The mathematics curriculum and pupils' Thinking processes: Learning to solve algebraic problems in year 7 and year 8 in England and Thailand

Narumon Sakpakornkan

The Mathematics Curriculum and Pupils' Thinking Processes: Learning to Solve Algebraic Problems in Year 7 and Year 8 in England and Thailand

Narumon Sakpakornkan

A Thesis Submitted for the Degree of Doctor of Philosophy

University of Durham
School of Education

2004

ABSTRACT

Previous studies of pupils' learning of algebra have been inclined to study errors in given answers. The present study attempts to investigate pupils' thinking processes in the early stages of learning algebra by examining and comparing responses made by English and Thai pupils to the researcher's algebra test.

The research was conducted among pupils during their "normal" lessons in secondary school algebra. Pupil participants were in the first two years of secondary education. Data collection included lesson observations, interviews, and the researcher's algebra test. The thinking processes were first categorised from the interview data to provide a framework for analysing pupils' written responses to the researcher's test. Later, a codebook was kept in which pupils' responses to the researcher's test were coded. The study goes on to analyse the outcomes from this coding procedure.

The research indicates that the differences in the way pupils think appear to be closely related to the teaching and curriculum provided. In both countries, success in algebra is dependent on having good arithmetic skills. Also the reluctance of pupils to use algebra to solve easy problems results in algebraic skills being inadequately developed to solve more difficult problems.

An implication for practice is that the Thai school should consider the bearing which the understanding of simplification of like terms has upon a pupil's ability to solve linear equations. Both schools could consider ways of making effective use of patterns and sequences to develop algebraic thinking. The codebook developed in this research could serve as a tool for mathematics teachers in helping them to understand the complexity of their pupils' thinking processes in solving algebraic problems. An investigation involving more schools in other settings could follow this.

Declaration

I declare that this thesis, which I submit for the degree of Doctor of Philosophy at the University of Durham, results entirely from my work and has not previously been submitted for a degree at this or any other university.

Statement of Copyright

Acknowledgements

First of all I would like to thank my supervisors, Professor Barry Cooper and Dr. Tony Harries, for their invaluable comments, support and encouragement during my study in Durham.

I am greatly indebted to the groups of pupils from the English and Thai schools who were always willing to assist me. I also wish to thank the Head Teachers, Head of Mathematics Departments and in particular the four mathematics teachers from each of the schools who allowed me to observe their lessons.

My gratitude also goes out to all my friends in the Department of mathematics and statistics at Buriram Rajabhat University, the participating Thai school, and also to my friends in Newcastle and Durham, in particular to Dr. Angela R. Penny, Estée-Wale Ricardo and Kosoom Kreetong for their supports. I also wish to thank Siriwan Paennarin for her assistance in checking the translated documents. I would also like to thank all members of the Old Elvet Methodist Church house group, in particular to Chris Webb for his kindness and support.

I am grateful to the Thai government for their financial assistance through a scholarship that made my study in England possible.

Finally, I would like to express my warmest appreciation to my parents, aunts, and brothers for their moral supports and encouragement throughout my study in England.

TABLE OF CONTENTS

LIST OF TABLES, FIGURES, AND APPENDICES

TABLES

FIGURES

APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Research Background

Thailand is now confronting the most drastic social changes both from within itself and from its interconnectedness with a complex and rapidly changing world. Reform of the education system is one of the most important areas of social reform since it is believed that education is a very important part of the process needed to enhance individual development within the country. One of the goals of the national policy directives is to improve curricular content and teaching-learning processes at all levels and types of education (ONCE, 2001). The changes in the greater need for mathematics in an information-age world, changes in how mathematics is used and changes in the role of technology currently push the need for reform of mathematics curricula in Thailand.

In Britain, the Blair's government's plans for the future of education centres on the creation of a 'post comprehensive' climate. The government has proposed a radical reform of secondary education (DfES, 2001a), which will lead to the conversion of half of the country's comprehensive schools into specialist institutions by the year 2006.

Two case studies were pursued in order to gain knowledge about the similarities and differences of pupils' thinking processes in solving algebraic problems between Thailand and England. According to the results from the repeat of the Third International Mathematics and Science Study (TIMSS-R), the most difficult content area for Thai students was algebra with average scores significantly lower than the international average (Klainin, 2003a). Therefore the area of mathematics chosen for the study was algebra. Two sets of Year 7 pupils and two other sets in Year 8 in one school in the Northeast of England and broadly comparable groups in Thailand were studied in depth. The results of the study informed the issue of "how mathematics curricula might be interrelated with the pupils' thinking processes in solving algebraic problems?". Through the study it was possible to gain a better understanding of teaching-learning processes and the relationship between the mathematics curriculum and pupils' thinking processes. The English and Thai schools started to teach algebra during the same school year but there were different ways in which it was delivered. The investigation of pupils' thinking processes in solving algebraic problems, in these two schools, will be useful for the mathematics teachers. The comparative approach can help to inform mathematics curriculum change in Thailand.

1.2 Methodology

The present study is based on research investigating pupils' thinking processes in solving algebraic problems in the English and the Thai schools. The research was designed to consist of two main studies, one qualitative in nature, and the other quantitative. Qualitative data was obtained from observing algebra lessons, semi-structured interviews, and pupils' written responses to an algebra test administered by the researcher. Quantitative aspects involved the proportion of achievement scores, and proportion scores of the use of generalisable and other processes in pupils' responses to the algebra test.

1.3 Describing the chapters

This section presents an overview of chapters that form this study.

Chapter 2 discusses the background rationale for using a comparative case study in investigating pupils' thinking processes when solving algebraic problems. This chapter also presents an overview of education in England and Thailand before higher education. It looks at the education reform movement and mathematics curricula in both countries. The algebra results from the repeat of the Third International Mathematics and Science Study (TIMSS-R) and the mathematical literacy scores from the Programme for International Student Assessment (PISA) are looked at. Mathematics curricula and algebra curricula used in the participating schools are also presented.

Chapter 3 considers algebraic thinking and findings from previous research, which has addressed pupils' difficulties with learning algebra. The approach to algebra adopted in the present study is also discussed.

Chapter 4 addresses research design including the ways of choosing comparable research sites, the case study schools, ethical considerations, data sources, instrumentation, and the researcher's roles. It also looks at preparation for data collection relating to lesson observations, interviews, and the researcher's test. An evolution of method of analysing data and development of a codebook are addressed.

Chapter 5 presents the quantitative results of the algebra test by comparing pupils' mean proportion achievement scores. Also a measure of pupils' thinking processes is developed in order to make comparisons between the two case study schools.

Chapter 6 discusses the qualitative results and findings from the algebra test.

Chapter 7 gives conclusions and implications for further research.

1.4 The research aims

The aims of the research were to:

- investigate the mathematics curricula in English and Thai schools as they relate to the pupils' thinking processes in solving algebraic problems,
- analyse the pupils' thinking processes in solving algebraic problems,
- relate the pupils' thinking processes in solving algebraic problems to their experience in algebra lessons in their own country.

1.5 The research questions

A major concern of mathematics is problem solving and the way in which understanding is gained by working through exercises. Researchers and scholars consider this process and indicate some important sources of pupils' difficulties with mathematics. For example, Gray and Tall (1994) suggest that "ambiguity of notation allows the successful thinker the flexibility in thought to move between the process to carry out a mathematical task and the concept to be mentally manipulated as part of a wider mental schema" (p. 116). From this schema there are many studies that assume that arithmetic precedes algebra. For instance, Filloy and Rojano (1989) address the pupils' transition from arithmetic to algebra. They introduce the notion of "didactic cut" between arithmetic and algebra, which arises when the pupil's arithmetic resources break down in tackling linear equations. Meanwhile, Herscovics and Linchevski (1994) introduce the notion of a "cognitive gap", that is, pupils' inability to operate spontaneously with or on the unknown within the equations. They also claim that Filloy and Rojano's notion of didactic cut focuses on mathematical form rather than process. This is the direct opposite to Gattegno (1978) who sees algebra preceding arithmetic. He claims that school education favours verbal description resulting in an over emphasis on algebraic ways of thinking (p. 74). Similarly, Mason (1996) acknowledges that the reductionism implicit in emphasising issues of transition through more difficult forms of algebraic equations, draws attention away from the underlying principle of algebra. These points of views in learning processes highlight the present research questions as follows:

(1) How do pupils in England and in Thailand solve algebraic problems?

(2) How different are their thinking processes when solving algebraic problems?

(3) How might mathematics curricula be interrelated with pupils' thinking processes in solving algebraic problems?

CHAPTER 2

RESEARCH BACKGROUND

2.1 Introduction

This chapter is organised in four sections. Section one gives the definition, the purposes and the methods of comparative education, and the justification for doing a comparative case study. Section two discusses the education system and its reform movement in England and Thailand. Section three looks at the English and Thai secondary school mathematics curricula. Section four presents the scores in mathematics of English and Thai pupils in the Third International Mathematics and Science Study-Repeat (TIMSS-R) and the results for England and Thailand from the Organisation for Economic Co-operation (OECD): the Programme for International Student Assessment (PISA) 2000.

2.2 Comparative Case Study

A rationale behind international comparative research in education is that it may be possible to learn from other countries in efforts to improve schools and pupils' achievement. Thailand is currently in the process of reforming its educational system. The comparative study with England aims to help the reform of mathematics education in Thailand. The following section examines the comparative educationists' view of comparative education as it relates to the present study.

2.2.1 The definition

Postlethwaite (1988) in *The Encyclopedia of Comparative Education and National Systems of Education* states that to "compare" means to examine two or more entities by putting them side-by-side and looking for similarities and differences between or among them. In the field of education, this can apply both to comparisons between and within systems of education (p. xvii).

In *The International Encyclopedia of Education* Epstein (1994) notes that comparative education is primarily an academic and interdisciplinary pursuit. Comparativists are primarily scholars interested in explaining why educational systems and processes vary and how education relates to wider social factors and forces.

Broadfoot (1999) points out that "comparative education is definitely not travellers' tales, nor the basis for unsystematic policy-borrowing" (p. 29).

The present study examines pupils' thinking processes in solving algebraic problems. It aims to understand similarities and differences, not in terms of the socio-economic and political feature but rather in terms of similarities and differences in curricula. The pupils' processes were put side by side and the similarities and differences between the English and Thai case study schools (see Chapter 6 for details) were explored. The investigation intended to identify influences on learning and how algebraic thinking can be improved. The explanations of different processes related to the mathematics curriculum in each country are also addressed in Chapter 6 and Chapter 7.

2.2.2 Purposes and methods of comparative education

There are four major aims of comparative education. (1) Identifying what is happening elsewhere that might help improve our own system of education. (2) Describing similarities and differences in educational phenomena between education systems and interpreting why these exist. (3) Estimating the relative effects of variables (thought to be determinants) on outcomes (both within and between systems) of education. (4) Identifying general principles concerning educational effects (Postlethwaite, 1988, p. xix-xx).

In defence of the study of educational issues in a comparative context, Phillips (1999) argues that the comparative study of education:

- shows what is possible by examining alternatives to provision 'at home';
- offers yardsticks by which to judge the performance of education systems;

- describes what might be the consequences of certain courses of action, by looking at experience in various countries (i.e. in attempting to predict outcomes it can serve both to support and to warn against potential policy decisions);
- provides a body of descriptive and explanatory data which allows us to see various practices and procedures in a very wide context that helps to throw light upon them;
- contributes to the development of an increasingly sophisticated theoretical framework in which to describe and analyse educational phenomena;
- serves to provide authoritative objective data which can be used to put the less objective data of others (politicians and administrators, principally) who use comparisons for a variety of political and other reasons, to the test;
- has an important supportive and instructional role to play in the development of any plans for educational reform, when there must be concern to examine experience elsewhere;
- helps to foster co-operative and mutual understanding among nations by discussing cultural differences and similarities and offering explanations for them;
- is of intrinsic intellectual interest as a scholarly activity, in much the same way as the comparative study of religion, or literature, or government (p. 15-16).

A hierarchical classification of types of comparative studies that organises the range of approaches is: (1) Single-site studies: description and documentation that provide detailed empirical documentation of educational phenomena in a particular, typically national setting. (2) Comparative contextualized case studies which provide single-site studies but which are contextualized in term of the broader international debates/theoretical frameworks/empirical accounts of the issue. (3) Comparative empirical studies that are designed as explicitly comparative based on a coherent rationale for their selection in order to illuminate 'constants and contexts'. (4) Theoretically informed comparative studies that review the contexts being compared are themselves theorised as part of a wider social science debates on, for example, the relationship between system and action, power and control, culture and the creation of meaning. (5) Theoretically informing comparative studies that use comparative research to inform theory (Phillips, 1999, p. 23-24).

Often too, in looking at the particularities of educational provision in other countries, it proves to be the case that the very aspects attracting our attention are being subjected to close scrutiny in those countries. Indeed, we might find that there is reciprocal interest in what might be learnt from the features of our home system which we are desirous to reform. Policy-makers contemplating reform might learn much from such internal interest. (Phillips, 1999, p. 17)

Learning from others' experience is far removed from the simplistic notion of 'borrowing' in the context of comparative education. The agreement of policies and approaches in education that might be extracted from a foreign situation is very unlikely to succeed in a different context. However, the weighing of evidence from other countries in such a way as to inform and influence policy development at home should be a very natural part of any efforts to introduce change.

Bruner (1996) remarks that there are two interpretations of education: 'information processing', which he calls the 'computational' approach, and 'meaning making', which he calls the 'cultural' approach. Culture forms and makes possible the workings of a distinctively human mind. In this way, learning and thinking are always *situated* in a cultural setting and are always dependent upon the utilization of cultural resources (p. 1-4). Making meaning of lives is what education is about and that should also be the aim of educational research so it is in the realm of comparative education. This is particularly meaningful with the present challenges of globalisation, where traditional cultural values face foreign invasions. Unless we have a better understanding of the cultural specificity in education, many of the strengths accumulated by human wisdom will disappear. Comparative education could therefore have a very constructive role to play.

As accounts are based mainly on studies conducted in the USA and the United Kingdom, the Consortium for Cross-Cultural Research in Education felt that research should be more widely based so as to give access to an international perspective against which we can better understand our own problems. The results of the present study would also enable teachers in any of the countries concerned to compare their experiences with those of teachers elsewhere and thereby develop a cross-cultural perspective on their work as well as promoting a sense of international professional identity.

The results of international comparisons have been used for four complementary but very different purposes (Robitaille & Robeck, 1996): (1) making comparison by comparing the performance of students and the effects of different factors in different countries; (2) explaining any difference in achievement found between different group of students; (3) helping countries to understand their own educational systems better by drawing attention to their relative strengths and weaknesses compared with other countries; (4) identifying models and practices in other countries which may provide possible solutions to national problems.

2.2.3 Justification for the methodology of the present study

One comprehensive school in the Northeast of England and one state school in the Northeast of Thailand were chosen for the case studies. Pupils' thinking processes when solving algebraic problems was the focus of inquiry. The main purpose was to focus on pupils' thinking processes. Comparing Thailand with England, which has reformed its education system for more than two decades might help us to see the strengths and weaknesses of each. There are aspects of algebra and numeracy strategy where it would be helpful to seek more understanding of pupils' thinking processes when solving algebra problems.

As Alexander (1999) states "culture both drives and is everywhere manifested in what goes on in classrooms, from what you see on the walls to what you cannot see going on inside children's heads. Thus, any one school or classroom can tell us a great deal about a country and its education system. But this is only so, if the research methods used are sufficiently searching and sensitive to probe beyond the observable moves and countermoves of pedagogy to the values these embody"(p. 158). Moreover, Reynolds (1999) suggests that the use of another culture's 'lens' to better understand the limitations and strengths of one's educational practice also applies at the level of educational philosophy as well as educational practice.

2.3 English and Thai Education

2.3.1 English Education

The English educational philosophy is characterised by the development of three approaches; morality (the Christian ideal); individualism, and specialisation (McLean, 1990).

In England, the responsibility for the education service lies with the Department for Education and Skills (DfES). The inspection of schools in England is the responsibility of a separate, non-ministerial government department, the Office for Standards in Education (Ofsted), which also has responsibilities for the pre-school education and care, and for provision for 16- to 19-year-olds.

The local education authorities (LEAs) in England are responsible for organising publicly funded school education within their area. LEAs also have a responsibility for quality assurance in the schools that they maintain and for promoting high standards of education.

The legal framework for primary and secondary schools divides them into community, voluntary, and foundation schools. The majority of schools are community schools; schools established and fully funded by local education authorities (LEAs). Foundation schools are also funded by LEAs, but are owned by the school governing body or a charitable foundation. Voluntary bodies, mainly churches, which retain some control over their management, originally established voluntary schools. Such schools are mainly funded by the LEAs.

Pre-primary education

For children aged from three months to three years, provision is largely in the private and voluntary sectors, where parents pay fees. For children aged from three to five, publicly funded early year's education and childcare is currently being expanded and developed in co-operation with the private and voluntary sectors. All areas of England are working towards the Government aim of universal, free nursery provision for three-year-olds by 2004.

The Education Act 2002 formally established the Foundation Stage of education in England, which caters for children, aged three until the end of the reception class (usually aged five).

Compulsory education

Education is compulsory from age five to age 16. Many children in England start in the reception class of primary school at age four. Most pupils move from a primary school to a secondary school at age 11, although in some areas of England, pupils attend middle school from the age of 8 or 9 to 12 or 13. Many secondary schools also provide education for post-compulsory students aged 16 to 18.

Length of school day/week/year

School must be open for 190 days a year. The local education authority or school governing body, depending on the legal category of school, determines the actual dates. The school year generally runs from September to July. Schools normally operate five days a week (Monday to Friday). There is currently some movement towards the adoption of a standardised six-term school year that would be consistent year on year from 2003/4. However, the decision to adopt this new model remains with the LEA.

Minimum recommended weekly lesson times in England are 21 hours (for 5-to 7-year-olds), 23.5 hours (for 8-to 11-year-olds) and 24 hours (for 12-to 16-year-olds). Most schools provide more hours than the suggested minimum. The school day generally runs from around 09:00 hours to between 15:00 and 16:00 hours. The school determines the organisation of time within the school day.

Class size/student grouping

Class sizes for 5- to 7-year-olds are limited to 30 pupils. There are no requirements for other age groups. The organisation of teaching groups is a matter for the school. It is most common that pupils are taught in mixed-ability classes at primary level, although many teachers use some form of ability grouping within a mixed-ability class. Secondary schools commonly group pupils for some subjects according to ability in that particular subject (a practice known as 'setting'), whilst teaching other subjects in mixed-ability groups. Teachers are expected to ensure that there are sufficient opportunities for differentiated work for pupils of all abilities.

Curriculum control and content

In England, the Qualifications and Curriculum Authority (QCA), a non-departmental public body, advises the Secretary of State for Education and Skills on matters affecting the school curriculum.

Schools are required to provide a balanced and broad based curriculum and have discretion to develop the whole curriculum to reflect their particular needs and circumstances. There are also specific statutory requirements for particular subjects. These requirements are the same for all publicly funded schools, including selective schools.

The curriculum for compulsory education in England is divided into four key stages (KS); KS1 (ages 5 to 7), KS2 (ages 7 to 11), KS3 (ages 11 to 14) and KS4 (ages 14 to 16). The National Curriculum compulsory subjects for KS1-3 include English, mathematics, science, design and technology, ICT, physical education, history, geography, art and design, and music. A foreign language is compulsory at KS3. In September 2002, citizenship became a statutory requirement in England from KS3.

Although outside the National Curriculum framework, religious education is also compulsory from KS1, as is sex education from KS3. Personal, social and health education (PSHE) is not statutory in England, but schools are expected to provide it. At KS4, there are fewer compulsory subjects.

Assessment, progression and qualifications

There are statutory assessment arrangements on entry to primary school and at the end of key stages 1, 2, and 3. These arrangements include teacher assessment and externally set and externally marked or moderated tests. The tests at the end of key stages 1, 2 and 3 are commonly known as 'SATS'. The QCA is the statutory advisory body responsible for keeping these assessment arrangements under review.

The QCA also serves as the regulatory body for the qualifications taken at the end of compulsory education. Awarding bodies (independent organisations recognised by the regulatory authorities) offer a range of national qualifications. The majority of pupils take General Certificate of Secondary Education examinations (GCSEs) in a range of single general or vocational subjects. Assessment schemes vary but always include externally set and externally marked assessments; there may also be internally marked and externally moderated assessment. Assessment may include oral and practical as well as written examinations (Holt, et al., 2002).

2.3.2 Thai Education

Education in Thailand developed from the traditional education offered in the temple, the palace and the family to modernised education for national development in accordance with the National Scheme of Education and the National Education Development Plan. Since 1997, the beginning of the new era of Thailand's national education, the development of Thai education has started to move forward based on the provisions of the 1997 Constitution relating to education and the National Education Bill (NEC, 1999).

Pre-school Education

Pre-school education is provided for 3-5 year old children. The aims are to encourage the harmonious physical, intellectual, emotional and social development of children prior to formal education. Pre-school education can be provided in many ways, such as, childcare centres, nursery schools, and kindergartens. The Ministry of Education established a kindergarten in every provincial capital to serve as a model for the private ones. As this level of education is optional, the private sector has played a role, in that most pre-schools are private. These schools are under the supervision of the Office of the Private Education Commission in the Ministry of Education.

Primary Education

Primary education emphasises literacy, numeracy, communication skills, and abilities relevant to future occupational roles. At this level, education is compulsory and free of charge for children aged 6-11 including the disadvantaged ones. The primary school curriculum is an integrated one comprising five areas of learning experiences namely: basic skills developments, life experience, character development, work oriented education, and special experiences. The special experience option is offered to children in the last two grades at the primary level—Pratom 5 and 6. As pupils' backgrounds in the various parts of the country is different, a basic national core curriculum allows certain flexibility for regional diversification. Primary education is under several government agencies. Most government primary schools are under the Office of the National Primary Education Commission, Ministry of Education. There are also demonstration schools attached to some teachers' colleges and universities, and municipal schools under the Ministry of Interior.

Secondary Education

Secondary education is divided into two levels, each covering a period of three years. The lower secondary education comprises three years, called secondary 1, 2, and 3 (similar to Year 7, 8, and 9 in England). The three years of upper secondary education are called secondary 4, 5, and 6. The lower level places emphasis on pupil's intellect, ethics morality and basic skills. This allows pupils to explore their individual interests and aptitudes through a wide choice of academic and vocational subjects. At the upper level appropriate academic and vocational knowledge and skills corresponding with pupil's interests and aptitudes are provided. The knowledge and skills are considered beneficial for pupils to continue study at a higher level or to enter the professional world. The secondary curriculum covers five broad fields: language, science and mathematics, social studies, character development, and work education. A wide range of exploratory pre-vocational subjects is also available. Use of the credit system at this level facilitates flexibility in the teaching-learning process. Both the public and private sectors are involved in the organisation of secondary education. Public schools are chiefly under the Department of General Education, Ministry of Education.

Higher Education

Higher education aims at the full development of human intellectuality and the advancement of knowledge and technology. This level may be organised in the forms of colleges, universities, or institutions for specialised studies.

The education system in Thailand has long been based on "chalk and talk" pedagogy, rote learning, with importance placed on school education, and with teachers as the centre of teaching–learning activities (Kaewdang, 2001).

2.3.3 Education Reform

Reform movement in England

One of the primary characteristics of late 20th century education has been a drive to evaluate and assess the quality of education for perceived future national needs. However, the late 1970's were years of some confusion over future directions, with much dismay being expressed about threats to the competitiveness of Britain with decline in commonwealth markets and growing competition from Europe and the Americas. Coupled with national concerns, was anxiety that educational standards were slipping, that young people were not being adequately prepared for a changing workforce and that levels of literacy and numeracy were at all-time lows.

Resulting from the debate in Britain was the 1988 Education Reform Act which reversed many years of 'progressive' education in schools. Its historical antecedent was the Code Napoleon concept of a state mandated curriculum. The Reform Act created a National Curriculum (NC) leaving little room for innovation or initiative on the part of the teacher. Under this new regime, schools are encouraged to respond to market forces. Local systems of fiscal management and governance were set up, and using the benchmark of the NC testing and GCSE examination system, league tables of school performance were established. The climate of British schools, therefore, is much more test driven than was the case previously. Teachers and their schools are held accountable through mandated sharing of information with the community, as well as experiencing a very vigorous system of school accreditation and evaluation.

The present government's plans for the future of education in Britain centre on the creation of a 'post comprehensive' climate. In a February 2001 Green Paper (DfES, 2001b), the government proposed a radical reform of secondary education that will lead to the conversion of half of the country's comprehensive schools into specialist institutions by the year 2006. The purpose is to replace the culture of uniformity in secondary education with schools having a distinctive mission, ethos, and purpose, where diversity will not be the exception but the hallmark of secondary education. What is remarkable about the Labour government's re-thinking of the concept of secondary education and coming up with a solution that many encourage a more selective approach to education, is not the sea of change it represents but the vote of no confidence to a system of schooling which, since 1965, has been the centre of educational provision in the United Kingdom.

Reform movement in Thailand

In the past two decades, Thailand has undergone a rapid transformation from a predominantly agriculture-based, government-subsidised economy to an emerging industrial, market-driven economy. To sustain the growth and development of a market driven economy, however, new types of knowledge and skills and an increasing investment in human capital is required. The changing economic landscape of Thailand demands that workers have higher-level knowledge and skills including competencies in new technologies. Workers are increasingly expected to be life-long, autonomous and self-regulated learners and to have the ability to adapt readily to changing circumstances. Achieving these new human capabilities means that aspects of the current education system, in particular teaching and learning approaches, as well as educational management practices in Thailand would have to change significantly.

The need for Thai schools to develop other types of knowledge beyond technical knowledge, and a new approach to teaching and learning has been recognised in the new Education Act (NEC, 1999). Consequently, a national pilot study to introduce the new learning approaches has been commissioned (Piya-Ajarriya, 2001). The pilot project used an ambitious pioneering initiative of a school-based approach to training. This decentralised bottom-up model contrasted strongly with the existing traditional authoritative top-down college-based staff development system of in-service training. The project involved 253 schools, 10,094 teachers, and 224,471 students with a time line of approximately 9 months. The complex approach adopted in the project hoped to make the learning experience authentic and empower learners (master teachers, school administrators, students) to take a more active leadership role in implementing the reform envisaged by the Education Act.

Thailand has a long history of a teacher-centred approach and centralised management and monitoring. To adopt a student-centred learning as a singularly focused approach for the educational reform in Thailand as the pilot project did, may not necessarily produce the human capabilities that are most valued for a knowledge and information-based society (Pillay & Elliott, 2002). The trend in education reform around the world is to provide choices and alternative approaches to learning where teachers as professionals have the knowledge and skills to decide which approach to adopt when and for what reasons.

Learning reform, as stated in the 1999 National Education Act is concerned with the reform of contents, learning process, assessment, and teachers and pupils' roles. Under the National Education Act, education is decentralised and compulsory education has been extended from six years to nine years.

In practice, the educational process in Thai schools before the 1999 Act seemed to stress memorisation rather than problem solving and self-learning. Educational measurements and admission examinations seemed to be based mainly on memorisation of subject contents. As a result, students' weaknesses lie in thinking process, analysis, rational and systematic synthesis, creative thinking, and problem solving.

2.4 International comparisons in English and Thai Mathematics

There is limited research on the comparisons between English and Thai mathematics education. Some results of TIMSS and PISA are outlined in the following sections.

2.4.1 The Third International Mathematics and Science Study-Repeat (TIMSS-R) results

Testing in mathematics and sciences was administered in 1995 and again in 1999 as part of the TIMSS-R comparative assessments. Thirty-eight nations participated and administered testing to state sector school children of similar ages. England participated along with Asian countries including Thailand, Japan, Singapore, South Korea, and Hong Kong. Results were categorised into three 'bands': above the international average, average, and below the international average. In 1995 English (Year 5) and Thai (Year 4) pupils scored below the international average. In 1999 those same children, now in Year 9 (England) and Year 8 (Thailand) scored below the international average. The English pupils had made some incremental but relatively small gains.

In 1999, English (Year 9) pupils had a mean score of 496 in mathematics on the third International Mathematics and Science Study-Repeat (TIMSS-R). The average score of English pupils was higher than the average scores (467) of pupils in Thailand.

About 4 per cent of Thai pupils scored in the top 10 per cent of TIMSS-R international benchmarks in mathematics in 1999. A smaller proportion of Thai Year 8 reached the benchmark than Year 9 pupils in England, where 7 per cent of year 9 pupils reached this benchmark.

In Thailand, for pupils' achievement test conducted by TIMSS (1999), eight graders (rank 27th) had lower levels of mathematics achievement compared with other South East Asian pupils such as in Singapore (rank 1st) and Malaysia (rank 16th). Thai pupils had higher levels only compared with Indonesian (rank 34th) and the Philippines (rank 36th).

The most difficult content area for Thailand is algebra with average scores significantly lower than the international level. The second most difficult area is measurement which had significantly lower scores than the international average (Klainin, 2003a).

The international testing of students in mathematics and science is, however, only a very small aspect of educational provision in any country. What is clear is that the pressure on the educational system to 'deliver' what industry and commerce demands, and the correspondingly generated policy climate, necessitates some major responses on the part of schools.

2.4.2 Results for England from OECD PISA 2000

The Programme for International Student Assessment (PISA) is a collaborative study among the member countries of the Organisation for Economic Co-operation and Development (OECD). Its main purpose is to assess the knowledge and skills of 15 year olds in three broad areas of literacy: reading, mathematics, and science. The assessments measure how well young people can use basic knowledge and concepts learned at school and elsewhere in order to function adequately in their adult lives. In PISA, "mathematical literacy is the capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgements, and to engage in mathematics, in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen" (OECD, 1999).

Students in England scored an average of 523 points on the reading literacy scale, significantly higher than students in OECD countries as a whole, where the mean score was set at 500. In only two of 32 countries, Finland and Canada, do 15 year olds perform significantly better than in England.

Students in England also did significantly better than the OECD average in both mathematical and scientific literacy, averaging 529 and 523 points respectively. Only Japan and Korea did significantly better in mathematical literacy, and only Korea in scientific literacy.

The United Kingdom national statistics (2001) reports there was a high level of correlation between the achievement levels of a country's students in the three domains of literacy. Of the twelve countries that scored significantly higher than the OECD average in reading literacy, eleven were also significantly above average in mathematical literacy, and ten in science literacy. Similarly, nearly all the countries that were significantly below average in reading literacy were also significantly below average in mathematical and in science literacy.

Thailand was a non-member country of OECD, however part of the scoring was reported. Thai pupils were at an average of 431 points on the reading literacy scale, 432 points on mathematical literacy, and 436 points on scientific literacy. Pupils in Thailand participating in PISA 25% were at Level 1 in reading literacy, 37% reached Level 2, 20% reached Level 3, and about 4-5% reached Level 4. None of them reached Level 5, which is the highest level, compared with England where 16% of pupils reached Level 5.

2.4.3 Justification for comparative case study focusing on learning algebra

The present research aims to investigate and then compare pupils' thinking processes when solving algebraic problems. Pupils in Year 7 and 8 of a school in the Northeast of England and those of a school in the Northeast of Thailand were investigated and compared with each other for a number of reasons. These are:

- The purpose of the Thai 1999 National Education Act is to provide training in thinking in how to face various situations and in the management and application of knowledge for solving problems. Education in England is distinctive for training in thinking and problem solving. The PISA results confirm this as it measured how well young people can use basic knowledge and concepts learned at school and elsewhere in order to function in their adult lives.
- Thai students had lower levels of achievement, compared with other Asian students in the international tests. The system of entrance examination to higher education in Thailand is also a major hurdle to effective teaching/learning mathematics. The test is intended to emphasis both content and the learning process, but students have showed that they are only interested in passing the examination as a mean for university admission.
- Assessment in Thai schools has long been by multiple-choice test. In order to develop assessment beyond the multiple-choice type of tests, a recent Educational Act (1999) states that educational institutions shall assess learners' performance through observation of their development, personal conduct, learning behaviour, participation in activities and by the results of tests accompanying the teaching-learning process commensurate with levels and types of education. By contrast, assessments in England are generally of the short answer and open-ended kind and encourage explanation of pupils' work.
- The weakest area for Thai pupils in international test like TIMSS-R was algebra. England, since 1988, has undertaken many research projects in teaching and learning algebra. For example, 'the Strategies and Errors in Secondary Mathematics (Booth, 1984)', 'Teaching and Learning Algebra pre-19 (Sutherland,

1997)', 'Key Aspects of Teaching Algebra in Schools (Mason & Sutherland, 2002)', and 'A Comparative Study of Algebra Curricula (Sutherland, 2002)'.

2.5 English and Thai mathematics curricula

2.5.1 Comparison of mathematics curricula in the English and Thai schools

The comparison is limited to mathematics resources, classroom structures, and assessment practices in the English and Thai schools.

Mathematics Resources

In Thailand, the Ministry of Education examines textbooks and give them approval to be used in schools. Schools choose textbooks from the list of approved publications. Most schools tend to choose the textbooks published by the Ministry of Education. In contrast, there is no central approval required for publishers of textbooks in England. Different textbooks are available for each level of schooling. The authors of these books interpret the published curriculum drawing on the expertise and experience of teachers and academics. Schools are also able to select for purchase, whatever materials publishers make available to them. In practice, the mathematics department and individual teachers use these textbooks as guides to inform their planning and teaching.

In England, the first three years of secondary education is known as Key Stage 3. Key Stage 3 mathematics is one of three core subjects with approximately 90 hours per year. The mathematics curriculum conceived as content and process is divorced from pedagogy thus allows teachers and schools to determine their own schemes of work using any methodology they prefer. In the National Numeracy Strategy 2000 (Ofsted, 1999) the government claimed to have brought mathematics to the forefront of the education agenda and provided a comprehensive system of training and support. Most primary schools now teach mathematics lessons daily with emphasis on mental arithmetic skills. There are also numeracy courses such as summer numeracy schools and family numeracy courses to help children make the transition from the primary to secondary school, as well as pilot schemes for pupils as they start secondary school.

In Thailand, school curricula have been modified and revised in order to be responsive to changing socio-economic conditions as well as to advanced technologies. The development of primary and secondary school curricula is mainly the responsibility of the Ministry of Education, which publishes textbooks and teachers' guides used by most schools. Thai Lower Secondary School core mathematics is taught for approximately 90 hours per year. The main teaching style in Thai mathematic lessons is "chalk and talk". Pupils are given approved texts as a part of standardized curriculum implemented by teachers. They are instructed to pay attention and takes notes, and they usually do not make comments or ask questions (Giacchino-Baker, 2003).

Table 2.1 gives an idea of the mathematics content of Year 7 and 8 in the English school, and Secondary 1 and 2 in the Thai school.

Table 2.1 Percentage of mathematics content in each year

Contents	**English school**		**Thai school**	
	Year 7	**Year 8**	**Secondary 1**	**Secondary 2**
Numbers	**30.9 %**	**31.4 %**	**60.2 %**	**39.8 %**
Algebra	**27.6 %**	**24.8 %**	**21.3 %**	**15.7 %**
Shapes Space and Measures	**23.6 %**	**25.6 %**	**18.5 %**	**37.0 %**
Handling Data	**17.9 %**	**18.2 %**	**0.0 %**	**7.4 %**

(From: mathematics department scheme of work in the English school and teachers' guide in the Thai school)

As indicated in Table 2.1 there is a strong emphasis on the numbers topic in the Thai school in the first year, approximately twice that of the typical English school. There was a decrease in the percentage of algebra content in the second year in both English and Thai schools. There is slightly more emphasis on the shapes space and measures in the Thai school than in the English school. It also appears from information in Table 2.1 that little attention is paid to handling data in either the Thai or the English school, but especially in the Thai school.

In mathematics teaching, pupils must be enabled to build up mathematical concepts. Secondary school mathematics is one of teaching pupils the basic knowledge. In England, Key stage 3 pupils are taught mathematics whose contents, drawn from all the numbered sections of the program of study of the curriculum, are interwoven. In Thailand, lower secondary school pupils are taught mathematics contents in sequence.

Table 2.2 gives a breakdown of algebra contents at the first year of secondary level in the English and Thai schools.

Table 2.2 Number of algebra lessons at the first year of secondary level

Term	Algebra content	English school 3 term year (Year 7)				Thai school 2 term year (Secondary 1)			
		Top		Bottom		High		Low	
		N	%	N	%	N	%	N	%
1	Sequences/patterns	7	35	2	12	-		-	
	Functions/graphs	-		2	12	-		-	
	Word problems	1	5	-		-		-	
	Simplification	5	25	3	18	-		-	
	Substitution	1	5	2	12	-		-	
2	Functions/graphs	2	10	3	18	8	42	10	67
	Solving equations	-		2	12	8	42	5	33
	Word problems	-		-		3	16	-	
3	Solving equations	3	15	-					
	Word problems	1	5	-					
	Substitution	-		3	18				

As seen in Table 2.2, the academic year for the English school is divided into three terms but the Thai school adopts a two-term academic year. The English school offers algebra lessons in all three terms whereas the Thai school offers algebra lessons only in the second term. In Thai school, content such as substitution is taught under the solving equations topic. Sequences/patterns and simplification are not taught in the Thai school.

Table 2.3 gives a breakdown of algebra content offer in Year 8 at the English school and in the second year at the Thai school.

Table 2.3 Number of algebra lessons at the second year of secondary level

Term	Algebra content	English school (Year 8)				Thai school (Secondary 2)			
		Top		Bottom		High		Low	
		N	%	N	%	N	%	N	%
1	Word problems	2	14	-		-		-	
	Simplification	2	14	2	17	-		-	
	Substitution	-		1	8	-		-	
2	Functions/graphs	5	36	4	33	4	29	2	18
	Simplification	1	7	-		-		-	
	Solving equations	4	29	-		7	50	7	64
	Word problems	-		1	8	3	21	2	18
3	Solving equations	-		4	33				

Table 2.3 indicates that in the English school algebra lessons are taught in the first two terms to pupils in the top set. The solving of equations is taught to pupils in the bottom set in the third term. In the Thai school pupils were taught algebra only in the second term. Once again, in the Thai school substitution is taught under the topic solving equations.

Classroom Structure

Table 2.4 gives the number of pupils in the high ability and low ability groups in the English and Thai schools.

Table 2.4 Number of participants by ability grouping

Ability	England		Thailand	
	Year 7	Year 8	Secondary 1	Secondary 2
High	28	28	49	54
Low	22	25	46	37

It is clear from Table 2.4 that the number of pupils in the Thai classroom is twice the number of the English one. The Thai school sets a ceiling for the number of pupils up to 55 whereas for the English school the number is 28.

Figure 2-1 shows the English classroom configuration of Year 7 and Year 8.

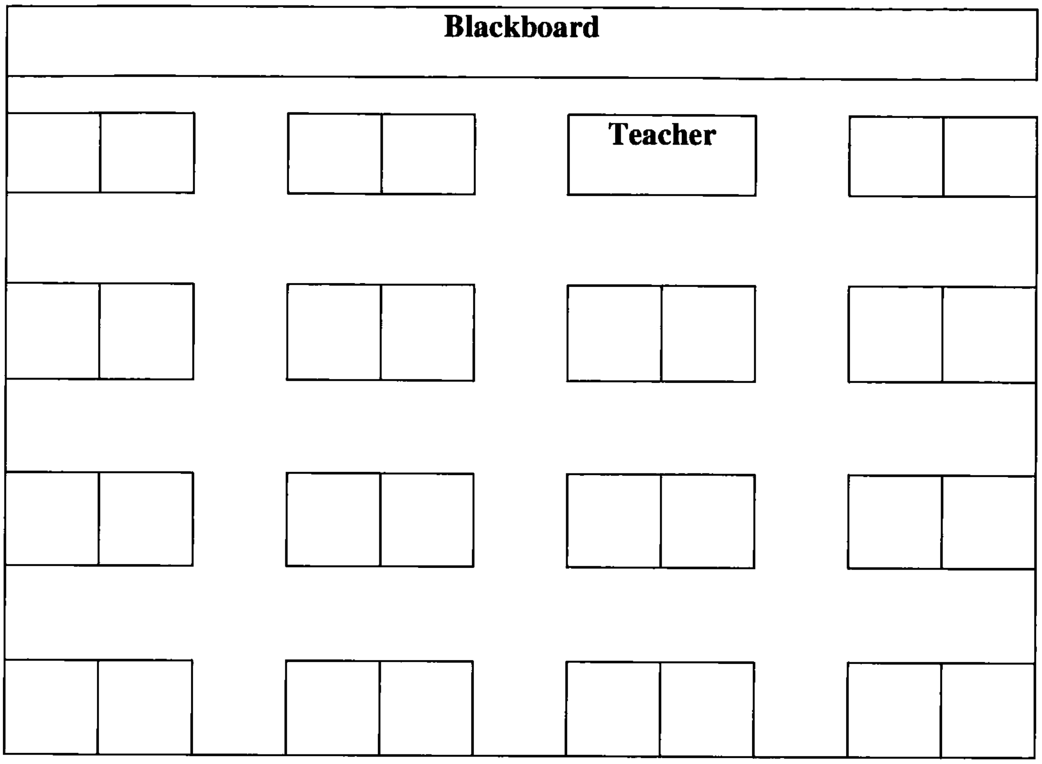


Figure 2-1 English school classroom configurations

In the English school the structure is that of whole class teaching with most interaction taking place only between the teacher and pupils. During lessons the teacher is able to support individual pupils easily by walking around. All pupils are accessible to the teacher.

Figure 2-2 shows a classroom configuration of secondary 1 in Thai school.

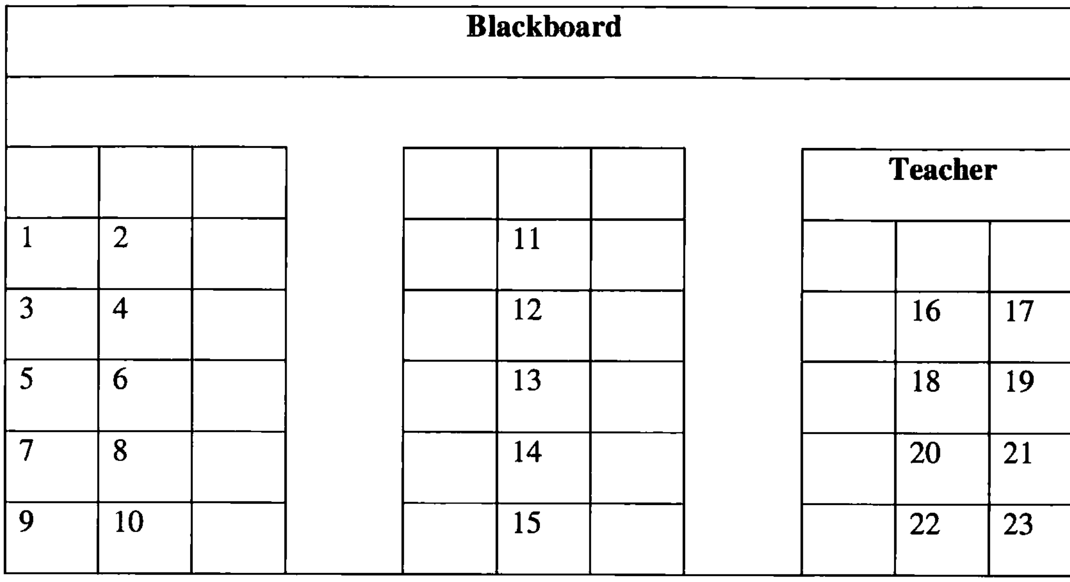


Figure 2-2 Thai school classroom configuration for Secondary 1

In the Thai school for Secondary 1 as in the English school the structure is that of whole class teaching with interaction taking place between the teacher and pupils. It is clear from Figure 2-2, which in the 23 locations indicated, the teacher does not have easy access to the pupils.

Figure 2-3 shows the classroom configuration for Secondary 2 in the Thai school.

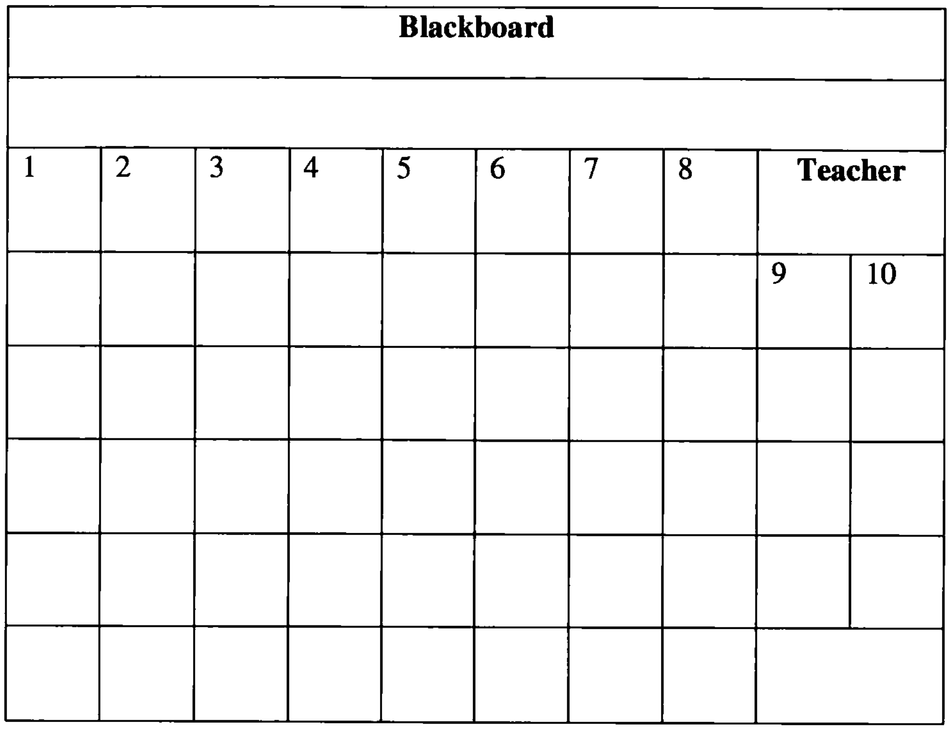


Figure 2-3 Thai school classroom configuration for Secondary 2

As can be seen from Figure 2-3, the teacher wishing to give support can access only the 10 pupil locations indicated.

Assessment Practices

Subject teachers assess pupils over a school year. In Thailand, at the end of a major topic, the teacher sets and marks a multiple-choice test. Over the school year, there is likely to be about 12 tests, consisting of 8 topic tests, two mid-term tests, and two end of term tests. In the English school, by contrast, there are four short answer tests consisting of 3 end of term tests and the end of year test. There is also a mental calculation test, which takes place at the end of each academic year.

Table 2.5 gives a sample of the school test items under the solving equation theme in both the English and Thai school.

Table 2.5 Sample of the English and Thai school test items

English school test items	Thai school test items
Year 7 • Solve the following equations: a) $p+4=7$ b) $4x=28$ c) $3x-7=23$ • I think of a number, multiply it by 4 and subtract 3. The answer is 33. Let x be the number I thought of. Write an equation to show this and then solve the equation.	**Secondary 1** • Which one is false? a) $45-x=10$, hence $x=35$ b) $x+20=48$, hence $x=18$ c) $4x=48$, hence $x=12$ d) $\frac{a}{5}=12$, hence $a=60$ • If $\frac{x}{3}+1=12$, find x. a) 35 b) 33 c) 32 d) 30 • If $12+x=27$, $y-x=7$, $\frac{y}{2}=c$, find c. a) 11 b) 10 c) 15 d) 22
Year 8 • Jack is 3 times as old as Peter. In 4 years time he will be twice as old. How old is Jack now? • Solve the equations: a) $x+4=12$ b) $3x-7=20$ c) $5x+6=2x-3$ d) $3(x+2)=x-4$	**Secondary 2** • If $2m-3=5$, find m^2. a. 4 b. 12 c. 16 d. 18 • If $7x-1=3x-21$, find x. a. -5 b. 5 c. -2 d. 2 • If $\frac{x}{2}-\frac{1}{5}=-2x$, find $-25x$. a. 2 b. -2 c. 3 d. -3

As shown in Table 2.5, the English school test items ask pupils to work out the answers. In contrast the Thai school test items ask pupils to choose the answer from choices given.

The following sections outline the algebra curricula in the English and Thai school.

2.6 English and Thai algebra curricula

2.6.1 Algebra objective in the English and Thai secondary school

As the study was conducted in the first two years at secondary school in England and Thailand, it would be interesting to examine the key objectives in their algebra curricula as shown in Table 2.6.

Table 2.6 Algebra key objectives in the English and Thai schools

English school algebra	Thai school algebra
Key objectives **Year 7** • Use letter symbols to represent unknown numbers or variables. • Know and use the order of operations and understand that algebraic operations follow the same conventions and order as arithmetic operations. • Plot the graphs of simple linear functions.	**Key objectives** **Secondary 1** • Solve equations and check their solutions • Use equations to solve word problems • Draw graphs of linear functions.
Year 8 • Simplify or transform linear expressions by collecting like terms; multiply a single term over a bracket. • Substitute integers into simple formulae. • Plot the graphs of linear functions, where y is given explicitly in terms of x; recognise that equations of the form $y = mx+ c$ correspond to straight-line graphs.	**Secondary 2** • Solve equations and check their solutions • Use equations to solve word problems. • Draw graphs of linear functions and simple curves, which are applied to some daily life situations and natural phenomena.
Year 9 • Generate terms of a sequence using term-to-term and position-to-term definitions of the sequence, on paper and using ICT; write an expression to describe the n^{th} term of an arithmetic sequence. • Given values for m and c, find the gradient of lines given by equations of the form $y = mx + c$. • Construct functions arising from real-life problems and plot their corresponding graphs; interpret graphs arising from real situations.	**Secondary 3** • Solve linear equations and inequalities in one variable. • Solve linear equations in two variables. • Solve quadratic equations. • Draw graph of equation in the form $y = ax^2 + bx + c; a \neq 0$.

Sources: Framework for teaching mathematics: Year7, 8 and 9 England and Mathematics curriculum for the lower secondary level Thailand

As illustrated in Table 2.6, the English school algebra in Year 7, under the first key objective, pupils are taught to use letter symbols, to generate and describe simple integer sequences from a given rule, and to describe the general term in simple cases (**patterns/sequences**). For the second objective, pupils are taught to use the same order of operations as arithmetic operations in order to simplify linear algebraic expressions of like terms (**simplification**). Within the same objective pupils are also taught to construct and solve simple linear equations with the unknown on one side (**solving equations**). For the third objective, pupils are taught to generate coordinate pairs that satisfy a simple linear rule, and plot the graphs of simple linear functions where y is given in the form $y = x+c$ (**graphs of linear functions**).

In Year 8, for the first key objective, pupils develop their ability to simplify or transform linear expressions by collecting like terms (**simplification**) and also begin to multiply a single term over a bracket. For the second objective, pupils are taught to substitute positive integers into simple linear expressions and formulae involving small powers (**substitution**). For the third objective, pupils are taught to generate points in all four quadrants and plot the graphs of linear functions, where y is given in the form $y = mx+c$. They are taught to recognise that equations of this form correspond to straight-line graph (**graphs of linear functions**).

In contrast, the Thai school the algebra content focused on solving equations and drawing graphs of linear functions in both the first and second years. In Secondary 1, for the first key objective, pupils are taught to **solve equations** with the unknown on one side by using explicit balancing and to check the solution using **substitution.** For the second objective, pupils are taught to construct linear equations in order to **solve word problems.** For the third objective, pupils are taught to plot the graphs of two sets of related quantities, and to interpret these graphs (**graphs of linear functions**).

In Secondary 2, for the first key objective, pupils are taught to **solve equations** with the unknown on one side, and also with the unknown on both sides, by using explicit balancing, and always to check the solution using **substitution.** For the second objective, pupils are taught to extend the work on **solving word problems**. For the third objective, pupils are taught to plot the **graphs of linear functions** with various conditions, and interpret graphs arising from real situations.

After consulting the algebra content of the English and Thai mathematics curricula in the English and Thai schools' teaching programme the six themes were identified. These themes are patterns/sequences, simplification, substitution, solving equations, graphs of linear functions, and word problems. The researcher felt that six themes cover both curricula, although it is recognised that each country places its own emphasis on each of the different themes.

The next chapter addresses the research on teaching/learning algebra and the difficulties faced by pupils in learning it.

CHAPTER 3

REVIEW OF THE ALGEBRA LITERATURE

3.1 Introduction

This chapter reviews the algebra research literature. As stated in Chapter 1, this study focuses on a comparison of the thinking processes when solving algebraic problems between English and Thai pupils. Of special interest to this study is research that has addressed pupils' strategies in approaching algebra and its implications to the teaching and learning of introductory algebra.

The chapter is organised into three sections. Section 1 presents researchers' views on the meaning of algebraic thinking. Section 2 considers some research findings that address pupils' difficulties with learning algebra and the issue of providing a theoretical background to the teaching and learning of algebra. Section 3 presents the conclusions derived from the reviews and discusses the approach to algebra within the six themes – general patterns and sequences, simplification of algebraic expressions, substitution, solving equations, graphs of linear functions, and word problems – adopted for the present study.

3.2 Algebraic Thinking

> Children come to school with the requisite powers to think mathematically, and in particular, to 'think' algebraically
>
> (Mason, 2002, p. 4)

Numerous studies in teaching and learning algebra discuss the meaning of algebraic thinking (e.g., Mason, 1992; Kaput, 1995; Herbert & Brown, 1997). Generally, algebraic thinking is defined as abstract arithmetic, as modelling, and as a language.

3.2.1 Algebraic thinking as abstract arithmetic

Algebra is often referred to as generalised or abstract arithmetic. Number properties and operations in arithmetic context can lay a solid foundation for the beginning of formal study of algebra. For example, interpretation of 3×14 as 3×(10+4) = (3×10)+(3×4) = 30+12 = 42. This process of arithmetic gives pupils networks of connections that they can draw on when they begin in the algebra context as $3(x+4) = (3\times x)+(3\times 4)=3x+12$.

Hewitt (1998) suggests that finding a way of structuring to help carry out something efficiently and effectively is the algebra used in order to carry out the arithmetic. He interprets the case of 19×16 to demonstrate that there are many different ways that people approach this task. For instance:

19×16 = 2×16×10-16

19×16 = 20×16-20+4

19×16 = (19×10)+(19×5)+19

19×16 = 19×2×2×2×2

Or for someone to use repeated addition:

19×16=19+19+19+19+19+19+19+19+19+19+19+19+19+19+19+19

19×16=16+16+16+16+16+16+16+16+16+16+16+16+16+16+16+16+16+
16+16

Hewitt proposes that whichever way this calculation is carried out, some kind of algebraic structure is being used. Even if repeated addition is used, there is still an algebraic structure:

$m\times n = m+m+\ldots+m$ (n times)

or

$m\times n = n+n+\ldots+n$ (m times)

Carraher, Schliemann and Brizuela (2001) state that arithmetic derives much of its meaning from algebra. For them, the expression, "+3", can represent both an operation for acting on a particular number and a relationship among a set of input values and a set of output values. This is borne out by the fact that we can use functional, mapping notation, "$n \rightarrow n+3$", to capture the relationship between two interdependent variables, n and n plus three (Schliemann, Carraher, & Brizuela, 2000). So the objects of arithmetic can be thought of as both particular (if $n = 5$ then $n+3 = 5+3 = 8$) and general (n can represent all numbers); arithmetic encompasses number facts and the general patterns that underlie the facts. Word story problems need not be merely about working with particular quantities but working with sets of possible values and hence about variables and their relations.

They propose that arithmetic also involves representing and performing operations on unknowns. This is easy to forget since arithmetic problems are typically worded so that pupils can invest minimal effort in using written notation to describe known relations. The relations tend to be expressed by pupils in final form, where the unknown corresponds to an empty space to the right of an 'equals' sign. Where arithmetic problems are sufficiently complex that pupils could not straightaway represent the relations in final form, it would become easier to appreciate how central algebraic notation is to solving arithmetic problems. Carraher, Schliemann and Brizuela (2001) also suggest that arithmetic can and should be infused with algebraic meaning of arithmetical operations. In this sense, algebraic concepts and notation are part of arithmetic and should be part of arithmetic curricula for pupils.

The idea of algebra as generalised arithmetic is a natural progression for some pupils (Thomas & Tall, 2001). This was demonstrated through the discussion with a pupil aged seven years and one month; who was required to explain the idea of using n to stand for a number and 'two n' or 'two times n' to stand for two times the number n. After giving and requesting a few examples for $n = 2, 3, 4$, and asking about the value of $2n+1$ for several values of n, the pupil was asked:

"Is two *n* always even? ... Or is it sometimes odd?"

[Three seconds pause.] **"Always even"**

"Why is it always even?"

"Well, if you add an even number with an even number, you end up with an even number."

"Right"

"If you add an odd number and an odd number, you come up with an even number, but if you add an even number with an odd number, you come up with an odd number."

[Chuckling:] "That's very good! Who told you that?"

"I worked it out myself."

Thomas and Tall (2001) explain that this pupil had shown a rich understanding of arithmetic and moved naturally from arithmetic to algebra because generalisation has taken place.

Liebenberg et al. (1998) point out that an important difference between arithmetic and algebra is that arithmetic could often bypass the conventions related to the algebraic structure. For example, if it had been agreed that every possible pair of brackets should be inserted in each arithmetic string, it could avoid the need for a convention about the order of operations in most cases. In algebra, however, even simple equations cannot be handled without a convention about the order of operations.

Thomas and Tall (2001) indicate that the shift from arithmetic in everyday situations to the synthetic symbolism of generalised arithmetic and algebra involves more complex expressions that cause a difficult transition for many. This transition is made more difficult by the change in meaning of the symbolism. In arithmetic, the expression 7+4 is an operational procept (the combination of process and concept) in the sense that it has a built-in counting procedure to give the result. In algebra, however, the symbol $7+x$ is first an expression for a process of evaluation that cannot be performed until x is known. The difficulty of conceiving an algebraic expression as the solution to a problem has been described as a perceived lack of closure (Collis, 1972).

Davis, Jockusch & McKnight (1978) made a similar observation that 'this is one of the hardest things for some seventh-graders to cope with; they commonly say, "but how can I add 7 to *x*, when I don't know what *x* is?"' In the same vein, Matz (1980) commented that, in order to work with algebraic expressions, pupils must "relax arithmetic expectations about well-formed answers, namely that an answer is a number".

Kieran (1981) similarly commented on some pupils' inability to "hold unevaluated operations in suspension". All of these can now be described as the problem of manipulating symbols that—for many pupils—represent potential processes (or specific procedures) that they cannot carry out, yet are expected to treat as manipulable entities. Essentially, even when pupils can handle general arithmetic, they may see algebra expressions as unencapsulated processes rather than manipulable procepts. Many pupils remain process-oriented (Thomas, 1994), thinking primarily in terms of mathematical processes and procedures, causing them to view equations in terms of the results of substitution into an expression (Kota & Thomas, 1998).

Arcavi (1994) describes 3 main features of the algebraic way of thinking. The first feature is an operational symbolism. Second is the preoccupation with mathematical relations rather than with mathematical objects. Relations determine the structures constituting the subject matter of modern algebra. The algebraic mode of thinking is based on relational rather than predicate logic. Finally, it is freedom from any ontological questions and commitments and, connected with this, abstractness rather than intuitiveness. Formulating problems algebraically (usually as equations) presents cognitive challenges far beyond the language aspects. For example, identifying the variables involved and noticing functional behaviour and necessary relationships are difficult steps requiring a new "algebraic" way of thinking not just an extension of arithmetic thinking into a domain of letters (Stacey & MacGregor, 1997). Radford (2000) emphasised in the framework of semiotic analysis "algebraic thinking is the specific way in which the pupils conceptually acted in order to carry out the actions required by the generalising task" (p. 258).

However, Blanton and Kaput (2000) emphasised algebraic reasoning as a way of thinking mathematically by using the term "habits of mind". They believed that pupils' elementary school experiences should extend beyond arithmetic proficiency to cultivate habits of mind that can support the increasingly complex mathematics of the new century (Kaput, 1999; National Council of Teachers of Mathematics [NCTM], 2000).

Cooper, Williams and Baturo (1999b) demonstrated that teaching episodes, which reflected on arithmetic to build algebra generally worked, but the arithmetic needed to lead straight to the algebra generalisations for each activity. This finding was incorporated in the teaching episodes and worksheets associated with the simplification of algebraic expressions. For algebraic simplification, the link between arithmetic and algebra may be thwarted when understanding of the arithmetic components (e.g., subtraction and division ideas) is missing or defective.

In contrast, Matz (1980) and Lins (1990) suggest that the transition from the arithmetical context to the algebraic context is not a direct one as argued by Booth (1988). According to Matz and Lins many of the obstacles in the algebraic context do not necessarily reflect difficulties in the numerical context; they probably reflect difficulties in interpreting the new context. This theory suggests that there are situations in which the correct knowledge from the numerical context will be transferred correctly to algebraic context and situations when it will be transferred incorrectly.

Sutherland (1991) makes a general observation that 'the emphasis on structure in algebraic thinking can be contrasted with an emphasis on process in arithmetic thinking. Algebraic thinking does not replace arithmetic thinking – it supersedes it, becoming a new vantage point from which to view arithmetic.' It could take the view that the transition from arithmetic to algebra 'is not initiation into decontextualised knowledge but initiation into another social practice' (p. 45). In this view, generalisation and abstraction are not processes or states of mind, but changes in practice that occur within a relevant setting (Kennewell, 2001).

The distinction between arithmetic thinking and algebraic thinking is that arithmetic thinking focuses on operations on known numbers. However algebraic thinking studies these operations per se, for example, working on the structure of arithmetic and deeper understanding of how arithmetic works (e.g. Sierpinska, 1995, Sfard, 1995).

3.2.2 Algebraic thinking as modelling

A study by Koedinger (1998) examined how to improve algebraic modelling by the inductive support strategy—use of concrete instances to help pupils induce algebraic sentences. The experiment carried on different approaches that might better aid pupils in learning to model with algebra symbols. One such example can be seen in a textbook problem, "Drane & Route Plumbing Co. charges $42 per hour plus $35 for the service call. 1) Create a variable for the number of hours the company work. Write an expression for the number of dollars you must pay them. 2) How much would you pay for the three hours service call? 3) What will the bill be for 4.5 hours? 4) Find the number of hours worked when you know the bill came out to $140."

1) 35 + 42h = d	1) 35 + 42*3 = d
2) 35 + 42*3 = d	2) 35 + 42*4.5 = d
3) 35 + 42*4.5 = d	3) 35 + 42h = d
4) 35 + 42h = 140	4) 35 + 42h = 140
Textbook (Symbolize first)	**Inductive support (solve & then symbolize)**

The results indicated that pupils in the inductive support experimental group learned significantly more from pre-test to post-test than pupils in the textbook control group.

The process of transformation within problem solving oriented situations is analysed in Boero (1993). The concern is with the process of problem transforming rather than with the process of algebraic expression transforming. He remarks that the different roles played by the transformation function imply specific and different cognitive engagements by learners. This issue was discussed in term of anticipation, which allows planning and continuous feedback. The process of problem transforming may happen without, before and/or after algebraic formalisation. In the case of transformations performed after formalisation, anticipation is based on some properties of the external algebraic representation. Pimm (1995) points out that anticipation could provide an alternative to the "blind" manipulation that is found in the beginning of facing algebraic problems. Gallo (1994) comments that formal transformation of expressions makes sense when they are inserted in a conceptually structured context. She discusses the adaptation of models activated by the pupils during the algebraic manipulation.

Bolea, Bosch and Gascon, (1999) establish a notion of algebra that allows them to interpret 'the study of algebra' in a given institution. This is used as the basis to generate a series of didactic phenomena related to what is commonly called "the learning process of algebra". They state that elementary algebra does not appear as a self-contained mathematical work comparable to other works studied in academic core courses (such as arithmetic, geometry, statistics, etc.), but rather as a modelling tool to be (potentially) used in all mathematical curricular works and which appears to be more or less used in them. The model of elementary algebra chosen as an alternative to 'generalised arithmetic' is based on the realisation that elementary algebra is in fact a mathematical tool, the algebraic tool, that can be used to study many different kinds of problems not only or exclusively pertaining to arithmetic (p. 138). They distinguishes algebraic modelling from other kinds of mathematics modelling as

- The algebraic modelling of a given mathematical work describes explicitly and materially all the techniques contained in the initial work, thus allowing for a quick development of these techniques, as well as for the explicitation of their interrelations and the unification of the related types of problems.
- An algebraic modelling may be considered as the answer to a technological questioning related to the initial work, such as the way in which to describe and justify the initial techniques, the condition under which they can be applied, the type of problems they can solve, etc.

- In algebraic modelling, all components of the initial work are modelled as a whole, and not as separate entities, a fact that tends to simplify the structure of the algebra work eventually obtained (p. 139-140).

3.2.3 Algebraic thinking as language

There are many kinds of algebra as language such as group theory, Boolean algebra and geometric algebra. In this case, the focus is on algebra as a language that has semantics and syntax. In what might be called traditional algebra, letters are used in algebra not for words but also for representing mathematical objects. School algebra is a symbol system with a syntax that allows particular conventions to be used for manipulating terms and simplifying expressions. The ability to understand the rules associated with a language is very important. A thorough understanding of the structural aspects of mathematical properties is necessary—the semantics of algebraic expressions. Booth (1989) is of the view that semantic problems occur as a result of a poor understanding of the relations and mathematical structures that underlie algebraic symbols and syntactic difficulties arise from the introduction and manipulation of the symbols in algebra.

The language of algebra with its semantics and syntax must be therefore presented properly in order to make conceptual understanding to occur. The rules of algebra can be reinforced through the teaching of concepts in conjunction with semantic and syntactic meaning. The sentence x *represents* y for example, means that the syntactical construction x represents the semantic object y. In algebra word problems, syntactic translation is the process of translating words into an equation by sequentially replacing key words by mathematical symbols.

Researchers and mathematics educators alike have expressed algebra as a language. This can be viewed from two perspectives, a language of mathematics (e.g., Mason et. al, 1985; Wagner & Kieran, 1989; Kieran, 1991; Van der Kooij, 2001) and a symbolic language of communication with computer (e.g., Boero, 1994; Sutherland & Rojano, 1993b). Mason et al. (1985) for example, declare: "algebra is firstly a language—a way of saying and communicating" (p. 1). They conclude that the pupils have already mastered the elements of algebra before they go to school in the sense that they learn to speak, read, and generally make sense of the world. The teaching of algebra is then, using and bringing them to saying and recording in a new context. Bell (1993) refers to algebra, as "it's more like a language than anything else". He proposes that the algebraic language be learned "in a way more similar to that in which the mother tongues is learnt" (p. 11). For Sutherland and Rojano (1993a) "algebra is the language of mathematics, a language which can be used to express ideas within mathematics itself or within other disciplines" (p. 2).

Kieran (1991) defines algebra "as a branch of mathematics that deals with symbolising general numerical relationships and mathematical structures and with operating on those structures" (p. 391). This implies that school algebra has both procedural and structural aspects. Procedural refers to arithmetic operations, such as evaluating the expression $3x+y$, where $x = 3$ and $y = 2$, the result is 11. A second example is solving an equation like $2x+3 = 7$ by substituting various values for x. The objects that are operated on are the numeric instantiations rather than the algebraic expressions (Kieran, 1991). The structural aspects include topics like simplifying and factoring expressions, solving equations by performing the same operation on both sides, and manipulating functional equations. Structural aspects refer to operations on algebraic expressions rather than on numbers, such as combining like terms in the expression $3x+2y+x$, which simplifies to $4x+2y$ or $2(2x+y)$. Hence, algebra with a procedural and structural foundation mirrors a language, where mathematical objects must be given meaning.

The view of algebra as a language in this case has been changed and broadened by technology. As Tall (1992a) declares "introducing algebraic symbolism by using it as a language of communication with the computer, through programming in a suitable computer language ... it develops a meaningful algebraic language which can be used to describe number patterns, and it gives a foundation for traditional algebra and its manipulation" (p. 38). The algebraic language is required in order to develop awareness of mathematical objects and relationships, many of which are impossible to manage without such a language. Without appropriate emphasis on symbolic language such essential ideas as algebraic equivalence cannot be learned (Sutherland, 1997).

Boero (1994) points out the role of algebraic language in mathematics and analyses how the transformation function of the algebraic code enters into action in different mathematical activities. What are the cognitive processes, and especially the prerequisites involved? What are the consequences of such analysis on the educational level?

Transforming algebraic expressions is framed in the more general perspective of transforming the problem in order to better manage it. A crucial aspect of some problem solving strategies is the transformation function of the algebraic code. This plays different roles in mathematical activities, according to different kinds of problems, and each role implies a specific engagement by students (Bazzini, Gallo and Lemut, 1996).

The process of construction and interpretation may be blocked if pupils consider the terms in a rigid way and do not grasp the underlying interrelation between sense and denotation of a given name (Arzarello, Bazzini and Chippini, 1994, 1995). In other words, there is evidence that the pupil is often not able to take the whole potential of the algebraic code, that is, the power of incorporating different properties within the name. The name is seen as a rigid designator, a source of obstacles for algebraic thinking. Consequently, growing difficulties appear in front of algebraic transformations, and their additional requirement of foreseeing and applying, guessing, and testing the effectiveness, is a continuous tension (Boero, 1994).

All these issues foster a careful analysis of the questions related to the learning of algebra as a language. Such questions are rooted, at an early school level, in the dialectic relation between semantics and syntax, procedures and structures, natural and symbolic language. The passage from natural language to symbolic language is a key point in the development of algebraic thinking and asks for special attention in teaching (Bazzini, 1999).

In a technological context, Sutherland's work with computers has shown that there are many ways in which algebra-like symbols can be used to mediate an algebraic approach (Healy, Pozzi & Sutherland, 1996, Sutherland, 1992). This mediating role can influence pupils' activity in both computer-based and paper-based settings. This is illustrated by the way pupils learn to accept the algebraic idea of transforming the unknown, an idea which most pupils find difficult. With spreadsheets pupils first use a spreadsheet cell to represent the unknown and move from referring to the unknown by a cell reference (for example A5) to referring to it by an algebraic name (for example x). Another example is the way in which a Logo variable name (for example SIDE, W) comes to represent a general number, which is similar to the spreadsheet example discussed earlier. Pupils use the computer-based language in their talk as they communicate with their peers and the computer. School mathematics does not usually take advantage of this mediating role of algebraic symbols, possibly because of a reaction to the meaningless symbol manipulation associated with the more traditional mathematics curriculum.

In the case of the Rectangular Field problem, pupils might construct the rule such as "LENGTH+WIDTH+LENGTH+WIDTH" that links very closely to the way they might point with their fingers to the sides of a rectangle to think about the idea of perimeter. Mouse pointing becomes a way of supporting pupils to express general relationships, which are then represented automatically in spreadsheet code. Pupils become aware of this spreadsheet code without explicit instruction and interact with it when they need to modify their constructions. They begin to use the spreadsheet code in their talk and can write it down when communicating with others. In this way the algebra-like spreadsheet code is learned effortlessly without explicit teaching. Pupils use the spreadsheet specific calculations to help in the construction of general rules and often verify their general rule with reference to specific numbers. In this way links between symbols and general numbers are established (Sutherland & Rojano, 1993b).

Note that in Logo, unlike other programming languages, there is a clear distinction between the name of a variable, "length, and the value assigned to it, :length. We could also write a procedure to take length as an input (Clements & Sarama, 1997).

3.3 Teaching and learning algebra

3.3.1 Pupils' learning difficulties

> There is a stage in the curriculum when the introduction of
> algebra may make simple things hard, but not teaching
> algebra will soon render it impossible to make hard things simple.
>
> (Tall & Thomas, 1991, p. 129)

Algebra has always been considered difficult to learn, and correspondingly hard to teach (Kennelwell, 2001). A considerable body of evidence has been assembled certifying to the difficulty of learning algebra. Traditionally, schools have delayed and restricted the algebra curriculum rather than seeking ways of overcoming the difficulties (Sutherland, 1997). Researchers (e.g., Boulton-Lewis et al., 1998; Linchevski and Herscovics, 1996) show that achievement rates in algebra are poor. Among the evidence, pupils' difficulties with learning algebra were categorised as the order of operations, accepting lack of closure, and syntactic difficulties.

Order of operations

Boulton-Lewis et al. (1997a) suggest a two-path instructional model to improve student learning of algebra. The model was based on the belief that understanding of complex algebra is the end product of a learning sequence of mathematical concepts. For example, 3×5 and $5+2$ in arithmetic are a pre-requisite for $3x$ and $x+2$ in algebra; $3\times5-4$ and $5+2-4$ in complex arithmetic forms are an important pre-requisite to understanding $3x-4$ and $(x+2)-4$ in complex algebra forms. Moreover, Thomas and Tall (2001) remark on the usual sequence of reading from left to right. The order of operation causes some difficulty moving from arithmetic to algebra. Similar findings (Norton & Cooper, 1999) concluded that many students had neither operational nor structural understanding of arithmetic and this will certainly make it difficult for them to develop operational and structural understandings of algebra concepts.

Algebra is an abstract system in which interactions reflect the structure of arithmetic (Cooper, Williams & Baturo, 1999a). Its processes are abstract schemas (Ohlsson, 1993) or structural conceptions (Sfard, 1991) of the arithmetic operations, equals, and operational laws, combined with the algebraic notion of variable (Cooper et al., 1997). Arithmetic does not operate at the same level of abstraction as algebra for, although they both involve written symbols and an understanding of operations (e.g., order of operations, inverse operations – Herscovics & Linchevski, 1994) arithmetic is limited to numbers and numerical computations.

