Stability and wave motion problems in continuous media with second sound

Jiratchaya Jaisaardsuetrong

Stability and Wave Motion Problems in Continuous Media with Second Sound

Jiratchaya Jaisaardsuetrong

A Thesis presented for the degree of

Doctor of Philosophy

Numerical Analysis

Department of Mathematical Sciences

University of Durham

England

July 2009

2 9 SEP 2009

Dedicated to

My family

Stability and Wave Motion Problems in Continuous Media with Second Sound

Jiratchaya Jaisaardsuetrong

Submitted for the degree of Doctor of Philosophy

July 2009

Abstract

In this thesis we investigate thermal convection and wave motion in models of second sound such as the Cattaneo model, Green and Laws model, Batra model, and Green and Naghdi model. For the Green and Laws and Batra models we also investigate questions of stability and uniqueness.

The term second sound means the transport of heat as a thermal wave. The models are all presented within the framework of continuum mechanics. We use a mathematical technique involving an acceleration wave to solve some problems. Furthermore, in one of the chapters we use a numerical method, namely a D^2 Chebyshev tau method to find eigenvalues of a thermal convection problem. This technique is a highly accurate method.

In Chapter two we study thermal convection with the Cattaneo model. The model is about thermal convection in a layer of fluid heated from below. We also employ D^2 Chebyshev tau method to obtain numerical results for the model.

In Chapter three we study various properties such as instability and uniqueness of the model of second sound which is derived by Green and Laws. We investigate the model of Green and Laws for which the generalized temperature ϕ depends on θ and $\dot{\theta}$. We also show differences between the results when the boundary and initial conditions have been changed.

In Chapter four we study uniqueness, instability and wave motion of a Batra model. In Chapter five we investigate thermal waves in a rigid heat conductor. This is a more recent model of heat transport in a rigid body, namely that derived by

Green and Naghdi.

In the final chapter, Chapter six, we consider a generalization of the theory of Chapter five, to include fluid mechanical behaviour. We adopt a special relation for the Helmholtz free energy in the model of Green and Naghdi. We analyse behaviour of an acceleration wave for the model.

Declaration

The work in this thesis is based on research carried out in the Numerical Analysis Group, the Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work with the exception of Chapter 5. Chapter 5 was written in collaboration with Prof. Brian Straughan of the University of Durham, England. The content of Chapter 5 is published in [81]

Acknowledgements

Contents

List of Figures

List of Tables

Chapter 1

Introduction

In this thesis we study several models for "second sound". The term second sound refers to the transport of heat as a thermal wave. In particular, we study models for second sound which have been derived using the principles of continuum thermodynamics. A lucid introduction to continuum mechanics may be found in pages 310-344 of the book by Fabrizio [46]. The mathematical techniques we use are based primarily on two ideas. The first is to analyse problems of linear instability in a hydrodynamics setting. This involves solving eigenvalue problems numerically and to this end we employ a D^2-Chebyshev tau method. The Chebyshev tau method is a highly accurate numerical method, which is described in Chapter 2. This technique has been very successful in yielding sharp results for hydrodynamic stability problems and suitable (mainly recent) references are Carr [17, 18], Chang *et al.* [23], Dongarra *et al.* [41], Hill [70, 71], Hill & Straughan [72–74], Orszag [125], Straughan [168–170], Straughan & Walker [174–176], Webber [188, 189].

The second mathematical technique we employ is that of acceleration waves. This technique has been in employment for some time, see Chen [24], Fabrizio and Morro [49], Ogden [124], Truesdell and Toupin [181], Truesdell and Noll [180]. However, it is still a very powerful technique which is much in current use. This is because it yields a means of analysing a fully nonlinear problem with no approximation and then yields valuable results which allow one to test the validity of the physical theory. To indicate just how important this technique is in the recent literature we list the following references, all of which deal with acceleration waves, or very similar

