star T\star T \quad .\end{aligned} \tag{1.12}$$

Furthermore using Proposition 1.2.15

$$\begin{aligned}\frac{d}{dt}\nabla_j T_{i_1\dots i_r} &= \frac{d}{dt}\left(\frac{\partial T_{i_1\dots i_r}}{\partial x^j} - \Gamma^{k_1}_{ji_1}T_{k_1 i_2,\dots i_r} - \dots - \Gamma^{k_r}_{ji_r}T_{i_1,\dots i_{r-1}k_r}\right)\\ &= \nabla_j\frac{dT_{i_1\dots i_r}}{dt} + A\star\nabla A\star T \quad .\end{aligned} \tag{1.13}$$

Now using this repeatedly for $T \in \mathfrak{T}^0_2(M)$ then

$$\begin{aligned}
\frac{d}{dt}\nabla_{k_1}\dots\nabla_{k_m}T_{ab} &= \nabla_{k_1}\frac{d\nabla_{k_2}\dots\nabla_{k_m}T_{ab}}{dt} + A\star\nabla A\star\nabla^{m-1}T \\
&= \nabla_{k_1}\nabla_{k_2}\frac{d\nabla_{k_3}\dots\nabla_{k_m}T_{ab}}{dt} + \nabla_{k_1}(A\star\nabla A\star\nabla^{m-2}T) \\
&\qquad + A\star\nabla A\star\nabla^{m-1}T \\
&= \nabla_{k_1}\nabla_{k_2}\frac{d\nabla_{k_3}\dots\nabla_{k_m}T_{ab}}{dt} + \nabla A\star\nabla A\star\nabla^{m-2}T \\
&\qquad + A\star\nabla^2 A\nabla^{m-2}T + A\star\nabla A\star\nabla^{m-1}T \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&= \nabla_{k_1}\dots\nabla_{k_m}\frac{dT_{ab}}{dt} + \sum_{\substack{a,c\geq 0,\ b\geq 1\\ a+b+c=m}} \nabla^a A\star\nabla^b A\star\nabla^c T \quad . \qquad (1.14)
\end{aligned}$$

We also need to calculate for $T \in \mathfrak{T}^0_2(M)$

$$\begin{aligned}
\Delta|\nabla^m T|^2 &= g^{ij}\nabla_i\nabla_j\left\langle \nabla_{k_1}\dots\nabla_{k_m}T_{ab}, \nabla_{k_1}\dots\nabla_{k_m}T_{ab}\right\rangle \\
&= 2g^{ij}\nabla_i\left\langle \nabla_j\nabla_{k_1}\dots\nabla_{k_m}T_{ab}, \nabla_{k_1}\dots\nabla_{k_m}T_{ab}\right\rangle \\
&= 2\left\langle \nabla_l\nabla_{k_1}\dots\nabla_{k_m}T_{ab}, \nabla_l\nabla_{k_1}\dots\nabla_{k_m}T_{ab}\right\rangle \\
&\qquad + 2g^{ij}\left\langle \nabla_i\nabla_j\nabla_{k_1}\dots\nabla_{k_m}T_{ab}, \nabla_{k_1}\dots\nabla_{k_m}T_{ab}\right\rangle \quad . \qquad (1.15)
\end{aligned}$$

We have from Corollary 1.1.13 and Remark 1.1.12 that $\nabla_i\nabla_j T - \nabla_j\nabla_i T = A\star A\star T$, which we now use repeatedly along with the product rule:

$$\begin{aligned}
&\nabla_j\nabla_{k_1}\dots\nabla_{k_s}T_{ab} \qquad (1.16)\\
&\quad = \nabla_{k_1}\nabla_j\nabla_{k_2}\dots\nabla_{k_s}T_{ab} + A\star A\star\nabla^{s-1}T \\
&\quad = \nabla_{k_1}\nabla_{k_2}\nabla_j\nabla_{k_3}\dots\nabla_{k_s}T_{ab} + \nabla_{k_1}(A\star A\star\nabla^{s-2}T) + A\star A\star\nabla^{s-1}T \\
&\quad = \nabla_{k_1}\nabla_{k_2}\nabla_j\nabla_{k_3}\dots\nabla_{k_s}T_{ab} + A\star\nabla A\star\nabla^{s-2}T + A\star A\star\nabla^{s-1}T \\
&\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\quad = \nabla_{k_1}\nabla_{k_2}\nabla_{k_3}\dots\nabla_{k_s}\nabla_j T_{ab} + \sum_{\substack{a,b,c\geq 0\\ a+b+c=s-1}} \nabla^a A\star\nabla^b A\star\nabla^c T \quad . \qquad (1.17)
\end{aligned}$$

We are now in a position to show the theorem. We will consider first $m = 1$ to give the idea of the proof before moving on to higher order cases. Using equations (1.12) and (1.13) and Proposition 1.2.13 we have

$$\begin{aligned}
\frac{d|\nabla A|^2}{dt} &= 2\left\langle \nabla A, \frac{d\nabla A}{dt}\right\rangle + A\star A\star\nabla A\star\nabla A\\
&= 2\left\langle \nabla A, \nabla\frac{dA}{dt}\right\rangle + A\star A\star\nabla A\star\nabla A\\
&= 2\left\langle \nabla_k h_{ij}, \nabla_k\left(\Delta h_{ij} + 2Hh_{jl}g^{lm}h_{mi} - h_{ij}|A|^2\right)\right\rangle + A\star A\star\nabla A\star\nabla A\\
&= 2g^{ab}\left\langle \nabla_k h_{ij}, \nabla_k\nabla_a\nabla_b h_{ij}\right\rangle + A\star A\star\nabla A\star\nabla A \quad .
\end{aligned}$$

But now using equations (1.15) and (1.17) we see

$$\begin{aligned}
\Delta|\nabla A|^2 &= 2|\nabla^2 A|^2 + 2g^{ab}\left\langle \nabla_a\nabla_b\nabla_k h_{ij}, \nabla_k h_{ij}\right\rangle\\
&= 2|\nabla^2 A|^2 + 2g^{ab}\left\langle \nabla_a\nabla_k\nabla_b h_{ij} + A\star A\star\nabla A, \nabla_k h_{ij}\right\rangle\\
&= 2|\nabla^2 A|^2 + 2g^{ab}\left\langle \nabla_a\nabla_k\nabla_b h_{ij}, \nabla_k h_{ij}\right\rangle + A\star A\star\nabla A\star\nabla A\\
&= 2|\nabla^2 A|^2 + 2g^{ab}\left\langle \nabla_k\nabla_a\nabla_b h_{ij}, \nabla_k h_{ij}\right\rangle + A\star A\star\nabla A\star\nabla A \quad .
\end{aligned}$$

Putting all this together gives as claimed

$$\left(\frac{d}{dt} - \Delta\right)|\nabla A|^2 = -2|\nabla^2 A|^2 + A\star A\star\nabla A\star\nabla A \quad .$$

So far so good. Now we come to the general case. The calculations here are effectively the same, although more protracted. From equations (1.12) and (1.14) and Proposition 1.2.13 we have as before

$$\begin{aligned}
\frac{d|\nabla^m A|^2}{dt} &= 2\left\langle \nabla_{i_1}\ldots\nabla_{i_m}h_{ab}, \frac{d}{dt}\nabla_{i_1}\ldots\nabla_{i_m}h_{ab}\right\rangle + A\star A\star\nabla^m A\star\nabla^m A\\
&= 2\left\langle \nabla_{i_1}..\nabla_{i_m}h_{ab}, \nabla_{i_1}..\nabla_{i_m}\frac{dh_{ab}}{dt}\right\rangle + \sum_{\substack{a,b,c\geq 0\\ a+b+c=m}} \nabla^a A\star\nabla^b A\star\nabla^c A\star\nabla^m A\\
&= 2\left\langle \nabla_{i_1}\ldots\nabla_{i_m}h_{ab}, \nabla_{i_1}\ldots\nabla_{i_m}\left(\Delta h_{ab} + 2Hh_{il}g^{lu}h_{uj} - h_{ij}|A|^2\right)\right\rangle\\
&\qquad + \sum_{\substack{a,b,c\geq 0\\ a+b+c=m}} \nabla^a A\star\nabla^b A\star\nabla^c A\star\nabla^m A\\
&= 2g^{cd}\left\langle \nabla_{i_1}\ldots\nabla_{i_m}h_{ab}, \nabla_{i_1}\ldots\nabla_{i_m}\nabla_c\nabla_d h_{ab}\right\rangle\\
&\qquad + \sum_{\substack{a,b,c\geq 0\\ a+b+c=m}} \nabla^a A\star\nabla^b A\star\nabla^c A\star\nabla^m A \quad .
\end{aligned}$$

As previously, we deal with the first term using the Laplacian and equation (1.15)

$$\Delta|\nabla^m A|^2 = 2|\nabla^{m+1}A|^2 + 2g^{cd}\left\langle \nabla_c\nabla_d\nabla_{i_1}\ldots\nabla_{i_m}h_{ab}, \nabla_{i_1}\ldots\nabla_{i_m}h_{ab}\right\rangle \quad .$$

But by equation (1.17)

$$\begin{aligned}
&\left\langle \nabla_c \nabla_d \nabla_{i_1} \dots \nabla_{i_m} h_{ab}, \nabla_{i_1} \dots \nabla_{i_m} h_{ab} \right\rangle \\
&\quad = \langle \nabla_c \nabla_{i_1} \dots \nabla_{i_m} \nabla_d h_{ab}, \nabla_{i_1} \dots \nabla_{i_m} h_{ab} \rangle \\
&\qquad\qquad + \left\langle \nabla_c \left(\sum_{\substack{a,b,c \geq 0 \\ a+b+c=m-1}} \nabla^a A \star \nabla^b A \star \nabla^c A \right), \nabla_{i_1} \dots \nabla_{i_m} h_{ab} \right\rangle \\
&\quad = \langle \nabla_c \nabla_{i_1} \dots \nabla_{i_m} \nabla_d h_{ab}, \nabla_{i_1} \dots \nabla_{i_m} h_{ab} \rangle + \sum_{\substack{a,b,c \geq 0 \\ a+b+c=m}} \nabla^a A \star \nabla^b A \star \nabla^c A \star \nabla^m A \\
&\quad = \langle \nabla_{i_1} \dots \nabla_{i_m} \nabla_c \nabla_d h_{ab}, \nabla_{i_1} \dots \nabla_{i_m} h_{ab} \rangle + \sum_{\substack{a,b,c \geq 0 \\ a+b+c=m}} \nabla^a A \star \nabla^b A \star \nabla^c A \star \nabla^m A \quad .
\end{aligned}$$

Putting all of this together

$$\left(\frac{d}{dt} - \Delta \right) |\nabla^m A|^2 = -2 |\nabla^{m+1} A|^2 + \sum_{\substack{a,b,c \geq 0 \\ a+b+c=m}} \nabla^a A \star \nabla^b A \star \nabla^c A \star \nabla^m A \quad .$$

□

1.2.6 Maximum principle

I include a maximum principle:

Lemma 1.2.17 (Weak Maximum Principle)**.** *Let M^n be a compact smooth manifold with boundary and let $F : M^n \times [0,T) \to \mathbb{R}$ be a function twice differentiable in space and once in time. Let $g_{ij}(x,t)$ be a metric on M^n which varies over time. Suppose F has the following properties:*

1. *At a stationary point (i.e. $\nabla F = 0$) we have*
$$\left(\frac{d}{dt} - \Delta \right) F(x,t) \leq 0 \qquad \forall (x,t) \in M^n \times [0,T) \quad .$$
2. *On the boundary if μ is an outward pointing unit normal then*
$$\nabla_\mu F \leq 0 \qquad \forall (x,t) \in \partial M \times [0,T) \quad .$$

Here Δ is the Laplace–Beltrami operator and ∇ is the Levi-Civita connection. Then $F \leq \max \left\{ \sup_{x \in M^n} F(x,0), 0 \right\}$.

Proof. This is similar to the proof of [26, Lemma 3.1]. We will consider positive maxima of $f = e^{-\alpha t}F$ for $\alpha > 0$. At a positive interior maximum of f we have $\nabla f = 0$ and $\nabla^2 f \leq 0$ and so $\Delta f \leq 0$. Substituting into the evolution equation we have that at a maximum

$$\frac{df}{dt} \leq -\alpha f < 0 \ .$$

We therefore have that f is decreasing at any interior maximum point.

Suppose we have a non-decreasing maximum of f at $(\mathbf{p}, t) \in \partial M^n \times [0, T)$, and we have that at this point $\nabla_\mu f \leq 0$. If $\nabla_\mu f < 0$ at this point then no boundary maximum is allowed. Hence we have that at $(\mathbf{p}, t)$, $\nabla_\mu F = 0$. We consider f at time t in local coordinates, chosen so that the preimage of a neighbourhood of $\mathbf{p}$ is a neighbourhood of $\mathbf{0} \in \{\mathbf{x} \in \mathbb{R}^n | x^n \geq 0\}$ so that the direction μ is $-\mathbf{e}_n$. Therefore in these coordinates $-Df \cdot \mathbf{e}_n = 0$. Since f is positive at $\mathbf{0}$ and $\left(\frac{d}{dt} - \Delta\right) f \leq -\alpha f$, there exists a neighbourhood $\mathbf{0} \in U \subset \{\mathbf{x} \in \mathbb{R}^n | x^n \geq 0\}$ such that $\left(\frac{d}{dt} - \Delta\right) f \leq 0$. Using this we see f satisfies

$$0 \leq \frac{df}{dt} \leq g^{ij}(x) D_{ij} f - g^{ij}(x) \Gamma^k_{ij}(x) D_k f \quad .$$

But now we may apply the *elliptic* Hopf Lemma (see for example [9, Lemma 3.4]) to get that $DF \cdot \mathbf{e}_n < 0$, a contradiction. Therefore a positive maximum of f at the boundary cannot be increasing.

Therefore we have

$$F(x, t) \leq e^{\alpha t} \max\{\sup_{x \in M_0} F, 0\} \quad .$$

Now sending $\alpha \to 0$ we have the estimate. □

Corollary 1.2.18 (Which will also be referred to as Weak Maximum Principle)**.** *Under the assumptions of the above Lemma we in fact have that*

$$F \leq \sup_{x \in M^n} F(x, 0) \quad .$$

Proof. Set $K = \sup_{x \in M^n} F(x, 0)$ and set $j = K + 1 + F$. Since the operator above always contains derivatives, we may apply Lemma 1.2.17 to j and so we are done. □

Corollary 1.2.19 (Again Weak Maximum Principle)**.** *Suppose M^n, g_{ij}, ∇ and Δ are as in Lemma 1.2.17, but this time F has the properties:*

1. *At a stationary point (i.e. $\nabla F = 0$) we have*

$$\left(\frac{d}{dt} - \Delta\right) F(x,t) \geq 0 \qquad \forall (x,t) \in M^n \times [0,T) \quad .$$

2. *On the boundary if μ is an outward pointing unit normal then*

$$\nabla_\mu F \geq 0 \qquad \forall (x,t) \in \partial M \times [0,T) \quad .$$

Then $F \geq \inf_{x \in M^n} F(x,0)$.

Proof. Apply Corollary 1.2.18 to $l = -F$. □

We will find the following relations useful:

Proposition 1.2.20 (Product and chain rules for the heat operator)**.** *Let $f, g : M^n \times [0,T) \to \mathbb{R}$ and let $\psi : \mathbb{R} \to \mathbb{R}$ be twice differentiable then*

$$\left(\frac{d}{dt} - \Delta\right) f \cdot g = f\left(\frac{d}{dt} - \Delta\right) g + g\left(\frac{d}{dt} - \Delta\right) f - 2\left\langle \nabla f, \nabla g\right\rangle$$
$$\left(\frac{d}{dt} - \Delta\right) \psi(f) = \frac{d\psi}{dx}(f)\left(\frac{d}{dt} - \Delta\right) f - \frac{d^2\psi}{dx^2}(f)|\nabla g|^2 \quad .$$

Proof. This is simply a matter of calculus. We see

$$\begin{aligned}
\Delta f \cdot g &= g^{ij}\left(\frac{\partial^2 fg}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial fg}{\partial x^k}\right) \\
&= g^{ij}\left(g\frac{\partial^2 f}{\partial x^i \partial x^j} - g\Gamma^k_{ij}\frac{\partial f}{\partial x^k} + 2\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} + f\frac{\partial^2 g}{\partial x^i \partial x^j} - f\Gamma^k_{ij}\frac{\partial g}{\partial x^k}\right) \\
&= g\Delta f + 2\left\langle \nabla f, \nabla g\right\rangle + f\Delta g
\end{aligned}$$

and the first result follows.

Similarly for the second

$$\begin{aligned}
\Delta\psi(f) &= g^{ij}\left(\frac{\partial^2\psi(f)}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial\psi(f)}{\partial x^k}\right) \\
&= g^{ij}\left(\frac{d\psi}{dx}(f)\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{d\psi}{dx}(f)\frac{\partial f}{\partial x^k} + \frac{d^2\psi}{dx^2}(f)\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\right) \\
&= \frac{d\psi}{dx}(f)\Delta f + \frac{d^2\psi}{dx^2}(f)|\nabla f|^2
\end{aligned}$$

and again the result follows. □

Chapter 2

Quasilinear existence theory

In this chapter I will deal with existence of solutions to Partial Differential Equations (PDEs) of quasilinear type with a Neumann boundary condition. This material is standard and what we write here is based on selected sections of [20] rewritten with minor alterations to better suit our purposes. It would be nice to prove existence from first principles. Unfortunately since existence theory is, by necessity, fairly lengthy to avoid simply copying out entire chapters we will assume results on *linear* PDEs and here concentrate on selected results bridging the gap from this to the quasilinear theory.

Let $\Omega \subset \mathbb{R}^n$ be a domain with a smooth boundary $\partial\Omega$ and outward pointing unit normal γ. By crossing with a time interval we define our parabolic domain to be $\Omega_T = \Omega \times [0, T) \subset \mathbb{R}^{n+1}$. We will write in capitals $X = (x, t),\ Y = (y, s)$ to indicate elements of the parabolic domain as opposed to $x,\ y$, elements of the domain Ω. For such X and Y we define the parabolic distance

$$|X - Y|_P = \max\{|x - y|, |t - s|^{\frac{1}{2}}\} \quad ,$$

and the parabolic cylinder

$$Q(X, R) = \{Y \in \mathbb{R}^n \times \mathbb{R} | s < t, |Y - X|_P < R\} \quad .$$

For the rest of the chapter we will drop the subscript on the parabolic distance – the difference between this and the absolute value will always be clear from context. The parabolic boundary $\mathcal{P}\Omega_T$ is the subset of all points in $X \in \overline{\Omega}_T$ such that for all

R, $Q(X,R)$ contains points not in Ω_T. For the Ω_T we have chosen this boundary may be split into three parts: The bottom, the corner and the side where

$$B(\Omega_T) = \Omega \times \{0\}, \quad C(\Omega_T) = \partial\Omega \times \{0\}, \quad S(\Omega_T) = \partial\Omega \times (0,T) \ .$$

Now I shall define the set of operators with which we are concerned:

Definition 2.0.21. A *quasilinear operator* P is defined by

$$Pu = -u_t + a^{ij}(X,u,Du)D_{ij}u + a(X,u,Du)$$

and we say P is *parabolic* in some subset $S \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$ if for $(X,z,p) \in S$

$$\lambda(X,z,p)|\eta|^2 \leq a^{ij}(X,z,p)\eta^i\eta^j \leq \Lambda(X,z,p)|\eta|^2$$

for some $\lambda > 0$. If in addition to this we have $\frac{\Lambda}{\lambda}$ uniformly bounded on S then P is *uniformly parabolic* on S.

We will assume from now on that $a^{ij}(X,z,p)$ and $a(X,z,p)$ are smooth in each of their coefficients.

Our boundary operator will be

$$Mu = Du \cdot \zeta = 0$$

for some ζ such that $\zeta \cdot \gamma > C_\gamma > 0$. We search for solutions u of the following

$$\begin{cases} Pu = 0 & \forall X \in \Omega_T \\ Mu = 0 & \forall X \in S(\Omega_T) \\ u = u_0 & \forall X \in B(\Omega_T) \cup C(\Omega_T) \end{cases} \tag{2.1}$$

where we additionally assume that on $C(\Omega_T)$ the initial data u_0 also satisfies $Mu_0 = 0$.

The method of proof for existence of a solution for all time is as follows:

1. Proof of existence for some short time.

2. Given short time existence, under the assumption $|u|_\delta$ is bounded* where $2 > \delta > 1$ proof of existence for all time.

3. Proof that given bounds on $|Du|$ and u we have a bound on $|u|_\delta$ for some $2 > \delta > 1$.

*See next section for the definition of this norm

2.1 Hölder norms

For convenience of the reader we now define the parabolic Hölder norm. For $\alpha \in (0,1]$ a function $f : \Omega_T \to \mathbb{R}$ we say f is *Hölder continuous at* X_0 *with exponent* α if:

$$[f]_{\alpha,X_0} = \sup_{X\in\Omega_T - X_0} \frac{|f(X) - f(X_0)|}{|X - X_0|^\alpha}$$

is finite. If

$$[f]_{\alpha,\Omega_T} = \sup_{X_0\in\Omega_T} [f]_{\alpha,X_0}$$

is finite then f is *uniformly Hölder continuous in* Ω_T.

Differentiability of a function implies it is Lipschitz and so the above applies with $\alpha = 1$. Therefore we may think of the α is as a fractional differentiability. We now use this to produce the parabolic Hölder norm – a norm suitably weighted to imitate the "one time derivative to two space derivatives" in parabolic equations.

First we get the equivalent of the above. For $\beta \in (0,2]$ define

$$\langle f\rangle_{\beta;X_0} = \sup\left\{\frac{|f(x_0,t) - f(X_0)|}{|t - t_0|^{\frac{\beta}{2}}} : (x_0,t) \in \Omega_T - X_0\right\}$$

and as previously

$$\langle f\rangle_{\beta;\Omega} = \sup_{X_0\in\Omega} \langle f\rangle_{\beta;X_0} \quad .$$

We define for $a = k + \alpha$ for $\alpha \in (0,1]$ and β a multi index

$$\begin{aligned}
\{f\}_{a;\Omega_T} &= \sum_{|\beta|+2j=k-1} \{D_x^\beta D_t^j f\}_{\alpha+1} \\
[f]_{a;\Omega_T} &= \sum_{|\beta|+2j=k-1} [D_x^\beta D_t^j f]_\alpha \\
|f|_{a;\Omega_T} &= \sum_{|\beta|+2j\le k-1} sup|D_x^\beta D_t^j f| + [f]_\alpha + \langle f\rangle_\alpha \quad .
\end{aligned}$$

We may quickly see that $| - |_a$ defines a norm on $H_a(\Omega_T) = \{f : |f|_a < \infty\}$ and under this $H_a(\Omega_T)$ is a Banach space. We will also use the *elliptic* Hölder norms on functions $\psi : \Omega \to \mathbb{R}$, that is functions on a domain without the "time" direction. Rather than rewriting the above, we in fact define $|\psi|_a = |\widetilde{\psi}|_a$ where $\widetilde{\psi} : \Omega_T \to \mathbb{R}$ is the function equal to ψ at $t = 0$, and constant in time.

2.2 Short time existence

To show short time existence we will need the following fixed point theorem:

Theorem 2.2.1 (Schauder Fixed Point Theorem). *Let Z be a compact convex subset of a Banach space. Let J be a continuous map of Z into itself. Then J has a fixed point.*

Also we will need the following linear existence theorem.

Theorem 2.2.2 ([20], Theorem 5.18). *Given a linear operator*

$$Lu = a^{ij}(X)D_{ij}u + b^i(X)D_i u + c(X) - u_t$$

and the boundary condition

$$\widetilde{M}u = \beta \cdot Du + \beta^0 u$$

and suppose $\partial\Omega \in H_{2+\alpha}$ and also

- $a^{ij}\eta_i\eta_j \geq \lambda|\eta|^2, \quad \mathcal{T} = a^{ii} \leq \Lambda$
- $|a^{ij}|_\alpha \leq A,\ |b^i|_\alpha \leq B,\ |c|_\alpha \leq c_1$
- $\beta \cdot \gamma \geq \chi > 0,\ |\beta^j|_{1+\alpha} \leq B_1\chi$

then for all $f \in H_\alpha$ and $\psi \in H_{1+\alpha}$ for initial data $u_0 \in H_{2+\alpha}(\Omega)$ with $Mu_0 = \psi$ on $C(\Omega_T)$ there exists a unique solution $u \in H_{2+\alpha}$ to $Lu = f$ in $\Omega \times [0,T]$ and $Mu = \psi$ on $\partial\Omega \times [0,T]$, and further there exists a constant $C_1(A, B, c_1, n, \alpha, B_1, \beta, \chi, \alpha, \lambda, \Omega)$ such that

$$|u|_{2+\alpha} \leq C_1(|f|_\alpha + |\psi|_{1+\alpha} + |u_0|_{2+\alpha})$$

Proof. The necessary estimates and existence arguments that lead to this theorem are contained in the greater part of the first five chapters of [20]. □

We are now in a position to give a short time existence theorem – the one given here is a modification of [20][Theorem 8.2] which deals with the Dirichlet boundary condition. Here we show the important estimates by hand rather than via an interpolation inequality.

Theorem 2.2.3. *Let P and M be as above such that P is uniformly parabolic and $u_0 \in H_{2+\alpha}(\Omega)$ such that $Mu_0 = 0$ on $\partial\Omega$ then there exists an $\epsilon > 0$ such that a solution to equation (2.1) exists on Ω_ϵ.*

Proof. The idea here is to show that for ϵ small enough we may make estimates to apply Theorem 2.2.1 to a particular map so that the resulting fixed point is the required solution. Let $\theta \in (1,2)$ we define $M_0 = 1 + |u_0|_\theta$ and let

$$Z = \{v \in H_\theta(\Omega_\epsilon)|\ |v|_\theta < M_0\} \quad .$$

We now define by map $J : Z \to H_\theta$ by $Jv = u$ if

$$\begin{cases} -u_t + a^{ij}(X, v, Dv)D_{ij}u + a(X, v, Dv) = 0 & \text{for } X \in \Omega_\epsilon \\ Mu = 0 & \text{for } X \in S(\Omega_\epsilon) \\ u(\cdot, 0) = u_0(\cdot) & \text{on } B(\Omega_\epsilon) \cup C(\Omega_\epsilon) \quad , \end{cases}$$

that is J is the "inverse" of a linear parabolic operator. We note here that if $J\widetilde{u} = \widetilde{u}$ then $\widetilde{u}$ satisfies equation (2.1). We know that for each v, Jv exists due to Theorem 2.2.2 and furthermore we know that

$$|u|_{2+\alpha} \leq C(|v|_\theta) \leq C(M_0) \tag{2.2}$$

and so certainly $u \in H_\theta$. To apply the Schauder fixed point theorem though, we need $u \in Z$.

We consider $|u - u_0|_\theta$ and wish to show this is bounded by some constant times ϵ to a positive power. Writing $\theta = 1 + \alpha$ and $f = u - u_0$ then

$$|f|_{1+\alpha} = \sum_{i=1}^{n} \sup |D_i f| + \sum_{i=1}^{n} [D_i f]_\alpha + \sup |f| + \langle f \rangle_{1+\alpha} \quad .$$

We may deal with the first and third terms respectively by using that $\langle D_i u\rangle_{1+\alpha}$ and $\sup |u_t|$ are summands in the definition of $|u|_{2+\alpha}$ and are therefore bounded by $C(M_0)$ by (2.2). Since $f(\cdot, 0) = 0$ then $|D_i f| \leq C(M_0)\epsilon^{\frac{1+\alpha}{2}}$ and $|f| \leq C(M_0)\epsilon$.

Similarly for the final term again using $|u_t| \leq C(M_0)$ and hence

$$\langle f \rangle_{1+\alpha} = \sup_{s \neq t} \frac{|f(x,t) - f(x,s)|}{|t-s|^{\frac{1+\alpha}{2}}} \leq C(M_0) \frac{|t-s|}{|t-s|^{\frac{1+\alpha}{2}}} = C(M_0)\epsilon^{\frac{1-\alpha}{2}} \quad .$$

Finally using the same trick this time using $\{D_i u\}_{1+\alpha} < C(M_0)$, we estimate

$$[D_i f]_\alpha = \sup_{\substack{X,Y\in\Omega_\epsilon \\ X\neq Y}} \frac{|D_i f(X) - D_i f(Y)|}{\left(\max\{|x-y|, |t-s|^{\frac{1}{2}}\}\right)^\alpha} \leq C(M_0)|t-s|^{\frac{2-\alpha}{2}} \leq C(M_0)\epsilon^{\frac{2-\alpha}{2}} \quad ,$$

and so assuming $\epsilon < 1$ we have $|f|_\theta < 2(n+1)C(M_0)\epsilon^{\frac{1-\alpha}{2}}$. Therefore setting ϵ sufficiently small we have $|u|_\theta \leq |u_0|_\theta + 2(n+1)C(M_0)\epsilon^{\frac{1-\alpha}{2}} < M_0$. Therefore J maps Z into Z and we may apply the Schauder fixed point theorem. □

2.3 Long time existence

We now give a condition for quasilinear PDEs to last for all time. This is a modified version of [20][Theorem 8.3].

Theorem 2.3.1. *Suppose we have short time existence (i.e. Theorem 2.2.3) to equation (2.1) and know that for all time a solution exists there are constants $\delta \in (1,2)$ and $C_\delta > 0$ such that*

$$|u|_\delta \leq C_\delta$$

then a solution exists for all time.

Proof. Suppose that a solution to (2.1) exists for some maximal open time interval $[0,T)$ where T is finite.

Assuming first the linear boundary condition, we know that a solution of equation (2.1) is also a solution of a *linear* operator: Let

$$Lv = a^{ij}(X,u,Du)D_{ij}v + a(X,u,Du) \quad .$$

then u satisfies $Lu = 0$ in Ω_T, $Mu = 0$ on $S(\Omega_T)$ and $u(\cdot,0) = u_0(\cdot)$ on $B(\Omega_T) \cup C(\Omega_T)$. Now writing $\delta = 1+\alpha$, by our bound on $|u|_\delta$ we have a bound on $|a^{ij}(X,u,Du)|_\alpha$ and $|a(X,u,Du)|_\alpha$ depending on C_δ. Therefore by the Schauder estimate in Theorem 2.2.2 we have the *uniform* estimate

$$|u|_{2+\alpha} \leq C_1(C_\delta)|u_0|_{2+\alpha} = C_2 \tag{2.3}$$

for $t \in [0,T)$.