Kieran (1989b) emphasises that an important aspect of the difficulty is pupils' difficulty to recognise and use *structure*. Structure includes the 'surface' structure (e.g. that the expression $3(x+2)$ means that the value of x is added to 2 and the result is then multiplied by 3) and the 'systemic' structure (the equivalent forms of an expression according to the properties of operations (e.g. that $3(x+2)$ can be expressed as $(x+2)\times3$ or as $3x+6$). Kieran also sees algebra as the formulation and manipulation of general statements about numbers, and hence hypothesises that pupils' prior experience with the structure of numerical expressions in primary school should have an important effect on their ability to make sense of algebra. Booth (1989) expresses a similar view:

> ...a major part of students' difficulties in algebra stems precisely from their lack of understanding of arithmetical relations. The ability to work meaningfully in algebra, and thereby handle the notational conventions with ease, requires that students first develop a semantic understanding of arithmetic. (p. 58)

First and foremost, there is considerable cognitive conflict between the deeply ingrained implicit understanding of natural language and the symbolism of algebra. In most western civilizations, both algebra and natural language are spoken, written and read sequentially from left to right. In algebra, the letter is not always processed from left to right. This difficulty of unravelling the sequence in which the algebra must be processed, conflicts with the sequence of natural language. Tall and Thomas (1991) term this the *parsing* *obstacle.* It manifests itself in various ways, for example the pupil may consider that ab means the same as $a+b$, because they read the symbol '*ab*' as '*a*' *and* '*b*', and interpret it as $a+b$. Or the pupil may read the expression $2+3a$ from left to right as 2+3 giving 5, and consider the full expression to be the same as $5a$.

Accepting lack of closure

Tirosh, Even and Robinson (1998) point out that pupils frequently face cognitive difficulty in 'accepting lack of closure' (Collis, 1972). For instance, for the symbol $5x+8$, pupils tend to 'add' these two terms to 'complete' or 'finish' them (Booth, 1988; Collis, 1975; Davis, 1975). For pupils, it seems to be reasonable to get expressions such as $13x$ or 13.

In examining the difficulties pupils encounter in moving from arithmetic to algebra, Sfard (1991, 1994), Kieran (1989a, 1992), and Herscovics (1989) describe a number of obstacles that can be connected directly to the difficulty of reification as described by Sfard. For example, pupils usually have difficulty accepting an algebraic expression as an answer; they see an answer as a specific number, a numerical product of a computational operation. Furthermore, the equal sign is usually interpreted as requiring some action rather than signifying equivalence between two expressions, leading to the misconception that $x+8 = 8x$.

Wagner, Rachlin and Jensen (1984) found that many algebra pupils tried to add "= 0" to expressions they were asked to simplify. One explanation may lie in the unwillingness of pupils to accept 'lack of closure' as suggested by Hoyles and Sutherland (1992). Previous studies have found that many pupils cannot accept that an unclosed algebraic expression is an algebraic object. So, for example, pupils are unable to accept that an expression of the form $x+3$ could possibly be the solution of a problem.

e.g. $2a+a+3$

$= 3a+3 = 0$

$= 3a = -3$

$= a = -1$

and $x^2 + 5x + 6$

$= (x+3)(x+2) = 0$

$x = -3$ or -2

Hall (2002) says that this kind of error may indicate an absence of knowledge of the difference in meaning of an expression and an equation. Such a "lack of closure" experienced by pupils may be a contributing factor to the production of errors, or at least a misunderstanding of the very objective of trying to simplify an expression. (e.g. Tirosh, Even & Robinson, 1998).

Prior to the introduction of algebra, pupils become accustomed to working in mathematical environments where they solve problems by producing a numerical 'answer', leading to the expectation that the same will be true for an algebraic expression (Kieran, 1981). An arithmetic expression such as 3+2 is successfully interpreted as an invitation to calculate the answer 5, whereas the algebraic expression $3+2a$ cannot be calculated until the value of a is known. Tall and Thomas (1991) defined this unfulfilled and erroneous expectation as the *lack of answer obstacle.* This causes a related difficulty, which Tall and Thomas term the *lack of closure obstacle,* in which the pupil experiences discomfort attempting to handle an algebraic expression, which represents a process that s/he cannot carry out.

Another closely related dilemma is the *process-product obstacle,* caused by the fact that an algebraic expression such as $2+3a$ represents both the *process* by which the computation is carried out and also the *product* of that process. To a pupil who thinks only in terms of process, the symbols $3(a+b)$ and $3a+3b$ (even if they are understood) are quite different, because the first requires the addition of a and b before the multiplication of the result by 3, but the second requires each of a and b to be multiplied by 3 and then the results added. Yet such a pupil is asked to understand that the two expressions are essentially equivalent, because they always give the same product. The pupil must face the problem of realising that the symbol $3a+6$ represents the implied product of any process whereby one takes a number, multiplies it by 3 and then adds 6 to the result. This requires the encapsulation of the process as an object so that one can talk about it without the need to carry out the process with particular values for the variable. When the encapsulation has been performed, two different encapsulated objects must then be coordinated and regarded as the 'same' object if they always give the same product—a task of considerable complexity.

Research on pupils' interpretations of algebraic equations and the process of solving these equations reveal that there are many conceptual difficulties. Booth (1988) says "in algebra, the focus is on the derivation of procedures and relationships and the expression of these in generalised, simplified form" (p. 21). Pupils have difficulty accepting algebraic expressions as "answers" preferring to pick values for the variables in order to give a numerical answer.

Kieran (1981) and Wagner (1977) show that secondary school pupils typically regard the equals sign operationally as "a unidirectional symbol preceding a numerical answer" (Booth, 1988, p. 24), instead of relationally indicating that two quantities are the same. Kieran (1988) also reported that when solving equations, beginning algebra pupils tend to rely on a memorised procedure that appears to disregard the role of the equal sign in the equation.

Wagner and Parker (1988) describe the difficulty that pupils with an operational view of equality often face when solving equations in algebra. Most solution methods assume no relational view of the equals sign, so that pupils must work with the entire relation as they transform it into equivalent relations. They state, "Few pupils fully appreciate the fact that solving an equation is finding the value(s) of the variable for which the left- and right-hand sides are equal" (p. 333). A fundamental requirement of algebra is an understanding that the equal sign indicates equivalence and that information can be processed in either direction (Kieran, 1981; Linchevski, 1995). It has been noted previously that many pupils' understanding of equals is action indication (e.g., "makes or gives" Stacey & MacGregor, 1997) or syntactic (showing the place where the answer should be written – Filloy & Rojano, 1989). Misconceptions relating to the equal concept make it very difficult for pupils to transform and solve equations.

A study by Norton and Cooper (2001), found that pupils showed poor understanding of the concept of equal, order conventions where brackets are not central, operation laws and directed numbers operations. In contrast, pupils showed good understandings of the order convention where brackets were present. Interestingly, many of the deficiencies are such that they would cause difficulties in arithmetic as well as algebra. However, others (concept of equals, application of distributive and associative laws and directed number concept) are such that many arithmetic procedures may not be affected. As argued by Kieran (1992), they may cause difficulties in the transition to algebra. It should be noted that weaknesses such as those with respect to the concept of equals would only affect algebraic manipulations of equations. It is possible for pupils with poor understanding of equals to solve algebraic equations by bracketing (working backwards) or trial and error (Boulton-Lewis et al., 1997b).

Syntactic difficulties

Kieran (1992) and Kücheman (1978, 1981) propose that many pupils have difficulty viewing a letter as a generalised number or unknown. MacGregor and Stacey (1997) have shown that pupils' interpretations of letters and algebraic expressions are based on intuition and guessing, on analogies with other systems they know or on a false foundation created by misleading teaching materials. They state that misinterpretations lead to difficulties in making sense of algebra and may persist for several years if not recognised and corrected. Moreover, they also suggest that younger pupils' misinterpretations are not indicators of low levels of cognitive development but thoughtful attempts to make sense of a new notation or transfer of meanings from other contexts.

The difficulties pupils who study algebra face without adequate arithmetic prerequisite knowledge can be easily seen in the following Year 9 task:

"Solve for x: $2(x-1)+2 = -(4-3x)$". Completing this algebra task requires understanding of the equal concept, orders conventions, operational laws and directed numbers.

Stacey and MacGregor (1997) cite several causes for the misunderstandings pupils commonly have:

i) Pupils' interpretations of algebraic symbolism are based on other experiences that are not helpful,
ii) The use of letters in algebra is not the same as their use in other contexts,
iii) The grammatical rules of algebra are not the same as ordinary language rules,
iv) Algebra cannot say a lot of the things that students want it to say (p. 110).

They found that many eleven-year-olds who had never been taught algebra thought that the letters were abbreviations for words – such as *h* for height – or for specific numbers. These numbers were the "alphabetical value" of the letter – such as $h = 8$ because it was the eighth letter of the alphabet. Another interpretation stems from Roman numerals. For example, $10h$ would be interpreted as "ten less than *h*" because IV means "one less than five".

Another misunderstanding comes from pupils being told that letters represent numbers in algebra. However, pupils are familiar with letters standing for words or labels – such as "p. 10 means page 10" and "<ABC is named using letters to represent points". Another problem is when quantities are represented using the beginning letter of their names. Teachers discuss '*t*' as "time," '*d*' as "distance", and '*s*' as "speed". They make statements such as "Let '*r*' denote the radius" and "We'll use '*t*' to stand for the total." Teachers realise that these letters stand for quantities and measurements, but some pupils see them as standing for the words themselves. Pupils use their prior experiences in arithmetic to interpret equations. Many times pupils have been taught that an equal sign means "gives" or "makes" as in "2 plus 3 gives 5". When given an equation such as $a = 20+b$ many problems arise. Some pupils do not think they can solve for a since b is unknown. Likewise, they cannot tell which variable would have the larger value. Language also presents problems. Rules from language do not apply in mathematical

expressions. The interpretation of $a = 20+b$ for some pupils would be "$a = 20$, then add b". In this interpretation, pupils are trying to apply the rules of English in that the events occur in the order they are presented because there is nothing to signal a change in the order. Another problem arises when pupils try to put what they talk about into algebraic sentences. For example, when given an x-y chart with values and asked to describe the pattern, pupils may know how to do this. However, if asked to represent the pattern algebraically, pupils may not know how to proceed (Stacey & MacGregor, 1997).

Research indicates a number of concepts need to be understood before pupils can begin algebra study. These include: the concept of equal, that is, both sides are equivalent and that information can be processed in either direction in a symmetrical fashion (Linchevski, 1995). It has been noted that some pupils understand equal to be a place where something should be written (Filloy & Rojano, 1989), or as "makes or gives" (Stacey & MacGregor, 1997).

Boulton-Lewis et al. (1998) study of 33 pupils over three years from grade 7 to 9 using interview techniques revealed that by Year 9 most pupils had sufficient understanding of these concepts to operate operationally on algebra problems, that is, they were able to use arithmetic operations to gain closure. The findings also showed that about half the pupils still did not understand equals in the algebraic sense as equivalence/balancing. The researchers conclude that pre-algebra instruction should include the focus on operational laws, equality as equality of sides leading to equivalence, inverse procedures and the use of letters to represent unknowns.

Linchevski and Herscovics (1996) show that pupils' interpretation of mathematical structures in a numerical context is often related to the specific numerical combinations. For example, the research found that the following three expressions 27-5+3; 167-20+10+30 and 50-10+10+10, while having the same structure, triggered different rates of detachment (adding all the numbers after the subtraction and then subtracting the answer from the first number). It was found that 50-10+10+10 triggered the highest rate of detachment since many pupils over-generalised the primitive model of multiplication as repeated addition.

Tall and Thomas (1991) believe that, whilst initial difficulties cannot be totally avoided, they are exaggerated by the teaching of algebra in a context in which the symbolism does not make sense to the vast majority of pupils. It is strongly believed that the success rate can be significantly improved by giving a coherent meaning to the concepts by using a computer.

Sutherland (1991) found that pupils working with Logo and spreadsheets accept "unclosed" expressions such as $x+7$ without difficulty. This led her to question the claim that the need for closure is a major obstacle in learning algebra. Tall and Thomas (1991), also working in a computer environment, note that there needs to be "a reassessment of fondly held beliefs of what is hard and what is easy" (p. 145).

After carefully documenting the difficulties of algebra (e.g., Booth, 1984; Filloy & Rojano, 1989; Kieran, 1985a, 1989a; Sfard & Linchevski, 1994), the field of mathematics education has gradually embraced the idea that algebra need not be postponed until adolescence (e.g., Davis, 1985, 1989; Kaput, 1995). Increasingly, researchers have come to conclude that young pupils can understand mathematical concepts assumed to be fundamental to the learning of algebra (e.g., Carraher, Schliemann & Brizuela, 1999).

From this view, Lawson (1990) states that the study of algebra is a key component in understanding mathematical systems and "should not await high school freshman or precocious eighth graders – as if they are required to master computation before being introduced to algebraic concepts" (p. 1). The introduction of equations in elementary schools helps to set up pupils for being successful in algebra. By waiting until a pupil is taking algebra to introduce equations, problems may arise. Introductory chapters in algebra tend to move very quickly and ask problems which could easily be solved without the use of algebra. As a result, many pupils do not take the beginning chapters in algebra seriously and later realise that they should have. Another problem is that some ninth grade pupils show an aversion to using letters instead of numbers, especially when they know what the number should be (Nibbelink, 1990). A pupil's understanding of variables is vital for their success in algebra. The idea of using a letter to represent a number or other mathematical object is very mysterious to pupils. If a pupil's first exposure to variables is allowing a letter to represent an unknown number, then that pupil is going to be limited in his/her understanding of variables. Difficulty will arise when faced with understanding sentences that begin "For all real numbers, *x*, ..." and "For any real numbers *a*, *b*, ..." (Leitzel, 1989). If pupils "do not view letters as representing numbers, then performing arithmetic operations with them is a meaningless task" (Chalouh & Herscovics, 1988, p. 34).

Research into teaching and learning algebra has demonstrated that one of the fundamental problems is pupils' difficulty in being able to manage a formula and its meaning at the same time (e.g. Arzarello, Bazzini & Chiappini, 1995; Sfard, 1991; Linchevski & Herscovics, 1996).

Given the gulf between arithmetic and algebra, it is no surprise that research in mathematics education has consistently found that pupils have enormous difficulties with algebra (see, for instance, Booth, 1984; Filloy & Rojano, 1989; Kieran, 1985a, 1989a; Sfard & Linchevski, 1994; Vergnaud, 1985; Wagner, 1981). To help pupils overcome the difficulties encountered in the transition from arithmetic to algebra, researchers such as Herscovics and Kieran (1980), and Kieran (1985b) have developed teaching approaches that seek to gradually transform seventh and eighth graders' knowledge of arithmetic, thus allowing them to build an understanding of equations.

Previous research has highlighted pupils' difficulties in solving equations when unknown quantities appear on both sides of the equality (e.g., Filloy & Rojano, 1989; Herscovics & Linshevski, 1994). Many attributed such findings to developmental constraints and the inherent abstractness of algebra, concluding that even adolescents were not ready to learn algebra (Collis, 1975; Filloy & Rojano, 1989; Herscovics & Linchevski, 1994; Linchevski, 2001; MacGregor, 2001; Sfard & Linchevski, 1994). Furthermore, some have claimed that pupils are engaging in algebra only if they can understand and use the syntax of algebra and solve equations with variables on both sides of the equals sign (Filloy & Rojano, 1989).

3.3.2 Theoretical approaches to the teaching and learning of algebra

Many studies have been carried out with the aim of providing theoretical background to the teaching and learning of algebra. Some of them are as follows:

In examining the difficulties pupils encounter in moving from arithmetic to algebra, Sfard (1991, 1994), Kieran (1989a, 1992), and Herscovics (1989) describe a number of obstacles that can be connected directly to the difficulty in reification as described by Sfard. For example, pupils usually have difficulty accepting an algebraic expression as an answer; they see an answer as a specific number, a numerical product of a computational operation. The equal sign is usually interpreted as requiring some action rather than signifying equivalence between two expressions.

Kieran (1992) proposes that the problem with modern algebra is that we impose symbolic algebra on pupils without taking them through the stages of rhetorical and syncopated algebra. Thus, as many educators and pupils have observed, pupils often emerge from algebra with a feeling that they have been taught an abstract system of operations on letters and numbers with no meaning. Herscovics (1989) describes the situation by stating that the pupils have been taught the syntax of a language without the semantics. In other words, they know all the rules of grammar but do not understand the meaning of the words. Sfard and Kieran would argue that this situation has resulted from jumping to symbolic algebra without exploring rhetorical and syncopated algebra.

Sfard's three-stage process seems to repeat itself historically and perhaps cognitively in the development of understanding of other mathematical concepts. Negative numbers, for example were originally considered the abstract result of the process of subtracting a larger number from a smaller one. It took hundreds of years for mathematicians to see negative numbers as objects representing direction rather than the waste products from a process on counting numbers. Complex numbers, again originally defined in terms of a process, for 300 years appeared to be useless to algebra learners but interpreting the numbers as a way of referencing the plane—visualising these numbers as objects—they eventually became indispensable in engineering.

Reification was thus a historically difficult process; it is no wonder that it is a difficult process in the classroom. Sfard (1994) admits that her research indicates reification does not build slowly over time but is a sudden flash of insight, a "big bang," a "discontinuity" (p. 54). Freudenthal (1978), a leading philosopher of mathematics education in The Netherlands, claims "what matters in the learning process are discontinuities" (p. 165).

Sutherland and Rojano (1993b) have designed activities that allow them to investigate the potential of the spreadsheet in helping young pupils (10-15-year-olds) move from non-algebraic strategies to more algebraic approaches when coping with negotiating algebra word problems solutions. On the basis of their previous results using computer environments, Sutherland and Rojano have been trying out different strategies to help overcome pupils' reluctance to spontaneously work with the unknown when facing situations involving generality. Their results have shown that work with the spreadsheet helped pupils to accept the idea of working with the unknown. Their findings suggest that the algebra-like spreadsheet symbolic code may be used to mediate the algebraic approach. They argue that, in a spreadsheet, a critical feature in helping children move from a non-algebraic approach to a more algebraic strategy is that pupils first use a cell to represent the unknown by a cell reference (for example, x), then other mathematical relationships are expressed in terms of this unknown. Then pupils can use pointing with the mouse to support the expression of mathematical relationships. When a given problem has been expressed in the spreadsheet code pupils can vary the unknown either by copying down the rules or by changing the number in the cell representing the unknown. This method has shown encouraging results.

The view of algebra as a language has been changed and broadened by technology. The availability of different representations for expressing quantitative relationships such as graphics and tables has influenced the ways in which mathematics educators conceive the teaching and learning of algebra. From this view, algebra can be seen as a language with various dialects: symbols, graphs and tables. Particularly, new technology seems to strengthen the view of algebra as a language for generalising arithmetic.

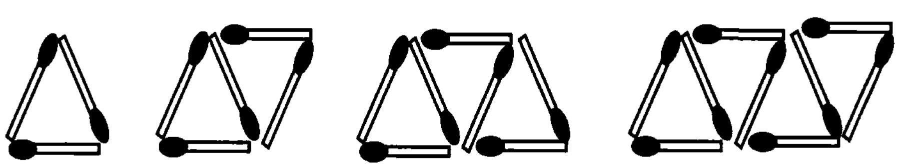
3.4 Solving algebraic problems

The present study investigated pupils' thinking processes when solving algebraic problems. The study intended to cover most of the algebra content at the first two years of secondary education as outlined in the mathematics curricula in both England and Thailand. The content was categorised into six themes – patterns/sequences, simplification, substitution, solving equations, graphs of linear functions, and word problems. The next sections focus on previous findings concerning these six themes.

3.4.1 Patterns and sequences

Much of the available research on pupils' thinking processes in generalisation reports on pupils' strategies in abstracting number patterns and formulating general relationships between the variables in the situation (e.g., Garcia-Cruz and Martinon, 1997; MacGregor & Stacey, 1993b; Orton and Orton, 1994; Taplin, 1995).

Linchevski et al. (1998) presented grade 7 pupils with a match problem as follow.



picture 1 picture 2 picture 3 picture 4

The table shows how many matches are used for the different pictures. Complete the table.

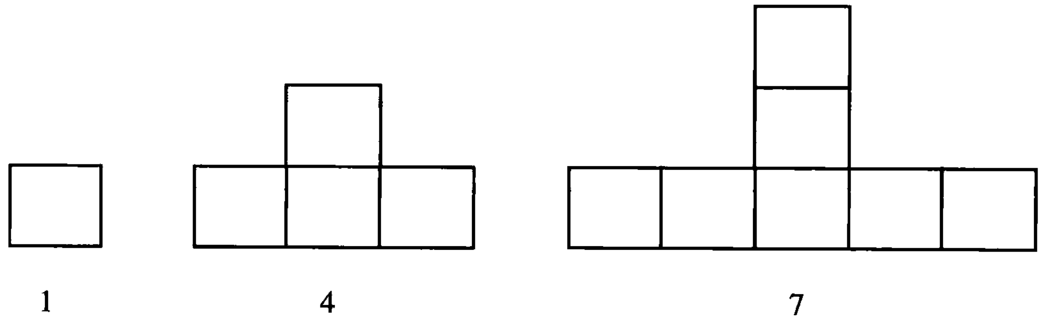
Picture number	1	2	3	4	5		20		100		n
Number of matches	3	5	7	9							

Few pupils managed to construct a function rule to find function values. Rather, they focused on recursion (e.g. $f(n+1) = f(n) + 2$ in the problem above), which led to many mistakes as they tried to find a manageable method to calculate larger function values. The most common, nearly universal mistake was to use the proportionality property that if $n_2 = k \times n_1$, then $f(n_2) = k \times f(n_1)$. This is illustrated in the problem above; from $f(5) = 11$ they deduced that $f(20) = 4\times11 = 44$. Although this property applies only to functions of the type $f(n) = an$, pupils erroneously applied it to any function. The use of "seductive numbers" in a sequence like n = 5, 20, and 100 *stimulated* the error and they found that most pupils' generalisations and justification methods were invalid. Pupils were not aware of the role of the database in the process of generalisation and validation. An example of this is seen in the problem above, pupils did not, and seemed unable, to verify their generalisations against the given data pairs (1; 3), (2; 5), (3; 7), (4; 9).

They also found that pupils worked nearly exclusively in the number context and did not use the structure of the pictures at all.

Radford (1996) considers an analysis of the logical base inherent in the generalisations of number patterns. This analysis begins by considering the goal of such generalisations which is to "see a pattern" in the set of data ("observed facts") and to obtain a "new result" (conclusion or rule). Firstly, the recognition of a pattern can lead to different kinds of representations due to the way in which the pattern is perceived or interpreted, for example:

"observed facts":



This will lead to "seeing the facts" in different ways and the emergence of new representational systems of these facts, for example:

1; 1+3×1; 1+3×2; ...

or 3×1 −2; 3×2 −2; 3×3 −2; ...

In finding the number of squares in the 100^{th} picture, the generalisation involves extracting what is variant and invariant from the syntactic structure of these new representations.

The review of past literature related to pupils' thinking processes in 'seeing number patterns' found that pupils who used the proportionality property (scaling up) process, which could apply to the function of type $f(n) = an$, tended to apply it to any functions. In the present study, the algebra test items were designed to investigate how pupils worked on functions of type $f(n) = an$ and $f(n) \neq an$ (e.g., item 1, item 13 in Section 6.2).

3.4.2 Simplification of algebraic expressions

Cooper, Williams and Baturo (1999b) concluded from their research findings that the link between arithmetic and algebra seemed generally successful for algebraic simplification. The processes of simplification have to be extracted away from the particular instances in which they appear. However, the process is arduous for pupils and is easily complicated by missing or defective arithmetic components.

Demby (1997) reports the traditional emphasis in the curriculum on 'finding the answer' allows pupils to get by with informal and intuitive procedures in arithmetic. In algebra they are required to recognise the structure that they have been able to avoid in arithmetic. Matz (1982) argues that it is not unreasonable that pupils should interpret the algebraic expression $3x$ as $3+x$ according to their experiences such as $3\frac{3}{4}$ being interpreted as $3+\frac{3}{4}$. Thus, there may be room for confusion and misconception in the initial stages of simplifying an expression. He is concerned with 'degenerates formalism' characterised by thoughtless, 'slapdash' manipulation of symbols. Tirosh, Even and Robinson (1998) explain the dual nature of mathematical notations: process and object e.g., $3x+5$ might be viewed as the process 'add three times x and five' and for an object. They state that pupils tend to grasp it only as a process and finish the expression as $8x$ or 8 (see accepting lack of closure mentioned earlier).

The literature points to many complex psychological processes involved in gaining an understanding and avoiding a misinterpretation of algebra rules. For instance, Kieran (1992) reports that only a very small percentage of 13- to 15- year-old pupils is able to consider the letter as a generalised number. Also, Küchemann (1981) concludes that the majority of 13- to 15- year-old pupils were unable to cope with algebraic letters as unknowns or generalised numbers. He identified pupils' understanding of algebraic letters into six levels; letter evaluated, letter not used, letter used as an object, letter used as a specific unknown, letter used as a generalised number, and letter used as a variable. This understanding is important in the process of the simplification of algebraic expressions where both 'the question and the answer' involve letters. The understanding of a letter as a generalised number has implications in the checking of work. There is a big difference between checking the results of an equation and an expression. For instance, in the case of the equation $x+2 = 5$, the result is $x = 3$, the checking require only $3+2 = 5$, while for the expression $a+2a = 3a$, need to check that the result works for any number e.g., $a = 2, 5, 10$ .... Thus, checking the simplification of an expression seems to be harder than checking the solution of an equation.

The previous research reported that pupils have difficulty in viewing a letter as a generalised number or unknown, and in accepting lack of closure. To investigate pupils acceptance of lack of closure the test items were designed to include expression containing unlike terms as well as the more straightforward combinations of like terms (e.g., item 2, item 8 in Section 6.5).

3.4.3 Substitution

Radford (1997) defines the trial and error method as a simple method, which has the advantage of requiring knowledge of only simple arithmetic concepts. She states that this method has the disadvantage that it can take a long time to find the answer, depending on the complexity of the numbers involved. In this method one simply repeats the same procedure with different quantities until one obtains the correct answer.

Tall (2001) points out that the National Curriculum in England intended to use arithmetic problems such as the following as a precursor of algebra:

(1): 3+4 = □, (2): 3+ □ = 7, (3): □ +3 = 7.

Although these *look* like algebra, they are certainly not. Pupils perform them using their repertoire of methods of counting and deriving or knowing facts. Question (1) can be done by counting method; (2) can be done by 'count-on' from 3 to find how many are counted to get to 7. Question (3) is more subtle. If the pupil senses that the order of addition does not matter, the problem is essentially the same as (2); and can be solved by count-on from 3. If not, the pupil who counts has a far more difficult task to find out 'at what number do I start to count-on 3 to get 7?' This involves trying various starting points to count-upto using a 'guess-and-test' strategy.

Foster (1994) used these three types of questions in a study of 'typical' pupils in the first three years of an English Primary School. He found a significant spectrum of performance in the first year where the lower third were almost totally unable to respond to questions of types (2) and (3). By the third year the top two-thirds of the class obtained almost 100% correct responses but the lower third obtained 93% correct on type (1), 73% correct on type (2) and 53% on type (3).

Carraher, Schliemann and Brizuela (2001) ask in their title: 'can young children operate on unknown?' The evidence they provide reveals that their approach has absolutely *no* operation *on* unknowns in the sense of symbol manipulation. There is evidence of evaluation by substitution (as a by-product rather than a direct focus of the activity). In general, the pupils' activity involves arithmetic operations on arithmetic symbols.

Demby (1997) identifies seven types of procedures used by pupils, labelled: (A) *Automatization,* (F) *Formulas,* (GS) *Guessing-Substituting,* (PM) *Preparatory Modification* of the expression, (C) *Concretization,* (R) *Rules,* (QR) *Quasi-rules.* Demby reported in grade 7, 3 of 108 pupils solved problem 2 correctly (Find the numerical value of expressions (g) $2x+3-3x$ and (h) $-x+2-x^2+1$ for $x = -5$); 5 pupils did not attempt it. Nevertheless, 85% of pupils (90% of those who tried to solve the problem) manifested elementary understanding of substitution. The errors on computations with negative numbers caused serious troubles. Only a quarter of seventh graders manifested substituting in the simplified version of the expression though it had been explained many times in the classroom.

Linnecor (1999) points out that a number of misconceptions can arise when asking pupils to collect terms and substitute in values. One common misconception is that they believe answers should always be single terms and numerical. For example, if an answer is $a+b$, pupils would replace this with a co-joined term ab and then substitute numerical values into this. This co-joined term may be read in a 'place value' sense as in arithmetic. For example, if $y = 3$, the term '$4y$' may be interpreted as 43 (Booth, 1989).

Research in pupils' early learning of algebra found that pupils could substitute values for letters. Some pupils made errors in treating co-joined terms as 'place value' in arithmetic and in computing directed numbers. These informed the design of the algebra test items to examine pupils' thinking processes in substituting positive and negative numbers and to observe how pupils deal with co-joined terms (e.g., item 3, item 9 in Section 6.8).

3.4.4 Solving Equations

In the construction of algebraic thinking, the ability to write and to solve equations is important (Reggiani, 1994). Numerous studies have considered the capacity to write and to solve equations. The concept of equivalence has been researched in the context of using the equal sign in its relational sense (e.g., Behr, Erlwanger & Nichols, 1980; Booth, 1982; Sfard, 1994; Liebenberg, Sasman & Olivier, 1999). Ursini & Trigueros (1997) studied the various uses of letters as unknown quantities and as parameters.

Boulton-Lewis et al. (1998) studied 33 pupils over three years from grade 7 to 9 using interview techniques. They probed pupils' understanding of commutative and inverse laws of operations, meaning of equal, meaning of unknown, variable concept and solutions of linear equations. The results indicated that most ninth graders had sufficient understanding of these concepts to operate operationally on algebra problems. Pupils were able to use arithmetic operations to gain closure. However, the authors noted that about half of the pupils did not understand the equals sign in the algebraic sense to need to do the same operations to both sides to maintain equivalence.

A study by Norton and Cooper (1999) followed on that by Boulton-Lewis et al. (1998) in exploring the nature of pupil awareness to begin algebra. He observed 45 Year 9 pupils and 9 Year 10 pupils over 20 lessons. The findings showed that many pupils had neither operational nor structural understanding of arithmetic. This result contradicted the study by Boulton-Lewis et al. (1998) who reported that "by grade 9 most pupils had sufficient understanding of the commutative law to apply this to linear equations, the majority of pupils also had displayed a satisfactory understanding of inverse procedures and of correct order of operations … most pupils had satisfactory arithmetic understandings to enable them to apply these principles to algebra" (p. 149).

Herscovics and Kieran (1980) asked pupils to build numerical expressions with more than one operation on each side of the equal sign in an effort to expand their understanding of the equal sign. In the latter research, pupils realised that the concept of equation indicated that the numerical expressions on each side had the same numerical value. However, the expressions they constructed were often not equivalent. Booth (1982) conducted research that provided information on the kinds of expressions that pupils would perceive as being equivalent. It was found that pupils regarded expressions such as $5\times e+2$ and $5\times(e+2)$ as being equivalent and that the pupils' interpretation of these expressions changed depending on the context.

Other studies investigated how pupils judged the equivalence of numerical expressions *without computing the answer* (e.g. Collis, 1975). The findings suggest that pupils are not in a position to judge the equivalence of numerical expressions without computing. As in Kieran's (1989a) study, the indications are that pupils are not aware of the underlying structure of arithmetic operations and their properties. This situation is most likely due to a predominantly computational focus in the earlier grades.

Liebenberg, Sasman and Olivier (1999) developed two dimensions of understanding the equivalence of algebraic expressions. The first dimension of understanding is that two algebraic structures are equivalent if the numerical expressions are equal for ***all*** values of the variable. The second dimension of understanding involves the function or *usefulness* of algebraic equivalence so that the transformation of one algebraic expression into another becomes meaningful for the pupils. They stated that pupils do not simply engage in simplifying algebraic expressions but focus explicitly on the properties of the operations that make it possible to carry out transformations.

When considering in particular the capacity to solve equations, a number of studies have brought to light various other aspects that are included in this process. In particular, studies have been carried out concerning certain problems linked with using the equal sign in its relational sense (Sfard 1994) and the various uses of letters as unknown quantities and as parameters (Ursini & Trigueros, 1997). Tall (1995) found differences between flexible thinkers at all ages using symbols dually as process or concept and those relying on symbolism to cue routine procedures. In algebra, those who saw the symbols as procedures to be carried out are less likely to grasp the meaning of the symbolism. Pupils conceiving of $3+2x$ as a *process* do not see it making sense unless x is known to have a value, but if x is known, there seems no reason to complicate matters by using the symbol x. An equation such as $5x+1 = 11$ might make sense as a problem where five times a number plus one is eleven, so five times the number is ten, and the number is two. But the equation $5x+1 = 3x+5$ would be less likely to make sense because the equals sign no longer means "makes" and there are now two processes to carry out, one on each side. The flexible thinker has a meaningful way of manipulating equations to obtain a solution, but the procedural thinker is more likely to learn mechanical routine (Tall & Thomas, 1991).

The most common method of introduction for linear equations is an example of the first alternative, that of 'the equation as a balance' (Pirie & Martin, 1997). Typically, pictures of a weighing machine with two balancing scale pans are presented with objects and weights in the scale pans. The problem is to find the weight of a single object. Initially pupils solve the early simple problems intuitively; they can 'see' the answers. More difficult examples are offered with objects and weights mixed together in the scale pans and the suggestion is made that they take things off (pseudo-physically) each scale pan until they have an answer. An immediate difficulty arises: unless the pupils are to keep drawing pictures of scale pans, objects and weights, they must invent, or be taught, a symbolic representation of the problem. The 'equals sign' (=) is taken to represent the pivot of the balance and the solution of the problem is achieved by 'taking the same things away from both sides, to preserve the balance'. The clear link is made between physical removal and subtraction. This does, however, add to the complexity of coming to understand the concept of linear equations the need for the ability to symbolise from a verbal problem.

Singer (2001) states that for the conditional equations, which they were able to solve by working back, it was possible to start with the equation and find a sequence of equivalent equations, the last of which clearly indicated a single value that satisfied the equation. This method of working back enables us to have a fairly routine method for finding roots to a wide variety of equations and for most such equations working back is the more efficient method to use. Equations that cannot be solved by some form of working back are usually not included in most introductory books on ordinary algebra and so many algebra books only use the method of working back.

The previous studies pointed out that pupils showed poor understanding of the concept of the equals sign in the algebraic sense, operation laws and use of directed numbers. Research reports pupil difficulties in solving equations when the unknown appears on both sides of the equality. These findings informed the design of the algebra test items to investigate how pupils find the unknown quantities with positive and negative signs in different positions in the equation and to observe how they maintain equivalence (e.g., item 4, item 16 in Section 6.11).

3.4.5 Graphs of linear functions

The National Council of Teachers of Mathematics (1989) refers to the concept of function as "an important unifying idea in mathematics" (p. 154). Alongside the statements emphasising the importance of functions are recommendations on how the function concept should be taught. Some recommendations have been based on consideration of pupils' cognitive processes in constructing concepts about functions. Sfard (1989) for example, observes that pupils first develop an *operational* conception of function, in which they think of the computational processes associated with functions. This is sometimes followed by a *structural* conception in which they think of functions as objects. She proposes that mathematical concepts like function should not be introduced by means of structural descriptions, such as that described by the definition of function as a set of ordered pairs. Rather, introduction should be by operational descriptions, such as the definition of function as a dependence of one varying quantity on another. Dreyfus and Eisenberg (1982) similarly suggest that functions should be introduced in such a way that pupils' *intuitions* and experiences are utilised. Dubinsky, Hawkes and Nichols (1989) proposed a model for the learning of functions by college students. In the context of this model they suggested that certain computer activities might assist pupils in constructing function concepts.

The concept of function is very complex. There are several reasons for this. First, there are many common ways to represent functions, including graphs, formulas, tables, mappings, and descriptions. Meaningful understanding requires individuals to construct multiple representations as well as operations for transforming from one representation to another. Second, the notion of function involves many other concepts. A few of the sub-concepts associated with it are domain, range, inverse, and composition. Other concepts closely related to function are quantity, variable and ratio. It is difficult to discuss functions without referring to some of these sub-concepts. Third, there are several accepted definitions for function (e.g., dependence relation, rule, mapping, and set of ordered-pairs). Although these definitions are equivalent (or nearly equivalent) mathematically, they differ conceptually (e.g., Vinner & Dreyfus, 1989).

The function concept has been a major focus of attention for the mathematics education research community over the past decade (for example, Dubinsky & Harel, 1992). Schwingendorf, Hawks and Beineke (1992) contrast the vertical development of the concept in which the process aspect is encapsulated as a function concept and the horizontal development relating different representations. DeMarois & Tall (1996) refer to these as depth and breadth respectively (noting that increasing depth here means higher levels of cognitive abstraction) and investigate the way in which the pupils' concept image of function can be described in terms of these two dimensions.

DeMarois and Tall (1999) studied the complexity of the function concept using a function machine. The function machine provides a primitive idea that the majority of the pupils recognised at the beginning of the course, at least at a procedural level. It has an inner procedure that can be viewed externally as an interiorised process and potentially as a mental object that can be operated upon. In this sense the function machine can operate as a cognitive root for the function concept itself. However, they stated that for many pupils, the complexity of the function concept is such that the making of direct links between all the different representations is a difficult long-term task.

The review of past literature related to pupils' thinking processes in constructing function concepts found that pupils have difficulty in linking the different representations of a function. This informed the design of the algebra test items to investigate how pupils connect a choice of graphs with a given function (e.g., item 17, item 23 in Section 6.14).

3.4.6 Word problems

Learning to solve problems using algebra is hard. It is well known that students often have difficulty in writing algebraic equations to represent the information given in word problems and that it is hard to learn the ways in which the equations must be solved to get solutions (Stacey & MacGregor, 2000). There have been many studies of the processes of comprehension of word problems (Just & Carpenter, 1989, 1992; Mayer, Lewis & Hegarty, 1992; Nathan, Kintsch & Young, 1992). Stacey & MacGregor (2000) state that there is no easy transition from comprehending a problem to formulating an equation or set of equations—in fact it is a major site of difficulty that operates differently in solving problems arithmetically or algebraically.

There are many published reports of pupils' errors in writing simple algebraic equations (e.g., Clement, Lochhead & Monk, 1981; Cooper, 1985; Kaput & Sims-Knight, 1983; Mestre, 1988). It is widely accepted that pupils make errors because of:

- The use of algebraic letters as abbreviated words (e.g., *a* means "apple", not "number of apples").
- Attempting to translate directly from key words to mathematical symbols, from left to right, without concern for meaning (e.g.,

"There are six times as many cats as dogs" is translated incorrectly as

$6 \times c = d$).

- Use of the "equal" sign to indicate that what is on the left is loosely associated with what is on the right (e.g., $20p = t$ could mean "There are 20 pupils for every teacher").
- The misleading influence of mental pictures (e.g., groups of 20 pupils and individual teachers seen in the mind's eye, and represented on paper as $20p + t$, $20p = t$ or $20p$: t).

The type of error shown in the example above, where the numerical value is associated with the wrong variable in a simple linear equation is referred to in the literature as the reversal error. It is accepted (Herscovics, 1989; Laborde, 1990; Mestre, 1988) that a major cause of reversal error is the attempt to translate directly from words to symbols.

Previous research has highlighted pupils' difficulty in writing algebraic equations to represent the information given in word problems. Pupils for example, used letters as abbreviated words, and translated directly from left to right without concern for meaning. The purpose of the algebra test items was to investigate how pupils transform the word problems to the equations in different given situations (e.g., item 12, item 18 in Section 6.17). The test did not include the typical "student/professor" problem because this did not appear in the early stages of teaching algebra in the mathematics curriculum of both England and Thailand.

3.4.7 Concluding remarks

From the literature reviewed, researchers see algebraic thinking as a combination of abstract arithmetic, modelling and language. In the classroom these strands are not distinct but are developed across a number of curriculum themes: patterns/sequences, simplification, substitution, solving equations, graphs of linear functions, and word problems.

The main issues that reflect pupils' successful processes in learning early algebra can be summarised as:

- Pupils showed good understandings of the order of operations where brackets were present.
- Success rates can be significantly improved by using a computer to teach algebra.
- Pupils working with Logo and spreadsheets accept 'unclosed' expressions without difficulty.

The difficulty in learning early algebra can be summarised as:

- The sequence of reading from left to right
- The order of operations
- Accepting lack of closure
- The role of the equals sign in equations
- Viewing letters
- The unknown quantity appears on both sides of an equation
- Computing negative numbers
- Using letters as abbreviated words
- Translating directly from key words to mathematical symbols.

Comparing the thinking processes of pupils with different curricular experiences, as in the English and Thai schools participating in this research, might help us to gain a better understanding of how to cultivate important basic concepts and make algebra more accessible to the novice.

The next chapter presents the research design and methodology developed in the present study as influenced by the literature reviewed in Chapter 2 and Chapter 3.

CHAPTER 4

RESEARCH DESIGN AND METHODOLOGY

4.1 Introduction

This study compares the work of English and Thai pupils in mathematics learning in the first two years in secondary school. It investigates the thinking processes employed by the pupils in solving algebraic problems and compares them. The processes used are then related to the curriculum and how it is delivered in both contexts.

In the previous chapters, literature relating to algebraic thinking, difficulties in learning algebra, and to comparative case studies were reviewed. This chapter looks at the choice of the broadly comparable case study schools, ethical considerations, data sources, instrumentation, and the researcher's roles. It then outlines how lessons were observed and how tests and individual interviews were conducted. The chapter closes by presenting the evolution of the algebra test, the methods used to analyse the data, development of the codebook and some examples of coding the algebra test items.

4.2 Research design

This study was designed to use two main methodologies, one qualitative, and the other quantitative. Qualitative data was obtained from algebra lesson observations, semi-structured interviews and pupils' written responses to the algebra test. Quantitative data involved calculation of the proportion of responses indicating use of generalisable and other processes in pupil responses to the algebra test. These features characterise the present study as a qualitative research on the cognitive nature of the phenomena and a quantitative research on the cognitive achievement.

The aims of the research were to investigate the mathematics curricula in the English and Thai schools, analyse pupils' thinking processes in solving algebraic problems, and relate the pupils' thinking processes to their experience in their own country's algebra lessons.

To achieve these aims the research questions were (1) how do pupils in an English school and in a Thai school solve algebraic problems? (2) how different are their thinking processes when solving algebraic problem? and (3) how might mathematics curricula be interrelated with pupils' thinking processes in solving algebraic problems?

4.2.1 Choosing comparable research sites

A comprehensive school in Northeast England was purposively selected to conduct the investigation because this kind of school is similar to the state school in Thailand. In the first instance, two comprehensive schools were visited but only one accepted the invitation to become involved in the study. With most schools in Thailand being state schools, a broadly comparable state school in Northeast Thailand was selected to facilitate comparisons, in line with the aims of the study. For example, a large proportion of pupils of high ability, and a similar number of sets of pupils in each of Year 7 and Year 8. Analysis by *t*-test was used to find broadly comparable groups as shown in Table 4.1.

Table 4.1 Comparison of the selected groups

Groups	**English school**		**Thai school**			
	Mean	**SD**	**Mean**	**SD**	***t***	***p***
Year 7/Secondary 1						
Top set/High ability	81.13	5.56	80.94	6.68	0.12	.91
Bottom set/Low ability	33.94	10.53	37.62	7.89	-1.51	.14
Year 8/Secondary 2						
Top set/High ability	56.26	8.95	61.30	18.60	-1.60	.11
Bottom set/Low ability	22.55	8.64	25.65	4.49	-1.56	.13

As illustrated in Table 4.1, the mean score of the test for Secondary 1 high ability group in Thai school was 80.94 (SD = 6.68). The corresponding score in the English Year 7 top set was 81.13 (SD = 5.56). The difference in these scores is not statistically significant at the two-tailed 5% level ($p = .91$). The mean score of the test for Secondary 1 low ability group in Thai school was 37.62 (SD = 7.89). The corresponding scores in the English

Year 7 bottom set was 33.94 (SD = 10.53). The difference in these scores is not statistically significant at the two-tailed 5% level ($p = .14$).

The mean score of the test for Secondary 2 high ability group in Thai school was 61.30 (SD = 18.60). The corresponding score in the English Year 8 top set was 56.26 (SD = 8.95). The difference in these scores is not statistically significant at the two-tailed 5% level ($p = .11$). The mean score of the test for Secondary 2 low ability group in Thai school was 25.65 (SD = 4.49). The corresponding score in the English Year 8 bottom set was 22.55 (SD = 8.64). The difference in theses scores is not statistically significant at the two-tailed 5% level ($p = .13$).

Entry to the comprehensive school in England was gained in June 2001 following a meeting with the head of the mathematics department and letter to the head teacher of the school (Appendix A). In July 2001 arrangements with the state school in Thailand were also finalised using similar processes. In both cases the purpose of the research was clearly outlined and assurances of anonymity and confidentiality given. Following this, the heads of departments negotiated with four teachers to allow the researcher to observe their lessons for the specified period.

Participants

The 103 pupils in the English school and 186 pupils in the Thai school were participants in the present study. Breakdowns by sex and ability groupings are shown in Table 4.2.

Table 4.2 Number of pupil participants

			sex		
country			boys	girls	Total
Eng	ability	high	28	28	56
		low	18	29	47
	Total		46	57	103
Thai	ability	high	41	62	103
		low	41	42	83
	Total		82	104	186

As indicated in Table 4.2, a total of 103 pupils in the English school with 56 high ability and 47 low ability, 46 boys and 57 girls participated in the present study. For the Thai school, 186 pupils in total with 103 high ability and 83 low ability, 82 boys and 104 girls participated.

All 103 English pupils in the four sets—a top set of Year 7, a top set of Year 8, a bottom set of Year 7, and a bottom set of Year 8—participated in the study for the purposes of field observation and taking the test. Comparable groups of 186 Thai pupils, a high ability group of Secondary 1, a high ability group of Secondary 2, a low ability group of Secondary 1, and a low ability group of Secondary 2 participated for the same purposes. Four pupil participants from each group were selected, based on the school test scores and close observations in their lessons, to take part in the individual interviews (see Section 4.3.5).

Pupils' and teachers' verbalisations in lessons were audio taped. One Thai pupil in the Secondary 2 low ability group allowed no audiotape recording during interviews. In this case interview notes were transcribed immediately after the interviews.

4.2.2 English case study school

The English school is a mixed 11-18 years comprehensive school in County Durham in the Northeast of England. It was established from the amalgamation of a former selective grammar school with two non-selective secondary schools. The school numbers have steadily increased over the years and there are some 1,500 pupils attending the school of which over 300 are studying Post-16.

Pupils entering the school are drawn from areas covering a wide range of socio-economic backgrounds. The majority, however, are from households with higher than average educational advantage. Pupils' attainment on entry to the school is spread over the full ability range but with a larger proportion of pupils at the higher levels than the national average (Ofsted, 1999).

In England, the school year begins in September and ends in July. With the researcher having no experience of the National Curriculum in England, data collection was conducted over the full school year in mathematics lessons. This allowed for observation of the complete mathematics curriculum for Year 7 and Year 8 but the focus was on the algebra content.

4.2.3 Thai case study school

The Thai school is a mixed school with 3 years of lower secondary and 3 years of upper secondary levels in Buriram Province in the Northeast of Thailand. It is the oldest secondary school in the Province from which a number of pupils pass the national entrance examination to study in the major universities each year. There are over 3,000 pupils attending the school. Although this number of pupils is approximately twice that in the English case study school the number of sets of pupils is similar because Thai classes are about twice as large (see Table 2.2). Pupils entering the school were drawn from areas covering a wide range of socio-economic backgrounds. Pupils' attainment on entry to the school covers the full ability range but there is a large proportion of pupils of high ability.

In Thailand, the school year begins in May and ends in March. The case study school started the academic year in mid-May so data collection was conducted from October 29th, 2002 to February 14th, 2003 in algebra lessons. Due to the fact that the researcher taught for 13 years in the local secondary school and spent five years in training mathematics student teachers at Buriram Rajabhat University, there was much familiarity with the mathematics curriculum at the time of collecting data.

4.2.4 Ethical considerations

As the researcher was interested in learning from participants at the schools it was felt necessary to contribute to the collaborating schools. During classroom observation, therefore, the researcher assisted pupils in their exercises and provided additional tutoring for pupils who were struggling with mathematics. Assistance was also offered to the teachers in checking their pupils' worksheets and exercise books. In carrying out the research, teachers' and pupils' rights with regard to continuing participation and anonymity in the final thesis were strictly observed. At the end of the study and the examination process, all tape recordings relating to the data collection were destroyed.

4.2.5 Data sources

Regianni (1994) suggests that classroom observations provide a rich source of information. In this study, the primary sources of data were lesson observations, interviews with pupil participants, and analysis of pupils' responses to the algebra test items. Other pupils' written works were also collected to provide a broader view of the pupils' thinking processes.

4.2.6 Instrumentation

Apart from observing algebra lessons, the use of field notes, and interview transcripts, data was also collected through regular school tests, and an algebra test developed by the researcher.

4.2.7 Supporter/researcher

Due to the interactions with participants, the researcher more or less played two roles—supporter and researcher—which improved both the relationship with and understanding of the participants. Supporting the teacher and pupil participants in their lessons not only provided detailed knowledge of the context for the interviews but also helped in knowing the pupils much better than if lessons were simply observed from the back of the classroom and a few selected for close observation and interviews.

Considering the goals of the study, validity and reliability are ensured during analysis through thematic analysis, as described under the section method of analysis. In essence, the conduct of the study and the method of analysis were designed to take advantage of the opportunities provided by the researcher's role as supporter/researcher.

4.3 The Study

The study was conducted from September 2001 to July 2002 in the English school and from October 2002 to February 2003 in the Thai school. Qualitative data was collected through classroom observations recorded via field notes and audiotape recording, and informal discussion with pupils and teachers. Four pupils of varying academic ability in each of year 7 and year 8 were selected for semi-structured interviews and close observations. In addition, work samples, samples of exercise books, and other curriculum materials with pupils' answers were collected. The semi-structured interviews with pupils were used to clarify their thinking processes and to generate additional data from their interpretation of events.

The semi-structured interviews focused on pupils' thinking processes for getting the answers to algebra items. The analysis of the written responses to the algebra test also provided an opportunity to gain insights into pupils' thinking processes. Informal conversations focused on clarifying ambiguities or checking explanations of the responses to the algebra test.

The combination of field notes, interview transcripts and the responses to the algebra test provided evidence for analysis of both the thinking processes and the relation to mathematics curricula in the case study schools.

4.3.1 The English school lesson observations

As stated earlier, heads of mathematics department negotiated with four mathematics teachers in each study site for access to their mathematics lessons on the following basis:

Ms. Great taught five classes of mathematics, one of Year 7, two of Year 8, one of Year 10, and one of Sixth Form classes. The selected class for observing was Year 7x1 (Year 7, x-band, top set 1). The mid-point of a term two test (see Appendix B Year 7 Test half term 2), which had been translated into Thai, was used to find the comparable group in the Thai school. The mean score of this set was 81.13 (SD = 5.56). Mrs. Smart taught two classes of mathematics, one of Year 7, and one of Year 8. The class selected for observation was Year 7y4 (Year 7, y-band, bottom set 4). The mean score of the test at the mid-point of term two was 33.94 (SD = 10.53). Mrs. Angel taught four classes of mathematics, two of Year 7 and two of Year 8. The class for observation was Year 8x2 (Year 8, x-band, top set 2). The test at the mid-point of term one (see Appendix B Year 8 Test 1), translated into Thai, and was used to find the comparable group in the Thai school. The mean score of this set was 56.26 (SD = 8.95). Miss Bright taught seven classes of mathematics, One of Year 7, one of Year 8, one of Year 9, one of Year 10, one of Year 11, and two Sixth Form classes. The Year 8y4 (Year 8, y-band, bottom set 4) was the class selected for observation. The mean score of the test at the mid-point of term one was 22.55 (SD = 8.64).

Table 4.3 gives detail of timetable and rooms for observations in the English school.

Table 4.3 An English school observation timetable

Date	1 9.15	2 10.15	3 11.25	4 1.10	5 2.20
Mon1	-	7x1 w4	8x2 w14	8y4 w13	7y4 w16
Tue1	-	-	-	-	-
Wed1	7y4 w16	8y4 w13	-	-	-
Thur1	-	-	-	7x1 w5	8x2 w14
Fri1	8y4 w13	7y4 w16	-	8x2 w14	7x1 w16
Date	1 9.15	2 10.15	3 11.25	4 1.10	5 2.20
Mon2	-	7x1 w7	8x2 w14	8y4 w13	7y4 w16
Tue2	-	-	-	-	-
Wed2	7y4 w16	8y4 w13	-	-	-
Thur2	-	-	-	7x1 w16	8x2 w14
Fri2	8y4 w13	7y4 w16	-	8x2 w14	7x1 w15

Note: first number = year group,
x1 – x4 = x band (top set) group 1 - 4,
y1 – y4 = y band (bottom set) group 1 - 4

The English school arranged pupils into eight sets with two bands x and y, four sets in each band – two top sets and two bottom sets. Thus there were four top sets (x1, x2, y1, y2) and four bottom sets (x3, x4, y3, y4). For example, '7x1' means Year 7, x-band, top set 1; and '8y4' means Year 8, y-band, bottom set 4. The school labels their classroom as 'w1-w16'.

The two-week timetable shown in Table 4.3 was used through an academic year (September 2001 - July 2002) in the English school.

The algebra content to be concerned by this study was defined as follows: theme 1 patterns/sequences; theme 2 simplification; theme 3 substitution; theme 4 solving equations; theme 5 graphs of linear functions; and theme 6 word problems. These themes were a combination of the algebra content as outlined in both the National Numeracy Strategy: Framework for teaching mathematics year 7, 8, and 9 in England and the Mathematics Curriculum for lower secondary level in Thailand as mention in Chapter 2.

The English school algebra lessons in each set were spread over all three terms of the school academic year as shown in Table 4.4.

Table 4.4 An English school algebra lessons allocation

Sets	**September 2001**	**October 2001**	**November 2001**	**January 2002**	**February 2002**	**March 2002**	**April 2002**	**May 2002**
7x1	1111111		6222232			55	44	46 ■
7y4	11	2	2233 55		44	555		333 ■
8x2		66 22			555554	4424	■	
8y4		223		5	56 55		4444■	

Note: 1-6 = theme 1-theme 6, ■ = researcher's test

In Year 7 the English school taught the basic concepts of theme 1, patterns/sequences, theme 2, simplification, and theme 3, substitution before moving to theme 4, solving equations, theme 5, graphs of linear functions, and theme 6, word problems. Year 8 pupils spent more time on the solving equations and graphs of linear functions themes than simplification, substitution, and word problems themes. Theme 1, patterns/sequences, was omitted in Year 8.

In writing the algebra test items, terms found in mathematics textbooks, lessons and in the English national curriculum documents were used.

4.3.2 The Thai school lesson observations

The target groups in the Thai school were selected to compare with the English groups (see section 4.2.1). This was done before the school timetable was organised. The timetables were then asked to arrange the observation schedule shown in Table 4.5.

Ms. Supashin taught four classes of Secondary 1 mathematics, two high ability groups and two low ability groups. The class selected for observation was Secondary 1/01 (high ability group). Mrs. Surachai taught six classes of mathematics, two of Secondary 1 and four of Secondary 2. In this case the class selected for observation was Secondary 1/09 (low ability group).

Mrs. Pachakan taught five classes of mathematics, two of Secondary 1 and three of secondary 2. A high ability group Secondary 2/04 was selected for observation. Miss Nongchai taught five classes of mathematics, two of Secondary 1 and three of Secondary 2. The selected group was Secondary 2/10 (low ability group).

Table 4.5 gives detail of timetable and the rooms for observations in the Thai school.

Table 4.5 Thai school observation timetable

Date	1 8:25	2 9:20	3 10:15	4 11:10	5 12:00	6 12:55	7 13:50	8 14:45
Mon	1/01 535	1/09 536					2/04 534	
Tue	2/04 534		2/10 533					
Wed	1/01 535		1/09 533		2/10 533			
Thu					2/10 533			
Fri	1/01 535		-			2/04 534	1/09 536	

Note: number/ = year group
/number = ability group

The Thai school arranged pupils into ten groups. The first four high ability groups, group '01' to '04', and six mixed ability groups, group '05' to '10'. For example, '1/01' means Secondary 1 high ability group 1, and '2/10' means Secondary 2 mixed ability group 10.

The school codes their classroom as '531-538', '531 means building 5 on the third floor, room 1.

This timetable was used in the second term (October 2002-March 2003) in the Thai school.

The Thai school algebra lessons in each group were allocated to the second term of the two-term school year as shown in Table 4.6.

Table 4.6 Thai school algebra lessons allocation

Groups	**October 2002**	**November 2002**	**December 2002**	**January 2003**	**February 2003**
1/01			444444	6664555545	555 ■
1/09				55555555	5544444 ■
2/04		4444444666		5555 ■	
2/10	4	44444466		55 ■	

Note: 1-6 = theme 1-theme 6, ■ = researcher's test

The Thai school taught the algebra content in two main headings: theme 4 – solving equations and theme 5 – graphs of linear functions. Theme 3 – substitution and theme 6 – word problems were taught under the solving equations theme. Theme 1 – patterns and sequences and theme 2 – simplification were omitted as a specific topic. It was included as a part of solving equations. In Secondary 2 solving equations was given greater emphasis than the work on graphs of linear functions.

Lesson observations from both the English and Thai schools were intended as a primary source of data because during instruction pupils' initial and exploratory ideas could be gathered from their verbalisations. The purposes of conducting classroom observations are outlined below:

- To become familiar with the participants and to facilitate formal and informal discussion,
- To provide a base from which to develop an understanding of individual pupils' thinking processes when solving algebraic problems, and

- To facilitate selection of pupils for the interviews.

In most lessons teachers in both countries gave the pupils examples and then practice through exercises. During this time the researcher was able to move among pupils with the purpose of assisting them and obtaining information about their thinking processes. The approach adopted was to question pupils on how they go about solving different algebraic problems. This provided useful information for analysing and categorising their written responses in the algebra test. The test provided a means for triangulation to gain insights into pupils' thinking processes.

The researcher's algebra test (see Appendix E) was administered to all pupil participants in both countries after the last algebra lesson of the school academic year.

4.3.3 The English school tests

For the academic year, September 2001-July 2002, the English school administered three short answer mid-term tests, one end of year test and one mental calculation test also administered at the end of the year. The tests used as the bases for individual interviews were the three mid-term tests (see Figure 4-1).

Year	Term 1	Term 2	Term 3
7	▲☺	▲☺	▲ ■
8	▲☺		▲☺ ■

▲: school test ☺: interviews ■: researcher's test

Figure 4-1 English school tests allocation

The first mathematics test in Year 7 took place at the mid-point of term one as shown in Figure 4-1, this lasted 45 minutes. Pupils were not allowed to use calculators. The test consisted of 35 items, five of which involved algebra. Among these five items, were three physical pattern and two sequence number items. Eight selected pupils (see Section 4.3.5), four from each set were interviewed about two selected items (within which were four sub-items), one from physical pattern and the other from sequence number, to find out how they went about solving the problems.

The second test took place at the mid-point of term two and lasted 45 minutes. In this case pupils were allowed to use calculators. The test comprised 29 items with three algebraic problems. All three algebra items were used as a basis for interviews with pupils.

The third test was administered almost at the end of term three. The test consisted of 32 items, nine of which involved algebra. The bottom set (7y4) did not take this test but pupils selected by the researcher were asked to do only the algebra items in this test.

For Year 8 groups, the first test was administered at the mid-point of term one. The test comprised 24 items, eight of which were algebra with 38 sub items. Twelve of these sub items formed the basis for the interviews with pupils. The second test contained no algebra items but the third test, given at the mid-point of term three, included 7 algebra items.

4.3.4 The Thai school tests

For the academic year, May 2002-March 2003, the Thai school administered 10 topic specific tests, two mid term and two end of term tests to Secondary 1. Among these tests were two topic specific tests and one end of term test involving algebraic problems. For Secondary 2 groups there were 13 topic specific tests, two mid term tests and two end of term tests. Among the tests were two topics specific, one mid term and one term test involving algebra (see Figure 4-2).

Secondary	Term 1	Term 2
1		▲☺▲☺ ■
2		▲☺ ▲☺ ▲☺ ■

▲: school test ☺: interviews ■: researcher's test

Figure 4-2 Thai school tests allocation

The first test involving algebra for Secondary 1, located near the end of term two as shown in Figure 4-2, was the ordered pairs and graph specific test. The test consisted of four open-ended items. The second test was the equation specific test and consisted of four open-ended items. The mid term test had 40 multiple-choice items each with four possible answers. None were algebraic problems.

The first test involving algebra for Secondary 2 was the equations and inequality specific test. It had 30 multiple-choice items each with four possible answers. Five of these items were used as the basis for an interview with individual pupils. The second test was a two-part linear function graphs specific test. The first part had 20 multiple-choice items each with four possible answers. The second part was an open-ended item. The mid term test had 40 multiple-choice items each with four possible answers. Of these 11 involved algebra, five of which were used in the interview with pupils.

4.3.5 The individual interviews

As planned, individual interviews with pupils were conducted after school tests in both English and Thai schools as shown in Figures 4-1 and 4-2. Selecting pupils for interviews took into account test scores and lesson observations in each group. The test scores were arranged from the highest to the lowest. Four pupils, two boys and two girls were selected on the basis of their first school test scores and their willingness to participate, one boy and one girl from the upper half and one boy and one girl from the lower half.

These interviews were conducted during the lunch period at the English school. In the Thai school these took place after testing, during the lunch period, the next morning and before their next mathematics lesson. All interviews discussed in this study were done on a one-on-one basis. As mentioned already, the purpose of the semi-structured interviews was to gain insights into pupils' thinking processes in solving algebraic problems. Questions included "please explain to me, how you did this one", "how did you get that?", and "how did you work out this?". Interviews were audio taped and then fully transcribed. Tapes were listened to on several occasions and transcripts reviewed accordingly. The analysis sought relevant examples to outline a framework for coding pupils' written responses to the researcher's algebra test.

Pupils' thinking processes from interview data analysis are summarised in Tables 4.7 – 4.10. As a result of the interviews, it was possible to record the processes that pupils used in order to complete the test problems. These processes were then categorised into two broad categories – "generalisable process" and "other process". Generalisable processes are those in which indicate the proper use of mathematical rules, or, when dealing with higher level items, show methods which can be recognised as approaching this. At this point, all other responses were categorised as "other process". At this stage it was difficult to specify pupils' thinking processes. These processes were analysed from few items related to algebra topics in the school tests with only a small group of pupils. The aims here were to inform the design of the algebra test and to help the analysis of the pupils' responses to it.

Table 4.7 shows generalisable and other processes used by eight pupils in Year 7, four pupils from top set and four from bottom set, in the English school tests 1 and 2. The sample of interview data and questions for interview are illustrated in Appendix C Table 1 and Table 2.

Table 4.7 Year 7 pupils' interview data from the English school tests

Generalisable processes	Other processes
Test 1	
Repeated operation	Repeated operation-like
Inverse operation	
Test 2	
	Letter ignored
Simplify like terms (counting letters)	Simplify unlike terms
Simplify like terms (grouping)	Substitution-like (e.g. plus, $abc = 4+2+3$)
Substitution	Substitution-like (e.g. replace, $abc = 423$)
Multiply out brackets	Power ($x^2 = 4+4$, $x^2 = 4\times2$)
	Add first term in the brackets ($4+x = 4x$)
	Choose a number for x, power ($x^2 = x\times2$),
	Ignored brackets and signs

Table 4.8 illustrates the generalisable and other processes used by eight pupils in Year 8, four pupils from top set and four from bottom set, in the English school test 1 and test 3. The interview questions are shown in Appendix C Table 3 and Table 4.

Table 4.8 Year 8 pupils' interview data from the English school tests

Generalisable processes	Other processes
Test 1	
Substitution	Substitution-like (e.g. $2x$-$3y$ = 24 - 33)
Count letters	Incorrect operation
Simplify like terms	Incorrect grouping
Modelling	(e.g. 10-2(x+y) = 10-2(5+3), 8(8) = 64 and
Multiply out bracket	10-2(x+y) = 5+3 = 8, 10-2 = 8, 8-8 = 0)
	Times zero xyz = 0×4×5 = 20
	Substitution-like xyz = 0+4+5 = 9
	Ignored zero
	Incorrect operation (power)
	$4^2 = 8$
	Letter ignored
	n×m = 10+5 = 15 (I think of a number, double
	it and add 5)
	Simplify unlike terms
	x×x = 2x
Test 3	
Substitution	Simplify unlike terms
Power	
Incorrect operation (e.g. -3× -3 = -9)	
Trial and error	
Inverse operation	
Implicit balancing	
Change sides change signs	

Table 4.9 shows the generalisable and other processes used by eight pupils in Secondary 1, four pupils from the high ability group and four from the low ability group, in the Thai school tests. Questions for interview are included in Appendix C Tables 5-8.

Table 4.9 Secondary 1 pupils' interview data from the Thai school tests

Generalisable processes	Other processes
Solving equations Implicit balancing Explicit balancing Substitution Modelling Arithmetic approach **Graphs** Ordered pairs recognition Substitution Plotting graphs Drawing graphs Reading graphs	Balancing-like Substitution-like

Table 4.10 gives the generalisable and other processes used by eight pupils in Secondary 2, four pupils from the high ability group and four from the low ability group, in the Thai school tests. The interview questions are shown in Appendix C Table 9-11.

Table 4.10 Secondary 2 pupils' interview data from the Thai school tests

Generalisable processes	Other processes
Solving equations Explicit balancing Multiply out bracket Implicit balancing Grouping Change sides change signs Substitution Simplify like terms	Letter ignored Bracket ignored Balancing-like Count letters
Midterm test Explicit balancing Implicit balancing Substitution Multiply out bracket Grouping Change sides change signs Modelling Arithmetic approach	Power $4^2=4\times2$
Graphs Substitution Drawing graphs	

The generalisable and other processes as found from the interview data following the Year 7 and Year 8 school test 1 (Tables 4.7 and 4.8) were used to help design the algebra test. For example, grouping inside and outside the brackets, and using substitution-like processes informs test item 15 "if p = 5, r = 3, find the value of 2(p+3r)-8".

Interview data in Table 4.7-4.10 were used to help in the stages of analysing pupils' thinking processes from their methods/explanations responses to the algebra test. For instance, for item 15 above if a pupil responded 2-8 = -6, (5+33), 38-6 =32, the analysis would be grouping inside (5+33) and outside (2-8) the brackets, and substitution-like (33 for 3r) processes.

4.4 The Algebra Test

The algebra test administered by the researcher was prepared in two versions—English and Thai. It was designed after consulting the English National Numeracy Strategy: Framework for Teaching Mathematics Year 7, 8, and 9, the Thai Mathematics Curriculum for the Lower Secondary Level, the mathematics textbooks in each context, the English school lesson observations, and the interviews regarding the English school test 1. It was completed after the first term of the study in the English school. The test was structured on the basis of six themes—patterns/sequences, simplification, substitution, solving equations, graphs of linear functions, and word problems—based on the curricula in both countries as mention in Chapter 2. Within each theme there were intended to be four levels of difficulty questions ranging from the easiest to the most difficult level. The items in the test itself were arranged in order of increasing *expected* difficulty grouping from level one through level four of each theme as discuss in section 4.4.1.

4.4.1 Nature and structure of the algebra test

The main purpose of the present study is to examine and compare the thinking processes used by English and Thai pupils in solving algebraic problems rather than to test the achievement. The study also explores how this relates to the curricula delivered in the two countries. This section outlines the nature and structure of the algebra test paper. The key characteristics are summarised for the purpose of comparison.

The algebra test paper was intended to cover most of the algebra content of the first two years of secondary education as outlined in the mathematics curricula in both England and Thailand. Since the test was to be given in the normal mathematics lesson, it was constructed so that pupils should be able to complete within one 50 minutes session. This minimises any disruption to the pupils' education. Six themes of algebra content were included. The design of the theme 4 solving equations test items is in Table 4.11.

Table 4.11 Theme 4 solving equations test items design

Key issues across the areas of influence	Test items
Unknown on one side Working back method/inverse operation Concept of equal as "makes" Negative number	Level 1 The unknown in the first term $5a-2 = 8$
Unknown on one side Unknown in the middle term Working back method/inverse operation Read from left to right Concept of equal Negative number	Level 2 The unknown in the middle term $5-2b = 1$
Unknown on both sides Concept of equal as "equivalence" Negative number Simplify like terms	Level 3 The unknown on both sides $3y-6 = y-2$
Unknown on one side with brackets Concept of equal Multiply out brackets Negative number Simplify like terms	Level 4 The unknown in brackets $2(3x-1)-(x+4) = 9$

Table 4.11 showed theme 4 solving equations test items design. It consisted of four questions. Level 1 item was designed to investigate how pupils find out the unknown quantity in the first term of equation. The English school curriculum suggests teaching this topic using inverse operation (working back method) whereas Thai school maintained the equivalence. Level 2 item examined pupils' thinking processes in finding the unknown when it appeared as the middle term of an equation. The key issues, which came from previous research, stated the difficulty in reading from left to right. Level 3 item probed the pupils' thinking processes in managing the unknown when it appeared on both sides of the equal sign. Level 4 item was created to gain insight into the pupils'

thinking processes when the unknown is in brackets. Clearly within each theme and level a number of possible test items were available. A typical question within each theme/level was chosen. The theme 4 level 1 item for example was chosen on the basis of the English school curriculum. This kind of question is introduced in the first place when solving equations using inverse operations (working back method). The Thai curriculum solves this kind of equation using the explicit balancing process (concept of equal).

A similar process was used to design the test items for all six themes as summarised in Table 4.12. As can be seen in Table 4.12, within each of the six themes, test items were organised into four levels of *expected* difficulties based on the examination of curricula, previous research findings (see Chapter 3), classroom observations, pupil interviews, and the researcher's experience. Issues which arose with these areas of influence, inevitably overlapped (see Appendix D).

Table 4.12 Level of expected difficulty

Theme	Level 1	Level 2	Level 3	Level 4
1.patterns/ sequences	Continue concrete objects	Before generalise concrete objects, abstract objects	Generalise concrete objects	Generalise abstract objects
2.simplification	Simplify one variable	Simplify two variables	Simplify two variables with brackets	Simplify two variables with second order and brackets
3.substitution	Substitute positive number	Substitute positive and negative numbers	Substitute positive numbers with brackets	Substitute positive numbers in a two variable expression with second order and brackets
4.solving equations	The unknown in 1st term	The unknown in middle term	The unknown in both sides	The unknown in brackets
5.graphs of linear functions	Graph of $x+y = c$	Graph of $y = mx+c$ cross x-axis	Graph of $y = x+c$ cross x-axis and y-axis	Graph of $y = mx+c$ cross x-axis and y-axis
6.word problems	One variable in one step, and in two steps	One variable in two steps with brackets and positive numbers	One variable in two steps with brackets and negative numbers	One variable of second order

Theme 1, patterns/sequences, consisted of eight questions. The key ideas from previous research found that pupils tended to use the proportionality property (scaling up) to any function. Test items within this theme were designed to investigate both kinds of functions $f(n) = an$ and $f(n) \neq an$. Level 1, item 1a, 1b, and 13a were designed to investigate how pupils continue a physical pattern in which the term value is a multiple of the term number and also when the term value is not a multiple of the term number. Level 2, item 7, and 19a were designed to provide information on how pupils continue the number sequences in the cases where the first term is 1 and the first term is not 1. Item 13b tests the pupils' method of extending the work in 13a. Level 3, item 13c was included to determine how pupils worked out a general formula from a physical pattern. Level 4, item 19b was to examine how the pupils worked out a general formula from number sequences.

Theme 2, simplification, was tested using four questions. The design of test items was influenced by key issues in reading from left to right, acceptance of lack of closure, letters ignored, and multiply out brackets. Level 1, item 2 was designed to observe how pupils simplify a one variable expression. Level 2, item 8 was used to investigate how pupils manipulate an expression with two variables. Level 3, item 14 was added to examine pupils' thinking processes when faced with an expression with brackets. Level 4, item 20 sought to gain insight into how pupils' manipulate variables with second order terms and brackets.

Theme 3, substitution, consisted of four questions. Key issues across the areas of influence in designing test items were substituting positive and negative numbers, use of powers, replacing in a co-joined term, and multiplying out of brackets. Level 1, item 3 was created to investigate how pupils substitute positive numbers to evaluate an expression. Level 2, item 9 sought to examine pupils' thinking processes when they substitute positive and negative numbers. Level 3, item 15 was intended to observe pupils' thinking processes when they substitute positive numbers with brackets. Level 4, item 21was added to gain insight into pupils' thinking processes when variables of the second order and brackets are present.

Theme 5, graphs of linear functions, consisted of five questions. Key issues across the areas of influence were generating coordinate pairs, plotting and interpreting the graphs of linear functions, linking different representations of functions, and finding the x-intercept, and the y-intercept. Level 1, item 5 was designed to investigate how pupils find and plot the coordinates of the line with equation $x+y = 4$. Level 2, item 11 was designed to observe how pupils worked out the coordinate of where the graph $y = 2x-6$ crossed the x-axis. Level 3, item 17 investigated how pupils connect a choice of graphs with the function $y = x+5$. Level 4, item 23 was to investigate how they connect a choice of graphs with the function $y = 2x+6$.

Theme 6, word problems, consisted of five questions. Key issues across the areas of influence in designing test items were writing equations, solving by working back, translating word problems from left to right, and methods for solving equations. Level 1, items 6a, 6b were designed to investigate how pupils find out the original number in the given situations using one variable. Level 2, item 12 was to investigate pupils' thinking processes when facing the word problem that could transform to the equation such as $x+a = 2(x+b)$. Level 3, item 18 was to probe pupils' thinking processes when facing the word problem that could transform to the equation such as $2x = 5(14-x)$ or $5y = 2(14-y)$. Level 4, item 24, was designed to search their thinking processes in the word problem in the familiar geometric situation that could transform to the equation such as $ax^2 = b$.