analyses such as these involving other discontinuity waves, Chen [24], Christov *et al.* [27, 28], Ciarletta & Iesan [30], Ciarletta & Straughan [31-33], Ciarletta *et al.* [34], Curro *et al.* [39], Eremeyev [45], Fabrizio & Morro [49], Fu & Scott [53-55], Gultop [66], Iesan & Scalia [80], Jordan [83-88], Jordan & Christov [89], Jordan and Feuillade [90], Jordan & Puri [91, 92], Jordan & Straughan [95], Kameyama & Sugiyama [100], Lin & Szeri [103], Mariano & Sabatini [112], Marasco [110], Marasco & Romano [111], Mentrelli *et al.* [114], Ogden [124], Ostoja-Starzewski & Trebicki [126], Rai [156], Rajagopal & Truesdell [157], Ruggeri & Sugiyama [160], Sabatini & Augusti [161], Straughan [170, 172], Sugiyama [179], Truesdell & Toupin [181], Truesdell & Noll [180], Valenti *et al.* [186], Weingartner *et al.* [190, 191], Whitham [192].

The topic of second sound is a very hot one in the current applied mathematical literature. There are many theories for heat propagation as a wave in rigid bodies, in fluids, in gases, in elastic bodies, and even in materials with much more exotic structure. A thorough review of second sound theories, generalised heat conduction theories, and acceleration waves is contained in the book by Straughan [170] and in the forthcoming book by Straughan [171]. The material for this introduction and the cited references are taken from the books of Straughan [170, 171]. To give an idea of the extent of interest we quote the following references, all of which are dealing with some aspect of second sound, Alvarez *et al.* [1], Alvarez *et al.* [2], Anile & Romano [6], Bargman & Steinmann [8-10], Bargmann *et al.* [11], Bargmann *et al.* [7], Brusov *et al.* [14], Buishvili *et al.* [15], Cai *et al.* [16], Cattaneo [19], Caviglia *et al.* [20], Chandrasekhariah [21, 22], Chen & Gurtin [25], Christov & Jordan [26], Cimmelli & Frischmuth [35], Ciancio & Quintanilla [29], Coleman *et al.* [36, 37], Coleman & Newman [38], De Cicco & Diaco [40], Dreyer & Struchtrup [43], Duhamel [44], Fabrizio *et al.* [48], Fabrizio *et al.* [47], Fichera [50], Franchi & Straughan [52], Green [57], Green & Laws [58], Green & Naghdi [60-65], Gurtin & Pipkin [67], Han *et al.* [68], Hetnarski & Ignaczak [69], Horgan & Quintanilla [75], Iesan [76-78], Iesan & Nappa [79], Jaisaardsuetrong & Straughan [81], Johnson *et al.* [82], Jordan & Puri [94], Joseph & Preziosi [96, 97], Jou *et al.* [98], Jou & Criado [99], Kaminski [101], Lindsay & Straughan [105, 106], Lin & Payne [102], Linton-Johnson

et al. [107], Loh *et al.* [108], Metzler & Compte [115], Meyer [116], Mitra *et al.* [117], Morro *et al.* [118], Morro & Ruggeri [119, 120], Payne & Song [130–136], Puri & Jordan [137–140], Puri & Kythe [141, 142], Quintanilla [143–145], Quintanilla & Racke [146–149], Quintanilla & Straughan [150–155], Roy *et al.* [158], Ruggeri [159], Saleh & Al-Nimr [162], Sanderson *et al.* [163], Serdyukov [164], Serdyukov *et al.* [165], Shnaid [166], Straughan [169–171], Straughan & Franchi [173], Su *et al.* [178], Su & Dai [177], Tzou [182, 183], Vadasz [185], Vadasz *et al.* [184], Vedavarz *et al.* [187], Zhang & Zuazua [193].