We now take a sequence of times $t_i \to T$, and define $\widetilde{u}_i(\cdot) = u(\cdot, t_i)$. Due to the bound we have equicontinuity of the $\widetilde{u}_i$ and therefore there exists a subsequence, which by abuse of notation we shall also write $\widetilde{u}_i$, such that the $\widetilde{u}_i \to \widetilde{u}$ uniformly as $i \to \infty$. Furthermore by (2.3) we have equicontinuity of $D_j\widetilde{u}_i$, $D_{jk}\widetilde{u}_i$ and $\widetilde{u}_{i,t}$ and therefore may assume (taking further subsequences and abuse) $D_j\widetilde{u}_i \to D_j\widetilde{u}$, $D_{jk}\widetilde{u}_i \to D_{jk}\widetilde{u}$ and $\widetilde{u}_{i,t} \to \widetilde{u}_t$ uniformly where we define $\widetilde{u}_t$ here to be $a^{ij}(X, \widetilde{u}, D\widetilde{u})D_{ij}\widetilde{u} + a(X, \widetilde{u}, D\widetilde{u})$. Therefore using $\widetilde{u}$ we extend u to the interval $[0, T]$. The bound on $|u|_\delta$ still holds by the C^2 convergence of $\widetilde{u}_i$ to $\widetilde{u}$, and so by the continuity of P and M we have that u is a solution of (2.1) on $[0, T]$. We now see that in fact $\widetilde{u}$ is $C^{2+\alpha}$:

Suppose we are given $x, y \in \Omega$, and write D^2u as shorthand for any particular $D_{jk}u$. Using the uniform convergence of the second derivatives we choose t sufficiently close to T that $|D^2\widetilde{u}(\cdot) - D^2u(\cdot, t)| < \epsilon < |x - y|$. Then

$$\frac{|D^2u(x) - D^2u(x)|}{|x-y|^\alpha} \leq \frac{|D^2u(x,t) - D^2u(y,t)| + 2\epsilon}{\max\{|x-y|, |T-t|^{\frac{1}{2}}\}^\alpha} \leq 2 + C_2$$

due to the bound on $[D^2u]_\alpha$ for $t < T$. Taking suprema we have that $|\widetilde{u}|_{2+\alpha} < 2+C_2$.

We may now apply our short time existence Theorem to equation (2.1) but with $u_0 = \widetilde{u}$ and get a solution $\widehat{u}$ in Ω_ϵ. But now we define

$$w(x,t) := \begin{cases} u(x,t) & \text{for } (x,t) \in \Omega \times [0,T] \\ \widehat{u}(x, t-T) & \text{for } (x,t) \in \Omega \times (T, T+\epsilon) \end{cases} .$$

Since $u_t(\cdot, s)$ converges $\widehat{u}_t(0)$ as $s \to T$, w is twice differentiable in space and once differentiable in time and satisfies $Pw = 0$ and $Mw = 0$. Furthermore by strong maximum principle it is the unique solution $Lw = 0$ implying that by Theorem 2.2.2 that it is in fact $w \in H^{2+\alpha}(\Omega_{T+\epsilon})$. This contradicts the definition of T. □

2.4 Further remarks

I have not yet said anything about part 3 of the proof of existence of a solution: Given an upper bound on u and $|Du|$ can we get an estimate on $[D_iu]_\alpha$? The answer is under very general conditions, yes. To prove this here I would end up rewriting most of Chapter 12 of [20], and therefore I simply cite [20, Theorem 12.3] for estimates on the interior of the domain and [20, Theorem 12.10] for estimates at the boundary. The proofs of these Theorems rest upon showing that $D_i u$ is a weak supersolution of certain linear PDEs and then using either Harnack estimates or the theory of strong solutions.

Another issue I have not mentioned is that of uniqueness. In the cases of equations as in Chapter 4 and 5 we have that in the operator Pu, $a^{ij}(X, u, Du)$ and $a(X, u, Du)$ do not depend upon u. In these cases we may get uniqueness from a parabolic comparison principle similar to [20, Theorem 9.2], although here we must use Neumann boundary conditions. To get around this we may use a proof almost identical to that in [9, Theorem 9.2].

Chapter 3

Spacelike mean curvature flow inside timelike cones

In this Chapter we will be concerned with equation (1.9), where the boundary manifold Σ is chosen to be a timelike cone – a cone in Minkowski space with its apex at $\mathbf{0}$ with the property that each position vector is timelike (see section 3.2 for full details). We will be flowing a manifold which is topologically an n-ball, that is, $M^n = B^n$, which will be flowed by its mean curvature within the interior of the cone. We recall we wish to find $\mathbf{F} : M^n \times [0,T] \to \mathbb{R}^{n+1}_1$ such that

$$\begin{cases} \frac{d\mathbf{F}}{dt} = \mathbf{H} = H\nu & \forall (x,t) \in M^n \times [0,T] \\ \mathbf{F}(\cdot,0) = \mathbf{F}_0(\cdot) & \\ \mathbf{F}(x,t) \subset \Sigma & \forall (x,t) \in \partial M^n \times [0,T] \\ \langle \nu, \mu \rangle (x,t) = 0 & \forall (x,t) \in \partial M^n \times [0,T] \end{cases} \tag{3.1}$$

where $\mathbf{F}_0 : M^n \to \mathbb{R}^{n+1}_1$ is an initial spacelike embedding which satisfies the boundary condition.

Remark 3.0.1. Under these conditions we note that we have a special solution to mean curvature flow: In Example 1.2.5 we saw homothetically expanding hyperbolic hyperplanes as examples of mean curvature flow. In fact, such solutions satisfy the boundary condition described above. A hyperbolic hyperplane is normal to its position vector, while for any cone Σ centred at the origin the position vector is in the tangent space, that is $\mathbf{p} \in T_{\mathbf{p}}\Sigma$. Therefore the normal of any hyperbolic hyperplane

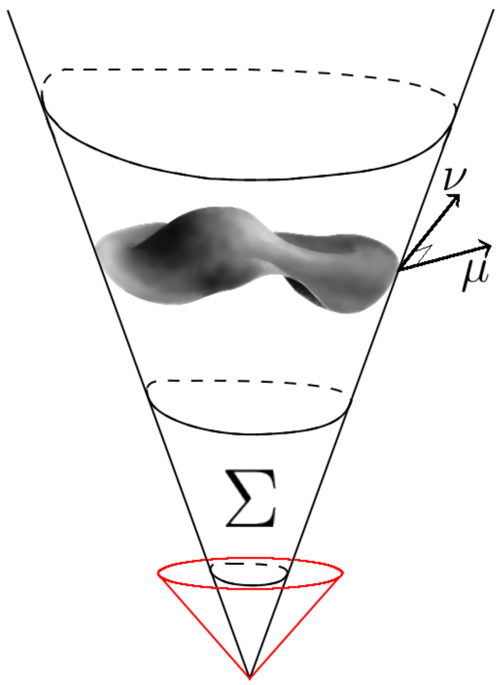


Figure 3.1: The flowing manifold sits inside the boundary manifold, Σ, which in turn sits inside the light cone (again in red)

is contained in the tangent space of Σ and the boundary condition $\langle \nu, \mu \rangle = 0$ is automatically satisfied.

Therefore I make the following definition:

Definition 3.0.2. Define the *expanding hyperbolic hyperplane inside the cone* $\mathbf{G}_k$ to be the solution to (3.1), starting with the section of hyperbolic plane of initial "radius" k inside the cone Σ. That is at time $t = 0$, $\langle \mathbf{G}_k, \mathbf{G}_k \rangle = -k^2$ with $(\mathbf{G}_k)_{n+1} > 0$. We saw in Example 1.2.5 that

$$-\langle \mathbf{G}_k, \mathbf{G}_k \rangle = k^2 + 2nt \quad .$$

In this chapter we will begin by proving the following long time existence result:

Theorem 3.0.3. *Let Σ be a convex cone. Given that M_0 is initially spacelike then a solution to equation (3.1) exists for all time. Furthermore this solution stays between two homothetic solutions $\mathbf{G}_{C_0}$ and $\mathbf{G}_{C_1}$ where C_0 an C_1 are the minimum and maximum values of $\sqrt{-\langle \mathbf{F}, \mathbf{F} \rangle}$ at time $t = 0$. Mean convexity is preserved by the flow (i.e. if $H > 0$ initially then it will remain so for all time).*

Due to the comparison with the solution $\mathbf{G}_{C_0}$ in the above, we know that in solutions to equation (3.1), $x_{n+1} \to \infty$. Therefore for further convergence results we need a suitable notion of renormalisation. To this end we define $\widehat{M}$ to be the blowdown of M, where $\widehat{M}$ is M renormalised by dilations so that $\widehat{M}$ has constant unit area. By defining convergence "at infinity" to be the convergence of $\widehat{M}$, we get the following:

Theorem 3.0.4. *Any initially spacelike solution of equation (3.1) with a convex cone boundary condition under renormalisation will converge to a homothetic solution in the C^1 norm. Further, there exists an increasing sequence of t_i such that $\widehat{M}_{t_i}$ converge to the solution on the interior of M in the C^∞ topology.*

3.1 A reparametrisation

For simplicity we may reparametrise the above system as a graph over a topological disc $D \subset B_1^n(\mathbf{0})$ defined by the intersection of the interior of Σ with the hyperplane perpendicular to $\mathbf{e}_{n+1}$ and intersecting $(0, \ldots, 0, 1)$. We may then describe a spacelike manifold M inside Σ as follows: At a point $\mathbf{x} \in D$, if we take the ray from $\mathbf{0}$ through $\mathbf{x}$ then the ray will intersect M only once. If $\mathbf{p}$ is that point of intersection then let $u(\mathbf{x}) = \sqrt{-\langle \mathbf{p}, \mathbf{p} \rangle}$. The graph u now parametrises M by $\mathbf{F}(x) = u(x)\frac{\mathbf{x}+\mathbf{e}_{n+1}}{\sqrt{1-|\mathbf{x}|^2}}$. Standard calculations give geometric quantities (see Proposition A.0.17), for example:

$$g_{ij} = \frac{u^2}{1-|\mathbf{x}|^2}\left(\delta_{ij} + \frac{x_i x_j}{1-|\mathbf{x}|^2}\right) - D_i u D_j u$$

and

$$\nu = \frac{(1-|\mathbf{x}|^2)\mathbf{D}u + u\mathbf{x} + (Du \cdot x(1-|\mathbf{x}|^2) + u)\mathbf{e}_{n+1}}{(1-|\mathbf{x}|^2)v}$$

where v is a gradient-like function

$$v = \sqrt{\frac{u^2}{1-|\mathbf{x}|^2} + (Du.\mathbf{x})^2 - |Du|^2}\ .$$

Similarly we see that a solution to MCF is equivalent to a solution to the following parabolic quasilinear PDE: For $u : D \times [0, T) \to \mathbb{R}$ then

$$\begin{cases} \frac{du}{dt} = \frac{vH\sqrt{1-|\mathbf{x}|^2}}{u} = g^{ij}D_{ij}u + \frac{n+1}{u} - \frac{1}{v^2}\left(\frac{u}{1-|\mathbf{x}|^2} + 2Du.\mathbf{x}\right) & \forall \mathbf{x} \in D \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) & \forall \mathbf{x} \in D \\ Du \cdot (\gamma - \gamma \cdot x\mathbf{x}) = 0 & \forall \mathbf{x} \in \partial D \end{cases} \tag{3.2}$$

where

$$g^{ij} = \frac{1-|\mathbf{x}|^2}{u^2}\left(\delta_{ij} + \frac{1}{v^2}\left[\left(|Du|^2 - \frac{u^2}{1-|\mathbf{x}|^2}\right)x_i x_j + D_i u D_j u - Du.\mathbf{x}\left(x_i D_j u + x_j D_i u\right)\right]\right)$$

is the inverse of the metric and γ is the outward pointing unit normal to D. These calculations though standard are fairly lengthy, and so are not included here – for those who are interested they are written up in full: See Appendix A for 11 pages of differentiation and linear algebra.

Long-term existence is equivalent to uniform parabolicity of the above equation and C^1 bounds on u (see Chapter 2). By calculating eigenvalues of the metric g_{ij} (see Proposition A.0.18) we see that uniform parabolicity is equivalent bounding $\max\left\{\frac{1}{v^2}, \frac{1}{u^2}, u^2\right\}$ from above.

In fact uniform parabolicity is stronger than the gradient estimate: Suppose we have uniform parabolicity. Then $v^2 > C > 0$ and so

$$\frac{u^2}{1-|\mathbf{x}|^2} - C > |Du|^2 - (Du.\mathbf{x})^2 > |Du|^2(1-|\mathbf{x}|^2)$$

by Cauchy – Schwarz. Therefore

$$|Du|^2 < \frac{u^2}{(1-|\mathbf{x}|^2)^2} < \widetilde{C}$$

which gives the gradient estimate. Hence for existence on an interval $[0, T]$ we need only find an upper bound on $\frac{1}{v^2}$ and upper and lower bounds on u^2 on that interval.

3.2 The boundary manifold

Here I will define more rigorously the boundary manifold Σ and state formulae for its curvature. Let $\widetilde{\mathbf{S}} : S^n \to B_1(0) \subset \mathbb{R}^n$ be a smooth embedding of a sphere into the open unit ball centred at the origin with outward unit normal $\mathbf{n}$. Then we may define a boundary cone $\Sigma_{\widetilde{\mathbf{S}}}$ (later the subscript will be dropped) by embedding $\mathbb{R}^n$ into $\mathbb{R}^{n+1}_1$ at height 1 and then defining $\Sigma_{\widetilde{\mathbf{S}}}$ to be the set of all rays going through the origin and some point $(\widetilde{\mathbf{S}}(x), 1)$. More explicitly we may give a parametrisation $\mathbf{S} : (0, \infty) \times S^n \to \mathbb{R}^{n+1}_1$ of $\Sigma_{\widetilde{\mathbf{S}}}$ by

$$(l, x) \mapsto l\widetilde{\mathbf{S}}(x) + l\mathbf{e}_{n+1}$$

Now we may calculate all quantities needed. For example, we may see that in these coordinates

$$A^{\Sigma}\left(\cdot, \frac{\partial}{\partial l}\right) = 0 \ , \qquad A^{\Sigma}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{lA^{\widetilde{S}}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)}{\sqrt{1 - \left\langle \widetilde{\mathbf{S}}, \mathbf{n} \right\rangle^2}} \quad .$$

Therefore, for an orthonormal set of vectors $\mathbf{e}_l, \mathbf{e}_1, \ldots, \mathbf{e}_{n-1} \in T_{\mathbf{p}}\Sigma$ obtained by picking orthonormal coordinates on $\widetilde{S}$ and renormalising,

$$A^{\Sigma}(\mathbf{e}_i, \mathbf{e}_j) = \frac{A^{\widetilde{S}}(\mathbf{e}_i, \mathbf{e}_j)}{l\sqrt{1 - \left\langle \widetilde{\mathbf{S}}, \mathbf{n} \right\rangle^2}} \quad . \tag{3.3}$$

Hence we can see that, as we would expect, weak convexity of Σ – that is $A^{\Sigma}(\cdot, \cdot) \geq 0$ – is equivalent to convexity of the embedding $\widetilde{\mathbf{S}}$, the second fundamental form has a zero eigenvector along the timelike rays from the origin and the second fundamental form decreases linearly as you move up the cone.

3.3 Evolution equations

We need the evolution of a few more quantities to those in Section 1.2.5 which we will derive here by straightforward calculation. I define the following:

$$F^2 = -\langle \mathbf{F}, \mathbf{F} \rangle \ > 0$$

$$S = -\langle \mathbf{F}, \nu \rangle > F > 0 \quad .$$

We may think of these as in some sense C^0 and C^1 measures of how far our flowing manifold is from a homotheic solution $\mathbf{G}_k$.

Lemma 3.3.1. *Under MCF we have*

$$\left(\frac{d}{dt} - \Delta\right) F^2 = 2n \quad .$$

Proof. We see

$$\frac{dF^2}{dt} = -2\left\langle \frac{d\mathbf{F}}{dt}, \mathbf{F} \right\rangle = -2H\left\langle \nu, \mathbf{F} \right\rangle = 2HS$$

and further

$$\begin{aligned} \Delta F^2 &= -2g^{ij}\left(\left\langle \frac{\partial^2 \mathbf{F}}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial \mathbf{F}}{\partial x^k}, \mathbf{F}\right\rangle + \left\langle \frac{\partial \mathbf{F}}{\partial x^i}, \frac{\partial \mathbf{F}}{\partial x^j}\right\rangle\right) \\ &= -2g^{ij}\left(h_{ij}\left\langle \nu, \mathbf{F}\right\rangle + g_{ij}\right) \\ &= 2HS - 2n \quad . \end{aligned}$$

So we are done. □

Lemma 3.3.2. *On the interior of the flowing manifold we have*

$$\left(\frac{d}{dt} - \Delta\right) S = 2H - S|A|^2$$

Proof. Using Lemma 1.2.10 we get

$$\begin{aligned} \frac{dS}{dt} &= -\left\langle \frac{d\mathbf{F}}{dt}, \nu \right\rangle - \left\langle \mathbf{F}, \frac{d\nu}{dt} \right\rangle \\ &= -H\left\langle \nu, \nu\right\rangle - \left\langle \mathbf{F}^\top, \nabla H\right\rangle \\ &= H - \left\langle \mathbf{F}^\top, \nabla H\right\rangle \end{aligned}$$

and

$$\begin{aligned} \Delta S &= -g^{ij}\left(\frac{\partial}{\partial x^i}\left[A\left(\mathbf{F}^\top, \frac{\partial}{\partial x^j}\right)\right] - A\left(\mathbf{F}^\top, \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right)\right) \\ &= -g^{ij}\left(\nabla_{\frac{\partial}{\partial x^i}} A\left(\mathbf{F}^\top, \frac{\partial}{\partial x^j}\right) + A\left(\nabla_{\frac{\partial}{\partial x^i}}\mathbf{F}^\top, \frac{\partial}{\partial x^j}\right)\right) \\ &= -\nabla_{\mathbf{F}^\top} H - g^{ij}A\left(\nabla_{\frac{\partial}{\partial x^i}}\mathbf{F}^\top, \frac{\partial}{\partial x^j}\right) \quad . \end{aligned}$$

Now we calculate

$$\begin{aligned} g^{ij}A\left(\nabla_{\frac{\partial}{\partial x^i}}\mathbf{F}^\top, \frac{\partial}{\partial x^j}\right) &= g^{ij}A\left(\left[\overline{\nabla}_{\frac{\partial}{\partial x^i}}(\mathbf{F} - S\nu)\right]^\top, \frac{\partial}{\partial x^j}\right) \\ &= g^{ij}A\left(\left[\frac{\partial \mathbf{F}}{\partial x^i} - \frac{\partial S}{\partial x^i}\nu - S\frac{\partial \nu}{\partial x^i}\right]^\top, \frac{\partial}{\partial x^j}\right) \\ &= g^{ij}A\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) - Sg^{ij}A\left(\frac{\partial}{\partial x^j}, h_{ik}g^{kl}\frac{\partial}{\partial x^l}\right) \\ &= H - S|A|^2 \end{aligned}$$

which shows

$$\Delta S = -\nabla_{\mathbf{F}^\top} H - H + S|A|^2 \quad .$$

Hence

$$\left(\frac{d}{dt} - \Delta\right) S = H - \left\langle \mathbf{F}^\top, \nabla H \right\rangle + \nabla_{\mathbf{F}^\top} H + H - S|A|^2 = 2H - S|A|^2 \quad .$$

□

3.4 Boundary derivatives

To apply Hopf maximum principle we also need to consider derivatives of functions at the boundary in the direction of μ, the normal to Σ. As in the case of Stahl [25] these identities come from derivatives of the boundary condition. We first demonstrate the following simple result.

Lemma 3.4.1. *For* $\mathbf{p} \in \partial M^n \times [0, T)$ *we have*

$$\left\langle \nabla F^2, \mu \right\rangle = 0$$

Proof. We know that $\nabla F^2 = (\overline{\nabla} F^2)^\top$. Furthermore we have that $\overline{\nabla} F^2 \in T_{\mathbf{p}}\Sigma$ and so we have

$$\begin{aligned} \left\langle \mu, \nabla F^2 \right\rangle &= \left\langle \mu, \overline{\nabla} F^2 + \left\langle \nu, \nabla F^2 \right\rangle \nu \right\rangle \\ &= \left\langle \mu, \overline{\nabla} F^2 \right\rangle \\ &= 0 \quad . \end{aligned}$$

□

Now we take spatial derivatives of the boundary condition:

Lemma 3.4.2. *For* $W \in T_p M_t \cap T_p \Sigma$ *then*

$$A(\mu, W) = -A^\Sigma(\nu, W) \quad .$$

Proof. For such a W

$$\begin{aligned}
0 &= W \langle \nu, \mu \rangle_{\mathbb{R}^{n+1}_1} \\
&= \left\langle \overline{\nabla}_W \nu, \mu \right\rangle + \left\langle \nu, \overline{\nabla}_W \mu \right\rangle \\
&= \left\langle \overline{\nabla}_W \nu, \mu \right\rangle + \left\langle \nu, \overline{\nabla}_W \mu \right\rangle \\
&= A(W, \mu) + A^{\Sigma}(\nu, W) \quad .
\end{aligned}$$

□

For our gradient estimate we also need

Lemma 3.4.3. *For $p \in \partial M^n \times [0, T)$ we have*

$$\langle \nabla S, \mu \rangle = -A^{\Sigma}(\mathbf{F}^{\top}, \nu) \quad .$$

Proof. Look:

$$\begin{aligned}
\nabla S &= \frac{\partial S}{\partial x^i} g^{ij} \frac{\partial}{\partial x^j} \\
&= -\left(\left\langle \frac{\partial \mathbf{F}}{\partial x^i}, \nu \right\rangle + \left\langle \mathbf{F}, \frac{\partial \nu}{\partial x^i} \right\rangle \right) g^{ij} \frac{\partial}{\partial x^j} \\
&= -A\left(\mathbf{F}^{\top}, \frac{\partial}{\partial x^i} \right) g^{ij} \frac{\partial}{\partial x^j} \quad .
\end{aligned}$$

Since $\left\langle \mathbf{F}^{\top}, \mu \right\rangle = 0$ we may apply Lemma 3.4.2 to give

$$\begin{aligned}
\langle \nabla S, \mu \rangle &= -A(\mathbf{F}^{\top}, \mu) \\
&= A^{\Sigma}(\mathbf{F}^{\top}, \nu) \quad .
\end{aligned}$$

□

Corollary 3.4.4. *At a point* $\mathbf{p}$ *as above*

$$\langle \nabla S, \mu \rangle = -S A^{\Sigma}(\nu, \nu) \quad .$$

Proof. We know that the second fundamental form of Σ has a zero eigenvector in the direction $\mathbf{F}$, and so we may calculate

$$\begin{aligned}
A^{\Sigma}(F^{\top}, \nu) &= A^{\Sigma}(\mathbf{F} - S\nu, \nu) \\
&= A^{\Sigma}(\mathbf{F}, \nu) - S A^{\Sigma}(\nu, \nu) \\
&= -S A^{\Sigma}(\nu, \nu) \quad .
\end{aligned}$$

□

Now differentiating the boundary condition with respect to time:

Lemma 3.4.5. *For* $p \in \partial M^n \times [0, T)$ *we have*

$$\langle \nabla H, \mu \rangle = -HA^{\Sigma}(\nu, \nu) \quad .$$

Proof. Using Lemma 1.2.10

$$\begin{aligned} 0 &= \frac{d}{dt} \left\langle \nu, \mu_{|\mathbf{F}} \right\rangle_{\mathbb{R}_1^{n+1}} \\ &= \langle \nabla H, \mu \rangle + \langle \nu, D\mu(H\nu) \rangle_{\mathbb{R}_1^{n+1}} \\ &= \langle \nabla H, \mu \rangle + HA^{\Sigma}(\nu, \nu) \quad . \end{aligned}$$

□

Remark 3.4.6. We note that if Σ is convex then the normal derivatives of both H and S at the boundary are negative.

On the other hand regardless of the boundary we are able to get

Corollary 3.4.7. *For* $p \in \partial M^n \times [0, T)$ *we have*

$$\left\langle \nabla \frac{H}{S}, \mu \right\rangle = 0 \quad .$$

3.5 Gradient estimate

We now obtain a gradient estimate, that is to say, a lower bound on v. Note that in the graphical notation of equation (3.2)

$$S = \frac{u^2}{v\sqrt{1 - |\mathbf{x}|^2}} \quad .$$

Hence it is sufficient to find a suitable upper bound on S and a lower bound on $u^2 = F^2$. We will need an assumption:

Assumption 3.5.1. We will assume from here on that Σ is convex.

We will also need the weak maximum principle of Corollaries 1.2.18 and 1.2.19.

Using Lemmas 3.3.1 and 3.4.1, we see we can immediately apply the above to both $F^2 - 2nt$ and $2nt - F^2$ to give

$$C_1(M_0) \leq F^2 - 2nt \leq C_2(M_0) \quad . \tag{3.4}$$

This may be interpreted as if our manifold lies between two copies of a hyperbolic solution $\mathbf{G}_{C_1}$ and $\mathbf{G}_{C_2}$ initially, then it will do so for all time. It also gives the required bounds on u, and further ensures that M_0 stays away from the singularity of Σ for all the time a solution exists.

Now using equation (1.10) and Lemma 3.3.2 we consider the evolution of $\frac{H}{S}$:

$$\begin{aligned}\frac{d\frac{H}{S}}{dt} &= \frac{1}{S}\frac{dH}{dt} - \frac{H}{S^2}\frac{dS}{dt} \\ &= \frac{1}{S}\Delta H - \frac{H|A|^2}{S} - \frac{H}{S^2}\Delta S - 2\frac{H^2}{S^2} + \frac{H|A|^2}{S} \\ &= -2\left(\frac{H}{S}\right)^2 + \Delta\frac{H}{S} - 2\left\langle\frac{\nabla S}{S}, \nabla\frac{H}{S}\right\rangle \quad .\end{aligned}$$

So at a point where $\nabla\frac{H}{S} = 0$ we have

$$\left(\frac{d}{dt} - \Delta\right)\frac{H}{S} = -2\left(\frac{H}{S}\right)^2 \quad .$$

Hence from this, Lemma 3.4.7 and the weak maximum principle we immediately get

$$\frac{H}{S} \leq C_3(M_0)$$

In fact, we can do better. At a stationary point of $\frac{H}{S}(C+2nt)$ we get

$$\left(\frac{d}{dt} - \Delta\right)\frac{H}{S}(C+2nt) = \frac{H}{S}\left(2n - 2(C+2nt)\frac{H}{S}\right) \quad .$$

Hence given that $H > 0$ on M_0 and again applying weak maximum principle we have for $C_3, C_4 > 0$

$$\frac{C_3}{C+2nt} \leq \frac{H}{S} \leq \frac{C_4}{C+2nt}$$

or for $\widehat{C}_3, \widehat{C}_4 > 0$

$$\widehat{C}_3 \leq \frac{H}{S}F^2 \leq \widehat{C}_4 \quad . \tag{3.5}$$

If $H \geq 0$ on M_0 then the constant C_3 is zero. This estimate implies preservation of weak or strict mean convexity since

$$H \geq C_4\frac{S}{C+2nt} \geq 0 \quad .$$

If we neglect the assumption of initial mean convexity, estimate (3.5) still holds, although $\widehat{C}_3 \leq -n$.

Until now we have not used our assumption, and this is the point at which it comes in, in the form of a sign of the boundary derivative on H (and later S). Using equation 1.10 we get that on the interior of M

$$\left(\frac{d}{dt} - \Delta\right) H^2 = -2H^2|A|^2 - 2|\nabla H|^2 \quad ,$$

and from Lemma 3.4.5 and our assumption, $\nabla_\mu H = -2H^2 A(\nu,\nu) \leq 0$. By the weak maximum principle we therefore have

$$H^2 < C_5 \quad .$$

Now using Lemmas 3.3.1, 3.3.2, 3.4.1 and 3.4.3 we calculate for $f = S - \frac{\sqrt{C_5}}{n} F^2$ that

$$\left(\frac{d}{dt} - \Delta\right) f = 2H - 2\sqrt{C_5} - S|A|^2 \leq -S|A|^2 \leq 0$$

and

$$\langle \nabla f, \mu \rangle = \langle \nabla S, \mu \rangle \leq 0 \quad .$$

Again applying the weak maximum principle we see

$$S \leq C_6(M_0) + \frac{\sqrt{C_5}}{n} F^2$$

and hence we get

$$v > \frac{F^2}{\sqrt{1 - |\mathbf{x}|^2}\left(C_6 + \frac{\sqrt{C_5}}{n} F^2\right)} > 0 \quad .$$

We have the estimates required, and give the following summary:

Theorem 3.5.2. *Given that M_0 is spacelike, a solution to equation (3.1) exists for all time. Mean convexity is preserved by the flow and if the solution is initially bounded by $\mathbf{G}_{C_1}$ and $\mathbf{G}_{C_2}$, it will remain so for all time.*

Proof. From the above bounds we see that for any finite time interval $[0, T]$ we have uniform parabolicity, and therefore existence of a smooth solution from the standard results on quasilinear PDEs in Chapter 2. Therefore we have existence of a solution on the interval $[0, \infty)$, since otherwise we would have non existence at a finite $t < T = t + 1$. Therefore long time existence is proved. □

3.6 Improvements to estimates

We expect our solution to move towards an expanding hyperbolic hyperplane $\mathbf{G}_k$, but if this is so, easy calculations imply that the estimates from the previous section are not optimal. We currently have that $F \leq S \leq C_6 + C_7F^2$ while on a special solution $S = F$. Also we only have $\frac{C_3}{\sqrt{C+2nt}} \leq H \leq \sqrt{C_5}$ while on a special solution we know $H = \frac{n}{F}$. However our ratio of H to S, estimate (3.5), is of the right order.