The test items were arranged into four groups in order of increasing *expected* difficulty. The first group consisted of level 1 items, the second of level 2 items, the third of level 3 items, and the fourth of level 4 items, across each of the six themes. Pupils' thinking was observed across four levels of *expected* difficulty in order to allow the recognition of significant variation.

4.4.2 The algebra test development

In developing the test, two English mathematicians were asked to review the English version of the test. This was necessary to update terms, clarify confusing items, and consider the validity of the test items. After examination by these individuals, several items were changed. Theme 1 patterns/sequences in item 1 the term "explain how you know" for instance was changed to "explain how you work it out", and item 7 the term "Fill in the missing number" was changed to "Fill in the blanks in this sequence". Theme 2, "simplify $6+3x-y-6x-y-2$" was changed to "Simplify the expression $6+3b-c-6b-c+2$". The letters was changed from x and y to b and c to show any letters could be used in this context.

After one term of observation in the English school, the test items were revised before piloting the test. The term "remove the brackets" was changed to "multiply out the brackets". The first expression was used in the textbook but in the real lessons teachers used the second expression. The researcher changed the expression to minimise the pupils' confusion.

Two English pupils in Year 8 top set who did not participate in the present study were asked to pilot the test. The aims here were to clarify terms and to determine the time required to complete the test. These pupils were asked to comment and to clarify items that were not easily understood. No items were changed because at least these pupils had no difficulty understanding.

In the same way, two Thai mathematicians reviewed the Thai version of the test. An English Language teacher was asked to review the translation from English to Thai. The test was revised to update the expressions "รูปแบบ" was changed to "แบบรูป", and "ถอดวงเล็บ" was changed to "คูณเข้าวงเล็บ". Both expressions were translated from English words. In the first case, the first word translated from 'pattern' and it was the same as a translation of the word "form". The second word was a more appropriate translation. In the second case, the first word translated from 'remove the brackets', and made numbers of pupils confused with "take the brackets off without multiplying". Therefore this word was changed to the word that means "multiply out the brackets".

Test analysis was performed by giving the test to one group of Secondary 2 pupils in a Thai school that was not involved in the present study. In total 47 pupils were tested. The alpha coefficient for reliability of the test was 0.87, indicating that the test scores may be trusted to represent pupils' performance on the concepts and skills measured by the test.

To analyse item difficulty index (p) and item discrimination index (d) 27% of upper score group and 27% of lower score group were used. Four of the 30 items were very easy ($p > 0.80$). A further 16 items were appropriate (p = 0.20-0.80) and 10 items were very difficult ($p < 0.20$) (see Appendix F). The easy items relating to the more difficult items were included in the same question. The more difficult items aimed to observe pupils' thinking processes when moving between those levels of expected difficulty.

For item discrimination index (d), 14 of 30 items were very good ($d > 0.39$) in separating high and low performance, 7 items were good (d = 0.20-0.39) and 9 items were not so good ($d < 0.20$) in separating high and low performance (see Appendix F).

The order of difficulty of the items within each theme was found to be as expected with the one exception of the level 4 item in theme 6. However, the item discrimination index for this item was very good ($d = 0.85$) in separating high and low performance. In theme 2 simplification, levels 2, 3 and 4 there were no pupils who obtained the right answer. These items were still included in the test because although this topic was not taught in Thai schools it was delivered in English schools and is an important basic concept in learning algebra as shown in the previous research findings (see Chapter 3). The researcher carefully revised the questions with reference to the English school curriculum and the previous research findings.

4.4.3 Evolution of the Method of analysing data

In the research proposal for this study, the research questions included the following: "How do pupils in English school and in Thai school solve algebraic problems? How different are their thinking processes? To what extent do pupils' thinking processes relate to the mathematics curricula?" These questions were considered sufficient to guide the lesson observations, interviews, and preparation of the algebra test.

A detailed analysis of each of the algebra test items was carried out in two stages. The algebra test scores were analysed and compared by country, sex and ability using *t*-test adjusted alpha level of .05. Factor analysis was used to inspect the correlation between themes in all and in each country. Analysis of variance (ANOVA) was conducted to explore the impact of country, sex and ability on pupils' fully generalisable and other processes scores. The fully generalisable and other processes are defined in each theme of the test as shown in codebook (Appendix G). Comparisons of the proportion of fully generalisable and other processes between countries on the six themes of the algebra test were analysed and compared using *t*-test adjusted alpha levels of .05. Second, comparison of pupils' explanations in the algebra test and field notes from lesson observations was annotated to clarify the references of what the pupil was thinking. Coding of responses was developed to focus on describing thinking processes in the short term. The scheme had categories such as repeated operation, draw or count, scaling up, and letter ignored.

4.4.4 Codebook development

As Boyatzis (1998) states a good thematic code consists of five elements: a label, a definition of what the theme concerns, a description of how to know when the theme occurs, a description of any qualifications or exclusions to the identification of the theme, and finally, examples to eliminate possible confusion when looking for the theme.

In the first stage of categorising pupils' thinking processes, responses to the algebra test were labelled according to their explanations using words from lesson observations, interviews, and words used in the algebra literature. At this stage a simple listing of pupils' explanations with key words for quick reference was performed.

The second stage placed the same key words and the pupils' explanations under the heading containing the question as an example shown in the first column in Table 4.13. Repetitious strategies were numbered (see Table 4.13) to avoid losing track of individual pupils' thinking processes. This was important because repetitious strategies are more inclined to evidence conceptual understanding whereas single answers may involve a simple error. In these stages code 1 and code 2 were labelled. An example of this is shown in table 4.13 were labelled. The labels used in each item were terms that can be linked to other items in the codebook. This allows the researcher to group and categorise codes and add new codes to the groups and categories.

The next stage was to categorise the thinking processes into generalisable and other processes as shown in the code 3 column in Table 4.13. Generalisable processes are those in which indicate proper use of mathematical rules, or, when dealing with higher level items, show methods which can be recognised as approaching this. These processes were defined in each theme of the algebra test in the codebook (see Appendix G). Other processes were the methods/explanations that could not be recognised as above and included inappropriate strategies or wrongly perceived situations. These processes are also defined in each theme of the algebra test in the codebook (see Appendix G). Explanations that showed the generalisable processes were grouped whether they obtained the correct answer or not. Explanations that showed the other processes were grouped separately. Answers given following an unidentified process or with incomplete work were categorised separately.

Table 4.13 gives an example in the coding development process for item 1a "How many matchsticks are needed for the 4th pattern in this series? (see Appendix G).

Table 4.13 Coding development for theme 1 level 1 item 1a

Processes/explanations	Code 1	Code 2	Code 3
The expression is 3n (n term) 3 times table (times 3) Times the number of pattern by 3 (times 3) Times 3 from the last answer (times 3) 3x4 (times 3) 1x3 = 3, 2x3 = 6, 3x3 = 9, 4x3 =12 (times 3)	Alg Alg Alg Alg Alg Alg	g	Ag
Add another 3 matchsticks (add 3) Keep adding 3 (add 3) Add one more stick on the end then two on the side (add 3) Each pattern increases by 3 (add 3) 9+3 (add 3) The pattern is going up in 3's (up 3)	Reo Reo Reo Reo Reo Reo	re	Are
Count 3 more (count 3) Counting the matchsticks (count) Draw the 4th pattern (draw)	Drcu Drcu Drcu	d	Ad
Add 2 more on (add 2) The 4th is double the 2nd (scaling up)	Oth Oth	d sc	Od Osc

Notes: For Code 1 column, Alg = algebra process, Reo = repeated operation, Drcu = draw or count, Oth = other
For Code 2 column, g = generalisation, re = repeated operation, d = draw or count, sc = scaling up
For Code 3 column, A = generalisable process, g = generalisation, re = repeated operation, d = draw or count, O = other, sc = scaling up

For Code 1 Alg (algebra process) was used to code those processes which indicated an operational link between the term number and the term value. Reo (repeated operation process) was used to code processes which recognised the connection between consecutive terms. Drcu (draw or count) was used to code methods which method up on drawing or counting. Oth (Other process) was used for any other methods.

After devising code 1 for all six themes, Code 2 was developed. The aims of this stage were to link the terms through all six themes and define sub-processes of "other process". Code 3 resulted in a renaming of the processes into "generalisable (A)" and "other (O)" categories as in Table 4.13. Two categories W (unidentified process) and R (incomplete work/no response) are required to complete the coding in all six themes. Checking consistency of the items within each theme was carried out in vertical and horizontal directions to review coding.

The procedure used to analyse the English pupils' thinking processes was also applied to organise, code, and categorise Thai pupils' thinking processes. Processes that did not appear in the explanation for English pupils were added to the codebook. The coding process was first attempted after collecting data from the English school (Code 1-3). The second coding was made after collecting the Thai school data (Code 3). The third coding ran through all data from both schools and this was redone once a month on two more occasions (both Code 3). The different processes coded (inconsistency) in the last three times of coding were revised and discussed with experts to justify the coding.

The codebook was reviewed to determine whether the inconsistencies were due to guidelines or problems with the code definitions such as overlapping or ambiguous inclusion criteria that make it difficult to distinguish between two codes. These types of problems are generally discussed with experts. For example, theme 1 level 1 item 1a "How many matchsticks are needed for the 4^{th} pattern?" one pupil (N=286) worked as "the 4^{th} is double the 2^{nd}" (scaling up). Although this is a good method to solve the problem it could not lead to the general rule, so it was categorised as 'other process'. In another case, theme 1 level 2 item 19a "the 7^{th} term of this sequence 2, 5, 8, 11, 14, 17, …is …", one pupil (N=286) explained the process as "2*n*+ number of term before". This item is easily categorised as "generalisable process" but the following item (19b) asked for the general term of item 19a. When investigating item 19b "the n^{th} term of this sequence is….", the same pupil answered "2*n*+number before" and explained the process as "2*n*+number before". The process "2*n*+number before" may be a small slip in the answer and explanation to item 19b. However, the "2*n*+number before" appeared twice in item 19b that asked for a general rule and totally different to "2*n*+ number of term before" explained in item 19a. The phrase "number before" interprets the term value whereas "number of term before" states the term number. For example, number before 6^{th} term is 14 but number of term before is 5. This case categorised the process as "other process". Once the problems were identified and the codebook clarified, all previous coding was reviewed and recoded so that it was consistent with the revised definitions. This iterative coding process continues until all pupils' explanations have been satisfactorily coded. Difficulties such as the above were seen in only a very small number of cases. Thus the overall effect did not significantly affect the coding processes.

Successful work requires good understanding of concepts and accurate manipulative skills. Solutions involve many steps, some of which can often be carried out mentally. Where this happens and steps in working are omitted it may not be possible to categorise a response with any certainty. For example, consider 5a-2 = 8, a pupil who gains 5a = 10 as the first step in his/her solution may be thinking 10-2 = 8 → 5a = 10 (using *arithmetic knowledge of number bonds*) or 5a = 8+2 =10 (using *implicit balancing process*) and there is no way of knowing which.

Difficulties can also arise where the arithmetic calculations are incorrect. Here, it may be impossible to decide whether the errors are caused by carelessness, ignorance or by misunderstanding. For example, a pupil who follows 5a = 10 with a = 5 may be thinking 5×2 = 10 but carelessly writes down 5 instead of 2, or may be thinking 5a means 5+a. There is no way of telling simply by looking at the written response.

A further problem arose where pupils had made some progress but had not completed an item. The researcher chose to place all such unfinished work in a separate category "incomplete response". Clearly, as an alternative, it would have been possible to look at the work in these responses, and, according to the amount of correct work included, form some other appropriate sub-categories. The researcher decided not to attempt this further categorisation having seen that, although some of this work appeared superficially correct, on closer inspection it was found to be lacking. For example, in Item 3, 4+5×3 was seen with no further working. There is insufficient evidence here to know whether the pupil understands what s/he has written down (the possibility of 9×3 cannot be excluded). It was because of situations such as this that the category "incomplete response" was retained.

Because there are so many stages along the way to a correct answer the researcher decided to use the broad categories "generalisable process", "other process", "unidentified process", and "incomplete response".

A codebook was used as indicator during the coding process. Each item was marked as "1" for the correct answer and "0" for incorrect, incomplete work, and no response. The un-reached items were omitted from the analysis as shown in Figure 4-3.

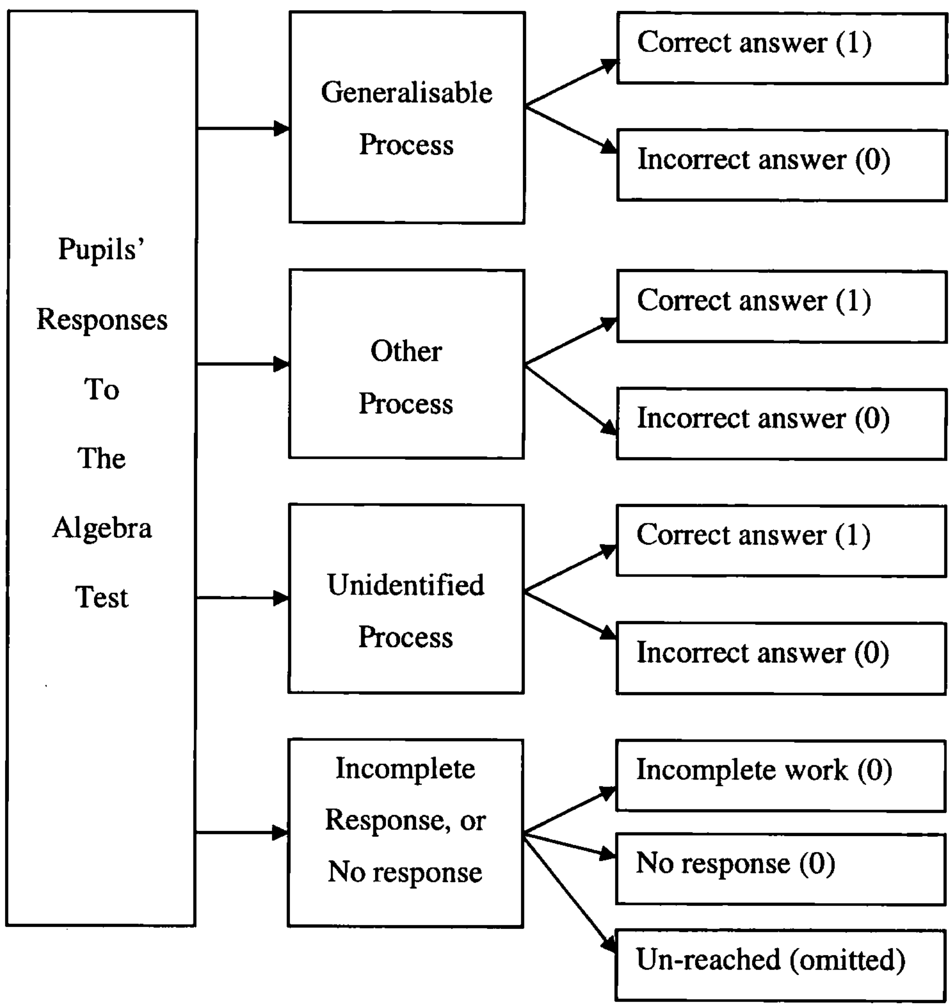


Figure 4-3 Category of pupils' thinking processes

Coding the algebra test: some examples

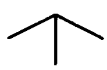
The codebook was developed and used to analyse pupils' explanations in response to questions in the algebra test. Its structure has evolved into the four categories generalisable process, other process, unidentified process, and incomplete response process for each theme as stated in evolution of the method of analysing data section.

Within each process there were different sub-processes of the generalisable and other processes.

For example, theme 1 patterns/sequences, item 1 was coded as shown in Tables 4.14 and 4.15.

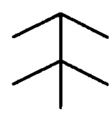
Item 1. Look at the number of matchsticks in each pattern.

1st pattern



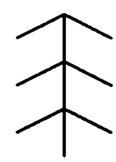
3 matchsticks

2nd pattern



6 matchsticks

3rd pattern



9 matchsticks

a. How many matchsticks are needed for the 4th pattern in this series? *(Level 1 concrete objects)*

b. How many matchsticks are needed for the 10th pattern in this series? *(Level 1 concrete objects)*

Table 4.14 Coding the Level 1 question, 1a, in the patterns and sequences theme

Processes Theme 1 Level 1 (item 1a)	**Examples**	**Code**
Generalisable process		
Generalisation	Times the pattern by 3	Ag
Repeated operation	Adding on 3	Are
Draw or count	Counted 3 more, draw the 4th pattern	Ad
Other process		
Scaling up	The 4th is double the 2nd	Osc
Draw or count incorrectly	Count 2 more on	Od
Unidentified process		
No process		W
Incomplete response		
No response		R9

Table 4.15 Coding the Level 1 question, 1b, in the patterns and sequences theme

Processes **Theme 1** **Level 1(item 1b)**	**Examples**	**Code**
Generalisable process		
Generalisation	Times the number pattern by 3	Ag
Repeated operation	Added another 3	Are
Draw or count	Drawing the 10^{th} pattern	Ad
Other process		
Generalisation-like	$1^{st}=3$, $2^{nd}=6$, $3^{rd}=9$, $10^{th}=(9/3)\times10$	Og
Scaling up	$2^{nd}+3^{rd}=5^{th}$, $6+9=15$, $15\times2=30$	Osc
Draw or count incorrectly	drawing the pattern and count matchsticks	Od
Unidentified process		
No process		W
Incomplete response		
No response		R9

Within the generalisable process group, there are 3 sub-processes.

(1) *The generalisation process* (**Ag**) in which the pupils used the rule to find out the solution.

(2) *The repeated operation process* (**Are**) refers to those pupils who had some idea what the operation of the previous solution was and then re-used it.

(3) *The draw or count process* (**Ad**) reflects an easier way to get the answer from basic arithmetic processes.

There are 4 sub-processes used within the other process group.

(1) *The generalisation-like process* (**Og**) is an attempt to perform the rule incorrectly.

(2) *The repeated operation-like process* (**Ore**) is an attempt to use the previous solution but in an incorrect pattern.

(3) *The scaling up process* (**Osc**) is an attempt to find the answer by using the prior pattern number.

(4) *The draw or count incorrectly process* (**Od**) is that showing the basic arithmetic process in drawing or counting in an incorrect pattern.

The unidentified process (**W**) group gave the result without showing working.

There are 3 sub-processes used in the incomplete response group.

(1) *The incomplete work* (**R7**): pupils showed an attempt to work it out but did not reach completion.

(2) *No response* (**R9**): pupils made no attempt.

(3) *Un-reached* (**Ru**): pupils did not reach that question because of the time limit.

4.5 Summary and Conclusion

This chapter described the specific research design and methods that were used to conduct the research in the present study. Both quantitative and qualitative data were collected. Quantitative data were analysed using the *t*-test, factor analysis and analysis of variance. Qualitative data were analysed using thematic analysis procedures to code the data and then investigated in more depth by comparing pupils' thinking processes between the two case study schools.

The results of the data collection and analyses are presented in Chapter 5, and Chapter 6.

CHAPTER 5

QUANTITATIVE RESULTS OF THE ALGEBRA TEST

5.1 Introduction

This chapter gives the quantitative results of the study and is organised into four sections. Section one presents the algebra test scores by country, sex, and ability using *t*-test. Section two presents the correlation among the six themes using factor analysis. Section three shows a comparison of the proportion of pupils using the generalisable and the other processes by country with sex and ability using ANOVA. Section four gives a further comparison of pupils using the generalisable and the other processes by country in each theme of the algebra test using *t*-test. These results are drawn from the algebra test given to 103 English pupils and 186 Thai pupils in the case study school in each country.

Thematic analysis was used to categorise and to code pupils' thinking processes as generalisable process, the other process, unidentified process, and incomplete response (see Chapter 4). Transformation of data coding to find the proportion of pupils' thinking process at each category score 1 for the target thinking process and 0 for the rest processes.

5.2 The algebra test scores

The number of test items in each theme was different—theme 1 eight items, theme 2 four items, theme 3 four items, theme 3 four items, theme 4 four items, theme 5 five items, and theme 6 five items. Therefore the proportion scores for each theme reported the results. Pupils' raw scores (number correct) are translated into the proportion of achievement scores for each theme and then compared to proportions of achievement scores by country, sex, and ability.

When the evidence showed that pupils did not reach certain questions then these questions are not included when finding percentages or proportions.

Table 5.2 English and Thai pupils' mean proportion achievement scores by sex

Themes	Boys		Girls			
	Mean	SD	Mean	SD	*t*	*p*
Patterns/Sequences						
English	0.52	0.21	0.58	0.17	-1.48	.14
Thai	0.64	0.15	0.66	0.14	-0.95	.34
Simplification						
English	0.24	0.23	0.35	0.29	-1.98	.051
Thai	0.11	0.18	0.12	0.20	-0.48	.63
Substitution						
English	0.61	0.35	0.61	0.32	0.05	.96
Thai	0.68	0.36	0.79	0.34	-2.05	.04
Solving Equations						
English	0.47	0.37	0.36	0.34	1.53	.13
Thai	0.44	0.28	0.50	0.27	-1.32	.19
Graph of linear functions						
English	0.13	0.16	0.14	0.18	-0.19	.85
Thai	0.29	0.28	0.39	0.28	-2.61	.01
Word Problems						
English	0.53	0.28	0.49	0.23	0.93	.35
Thai	0.52	0.23	0.53	0.22	-0.02	.99

As presented in Table 5.2, the mean proportion achievement scores of English boys and girls are not considered significantly different in all six themes—pattern (t = -1.48, df = 101, p = .14), simplification (t = -1.98, df = 101, p = .051), substitution (t = .05, df = 101, p = .96), solving equation (t = 1.53, df = 101, p = .13), graph (t = -0.19, df = 101, p = .85) and word problem (t = 0.93, df = 101, p = .35) themes.

The mean proportion achievement scores of Thai girls are significantly higher than those of Thai boys in substitution (t = -2.05, df = 184, $p < .05$), and graph (t = -2.61, df = 184, $p < .05$) themes. The mean proportion achievement scores of Thai boys and girls are not considered significantly different in four themes—pattern (t = -0.95, df = 184, p = .34), simplification (t = -0.48, df = 184, p = .63), solving equation (t = -1.32, df = 184, p = .19), and word problem (t = -0.02, df = 184, p = .99) themes.

Table 5.3 presents a comparison of pupils' proportion scores by ability on six themes.

Table 5.3 English and Thai pupils' mean proportion achievement scores by ability

Themes	High ability		Low ability			
	Mean	SD	Mean	SD	t	p
Patterns/Sequences						
English	0.64	0.13	0.45	0.19	5.89	.00
Thai	0.70	0.12	0.58	0.15	6.18	.00
Simplification						
English	0.46	0.25	0.11	0.15	8.88	.00
Thai	0.20	0.22	0.01	0.05	8.20	.00
Substitution						
English	0.77	0.24	0.41	0.32	6.38	.00
Thai	0.92	0.18	0.51	0.38	8.95	.00
Solving Equations						
English	0.66	0.28	0.11	0.15	13.05	.00
Thai	0.61	0.22	0.30	0.22	9.71	.00
Graph of linear functions						
English	0.11	0.16	0.16	0.18	-1.54	.13
Thai	0.46	0.30	0.21	0.19	6.97	.00
Word Problems						
English	0.66	0.20	0.33	0.20	8.35	.00
Thai	0.63	0.21	0.40	0.16	8.21	.00

As is evident in Table 5.3, the mean proportion achievement scores of English high and low ability groups are not considered significantly different in the graph theme (t = -1.54, df = 101, p = .13). The mean proportion achievement scores of English high ability groups are significantly higher than those of low ability groups in five themes—patterns (t = 5.89, df = 101, p < .001), simplification (t = 8.88, df = 101, p < .001), substitution (t = 6.38, df = 101, p < .001), solving equations (t = 13.05, df = 101, p < .001), and word problem (t = 8.35, df = 101, p < .001).

The mean proportion achievement scores of Thai high ability groups are significantly higher than those of low ability groups in all six themes—pattern (t = 6.18, df = 184, p < .001), simplification (t = 8.20, df = 184, p < .001), substitution (t = 8.95, df = 184, p < .001), solving equation (t = 9.71, df = 184, p < .001), graph (t = 6.97, df = 184, p < .001) and word problem (t = 8.21, df = 184, p < .001).

5.3 The proportion of pupils using generalisable process

The proportion of the use of the generalisable process was calculated by scoring 1 for generalisable process and 0 for the rest processes. For example, the coding generalisable process "A", the other process "O", unidentified process "W", and incomplete response "R", pupil response to theme 1 (eight items) as "AAORORWO" transferred to "11000000" for generalisable process and the proportion was "0.25" (2÷8). This transformation procedure also applied to Section 5.5 and 5.6.

The factor structures of test items in each theme were explored through SPSS using principal components extraction and varimax rotation. Inspection of the correlation matrix revealed the presence of many coefficients of 0.3 and above. The Kaiser-Meyer-Oklin values for the total, the English and the Thai generalisable process groups were 0.80, 0.82, and 0.79 respectively, exceeding the recommended value of 0.6 (Tabachnick and Fidell, 1996). The Bartlett's Test of Sphericity (Bartlett, 1954) reached statistical significance ($p < .001$), supporting the factorability of the correlation matrix.

Table 5.4 shows the coefficients of correlation expressing the degree of linear relationship between the row and column variables of the matrix.

Table 5.4 Coefficients of correlation between variables in both countries

Correlation Matrix

		patternA	simpliA	substiA	equaA	graphA	wordA
Correlation	patternA	1.000	.182	.415	.378	.277	.431
	simpliA	.182	1.000	.241	.399	.035	.295
	substiA	.415	.241	1.000	.588	.349	.434
	equaA	.378	.399	.588	1.000	.332	.442
	graphA	.277	.035	.349	.332	1.000	.183
	wordA	.431	.295	.434	.442	.183	1.000

As contained in Table 5.4, the percent variation in common for the data on two variables is the square of this coefficient multiplied by 100. For example, the correlation of 0.415 between 'pattern' and 'substitution' gives 17.22% $(0.415^2 \times 100)$. Thus, the values on one of these two variables accounts for 17.22 % of the variance in the values on the other variable. Similarly, the correlation of 0.399 between 'simplification' and 'solving equation' means that 15.92% $(0.399^2 \times 100)$ of the variance in solving equation scores can be "explained" from their simplification scores and vice versa.

Table 5.5 contains the loadings of each theme on the six components for pupils from the English and Thai schools.

Table 5.5 Percentage of variance explained for each test theme

Total Variance Explained

	Initial Eigenvalues			Extraction Sums of Squared Loadings			Rotation Sums of Squared Loadings		
Component	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
1	2.731	45.517	45.517	2.731	45.517	45.517	2.042	34.028	34.028
2	1.005	16.748	62.266	1.005	16.748	62.266	1.694	28.238	62.266
3	.761	12.677	74.942						
4	.593	9.885	84.827						
5	.529	8.820	93.647						
6	.381	6.353	100.000						

Extraction Method: Principal Component Analysis.

It is clear from Table 5.5, the factor solution for the first two components recorded eigenvalues above 1 (2.042, 1.694). These two components explained a total of 62.27% of the total variance.

Table 5.6 shows the communalities that represent the amount of variance in a variable explained by the components given in Table 5.7

Table 5.6 Communality in both schools

Communalities

	Initial	Extraction
patternA	1.000	.481
simpliA	1.000	.737
substiA	1.000	.633
equaA	1.000	.660
graphA	1.000	.692
wordA	1.000	.533

Extraction Method: Principal Component Analysis.

← $(0.623)^2 + (0.303)^2$

Table 5.7 The analysis of principal components in both schools

Rotated Component Matrix(a)

	Component	
	1	2
patternA	.623	.303
simpliA	-.056	.857
substiA	.690	.396
equaA	.569	.579
graphA	.810	-.192
wordA	.441	.582

Extraction Method: Principal Component Analysis. Rotation Method: Varimax with Kaiser Normalization.
a Rotation converged in 3 iterations.

By default, this is the squared multiple correlations obtained when each theme is regressed on all the other themes. For example, Communality of pattern theme is .481 calculated from Table 5.7 component matrix, the sum of squared loadings over components $[(0.623)^2 + (0.303)^2] = 0.48]$. Thus, it indicates 48.0% of the variance in the patterns theme is explained by these two components.

As can be seen in Table 5.7, component 1 loads heavily for patterns, substitution, and graphs themes, while component two loads heavily on simplification, equation and word problem themes.

5.3.1 The proportion of English pupils using generalisable process

Table 5.8 shows the coefficients of correlation expressing the degree of linear relationship between the row and column themes of the matrix. For example, the correlation of 0.529 between 'pattern theme' and 'word problem theme' gives $27.98\% (0.529^2 \times 100)$. Thus the values of one of these two themes accounts for 27.98% of the variance in the values on the other theme. Similarly, the correlation of 0.537 between 'word problem theme' and 'solving Equation theme' means that $28.84\% (0.537^2 \times 100)$ of the variance in 'word problem theme' scores can be explained from their 'solving equation theme' scores and vice versa.

Table 5.8 Coefficients of correlation between themes in the English school

Correlation Matrix (a)

		patternA	simpliA	substiA	equaA	graphA	wordA
Correlation	patternA	1.000	.337	.457	.478	.170	.529
	simpliA	.337	1.000	.390	.459	-.014	.356
	substiA	.457	.390	1.000	.462	.010	.485
	equaA	.478	.459	.462	1.000	-.154	.537
	graphA	.170	-.014	.010	-.154	1.000	.029
	wordA	.529	.356	.485	.537	.029	1.000

a Only cases for which country = Eng are used in the analysis phase.

As is evident from Table 5.8 the equation theme is highly correlated with the pattern, simplification, substitution, and word problem themes. The graph theme has a low correlation with all themes. In the English curriculum, the pattern theme is introduced to pupils when they first move from arithmetic to algebra. The simplification and substitution themes are taught as basic skills for use in solving algebraic problems. Questions in the word problem theme expected pupils to transform words to an equation form and then to solve using methods in solving equations. Not surprisingly, solving equations showed a cluster of highly correlation themes.

For the 'graph theme', English pupils are taught separately from the other themes of the algebra lessons. It was the school's scheme of work to teach the algebra content in all three terms of the academic year. There also seems to be less emphasis on the drawing straight-line graphs topic—there were only two lessons in Year 7 top set and one lesson in bottom set; for Year 8, one lesson in top set and two lessons in bottom set.

Table 5.9 shows the loadings of each of the themes on the six components for pupils' from the English school.

Table 5.9 The percentage of the variance in the English school

Total Variance Explained

Component	Initial Eigenvalues			Extraction Sums of Squared Loadings			Rotation Sums of Squared Loadings		
	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
1	2.805	46.752	46.752	2.805	46.752	46.752	2.803	46.722	46.722
2	1.099	18.323	65.075	1.099	18.323	65.075	1.101	18.353	65.075
3	.689	11.483	76.558						
4	.553	9.210	85.768						
5	.452	7.536	93.303						
6	.402	6.697	100.000						

Extraction Method: Principal Component Analysis.

a. Only cases for which COUNTRY = Eng are used in the analysis phase.

As can be seen in Table 5.9, the first two components, for a rotated factor solution, recorded eigenvalues above 1 (2.803, 1.101). These two components explain a total of 65.08% of the variance.

Table 5.10 shows the communalities that represent the amount of variance in a variable explained by the retained components as given in Table 5.11. For example, communality of the patterns theme is the sum of the squared loadings over components $[(0.744)^2 + (0.322)^2 = 0.656]$. Thus, it indicates 65.6% of the variance in the patterns theme is explained by these two components.

Table 5.10 Communality in the English school

Communalities(a)

	Initial	Extraction
patternA	1.000	.657
simpliA	1.000	.448
substiA	1.000	.559
equaA	1.000	.697
graphA	1.000	.922
wordA	1.000	.622

← $(0.744)^2 + (0.322)^2$

Extraction Method: Principal Component Analysis.

a Only cases for which country = Eng are used in the analysis phase.

Table 5.11 The analysis of principal components in the English school

Rotated Component Matrix (a,b)

	Component	
	1	2
patternA	.744	.322
simpliA	.662	-.100
substiA	.747	.037
equaA	.799	-.241
graphA	-.016	.960
wordA	.785	.079

Extraction Method: Principal Component Analysis. Rotation Method: Varimax with Kaiser Normalization.
a Rotation converged in 3 iterations.
b Only cases for which country = Eng are used in the analysis phase.

As can be seen in Table 5.11, component 1 loads heavily for the patterns, simplification, substitution, equations, and word problem themes. At the same time, component 2 loads heavily on the graph theme, confirming the taught experience in the English school where few lessons on drawing graphs were taught and delivered separately from other themes in the algebra area.

5.3.2 Proportion of Thai pupils using generalisable process

Table 5.12 shows that the 'equations theme' has high correlation with the simplification, substitution, and graph themes.

Table 5.12 Coefficients of correlation between variables in Thai school

Correlation Matrix (a)

		patternA	simpliA	substiA	equaA	graphA	wordA
Correlation	patternA	1.000	.349	.339	.257	.206	.342
	simpliA	.349	1.000	.338	.521	.385	.371
	substiA	.339	.338	1.000	.674	.409	.394
	equaA	.257	.521	.674	1.000	.553	.353
	graphA	.206	.385	.409	.553	1.000	.227
	wordA	.342	.371	.394	.353	.227	1.000

a Only cases for which country = Thai are used in the analysis phase.

In the Thai curriculum, the 'patterns theme' and the 'simplification theme' were not introduced in algebra lessons when pupils first moved from arithmetic to algebra as was done in the English school. The simplification and substitution themes were used for solving and checking the result of algebraic equations. The common factor approach was used to simplify like terms.

The graph theme, this is a main topic in algebra lessons in the Thai curriculum. The Thai pupils encounter the graph theme as a part of solving equations unlike in the English school, where the graph theme consisted of only a few lessons and was taught separately from the other themes.

Table 5.13 shows the loadings of each of the themes on the six components for pupils' from the Thai school.

Table 5.13 The percentage of the variance in the Thai school

Total Variance Explained[a]

Component	Initial Eigenvalues			Extraction Sums of Squared Loadings			Rotation Sums of Squared Loadings		
	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
1	2.945	49.088	49.088	2.945	49.088	49.088	2.247	37.452	37.452
2	.937	15.614	64.703	.937	15.614	64.703	1.635	27.251	64.703
3	.685	11.417	76.120						
4	.647	10.780	86.899						
5	.529	8.818	95.717						
6	.257	4.283	100.000						

Extraction Method: Principal Component Analysis.

a. Only cases for which COUNTRY = Thai are used in the analysis phase.

Presented in Table 5.13, the choice of the two components is justified by the eigenvalue and inspection of the scree plot. The first two components, for a rotated factor solution, recorded eigenvalues (2.247, 1.635). These two components are explained a total of 64.7% of the variance.

Table 5.14 shows the communalities that represent the amount of variance in a variable explained by the retained components (Table 5.15). For example, Communality of 'simplification theme' is the sum of the squared loadings over components $[(0.525)^2 + (0.483)^2 = 0.509]$. Thus, it represents 50.9% of variance in the simplification theme explained by these two components.

Table 5.14 Communalities in the Thai school

Communalities(a)

	Initial	Extraction
patternA	1.000	.693
simpliA	1.000	.509
substiA	1.000	.619
equaA	1.000	.805
graphA	1.000	.665
wordA	1.000	.592

← $(0.525)^2 + (0.483)^2$

Extraction Method: Principal Component Analysis.
a Only cases for which country = Thai are used in the analysis phase.

Table 5.15 The analysis of principal components in the Thai school

Rotated Component Matrix (a,b)

	Component	
	1	2
patternA	.080	.829
simpliA	.525	.483
substiA	.702	.354
equaA	.868	.228
graphA	.815	.024
wordA	.233	.733

Extraction Method: Principal Component Analysis. Rotation Method: Varimax with Kaiser Normalization.
a Rotation converged in 3 iterations.
b Only cases for which country = Thai are used in the analysis phase.

Table 5.15 illustrated that component 1 loaded heavily for the simplification, substitution, equation, and graph themes. Component 2 loads most heavily on the patterns and word problem themes. The first factor, whatever it is, captures the form of covariation between the cluster of simplification, substitution, equation and graph themes. The second factor captures the form of covariation between the cluster of patterns and word problems themes.

As can be seen in Table 4.3, English school pupils are taught the graphs of linear functions separately from the other themes in algebra area. In contrast, the graphs content for Thai school pupils are delivered as one of two chapters in the algebra area (see Table 4.6). The other chapter of algebra in Thai school was solving equations and inequalities. This included substitutions, solving equations, and solving word problems. As a result Thai pupils scored in the second component captures of covariation between patterns and word problems might be explained as these two themes take longer read than other themes and they often do not want to.

5.4 A comparison of pupils using generalisable process and other process by country with sex and ability

5.4.1 The proportion of the pupils using generalisable process

For the proportion of the use of generalisable process, a two-way unrelated analysis of variance (ANOVA) was conducted to explore the impact of country, sex, and ability on pupils' correct conceptions process scores, as measured by the algebra test.

From Table 5.16 it is evident that there was no statistically significant difference in the effect of sex on the proportion of the use of generalisable process for English and Thai pupils.

Table 5.16 ANOVA for the proportion of the use of generalisable process

Source of variation	Sum of Squares	*df*	Mean Square	*F*	*p*
COUNTRY	.33	1	.33	20.74	.00
SEX	.04	1	.04	2.71	.10
ABILITY	3.28	1	3.28	205.52	.00
COUNTRY*SEX	.00	1	.00	.00	.99
COUNTRY*ABILITY	.09	1	.09	5.52	.02
SEX*ABILITY	.00	1	.00	.13	.72
COUNTRY*SEX*ABILITY	.00	1	.00	.07	.79
ERROR	4.48	281	.00		

There were statistically significant main effects for country ($F_{1,281} = 20.74$, $p < .001$) and for ability ($F_{1,281} = 205.52, p < .001$) but not for sex ($F_{1,281} = 2.71, p = .10$). The significant effect was obtained for the interaction for country*ability ($F_{1,281} = 5.52$, $p < .05$) but not for country*sex ($F_{1,281} = .00, p = .99$) and not for country*sex*ability ($F_{1,281} = .07, p = .79$). This means that the effect of country varied across ability groups.

There was a statistically significant difference for English and Thai pupils in terms of their generalisable process scores, and that there were statistically significant differences in scores for ability. The effect of sex and ability do not vary by country.

5.4.2 The proportion of the pupils using other process

The proportion of the use of other process is calculated by scoring 1 for the "other process" and 0 for the rest processes. For example, the coding the generalisable process "A", the other process "O", unidentified process "W", and incomplete response "R", pupil response to theme 1 (eight items) as "AAORORWO" transferred to "00101001" for the other process and the proportion was 0.38 (3÷8).

For the proportion of the use of the other process, a two-way unrelated analysis of variance (ANOVA) was also conducted to explore the impact of country, sex and ability on pupils' the other process scores, as measured by the algebra test.

In Table 5.17, ANOVA results show that there was a statistically significant difference for English and Thai pupils in terms of their other process scores, and that there were no statistically significant differences in scores for sex and for ability. The effect of sex and ability do vary by country.

Table 5.17 ANOVA for the proportion of the use of the other process

Source of variation	Sum of Squares	*df*	Mean Square	*F*	*p*
COUNTRY	.05	1	.05	6.8	.01
SEX	.02	1	.02	2.61	.11
ABILITY	.00	1	.00	.76	.39
COUNTRY*SEX	.11	1	.11	14.22	.00
COUNTRY*ABILITY	.08	1	.08	10.52	.00
SEX*ABILITY	.00	1	.00	.36	.55
COUNTRY*SEX*ABILITY	.04	1	.04	6.28	.01
ERROR	2.21	281	.01		

As shown in Table 5.17, there was a statistically significant main effect for country $(F_{1,281} = 6.80, p = .01)$ but not for sex $(F_{1,281} = 2.61, p = .11)$ and not for ability $(F_{1,281} = 0.76, p = .39)$. The significant effects were obtained for their interaction for country*sex $(F_{1,281} = 14.22, p < .001)$, country*ability $(F_{1,281} = 10.52, p < .001)$, and country*sex*ability $(F_{1,281} = 6.28, p = .01)$. That means the effect of country varied across sex and ability groups. There was a statistically significant difference for English and Thai pupils in terms of the use of other process scores, but that there were no statistically significant differences in scores for sex and for ability.

5.5 A comparison of the proportion of pupils using the generalisable process and the other process between the English school and the Thai school

Table 5.18 shows a comparison of the proportion of the generalisable and the other process used for each theme of the algebra test between English and Thai pupils in the case study schools.

Table 5.18 Comparison of the proportion of the generalisable and other process used by country on each theme

Themes	England		Thailand			
	Mean	SD	Mean	SD	*t*	*p*
Patterns/Sequences						
Generalisable process	0.56	0.18	0.66	0.14	-4.89	.00
Other process	0.13	0.13	0.11	0.13	0.97	.33
Simplification						
Generalisable process	0.36	0.33	0.12	0.21	6.40	.00
Other process	0.13	0.22	0.15	0.24	-0.83	.41
Substitution						
Generalisable process	0.62	0.33	0.76	0.35	-3.30	.00
Other process	0.13	0.19	0.03	0.12	4.64	.00
Solving Equations						
Generalisable process	0.42	0.36	0.49	0.28	-1.78	.08
Other process	0.10	0.14	0.10	0.17	-0.46	.65
Graph of linear functions						
Generalisable process	0.14	0.18	0.36	0.29	-8.01	.00
Other process	0.30	0.27	0.20	0.20	3.45	.00
Word Problems						
Generalisable process	0.53	0.26	0.57	0.22	-1.45	.15
Other process	0.17	0.19	0.13	0.16	1.77	.08

5.5.1 Theme 1 Patterns/Sequences

Looking at Table 5.18, the mean score for the generalisable process group for Thai pupils (M = 0.66, SD = 0.14) is significantly higher (t = -4.89, df = 287, two-tailed $p < .001$) than that of the English pupils (M = 0.56, SD = 0.18). The mean score of the other process group, of the English pupils is 0.13 (SD = 0.13) and that of the Thai pupils is 0.11 (SD = 0.13). The difference is not considered statistically significant at the 5% level (t = 0.97, df = 287, p = .33).

As presented in Chapter 4 the lesson observations revealed that English Year 7 pupils have experienced the patterns/sequences at an early stage in the introduction of algebra. However, in the Thai school, pupils have no experience in these lessons. The empirical evidence suggests that a minority of pupils in both the English school and the Thai school could solve the level 3 and 4 questions.

5.5.2 Theme 2 Simplification

From Table 5.18 it can be seen that the mean score of the English pupils in the generalisable process group (M = 0.36, SD = 0.33) is significantly higher (t = 6.40, df = 287, p < .001) than that of the Thai pupils (M = 0.12, SD = 0.21). The mean score of the English pupils in the other process group is 0.13 (SD = 0.22) and that of the Thai pupils is 0.15 (SD = 0.24). The difference is not statistically significant at the 5% level (t = -0.83, df =287, p = .41).

As mentioned (see Chapter 2), simplifying like terms was taught in both Year 7 (25% in the top set and 18% in the bottom set) and Year 8 (14% in the top set and 17% in the bottom set), whilst this topic does not appear in the Thai mathematics curriculum in either Secondary 1 or 2. The Thai school used the common factor approach to deal with like terms.

5.5.3 Theme 3 Substitution

The mean score of pupils using generalisable process, for Thai pupils (M = 0.76, SD = 0.35) is statistically significantly higher (t = -3.30, df = 287, p < .01) than that of the English pupils (M = 0.62, SD = 0.33). The mean score of the pupils using other process, of the English pupils (M = 0.13, SD = 0.19) is statistically significantly higher (t = 4.64, df = 287, p < .001) than that of the Thai pupils (M = 0.03, SD = 0.12).

As stated in chapter 2, the substitution process was taught in the English school (5% in Year 7 the top set, 30% in the bottom set, and 8% in Year 8 the bottom set). In the Thai school substitution was used to check the solutions under the topic of solving equations in both Secondary 1 and 2.

5.5.4 Theme 4 Solving Equations

Referring again to Table 5.18, the mean score of the English pupils using generalisable process is 0.42 (SD = 0.36) and that of the Thai pupils is 0.49 (SD = 0.28). The difference is not statistically significant at the 5 % level (t = -1.78, df = 287, p = .08). The mean score of the English pupils using the other process is 0.10 (SD = 0.14) and that of Thai pupils is 0.10 (SD = 0.17). The difference is not statistically significant at 5% level ($t = -0.46$, $df = 287$, $p = .65$).

The use of the generalisable and the other processes were similar even though English pupils' experience of solving equations was less than half of that of the Thai pupils. As pointed out earlier, the English school pupils were taught the contents separately as simplifying like terms, substitutions, solving equations and word problems over three terms, unlike the Thai school pupils where these contents were taught in one topic.

5.5.5 Theme 5 Graphs of linear functions

For the graphs of linear functions theme Table 5.18 shows that the mean score of the Thai pupils using generalisable process (M = 0.36, SD = 0.29) is statistically significantly higher ($t = -8.01$, $df = 287$, $p < .001$) than that of the English pupils (M = 0.14, SD = 0.18). The mean score of the English pupils using the other process (M = 0.30, SD = 0.27) is statistically significantly higher ($t = 3.45$, $df = 287$, $p < .001$) than that of Thai pupils (M = 0.20, SD = 0.20).

As pointed out earlier, the English school pupils received instruction in the graphs of linear functions separately. The algebra area was taught in three terms unlike the Thai school pupils where graphs of linear functions were taught in one chapter and the other chapter of algebra was delivered in the same term, both in Secondary 1 and 2.

5.5.6 Theme 6 Word Problems

The mean score of the pupils using generalisable process as contained in Table 5.18, for Thai pupils is 0.57 (SD = 0.22) and for English pupils is 0.53 (SD = 0.26). The difference is not statistically significant at the 5% level ($t = -1.45$, $df = 287$, $p = .15$). The mean scores of the English pupils in the other process group is 0.17 (SD = 0.19), and that of Thai pupils is 0.13 (SD = 0.16). The difference is not statistically significant at the 5% level ($t = 1.77$, $df = 287$, $p = .08$).

The proportions used of the generalisable and the other processes were similar. This topic was taught in two lessons to each Year 7 and 8 top set pupils in the English school and one lesson to Year 8 bottom set. In the Thai school, three lessons were delivered to each Secondary 1 and 2 high ability group, and two lessons to the Secondary 2 low ability group. It could be argued that limited emphasis on this topic in both schools forced pupils to solve these problems without using algebraic methods.

5.6 Summary and conclusion

The achievement mean scores, for English pupils are statistically significantly higher than those of Thai pupils in the simplification theme. However, Thai pupils' mean score is higher in patterns, substitution, and graph of linear function themes. For the solving equation and word problem themes, there are no real differences in achievement for English and Thai pupils. The substitution theme means scores of Thai girls are statistically significantly higher than Thai boys. The graphs of linear functions theme mean scores of English high ability and low ability groups are not considered significantly difference.

For both countries, factor analysis revealed the presence of two components with eigenvalues exceeding 1, explaining 62.27% of the variance. For the English pupils, it presents two components for a rotated factor solution, with recorded eigenvalues greater than 1 (2.80, 1.10). These two components explained a total of 65.08% of the variance. For the Thai pupils, the first two components for a rotated factor solution, recorded eigenvalues (2.25, 1.64). These two components explained a total of 64.70% of the variance.

A comparison of the proportion of pupils using generalisable process by country with sex and ability showed that there was a statistically significant difference between English and Thai pupils in terms of the use of generalisable process. There was a significant difference in the interaction with ability but not with sex. There was also a significant difference between English and Thai pupils in terms of the use of other process.

Moreover, there was statistically significant difference in their interaction for sex and for ability.

The mean score for English pupils of the generalisable process group is statistically significantly higher than those of Thai pupils in the simplification theme. However, Thai pupils' mean scores are higher in patterns, substitution, and graphs of linear functions themes. For the other process group, English pupils' mean scores are statistically significantly higher than those of Thai pupils in substitution and graphs of linear functions themes.

The next chapter presents pupils' thinking processes in more detail to clarify the phenomena. The pupils' thinking process used and outcomes of each item were categorised as the generalisable process, the other process, unidentified process, and incomplete response. Each process is defined in the next chapter.

CHAPTER 6

A COMPARISON OF PUPILS' THINKING PROCESSES IN SOLVING ALGEBRAIC PROBLEMS BETWEEN ENGLISH SCHOOL AND THAI SCHOOL

6.1 Introduction

This chapter continues with the analysis and discussion of key findings concerning pupils' thinking processes in solving algebraic items of the researcher's algebra test. The chapter also includes a discussion of the mathematics curriculum contents in England and Thailand as related to the pupils' thinking processes.

Thematic analysis, as stated in Chapter 4, was used to categorise pupils' thinking processes when solving each question. The thinking processes were categorised from pupils' written responses described in the codebook (Appendix G). These four categories are generalisable process, other process, unidentified process, and incomplete response as mentioned in Chapter 4.

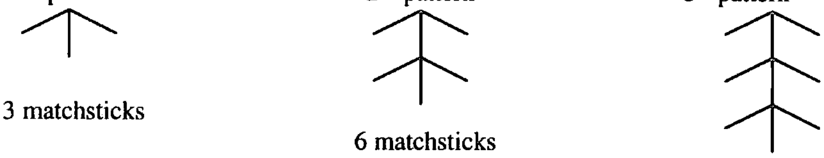
6.2 Theme 1 Patterns and Sequences

The first theme of the researcher's algebra test is patterns/sequences, organised into four levels of expected difficulty. It consisted of eight questions designed to investigate pupils' thinking processes as they find a general rule. The questions are shown in Figure 6-1.

Patterns/sequences

Item 1. Look at the number of matchsticks in each pattern.

1^{st} pattern — 2^{nd} pattern — 3^{rd} pattern …

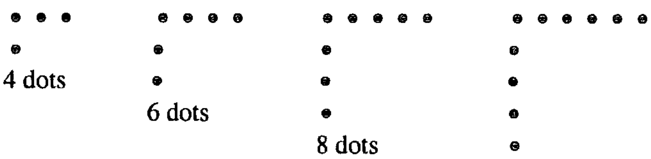


3 matchsticks — 6 matchsticks — 9 matchsticks

a. How many matchsticks are needed for the 4^{th} pattern in this series? *(Level 1 concrete objects)*
b. How many matchsticks are needed for the 10^{th} pattern in this series? *(Level 1 concrete objects)*

Item 13. Look at the number of dots in each pattern.

1^{st} pattern — 2^{nd} pattern — 3^{rd} pattern — 4^{th} pattern …



4 dots — 6 dots — 8 dots — 10 dots

a. How many dots are there in the 5^{th} pattern? *(Level 1 concrete objects)*
b. How many dots are there in the 20^{th} pattern? *(Level 2 concrete objects)*
c. How many dots are there in the n^{th} pattern? *(Level 3 generalise concrete objects)*

Item 7. Fill in the blanks in this sequence. *(Level 2 abstract objects)*
1, 2, 4, 8, 16, 32, ……., …….

Item 19. Look at this sequence.
2, 5, 8, 11, 14, 17, …
a. The 7^{th} term of this sequence is ………. *(Level 2 abstract objects)*
b. The n^{th} term of this sequence is ………. *(Level 4 generalise abstract objects)*

Figure 6-1 Patterns and sequences test items

Pupils' thinking processes in solving pattern and sequence problems were categorised from pupils' responses as *generalisable* process, *other process*, *unidentified process,* and *incomplete response* process.

Generalisable process is the methods that reflect the way of generalising rules. These ways of thinking include *generalisation*, *repeated operations* and *draw/count* strategies.

Other process is that in which pupils attempt to obtain general rules from wrongly perceived situations. These include inappropriate scaling up strategies and attempts to draw or count from incorrect patterns.

Unidentified processes are those that give the answer without showing working. Some correct answers appeared without working.

Incomplete response processes are those that showed an attempt to work it out but did not reach completion. Also included are those that made no response to the question.

6.3 A comparison of pupils' thinking processes in searching for patterns/sequences between the English and Thai schools

Figures 6-2 and 6-3 give a breakdown of the processes that the English and Thai pupils used in approaching these problems at each level of difficulty.

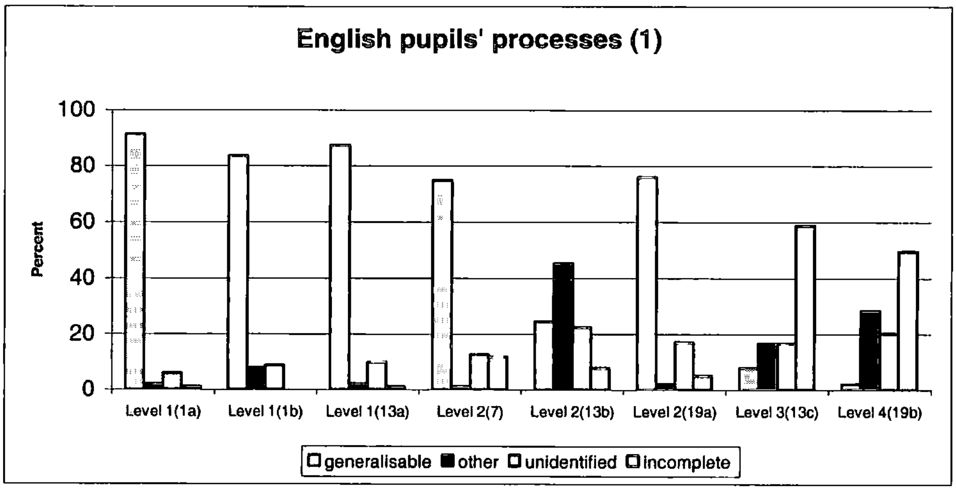


Figure 6-2 Percentage of process used in theme 1 by English pupils

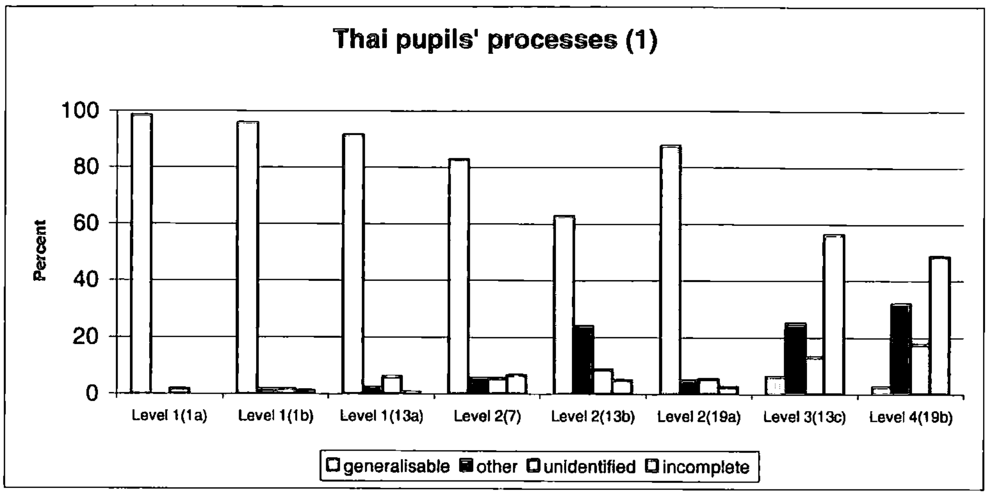


Figure 6-3 Percentage of process used in theme 1 by Thai pupils

As shown in the figures 6-2 and 6-3, pupils mainly used a generalisable process to solve the level 1 and level 2 problems. There was a sharp drop in using the generalisable process when facing the harder questions at level 3 and level 4.

Table 6.1 gives the actual percentage of each process used and corresponding outcomes at each level of difficulty.

Table 6.1 Percentage of process used and outcomes for theme 1

Country	Level (item)	Processes							
		Generalisable		Other		Unidentified process		Incomplete responses	
		Used	% correct	Used	% correct	Used	% correct	Used	% correct
England (n=103)	**1 (1a)**	**91.3**	**98.9**	**1.9**	**50.0**	**5.8**	**83.3**	**1.0**	**0.0**
	1 (1b)	**83.5**	**94.2**	**7.8**	**100.0**	**8.7**	**66.7**	**0.0**	**0.0**
	1 (13a)	**86.4**	**97.8**	**2.0**	**0.0**	**9.8**	**100.0**	**1.0**	**0.0**
	2 (7)	**87.3**	**94.8**	**1.0**	**0.0**	**12.6**	**69.2**	**11.7**	**0.0**
	2 (13b)	**24.5**	**80.0**	**45.1**	**2.2**	**22.5**	**17.4**	**7.8**	**0.0**
	2 (19a)	**76.0**	**97.4**	**2.0**	**100.0**	**17.0**	**100.0**	**5.0**	**0.0**
	3 (13c)	**7.8**	**100.0**	**16.7**	**0.0**	**16.7**	**17.6**	**58.8**	**0.0**
	4 (19b)	**2.0**	**50.0**	**28.3**	**0.0**	**20.2**	**5.0**	**49.5**	**0.0**
Thailand (n=186)	**1 (1a)**	**98.4**	**99.5**	**0.0**	**0.0**	**1.6**	**66.7**	**0.0**	**0.0**
	1 (1b)	**95.7**	**97.8**	**1.6**	**66.7**	**1.6**	**100.0**	**1.1**	**0.0**
	1 (13a)	**91.4**	**99.4**	**2.2**	**0.0**	**5.9**	**100.0**	**0.5**	**0.0**
	2 (7)	**82.8**	**96.8**	**5.4**	**0.0**	**5.4**	**50.0**	**6.5**	**0.0**
	2 (13b)	**62.7**	**65.5**	**23.8**	**0.0**	**8.6**	**31.3**	**4.9**	**0.0**
	2 (19a)	**87.6**	**95.9**	**4.7**	**62.5**	**5.3**	**88.9**	**2.4**	**0.0**
	3 (13c)	**5.9**	**81.8**	**24.9**	**0.0**	**13.0**	**12.5**	**56.2**	**0.0**
	4 (19b)	**2.4**	**100.0**	**31.7**	**0.0**	**17.4**	**6.9**	**48.5**	**0.0**

As can be seen in Table 6.1, the percentage of English pupils showing the generalisable process for level 1 items are: 1a, 91.3%; 1b, 83.5%; 13a, 86.4% and of those 98.9%, 94.2%, and 97.8% gained the correct answers. The percentages of Thai pupils showing the generalisable process are 1a, 98.4%; 1b, 95.7%; 13a, 91.4% and of those 99.5%, 97.8%, and 99.4% gained the correct solutions.

There was a decrease between level 1 and level 2 of those making up the generalisable process group in both countries. English pupils showed the generalisable process for level 2 items 7, 87.3%; 13b, 24.5%; 19a, 76.0% and of those 94.8%, 80.0%, and 97.4% gained the correct answers. Thai pupils showed the generalisable process for items 7, 82.8%; 13b, 62.7%; 19a, 87.6% and of those 96.8%, 65.5%, and 95.9% gained the correct answers.

There was a large drop between level 2 and level 3 for the generalisable process groups in both countries. Of English pupils 7.8% showed the generalisable process to item 13c and of those 100% gained the correct solution. Of Thai pupils 5.9% showed the generalisable process to item 13c and of those 81.8% gained the correct solution.

For the level 4 question, 2.0% of English pupils showed the generalisable process to item 19b and of those 50.0% gained the correct solution. Of Thai pupils 2.4% showed the generalisable process and of those 100% gained the correct solution.

The following sections describe the sub-processes that pupils used at each level of difficulty.

Within the generalisable process group there are 3 sub-processes:

(1) *The generalisation process* in which pupils perform the rule to find out the solution.

(2) *The repeated operation process* refers to some knowledge of the operation for the previous solution and which is then re-used.

(3) *The draw or count process* reflects the empirical approach rather than looking for a rule.

There are 4 sub-processes within the other process group.

(1) *The generalisation-like process* is an attempt to perform the rule incorrectly.
(2) *The repeated operation-like process* is an attempt to use the previous solution but in the **incorrect** pattern.
(3) *The inappropriate scaling up process* is an attempt to find the answer by using the prior pattern number.
(4) *The draw or count incorrectly process* is that showing the basic process to be drawing or counting with an incorrect pattern.

The unidentified process group gave the result without showing working. Some of these pupils described their thinking processes as "a guess".

There are 3 sub-processes in the incomplete responses group.

(1) The *incomplete* work showed an attempt to work it out but did not reach completion.

(2) *No response:* pupils made no attempt.

(3) *Un-reached*: pupils did not reach that question because of the limit of time.

For the remainder of this chapter *the unidentified process* and *the incomplete response* groups are defined as stated above.

6.3.1 Process used and outcomes for theme 1 level 1 item 1a

This level 1 item 1a "How many matchsticks are needed for the 4th pattern?" was designed to investigate how pupils worked out the next formula from a physical pattern. Pupils' responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.2 Percentage of process used and outcomes for theme 1 level 1 item 1a

Processes Theme 1 Level 1 (1a)	English school		Thai school	
	Used	%correct	Used	%correct
Generalisable process	**91.3**	**98.9**	**98.4**	**99.5**
Generalisation	29.1	100.0	30.6	100.0
Repeated operation	58.3	98.3	45.7	98.8
Draw or count	3.9	100.0	22.0	100.0
Other process	**1.9**	**50.0**	**0.0**	**0.0**
Scaling up	1.0	100.0	0.0	0.0
Draw or count incorrectly	0.0	0.0	0.0	0.0
Unidentified process	**5.8**	**83.3**	**1.6**	**66.7**
No process	5.8	83.3	1.6	66.7
Incomplete response	**1.0**	**0.0**	**0.0**	**0.0**
No response	1.0	0.0	0.0	0.0

As can be seen in Table 6.2, the most common process used in the generalisable process group was *the repeated operation process.* Of English pupils 58.3% used *the repeated operation process* and of those 98.3% gained the correct answer. The corresponding percentages for Thai pupils were 45.7% and 98.8%. For example, most English and Thai pupils who used this process showed their processes as

"keep adding 3"

"the pattern is going up in 3s" and

"increase 3 each time".

The second most common process used was *the generalisation process.* Of English pupils 29.1% and of Thai pupils 30.6% used this process and of those all gained the correct answer. For example, they showed their processes as

"times the pattern by 3"

"the 3 times table" and

"multiples of 3".

In the other process group, only 1.0% of English pupils attempted to get the answer using *the scaling up process* and the other 1.0% counted the pattern incorrectly. An English girl showed *the scaling up process* as

"4^{th} is double 2^{nd}".

Another English pupil showed the process as

"count 2 more on".

In *the unidentified process* group, 5.8% of English pupils gave the answer without showing working and of those 83.3% gained the correct answer. The corresponding percentages for Thai pupils were 1.6% and 66.7%.

In the incomplete response group, only 1.0% of English pupils made no attempt at this question.

6.3.2 Process used and outcomes for theme 1 level 1 item 1b

The level 1 item 1b "How many matchsticks are needed for the 10^{th} pattern?" was designed to investigate how pupils worked out the formula from a physical pattern. As before, pupils' responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.3 Percentage of process used and outcomes for Theme 1 level 1 item 1b

Processes Theme 1 Level 1 (1b)	English school		Thai school	
	Used	%correct	Used	%correct
Generalisable process	**83.5**	**94.2**	**95.7**	**97.8**
Generalisation	70.9	100.0	66.1	100.0
Repeated operation	8.7	66.7	17.7	90.9
Draw or count	3.9	50.0	11.8	95.5
Other process	**7.8**	**100.0**	**1.6**	**66.7**
Generalisation-like	0.0	0.0	0.5	100.0
Scaling up	7.8	100.0	0.5	100.0
Draw or count incorrectly	0.0	0.0	0.5	0.0
Unidentified process	**8.7**	**66.7**	**1.6**	**100.0**
No process	8.8	66.7	1.6	100.0
Incomplete response	**0.0**	**0.0**	**1.1**	**0.0**
No response	0.0	0.0	1.1	0.0

As presented in Table 6.3, the most common process used in the generalisable process group was *the generalisation process.* For English pupils 70.9% used *the generalisation process* and of those 100% gained the correct answer. The corresponding percentages for Thai pupils were 66.1% and 100%. For example, most of English pupils who used this process showed their processes as

"Times whatever term you want by 3",
"the expression is 3*n*", and
"3 times table".

Thai pupils showed their processes as

"3 times 10"
"3 times table" and
"the first pattern times 10".

In the other process group, 7.8% of English pupils used *the scaling up process* and of those 100% gained the correct solution. Of Thai pupils 0.5% used this process with of those 100% gained the correct solution. For example, English other process group showed *the scaling up process* as

"double 5th pattern " and
"2nd pattern times 5"

A Thai pupil attempted to perform the rule incorrectly as

"$1^{st} = 3, 2^{nd} = 6, 3^{rd} = 9, 10^{th} = \frac{9}{3} \times 10$".

A Thai pupil gave the answer as "23" and showed *the draw or count incorrectly process* as "drawing the pattern and then count the matchsticks".

In the unidentified process group, 8.7% of English pupils gave the answer without showing working and of those 66.7% gained the correct answer. The corresponding percentages for Thai pupils were 1.6% and 100%.

In the incomplete response group, only 1.1% of Thai pupils made no attempt at this question.

6.3.3 Process used and outcomes for theme 1 level 1 item 13a

For this level 1 item 13a "How many dots are there in the 5th pattern?" was designed to investigate how pupils worked out the next formula from a physical pattern. Pupils' responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.4 Percentage of process used and outcomes for theme 1 level 1 item 13a

Processes Theme 1 Level 1 (13a)	English school		Thai school	
	Used	%correct	Used	%correct
Generalisable process	**87.3**	**97.8**	**91.4**	**99.4**
Generalisation	4.9	100.0	2.7	100.0
Repeated operation	76.5	97.4	68.6	100.0
Draw or count	5.9	100.0	20.0	97.3
Other process	**2.0**	**0.0**	**2.2**	**0.0**
Draw or count incorrectly	2.0	0.0	2.2	0.0
Unidentified process	**9.8**	**100.0**	**5.9**	**100.0**
No process	9.8	100.0	5.9	100.0
Incomplete response	**1.0**	**0.0**	**0.5**	**0.0**
No response	1.0	0.0	0.5	0.0

As reported in Table 6.4, the most common process used in the generalisable process group was *the repeated operation process.* Of English pupils 76.5% used this process and of those 97.4% gained the correct answer. The corresponding percentages for Thai pupils were 68.6% and 100%. Most of English and Thai pupils who used this process showed the processes as

"two times table"
"the pattern is going up in 2s" and
"increase 2 each time".

The second most common was *the draw or count process.* Of English pupils 5.9% used this process and of those 100% gained the correct answer. The corresponding percentages for Thai pupils were 20.0% and 97.3%. Processes showed by pupils included,

"count 2 more"
"draw 5^{th} pattern" and
"draw one dot more each side".

In the other process group, 2.0% of English pupils and 2.2% of Thai pupils *drew or counted incorrectly.* The processes they showed included,

"The top row has the ratio 1:2 and the side ratio 1:3, 6+7=13",
"1×5 = 5".

In the unidentified process group, 9.8% of English pupils gave the answer without showing working and of those 100% gained the correct answer. The corresponding percentages for Thai pupils were 5.9% and 100%.

In the incomplete response group, only 1.0% of English pupils made no attempt at this question. The corresponding percentage for Thai pupils was 0.5%.

6.3.4 Process used and outcomes for theme 1 level 2 item 7

The level 2 item 7 "Fill in the blanks in this sequence 1, 2, 4, 8, 16, 32, ..., ..." was designed to provide information on how they worked out the next formula from number sequences (consecutive). As before, pupils' processes were categorised as generalisable process, other process, unidentified process and incomplete response.

Table 6.5 Percentage of process used and outcomes for theme 1 level 2 item 7

Processes Theme 1 Level 2 (7)	English school		Thai school	
	Used	%correct	Used	%correct
Generalisable process	**74.8**	**94.8**	**82.8**	**96.8**
Repeated operation	74.8	94.8	81.7	96.7
Draw or count	0.0	0.0	1.1	100.0
Other process	**1.0**	**0.0**	**5.4**	**0.0**
Repeated operation-like	0.0	0.0	3.8	0.0
Draw or count incorrectly	1.0	0.0	1.6	0.0
Unidentified process	**12.6**	**69.2**	**5.4**	**50.0**
No process	12.6	69.2	5.4	50.0
Incomplete response	**11.7**	**0.0**	**6.5**	**0.0**
No response	11.7	0.0	6.5	0.0

As shown in Table 6.5, the most common process used in the generalisable process group was *the repeated operation process.* Of English pupils 74.8% used *the repeated operation process* and of those 94.8% gained the correct answer. The corresponding percentages for Thai pupils were 81.7% and 96.7%. Most of the English and Thai pupils who used this process showed their processes as

"double it each time"

"times 2 of the number before" and

"1+1 = 2, 2+2 = 4, 4+4 = 8, ..., 32+32 = 64, 64+64 = 128".

In the other process group, 3.8% of Thai pupils used *the repeated operation-like process.* Of English pupils 1.0% and of Thai pupils 1.6% *draws or counts incorrectly.* Some Thai pupils used *the repeated operation-like process* and showed their processes as

"8 times table", and "16+32 = 48, 16+48 = 74"

English and Thai pupils showed *the draw or count incorrectly* as

"increase 2 and then increase 8", and "just add 2 on".

In the unidentified process group, 12.6% of English pupils gave the answer without showing working and of those 69.2% gained the correct answer. The corresponding percentages for Thai pupils were 5.4% and 50.0%.

In the incomplete response group, 11.7% of English pupils and 6.5% of Thai pupils made no attempt at this question.

6.3.5 Process used and outcomes for theme 1 level 2 item 13b

The level 2 item 13b "How many dots are there in the 20th pattern?" was designed to provide information on how pupils worked out the formula from the sequence of numbers. As before, pupils' processes were categorised as generalisable process, other process, unidentified process and incomplete response.

Table 6.6 Percentage of process used and outcomes for theme 1 level 2 item 13b

Processes Theme 1 Level 2 (13b)	English school		Thai school	
	Used	**%correct**	**Used**	**%correct**
Generalisable process	**24.5**	**80.0**	**62.7**	**65.5**
Generalisation	11.8	91.7	8.6	93.8
Repeated operation	1.0	100.0	1.1	50.0
Draw or count	11.8	66.7	53.0	61.2
Other process	**45.1**	**2.2**	**23.8**	**0.0**
Generalisation-like	0.0	0.0	0.5	0.0
Repeated operation-like	15.7	0.0	10.3	0.0
Scaling up	27.5	3.6	8.6	0.0
Draw or count incorrectly	2.0	0.0	4.3	0.0
Unidentified process	**22.5**	**17.4**	**8.6**	**31.3**
No process	22.5	17.4	8.6	31.3
Incomplete response	**7.8**	**0.0**	**4.9**	**0.0**
No response	7.8	0.0	4.9	0.0

As can be seen in Table 6.6, the most common processes used among English pupils in the generalisable process group were *the generalisation* and *the draw or count processes.* Of English pupils 11.8% used *the generalisation process* and of those 91.7% gained the correct answer. Another 11.8% of English pupils used *the draw or count process* and of those 66.7% gained the correct answer. The corresponding percentages for Thai pupils were 53.0% and 61.2%. The majority of English pupils who used *the draw or count process* showed their processes as "keep adding 2", and "12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42".

Thai pupils showed *the draw or count processes* as

"1 4, 2 6, 3 8, 4 10, 5 12, 6 14, 7 16, 8 18, 9 20, 10 22,..., 19 40, 20 42"

"increase 2 each time" and "count on in 2s".

In the other process group, 27.5% of English pupils used *the scaling up process* and of those 3.6% gained the correct solution. Of Thai pupils 10.3% used *the repeated operation-like process* of which none gained the correct solution. As an example, English pupils showed *the scaling up process* as

"times 12 dots from 5^{th} pattern by 4",

"double 10^{th} pattern" and

"$5^{th} = 12$, $10^{th} = 22$, $15^{th} = 32$, $20^{th} = 42$".

Thai pupils showed *the repeated operation-like process* as

"times term by 2", and

"times 1^{st} pattern by 20"

In the unidentified process group, 22.5% of English pupils gave the answer without showing working and of those 17.4% gained the correct answer. The corresponding percentages for Thai pupils were 8.6% and 31.3%.

In the incomplete response group, 7.8% of English pupils and 4.9% of Thai pupils made no attempt at this question.

6.3.6 Process used and outcomes for theme 1 level 2 item 19a

In the level 2 item 19a "The 7^{th} term of this sequence 2, 5, 8, 11, 14, 17, ... is ..." was designed to provide information on how they worked out the next formula from sequence of numbers. As before, pupils' processes were categorised as generalisable process, other process, unidentified process and incomplete response.

Table 6.7 Percentage of process used and outcomes for theme 1 level 2 item 19a

Processes Theme 1 Level 2 (19a)	English school		Thai school	
	Used	**%correct**	**Used**	**%correct**
Generalisable process	**76.0**	**97.4**	**87.6**	**95.9**
Generalisation	2.0	100.0	0.6	100.0
Repeated operation	74.0	97.3	69.8	94.9
Draw or count	0.0	0.0	17.2	100.0
Other process	**2.0**	**100.0**	**4.7**	**62.5**
Generalisation-like	1.0	100.0	0.0	0.0
Repeated operation-like	1.0	100.0	3.0	60.0
Draw or count incorrectly	0.0	0.0	1.8	66.7
Unidentified process	**17.0**	**100.0**	**5.3**	**88.9**
No process	17.0	100.0	5.3	88.9
Incomplete response	**5.0**	**0.0**	**2.4**	**0.0**
No response	5.0	0.0	2.4	0.0

As shown in Table 6.7, the most common process used in the generalisable process group was *the repeated operation process.* Of English pupils 74.0% used *the repeated operation process* and of those 97.3% gained the correct answer. The corresponding percentages for Thai pupils were 69.8% and 94.9%. A large number of English and Thai pupils showed their processes as

"It is going up in 3s", "Add on 3" and "increase 3 each time".

In the other process group, 3.0% of Thai pupils used *the repeated operation-like process* and of those 60.0% gained the correct answer. Of English pupils 1.0% used *the generalisation-like process,* the other 1.0 % used *the repeated-like process* and of those all gained the correct answer. One English pupil showed *the generalisation-like process* as "2*n*+ number of term before".

Pupils tended to use *the repeated operation-like process* as

"times 3 every time", and "times 3 seven times".

In the unidentified process group, 17.0% of English pupils gave the answer without showing working and of those 100% gained the correct answer. The corresponding percentages for Thai pupils were 5.3% and 88.9%.

In the incomplete response group, 5.0% of English and 2.4% of Thai pupils made no attempt at this question.

6.3.7 Process used and outcomes for theme 1 level 3 item 13c

The level 3 item 13c "How many dots are there in the n^{th} pattern?" was designed to observe how pupils worked out the formula from a physical pattern in general form. As before, pupils' responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.8 Percentage of process used and outcomes for theme 1 level 3 item 13c

Processes Theme 1 Level 3 (13c)	English school		Thai school	
	Used	%correct	Used	%correct
Generalisable process	**7.8**	**100.0**	**5.9**	**81.8**
Generalisation	7.8	100.0	5.9	81.8
Other process	**16.7**	**0.0**	**24.9**	**0.0**
Generalisation-like	0.0	0.0	1.1	0.0
Repeated operation-like	10.8	0.0	21.6	0.0
Draw or count incorrectly	5.9	0.0	2.2	0.0
Unidentified process	**16.7**	**17.6**	**13.0**	**12.5**
No process	16.7	17.6	13.0	12.5
Incomplete response	**58.8**	**0.0**	**56.2**	**0.0**
No response	58.8	0.0	56.2	0.0

Table 6.8 showed that the generalisable process group, 7.8% of English pupils used *the generalisation process* and of those 100% gained the correct answer. The corresponding percentages for Thai pupils were 5.9% and 81.8%. Generally, pupils showed *the generalisation process* as

"times the n by 2 and add 2",

"2×2+2 = 6, 2×3+2 = 8, 2×4+2 = 10, 2×5+2 = 12", and

"$2n$+2".

In the other process group, 10.8% of English pupils and 21.6% of Thai pupils used *the repeated operation-like process.*

For example, pupils showed *the repeated operation-like process* as

"Adding on 2 each time",
"times the pattern by 2",
"n = 14 (in English consonants), 13th =28, 14th =28+2=30".

In the unidentified process group, 16.7% of English pupils gave the answer without showing working and of those 17.6% gained the correct answer. The corresponding percentages for Thai pupils were 13.0% and 12.5%.

In the incomplete response group, 58.8% of English pupils and 56.2% Thai pupils made no attempt at this question.

6.3.8 Process used and outcomes for theme 1 level 4 item 19b

This level 4 item 19b "The n^{th} term of this sequence 2, 5, 8, 11, 14, 17, ... is ..." was designed to examine how pupils worked out the formula from number sequence in general form. As before, pupils' responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.9 Percentage of process used and outcomes for theme 1 level 4 item 19b

Processes Theme 1 Level 4 (19b)	English school		Thai school	
	Used	**%correct**	**Used**	**%correct**
Generalisable process	**1.0**	**50.0**	**2.4**	**100.0**
Generalisation	1.0	50.0	2.4	100.0
Other process	**28.3**	**0.0**	**31.7**	**0.0**
Generalisation-like	1.0	0.0	1.2	0.0
Repeated operation-like	22.2	0.0	22.8	0.0
Scaling up	0.0	0.0	0.6	0.0
Draw or count incorrectly	5.1	0.0	7.2	0.0
Unidentified process	**20.2**	**5.0**	**17.4**	**6.9**
No process	20.2	5.0	17.4	6.9
Incomplete response	**49.5**	**0.0**	**48.5**	**0.0**
No response	49.5	0.0	48.5	0.0

As reported in Table 6.9, in the generalisable process group, 2.0% of English pupils used *the generalisation process* and of those 50.0% gained the correct answer. The corresponding percentages for Thai pupils were 2.4% and 100%. Pupils were likely to show *the generalisation process* as "times by three -1" and "3*n*-1".

In the other process group, 22.2% of English pupils and 22.8% of Thai pupils used *the repeated operation-like process.*

For instance, the pupils showed *the repeated operation-like process* as

"adding on 3s","increase 3 each time", and "going up in 3 twice more".

In the unidentified process group, 20.2% of English pupils gave the answer without showing working and of those 5.0% gained the correct answer. The corresponding percentages for Thai pupils were 17.4% and 6.9%.

In the incomplete response group, 49.5% of English pupils and 48.5% of Thai pupils made no attempt at this question.

6.4 Summary and discussion of findings Theme 1

For English pupils in the generalisable process group, the tendency was to use *the repeated operation process* in the level 1 (1a, 13a), and level 2 (7, 19a) questions. *The generalisation process* was used to extend the level 1 (1b), level 2 (13b) patterns, and to create the rules in the level 3 and level 4. The main processes used in tackling the level 1 and level 2 questions were in using *the inappropriate scaling up process* and *drawing or counting incorrectly. The repeated-like process* was commonly used in the level 3 and level 4 questions. The unidentified process group gave the answer without showing working. The incomplete response group in each of the eight questions comprised predominantly those who made no response at all.

Thai pupils in the generalisable process group also used *the repeated operation process* in the level 1 (1a, 13a), and level 2 (7, 19a) questions. In general, they used *the generalisation process* in the level 1 (1b), level 3, and level 4. The other process group commonly used *the drawing or counting process incorrectly* in the level 1 questions. They frequently used *the repeated operation-like process* in the level 2, level 3, and level 4 questions. The unidentified process group gave the answer without showing working.

Those in the incomplete response group mainly made no response at all or did not reach the questions because of the limit of time.

The results generally showed that English and Thai pupils used similar processes to approach the problems. The empirical data suggested that majority of pupils had the basic concept of continuing patterns/sequences (levels 1 and 2). Only a few pupils succeeded to construct a rule (levels 3 and 4).

6.4.1 Using other process to obtain the correct solution

In items 1a and 1b, some English pupils used *the scaling up process* in which the number of matchsticks in pattern kn was taken to be k times the number of matchsticks in pattern n. For example, number of matchsticks in pattern 10 is 5 times number of matchsticks in pattern 2, and number of matchsticks in pattern 20 is 4 times number of matchsticks in pattern 5. Some pupils also noted the increment of 3 from one pattern to the next. With these two ideas they were able to generate the correct answer. Typical of their responses were:

item 1a "4th is double 2nd"

item 1b "add 3 matchsticks to the 4th pattern and double it",

"the 2nd pattern has 6 matchsticks and 6×5 = 30", and

"15 matchsticks = 5th pattern times 2 = 10th".

The use of this process resulted in the correct answer because it was a question in which the number of matchsticks was indeed a multiple of the pattern number. However, this approach failed in item 13 because the number of dots was not a multiple of the pattern number. Some pupils were able to make an adjustment to achieve the correct answer. For instance, item 13b "Double the number of dots in the 10th pattern then minus 2 dots" was the explanation given by one pupil to achieve the correct answer. In this case pupils not only used *the inappropriate scaling up process* but also tested the solution as well. Linchevski et al. (1998) stated the similar result where pupils use of "seductive numbers" in a sequence like n = 5, 20, and 100 stimulated the error and found that pupils were not aware of the role of the database in the process of generalisation and validation.

The explanations of the process used for solving item 19a included "2n+number of term before" (*generalisation-like*), and "Times 3 every time" (*repeated operation-like*) for which the correct solution was obtained. The evidence shows that pupils were able to continue the sequence and attempted to explain their rules but without being fully correct.

Thai pupils who used the other process and gained the correct solution explained their processes to item 1b as "$1^{st} = 3$, $2^{nd} = 6$, $3^{rd} = 9$, $10^{th} = \frac{9}{3} \times 10$" (*generalisation-like*), and "5^{th} pattern is 15, then double it" (*scaling up*). The first explanation indicated the use of pattern 3 to gain the correct answer. The second explanation obtained the correct answer because the number of matchsticks was a multiple of the pattern number as mentioned above. On item 19a, explanations were "add 2 each time" (*repeated operation-like*), and "count 2 each time" (*draw or count incorrectly*). This clearly suggests that pupils were able to continue the sequence but explained their rules incorrectly.

6.4.2 The increased in using other process among English pupils at level 2 theme 1 item 13b

As reported in Table 6.6, 11.7% of English pupils in the generalisable process group used *the generalisation process,* and another 11.7% used *the draw or count process.* Of English pupils 27.2% used *the scaling up process* and thus fell into the other process group. English pupils were more likely to use *the scaling up process* than *the repeated operation process.* They may have seen that the number of dots in the pattern was not an exact multiple of the pattern number or used the process that was successful with the earlier item (1b).

Noticeably, English pupils were more willing to look for short cuts to achieve the solution because they were not prepared to spend a long time in carrying out *the draw or count process.* It is also possible that English pupils felt that this *draw or count process* was too basic and not "proper" mathematics. Zazkis and Liljedahl (2002a, 2002b) reported similar results in their investigations of arithmetic sequences with pre-service elementary school teachers. In these studies pupils were provided with the first 4 or 5 elements in an arithmetic sequence and were asked to provide examples of large numbers in this sequence and to determine whether certain numbers belonged to the sequence if it continued infinitely. The direct proportion (scale factor) approach, appropriate to a sequence of multiples (e.g., 3, 6, 9, 12,...), was also extended and applied to sequences of so called 'non-multiples' (e.g., 2, 5, 8, 11,...).

By contrast, Thai pupils in the generalisable process group 53.0% used *the draw or count process.* They used this process although it takes a long time. It seems that it is more important for these pupils to get the correct answer than it is to use a more advanced approach. It could be argued that Thai pupils have no experience of generalised patterns/sequences lessons at all. They attempted to make sense in the new context using prior knowledge as mentioned by MacGregor and Stacey (1997), and Blanton and Kaput (2000).

6.4.3 The large drop from theme 1 level 2 to levels 3 and 4

"Algebra in Key Stage 3 is generalised arithmetic" (DfEE, 2001, p. 14). Using generalised arithmetic ideas through number patterns was introduced in the early algebra lessons in the English school, 5 of 20 algebra lessons to Year 7 top set and 2 of 17 algebra lessons to Year 7 bottom set. There were no patterns and sequences lessons in Year 8. In the Thai school there were no patterns and sequences lessons in either secondary 1 or secondary 2 (see Chapter 2).

The basic concepts of patterns and sequences were taught in the primary school (Year 5, 6) in both countries. In England, the n^{th} term rules should be covered in Year 7 and 8 as stated in the National Numeracy Strategy: Framework for teaching mathematics Year 7, 8, and 9. In practice, as mentioned earlier, lessons were taught early in Year 7 and none in Year 8. In the case of Thailand this topic does not appear until Secondary 4 (Year 10). The results confirm Blanton and Kaput (2000) who believed that pupils' elementary school algebra experience should extent beyond arithmetic proficiency to support the more complex mathematics.

Not surprisingly, most pupils in both countries could not reach levels 3 and 4 of this theme. Levels 3 and 4 were designed to investigate the processes of pupils' thinking as they search for a general rule. This result indicates pupils' ability to continue the pattern and arithmetic sequence but not to generalise a rule or find 'the n^{th} term'. For example, explanations on items 13c and 19b were "you can have any number you like", and "n = 14th" ($a = 1$, $b = 2$, $c = 3$, …, $m = 13$, $n = 14$). The first comment reflects the experience of hearing expressions such as "n can be any number" and the second is merely the numeric ordering of the letters in the English alphabet. It has been noted that many pupils have difficulty viewing a letter as generalised number or unknown (Küchemann, 1978, 1981; Kieran, 1992). MacGregor and Stacey (1997) also suggest that pupils attempt to make sense of a new notation by transfer of meanings from other contexts are not indicative of low level of cognitive development.

Orton and Orton (1999) reported similar results when they investigated pupils' patterning abilities and found that the ability to continue a pattern comes well before the ability to describe the general term. Lee (1996) noted that students participating in her study had difficulty, not with spotting a pattern, but with recognising an algebraically useful pattern.

Threlfall and Frobisher (1999) state performing generalisation of pattern is significant in study of mathematics. A clear understanding in this patterns/sequences theme at the early stage in learning algebra is necessary. To help the novice, more emphasis on the bridging from arithmetic to algebra has to be cultivated carefully. Ignorance of this stage might cause pupils more difficulty at higher level of algebra.

6.5 Theme 2 Simplification

The second theme of the test, simplification, is the process of adding and subtracting like terms in an expression. Like terms are those having exactly the same letters and exponents. They may differ only in their coefficients. This theme was tested using four questions, designed to observe the pupils' thinking processes as they manipulated the like terms in different forms of expression. The questions are shown in figure 6-4.

Simplification

Item 2 Simplify the expression $2a - a + 3a$. *(Level1 simplify one variable)*

Item 8 Simplify the expression $6 + 3b - c - 6b - c + 2$. *(Level2 simplify two variables)*

Item 14 Simplify $3p + 5(p-3) - 2(q-4)$. *(Level3 simplify two variables with brackets)*

Item 20 Multiply out the bracket and then simplify $x^2 + 2xy - 3(xy - 2x^2)$. *(Level 4 simplify two variables with second order and brackets)*

Figure 6-4 Simplification test items

The thinking processes in simplifying the algebraic expressions were categorised from participants' responses as *generalisable process, other process, unidentified process* and *incomplete response.*

Generalisable processes are the methods that showed the correct way to simplify like terms in the expression and multiply out the brackets whether they obtained the correct answer or not.

Other processes are those in which pupils attempt to simplify unlike terms, omit brackets, multiply only the first term in the brackets on attempt to set up an equation or carry out substitution. In these processes, they obtained the incorrect answers.

The *unidentified process* and the *incomplete response* are as defined earlier.

6.6 A comparison of pupil's thinking processes in simplifying algebraic expressions between the English and Thai schools

Figures 6-5 and 6-6 show the percentage of processes used by the English and Thai pupils in approaching theme 2 at each level of difficulty.

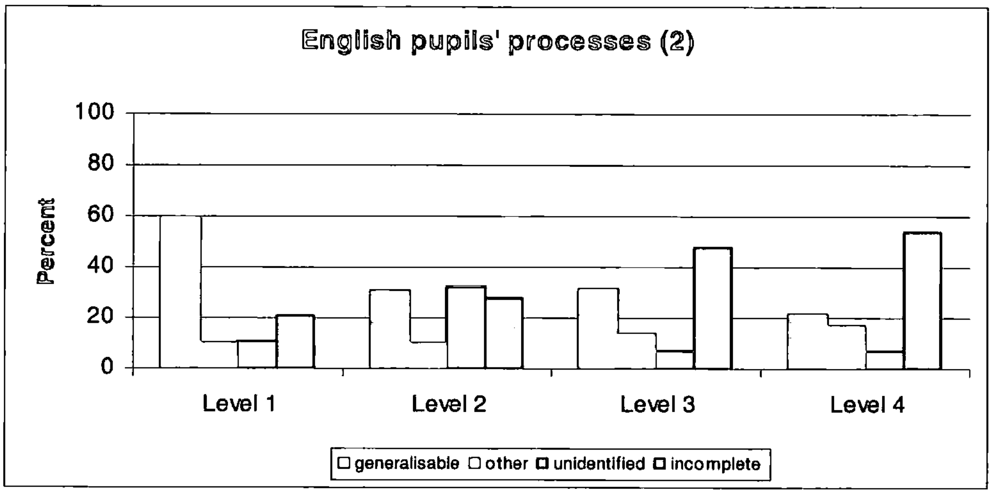


Figure 6-5 Percentage of process used in theme 2 by English pupils

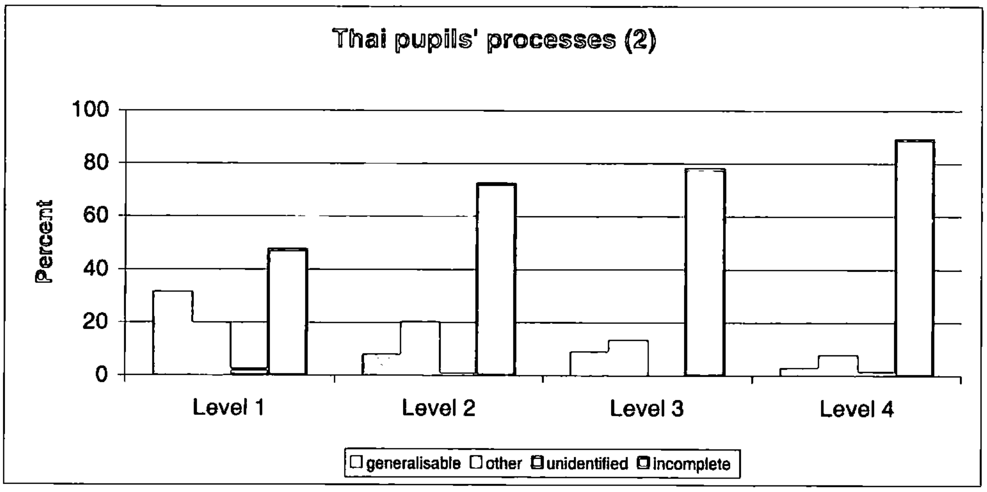


Figure 6-6 Percentage of process used in theme 2 by Thai pupils

As illustrated, a small number of Thai pupils used the generalisable process to simplify the expressions. More than 50% of English pupils used the generalisable process for level 1 but had far less success at the higher levels.

Table 6.10 gives the actual percentage of each process and corresponding outcomes at each level.

Table 6.10 Percentage of process used and outcomes for theme 2

Country	Level (item)	Processes							
		Correct conception		Misconception		Unidentified process		Incomplete response	
		Used	% correct	Used	% correct	Used	% correct	Used	% correct
England (n=103)	1 (2)	59.2	96.7	9.7	0.0	10.7	54.5	20.4	0.0
	2 (8)	30.4	51.6	9.8	0.0	32.4	27.3	27.5	0.0
	3 (14)	31.7	40.6	13.9	0.0	6.9	0.0	47.5	0.0
	4 (20)	21.9	33.3	16.7	0.0	7.3	0.0	54.2	0.0
Thailand (n=186)	1 (2)	31.2	84.5	19.4	0.0	2.2	25.0	47.3	0.0
	2 (8)	7.5	57.1	19.9	0.0	0.5	0.0	72.0	0.0
	3 (14)	8.6	33.3	13.0	0.0	0.0	0.0	77.8	0.0
	4 (20)	2.5	50.0	7.4	0.0	1.2	0.0	89.0	0.0

Table 6.10 reported that 59.2% of English pupils and 31.2% of Thai pupils used the generalisable process to solve the level 1 question. There was a large drop between level 1 and level 2 of those making up the generalisable process group in both countries. Of English pupils, 30.4%, and of Thai pupils, 7.5% used generalisable process to approach the level 2 question. There was a minimal increase to 31.7% among English pupils, and increase to 8.6% among Thai pupils, using the generalisable process to solve the level 3 question. For the level 4 question, Thai pupils used the generalisable process in only 2.5% of cases compared with English pupils in 21.9% of cases. The details of each process are described in the next section.

The following sections describe the sub-processes, which pupils used at each level of difficulty.

Within the generalisable process group there are 4 sub-processes:

(1) *The generalisable incorrect operation process* is working with different operations from those given in the question given or wrong order of operating.

(2) *The generalisable left to right computing* responds to a question as it set up by multiplying out brackets and then simplifying the first term with the next like term.

(3) *The letter temporary ignored computing process* refers to those who tried to work with coefficients only.

(4) *The plus to minus computing process* refers to those who deal with the positive term and then negative term.

There are 4 sub-processes within the other process group.

(1) *The other process incorrect operation*, shows the processes to omit the brackets or multiplied only the first term in the bracket, and minus sign confused.

(2) *The other process letter ignored computing* addresses the processes of computing only the numbers appeared in the expression, or simplifying unlike terms.

(3) *The other process grouping strategy* operates the terms inside and outside brackets separately.

(4) *The other process substitution*, in which a particular value is assumed and hence a numerical answer obtained.

The *unidentified process* and the *incomplete response* groups were as defined earlier.

6.6.1 Process used and outcomes for theme 2 level 1 item 2

The level 1 item 2 "Simplify the expression $2a$-a+$3a$" was designed to examine pupils' thinking processes when manipulating a one-variable expression. Pupils' responses were categorised into four groups as generalisable process, other process, unidentified process, and incomplete response.

Table 6.11 shows the percentage of processes used and percentage correct in the level 1 simplification question.

Table 6.11 Percentage of process used and outcomes for theme 2 level 1 item 2

Processes Theme 2 Level 1 (2)	**English school**		**Thai school**	
	Used	**%correct**	**Used**	**%correct**
Generalisable process	**59.2**	**96.7**	**31.2**	**84.5**
Incorrect operation	1.0	0.0	4.3	0.0
Letter temporary ignored	3.9	100.0	0.0	0.0
Left to right	48.5	100.0	25.3	100.0
Plus to minus	5.8	83.3	1.6	66.7
Other process	**9.7**	**0.0**	**19.4**	**0.0**
Incorrect operation	1.0	0.0	5.9	0.0
Letter ignored	6.8	0.0	5.4	0.0
Substitution	1.9	0.0	8.1	0.0
Unidentified process	**10.7**	**54.5**	**2.2**	**25.0**
No process	10.7	54.5	2.2	25.0
Incomplete response	**20.4**	**0.0**	**47.3**	**0.0**
Incomplete	15.5	0.0	2.7	0.0
No response	4.9	0.0	44.6	0.0

As shown in Table 6.11, the most common process used in the generalisable process group was *the generalisable left to right process.* English and Thai pupils using this process all gained the correct answer. For example, the pupils showed *the left to right computing process* as

"($2a$-a) = 1a+3a = 4a", and

"2a-a = a, a+3a = 4a".

In the other process group, 6.8% of English pupils *ignored the letters* while 8.1% of Thai pupils used *the other process substitution.* For example, English pupils illustrated *the letter ignored process* as

"2a-a = 1a+3 = 4 (letter ignored), and

"2a-a = 2+3 = 5+a = 5a (number ignored, letter ignored, incorrect operation)".

Thai pupils showed this process as

"2a-a+3a, 5a-a, a-a, a = 5 (plus, number ignored, numerical answer)"and

"2a-a+3a = 2+3 = 5 (letter ignored, combined numbers appear in the expression)".

The other process substitution was common among Thai other process group. They responded to the question as

"(2×1)-1+3×1 = (2-1)+3=1+3=4 (substitute a = 1)",

"(2×2)-2+(3×2) = (4-2)+5 = 2+5 = 7 (substitute a = 2)", and

"(2×4)-4+(3×4) = 8-4+12 = 4+12 = 16 (substitute a = 4)".

In the unidentified process group, 10.7% of English pupils gave the answer without showing working and of those 54.5% gained the correct answer. The corresponding percentages for Thai pupils were 2.2% and 25.0%.

In the incomplete response group, a large number (44.6%) of Thai pupils made no attempt. Of English pupils 15.5% made only a partial attempt. English pupils in this group attempted to simplify as

"$2a+3a$, $5a-a$",

"$2a-a = a$, $a+3a$", and

"$3a(2a-a)$".

The results indicate that about half of English and only around a third of Thai pupils had abilities to simplify like terms. This suggests that they are likely to have even more problems on the harder level of difficulty.

6.6.2 Process used and outcomes for theme 2 level 2 item 8

The level 2 item 8 "Simplify the expression $6+3b-c+2$" was designed to investigate how pupils manipulate a two-variable expression. As before, pupils' responses were categorised as generalisable process, other process, unidentified process, and incomplete response.

Table 6.12 shows the percentage of processes used and percentage correct in the level 2 simplification question.

Table 6.12 Percentages of processes used and outcomes for theme 2 level 2 item 8

Processes Theme 2 Level 2 (8)	**English school**		**Thai school**	
	Used	**%correct**	**Used**	**%correct**
Generalisable process	**30.4**	**51.6**	**7.5**	**57.1**
Incorrect operation	6.9	0.0	2.7	0.0
Left to right	1.0	100.0	0.0	0.0
Grouping	22.5	65.2	4.8	88.9
Other process	**9.8**	**0.0**	**19.9**	**0.0**
Incorrect operation	1.0	0.0	3.8	0.0
Letter ignored	7.8	0.0	13.4	0.0
Substitution	1.0	0.0	2.7	0.0
Unidentified process	**32.4**	**27.3**	**0.5**	**0.0**
No process	32.4	27.3	0.5	0.0
Incomplete response	**27.5**	**0.0**	**72.0**	**0.0**
Incomplete	5.9	0.0	1.6	0.0
No response	21.6	0.0	70.4	0.0

As shown in Table 6.12, the most common process used in the generalisable process group was *the grouping process*. Of English pupils 22.5% used this process with of those 65.2% gaining the correct answer. The corresponding percentages for Thai pupils were 4.8% and 88.9%. For example, the pupils showed *the grouping process* as "6+2+3*b*-6*b*-*c*-*c*" and then simplified them. The less successful pupils tended to make the incorrect simplification of -*c*-*c*, which ignored the first minus sign. They simplified *c*-*c* = 0 instead of -*c*-*c* = -2*c*. These responses indicate the error in arithmetic rather than algebra itself.

In the other process group, 7.8% of English pupils and 13.4% of Thai pupils showed their processes as *the other process letter ignored*. Most of them tended to combine the first two terms and then compute the rest. For instance, they addressed the processes as "6+3*b* = 9*b*-*c* = 8*b*-6*b* = 2*b*-*c* = 1*b*+2 = 3*b*".

Only Thai pupils used *the other process incorrect operation* by treating as an equation and attempting balancing. For example, Thai pupils showed the processes as "6+3*b*-*c*-6*b*-*c*+2, 8-3*b* (simplify like term, minus sign confused as *c*-*c* = 0), $\frac{-3b}{-3}=\frac{8}{-3}$ (set up an equation, balancing confused),

$b=-2\frac{2}{3}$", and

"6+3-*c*-6-*c*+2 (cancelling *b*),

6+3-*c*+*c*-6-*c*+*c*+2 (balancing confused),

6+3- -6- +2 (minus sign confused),

6+3+6+2 = 9+6+2 = 17" (a numerical answer).

In the unidentified process group, 32.4% of English pupils gave the answer without showing working and of those 27.3% gained the correct answer. The corresponding percentages for Thai pupils were 0.5% and 0.0%.

In the incomplete response group, 21.6% of English pupils and 70.4% Thai pupils made no attempt to this question. English pupils who attempted to simplify this problem and did not reach completion showed the processes as

"3*b*-6*b*-*c*-*c*+6+2 (grouping like things),

-3*b*-*c*-*c*+8", and

"9×*b*-*c*-6×*b*".

Otherwise, Thai pupils showed the processes as

"9*b*-*c*-6*b*-*c*+2=3*b*-*c*-*b*-*c*+2", and

"3*b*-6*b*-*c*-*c*+2+6".

These results confirm difficulties pupils tend to have in simplifying algebraic expressions with negative signs.

6.6.3 Process used and outcomes for theme 2 level 3 item 14

The level 3 item 14 "Simplify 3p+5(p-3)-2(q-4)" was designed to observe how pupils multiply out the brackets and simplify expression. As before, pupils' responses were categorised into four groups as generalisable process, other process, unidentified process, and incomplete response.

Table 6.13 shows the percentage of processes used and percentage correct in the level 3 simplification question.

Table 6.13 Percentage of process used and outcomes for theme 2 level 3 item 14

Processes Theme 2 Level 3 (14)	English school		Thai school	
	Used	%correct	Used	%correct
Generalisable process	**31.7**	**40.6**	**8.6**	**33.3**
Incorrect operation	29.7	36.7	6.3	9.1
Left to right	2.0	100.0	2.3	100.0
Other process	**13.9**	**0.0**	**13.0**	**0.0**
Incorrect operation	1.0	0.0	3.4	0.0
Letter ignored	8.9	0.0	3.4	0.0
Grouping	3.0	0.0	2.8	0.0
Substitution	1.0	0.0	3.4	0.0
Unidentified process	**6.9**	**0.0**	**0.0**	**0.0**
No process	6.9	0.0	0.0	0.0
Incomplete response	**47.5**	**0.0**	**77.8**	**0.0**
Incomplete	4.0	0.0	3.4	0.0
No response	43.5	0.0	74.4	0.0

Table 6.13 show that the most common process used in the generalisable process group was *the incorrect operation process*. Of English pupils 29.7% did this and of those 36.7% fortuitously gained the correct answer. The corresponding percentages for Thai pupils were 6.3% and 9.1%. Most pupils in the generalisable process group showed the process with *the incorrect operation process*. The confusion they faced was in working with negative signs such as

"$3p+5p-15-2q-8 = 8p-7-2q$", and

"$3p+5p-15-2q-8 = 8p-2q-23$".

The first strategy, which gained the correct answer ignored the first minus sign and then computed 15-8 = 7. The second method gained the incorrect answer with correct operated as -15-8 = -23. These results indicate lack of understanding of working with negative numbers as shown by pupils in simplifying the level 2 problem.

In the other process group, 8.9% of English pupils used *the letter ignored process*. For Thai pupils, the percentage in each of *incorrect operation, letter ignored, and substitution* *process* was 3.4%. The common process used within the other process group in both countries was *the letter ignored computing*. For instance, they showed the processes as "$3p+5\times-3p-2\times-4q$, $3p-15p-8q$" and "$3p+(5p-15)-(2q-8)$, $3p+ -10p-(-6q)$, $-7p-(-6q)$, $-7p+6q$".

The first example indicated the process of summing the terms in the brackets first as $p-3 = -3p$, $q-4 = -4q$ and then sum the rest with minus sign confusion. The second one showed ignorance of letters when they summed the terms in brackets.

In the unidentified process group, 6.9% of English pupils gave the answer without showing working and of those none gained the correct answer.

In the incomplete response group, 43.5% of English pupils and 74.4% of Thai pupils made no attempt to this question. For instance, English pupils showed the incomplete working as
"$3p+5p-15-2q-8 = 3p+5p-15-8-2q$", and
"$3+5p-15-2q-8$".

Thai pupils showed incomplete working as

"$3p+5p-15-2q-8 = 8p-15-2q-8$" and
"$3p+(5p-15)-(2q-8)$".

The results indicate the problems pupils had with multiplying out brackets and computing negative numbers. Some pupils viewed "p" as "q", and vice versa.

6.6.4 Process used and outcomes for theme 2 level 4 item 20

The level 4 item 20 "Multiply out the bracket and then simplify $x^2+2xy-3(xy-2x^2)$" was designed to gain insight into how pupils multiply out the brackets and simplify the like terms in different forms of two variables. Again, pupils' responses were categorised into three groups as generalisable process, other process, unidentified process, and incomplete response.

Table 6.14 shows the percentage of processes used and percentage correct in the level 4 simplification question.

Table 6.14 Percentage of process used and outcomes for theme 2 level 4 item 20

Processes Theme 2 Level 4 (20)	English school		Thai school	
	Used	%correct	Used	%correct
Generalisable process	**21.9**	**33.3**	**2.5**	**50.0**
Incorrect operation	18.8	22.2	1.2	0.0
Left to right	3.1	100.0	1.2	100.0
Other process	**16.7**	**0.0**	**7.4**	**0.0**
Incorrect operation	13.5	0.0	5.5	0.0
Substitution	3.1	0.0	1.8	0.0
Unidentified process	**7.3**	**0.0**	**1.2**	**0.0**
No process	7.3	0.0	1.2	0.0
Incomplete response	**54.2**	**0.0**	**89.0**	**0.0**
Incomplete	4.2	0.0	3.1	0.0
No response	50.0	0.0	85.9	0.0

As can be seen in Table 6.14, the common process used in the generalisable process group was *the incorrect operation.* In both the generalisable process and other process groups, pupils worked the question with *the incorrect operation process.* Of English pupils in the generalisable process group 18.8% used *the incorrect operation* and of those 22.2% gained the correct answer. For instance, the generalisable process group showed the processes with *the incorrect operation* as

"x^2+2xy-3xy-6x^2, 7x^2+ -xy"

"x^2+2xy-3xy-6x^2, -5x^2+ -xy" and

"x^2+2xy-3xy-6x^2, x^2-6x^2+2xy-3xy, 2x-12x+5xy, 10x+5xy".

The first example gained the correct answer with twice minus sign confused when multiplying out the brackets and when simplifying like terms. The second example gained the wrong answer with one error with the minus sign when expanding brackets. The third one showed confusion not only with negative signs but also indices.

The other process group addressed the process with *the incorrect operation* as

"-x^2-4x^4, 2y-3y = y, 2x-4x^6-y" and

"$x^2+2x\times2y-(3x\times3y-6x^2)$, $7x^4+(-x)\times(-y)$, $7x^4+xy$, $8x^4+y$".

The first example showed the processes of simplifying as

"$x^2+2x+2y-3x-3y-4x^4$, $x^2-3x= -x^2$".

The second one showed that they saw $2xy$ as $2x$ times $2y$ and the same for $3xy$, and then ignored the multiply signs when attempting simplification.

In the unidentified process group, 7.3% of English and 1.2% of Thai pupils gave the answer without showing working and of those none gained the correct answer.

In the incomplete response group, many pupils in both countries made no attempt to answer this question, 50.0% of English and 85.9% of Thai pupils. For example, English pupils showed the incomplete works as

"x^2+, x^2-1xy, $6x^2-1xy$", and

"$x^2+2xy-3xy-6x^2= x^2-1xy-6x^2$".

Thai pupils showed the process as

"$x^2+2xy-3xy+6x^2$", and

"$x^2+2xy-(3xy-6x^2)$".

The less successful pupils attempt in solving level 4 item 20 included multiplying out the bracket and then simplifying $x^2+2xy-3(xy-2x^2)$. This seems to confirm pupils' inabilities to deal with the like terms and with negative signs.

6.7 Summary and discussion of findings Theme 2

English pupils' processes: the generalisable process group commonly used *the left to right process* to simplify the level 1 question. *The generalisable grouping process* was mainly used to approach the level 2 item. These pupils frequently used *the incorrect operation* at the level 3 and level 4 questions. The other process group commonly used *the letter ignored process* to solve the level 1, 2, and 3 questions. They primarily made *the other process incorrect operation* at the level 4. Pupils in the incomplete response group frequently made *the incomplete works* to the level 1 question. They commonly gave the answer without working on the level 2 and made *no response* at the level 3 and 4.

Thai pupils' processes: the generalisable process group commonly used *the left to right process* to simplify the level 2 question. *The generalisable grouping process* was frequently used to approach the level 2 problem. They primarily made *the generalisable incorrect operation process* on level 3 and level 4. The other process group commonly used *the other process substitution* to simplify the level 1 problem. *The other process letter ignored* was mainly used to approach the level 2 question. These pupils frequently used *the other process incorrect operation* to deal with the level 3 and level 4 expressions. Those in the incomplete response group frequently made *no response* at all questions.

The results indicate that a large number of pupils in both countries made mainly incomplete responses to the level 2, level 3, and level 4 questions.

From the results, it can be seen that English and Thai pupils in the generalisable process group used similar processes to simplify the expressions. The main difficulties were again dealing with like terms and with negative signs.

6.7.1 The other process used at theme 2 level 1 (simplify $2a-a+3a$)

The other process not seen in the English pupils' responses but which appeared in the work of Thai pupils was *the other process "incorrect operation"* in which they tried to **set up an equation**. *The substitution process* was commonly used among Thai pupils in the other process group. This reflects the taught experiences in the Thai school, where the algebra content was introduced by work on solving equations. The Thai curriculum delivered the solving of equations **without** the concept of simplifying like terms. The process of simplification has been ignored and the balancing process of operating equally on both sides was used in solving equations. This led to the use of other process in simplifying like terms among Thai pupils.

The result is similar to that of Linnecor (1999) who found that pupils believe answers should always be single term and numerical when asking them to collect terms and substitute in values. It also supports Wagner, Rachlin and Jenson (1984) who find that pupils tried to add "= 0" to the expression they were asked to simplify. Thai pupils complained that the problems were incomplete and asked the invigilator, during the test, for the item to be completed by addition of "= a number" on the right hand side. In contrast, the English pupils were able to work with the expression. There were some incomplete responses such as "$5a$-a"and "a+$3a$", which showed the first step towards the correct solution. The other process among English pupils was *letter ignored.* Some of them still have difficulty to accept "lack of closure" (Collis, 1975; Hoyles & Sutherland, 1992). Unlike Cooper, Williams and Baturo (1999b) who find that the link between arithmetic and algebra seemed generally successful for algebraic simplification.

6.7.2 Incorrect operations but obtained the correct solution

Level 3, item 14 "simplify $3p+5(p-3)-2(q-4)$". English and Thai pupils showed *the incorrect operation* but obtained the correct solution such as "$3p+5p-15-2q-8 = 8p-7-2q$". This happened among pupils in top sets. They viewed '-15-8' as '-(15-8)' and may have got the correct answer fortuitously. The problem with multiplying out the brackets was commonly confused when dealing with the negative numbers. Similarly, Booth (1989) stated that a major part of pupils' difficulties in algebra stems from the lack of understanding of arithmetic.

For level 4, item 20 "simplify $x^2+2xy-3(xy-2x^2)$ ". English pupils showed *the incorrect operation* but obtained the correct solution as" $3xy+6x^2-2xy+x^2=5xy+7x^2$ " (dealing with brackets first then write from **right to left** and thus introducing the negative sign when "$2xy$-" becomes "-$2xy$" with a further incorrect operation when simplifying $3xy$-$2xy$). Also seen was " $x^2+2xy-3xy-6x^2=7x^2+5xy$ " (multiply out brackets incorrectly [$-6x^2$] and incorrectly simplifying $2xy$-$3xy$ [= +$5xy$]).

6.7.3 The large drop from theme 2 level 1 to levels 2, 3, and 4

Level 2, item 8 "simplify the expression 6+3*b*-*c*-6*b*-*c*+2". The common confusion among English pupils was "-*c*-*c*" in which appeared as "-(*c*-*c*)" and got "-0". The evidence showed that they could simplify like terms but lacked understanding of operating with negative numbers.

Thai pupils asked for the values of variables, making comments such as "the problem does not tell the values of *b* and *c*". This clearly indicates they wanted to substitute the numbers instead of the letters. Moreover, the attempt to find the value of *b*, *c* or *bc* shows misconceptions about simplifying like terms. As mentioned earlier, Thai pupils had no experience in simplifying like terms. They attempted to use their experience of solving equations.

As stated in Section 6.7.2 item 14 (level 3), pupils from both countries were confused when operating with negative numbers in multiplying out the brackets. For example, many of them showed the process as "3*p*+5*p*-15-2*q*-8 = 8*p*-2*q*-15-8 = 8*p*-2*q*-23". Although the English pupils had experience in simplifying like terms and multiplying out the brackets, a high percentage (42.7%) made no response to this question. Thai pupils had no experience of these topics. It was not surprising that 71% of them made no attempt at this item. Some of them explained their reasons as "I could not find *p* unless I knew the value of *q*" or "I could not make it into an equation". They wanted to link with the solving of equations delivered in their lessons.

For item 20 (level 4), the English pupils in the top sets solved this question using *the incorrect operation process.* The common confusion arose in dealing with negative sign and powers when attempting to multiply out the brackets. Most pupils made no response to this question.

Similarly, Williams and Cooper (2001) state that the process of simplification is difficult for the pupils and is easily complicated by missing arithmetic components. Understanding of algebraic letters as unknowns or generalised numbers is important (Küchemann, 1981; Kieran, 1992). A clear understanding of this process is necessary. To help the novice, more emphasis on manipulating like terms and dealing with negative signs has to be cultivated carefully. Ignorance at this stage will cause pupils' difficulties in dealing with higher levels of algebra.

6.8 Theme 3 Substitution

The third theme of the test is that of substitution and it was organised into four levels of expected difficulty. It consisted of 4 questions, designed to observe the processes of pupils thinking as they substitute the numbers instead of the letters. The questions are shown in Figure 6-7.

Substitution

Item 3 If a=4, b=3, find the value of $a+5b$. *(Level 1 substitute positive numbers)*

Item 9 If s=2, t= -1, find the value of $5s+3t$. (Level 2 substitute positive and negative numbers)

Item 15 If p=5, r=3, find the value of $2(p+3r)-8$. (Level 3 substitute positive numbers with brackets)

Item 21 If x=2, y=3, find the value of $3x^2-xy+2y^2-10$. *(Level 4 substitute positive numbers in a two* variable expression with second order and brackets.)

Figure 6-7 Substitution test items

Pupils' thinking processes in handling substitution problems were categorised from their responses as correct substitution processes, incorrect substitution processes, unidentified process, and incomplete response.

Correct substitution processes are the strategies that showed the way to replace the given numbers instead of the letters into the expression correctly.

Incorrect substitution processes are those in which values were replaced without due concern for the operations or numbers different from those given were inserted.

There is also the unidentified process and incomplete response process as defined earlier.

6.9 A comparison of pupil's thinking processes in substituting algebraic expressions between the English and Thai schools

Figures 6-8 and 6-9 give a breakdown of the processes that English and Thai pupils used in approaching these problems at each level of difficulty.

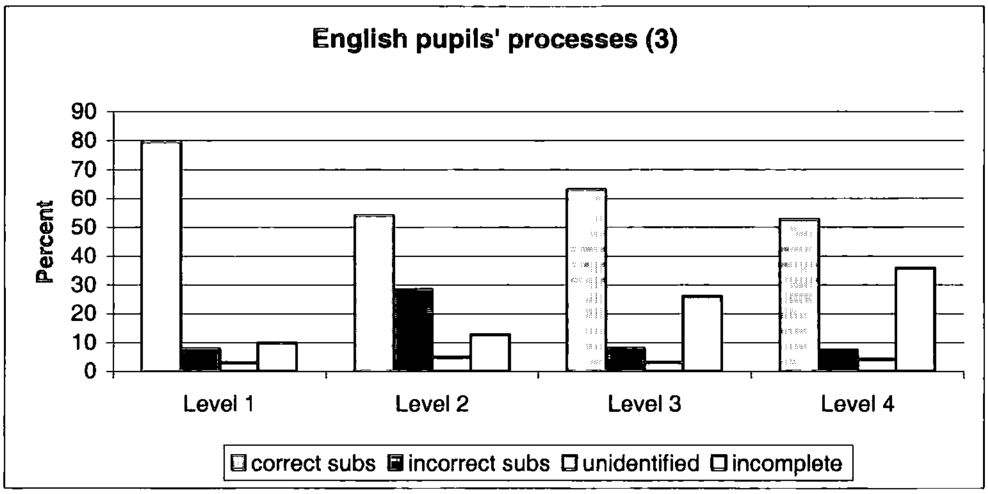


Figure 6-8 Percentage of process used in theme 3 by English pupils

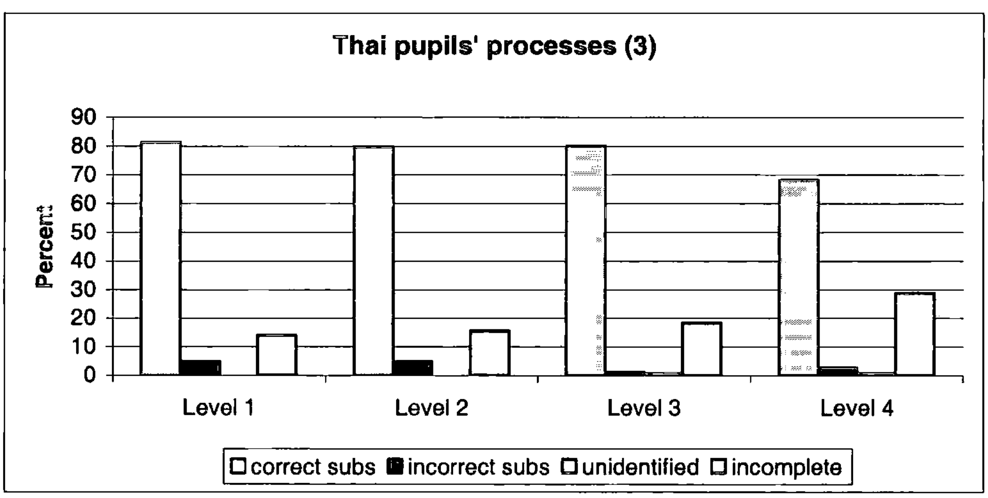


Figure 6-9 Percentage of process used in theme 3 by Thai pupils

As shown in Figures 6-8 and 6-9, for the most part English and Thai pupils used the correct substitution process to approach the problems at all levels.

Table 6.15 gives the actual percentage of each process and corresponding outcomes at each level of difficulty.

Table 6.15 Percentage of process used and outcomes for theme 3

Country	Level (item)	Processes							
		Correct Substitution		Incorrect Substitution		Unidentified process		Incomplete response	
		Used	% correct	Used	% correct	Used	% correct	Used	% correct
England (n=103)	1 (3)	79.6	90.2	7.8	12.5	2.9	33.3	9.7	0.0
	2 (9)	53.9	70.9	28.4	0.0	4.9	60.0	12.7	0.0
	3 (15)	63.0	46.0	8.0	0.0	3.0	0.0	26.0	0.0
	4 (21)	52.6	14.0	7.4	0.0	4.2	0.0	35.8	0.0
Thailand (n=186)	1 (3)	81.2	87.4	4.8	11.1	0.0	0.0	14.0	0.0
	2 (9)	79.6	83.1	4.8	0.0	0.0	0.0	15.6	0.0
	3 (15)	80.0	64.3	1.1	0.0	0.6	0.0	18.3	0.0
	4 (21)	68.1	55.9	2.5	0.0	0.6	100.0	28.8	0.0

As indicated in Table 6.15, for level 1 question, 79.6% of English pupils showed *the correct substitution process* and of those 90.2% gained the correct solution. The corresponding percentages for Thai pupils were 81.2% and 87.4%.

There was a large drop between level 1 and level 2 of those making up the correct substitution group in England. Of English pupils 53.9% showed *the correct substitution process* and of those 70.9% gained the correct answer. There was a slight decrease for Thai pupils. For Thai pupils 79.6% showed their work as *the correct substitution process* and of those 83.1% gained the correct solution.

At level 3 there was an increase to 63.0% of English pupils showing *the correct substitution processes* and of those 46.0% gained the correct solution. The percentage of Thai pupils using this process decreased. Of Thai pupils 80.0% used *the correct substitution process* and of those 64.3% gained the correct answer. At level 4, 52.6% of English pupils showed *the correct substitution process* but only 14.0% obtained the correct answer. The corresponding percentages of Thai pupils were 68.1% and 55.9%.

The following sections describe the sub-processes that pupils used at each level of difficulty.

Within the correct substitution group there are 2 sub-processes:

(1) *The correct arithmetic process* is the response that replaces the numbers given instead of the letters and then evaluates correctly.

(2) *The incorrect arithmetic process* refers to the case when the given values are inserted into the expression correctly but a mistake appears in carrying out the arithmetic operations.

There are 2 sub-processes used within the incorrect substitution group.

(1) *The correct arithmetic process* is the response in which replaced the value given such as "if a = 4, b = 3, find the value of a+5b" 5b becomes 53 or replaced the different value given such as 5b is 5×b but b≠3 followed by the correct computation.

(2) *The incorrect arithmetic proc*ess replaced the value as the correct arithmetic process but followed by incorrect computation.

The unidentified process and the incomplete response groups were as defined earlier.

6.9.1 Process used and outcomes for theme 3 level 1 item 3

Table 6.16 shows the percentage of processes used and percentage correct in the level 1 question, item 3, of substitution theme.

Table 6.16 Percentage of process used and outcomes for theme 3 level 1 item 3

Processes Theme 3 Level 1 (3)	English school		Thai school	
	Used	**%correct**	**Used**	**%correct**
Correct substitution	**79.6**	**90.2**	**81.2**	**87.4**
Correct arithmetic	71.8	100.0	71.0	100.0
Incorrect arithmetic	7.8	0.0	10.2	0.0
Incorrect substitution	**7.8**	**12.5**	**4.8**	**11.1**
Correct arithmetic	5.8	16.7	3.8	14.3
Incorrect arithmetic	1.9	0.0	1.1	0.0
Unidentified process	**2.9**	**33.3**	**0.0**	**0.0**
No process	2.9	33.3	0.0	0.0
Incomplete response	**9.7**	**0.0**	**14.0**	**0.0**
Incomplete	6.8	0.0	0.0	0.0
No response	2.9	0.0	14.0	0.0

As is evident in Table 6.16, the majority of English and Thai pupils showed their processes as *correct substitution and correct arithmetic*. Of English pupils 71.8% used this process and of those 100% gained the correct answer. The corresponding percentages of Thai pupils were 71.0% and 100%. For example, they showed the correct substitution and correct arithmetic processes as

"4+(5×3) = 19",

"4+5×3 = 19", and

"4+5(3) = 4+15=19".

Most pupils who used *the incorrect arithmetic process* showed their work, reading from left to right as

"4+5×3 = 27".

In the incorrect substitution group, 5.8% of English pupils showed *the correct arithmetic process* and of those 16.7% gained the correct answer. The corresponding percentages of Thai pupils were 3.8% and 14.3%. For example, the incorrect substitution group showed their work with *the correct arithmetic process* as

"4+53 = 57" (5*b* as 53), and

"4+(5×4) = 4+20 = 24" ($b \neq 3$).

In the unidentified process group, 2.9% of English pupils gave the answer without showing working and of those 33.3% gained the correct answer.

In the incomplete response group, 14.0% of Thai pupils made no attempt. Of English pupils 6.8% made only a partial attempt. For instance, English pupils in this group attempted to work as far as

"3×5 = 15", and
"4+5×3".

6.9.2 Process used and outcomes for theme 3 level 2 item 9

The level 2 item 9 "If s = 2, t = -1, find the value of $5s+3t$" was designed to investigate pupils' processes of substituting positive and negative numbers. As before, pupils' responses were categorised as correct substitution, incorrect substitution, unidentified process, and incomplete response.

Table 6.17 shows the percentage of processes used and percentage correct in the level 2 question, item 9, of substitution theme.

Table 6.17 Percentage of process used and outcomes for theme 3 level 2 item 9

Processes Theme 3 Level 2 (9)	**English school**		**Thai school**	
	Used	**%correct**	**Used**	**%correct**
Correct substitution	**53.9**	**70.9**	**79.6**	**83.1**
Correct arithmetic	38.2	100.0	66.1	100.0
Incorrect arithmetic	15.7	0.0	13.4	0.0
Incorrect substitution	**28.4**	**0.0**	**4.8**	**0.0**
Correct arithmetic	19.6	0.0	2.2	0.0
Incorrect arithmetic	8.8	0.0	2.7	0.0
Unidentified process	**4.9**	**60.0**	**0.0**	**0.0**
No process	4.9	60.0	0.0	0.0
Incomplete response	**12.7**	**0.0**	**15.6**	**0.0**
Incomplete	6.9	0.0	0.5	0.0
No response	5.9	0.0	15.1	0.0

As shown in Table 6.17, in the correct substitution group, 38.2% of English pupils and 66.1% of Thai pupils showed *the correct arithmetic process* and of those 100% gave the correct answer.

The correct arithmetic process was used in the correct substitution group in both countries. For example, they showed their processes as

"(5×2)+(3×-1) = 10+ -3 = 7",
"5×2 + 3×-1 = 10+-3 = 7", and
"5(2)+3(-1) = 10+(-3) = 7".
The pupils who used *the incorrect arithmetic process* showed their work as
"5×2 = 10, 3×-1 = -3, 10+-3 = -13", and
"(5×2)+(3×-1) = 10+(-3) = -7".

In the incorrect substitution group, 19.6% of English pupils showed *the correct arithmetic process*. Of Thai pupils 2.7% showed *the incorrect arithmetic process*. For instance, the incorrect substitution group showed their work with *the correct arithmetic process* as

"5×2 = 10+3×1 = 3, 10+3 = 13",
"52+2 = 54", and
"5×2+3-1 = 12".

In the unidentified process group, 4.9% of English pupils gave the answer without showing working and of those 60.0% obtained the correct answer.

In the incomplete response group, 6.9% of English pupils made a partial attempt. Of Thai pupils 15.1% made no response to this question. For example, English pupils in this group attempted to work as far as

"5×2+3×-1", and
"5×2+3×-1, 10+ -3".

6.9.3 Process used and outcomes for theme 3 level 3 item 15

This level 3 item 15 "If $p = 5$, $r = 3$, find the value of $2(p+3r)-8$" was designed to observe how pupils substituted particular values into an expression with brackets. As before, pupils' responses were categorised into four groups as correct substitution, incorrect substitution, unidentified process and incomplete response.

Table 6.18 shows the percentage of processes used and percentage correct in the level 3 question, item 15, of substitution theme.

Table 6.18 Percentage of process used and outcomes for theme 3 level 3 item 15

Processes Theme 3 Level 3 (15)	English school		Thai school	
	Used	**%correct**	**Used**	**%correct**
Correct substitution	**63.0**	**46.0**	**80.0**	**64.3**
Correct arithmetic	28.0	100.0	51.4	100.0
Incorrect arithmetic	35.0	2.9	28.6	0.0
Incorrect substitution	**8.0**	**0.0**	**1.1**	**0.0**
Correct arithmetic	1.0	0.0	0.0	0.0
Incorrect arithmetic	7.0	0.0	1.1	0.0
Unidentified process	**3.0**	**0.0**	**0.6**	**0.0**
No process	3.0	0.0	0.6	0.0
Incomplete response	**26.0**	**0.0**	**18.3**	**0.0**
Incomplete	6.0	0.0	1.1	0.0
No response	20.0	0.0	17.1	0.0

As can be seen in Table 6.18, in the correct substitution group, 35.0% of English pupils showed *the incorrect arithmetic process* and of those 2.9% gained the correct answer. Of Thai pupils 51.4% showed *the correct arithmetic process* and of those 100% gained correct answer.

The correct arithmetic process was commonly used among the correct substitution group in Thailand. For example, they showed their processes as

"2(5+3×3)-8 = 10+18-8 = 10+10 = 20",

"2(5+3×3)-8 = 2(5+9)-8 = 2(14)-8 = 28-8 = 20", and

"(2×5)+(2×3×3)-8 = 10+18-8 = 28-8 = 20".

English pupils showed *the incorrect arithmetic* process as

"2×5 = 10+9 = 19-8 = 11", and

"2(5+3×3)-8 = 5+3×3 = 14-8 = 6×2 = 12".

In the incorrect substitution group, 7.0% of English pupils and 1.1% of Thai pupils showed *the incorrect arithmetic process*. The incorrect substitution group showed their work with *the incorrect arithmetic process* as

"2(5+3×5)-8 = 2×40-8 = 72",

"2+5+3-8 = 2", and

"2×5p+3×3r-8 = 2p+9r-8".

In the unidentified process group, 3.0% of English pupils and 0.6% of Thai pupils gave the answer without showing working.

In the incomplete response group, 20.0% of English pupils and 17.1% of Thai pupils made no attempt.

6.9.4 Process used and outcomes for theme 3 level 4 item 21

The level 4 item 21 "If $x = 2$, $y = 3$, find the value of $3x^2$-xy+$2y^2$-10" was designed to gain insight into how pupils substituted numbers for variables of the second order. Again, pupils' responses were categorised into four groups as correct substitution, incorrect substitution, unidentified process, and incomplete response.

Table 6.19 shows the percentage of processes used and percentage correct in the level 4 question, item 21, of substitution theme.

Table 6.19 Percentage of process used and outcomes for theme 3 level 4 item 21

Processes Theme 3 Level 4 (21)	English school		Thai school	
	Used	%correct	Used	%correct
Correct substitution	**52.6**	**14.0**	**68.1**	**55.9**
Correct arithmetic	7.4	100.0	38.0	100.0
Incorrect arithmetic	45.3	0.0	30.1	0.0
Incorrect substitution	**7.4**	**0.0**	**2.5**	**0.0**
Correct arithmetic	0.0	0.0	0.0	0.0
Incorrect arithmetic	7.4	0.0	2.5	0.0
Unidentified process	**4.2**	**0.0**	**0.6**	**100.0**
No process	4.2	0.0	0.6	100.0
Incomplete response	**35.8**	**0.0**	**28.8**	**0.0**
Incomplete	7.4	0.0	0.0	0.0
No response	28.4	0.0	28.8	0.0

As presented in Table 6.19, in the correct substitution group, 45.3% of English pupils showed *incorrect arithmetic process*. Of Thai pupils 38.0% showed *correct arithmetic process* and of those 100% gained the correct answer.

A correct arithmetic process was usually used among the correct substitution group in Thailand. For instance, they showed their processes as

"$(3\times2^2)-(2\times3)+(2\times3^2)-10 = (3\times4)-6+(2\times9)-10 = 12-6+18-10 = 6+8 = 14$", and
"$3\times2^2-(2\times3)+2\times3^2-10 = (12-6)+(18-10) = 6+8 = 14$".
English pupils showed *the incorrect arithmetic* process as
"$3\times2^2 = 6^2 = 12-2\times3 = 6+2\times3^2 = 12-10 = 12-6+12-10 = 8$", and
"$3\times2^2-2\times3+2\times3^2-10 = 36-6+36-10 = 30+26 = 56$".

In the incorrect substitution group, 7.4% of English pupils and 2.5% of Thai pupils showed *the incorrect arithmetic process*. The incorrect substitution group showed their work with *the incorrect arithmetic process* as

"34-23+26-10 = 11+16 = 27", and
"$9x+8x-y+4x-10 = 17x+3y-10 = 34+9-10 = 33$".

In the unidentified process group, 4.2% of English pupils gave the answer without showing working. Of Thai pupils 0.6% gave the answer without showing working and of those 100% gained the correct solution.

In the incomplete response group, 28.4% of English pupils and 28.8% of Thai pupils made no attempt.

6.10 Summary and discussion of findings Theme 3

English pupils' thinking processes: the correct substitution group mainly used *the correct arithmetic process* to evaluate the level 1 and level 2 questions. They primarily used *the incorrect arithmetic process* on the level 3 and level 4 problems. The *incorrect substitution* group frequently showed the correct process on the level 1 and level 2 questions. They commonly used *the incorrect arithmetic process* on the level 3 and level 4 expressions. Those in the incomplete response group often showed incomplete work to the level 1 and level 2 questions. Quite frequently they made *no response* to the level 3 and level 4 questions.

Thai pupils' thinking processes: the correct substitution group mainly showed *the correct arithmetic process* in all expressions. The incorrect substitution group commonly used *the correct arithmetic process* in the level 1 problem. *The incorrect arithmetic process* was mainly used to evaluate the levels 2, 3, and 4 questions. It was quite common for pupils in the incomplete response group to make *no response* to all expressions.

As the results indicate, English and Thai pupils used similar processes to approach the level 1 question. Differences in processes used increased when faced with the harder items. The main difficulties were dealing with negative signs, understanding exponents and expanding the brackets.

6.10.1 Correct answer from incorrect substitution with correct arithmetic

One English pupil showed the process (level 1, item 3 "if $a = 4$, $b = 3$, find the value of $a+5b$") to obtain the correct solution as "$a+5b$ = 4+53, 4+15 = 19". This pupil wrote 53 but calculated as 5×3. Similarly, one Thai pupil showed the process as "$a+5b$ = 4+(53) = 4+15 = 19".

The evidence shows that these two pupils (one in English school and the other in Thai school) had realised "$5b$ means 5 times b". However, they wrote *the incorrect substitution process* as "53 = 15". This is similar to Booth's finding (1989) that pupils' no longer misconception as 53 (fifty three).

6.10.2 Correct answer from correct substitution but incorrect arithmetic

English pupils showed the process (level 3, item 15 "if $p = 5$, $r = 3$, find the value of $2(p+3r)-8$") to obtain the correct answer as "5+3×3 = 14, 2-8 = 6, 14+6 = 20". This reflects the pupils' experience of hearing the advice "do the brackets first". This advice may also have led to errors in the steps of multiplying out the brackets in the later questions.

6.10.3 The large drop from theme 3 level 1 to levels 2, 3 and 4

For the level 2 item 9 "if $s = 2$, $t = -1$, find the value of $5s+3t$", the incorrect substitution group of English pupils replaced the value of t but ignored the minus sign. For example, they showed the process as "5×2 = 10, 3×1 = 3, 10+3 = 13". Others replaced the negative sign but made an incorrect computation. Thus, they showed the process as "5×2 = 10, 3×-1 = -3, = -13" or " = 13". This seems to confirm the confusion in computing with negative numbers mentioned earlier. Likewise, Demby (1997) reported that most errors concerned computations on negative numbers when grade 7 pupils were asked to find the numerical value of expressions $2x+3-3x$ and $-x+2-x^2+1$ for $x = -5$.

For the level 3 item 15 "if $p = 5$, $r = 3$, find the value of $2(p+3r)-8$", the majority of English pupils gained the answer of "11" by showing their processes as

"2×5 + 3×3 – 8 = 10+9-8 = 11". They multiplied only the first term in the brackets and ignored the second term in the brackets. This reflects the experience of hearing the advice "expand the brackets **first** then add or subtract any like terms".

For the level 4 item 21 "if $x = 2$, $y = 3$, find the value of $3x^2 - xy + 2y^2 - 10$" incorrect answers were mainly caused by errors in computation. The evidence showed that the pupils could substitute the numbers given into the expression. Most errors were in dealing with the index notation. English and Thai pupils who made errors with the index notation showed their processes as

"$3 \times 2^2 = 6^2 = 12 - 2 \times 3 = 6 + 2 \times 3^2 = 12 - 10$, 12-6+12-10 = 8",

"$3 \times 2^2 - 2 \times 3 + 2 \times 3^2 - 10$, 36-6+36-10, 30+26 = 56".

In the first example, the pupils viewed 3×2^2 as $(3 \times 2)^2$, 2×3^2 (which is incorrect) and then made the second error of "$6^2 = 6 \times 2$". These pupils had misconceptions about the index notation. In the second example, the pupils also viewed, 3×2^2 as $(3 \times 2)^2$, 2×3^2 (which is incorrect) but correctly evaluated "$6^2 = 36$". This group of pupils has the correct conception of the square notation but dealt incorrectly with the coefficient as they read "three times two squared" and "two times three squared".

The accuracy in evaluating expressions was greater when the pupils used parentheses. Norton and Cooper (2001) also found that pupils showed good understanding of the order convention where brackets were present. Thai pupils and the English high ability group tend to use the brackets to remind themselves of the order in solving the problems. This appeared less among the English low ability group. The lack of knowledge of using brackets when substituting suggests they are less likely to succeed at the higher levels of mathematics. It seems that calculating with negative numbers, understanding of the index notation and expanding brackets are topics in which there is need for more careful attention in both countries.

6.11 Theme 4 Solving Equations

The fourth theme of the test is that of solving equations and it was organised into four levels of expected difficulty. It consisted of four questions, designed to determine the pupils' thinking processes as they solved the given equations. The questions are shown in Figure 6-10.

Solving equations

Item 4 Solve the equation $5a$-2 = 8. *(Level1 The unknown in the first term)*

Item 10 Solve the equation 5-2b = 1. *(Level2 The unknown in middle term)*

Item 16 Solve the equation 3y-6 = y-2. *(Level3 The unknown in both sides)*

Item 22 Solve the equation 2(3x-1)-(x+4) = 9. *(Level4 The unknown in brackets)*

Figure 6-10 Solving equations theme test items

The pupils' thinking processes in solving equations were categorised from pupils' responses as *generalisable process*, *other process*, *unidentified process* and *incomplete response.*

Generalisable processes are methods that show the way to solve the equation following the rules. These rules include balancing, substitution, inverse techniques, multiplying out brackets and simplifying like terms.

Other processes are those in which pupils attempt to solve the equations following only "partial" rules. These "partial" rules include an attempt at balancing, substitution and inverse techniques. The use of other process in expanding brackets included multiplying only the first term of the bracket, combining unlike terms within the brackets and applying the multiplying factor to an extra bracket.

As before the unidentified process and the incomplete response are also considered.

6.12 A comparison of pupils' thinking processes in solving equations between the English and Thai schools

Figures 6-11 and 6-12 give a breakdown of the processes that the English and Thai pupils used in approaching these problems at each level of difficulty.

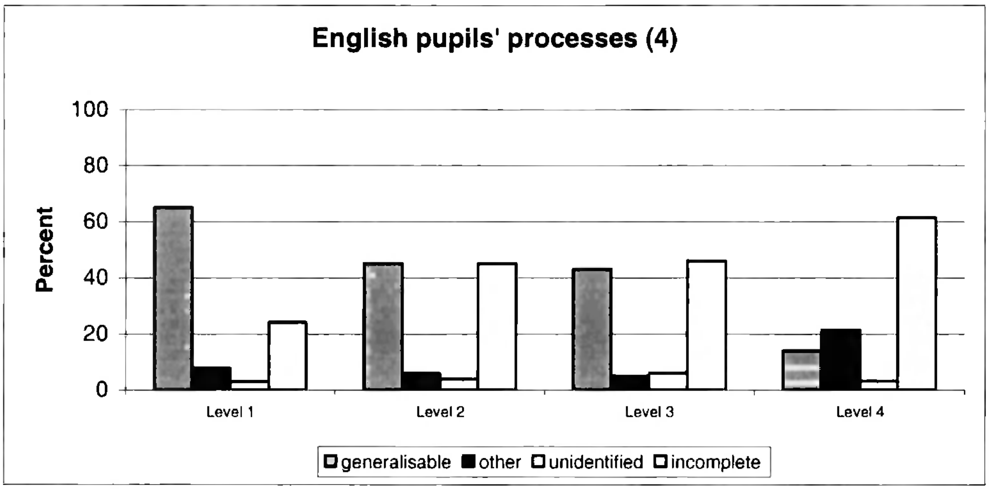


Figure 6-11 Percentage of process used in theme 4 by English pupils

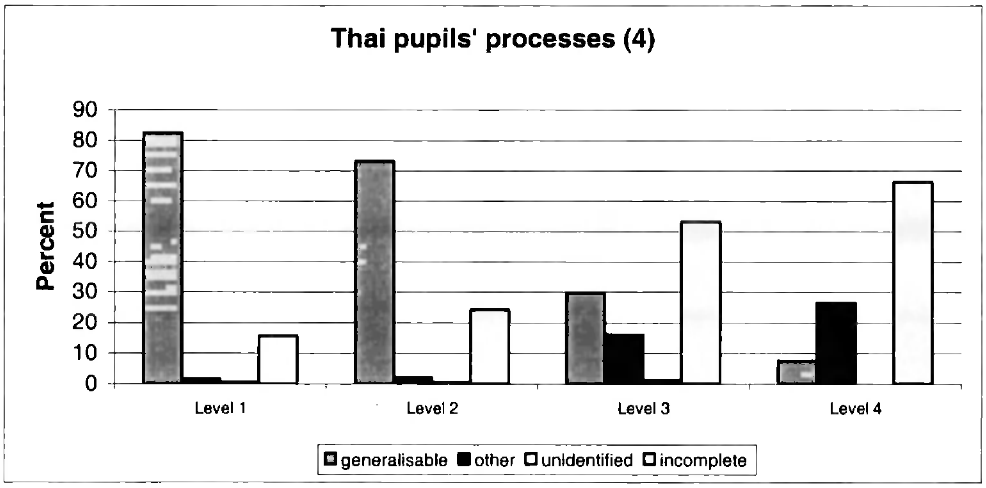


Figure 6-12 Percentage of process used in theme 4 by Thai pupils

As shown in Figures 6-11 and 6-12, Thai pupils mainly used *the generalisable process* to solve the level 1 and level 2 problems. English pupils frequently used *the generalisable process* when they faced the level 1, level 2 and level 3 questions.

Table 6.20 gives the actual percentage of each process and corresponding outcomes at each level of difficulty.

Table 6.20 Percentage of process used and outcomes for theme 4

Country	Level (item)	Processes							
		Generalisable process		Other process		Unidentified process		Incomplete response	
		Used	% correct	Used	% correct	Used	% correct	Used	% correct
England (n=103)	1 (4)	**65.0**	**94.0**	**7.8**	**0.0**	**2.9**	**66.7**	**24.3**	**0.0**
	2 (10)	**45.1**	**37.0**	**5.9**	**16.7**	**3.9**	**50.0**	**45.1**	**0.0**
	3 (16)	**43.0**	**69.8**	**5.0**	**60.0**	**6.0**	**0.0**	**46.0**	**0.0**
	4 (22)	**14.0**	**30.8**	**21.5**	**5.0**	**3.2**	**0.0**	**61.3**	**0.0**
Thailand (n=186)	1 (4)	**82.3**	**96.7**	**1.6**	**0.0**	**0.5**	**100.0**	**15.6**	**0.0**
	2 (10)	**73.1**	**45.6**	**2.2**	**75.0**	**0.5**	**0.0**	**24.2**	**0.0**
	3 (16)	**29.7**	**90.4**	**16.0**	**28.6**	**1.1**	**0.0**	**53.1**	**0.0**
	4 (22)	**7.5**	**58.3**	**26.3**	**0.0**	**0.0**	**0.0**	**66.3**	**0.0**

As presented in Table 6.20, level 1 question, 65.0% of English pupils used *the generalisable process* and of those 94.0% gained the correct answer. The corresponding percentages for Thai pupils were 82.3% and 96.7%.

There was a decrease between level 1 and level 2 of those making up *the generalisable process* in both countries. Of English pupils 45.1% used *the generalisable process* and of those only 37.0% gained the correct answer. The corresponding percentages for Thai pupils were 73.1% and 45.6%.

At level 3 there was a minimal decrease to 43.0% of English pupils using *the generalisable process* and of those 69.8% gained the correct solution. There was a large drop between level 2 and level 3 for the corresponding group in Thailand. Of Thai pupils 29.7% used *the generalisable process* and of those 90.4% gained the correct answer.

For the level 4 item, 14.0% of English pupils used *the generalisable process* and of those 30.8% gained the correct solution. Of Thai pupils 7.5% used *the generalisable process* but of those 58.3% gained the correct answer.

The following sections describe the sub-processes pupils used at each level of difficulty.

Within the generalisable process group there are 4 sub-processes:

(1) *The balancing process* describes responses in which pupils perform the same operation to both sides of the equation or move a number to the opposite side of the equation with the inverse operation.

(2) *The substitution process* refers to those responses in which replace the letter by a number in an attempt to make both sides of the equation has equal value.

(3) *The inverse process* reflects the reverse of those steps of the equation from the right hand side to the left hand side.

(4) *The multiply out brackets process* includes expansion of brackets and simplification of like terms.

There are 5 sub-processes used within the other process group.

(1) *The balancing-like process* moves a number to the opposite side of the equation with the same operation.

(2) *The substitution-like process* attempts to replace the letter by a number without concern that the equation is true.

(3) *The inverse-like process* is used to describe those attempts, which used an inverse operation even though it is inappropriate.

(4) *The incorrect operation process* covers responses in which pupils' work does not appear to have any relevance to solving the equation.

(5) *The multiply out brackets-like process* showed an attempt to simplify unlike terms in the brackets, multiply only the first term of the brackets, or applying the factor to an extra terms.

As before, there also were the unidentified process and the incomplete response groups.

6.12.1 Process used and outcomes for theme 4 level 1 item 4

The level 1 item 4 "Solve the equation $5a$-2 = 8" was designed to investigate how pupils find out the unknown quantity that fits the equation. Pupils' responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.21 shows the percentage of process used and percentage correct in the level 1 question, item 4, of solving equations theme.

Table 6.21 Percentage of process used and outcomes for theme 4 level 1 item 4

Processes Theme 4 Level 1 (4)	English school		Thai school	
	Used	%correct	Used	%correct
Generalisable process	**65.0**	**94.0**	**82.3**	**96.7**
Balancing	47.6	98.0	75.3	97.1
Substitution	13.6	78.6	5.4	90.0
Inverse	3.9	100.0	1.6	100.0
Other process	**7.8**	**0.0**	**1.6**	**0.0**
Balancing-like	1.0	0.0	0.0	0.0
Substitution-like	5.8	0.0	0.5	0.0
Incorrect operation	1.0	0.0	1.1	0.0
Unidentified process	**2.9**	**66.7**	**0.5**	**100.0**
No process	2.9	66.7	0.5	100.0
Incomplete response	**24.3**	**0.0**	**15.6**	**0.0**
Incomplete	6.8	0.0	1.6	0.0
No response	17.5	0.0	14.0	0.0

As can be seen in Table 6.21, the most common process used in the generalisable process group was *the balancing process.* Of English pupils 47.6% used *the balancing process* and of those 98.0% gained the correct answer. The corresponding percentages for Thai pupils were 75.3% and 97.1%.

For example, most of English pupils who used this process showed their work as

"$5a$-2 = 8, $5a$ = 8+2, $5a$ = 10, a = 2".

Thai pupils showed their work as

"$5a$-2 = 8, $5a$-2+2 = 8+2, $5a$ = 10, $\frac{5a}{5}=\frac{10}{5}$, a = 2".

The second most common was *the substitution process.* This process was used among the generalisable process group in both countries. For instance, they showed their processes as

"$5\times2 = 10\text{-}2 = 8$", "$5\times2\text{-}2 = 8$" and
"$5\times1 = 5\text{-}2 = 3$, $5\times2 = 10\text{-}2 = 8$".

In the other process group, 5.8% of English pupils substituted the number without any concern about the equals sign. Of Thai pupils 1.1% solved the equation using *the incorrect operation process.*

For example, English pupils showed *the substitution-like process* as

"$5a\text{-}2 = 8$, $5+4\text{-}2 = 7$",
"$5a\text{-}2 = 8$, $5\times4\text{-}2 = 8$" and
"$5\times4\text{-}2 = 18$".

A Thai pupil showed *the substitution-like process* as

"$5a\text{-}2 = 8$, $5+5\text{-}2 = 8$, $8 = 8$, $a = 5$".