To improve our estimates we consider

$$\frac{|\nabla F^2|^2}{F^2} = 4\frac{|F^\top|^2}{F^2} = \frac{4}{F^2}\langle F - S\nu, F - S\nu\rangle = 4\frac{S^2 - F^2}{F^2} \quad .$$

Note that this quantity is invariant under scaling and is zero on our special solution. We wish to show that this will in fact asymptote to zero. We calculate

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)(S^2 - F^2) &= 2S\left(\frac{d}{dt} - \Delta\right)S - 2|\nabla S|^2 - \left(\frac{d}{dt} - \Delta\right)F^2 \\ &= 4SH - 2S^2|A|^2 - 2|\nabla S|^2 - 2n \quad .\end{aligned}$$

and so for $J = \frac{S^2-F^2}{F^2}$ we have

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)J &= \frac{1}{F^2}\left(\frac{d}{dt} - \Delta\right)(S^2 - F^2) + \frac{2}{F^4}\left\langle \nabla F^2, \nabla(S^2 - F^2)\right\rangle \\ &\quad + (S^2 - F^2)\left(-\frac{1}{F^4}\left(\frac{d}{dt} - \Delta\right)F^2 - 2\frac{|\nabla F^2|^2}{F^6}\right) \\ &= \frac{1}{F^2}\Big[4SH - 2S^2|A|^2 - 2|\nabla S|^2 - 2n \\ &\qquad - 2nJ - 8J^2 + \frac{2}{F^2}\left\langle \nabla F^2, \nabla(S^2 - F^2)\right\rangle\Big] \quad .\end{aligned}$$

Since $|A|^2 \geq \frac{H^2}{n^2}$ we estimate

$$4SH - 2S^2|A|^2 \leq 4SH - \frac{2}{n^2}S^2H^2 \leq 2n^2 - \frac{2}{n^2}\left(SH - n^2\right)^2 \leq 2n^2 \quad .$$

By Cauchy–Schwarz and Young's inequalities we also see

$$\begin{aligned}\frac{1}{F^2}\left\langle \nabla F^2, \nabla(S^2 - F^2)\right\rangle - |\nabla S|^2 &\leq 2\frac{S}{F}\frac{|\nabla F^2|}{F}|\nabla S| - \frac{|\nabla F^2|^2}{F^2} - |\nabla S|^2 \\ &\leq \frac{S^2}{F^2}\frac{|\nabla F^2|^2}{F^2} - \frac{|\nabla F^2|^2}{F^2} \\ &= 4J^2 \quad .\end{aligned}$$

Applying these estimates to the evolution equation for J we have

$$\left(\frac{d}{dt} - \Delta\right) J \leq \frac{2n}{F^2}\left[n - 1 - J\right]$$

which implies since the boundary derivative of J is $\langle \nabla J, \mu \rangle = 2\frac{S}{F^2}\langle \nabla S, \mu \rangle < 0$ that J is bounded by the maximum of its initial value and $n-1$. But we need convergence as $t \to \infty$ and so for $C \geq C_2$ (see equation (3.4)) we consider

$$\begin{aligned}
&\left(\frac{d}{dt} - \Delta\right) J \log(C + 2nt) \\
&\qquad = \log(C + 2nt)\left(\frac{d}{dt} - \Delta\right) J + \frac{2nJ}{C + 2nt} \\
&\qquad \leq \frac{2n\log(C + 2nt)}{F^2}\left[n - 1 - J\right] + \frac{2nJ}{C + 2nt} \\
&\qquad \leq \frac{2n\log(C + 2nt)}{F^2}\left[n - 1 - \left(1 - \frac{1}{log(C + 2nt)}\right) J\right]
\end{aligned}$$

and by choosing C sufficiently large such that, for example $C > \max\{e^2, C_2\}$ then we have the following:

Proposition 3.6.1. *There exists constants $C_S, D_S, \widetilde{D}_S > 0$ depending only on n and M_0 such that*

$$\frac{|\nabla F^2|^2}{F^2} \leq 4\frac{C_S}{\log(D_S + 2nt)} \leq 4\frac{C_S}{\log(\widetilde{D}_S + F^2)}$$

or equivalently

$$\frac{S^2}{F^2} \leq 1 + \frac{C_S}{\log(D_S + 2nt)} \leq 1 + \frac{C_S}{\log(\widetilde{D}_S + F^2)}$$

Now the estimate (3.5) implies the following:

Corollary 3.6.2. *There exist constants C_1^H and $C_2^H > 0$ such that*

$$C_1^H \leq HF \leq C_2^H$$

Where C_1^H is positive if M_0 is initially mean convex.

Remark 3.6.3. Although we may not use this to say anything more about exactly what H tends towards we may say what the *average* of H will be asymptotically if we assume that that H is initially positive. From the proof of Lemma 3.3.1 we have that $\Delta F^2 = 2HS - 2n$ and therefore we see using Lemma 3.4.1 and Divergence Theorem that

$$\int_M HS d\mu = n \int_M d\mu \quad .$$

Therefore since $H > 0$ we may estimate

$$n \leq \frac{\int_M HF d\mu}{\int_M d\mu} \leq n\sqrt{1 + \frac{C_S}{\log(D_S + 2nt)}}$$

which asymptotically corresponds to what we would expect on our special solution.

3.7 Interior curvature estimates

Estimates on $|A|^2$ on the entirety of M are difficult. It is true that as in [25] we are able to get estimates on $\nabla_\mu h_{ij}$ at the boundary by differentiating twice in space and using the estimates already mentioned in Section 3.4. However, these give a mixture of Dirichlet and Neumann conditions for h_{ij}, which are unpleasant even in the simplest situation of the cone having the cross-section of a round sphere. Instead we obtain some interior estimates on $|A|^2$ and its derivatives. Note that on the homothetic solution we know that $|A|^2 = \frac{n}{F^2}$, and we search for estimates of a similar order.

We use an argument similar to Ecker's interior estimates in [3], with the difference that here we also want suitable renormalisation. The main issue becomes the question of a cutoff function. To construct this, first suppose $\overline{K} : \mathbb{R}^{n+1}_1 \to \mathbb{R}$ and define $K : M^n \times [0,T) \to \mathbb{R}$ by $K(x,t) = \overline{K}(\mathbf{F}(x,t))$.

Lemma 3.7.1. *On the flowing manifold*

$$\left(\frac{d}{dt} - \Delta\right) K = -\overline{\nabla}_\nu \overline{\nabla}_\nu \overline{K}\big|_{\mathbf{F}} - \overline{\Delta}\,\overline{K}\big|_{\mathbf{F}} \quad .$$

Proof. First

$$\frac{d}{dt}K = \left\langle \overline{\nabla}\,\overline{K}, \frac{d\mathbf{F}}{dt}\right\rangle = H\overline{\nabla}_\nu\overline{K} \quad .$$

We also calculate

$$\begin{aligned}\Delta K &= g^{ij}\left(\left\langle \overline{\nabla}\,\overline{K}, \frac{\partial^2\mathbf{F}}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial \mathbf{F}}{\partial x^k}\right\rangle + \overline{\nabla}_{\frac{\partial \mathbf{F}}{\partial x^i}}\overline{\nabla}_{\frac{\partial \mathbf{F}}{\partial x^j}}\overline{K}\right)\\ &= H\overline{\nabla}_\nu\overline{K} + g^{ij}\overline{\nabla}_{\frac{\partial \mathbf{F}}{\partial x^i}}\overline{\nabla}_{\frac{\partial \mathbf{F}}{\partial x^j}}\overline{K}\end{aligned}$$

using the Weingarten relations, equation (1.4). But now by considering locally in a suitable orthonormal coordinate system and noting the sign of ν we see

$$\Delta K = H\overline{\nabla}_\nu \overline{K} + \overline{\Delta}\,\overline{K} + \overline{\nabla}_\nu \overline{\nabla}_\nu \overline{K}$$

which gives us the Lemma. □

Now we stipulate an additional condition on $\overline{K}$, namely that $\overline{\nabla}_{\mathbf{F}}\overline{K} = 0$. That is, the cutoff function is defined on a hyperbolic plane and remains constant on rays from the origin. As in Example 1.2.5 we define $Y_\lambda = \{\mathbf{x} |\, \langle \mathbf{x}, \mathbf{x}\rangle = -\lambda^2, x_{n+1} > 0\}$, that is Y_λ is a spacelike embedding of the hyperbolic plane of "radius" λ.

Corollary 3.7.2. *Under the condition $\overline{\nabla}_{\mathbf{F}}\overline{K} = 0$ we have at $\mathbf{p} \in M$*

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right) K &\le |\nabla^{Y_F\,2}\overline{K}|^{Y_F}(\mathbf{p}) \left(\frac{S^2}{F^2} - 1\right) - \Delta^{Y_F}\overline{K}(\mathbf{p}) \\
&= |\nabla^{Y_1\,2}\overline{K}|^{Y_1}\left(\frac{\mathbf{p}}{F}\right) \frac{\frac{S^2}{F^2} - 1}{F^2} - \frac{1}{F^2}\Delta^{Y_1}\overline{K}\left(\frac{\mathbf{p}}{F}\right) \\
&\le \frac{\widetilde{C}_K}{F^2} \quad .
\end{aligned}$$

Proof. Since Y_F is perpendicular to $\mathbf{F}$ and $\overline{\Delta}\,\overline{K}$ has no contribution from the $\mathbf{F}$ direction we immediately have $\overline{\Delta}\,\overline{K} = \Delta^{Y_F}\overline{K}$. Similarly we have

$$\begin{aligned}
|\overline{\nabla}_\nu \overline{\nabla}_\nu \overline{K}\big|_{\mathbf{F}}| &= |\overline{\nabla}_{\nu - \frac{\langle \nu, \mathbf{F}\rangle}{F^2}\mathbf{F}} \overline{\nabla}_{\nu - \frac{\langle \nu, \mathbf{F}\rangle}{F^2}\mathbf{F}} \overline{K}\big|_{\mathbf{F}}| \\
&= |\overline{\nabla}^{Y_F}_{\nu + \frac{S}{F^2}\mathbf{F}} \overline{\nabla}^{Y_F}_{\nu + \frac{S}{F^2}\mathbf{F}} \overline{K}\big|_{\mathbf{F}}| \\
&\le |\nabla^{Y_F\,2}\overline{K}|^{Y_F}\left(\frac{S^2}{F^2} - 1\right)
\end{aligned}$$

by Cauchy–Schwarz, giving the first inequality.

The second is using the scaling of K on Y_F. This allows us to estimate over Y_1 rather that Y_F where F may vary from point to point. This inequality is simply from properties of dilations and the constancy of $\overline{K}$ on rays from $\mathbf{0}$: Keeping a function constant but dilating the manifold by λ while keeping $\overline{K}$ the same we get that g_{ij} becomes $\lambda^2 g_{ij}$, g^{ij} becomes $\lambda^{-2}g^{ij}$, Γ^k_{ij} remains Γ^k_{ij} and so on. This gives the stated formula.

The third of these comes from estimating second derivatives of $\overline{K}$ on Y_1 and Proposition 3.6.1. □

The final two conditions we wish $\overline{K}$ to have in addition to that of the above Corollary are:

- $0 \le \overline{K} \le C$ where $\overline{K}$ restricted to Y_1 has compact support and
- $\frac{|\nabla^{Y_1} K|^2}{K} \le C_K$ for some $C_K > 0$.

The question of whether such a function exists is easily solved. For example, if we take the Poincaré model of hyperbolic space (which is isometric to Y_1) we could take K to be the radial function $K(r) = (1 - Er^2)^3_+$. Then we calculate in this metric

$$\frac{|\nabla^{\text{Poin}} K|^2}{K} = \frac{\left(\frac{dK}{dr}\right)^2 (1-r^2)^2}{4K} = 9E^2 r^2 (1 - Er^2)_+ (1 - Er^2)$$

which is clearly bounded (depending on E) on the unit ball. Furthermore this function is zero outside a hyperbolic ball and bounded by 1, and by changing E we may choose the radius of the hyperbolic ball which is $\text{supp}(K)$.

For K satisfying the above we know

$$\begin{aligned}|\nabla K|^2 &= |\overline{\nabla}\,\overline{K}|^2 + \left\langle \overline{\nabla} K, \nu \right\rangle^2 \le |\overline{\nabla}\,\overline{K}|^2 + \left\langle \nabla^{Y_F} K, \nu - \frac{S}{F^2}\mathbf{F} \right\rangle^2 \\ &\le \frac{S^2}{F^2} |\overline{\nabla}\,\overline{K}|^2 \le C |\overline{\nabla}\,\overline{K}|^2 \le \frac{\widetilde{C}_S |\nabla^{Y_1} K|^2}{F^2}\end{aligned}$$

for $\widetilde{C}_S = 1 + \frac{C_S}{\log D_S} > 0$ where we used the Cauchy–Schwarz inequality on the hyperbolic plane and Proposition 3.6.1. Therefore, for such a function we have that at a maximum of fK, that is $f\nabla K + K \nabla f = 0$ then

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) fK &\le \frac{f\widetilde{C}_K}{F^2} + K\left(\frac{d}{dt} - \Delta\right) f - 2\left\langle \nabla K, \nabla f \right\rangle \\ &\le \frac{f\widetilde{C}_K}{F^2} + \widetilde{C}_S f \frac{|\nabla^{Y_1} K|^2}{F^2 K} + K\left(\frac{d}{dt} - \Delta\right) f \\ &\le \frac{f\widehat{C}_K}{F^2} + K\left(\frac{d}{dt} - \Delta\right) f \quad . \end{aligned} \tag{3.6}$$

Now that we have a suitable cutoff function we are ready to get some estimates:

Lemma 3.7.3. *Let $L \subset \mathbb{R}^{n+1}_1$ be such that if $\mathbf{x} \in L$ then $\lambda x \in L \quad \forall \lambda \in \mathbb{R}$, and so that $Y_1 \cap L$ is a compact set of minimum hyperbolic distance $d > 0$ from Σ with a smooth boundary. Then on $M_t \cap L$*

$$|A|^2 \le \frac{C_A}{F^2}$$

where the constant $C_A > 0$ depends on d, the second derivatives of the boundary of $Y_1 \cap L$, n and M_0.

Proof. Since we have suitable cutoff functions (if necessary just using the radial one with sufficiently large E) then the proof comes down to suitable evolution equations. We calculate using equation (1.11) that for $f_0 = |A|^2(D + 2nt)$:

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) f_0 &= 2n|A|^2 - 2(D + 2nt)(|A|^4 + |\nabla A|^2) \\ &\leq \frac{2nf_0}{D + 2nt} - \frac{2f_0^2}{D + 2nt} \quad .\end{aligned}$$

Now applying equation (3.6) we have by choosing D large enough

$$\left(\frac{d}{dt} - \Delta\right) Kf_0 \leq \frac{f_0}{F^2}\left[\widehat{C}_K + 2nK - Bf_0K\right]$$

for some $B > 0$ depending on C_1 and C_2 (see equation (3.4)). Therefore since $K = 0$ at the boundary we have the Lemma. □

Lemma 3.7.4. *For L as in the previous Lemma we have that for all $m \geq 1$ there exists a constant $C_{A,m}$ depending on m, n, L, d, M_0 and the second derivatives of the boundary of $Y_1 \cap L$ such that*

$$|\nabla^m A|^2 \leq \frac{C_{A,m}}{F^2}$$

Proof. The proof is by induction. Writing $J_1 = |A|^2(D + 2nt) + E < C_A + E$ where $E > 0$ is a constant yet to be chosen, we define

$$f_1 = (D + 2nt)|\nabla A|^2 J_1 \quad .$$

Using Proposition 1.2.16, the Cauchy Schwarz inequality and the above Lemma, writing C_n for any positive constant depending only on n and M_0 then

$$\begin{aligned}
\left(\frac{d}{dt}-\Delta\right) f_1 \leq (D+2nt)\Bigg[&\frac{2n}{D+2nt}|\nabla A|^2 J_1 + J_1\left(-2|\nabla^2 A|^2 + C_n|A|^2|\nabla A|^2\right) \\
&+ |\nabla A|^2\left(2n|A|^2 - 2(D+2nt)(|A|^4+|\nabla A|^2)\right) \\
&- 2(D+2nt)\left\langle \nabla|\nabla A|^2, \nabla|A|^2\right\rangle\Bigg] \\
\leq (D+2nt)\Bigg[&-2J_1|\nabla^2 A|^2 - 2(D+2nt)|\nabla A|^4 \\
&+ \frac{C_n(E+1)}{D+2nt}|\nabla A|^2 + 8(D+2nt)|\nabla^2 A||\nabla A|^2|A|\Bigg] \\
\leq (D+2nt)\Bigg[&-2J_1\left(|\nabla^2 A| - 2\frac{(D+2nt)}{J_1}|\nabla A|^2|A|\right)^2 \\
&+ \frac{4|\nabla A|^4|A|^2(D+2nt)^2}{(|A|^2(D+2nt)+E)^2} - 2(D+2nt)|\nabla A|^4 \\
&+ \frac{C_n(E+1)}{D+2nt}|\nabla A|^2\Bigg] \\
\leq (D+2nt)\Bigg[&\frac{C_n}{E^2}|\nabla A|^4(D+2nt) - 2(D+2nt)|\nabla A|^4 \\
&+ \frac{C_n(E+1)}{D+2nt}|\nabla A|^2\Bigg].
\end{aligned}$$

We now choose E sufficiently large that the coefficient $\frac{C_n}{E^2} \leq \frac{1}{2}$ and therefore

$$\begin{aligned}
\left(\frac{d}{dt}-\Delta\right) f_1 &\leq (D+2nt)\left[-\frac{3}{2}(D+2nt)|\nabla A|^4 + \frac{C_n}{D+2nt}|\nabla A|^2\right] \\
&\leq \frac{C_n f_1}{D+2nt} - \delta_1 f_1^2
\end{aligned}$$

for some $\delta_1 > 0$, where here again we used the bound on J_1. Substituting into equation (3.6) we see:

$$\left(\frac{d}{dt}-\Delta\right) f_1 K \leq f_1\left(\frac{C_n}{F^2} - \delta_1 K f_1\right)$$

Now we assume the Lemma holds for up to $m-1$. We define

$$J_m = |\nabla^{m-1}A|^2(D+2nt) + E < C_n + E$$

and set $f_m = (D+2nt)|\nabla^m A|^2 J_m$. Exactly as with f_1 we calculate, using the inductive hypothesis,

$$
\begin{aligned}
&\left(\frac{d}{dt}-\Delta\right)f_m \\
&\qquad \leq (D+2nt)\Bigg[-2J_m|\nabla^{m+1}A|^2+C_n(1+E)\left(\frac{|\nabla^m A|^2}{D+2nt}+\frac{|\nabla^m A|}{(D+2nt)^{\frac{3}{2}}}\right) \\
&\qquad\qquad +|\nabla^m A|^2\left(\frac{C_n}{(D+2nt)^2}-2|\nabla^m A|^2\right) \\
&\qquad\qquad +8(D+2nt)|\nabla^{m+1}A||\nabla^m A|^2|\nabla^{m-1}A|\Bigg] \\
&\qquad \leq (D+2nt)\Bigg[\frac{C_n}{E^2}|\nabla^m A|^4(D+2nt)-2|\nabla^m A|^4 \\
&\qquad\qquad +C_n(1+E)\left(\frac{|\nabla^m A|^2}{D+2nt}+\frac{|\nabla^m A|}{(D+2nt)^{\frac{3}{2}}}\right)\Bigg] \quad .
\end{aligned}
$$

Again choosing E sufficiently large and using our bounds on J_m to get

$$\left(\frac{d}{dt}-\Delta\right)f_m \leq -2\delta_m f_m^2 + C_n\frac{f_m+\sqrt{f_m}}{D+2nt} \quad .$$

for some $\delta_m > 0$ and so by equation (3.6)

$$\left(\frac{d}{dt}-\Delta\right)Kf_m \leq f_m\left(\frac{C_n}{D+2nt}-\delta_m K f_m\right)+K\left(C_n\sqrt{f_m}-\delta_m f_m^2\right) \quad .$$

For f_m larger that some constant P (depending only on n and M_0) the second bracket is negative, while the first becomes negative if Kf_m becomes large. Hence at every point on the support of K where $f_m > P$, Kf_m can have no increasing maxima, and therefore Kf_m is bounded. Therefore we have an interior bound on $\frac{|\nabla^m A|^2}{F^2}$. □

3.8 Convergence and renormalisation

The purpose of this section is to define the shape of the solution as $t \to \infty$. For this some notion of blowdown will be needed.

Definition 3.8.1. If $\mathbf{F} : M^n \times [0,\infty) \to \mathbb{R}^{n+1}_1$ satisfies equation (3.1) then let $\widehat{\mathbf{F}} = \psi(t)\mathbf{F}$ where $\psi(t)$ is some factor such that the area of $\widehat{\mathbf{F}}(M)$ is 1. For any geometric quantity f on $\mathbf{F}$ we will denote the same quantity $\widehat{f}$ on $\widehat{\mathbf{F}}$

If $\mathbf{G} : M^n \times [0,\infty) \to \mathbb{R}^{n+1}_1$ then $\mathbf{F} \to \mathbf{G}$ as $t \to \infty$ in C^0, C^1, ... if $\widehat{\mathbf{F}} \to \widehat{\mathbf{G}}$ as $t \to \infty$ in C^0, C^1, ...

Remark 3.8.2. It is usual (as in [13]) to renormalise time. Indeed we may do so here by defining $s = \int_0^t \psi(r)^2 dr$. We then obtain

$$\frac{d\widehat{\mathbf{F}}}{ds} = \frac{d\widehat{\mathbf{F}}}{dt}\frac{dt}{ds} = \psi^{-2}\left(\psi H\nu - \psi^{-1}\frac{1}{n}\frac{\int_M H^2 d\mu}{\int_M d\mu}\mathbf{F}\right) = \widehat{H}\nu + \frac{\widehat{\mathbf{F}}}{n}\int_{\widehat{M}} \widehat{H}^2 d\widehat{\mu} \quad .$$

In actual fact Lemma 3.8.3 will show that $s \geq C\,\log(t)$ and hence we need not make a distinction between $s \to \infty$ and $t \to \infty$.

We now estimate the quantity ψ.

Lemma 3.8.3. *There exist constants $C_Y, \widetilde{C}_Y > 0$ such that*

$$\frac{C_Y}{F} \leq \psi(t) \leq C_Y \frac{\sqrt{1 + \frac{C_S}{\log(\widetilde{D}_S + F^2)}}}{F} \leq \frac{\widetilde{C}_Y}{F} \quad .$$

Proof. Let $\mathbf{Y}$ be a parametrisation of Y_1. Then any spacelike manifold contained within the lightcone may be written as $\mathbf{Z} = u(x)\mathbf{Y}(x)$. Hence we get the following induced metric:

$$g^Z_{ij} = \left\langle u\frac{\partial \mathbf{Y}}{\partial x^i} + D_i u\mathbf{Y}, u\frac{\partial \mathbf{Y}}{\partial x^j} + D_j u\mathbf{Y}\right\rangle = u^2 g^Y_{ij} - D_i u D_j u \quad .$$

We see that

$$\begin{aligned} g^Z_{ij}&\left(g_Y^{jk} + \frac{D_a u g_Y^{aj} D_b u g_Y^{bk}}{u^2 - |\nabla^Y u|^2}\right) \\ &= u^2\delta_i^k + D_i u D_p u g^{pk}\left(\frac{u^2}{u^2 - |\nabla^Y u|^2} - 1 - \frac{|\nabla^Y u|^2}{u^2 - |\nabla^Y u|^2}\right) \\ &= u^2 \delta_i^k \quad . \end{aligned}$$

Therefore

$$g^{ij}_Z = \frac{1}{u^2}\left(g_Y^{ij} + \frac{D_a u g_Y^{ai} D_b u g_Y^{bj}}{u^2 - |\nabla^Y u|^2}\right) \quad .$$

We calculate for $A_i^j = \frac{1}{u^2} g^Z_{ia} g_Y^{aj} = \delta_i^j - \frac{D_i u D_a u g_Y^{aj}}{u^2}$ that there are $n-1$ eigenvectors of eigenvalue 1 while the remaining eigenvector is in the direction $D_a u g_Y^{ai}$ and we see

$$A_i^j g_Y^{ia} D_a u = D_a u g_Y^{aj}\left(1 - \frac{|\nabla^Y u|^2}{u^2}\right)$$

and therefore calculate

$$\det g^Z_{ij} = (u^2)^n \det(A_i^j)\det(g^Y_{ij}) = (u^2)^{n-1}(u^2 - |\nabla^Y u|^2)\det(g^Y_{ij}) \quad .$$

But now we calculate on our manifold:

$$\begin{aligned}|\nabla u^2|^2 &= 4u^2 D_i u g_Z^{ij} D_j u\\ &= 4\left(|\nabla^Y u|^2 + \frac{|\nabla^Y u|^4}{u^2 - |\nabla^Y u|^2}\right)\\ &= \frac{4u^2|\nabla^Y u|^2}{u^2 - |\nabla^Y u|^2}\end{aligned}$$

this gives

$$|\nabla^Y u|^2 = \frac{u^2|\nabla u^2|^2}{4u^2 + |\nabla u^2|^2} \quad .$$

We may write the area of the manifold as the integral over the interior of Σ intersected with Y_1. We call this set B. Therefore

$$\begin{aligned}\int_Z d\mu &= \int_B u^n \sqrt{1 - \frac{|\nabla^Y u|^2}{u^2}} d\mu_{Y_1} = \int_B u^n \sqrt{1 - \frac{|\nabla u^2|^2}{4u^2 + |\nabla u^2|^2}} d\mu_{Y_1}\\ &= \int_B u^n \sqrt{\frac{4u^2}{4u^2 + |\nabla u^2|^2}} d\mu_{Y_1} = \int_B u^n \sqrt{\frac{1}{1 + \frac{|\nabla u^2|^2}{4u^2}}} d\mu_{Y_1} \quad .\end{aligned}$$

Applying this to our flowing manifold, then we have $u = F$ and so using Proposition 3.6.1 and equation (3.4) we see for t large enough

$$C_Y^{-\frac{1}{n}} F^n \sqrt{\frac{1}{1 + \frac{C_S}{\log(\widetilde{D}_S + F^2)}}} \leq \int_M d\mu \leq C_Y^{-\frac{1}{n}} F^n$$

where $C_Y = \left(\int_B d\mu_{Y_1}\right)^{-n}$. Noting that $\int_{\lambda M} d\mu = \lambda^n \int_M d\mu$ which implies $\psi = \left(\int_M d\mu\right)^{-\frac{1}{n}}$ we have the Lemma. □

Theorem 3.8.4. *Any initially spacelike solution of equation (3.1) with a convex cone boundary condition will converge as time tends towards infinity to some* $\mathbf{G}_{R_\infty}$ *in the* C^1 *norm. Furthermore, on any interior set uniformly away from the boundary there exists an increasing sequence of* $t_i \to \infty$ *such that* $\widehat{M}_{t_i}$ *converge to the solution on the interior in the* C^∞ *topology.*

Proof. Under $\mathcal{D}_\lambda$, a dilation by a factor λ, we have $\mathcal{D}_\lambda(F^2) = \lambda^2 F^2$, $\mathcal{D}_\lambda S = \lambda S$, and so on. Hence from equation (3.4), Proposition 3.6.1, Corollary 3.6.2 and the above estimates on the dilation factor then we get

$$\begin{array}{ccccc}\widehat{C}_1 & \leq & \widehat{F}^2 & \leq & \widehat{C}_2\\ 0 & \leq & |\nabla \widehat{F}^2|^2 & \leq & \frac{\widehat{C}_S}{\log(\widetilde{D}_S + F^2)}\\ \widehat{C}_1^H & \leq & \widehat{H} & \leq & \widehat{C}_2^H\end{array}$$

where $\widehat{C}_1, \widehat{C}_2, \widehat{C}_S, \widehat{C}_2^H > 0$ and $\widehat{C}_1^H > 0$ if $C_1^H > 0$ and all of these constants depend only on n and M_0. The first of these is boundedness of the renormalised hypersurface, the second implies that in fact we have C^1 convergence to a hypersurface with $|\nabla F^2| = 0$. Therefore these estimates imply C_1 convergence of $\widehat{M}$ to Y_R, the hyperbolic hyperplane of radius

$$R_\infty = \left(\int_B d\mu_{Y_1}\right)^{-\frac{1}{n}}$$

where B is as in the above Lemma.

By Lemmas 3.7.3 and 3.7.4 we have that for L as in Lemma 3.7.3, then for all time on $M_t \cap L$

$$\begin{array}{rcccl} \max\left\{\frac{\widehat{C}_1^H}{n}, 0\right\}^2 & \leq & |\widehat{A}|^2 & \leq & \widehat{C}_A \\ 0 & \leq & |\nabla^m \widehat{A}|^2 & \leq & \widehat{C}_{A,m} \end{array}$$

where the constant depends on the boundary of L and how far L is from Σ.

We can now use Arzelá–Ascoli and a diagonal argument to complete the theorem:

Let U_j be an open interior set of Y_1 such that the boundary is of at most hyperbolic distance $2^{-(j+k)}$ from the Σ and at least $2^{-(j+k)-1}$ such that the boundary of U_j is C^∞. We note $U_j \subset U_{j+1}$ and choose k sufficiently large that $U_1 \neq \emptyset$. We define $L_i = \{\xi \mathbf{x} | \mathbf{x} \in U_i, \xi > 0\}$ and we now construct for a 2 parameter set of sequences $t_i^{(a,b)}$ with the properties:

1. $t_i \to \infty$ as $i \to \infty$
2. $|\nabla^b \widehat{A}|^2\big|_{L_a \cap M_{t_i^{(a,b)}}}$ converges to the corresponding value on Y_R (that is 0 for $b > 0$ and $\frac{n}{R^2}$ for $b = 0$) as $i \to \infty$ and
3. If we write $\sqsubset$ for "is a subsequence with respect to i of" then the following diagram holds

$$\begin{array}{ccc} f_i^{(a+1,b+1)} & \sqsubset & f_i^{(a+1,b)} \\ \sqcap & & \sqcap \\ f_i^{(a,b+1)} & \sqsubset & f_i^{(a,b)} \end{array} .$$

First, by Arzelá–Ascoli and the equicontinuity of $\widehat{h}_{ab}\big|_t$ on L_1 (which comes from the bound on $|\nabla \widehat{A}|^2$) we know there exists an increasing sequence of $t_i^{(1,1)}$, such that $\widehat{h}_{ab}\big|_{t_i^{(1,1)}}$ uniformly converges to a C_0 function on L_1. Furthermore, this function must be $\frac{1}{R} g_{ij}^{Y_R}$ since otherwise integrating would contradict the C^1 convergence to Y_{R_∞}.

We proceed by induction. Suppose that a $m \times m$ "square" of inclusions is defined such that conditions 1–3 above are all satisfied. The most restricted subsequence so far will be $t_i^{(m,m)}$.