In the unidentified process group, 2.9% of English pupils gave the answer without showing working and of those 66.7% gained the correct answer. The corresponding percentages for Thai pupils were 0.5% and 100%.

In the incomplete response group, 17.5 % of English pupils and 14.0% of Thai pupils made no attempt at this question. For example, the English pupils showed their incomplete work as

"$5a\text{-}2 = 8$, $5a = 10$", "$5a\text{-}2 = 8$, $5a = 6$"and
"$5a\text{-}2 = 8$, $5\times2\text{-}2$".

Thai pupils show their incomplete work as

"$5a\text{-}2 = 8$, $5\times a = 8$, $5\times2 = 10\text{-}2=8$",

"$5a\text{-}2 = 8$, $5a\text{-}2+2 = 8+2$, $\frac{5a}{5}=\frac{10}{5}$, $5a = 5$" and

"$5a\text{-}2 = 8$, $5a\text{-}2+2 = 8+2$, $5a = 10$".

6.12.2 Process used and outcomes for theme 4 level 2 item 10

The level 2 item 10 "Solve the equation $5\text{-}2b = 1$" was designed to examine pupils' thinking processes in managing the unknown appearing in the middle term and dealing with the negative sign. As before, pupils' processes were categorised as generalisable process, other process, unidentified process and incomplete response groups.

Table 6.22 shows the percentage of process used and percentage correct in the level 2 item 10, of solving equations theme.

Table 6.22 Percentage of process used and outcomes for theme 4 level 2 item 10

Processes Theme 4 Level 2 (10)	English school		Thai school	
	Used	%correct	Used	%correct
Generalisable process	**45.1**	**37.0**	**73.1**	**45.6**
Balancing	39.2	30.0	67.7	42.9
Substitution	5.9	83.3	5.4	80.0
Other process	**5.9**	**16.7**	**2.2**	**75.0**
Balancing-like	0.0	0.0	1.1	0.0
Substitution-like	2.0	50.0	0.5	100.0
Inverse-like	1.0	0.0	0.0	0.0
Incorrect operation	2.9	0.0	0.5	100.0
Unidentified process	**3.9**	**50.0**	**0.5**	**0.0**
No process	3.9	50.0	0.5	0.0
Incomplete response	**45.1**	**0.0**	**24.2**	**0.0**
Incomplete	22.5	0.0	2.2	0.0
No response	22.5	0.0	22.0	0.0

As shown in Table 6.22, the most common process used in the generalisable process group was *the balancing process.* Of English pupils 39.2% used *the balancing process* and of those 30.0% gained the correct answer. The corresponding percentages for Thai pupils were 67.7% and 42.9%.

For example, the English pupils showed their processes as

"$5\text{-}2b = 1$, $\text{-}2b = 1\text{-}5$, $\text{-}2b = \text{-}4$, $b = 2$",

"$5\text{-}2b = 1$, $2b = 1+5$, $2b = 6$, $b = 3$", and
"$5\text{-}2b = 1$, $2b = 1\text{-}5$, $2b = \text{-}4$, $b = \text{-}2$".

Thai pupils showed their processes as

$5\text{-}2b = 1$, $5\text{-}2b+5 = 1+5$, $2b = 6$, $\frac{2b}{2} = \frac{6}{2}$, $b = 3$",

"$5\text{-}2b = 1$, $5\text{-}2b\text{-}5 = 1\text{-}5$, $\frac{-2b}{-2} = \frac{-4}{-2}$, $b = 2$" and

"$5\text{-}2b = 1$, $5\text{-}2b\text{-}5 = 1\text{-}5$, $2b = \text{-}4$, $\frac{2b}{2} = \frac{-4}{2}$, $b = \text{-}2$".

In the other process group, 2.0% of English pupils and 0.5% of Thai pupils used *the substitution-like process*. Correct answers were sometimes gained fortuitously. For instance,

$5\text{-}2b = 1$, $5\text{-}b = 1+2$, $5\text{-}b = 3$, $b = 5\text{-}3$, $b = 2$.

Thai pupils showed *the substitution-like process* as

"$5\text{-}2b = 1$, $5\text{-}(2+2) = 1$, $5\text{-}4 = 1$, $b = 2$" and
"$5\text{-}2b = 1$, $b = 5$, $5\text{-}5 = 1$".

In the unidentified process group, 3.9% of English pupils gave the answer without showing working and of those 50.0% gained the correct answer. Of Thai pupils 0.5% did not show working and gave the incorrect answer.

In the incomplete response group, 22.5% of English pupils made a partial attempt and a further 22.5% made no attempt. Of Thai pupils 22.0% made no attempt at this problem.

For example, the English pupils show their incomplete work as

"$5\text{-}2b = 1$, $\text{-}2b = 1\text{-}5$, $\text{-}2b = \text{-}4$, $\text{-}b = \text{-}2$",

"$5\text{-}2b = 1$, $5 = 1+2b$, $5 = 3b$, $= 3b\text{-}5$, $= \frac{3b}{5}$" and

"$5\text{-}2\times2 = 1$, $5\text{-}4 = 1$".

Thai pupils showed their incomplete work as

"5-2*b* = 1, 5-(2×2) = 1, 5-4 = 1" and

"5-2*b* = 1, 5-2*b*+5 = 1+5, 2*b* = 6".

6.12.3 Process used and outcomes for theme 4 level 3 item 16

The level 3 item 16 "Solve the equation 3y-6 = y-2" was designed to observe pupils' thinking processes when facing the unknown in both sides. Responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.23 shows the percentage of process used and percentage correct in the level 3 question, item 16, of solving equations theme.

Table 6.23 Percentage of process used and outcomes for theme 4 level 3 item 16

Processes Theme 4 Level 3 (16)	English school		Thai school	
	Used	%correct	Used	%correct
Generalisable process	**43.0**	**69.8**	**29.7**	**90.4**
Balancing	43.0	69.8	29.1	90.2
Substitution	0.0	0.0	0.6	100.0
Other process	**5.0**	**60.0**	**16.0**	**28.6**
Balancing-like	4.0	75.0	9.1	43.8
Substitution-like	0.0	0.0	1.7	0.0
Inverse-like	1.0	0.0	0.0	0.0
Incorrect operation	0.0	0.0	5.1	11.1
Unidentified process	**6.0**	**0.0**	**1.1**	**0.0**
No process	6.0	0.0	1.1	0.0
Incomplete response	**46.0**	**0.0**	**53.1**	**0.0**
Incomplete	8.0	0.0	6.9	0.0
No response	38.0	0.0	46.3	0.0

As shown in Table 6.23, in the generalisable process group, 43.0% of English pupils used *the balancing process* and of those 69.8% gained the correct answer. The corresponding percentages for Thai pupils were 29.1% and 90.2%.

For example, the English pupils showed *the balancing process* as

"3*y*-6 = *y*-2, 3*y* = *y*-2+6, 3*y*-*y* = 8, 2*y* = 8, *y* = 4" and

"$3y$-6 = y-2, $3y$ = y-2+6, $3y$-y = 4, $2y$ = 4, y = 2".

Thai pupils showed their processes as

"$3y$-6 = y-2, $3y$-y = -2+6, $\frac{2y}{2}=\frac{4}{2}$, y = 2" and

"$3y$-6 = y-2, $4y$-6+6 = -2+6, $4y$ = 4, y = 1".

In the other process group, 4.0% of English pupils used *the balancing-like process* to approach this problem and of those 75.0% gain the correct answer. Of Thai pupils 9.1% used *the balancing-like process* and of those 43.8% gained the correct answer.

Only one of the other process English pupils used *the inverse-like process* to solve this problem. This particular pupil showed the process as

"$3y$-6 = y-2, 3×y-6 = y-2, 3÷y+6 = y+2, y = 4".

Thai pupils in the other process group commonly used *the balancing-like process.*

For example they showed their processes as

"$3y$-6 = y-2, $3y$-6+6 = y-2+6, $3y$ = y-8+8, $3y$-8+8 = y-8+8, $\frac{3y}{y}=y$, 3 = y" and

"$3y$-6 = y-2, $3y$-6+6 = $2y$-2+2, $\frac{3y+6}{3}=\frac{y+2}{3}$, y = 8".

In the unidentified process group, 6.0% of English pupils and 1.1% of Thai pupils gave the answer without showing working.

In the incomplete response group, 38.0% of English pupils and 46.3% of Thai pupils made no attempt at this question.

For example, the English pupils showed their incomplete work as

"$3y$-6 = y-2, $3y$+y = y-8, $4y$ = 8",

"$3y$-6 = y-2, 3×3 = 9-6 = 3" and

"3*y*-6 = *y*-2, 3*y*-6-2 = *y*".

Thai pupils showed their incomplete work as

"3*y*-6 = *y*-2, 3*y*-6+6 = *y*-2+6, 3*y*+2 = *y*-4", and

"3*y*-6 = *y*-2, 3*y*-6+2 = *y*-2+2, 3*y*-8 = *y*".

6.12.4 Process used and outcomes for theme 4 level 4 item 22

The level 4 item 22 "Solve the equation 2(3*x*-1)-(*x*+4) = 9" was designed to gain insight into how pupils simplify the equation when the unknown is in brackets. As before, pupils' responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.24 shows the percentage of process used and percentage correct in the level 4 question, item 22, of solving equations theme.

Table 6.24 Percentage of process used and outcomes for theme 4 level 4 item 22

Processes Theme 4 Level 4 (22)	English school		Thai school	
	Used	%correct	Used	%correct
Generalisable process	**14.0**	**30.8**	**7.5**	**58.3**
Multiply out bracket	14.0	30.8	7.5	58.3
Other process	**21.5**	**5.0**	**26.3**	**0.0**
Balancing-like	0.0	0.0	11.3	0.0
Substitution-like	4.3	25.0	2.5	0.0
Multiply out bracket-like	17.2	0.0	12.5	0.0
Unidentified process	**3.2**	**0.0**	**0.0**	**0.0**
No process	3.2	0.0	0.0	0.0
Incomplete response	**61.3**	**0.0**	**66.3**	**0.0**
Incomplete	12.9	0.0	8.8	0.0
No response	48.4	0.0	57.5	0.0

As illustrated in Table 6.24, in the generalisable process group, 14.0% of English pupils showed the process *multiplying out the brackets* and of those 30.8% gained the correct answer. The corresponding percentages for Thai pupils were 7.5% and 58.3%.

The multiply out brackets process was done by multiplying over the brackets by the factor and then simplifying like terms. The balancing process or the substitution process then followed this. The English pupils showed their processes as

"2(3x-1)-(x+4) = 9, 6x-2-x+4 = 9, 6x-x+4 = 11, 5x = 15, x = 3",

"2(3x-1)-(x+4) = 9, 7x+6 = 9, 7x = 15, x = $2\frac{1}{7}$" and

"2(3x-1)-(x+4) = 9, 5x-2 = 9, 5x = 11, x = 2.2".

Thai pupils showed their work as

"2(3x-1)-(x+4) = 9, (6x-2)-(x-4) = 9, 6x-2-x-4 = 9, 6x-x = 9+2+4, 5x = 15, x = 3"and

"2(3x-1)-(x+4) = 9, 6x-2-x+4 = 9, 6x-x+2-2 = 9-2, $\frac{5x}{5}=\frac{7}{5}$, x = $1\frac{2}{5}$".

In the other process group, 17.2% of English pupils and 12.5% of Thai pupils, errors arose in multiplying out only the first term in the brackets, multiplying both brackets, or simplifying unlike terms in the brackets. For example, the English pupils showed their processes as

"2(3x-1)-(x+4) = 9 2×**2*x*-4*x*** = 9, 9-2 = 2x-4x, 7 = -2x, 0.28 = x",

"2(3x-1)-(x+4) = 9, **6*x*-2-2*x*+8** = 9, 8x+6 = 9, 8x = 3, x = $\frac{3}{8}$" and

"2(3x-1)-(x+4) = 9, **6*x*-1**-x-4 = 9, 5x-5 = 9, 5x = 14, x = $2\frac{4}{5}$".

Thai pupils showed their processes as

"2(3x-1)-(x+4) = 9, 2(**2*x*-4*x***) = 9, 2(-2x) = 9, x = ($\frac{9}{2}$)×2, x = 9",

"2(3x-1)-(x+4) = 9, **6*x*-2-2*x*+8** = 9, 4x+6 = 9, 4x+6-6 = 9-6, $\frac{4x}{4}=\frac{3}{4}$, $x=\frac{3}{4}$" and

"2(3x-1)-(x+4) = 9, 6x-2-**4*x*** = 9, 2x-2+2 = 9+2, 2x = 11, $\frac{2x}{2}=\frac{11}{2}$, $x=5\frac{1}{2}$".

In the unidentified process group, 3.2% of English pupils gave the answer without working shown.

In the incomplete response group, 48.4% of English pupils and 57.5% of Thai pupils made no attempt to this question. Thai pupils showed their incomplete work as

"$2(3x-1)-(x+4) = 9$, $(6x-2)-(x+4) = 9$".

6.13 Summary and discussion of findings Theme 4

English pupils' thinking processes: the generalisable process group mainly used *the balancing process* in the level 1, 2, and 3 equations. They frequently used *the multiplying out brackets* followed by *the balancing process* in the level 4 equation. The other process group commonly used *the substitution-like process* in the level 1 equation. *The substitution-like process* and *the incorrect operation process* were frequently used in the level 2 equation. Only one of the English pupils used *the inverse-like process* in the level 3 equation. They commonly used *the multiply out brackets-like process* in the level 4 equation. *The unidentified process* group gave the answer without showing any working at all. Those in *the incomplete response* group mainly made ***no** response* at all.

Thai pupils' thinking processes: the generalisable process group commonly used *the balancing process* in the level 1, 2 and 3 equations. They frequently used *the multiply out brackets* followed by *the balancing process* in the level 4 equation. The other process group commonly used *the incorrect operation* in the level 1 equation. They frequently made *the substitution-like process* in the level 2 equation. *The balancing-like process* was mainly used in the level 3 equation. In the level 4 equation, they frequently used the *multiply out brackets-like process.* A small number of Thai pupils gave the answer without showing working in the level 1, 2 and 3 equations. Those in *the incomplete response* group commonly made ***no** response* at all.

From the results, it can be seen that English and Thai pupils in the generalisable process groups used a similar approach to solve equations. The main difficulties were dealing with negative signs and multiplying out brackets.

The results indicated that the Thai pupils' success rate was more than 50% in solving the level 1 and level 2 equations. The English pupils' success rate was more than 50% in solving the level 1 equation.

The balancing process for solving equations was carried out explicitly and strongly emphasised in the Thai school, whilst *the inverse operation* was highly emphasised in the English school. *The balancing process* was seen implicitly in the English school.

6.13.1 Using other process but obtained the correct answer

For the level 2 item 10 "solve the equation 5-2b = 1" one English pupil showed *the substitution-like process* as "5-2-b = 1, b = 2". This pupil saw 5-2 = 3, and then took away two to get "= 1" on the right hand side. This pupil ignored the meaning of 2b and did not use algebraic thinking to solve the equation.

A Thai pupil showed *the substitution-like process* as"5-(2+2) = 1, 5-4 = 1". This pupil viewed 2b as 4 then got 2+2 = 4 to take away from 5 to make it "= 1" on the right hand side. The other process was taking '2b' as '2+b'.

For the level 4, item 22 "solve the equation 2(3x-1) - (x+4) = 9" an English pupil showed *the substitution-like process* as "2(3x-1) - (x+4) = 9, 2(8) – (6) = 9, x = 3". The evidence showed that the pupil seems to do "trial and error" implicitly.

6.13.2 The large drop from theme 4 level 2 to levels 3 and 4

For the level 3, item 16 "solve the equation 3y-6 = y-2" the Thai pupils used *the explicit balancing process* in the solving of equations. The high ability group tended to show *the explicit balancing process* on numbers but balancing on letters was implicit. For example, "3y-6+6 = y-2+6, 3y = y+4, 3y-y = 4, 2y = 4, $y = \frac{4}{2}$, y = 2". The most common errors were operating with negative numbers. These appeared in both explicit and implicit *balancing processes*. For example, the explicit balancing users wrote "3y-6+6 = y-2+6, 3y = y-8, 3y+8 = y-8+8, y = 2". The pupils viewed "-2+6" as "-(2+6)". They tended to ignore the first sign (whether plus or minus) and used the operation **between** those numbers.

The implicit balancing users wrote "3*y*-6+*y* = -2, 3*y*+*y* = -2+6, 4*y* = -8, *y* = -2" (errors appeared both when balancing letters and operating on numbers), and "3*y*-*y* = 6+2, 4*y* = 8, *y* = 2" (errors appeared in simplifying like terms and negative numbers). It should be noted that the Thai pupils had no experience in simplifying like terms before solving equations.

Large numbers of the low ability groups made no attempt at this question. The difficulty they faced was when the letters appeared on both sides. There were some pupils who attempted to simplify the letters and put them on one side. They then "solved" this item as "3-6 = 2*y*-2, 3 = 2*y*-2, 2+3 = 2*y*-2+2, 5 = 2*y*, $\frac{5}{2}=\frac{2y}{2}$, 2.5 = *y*" (two of the letter "*y*" on one side at the first step) and

"3*y*-6+6 = *y*-2+6, 3*y* = *y*-8, 3*y*+8 = *y*-8+8, 11*y* = *y*, *y* = 11" (sum -2+6 as -(2+6) and ignored the letter on one side at the last step).

These two examples demonstrate a lack of understanding in simplifying like terms.

By contrast, a minimal decrease in the generalisable process group between levels 2 and 3 of English pupils was seen. However, they could not use *the inverse operation process* (working back) to solve this problem as it was only taught in their algebra lessons for the case when the unknown is alone on one side. The implicit balancing process was used to solve this item among pupils in the top sets. For example, they showed the process as "3*y*-6 = *y*-2, 3*y*-*y*-6 = -2, 3*y*-*y* = -2+6, 2*y* = 4, *y* = 2". The most common errors were with operation signs when balancing was done implicitly (e.g. 3*y* = *y*-2+6, 3*y* = *y*+4, 4*y* = 4, *y* = 1). Given this situation it might be better to make the balancing process explicit in the English school teaching.

For the level 4 item 22 "solve the equation 2(3*x*-1) – (*x*+4) = 9" the English pupils commonly made errors in multiplying out the brackets. The evidence showed the first other process as "*multiply both brackets*". For example, "2(3*x*-1)-(*x*+4) = 9, 6*x*-2-**2*x*+8**,

$4x+6 = 9$, $4x = 3$, $x = \frac{3}{4}$". The second other process was "*multiply out the first term in the brackets*". For instance, "2(3*x*-1)-(*x*+4) = 9, 6*x*-1-*x*-4 = 9, 5*x*-5 = 9, 5*x* = 14, $x = 2\frac{4}{5}$".

Finally, the third other process was "*does in the brackets first*". For example, "2(3*x*-1)-(*x*+4) = 9, 2×2*x*-4*x* = 9, 9-2 = 2*x*-4*x*, 7 = -2*x*, *x* = 0.28".

These three kinds of other process were seen among Thai pupils. Moreover, Thai pupils used a further 'other process' of *balancing process*. For example, "2(3*x*-1)-(*x*+4) = 9, (3*x*-1)-(*x*-4) = $\frac{9}{2}$, (3*x*+*x*)-(-1-4) = $\frac{9}{2}$, 4*x*-(-5) = $\frac{9}{2}$, *x*-(-5) = $\frac{9}{2\times4}$, *x*+5 = 18, *x* = 18-5, *x* = 13", and "2(3*x*-1)-(*x*+4) = 9,(6*x*-2)-(*x*+4) = 9, (6*x*-2)-(*x*+4-4) = 9-4, (6*x*-2+2)-*x* = 5+2, $\frac{7x}{7} = \frac{7}{7}$ = 1".

To appreciate the other process in expanding or multiplying out the brackets, it should be recalled that both the English and Thai pupils have had the experience of hearing expressions such as "do the brackets first" and "multiply all terms in the brackets". As mentioned earlier in the simplification theme, Thai pupils were taught to solve equations using *the balancing process* without the experience of simplifying like terms. Furthermore, the work of some Thai pupils showed a lack of understanding of equivalence constraints.

The results support Herscovics and Kieran (1980) who conducted research in an effort to expand pupils' understanding of the equal sign. They found that the expressions pupils constructed were often not equivalent and contradicted the order of operations. Also Kieran's (1989a) study indicated that pupils are not aware of the underlying structure of arithmetic operations and their properties. Boulton-Lewis et al. (1998) also showed that about half of the pupils in their study did not understand "equals" in the algebraic sense as equivalence/balancing. The researcher's findings contradict those of Boulton-Lewis et al. (1998) who stated that pupils had a satisfactory understanding of inverse procedures and of correct order of operations and were able to apply arithmetic principles to algebra.

6.14 Theme 5 Graphs of linear functions

The fifth theme of the test was graphs of linear functions, organised into four levels of expected difficulty. It consisted of four questions, designed to investigate pupils' thinking processes when graphing linear functions. The questions are shown in Figure 6-13.

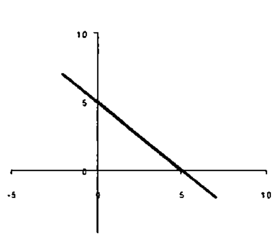
Graphs of linear functions

Item 5 Plot three coordinates and draw the line of $x+y = 4$. (Level1 Graph of the equation $x+y = c$.)

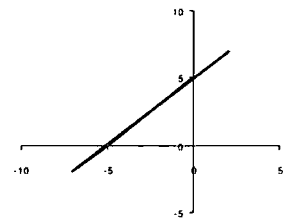
Item 11 Where does the graph of the equation $y = 2x$-6 cross the x-axis? (Level2 Graph of the equation $y = 0$, $y = mx+c$.)

Item 17 Which of the following could be part of the graph of $y = x+5$? (Level3 Graph of the equation $x = 0$, $y = 0$, $y = x+c$.)

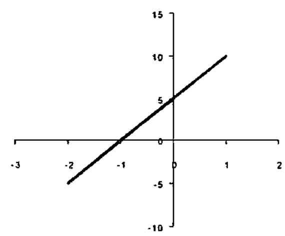
a)



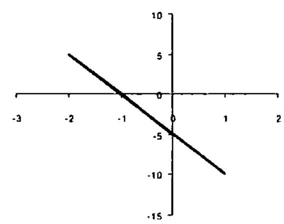
b)



c)

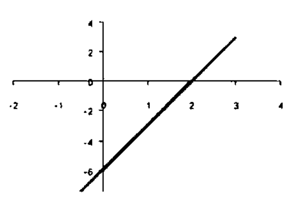


d)

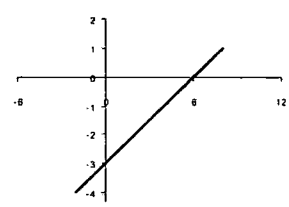


Item 23 Which of the following could be part of the graph of $y = 2x+6$? (Level4 Graph of the equation $x = 0$, $y = 0$, $y = mx+c$.)

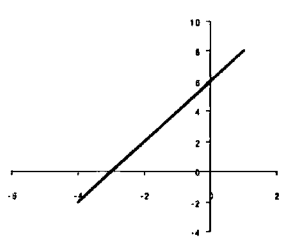
a)



b)



c)



d)

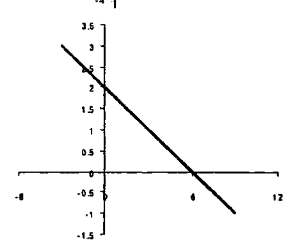


Figure 6-13 Graphs of linear functions theme test items

Pupils' thinking processes in approaching graphs of linear functions problems were categorised from pupils' responses as *generalisable process, other process, unidentified process, and incomplete response.*

Generalisable processes are those methods that reflect the way to explore functional relationships. These ways of thinking include ordered pairs recognition and graph construction strategies.

Other processes are those in which pupils incorrectly attempt to explore functional relationships. These attempts include ordered pairs recognition-like, using the constants appearing in the equation, and drawing the line in the wrong direction.

The unidentified process and the incomplete response are as defined earlier.

6.15 A comparison of pupils' thinking processes in graphing linear functions between the English and Thai schools

Figures 6-14 and 6-15 present a breakdown of the processes that the English and Thai pupils used in approaching these problems at each level of difficulty.

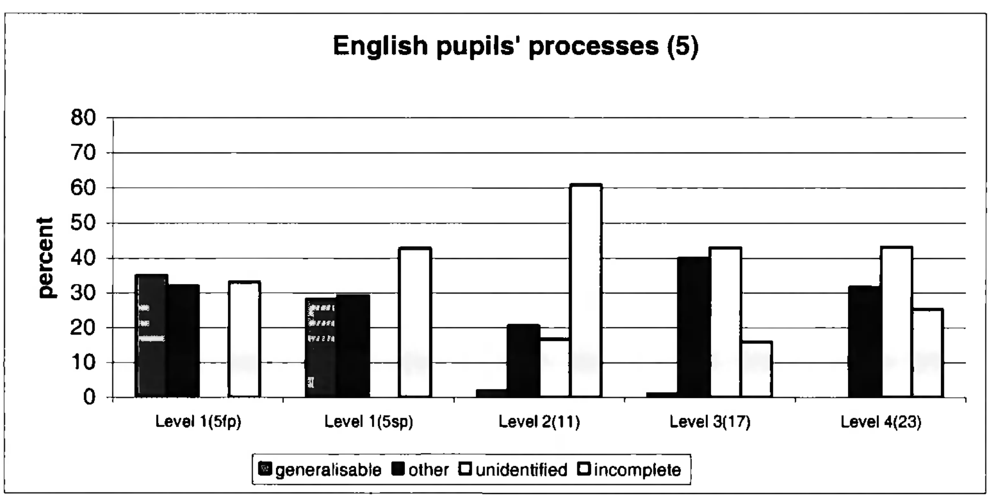


Figure 6-14 Percentage of process used in theme 5 by English pupils

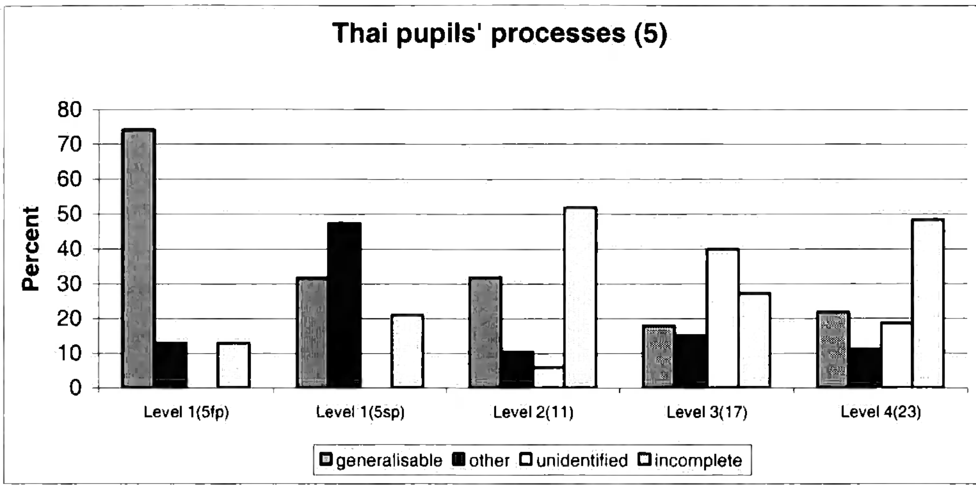


Figure 6-15 Percentage of process used in theme 5 by Thai pupils

As shown in Figure 6-14 and 6-15, Thai pupils commonly used *the generalisable process* to solve the level 1 (first part) problem. There was a large drop in using *the generalisable process* when solving the levels 2, 3, and 4 items among English pupils.

Table 6.25 gives the actual percentage of each process and corresponding outcomes at each level of difficulty.

Table 6.25 Percentage of process used and outcomes for theme 5

Country	Level (item)	Processes							
		Generalisable process		Other process		Unidentified process		Incomplete response	
		Used	% correct	Used	% correct	Used	% correct	Used	% correct
England (n=103)	1 (5fp)	35.0	100.0	32.0	15.2	0.0	0.0	33.0	0.0
	1 (5sp)	28.2	100.0	29.1	0.0	0.0	0.0	42.7	0.0
	2 (11)	2.0	50.0	20.6	0.0	16.7	0.0	60.8	0.0
	3 (17)	1.0	100.0	40.0	2.5	43.0	14.0	16.0	0.0
	4 (23)	0.0	0.0	31.6	6.7	43.2	9.8	25.3	0.0
Thailand (n=186)	1 (5fp)	74.2	100.0	12.9	33.3	0.0	0.0	12.9	0.0
	1 (5sp)	31.7	100.0	47.3	0.0	0.0	0.0	21.0	0.0
	2 (11)	31.8	59.3	10.3	0.0	5.9	9.1	51.9	0.0
	3 (17)	17.9	80.6	15.0	3.8	39.9	23.2	27.2	0.0
	4 (23)	21.9	76.5	11.0	0.0	18.7	41.4	48.4	0.0

As can be seen in Table 6.25, level 1 item 5 (first part), 35.0% of English pupils used *the generalisable process* and of those 100% gained the correct answer. For the second part, 28.2% used *the generalisable process* and of those 100% gained the correct answers. The corresponding percentages for Thai pupils were 74.2% (first part), 31.7% (second part), of whom100% gained the correct answer in each case.

There were only a small number of English pupils using the generalisable process at levels 2, 3, and 4. Of Thai pupils, 31.8% used *the generalisable process* in the level 2 question and of those 59.3% gained the correct answer. There was a large drop between level 2 and level 3 for the Thai generalisable process groups. Of Thai pupils, 17.9% showed *the generalisable process* and of those 80.6% gained the correct answer. At level 4 there was a minimal increase to 21.9% of Thai pupils showing *the generalisable process* and of those 76.5% gained the correct solution.

The following sections describe the sub-processes that pupils used at each level of difficulty.

Within the generalisable process group there are 2 sub-processes:

(1) *The ordered pair recognition process* is one in which the pupils move from the equation to ordered pairs.

(2) *The drawing graph process* is where pupils plotted some coordinates and then drew the line until it crossed the x-axis.

There are 3 sub-processes used within the other process group.

(1) *The ordered pair recognition-like process*: pupils moved from an equation to ordered pairs but these did not represent the given equation.

(2) T*he drawing graph incorrectly process*: pupils plotted the coordinates and drew a line which did not reach the x-axis or which did not represent the given function.

(3) *The constant using process*: there is an attempt to use the constant appearing in the equation.

Unidentified process and incomplete response are as defined earlier.

6.15.1 Process used and outcomes for theme 5 level 1 item 5

The level 1 item 5 "Plot three coordinates and draw the line if $x+y = 4$" was designed to investigate how the pupils find the coordinates and draw a line through them. Pupils' responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.26 shows the percentage of processes used and percentage correct in the level 1 question, item 5 (first part), of the graphs of linear functions theme.

Table 6.26 Percentage of process used and outcomes for theme 5 level 1 item 5 (1st part)

Processes Theme 5 Level 1 (5 first part)	English school		Thai school	
	Used	%correct	Used	%correct
Generalisable process	**35.0**	**100.0**	**74.2**	**100.0**
Ordered pairs recognition	35.0	100.0	74.2	100.0
Other process	**32.0**	**15.2**	**12.9**	**33.3**
Ordered pairs recognition-like	32.0	15.2	12.9	33.3
Unidentified process	**0.0**	**0.0**	**0.0**	**0.0**
Incomplete response	**33.0**	**0.0**	**12.9**	**0.0**
No response	33.0	0.0	12.9	0.0

As is evident from Table 6.26, the process used in the generalisable process group was *the ordered pair recognition process.* Of English pupils, 35.0% used this process and of those 100% gained the correct answer. The corresponding percentages for Thai pupils were 74.2% and 100%.

Pupils from both countries showed their set of ordered pairs, for example:

"(0, 4), (2, 2), (4, 0)",

"(1, 3), (2, 2), (3, 1)", and

"(1, 3), (-1, 5), (-2, 6)".

In the other process group, 32.0% of English pupils and 12.9% of Thai pupils showed at least one ordered pairs for which $x+y \neq 4$.

The ordered pair recognition-like process was used within the other process group. They found a set of ordered pairs at least one of which did not satisfy $x+y = 4$. Some attempted to form ordered pairs but showed no values. For instance, they showed their set of ordered pairs as

"(0, 4), (4, 0), **(4, 4)**",

"**(4, 3)**, **(4, 2)**, **(4, 1)**",

"(x+, y+), (y-, x+), (y-, x-)".

The incomplete response group comprised 33.0% of English pupils and 12.9% of Thai pupils, all of whom made no attempt.

Table 6.27 shows the percentage of process used and percentage correct in the level 1 question, item 5 (second part), of the graphs of linear functions theme.

Table 6.27 Percentage of process used and outcomes for theme 5 level 1 item 5 (2nd part)

Processes Theme 5 Level 1 (5 second part)	**English school**		**Thai school**	
	Used	**%correct**	**Used**	**%correct**
Generalisable process	**28.2**	**100.0**	**31.7**	**100.0**
Drawing graph	28.2	100.0	31.7	100.0
Other process	**29.1**	**0.0**	**47.3**	**0.0**
Drawing graph incorrectly	29.1	0.0	47.3	0.0
Unidentified process	**0.0**	**0.0**	**0.0**	**0.0**
Incomplete response	**42.7**	**0.0**	**21.0**	**0.0**
No response	42.7	0.0	21.0	0.0

As indicated in Table 6.27, the process used in the generalisable process group was *the drawing graph process*. Of the English pupils 28.2% used this process and of those 100% gained the correct answer. The corresponding percentages for Thai pupils were 31.7% and 100%.

Within the generalisable process group, *the drawing graph process* was used among pupils in both countries. In this process the pupils are able to obtain and plot a correct set of ordered pairs and draw a single straight line through them. For example, they showed their graphs as Figure 6-16.

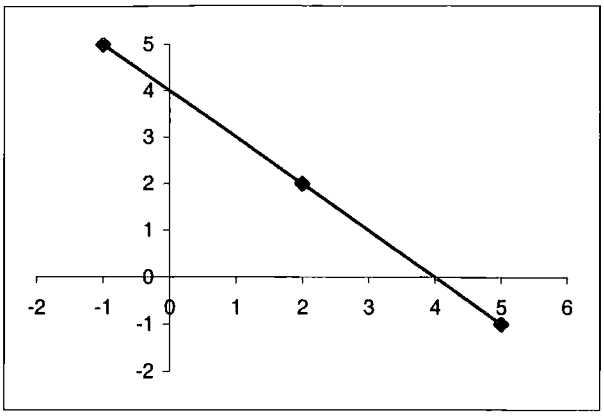


Figure 6-16 Graph of $x+y = 4$

In the other process groups were found 29.1% of English pupils and 47.3% of Thai pupils. *The drawing graph incorrectly process* was used within this group. They plotted the coordinates without drawing a line or found that their points did not lie on a single straight line. For instance, they showed their graphs as Figures 6-17 and 6-18.

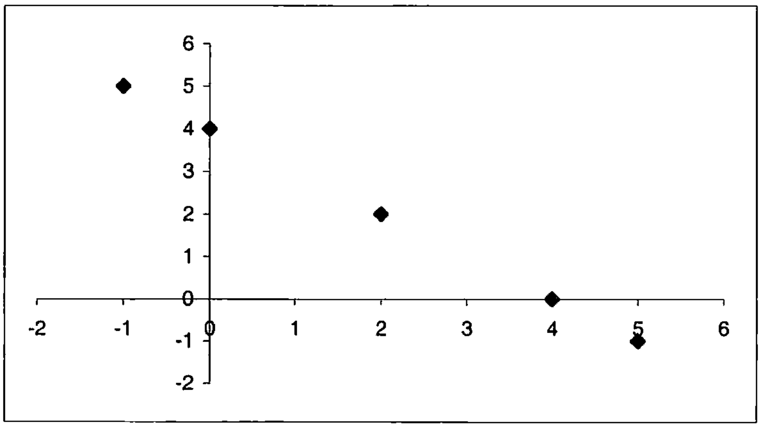


Figure 6-17 Plotting of ordered pairs

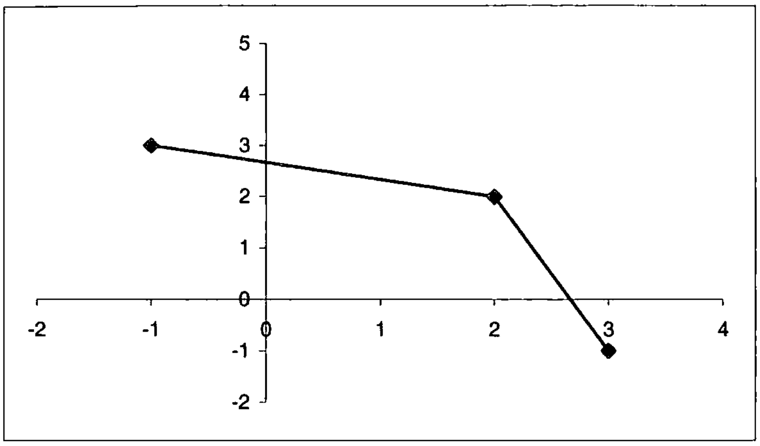


Figure 6-18 Graph did not lie on a single straight line

The incomplete response groups comprised 42.7% of English and 21.0% of Thai pupils, all of whom made no attempt.

6.15.2 Process used and outcomes for theme 5 level 2 item 11

The level 2 item 11 "Where does the graph of the equation $y = 2x-6$ cross the x-axis?" was designed to observe how they worked out the coordinates and whether they understood the meaning of "cross the *x-axis*". As before, pupils' responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.28 shows the percentage of process used and percentage correct in the level 2 question, item 11, of the graphs of linear functions theme.

Table 6.28 Percentage of process used and outcomes for theme 5 level 2 item 11

Processes Theme 5 Level 2 (11)	**English school**		**Thai school**	
	Used	**%correct**	**Used**	**%correct**
Generalisable process	**2.0**	**50.0**	**31.9**	**59.3**
Ordered pairs recognition	2.0	50.0	31.4	58.6
Drawing graph	0.0	0.0	0.5	100.0
Other process	**20.6**	**0.0**	**10.3**	**0.0**
Ordered pairs recognition-like	2.0	0.0	7.0	0.0
Drawing graph incorrectly	6.9	0.0	2.7	0.0
Constants using	11.8	0.0	0.5	0.0
Unidentified process	**16.7**	**0.0**	**5.9**	**9.1**
No process	16.7	0.0	5.9	9.1
Incomplete response	**60.8**	**0.0**	**51.9**	**0.0**
Incomplete response	1.0	0.0	0.0	0.0
No response	59.8	0.0	51.9	0.0

Table 6.28 shows that the most common process used in the generalisable process group was *the ordered pair recognition process.* Of the English pupils 2.0% used *the ordered pair recognition process* and of those 50.0% gained the correct answer. The corresponding percentages for Thai pupils were 31.4% and 58.6%.

Pupils from both countries showed *the ordered pairs recognition process* as

"3×2 = 6-6 = **0**",

"let y = **0**, 0 = 2x-6, 6 = 2x, **3** = x",

"(x, y), (**3**, **0**), (4, 2), (5, 4)".

In the other process group, 11.8% of English pupils used the constants appearing in the equation to find the solution. Of Thai pupils 7.0% used *the ordered pair recognition-like process.* The English other process group showed *the constant using process* as

"It crosses on the constant of the equation",

"2x-6 = -4", and

"**-6** must cross the x-axis to be y = 2x-**6**".

The Thai pupils showed *the ordered pairs recognition-like process* as "let $x = 6$, $2x\text{-}6 = 6$, $x = 6$, $y = 6$",

"let $x = 0$, $y = 2x\text{-}6$, $y = 2\times0\text{-}6$, $y = 0\text{-}6$, $y = \text{-}6$", and

"let $x = 1$, $y = 2(1)\text{-}6$, $y = 2\text{-}6$, $y = \text{-}4$; $y = 1, 1 = 2x\text{-}6$, $1+6 = 2x$, $7 = 2x$, $\frac{7}{2} = \frac{2x}{2}, 3\frac{1}{2} = x$".

In the unidentified process group, 16.7% of English pupils gave the answer without showing working and all the answers were wrong. Of Thai pupils 5.9% showed no working but 9.1% of these gained the correct answer.

In the incomplete response group, 59.8% of English pupils and 51.9% of Thai pupils made no attempt at this question.

6.15.3 Process used and outcomes for theme 5 level 3 item 17

The level 3 item 17 "Which of the following could be part of the graph of $y = x+5$" was designed to look at pupils' thinking processes when looking at a part of a graph. As before, pupils' responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.29 shows the percentage of process used and percentage correct in the level 3 question, item 17, of the graphs of linear functions theme.

Table 6.29 Percentage of process used and outcomes for theme 5 level 3 item 17

Processes Theme 5 Level 3 (17)	**English school**		**Thai school**	
	Used	**%correct**	**Used**	**%correct**
Generalisable process	**1.0**	**100.0**	**17.9**	**80.6**
Ordered pairs recognition	1.0	100.0	17.9	80.6
Other process	**40.0**	**2.5**	**15.0**	**3.8**
Ordered pairs recognition-like	2.0	0.0	9.2	6.3
Drawing graph incorrectly	3.0	0.0	1.2	0.0
Constants using	35.0	2.9	4.6	0.0
Unidentified process	**43.0**	**14.0**	**39.9**	**23.2**
No process	43.0	14.0	39.9	23.2
Incomplete response	**16.0**	**0.0**	**27.2**	**0.0**
No response	16.0	0.0	27.2	0.0

From Table 6.29, it can be seen that the process used in the generalisable process group was *the ordered pair recognition process.* Of English pupils 1.0% (only one pupil) used *the ordered pair recognition process* and gained the correct answer (100% success). The corresponding percentages for Thai pupils were 17.9% and 80.6%. For example, the pupils showed their sets of ordered pairs as

"All of $y = x+5$ are (0, 5), (1, 6), ..., (-4, 1), (-5, 0)",
"If $y = x+5$, the $x = y-5$ so, on the x-axis it is +5, on the y-axis it is -5", and
"Cross x at $y = 0$; $0-5 = x+5-5$, $-5 = x$; cross y at $x = 0$, $y = 5$".

In the other process group, 35.0% of English pupils used the constants appearing in the equation to find the answer. For example, English other process group showed *the constant using process* as

"5 is the constant that means the line travels through 5",

"Go 5 across and 5 up", and

"5 is not a minus".

Of Thai pupils 9.2% used *the ordered pair recognition-like processes* and of those 6.3% gained the correct answer.

Thai pupils showed *the ordered pairs recognition-like process* as

"(5, 5), $y = x+5$, $5 = 5+5$, $5 = 10$ false; (-5, 5), $5 = -5+5$, $5 = 0$ false", and

"Substitute x, y, so a) is true".

In the unidentified process group, 43.0% of English pupils gave the answer without showing working and of those 14.0% gained the correct answer. The corresponding percentages for Thai pupils were 39.9% and 23.2%.

In the incomplete response group, 16.0% of English pupils and 27.2% of Thai pupils made no attempt.

6.15.4 Process used and outcomes for theme 5 level 4 item 23

The level 4 item 23 "Which of the following could be part of the graph of y = 2x+6" was designed to investigate how the pupils find the relationship between the graph and the given function. As before, pupils' responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.30 shows the percentage of processes used and percentage correct in the level 4 question, item 23, of graphs of linear functions theme.

Table 6.30 Percentage of process used and outcomes for theme 5 level 4 item 23

Processes Theme 5 Level 4 (23)	**English school**		**Thai school**	
	Used	**%correct**	**Used**	**%correct**
Generalisable process	**0.0**	**0.0**	**21.9**	**76.5**
Ordered pairs recognition	0.0	0.0	21.9	76.5
Other process	**31.6**	**6.7**	**11.0**	**0.0**
Ordered pairs recognition-like	2.1	0.0	4.5	0.0
Drawing graph incorrectly	9.5	0.0	3.9	0.0
Constants using	20.0	10.5	2.6	0.0
Unidentified process	**43.2**	**9.8**	**18.7**	**41.4**
No process	43.2	9.8	18.7	41.4
Incomplete response	**25.3**	**0.0**	**48.4**	**0.0**
No response	25.3	0.0	48.4	0.0

As can be seen in Table 6.30, the process used among the Thai generalisable process group was *the ordered pair recognition process.* Of Thai pupils 21.9% used *the ordered pair recognition process* and of those 76.5% gained the correct answer. For example, the pupils showed their sets of ordered pairs as

"(x, y), (0, 6), (1, 8), (2, 10)",

"Substitute x, y values into the equation", and

"Cross x at $y = 0$, (-3, 0); cross y at $x = 0$, (0, 6)".

There were no English pupils in the generalisable process group.

In the other process group, 20.0% of English pupils used the constants appearing in the equation to find the answer. Of Thai pupils 4.5% used *the ordered pair recognition-like processes.* The English pupils in other process group showed *the constant using process* as

"Used the last number in the equation to work out",

"Because of 6 being the add", and

"Find the number, which have 2 and 6 as not minus numbers".

Thai pupils showed *the ordered pairs recognition-like process* as

"a) (2, -6), y = $2x$+6, So –6 = 10, b) (6, -3) → -3 = 0, c) (-3, 6) → 6 = 0, d) (6, 2) → 2 = 18".

In the unidentified process group, 43.2% of English pupils gave the answer without showing working and of those 9.8% gained the correct answer. The corresponding percentages for Thai pupils were 18.7% and 41.4%.

In the incomplete response group, 25.3% of English pupils and 48.4% of Thai pupils made no attempt.

6.16 Summary and discussion of findings Theme 5

English pupils' processes: the generalisable process group commonly used *the ordered pairs recognition process* in the level 1 (first part), 2 and 3 problems. None of these pupils used the generalisable process in the level 4 problem. They frequently used the *drawing graph process* in the level 1 question (second part). The main process used in the other process group in tackling the level 1 question (first part) was in using *the ordered pair recognition-like process. The drawing graph incorrectly process* was mainly used in the level 1 question (second part). *The constant using process* was frequently used in the level 2, level 3, and level 4 questions. *The unidentified process* group gave the answer without showing working to the levels 2, 3, and 4 questions. *The incomplete response* group in each of the 4 questions comprised predominantly those who made no response at all.

Thai pupils' processes: the generalisable process group mainly used *the ordered pairs recognition process* in the levels 1 (first part), 2, 3, and 4 questions. They frequently used *the drawing graph process* in the level 1 question (second part). The main process used in the other process group in tackling the levels 1 (first part), 2, 3, and 4 items was *the ordered pairs recognition-like process. The drawing graph incorrectly process* was commonly used in the level 1 question (second part). *The unidentified process* group gave the answer without showing working in the levels 2, 3, and 4 questions. *The incomplete response* group in each of four questions again comprised predominantly those who made no response at all.

The results suggest that English and Thai pupils in the generalisable process groups commonly used a similar process to approach the problems. A small number of English pupils used the generalisable process in the levels 2 and 3 questions. The Thai pupils in the other process group tended to *draw the graph incorrectly. The constant using process* was that most commonly used among the English pupils in the other process group.

6.16.1 Using other process but obtained the correct solution

For the level 1 item 5 "plot three coordinates and draw the line of $x+y = 4$", the first part of this item asked for three pairs of coordinates. Finding two of three correct ordered pairs earned the mark but it was regarded as other process. For example they gave the correct ordered pairs (4, 0), (0, 4) and an incorrect third ordered pair such as (-4, 0), or (-4, 4) that indicated their other process.

For the level 3 item 17 "which of the following could be part of the graph of $y = x+5$?", one English pupil who used the constant appearing in the equation to get the answer and gain the correct solution explained the process as "+5 more of $x = y$". The evidence shows the pupil could draw the graph of $x = y$. In drawing graph lessons English pupils were taught to draw the graphs of $x = \pm$ c, $y = \pm$c (c is a constant), and

$y = x$. This pupil used the familiar graph to make sense of the new situation and gained the correct answer.

For the level 4 item 23 "which of the following could be part of the graph of $y = 2x+6$?", an English pupil who used the constant appearing in the equation to find the solution explained the process as "used the last number in the equation to work out". There was only one choice of the graphs in which $y = 6$. Thus the correct answer is gained.

6.16.2 The large drop from theme 5 level 1 first part to the second part and to levels 2, 3, 4

The Thai pupils showed the graph of $x+y = 4$ only plotting the ordered pairs. They did not draw the line as the question asked them to do. The use of other process in drawing graph among the Thai pupils in the second part of level 1 item 5 reflects the taught experience in the Thai school, which is very different from that in the English school. In the first year, pupils faced with equations, moved from the equation to a set of ordered pairs, plotted those ordered pairs on the graph, and then drew a line but only under certain conditions. The Thai school placed very strong emphasis on these conditions (see Chapter 4). For example, pupils have to plot the points and draw a line only when it is given that x, y are **real** numbers; they are taught not to join the line when x, y are **integers**. They are also taught to draw line segments when x is more/less than a **given number**. These sophisticated steps and details could well lead to the use of other process and confusion among Thai pupils and the consequent drop in success from the first to the second part of item 5.

By contrast, the English pupils plotted the ordered pairs and drew the line **without any conditions**. The English pupils were less successful in the first part and inevitably there was less opportunity to draw the correct straight line.

For the levels 2, 3, and 4, the English pupils used the constants in the equations to find their solutions. The other process in *using constants* among the English pupils reflects the taught experience in the English school. Plotting points and joining them were the exercises for Year 7 pupils. Writing down the coordinates and drawing straight-line graphs formed the practice for Year 8 pupils. There were lessons on the intersection of two straight-line graphs and intercepts on the x-axis and the y-axis (e.g. $x = -4$, $y = 7$, these two lines crosses at (-4, 7), and graph $y = 2$ crosses y-axis at 2). It could be argued that the English pupils tried to use the numbers appearing in the equations to find their answers because of their experience of the lessons mentioned above.

The Thai pupils responded to the level 2 item for finding "the coordinates of the point where graph $y = 2x$-6 crosses the x-axis" by wrongly substituting $x = 0$ in the equation and finding the value of y. In their algebra lessons, much emphasis was placed on "crossing x when $y = 0$, and crossing y when $x = 0$". The evidence of using substitution $x = 0$ in this item indicated some memory of what they were taught, but without understanding. The x-intercept, y-intercept content was taught to the high ability group but not in the low ability group.

For the levels 3 and 4, the Thai pupils attempted to check all the choices given by substituting x values to find y values. However, the numbers they used were not appropriate and therefore they could not find the correct choices. These pupils made conclusions such as "no correct choice given". The most common numbers they used to substitute were the numbers appearing on the graph in each choice. For example, $x = 5$, $y = 5$ for choice (a) of the level 3 question, and $x = 2$, $y = -6$ for choice (a) of the level 4 question.

The approach to graphs of linear functions in the Thai school seems to be contrary to a recommendation from Sfard (1989) that function concept should not be introduced by a set of ordered pairs but rather by a dependence of one varying quantity on another. For the English school the "Function machine" provides a primary idea of the function concept. The study in the complexity of the function concept using the function machine of DeMarois and Tall (1999) show that for many pupils the complexity of the function concept such that the making of direct links between all the different representations is a difficult long-term task.

6.17 Theme 6 Word problems

The sixth theme of the test was word problems, also organised into four levels of expected difficulty. It consisted of five questions, designed to observe the pupils' thinking processes as they solved word problems. The questions are shown in Figure 6-19.

Word problems

Item 6a I think of a number, times it by 4. The answer is 20. What was my original number? (Level 1 one variable in one step)

Item 6b I think of a number, times it by 3, and then take away 5. The answer is 16. What was my original number? (Level 1 One variable in two steps)

Item 12 David is 21 years old. Susan is 3 years old. When will David be exactly twice as old as Susan? (Level 2 One variable in two steps with brackets and positive numbers)

Item 18 The Old Elvet Centre gym has 2-kilogram and 5-kilogram disks for weight lifting. Due to their budget, this year they only have fourteen disks in all. The total weight of the 2-kilogram disks is the same as the total weight of the 5-kilogram disks. What is the total weight of all the disks? (Level 3 One variable in two steps with brackets and negative numbers)

Item 24 The length of a rectangle is twice as long as its width. The area of the rectangle is 32 square metres. What is the width and the length of this rectangle? (Level 4 One variable of second order)

Figure 6-19 Word problems theme test items

As with the other themes pupils' thinking processes in approaching word problems were categorised from their responses as *generalisable process, other process, unidentified process* and *incomplete response.*

Generalisable processes are methods that show the correct way to solve word problem using arithmetic or algebraic processes. These processes include modelling, inverse operations, and repeated operations (trial and error) methods.

Other processes are those in which pupils attempted to make sense of each situation using arithmetic or algebraic processes which were incomplete or only partially correct. These attempts include modelling-like, inverse operation-like, and repeated operation-like methods.

As before, there is also the unidentified process and the incomplete response processes.

6.18 A comparison of pupils' processes in solving word problems between the English and Thai schools

Figures 6-20 and 6-21 give a breakdown of the processes that the English and Thai pupils used in approaching these problems at each level of difficulty.

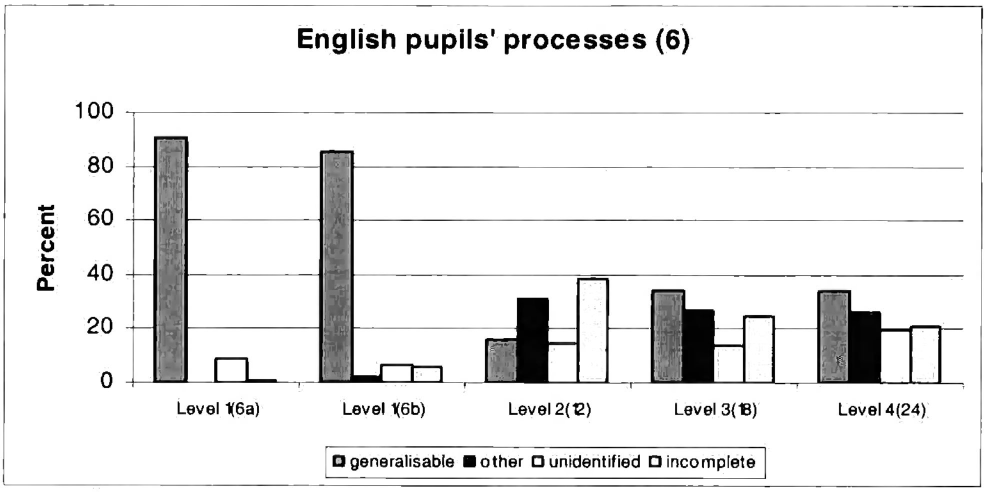


Figure 6-20 Percentage of process used in theme 6 by English pupils

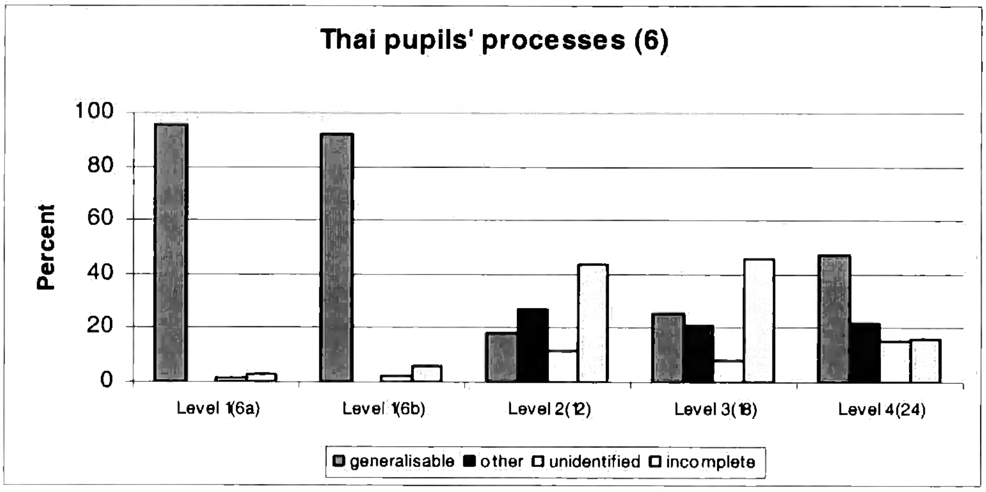


Figure 6-21 Percentage of process used in theme 6 by Thai pupils

As displayed in Figures 6-20 and 6-21, a large number of English and Thai pupils mainly used *the generalisable process* to solve the level 1 items 6a, 6b. There was a large drop in using *the generalisable process* at levels 2, 3 and 4 questions.

Table 6.31 gives the actual percentage of each process and corresponding outcomes at each level of difficulty.

Table 6.31 Percentage of process used and outcomes for theme 6

Country	Level (item)	Processes							
		Generalisable process		Other process		Unidentified process		Incomplete response	
		Used	% correct	Used	% correct	Used	% correct	Used	% correct
England (n=103)	1(6a)	90.3	97.8	0.0	0.0	8.7	77.8	1.0	0.0
	1(6b)	85.4	95.5	1.9	0.0	6.8	71.4	5.8	0.0
	2(12)	15.7	68.8	31.4	40.6	14.7	6.7	38.2	0.0
	3(18)	34.0	82.4	27.0	0.0	14.0	21.4	25.0	0.0
	4(24)	33.7	96.8	26.1	8.3	19.6	16.7	20.7	0.0
Thailand (n=186)	1(6a)	95.7	99.4	0.0	0.0	1.1	100.0	3.2	0.0
	1(6b)	91.9	93.6	0.0	0.0	2.2	100.0	5.9	0.0
	2(12)	17.8	90.9	27.0	8.0	11.9	22.7	43.2	0.0
	3(18)	25.3	100.0	21.2	5.6	8.2	21.4	45.3	0.0
	4(24)	47.0	87.1	22.0	27.6	15.2	80.0	15.9	0.0

As reported in Table 6.31, level 1 item 6a, 90.3% of English pupils used *the generalisable process* and of those 97.8% gained the correct answer. The corresponding percentages for Thai pupils were 95.7% and 99.4%. There was a minimal decrease between level 1 item 6a and item 6b of those making up the generalisable process group in both countries. Of English pupils 85.4% used *the generalisable process* and of those 95.5% gained the correct answer. Of Thai pupils 91.9% used *the generalisable process* and of those 93.6% gained the correct solution.

For the level 2 item, there was a sharp drop in those using *the generalisable process*. Of English pupils 15.7% used *the generalisable processes* and of those 68.8% gained the correct answer. The corresponding percentages for Thai pupils were 17.8% and 90.9%.

For the level 3 item, there was an increase to 34.0% of English pupils using *the generalisable process* and of those 82.4% gained the correct answer. The corresponding percentages for Thai pupils were 25.3% and 100%.

For the level 4 item, there was a minimal decrease to 33.7% of English pupils using *the generalisable process* and of those 96.8% gained the correct answer. There was an increase between level 3 and level 4 for the corresponding group in Thailand. Of Thai pupils 47.0% used *the generalisable process* and of those 87.1% gained the correct solution.

The following sections describe the sub-processes, which pupils used at each level of difficulty.

Within the generalisable process group there are 3 sub-processes:

(1) *The modelling process* in which the pupils translate from words to an equation and then solve the equation.

(2) *The inverse operation process* reflects the way of working as the opposite operation from that given in the question.

(3) *The repeated operation process* refers to those who used some form of trial and error with correct substitutions.

There are 3 sub-processes used within the other process group.

(1) *The modelling-like process* in which pupils attempt to translate from words to equation but in different forms of situation given.

(2) *The inverse operation-like process* is where the pupils attempt to do the opposite operations but in the wrong order.

(3) *The repeated operation-like process* is where the pupils attempt a trial and error solution but with incomplete/incorrect substitution.

The unidentified process and the incomplete response groups are defined as earlier.

6.18.1 Process used and outcomes for theme 6 level 1 item 6a

The level 1 Item 6a "I think of a number, times it by 4. The answer is 20. What was my original number?" was designed to investigate how pupils find the original number when the equation formed is expected to be of the type $ax = b$. Pupils' responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.32 shows the percentage of process used and percentage correct in the level 1 question, item 6a, of the word problems theme.

Table 6.32 Percentage of process used and outcomes for theme 6 level 1 item 6a

Processes Theme 6 Level 1 (6a)	English school		Thai school	
	Used	%correct	Used	%correct
Generalisable process	**90.3**	**97.8**	**95.7**	**99.4**
Modelling	35.9	97.3	65.1	99.2
Inverse operations	43.7	100.0	25.8	100.0
Repeated operations	10.7	90.9	4.8	100.0
Other process	**0.0**	**0.0**	**0.0**	**0.0**
Unidentified process	**8.7**	**77.8**	**1.1**	**100.0**
No process	8.7	77.8	1.1	100.0
Incomplete response	**1.0**	**0.0**	**3.2**	**0.0**
No response	1.0	0.0	3.2	0.0

As shown in Table 6.32, the most common process used among the English generalisable process group was *the inverse operation process.* Of English pupils 43.7% used *the inverse operation process* and of those 100% gained the correct answer. The Thai generalisable process group used *the modelling process.* Of Thai pupils 65.1% used this process and of those 99.2% gained the correct answer.

For example, English generalisable process pupils showed *the inverse operation process as*

"Divide 20 by 4",
"Did it backwards", and
"Do the reverse, 20/4".

Thai generalisable process group showed *the modelling process* as

"$x \times 4 = 20$, $x \times \frac{4}{4} = \frac{20}{4}, x = 5$",

"$a \times 4 = 20$, $a = \frac{20}{4}, a = 5$", and

"Make it an equation".

In *the unidentified process* group, 8.7% of English pupils gave the answer without showing working and of those 77.8% gained the correct solution. The corresponding percentages for Thai pupils were 1.1% and 100% (two pupils).

In *the incomplete response* group, only 1.0% of English pupils and 3.2% of Thai pupils made no attempt at this question.

6.18.2 Process used and outcomes for theme 6 level 1 item 6b

The level 1 Item 6b "I think of a number, times it by 3, and then take away 5. The answer is 16. What was my original number?" was designed to investigate how pupils find the original number when the equation formed is expected to be of the type $ax + b = c$. As before, pupils' responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.33 shows the percentage of process used and percentage correct in the level 1 question, item 6b, of the word problems theme.

Table 6.33 Percentage of process used and outcomes for theme 6 level 1 item 6b

Processes Theme 6 Level 1(6b)	**English school**		**Thai school**	
	Used	**%correct**	**Used**	**%correct**
Generalisable process	**85.4**	**95.5**	**91.9**	**93.6**
Modelling	35.9	91.9	73.7	93.4
Inverse operations	42.7	100.0	15.6	100.0
Repeated operation	6.8	85.7	2.7	60.0
Other process	**1.9**	**0.0**	**0.0**	**0.0**
Inverse operation-like	1.0	0.0	0.0	0.0
Repeated operation-like	1.0	0.0	0.0	0.0
Unidentified process	**6.8**	**71.4**	**2.2**	**100.0**
No process	6.8	71.4	2.2	100.0
Incomplete response	**5.8**	**0.0**	**5.9**	**0.0**
No response	5.8	0.0	5.9	0.0

As can be seen in Table 6.33, the most common process used among the English generalisable process group was *the inverse operation process.* Of English pupils 42.7% used the inverse operation process and of those 100% gained the correct answer. For example, the English generalisable process group showed *the inverse operation process* as

"Add 5 to 16 then divide the number you get by 3",

"16+5 = 21, $\frac{21}{3} = 7$", and

"Did the sum backwards".

The Thai pupils in the generalisable process group commonly used *the modelling process.* Of the Thai pupils 73.7% used this process and of those 93.4% gained the correct answer. They showed their processes as

"$x \times 3-5 = 16$",

"$a3-5 = 16$", and

"$(x \times 3)-5 = 16$, $(x \times 3)-5+5 = 16+5$, $x \times 3 = 21$, $\frac{(x \times 3)}{3} = \frac{21}{3}$, $x = 7$".

In the other process group, only one English pupil used *the inverse operation-like process* and the other *used the repeated operation-like process* (trial and error) and showed the process as "$\frac{16}{3}+5=10.1$". Another English pupil showed *the repeated operation-like process* as

"___× by 3-5=16, it is below 0 and then found -4 and it worked".

In *the unidentified process* group, 6.8% of English pupils gave the answer without showing working and of those 71.4% gained the correct solution. The corresponding percentages for Thai pupils were 2.2% and 100%.

In *the incomplete response* group, 5.8% of English pupils and 5.9% of Thai pupils made no attempt at this question.

6.18.3 Process used and outcomes for theme 6 level 2 item 12

The level 2 item 12 "David is 21 years old. Susan is 3 years old. When will David be exactly twice as old as Susan?" investigated how pupils found the solution when the expected equation is of the type $x+a = 2(x+b)$. As before pupils' responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.34 shows the percentage of process used and percentage correct in the level 2 question, item 12, of the word problems theme.

Table 6.34 Percentage of process used and outcomes for theme 6 level 2 item 12

Processes Theme 6 Level 2 (12)	English school		Thai school	
	Used	%correct	Used	%correct
Generalisable process	**15.7**	**68.8**	**17.8**	**90.9**
Modelling	0.0	0.0	1.1	100.0
Repeated operations	15.7	68.8	16.8	90.3
Other process	**31.4**	**40.6**	**27.0**	**8.0**
Modelling-like	31.4	40.6	27.0	8.0
Unidentified process	**14.7**	**6.7**	**11.9**	**22.7**
No process	14.7	6.7	11.9	22.7
Incomplete response	**38.2**	**0.0**	**43.2**	**0.0**
No response	38.2	0.0	43.2	0.0

From Table 6.34, it is clear that the most common process used in the generalisable process group was *the repeated operation process.* Of English pupils 15.7% used *the repeated operation process* and of those 68.8% gained the correct answer. The corresponding percentages for Thai pupils were 16.8% and 90.3%.

For example, the generalisable process groups from both countries showed *the repeated operation process* as

"22 4, 23 5, 24 6, ..., 35 17, 36 18 ",

"Add up until David's is twice", and

"Adding 15 onto each person's".

In the other process group, 31.4% of English pupils used *the modelling-like process* and of those 40.6% gained the correct answer. The corresponding percentages for Thai pupils were 27.0% and 8.0%.

For example, the pupils in the other process group showed *the modelling-like process* as

"Double the numbers",

"21-3=18×2=36", and

"times 21 by 3 then halved it"

In the *unidentified process* group, 14.7% of English pupils gave the answer without showing working and of those 6.7% gained the correct solution. The corresponding percentages for Thai pupils were 11.9% and 22.7%.

In the incomplete response group, 38.2% of English pupils and 43.2% of Thai pupils made no attempt at this question.

6.18.4 Process used and outcomes for theme 6 level 3 item 18

The level 3 item 18 "The Old Elvet Centre gym has 2-kilogram and 5-kilogram disks for weight lifting. Due to their budget, this year they only have fourteen disks in all. The total weight of the 2-kilogram disks is the same as the total weight of the 5-kilogram disks. What is the total weight of all the disks?" was designed to probe pupils' thinking processes in solving a word problem that related to a real world situation; the expected equation being of the form $2x = 5(14-x)$ or $5y = 2(14-y)$. As before, pupils' responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.35 shows the percentage of process used and percentage correct in the level 3 item 18, of the word problems theme.

Table 6.35 Percentage of process used and outcomes for theme 6 level 3 item 18

Processes Theme 6 Level 3 (18)	English school		Thai school	
	Used	%correct	Used	%correct
Generalisable process	**34.0**	**82.4**	**25.3**	**100.0**
Repeated operations	34.0	82.4	25.3	100.0
Other process	**27.0**	**0.0**	**21.2**	**5.6**
Modelling-like	27.0	0.0	21.2	5.6
Unidentified process	**14.0**	**21.4**	**8.2**	**21.4**
No process	14.0	21.4	8.2	21.4
Incomplete response	**25.0**	**0.0**	**45.3**	**0.0**
Incomplete work	0.0	0.0	0.6	0.0
No response	25.0	0.0	44.7	0.0

As presented in Table 6.35, the process used in the generalisable process groups in both countries was *the repeated operation process.* Of English pupils 34.0% used *the repeated operation process* and of those 82.4% gained the correct answer. The corresponding percentages for Thai pupils were 25.3% and 100%. For example, the generalisable process group showed *the repeated operation process* as

"3×5 = 15, 11×2 = 22, 4×5 = 20, 10×2 = 20 ",

"2×10 = 20, 5×4 = 20", and

"14 = 1+13, 2+12, 3+11, 4+10, 5×4 = 2×10".

In the other process group, 27.0% of English pupils used *the modelling-like process* with no success. Of Thai pupils 21.2% used *the modelling-like process* and of those 5.6% gained the correct solution. For example, the pupils showed *the modelling-like process* as

"5×7 = 35, 2×7 = 14",

"2×14 = 28, 2×5 = 10", and

"Half 14 is 7, 2kg = 7×2, 5kg = 7×5, total 14+35 = 49".

In the unidentified process group, 14.0% of English pupils gave the answer without showing working and of those 21.4% gained the correct solution. The corresponding percentages for Thai pupils were 8.2% and 21.4%.

In the incomplete response group, 25.0% of English pupils and 44.7% of Thai pupils made no attempt at this question.

One Thai pupil showed the incomplete work as "14×2 = 28, 14×5 = 70".

6.18.5 Process used and outcomes for theme 6 level 4 item 24

The level 4 item 24 "The length of a rectangle is twice as long as its width. The area of the rectangle is 32 square metres. What are the width and the length of this rectangle?" was designed to look at pupils' thinking processes in solving a familiar mensuration problem, the expected equation being of the form $ax^2 = b$. As before, pupils' responses were categorised into four groups as generalisable process, other process, unidentified process and incomplete response.

Table 6.36 shows the percentage of process used and percentage correct in the level 4 item 24, of the word problems theme.

Table 6.36 Percentage of process used and outcomes for theme 6 level 4 item 24

Processes Theme 6 Level 4 (24)	English school		Thai school	
	Used	%correct	Used	%correct
Generalisable process	**33.7**	**96.8**	**47.0**	**87.1**
Modelling	3.3	66.7	15.9	71.4
Repeated operations	30.4	100.0	31.1	95.1
Other process	**26.1**	**8.3**	**22.0**	**27.6**
Modelling-like	22.8	9.5	21.2	25.0
Repeated operation-like	3.3	0.0	0.8	0.0
Unidentified process	**19.6**	**16.7**	**15.2**	**80.0**
No process	19.6	16.7	15.2	80.0
Incomplete response	**20.7**	**0.0**	**15.9**	**0.0**
No response	20.7	0.0	15.9	0.0

As presented in Table 6.36, the most common process used among the generalisable process group in both countries was *the repeated operation process.* Of English pupils 30.4% used *the repeated operation process* and of those 100% gained the correct answer. The corresponding percentages for Thai pupils were 31.1% and 95.1%. For example, the pupils showed *the repeated operation process* as

"Guess numbers until got two numbers that had the smaller one half the big one",

"2x16 = 32, 4x8 = 32, 4 = half 8", and

"$2x \times x = 32$, 12x6, 6x3, 8x4".

In the other process group, 22.8% of English pupils used *the modelling-like process* and of those 9.5% gained the correct answer. The corresponding percentages for Thai pupils were 21.2% and 25.0%. For example, the other process group showed *the modelling-like process* as

"Divide 32 by 2 and divide by 2 again",

"$\frac{32}{4} = 8$", and

"$2x\times 2 = 32$, $\frac{2x\times 2}{2} = \frac{32}{2}$, $2x = 16, x = 8$".

In the unidentified process group, 19.6% of English pupils gave the answer without showing working and of those 16.7% gained the correct solution. The corresponding percentages for Thai pupils were 15.2% and 80.0%.

In the incomplete response group, 20.7% of English and 15.9% of Thai pupils made no attempt at this question.

6.19 Summary and discussion of findings Theme 6

The English pupils' thinking processes: the generalisable process group mainly used *the inverse operations process* to solve the level 1 (6a, 6b) questions. They frequently used *the repeated operations process* in the levels 2, 3, and 4 questions. Only two pupils showed other process in tackling the level 1 (6b) question. *The modelling-like process* was commonly used in the levels 2, 3, and 4 questions. *The unidentified process* group gave the answer without showing working. *The incomplete response* group in each of the five questions comprised predominantly those who made no response at all.

The Thai pupils' thinking processes: the generalisable process group commonly used *the modelling process* to solve level 1 (6a, 6b) question. They frequently used *the repeated operations process* on the levels 2, 3, and 4 questions. The main using of other process in tackling the levels 2, 3, and 4 questions arose in the use of *the modelling-like process. The unidentified process* group gave the answer without showing working. *The incomplete response* group in the levels 1, 2 and 3 comprised predominantly those who made no response at all.

From the results it can be seen that English and Thai pupils in the generalisable process group used different processes to solve the level 1 (6a, 6b) question. They used similar processes when facing the harder questions in the levels 2, 3, and 4. They made similar use of other process throughout the four levels.

To solve the level 1 (6a, 6b) question the English generalisable process group slightly preferred *the inverse operation process* to *the modelling process.* The Thai pupils strongly preferred *the modelling process.* This empirical evidence reflected the lessons taught in the English and the Thai case study schools. In the English school, the lessons in solving these kinds of question emphasised *the reverse process*. In Thailand, *the modelling process* was the only process used to solve this type of problem.

In the English school, word problems content was taught for only one lesson out of 20 in the Year 7 top set and one of 12 in the Year 8 bottom set. In the Thai school, this topic represented about 20% of the algebra content in secondary 1 high ability group (4 lessons out of 20), none in low ability group, 23.1% in secondary 2 high ability group (3 lessons out of 13), and 11.1% in low ability group (one lesson out of nine). Although Thai pupils had more experience on this topic, the processes used to approach the levels 2, 3, and 4 problems were similar.

6.19.1 Using other process but obtained the correct solution

For the level 2 item 12 "David is 21 years old. Susan is 3 years old. When will David be exactly twice as old as Susan?" the pupils who were unable to form an equation but gained the correct answer explained their processes as "18 years between S and D, $S = 18$, $D = 36$", and "21-3 = 18, 21+15 = 36, 3+15 = 18".

The first approach has been successful because it could show that the difference in age is x (18), the younger is x (18), the older is $2x$ (36). In this case pupils used the numbers appearing in the question as 21-3 = 18, giving Susan's age. Susan's age times two (18x2) gives David's age.

The second approach was successful because it could show that the equation is of the form '$a-b = b+y$', where a: older age, b: younger age, y: next period of time. As follows:

$a+y = 2(b+y) \rightarrow a-2b = y \rightarrow a-b = b+y$

$21+15 = 36 \qquad\qquad \rightarrow 21-3 = 18.$

Pupils did not formulate an equation and used an arithmetic approach to gain the answer. Similarly, MacGregor and Stacey (1993b) demonstrated that the majority of pupils do not use a syntactic translation procedure to write algebraic equations.

For the level 3 item 18 "The Old Elvet Centre gym has 2-kilogram and 5-kilogram disks for weight lifting. Due to their budget, this year they only have fourteen disks in all. The total weight of the 2-kilogram disks is the same as the total weight of the 5-kilogram disks. What is the total weight of all the disks?", the explanations of the Thai pupils who used *the modelling-like process* and gained the correct answer were: "Find the least common multiple of 2 and 5 that add up to 14, that is 10 and 4, then times 2 *(10×2)* and 5 *(4×5)*, add them makes 40" and "Thinking out the two parts of 2-kilogram and 5-kilogram, suppose they are 20, then divided by 2-kilogram and 5-kilogram *(20÷2, 20÷5)*". The pupils explained their processes in words supporting Nibbelink (1990) who states that pupils show an aversion to using letters instead of numbers, especially when they know what the number should be. Also MacGregor and Stacey (1997) have shown that pupils interpretations of letters and algebraic expressions are based on intuition and guessing and on analogies with other systems they know.

For the level 4 item 24 "the length of a rectangle is twice as long as its width. The area of the rectangle is 32 metres square. What are the width and the length of this rectangle?", the English pupils who used *the modelling-like process* showed their working as "$\frac{32}{4}=8, \frac{8}{2}=4$", "$2x+2x\times4=32$, $4x=\frac{32}{4}$, $x=8$", and "$\frac{32}{2}=16, \frac{16}{2}=8$".

The Thai pupils showed their working as
"$4x = 32, x = \frac{32}{4}, x = 8$, length = twice width, $\frac{8}{2} = 4$, width = 4",
"$2x\times2=32, \frac{2x\times2}{2}=\frac{32}{2}, 2x=16, \frac{2x}{2}=\frac{16}{2}, x = 8$".

Finding the area and perimeter of a rectangle were familiar topics in both the English and the Thai schools. Not surprisingly, pupils obtained the correct answers with the explanation above. The evidence showed that these pupils knew the correct answers but realised they were expected to set up an equation.

6.19.2 The large drop from theme 6 level 1 to levels 2, 3, and 4

The levels 2, 3, and 4 questions were problems that could not translate directly word by word to algebraic symbols as in the level 1 questions.

For the level 2 item 12, the pupils in both countries tried to use the numbers in the question such as "21+6 and 3+6", "added 21 to 3 then double it", "21×2 = 42, 3×2 = 6", or " the age difference is 18, times by 2 gives 36".

The first example reflects their thinking of "twice the age of Susan = 3×2". This group of pupils viewed David's age as "more than Susan's age by twice her age" instead of "exactly twice". They got 27 as David's age and 9 as Susan's age, which means the difference in their ages is twice Susan's age. However, the question asked when David would be **exactly twice** as old as Susan. The second and the third ignored the "exactly twice". The fourth was as explained above in Section 6.19.1.

However, the most successful approach was "keep adding one to each age until one number is twice as big as the other". This proved to be a simple way to get the answer.

For the level 3 item 18, the pupils in both countries tended to use the numbers appearing in the question such as "7×5 = 35, 7×2 = 14, 35+14 = 49", "2kg×5kg = 10, 14×10kg = 140", "2+5 = 7kg, 14×7kg = 98kg". The first example reflects their thought as "the **number** of disks of each kind is the same". However, the question asked for "the **weight** of disks to be the same". The second example simply multiplied all the numbers to get the answer. The third example simply added the two weights and then multiplies by 14. The second and third examples have no merit. However, the successful calculation was "5×2kg = 10kg, 2×5kg = 10kg 10×2kg = 20kg, 4×5kg = 20kg", what is called *the repeated operations process* in the present study.

The level 4 item 24, requires knowledge of the formula for finding the area of a rectangle. Some English pupils viewed 32 as the perimeter and showed the working as "x+x+2x+2x = 32, 6x = 32". The process "divide 32 by 2 and by 2 again" was used to deal with "length is twice as long as width" by both the English and Thai pupils. Some

Thai pupils tried to set up an equation such as "$x \times y = 32$" and followed by 'trial and error'. Others formed equations such as "$2x \times x = 32$" or "$a \times 2a = 32$" and followed by solving the equation.

The most successful process used among pupils in both countries was "*the repeated operation*". They found the product of two numbers until they got the answer (8×4 = 32). Some pupils drew the rectangle and tried to multiply two numbers until the correct pair found (e.g."16×2, 8×4").

Only a small number of pupils in each country solved the problems with the use of algebra. Again, this supports Nibbelink's (1990) views that introductory chapters in algebra tend to move very quickly and ask problems which could easily be solved without the use of algebra. As a result, many pupils do not take the early chapters in algebra seriously and later realize that they should have.

CHAPTER 7

CONCLUSIONS AND IMPLICATIONS

7.1 Introduction

The present study is an introduction to the understanding of pupils' thinking processes in the early stages of learning algebra. The study has shown the need for further research on the way pupils think about algebra. As discussed in chapters 2 and 3, past research has been inclined to study errors in given answers. However this research has tried to learn from pupils' explanations and thereby enhance our understanding of their thinking.

First the chapter describes how the research questions posed in Chapter 1 were answered and enabled the purpose of the study to be fulfilled by suggesting possibilities for teaching and learning. The chapter then reflects on possible limitations arising in the research and their consequent influences on the understanding of pupils' thinking when solving algebraic problems. The chapter closes by examining the scope for future research in algebra and suggests research questions that have arisen out of the present study.

7.2 Answers to the research questions

Chapter 1 outlined the three research questions for this study. This investigation has, to an extent, provided answers to these questions. Although answers may only be partial they do appropriately lead to new and useful research questions that other researchers may wish to investigate further. This section outlines the answer to the questions as posed in Chapter 1.

7.2.1 English and Thai pupils' thinking processes in solving algebraic problems

Table 7.1 summarises the findings of Chapter 6. It shows a comparison of pupils' thinking processes when solving each item of the algebra test developed by the researcher. Additional explanation of the terms used follows the table.

Table 7.1 Comparison of pupils' thinking processes between the English and Thai schools

Theme	Level (item)	The English pupils' processes		The Thai pupils' processes	
		Generalisable	Other	Generalisable	Other
1	1(1a)	Repeated operation 58.3%	-	Repeated operation 45.7%	-
		Generalisation 29.1%	-	Generalisation 30.6%	-
		-	-	Draw or count 22.0%	-
	1(1b)	Generalisation 70.9%	-	Generalisation 66.1%	-
		-	-	Repeated operation 17.7%	-
		-	-	Draw or count 11.8%	-
	1(13a)	Repeated operation 76.5%	-	Repeated operation 68.6%	-
		-	-	Draw or count 20.0%	-
	2(7)	Repeated operation 74.8%	-	Repeated operation 81.7%	-
	2(13b)	Generalisation 11.8%	Scaling up 27.5%	Draw or count 53.0%	Repeated …-like 10.3%
		Draw or count 11.8%	Repeated…-like 15.7%	-	-
	2(19a)	Repeated operation 74.0%	-	Repeated operation 69.8%	-
		-	-	Draw or count 17.2%	-
	3(13c)	-	Repeated…-like 10.8%	-	Repeated…-like 21.6%
	4(19b)	-	Repeated…-like 22.2%	-	Repeated…-like 22.8%
2	1(2)	Left to right 48.5%	-	Left to right 25.3%	-
	2(8)	Grouping 22.5%	-	-	Letter ignored 13.4%
	3(14)	Incorrect operation 29.7%	-	-	-
	4(20)	Incorrect operation 18.8%	Incorrect operation 13.5%	-	-
3	1(3)	Correct arithmetic 71.8%	-	Correct arithmetic 71.0%	-
		-	-	Incorrect arithmetic 10.2%	-
	2(9)	Correct arithmetic 38.2%	Correct arithmetic 19.6%	Correct arithmetic 66.1%	-
		Incorrect arithmetic 15.7%	-	Incorrect arithmetic 13.4%	-
	3(15)	Incorrect arithmetic 35.0%	-	Correct arithmetic 51.4%	-
		Correct arithmetic 28.0%	-	Incorrect arithmetic 28.6%	-
	4(21)	Incorrect arithmetic 45.3%	-	Correct arithmetic 38.0%	-
		-	-	Incorrect arithmetic 30.1%	-

Note: Presented process used from 10% and more

Table 7.1 (continue)

Theme	Level (item)	The English pupils' processes		The Thai pupils' processes	
		Generalisable	Other	Generalisable	Other
4	1(4)	Balancing 47.6%	-	Balancing 75.3%	-
		Substitution 13.6%	-	-	-
	2(10)	Balancing 39.2%	-	Balancing 67.7%	-
	3(16)	Balancing 43.0%	-	Balancing 29.1%	-
	4(22)	Multiply out bracket 14.0%	Multiply out…-like 17.2%	-	Multiply out…-like 12.5%
				-	Balancing-like 11.3%
5	1(fp)	Ordered pairs 35.0%	Ordered pairs-like 32.0%	Ordered pairs 74.2%	Ordered pairs-like 12.9%
	1 (sp)	Drawing graph 28.2%	Drawing graph incorrect 29.1%	Drawing graph 31.7%	Drawing graph incorrect 47.3%
	2(11)	-	Constant using 11.8%	Ordered pairs 31.4%	-
	3(17)	-	Constant using 35.0%	Ordered pairs 17.9%	-
	4(23)	-	Constant using 20.0%	Ordered pairs 21.9%	-
6	1(6a)	Inverse operation 43.7%	-	Modelling 65.1%	-
		Modelling 35.9%	-	Inverse operation 25.8%	-
		Repeated operation 10.7%	-		-
	1(6b)	Inverse operation 42.7%	-	Modelling 73.7%	-
		Modelling 35.9%	-	Inverse operation 15.6%	-
	2(12)	Repeated operation 15.7%	Modelling-like 31.4%	Repeated operation 16.8%	Modelling-like 27.0%
	3(18)	Repeated operation 34.0%	Modelling-like 27.0%	Repeated operation 25.3%	Modelling-like 21.2%
	4(24)	Repeated operation 30.4%	Modelling-like 22.8%	Repeated operation 31.1%	Modelling-like 21.2%
				Modelling 15.9%	

Note Presented process used from 10% and more

Table 7.1 shows the process mainly used to tackle each item by theme. Only those processes with at least 10% use are included. The blank spaces may be interpreted as no generalisable/other processes used, less than 10% use, or no response.

Looking at theme 1 patterns/sequences, it is seen that the most common process used by both English and Thai pupils is the same for all items with the exception of level 2, 13b, and trivially, the level 3 and 4 items where the use of a generalisable process was less than 10%. In the case of item 13b, for Thai pupils who had no experience of this topic (patterns/sequences) in their curriculum, it is not surprising that about half of them resorted to the elementary approaches of drawing and counting. Some English pupils (11.8%) also used this approach but an equal number used a generalisation process showing that they were starting to think in an algebraic way. Additionally, about a quarter of English pupils used a *scaling up process,* which although not give the correct answer, suggests that a more sophisticated level of thinking was being used.

Again in theme 2 simplification, Thai pupils were unfamiliar with the simplification of like terms and so the use of a generalisable process was below 10% in levels 2, 3 and 4. In contrast about a quarter of English pupils showed a generalisable process at these levels. This difference simply reflects their curriculum differences.

In theme 3 substitution, the performance of Thai pupils seems to be much better than that of English pupils. In items 9, 15 and 21, not only is there a greater use of generalisable processes but also Thai pupils carry out arithmetic accurately. Substitution is a process that is widely used in the Thai curriculum, so it is not surprising that Thai pupils do better in this theme.

In theme 4 solving equations, Thai pupils used *the balancing process* throughout. This was the only method taught in the Thai school. It proved to be quite successful at levels 1 and 2 but less so at levels 3 and 4. English pupils also predominantly used *the balancing process* and interestingly, were more successful than the Thai pupils at levels 3 and 4, even though they were less successful than the Thai pupils at levels 1 and 2.

In theme 5 graphs of linear functions, a little over a quarter of English pupils used generalisable processes at level 1, and at levels 2, 3 and 4 fewer than 10% did so. At these higher levels English pupils attempted to use the constants appearing in the equation but did so inappropriately. Thai pupils could find ordered pairs with decreasing success at higher levels.

Looking at theme 6 word problems, the level 1 items 6a and 6b proved to be easily completed by almost all pupils. At the higher levels a "trial and error" (*repeated operation*) approach was the recognised generalisable process. "Other" process was "*modelling-like*" in which an attempt was made to translate from words to an equation but not, in fact, to one representing the given situation. Although percentages varied the approaches of the English and Thai pupils were similar.

7.2.2 Commonalities and differences in pupils' thinking processes to solve algebraic problems

Commonalities

The similar high success rates in levels 1 and 2 of patterns/sequences theme may be because pupils could reach the answer easily simply by using arithmetic procedures (number bonds) with no algebra being needed. In the level 3 and 4 questions, where algebra is clearly required for the n^{th} term expression, pupils from both countries found that their understanding was inadequate. In the case of Thai pupils, who had received no teaching on the n^{th} term, this is not surprising and although English pupils had received some teaching on this topic, the impression gained by the researcher was that little emphasis was placed on it. Also the researcher's test was taken about eight months after the topic was taught, giving time to forget it. In general, pupils seem to have grasped the basic concept of continuing patterns and sequences. The generalisable process groups primarily used *the repeated operation process* to approach the problems.

English and Thai pupils more or less used similar processes to approach the level 1 question of substitution theme. The content area substitution was taught separately in the English school in both Year 7 and Year 8 while the Thai school did not teach this area independently but offered it under the topic of solving equations where it was used to check answers in both Secondary 1 and Secondary 2.

The high success rate at level 1 and failure to achieve in both countries at levels 2, 3, and 4 of the word problems theme suggests that most pupils were not able to link algebraic methods to the solution of word problems.

The more successful pupils in the Thai school tended to use brackets to remind themselves of the order of operations required to solve equations. In spite of pupils having been taught these topics in both countries, they had some difficulty in calculating with negative numbers (e.g. viewed $-2+6$ as $-(2+6)$), understanding the index notation (e.g. viewed 4^2 as 4×2), and multiplying out brackets (see Section 6.13.2).

Differences

Simplification of like terms was taught as a specific topic in the English school both in Year 7 and Year 8. The Thai school did not teach this content area independently but covered it under the topic of solving equations where a common factor approach [$2x+5x=(2+5)x$] was used in both Secondary 1 and Secondary 2 for dealing with like terms. English pupils were far more successful than their Thai counterparts. Thai pupils, whose only experience of simplification occurred in the context of solving equations, had difficulty dealing with questions asking only to simplify. Thai pupils persisted in trying to get a numerical answer i.e. set up an equation and solve it.

The differences in percentage of generalisable process groups between English and Thai pupils were increased when facing the harder questions, with the Thai pupils being significantly better in the substitution theme. Perhaps a reason for the better performance of the Thai pupils is in the frequent use made of substitution when working with equations and graphs. Thai pupils used substitution to check solutions to equations and also to calculate coordinates for graphs. This work was strongly emphasised in the Thai school but the same link between substitution, equations and graphs was not made in the English school. Again the Thai generalisable process group tended to make use of brackets when substituting negative numbers.

Solving equations using *inverse operations* and *implicit balancing* was covered in both Year 7 and Year 8 in the English school. *Explicit balancing* was used to teach this topic in the Thai school in both Secondary 1 and Secondary 2. The results seem to suggest that Thai pupils were more successful in solving the level 1 and 2 questions than the English pupils. However, English pupils were more successful in solving the level 3 and 4 questions. The relatively poor performance of Thai pupils on the level 3 and 4 questions indicates that the understanding of *the balancing method* has broken down and the pupils have not been able to transfer their techniques from the easy equations to the harder ones. It may be that Thai pupils have been taught to respond by memorising rather than by understanding. In fact, *the explicit balancing* method, as taught in the Thai school did not require understanding of the meaning, but only knowledge of the appropriate "moves". On the other hand English pupils, where lessons emphasised understanding of the concepts introduced at each stage, maintained a more constant level of performance when solving the equations.

The graphs of linear functions content occupied only a few lessons in the English school in Year 7 and Year 8. The Thai school placed much more emphasis on this area in Secondary 1 but spent only a few lessons in Secondary 2. For Secondary 2, the graphs lessons came towards the end of term and were often rushed in order to complete the curriculum on time. As stated in section 6.16.2, the English school taught pupils to find the intersection of two straight-line graphs but taught only the graphs that are parallel to the x-axis and to the y-axis (e.g. $x = -4$, $y = 7$, these two lines crosses at $(-4, 7)$). This led to the use of "other process" among English pupils in trying to use the constant appearing in the question. Pupils who were taught to draw the graphs without experience of x-intercept, y-intercept content tended to ignore the questions.

Little time was spent on word problems in both the English and Thai schools. The English school taught pupils to solve word problems with *inverse operation* and *modelling process* (translating from words to an equation). In contrast, the Thai school delivered only one process, *the modelling process*. The English generalisable process group slightly preferred *the inverse operation process* to *the modelling process*. Thai pupils were restricted to *the modelling process*.

The timing of the test in relation to delivery and completion of the various algebra topics may have advantaged the Thai pupils. The scheme of work used in the English school to deliver the algebra content was very different from that used in the Thai school. The English school chose to break up the content into a number of fragmented periods of work spread over all three of the school terms. In the Thai school the work was presented in a more consolidated fashion and was delivered in two longer spells in the second half of the academic year. The researcher's test was taken towards the end of the academic year. For the Thai pupils this followed almost immediately on the work they had been doing in class but for the English pupils much of the work had been done earlier in the first half of the academic year. Consequently, in spite of some revision, recall of the English pupils may have deteriorated.

The factor analysis carried out in Chapter 5 would appear to lend some support to the above observations. In the English school the first component loads heavily for five of six themes: patterns/sequences, simplification, substitution, solving equations and word problems. In second component loads heavily on the missing theme in the first component, graphs of linear functions. The researcher observes that their results reflect the arrangement of teaching in the English school where the graphs of linear functions theme is taught and delivered quite separately from the other themes.

In the Thai school the first component loads heavily in simplification, substitution, solving equations and graphs of linear functions. The second component loaded heavily on patterns/sequences and word problems. The researcher believes that the themes in the first component are those which suit the Thai tradition of using memorisation, whereas the patterns/sequences and word problems themes require greater intuition and awareness of process.

However, as Gould (1996) comments interpretation of factor analysis is always difficult and any interpretation of the factors uncovered is always open to an alternative by another reader.

7.3 Limitations of the study

A number of limitations of the present study are likely to have an impact on the results. These limitations are largely a consequence of the case study schools and methodology adopted for the study. Although the environment in the English school was mostly positive a major difficulty experienced was that pupils were only allowed to stay in the building during lessons. This greatly limited the opportunities for conducting interviews. As a result interviews could only be conducted in the after lunch period, about 20-30 minutes.

It was not possible to cover the whole range of ability in the available time. Therefore some decisions about which groups to be observed had to be made. The researcher made the decision to introduce as strong a contrast as possible by using high and low ability groups and in particular concentrate on the algebra taught in the first two years.

Another limitation arises from the coding system employed as described in Chapter 4 section 4.4.4. Trying to understand pupils' thinking processes from their written responses only is difficult because any given response may have been reached by many different approaches. However, asking for explanations of working brings its own problems. It has to be recognised that pupils may have greater difficulty in explaining their methods of solution than in actually carrying out the solution itself. Thus, when the researcher asked pupils to explain their working it is quite possible that the understanding may be correct, but that the pupils were unable to adequately explain what they have done. This situation is more likely to arise in the patterns/sequences and word problems themes than in the other themes.

The researcher's work in observing lessons and holding conversations with pupils in which the words "Tell me how you work it out" "Explain how you got this" were commonly used, increases the likelihood that the researcher will not misinterpret the pupils' responses. However, there remains the possibility that the metacognitve ability to explain their working lags behind the ability to carry out the mathematical process.

Therefore it may be possible that in some cases the analysis underestimated the quality of the pupils' thinking processes.

A further limitation is that the study was restricted to only one school in England and one in Thailand. Only eight teachers, four in each school participated. Any variations in the quality of the teaching are likely to have an important effect. This may limit the generalised application of findings to pupils in other schools.

7.4 Implications of the study and suggestions for future research

This research found that the generalisable process groups in both the English and Thai schools used the arithmetic operations effectively. When moving from arithmetic to algebra, these groups tended to remain with simpler operations, which could be repeated rather than move on to more advanced ones. For example, repeated addition was carried out where multiplication would be more effective, and when looking at sequences, later terms could only be reached by calculating all the preceding terms. With only a few exceptions, generalisation to the n^{th} term could not be made. For pupils in the generalisable process groups, greater accuracy was achieved by the use of brackets to remind themselves of the correct order of operations required and of the stages involved in the substitution of negative numbers.

The patterns/sequences and simplification themes were missing from the first two years in the Thai secondary school mathematics curriculum. Thai pupils do have experience in primary school of working with patterns and sequences in a concrete way. However, the secondary school curriculum, at the time of this research, does not make use of these early experiences to assist in the understanding of the related algebra leading to n^{th} term formulae. Also difficulties arose in the process of solving equations when pupils could not simplify like terms. This research suggests that a small scale research to investigate the connection between the patterns/sequences and solving equations, and also between simplification of like terms and solving equations, be carried out. For the graphs of linear functions theme, the sophisticated steps in drawing graphs **with conditions** (e.g. x, y are integers, are real numbers) in Secondary 1 led to confusion. This research suggests that if conditions were not introduces at such on early stage then this confusion would be reduced. Further research needs to be carried out to confirm these suggestions. What seems to be needed are some small scale researches to evaluate this suggestion. For example, investigate

- Teaching patterns/sequences to Secondary 1 and secondary 2 pupils before solving equations.
- Teaching simplification of like terms to Secondary 1 and Secondary 2 pupils before solving equations.
- Teaching to draw graphs of linear functions without conditions to Secondary 1 pupils.

In the English school, the patterns/sequences theme having been done in year 7 was ignored in year 8. Time was spent on simplification and substitution at the expense of graphs of linear functions and word problems. This research suggests that teachers should try to include patterns/sequences topic in Year 8 and allow more time for the teaching of graphs of linear functions and word problems. Ideally the algebra content should be seen as an integrated whole, even though its teaching is spread over the three terms, and the connections between the various themes should be pointed out whenever possible.

The use of inverse operations in solving equations, as in the English National Curriculum, led to difficulties when faced with the unknown on both sides. This research suggests that further research be carried out into the solution of equations using explicit balancing with understanding of equivalence.

A further finding is that pupils' main difficulties arise from an inadequate knowledge of fundamental number operations. It is important that more research be carried out on the topic of numbers with emphasis on operating with negative numbers, and understanding exponents.

The research reported here confirms the difficulty of moving from arithmetic to algebra. Pupils are not willing to give up arithmetic methods in favour of algebra when they can "see" the answer without using algebra. Unfortunately when problems are too difficult for the answer to be "obvious", the pupils' algebra has not developed sufficiently to be used effectively to find a solution. The remaining task is choosing the test items that bridge this difficulty and help pupils' transition from arithmetic to algebra.

The codebook provides an extensive way of coding pupils' thinking processes in solving algebraic problems. It shows a way to understand pupils' thinking processes in approaching algebraic problems. Hopefully it will serve as a tool for mathematics teachers in helping to understand the complexity of their pupils' thinking processes.

This study was carried out during "normal" lessons and within this context certain elements were beyond the control of the researcher. However, it is believed that only by carrying out research in the classroom situation is it possible to provide results that may truly be useful for classroom practice.

In a research project it is usual to start with a small-scale investigation before moving to a medium sized one and eventually to one on a large scale. The researcher worked with pupils in small groups before developing the algebra test, which was then used in the classroom situation in the two schools. A large-scale investigation involving more schools could follow this.

Appendix A Letter to Administrators

School of Education
University of Durham

My name is Narumon Sakpakornkan, a PhD student at University of Durham. My research project is related to secondary school mathematics curriculum and the processes of pupils' thinking in learning mathematics. The focus of this research project will provide insight in the processes in pupils' thinking in solving mathematics problems. The methods of collecting data will comprise classroom observations and semi- structured interview with participants. Throughout this research project teachers' and pupils' rights with regard to continuing participation and anonymity in final thesis will be observed. I will not disturb the daily routines of the school. The participants have the right to not answer questions and they may withdraw from the research at any time.

In order to conduct this research on mathematics curriculum and the processes of pupils' thinking in solving mathematics problems, I would like to ask for your permission to observe year7 and year8 mathematics lessons, and collecting the data from October 2001 to July 2002 in your prestigious school.

Yours sincerely,

Miss Narumon Sakpakornkan
PhD student
School of Education
University of Durham

University of Durham

School of Education
University of Durham
Leazes Road
Durham DH1 1TA
United Kingdom
Main Switchboard: 0191 374 2000
Direct Line: 0191 374
Fax: 0191 374 3506
International: +44 191 374
E-mail:

October 2002

To
Director
Buriram Provincial General Education Office
Buriram
Thailand

Dear Director,

Ms. Narumon Sakpaknorkan is a PhD student at the University of Durham. She is working on a research project related to the teaching and learning of algebra in secondary schools. Her intention is to compare the processes of pupil thinking in algebra in a school in Durham, UK and in a school in Thailand.

Thus far she has collected and analysed data from the school in Durham. Now she is returning to Thailand and wishes to collect data from Buriram Pittayakhom School in Buriram Province. I would like to ask for permission for her to observe years 1 and 2 lessons at the above school.

I would be very grateful for your assistance in this matter as I believe that the research being conducted will make a significant contribution to the thinking about mathematics education.

Your faithfully

Tony Harries

Dr. Tony Harries
(Lecturer in Mathematics Education
Course Leader for Mathematics Education Courses
PhD supervisor)

ที่ ศธ 1524.01/ 2430

สถาบันราชภัฏบุรีรัมย์
ถนนจิระ อำเภอเมืองบุรีรัมย์
จังหวัดบุรีรัมย์ 31000

25 ตุลาคม 2545

เรื่อง ขออนุญาตให้บุคลากรเข้าดำเนินการเก็บข้อมูลเพื่อการวิจัย

เรียน ผู้อำนวยการสามัญศึกษาจังหวัดบุรีรัมย์

สิ่งที่ส่งมาด้วย หนังสือขออนุญาตดำเนินการวิจัย

ด้วย นางสาวนฤมล ศักดิ์ปกรณ์กานต์ อาจารย์ 2 ระดับ 6 สังกัดโปรแกรมวิชาคณิตศาสตร์ สถาบันราชภัฏบุรีรัมย์ ขณะนี้กำลังศึกษาต่อระดับปริญญาเอก สาขาการศึกษาคณิตศาสตร์ ณ University of Durham ประเทศอังกฤษ มีความประสงค์จะขออนุญาตดำเนินการเก็บข้อมูลเพื่อการวิจัย ณ โรงเรียนบุรีรัมย์พิทยาคม อำเภอ เมือง จังหวัดบุรีรัมย์ เนื่องจากสถานที่ดังกล่าวมีคุณลักษณะที่น่าสนใจในการศึกษาข้อมูล

ดังนั้น สถาบันฯจึงขออนุญาตให้ นางสาวนฤมล ศักดิ์ปกรณ์กานต์ เข้าเก็บรวบรวมข้อมูลเพื่อการวิจัย ตั้งแต่วันที่ 28 ตุลาคม 2545 ถึงวันที่ 21 กุมภาพันธ์ 2546

จึงเรียนมาเพื่อโปรดพิจารณา สถาบันฯหวังเป็นอย่างยิ่งว่าคงได้รับความอนุเคราะห์จากท่านด้วยดี ขอขอบคุณเป็นอย่างสูงมา ณ โอกาสนี้

ขอแสดงความนับถือ

(ผู้ช่วยศาสตราจารย์ ดร.ปราโมทย์ เบญจกาญจน์)
อธิการบดีสถาบันราชภัฏบุรีรัมย์

สำนักงานอธิการบดี
โทร. 0 4461 1221 ต่อ 3102 , 3103

ที่ ศธ 0840 / [illegible]

สำนักงานสามัญศึกษาจังหวัดบุรีรัมย์
ถนนนิวาศ อำเภอเมือง บร 31000

28 ตุลาคม 2545

เรื่อง ขออนุญาตให้บุคลากรเข้าดำเนินการเก็บข้อมูลเพื่อการวิจัย

เรียน ผู้อำนวยการโรงเรียนบุรีรัมย์พิทยาคม

ด้วยสถาบันราชภัฏบุรีรัมย์แจ้งว่า นางสาวนฤมล ศักดิ์ปกรณ์กานต์ ตำแหน่งอาจารย์ 2 ระดับ 6 สังกัดโปรแกรมวิชาคณิตศาสตร์ สถาบันราชภัฏบุรีรัมย์ ขณะนี้กำลังศึกษาต่อระดับปริญญาเอก ณ University of Durham ประเทศอังกฤษ มีความประสงค์ขออนุญาตดำเนินการเก็บข้อมูลเพื่อการวิจัย ที่โรงเรียนบุรีรัมย์พิทยาคม ระหว่างวันที่ 28 ตุลาคม 2545 ถึงวันที่ 21 กุมภาพันธ์ 2546

สำนักงานสามัญศึกษาจังหวัดบุรีรัมย์ ใคร่ขอความอนุเคราะห์ท่านได้ให้ข้อเสนอแนะ และความสะดวกในการดำเนินการเก็บข้อมูลของบุคลากรดังกล่าวด้วย จักขอบคุณยิ่ง

จึงเรียนมาเพื่อโปรดทราบและดำเนินการต่อไป

ขอแสดงความนับถือ

(นายสันต์ชัย พุทธบุญ)
ผู้อำนวยการสามัญศึกษาจังหวัดบุรีรัมย์

ฝ่ายพัฒนาการศึกษา
โทร. 0 4461 2408
โทรสาร 0 4461 4362

Appendix B The English and Thai schools' tests

Maths Test Year 7 (half term one)
45 Minutes NO calculators
Work as fast as you can.

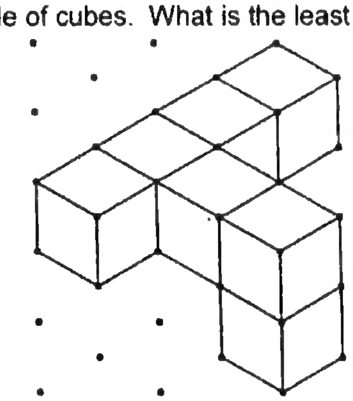
1. Put these sets of numbers in order from smallest to biggest.
   a) 71, 23, 17, 32. b) 84, 73, 91, 89, 18

2. Write in words:
   a) 36 b) 4.7 c) 12.68

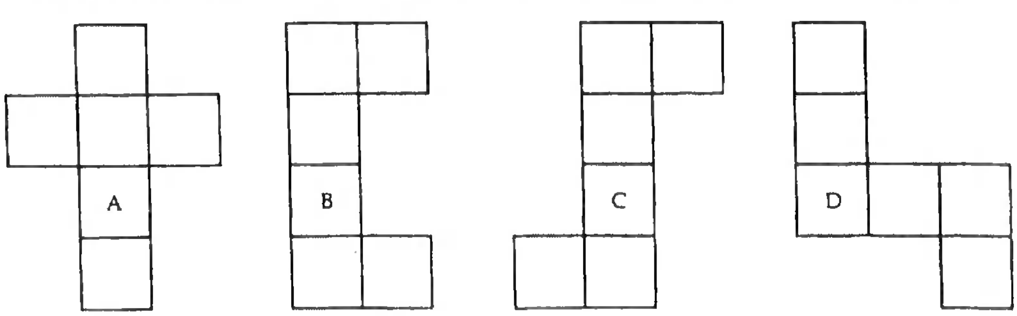
3. Write in figures:
   a) Fifty nine b) One point four
   c) Eleven point zero five six

4. Write down the next 3 ODD numbers bigger than 10.

5. Here is a shape made of cubes. What is the least number of cubes needed to make the shape?



6. Here are four patterns of six squares. Which two patterns are nets of cubes?

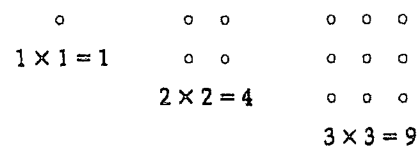


7. a) 32 + 46 = b) 43 – 24 =
   c) 327 + 286 = d) 436 – 253 =

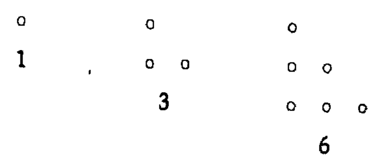
8. Put these sets of numbers in order from smallest to biggest.
   a) 7, 0.3, 12, 0.14, 2 b) 0.115, 0.23, 3, 0.1

9. In the number 9632.785 the 7 stands for 7 tenths. What does the
   a) 6 stand for? b) 8 stand for?
   c) 9 stand for? d) 5 stand for?

10. Here are three square patterns. Draw the next square pattern and write down its multiplication.



11. Here are three triangle patterns. Draw the next triangle pattern and write down the number of dots.



12. Write down the next TWO numbers in these sequences
   a) 3, 6, 9, 12 b) 4, 9, 14, 19
   c) 46, 39, 32, 25, 18 d) 2, 3, 5, 8, 12, 17

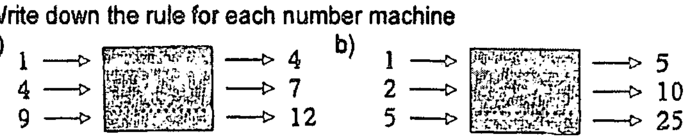
13. a) 16 x 10 = b) 23 x 100 =
   c) 450 ÷ 10 = d) 3600 ÷ 100 =

14. a) 2.53 + 8.89 = b) 3.2 + 4.63 + 7 =
   c) 4.2 – 2.16 =

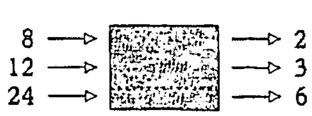
15. Using these digits only 2, 6, 3, 4 make
   a) the largest possible number
   b) the smallest possible number

16. Write down the rule for each number machine

a) b)

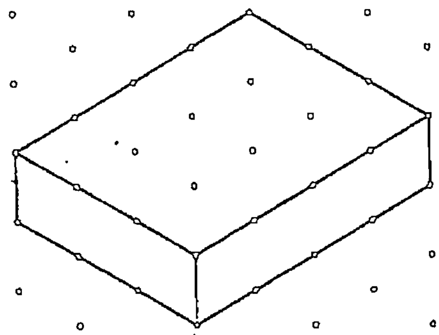


c)



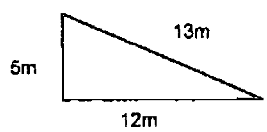
17. What needs to be added to a) 16.39 to make 26.39?
   b) 23.41 to make 23.51?

18. Here is a cuboid drawn on 1cm isometric paper. The lines which show its edges are drawn their correct lengths. Write down these sizes for the cuboid:
 a) length b) width c) height

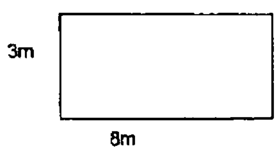


19. Find the <u>perimeter</u> of these shapes

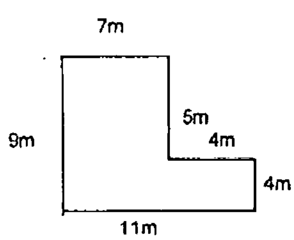
a)



b)



c)



d) a square of side 6m.

20. a) 26.3 x 10 = b) 246.5 ÷ 10 =
 c) 879 ÷ 100 = d) 0.093 x 1000 =

21. a) 0.9 ÷ 0.1 = b) 0.99 ÷ 0.01 =

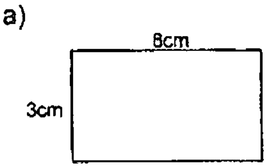
22. a) What is 0.1 less than 2.0?
 b) What is 0.01 more than 2.09?

23. In each case find the value of X.
 a) 10 x 0.4 = X. b) 0.4 x X = 400
 c) 0.4 ÷ 10 = X

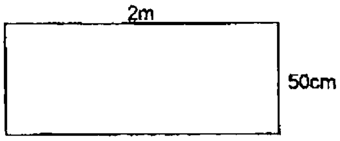
24. a) $9^2 =$ b) $3^2 + 4^2 =$

25. Find the area of these shapes.

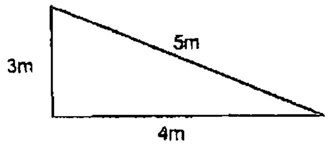
a)



b)



c)



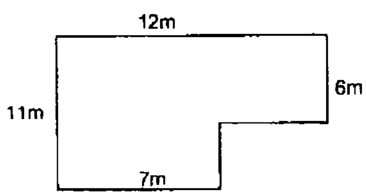
26. Place one of these signs >, =, <, between each pair of numbers.

a) 2.1 1.8 b) 0.4 0.23 c) 12.405 12.45

27. The n^{th} term of a sequence is n+5. Write the first 4 terms.

28. a) Which is warmer, -8°C or 3°C?
b) Which is warmer, -15°C or -5°C?

29. Find the area of this shape:

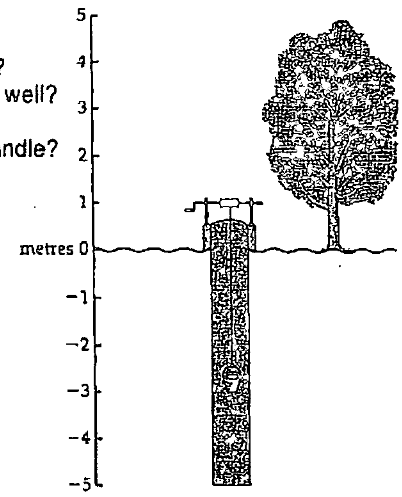


30. Find the number half way between
a) 7 and 8
b) 6.5 and 6.6
c) -3 and 6

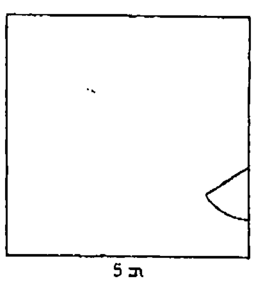
31. Write down these temperatures in order. Start with the coldest.
-4°C, 2°C, -1°C, 0°C, -7°C, 5°C

32. The temperature in an igloo is 7°C. The temperature outside is -25°C.
a) What is the difference between the inside and the outside temperatures?
b) The inside temperature goes up by 4°C. What is the new temperature?
c) The outside temperature goes down 13°C. What is the new temperature?

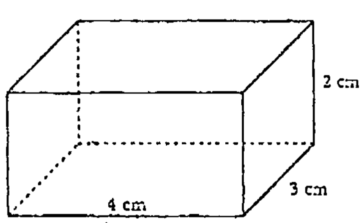
33. The diagram shows a picture of a tree and a well.
  a) How far below the top of the tree is the well handle?
  b) How far is the top of the tree from the bottom of the well?
  c) How far is the bucket from the bottom of the well?
  d) How far is the bottom of the bucket from the well handle?



34. A room is a square of side 5m.
  a) Find the area of the room.
  b) If a carpet costs £15 for $1m^2$, how much will it cost to carpet the room?
  c) There is skirting board round the room with a 1m gap for the door. How long is the skirting board?



35. Find the total surface area of this cuboid.



MATHEMATICS DEPARMENT

YEAR 7 TEST

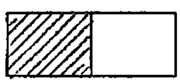
HALF TERM 2 45 minutes.

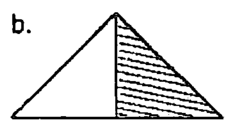
CALCULATORS MAY BE USED

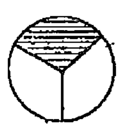
Level 3

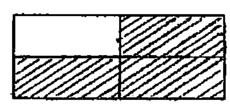
1. Work out the following -
   a. 846 + 195 b. 348 – 179
   c. 476 ÷ 4 d. 237 x 5

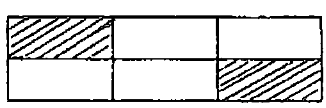
2. What fraction of each of the following shapes is shaded.

a.


b.


c.


d.


e.


3. Cancel each of the following fractions to its simplest form.
   a. $\frac{5}{10}$ b. $\frac{24}{30}$ c. $\frac{9}{36}$ d. $\frac{27}{30}$

Level 4

4. Ten boxes of matches were taken and the number of matches in each box were counted and found to be as follows -
   48, 49, 49, 49, 49, 50, 51, 51, 52, 53.
   What is
   a. The mode b. The range
   of this group of numbers.

5. On the axes provided plot and label the points
   A (4, 0) B (4, 3) C (2, 3) D (2, 0).
   Join up the points.
   What is the name of the shape you have drawn?

6. Simplify each of the following.
   a. $p + p + p$ b. $2x + 3y + 5x - y$

   c. $3x^2 + 2x + x^2 + 5x$

Level 5

7. If $a = 4$, $b = 2$, $c = 3$, work out the value of the following.
   a. $a + b$. b. $a - c$. c. abc.

   d. $ab + bc$ e. $3c$. f. a^2

8. What is the probability of something definitely not happening?

9. If a fair dice is thrown what is the probability of getting a 4?

10. On the axes provided plot and label the points
    X (3, 1) Y (-1, 1) Z (-1, -2) .
    These are three corners of a rectangle. What are the co-ordinates of the fourth corner?

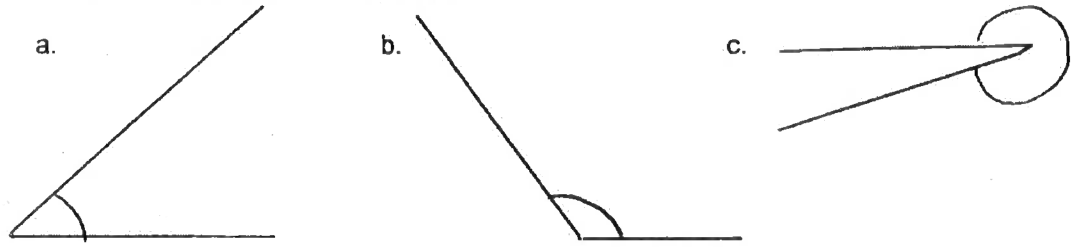
11. Write each of the following fractions as a mixed number, simplifying your answers where possible.
    a. $\frac{9}{8}$ b. $\frac{13}{6}$ c. $\frac{14}{4}$

12. Write each of the following as a top heavy fraction
    a. $1\frac{1}{4}$ b. $3\,{}^{4}/_{5}$

13. Find the median age of a group of pupils who are:
    12, 12, 13, 14, 14.

14. The number of letters in each word of a sentence are:
    3, 6, 2, 7, 2, 4
    What is the mean number of letters per word.

15. Measure each of the following angles.



16. On the answer sheet draw an angle which measures
a. 50°   b. 165°

17. Multiply out the following and simplify your answers where possible.
a. $4(x + 2y)$ b. $2(x^2 + 2y^2)$

c. $3x(x + 2y) + 4x(3x - y)$

18. Work out
a. ½ of 12   b. ¼ of 20   c. $^3/_5$ of 30

19. Of what number is 9 a quarter?

20 Work out the following, simplifying your answers where possible.
a. $\frac{2}{9} + \frac{5}{9}$ b. $\frac{7}{8} - \frac{5}{8}$ c. $\frac{9}{16} + \frac{11}{16}$

Level 6

21. Ali has six cards

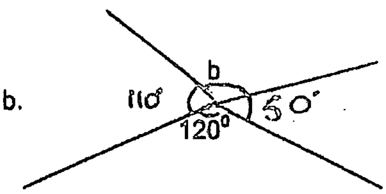
10	10	10	10	?	?

The mean of the cards is ten.
The range of the cards is 4.
What are the numbers on the other 2 cards?

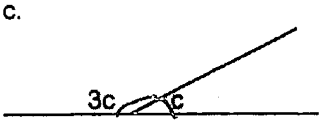
22. Calculate the size of the angles given by a letter.

a.

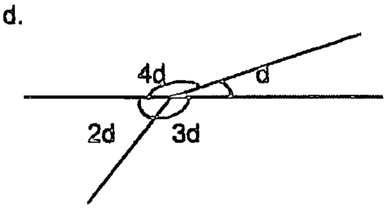
b.



c.



d.



23. Write each of the following as a decimal.
a. $\frac{1}{2}$ b. $\frac{7}{10}$ c. 40%

24. Write each of the following as a percentage.
a. $\frac{1}{4}$ b. 0.19 c. $\frac{9}{100}$

25. Write each of the following as a fraction in its simplest form.
a. 0.7 b. 30% c. 0.08

26. If the probability that it will rain tomorrow is 0.7 what is the probability that it will not rain tomorrow.

27. Put each of these sets of numbers in order from the smallest to biggest.
a. $\frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}$ b. $0.2, \frac{1}{10}, \frac{3}{100}, 0.5$

c. $\frac{1}{10}, \frac{2}{5}, \frac{3}{20}, \frac{7}{20}$

28. In a survey to find the number of pints of milk delivered to 20 houses in a street the results were as follows:

2	3	4	2	1	0	2	2	1	1
1	2	1	2	3	1	2	1	0	1

On the answer grid complete the frequency table for the above information.

29. The following frequency table shows the number of cars owed by families in a street.

Number of Cars	Frequency
0	6
1	7
2	9
3	3

Calculate the mean number of cars per household.

Mathematics Department

Year 7 Test 3

Time: 45 minutes

You may use a calculator unless the question tells you not to.

Level 4

1. Round each of the following to the nearest ten:
   a) 894 b) 69 c) 299

2. What is the remainder when
   a) 58 is divided by 6 b) 101 is divided by 8.

3. Round each of the following numbers to 1 decimal place:
   a) 6.74 b) 200.19 c) 85.98

4. Multiply together the following numbers. **Do not use a calculator.** Show your working clearly in the space on the answer grid.

   a) 34 x 9 b) 63 x 19

5. Work out a) 5^2 b) 2^3 c) 0.3^2

6. Find a) $\sqrt{36}$ b) $\sqrt{0.04}$

7. A survey was carried out to discover how many pints of milk were delivered to each of 20 houses in a street. The results were as follows:

4	3	2	1	4
0	2	3	2	2
5	4	3	2	0
1	2	1	1	2

   Show how you would record these results in a frequency table.

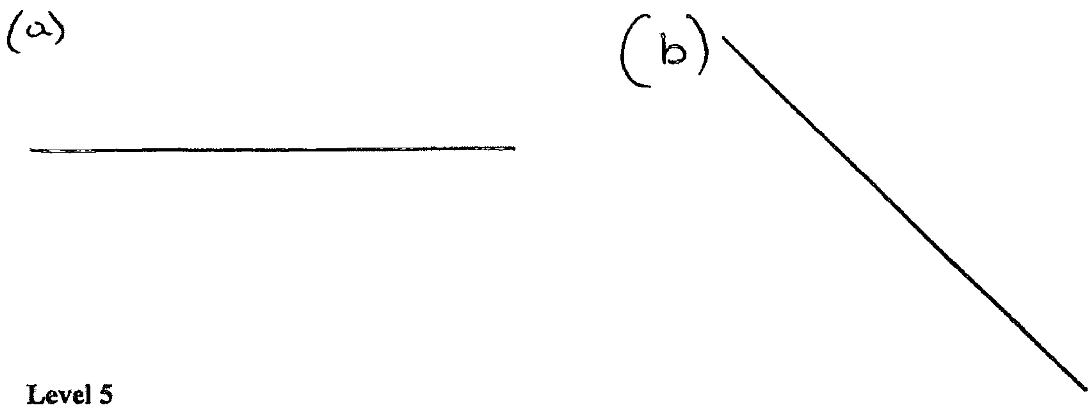
8. List all the factors of 12.
   Which of these are prime numbers?

9. 3, 9, 12, 17, 25, 36
   Which of the above numbers are
   a) prime b) square c) triangular d) multiples of 3.

10. List all the factors of a) 30 b) 45
    What are the common factors of 30 and 45?
    What is the highest common factor of 30 and 45?

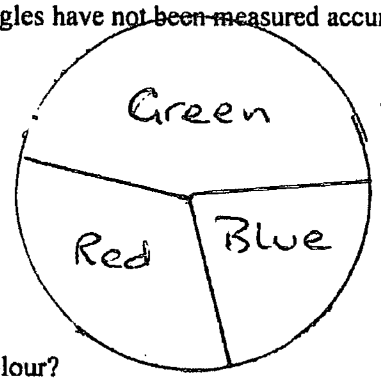
11. To find the numbers in a given sequence it is necessary to add 5 to the previous term. If the first term is 6 what are the next 3 terms?

12. Measure each of the following lines giving your answer to the nearest millimetre.



Level 5

13. The following pie chart shows the results of a survey to investigate the favourite colours of a class of 30 pupils. The angles have not been measured accurately.



What was the most popular colour?
6 pupils chose blue. What angle should represent blue?

14. Change each of the following to grammes:
a) 2kg b) 600 mg c) 2.7kg

15. **On your answer grid** put in brackets to make each of the following a true statement:
a) 6 + 2 x 5 = 40 b) 8 + 4 x 6 + 3 = 44 c) 9 x 8 –3 + 11 = 58

16. Simplify each of the following ratios:
a) 6:12 b) 20:15 c) £3 : 50p

17. Alan and Briony have £800 given to share between them. They have to divide it in the ratio 5:3.How much do they each receive?

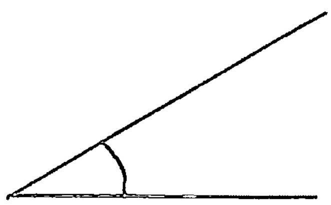
18. The nth term of a sequence is given as 3n-4. What are the first 4 terms of the sequence?

19. In a class the ratio of boys to girls is 3: 4. If there are 16 girls how many pupils are there in the class altogether?

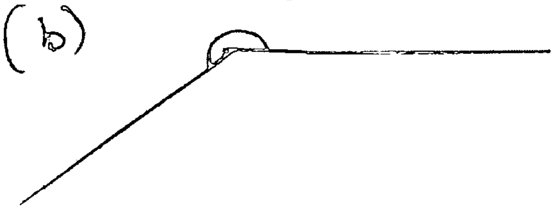
20. Two prime numbers are added together. Their total is 21. What are the two numbers?

21. Measure each of the following angles. Write your answer on the answer grid.

(a)



(b)



22. Construct a triangle ABC where $\angle BAC = 60°$, $\angle ABC = 55°$ and AB = 7cm.

23. Use a calculator to work out $\dfrac{23.55 \times 36.34}{45.4 + 34.6}$.

24. Work out each of the following:
   a) (2+5)x4 b) 3 + 7x3 c) (4 + 7) x (8 – 3)

25. Change these into centimetres
   a) 4 metres b) 60 millimetres c) 0.5 kilometres.

26. Gather together the same sorts of terms
   a) $3x + 2y + 4xy + 3xy - 2y - x$
   b) $4(x+y) - 2x$
   c) $2(x^2 + x) - (x^2 - x)$

Level 6

27. Give 3 points which satisfy the equation $y = x + 4$

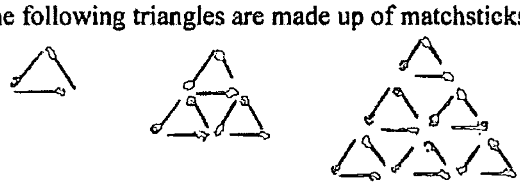
28. Solve the following equations:
   a) $p + 4 = 7$ b) $4x = 28$ c) $3x - 7 = 23$

29. In the diagram on the answer grid line A has the equation $y = 3$. What is the equation of line B?

30. On the answer grid draw the lines with equations a) $x = 2$, b) $y = -1$.
   What are the coordinates of the point where they intersect?

31. I think of a number, multiply it by 4 and subtract 3. The answer is 33.
   Let x be the number I thought of. Write an equation to show this and then solve the equation.

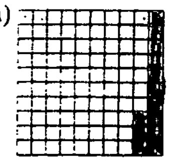
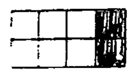
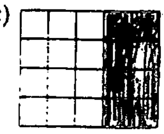
32. The following triangles are made up of matchsticks.



How many will be needed to make the next triangle?
What is the formula for the number of matches needed for the nth triangle?

Year 8 Test 1 You will need a calculator for some of the questions.

1 Write down the fraction and the percentage that is shaded in each shape.

(a)  (b)  (c) 

(a) fraction ______ (b) fraction ______ (c) fraction ______

percentage ____% percentage ____% percentage ____%

2 Complete the following:

(a) $25\% = \frac{\quad}{100} = \frac{\quad}{20} = \frac{\quad}{4}$ (b) $80\% = \frac{\quad}{100} = \frac{\quad}{50} = \frac{\quad}{20} = \frac{\quad}{10} = \frac{\quad}{5}$

(c) 0.7 = _____% (d) 0.75 = _____% (e) 0.09 = _____%

(f) 0.9 = _____% (g) 25% = 0._____ (h) 50% = 0._____

(i) 3% = 0._____ (j) 13% = 0._____

3 The results of a survey of the favourite dinners of 600 pupils in a school were as follows:

Sausage, beans & chips 25% Fish & chips 20% Cheeseburger 10% Salads 5%
Beefburger & chips 11% Beef & veg 7% Others ______%

(a) Fill in the percentage that voted for Others.
(b) How many pupils voted for Sausage, beans & chips? ______ pupils
(c) Calculate the number of pupils that voted for the remainder:

Fish & chips ______ pupils Cheeseburger ______ pupils

Beefburger & chips ______ pupils Salads ______ pupils

Beef & veg ______ pupils Others ______ pupils

4 Calculate the following. DO NOT FORGET to write in the *units* of the answer.

(a) 3% of £5 = ______ (b) 5% of £30 = ______

(c) 13% of £300 = ______ (d) 75% of 8 kg = ______

5 In a class of 30 pupils there are 18 boys.
What percentage of the class are boys?

______

6 VAT at 15% is added to telephone bills in order to find the total cost.
Calculate the total cost of a £33.60 telephone bill.

Total cost = £______

7 Cliff has won the football pools. He decides to give away 9% to charity, keep 25% for himself and divide the remainder equally between his two sons.

(a) What percentage do each of Cliff's sons receive?

Each son receives ______%

(b) Cliff keeps £1255 for himself. How much was his pools win?

Cliff won £______

(c) How much did Cliff give to charity?

Cliff gave £______

8 3% of the population of a country are above the age of 80. How many persons per million is this?

__________ per million

9 'High Tech Electronics' are holding a sale. Complete the missing details on the price tickets.

TELEVISION 20% OFF	HI-FI ______% OFF
Normal price £340	Normal price £450
SALE PRICE £______	SALE PRICE £405

10 (a) 5^2 = ____ (b) 7^2 = ____ (c) 12^2 = ____ (d) 8^2 = ____

(a) 2^3 = ____ (b) 4^3 = ____ (c) 2^5 = ____ (d) 3^4 = ____

(e) 0.1^2 = ____ (f) 0.5^2 = ____ (g) 10^3 = ____ (h) 20^3 = ____

(a) $\sqrt{25}$ = ____ (b) $\sqrt{81}$ = ____ (c) $\sqrt{100}$ = ____ (d) $\sqrt{225}$ = ____

(e) $\sqrt[3]{8}$ = ____ (f) $\sqrt[3]{1000}$ = ____ (g) $\sqrt[3]{27}$ = ____ (h) $\sqrt[4]{16}$ = ____

11 Since $5^2 = 25$ and $6^2 = 36$ then $\sqrt{29}$ lies between 5 and 6.
Between which pairs of whole numbers does the square root of the following lie:

(a) 40 Square root lies between ____ and ____

(b) 95 Square root lies between ____ and ____

(c) 12 Square root lies between ____ and ____

(d) 55 Square root lies between ____ and ____

12 Express each of the following as the product of primes:

(a) 36 = __________ (b) 98 = __________

(c) 16 = __________ (d) 80 = __________

13 Use a calculator to find the values of the following (if your answer is a decimal, write this to 2 decimal places).

(a) $\sqrt{26}$ = __________ (b) $\sqrt{3.56}$ = __________

(c) $\sqrt{0.95}$ = __________ (d) $\sqrt{5000}$ = __________

14 Write down the value of the following:

(a) $5^2 \times 2^3 =$ ______ (b) $4^3 \times 6^2 =$ ______

(c) $(5 \times 7)^2 =$ ______ (d) $2^3 \times 2^3 \times 2^5 =$ ______

15 Calculate the length of the sides of squares that have these areas:

(a) Area = 25 cm^2

Side length = ______ cm

(b) Area = 45 m^2

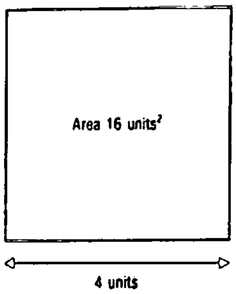
Side length = ______ m

(c) Area = 105 mm^2

Side length = ______ mm

(d) Area = 0.5 m^2

Side length = ______ m



16 (a) Fill in the missing values in the multiplication table.

(b) Look for the pattern in the numbers.
What number squared will give the value 123 456 787 654 321?

______

1 × 1	=	1
11 × 11	=	121
111 × 111	=	______
1111 × 1111	=	1234321
11111 × 11111	=	______
______	=	______
______	=	______

17 Write down the value when $x = 4$ and $y = 3$:

(a) xy ______ (b) $3x$ ______

(c) $x + y$ ______ (d) $2x - 3y$ ______

18 Write down the value when $x = 5$, $y = 3$ and $z = -2$:

(a) $2(x + y)$ ______ (b) $x + z$ ______

(c) $y - z$ ______ (d) $10 - 2(x + y)$ ______

19 Write down the value when $x = 0$, $y = 4$ and $z = 5$:

(a) y^2 ______ (b) xyz ______

(c) $3x$ ______ (d) $(x + y)^2$ ______

20 Simplify:

(a) $a + a + a + a + a$ ________ (b) $a + 2a + 3a$ ________

(c) $3b + 2b - b$ ________ (d) $5b - 3b + 4b - 5$ ________

(e) $3c + 2d - c + d$ ________ (f) $4c + 5c - d + 8$ ________

(g) $e - 5e - 2e + 10e$ ________ (h) $2e - 3f + 3e + f$ ________

21 Simplify:

(a) $2(a + 3)$ ________ (b) $3(a + b)$ ________

(c) $a(b + 2)$ ________ (d) $4(b - 3)$ ________

(e) $6 + 2(c + 3)$ ________ (f) $7 - 3(c + 2)$ ________

(g) $3(a - b) + 2(a + b)$ ________ (h) $10 - 5(d - 2)$ ________

22 Write a simple expression for these statements; use n for 'number' and m for 'another number'.

(a) I think of a number, double it and add 5. ________

(b) I think of a number, add 5 and then double it. ________

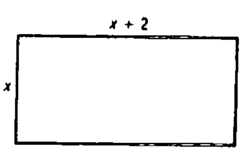
(c) I think of a number, multiply it by five and then add twice another number.

____

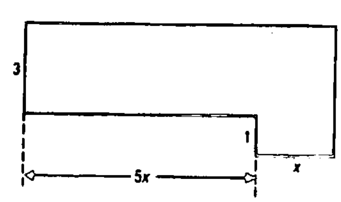
(d) I think of a number and then subtract twice another number. . .

23 Find the perimeter and the area of each shape.

(a)



(b)



(a) Perimeter = ________ (b) Perimeter = ________

Area = ________ Area = ________

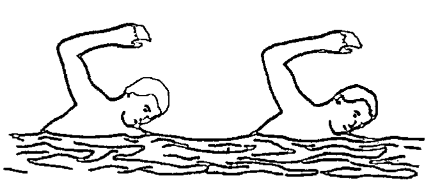
24 Stephen can swim twice as far as Jason and y more lengths than Jenny. Jason can swim x lengths which is 20 lengths more than Adam can swim.
Write an expression for:

(a) how far Stephen can swim ____

(b) how far Adam can swim ____

(c) how far Jenny can swim ____

(d) If Adam can swim 15 lengths, how far can Stephen swim?



Mathematics Department

Year 8 Test 3

Time: 45 minutes.
You may use a calculator unless the question tells you not to.

Level 4

1. Write the following in order starting with the smallest:
   0.2, 0.14, 0.06, 0.22

2. Round each of the following to the nearest 10:
a) 234 b) 657 c) 18.9

3. Round each of the following to 2 decimal places:
a) 15.255 b) 19.2345

4. Work out the following **without a calculator**. Show your working on the answer sheet:
a) 4 x 6 b) 4 x 0.6 c) 12.02 + 1.89 d) $\frac{34.5}{0.5}$

5. On your answer sheet reflect the triangle in the mirror line.

Level 5

6. Use a calculator to work out:
a) 32^2 b) $\sqrt{45}$ to 2d.p. c) $\sqrt[3]{50}$ to 2d.p. d) $\frac{49 \times (34-23)}{5.5}$

7) Complete the mapping for the following function on your answer sheet:
$x \rightarrow x+5$
Input
1
2
3
4

8) If a = 2, b = -3 and c = 5 work out
a) b + c b) abc c) 3a + 2b d) b^2

9. a) On the answer sheet work out the value of y in each case for the function y = 2x+1.
b) On the axes plot the points from your table and join them up.
c) Where does this line cross the y axis?

10. Cancel down each of the following ratios:
a) 4 : 6 b) 40p : £2 c) 35 : 40 : 25

11. If there are 1.6 Euros to every pound,
a) how many Euros will Fred get for £25?
b) how many pounds will be the same as 20 Euros?

12. The ratio of boys to girls in a school is 2:3. What fraction of the school are girls?

Level 6

13. Find a rule for the following function machine:

Input	Output
1	2
2	5
3	8
4	11

14. On your answer paper plot the points A (2,3), B (2, 0) and C (3,3). Join them up.
Draw in the line x = 1.
Reflect your shape in the line x = 1. Label your new shape $A'B'C'$
Now rotate $A'B'C'$ through 90 degrees anti-clockwise about (0,0).
Label your new shape $A''B''C''$.

15. Jack is 3 times as old as Peter. In 4 years time he will be twice as old. How old is Jack now?

16. Three directors of a company own 30%, 45% and 25% respectively. The profits of £10000 are shared between them in the ratio of their share of the firm. How much should each receive?

17. On the answer sheet complete the tables for the equations
$y = 2x$ and $y = 2x + 3$

On the axes provided draw the graphs of
$y = 2x$ and $y = 2x + 3$ and label them.

What do you notice?

18. Solve the equations
a) $x + 4 = 12$ b) $3x - 7 = 20$ c) $5x + 6 = 2x - 3$ d) $3(x + 2) = x - 4$

Level 7

19. Write each of the following numbers in standard form:
a) 1 230 000 b) 0.003 44

20. Work out the following and give your answer in standard form:

a) $(2.3 \times 10^7) + (3.4 \times 10^5)$ b) $(5.1 \times 10^4) \times (7.3 \times 10^3)$

1/1

การสอบจุดประสงค์เรื่องสมการ ชื่อ.. เลขที่......... ชั้น.........

1. จงหาค่า x จากสมการ $\frac{3x - 1}{5} = 5$ และแสดงวิธีตรวจสอบด้วย

<u>วิธีทำ</u> ....

....

....

....

....

....

....

2. จงหาค่า x จากสมการ $\frac{4}{5}(3x + 2) = 16$ และแสดงวิธีตรวจสอบด้วย

<u>วิธีทำ</u> ....

....

....

....

....

....

3. ครึ่งหนึ่งของผลบวกของจำนวนจำนวนหนึ่งกับ 42 มีค่าเท่ากับ 56 แล้วสามเท่าของจำนวนนั้นมีค่าเท่าใด

<u>วิธีทำ</u> ....

....

....

....

....

....

....

4. เมื่อนำจำนวนจำนวนหนึ่งลบด้วย 13 แล้วสามในสี่ของ[illegible]ของผลลบนั้นมีค่าเท่ากับ 27 จงหาจำนวนจำนวนนั้น

<u>วิธีทำ</u> ....

....

....

....

....

....

....

....

การสอบจุดประสงค์เรื่องสมการ ค 301 ชื่อ.................................เลขที่.....ชั้น......

1\. จงหาค่า x จากสมการและแสดงวิธีตรวจสอบด้วย สมการคือ $\frac{2x-5}{7} = 7$

<u>วิธีทำ</u> ....

....

....

....

....

....

2\. จงหาค่า k จากสมการ $3(k + 4) = 12$ แสดงวิธีตรวจสอบด้วย

<u>วิธีทำ</u> ....

....

....

....

3\. สี่เท่าของจำนวนจำนวนหนึ่งเมื่อหักออก 1 แล้วแบ่งออกเป็น 3 ส่วน แต่ละส่วนมีค่าเท่ากับ 5 จำนวนนั้นมีค่าเท่าใด

<u>วิธีทำ</u> ....

....

....

....

....

....

4\. สามในห้าของผลบวกของจำนวนจำนวนหนึ่งกับ 13 มีค่าเท่ากับ 21 จงหาจำนวนนั้น

<u>วิธีทำ</u> ....

....

....

....

....

....

....

....

....

....

....

การสอบจุดประสงค์เรื่องสมการ ค 301 ชื่อ.................................... เลขที่.......ชั้น.......

1. จงหาค่า x จากสมการ $\frac{3}{5}(2x-5) = 15$ และแสดงวิธีตรวจสอบด้วย

<u>วิธีทำ</u> ....

....

....

....

....

....

2. จงหาค่า x จากสมการ $\frac{2x+7}{5} = 9$ และแสดงวิธีตรวจสอบด้วย

<u>วิธีทำ</u> ....

....

....

....

....

....

....

3. เมื่อนำ 8 ไปบวกกับสองเท่าของจำนวนจำนวนหนึ่ง แล้ว 5 เท่าของจำนวนนั้นผลบวกนั้นมีค่าเป็น 105 จงหาจำนวนนั้น

<u>วิธีทำ</u> ....

....

....

....

....

....

....

....

4. สองในสามของจำนวนนักเรียนห้องหนึ่งเป็นหญิง ถ้านักเรียนห้องนี้ทั้งหมด 50 คน นักเรียนห้องนี้จะมีนักเรียนชายมากกว่าหรือน้อยกว่านักเรียนหญิงกี่คน

<u>วิธีทำ</u> ....

....

....

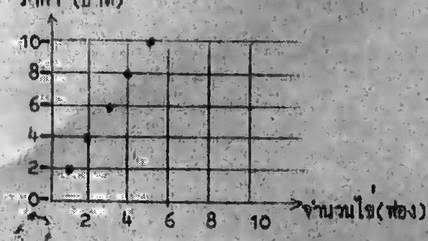
....

....

....

การสอบจุดประสงค์เรื่องคู่อันดับและกราฟ ชื่อ...เลขที่..........ชั้น.....

1. จงเขียนคู่อันดับ ตาราง แผนภาพ ของกราฟข้างล่างนี้



วิธีทำ ....

....

....

....

....

....

....

....

2. จงเขียนตาราง คู่อันดับ แผนภาพ และกราฟของประโยค "จำนวนนับสองจำนวนรวมกันเท่ากับห้า"

วิธีทำ กำหนดให้จำนวนที่ 1 เป็น .... และจำนวนที่ 2 เป็น ....

สมการ ....

....

....

....

....

....

....

3. กราฟของสมการ $y = 2x + 1$ และ $x - y = 0$ ตัดกันที่จุดใด และกราฟทั้ง 2 ตัดแกน x และแกน y ที่จุดใด เมื่อ x, y เป็นจำนวนใดๆ

วิธีทำ ....

....

....

....

....

....

....

....

....

4. จงตรวจสอบว่าคู่อันดับที่กำหนดให้เป็นคำตอบของสมการที่กำหนดให้หรือไม่ (แสดงวิธีตรวจสอบด้านหลัง)

4.1 คู่อันดับคือ (0, 0) กับ (2,3) สมการคือ $y + 1 = 2x$

4.2 คู่อันดับคือ (1,3) กับ (2,2) สมการคือ $y = 4 - x$

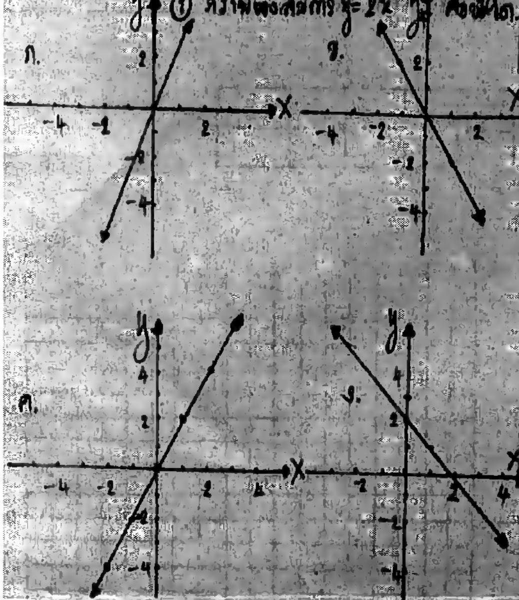
โรงเรียนบุรีรัมย์พิทยาคม

ข้อสอบจุดประสงค์ เรื่อง กราฟสมการเส้นตรง คะแนนเต็ม 10 คะแนน ชั้น ม.2

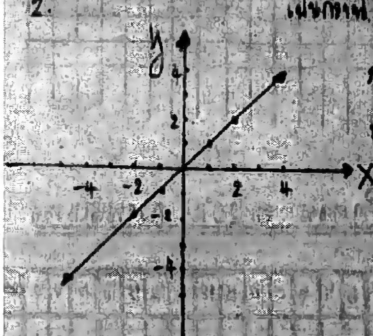
ตอนที่ 1 จงเลือกข้อที่ถูกที่สุดเพียงข้อเดียว

1. กราฟของสมการ $y = 2x$ คือข้อใด

ก. ข. ค. ง.