Also by Arzelá–Ascoli we may take a subsequence $t_i^{(m+1,1)} \sqsubset t_i^{(m,m)}$ such that h_{ab} converges uniformly on L_{m+1}. Now taking a subsequence of this, $t_i^{(m+1,2)}$ we have convergence of $\nabla_c h_{ab}$ on $L_{m+1} \cap M_{t_i^{(m+1,2)}}$. We may continue along this "row" up to the convergence of $\nabla^m A$ under the subsequence $t_i^{(m+1,m)}$. Note that condition 3 of the above will automatically be satisfied by these sequences, by our choice of initial sequence.

$$
\begin{array}{ccccc}
f_i^{(m+1,m)} & \sqsubset & \dots & \sqsubset & f_i^{(m+1,1)} \\
\sqcap & & & & \sqcap \\
f_i^{(m,m)} & \sqsubset & \dots & \sqsubset & f_i^{(m,1)} \\
\sqcap & & & & \sqcap \\
\vdots & & \ddots & & \vdots \\
\sqcap & & & & \sqcap \\
f_i^{(1,m)} & \sqsubset & \dots & \sqsubset & f_i^{(1,1)}
\end{array} .
$$

Next we deal with the column. Again we start with a subsequence of the most restricted subsequence, that is, we choose $t_i^{(1,m+1)} \sqsubset t_i^{(m+1,m)}$ such that $\nabla^{m+1} A$ converges to zero on L_1. Taking $t_i^{(2,m+1)} \sqsubset t_i^{(1,m+1)}$ and so on, we define up to $t_i^{(m+1,m+1)}$. The condition 3 is automatically satisfied and so the construction is complete by induction and repeated use of Arzelá Ascoli.

Now by abuse of notation, choose $t_i = t_i^{(i,i)}$ then as $i \to \infty$, $t_i \to \infty$ and $\widehat{M}_{t_i} \to Y_R$ in C^p for all p. $\square$

Chapter 4

Graphs in Minkowski space

In this chapter we deal with graphical mean curvature flow in Minkowski space with a Neumann boundary condition. This gives that the boundary manifold Σ is a cylinder.

This work came about as an attempt to take the graphical results on mean curvature flow with a Neumann boundary condition from Euclidean space into Minkowski space. In an ideal world it would be nice to imitate the integral methods of Huisken in [14] and apply Stampaccia iteration. Indeed this is the method I initially followed. Alas, due to issues at the boundary (see Section 4.3) it became necessary to impose weak convexity on the domain. Given this condition, these methods yielded the first proof of the gradient estimate I give here. In actual fact this assumption means that there is a much simpler maximum principle argument which gives the same result. This is the second proof of the gradient estimate.

Let Ω be a compact domain in $\mathbb{R}^n$ with the Euclidean metric considered as a subspace of Minkowski space with the induced metric. For example in coordinates as in Example 1.1.4 we may take $\Omega \subset \text{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Let $\partial\Omega$ be $C^{2,\alpha}$ and define $\Sigma = \{(\mathbf{x}, y) \in \mathbb{R}^{n+1}_1 | \mathbf{x} \in \partial\Omega,\ y \in \mathbb{R}\}$ to be the cylinder over Ω. We will define γ to be the outward pointing normal to $\partial\Omega$ so that $\mu = (\gamma^1, \ldots, \gamma^n, 0)$ is the outward unit normal to Σ.

As in the previous chapter we let $M^n = B^n$, and wish to consider solutions to equation (1.9) with the Σ specified above and $\beta = 0$. Again we will require that our initial embedding $\mathbf{F}_0 : M^n \to \mathbb{R}^{n+1}_1$ is spacelike.

The spacelikeness condition implies we are able to consider $\mathbf{F}_0$ as a graph $u_0 : \Omega \to \mathbb{R}$ initially with the derivative bound $|Du_0| < 1$. Using notation similar to [8] and [14], we define

$$
\begin{aligned}
v &= \sqrt{1-|Du|^2} \\
\mathbf{a}(\mathbf{q}) &= \frac{\mathbf{q}}{\sqrt{1-|\mathbf{q}|^2}} \\
a^{ij} = \frac{\partial a^i}{\partial q^j} &= \frac{\delta_{ij}}{\sqrt{1-|\mathbf{q}|^2}} + \frac{q^i q^j}{(1-|\mathbf{q}|^2)^{\frac{3}{2}}} \ .
\end{aligned}
$$

We will always take $\mathbf{a} = \mathbf{a}(Du)$ and $a^{ij} = a^{ij}(Du)$, that is $\mathbf{q} = Du$ in the above. The function v is called the gradient function (equivalent functions are commonly used in Euclidean space, see for example [4]). Using these quantities we may rewrite equation (1.9) in graphical coordinates as follows:

$$
\begin{cases}
\frac{du}{dt}(\mathbf{x},t) = vD_i\left(a^i\right)(\mathbf{x},t) & \forall(\mathbf{x},t) \in \Omega \times [0,T] \\
u(\mathbf{x},0) = u_0(\mathbf{x}) & \forall \mathbf{x} \in \Omega \\
\gamma^i a^i(\mathbf{x},t) = 0 = D_i u \gamma^i & \forall(\mathbf{x},t) \in \partial\Omega \times [0,T] \quad .
\end{cases} \tag{4.1}
$$

The derivation of this is contained in Appendix B. While we will not do these standard computations here, we state that

$$
\nu = a^i \mathbf{e}_i + \frac{1}{v}\mathbf{e}_{n+1}, \quad g^{ij} = v a^{ij} \text{ and } H = D_i(a^i) \quad .
$$

Remark 4.0.5. The uniform parabolicity of equation (4.1) is equivalent to the bound $|Du| < 1$ which is in turn equivalent to spacelikeness. Therefore if we have such an estimate on $[0,T]$ for bounded u we have both uniform parabolicity of the above and a gradient estimate and may apply the quasi linear existence theory of Chapter 2 to get existence of a smooth solution.

We will show the following:

Theorem 4.0.6. *A smooth solution to equation (4.1) exists for all time and converges to a constant solution $u = C$ as $t \to \infty$.*

4.1 A gradient estimate

We will need an equation for the evolution of the gradient, v.

Lemma 4.1.1. *If we consider v as a function over Ω then*

$$\frac{\partial v}{\partial t} = -a^i D_i(Hv) \tag{4.2}$$

$$= -Ha^i D_i v + vD_j(a^{jk} D_k v) + va^{jk} D_{kr} u a^{rt} D_{tj} u \quad . \tag{4.3}$$

Proof. By definition of v we have

$$\frac{\partial v}{\partial t} = -\frac{D_i(\frac{du}{dt})D_i u}{\sqrt{1-|Du|^2}}$$
$$= -a^i D_i(Hv) \quad .$$

The second equality comes from expressing H as $D_i(a^i)$ and interchanging derivatives:

$$-a^i D_i(Hv) = -Ha^i D_i v - va^i D_i(D_j a^j)$$
$$= -Ha^i D_i v - va^i D_j(a^{jk} D_{ki} u) \quad .$$

But

$$D_j(a^{jk} D_k v) = -D_j(a^{jk} D_{kr} u a^r) = -a^i D_j(a^{jk} D_{ki} u) - a^{jk} D_{kr} u a^{rt} D_{tj} u$$

hence

$$\frac{\partial v}{\partial t} = -Ha^i D_i v + vD_j(a^{jk} D_k v) + va^{jk} D_{kr} u a^{rt} D_{tj} u \quad .$$

□

A vital question is what happens at the boundary, answered by the following boundary Lemma:

Lemma 4.1.2. *Suppose Ω is convex then we have that for all $\mathbf{x} \in \partial\Omega$ that*

$$\gamma^i a^{ik} D_k v \geq 0 \quad .$$

Proof. This proof is based on [8, Lemma 1.2], where here we additionally use that convexity of the domain implies that the second fundamental form with of $\partial\Omega$ with respect to γ is positive definite. At a point $x \in \partial\Omega$ via a linear orthogonal transformation in $\mathbb{R}^n$ we may take $\gamma(x)$ to be $\mathbf{e}_n$. We want to differentiate the boundary condition

$$a^i \gamma^i = 0 \quad .$$

Locally we may consider the boundary as a graph $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$. We know for $j = 1, \ldots, n-1$ that at the point x

$$0 = D_j(a^i\gamma^i) = D_{kj}ua^{ik}\gamma^i + a^iD_j\gamma^i \quad .$$

Furthermore, by the boundary condition we know that at x, $a^n = 0$ and so

$$\begin{aligned} 0 &= a^jD_j(a^i\gamma^i) \\ &= a^jD_{jk}ua^{ki}\gamma^i + a^iD_j\gamma^ia^j \\ &= -D_kva^{ik}\gamma^i + a^iD_j\gamma^ia^j \quad . \end{aligned}$$

The Lemma is reduced to the question of a sign on $a^iD_j\gamma^ia^j$. Again using that $a^n = 0$ at x, we see that we only need consider $a^\alpha D_\beta\gamma^\alpha a^\beta$ where Greek indices imply summations up to $n-1$. Using the graph ω we calculate

$$\gamma = \frac{1}{\sqrt{1+|D\omega|^2}}(-D_\alpha\omega e_\alpha + e_n)$$

so

$$D_\alpha\gamma^\beta = \frac{-D_{\alpha\beta}\omega}{\sqrt{1+|D\omega|^2}} + \frac{D_\alpha\omega D_{\beta\eta}\omega D_\eta\omega}{(1+|D\omega|^2)^{\frac{3}{2}}} = h^\Sigma_{\alpha\eta}g^{\eta\beta}_\Sigma \quad .$$

At our point x, $D\omega = 0$ from choice of coordinates and hence at this point $g^{\eta\beta}_\Sigma = \delta_{\eta\beta}$ and so by convexity $a^\alpha D_\beta\gamma^\alpha a^\beta \geq 0$, and the Lemma holds at x. Since x was an arbitrary point this is true on all the boundary. □

We will need the following Lemma on the time derivatives of certain L^p norms:

Proposition 4.1.3. *Let M^n be a compact n-dimensional manifold with boundary and let $\mathbf{F} : M^n \times [0,T) \to \mathbb{R}^{n+1}_1$ be a smoothly varying family of smooth space-like embeddings. Suppose $g : M^n \times [0,T) \to \mathbb{R}$ is a positive function differentiable in time and continuous in space and $p > 0$ is a constant. Set $g_k = (g-k)_+$ and $G(t) = \sup_{x\in M^n} g(x,t)$. Suppose that there exists an $\epsilon > 0$ such that* for all $k \in ((G(t)-\epsilon)_+, G(t))$

$$\frac{d}{dt}\left(\int_{M^n} g^p_k d\mu\right) \leq 0 \tag{4.4}$$

then $G(t)$ is non increasing for all $t \in (0,T)$.

Proof. Suppose not. Then at some time $\tau > 0$ there exists a δ and an $0 < \widetilde{\epsilon} < \frac{\epsilon}{4}$ such that

$$G(\tau + \delta) = G(\tau) + \widetilde{\epsilon}$$

and for $t \in (\tau, \tau+\delta)$, $G(t) \in (G(\tau), G(\tau+\delta))$ (we know that G is continuous). Since M^n is compact then there exists an $x \in M^n$ such that $g(x, \tau + \delta) = G(\tau + \delta)$. We will show that g cannot be continuous in space at $(x, \tau + \delta)$.

Set

$$X = \left\{ x \in M^n : g(x, \tau + \delta) > G(\tau) + \frac{\widetilde{\epsilon}}{2} \right\} \quad .$$

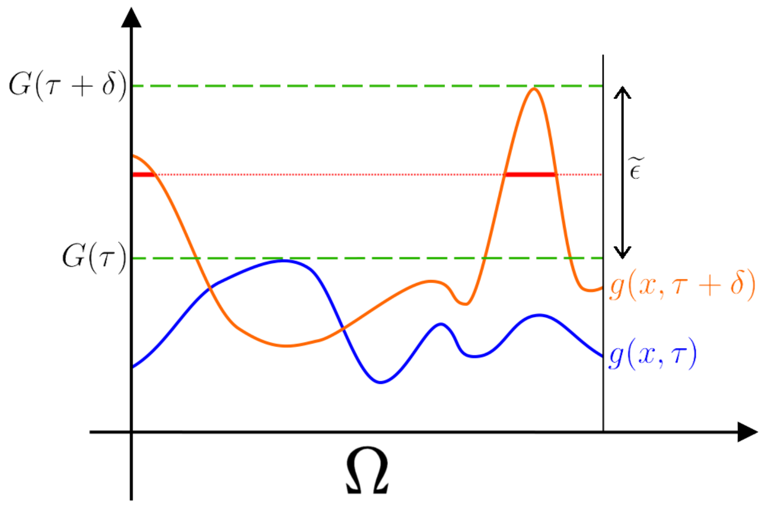


Figure 4.1: A picture of an impossible situation: Here we see the set up for the contradiction. The set X is shown in thick red.

Due to our choice of $\widetilde{\epsilon}$ we have that equation (4.4) holds for all $(k, t) \in (G(\tau) - \frac{\epsilon}{2}, G(\tau)) \times (\tau, \tau + \delta)$. Hence for k in this region we may integrate:

$$\begin{aligned} \int_\Omega g_k^p d\mu \Big|_{t=\tau} &\geq \int_\Omega g_k^p d\mu \Big|_{t=\tau+\delta} \\ &\geq \left(G(\tau) + \frac{\widetilde{\epsilon}}{2} - k \right)^p \mu|_{t=\tau+\delta}(X) \quad . \end{aligned}$$

Hence since $G(\tau) > k$ we have the estimate

$$\int_\Omega g_k^p d\mu \Big|_{t=\tau} \geq \left(\frac{\widetilde{\epsilon}}{2} \right)^p \mu|_{t=\tau+\delta}(X) \quad .$$

The righthand side of this does not depend on k, while the left hand side tends towards 0 as $k \to G(\tau)$. Hence X is of measure zero with respect to μ at time $\tau+\delta$ and so cannot contain any open sets, implying g is not continuous in space. $\square$

We may now use this to give the following.

Proposition 4.1.4. Gradient Estimate *Given spacelike u_0 over a compact convex domain Ω then for all the time a solution exists the gradient is bounded from below by its initial minimum.*

Proof. The above is equivalent (see e.g. [20]) to finding a bound such that $|Du| < 1$ for all time. We define

$$w = log\left(\frac{1}{v}\right) = -log(v)$$

and

$$w_k = \max\{0, w-k\} \quad .$$

We wish to bound w from above. To this end, defining $A(k) = \{x \in \Omega | w_k(x) > 0\}$ then

$$\begin{aligned}
\frac{d}{dt}\int_\Omega w_k^2 dH_n &= \frac{d}{dt}\int_\Omega w_k^2 v dx \\
&= 2\int_\Omega w_k \frac{dw}{dt} v dx + \int_\Omega w_k^2 \frac{dv}{dt} dx \\
&= -2\int_\Omega w_k \frac{dv}{dt} dx + \int_\Omega w_k^2 \frac{dv}{dt} dx \\
&= -2\int_\Omega w_k(-Ha^i D_i v + vD_j(a^{jp}D_p v) + va^{jq}D_{qr}ua^{rt}D_{tj}u)dx \\
&\qquad - \int_\Omega w_k^2 a^i D_i(Hv)dx \\
&= 2\int_\Omega w_k Ha^i D_i v dx - 2\int_{A(k)} w_k v D_j(a^{jp}D_p v)dx \\
&\qquad - 2\int_\Omega w_k va^{jq}D_{qr}ua^{rt}D_{tj}udx - \int_{A(k)} w_k^2 a^i D_i(Hv)dx \quad .
\end{aligned}$$

Now using Divergence Theorem on the second and fourth terms (and using the boundary condition) then

$$\begin{aligned}\frac{d}{dt}\int_\Omega w_k^2 dH_n &= 2\int_\Omega w_k H a^i D_i v dx + 2\int_{A(k)} D_j(w_k v) a^{jp} D_p v dx \\ &\quad - 2\int_{\partial\Omega\cap A(k)} \gamma^j a^{jl} D_l v w_k v dH_{n-1} - 2\int_\Omega w_k v a^{js} D_{sr} u a^{rt} D_{tj} u dx \\ &\quad + \int_{A(k)} D_i(w_k^2 a^i) H v dx \\ &= 2\int_\Omega w_k H a^i D_i v dx - 2\int_{A(k)} D_j v a^{jp} D_p v dx + 2\int_\Omega w_k D_j v a^{jp} D_p v dx \\ &\quad - 2\int_{\partial\Omega} \gamma^j a^{jl} D_l v w_k v dx - 2\int_\Omega w_k v a^{js} D_{sr} u a^{rt} D_{tj} u dx \\ &\quad - 2\int_\Omega w_k H a^i D_i v dx + \int_\Omega w_k^2 H^2 v dx \\ &= \int_\Omega w_k^2 H^2 v dx + 2\int_\Omega w_k \frac{|\nabla v|^2}{v} dx - 2\int_{\partial\Omega} \gamma^j a^{jl} D_l v w_k v dx \\ &\quad - 2\int_{A(k)} \frac{|\nabla v|^2}{v} dx - 2\int_\Omega w_k v |A|^2 dx \quad .\end{aligned}$$

But now using Lemma 4.1.2, that is $\gamma^j a^{jk} D_k v > 0$, and the inequality $\frac{H^2}{n^2} \leq |A|^2$ then

$$\frac{d}{dt}\int_\Omega w_k^2 dH_n \leq \int_\Omega w_k(w_k - \frac{2}{n^2}) H^2 v dx + 2\int_{A(k)} (w_k - 1)\frac{|\nabla v|^2}{v} dx \quad .$$

Hence if $w_k \leq \frac{2}{n^2}$

$$\frac{d}{dt}\int_\Omega w_k^2 dH_n \leq 0 \quad .$$

Now we can apply Proposition 4.1.3 (with $\epsilon = \frac{1}{n^2}$) to give that $W(t) = \sup_{x\in\Omega} w(x,t)$ is nonincreasing, which is to say that the gradient is bounded by its initial value. □

4.2 Gradient estimate via maximum principle

The above estimate is also attainable using a maximum principle method. Here we use such a method to find an upper bound on $Y = \frac{1}{v}$. The evolution equation of Y on the flowing manifold was calculated in [3], namely

$$\left(\frac{d}{dt} - \Delta\right) Y = -|A|^2 Y \quad . \tag{4.5}$$

As usual we need a boundary derivative and since

$$\frac{\partial}{\partial x^i}\left(-\langle e_{n+1},\nu\rangle\right) = -\left\langle \overline{\nabla}\nu, e_{n+1}^T\right\rangle - \left\langle \nu, \overline{\nabla} e_{n+1}\right\rangle$$
$$= -A(e_{n+1}^T, \frac{\partial}{\partial x^i})$$

we get

$$\nabla_\gamma Y = -A(e_{n+1}^T,\gamma) \quad .$$

As in Lemma 3.4.2 at a point $\mathbf{p}\in\partial M$ we differentiate the boundary condition in direction $W\in T_{\mathbf{p}}\partial M$ to get

$$0 = \langle\nabla_W\nu,\gamma\rangle + \langle\nu,\nabla_W\gamma\rangle = A(W,\gamma) + A^\Sigma(W,\nu) \quad .$$

We see

$$\nabla_\gamma Y = A^\Sigma(e_{n+1}^T,\nu) = A^\Sigma(e_{n+1}-Y\nu,\nu) = -YA^\Sigma(\nu,\nu)\le 0 \tag{4.6}$$

where we used that $\mathbf{e}_{n+1}$ is a zero eigenvector of $A^\Sigma(\cdot,\cdot)$ and the convexity of the boundary manifold. Applying maximum principle (Lemma 1.2.17) immediately gives the an alternative proof of Proposition 4.1.4.

4.3 Boundary issues on more general domains

The original proof was an attempt at getting gradient estimates on more general domains, as in [14]. This method doesn't work here for reasons of the ambient space: The lengths of projected vectors have to be estimated using $Y=\frac{1}{v}$, and it is necessary to project at the boundary. We consider the boundary integral from the proof of Theorem 4.1.4 and using equation (4.6)

$$-\int_{\partial\Omega} 2w_k D_j v a^{jl}\gamma^l v d\mathcal{H}_{n-1} = -\int_{\partial M} 2w_k\frac{\nabla_\gamma v}{v}d\mu_{\partial M} = -\int_{\partial M} 2w_k A^\Sigma(\nu,\nu)d\mu_{\partial M}$$

where we used that v is the volume element of the boundary due to the boundary condition. We must estimate this from above. Using that $\mathbf{e}_{n+1}$ is an eigen vector of $A^\Sigma(\cdot,\cdot)$, we estimate $A^\Sigma(\nu,\nu)\Big|_{\mathbf{x}\in\partial M}\ge f(\mathbf{x})|\nu-\frac{1}{v}\mathbf{e}_{n+1}|^2 = f(\mathbf{x})\frac{1-v^2}{v^2}$. Therefore we must estimate

$$-\int_{\partial\Omega} w_k A^\Sigma(\nu,\nu)v d\mathcal{H}_{n-1} \le C\int_{\partial\Omega} w_k\frac{1-v^2}{v}d\mathcal{H}_{n-1} = C\int_{\partial\Omega} w_k e^{w_k+k} - vw_k d\mathcal{H}_{n-1} \quad .$$

This estimate is a good one, for example for $n = 2$ in the domain pictured supposing $|Du| = 0$ at the boundary on the portions marked in blue we have equality in this estimate for $C = 1$.

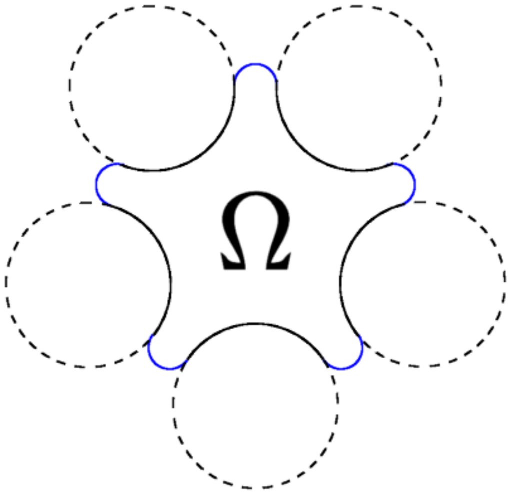


Figure 4.2: On the holly leaf domain $\Omega \subset \mathbb{R}^2$ we consider u with $Du = 0$ on the blue regions while Du may be large elsewhere

From here we must estimate this on the interior using some Lemma similar to [8, Lemma 1.4]. The above boundary integral makes this extremely difficult – estimating into the interior using methods like the mentioned Lemma would give terms like $\int_\Omega w_k e^{w_k+k} d\mu$ – a term that is of exponentially larger order than anything we can get from the evolution of w_k. This anecdotal evidence suggests that to get gradient estimates we must use either conditions on the boundary such as convexity, or further conditions on the initial manifold to get around this problems.

The issue here is exactly one of the geometry of the ambient space: In Euclidean space we may estimate $A^\Sigma(\nu,\nu) < C$ while here, due to the fact that the projection of a vector can be longer than the original vector we must estimate $A^\Sigma(\nu,\nu) < \frac{C}{v^2}$ which is too large. I suspect this estimate will always be a problem when using integral methods to get gradient estimates with boundary manifolds of indefinite metric.

4.4 Long time existence and convergence

I include the proof of a result from analysis which we will need.

Lemma 4.4.1. *Suppose Ω is compact and $f : \Omega \times [0,\infty) \to \mathbb{R}$ is a C^1 function such* *that there exists constants $C, \overline{C} > 0$ so that $|Df|(x,t) < \overline{C}$ and*

$$\int_0^\infty \int_\Omega |Df|^2 + \left(\frac{df}{dt}\right)^2 dx\, dt \leq C \tag{4.7}$$

then $f(\cdot, t)$ tends uniformly towards $f_\Omega(t)$, the average of f over Ω at each time, as $t \to \infty$.

Proof. Suppose not. Then there exists an $\epsilon > 0$ a sequence $\widehat{t_j}$ such that $(f - f_\Omega)|_{\widehat{t_j}} > \epsilon$. Without loss of generality we may assume that $\widehat{t}_{j+1} - \widehat{t_j} > 1$.

I now claim there is a sequence $t_i \in [\widehat{t_i}, \widehat{t_i} + 1]$ such that $\int_\Omega |Df|^2 dx|_{t_i} \to 0$. For otherwise there exists a subsequence $\widehat{t}_{j(i)}$ such that for all $t \in [\widehat{t}_{j(i)}, \widehat{t}_{j(i)} + 1]$, $\int_\Omega |Df|^2 dx|_t > \epsilon$. But then we have

$$\int_0^\infty \int_\Omega |Df|^2 dx\, dt \geq \sum_{i=1}^\infty \int_{\widehat{t}_{j(i)}}^{\widehat{t}_{j(i)}+1} \int_\Omega |Df|^2 dx\, dt \geq \sum_{i=1}^\infty \epsilon$$

contradicting (4.7).

By the Poincaré inequality (see [21, Lemma 1.65], for example) for some $C > 0$ depending on Ω,

$$\int_\Omega (f_\Omega - f)^2 dx|_{t_i} \leq C \int_\Omega |Df|^2 dx|_{t_i} \to 0 \quad .$$

Hence we have that $f(\cdot, t_i) \to f_\Omega(t_i)$, firstly almost everywhere, but then by the bound on space derivatives, uniformly. We now see

$$\left(\int_\Omega f(\cdot, \widehat{t_i}) - f(\cdot, t_i) dx\right)^2 \leq \left(\int_{\widehat{t_i}}^{\widehat{t_i}+1} \int_\Omega \left|\frac{df}{dt}\right| dx\, dt\right)^2 \leq |\Omega| \int_{\widehat{t_i}}^{\widehat{t_i}+1} \int_\Omega \left(\frac{df}{dt}\right)^2 dx\, dt$$

by the Hölder inequality. Again by summing over i to avoid contradicting (4.7) we have that this integral tend to 0 as $i \to \infty$. Hence we have that $f(\cdot, \widehat{t_i}) \to f(\cdot, t_i)$, first almost everywhere, then as before uniformly which in turn implies that $f(\cdot, \widehat{t_i}) \to f_\Omega(t_i)$ uniformly. Therefore $f_\Omega(\widehat{t_i}) \to f_\Omega(t_i)$ and so $f(\cdot, \widehat{t_i}) \to f_\Omega(\widehat{t_i})$, a contradiction. We conclude there exists no such $\widehat{t_i}$. □

We will now address the question of convergence.

Theorem 4.4.2. *Equation (4.1) has a smooth solution for all time converging to a flat solution $u = c$ for some constant c.*

Proof. This is a very similar argument to in [14]. We already have bounds on the gradient of u. A C^0 bound on u is rapidly obtained by quasi-linear comparison principle (proof of this almost identical to [9, Theorem 9.2]) with the maximal surface solutions $\hat{u} = c$ where c is some constant. By standard quasi-linear existence theory (again see Chapter 2) we have a smooth solution to equations 4.1 with $T = \infty$. By Divergence Theorem and Lemma 4.1.1

$$\begin{aligned}\frac{d}{dt}\int_\Omega v dx &= -\int_\Omega a^i D_i(Hv) dx \\ &= \int_\Omega H^2 v dx \quad .\end{aligned}$$

Hence since $v \leq 1$, we have

$$\begin{aligned}\int_0^T \int_\Omega |\frac{du}{dt}|^2 dx dt &= \int_0^T \int_\Omega H^2 v^2 dx dt \\ &\leq \int_0^T \int_\Omega H^2 v dx dt \\ &= \int_\Omega v dx\bigg|_{t=T} - \int_\Omega v dx\bigg|_{t=0} \\ &\leq \int_\Omega v dx\bigg|_{t=T} \\ &\leq C_1\end{aligned}$$

where C_1 is a constant depending on the gradient estimate and $|\Omega|$. Again using the gradient estimate we calculate the following:

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^2 v dx &= 2\int_\Omega uHv^2 dx - \int_\Omega u^2 a^l D_l(Hv) dx \\ &= 2\int_\Omega uHv^2 dx + \int_\Omega u^2 H^2 v dx + \int_\Omega 2uHva^i D_i u dx \\ &= 2\int_\Omega Hu dx + \int_\Omega u^2 H^2 v dx \\ &= 2\int_\Omega D_i(a^i) u dx + \int_\Omega u^2 H^2 v dx \\ &= \int_\Omega u^2 H^2 v dx - 2\int_\Omega \frac{|Du|^2}{v} dx\end{aligned}$$

where we used Divergence Theorem on the first and fourth lines. Integrating with time and using our C^0 bound on u we have

$$\begin{aligned}\int_0^T\int_\Omega \frac{|Du|^2}{v}dxdt &\leq \int_0^T\int_\Omega u^2H^2vdx + \int_\Omega u^2vdx\Big|_{t=0}\\ &\leq C_2\int_0^T\int_\Omega H^2vdxdt + \int_\Omega u^2vdx\Big|_{t=0}\\ &\leq C_2\int_\Omega vdx\Big|_{t=T} + \int_\Omega u^2vdx\Big|_{t=0}\\ &\leq C_3 \quad .\end{aligned}$$

Since none of the constants above depend on T we deduce for some $\widehat{C} > 0$,

$$\int_0^\infty\int_\Omega |Du|^2 + |\frac{du}{dt}|^2dxdt < \widehat{C} \quad .$$

Therefore, by Lemma 4.4.1 u converges uniformly to some constant $C(t) = u_\Omega(t)$, possibly varying in time. Since by the comparison principle, $\inf_{x\in\Omega} u(x,t)$ is nondecreasing and $\sup_{x\in\Omega} u(x,t)$ is nonincreasing, we in fact see that $C(t)$ must converge uniformly to a constant function c and we are done. $\square$

Chapter 5

The constant prescribed boundary angle problem for mean curvature flow

In this chapter I will give some results on mean curvature flow in Euclidean space, although this time the boundary condition is not necessarily perpendicular: The angle between ν and μ will be specified to be some constant to close to $\frac{\pi}{2}$.

Let M^n be a smooth manifold with boundary ∂M. We look for $\mathbf{F} : M^n \times [0,T] \to \mathbb{R}^{n+1}$ such that

$$\begin{cases} \mathbf{F}(0,x) = \mathbf{F}_0(x) & x \in M^n \\ \left\langle \frac{d\mathbf{F}}{dt}, \nu \right\rangle = -H & x \in M^n \times [0,T) \\ \mathbf{F}(x,t) \subset \Sigma & \forall x \in \partial M^n \times [0,T] \\ \langle \nu, \mu \rangle (x,t) = -\beta & x \in \partial M^n \times [0,T) \end{cases} \tag{5.1}$$

where $1 > \beta \geq 0$ is a constant. We have two choices to make – the choice of Σ and the choice of the initial manifold.