2. เป็นกราฟของสมการใด



ก. $x + y = 0$

ข. $-x + y = 0$

ค. $x - y = 0$

ง. $-x - y = 0$

3. กราฟของสมการ $y = 3x$ และ $y = x$ ตัดกันที่จุดใด

ก. $(0, 0)$ ข. $(0, 1)$

ค. $(0, -1)$ ง. $(1, 0)$

4. กราฟของสมการใด ที่ลากผ่านจุด $(-3, 4)$ และขนานกับแกน X

ก. $x = -3$ ข. $x = 3$

ค. $y = -4$ ง. $y = 4$

5. กราฟของสมการใด ที่ลากผ่านจุด $(-1, -2)$ และขนานกับแกน y

ก. $x + 1 = 0$

ข. $x - 1 = 0$

ค. $y + 2 = 0$

ง. $y - 2 = 0$

6. คู่อันดับใดอยู่บนกราฟของสมการ

$-2x + 2y - 12 = 0$

ก. $(1, 7)$ ข. $(-1, 7)$

ค. $(7, 1)$ ง. $(-7, 1)$

7. คู่อันดับในข้อใดอยู่บนกราฟของสมการ $2x - y = 1$

ก. $(-1, -3)$ ข. $(1, 2)$

ค. $(0, 1)$ ง. $(2, -3)$

8. ถ้าจุด $(-3, 1)$ อยู่บนกราฟของเส้นตรง

$-x + ky - 8 = 0$ แล้ว k มีค่าเท่าใด

ก. 5 ข. −5

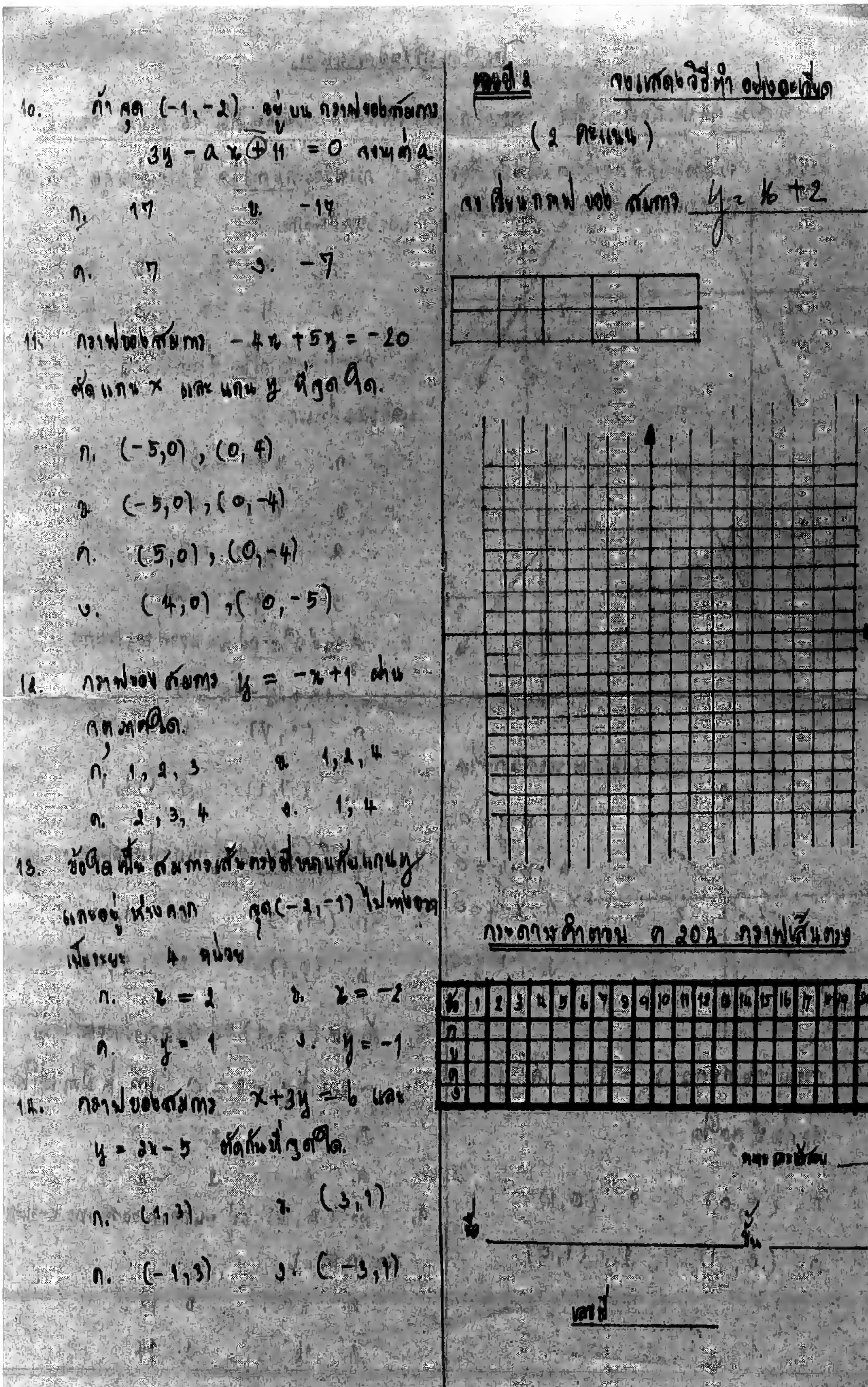
ค. 4 ง. −4

9. ถ้า $(x, 2)$ อยู่บนกราฟของสมการ $x - 3y = 12$

จงหาค่าของ x

ก. 20 ข. 18

ค. 16 ง. 14



10. ถ้าจุด (-1, -2) อยู่บนกราฟของสมการ $3y - ax + 11 = 0$ จงหาค่า a

ก. 17 ข. -17

ค. 7 ง. -7

11. กราฟของสมการ $-4x + 5y = -20$ ตัดแกน x และแกน y ที่จุดใด

ก. (-5, 0), (0, 4)

ข. (-5, 0), (0, -4)

ค. (5, 0), (0, -4)

ง. (4, 0), (0, -5)

12. กราฟของสมการ $y = -x + 1$ ผ่านจตุภาคใด

ก. 1, 2, 3 ข. 1, 2, 4

ค. 2, 3, 4 ง. 1, 4

13. ข้อใดเป็นสมการเส้นตรงที่ขนานกับแกน y และอยู่ห่างจากจุด (-2, -1) ไปทางขวาเป็นระยะ 4 หน่วย

ก. $x = 2$ ข. $x = -2$

ค. $y = 1$ ง. $y = -1$

14. กราฟของสมการ $x + 3y = 6$ และ $y = 2x - 5$ ตัดกันที่จุดใด

ก. (1, 3) ข. (3, 1)

ค. (-1, 3) ง. (-3, 1)

ตอนที่ 2 จงแสดงวิธีทำอย่างละเอียด

(2 คะแนน)

จงเขียนกราฟของสมการ $y = \frac{1}{6} + 2$

กระดาษคำตอบ ค 204 กราฟเส้นตรง

ข้อ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
ก																				
ข																				
ค																				
ง																				

คะแนนที่ได้ ______

ชื่อ ______ ชั้น ______

เลขที่ ______

[illegible]

1. [illegible]

ก. $\frac{7}{8}(x-5) < 10$ [illegible]

2. [illegible]

3. [illegible] $2m$ [illegible] m^2 [illegible]

ก. 4 [illegible] 12 [illegible] 16 [illegible] 18

4. [illegible] $5 - 2y$ [illegible]

ก. 10 [illegible]

5. [illegible]

6. [illegible] $7x - 1 = 3x - 21$ [illegible]

ก. −5 [illegible] 5 [illegible]

7. [illegible] $\frac{x}{2} - \frac{1}{5}$ [illegible]

ก. 2 [illegible] 3 [illegible]

8. [illegible] $6y + 2(y - 1)$ [illegible]

ก. −3 [illegible]

9. [illegible] $\frac{x-12}{4} = \frac{2x+1}{3}$ [illegible]

ก. −8 [illegible] 8 [illegible]

10. [illegible] $\frac{x}{4} - \frac{2x}{3} = \frac{5x}{6} + 1$ [illegible] $6x$ [illegible]

ก. −4 [illegible]

11. [illegible]

ก. 20 [illegible] 30 [illegible] 35

12. [illegible]

ก. 29 [illegible] 21 [illegible]

13. [illegible]

ก. 26 [illegible] 27 [illegible] 29 [illegible] 31

14. [illegible] 3 [illegible] $\frac{2}{5}$ [illegible] $\frac{2}{7}$ [illegible] 450 [illegible]

ก. 1,575 ข. [illegible] ค. [illegible] ง. [illegible]

15. [illegible]

ก. 2.5 : 4.9 = 5 : 7 ข. [illegible]

ค. 1.8 : 6 = 24 : 80 ง. [illegible]

16. $3\frac{1}{4} : 4\frac{1}{3} = x : 12$, x [illegible]

ก. 8 ข. [illegible] ค. [illegible] ง. [illegible]

17. A : B = 4 : 5 , A : C = 3 : 6 [illegible] A : B : C [illegible]

ก. 4 : 5 : 3 ข. [illegible] ค. 12 : 15 : 20 ง. [illegible]

18. m : n = 2 : 3, n : p = 3 : 4, p : x = 4 [illegible]

ก. 6 ข. [illegible]

19. [illegible]

1,500 [illegible]

ก. 450 ข. [illegible] ค. [illegible] ง. [illegible]

20. [illegible] 0.27 [illegible]

[illegible]

ก. 4.5 [illegible] ข. [illegible] ค. [illegible] ง. [illegible]

21. [illegible] 10.21 [illegible]

ก. 50 ข. 98 ค. 102 ง. 153

22. [illegible]

ก. 139.5 [illegible]

ข. 142 [illegible]

ค. 146 [illegible]

ง. 155 [illegible]

23. [illegible]

ก. $8\frac{1}{3}\% = 1 : 12$ ข. [illegible] % = 21 : 500

ค. 2.25 % = 9 : 400 ง. [illegible] % = 1 : 250

24. 13.4 : 11.7 = 93.8 : X [illegible] X [illegible]

ก. 81.9 ข. [illegible] ค. [illegible] ง. [illegible]

25. $\frac{5}{24}$ [illegible]

ก. $2\frac{5}{6}\%$ ข. $20\frac{5}{6}\%$ [illegible]

26. [illegible] 800 บาท [illegible] 20 % [illegible] 7% [illegible]

ก. 1,0[illegible] บาท ข. [illegible] บาท ค. 1,0[illegible] บาท ง. 1,0[illegible] บาท

27. [illegible] 192 [illegible]

ก. 1,[illegible] ข. [illegible] ค. 2,040 ง. 2,[illegible]

28. [illegible] 360 บาท [illegible]

ก. 9,000 ข. [illegible] ค. [illegible] ง. [illegible]

29. [illegible]

ก. [illegible] ข. 1,[illegible] ค. [illegible] ง. [illegible]

30. [illegible] 80,000 บาท [illegible] 2.5 [illegible]

ก. 300 ข. 1,[illegible] ค. 2,000 ง. 2,400

31. [illegible]

ก. 2,240
ข. 2,[illegible]
ค. 3,204
ง. 3,[illegible]

32. [illegible]

ก. [illegible]
ข. [illegible]
ค. 1,[illegible]
ง. 1,720

33. [illegible] 18 [illegible] 12 [illegible] 1,944 [illegible]

ก. 7 ข. 8 ค. 9 ง. 10

34. [illegible]

ก. 1 [illegible] 2 [illegible] ข. 1 [illegible] 100 [illegible]
ค. 1 [illegible] 200 [illegible] ง. 1 [illegible] 200 [illegible]

35. [illegible] 20 [illegible] 60 [illegible]

ก. 14 ข. 16 ค. 18 ง. 20

36. [illegible] 1 [illegible] 1.5 [illegible] 1 [illegible]

ก. 75 ข. 150 ค. 750 ง. 1,500

37. [illegible] 4.5 [illegible] 2 [illegible] 1.2 [illegible]

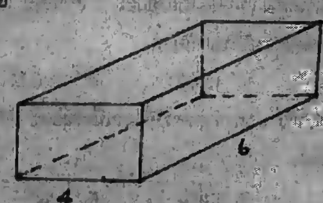
ก. 4.5 ข. [illegible] ค. [illegible] ง. [illegible]

38. ปริซึมสามเหลี่ยมมุมฉากยาว 11 เซนติเมตร มีด้านประกอบมุมฉากยาว 8 และ 6 เซนติเมตร ปริซึมนี้มีพื้นที่ผิวทั้งหมดกี่ตารางเซนติเมตร

ก. 132 ข. 213 ค. 312 ง. 321

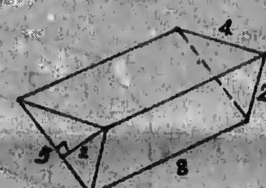
39. จากรูปมีพื้นที่ผิวทั้งหมดกี่ตารางหน่วย

ก. 84
ข. 96
ค. 108
ง. 123



40. จากรูปมีพื้นที่ผิวทั้งหมดกี่ตารางหน่วย

ก. 88
ข. 98
ค. 108
ง. 118



ม. 2

โรงเรียนบุรีรัมย์พิทยาคม จังหวัดบุรีรัมย์

ข้อสอบจุดประสงค์เรื่องสมการและอสมการ เวลา 1 ชั่วโมง 20 คะแนน

6

<u>คำสั่ง</u> จงเลือกข้อที่ถูกที่สุดเพียงข้อเดียว

1. $-2a - 3a = -5a$ ดังนั้น $-5a = -2a - 3a$ เป็นสมบัติการเท่ากันข้อใด
ก. การสลับที่ ข. การสมมาตร
ค. การแจกแจง ง. การถ่ายทอด

2. "3 ใน 5 ของผลต่างจำนวนจำนวนหนึ่งกับ 4 มีค่าไม่เกิน 10" เขียนเป็นประโยคสัญลักษณ์ได้อย่างไร
ก. $\frac{3}{5}(x-4) \geq 10$ ข. $\frac{3}{5}(x-4) \leq 10$
ค. $\frac{3}{5}x - 4 \geq 10$ ง. $\frac{3}{5}x - 4 \leq 10$

3. $\frac{m}{3} - 1 > -2$ เขียนกราฟแสดงคำตอบได้อย่างไร

4. 

กราฟแสดงจำนวนใด
ก. จำนวนทุกจำนวนระหว่าง -9 กับ -6
ข. จำนวนทุกจำนวนตั้งแต่ -9 ถึง -6
ค. จำนวนทุกจำนวน ยกเว้น -9 และ -6
ง. จำนวน -9 และ -6

5. $x + 5 = 12$ $x = 12 - 5$ เป็นสมบัติข้อใด
ก. การย้ายข้าง ข. การบวก
ค. การลบ ง. การถ่ายทอด

6. $2 - 3m = -10$ $m = ?$
ก. 4 ข. -4 ค. 6 ง. -6

7. $6(x-1) = -54$ $x = ?$
ก. 8 ข. -8 ค. 9 ง. -9

8. $7x - 1 = 3x - 21$ $x = ?$
ก. 2 ข. -2 ค. 5 ง. -5

9. $\frac{a}{2} - \frac{3}{4} = \frac{a}{8} - 1$ แล้ว $6a = ?$
ก. 8 ข. -8 ค. 4 ง. -4

10. $5x + 2(x-1) = 61$ $x = ?$
ก. $\frac{59}{7}$ ข. $-\frac{59}{7}$ ค. 9 ง. -9

11. $\frac{1-x}{2} = \frac{2-3x}{5}$ $x = ?$
ก. 1 ข. -1 ค. 10 ง. -10

12. $1 - \frac{x}{2} = 2 - \frac{3x}{5}$ $x = ?$
ก. 1 ข. -1 ค. 10 ง. -10

13. 5 ใน 6 ของจำนวนหนึ่ง มากกว่า 1 ใน 2 ของจำนวนนั้นอยู่ 20 จงหาเลขจำนวนนั้น
ก. 60 ข. -60 ค. 20 ง. -20

14. 3 เท่าของจำนวนหนึ่ง มีค่าเท่ากับ 6 ใน 7 ของจำนวนนั้นบวกกับ 30 จงหาเลขจำนวนนั้น
ก. 2 ข. 4 ค. 14 ง. 18

15. พี่มีอายุมากกว่าน้อง 3 ปี ถ้าอายุของพี่และน้องรวมกันได้ 27 ปี จงเขียนสมการ เมื่อ x แทนอายุพี่
ก. $2(x-3) = 27$ ข. $2x - 3 = 27$
ค. $2(x+3) = 27$ ง. $2x + 3 = 27$

16. สนาม □ ผืนผ้า มีด้านยาว ยาวกว่า 2 เท่าของด้านกว้างอยู่ 3 เมตร ถ้าด้านยาวยาว 35 เมตร จงเขียนสมการเพื่อหาด้านกว้าง
ก. $2x - 3 = 35$ ข. $2x + 3 = 35$ ค. $2(x-3) = 35$ ง. $2(x+3) = 35$

17. มีส้มอยู่จำนวนหนึ่ง แบ่งเด็กไป $\frac{4}{7}$ ของที่มีอยู่ เหลือส้มอยู่ 84 ผล จงหาส้มทั้งหมด
ก. 48 ข. 36 ค. 147 ง. 196

ข้อสอบมีข้อละ 1 คะแนน ผู้สอน [illegible]

Appendix C Interviews data

Individual interviews on English school tests

Sample of individual interviews on English school test were transcribed and analysed their thinking processes. The first interview episode carried to a top set pupil participant. **N**: is interviewer, and **A**: is interviewee.

N: Hi, Anny. Sit down please. How about your test this time?

A: It's hard.

N: Yes. It's harder than the first time. Could we talk about some of them?

A: Yes.

N: Please explain to me, how you did this one. $(p+p+p)$

A: $p3$

N: Yes, how you got that.

A: $p+p+p$ it's $p3$. **(Simplify like terms)**

N: Yes, but we write the number before letter. Then we write $3p$. Let see b.

$(2x+3y+5x-y)$

A: $7x+2y$

N: How did you get $7x+2y$?

A: $2x+5x = 7x$ **(Simplify like terms)**

3y-y=2y **(Simplify like terms)**

N: Yes. How about this one? $(3x^2+2x+x^2+5x)$

A: $4x^2+7x$

$3x^2+x^2=4x^2$, $2x+5x=7x$ **(simplify like terms)**

N: Good. This one please (If $a=4$, $b=2$, $c=3$ how do you work out $a+b$?)

A: 4+2 **(Substitution)**

N: a-c?

A: 4-3 **(Substitution)**

N: abc?

A: a times b times c, 4×2×3 **(substitution)**

N: $ab+bc$?

A: $a \times b + b \times c$

N: $3c$?

A: $3 \times c = 3 \times 3$ **(substitution)**

N: a^2?

A: 4×4 **(Substitution)**

N: Great. Move to No.17. How about this one *[4(x+2y)]*

A: $4+x=4x$ **(Simplify unlike terms, add to the first term in bracket)**

$4x+2y$

N: b? $[2(x^2+2y^2)]$

A: $2+x^2=2x^2$ **(Add to the first term in bracket)**

$2x^2+2y^2$

N: c? [$3x(x+2y)+4x(3x-y)$]

A: $7x$.....$3x+4x = 7x$ **(Simplify terms outside the brackets)**

$4x+2y+7x-y$ **(simplify terms inside the brackets)**

N: Can I explain to you how to do this 17?

A: Yes.

The second interview episode with a bottom set pupil participant.

N: Hello Will, how are you today?

W: I'm fine.

N: Good. Do you know why I want to talk to you again?

W: No.

N: Because you are a very good behaviour in mathematics class. I see you do maths work in lessons, you can do it very well.

W: I like maths.

N: Great. Could we talk a bit about how you did in the test?

W: Yes.

N: This one first, *(p+p+p)*

W: 3p

N: Please explain to me how you got 3p.

W: 1p 2p 3p *(He points at each p)* (Count)

N: How about this one? *(2x+3y+5x-y)*

W: 2x+5x = 7x (Simplify like terms)

3y-y = 2y (Simplify like terms)

N: And this? $(3x^2+2x+x^2+5x)$

W: $3x^2+x^2 = 4x^2$ (Simplify like terms)

2x+5x = 7x (Simplify like terms)

$4x^2+7x$

N: Great! Look at No.7 if a=4, b=2, c=3, how did you work out a+b?

W: a+b = 4+2 = 6 (Substitution)

N: a-c?

W: a-c = 4-3 = 1 (Substitution)

N: abc?

W: 4+2+3 (viewed "abc = a+b+c")

N: You do adding?

W: Yes.

N: ab+bc?

W: 4+2+2+3 (viewed "ab = a+b, bc = b+c")

N: 3c?

W: 3+3 **(viewed "3c = 3+c")**

N: a^2?

W: 4+4 **(view "power" as addition)**

N: One more. No.17. This. *[4(x+2y)]*

W: $4+x+2y = 4x2y$ **(multiply out brackets by writing all terms)**

The interviews carried out with four pupils in the top set and other four in the bottom set. The pupils' thinking processes are summarised in Table 5.10.

Table 1 The process used in the English school test 2 Year 7

Generalisable process	Other process
Simplify $p+p+p$ Simplify like terms (counting letters)	Substitution-like
Simplify $2x+3y+5x-y$ Simplify like terms (grouping)	Letter ignored, substitution-like
Simplify $3x^2+2x+x^2+5x$ Simplify like terms (grouping)	Substitution-like, letter ignored, power, Simplify unlike terms
If $a = 4, b = 2, c = 3$, **work out** $a+b$ Correct substitution	
If $a = 4, b = 2, c = 3$, **work out** abc Correct substitution	Substitution-like (plus, 4+2+3) Substitution-like (replace, 423)
If $a = 4, b = 2, c = 3$, **work out** $ab+bc$ Correct substitution	Substitution-like (plus, replace)
If $a = 4, b = 2, c = 3$, **work out** $3c$ Correct substitution	Substitution-like (plus)
If $a = 4, b = 2, c = 3$, **work out** a^2 Correct substitution	Power (4+4, 4×2)
Multiply out and simplify your answer where possible. $4(x+2y)$ Multiply out brackets correctly	Add first term in the brackets ($4+x = 4x$) Substitution (a number + 2) + 4
Multiply out and simplify your answer where possible. $2(x^2+2y^2)$ Multiply out brackets correctly	Add first term in the brackets, choose a number for x, power ($x^2=x\times2$), letter ignored
Multiply out and simplify your answer where possible. $3x\,(x+2y)+4x(3x-y)$ Multiply out brackets correctly	Choose a number for x, ignored brackets and signs

Table 2 The process used in the English school test 1Year 7

Generalisable process	Other process
Write down the rule for each number machine 1→ □ →4 4→ □ →7 9→ □ →12 Repeated operation	
Write down the rule for each number machine 1→ □ →5 2→ □ →10 5→ □ →25 Repeated operation	Repeated operation-like
Write down the rule for each number machine 8→ □ →2 12→ □ →3 24→ □ →6 Inverse operation	Repeated operation-like
The nth term of a sequence is n+5. Write the first 4 terms. Repeated operation	

Table 3 The process used in the English school test 1Year 8

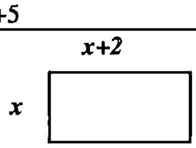
Generalisable process	Other process
Write down the value when $x = 4$ and $y = 3$, **Find $3x$** Substitution	
Write down the value when $x = 4$ and $y = 3$, **Find $2x$-$3y$** Substitution	Substitution-like (24 - 33)
Write down the value when $x = 5, y = 3$ and **$z = -2$, Find $x+z$** Substitution	
Write down the value when $x = 5, y = 3$ and **$z = -2$, Find 10-2($x+y$)** Substitution Multiply out bracket	Incorrect operation Incorrect grouping 10-2(5+3) 8 (8) = 64 5+3 = 8, 10-2 = 8, 8-8 = 0
Write down the value when $x = 0, y = 4$ and **$z = 5$, Find xyz** Substitution	Times zero 0×4×5 = 20 Substitution-like 0+4+5 = 9 Ignored zero
Write down the value when $x = 0, y = 4$ and **$z = 5$, Find $(x+y)^2$** Substitution & power	Incorrect operation power $4^2 = 8$
Simplify $a+a+a+a+a$ Count Simplify like terms	Incorrect operation a^5
Simplify $3b+2b-b$ Simplify like terms from left to right	Letter ignored
I think of a number, double it and add 5 Modelling n×2 +5	n×m = 10+5 = 15
 **Perimeter = ..., Area =** Count Simplify like terms Multiply out bracket	Letter ignored Simplify unlike terms x×x = 2x
Stephen can swim twice as far as Jason and y more lengths than Jenny. Jason can swim x lengths, which is 20 lengths more than Adam can swim. Write an expression how far Stephen can swim Modelling	
Write the expression how far Adam can swim Modelling	

Table 4 The process used in the English school test 3Year 8

Generalisable process	Other process
Complete the mapping for the following function on your answer sheet $x \rightarrow x+5$ **Input Output** **1** **2** **3** **4** Substitution	
If $a = 2$, $b = -3$ and $c = 5$ work out $b+c$ Substitution	
If $a = 2$, $b = -3$ and $c = 5$ work out abc Substitution	
If $a = 2$, $b = -3$ and $c = 5$ work out $3a+2b$ Substitution	
If $a = 2$, $b = -3$ and $c = 5$ work out b^2 Substitution Power Incorrect operation $-3\times -3 = -9$	
On the answer sheet work out the value of y in each case for the function $y=2x+1$ Substitution	
Find a rule for the following function machine **Input Output** **1 2** **2 5** **3 8** **4 11** Trial and error	
Solve the equation $x+4 = 12$ Inverse operation Substitution Incorrect operation	
Solve the equation $3x-7 = 20$ Implicit balancing Change side change sign	
Solve the equation $5x+6 = 2x-3$ Change side change sign Trial and error	
Solve the equation $3(x+2) = x-4$ Trial and error	Simplify unlike terms

The individual interview on the Thai school tests

Secondary 1 girl high ability group

N: Hello Ariya, which paper test did you get?

A: This one Miss.

N: Is it difficult?

A: No.

N: Good, tell me how you did No. 1? (Find x from $\frac{3x-1}{5}=5$ and check the result)

A: Times 5 on both sides, $\frac{3x-1}{5}\times 5=5\times 5$ **(Explicit balancing)**

$3x\text{-}1 = 25$

$3x\text{-}1\text{+}1 = 25\text{+}1$ **(Explicit balancing)**

$3x = 26$

$\frac{3x}{3}=\frac{26}{3}$ **(Explicit balancing)**

$x=8\frac{2}{3}$

Check the result,

$\frac{3\times\frac{26}{3}}{5}-1=5$ **(Substitution)**

$\frac{26-1}{5}=5$

$\frac{25}{5}=5$

$5=5$

N: Very good, No. 2 please (Find x from $\frac{4}{5}(3x+2)=16$ and check the result)

A: $\frac{4}{5}(3x+2)\div\frac{4}{5}=16\div\frac{4}{5}$ **(Explicit balancing)**

$3x+2=16\times\frac{5}{4}$

$3x+2 = 20$

$3x+2-2 = 20-2$ **(Explicit balancing)**

$3x = 18$

$\frac{3x}{3} = \frac{18}{3}$ **(Explicit balancing)**

$x = 6$

Check the answer,

$\frac{4}{5}[(3\times6)+2]=16$ **(Substitution)**

$\frac{4}{5}(18+2)=16$

$\frac{4}{5}\times20=16$

$4\times4 = 16$

$16 = 16$

N: Good. Move to No. 3 please (Half of the sum of a number and 42 is 56. Find three times of that number.)

A: Let the number is x,

Half of the sum of a number and 42 = $\frac{1}{2}\times(x+42)$

Equation: $\frac{1}{2}\times(x+42)=56$

$\frac{1}{2}\times(x+42)\div\frac{1}{2}=56\div\frac{1}{2}$ **(Explicit balancing)**

$x+42 = 56\times2 = 112$

$x+42-42 = 112-42$ **(Explicit balancing)**

$x = 70$

Three times of the number = $3\times70 = 210$

Check the answer,

$\frac{1}{2}(70+42)=56$ **(Substitution)**

$\frac{1}{2}\times112=56$

$56 = 56$

N: Very good, No. 4 please (The number subtracted by 13 , three fourth of the sum is 27. Find that number.)

A: Let the number is x.

The number subtracted by 13 = x-13

Three fourth of the sum is 27 = $\frac{3}{4}\times(x-13)=27$ (Modelling)

Equation is $\frac{3}{4}\times(x-13)=27$

$$\frac{3}{4}\times(x-13)\div\frac{3}{4}=27\div\frac{3}{4}$$ (Explicit balancing)

$$x-13=27\times\frac{4}{3}$$

x-13 = 36

x-13+13 = 36+13 = 49 (Explicit balancing)

$x = 49$

Check the answer,

$$(49-13)\times\frac{3}{4}=27$$ (Substitution)

$$36\times\frac{3}{4}=27$$

9×3 = 27

27=27

N: Very good, thanks.

Table 5 The process used in the Thai school equation test of Secondary 1H

Generalisable process	Other process
Find x from $\frac{3x-1}{5} = 5$ and check the result **Find x from $\frac{2x-5}{7} = 7$ and check the result** **Find x from $\frac{3}{5}(2x-5) = 15$ and check the result.** Implicit balancing Explicit balancing Substitution	
Find x from $\frac{4}{5}(3x+2) = 16$ and check the result **Find x from 3(x+4) = 12 and check the answer** **Find x from $\frac{2x+7}{3} = 9$ and check the result** Implicit balancing Explicit balancing Substitution	
Half of the sum of a number and 42 is 56. Find three times of that number. **Four times of a number when subtract 1 and then divided by 3 is 5. Find that number.** **Add 8 to twice of a number, five times of the sum is 105. Find that number.** Modelling Explicit balancing Implicit balancing Substitution	
A number subtracted by 13, three fourth of the result is 27. Find that number. **Three fifth of the sum of a number and 13 is 21. Find that number.** **Two third of pupils in the class are girls. If the pupils in this class altogether are 50. Find the boys and how many more or less than the girls.** Modelling Implicit balancing Explicit balancing Arithmetic approach Substitution	

Table 6 The process used in the Thai school equation test of Secondary 1L

Generalisable process	Other process
Solve an equation $2x$-1 = 11 and check the result. Explicit balancing Substitution	Substitution-like
Solve an equation $\frac{x-5}{3} = 4$ and check the result Explicit balancing Substitution	Balancing-like Substitution-like
Solve an equation $\frac{x}{3} - 5 = 4$ and check the result Explicit balancing Substitution	Substitution-like

Secondary 1 girl low ability group

N: Hello Miranee, could you tell me how did you do No.1? (Draw graph of equation "Total money of Dang and Dam is 8 baht")

M: I let x is Dang's money

y is Dam's money

N: Yes.

M: I put x is 1, 2, 3,... and found the number add up to 8. **(Substitution)**

x+y = 8

1+7 = 8

2+6 = 8

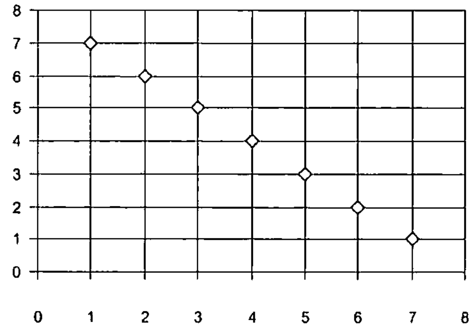
3+5 = 8

4+4 = 8

5+3 = 8

6+2 = 8

7+1 = 8



(Plotting graph)

x	1	2	3	4	5	6	7
y	7	6	5	4	3	2	1

N: Good. No. 2 please. (Draw a graph of equation $y = x+1$ when x are any numbers from 0.)

M: I put the number x and add 1 to get y. (**Substitution**)

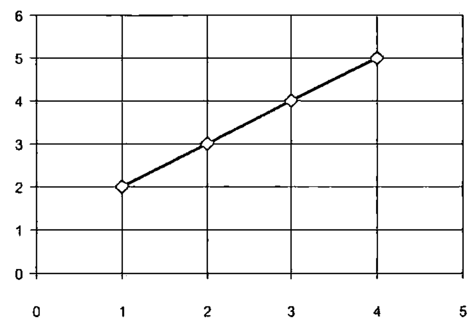
$y = x+1$

$2 = 1+1$

$3 = 2+1$

$4 = 3+1$

$5 = 4+1$



x	1	2	3	4
y	2	3	4	5

(**Drawing graph**)

N: Excellent. No.3 please. (Draw graph of equation x-y = 1 when x is integer.)

M:

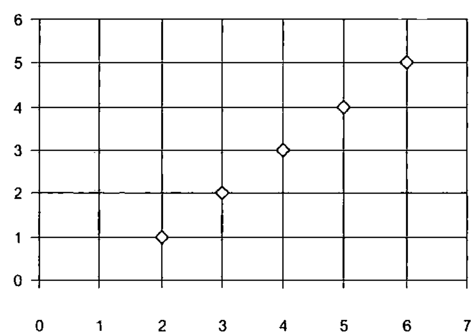
$x-y = 1$

$2-1 = 1$

$3-2 = 1$

$4-3 = 1$

(**Substitution**)



x	2	3	4	5	6
y	1	2	3	4	5

(**Plotting graph**)

N: Well done. Thanks.

Table 7 The process used in the Thai school graph test of Secondary 1L

Generalisable process	Other process
Draw graph of equation "Total money of Dang and Dam is 8 baht" Substitution Plotting graph	
Draw a graph of equation $y = x+1$ when x are any numbers from 0. Substitution Drawing graph	
Draw graph of equation x-y = 1 when x are integer. Substitution Plotting graph	

Table 8 The process used in the Thai school graph test of Secondary 1H

Generalisable process	Other process
Draw table, write ordered pairs and draw graph from the given diagram. 1 → 3 2 → 6 3 → 9 4 → 12 5 → 15 Ordered pairs competition Plotting graph Substitution	
Where graphs of x-y =4 and x+y crosses and where they cross x-axis, y-axis? Substitution Drawing graph Reading graphs	

Table 9 The process used in the Thai school equation test of Secondary 2

Generalisable process	Other process
2-3*m* = -10, *m* =? a. 4 b. –4 c. 6 d. –6 Explicit balancing	Letter ignored
6(*x*-1) = -54, *x* =? a. 8 b. –8 c. 9 d. –9 Multiply out bracket Explicit balancing Implicit balancing	Bracket ignored Balancing-like
7*x*-1 = 3*x*-21, *x* =? a. 2 b. –2 c. 5 d. –5 Explicit balancing Implicit balancing Grouping Change sides change signs Substitution	
$\frac{a}{2}-\frac{3}{4}=\frac{a}{8}-1$, 6*a* =? a. 8 b. –8 c. 4 d. –4 Explicit balancing Grouping Implicit balancing Simplify like terms Substitution	
5*x*+2(*x*-1) = 61, *x* =? a. $\frac{59}{7}$ b. $-\frac{59}{7}$ c. 9 d. -9 Multiply out bracket Explicit balancing Simplify like terms	Balancing-like Count *x*

Table 10 The process used in the Thai school midterm test of Secondary 2

Generalisable process	Other process
If $2m$-3 = 5, then m^2=? **a. 4 b. 12 c. 16 d. 18** Explicit balancing Implicit balancing Substitution	Power 4^2=4×2
If 5-2y = -15, then y=? **a. 10 b. –10 c. 5 d. –5** Explicit balancing Implicit balancing	
If 8(x-1) = -72, then x=? **a. 8 b. –8 c. 10 d. –10** Multiply out bracket Explicit balancing	
If 7x-1 = 3x-21, x=? **a. –5 b. 5 c. –2 d. 2** Implicit balancing Grouping Explicit balancing Change sides change signs	
Chalee older than Nudi 3 years. Both of the ages add up to 43 years. How old is Nudi? **a. 20 b. 21 c. 22 d. 23** Modelling Implicit balancing Substitution Arithmetic approach	

Table 11 The process used in the Thai school graph test of Secondary 2

Generalisable process	Other process
Where does graph of equation $y = 3x$ cross $y=x$? **a. (0, 0) b. (0, 1) c. (0, -1) d. (1, 0)** **Where does graph of equation $y = 3x$ cross $y = x$?** **a. (0, 3) b. (3, 0) c. (0, 0) d. (0, 1)** Substitution Drawing graphs	
Which graph pass (3, 4) and parallel to x-axis? **a. $x = -3$ b. $x = 3$ c. $y = -4$ d. $y = 4$** **Which graph pass (-2, 7) and (-1, 6)?** **a. $2y-x+4 = 0$ b. $y+x-5 = 0$ c. $3x-y+9 = 0$ d. $2x-5y-3 = 0$** Substitution Drawing graph	
Draw graph of equation $y = x+2$ **Draw graph of equation $y = 2x-1$** Substitution Drawing graph	

Appendix D Key issues across the areas of influences

Key issues across the areas of influences	Test items (Theme 4)
Unknown on one side Working back method/inverse operation Concept of equal Negative number	Level 1 The unknown in the first term $5a\text{-}2 = 8$
Unknown on one side Unknown in the middle term Working back method/inverse operation Read from left to right Concept of equal Negative number Order of operations Simplify like terms	Level 2 The unknown in the middle term $5\text{-}2b = 1$
Unknown on both sides Concept of equal Negative number Order of operations Simplify like terms	Level 3 The unknown on both sides $3y\text{-}6 = y\text{-}2$
Unknown on one side with brackets Concept of equal Multiply out brackets Negative number Simplify like terms	Level 4 The unknown in brackets $2(3x\text{-}1)\text{-}(x\text{+}4) = 9$

Key issues across the areas of influences	Test items (Theme 1)
$f(n) = an$ Repeated operation	Level 1 continue physical pattern (multiple of term number) How many matchsticks are needed for the 4^{th} pattern in this series?
$f(n) = an$ Position-to-term rule Repeated operation	Level 1 continue physical pattern before n^{th} term (multiple of term number) How many matchsticks are needed for the 10^{th} pattern in this series?
$f(n) \neq an$ Position-to-term rule Repeated operation	Level 1 continue physical pattern (not a multiple of term number) How many dots are there in the 5^{th} pattern?
$f(n) \neq an$ Position-to-term rule Repeated operation	Level 2 continue physical pattern before n^{th} term (not multiple of term number) How many dots are there in the 20^{th} pattern?
Given a rule Describe the general term in a simple case $f(n) \neq an$ n^{th} term formula	Level 3 general form of physical pattern How many dots are there in the n^{th} pattern?
Generate simple integer sequences Repeated operation	Level 2 number sequence with first term is 1 1 Fill in the blanks in this sequence. 1, 2, 4, 8, 16, 32, …, …
General term of linear sequence Repeated operation	Level 2 number sequence with first term is not 1 The 7^{th} term of this sequence (2, 5, 8, 11, 14, 17, …) is …..
Use linear expressions to describe the n^{th} term n^{th} term formula	Level 4 general form of number sequence The n^{th} term of this sequence (2, 5, 8, 11, 14, 17, …) is …

Key issues across the areas of influences	Test items (Theme 2)
Collecting like terms Accepting lack of closure Reading left to right	Level 1 simplify one variable Simplify the expression $2a$-a+$3a$
Collecting like terms Accepting lack of closure Reading left to right Letter ignored	Level 2 simplify two variables Simplify the expression $6+3b-c-6b-c+2$
Collecting like terms Multiply over a bracket Accepting lack of closure Grouping inside and outside brackets	Level 3 simplify two variables with brackets Simplify $3p+5(p-3)-2(q-4)$
Collecting like terms Multiply over a bracket Accepting lack of closure Index numbers	Level 4 simplify two variables with second order and brackets Multiply out the bracket and then simplify $x^2+2xy-3(xy-2x^2)$

Key issues across the areas of influences	Test items (Theme 3)
Substitute positive integers into simple linear expression Replace co-joined term	Level 1 substitute positive number If a=4, b=3, find the value of $a+5b$.
Substitute letter in value Computing directed numbers Substitute positive and negative number Replace co-joined term	Level 2 substitute positive and negative numbers If s=2, t=-1, find the value of $5s+3t$.
Grouping inside and outside the brackets Replace co-joined term Multiply out brackets	Level 3 substitute positive numbers with brackets If p=5, r=3, find the value of $2(p+3r)-8$.
Substitute positive integers into expression involving small powers Replace co-joined term	Level 4 substitute positive numbers in a two variable expression with second order and brackets If x=2, y=3, find the value of $3x^2-xy+2y^2-10$

Key issues across the areas of influences	Test items (Theme 5)
Generate coordinate pairs Plot the graphs of linear function Recognise straight-line graph	Level 1 graph of the equation $x+y=c$ Plot three coordinates and draw the line of $x+y=4$.
Plot and interpret the graph of linear function x-intercept, y-intercept	Level 2 graph of the equation $y=0$, $y=mx+c$ Where does the graph of the equation $y=2x-6$ cross the x-axis?
Linking different representations of functions Connect a choice of graphs with the given functions x-intercept, y-intercept	Level 3 Graph of the equation $x=0$, $y=0$, $y=x+c$ Which of the following could be part of the graph of $y=x+5$?
Linking different representations of functions Connect a choice of graphs with the given functions x-intercept, y-intercept	Level 4 Graph of the equation $x=0$, $y=0$, $y=mx+c$ Which of the following could be part of the graph of $y=2x+6$?

Key issues across the areas of influences	Test items (Theme 6)
Writing equation Working back Solving word problem using solving equation methods Using letters to represent unknown numbers	Level 1 one variable in one step I think of a number, times it by 4. The answer is 20. What was my original number?
Writing equation Working back Solving word problem using solving equation methods Using letters to represent unknown numbers	Level 1 one variable in two steps I think of a number, times it by 3, and then take away 5. The answer is 16. What was my original number?
Writing equation Solving word problem arising from real-life using solving equation methods Translate from left to right Using letters to represent unknown numbers	Level 2 one variable in two step with brackets and positive number David is 21 years old. Susan is 3 years old. When will David be exactly twice as old as Susan?
Writing equation Solving word problem arising from real-life using solving equation methods Translate from left to right Using letters to represent unknown numbers	Level 3 One variable in two steps with brackets and negative number The Old Elvet Centre Gym has 2-kilogram disks and 5-kilogram disks for weight lifting. Due to their budget, this year they only have fourteen disks in all. The total weight of the 2-kilogram disks is the same as the total weight of the 5-kilogram disks. What is the total weight of all the disks?
Writing equation Solving word problem in familiar geometric situation using solving equation methods Translate from left to right Using letters to represent unknown numbers	Level 4 one variable of second order The length of a rectangle is twice as long as its width. The area of the rectangle is 32 metres square. What is the width and the length of this rectangle?

Appendix E The Algebra Test

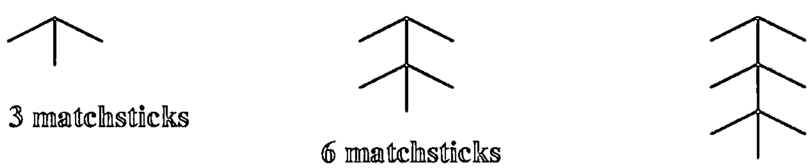
Name...Maths Set...................

Algebra Test

Please write all your answers and working on the test paper – do not use any rough paper.

1. Look at the number of matchsticks in each pattern.

1st pattern 2nd pattern 3rd pattern ...



3 matchsticks 6 matchsticks 9 matchsticks

(a) How many matchsticks are needed for the 4th pattern in this series?

Answer .... matchsticks.
Explain how you work it out.

(b) How many matchsticks are needed for the 10th pattern in this series?

Answer .... Matchsticks.
Explain how you work it out.

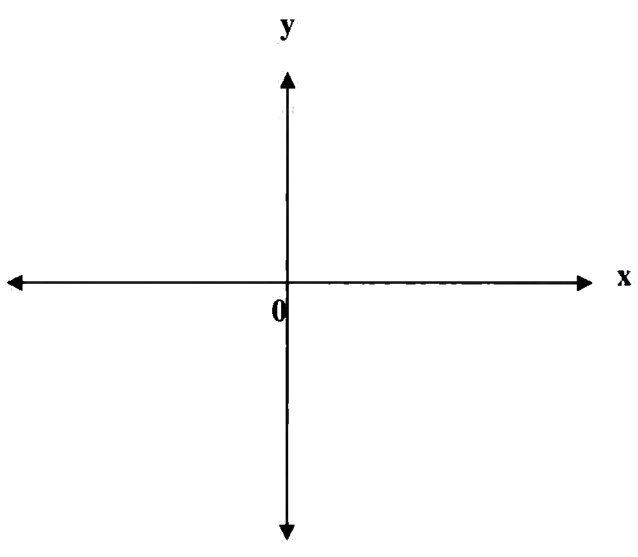
2. Simplify the expression $2a - a + 3a$.
Show your working.

3. If a = 4, b = 3, find the value of a + 5b.
   Show your working.

4. Solve the equation 5a – 2 = 8.
   Show your working.

5. Plot three coordinates and draw the line of x + y = 4.

(….., ……), (……, …….), (……., ……..)



6. (a) I think of a number, times it by 4. The answer is 20.
What was my original number?

Answer ....
Explain how you work it out.

(b) I think of a number, times it by 3, and then take away 5. The answer is 16.
What was my original number?

Answer ....
Explain how you work it out.

7. Fill in the blanks in this sequence.

1, 2, 4, 8, 16, 32, ..., ...
Explain how you work it out.

8. Simplify the expression $6 + 3b - c - 6b - c + 2$
Show your working.

9. If $s = 2, t = -1$, find the value of $5s + 3t$.
Show your working.

10. Solve the equation $5 - 2b = 1$.
Show your working.

11. Where does the graph of the equation $y = 2x - 6$ crosses the x-axis?

Answer ....
Explain how you work it out.

12. David is 21 years old. Susan is 3 years old. When will David be exactly twice as old as Susan?

Answer David will be ....years old.

Susan will be ....years old.
Explain how you work it out.

13. Look at the number of dots in each pattern.

1^{st} pattern	2^{nd} pattern	3^{rd} pattern	4^{th} pattern	…
o o o	o o o o	o o o o o	o o o o o o	
o	o	o	o	
4 dots	o	o	o	
	6 dots	o	o	
		8 dots	o	
			10 dots	

(a) How many dots are there in the 5^{th} pattern?

Answer ....dots

Explain how you work it out.

(b) How many dots are there in the 20^{th} pattern?

Answer ....dots

Explain how you work it out.

(c) How many dots are there in the n^{th} pattern?

Answer .... dots

Explain how you work it out.

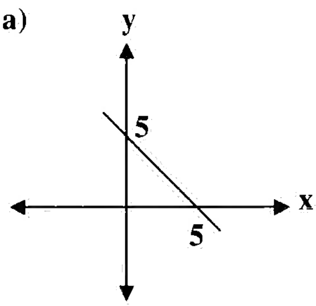
14. Multiply out the brackets and then simplify $3p + 5(p - 3) - 2(q - 4)$.
Show your working

15. If $p = 5$, $r = 3$, find the value of $2(p + 3r) - 8$.
Show your working

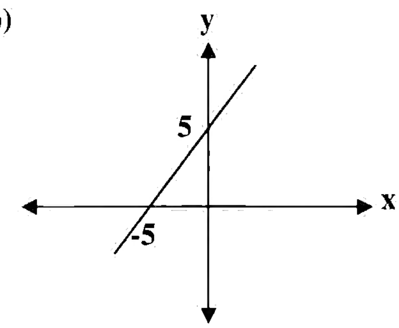
16. Solve the equation $3y - 6 = y - 2$.
Show your working

17. Which of the following could be part of the graph of y = x + 5?

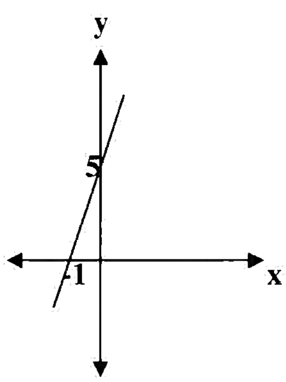
a)



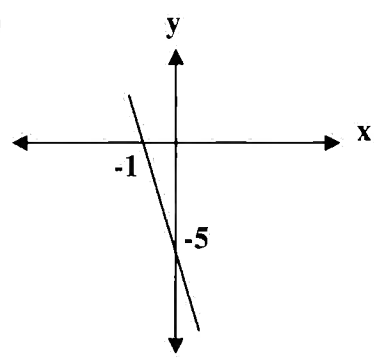
b)



c)



d)



Answer ....

Explain how you work it out.

18. The Old Elvet Centre gym has 2-kilogram and 5-kilogram disks for weight lifting. Due to their budget, this year they only have fourteen disks in all. The total weight of the 2-kilogram disks is the same as the total weight of the 5-kilogram disks. What is the total weight of all the disks?

Answer....................

Show your working.

19. Look at this sequence.

2, 5, 8, 11, 14, 17, ...

(a) The 7th term of this sequence is ....

Explain how you work it out.

(b) The nth term of this sequence is ....

Explain how you work it out.

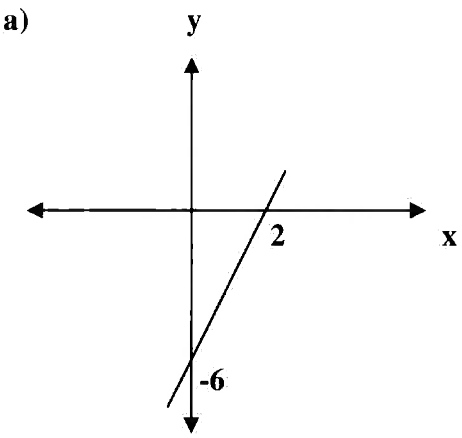
20. Multiply out the bracket and then simplify $x^2 + 2xy - 3(xy - 2x^2)$.
Show your working.

21. If $x = 2$, $y = 3$, find the value of $3x^2 - xy + 2y^2 - 10$.
Show your working.

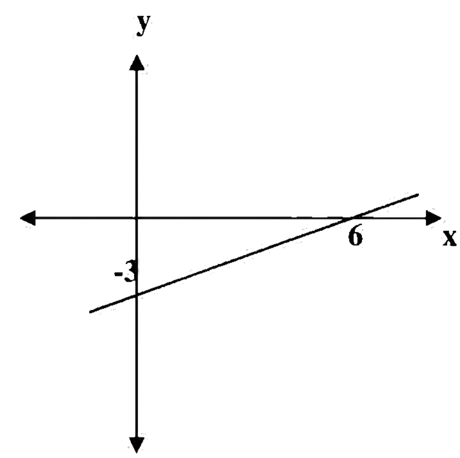
22. Solve the equation $2(3x - 1) - (x + 4) = 9$.
Show your working.

23.Which of the following could be part of the graph of y = 2x + 6?

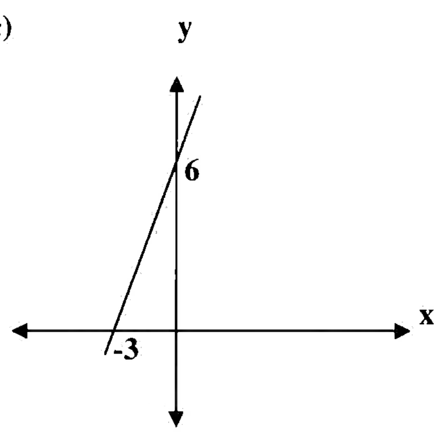
a)



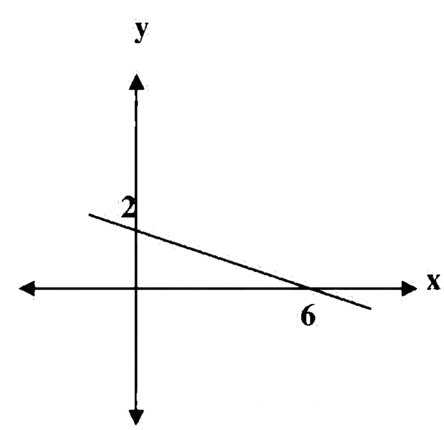
b)



c)



d)



Answer ....
Explain how you work it out.

24. The length of a rectangle is twice as long as its width. The area of the rectangle is 32 metres square. What is the width and the length of this rectangle?

Answer The width = ...metres
The length = ...metres
Explain how you work it out.

ชื่อ-สกุล..ชั้น ม.....................

แบบทดสอบพีชคณิต (จำนวน 12 หน้า)

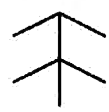
คำชี้แจง **ให้เขียนคำตอบ, แสดงวิธีทำ และทดลงในแบบทดสอบ – ห้ามใช้กระดาษอื่นทด**

1. พิจารณาจำนวนก้านไม้ขีดในแต่ละแบบรูป แล้วตอบคำถาม ข้อ (ก) และ ข้อ (ข)

แบบรูปที่ 1

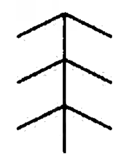
ไม้ขีด 3 ก้าน

แบบรูปที่ 2



ไม้ขีด 6 ก้าน

แบบรูปที่ 3 ...



ไม้ขีด 9 ก้าน

(ก) แบบรูปที่ 4 ต้องใช้ไม้ขีดจำนวนกี่ก้าน

ตอบ............... **ก้าน**

อธิบายวิธีหาคำตอบ

(ข) แบบรูปที่ 10 ต้องใช้ไม้ขีดจำนวนกี่ก้าน

ตอบ ....**ก้าน**

อธิบายวิธีหาคำตอบ

2. **จงหาผลลัพธ์ของ $2a - a + 3a$**

วิธีทำ

3. ถ้า $a = 4, b = 3$, จงหาค่าของ $a + 5b$

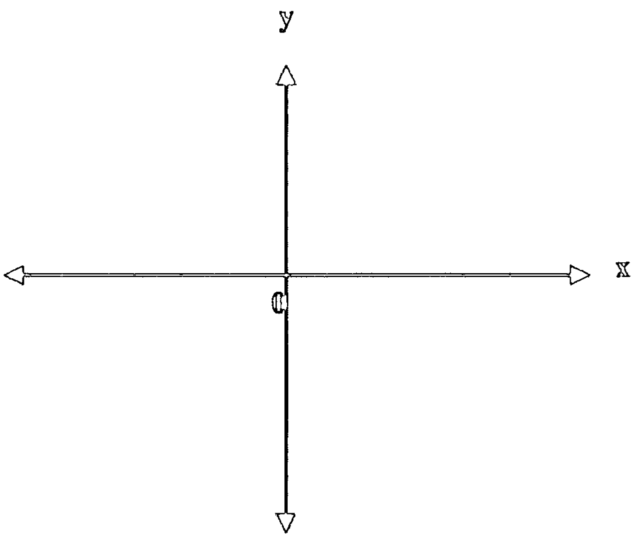
วิธีทำ

4. จงแก้สมการ $5a - 2 = 8$

วิธีทำ

5. จงเขียนคู่อันดับ 3 คู่ และเขียนกราฟของสมการ $x + y = 4$

(……, ……), (……, ……), (……, ……)



๖. (ก) จำนวนจำนวนหนึ่ง คูณกับ 4 ได้ผลลัพธ์เป็น 20 จำนวนนั้นเป็นเท่าไร

ตอบ ……………

อธิบายวิธีหาคำตอบ

(ข) จำนวนจำนวนหนึ่ง คูณกับ 3 แล้ว หักออก 5 ได้ผลลัพธ์เป็น 16 จำนวนนั้นเป็นเท่าไร

7. จงเติมจำนวนถัดไป ลงในช่องว่างให้สอดคล้องกันๆ

1, 2, 4, 8, 16, 32, ..., ...
อธิบายวิธีหาคำตอบ

8. จงหาผลลัพธ์ของ $6 + 3b - c - 6b - c + 2$
วิธีทำ

9. ถ้า $s = 2, t = -1$, จงหาค่าของ $5s + 3t$
วิธีทำ

10. จงแก้สมการ $5 - 2b = 1$

11. **กราฟของสมการ $y = 2x - 6$ ตัดแกน x ที่จุดใด**

ตอบ ....

อธิบายวิธีหาคำตอบ

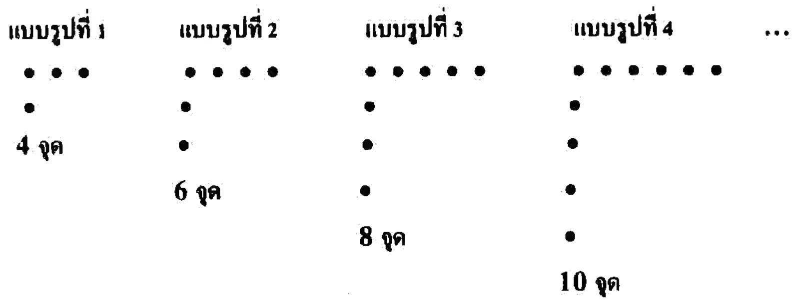
12. **ปัจจุบันเดวิดอายุ 21 ปี ซูซานอายุ 3 ปี เมื่อใดที่เดวิดมีอายุเป็นสองเท่าของอายุซูซาน**

ตอบ เมื่อเดวิดมีอายุ ....ปี

และ ซูซานมีอายุ ....ปี

อธิบายวิธีหาคำตอบ

13. พิจารณาจำนวนจุดในแต่ละแบบรูป



(ก) แบบรูปที่ 5 มีจำนวนจุดเป็นเท่าใด

ตอบ.....................จุด

อธิบายวิธีหาคำตอบ

(ข) แบบรูปที่ 20 มีจำนวนจุดเป็นเท่าใด

ตอบ ....จุด

อธิบายวิธีหาคำตอบ

(ค) แบบรูปที่ n มีจำนวนจุดเป็นเท่าใด

ตอบ .... จุด

อธิบายวิธีหาคำตอบ

14. จงคูณเข้าวงเล็บแล้วหาผลลัพธ์ของ $3p + 5(p - 3) - 2(q - 4)$

วิธีทำ

15. ถ้า $p = 5$, $r = 3$, จงหาค่าของ $2(p + 3r) - 8$

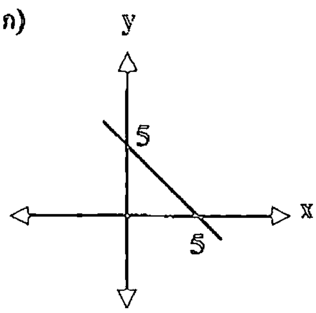
วิธีทำ

16. จงแก้สมการ $3y - 6 = y - 2$

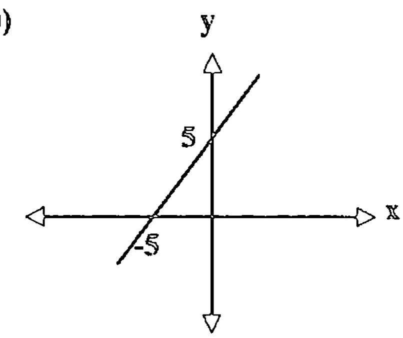
วิธีทำ

17. กราฟในข้อใด เป็นส่วนหนึ่งของสมการ $y = x + 5$

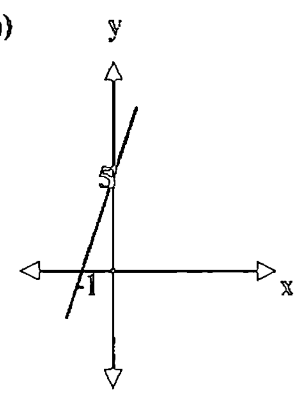
ก)



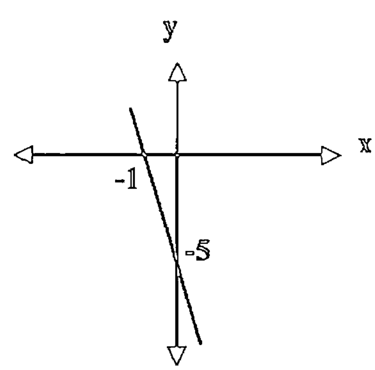
ข)



ค)



ง)



ตอบ ....

อธิบายวิธีหาคำตอบ

18. โรงยิมส์แห่งหนึ่ง มีลูกเหล็กยกน้ำหนักแบบ 2 กิโลกรัม และแบบ 5 กิโลกรัม เนื่องจากปีนี้มีงบประมาณจำกัด โรงยิมส์จึงซื้อลูกเหล็กยกน้ำหนักจำนวน 14 ลูก โดยน้ำหนักรวมของลูกเหล็กยกน้ำหนักแบบ 2 กิโลกรัม และแบบ 5 กิโลกรัม *มีน้ำหนักเท่ากัน* จงหาน้ำหนักรวมทั้งสิ้นของลูกเหล็กยกน้ำหนักทั้ง 14 ลูก

ตอบ.....................

อธิบายวิธีหาคำตอบ

19. พิจารณาชุดตัวเลข แล้วตอบคำถาม ข้อ (ก) และ ข้อ (ข)

2, 5, 8, 11, 14, 17, ...

(ก) ตัวเลขในอันดับที่ 7 คือ ....

อธิบายวิธีหาคำตอบ

(ข) ตัวเลขในอันดับที่ n คือ ....

อธิบายวิธีหาคำตอบ

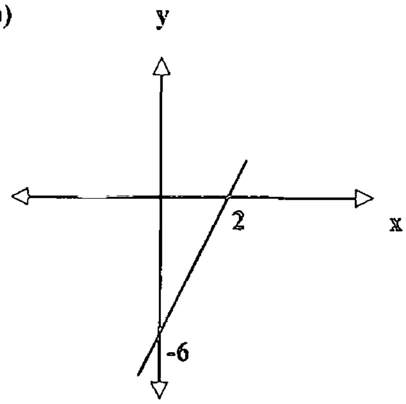
20. จงคูณเข้าวงเล็บและหาผลลัพธ์ของ $x^2 + 2xy - 3(xy - 2x^2)$
วิธีทำ

21. ถ้า $x = 2$, $y = 3$, จงหาค่าของ $3x^2 - xy + 2y^2 - 10$
วิธีทำ

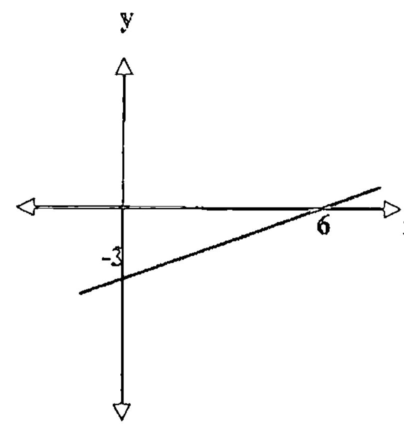
22. จงแก้สมการ $2(3x - 1) - (x + 4) = 9$
วิธีทำ

23. กราฟในข้อใด เป็นส่วนหนึ่งของสมการ $y = 2x + 6$

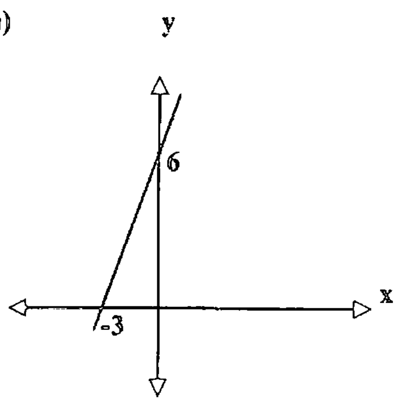
ก)



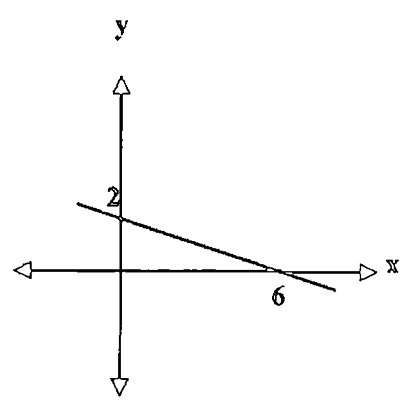
ข)



ค)



ง)



ตอบ ....

อธิบายวิธีหาคำตอบ

24. **สี่เหลี่ยมผืนผ้ารูปหนึ่ง มีด้านยาว ยาวเป็นสองเท่าของด้านกว้าง ถ้าสี่เหลี่ยมผืนผ้ารูปนี้มีพื้นที่ 32 ตารางเมตร จงหาความยาวของด้านกว้างและด้านยาวของสี่เหลี่ยมผืนผ้ารูปนี้**

ตอบ ด้านกว้าง = ...เมตร

ด้านยาว = ...เมตร

อธิบายวิธีหาคำตอบ

Appendix F Difficulty and discrimination index

Difficulty and discrimination index of the algebra test

Theme (Level)	Item	P	Q	P+Q	Difficulty	P-Q	Discrimination
1(1)	1a	13	12	25	0.96	1	0.08
1(1)	13a	13	11	24	0.92	2	0.15
1(1)	1b	13	10	23	0.88	3	0.23
1(2)	19a	12	10	22	0.85	2	0.15
1(2)	7	13	5	18	0.69	8	0.62
1(2)	13b	4	1	5	0.19	3	0.23
1(3)	13c	3	0	3	0.12	3	0.23
1(4)	19b	1	0	1	0.04	1	0.08
2(1)	2	11	0	11	0.42	11	0.85
2(2)	8	0	0	0	0	0	0
2(3)	14	0	0	0	0	0	0
2(4)	20	0	0	0	0	0	0
3(1)	3	13	3	16	0.62	10	0.77
3(2)	9	12	0	12	0.46	12	0.92
3(3)	15	11	0	11	0.42	11	0.85
3(4)	21	9	0	9	0.35	9	0.69
4(1)	4	13	4	17	0.65	9	0.69
4(2)	10	11	0	11	0.42	11	0.85
4(3)	16	9	0	9	0.35	9	0.69
4(4)	22	2	0	2	0.08	2	0.15
5(1)	5f	13	6	19	0.73	7	0.54
5(1)	5s	7	2	9	0.35	5	0.38
5(2)	11	9	0	9	0.35	9	0.69
5(3)	17	4	1	5	0.19	3	0.23
5(4)	23	4	0	4	0.15	4	0.31
6(1)	6a	13	11	24	0.92	2	0.15
6(1)	6b	13	7	20	0.77	6	0.46
6(2)	12	6	0	6	0.23	6	0.46
6(3)	18	5	0	5	0.19	5	0.38
6(4)	24	11	0	11	0.42	11	0.85

Notes: P = number of high performers who got question right
Q = number of low performers who got question right
N_P = number of high performers
N_Q = number of low performers

Item difficulty index (p) = $\frac{P+Q}{N_P+N_Q}$, Item discrimination index (d) = $\frac{P-Q}{N_P}$

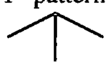
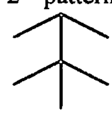
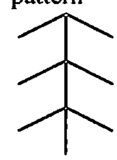
Difficulty index: the smaller the percentage figure, the more difficult the item.
Discrimination index: the higher the discrimination, the better in separating high and low performance.

Appendix G Codebook

Theme1 Patterns and Sequences

Patterns/sequences

Item1. Look at the number of matchsticks in each pattern.

1st pattern	2nd pattern	3rd pattern ...
		
3 matchsticks	6 matchsticks	9 matchsticks

a. How many matchsticks are needed for the 4th pattern in this series? *(Level 1 concrete objects)*
b. How many matchsticks are needed for the 10th pattern in this series? *(Level 1 concrete objects)*

Item13. Look at the number of dots in each pattern.

1st pattern	2nd pattern	3rd pattern	4th pattern ...
o o o	o o o o	o o o o o	o o o o o o
o	o	o	o
4 dots	o	o	o
	6 dots	o	o
		8 dots	o
			10 dots

a. How many dots are there in the 5th pattern? *(Level 1 concrete objects)*
b. How many dots are there in the 20th pattern? *(Level 2 concrete objects)*
c. How many dots are there in the nth pattern? *(Level 3 generalise concrete objects)*

Item7. Fill in the blanks in this sequence. *(Level 2 abstract objects)*
1, 2, 4, 8, 16, 32, ..., ...

Item19. Look at this sequence.
2, 5, 8, 11, 14, 17, ...
a. The 7th term of this sequence is .... *(Level 2 abstract objects)*
b. The nth term of this sequence is .... *(Level 4 generalise abstract objects)*

Generalisable processes (A) are the methods that reflect the way of generalising rules. These ways of thinking include *generalisation*, *repeated operations* and *draw/count* strategies.

Other processes (O) are those in which pupils attempt to obtain general rules from wrongly perceived situations. These include inappropriate scaling up strategies and attempts to draw or count from incorrect patterns.

Unidentified processes (W) are those that give the answer without showing working. Some correct answers appeared without working.

Incomplete response processes (R) are those that showed an attempt to work it out but did not reach completion. Also included are those that made no response to the question.

Within the generalisable process group there are 3 sub-processes:

(1) *The generalisation process* (Ag) in which pupils perform the rule to find out the solution.

(2) *The repeated operation process* (Are) refers to some knowledge of the operation for the previous solution and which is then re-used.

(3) *The draw or count process* (Ad) reflects the empirical approach rather than looking for a rule.

There are 4 sub-processes within the other process group.

(1) *The generalisation-like process* (Og) is an attempt to perform the rule incorrectly.
(2) *The repeated operation-like process* is an attempt to use the previous solution but in the incorrect pattern.
(3) *The scaling up process* (Osc) is an attempt to find the answer by using the prior pattern number.
(4) *The draw or count incorrectly process* (Od) is that showing the basic process to be drawing or counting with an incorrect pattern.

The unidentified processes (W) group gave the result without showing working. Some of these pupils described their thinking processes as "a guess".

There are 3 sub-processes in the incomplete response group.

(1) The *incomplete* (R7) work showed an attempt to work it out but did not reach completion.

(2) *No response* (R9): pupils made no attempt.

(3) *Un-reached* (Ru): pupils did not reach that question because of the limit of time.

For the remainder of this appendix *the unidentified processes* and *the incomplete response* groups are defined as stated above.

Processes **Theme 1** **Level 1 (1a)**	**Examples**	**Code**
Generalisable process		
Generalisation	Times the pattern by 3	Ag
Repeated operation	Adding on 3	Are
Draw or count	Counted 3 more, draw the 4th pattern	Ad
Other process		
Scaling up	The 4th is double the 2nd	Osc
Draw or count incorrectly	Count 2 more on	Od
Unidentified process		
No process		W
Incomplete response		
No response		R9

Processes **Theme 1** **Level 1(1b)**	**Examples**	**Code**
Generalisable process		
Generalisation	Times the number pattern by 3	Ag
Repeated operation	Added another 3	Are
Draw or count	Drawing the 10th pattern	Ad
Other process		
Generalisation-like	1st =3, 2nd =6, 3rd =9, 10th=(9/3)×10	Og
Scaling up	2nd+3rd =5th, 6+9=15, 15×2=30	Osc
Draw or count incorrectly	drawing the pattern and count matchsticks	Od
Unidentified process		
No process		W
Incomplete response		
No response		R9

Processes **Theme 1** **Level 2(7)**	**Examples**	**Code**
Generalisable process		
Repeated operation	Double it each time	Are
Draw or count	Count twice each time	Ad
Other process		
Repeated operation-like	8×2=16, 8×4=32, 8×6=48, 8×8=64	Ore
Draw or count incorrectly	Increase 2, and then increase 8	Od
Unidentified process		
No process		W
Incomplete response		
No response		R9

Processes Theme 1 Level 1(13a)	Examples	Code
Generalisable process		
Generalisation	Double pattern number and add 2	Ag
Repeated operation	The 2 times table, goes up in 2s	Are
Draw or count	Add one dot to each side	Ad
Other process		
Draw or count incorrectly	Ratio 1:3, 5:6, 6:7, 6+7=13	Od
Unidentified process		
No process		W
Incomplete response		
No response		R9
Un-reached		Ru

Processes Theme 1 Level 2(13b)	Examples	Code
Generalisable process		
Generalisation	$2\times20+2$, $20+1=21\rightarrow 21\times2=42$	Ag
Repeated operation	$20-4=16$, $16\times2=32$, $32+10=42$	Are
Draw or count	Keep adding 2	Ad
Other process		
Generalisation-like	1^{st} pattern=4, 4^{th}=10, $20^{th}=(20\times2)+4$	Og
Repeated operation-like	Times term by 2	Ore
Scaling up	Times the 5^{th} pattern by 4	Osc
Draw or count incorrectly	Count on 1 dot each time	Od
Unidentified process		
No process		W
Incomplete response		
No response		R9
Un-reached		Ru

Processes Theme 1 Level 3(13c)	Example	Code
Generalisable process		
Generalisation	Double n then add 2	Ag
Other process		
Generalisation-like	$n\times n+2$, $(n\times2)-1$	Og
Repeated operation-like	Increase 2 each time	Ore
Draw or count incorrectly	Count on 2	Od
Unidentified process		
No process		W
Incomplete response		
No response		R9
Un-reached		Ru

Processes Theme 1 Level 2(19a)	Examples	Code
Generalisable process		
Generalisation	Term ×3-1	Ag
Repeated operation	Add on 3 each time	Are
Draw or count	Count on 3 each time	Ad
Other process		
Generalisation-like	$2n$ + No. of term before	Og
Repeated operation-like	Times 3 every time	Ore
Draw or count incorrectly	The differences are 2 and 3	Od
Unidentified process		
No process		W
Incomplete response		
No response		R9
Un-reached		Ru

Processes Theme 1 Level 4(19b)	Example	Code
Generalisable process		
Generalisation	$(n\times3)-1$	Ag
Other process		
Generalisation-like	$2n$ + No. before	Og
Repeated operation-like	going up in 3s	Ore
Scaling up	$7^{th}=20$. $n^{th}=40$	Osc
Draw or count incorrectly	Count on 3	Od
Unidentified process		
No process		W
Incomplete response		
No response		R9
Un-reached		Ru

Theme 2 Simplification

Simplification
Item2 Simplify the expression $2a - a + 3a$. *(Level1 simplify one variable)*
Item8 Simplify the expression $6 + 3b - c - 6b - c + 2$. *(Level2 simplify two variables)*
Item14 Simplify $3p + 5(p-3) - 2(q-4)$. *(Level3 simplify two variables with brackets)*
Item20 Multiply out the bracket and then simplify $x^2 + 2xy - 3(xy - 2x^2)$. *(Level 4 simplify two variables with second order and brackets)*

Generalisable processes (A) are the methods that showed the correct way to simplify like terms in the expression and multiply out the brackets whether they obtained the correct answer or not.

Other processes (O) are those in which pupils attempt to simplify unlike terms, omit brackets, and multiply only the first term in the brackets on attempt to set up an equation or carry out substitution. In these processes, they obtained the incorrect answers.

The unidentified process (W) and the incomplete response processes (R) are as defined earlier.

Within the generalisable process group there are 4 sub-processes:

(1) *The generalisable incorrect operation process* (Aio) is working with different operations from those given in the question given or wrong order of operating.

(2) *The generalisable left to right computing process* (Alr), responded to a question as it set up by multiplying out brackets and then simplifying the first term with the next like term.

(3) *The letter temporary ignored computing process* (Alg) refers to those who tried to work with coefficients only.

(4) *The plus to minus computing process* (Apm) refers to those who deal with the positive term and then negative term.

There are 4 sub-processes within the other process group.

(1) *The other process incorrect operation* (Oio), showed the processes to omit the brackets or multiplied only the first term in the bracket, and minus sign confused.

(2) *The other process letter ignored* (Olg), addressed the processes of computing only the numbers appeared in the expression, or simplifying unlike terms.

(3) *The other process grouping strategy* (Ogr) operated the terms inside and outside brackets separately.

(4) *The other process substitution* (Os), in which a particular value is assumed and hence a numerical answer obtained.