Remark 5.0.3. The inequality $\beta \geq 0$ is not a condition – suppose we want negative β, then by simply flipping the normal ν then we have a boundary condition with positive β. This change of sign on ν does not affect the rest of the evolution equations.

Remark 5.0.4. In this definition we use *reparametrised* mean curvature flow: Since the boundary angle is now no longer perpendicular at the boundary the manifold may be flowing in or out of Σ which in the usual form would require dynamics on the boundary of the domain M^n. In fact by the usual methods (see [26, Section 2]) we may also consider this as mean curvature flow proper on the interior, but we must bear in mind that at the boundary we need to make some provisions for the flow out of the manifold (see proof of Lemma 5.1.10).

Although some study has been done on the case $\beta = 0$ by Huisken [14], Stahl [25] [26] and Buckland [2], less is known for non-perpendicular angles, we refer to Altchuler and Wu [1] and Freire [6]. In [2] [6] [25] [26], the authors are concerned with singularities that occur after some finite time. In this chapter we will be looking for situations closer to [14] and [1] where long time existence is obtained. In these papers graphs within cylindrical boundaries are considered: In [14], $\beta = 0$ over a general domain, and long time existence was shown along with convergence to a flat plane. In [1] the results are restricted to $n = 2$ but β is a function over the boundary of the domain and this time we have convergence to translating solutions. We will be looking for situations which locally look like diffeomorphic cylinders, and our goal is suitable gradient estimates and possible criteria for long time existence.

Let E be a vector field on $\mathbb{R}^{n+1}$ which is smooth away from a finite set of singularities. We choose Σ to be a smooth hypersurface made up of integral curves of E, such that Σ divides $\mathbb{R}^{n+1}$ into an interior and an exterior, with the property that any hypersurface S with normal parallel to E contained in the interior of Σ is compact. We shall also assume that S is diffeomorphically a disc. Then our initial condition on M_0 is being *graphical with respect to* E, that is $\langle \nu, E \rangle > 0$ on all of M_0. We will show that for suitable vector fields E, the graph property is preserved indicating possible long time existence.

Remark 5.0.5. It is clear that there must be conditions on the vector field E. The graph property is a gradient estimate, and leads to long time existence. As a counter example we may take any manifold that will lead to a singularity by mean curvature flow, take a vector field perpendicular to it and extend. This is initially a graph, but we cannot hope to get the above estimate.

We choose assumptions on E to allow gradient estimates. Our assumptions will be conditions to give "nice" evolution equations (Assumption 5.1.3) and boundary conditions (Assumption 5.1.7). These assumptions indicate a set of interesting boundary problems. One which had not been studied until now was a general rotational torus, which is explored in Section 5.4. We modify the iteration argument of Huisken [14] to give both long time existence and convergence in this case (see Theorem 5.4.15).

We extend μ, the unit normal outwards pointing vector field on Σ, to the interior of Σ by defining it be $-\nabla\phi(d)$ where d is the minimum distance to the boundary function where $\phi : \mathbb{R} \to \mathbb{R}$. We choose

$$\phi(x) = \begin{cases} x & x \in [0, \epsilon] \\ \text{smooth, monotonic} & x \in (0, 10\epsilon] \\ 0 & x > 10\epsilon \end{cases}$$

such that ϕ is smooth everywhere. By choosing ϵ small enough we have that μ is extended smoothly to the interior of Σ. We note that the extension has been chosen so that $\overline{\nabla}_\mu \mu = 0$ at Σ.

Definition 5.0.6. For any vector $X \in T_{\mathbf{p}}\mathbb{R}^{n+1}$ then if $\mathbf{p} \in M$

$$X^\top = X - \langle X, \nu \rangle \nu$$

while if $\mathbf{p} \in \Sigma$

$$X^\Sigma = X - \langle X, \mu \rangle \mu \quad .$$

Remark 5.0.7. Unfortunately it is standard to use μ both as the outward unit normal to Σ and as the volume measure on the manifold. Although it will always be completely clear from context we will write the volume measure on M_t as $\check{\mu}$ to make a distinction. The volume measures on the boundary ∂M_t will be written $\check{\mu}_\partial$ and the Lebesgue measure on a portion of $\mathbb{R}^n$ will be written dx.

5.1 Evolution equations and boundary derivatives

We have the following well known evolution equations:

Lemma 5.1.1. *On the interior of a manifold moving by mean curvature flow the following hold*

$$\frac{d\nu}{dt} = \nabla H \tag{5.2}$$
$$\frac{dg_{ij}}{dt} = -2Hh_{ij} \tag{5.3}$$
$$\left(\frac{d}{dt} - \Delta\right) H = H|A|^2 \tag{5.4}$$

Proof. See for example [13] □

It will be useful to have the following:

Lemma 5.1.2. *For a smooth vector field Z in $\mathbb{R}^{n+1}$, on the flowing manifold we have*

$$\left(\frac{d}{dt} - \Delta\right) \langle Z, \nu\rangle = \langle Z, \nu\rangle |A|^2 - 2g^{ij} \left\langle \overline{\nabla}_i \nu, \overline{\nabla}_j Z\right\rangle - g^{ij} \left\langle \overline{\nabla}^2_{ij} Z, \nu\right\rangle \quad .$$

Proof. First we calculate the time derivative:

$$\frac{d\langle Z, \nu\rangle}{dt} = \langle \nabla H, Z\rangle - H \left\langle \nu, \overline{\nabla}_\nu Z\right\rangle \quad .$$

As usual we will use the Laplacian to get rid of the highest order terms. We calculate

$$\begin{aligned}\Delta \langle \nu, Z\rangle = g^{ij} &\left\langle \overline{\nabla}_i(\overline{\nabla}_j \nu) - \overline{\nabla}_{\nabla_i \frac{\partial}{\partial x^j}} \nu, Z\right\rangle \\ &+ 2g^{ij} \left\langle \overline{\nabla}_i \nu, \overline{\nabla}_j Z\right\rangle + g^{ij} \left\langle \nu, \overline{\nabla}_i(\overline{\nabla}_j Z) - \overline{\nabla}_{\nabla_i \frac{\partial}{\partial x^j}} Z\right\rangle \quad .\end{aligned}$$

For the first of these terms, take a orthonormal basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ at a point $\mathbf{p} \in M$. We extend this to give orthogonal geodesic coordinates at $\mathbf{p}$. We calculate that at $\mathbf{p}$,

$$\begin{aligned}g^{ij} \left\langle \overline{\nabla}_i(\overline{\nabla}_j \nu) - \overline{\nabla}_{\nabla_i \frac{\partial}{\partial x^j}} \nu, Z\right\rangle &= g^{ij} \langle \mathbf{f}_j(\mathbf{f}_i \nu), Z\rangle \\ &= g^{ij} \left\langle \mathbf{f}_j(h_{il} g^{lk} \mathbf{f}_k), Z\right\rangle \\ &= g^{ij} \nabla_j h_{il} g^{lk} \langle \mathbf{f}_k, Z\rangle - g^{ij} h_{il} g^{lk} \langle h_{jk} \nu, Z\rangle \\ &= \nabla_{Z^\top} H - \langle \nu, Z\rangle |A|^2\end{aligned}$$

where we used the Weingarten and Codazzi equations. Since this does not depend on the coordinate system this holds for all $\mathbf{p} \in M$.

For the final term in the Laplacian we have

$$g^{ij}\left\langle \nu, \overline{\nabla}_i(\overline{\nabla}_j Z) - \overline{\nabla}_{\nabla_i \frac{\partial}{\partial x^j}} Z \right\rangle = g^{ij}\Bigg(\left\langle \nu, \overline{\nabla}_i(\overline{\nabla}_j Z) - \overline{\nabla}_{\overline{\nabla}_i \frac{\partial}{\partial x^j}} Z \right\rangle + \left\langle \overline{\nabla}_i \frac{\partial}{\partial x^j}, \nu \right\rangle \langle \nu, \overline{\nabla}_\nu Z\rangle \Bigg)$$
$$= g^{ij}\left\langle \overline{\nabla}^2_{ij} Z, \nu \right\rangle - H\langle \nu, \overline{\nabla}_\nu Z\rangle \quad .$$

Putting these identities together gives the lemma. □

We wish to show that the property $W = \langle E, \nu\rangle > 0$ is preserved. To do this we will consider $Q = -log(W)$, or $V = \frac{1}{W} = e^Q$ and we will require that these quantities have suitable evolution equations. This becomes a condition on E. From the above we see that

$$\left(\frac{d}{dt} - \Delta\right) Q = -\frac{1}{W}\left(\frac{d}{dt} - \Delta\right) W - \frac{|\nabla W|^2}{W^2}$$
$$= -|A|^2 + \frac{1}{W}\left[2g^{ij}\langle \overline{\nabla}_i \nu, \overline{\nabla}_j E\rangle + g^{ij}\left\langle \overline{\nabla}^2_{ij} E, \nu\right\rangle\right] - \frac{|\nabla W|^2}{W^2}$$

and therefore I stipulate the following:

Assumption 5.1.3. From now on we require that E may be shown to satisfy

$$2g^{ij}\langle \overline{\nabla}_i \nu, \overline{\nabla}_j E\rangle + g^{ij}\left\langle \overline{\nabla}^2_{ij} E, \nu\right\rangle \leq C_1^E W + \langle X^E, \nabla W\rangle$$

for some C_1^E and bounded vector field X^E.

For such vector fields by Young's inequality we have that

$$\left(\frac{d}{dt} - \Delta\right) Q \leq C_Q - |A|^2 - \frac{1}{2}|\nabla Q|^2 \quad . \tag{5.5}$$

We note that in the case that $E = \mathbf{e}_{n+1}$ then V is exactly v, the well known gradient function as used in for example [4].

We will also need an identity originally derived by Stahl for perpendicular boundary equations in [25]. This comes about by differentiating the boundary conditions once in space, and exactly the same proof applies for constant angle boundary conditions.

Lemma 5.1.4 (Stahl)**.** *For at a boundary point $\mathbf{p} \in \partial M$ and $X \in T_\mathbf{p}M \cap T_\mathbf{p}\Sigma$ we have*

$$A^\Sigma(X, \nu^\Sigma) + A(X, \mu^\top) = 0$$

Proof. See [25, Proposition 2.2] □

To get an estimate on the gradient and apply a Hopf maximum principle we will require the derivative of the gradient at the boundary. We note that at the boundary since $\nu^\Sigma = \nu + \beta\mu = \nu(1-\beta^2) + \beta\mu^\top$ that

$$\nu = \frac{1}{1-\beta^2}\left(\nu^\Sigma - \beta\mu^\top\right), \quad \mu = \frac{1}{1-\beta^2}\left(\mu^\top - \beta\nu^\Sigma\right) \quad .$$

Lemma 5.1.5. *At the boundary*

$$\nabla_{\mu^\top} W = \left\langle \nu, \overline{\nabla}_{\mu^\top} E\right\rangle - A^\Sigma(\nu^\Sigma, E) + \frac{W}{1-\beta^2}\left[\beta A(\mu^\top, \mu^\top) + A^\Sigma(\nu^\Sigma, \nu^\Sigma)\right] \quad .$$

Proof. We have

$$\begin{aligned}\nabla_{\mu^\top} W &= \left\langle \overline{\nabla}_{\mu^\top}\nu, E\right\rangle + \left\langle \nu, \overline{\nabla}_{\mu^\top} E\right\rangle \\ &= A(\mu^\top, E^\top) + \left\langle \nu, \overline{\nabla}_{\mu^\top} E\right\rangle \quad .\end{aligned}$$

For the first term, using that $E \in T_p\Sigma$ and Lemma 5.1.4

$$\begin{aligned}A(\mu^\top, E^\top) &= A(\mu^\top, E - W\nu) \\ &= A(\mu^\top, E - \frac{W}{1-\beta^2}\nu^\Sigma + \frac{W\beta}{1-\beta^2}\mu^\top) \\ &= -A^\Sigma(\nu^\Sigma, E - \frac{W}{1-\beta^2}\nu^\Sigma) + \frac{W\beta}{1-\beta^2}A(\mu^\top, \mu^\top) \\ &= -A^\Sigma(\nu^\Sigma, E) + \frac{W}{1-\beta^2}A^\Sigma(\nu^\Sigma, \nu^\Sigma) + \frac{W\beta}{1-\beta^2}A(\mu^\top, \mu^\top) \quad .\end{aligned}$$

□

Corollary 5.1.6. *On ∂M,*

$$\nabla_{\mu^\top} Q = \frac{1}{W}\left[A^\Sigma(\nu^\Sigma, E) - \left\langle \nu, \overline{\nabla}_{\mu^\top} E\right\rangle\right] - \frac{1}{1-\beta^2}A^\Sigma(\nu^\Sigma, \nu^\Sigma) - \frac{\beta}{1-\beta^2}A(\mu^\top, \mu^\top) \quad .$$

The (potentially) largest term here comes from the square bracket, and suitable bounding of this becomes a boundary assumption on the vector field E.

Assumption 5.1.7. Henceforth, we will assume that at the boundary E has the property that

$$A^\Sigma(\nu^\Sigma, E) - \left\langle \nu, \overline{\nabla}_{\mu^\top} E\right\rangle < C_2^E W \quad .$$

In practice the the first term of this will be fulfilled by assuming that E is an eigenvector of $A^\Sigma(\cdot,\cdot)$. The second is more restrictive, see the examples later (Section 5.3). Given that the above is fulfilled then

$$\nabla_{\mu^\top} Q = C_2^E - \frac{1}{1-\beta^2} A^\Sigma(\nu^\Sigma,\nu^\Sigma) - \frac{\beta}{1-\beta^2} A(\mu^\top,\mu^\top) \quad . \tag{5.6}$$

We must now find a way of estimating $A(\mu^\top,\mu^\top)$ at the boundary. The solution to this problem that we shall pursue is to use another gradient-like quantity namely the extension of the boundary condition to the interior of the manifold, $I = \langle \nu, \mu\rangle$.

Lemma 5.1.8. *At the boundary we have*

$$\nabla_{\mu^\top} I = A(\mu^\top,\mu^\top) + \beta A^\Sigma(\nu^\Sigma,\nu^\Sigma) \quad .$$

Proof. Using properties of the extension of μ,

$$\begin{aligned}\nabla_{\mu^\top} I &= \left\langle \overline{\nabla}_{\mu^\top}\nu, \mu\right\rangle + \left\langle \nu, \overline{\nabla}_{\mu^\top}\mu\right\rangle \\ &= A(\mu^\top,\mu^\top) + \beta\left\langle \nu, \overline{\nabla}_{\nu^\Sigma}\mu\right\rangle \\ &= A(\mu^\top,\mu^\top) + \beta A^\Sigma(\nu^\Sigma,\nu^\Sigma) \quad .\end{aligned}$$

□

We also need the evolution equation of I.

Lemma 5.1.9. *On the interior of M we may estimate*

$$\left(\frac{d}{dt} - \Delta\right) I \leq I|A|^2 + C_\mu(|A|+1)$$

where C_μ depends only on n and the first and second derivatives of Σ.

Proof. We have from Lemma 5.1.2 that

$$\left(\frac{d}{dt} - \Delta\right) I = I|A|^2 - 2g^{ij}\left\langle \overline{\nabla}_i\nu, \overline{\nabla}_j\mu\right\rangle - g^{ij}\left\langle \nu, \overline{\nabla}^2_{ij}\mu\right\rangle \quad .$$

The Lemma immediately follows by applying Cauchy–Schwarz, and using that the first and second derivatives of μ are bounded. □

Similarly to [25, Proposition 2.1] we use the time derivative of the boundary condition to calculate derivatives of H at the boundary.

Lemma 5.1.10. *At the boundary we have:*

$$\nabla_{\mu^\top}\frac{H}{W} = \nabla_{\mu^\top}HV = \frac{H}{W^2}\left[A^\Sigma(E,\nu^\Sigma) - \langle \nu, \overline{\nabla}_{\mu^\top}E\rangle\right]$$

and

$$\nabla_{\mu^\top}H = \frac{H}{1-\beta^2}\left[\beta A(\mu^\top,\mu^\top) + A^\Sigma(\nu^\Sigma,\nu^\Sigma)\right] \quad .$$

Proof. Let $\gamma(t) = \mathbf{F}(\mathbf{p}(t))$ be the position of ∂M_t at time t on some particular integral curve of E in Σ. We calculate

$$\begin{aligned}\frac{d\gamma}{dt} &= \frac{d}{dt}\mathbf{F}(\mathbf{p}(t))\\ &= \frac{d\mathbf{F}}{dt}\Big|_{\mathbf{p}(t)} + \frac{\partial \mathbf{F}}{\partial x^i}\frac{d\mathbf{p}^i}{dt}\\ &= -H\nu + \frac{\partial \mathbf{F}}{\partial x^i}\frac{d\mathbf{p}^i}{dt} \quad .\end{aligned}$$

We note that $\frac{\partial \mathbf{F}}{\partial x^i}\frac{d\mathbf{p}^i}{dt} \in T_{\mathbf{p}}M_t$ and that the constraint that γ is on an integral curve implies

$$-H\nu + \frac{\partial \mathbf{F}}{\partial x^i}\frac{d\mathbf{p}^i}{dt} = \lambda(t)E \quad ,$$

that is, $\frac{d\gamma}{dt}$ is the unique projection of $-H\nu$ into the direction E by vectors in the tangent space (we are assuming that M is graphical with respect to E). This implies that

$$\frac{\partial \mathbf{F}}{\partial x^i}\frac{d\mathbf{p}^i}{dt} = -\frac{H}{W}E^\top = -\frac{H}{W}(E - W\nu)$$

and so $\frac{d\gamma}{dt} = -\frac{H}{W}E$. Now using this and Lemma 5.1.1 then

$$\begin{aligned}\frac{d}{dt}\nu(\mathbf{p}(t),t) &= \frac{\partial \nu}{\partial x^i}\frac{dp^i}{dt} + \nabla H\big|_{\mathbf{p}(t)}\\ &= -\frac{H}{W}A(E^\top, \frac{\partial}{\partial x^i})g^{ij}\frac{\partial}{\partial x^j} + \nabla H\big|_{\mathbf{p}(t)} \quad .\end{aligned}$$

Similarly

$$\frac{d}{dt}\mu(\gamma(t)) = -\frac{H}{W}\overline{\nabla}_E\mu$$

and so

$$0 = \frac{d}{dt}\langle \nu, \mu\rangle = \nabla_{\mu^\top}H - \frac{H}{W}A(E^\top,\mu^\top) - \frac{H}{W}A^\Sigma(E,\nu^\Sigma) \quad .$$

The first identity comes from the fact that from Lemma 5.1.5

$$\nabla_{\mu^\top}W = A(E^\top,\mu^\top) + \langle \nu, \overline{\nabla}_{\mu^\top}E\rangle$$

and hence

$$\nabla_{\mu^\top} H - \frac{H}{W}\nabla_{\mu^\top} W = \frac{H}{W}\left[A^\Sigma(E,\nu^\Sigma) - \left\langle \nu, \overline{\nabla}_{\mu^\top} E\right\rangle\right] \quad .$$

Therefore

$$\nabla_{\mu^\top}\frac{H}{W} = \frac{\nabla_{\mu^\top} H}{W} - \frac{H\nabla_{\mu^\top} W}{W^2} = \frac{H}{W^2}\left[A^\Sigma(E,\nu^\Sigma) - \left\langle \nu, \overline{\nabla}_{\mu^\top} E\right\rangle\right] \quad .$$

Demonstrating the second identity is similar to the proof of Lemma 5.1.5:

$$\begin{aligned}
\nabla_{\mu^\top} H &= \frac{H}{W}\left[A(E^\top,\mu^\top) + A^\Sigma(E,\nu^\Sigma)\right] \\
&= \frac{H}{W}\Bigg[A(E - \frac{W}{1-\beta^2}\nu^\Sigma + \frac{W\beta}{1-\beta^2}\mu^\top, \mu^\top) \\
&\qquad\qquad + A^\Sigma(E - \frac{W}{1-\beta^2}\nu^\Sigma, \nu^\Sigma) + \frac{W}{1-\beta^2}A^\Sigma(\nu^\Sigma,\nu^\Sigma)\Bigg] \\
&= \frac{H}{1-\beta^2}\left[\beta A(\mu^\top,\mu^\top) + A^\Sigma(\nu^\Sigma,\nu^\Sigma)\right]
\end{aligned}$$

where we used Lemma 5.1.4 on the second line. □

5.2 Estimates via the maximum principle

We are now ready to prove our gradient estimate.

Proposition 5.2.1 (Gradient estimate on convex domains)**.** *Under the assumptions that Σ is convex, $C_2^E \leq 0$ and $\beta < \frac{\sqrt{5}-1}{2}$ then while the flowing manifold stays away from all singularities of E,*

$$Q \leq \sup_{\mathbf{x}\in M} Q(\mathbf{x},0) + \frac{|\beta|}{1-\beta^2} + C^{\beta,\Sigma} t \quad .$$

Proof. We consider the function $P = Q + \frac{\beta}{1-\beta^2} I$. This has been chosen to get rid of unpleasant boundary terms: We see that under the above assumptions

$$\begin{aligned}
\nabla_{\mu^\top} P &= \nabla_{\mu^\top} Q + \frac{\beta}{1-\beta^2}\nabla_{\mu^\top} I \\
&\leq \frac{1}{1-\beta^2}\left[-\beta A(\mu^\top,\mu^\top) - A^\Sigma(\nu^\Sigma,\nu^\Sigma) + \beta A(\mu^\top,\mu^\top) + \beta^2 A^\Sigma(\nu^\Sigma,\nu^\Sigma)\right] \\
&= -A^\Sigma(\nu^\Sigma,\nu^\Sigma) \\
&\leq 0 \quad .
\end{aligned}$$

On the interior for some $\widetilde{C}_P > 0$ we have

$$\left(\frac{d}{dt} - \Delta\right) P = \left(\frac{d}{dt} - \Delta\right) Q + \frac{\beta}{1-\beta^2}\left(\frac{d}{dt} - \Delta\right) I$$
$$\leq \widetilde{C}_P(1+|A|) - \left(1 - \frac{\beta}{1-\beta^2} I\right)|A|^2 - \frac{1}{2}|\nabla Q|^2 \quad .$$

Due to our choice of β we have that since $|I| < 1$, there exists an $\epsilon(\beta) > 0$ such that $1 - \frac{\beta}{1-\beta^2} I > 2\epsilon$, and therefore by Young's inequality

$$\left(\frac{d}{dt} - \Delta\right) P \leq C_Q - \epsilon |A|^2 - \frac{1}{2}|\nabla Q|^2 \quad .$$

We also estimate with respect to $|\nabla P|$. Again using Young's inequality we see that

$$|\nabla I|^2 = \left| A(\mu^\top, \frac{\partial}{\partial x^i}) g^{ij} \frac{\partial}{\partial x^j} + \left\langle \overline{\nabla}_{\frac{\partial}{\partial x^l}} \mu, \nu \right\rangle g^{kl} \frac{\partial}{\partial x^l} \right|^2 \leq |A|^2 + C_I \quad ,$$

and therefore we see that for some $\widehat{C}_P,\ \delta > 0$

$$\left(\frac{d}{dt} - \Delta\right) P \leq \widehat{C}_P - \delta |\nabla P|^2 \quad .$$

Now for a constant $U > 0$,

$$\left(\frac{d}{dt} - \Delta\right)(P - Ut) = \left(\frac{d}{dt} - \Delta\right) P - U$$
$$\leq \widehat{C}_P - U - \delta|\nabla P|^2 \quad ,$$

and so by letting $U \geq \widehat{C}_P$ then we have the Proposition. □

In fact with a little more work we may remove two of the conditions from the above Proposition. First we define a set of functions on $\psi_D : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $1 \leq \psi_D \leq 2$ inside Σ, with the property that at the boundary $\psi = 1$ and $\overline{\nabla}\psi_D = \overline{\nabla}_\mu \psi_D\, \mu = -D\mu$. Such functions are easily constructed using the minimum distance function.

Lemma 5.2.2. *On the interior of M we have*

$$\left(\frac{d}{dt} - \Delta\right)\psi_D = -g^{ij}\overline{\nabla}^2_{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x^j}} \psi_D \leq C_D \quad .$$

Proof. As usual

$$\frac{d\psi_D}{dt} = -H \left\langle \nabla \psi_D, \nu \right\rangle$$

and

$$\begin{aligned}\Delta\psi_D &= g^{ij}\left(\frac{\partial}{\partial x^i}\left\langle\nabla\psi_D,\frac{\partial}{\partial x^j}\right\rangle - \left\langle\nabla\psi_D,\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right\rangle\right)\\ &= g^{ij}\left(\left\langle\overline{\nabla}_{\frac{\partial}{\partial x^i}}\overline{\nabla}\psi_D,\frac{\partial}{\partial x^j}\right\rangle - h_{ij}\left\langle\nabla\psi_D,\nu\right\rangle\right)\\ &= g^{ij}\overline{\nabla}^2_{\frac{\partial}{\partial x_i},\frac{\partial}{\partial x^j}}\psi_D - H\left\langle\nabla\psi_D,\nu\right\rangle\end{aligned}$$

and we are done. □

We now put this to work:

Proposition 5.2.3. *If $\beta < \frac{\sqrt{5}-1}{2}$ then while the flowing manifold stays away from all singularities of E,*

$$Q \leq \sup_{\mathbf{x}\in M} Q(\mathbf{x},0) + \frac{|\beta|}{1-\beta^2} + C^{\beta,\Sigma}t \quad .$$

Proof. As in the proof of Lemma 5.2.1 we have

$$\left(\frac{d}{dt}-\Delta\right)P \leq \widehat{C}_P - \delta|\nabla P|^2, \quad \nabla_{\mu^\top}P = C_2^E - A^\Sigma(\nu^\Sigma,\nu^\Sigma) \leq C_{\Sigma,E} \quad .$$

Choosing $D \geq \frac{C_{\Sigma,E}}{1-\beta^2}$ we calculate

$$\left(\frac{d}{dt}-\Delta\right)(P+\psi_D) \leq C_V + C_D$$

We also have

$$\nabla_{\mu^\top}(P+\psi_D) = \nabla_{\mu^\top}P - D\left\langle\mu,\mu^\top\right\rangle \leq C_\Sigma - D(1-\beta^2) \leq 0 \quad .$$

Hence as previously by removing a large enough multiple of time again gives the Proposition. □

Lemma 5.2.4 (Preservation of mean convexity)**.** *Suppose that $\beta < \frac{\sqrt{5}-1}{2}$, $A^\Sigma(\cdot,\cdot)$ is either bounded or positive definite and initially $H > C_H > 0$. Then for all the time a solution exists $H > e^{\left[C_1^{\Sigma,\beta} - C_2^{\Sigma,\beta}t\right]}$ where $C_1^{\Sigma,\beta}, C_2^{\Sigma,\beta} > 0$ depend on Σ and β.*

Proof. This is almost identical to the previous Lemma. We see from Lemmas 5.1.8 and 5.1.10 that for $l = \log H - \frac{\beta}{1-\beta^2}I$ that at the boundary

$$\begin{aligned}\nabla_{\mu^\top}l &= \frac{1}{1-\beta^2}\left[\beta A(\mu^\top,\mu^\top) + A^\Sigma(\nu^\Sigma,\nu^\Sigma) - \beta A(\mu^\top,\mu^\top) - \beta^2 A^\Sigma(\nu^\Sigma,\nu^\Sigma)\right]\\ &= A^\Sigma(\nu^\Sigma,\nu^\Sigma) \quad .\end{aligned}$$

In the interior from Lemmas 5.1.1 and 5.1.9 and using our bound on β as before then

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)l &\geq |A|^2+\frac{|\nabla H|^2}{H^2}-\frac{\beta}{1-\beta^2}I|A|^2-C_\mu(|A|+1)\\ &\geq \epsilon|A|^2-C_\mu(|A|+1)\\ &\geq -C_1\end{aligned}$$

where $C_1>0$ depends on β and the extension of μ. For convex boundary we now apply maximum principle to $l+C_1t$ to give the Lemma.

For non convex boundary we again use ψ_D where D is sufficiently large that $D(1-\beta^2)\geq \left|A^\Sigma(\nu^\Sigma,\nu^\Sigma)\right|$. Therefore $\widetilde{l}=l-\psi_D$ has a good boundary derivative and $\left(\frac{d}{dt}-\Delta\right)\widetilde{l}\geq -C_2$. Then $\left(\frac{d}{dt}-\Delta\right)\widetilde{l}+C_2t\geq 0$ and we are done. □

Remark 5.2.5. The above proof does not require either of Assumptions 5.1.3 or 5.1.7. The only place that the graph property is used is in the derivation of the gradient of H at the boundary – we need the integral curve of E at the boundary to not be in the tangent space of the flowing manifold.

5.3 Examples

A natural question is how common are vector fields E that may be shown to satisfy Assumptions 5.1.3 and 5.1.7? Here I give three examples of situations in which these conditions hold.

5.3.1 Cylinders

Example 5.3.1. Let $E=\mathbf{e}_{n+1}$, the constant upwards pointing unit vector. In this case Σ becomes cylinders over domains $\Omega\subset\mathbb{R}^n$ where $\partial\Omega$ is smooth.

Proof. In this case the interior assumption (Assumption 5.1.3) is readily satisfied: We note that since this is a vector field that does not change in space we have $\overline{\nabla}_X E=0$ for all $X\in T_{\mathbf{p}}\mathbb{R}^{n+1}$. Therefore both terms disappear to give $C_1^E=0$ and $X^E=\mathbf{0}$.

Similarly at the boundary Assumption 5.1.7 we have $\left\langle \nu, \overline{\nabla}_{\mu^\top} E \right\rangle = 0$. We note that the second fundamental form of any cylinder of this kind has a zero eigenvector in the direction E, and therefore the term $A^\Sigma(\nu^\Sigma, E) = 0$. □

This first example is already well studied by Huisken [14] and Altschuler and Wu [1].

The results of the previous section generalise the long time existence result of Altschuler and Wu to dimensions $n \geq 3$ (and to more general domains in $n = 2$), but only with constant angle $\beta \leq \frac{\sqrt{5}-1}{2}$. Sadly for convergence we need a stronger estimate than those given: Altschuler and Wu's strong maximum principle argument (for example), requires a limit solution to the sequence of graphs $u_i = u(x, t+i)$. We would need a gradient estimate that is constant in time to be able to apply the Arzela–Ascoli Theorem here.

5.3.2 Cones

Example 5.3.2. $E = \mathbf{p}$, the position vector. This implies Σ is a cones in $\mathbb{R}^{n+1}$. The Assumptions are satisfied for some short time T if we specify $H, W > 0$ initially.

Proof. In this case we have $\overline{\nabla}_X E = X$ and $\overline{\nabla}^2 E = \overline{\nabla}_Y X - \overline{\nabla}_{\overline{\nabla}_Y X} E = 0$.

For the boundary assumption as in the previous Example we use that on a cone the second fundamental form has a zero eigenvector in the E direction. Therefore

$$A^\Sigma(\nu^\Sigma, E) - \left\langle \nu, \overline{\nabla}_{\mu^\top} E \right\rangle = -\left\langle \nu, \mu^\top \right\rangle = 0 \quad .$$

Applying this in Assumption 5.1.3 we have

$$2g^{ij} \left\langle \overline{\nabla}_i \nu, \overline{\nabla}_j E \right\rangle + g^{ij} \left\langle \overline{\nabla}^2_{ij} E, \nu \right\rangle = 2H \quad .$$

To show the assumption we need a little more work. Lemma 5.1.2 gives us

$$\left(\frac{d}{dt} - \Delta \right) W = W|A|^2 - 2H \quad .$$

We specify the condition that initially $H > 0$, and so by Lemma 5.2.4 this will remain true for as long as a solution exists. Therefore we may calculate

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)\frac{W}{H} &= \frac{W}{H}|A|^2 - 2 - \frac{W}{H}|A|^2 - 2W\frac{|\nabla H|^2}{H^3} + \frac{2}{H^2}\left\langle \nabla W, \nabla H\right\rangle \\ &= -2 + 2\left\langle \frac{\nabla H}{H}, \frac{H\nabla W - W\nabla H}{H^2}\right\rangle \\ &= -2 + 2\left\langle \nabla \log H, \nabla \frac{W}{H}\right\rangle \quad .\end{aligned}$$

At the boundary we have from Lemma 5.1.10

$$\nabla_{\mu^\top}\frac{W}{H} = -\frac{W^2}{H^2}\nabla_{\mu^\top}\frac{H}{W} = 0 \quad .$$

and so by considering the evolution of $\frac{W}{H} + 2t$ we see

$$H < \frac{W}{C - 2t} \quad .$$

□

The finite time under which this holds is not surprising. For example if we have $\beta = 0$ then we have a special solution – the homothetically shrinking sphere centred at the point of the cone. By Stahl's comparison theorem [26, Theorem 4.1], this acts as a comparison solution for all solutions with $\beta = 0$. Therefore since the sphere solution shrinks to a point at the singularity of Σ and E then we do not expect long time existence.

This case has been considered for $\beta = 0$ in the special case of the cone being a flat plane $\mathbb{R}^n$ by Stahl [25] where it was shown that an initially convex hypersurface shrinks to a round sphere at some point $\mathbf{p} \in \mathbb{R}^n$.

5.3.3 Tori

At a point $\mathbf{p} = (p_1, \ldots, p_{n+1}) \in \mathbb{R}^{n+1}$ we define

$$r = \sqrt{p_n^2 + p_{n+1}^2}, \quad \mathbf{r} = \frac{1}{r}(0, \ldots, 0, p_n, p_{n+1}), \quad \mathbf{t} = \frac{1}{r}(0, \ldots, 0, -p_{n+1}, p_n) \quad .$$

Example 5.3.3. Let $E = r\mathbf{t}$ then Assumptions 5.1.3 and 5.1.7 hold and Σ is a torus made from an embedding of $\mathbb{S}^{n-1}$ into $\text{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ then rotated in the $\mathbf{e}_n, \mathbf{e}_{n+1}$ plane.

Proof. We see that by standard calculations

$$\overline{\nabla} r = \mathbf{r},$$

$$\overline{\nabla}_X \mathbf{r} = \langle X, \mathbf{t}\rangle \overline{\nabla}_{\mathbf{t}} \mathbf{r} = \langle X, \mathbf{t}\rangle \frac{1}{r}\mathbf{t}$$

$$\overline{\nabla}_X \mathbf{t} = \langle X, \mathbf{t}\rangle \overline{\nabla}_{\mathbf{t}} \mathbf{t} = -\langle X, \mathbf{t}\rangle \frac{1}{r}\mathbf{r} \quad .$$

and so $\overline{\nabla}_X E = \overline{\nabla}_X(r\mathbf{t}) = \langle X, \mathbf{r}\rangle \mathbf{t} - \langle X, \mathbf{t}\rangle \mathbf{r}$. Further we have

$$\begin{aligned}\overline{\nabla}^2_{YX} E &= \overline{\nabla}_Y(\langle X, \mathbf{r}\rangle \mathbf{t} - \langle X, \mathbf{t}\rangle \mathbf{r}) - \overline{\nabla}_{\overline{\nabla}_Y X} E \\ &= \left\langle X, \overline{\nabla}_Y \mathbf{r}\right\rangle \mathbf{t} + \langle X, \mathbf{r}\rangle \overline{\nabla}_Y \mathbf{t} - \left\langle X, \overline{\nabla}_Y \mathbf{t}\right\rangle \mathbf{r} - \langle X, \mathbf{t}\rangle \overline{\nabla}_Y \mathbf{r} \\ &= \frac{\langle Y, \mathbf{t}\rangle}{r}\left[\langle X, \mathbf{t}\rangle \mathbf{t} - \langle X, \mathbf{r}\rangle \mathbf{r} + \langle X, \mathbf{r}\rangle \mathbf{r} - \langle X, \mathbf{t}\rangle \mathbf{t}\right] \\ &= 0 \quad .\end{aligned}$$

For Assumption 5.1.3 we see that

$$\begin{aligned}2g^{ij}\left\langle \overline{\nabla}_i \nu, \overline{\nabla}_j E\right\rangle + g^{ij}\left\langle \overline{\nabla}^2_{ij} E, \nu\right\rangle &= 2g^{ij}\left\langle \overline{\nabla}_i \nu, \left\langle \frac{\partial \mathbf{F}}{\partial x^j}, \mathbf{r}\right\rangle \mathbf{t} - \left\langle \frac{\partial \mathbf{F}}{\partial x^j}, \mathbf{t}\right\rangle \mathbf{r}\right\rangle \\ &= 2\left(A(\mathbf{r}^\top, \mathbf{t}^\top) - A(\mathbf{t}^\top, \mathbf{r}^\top)\right) \\ &= 0 \quad .\end{aligned}$$

For the boundary assumption, Assumption 5.1.7, we need some facts about the second fundamental form for such boundary manifolds Σ. Let $\mathbf{J} : \mathbb{S}^{n-1} \to \mathbb{R}^n$ be smooth such that $\langle \mathbf{J}, \mathbf{e}_n\rangle > 0$. Then set

$$\mathbf{G} = \mathbf{J} - \langle \mathbf{J}, \mathbf{e}_n\rangle \mathbf{e}_n + \langle \mathbf{J}, \mathbf{e}_n\rangle [\cos\theta \mathbf{e}_n + \sin\theta \mathbf{e}_{n+1}] \quad ,$$

which is a parametrisation of a general Σ of this kind. If ν^J is the normal to $\mathbf{J}$ in $\mathbb{R}^n$ then the normal to $\mathbf{G}$ in $\mathbb{R}^{n+1}$ is

$$\mu = \nu^G = \nu^J - \left\langle \nu^J, \mathbf{e}_n\right\rangle \mathbf{e}_n + \left\langle \nu^J, \mathbf{e}_n\right\rangle [\cos\theta \mathbf{e}_n + \sin\theta \mathbf{e}_{n+1}] \quad .$$

We may easily see that

$$\frac{\partial^2 \mathbf{G}}{\partial x^i \partial \theta} = \left\langle \frac{\partial \mathbf{J}}{\partial x^i}, \mathbf{e}_n\right\rangle [-\sin\theta \mathbf{e}_n + \cos\theta \mathbf{e}_{n+1}] \quad ,$$

which is perpendicular to ν^G. Therefore we know that the direction $\frac{\partial \mathbf{G}}{\partial \theta} = E$ is an eigenvector of $A^\Sigma(\cdot,\cdot)$. We also calculate the curvature in the direction E: Since

$$\frac{\partial^2 \mathbf{G}}{\partial^2 \theta} = -\langle \mathbf{J}, \mathbf{e}_n \rangle [\cos\theta \mathbf{e}_n + \sin\theta \mathbf{e}_{n+1}]$$

we have

$$A^\Sigma(E,E) = -\left\langle \nu^G, \frac{\partial^2 \mathbf{G}}{\partial^2 \theta} \right\rangle = \left\langle \nu^J, \mathbf{e}_n \right\rangle \langle \mathbf{J}, \mathbf{e_n} \rangle = \left\langle \nu^G, \mathbf{r} \right\rangle r = \langle \mu, \mathbf{r} \rangle r \quad .$$

Using this we see that

$$\begin{aligned} A^\Sigma(\nu^\Sigma, E) - \left\langle \nu, \overline{\nabla}_{\mu^\top} E \right\rangle &= \frac{\langle \nu, E \rangle}{r^2} A^\Sigma(E,E) - \left\langle \nu, \left\langle \mu^\top, \mathbf{r} \right\rangle \mathbf{t} - \left\langle \mu^\top, \mathbf{t} \right\rangle \mathbf{r} \right\rangle \\ &= \frac{W}{r} \langle \mu, \mathbf{r} \rangle - \frac{W}{r} \langle \mu, \mathbf{r} \rangle - \beta \langle \nu, \mathbf{r} \rangle \langle \nu, \mathbf{t} \rangle + \beta \langle \nu, \mathbf{t} \rangle \langle \nu, \mathbf{r} \rangle \\ &= 0 \end{aligned} \tag{5.7}$$

and so the Assumption is satisfied. □

To the authors knowledge this situation is not yet studied.

Proposition 5.2.3 gives long time existence for $\beta < \frac{\sqrt{5}-1}{2}$ and if $\beta > 0$ then Lemma 5.2.4 gives preservation of mean convexity. In fact for $\beta = 0$ this is not a useful condition: For a vector field $X \in \mathfrak{X}(\mathbb{R}^{n+1})$ then on M $\operatorname{div}(X^\top) = g^{ij}\left\langle \overline{\nabla}_{\frac{\partial}{\partial x^i}} X, \frac{\partial}{\partial x^j} \right\rangle - H\langle \nu, X \rangle$. Applying this to the vector field $E = r\mathbf{t}$ we have

$$0 = -\int_{\partial M} r \left\langle \mu, \mathbf{t}^\top \right\rangle d\breve{\mu} = -\int_M \operatorname{div}(E^\top) d\breve{\mu} = \int_M HW d\breve{\mu}$$

since at the boundary $\mathbf{t}^\top$ and μ are perpendicular for $\beta = 0$. We therefore see that either $H = 0$ or H is both positive and negative on M. Therefore (weak) mean convexity here implies a minimal surface, which does not make a good initial condition.

Remark 5.3.4. We have a special solution to this flow for $\beta = 0$: Any flat hyperplane going through $\mathbf{0}$ and perpendicular to $\mathbf{t}$ satisfies the boundary conditions and is a stationary solution to equation (5.1).

Regardless of β we may get relations between H and W. This is similar to the relations obtained between H and S in Chapter 3. We have $\left(\frac{d}{dt} - \Delta\right) W = W|A|^2$

and so

$$\left(\frac{d}{dt}-\Delta\right)\frac{H^2}{W^2} = 2\frac{H^2}{W^2}|A|^2 - 2\frac{|\nabla H|^2}{W^2} - 2\frac{H^2}{W^2}|A|^2 - 6\frac{H^2|\nabla W|^2}{W^4} + 8\frac{H}{W^3}\langle \nabla H, \nabla W\rangle$$
$$= \left[-2\frac{|\nabla H|^2}{W^2} + 2\frac{1}{W^2}\left\langle \nabla H, \frac{H}{W}\nabla W\right\rangle\right]$$
$$+ \left[6\frac{H}{W^4}\langle W\nabla H, \nabla W\rangle - 6\frac{H^2|\nabla W|^2}{W^4}\right] \quad .$$

At a positive stationary point of $\frac{H^2}{W^2}$ we have $H\nabla W = W\nabla H$, and therefore both of the brackets disappear. From equation (5.7) and Lemma 5.1.10 we see that $\nabla_{\mu^\top}\frac{H^2}{W^2} = 0$ and we may apply maximum principle to give $H^2 \leq CW^2$ for some $C > 0$. Since $|W| \leq 1$ we therefore have a bound on $|H|$.

The torus situation is clearly very interesting, and we go into more details with $\beta = 0$ in the next section.

5.4 Gradient estimate via integral methods for tori

We now look for better gradient estimates via integral methods in the case of tori for $\beta = 0$. We will see that the Stampaccia iteration method used by Huisken in [14] may be modified to apply to this case. This will lead to a gradient estimate uniform in time which will be enough to show convergence as in Chapter 4. Before going into this we look at a bound on the region in which M_t may move using maximum principles:

We may get estimates on the region in which M_t is contained.

Lemma 5.4.1. *Let u be the angle around the torus, taken from some arbitrary point. Then*

$$\left(\frac{d}{dt}-\Delta\right)u = \frac{2}{r}\left\langle \mathbf{r}^\top, \nabla u\right\rangle = -\frac{2}{r^2}\langle \nu, \mathbf{t}\rangle\langle \nu, \mathbf{r}\rangle \quad .$$

Proof. Using cylindrical coordinates on $\mathbb{R}^{n+1}$ we see that

$$\overline{\nabla} u = \frac{\mathbf{t}}{r}$$

and from this we may calculate the evolution equation of u. We see that

$$\frac{du}{dt} = -\frac{H}{r}\langle \nu, \mathbf{t}\rangle$$

and

$$\begin{aligned}\Delta u &= g^{ij}\left(\frac{\partial}{\partial x^i}\left\langle\frac{\mathbf{t}}{r},\frac{\partial}{\partial x^j}\right\rangle - \left\langle\frac{\mathbf{t}}{r},\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right\rangle\right)\\ &= g^{ij}\Bigg(\left\langle-\left\langle\mathbf{r},\frac{\partial}{\partial x^i}\right\rangle\frac{\mathbf{t}}{r^2}-\left\langle\frac{\partial}{\partial x^i},\mathbf{t}\right\rangle\frac{\mathbf{r}}{r^2},\frac{\partial}{\partial x^j}\right\rangle+\left\langle\frac{\mathbf{t}}{r},\overline{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right\rangle\\ &\qquad\qquad -\left\langle\frac{\mathbf{t}}{r},\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right\rangle\Bigg)\\ &= -\frac{2}{r^2}\left\langle\mathbf{r}^\top,\mathbf{t}^\top\right\rangle - H\left\langle\frac{\mathbf{t}}{r},\nu\right\rangle \quad .\end{aligned}$$

Therefore

$$\left(\frac{d}{dt}-\Delta\right)u = \frac{2}{r^2}\left\langle\mathbf{r}^\top,\mathbf{t}^\top\right\rangle = \frac{2}{r}\left\langle\mathbf{r}^\top,\nabla u\right\rangle \quad .$$

□

At a stationary point we have $\nabla u = \frac{1}{r}\mathbf{t}^\top = 0$ and therefore at such a point $\left(\frac{d}{dt}-\Delta\right)u = 0$. At the boundary, again using that $\mathbf{t}^\top$ and μ are perpendicular we have $\overline{\nabla}_\mu u = 0$ and so we may apply maximum principle and we get that u is bounded above and below by its initial values. Therefore, M_t may not twist itself around the torus any more than it is initially twisted. We note that it does not matter if the initial manifold goes around by more than 2π.

Remark 5.4.2. It will also be useful to bear in mind for future estimates that we are assuming that, simply by the bounds imposed in space by the boundary torus that for some r_0 and r_1 we know $0 < r_0 < r < r_1$.

5.4.1 Integral lemmas

As is standard in proofs via integral estimates in mean curvature flow we will require the Michael–Simon Sobolev inequality [18]. While this holds in much more general situations, we will only require M to be smooth embedded n-dimensional manifolds in $\mathbb{R}^{n+1}$.

Lemma 5.4.3 (The Michael–Simon–Sobolev inequality). *There exists a constant $C_S > 0$ depending only on n such that for any function $f \in C^1(\overline{M})$ such that f has compact support, we have*

$$\left(\int_M |f|^{\frac{n}{n-1}}d\check{\mu}\right)^{\frac{n-1}{n}} \leq C_S\int_M |\nabla f| + |H||f|d\check{\mu}$$

Since we will require such an inequality not just on functions of compact closure, but functions that may be non-zero at the boundary ∂M. We give a proof which is in essence [8, Lemma 1.1], although no longer graphical:

Lemma 5.4.4. *For any compact manifold M with boundary ∂M and for any function $f \in C^1(\overline{M})$ we have*

$$\left(\int_M |f|^{\frac{n}{n-1}} d\breve{\mu}\right)^{\frac{n-1}{n}} \leq C_S \left[\int_M |\nabla f| + |H||f| d\breve{\mu} + \int_{\partial M} |f| d\breve{\mu}_\partial\right]$$

where the constant C_S depends only on n.

Proof. Set $d : D \to \mathbb{R}$ to be the minimum distance *inside the manifold* to the boundary function. This is smooth close enough to the boundary. We define for k large enough $\widetilde{\eta}_k = \min\{1, kd\}$, and let η_k be a C^1 smoothing of this – the specifics of this smoothing do not matter in the following so long as it is close in the C^1 norm to $\widetilde{\eta}_k$, and we shall estimate one with the other. Set $f_k = \eta_k f$, and we consider the sequence f_i for $i \in \mathbb{N}$. Since $\mu(\{x | f(x) \neq f_i\}) \to 0$ as $i \to \infty$ we also have that

$$\left(\int_M |f_i|^{\frac{n}{n-1}} d\breve{\mu}\right)^{\frac{n-1}{n}} \to \left(\int_M |f|^{\frac{n}{n-1}} d\breve{\mu}\right)^{\frac{n-1}{n}}, \quad \int_M |H||f_i| d\breve{\mu} \to \int_M |H||f| d\breve{\mu} \quad .$$

For the remaining term:

$$\int_M |\nabla f_i| d\breve{\mu} \leq \int_M |\nabla f| \eta_i d\breve{\mu} + \int_M |f||D\eta_k| d\breve{\mu}$$

The first of the above may be estimated as the other terms. For the final term we choose a special parametrisation of the collar, some neighbourhood of ∂M. We parametrise so that $\mathbf{F} : \partial M \times [0, \epsilon] \to \mathbb{R}^{n+1}$ where $\mathbf{F}(\cdot, s)$ is a parametrisation of the level set $\{x \in M | d(x) = s\}$. Therefore $\frac{\partial}{\partial x^n} = \frac{\partial}{\partial s} = \nabla d$ and therefore the metric induced by $\mathbf{F}$ has $g_{in} = \delta_{in}$ at the boundary. Therefore for k large enough

$$\begin{aligned}
\int_M |f||D\eta_k| d\breve{\mu} &\leq \int_{\{x \in M | d(x) \leq \frac{1}{k}\}} k|f| d\breve{\mu} \\
&= k \int_0^{\frac{1}{k}} \int_{\partial M^n} |f| \sqrt{\det(g_{ij}(x, s))} dx ds \\
&\to \int_{\partial M^n} |f| \sqrt{\det(g_{ij}(x, 0))} dx
\end{aligned}$$

as $k \to \infty$. Due to the properties of g_{ij} we have the Lemma. □

We will also need the following, which is based on [8, Lemma 1.4]:

Lemma 5.4.5. *For M a compact manifold with boundary, then for all $f \in W^{1,\infty}(M)$ we have*

$$\int_{\partial M} |f| d\check{\mu}_\partial \leq C_\Sigma \int_M |\nabla f| + (|H| + 1)|f| d\check{\mu}$$

where the constant $C_\Sigma > 0$ depends only on Σ.

Proof. This is essentially just divergence theorem. We now use $\overline{d}$, the minimum distance to Σ and note that at Σ, $\overline{\nabla}\,\overline{d} = -\mu$. We take a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi'(0) = -1$ and $\phi(x) = 0$ for $x > R$ where R is less than the minimum focal distance of Σ. We define $\overline{\phi} = \phi(\overline{d})$ – a smooth function on $\mathbb{R}^{n+1}$. Then from the proof of Lemma 5.2.2 we know

$$\Delta\overline{\phi} \leq \widehat{C}_{\overline{\phi}} - H \left\langle \nabla\overline{\phi}, \nu \right\rangle \quad .$$

Therefore

$$\begin{aligned}
\int_{\partial M} f d\check{\mu}_\partial &= \int_M \operatorname{div}(f\nabla\overline{\phi}) d\check{\mu} \\
&= \int_M \left\langle \nabla f, \nabla\overline{\phi} \right\rangle + f\Delta\overline{\phi} d\check{\mu} \\
&\leq C_{\overline{\phi}} \int_M |\nabla f| + f(|H| + 1) d\check{\mu} \quad .
\end{aligned}$$

□

Corollary 5.4.6. *For all $f \in C^1(\overline{M})$ there exists a constant $\overline{C}_S$ depending on n and Σ such that*

$$\left(\int_M |f|^{\frac{n}{n-1}} d\check{\mu} \right)^{\frac{n-1}{n}} \leq \overline{C}_S \int_M |\nabla f| + (|H| + 1)|f| d\check{\mu}$$

We will need the following:

Lemma 5.4.7. *Suppose $f : M^n \times [0, T) \to \mathbb{R}$ is once differentiable in time such that $\frac{df}{dt}, f \in L^1(M_t)$. Then the following holds for $t > 0$ and $\beta = 0$:*

$$\frac{d}{dt} \int_{M_t} f d\check{\mu} = \int_{M_t} \frac{df}{dt} - H^2 f d\check{\mu}$$

Proof. This is the opposite case to Remark 5.0.4: If $\beta = 0$ then the manifold does not flow out through the boundary. Specifically we know that in the parametrisation defined by $\mathbf{F}$,

$$\int_{M_t} f d\breve{\mu} = \int_{M^n} f\sqrt{\det(g_{ij}(x,t))}dx \quad .$$

Since the boundary of M^n does not change with time we only need the derivative of the volume form. Using the well known determinant derivative formula we have using Lemma 5.1.1

$$\frac{d\sqrt{\det(g_{ij})}}{dt} = \frac{\frac{d\det(g_{ij})}{dt}}{2\sqrt{\det(g_{ij})}} = \frac{1}{2\sqrt{\det(g_{ij})}}\left[-2Hg^{ij}h_{ij}\det(g_{ij})\right] = -H^2\sqrt{\det(g_{ij})}$$

and therefore we have the Lemma. □

Corollary 5.4.8. *For $\beta = 0$, the area of M satisfies*

$$|M_t| = \int_{M_t} d\breve{\mu} \leq |M_0| \quad .$$

Proof. We immediately see

$$\frac{d}{dt}\int_M d\breve{\mu} = -\int_M H^2 d\breve{\mu} \leq 0 \quad .$$

□

We will also need the following iteration Lemma from [27, Lemma 4.1 i)]:

Lemma 5.4.9. *Suppose $\phi : (k_0, \infty) \to \mathbb{R}$ is a non–negative non–increasing function such that for all $h > k \geq k_0$ then*

$$\phi(h) \leq \frac{C}{(h-k)^\alpha}(\phi(k))^\beta$$

where C, α and β are positive constants. Then if $\beta > 1$ then $\phi(k_0 + d) = 0$ for

$$d^\alpha = C[\phi(k_0)]^{\beta-1}2^{\alpha\frac{\beta}{\beta-1}} \quad .$$

Proof (Translation from the French). We consider the sequence $k_s = k_0 + d - \frac{d}{2^s}$. Then $k_{s+1} = k_0 + d - \frac{d}{2^{s+1}} = k_s + \frac{d}{2^s} - \frac{d}{2^{s+1}} = k_s + \frac{d}{2^{s+1}}$. From the assumption on ϕ we know

$$\phi(k_{s+1}) = \frac{C2^{\alpha(s+1)}}{d^\alpha}(\phi(k_s))^\beta \quad .$$

We show by induction that $\phi(k_s) \leq \frac{\phi(k_0)}{2^{-s\mu}}$ where $\mu = \frac{\alpha}{1-\beta}$. This is clearly true for $s = 0$. Suppose this is true up to s. Then

$$\phi(k_{s+1}) \leq \frac{C2^{\alpha(s+1)}}{d^\alpha}(\phi(k_s))^\beta \leq \frac{C2^{\alpha(s+1)}}{d^\alpha}\left(\frac{\phi(k_0)}{2^{-s\mu}}\right)^\beta = \frac{C2^{\alpha\left(s+1+\frac{\beta}{1-\beta}s\right)}(\phi(k_0))^\beta}{C[\phi(k_0)]^{\beta-1}2^{\alpha\frac{\beta}{\beta-1}}}$$
$$= 2^{\alpha(s+1)(1+\frac{\beta}{1-\beta})}\phi(k_0) = \frac{\phi(k_0)}{2^{-(s+1)\mu}} \quad .$$

Therefore we see that $\phi(k_0 + d) = 0$ as required. □

5.4.2 The iteration argument

To prove this we will again consider $Q = \log V$. We recall

$$\left(\frac{d}{dt} - \Delta\right) Q \leq -|A|^2 - |\nabla Q|^2 \text{ on the interior of } M_t, \quad |\nabla^\top_\mu Q| \leq C_\Sigma \text{ on } \partial M_t \quad .$$

We now define $Q_k = (Q - k)_+$ and $A(k) = \{x \in M | Q_k > 0\}$. Note that Q_k^p is $p - 1$ times differentiable everywhere and may be assumed to be smooth inside $A(k)$. We define

$$\|A(k)\| = \int_0^T \int_{A(k)} d\breve{\mu}\, dt$$

and we look for estimates on this quantity. We show the following:

Proposition 5.4.10 (Partial Gradient Estimate)**.** *For all $t \in [0, T]$ we get the following gradient estimate inside tori:*

$$Q \leq k + C\|A(k)\|^{\frac{1}{4(n+1)}}$$

where $C > 0$ depends on Σ and M_0 and $k > k_1 > \sup_{M_0} Q$.

This partial gradient estimate will follow from applying Lemma 5.4.9 with $\phi(k) = \|A(k)\|$, showing that for some d, $\|A(d)\| = 0$. Then, to complete the gradient bound we will need a final estimate on $\|A(k)\|$, see Section 5.4.3. First though to prove the Proposition we will need an estimate on L^p norms of Q_k:

Lemma 5.4.11. *We may estimate for all $k > k_1$ and even p that*

$$\int_0^T \int_M Q_k^p d\breve{\mu}\, dt \leq C_Q(p)\|A(k)\|$$

where $C_Q > 0$ depends on the power p.

Proof. Using Lemma (5.4.1) we have that

$$\left(\frac{d}{dt}-\Delta\right)e^{\lambda u} = \lambda e^{\lambda u}\left[-\frac{2}{r^2}\langle \nu, \mathbf{t}\rangle\langle \nu, \mathbf{r}\rangle - \lambda|\nabla u|^2\right]$$
$$\leq \lambda e^{\lambda u}\left[\frac{W}{r^3}C_1^u - \lambda(1-\frac{W^2}{r^2})\right]$$

for $C_1^u > 0$. At the boundary this function has zero derivative in the μ direction. On $A(k)$, $W < e^{-k}$. Given a k_0 large enough we may choose $\lambda > 0$ large enough such that on $A(k)$ for $k > k_0$ we estimate

$$\left(\frac{d}{dt}-\Delta\right)e^{\lambda u} \leq -C_2^u|\nabla u|^2$$
$$\leq -3C_3^u$$

where we used our bounds on u. Writing C_n for any bounded positive constant depending only on M_0, p, Σ and n we calculate for $p > 2$ and $k > k_0$:

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)e^{\lambda u}Q_k^p &\leq -C_2^u|\nabla u|^2 Q_k^p - pe^{\lambda u}Q_k^{p-1}|\nabla Q|^2 \\ &\quad - e^{\lambda u}p(p-1)Q_k^{p-2}|\nabla Q|^2 - 4pQ_k^{p-1}\left\langle \nabla Q, \nabla e^{\lambda u}\right\rangle \\ &\leq Q_k^{p-2}[-C_2^u Q_k^2|\nabla u|^2 - pe^{\lambda u}Q_k|\nabla Q|^2 \\ &\quad - e^{\lambda u}p(p-1)|\nabla Q|^2 + C_n p Q_k|\nabla Q||\nabla u|] \\ &\leq Q_k^{p-2}\left[C_nQ_k - C_2^uQ_k^2|\nabla u|^2 - C_nQ_k|\nabla Q|^2 - C_n|\nabla Q|^2\right] \\ &\leq C_nQ_k^{p-2} - 2C_3^uQ_k^p - C_nQ_k^{p-1}|\nabla Q|^2 - C_nQ_k^{p-2}|\nabla Q|^2\end{aligned}$$

where we used Young's inequality of the form $ab = \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$ repeatedly. We may now use Lemma 5.4.7 and divergence Theorem to see that

$$\begin{aligned}\frac{d}{dt}\int_M e^{\lambda u}Q_k^p d\breve{\mu} &\leq \int_M \left(\frac{d}{dt}-\Delta\right)e^{\lambda u}Q_k^p - H^2e^{\lambda u}Q_k^p d\breve{\mu} \\ &\quad + \int_{\partial M} pC_nC_\Sigma Q_k^{p-1}d\breve{\mu}_\partial\end{aligned}$$

Estimating as above and using Lemma 5.4.5,

$$\begin{aligned}&\frac{d}{dt}\int_M e^{\lambda u}Q_k^p d\breve{\mu} \\ &\quad\leq \int_M \left(\frac{d}{dt}-\Delta\right)e^{\lambda u}Q_k^p - H^2e^{\lambda u}Q_k^p + C_nQ_k^{p-1} + C_nQ_k^{p-2}|\nabla Q|d\breve{\mu} \\ &\quad\leq \int_M Q_k^{p-2}\left[C_n - 2C_3^uQ_k^2 - C_nQ_k|\nabla Q|^2 - C_n|\nabla Q|^2 + C_nQ_k + C_n|\nabla Q|\right]d\breve{\mu} \\ &\quad\leq \int_M Q_k^{p-2}\left[C_n - C_3^uQ_k^2\right]d\breve{\mu} \quad .\end{aligned}$$

Hence choosing $k > k_1 = \max\{k_0, \sup_{M_0} Q + 1\}$ we may integrate to get

$$\int_0^T \int_{A(k)} Q_k^p d\breve{\mu}\, dt \le \widetilde{C} \int_0^T \int_{A(k)} Q_k^{p-2} d\breve{\mu}\, dt \quad .$$

We also in fact want the above to hold for $p = 2$. We see that

$$\begin{aligned}\frac{d}{dt}\int_M e^{\lambda u} Q_k^2 d\breve{\mu} &= \int_M 2e^{\lambda u} Q_k \left(\lambda Q_k \frac{du}{dt} + \frac{dQ}{dt}\right) - H^2 e^{\lambda u} Q_k^2 d\breve{\mu} \\ &= \int_{A(k)} \frac{de^{\lambda u} Q_k^2}{dt} - H^2 e^{\lambda u} Q_k^2 d\breve{\mu} \quad .\end{aligned}$$

On the open set $A(k) = \{x \in M | Q(x) > k\}$ we know that $u^2 Q_k^2$ is smooth we may write

$$\frac{d}{dt}\int_M e^{\lambda u} Q_k^2 d\breve{\mu} = \int_{A(k)} \Delta e^{\lambda u} Q_k^2 + \left(\frac{d}{dt} - \Delta\right) e^{\lambda u} Q_k^2 - H^2 e^{\lambda u} Q_k^2 d\breve{\mu}$$

where $\Delta e^{\lambda u} Q_k^p$ is calculated only inside $A(k)$. We now wish to apply the Divergence Theorem as before:

$$\int_{A(k)} \Delta e^{\lambda u} Q_k^2 d\breve{\mu} = \int_{A(k)} \mathrm{div}(\nabla(e^{\lambda u} Q_k^2)) d\breve{\mu}$$

But since $\nabla(e^{\lambda u} Q_k^2) = \lambda e^{\lambda u} Q_k^2 \nabla u + 2e^{\lambda u} Q_k \nabla Q$ is a smooth vector field on the interior and continuous up to the boundary of $A(k)$ we may still apply divergence theorem (if necessary by estimating $A(k)$ from the interior and taking the limit – continuity them implies the limits are the same). Furthermore we have that away from ∂M this vector field is 0 and so we do not get extra terms:

$$\int_{A(k)} \Delta e^{\lambda u} Q_k^2 d\breve{\mu} = \int_{\partial M \cap A(k)} \langle 2uQ_k \nabla(uQ_k), \mu\rangle\, d\breve{\mu}$$

and from here the rest of the above proof holds.