Processes Theme 2 Level 1(2)	Examples	Code
Generalisable process		
Incorrect operation	*2a-a+3a=6a-a=5a*	Aio
Letter ignored	2-1+3=4, 4*a*	Alg
Left to right	*2a-a+3a,2a-a=a+3a=4a*	Alr
Plus to minus	*2a-a+3a,2a+3a=5a-a=4a*	Apm
Other process		
Incorrect operation	*2a-a+3a-3a, 2a-a+a, 2a/2=a, a=*1	Oio
Letter ignored	*2a-a=*2, 2+3*a*=5*a*	Olg
Substitution	*a*=2, 2*a*=4, -2=2, +3*a*=8	Os
Unidentified process		
No process		W
Incomplete response		
Incomplete		R7
No response		R9

Processes Theme 2 Level 2(8)	Examples	Code
Generalisable process		
Incorrect operation	4+3*b-c*-6*b-c*, 4+9*b-c-c*, 4+9*b*-2*c*	Aio
Left to right	6+3*b-c*-6*b-c*+2, 8+3*b-c*-6*b-c*=8+-3*b*-2*c*	Alr
Grouping	3*b*-6*b*=-3*b*, 6+2=8, -*c*-*c*=-2*c*, -3*b*+8-2*c*	Agr
Other process		
Incorrect operation	6+3*b*-6*b*+2, 3*b*+*b*+2, 4*b*+2	Oio
Letter ignored	6-3*b*=9*b*-6*b*=3*b*, *c*+2=2*c*-*c*=*c*, 3*b*-*c*	Olg
Substitution	*b*=2, *c*=2, 6+3×2-2-6×2-2+2=4	Os
Unidentified process		
No process		W
Incomplete response		
Incomplete		R7
No response		R9
Un-reached		Ru

Processes **Theme 2** **Level 3(14)**	**Examples**	**Code**
Generalisable process		
Incorrect operation	$3p+5p$-15-$2q$-8, $8p$-$2q$-23	Aio
Left to right	$8p+5p$-15-$2q$+8, $8p$-7-$2q$	Alr
Other process		
Incorrect operation	$5\times p=5p-3$, $2\times q=2q-4$, $3p+5p=8p$, 4-3=1, $8p+1-2q$	Oio
Letter ignored	$3p+5\times-3p-2\times-4q$, $3p-15p-8q$	Olg
Grouping	p-3, q-4, $3p$+5-2; 6-1-3, 5-3, $2p$	Ogr
Substitution	$p=1$, $q=2$, $3\times1+5(\times1-3)-2(\times2-4)=2$	Os
Unidentified process		
No process		W
Incomplete response		
Incomplete		R7
No response		R9
Un-reached		Ru

Processes **Theme 2** **Level 4(20)**	**Examples**	**Code**
Generalisable process		
Incorrect operation	$+x^2-2xy-3xy-6x^2$, $+x^2-6x^2= -5x^2$, $-2xy-3xy=-5xy$	Aio
Left to right	$x^2+2xy-3xy+6x^2$, $7x^2-xy$	Alr
Other process		
Incorrect operation	$x^2+2xy-3xy-6x^2$, $x^2-6x^2+2xy-3xy=2x-12x+5xy$	Oio
Substitution	$2x^2=2\times2=4$, $4\times4=16+12=18-3=15$	Os
Unidentified process		
No process		W
Incomplete response		
Incomplete		R7
No response		R9
Un-reached		Ru

Theme 3 Substitution

Substitutions
Item3 If a=4, b=3, find the value of $a+5b$. *(Level1 substitute positive numbers)*
Item9 If s=2, t= -1, find the value of $5s+3t$. *(Level2 substitute positive and negative numbers)*
Item15 If p=5, r=3, find the value of $2(p+3r)-8$. *(Level3 substitute positive numbers with brackets)*
Item21 If x=2, y=3, find the value of $3x^2-xy+2y^2-10$. *(Level4 substitute positive numbers in a two variable expression with second order and brackets.)*

Correct substitution processes (As) are the strategies that showed the way to replace the given numbers instead of the letters into the expression correctly.

Incorrect substitution processes (Os) are those in which values were replaced without due concern for the operations or numbers different from those given were inserted.

There is also the unidentified process and incomplete response process as defined earlier. Within the correct substitution group there are 2 sub-processes:

(1) *The correct arithmetic process* (Asca) is the response that replaces the numbers given instead of the letters and then evaluates correctly.
(2) The incorrect arithmetic process (Asia) refers to the case when the given values are inserted into the expression correctly but a mistake appears in carrying out the arithmetic operations.

There are 2 sub-processes used within the incorrect substitution group.

(1) *The correct arithmetic process* (Osca) is the response in which replaced the value given such as "if a = 4, b = 3, find the value of a+5b" 5b becomes 53 or replaced the different value given such as 5b is 5×b but b≠3 followed by the correct computation.
*(2) The incorrect arithmetic pro*cess (Osia) replaced the value as the correct arithmetic process but followed by incorrect computation.

Processes Theme 3 Level 1(3)	Examples	Code
Correct substitution		
Correct arithmetic	4+(3×5)=4+15=19	Asca
Incorrect arithmetic	4+5×3=9×3=27	Asia
Incorrect substitution		
Correct arithmetic	4+53=57	Osca
Incorrect arithmetic	4+3=7, 7+5*b*=12*b*	Osia
Unidentified process		
No process		W
Incomplete response		
Incomplete		R7
No response		R9

Processes Theme 3 Level 2(9)	Examples	Code
Correct substitution		
Correct arithmetic	5×2=10+3×-1=-3, 10+-3=+7	Asca
Incorrect arithmetic	5×2, 3×-1, 10+-3=-13	Asia
Incorrect substitution		
Correct arithmetic	5×2=10, 3×1=3, 10+3=13	Osca
Incorrect arithmetic	5×2+3-1, 10+3=13-1=12	Osia
Unidentified process		
No process		W
Incomplete response		
Incomplete		R7
No response		R9
Un-reached		Ru

Processes Theme 3 Level 3(15)	Examples	Code
Correct substitution		
Correct arithmetic	2(5+3×3)-8, 10+18-8=20	Asca
Incorrect arithmetic	2×5+3×3-8, 10+9=19, 19-8=11	Asia
Incorrect substitution		
Correct arithmetic	2×5+33-8, 10+33-8, 43-8=35	Osca
Incorrect arithmetic	2+5+3-8=2	Osia
Unidentified process		
No process		W
Incomplete response		
Incomplete		R7
No response		R9
Un-reached		Ru

Processes Theme 3 Level 4(21)	Examples	Code
Correct substitution		
Correct arithmetic	3×2×2-2×3+2×3×3-10, 12-6+18-10, 6+8=14	Asca
Incorrect arithmetic	3×4-2×3+2×6-10, 12-6+12-10=8	Asia
Incorrect substitution		
Correct arithmetic		Osca
Incorrect arithmetic	3×2=6, 3×2=6, =12, 6-12=14, =26-10, =16	Osia
Unidentified process		
No process		W
Incomplete response		
Incomplete		R7
No response		R9
Un-reached		Ru

Theme4 Solving Equations

Solving equations
Item4 Solve the equation $5a$-2=8. *(Level1 The unknown in the first term)*
Item10 Solve the equation 5-2b=1. *(Level2 The unknown in middle term)*
Item16 Solve the equation 3y-6=y-2. *(Level3 The unknown in both sides)*
Item22 Solve the equation 2(3x-1)-(x+4)=9. *(Level4 The unknown in brackets)*

Generalisable processes (A) are methods that show the way to solve the equation following the rules. These rules include balancing, substitution, inverse techniques, multiplying out brackets and simplifying like terms.

Other processes (O) are those in which pupils attempt to solve the equations following only "partial" rules. These "partial" rules include an attempt at balancing, substitution and inverse techniques. The use of other process in expanding brackets included multiplying only the first term of the bracket, combining unlike terms within the brackets and applying the multiplying factor to an extra bracket.

Within the generalisable process group there are 4 sub-processes:

(1) *The balancing process* (Ab) describes responses in which pupils perform the same operation to both sides of the equation or move a number to the opposite side of the equation with the inverse operation.

(2) *The substitution process* (As) refers to those responses in which replace the letter by a number in an attempt to make both sides of the equation has equal value.

(3) *The inverse process* (Av) reflects the reverse of those steps of the equation from the right hand side to the left hand side.

(4) *The multiply out brackets process* (Am) includes expansion of brackets and simplification of like terms.

There are 5 sub-processes used within the other process group.

(1) *The balancing-like process* (Ob) moves a number to the opposite side of the equation with the same operation.

(2) *The substitution-like process* (Os) attempts to replace the letter by a number without concern that the equation is true.

(3) *The inverse-like process* (Ov) is used to describe those attempts, which used an inverse operation even though it is inappropriate.

(4) *The incorrect operation process* (Oio) covers responses in which pupils' work does not appear to have any relevance to solving the equation.

(5) *The multiply out brackets-like process* (Om) showed an attempt to simplify unlike terms in the brackets, multiply only the first term of the brackets, or applying the factor to an extra terms.

Processes **Theme 4** **Level 1(4)**	**Examples**	**Code**
Generalisable process		
Balancing	$5a$-2+2=8+2, $5a$=10, $\frac{5a}{5}=\frac{10}{5}$, a=2	Ab
Substitution	$5a$-2=8, 5×2=10, 10-2=8	As
Inverse	8+2=10/5=2, a=2	Av
Other process		
Balancing-like	$5a$=6, a=6	Ob
Substitution-like	5×4-2=18	Os
Incorrect operation	$5a$-2=3a=11	Ov
Unidentified process		
No process		W
Incomplete response		
Incomplete		R7
No response		R9

Processes **Theme 4** **Level 2(10)**	**Examples**	**Code**
Generalisable process		
Balancing	5-2*b*=1, 2*b*=1+5, 2*b*=6, *b*=6/2, *b*=3	Ab
Substitution	5-2*b*=1, 5-(2×2)=1, 5-4=1, 1=1	As
Other process		
Balancing-like	5-2*b*=1, 5-*b*=1+2, 5-*b*=3, *b*=5-3, *b*=2	Ob
Substitution-like	5-(2+2)=1, 5-4=1	Os
Inverse-like	5-2×*b*=1, 5+(2/*b*), *b*=3	Ov
Incorrect operation	5-2=1, total=3	Oio
Unidentified process		
No process		W
Incomplete response		
Incomplete		R7
No response		R9
Un-reached		Ru

Processes **Theme 4** **Level 3(16)**	**Examples**	**Code**
Generalisable process		
Balancing	3*y*=*y*-2+6, 3*y*-*y*=4, 2*y*=4, *y*=2	Ab
Substitution	*y*=2, (3×2)-6=2-2, 6-6=0, 0=0	As
Other process		
Balancing-like	3*y*+*y*=6+2, 4y=8, *y*=8/4, *y*=2	Ob
Substitution-like	3(-2)-6=-5-6=-11	Os
Inverse-like	3×*y*-6=*y*-2, 6/*y* +6=*y*+2, *y*=4	Ov
Incorrect operation	3*y*-2*y*=6, 1*y*=6, *y*=6	Oio
Unidentified process		
No process		W
Incomplete response		
Incomplete		R7
No response		R9
Un-reached		Ru

Processes **Theme 4** **Level 4(22)**	**Examples**	**Code**
Generalisable process		
Multiply out bracket	$3x\times2-2\times1-x+4=9$, $3x/3=9/3$, $x=3$	Am
Other process		
Balancing-like	$(3x-1)-(x+4)=9\times2$, $3x-1-x+4-4=18-4$, $3x-1-x=14+1$, $3x/3=15/3$, $x=5$	Ob
Substitution-like	$3\times2=6+6=12-1=11$, $11-2=9$, $x=2$	Os
Multiply out bracket-like	$2(3x-1)-4x=9$, $2\times2x-4x=9$, $4x-4x=9$, $x=9$	Om
Unidentified process		
No process		W
Incomplete response		
Incomplete		R7
No response		R9
Un-reached		Ru

Theme 5 Functions and Graphs

Functions/graphs
Item5 Plot three coordinates and draw the line of $x+y=4$. (Level1 Graph of the equation $x+y=c$.)
Item11 Where does the graph of the equation $y=2x-6$ cross the x-axis? (Level2 Graph of the equation $y=0$, $y=mx+c$.)
Item17 Which of the following could be part of the graph of $y=x+5$? (Level3 Graph of the equation $x=0$, $y=0$, $y=x+c$.)
Item23 Which of the following could be part of the graph of $y=2x+6$? (Level4 Graph of the equation $x=0$, $y=0$, $y=mx+c$.)

Generalisable processes (A) are those methods that reflect the way to explore functional relationships. These ways of thinking include ordered pairs recognition and graph construction strategies.

Other processes (O) are those in which pupils incorrectly attempt to explore functional relationships. These attempts include ordered pairs recognition-like, using the constants appearing in the equation, and drawing the line in the wrong direction.

Within the generalisable process group there are 2 sub-processes:

(1) *The ordered pair recognition process* (Aor) is one in which the pupils move from the equation to ordered pairs.

(2) *The drawing graph process* (Agh) is where pupils plotted some coordinates and then drew the line until it crossed the x-axis.

There are 3 sub-processes used within the other process group.

(1) *The ordered pair recognition-like process* (Oor): pupils moved from an equation to an ordered pairs but these did not represent the given equation.

(2) *The drawing graph incorrectly process* (Ogh): pupils plotted the coordinates and drew a line which did not reach the x-axis or which did not represent the given function.

(3) *The constant using process* (Ocn): there is an attempt to use the constant appearing in the equation.

Processes Theme 5 Level 1(5 first part)	Examples	Code
Generalisable process		
Order pair recognition	(1, 3), (0, 4), (2, 2)	Aor
Other process		
Order pair recognition-like	(4, 3), (4, 2), (4, 1)	Oor
Unidentified process		
Incomplete response		
No response		R9

Processes Theme 5 Level 1(5 second part)	Examples	Code
Generalisable process		
Drawing graph	\	Agh
Other process		
Drawing graph incorrectly	/, |	Ogh
Unidentified process		
Incomplete response		
No response		R9

Processes Theme 5 Level 2(11)	Example	Code
Generalisable process		
Order pair recognition	3×2=6-6=0	Aor
Drawing graph	Plotting the points	Agh
Other process		
Order pair recognition-like	$2x$-6=y, 2(1)-6=2-6, y=-4	Oor
Drawing graph incorrectly	Drawing graph	Ogh
Constants using	$2x$-6=4	Ocn
Unidentified process		
No process		W
Incomplete response		
Incomplete response		R7
No response		R9
Un-reached		Ru

Processes Theme 5 Level 3(17)	Example	Code
Generalisable process		
Order pair recognition	(x,y), (-1, 4), (0, 5), (1, 6), , (-5, 0)	Aor
Other process		
Order pair recognition-like	x= 1+5, y=1+5	Oor
Drawing graph incorrectly	ploting graph	Ogh
Constants using	It has to be crossing at 5	Ocn
Unidentified process		
No process		W
Incomplete response		
No response		R9
Un-reached		Ru

Processes Theme 5 Level 4(23)	Examples	Code
Generalisable process		
Order pair recognition	y=0, -3=x; x=0, y=6, cross at (-3, 0)	Aor
Other process		
Order pair recognition-like	a) (2, -6) b) 6, -3) c) (-3, 6) d) (6, 2)	Oor
Drawing graph incorrectly	The line goes through the diagonal	Ogh
Constants using	It crosses at (6, 2)	Ocn
Unidentified process		
No process		W
Incomplete response		
No response		R9
Un-reached		Ru

Theme 6 Word Problems

Word problems

Item6a I think of a number, times it by 4. The answer is 20. What was my original number? (Level1 one variable in one step)

Item6b I think of a number, times it by 3, and then take away 5. The answer is 16. What was my original number? (Level1 One variable in two steps)

Item12 David is 21 years old. Susan is 3 years old. When will David be exactly twice as old as Susan?
(Level2 One variable in two steps)

Item18 The Old Elvet Centre gym has 2-kilogram and 5-kilogram disks for weight lifting. Due to their budget, this year they only have fourteen disks in all. The total weight of the 2-kilogram disks is the same as the total weight of the 5-kilogram disks. What is the total weight of all the disks? (Level3 Two variable in two steps)

Item24 The length of a rectangle is twice as long as its width. The area of the rectangle is 32 metres square. What is the width and the length of this rectangle? (Level4 One variable and square root)

Generalisable processes (A) are methods that show the correct way to solve word problem using arithmetic or algebraic processes. These processes include *modelling, inverse operations, and repeated operations* (trial and error) methods.

Other processes (O) are those in which pupils attempted to make sense of each situation using arithmetic or algebraic processes which were incomplete or only partially correct. These attempts include *modelling-like, inverse operation-like, and repeated operation-like* methods.

Within the generalisable process group there are 3 sub-processes:

(1) *The modelling process* (Amo) in which the pupils translate from words to an equation and then solve the equation.

(2) *The inverse operation process* (Av) reflects the way of working as the opposite operation from that given in the question.

(3) *The repeated operation process* (Are) refers to those who used some form of trial and error with correct substitutions.

There are 2 sub-processes used within the other process group.

(1) *The inverse operation-like process* (Ov) is where the pupils attempt to do the opposite operations but in the wrong order.

(2) *The repeated operation-like process* (Ore) is where the pupils attempt a trial and error solution but with incomplete/incorrect substitution.

Processes Theme 6 Level 1(6a)	Examples	Code
Generalisable process		
Modelling	$4a$=20, $4a/4$=20/4, a=5	Amo
Inverse operations	Divide 20 by 4	Av
Repeated operations	Do 4×2, 3, 4, 5 until got 20	Are
Other process		
Unidentified process		
No process		W
Incomplete response		
No response		R9

Processes Theme 6 Level 1(6b)	Examples	Code
Generalisable process		
Modelling	x×3-5=16, 3x=21, x=7	Amo
Inverse operations	(16+5)/3=7	Av
Repeated operation	3×6=18-5=13, 3×8=24-5=19, 3×7=21-5=16	Are
Other process		
Inverse operation-like	16/3 +5=10.1	Ov
Repeated operation-like	_xby 3-5=16, It is below 0, found –4 worked	Ore
Unidentified process		
No process		W
Incomplete response		
No response		R9

Processes Theme 6 Level 2(12)	Examples	Code
Generalisable process		
Modelling	x+21=2(x+3)	Amo
Repeated operations	Try out the number again and again, n+15	Are
Other process		
Modelling-like	Added 21 to 3 then double it	Omo
Unidentified process		
No process		W
Incomplete response		
No response		R9
Un-reached		Ru

Processes **Theme 6** **Level 3(18)**	**Examples**	**Code**
Generalisable process		
Repeated operations	(2, 4, 6, 8, 10), (5, 10, 15, 20), 5×4=20, 2×10=20	Are
Other process		
Modelling-like	$2x$×7=14, $2x$=2, x=4	Omo
Unidentified process		
No process		W
Incomplete response		
Incomplete work		R7
No response		R9
Un-reached		Ru

Processes **Theme 6** **Level 4(24)**	**Examples**	**Code**
Generalisable process		
Modelling	$2x$×x=32,x×x=16, x=4, w=4, l=8	Amo
Repeated operations	x×y=32, 1×32=32, 2×16=32, 4×8, twice 4=8	Are
Other process		
Modelling-like	$2x$×2=32, $2x$=16, x=8	Omo
Repeated operation-like	Draw the box, from there got the answer (5,6)	Ore
Unidentified process		
No process		W
Incomplete response		
No response		R9
Un-reached		Ru

Appendix H Schools' scheme of work

Mathematics Department

KS3 National Strategy

Y7 Scheme Order of topics

	Topic	Main focus	Approx. time (lessons)	
Term 1	Algebra 1	Sequences	7	
	SSM 1	Perimeter and Area	5	
	Number 1	Place Value, Decimals, Neg. Nos.	7	
	Assessment Half Term 1			
	Algebra 2	Handling Letters	6	
	SSM 2	Lines and Angles	4	
	Number 2	Fractions, Decimals and Percentages	7	
	HD 1	Averages and Probability	7	(43)
Term 2	HD 2	Statistical Diagrams	6	
	Number 3	Calculation Methods, Units	9	
	Number 4	Ratio and Proportion	6	
	Algebra 3	Factors, Multiples, Functions	7	
	Algebra 4	Equations	5	
	SSM 3	Geometrical Properties, Drawing	6	(39)
Term 3	SSM 4	Transformations	7	
	SSM 5	Accurate Drawing, Nets	7	
	Algebra 5	Substitution, Graphs	9	
	Number 5	Estimation, Revision of Calculations	9	
	HD 3	Surveys	9	(41)

YEAR 7: AUTUMN TERM

Teaching objectives for the oral and mental activities

∘ Read and write whole numbers in figures and words. ∘ Multiply and divide whole numbers by 10, 100, 1000. ∘ Count on and back in steps of 0.1, 0.2, 0.25, ½, ¼... ∘ Round whole numbers to the nearest 10 or 100. ∘ Order, add and subtract positive and negative numbers in context. ∘ Recognise multiples and use simple tests of divisibility. ∘ Know pairs of factors of numbers to 100. ∘ Know or derive quickly prime numbers less than 30. ∘ Know or derive quickly squares to at least 12 × 12 and the corresponding roots. ∘ Convert between fractions, decimals and percentages. ∘ Find simple fractions of quantities. ∘ Know addition and subtraction facts to 20 and whole number complements of 100. ∘ Find two decimals (one decimal place) with a sum of 1. ∘ Add and subtract several small numbers or several multiples of 10, e.g. 50 – 40 + 80 – 100.	∘ Add and subtract pairs of numbers, e.g. 76 ± 38, 760 ± 380. ∘ Find doubles and halves of numbers, e.g. 670, 5.6. ∘ Recall multiplication facts to 10 × 10 and derive associated division facts. ∘ Multiply and divide a two-digit number by a one-digit number. ∘ Visualise, describe and sketch 2-D shapes in different orientations. ∘ Estimate and order acute and obtuse angles. ∘ Use metric units (length, mass, capacity) and units of time for calculations. ∘ Use metric units for estimation (length, mass, capacity). ∘ Convert between m, cm and mm, km and m, kg and g, litres and ml. ∘ Know rough metric equivalents of common imperial units. ∘ Apply mental skills to solve simple problems.

Teaching objectives for the main activities

	SUPPORT From the Y5 and Y6 teaching programmes	CORE From the Y7 teaching programme	EXTENSION From the Y8 teaching programme
Algebra 1 (6 hours) Sequences and functions (144–163)	∘ Recognise and extend number sequences formed by counting from any number in steps of constant size, extending beyond zero when counting back. ∘ Know squares to at least 10 × 10.	∘ Generate and describe simple integer sequences. ∘ Generate terms of a simple sequence, given a rule (e.g. finding a term from the previous term, finding a term given its position in the sequence). ∘ Generate sequences from practical contexts and describe the general term in simple cases. ∘ Express simple functions in words, then using symbols; represent them in mappings.	∘ Generate terms of a linear sequence using term-to-term and position-to-term definitions of the sequence, on paper and using a spreadsheet or graphical calculator. ∘ Begin to use linear expressions to describe the *n*th term of an arithmetic sequence. ∘ Represent mappings expressed algebraically.
Formulae and identities (112–113) Solving problems (32–35)		∘ **Use letter symbols to represent unknown numbers or variables.** ∘ Suggest extensions to problems by asking 'What if...?'; begin to generalise and to understand the significance of a counter-example.	

Mathematics Department

KS3 National Strategy

Y8 Scheme Order of topics

	Topic	Main focus	Approx. time (lessons)	
Term 1	Number 1	Integers, Powers, Roots	7	
	Number 2	Fractions, Decimals and Percentages	7	
	Algebra 2	Algebraic Manipulation	7	
	Assessment Half Term 1			
	HD 1	Probability	7	
	SSM 1	Lines and Angles, Constructions	7	
	SSM 2	Areas, Volumes, Units	7	(42)
Term 2	Number 3	Decimals	10	
	Algebra 3	Straight Line Graphs	7	
	Algebra 4	Equations	7	
	SSM 3	Transformations inc, Enlargement	7	
	HD 2	Charts	7	(38)
Term 3	HD 3	Surveys	8	
	Number 4	Calculations (Revision)	7	
	Algebra 5	Equations (Revision)	9	
	SSM 4	Plans and Elevations	10	
	Problems	Logic, Ratio and Proportion	7	(41)

Key Stage 3 National Strategy

Page numbers refer to the supplement of examples for the core teaching programme

	SUPPORT From the Y7 teaching programme	CORE From the Y8 teaching programme	EXTENSION From the Y9 teaching programme
Number 2 (6 hours) Fractions, decimals, percentages (60–77)	∘ Use fraction notation to express a smaller whole number as a fraction of a larger one; **simplify fractions by cancelling all common factors and identify equivalent fractions**; convert terminating decimals to fractions.	∘ Know that a recurring decimal is a fraction; use division to convert a fraction to a decimal; order fractions by writing them with a common denominator or by converting them to decimals.	
	∘ Add and subtract fractions with common denominators; calculate fractions of quantities (whole-number answers); multiply a fraction by an integer.	∘ Add and subtract fractions by writing them with a common denominator; calculate fractions of quantities (fraction answers); multiply and divide an integer by a fraction.	∘ Use efficient methods to add, subtract, **multiply and divide fractions**, interpreting division as a multiplicative inverse; cancel common factors before multiplying or dividing.
	∘ Understand percentage as the 'number of parts per 100'; calculate simple percentages.	∘ Interpret percentage as the operator 'so many hundredths of' and express one given number as a percentage of another; **use the equivalence of fractions, decimals and percentages to compare proportions; calculate percentages and find the outcome of a given percentage increase or decrease.**	∘ Solve problems involving percentage changes.
Calculations (82–85, 88–101)		∘ Understand addition and subtraction of fractions; use the laws of arithmetic and inverse operations.	
	∘ Consolidate the rapid recall of number facts, including positive integer complements to 100 and multiplication facts to 10 × 10, and quickly derive associated division facts.	∘ Recall known facts, including fraction to decimal conversions; use known facts to derive unknown facts, including products such as 0.7 and 6, and 0.03 and 8.	∘ Use known facts to derive unknown facts.
		∘ Consolidate and extend mental methods of calculation, working with decimals, fractions and percentages; solve word problems mentally.	∘ Extend mental methods of calculation, working with factors, powers and roots.
Algebra 2 (6 hours) Equations and formulae (112–119, 138–143)	∘ **Use letter symbols to represent unknown numbers or variables**; know the meanings of the words *term*, *expression* and *equation*.	∘ Begin to distinguish the different roles played by letter symbols in equations, formulae and functions; know the meanings of the words *formula* and *function*.	
		∘ Know that algebraic operations follow the same conventions and order as arithmetic operations; use index notation for small positive integer powers.	∘ Use index notation for integer powers and simple instances of the index laws.
	∘ Simplify linear algebraic expressions by collecting like terms.	∘ **Simplify or transform linear expressions by collecting like terms; multiply a single term over a bracket.**	∘ Simplify or transform algebraic expressions by taking out single term common factors.
		∘ Use formulae from mathematics and other subjects; **substitute integers into simple formulae**, and positive integers into expressions involving small powers (e.g. $3x^2 + 4$ or $2x^3$); derive simple formulae.	

ใบงานที่ 2

กำหนดสาระการเรียนรู้รายปี

ชั้น...ม.1.......

ผลการเรียนรู้ที่คาดหวังรายปี	สาระการเรียนรู้รายปี
24 วิเคราะห์ และ อธิบาย ความสัมพันธ์ของแบบรูป ที่กำหนดให้ได้	สาระที่ 4 พีชคณิต สมการเชิงเส้นตัวแปรเดียว. 25. แบบรูปและความสัมพันธ์ ตัวแปรเดียว
26 ระบุสิ่งที่เป็นตัวแปรของสมการเชิงเส้นตัวแปรเดียวได้	26 ตัวแปรของสมการ
27 แก้สมการเชิงเส้นตัวแปรเดียวอย่างง่าย โดยใช้สมบัติของการเท่ากันได้	27. การแก้สมการโดยใช้สมบัติของการเท่ากัน
28 เขียนสมการเชิงเส้นตัวแปรเดียวแทนสถานการณ์หรือปัญหาอย่างง่ายได้	28. การเขียนสมการเชิงเส้นตัวแปรเดียวแทนสถานการณ์ต่างๆ.
29 แก้โจทย์ปัญหาเกี่ยวกับสมการเชิงเส้นตัวแปรเดียวอย่างง่ายได้	29 การแก้โจทย์ปัญหาโดยใช้สมการ
30 ตระหนักถึงความสมเหตุสมผลของคำตอบที่ได้	—

ใบงานที่ 2

กำหนดสาระการเรียนรู้รายปี

ชั้น...ม.1.....

ผลการเรียนรู้ที่คาดหวังรายปี	สาระการเรียนรู้รายปี
	ความสัมพันธ์กราฟ
31 เขียนกราฟแสดงความเกี่ยวข้องของปริมาณสองชุดที่กำหนดให้ได้	31 การเขียนกราฟแสดงความสัมพันธ์ของปริมาณสองชุด
32 อ่านและแปลความหมายกราฟบนระนาบพิกัดฉากที่กำหนดให้ได้	32 การอ่านและแปลความหมายกราฟบนระนาบพิกัดฉากที่กำหนด
	สาระที่ 5 การวิเคราะห์ข้อมูลและความน่าจะเป็น (ต่อ)
33 บอกได้ว่าเหตุการณ์ที่กำหนดให้ เหตุการณ์ใดจะมีโอกาสเกิดขึ้นได้มากกว่ากัน	33 การคาดการณ์เกี่ยวกับเหตุการณ์ต่างๆ
	สาระที่ 6 ทักษะ/กระบวนการทางคณิตศาสตร์
34 ใช้วิธีการที่หลากหลายแก้ปัญหาได้	34 เทคนิคการแก้ปัญหา
35 ใช้ความรู้ ทักษะ กระบวนการทางคณิตศาสตร์ และเทคโนโลยีในการแก้ปัญหาในสถานการณ์ต่างๆ ได้อย่างเหมาะสม	35 กิจกรรมส่งเสริมการแก้ปัญหาโดยใช้ทักษะกระบวนการทางคณิตศาสตร์

ใบงานที่2-6_คณิตศาสตร์_ม1-3

แผนหลักวิชา คณิตศาสตร์ **รหัสวิชา** ค204

จำนวนคาบ/สัปดาห์ 3 คาบ จำนวนคาบเรียนเต็ม (20 สัปดาห์) 60 คาบ จำนวนคาบเรียนจริง (18 สัปดาห์) 54 คาบ จำนวนคาบ 80% 44 คาบ

จำนวนคาบ 60% 33 คาบ ขาดได้ไม่เกิน 10 คาบ อัตราส่วนคะแนนระหว่างภาค : ปลายภาค 60 : 40

เรื่อง	ข้อที่	จุดประสงค์การเรียนรู้	พฤติกรรม	จำนวนคาบ	น้ำหนักคะแนนการประเมินผล: รายจุดประสงค์ก่อน Mid-term	คะแนน Mid-term	รายจุดประสงค์หลัง Mid-term	วัดผลปลายภาค
สมการและอสมการ	1	สามารถแก้สมการ เขียนกราฟแสดงคำตอบ และสามารถใช้สมการแก้โจทย์ปัญหาได้	พุทธิพิสัย	10	5	7	-	5
อัตราส่วน ร้อยละ	2	สามารถแปลงอัตราส่วน สัดส่วน เป็นร้อยละ แปลงร้อยละเป็นอัตราส่วน และแก้โจทย์ปัญหาเกี่ยวกับอัตราส่วน ร้อยละ และสัดส่วนได้	พุทธิพิสัย	12	6	8		7
ปริมาตรและพื้นที่ผิว	3	สามารถหาปริมาตร, พื้นที่ผิวของรูป ◻ ปริซึม และแก้โจทย์ปัญหาได้	พุทธิพิสัย	6	5	5		6
เส้นขนาน	4	สามารถบอกสมบัติของเส้นขนาน และแก้โจทย์ปัญหาโดยใช้สมบัติของเส้นขนานได้	พุทธิพิสัย	6	-	3	3	3
การนำเสนอข้อมูล	5	สามารถนำเสนอข้อมูล แปลความหมายจากข้อมูล ตาราง แผนภูมิแท่ง/วงกลม, กราฟเส้น, และตารางแจกแจงความถี่ได้	พุทธิพิสัย	8	-	4	4	7
ความคล้าย	6	สามารถบอกความสัมพันธ์ของรูป ∆ คล้าย และนำไปแก้โจทย์ปัญหาได้	พุทธิพิสัย	5	-	-	3	5
กราฟ	7	สามารถบอกความสัมพันธ์ของกราฟเส้น และเขียนกราฟจากสมการเชิงเส้นสองตัวแปรได้	พุทธิพิสัย	7	-	-	4	7
		รวม		54	16	20	14	40

จิตพิสัย
- ความตั้งใจเรียน 2 คะแนน
- การมาเรียน 2 คะแนน
- การแต่งกาย 2 คะแนน
- ความเป็นระเบียบ 2 คะแนน
- ความซื่อสัตย์ 2 คะแนน

REFERENCES

Alexander, R. (1999) Comparing Classrooms and Schools, in: R. Alexander, P. Broadfoot & D. Phillips (Eds) *Learning from Comparing*, Vol. One (Oxford, Symposium Books).

Arcavi, A. (1994) Symbol sense: Informal sense-making in formal mathematics, *For the Learning of Mathematics*, 14(3), pp. 24-35.

Arzarello, F., Bazzini, L. & Chiappini, C. (1994) The process of naming in algebraic problem solving, in: J.P. Ponte & J.F. Matos (Eds) *Proceedings of the 18th International Conference for the Psychology of Mathematics Education* (University of Lisbon, Portugal).

Arzarello, F., Bazzini, L. & Chiappini, C. (1995) The construction of algebraic knowledge: Towards a socio-cultural theory and practice *Proceedings of the 19th International Conference for the Psychology of Mathematics Education* (Recife, Brazil).

Bartlett, M.S. (1954) A note on multiplying factors for various chi-squared approximations, *JRSSB*, 16, pp. 296-298.

Bazzini, L. (1999) On the construction and interpretation of symbolic expressions, in: I. Schwank (Ed) *Proceedings of the First Conference of the European Society for Research in Mathematics Education* (Germany, Osnabrueck-Hausohrbeck).

Bazzini, L., Gallo, E. & Lemut, E. (1996) Studies on Teaching and Learning Algebra, Italian Research in Mathematics Education 1988-1995, in: N.A. Malara, Menghini & M. Reggiani (Eds) *Seminario Nazionale di Ricerca in Didattica della Matematica* (Seville Spain, Consiglio Nazionale delle Ricerche).

Bednarz, N., Kieran, C. & Lee, L. (Eds.) (1996) *Approaches to algebra: Perspectives for research and teaching* (Dordrecht, The Netherlands: Kluwer).

Behr, M.J., Erlwanger, S. & Nichols, E. (1980) How children view the equals sign, *Mathematics Teaching*, 92, pp. 13-15.

Bell, A. (1993) Draft background paper, in: R. Sutherland (Ed) *Working conference of the ESRC seminar group* (London University, Institute of Education).

Bell, A. (1996a) Algebraic thought and the role of a manipulable symbolic language, in: N. Bednarz, C. Kieran & L. Lee (Eds) *Approaches to algebra* (Dordrecht, The Netherlands: Kluwer).

Bell, A. (1996b) Problem-solving approaches to algebra: Two aspects, in: N. Bednarz, C. Kieran & L. Lee (Eds) *Approaches to algebra: Perspectives for research and teaching* (Dordrecht, The Netherlands: Kluwer).

Bernard, J. & Cohen, M. (1988) An integration of equation-solving methods into a developmental learning sequence, in: A. Coxford & A. Schulte (Eds) *The Ideas of Algebra, K-12* (Reston, VA: NCTM).

Blanton, M. & Kaput, J. (2000) *Contexts that support students' algebraic reasoning* (Dartmouth, University of Massachusetts Dartmouth).

Boaler, J. (1997) *Experiencing school mathematics: Teaching styles, sex, and setting* (Suffolk, Open University Press).

Boero, P. (1993) About the transformation function of the algebraic code, in: R. Sutherland (Ed) *Algebraic processes and the role of symbolism* (London, Institute of Education).

Boero, P. (1994) About the role of algebraic language in mathematics and related difficulties, *Rendiconti del Seminario Matematico, University Torino*, 52, pp. 161-194.

Bolea, P., Bosch, M. & Gascon, J. (1999) The role of algebraization in the study of a mathematical organization, in: I. Schwank (Ed) *Proceedings of the First Conference of* *the European Society of Research in Mathematics Education* (Germany, Osnabrueck-Hauohrbeck).

Booth, L.R. (1982) Ordering your operations, *Mathematics in School*, 11(3), pp. 5-6.

Booth, L.R. (1984) *Algebra: Children's strategies and errors: A report of the strategies and errors in secondary mathematics project* (Berkshire, NFER-NELSON).

Booth, L.R. (1988) Children's difficulties in beginning algebra, in: A.F. Coxford (Ed) *The Ideas of Algebra, K-12 (1988 Yearbook)* (Reston, Virginia, VA: NCTM).

Booth, L.R. (1989) A question of structure, in: S. Wagner & C. Kieran (Eds) *Research issues in the learning and teaching of algebra* (Reston, VA: NCTM).

Boulton-Lewis, G., Cooper, T.J., Atweh, B., Pillay, H., Wilss, L. & Mutch, S. (1997a) Processing load and the use of concrete representations and strategies for solving linear equations, *Journal of Mathematical Behavior*, 16(4), pp. 379-397.

Boulton-Lewis, G., Cooper, T.J., Atweh, B., Pillay, H., Wilss, L. & Mutch, S. (1997b) The transition from arithmetic to algebra: A cognitive perspective, *International Group for the Psychology of Mathematics Education*, 21(2), pp. 185-192.

Boulton-Lewis, G., Cooper, T.J., Pillay, H. & Wilss, L. (1998) Pre-algebra: A cognitive perspective, *International Group for the Psychology of Mathematics Education*, 22(2), pp. 144-155.

Boyatzis, R.E. (1998) *Transforming Qualitative Information: Thematic analysis and code development* (USA, Sage Publications).

Breidenbach, D., Dubinsky, E., Hawks, J. & Nicolas, D. (1992) Development of the process conception of function, *Educational Studies in Mathematics*, 23, pp. 247-285.

Broadfoot, P. (1999) Not so much a context, more a way of life? Comparative education in the 1990s, in: R. Alexander, P. Broadfoot & D. Phillips (Eds) *Learning from Comparing*, Vol. 1 (Oxford, Symposium Books).

Bruner, J. (1996) *The culture of education* (Massachusetts, Harvard University Press).

Carpenter, T.P. & Levi, L. (2000) *Developing conceptions of algebraic reasoning in the primary grades* (Madison, WI: National Center for Improving Student Learning and Achievement in Mathematics and Science).

Carraher, D., Schliemann, A. & Brizuela, B. (1999) Bringing out the algebraic character of arithmetic *AERA Annual Meeting* (Montreal Canada, AERA).

Carraher, D., Schliemann, A. & Brizuela, B. (2001) Can young students operate on unknowns?, *Proceedings of the 25th Conference of the International Group of the Psychology of Mathematics Education* (Utrecht, The Netherlands: Freudenthal Institute).

Chaiklin, S. & Lesgold, S. (1984) Prealgebra students' knowledge of algebraic tasks with arithmetic expressions *The Annual Meeting of the American Educational Research Association* (New Orleans, AERA).

Chalouh, L. & Herscovics, N. (1988) Teaching algebraic expressions in a meaningful way, in: A.F. Coxford (Ed) *The ideas of algebra, K-12 (1988 Yearbook)* (Reston, VA: NCTM).

Clement, J. (1982) Algebra word problem solutions: Thought processes underlying a common misconception, *Journal for Research in Mathematics Education*, 13(1), pp. 16-30.

Clement, J., Lochhead, J. & Monk, G.S. (1981) Translation difficulties in learning mathematics, *American Mathematical Monthly*, 88, pp. 286-289.

Clements, D.H. & Sarama, J. (1997) Computer support algebraic thinking, *Teaching Children Mathematics*, 3(6), pp. 320-325.

Cobb, P., Yackel, E. & McClain, K. (2000) *Symbolizing and communicating in mathematics classrooms* (Mahwah, NJ: Erlbaum).

Collis, K.F. (1972) *A study of the relationship between formal thinking and combinations of operations* (Newcastle, NSW: University of Newcastle).

Collis, K.F. (1975) *The development of formal reasoning* (Newcastle, Australia: University of Newcastle).

Cooper, B. (1985) *Renegotiating Secondary School Mathematics: A study of curriculum change & stability* (Basingstoke, The Falmer Press).

Cooper, M. (1984) The mathematical "reversal error" and attempts to correct it, in: B. Southwell, R. Eyland, M. Cooper, J. Conroy & K. Collis (Eds) *Proceedings of the Eighth International Conference for the Psychology of Mathematics Education* (Sydney, PME).

Cooper, T.J., Boulton-Lewis, G., Atweh, B., Wilss, L. & Mutch, S. (1997) The transition from arithmetic to algebra: Initial understanding of equals, operations and variable, *International Group for the Psychology of Mathematics Education*, 21(2), pp. 89-96.

Cooper, T.J. & Williams, A.M. (1997) Teaching 3x, *Proceedings of the Annual Conference of the Australian Association for Research in Education* (Brisbane, AARE).

Cooper, T.J., Williams, A.M. & Baturo, A.R. (1999a) Equals, expressions, equations, and the meaning of variable: A teaching experiment, in: J.M. Truran & K.M. Truran (Eds) *Proceedings of the 22nd Conference of the Mathematics Education Research Group of Australia (MERGA), Making the difference* (Adelaide, MERGA).

Cooper, T.J., Williams, A.M. & Baturo, A.R. (1999b) *Simplifying algebraic expressions* (Queensland, Centre of Mathematics and Science Education, Queensland University of Technology).

Cortes, A. (1993) Analysis of errors and a cognitive model in the solving of equations, *Proceedings of the 17th International Conference for the Psychology of Mathematics Education* (Tsukuba, Japan, University of Tsukuba).

Crowley, L., Thomas, M.O.J. & Tall, D.O. (1994) Algebra, symbols and the translation of meaning, *Proceedings of the 18th International Conference for the Psychology of Mathematics Education* (Lisbon, Portugal, PME).

Cuoco, A.A. (1994) Multiple representations of functions, in: J. Kaput & E. Dubinsky (Eds) *Research issues in undergraduate mathematics learning* (Washington, D.C., Mathematical Association of America).

Davis, R.B. (1975) Cognitive processes involved in solving simple algebraic equations, *Journal of Children's Mathematical Behavior*, 1(3), pp. 7-35.

Davis, R.B. (1985) ICME-5 report: Algebraic thinking in the early grades, *Journal of Mathematical Behavior*, 4, pp. 195-208.

Davis, R.B. (1989) Research studies in how humans think about algebra, in: S. Wagner & C. Kieran (Eds) *Research agenda for mathematics education: Research issues in the learning and teaching of algebra* (Reston, VA: NCTM).

Davis, R.B., Jockusch, E. & McKnight, C. (1978) Cognitive processes in learning algebra, *Journal of Children's Mathematical Behavior*, 2(1), pp. 310-320.

DeMarois, P. & Tall, D. (1996) Facets and layers of the function concept, in: L. Puig & A. Gutierrez (Eds) *Proceedings of the 20th Annual Conference for the Psychology of Mathematics Education* (Valencia, Spain, University of Valencia).

DeMarois, P. & Tall, D. (1999) Function: Organizing principle or cognitive root?, *Proceedings of the 23rd International Conference for the Psychology of Mathematics Education* (Haifa, Israel, PME).

Demby, A. (1997) Algebraic procedures used by 13-to-15-year-olds, *Educational Studies in Mathematics*, 33, pp. 45-70.

Dettori, G., Garutti, R. & Lemut, E. (Eds.) (2001) *From arithmetic to algebraic thinking by using a spreadsheet* (Dordrecht, Kluwer Academic).

DfEE (2001) *Key Stage 3 National Strategy: Framework for teaching mathematics: Year 7, 8 and 9* (London, DfEE Publications).

DfES (2001a) Schools: Achieving success (London, The Stationery Office).

DfES (2001b) Schools: building on success (London, The stationery Office).

Dreyfus, T. & Eisenberg, T. (1982) Intuitive functional concepts: A baseline study on intuition, *Journal for Research in Mathematics Education*, 6(2), pp. 18-24.

Dreyfus, T. & Vinner, S. (1982) Some aspects of the function concept in college students and junior high school teachers, *Proceedings of the Sixth International Conference for the Psychology of Mathematics Education* (Antwerp, PME).

Dubinski, E., Hawkes, J. & Nichols, D. (1989) Development of the process conception of function by preservice teachers in a discrete mathematics course, *Proceedings of the 13th Annual International Conference for the Psychology of Mathematics Education* (Paris, PME).

Dubinsky, E. & Harel, G. (1992) The nature of the process conception of function, in: G. Harel & E. Dubinsky (Eds) *The concept of function aspects of epistemology and pedagogy* (Washington, D.C., Mathematics Association of America).

English, L.D. & Warren, E.A. (1998) Introducing the variable through pattern exploration, *Mathematics Teacher*, 91(2), pp. 166-170.

Epstein, E.H. (1994) Comparative and international education: Overview and historical development, in: Husen, Torsten, T.N. Postlethwaite & T. Neville (Eds) *The International Encyclopedia of Education* (Oxford, Pergamon & Elsevier).

Filloy, E. & Rajano, T. (1989) Solving equations: The transition from arithmetic to algebra, *For the Learning of Mathematics*, 9(2), pp. 19-25.

Filloy, E. & Sutherland, R. (1996) Designing curricula for teaching and learning algebra, in: A.J. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds) *International Handbook of Mathematics Education* (London, Kluwer Academic Press).

Foster, R. (1994) Counting on success in simple arithmetic tasks, *Proceedings of the18th Conference for the Psychology of Mathematics Education* (Lisbon, Portugal, PME).

Freudenthal, H. (1978) *Weeding and sowing: Preface to a science of mathematical education* (Dordrecht, Holland: D. Reidel).

Gallo, E. (1994) Algebraic manipulation as problem solving, in: N.A. Malara & L. Rico (Eds) *First Italian-Spanish Symposium on Mathematics Education* (Modena, Modena University Press).

Garcia-Cruz, J.A. & Martinon, A. (1997) Actions and invariant in schemata in linear generalising problems in: E. Pehkonen (Ed) *Proceedings of the 21st International Conference for the Psychology of Mathematics Education* (Lahthi, Finland, PME).

Gattegno, C. (1970) *What we owe children: The subordination of teaching to learning* (London, Routledge & Kegan Paul).

Gattegno, C. (1978) *The science of education part 2B: The awareness of mathematization* (New York, Educational Solutions).

Giacchino-Baker, R. (2003) *Adapting teaching styles to cultural differences: What do you do with Thai's after they smile?* (Pennsylvania, University of Pennsylvania).

Gill, B., Dunn, M. & Goddard, E. (2002) *Student achievement in England* (London, The Stationery Office).

Gould, S.J. (1996) *The mismeasure of man* (London, W.W. Norton & Company).

Gray, E. & Tall, D. (1991) Flexibility, duality and ambiguity in successful mathematical thinking, *Proceedings of the 15th International Conference for the Psychology of Mathematics Education* (Assisi, Italy, PME).

Gray, E. & Tall, D. (1993) Success and failure in mathematics: The flexible meaning of symbols as process and concept, *Mathematics Teaching*, 142, pp. 6-10.

Gray, E. & Tall, D. (1994) Duality, ambiguity, and flexibility: A proceptual view of simple arithmetic, *Journal for Research in Mathematics Education*, 26(2), pp. 115-141.

Hall, R. (2002) An analysis of thought processes during simplification of an algebraic expression, *Philosophy of Mathematics Education Journal*, 15 [Online], Available from: http://www.ex.ac.uk/~PErnest/pome15/processes.htm [accessed 20 September 2003].

Hart, K.M. (1981) *Children's understanding of mathematics: 11-16* (Great Britain, John Murray).

Healy, L., Pozzi, S. & Sutherland, R. (1996) Reflecting on the role of the computer in developing algebraic understanding, in: R. Sutherland, T. Rojano, A. Bell & R. Lins (Eds) *Algebraic Processes and Structure* (The Netherlands, Kluwer).

Herbert, K. & Brown, R. (1997) Patterns as tools for algebraic reasoning, *Teaching Children Mathematics*, 3, pp. 340-344.

Herscovics, N. (1989) Cognitive obstacles encountered in the learning of algebra, in: S. Wagner & C. Kieran (Eds) *Research issues in the learning and teaching of algebra*, Vol. 4 (Hillsdale, NJ: Lawrence Erlbaum).

Herscovics, N. & Kieran, C. (1980) Constructing meaning for the concept of equation, *Mathematics Teachers*, pp. 572-580.

Herscovics, N. & Linchevski, L. (1994) A cognitive gap between arithmetic and algebra, *Educational Studies in Mathematics*, 27, pp. 59-78.

Hewitt, D. (1998) Approaching arithmetic algebraically, *Mathematics Teaching*, 163, pp. 19-29.

HMI (1985) *Mathematics from 5 to 16* (London, Crown).

HMSO (1995) *Mathematics in the National Curriculum* (London, HMSO).

Holt, G., Andrews, C., Boyd, S., Harper, A., Loose, J., O' Donnell, S. & Sargent, C. (2002) *Education in England, Wales and Northern Ireland: A guide to the system* (Slough, National Foundation for Educational Research, NFER).

Hoyles, C. & Sutherland, R. (1992) *Logo mathematics in the classroom* (London, Routledge).

Hughes, M. (1986) *Children and number: Difficulties in learning mathematics* (Oxford, Blackwell).

Just, M.A. & Carpenter, P.A. (1989) The role of working memory in language comprehension, in: D. Klahr & K. Kotovsky (Eds) *Complex information processing* (Hillsdale, Erlbaum).

Just, M.A. & Carpenter, P.A. (1992) A capacity theory of comprehension: Individual differences in working memory, *Psychological Review*, 99(1), pp. 122-149.

Kaewdang, R. (2001) *Learning for the new century* (Bangkok, Office of the National Education Commission).

Kaput, J. (1995) Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K-12 curriculum, in: S. Fennel (Ed) *A framework for integrating algebra across K-12 mathematics* (Reston, VA: NCTM).

Kaput, J. (1999) Teaching and learning a new algebra, in: E. Fennema & T. Romberg (Eds) *Mathematics classrooms that promote understanding* (Mahwah, NJ: Erlbaum).

Kaput, J. & Sims-Knight, J.E. (1983) Errors in translations to algebraic equations: Roots and implications, *Focus on Learning Problems in Mathematics*, 5(3), pp. 63-78.

Kennewell, S. (2001) *Algebra in the head, on paper and on the screen* (Swansea, University of Wales Swansea).

Kieran, C. (1981) Concepts associated with the equality symbol, *Educational Studies in Mathematics*, 12, pp. 317-326.

Kieran, C. (1984) A comparison between novice and more-expert algebra students on tasks dealing with the equivalence of equations, in: J.M. Moser (Ed) *Proceedings of the Sixth Annual Meeting of PME-NA* (Madison, University of Wisconsin, PME).

Kieran, C. (1985a) Constructing meaning for equations and equation-solving, in: A. Bell, B. Low & J. Kilpatrick (Eds) *Theory, research & practice in mathematical education* (University of Nottingham, UK: Shell Center for Mathematical Education).

Kieran, C. (1985b) The equation-solving errors of novices and intermediate algebra students, *Proceedings of the Ninth International Conference Psychology of Mathematics Education* (Montreal, Canada, PME).

Kieran, C. (1988) Two different approaches among algebra learners, in: A.F. Coxford & A.P. Schulte (Eds) *The ideas of algebra, K-12* (Hillsdale, NJ: Lawrence Erlbaum).

Kieran, C. (1989a) A perspective on algebraic thinking, in: G. Vergnaud, J. Rogalski & M. Artigue (Eds) *Proceedings of the 13th International Conference for the Psychology of Mathematics Education* (Paris, Laboratoire PSYDEE).

Kieran, C. (1989b) The early learning of algebra: A structure perspective, in: S. Wagner & C. Kieran (Eds) *Research issues in the learning and teaching of algebra* (Hillsdale, NJ: Lawrence Erlbaum).

Kieran, C. (1991) A procedural-structural perspective on algebra research, *Proceedings of the 15th International Conference for the Psychology of Mathematics Education* (Assisi, Italy, PME).

Kieran, C. (1992) The learning and teaching of school algebra, in: D.A. Grouws (Ed) *Handbook of research on mathematics teaching and learning* (New York, NY: Macmillan).

Kieran, C. (1994) A functional approach to the introduction of algebra: Some pros and cons, in: J.P. da Ponte & J.F. Matos (Eds) *Proceedings of the 18th International Conference for the Psychology of Mathematics Education* (Lisbon, Portugal: PME).

Klainin, S. (2002) PISA: International student assessment project 2000-2009, *Science Mathematics and Technology Education*, 118, pp. 10-16.

Klainin, S. (2003a) Thai mathematics: Why is it not strong?, *Science Mathematics and Technology Education*, 125, pp. 18-24.

Klainin, S. (2003b) Thai Mathematics: Why is it not strong (2)?, *Science Mathematics and Technology Education*, 126, pp. 3-12.

Kline, P. (1993) *An Easy Guide to Factor Analysis* (Great Britain, Routledge).

Knuth, E. (2000) Understanding connections between equations and graphs, *The Mathematics Teacher*, 93(1), pp. 48-53.

Koedinger, K.R. (1998) Intelligent cognitive tutors as modeling tool and instructional model, *The NCTM Standards 2000 Technology Conference* (Arlington Virginia, NCTM).

Kota, S. & Thomas, M.O.J. (1998) Students' arithmetic preferences: Effect on problem solving ability, *Hiroshima Journal of Mathematics Education*, 6, pp. 33-48.

Küchemann, D. (1978) Children's understanding of numerical variables, *Mathematics in School*, 7(4), pp. 23-26.

Küchemann, D. (1981) *Algebra*, in: K.M. Hart (Ed) *Children's understanding of mathematics 11-16* (London, John Murray).

Laborde, C. (1990) Language and mathematics, in: P. Nesher & J. Kilpatrick (Eds) *Mathematics and Cognition* (Cambridge, Cambridge University Press).

Lawson, D.R. (1990) The problem, the issues that speak to change, in: E.L.J. Edwards (Ed) *Algebra for Everyone* (Reston, VA: The National Council of Teachers of Mathematics).

Lee, L. (1996) An initiation into algebraic culture through generalization activities, in: N. Bednarz, C. Kieran & L. Lee (Eds) *Approaches to algebra: Perspectives for research and teaching* (Dordrecht, Academic Publishers).

Lee, L. & Wheeler, D. (1987) *Algebraic thinking in high school students: Their conception of generalisation and justification* (Montreal, Canada: Concordia University, Mathematics Department).

Leitzel, J.R. (1989) Critical consideration for the future of algebra instruction, in: S. Wagner & C. Kieran (Eds) *Research issues in the teaching and learning of algebra* (Hillsdale, NJ: Lawrence Erlbaum).

Liebenberg, R., Linchevski, L., Olivier, A. & Sasman, M. (1998) Laying the foundation for algebra: Developing an understanding of structure, *Proceedings of the Fourth Annual Congress of the Association for Mathematics Education of South Africa* (Pieterburg, University of the North).

Liebenberg, R., Sasman, M. & Olivier, A. (1999) From numerical equivalence to algebraic equivalence, *Proceedings of the Fifth Annual Congress of the Association for Mathematics Education of South Africa* (Port Elizabeth: Port Elizabeth, Technikon).

Linchevski, L. (1995) Algebra with numbers and arithmetic with letters: A definition of pre-algebra, *Journal of Mathematical Behavior*, 14, pp. 113-120.

Linchevski, L. (2001) Operating on unknown: What does it really mean?, *Proceedings of the 25th International Conference for the Psychology of Mathematics Education* (Utrecht, The Netherlands: Freudenthal Instutute).

Linchevski, L. & Herscovics, N. (1996) Crossing the cognitive gap between arithmetic and algebra: Operating on the unknown in the context of equations, *Educational Studies in Mathematics*, 30(1), pp. 39-65.

Linchevski, L., Olivier, A., Sasman, M.C. & Liebenberg, R. (1998) Moments of conflict and moments of conviction in generalizing, in: A. Olivier & K. Newstead (Eds) *Proceedings of the 22nd International Conference for the Psychology of Mathematics Education* (Stellebosch, South Africa, PME).

Lincoln, Y.S. & Guba, E.G. (1985) *Naturalistic inquiry* (Beverly Hills, CA: Sage).

Linnecor, M. (1999) *Meeting algebra for the first time: Teaching algebra to year 7* (Manchester, Manchester Metropolitan University).

Lins, R. (1990) A framework of understanding what algebraic thinking is, in: G. Booker, P. Cobb & T.N. Mendicuti (Eds) *Proceedings of the 14th International Conference for the Psychology of Mathematics Education*, Vol. 2 (Mexico City, Mexico, PME).

Lochhead, J. (1980) Faculty interpretations of simple algebraic statements: The professor's side of the equation, *Journal of Mathematical Behavior*, 3(1), pp. 29-37.

Lochhead, J. & Mestre, J. (1988) From words to algebra: Mending misconceptions, in: A.F. Coxford (Ed) *The ideas of algebra, K-12 (1988 Yearbook)* (Reston, VA: NCTM).

Love, E. (1986) What is algebra?, *Mathematics Teaching*, 117, pp. 48-50.

MacGregor, M. (2001) Does learning algebra benefit most people?, in: H. Chick, K. Stacey & J. Vincent (Eds) *Proceedings of the 12th ICMI Study Conference* (Australia, University of Melbourne).

MacGregor, M. & Stacey, K. (1993a) Cognitive models underlying students' formulation of simple linear equations, *Journal for Research in Mathematics Education*, 24(3), pp. 217-232.

MacGregor, M. & Stacey, K. (1993b) Seeing a pattern and writing a rule, in: I. Hirabayashi, N. Nohda, K. Sheigemastu & F. Lin (Eds) *Proceedings of the 17th International Conference for the Psychology of Mathematics Education* (University of Tsukuba, Japan, PME).

MacGregor, M. & Stacey, K. (1997) Students' understanding of algebraic notation 11-15, *Educational Studies in Mathematics*, 33(1), pp. 1-19.

Malara, N.A. & Iaderosa, R. (1999) The interweaving of arithmetic and algebra: Some questions about syntactic and structural aspects and their teaching and learning, *European Research in Mathematics Education I.II* (Germany, Osnabrueck-Hausohrbeck).

Mason, J. (1987) What do symbols represent?, in: C. Janvier (Ed) *Problems of representation in the teaching and learning of mathematics* (NJ: Erlbaum/London, Hillsdale).

Mason, J. (1992) Frames for teaching, in: E. Love & J. Mason (Eds) *Teaching mathematics: Action and awareness* (Open University, Milton Keynes).

Mason, J. (1996) Expressing generality and roots of algebra, in: N. Bednarz, C. Kieran & L. Lee (Eds) *Approaches to algebra-perspectives for research and teaching* (Dordrecht, Kluwer Academic Publishers).

Mason, J. (2002) Generalisation and algebra: Exploiting children's powers, in: L. Haggerty (Ed) *Aspects of teaching Secondary Mathematics: Perspectives on practice* (London, Routledge Falmer).

Mason, J., Burton, L. & Stacey, K. (1982) *Thinking mathematically* (London, Addison Wesley).

Mason, J., Graham, A., Pimm, D. & Gowar, N. (1985) *Routes to roots of algebra* (Milton Keynes, Open University Press).

Mason, J. & Sutherland, R. (2002) *Key aspects of teaching algebra in schools* (Great Britain, Qualifications and Curriculum Authority).

Mathichon (2003) OECD Thai students low quality results, *Mathichon* (Bangkok, Mathichon).

Matz, M. (1980) Towards a computational theory of algebraic competence, *Journal of Mathematical Behavior*, 3(1), pp. 93-166.

Matz, M. (1981) Building metaphoric theory of mathematical thought, *Journal of Mathematical Behavior*, 3(1), pp. 93-166.

Matz, M. (1982) Towards a process model for high-school algebra errors, in: D. Sleeman & J.S. Brown (Eds) *Intelligent Tutoring Systems* (London, Academic Press).

Mayer, R.E., Lewis, A.B. & Hegarty, M. (1992) Mathematical misunderstanding: Qualitative reasoning about quantitative problems, in: J. Cambell (Ed) *The nature and origins of mathematical skills* (Amsterdam, North Holland).

McLean, M. (1990) *Britain and a single market Europe* (London, Kogan Page).

Meira, L. (1996) Students' early algebraic activity: Sense making and the production of meanings in mathematics, in: L. Puig & A. Gutierrez (Eds) *Proceedings of the 20th*

International Conference for the Psychology of Mathematics Education (Valencia, Spain, University of Valencia, PME).

Mestre, J. (1988) The role of language comprehension in mathematics and problem solving, in: R. Cocking & J. Mestre (Eds) *Linguistic and cultural influences on learning mathematics* (Hilldale, NJ: Erlbaum).

Ministry of Education (1992) *Lower Secondary School Mathematics Curriculum* (Bangkok, Kurusapa).

Nathan, M. & Koedinger, K. (2000) Moving beyond teachers' intuitive beliefs about algebra learning, *The Mathematics Teacher*, 93(3), pp. 218-223.

Nathan, M.J., Kintsch, W. & Young, E. (1992) A theory of algebra word-problem comprehension and its implication for the design of learning environments, *Cognition and Instruction*, 9(4), pp. 329-389.

National Education Commission (1999) *National Education Act of B.E.2542 (1999)* (Bangkok, Seven Printing Group).

National Statistics (2001) *First release: International student assessment, England 2000* [Online], Available from: http://www.statistics.gov.uk/releases [accessed 12 October 2003].

NCTM (1989) *Curriculum and evaluation standards for school mathematics* (Reston, VA: The Council).

NCTM (2000) *Principles and standards for school mathematics* (Reston, VA: Author).

Nibbelink, W.H. (1990) Teaching equations, *Arithmetic Teacher*, 38(3), pp. 48-51.

Norton, S.J. & Cooper, T.J. (1999) *Do our students really have the arithmetic knowledge to start algebra? Analysing misconceptions* (Queensland, The Queensland Association of Mathematics Teachers).

Norton, S.J. & Cooper, T.J. (2001) *Students' perceptions of the importance of closure in arithmetic: Implication for algebra* (Brisbane, Centre for Mathematics and Science Education, Australia).

Noss, R. (1986) Constructing a conceptual framework for elementary algebra through Logo programming, *Educational Studies in Mathematics*, 17, pp. 335-357.

Noss, R., Healy, L. & Hoyles, C. (1997) The construction of mathematical meanings: Connecting the visual with the symbolic, *Educational Studies in Mathematics*, 33, pp. 203-233.

Nunn, T.P. (1919) *The teaching of algebra* (London, Longman, Green and Company).

OECD (1999) *What PISA assesses: Mathematical literacy* [Online], Available from: http://www.pisa.oecd.org/pisa/math.htm [accessed 12 October 2003].

Ofsted (1999) *The national numeracy strategy* (London, Crown).

Ohlsson, S. (1993) Abstract schemas, *Educational Psychologist*, 28(1), pp. 51-66.

ONEC (2000a) *Education in Thailand 2001/2002* (Bangkok, Kurusapa Ladprao Press).

ONEC (2000b) *Learning reform: A learner-centred approach* (Bangkok, Author).

ONEC (2001) *Education in Thailand* (Bangkok, Author).

Orton, A. (1992) *Learning mathematics: Issues, theory and classroom practice* (London, Cassell).

Orton, A. & Orton, J. (1994) Students' perception and use of pattern and organization, *Proceedings of the 18th International Conference for the Psychology of Mathematics Education* (Lisbon, Portugal, PME).

Orton, A. & Orton, J. (1999) Pattern and the approach to algebra, in: A. Orton (Ed) *Pattern in the teaching and learning of mathematics* (London, Cassell).

Pallant, J. (2002) *SPSS survival manual* (Buckingham, Open University Press).

Papert, S. (1980) *Mindstorms: Children, computers and powerful ideas* (United States of America, Basic Books Inc).

Philipp, R.A. (1992) The many uses of algebraic variables, *Mathematics Teacher*, 85, pp. 557-561.

Phillips, D. (1999) On Comparing, in: R. Alexander, P. Broadfoot & D. Phillips (Eds) *Learning from Comparing*, Vol. One (Oxford, Symposium Books).

Pillay, H. & Elliott, R.G. (2002) Distributed learning: Understanding the workplace knowledge, *Journal of Interactive learning Research*, 13(1/2), pp. 95-112.

Pimm, D. (1995) *Symbols and meanings in school mathematics* (London, Routledge).

Pirie, S.E.B. & Martin, L. (1997) The equation, the whole equation and nothing but the equation! *Educational Studies in Mathematics*, 34, pp. 159-181.

Piya-Ajariya, L. (2001) *National pilot study on learning reform-schools for developing quality of learners* (Bangkok, ONEC).

Postlethwaite, T.N. (Ed.) (1988) *The encyclopedia of comparative education and national systems of education* (Oxford, Pergamon).

Radford, L. (1996) Some reflections on teaching algebra through generalization, in: N. Bednarz, C. Kieran & L. Lee (Eds) *Approaches to algebra - perspectives for research and teaching* (Dordrecht, Kluwer Academic Publishers).

Radford, L. (1997) On psychology, historical epistemology, and the teaching of mathematics: Towards a socio-cultural history of mathematics, *For the Learning of Mathematics*, 17(1), pp. 26-33.

Radford, L. (2000) Signs and meanings in students, emergent algebraic thinking: A semiotic analysis, *Educational Studies in Mathematics*, 42, pp. 237-268.

Reggiani, M. (1994) Generalization as a basis for algebraic thinking: Observations with 11-12 year old pupils, *Proceedings of the 18th International Conference for the Psychology of Mathematics Education* (Lisbon, Portugal: PME).

Reggiani, M. (1999) Syntactical and semantic aspects in solving equations: A study with 14 year old pupils, *Proceedings of the First Conference of the European Research in Mathematics Education* (Germany, Osnabrueck-Hauohrbeck).

Reynolds, D. (1999) Creating a new methodology for comparative educational research: The contribution of the international school effectiveness research project, in: R. Alexander, P. Broadfoot & D. Phillips (Eds) *Learning from Comparing*, Vol. 1 (Oxford, Symposium Books).

Robitaille, D.F. & Robeck, E.C. (1996) The character and the context of TIMSS, in: D.F. Robitaille & R.A. Garden (Eds) *Research questions and study design, TIMSS Monograph No. 2* (Vancouver, Pacific Educational Press).

Rojano, T. (1996) Developing algebraic aspects of problem solving within a spreadsheet environment, in: N. Bednarz, C. Kieran & L. Lee (Eds) *Approaches to algebra: Perspectives for research and teaching* (Dordrecht, The Netherlands: Kluwer).

Rojano, T., Sutherland, R., Ursini, S., Molyneux, S. & Jinich, E. (1996) Ways of solving algebra problems: The influence of school culture, *Proceedings of the Twentieth International Conference for the Psychology of Mathematics Education* (Valencia, Spain, PME).

Ryan, G.W. & Bernard, H.R. (2000) Data management and analysis methods, in: N. Denzin & Y. Lincoln (Eds) *Handbook of qualitative research* (Thousand Oaks, Sage Publications).

Sakpakornkan, N. & Harries, T. (2003a) Pupils' processes of thinking: Learning to solve algebraic problems in England and Thailand (simplification), in: J. Williams (Ed)

Proceedings of the British Society for Research into Learning Mathematics (University of Oxford, BSRLM).

Sakpakornkan, N. & Harries, T. (2003b) Pupils' processes of thinking: Learning to solve algebraic problems in England and Thailand (solving equations), *BERA Annual Conference* (Heriot-Watt University, Edinburgh, BERA).

Sasman, M., Olivier, A. & Linchevski, L. (1999) Factors influencing students' generalization thinking processes, in: O. Zaslavsky (Ed) *Proceedings of the 23rd International Conference for the Psychology of Mathematics Education* (Haifa, Israel Institute of Technology).

Schliemann, A.D., Carraher, D. & Brizuela, B. (2000) From quantities to ratio, functions and algebraic relations, *2000 AERA Meeting* (New Orleans, LA, AERA).

Schwingendorf, K., Hawks, J. & Beineke, J. (1992) Horizontal and vertical growth of the students' conception of function, in: G. Harel & E. Dubinsky (Eds) *The concept of function aspects of epistemology and pedagogy* (Washington, D.C., MAA).

Sfard, A. (1989) Transition from operational to structural conception: The notion of function revisited, in: G. Vergnaud, J. Rogalski & M. Artigue (Eds) *Proceedings of the 13th International Conference Psychology of Mathematics Education* (Paris, PME).

Sfard, A. (1991) On the dual nature of mathematics conceptions: Reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics*, 22, pp. 1-36.

Sfard, A. (1994) Reification as the birth of metaphor, *For the Learning of Mathematics*, 14(1), pp. 44-55.

Sfard, A. (1995) The development of algebra: Confronting historical and psychological perspectives, *Journal of Mathematical Behavior*, 14, pp. 15-39.

Sfard, A. & Linchevski, L. (1994) The gains and the pitfalls of reification - the case of algebra, *Educational Studies in Mathematics*, 26, pp. 191-228.

Sierpinska, A. (1995) *Understanding in mathematics* (London, Falmer Press).

Singer, R. (2001) *Understanding ordinary algebra unit: UV using variables* [Online], Available from: http://www.fractions-plus.com [accessed 24 September 2002].

Stacey, K. & MacGregor, M. (1997) Ideas about symbolism that students bring to algebra, *The Mathematics Teacher*, 90(2), pp. 110-113.

Stacey, K. & MacGregor, M. (2000) Curriculum reform and approaches to algebra, in: R. Sutherland, T. Rojano, A. Bell & R. Lins (Eds) *Perspective on school algebra* (Dordrecht/Boston/London, Kluwer Academic Publishers).

Steinberg, R., Sleeman, D. & Ktoza, D. (1990) Algebra students' knowledge of equivalence of equations, *Journal for Research in Mathematics Education*, 22(2), pp. 112-121.

Sutherland, R. (1991) Some unanswered research questions on the teaching and learning of algebra, *For the Learning of Mathematics*, 11(33), pp. 40-46.

Sutherland, R. (1992) Thinking algebraically: Pupil models developed in Logo and a spreadsheet environment, in: E. Lemut (Ed) *Cognitive models and intelligent learning environments for learning programming* (Berlin, Springer).

Sutherland, R. (1997) *Teaching and learning algebra pre-19* (London, Royal Society and Joint Mathematical Council of the United Kingdom).

Sutherland, R. (2002) *A comparative study of algebra curricula* (London, Qualifications and Curriculum Authority).

Sutherland, R. & Rojano, T. (1993a) *Algebraic processes and structure* (Dordrecht, The Netherland, Kluwer).

Sutherland, R. & Rojano, T. (1993b) A spreadsheet approach to solving algebra problems, *Journal of Mathematical Behavior*, 12(4), pp. 351-383.

Tabachnick, B.G. & Fidell, L.S. (1996) *Using multivariate statistics* (New York, Harper Collins).

Tall, D. (1992a) The transition from arithmetic to algebra: Number patterns, or proceptual programming? *New directions in algebra education* (Brisbane, Queensland University of Technology).

Tall, D. (1992b) The transition to advanced mathematical thinking: Functions, limits, infinity, and proof, in: D.A. Grouws (Ed) *Handbook of research on mathematics teaching and learning* (New York, Macmillan Publishing Company).

Tall, D. (1993) Just a matter of definition, *Educational Studies in Mathematics*, 25(3), pp. 261-277.

Tall, D. (1995) The psychology of symbols and symbol manipulators, *Proceedings of the Seventh Annual International Conference on Technology in Collegiate Mathematics Teaching* (Orlando, Addison-Wesley).

Tall, D. (2001) Reflections on early algebra, *Proceedings of the 25th International Conference for the Psychology of Mathematics Education* (Utrecht, The Netherlands, PME).

Tall, D. & Thomas, M. (1991) Encouraging versatile thinking in algebra using the computer, *Educational Studies in Mathematics*, 22, pp. 125-147.

Taplin, M. (1995) Special patterning: A pilot study of pattern formation and generalization, in: L. Meira & D. Carraher (Eds) *Proceedings of the 19th C International Conference for the Psychology of Mathematics Education* (Recife, Brazil, PME).

Thomas, M. (1994) A process-oriented preference in the writing of algebraic equations, in: G. Bell, B. Wright, N. Leeson & J. Geake (Eds) *Proceedings of the 17th Annual*

Conference of the Mathematics Education Research Group of Australia (Lismore, NSW: MERGA).

Thomas, M. & Tall, D. (2001) The long-term cognitive development of symbolic algebra, in: H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds) *Proceedings of the 12th ICMI Study Conference: The future of the teaching and learning of algebra* (Melbourne, University of Melbourne).

Threlfall, J. & Frobisher, L. (Eds.) (1999) *Patterns in processing and learning addition facts* (Wiltshire, Cassell).

Tirosh, D., Even, R. & Robinson, M. (1998) Simplifying algebraic expressions: Teacher awareness and teaching approaches, *Educational Studies in Mathematics*, 35, pp. 51-64.

Ursini, S. & Trigueros, M. (1997) Understanding of different uses of variable: A study with starting college students, *Proceedings of the 21st International Conference for Psychology of Mathematics Education* (Lahti, Finland, PME).

Van der Kooij, H. (2001) *Algebra: A tool for solving problems?* (Utrecht, Freudenthal Institute, Utrecht University).

Vance, J.H. (1998) Number operations from an algebraic prospective, *Teaching Children Mathematics*, 4, pp. 282-285.

Vergnaud, G. (1985) Understanding mathematics at the Secondary-School level, in: A. Bell, B. Low & J. Kilpatrick (Eds) *Theory, research & practice in mathematical aducation* (University of Nottingham, UK: Shell Centre for Mathematical Education).

Vinner, S. & Dreyfus, T. (1989) Images and definitions for the concept of function, *Journal for Research in Mathematics Education*, 20(4), pp. 356-366.

Wagner, S. (1977) Conservation of equation and function and its relationship to formal operational thought, *The Annual Meeting of the American Educational Research Association* (New York, AERA).

Wagner, S. (1981) Conservation of equation and function under transformations of variable, *Journal for Research in Mathematics Education*, 12, pp. 107-118.

Wagner, S. & Kieran, C. (Eds.) (1989) *Research issues in the learning and teaching of algebra* (Hillsdale, NJ: Lawrence Erlbaum).

Wagner, S. & Parker, S. (1988) Advancing algebra, in: B. Moses (Ed) *Algebraic thinking, grades K-12* (Reston, VA: NCTM).

Wagner, S., Rachlin, S.L. & Jensen, R.J. (1984) *Algebra learning project - final report* (Georgia, Department of Mathematics Education, University of Georgia).

Williams, A.M. & Cooper, T.J. (2001) Moving from arithmetic to algebra under the time pressures of real classrooms, in: H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds) *Proceedings of the 12th ICMI Study Conference: The future of the teaching and learning of algebra* (Melbourne, University of Melbourne).

Zazkis, R. & Liljedahl, P. (2002a) Arithmetic sequence as a bridge between conceptual fields, *Canadian Journal of Mathematics, Science and Technology Education*, 2(1), pp. 93-120.

Zazkis, R. & Liljedahl, P. (2002b) Generalization of patterns: The tension between algebraic thinking and algebraic notation, *Educational Studies in Mathematics*, 49, pp. 379-402.