We may therefore estimate for p even and $k > k_1$

$$\int_0^T \int_M Q_k^p d\breve{\mu}\, dt \le C_Q(p)\|A(k)\| \quad .$$

□

We are now in a position to show the gradient bound.

Proof of Proposition 5.4.10. We may calculate

$$\left(\frac{d}{dt}-\Delta\right)Q_k^p \leq pQ_k^{p-1}\left(\frac{d}{dt}-\Delta\right)Q - p(p-1)Q_k^{p-2}|\nabla Q|^2$$
$$\leq -pQ_k^{p-2}\left(Q_k|\nabla Q|^2+(p-1)|\nabla Q|^2\right) \quad .$$

Using C_n as in Lemma 5.4.11 we see using the bound on $|H|$:

$$\begin{aligned}\frac{d}{dt}\int_M Q_k^p d\check{\mu} &\leq \int_M \left(\frac{d}{dt}-\Delta\right)Q_k^p - H^2Q_k^p d\check{\mu} + pC_\Sigma\int_{\partial M} Q_k^{p-1}d\check{\mu}_\partial \\ &\leq \int_M \left(\frac{d}{dt}-\Delta\right)Q_k^p - H^2Q_k^p \\ &\qquad\qquad + C_n\left[(1+|H|)Q_k^{p-1}+(p-1)Q_k^{p-2}|\nabla Q|\right]d\check{\mu} \\ &\leq \int_M Q_k^{p-2}\Big[-pQ_k|\nabla Q|^2 - p(p-1)|\nabla Q|^2 - H^2Q_k^2 \\ &\qquad\qquad + C_nQ_k + C_n|\nabla Q|\Big]d\check{\mu} \\ &\leq \int_M Q_k^{p-2}\left[C_nQ_k^2+C_n\right]d\check{\mu} - \int_M pQ_k^{p-1}|\nabla Q| + (|H|+1)Q_k^p d\check{\mu} \\ &\leq C_2\int_M Q_k^{p-2}+Q_k^p\,d\check{\mu} - C_1\left[\int_M Q_k^{\frac{np}{n-1}}d\check{\mu}\right]^{\frac{n-1}{n}} \quad .\end{aligned}$$

where on the last line we used Corollary 5.4.6 with $f=Q_k^p$. Integrating with respect to time we have:

$$\sup_{t\in[0,T]}\int_M Q_k^p d\check{\mu} + C_1\int_0^T\left[\int_M Q_k^{\frac{np}{n-1}}d\check{\mu}\right]^{\frac{n-1}{n}}dt \leq C_2\int_0^T\int_M Q_k^{p-2}+Q_k^p d\check{\mu}\,dt \quad . \tag{5.8}$$

We now deal with the left hand side of this by the standard methods: By Young's inequality of the form $ab=\frac{a^c}{c}+\frac{b^d}{d}$ where $1=\frac{1}{c}+\frac{1}{d}$ we see

$$\begin{aligned}&\sup_{t\in[0,T]}\int_M Q_k^p d\check{\mu} + C_1\int_0^T\left[\int_M Q_k^{\frac{np}{n-1}}d\check{\mu}\right]^{\frac{n-1}{n}}dt \\ &\qquad\geq C_n\left(\sup_{t\in[0,T]}\int_M Q_k^p d\check{\mu}\right)^{\frac{1}{c}}\left(\int_0^T\left[\int_M Q_k^{\frac{np}{n-1}}d\check{\mu}\right]^{\frac{n-1}{n}}dt\right)^{\frac{1}{d}} \\ &\qquad\geq C_n\left(\int_0^T\left[\int_M Q_k^p d\check{\mu}\right]^{\frac{d}{c}}\left[\int_M Q_k^{\frac{np}{n-1}}d\check{\mu}\right]^{\frac{n-1}{n}}dt\right)^{\frac{1}{d}} \quad .\end{aligned}$$

Set $q=\frac{n}{n-1}$ then we now use Hölder's inequality and a careful choice of d. We choose $d = \frac{n+1}{n}$ which implies $c = n + 1$. Then

$$\begin{aligned}\int_M f^d d\breve{\mu} &= \int_M f^{d-1} f d\breve{\mu} \\ &\leq \left[\int_M f^{\frac{q(d-1)}{q-1}} d\breve{\mu}\right]^{1-\frac{1}{q}} \left[\int_M f^q d\breve{\mu}\right]^{\frac{1}{q}} \\ &= \left[\int_M f d\breve{\mu}\right]^{\frac{1}{n}} \left[\int_M f^q d\breve{\mu}\right]^{\frac{1}{q}} \\ &= \left[\int_M f d\breve{\mu}\right]^{\frac{d}{c}} \left[\int_M f^q d\breve{\mu}\right]^{\frac{1}{q}}\end{aligned}$$

and so

$$\sup_{t\in[0,T]} \int_M Q_k^p d\breve{\mu} + C_1 \int_0^T \left[\int_M Q_k^{\frac{np}{n-1}} d\breve{\mu}\right]^{\frac{n-1}{n}} dt \geq C_n \left(\int_0^T \int_M Q_k^{pd} d\breve{\mu}\, dt\right)^{\frac{1}{d}} .$$

Putting this, equation (5.8) and Lemma 5.4.11 together and choosing p to be even we see

$$\begin{aligned}\left(\int_0^T \int_M Q_k^{pd} d\breve{\mu}\, dt\right)^{\frac{1}{d}} &\leq C_n \int_0^T \int_M Q_k^{p-2} + Q_k^p d\breve{\mu}\, dt \\ &\leq C_3 \|A(k)\| .\end{aligned}$$

The Hölder inequality now implies

$$\left(\int_0^T \int_M Q_k^{pd} d\breve{\mu}\, dt\right)^{\frac{1}{d}} \geq \frac{\int_0^T \int_M Q_k^p d\breve{\mu}\, dt}{\|A(k)\|^{1-\frac{1}{d}}}$$

to give

$$|h-k|^p \|A(h)\| \leq \int_0^T \int_M Q_k^p d\breve{\mu}\, dt \leq C_3 \|A(k)\|^{2-\frac{1}{d}}$$

where $h > k$. The first inequality comes from estimating the middle term not over all of M but only over $A(h) \subset A(k)$. Since $2 - \frac{1}{d} = 1 + \frac{1}{n+1} > 1$ then applying Lemma 5.4.9 with $\phi(k) = \|A(k)\|$ we see that $\|A(k)\| = 0$ for $k = k_1 + D$ where

$$D^p = C_3 2^{\frac{2d-1}{d-1}} \|A(k_1)\|^{\frac{d-1}{d}}$$

and so choosing say, $p = 4$, the Proposition is proved. □

Remark 5.4.12. The above proof will hold with other vector fields E with evolution equation (5.5), on the condition that the "graph height" function, the equivalent of u is bounded above and below and has a suitable evolution equation. Here an example of "suitable" would be, for some constant C_u,

$$\left(\frac{d}{dt} - \Delta\right) u \leq C_u W \quad .$$

For such evolutions the above holds.

5.4.3 The final estimate

In [14], it was possible to get an estimate on the $\|A(k)\|$ simply using the equivalent of the function u. Unfortunately the evolution of u in the case of the tori can be both positive and negative and so the same proof doesn't hold. Instead we use the function r to estimate away the new terms in the evolution of u.

Lemma 5.4.13. *The function r evolves by*

$$\left(\frac{d}{dt} - \Delta\right) r = -\frac{|\mathbf{t}^\top|^2}{r} \quad .$$

Proof. Similarly to Lemma 5.4.1 we have $\frac{dr}{dt} = -H\left\langle \mathbf{r}, \nu \right\rangle$ and calculate

$$\begin{aligned}
\Delta r &= g^{ij}\left(\frac{\partial}{\partial x^i}\left\langle \mathbf{r}, \frac{\partial}{\partial x^j}\right\rangle - \left\langle \mathbf{r}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right\rangle\right) \\
&= g^{ij}\left(\left\langle \frac{1}{r}\left\langle \frac{\partial}{\partial x^i}, \mathbf{t}\right\rangle \mathbf{t}, \frac{\partial}{\partial x^j}\right\rangle + \left\langle \mathbf{r}, \overline{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right\rangle\right) \\
&= -H\left\langle \nu, \mathbf{r}\right\rangle + \frac{1}{r}|\mathbf{t}^\top|^2 \quad .
\end{aligned}$$

□

We now show that $\|A(k)\|$ is bounded for k large enough.

Proposition 5.4.14. *There exists a $k_2 > 0$ such that for all $k > k_2$ there exists a constant C depending only on M_0, Σ and n such that*

$$\|A(k)\| \leq C$$

Proof. We consider the function $f = u^2r^2$. We will calculate the derivative of the integral of this quantity over the manifold. First though we will need the evolution equation:

$$\left(\frac{d}{dt} - \Delta\right) f = r^2 \left[\frac{4u}{r} \langle \nabla r, \nabla u \rangle - 2|\nabla u|^2\right] + u^2 \left[-2|\mathbf{t}^\top|^2 - 2|\mathbf{r}^\top|^2\right] \\ - 8ur \langle \nabla r, \nabla u \rangle \quad .$$

At the boundary we have $\nabla_\mu f = u^2 \nabla_\mu r^2$. We estimate the integral over the boundary of this using divergence theorem:

$$\int_{\partial M} \nabla_\mu f d\breve{\mu}_\partial = \int_{\partial M} u^2 \nabla_\mu r^2 d\breve{\mu}_\partial = \int_M \operatorname{div}(u^2 \nabla r^2) d\breve{\mu} = \int_M \left\langle \nabla u^2, \nabla r^2 \right\rangle + u^2 \Delta r^2 d\breve{\mu} \\ = \int_M 4ur \langle \nabla u, \nabla r \rangle + u^2 \left[-2rH \langle \nu, \mathbf{r} \rangle + 2|\mathbf{t}^\top|^2 + 2|\mathbf{r}^\top|^2\right] d\breve{\mu} \quad .$$

Therefore, by Divergence Theorem,

$$\frac{d}{dt} \int_M f d\breve{\mu} = \int_M \left(\frac{d}{dt} - \Delta\right) f - H^2 f d\breve{\mu} + \int_{\partial M} \nabla_\mu f d\breve{\mu} \\ = \int_M 4ru \langle \nabla r, \nabla u \rangle - 2r^2|\nabla u|^2 - 2u^2|\mathbf{t}^\top|^2 - 2u^2|\mathbf{r}^\top|^2 - 8ru \langle \nabla r, \nabla u \rangle \\ + 4ur \langle \nabla u, \nabla r \rangle - 2u^2 rH \langle \nu, \mathbf{r} \rangle + 2u^2|\mathbf{t}^\top|^2 + 2u^2|\mathbf{r}^\top|^2 - H^2 f d\breve{\mu} \\ = \int_M -2r^2|\nabla u|^2 - 2u^2 rH \langle \nu, \mathbf{r} \rangle - H^2 f d\breve{\mu} \quad .$$

We note that $\langle \nu, \mathbf{r} \rangle^2 + \langle \nu, \mathbf{t} \rangle^2 \leq |\nu|^2 = 1$ and so $\langle \nu, \mathbf{r} \rangle^2 \leq 1 - \langle \nu, \mathbf{t} \rangle^2 = |\mathbf{t}^\top|^2$. Since $|\nabla u|^2 = \frac{|\mathbf{t}^\top|^2}{r^2}$, using Young's inequality we see

$$\frac{d}{dt} \int_M f d\breve{\mu} \leq \int_M -2|\mathbf{t}^\top|^2 + 2(u^2 r|H|)(|\langle \nu, \mathbf{r} \rangle|) - H^2 f d\breve{\mu} \\ \leq \int_M -|\mathbf{t}^\top|^2 + u^4 r^2 H^2 d\breve{\mu} \\ \leq \int_M -|\mathbf{t}^\top|^2 + C_1 H^2 d\breve{\mu} \quad .$$

for some $C_1 > 0$ by the boundedness of r and u. We have that

$$\frac{d}{dt} \int_M d\breve{\mu} = -\int_M H^2 d\breve{\mu}$$

and so integrating we have for any time interval $[0, T)$

$$\int_0^T \int_M H^2 d\breve{\mu}\, dt \leq |M_0| = C_H \quad . \tag{5.9}$$

Similarly we have

$$\int_0^T \int_M |\mathbf{t}^\top|^2 d\check{\mu}\, dt \le C_1 \int_0^T \int_M H^2 d\check{\mu}\, dt + \int_M u^2 r^2 d\check{\mu}\Big|_{t=0} \le C_1 C_H + C_2 = C_3$$

for some constant $C_2 > 0$ depending on the bounds on u^2, r^2 and $|M_0|$. On the region $A(k)$, $-\log W \ge k$ and so $\langle \nu, \mathbf{t}\rangle \le \frac{1}{r} e^{-k}$. Choosing k_2 large enough that $\langle \nu, \mathbf{t}\rangle \le \frac{1}{\sqrt{2}}$ then

$$\|A(k)\| = \int_0^T \int_{A(k)} d\check{\mu} dt \le \int_0^T \int_{A(k)} 2|\mathbf{t}^\top|^2 d\check{\mu}\, dt = 2\int_0^T \int_M |\mathbf{t}^\top|^2 d\check{\mu}\, dt \le 2C_3 \quad . \tag{5.10}$$

□

This completes the gradient estimate. We may now prove the following:

Theorem 5.4.15. *Suppose Σ is a torus of rotation and $\beta = 0$. Then for any initial disc M_0 satisfying the boundary condition which nowhere contains the vector field* $\mathbf{t}$ *in its tangent space, a solution to equation (5.1) with initial data M_0 exists for all time and converges uniformly to a flat cross-section of the torus.*

Proof. We take Ω to be a cross-section of the torus Σ and rewrite the manifold as a graph over the cross-section, parametrising a point in M_t by rotating it back around to hit Ω, the graph function being minus the angle by which we need to rotate. This is exactly the function u. Standard calculations (see Appendix C) imply that for both uniform parabolicity and a gradient estimate we need to bound the function $v = \frac{1}{\langle \mathbf{t}, \nu\rangle} = \frac{r}{W}$. Fortunately Propositions 5.4.10 and 5.4.14 give a constant upper bound on this,

$$v = \frac{r}{W} \le C_W$$

and so we have existence for all time.

For convergence we consider integrals of the derivatives of the graph over Ω. We have $\frac{du}{dt} = -\frac{Hv}{r} = -\frac{H}{W}$. Therefore using Appendix C

$$\int_0^T \int_\Omega \left(\frac{du}{dt}\right)^2 dx\, dt = \int_0^T \int_M \frac{H^2}{rW} d\check{\mu}\, dt \le C_1 \int_0^T \int_M H^2 d\check{\mu}\, dt \le C_2$$

where $C_1, C_2 > 0$ are constants and we used equation (5.9).

We see that in coordinates $|\mathbf{t}^\top|^2 = 1 - \langle \nu, \mathbf{t}\rangle^2 = \frac{v^2-1}{v^2} = \frac{r^2|Du|^2}{v^2}$. Therefore using the gradient estimate again

$$\int_0^T \int_\Omega |Du|^2 dx\, dt \leq C_3 \int_0^T \int_\Omega \frac{r^2|Du|^2}{v^2}\frac{1}{v} dx\, dt = C_3 \int_0^T \int_M |\mathbf{t}^\top|^2 d\check{\mu}\, dt \leq C_4$$

for constants $C_3, C_4 > 0$ where we used equation (5.10).

Therefore there exists a constant $C > 0$ such that

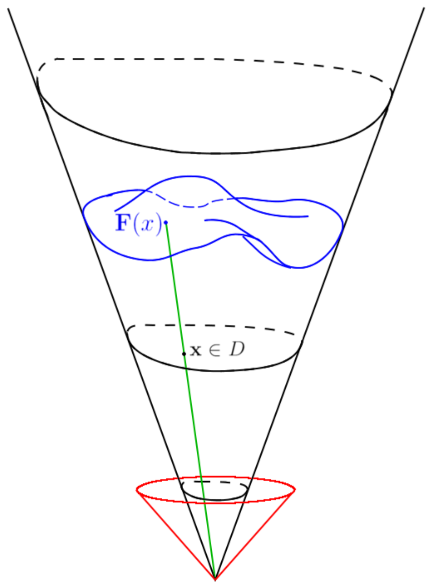
$$\int_0^\infty \int_\Omega \left(\frac{du}{dt}\right)^2 + |Du|^2 dx\, dt \leq C$$

and so we may apply Lemma 4.4.1. Therefore as in the proof of Theorem 4.4.2 (this time using that by maximum principle $\sup_{x\in M_t} u$ is non increasing and $\inf_{x\in M_t} u$ is non decreasing), M_t converges uniformly to some cross-section. □

Appendix A

Graphical coordinates inside the cone

We consider an isometric embedding of $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ at height 1 on the $\mathbf{e}_{n+1}$ axis, defining the origin of $\mathbb{R}^n$ to be the point $\mathbf{p}$ such that $|\mathbf{p}|^2 = -1$. Choose a domain contained in $D \subset B(1) \subset \mathbb{R}^n$ to be a smooth embedding with a smooth boundary ∂D ($B(1)$ here is the open unit ball in $\mathbb{R}^n$). We define the outward unit normal to D to be γ. Our boundary manifold Σ can now be constructed as the union of all rays from the origin going through $\partial\Omega$. By this construction we have that any spacelike manifold contained within Σ may be written as a graph $u : D \to \mathbb{R}_+$, and an explicit parametrisation of M is given by $\mathbf{F} : D \to \mathbb{R}^{n+1}_1$ where



$$\mathbf{F}(x) = \frac{u(x)(\mathbf{x} + \mathbf{e}_{n+1})}{\sqrt{1-|x|^2}}$$

We note that this parametrisation has been chosen so that $|\mathbf{F}|^2 = -u^2$. We see

$$\begin{aligned}\frac{\partial \mathbf{F}}{\partial x^i} &= \left[\frac{D_i u(x)}{\sqrt{1-|x|^2}} + \frac{x_i u(x)}{(1-|x|^2)^{\frac{3}{2}}}\right](\mathbf{x} + \mathbf{e}_{n+1}) + \frac{u(x)}{\sqrt{1-|x|^2}}\mathbf{e}_i \\ &= \frac{1}{\sqrt{1-|x|^2}}\left[\left(D_i u + \frac{x_i u}{1-|x|^2}\right)(\mathbf{x} + \mathbf{e}_{n+1}) + u\mathbf{e}_i\right] \quad .\end{aligned}$$

We may calculate g_{ij}:

$$\begin{aligned} g_{ij} &= \frac{1}{1-|x|^2}\Bigg[(|x|^2-1)\left(D_iu+\frac{x_iu}{1-|x|^2}\right)\left(D_ju+\frac{x_ju}{1-|x|^2}\right) \\ &\qquad + ux_j\left(D_iu+\frac{x_iu}{1-|x|^2}\right)+ux_i\left(D_ju+\frac{x_ju}{1-|x|^2}\right)+u^2\delta_{ij}\Bigg] \\ &= \frac{1}{1-|x|^2}\left[(|x|^2-1)D_iuD_ju+\frac{u^2x_ix_j}{1-|x|^2}+u^2\delta_{ij}\right] \\ &= \frac{u^2}{1-|x|^2}\left(\delta_{ij}+\frac{x_ix_j}{1-|x|^2}\right)-D_iuD_ju \quad . \end{aligned}$$

We define

$$\mathbf{V}=\mathbf{D}u+\frac{u}{1-|x|^2}(\mathbf{x}+\mathbf{e}_{n+1})+(Du\cdot x)\mathbf{e}_{n+1}$$

and note that

$$\begin{aligned} \left\langle \mathbf{V},\frac{\partial \mathbf{F}}{\partial x^i}\right\rangle &= \frac{1}{\sqrt{1-|x|^2}}\Bigg[Du\cdot x\left(D_iu+\frac{x_iu}{1-|x|^2}\right)+uD_iu-u\left(D_iu+\frac{x_iu}{1-|x|^2}\right) \\ &\qquad +\frac{u^2x_i}{1-|x|^2}-Du\cdot x\left(D_iu+\frac{x_iu}{1-|x|^2}\right)\Bigg] \\ &= 0 \quad . \end{aligned}$$

Since this is a normal vector we know that M is spacelike if $|V|^2<0$. We see this is equivalent to

$$\begin{aligned} |\mathbf{V}|^2 &= |Du|^2+Du\cdot x\left(2\frac{u}{1-|x|^2}-2\frac{u}{1-|x|^2}\right)-\frac{u^2}{1-|x|^2}-(Du\cdot x)^2 \\ &= |Du|^2-\frac{u^2}{1-|x|^2}-(Du\cdot x)^2 \\ &\le 0 \quad . \end{aligned}$$

We therefore see that on a spacelike hypersurface while V is non–zero it is in the upwards direction (with respect to $\mathbf{e}_{n+1}$): For in the $(n+1)^{\text{th}}$ direction we have

$$\frac{u}{1-|x|^2}+Du\cdot x\ge\frac{u}{1-|x|^2}-|Du|$$

by Cauchy–Schwarz. This is positive since spacelikeness implies $\frac{u^2}{1-|x|^2}\ge|Du|^2$ and so

$$\frac{u}{1-|x|^2}\ge\frac{u}{\sqrt{1-|x|^2}}\ge|Du| \quad ,$$

where the first inequality comes from $\frac{1}{\sqrt{1-|x|^2}} \geq 1$. Furthermore we see that in fact that $\mathbf{V} \neq \mathbf{0}$, for if it were then $\mathbf{D}u = -\frac{u}{1-|x|^2}\mathbf{x}$ and in this case the $\mathbf{e}_{n+1}$ coordinate would be

$$\frac{u}{1-|x|^2} + Du \cdot x = \frac{u(1-|x|^2)}{1-|x|^2} = u > 0 \quad ,$$

a contradiction. Hence the upwards pointing unit normal is

$$\nu = \frac{\mathbf{V}}{\sqrt{-|\mathbf{V}|^2}} = \frac{\mathbf{D}u + \frac{u}{1-|x|^2}(\mathbf{x} + \mathbf{e}_{n+1}) + (Du \cdot x)\mathbf{e}_{n+1}}{\sqrt{\frac{u^2}{1-|x|^2} + (Du \cdot x)^2 - |Du|^2}} \quad .$$

We also define

$$v = \sqrt{\frac{u^2}{1-|x|^2} + (Du \cdot x)^2 - |Du|^2}$$

to be a gradient–like function. This quantity will move towards zero as the normal moves towards the light cone, and therefore this can be viewed as a measure of spacelikeness.

We also need the second fundamental form. We have

$$\begin{aligned}
\frac{\partial^2 \mathbf{F}}{\partial x^i \partial x^j} &= \frac{\partial}{\partial x^j}\left(\frac{1}{\sqrt{1-|x|^2}}\left[\left(D_i u + \frac{x_i u}{1-|x|^2}\right)(\mathbf{x} + \mathbf{e}_{n+1}) + u\mathbf{e}_i\right]\right) \\
&= \frac{x_j}{1-|x|^2}\frac{\partial \mathbf{F}}{\partial x^i} + \frac{1}{\sqrt{1-|x|^2}}\frac{\partial}{\partial x^j}\left[\left(D_i u + \frac{x_i u}{1-|x|^2}\right)(\mathbf{x} + \mathbf{e}_{n+1}) + u\mathbf{e}_i\right] \\
&= \frac{x_j}{1-|x|^2}\frac{\partial \mathbf{F}}{\partial x^i} \\
&\quad + \frac{1}{\sqrt{1-|x|^2}}\left[\left(D_{ij}u + \frac{\delta_{ij}u + x_i D_j u}{1-|x|^2} + \frac{2x_i x_j u}{(1-|x|^2)^2}\right)(\mathbf{x} + \mathbf{e}_{n+1})\right. \\
&\qquad \left. + \left(D_i u + \frac{x_i u}{1-|x|^2}\right)\mathbf{e}_j + D_j u\mathbf{e}_i\right] \\
&= \frac{x_j}{1-|x|^2}\frac{\partial \mathbf{F}}{\partial x^i} + \frac{x_i}{1-|x|^2}\frac{\partial \mathbf{F}}{\partial x^j} \\
&\quad + \frac{1}{\sqrt{1-|x|^2}}\left[\left(D_{ij}u + \frac{\delta_{ij}u}{1-|x|^2} + \frac{x_i x_j u}{(1-|x|^2)^2}\right)(\mathbf{x} + \mathbf{e}_{n+1})\right. \\
&\qquad \left. + D_i u\mathbf{e}_j + D_j u\mathbf{e}_i\right] \\
&= \frac{x_j}{1-|x|^2}\frac{\partial \mathbf{F}}{\partial x^i} + \frac{x_i}{1-|x|^2}\frac{\partial \mathbf{F}}{\partial x^j} \\
&\quad + \frac{1}{\sqrt{1-|x|^2}}\left[\left(D_{ij}u + \frac{1}{u}(g_{ij} + D_i u D_j u)\right)(\mathbf{x} + \mathbf{e}_{n+1}) + D_i u\mathbf{e}_j + D_j u\mathbf{e}_i\right].
\end{aligned}$$

Therefore

$$
\begin{aligned}
h_{ij} &= -\left\langle \frac{\partial^2 \mathbf{F}}{\partial x^i \partial x^j}, \nu \right\rangle \\
&= \frac{-1}{v\sqrt{1-|x|^2}} \Bigg\langle \left(D_{ij}u + \frac{1}{u}(g_{ij} + D_i u D_j u) \right)(\mathbf{x} + \mathbf{e}_{n+1}) + D_i u \mathbf{e}_j + D_j u \mathbf{e}_i, \\
&\qquad\qquad \mathbf{D}u + \frac{u}{1-|x|^2}(\mathbf{x} + \mathbf{e}_{n+1}) + (Du \cdot \mathbf{x})\mathbf{e}_{n+1} \Bigg\rangle \\
&= \frac{-1}{v\sqrt{1-|x|^2}} \Bigg[-u \left(D_{ij}u + \frac{1}{u}(g_{ij} + D_i u D_j u) \right) + 2 D_i u D_j u \\
&\qquad\qquad + \frac{u}{1-|x|^2}(D_i u x_j + D_j u x_i) \Bigg] \\
&= \frac{1}{v\sqrt{1-|x|^2}} \left[u D_{ij}u + g_{ij} - D_i u D_j u - \frac{u}{1-|x|^2}(D_i u x_j + D_j u x_i) \right] .
\end{aligned}
$$

We will need the trace of the above and so will need the inverse of g_{ij}.

Claim A.0.16. In the above graphical coordinates

$$
\begin{aligned}
g^{ij} = \frac{1-|x|^2}{u^2} \Bigg(\delta^{ij} + \frac{1}{v^2} \Bigg[\left(|Du|^2 - \frac{u^2}{1-|x|^2} \right) x_i x_j \\
+ D_i u D_j u - Du \cdot x (x_j D_i u + x_i D_j u) \Bigg] \Bigg) .
\end{aligned}
$$

Proof. We will spare the reader the original calculation of the above, but simply show that $g^{ij} g_{jk} = \delta^i_k$. Let

$$
A^{ij} = \left[\left(|Du|^2 - \frac{u^2}{1-|x|^2} \right) x_i x_j + D_i u D_j u - Du \cdot x (x_j D_i u + x_i D_j u) \right]
$$

and we recall

$$
\begin{aligned}
g_{ij} &= \frac{u^2}{1-|x|^2} \left(\delta_{ij} + \frac{x_i x_j}{1-|x|^2} \right) - D_i u D_j u \\
v^2 &= \frac{u^2}{1-|x|^2} + (Du \cdot x)^2 - |Du|^2 .
\end{aligned}
$$

We see

$$\begin{aligned}
A^{ij}&\left(\delta_{jk}+\frac{x_jx_k}{1-|x|^2}-D_juD_ku\frac{1-|x|^2}{u^2}\right)\\
&=x_ix_k\left[\left(|Du|^2-\frac{u^2}{1-|x|^2}\right)\left(1+\frac{|x|^2}{1-|x|^2}\right)-\frac{(Du\cdot x)^2}{1-|x|^2}\right]\\
&\quad+x_iD_ku\Bigg[-\left(|Du|^2-\frac{u^2}{1-|x|^2}\right)Du\cdot x\frac{1-|x|^2}{u^2}\\
&\qquad\qquad-Du\cdot x+|Du|^2Du\cdot x\frac{1-|x|^2}{u^2}\Bigg]\\
&\quad+D_iux_k\left[\frac{Du\cdot x}{1-|x|^2}-Du\cdot x\left(1+\frac{|x|^2}{1-|x|^2}\right)\right]\\
&\quad+D_iuD_ku\left[1-|Du|^2\frac{1-|x|^2}{u^2}+(Du\cdot x)^2\frac{1-|x|^2}{u^2}\right]\\
&=\frac{-v^2}{1-|x|^2}x_ix_k+\frac{1-|x|^2}{u^2}v^2D_iuD_ku \quad .
\end{aligned}$$

Therefore we may calculate

$$\begin{aligned}
g^{ij}g_{jk}&=\left[\delta^{ij}+\frac{1}{v^2}A^{ij}\right]\left(\delta_{jk}+\frac{x_jx_k}{1-|x|^2}-D_juD_ku\frac{1-|x|^2}{u^2}\right)\\
&=\left(\delta^i_k+\frac{x_ix_k}{1-|x|^2}-D_iuD_ku\frac{1-|x|^2}{u^2}\right)-\frac{1}{1-|x|^2}x_ix_k+\frac{1-|x|^2}{u^2}D_iuD_ku\\
&=\delta^i_k \quad .
\end{aligned}$$

□

We now have the necessary quantities to work out the equations for non-parametric mean curvature flow. Non-parametric mean curvature flow is defined by

$$\left\langle\frac{d\mathbf{F}}{dt},\nu\right\rangle=-H \quad ,$$

that is, the movement in the normal direction is as in parametric mean curvature flow, but the manifold is allowed to move in tangent directions over time. We see that

$$\begin{aligned}
\left\langle\frac{d\mathbf{F}}{dt},\nu\right\rangle&=\frac{1}{v}\left\langle\frac{\frac{\partial u}{\partial t}(\mathbf{x}+\mathbf{e}_{n+1})}{\sqrt{1-|x|^2}},\mathbf{D}u+\frac{u}{1-|x|^2}(\mathbf{x}+\mathbf{e}_{n+1})+(Du\cdot x)\mathbf{e}_{n+1}\right\rangle\\
&=\frac{\frac{\partial u}{\partial t}}{v\sqrt{1-|x|^2}}\left(Du\cdot x-u-Du\cdot x\right)\\
&=\frac{-u\frac{\partial u}{\partial t}}{v\sqrt{1-|x|^2}}
\end{aligned}$$

and

$$\begin{aligned}
H &= g^{ij}h_{ij} \\
&= \frac{g^{ij}}{v\sqrt{1-|x|^2}}\left[uD_{ij}u + g_{ij} - D_iuD_ju - \frac{u}{1-|x|^2}(D_iux_j + D_jux_i)\right] \\
&= \frac{ug^{ij}D_{ij}u + n}{v\sqrt{1-|x|^2}} - \frac{g^{ij}}{v\sqrt{1-|x|^2}}\left(D_iuD_ju + \frac{u}{1-|x|^2}(D_iux_j + D_jux_i)\right) \quad .
\end{aligned}$$

Now using A^{ij} as above, that is

$$A^{ij} = \left[\left(|Du|^2 - \frac{u^2}{1-|x|^2}\right)x_ix_j + D_iuD_ju - Du\cdot x(x_jD_iu + x_iD_ju)\right]$$

then

$$\begin{aligned}
&A^{ij}\left(D_iuD_ju + \frac{u}{1-|x|^2}(D_iux_j + D_jux_i)\right) \\
&\quad = \left(|Du|^2 - \frac{u^2}{1-|x|^2}\right)(Du\cdot x)^2 + |Du|^4 - 2(Du\cdot x)^2|Du|^2 \\
&\qquad\qquad + \frac{2uDu\cdot x}{1-|x|^2}\left[\left(|Du|^2 - \frac{u^2}{1-|x|^2}\right)|x|^2 + |Du|^2 - |x|^2|Du|^2 - (Du\cdot x)^2\right] \\
&\quad = \left(|Du|^2 - \frac{u^2}{1-|x|^2}\right)(Du\cdot x)^2 - v^2|Du|^2 + \frac{u^2}{1-|x|^2}|Du|^2 - (Du\cdot x)^2|Du|^2 \\
&\qquad\qquad - \frac{2uDu\cdot x|x|^2}{1-|x|^2}v^2 + \frac{2uDu\cdot x}{1-|x|^2}\left[(1-|x|^2)|Du|^2 + (|x|^2-1)(Du\cdot x)^2\right] \\
&\quad = v^2\left(-|Du|^2 - \frac{2uDu\cdot x|x|^2}{1-|x|^2}\right) - \frac{u^2(Du\cdot x)^2}{1-|x|^2} + \frac{u^2}{1-|x|^2}|Du|^2 \\
&\qquad\qquad + 2uDu\cdot x|Du|^2 - 2u(Du\cdot x)^3 \\
&\quad = v^2\left(-|Du|^2 - \frac{2uDu\cdot x|x|^2}{1-|x|^2} - \frac{u^2}{1-|x|^2} - 2uDu\cdot x\right) \\
&\qquad\qquad + \frac{u^4}{(1-|x|^2)^2} + \frac{2u^3Du\cdot x}{1-|x|^2} \quad .
\end{aligned}$$

Therefore setting

$$f = \frac{u^2}{1-|x|^2} + 2uDu\cdot x = \frac{1-|x|^2}{u^2}\left(\frac{u^4}{(1-|x|^2)^2} + \frac{2u^3Du\cdot x}{1-|x|^2}\right)$$

then

$$\begin{aligned}H &= \frac{ug^{ij}D_{ij}u + n}{v\sqrt{1-|x|^2}} - \frac{\sqrt{1-|x|^2}}{vu^2}\Bigg(|Du|^2 + \frac{2uDu\cdot x}{1-|x|^2} - |Du|^2 - \frac{2uDu\cdot x|x|^2}{1-|x|^2} \\ &\qquad - \frac{u^2}{1-|x|^2} - 2uDu\cdot x + \frac{u^2 f}{(1-|x|^2)v^2}\Bigg) \\ &= \frac{ug^{ij}D_{ij}u + n + 1}{v\sqrt{1-|x|^2}} - \frac{f}{v^3\sqrt{1-|x|^2}} \\ &= \frac{ug^{ij}D_{ij}u + n + 1}{v\sqrt{1-|x|^2}} - \frac{\frac{u^2}{1-|x|^2} + 2uDu\cdot x}{v^3\sqrt{1-|x|^2}} \quad .\end{aligned}$$

So on the interior of D we have that

$$\begin{aligned}\frac{\partial u}{\partial t} &= g^{ij}D_{ij}u + \frac{n+1}{u} - \frac{f}{uv^2} \\ &= g^{ij}D_{ij}u + \frac{n+1}{u} - \frac{\frac{u}{1-|x|^2} + 2Du\cdot x}{v^2} \quad .\end{aligned}$$

We also require the boundary condition in graphical coordinates, and therefore need to calculate μ, the outwards pointing normal to Σ in terms of γ, the outwards pointing normal to D. Fortunately by the construction of Σ, μ is constant on rays from the origin; consequentially we only need calculate μ in terms of γ on $\partial D \subset \mathbb{R}^{n+1}_1$. At a point $\mathbf{x} \in \partial D$ we see that the position vector $W = \mathbf{x} + \mathbf{e}_{n+1}$, as one of the rays making up Σ, is in $T_{\mathbf{x}}\Sigma$. We see that

$$\langle W, \gamma + \gamma\cdot x\mathbf{e}_{n+1}\rangle = 0$$

and since $\mathbf{e}_{n+1} \perp \partial D$ then this must be in the direction of the outwards pointing normal. Therefore

$$\mu = \frac{\gamma + \gamma\cdot x\mathbf{e}_{n+1}}{\sqrt{1-(\gamma\cdot x)^2}} \quad .$$

The condition $\langle \nu, \mu\rangle = 0$ becomes

$$\begin{aligned}\langle \nu, \mu\rangle &= \frac{1}{v\sqrt{1-(\gamma\cdot x)^2}}\left\langle \mathbf{D}u + \frac{u}{1-|x|^2}(\mathbf{x}+\mathbf{e}_{n+1}) + (Du\cdot x)\mathbf{e}_{n+1}, \gamma + \gamma\cdot x\mathbf{e}_{n+1}\right\rangle \\ &= \frac{1}{v\sqrt{1-(\gamma\cdot x)^2}}\left(Du\cdot\gamma - Du\cdot x\gamma\cdot x\right) \\ &= 0 \quad .\end{aligned}$$

Hence our boundary condition is

$$Du\cdot(\gamma - \gamma\cdot x\mathbf{x}) = 0$$

where we note that since $|x| \leq Q < 1$ then by Cauchy–Schwarz $C = \gamma - \gamma.x\mathbf{x}$ is a bounded outward pointing vector field with the property $1 + Q^2 \leq |C| \leq 1 - Q^2$.

Putting all of this together we see that if our initial manifold may be written in this parametrisation as u_0 then $u : D \times [0, T) \to \mathbb{R}_+$ satisfies non-parametric mean curvature flow with a perpendicular boundary condition if

$$\begin{cases} \frac{\partial u}{\partial t} = g^{ij} D_{ij} u + \frac{n+1}{u} - \frac{\frac{u}{1-|x|^2} + 2Du\cdot x}{v^2} & \forall (x,t) \in D \times [0,T) \\ u(x,0) = u_0(x) & \forall x \in D \\ Du \cdot (\gamma - \gamma.x\mathbf{x}) = 0 & \forall (x,t) \in \partial D \times [0,T) \end{cases} \tag{A.1}$$

We now summarise the results so far:

Proposition A.0.17. *Under the parametrisation defined above we may express geometric quantities on the manifold in terms of u, explicitly*

$$\begin{aligned} g_{ij} &= \frac{u^2}{1-|x|^2}\left(\delta_{ij} + \frac{x_i x_j}{1-|x|^2}\right) - D_i u D_j u \\ \nu &= \frac{1}{v}\left(\mathbf{D}u + \frac{u}{1-|x|^2}(\mathbf{x} + \mathbf{e}_{n+1}) + (Du\cdot x)\mathbf{e}_{n+1}\right) \\ g^{ij} &= \frac{1-|x|^2}{u^2}\left(\delta^{ij} + \frac{1}{v^2}\left[\left(|Du|^2 - \frac{u^2}{1-|x|^2}\right) x_i x_j + \right.\right. \\ &\qquad\qquad \left.\left. D_i u D_j u - Du\cdot x(x_j D_i u + x_i D_j u)\right]\right) \\ h_{ij} &= \frac{1}{v\sqrt{1-|x|^2}}\left[u D_{ij} u + g_{ij} - D_i u D_j u - \frac{u}{1-|x|^2}(D_i u x_j + D_j u x_i)\right] \\ H &= \frac{u g^{ij} D_{ij} u + n + 1}{v\sqrt{1-|x|^2}} - \frac{\frac{u^2}{1-|x|^2} + 2u Du\cdot x}{v^3\sqrt{1-|x|^2}} \end{aligned}$$

where

$$v = \sqrt{\frac{u^2}{1-|x|^2} + (Du\cdot x)^2 - |Du|^2}\ .$$

We also see that non-parametric mean curvature flow with a perpendicular boundary condition becomes a PDE on $u : D \times [0,T) \to \mathbb{R}_+$, specifically u must satisfy

$$\begin{cases} \frac{\partial u}{\partial t} = g^{ij} D_{ij} u + \frac{n+1}{u} - \frac{\frac{u}{1-|x|^2} + 2Du\cdot x}{v^2} & \forall (x,t) \in D \times [0,T) \\ u(x,0) = u_0(x) & \forall x \in D \\ Du \cdot (\gamma - \gamma.x\mathbf{x}) = 0 & \forall (x,t) \in \partial D \times [0,T) \end{cases} .$$

Naturally we will wish to calculate when the equation (A.1) is parabolic so as to be able to apply parabolic existence theory. We look to bound the greatest and least eigenvalues of g^{ij} below from infinity and above zero respectively. Clearly since this is the inverse of the metric, if our manifold if the tangent space of our flowing manifold hits or goes beyond the light cone then we have no hope: At this point the metric will have a zero or negative eigenvector, removing any chance of positive definiteness of g^{ij}. This is the reason for stipulating that the initial manifold is spacelike.

As spacelikeness is clearly an issue to the parabolicity of this PDE we expect that parabolicity will depend in some way on v, our estimate of how close ν is to the light cone. The following comes as no surprise.

Proposition A.0.18. *For spacelike M the eigenvalues $\widetilde{\lambda}$ of g^{ij} are bounded by*

$$\frac{C_1}{u^2} \leq \widetilde{\lambda} \leq C_2 \max\left\{\frac{1}{u^2}, \frac{1}{v^2}\right\}$$

where C_1 and C_2 depend only on ∂D.

Proof. We wish to bound the eigenvalues of g^{ij} but looking at the formula for this (i.e. Claim A.0.16) would rather not do so directly. We instead choose to bound the eigenvalues of g_{ij} and use that if λ is an eigenvalue of g_{ij} then λ^{-1} is an eigenvalue of g^{ij}. Set

$$B_{ij} = \frac{1-|x|^2}{u^2} g_{ij} = \delta_{ij} + \frac{x_i x_j}{1-|x|^2} - \frac{(1-|x|^2) D_i u D_j u}{u^2} \quad .$$

We firstly note that $B_{ij} - \delta_{ij}$ is degenerate for $n > 2$: The image of $B_{ij} - \delta_{ij}$ is spanned by $\mathbf{x}$ and $\mathbf{D}u$ and so in fact the rank is at most 2. Therefore B_{ij} has $n - \dim(\text{span}\{\mathbf{x}, \mathbf{D}u\})$ eigenvectors perpendicular to $\mathbf{x}$ and $\mathbf{D}u$ of eigenvalue 1. We now deal with the remaining eigenvalues.

Suppose first $\dim(\text{span}\{\mathbf{x}, \mathbf{D}u\}) = 2$. The remaining eigenvectors must be written $a\mathbf{D}u + b\mathbf{x}$, and so the problem reduces down to solving the 2×2 matrix eigenvalue problem

$$\begin{pmatrix} 1 - \frac{|Du|^2(1-|x|^2)}{u^2} & \frac{Du.x}{1-|x|^2} \\ -\frac{Du.x(1-|x|^2)}{u^2} & \frac{1}{1-|x|^2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} \quad .$$

The determinant of the above matrix is

$$\frac{1}{u^2}\det\begin{pmatrix} \frac{u^2}{1-|x|^2} - |Du|^2 & Du.x \\ -Du.x & 1 \end{pmatrix} = \frac{v^2}{u^2} \quad .$$

The eigenvalues are then the solutions to

$$\lambda^2 - \left(1 - \frac{|Du|^2(1-|x|^2)}{u^2} + \frac{1}{1-|x|^2}\right)\lambda + \frac{v^2}{u^2} = 0 \quad .$$

Using the Cauchy–Schwarz inequality we have

$$\begin{aligned} \frac{v^2}{1-|x|^2} &= \frac{\frac{u^2}{1-|x|^2} + (Du.x)^2 - |Du|^2}{1-|x|^2} \\ &\leq \frac{u^2}{(1-|x|^2)^2} + \frac{(|x|^2-1)|Du|^2}{1-|x|^2} \\ &= \frac{u^2}{(1-|x|^2)^2} - |Du|^2 \end{aligned}$$

and so the trace is bounded by

$$T = \left(1 - \frac{|Du|^2(1-|x|^2)}{u^2} + \frac{1}{1-|x|^2}\right) \geq 1 + \frac{v^2}{u^2} \geq 0$$

where we are using weak spacelikeness for the last inequality. Since the square root in a concave function we know

$$\sqrt{C^2 + x} \leq |C| + \frac{x}{2|C|}$$

and so using these two estimates

$$\lambda_- = \frac{T - \sqrt{T^2 - 4\frac{v^2}{u^2}}}{2} \geq \frac{v^2}{Tu^2} \geq \frac{v^2}{2u^2}$$

where we estimated $T \leq 2$. Similarly

$$\lambda_+ = \frac{T + \sqrt{T^2 - 4\frac{v^2}{u^2}}}{2} \leq T - \frac{v^2}{u^2 T} \leq 2 \quad .$$

In the case $\dim(\text{span}\{\mathbf{x}, \mathbf{D}u\}) = 1$ we have one of three possibilities

1. $\mathbf{D}u$ is an eigenvector and $\mathbf{x} = \mathbf{0}$. This implies

$$1 \leq \lambda = 1 - \frac{|Du|^2}{u^2} = \frac{v^2}{u^2}$$

   from the definition of v.

2. $\mathbf{x}$ is an eigenvector and $\mathbf{D}u = \mathbf{0}$. Here we have

$$\lambda = 1 + \frac{|x|}{1-|x|^2} = \frac{1}{1-|x|^2}$$

which clearly bounded above and below, the upward bound depending on ∂D.

3. $\mathbf{D}u = \eta\mathbf{x}$ is an eigenvector and in this case

$$\lambda = 1 + \frac{|x|^2}{1-|x|^2} - \eta\frac{Du\cdot x(1-|x|^2)}{u^2} = \frac{1}{1-|x|^2} - \frac{\eta^2|x|^2(1-|x|^2)}{u^2} \quad .$$

We note that here $v^2 = \frac{u^2}{1-|x|^2} - \eta^2|x|^2(1-|x|^2)$ and so

$$\lambda = \frac{v^2}{u^2} \quad .$$

In the final case $\dim(\text{span}\{\mathbf{x}, \mathbf{D}u\}) = 0$, clearly $\mathbf{0} = \mathbf{x} = \mathbf{D}x$ and so $B_{ij} = \delta_{ij}$ and we are done.

Since given that λ is an eigenvector of B_{ij} then $\tilde{\lambda} = \frac{1-|x|^2}{u^2\lambda}$ is an eigenvalue of g^{ij} then we see that either

$$\frac{1-|x|^2}{2u^2} \leq \tilde{\lambda} \leq \max\left\{2\frac{1-|x|^2}{v^2}, \frac{1-|x|^2}{u^2}\right\} \quad .$$

Hence positive definiteness of g^{ij} is equivalent to a bound from below on v and a bound from above and below on u^2. □

Appendix B

Graphical coordinates in Minkowski space

Given $(n+1)$–dimensional Minkowski space parametrised as in Example 1.1.4 then we have an isometrically embedded copy of n-dimensional Euclidean space, $\mathbb{R}^n = \operatorname{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. We consider a spacelike manifold M which is a graph over $\mathbb{R}^n$ – that is M is parametrised by $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^{n+1}_1$ by

$$\mathbf{F}(\mathbf{x}) = \mathbf{x} + u(\mathbf{x})\mathbf{e}_{n+1}$$

where $u : \mathbf{R}^n \to \mathbf{R}$. Writing $\frac{\partial u}{\partial x^i} = D_i u$ then

$$\frac{\partial \mathbf{F}}{\partial x^i} = \mathbf{e}_i + D_i u \mathbf{e}_{n+1} \text{ and } g_{ij} = \left\langle \frac{\partial \mathbf{F}}{\partial x^i}, \frac{\partial \mathbf{F}}{\partial x^j} \right\rangle = \delta_{ij} - D_i u D_j u \quad .$$

Claim B.0.19. The inverse of the metric is

$$g^{jk} = \delta_{jk} + \frac{D_j u D_k u}{1 - |Du|^2} \quad .$$

Proof. This is easily verified:

$$\begin{aligned} g_{ij} g^{jk} &= (\delta_{ij} - D_i u D_j u) \left(\delta_{jk} + \frac{D_j u D_k u}{1 - |Du|^2} \right) \\ &= \delta_{ik} + \frac{D_i u D_k u}{1 - |Du|^2} - \left(D_i u D_k u + \frac{|Du|^2 D_j u D_k u}{1 - |Du|^2} \right) \\ &= \delta_{ik} + \frac{D_i u D_k u}{1 - |Du|^2} - \frac{D_i u D_k u}{1 - |Du|^2} \\ &= \delta_{ik} \quad . \end{aligned}$$

□

We define $v = \sqrt{1 - |Du|^2}$and we see that the upwards pointing unit normal is

$$\nu = \frac{1}{v}(Du + e_{n+1})$$

since $\langle Du + e_{n+1}, Du + e_{n+1}\rangle = |Du|^2 - 1$ and $\left\langle \frac{\partial \mathbf{F}}{\partial x^i}, \nu \right\rangle = 0$. Similarly we calculate

$$h_{ij} = -\langle \nu, D_{ij}F\rangle = -\langle \nu, D_{ij}u\mathbf{e}_{n+1}\rangle = \frac{D_{ij}u}{v} \quad .$$

As in Chapter 4 we define

$$\mathbf{a}(\mathbf{q}) = \frac{\mathbf{q}}{\sqrt{1-|\mathbf{q}|^2}}\Big|_{\mathbf{p}=Du} \text{ and } a^{ij} = \frac{\partial a^i}{\partial q^j}\Big|_{\mathbf{p}=Du} = \frac{\delta_{ij}}{\sqrt{1-|\mathbf{q}|^2}} + \frac{q^i q^j}{(1-|\mathbf{q}|^2)^{\frac{3}{2}}}\Big|_{\mathbf{p}=Du} \quad .$$

We therefore see that

$$H = \left(\frac{\delta_{jk}}{v} + \frac{D_j u D_k u}{v^3}\right) D_{ij}u = a^{ij}D_{ij}u = D_i\left(a^i\right) = D_i\left(\frac{D_i u}{\sqrt{1-|Du|^2}}\right) \quad .$$

For reparametrised mean curvature flow we require

$$-H = \left\langle \frac{d\mathbf{F}}{dt}, \nu \right\rangle = -\frac{\frac{du}{dt}}{v} \quad .$$

We therefore have that on the interior of a flowing manifold

$$\frac{du}{dt} = Hv = vD_i(a^i) = \sqrt{1-|Du|^2}D_i\left(\frac{D_i u}{1-|Du|^2}\right) = \left(\delta_{ij} + \frac{D_i u D_j u}{1-|Du|^2}\right) D_{ij}u \quad .$$

If we have a cylinder boundary (as in Chapter 4) we have $\langle \nu, \gamma\rangle = 0$ then we require

$$0 = \langle \gamma, \nu\rangle = \frac{1}{v}\langle \gamma, Du + \mathbf{e}_{n+1}\rangle = \frac{Du\cdot\gamma}{v} \quad .$$

In Chapter 4 we are interested in the parabolicity of this, and therefore we need to know when the coefficient matrix of the second derivatives is positive definite. This matrix is exactly $g^{ij} = \left(\delta_{ij} + \frac{D_i u D_j u}{1-|Du|^2}\right)$ but here I calculate the eigenvalues of this matrix's slightly simpler inverse $g_{ij} = \delta_{ij} - D_i u D_j u$ instead. Since $g_{ij} - \delta_{ij} = -D_i u D_j u$ has determinant zero and rank 1, we see that all but one of the eigenvalues is 1. We see that

$$g_{ij}D_j u = D_i u\left(1 - |Du|^2\right)$$

and therefore the remaining eigenvector is Du with eigenvalue $v^2 < 1$. Therefore uniform parabolicity is equivalent to a bound $v > C > 0$ for C some constant.

Appendix C

Graphical coordinates within tori in $\mathbb{R}^{n+1}$

Again this Appendix contains simple geometric calculations, this time in Euclidean space with Σ a torus. The results here are unsurprising, particularly given the previous two Appendices. Let $\Omega \subset \mathbb{R}^{n-1} \times \mathbb{R}_+$ be a compact domain with smooth boundary $\partial\Omega$ – this will be the cross section of the torus. Let $u : \Omega \to \mathbb{R}$. Then writing $\mathbf{x} = (x^1, \ldots, x^n) = (\mathbf{y}, r) = (y_1, \ldots, y_{n-1}, r) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$ we define $\mathbf{F} : \Omega \to \mathbb{R}^{n+1}$ by

$$\mathbf{F}(\mathbf{x}) = \mathbf{y} + r(\cos(u)\mathbf{e}_n + \sin(u)\mathbf{e}_{n+1}) \quad .$$

The function $\mathbf{F}$ takes Ω and wraps it around the inside of the torus by angle $u(x)$. We also define

$$\mathbf{r} = \cos(u)\mathbf{e}_n + \sin(u)\mathbf{e}_{n+1}, \quad \mathbf{t} = -\sin(u)\mathbf{e}_n + \cos(u)\mathbf{e}_{n+1} \quad .$$

We calculate

$$\frac{\partial \mathbf{F}}{\partial y^\alpha} = \mathbf{e}_\alpha + r\frac{\partial u}{\partial y^\alpha}\mathbf{t} \quad \text{and} \quad \frac{\partial \mathbf{F}}{\partial r} = \mathbf{r} + r\frac{\partial u}{\partial r}\mathbf{t} \quad .$$

For the duration of this Appendix let Greek $\alpha, \beta, \ldots$ imply indices in the range $\{1, \ldots, n-1\}$, then

$$g_{\alpha\beta} = \delta_{\alpha\beta} + r^2\frac{\partial u}{\partial y^\alpha}\frac{\partial u}{\partial y^\beta}, \quad g_{\alpha r} = r^2\frac{\partial u}{\partial y^\alpha}\frac{\partial u}{\partial r}, \quad g_{rr} = 1 + r^2\left(\frac{\partial u}{\partial r}\right)^2$$

and so in summary letting r be the n^{th} direction, and writing $D_i u = \frac{\partial u}{\partial x^i}$ then

$$g_{ij} = \delta_{ij} + r^2 D_i u D_j u \quad .$$

We now see that

$$g^{ij} = \delta_{ij} - \frac{r^2 D_i u D_j u}{1 + r^2|Du|^2}$$

since

$$\begin{aligned}\left(\delta_{ij} + r^2 D_i u D_j u\right)&\left(\delta_{jk} - \frac{r^2 D_j u D_k u}{1 + r^2|Du|^2}\right) \\ &= \delta_{ik} + r^2 D_i u D_k u \left[1 - \frac{1}{1 + r^2|Du|^2} - \frac{r^2|Du|^2}{1 + r^2|Du|^2}\right] \\ &= \delta_{ik} \quad .\end{aligned}$$

As in the previous Appendix we may quickly calculate that g_{ij} has $n-1$ eigenvalues equal to 1 and the remaining eigenvector, $\mathbf{D}u$, has eigen value $1+r^2|Du|^2$. Therefore we have

$$\sqrt{\det(g_{ij})} = \sqrt{1 + r^2|Du|^2} = v \quad .$$

We easily see that the unit normal to the graph is

$$\nu = \frac{-r(D_\alpha u \mathbf{e}_\alpha + D_r u \mathbf{r}) + \mathbf{t}}{v} \quad ,$$

because for example

$$v\left\langle \frac{\partial \mathbf{F}}{\partial y^\beta}, \nu \right\rangle = \left\langle \mathbf{e}_\beta + r\frac{\partial u}{\partial y^\beta}\mathbf{t}, v\nu \right\rangle = -rD_\beta u + rD_\beta u = 0 \quad .$$

Since

$$\begin{aligned}\frac{\partial^2 \mathbf{F}}{\partial y^\alpha \partial y^\beta} &= rD_{\alpha\beta}u\mathbf{t} - rD_\alpha u D_\beta u \mathbf{r} \\ \frac{\partial^2 \mathbf{F}}{\partial y^\alpha \partial r} &= \left(D_\alpha u + rD^2_{\alpha r}u\right)\mathbf{t} - rD_\alpha u D_r u \mathbf{r} \\ \frac{\partial^2 \mathbf{F}}{\partial^2 r} &= \left(2D_r u + rD^2_{rr}u\right)\mathbf{t} - r(D_r u)^2\mathbf{r}\end{aligned}$$

so in summary

$$\frac{\partial^2 \mathbf{F}}{\partial x^i \partial x^j} = \left(rD^2_{ij}u + \delta_{ni}D_j u + \delta_{nj}D_i u\right)\mathbf{t} - rD_i u D_j u \mathbf{r} \quad .$$

The second fundamental form may now be calculated

$$\begin{aligned}-vh_{ij} &= \left\langle \mathbf{t} - r(D_\alpha u \mathbf{e}_\alpha + D_r u \mathbf{r}), \frac{\partial^2 \mathbf{F}}{\partial x^i \partial x^j} \right\rangle \\ &= rD^2_{ij}u + \delta_{ni}D_j u + \delta_{nj}D_i u + r^2 D_r u D_i u D_j u \quad .\end{aligned}$$

Therefore

$$\begin{aligned}
-Hv = -g^{ij}h_{ij}v &= g^{ij}\left(rD^2_{ij}u + \delta_{ni}D_ju + \delta_{nj}D_iu + r^2D_ruD_iuD_ju\right)\\
&= rg^{ij}D^2_{ij}u + \left(\delta_{ij} - \frac{r^2D_iuD_ju}{1+r^2|Du|^2}\right)\left(\delta_{ni}D_ju + \delta_{nj}D_iu + r^2D_ruD_iuD_ju\right)\\
&= rg^{ij}D^2_{ij}u + 2D_ru + r^2D_ru|Du|^2 - \frac{r^2D_ru|Du|^2}{v^2}\left(2 + r^2|Du|^2\right)\\
&= rg^{ij}D^2_{ij}u + D_ru + r^2D_ru|Du|^2 + D_ru\left(1 - \frac{r^2|Du|^2}{v^2}\right) - r^2D_ru|Du|^2\\
&= rg^{ij}D^2_{ij}u + D_ru\left(1+\frac{1}{v^2}\right) \quad .
\end{aligned}$$

For reparametrised mean curvature flow we have

$$-H = \left\langle \frac{\partial \mathbf{F}}{\partial t}, \nu \right\rangle = \langle rD_tu\mathbf{t}, \nu\rangle = \frac{rD_tu}{v} \quad .$$

Therefore on the interior of the graph mean curvature flow is equivalent to

$$D_tu = g^{ij}D_{ij}u + \frac{D_ru}{r}\left(1+\frac{1}{v^2}\right) \quad .$$

In these coordinates $\mu = \gamma + \gamma^n(\mathbf{r} - \mathbf{e}_n)$, where $\gamma = (\gamma^1, \ldots, \gamma^n)$ is the outwards pointing unit normal to $\partial\Omega$. Then the boundary condition becomes

$$0 = \langle \nu, \mu\rangle = \frac{1}{v}\left(-rD_\alpha u\gamma^\alpha + D_ru\gamma^n\right) = \frac{Du\cdot\gamma}{v} \quad .$$

We see that uniform parabolicity is equivalent to the gradient estimate $v < C < \infty$. To get uniform parabolicity we therefore wish to bound the volume element from above, that is

$$v = \frac{1}{\langle \mathbf{t}, \nu\rangle} \quad ,$$

and since $v = \sqrt{1 + r^2|Du|^2}$ this estimate also supplies a gradient estimate.

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