1.1   Cattaneo theory

To give a precise application of acceleration wave analysis we begin with an example of heat propagation in a rigid solid. Straughan [170], pp. 349-360 describes these ideas in a porous medium when the heat flux laws are of Cattaneo type and then of dual phase lag type. We follow the presentation given there. The basic equations are those of an energy balance, and a constitutive equation for the heat flux q. Thus, let ε be the internal energy of the body per unit mass and let $\theta(x, t)$ be the temperature. For simplicity, we here restrict attention to one space dimension and so $\theta = \theta(x, t)$, where wave propagation will be in the x-direction. The energy balance law is in three-dimensions

$$\rho\frac{\partial \varepsilon}{\partial t} = -\frac{\partial q_i}{\partial x_i} \tag{1.1}$$

or

$$\rho\dot{\varepsilon} = -q_{i,i}. \tag{1.2}$$

Throughout, standard indicial notation is employed with a repeated index denoting summation over 1,2 or 1,2,3.

In the one-dimensional case (1.1) becomes

$$\rho\frac{\partial \varepsilon}{\partial t} = -\frac{\partial q}{\partial x}. \tag{1.3}$$

The general theory of acceleration waves and shock waves in nonlinear elastody-namics is covered in detail in Chen [24] and in Fabrizio & Morro [49], pages 518-532 for acceleration waves and pages 532-541 for shock waves. Truesdell & Toupin [181]

and Truesdell & Noll [180] cover many aspects of acceleration waves and singular surfaces, in general.

To illustrate the basic concepts of acceleration wave analysis, we mostly restrict attention to a plane acceleration wave moving in the direction of the x−axis, with one-dimensional motion. The precise definition of an acceleration wave depends on the material comprising the body we are examining. However, it will involve a surface $\mathcal{S}$ across which certain derivatives of the functions defining the problem will have discontinuities. A precise definition of an acceleration wave will be given in the context in which it occurs.

For a function $h(x, t)$ we define

$$h^+(x, t) = \lim_{x \to \mathcal{S}} h(x, t) \text{ from the right,}$$

$$h^-(x, t) = \lim_{x \to \mathcal{S}} h(x, t) \text{ from the left.}$$

In particular, h^+ is the value of h at $\mathcal{S}$ approaching from the region which $\mathcal{S}$ is about to enter. The jump of h at $\mathcal{S}$, written as $[h]$, is,

$$[h] = h^- - h^+. \tag{1.4}$$

A key relation in acceleration wave analysis (or generally in any discontinuity analysis) is the kinematic condition of compatibility, sometimes known as the Hadamard relation,

$$\frac{\delta}{\delta t}[f] = \left[\frac{\partial f}{\partial t}\right] + V\left[\frac{\partial f}{\partial x}\right] \tag{1.5}$$

where $\delta/\delta t$ denotes the time derivative at the wave. (The Hadamard relation is discussed in detail in Chen [24], Appendix 1, and also in Truesdell & Toupin [181], Section 180.)

From the definition of $[h]$ we may prove the relation for the jump of a product of functions g, h,

$$[gh] = g^+[h] + h^+[g] + [g][h]. \tag{1.6}$$

We use this relation extensively throughout this thesis when calculating the wave amplitudes.

When $K = K(\theta)$ and $\varepsilon = \varepsilon(\theta)$, equation (1.3) together with the Cattaneo law, Cattaneo [19], present us with the system of equations

$$\tau q_t + q = -K(\theta)\theta_x$$
$$\rho\varepsilon_\theta\theta_t = -q_x, \tag{1.7}$$

where τ is a relaxation time.

Let us define an acceleration wave $\mathcal{S}$ for system (1.7) to be a surface across which $\theta_t, \theta_x, q_t, q_x$ and their higher derivatives suffer a finite discontinuity, jump, but $\theta, q \in C^0(\mathbb{R} \times (0, \infty))$. Then taking the jump of (1.7) we see that

$$\tau[q_t] = -K[\theta_x]$$
$$\rho\varepsilon_\theta[\theta_t] = -[q_x]. \tag{1.8}$$

To progress beyond (1.8) we use the Hadamard relation to find

$$[q_t] = -V[q_x], \quad [\theta_t] = -V[\theta_x],$$

and then (1.8) become

$$-\tau V[q_x] = -K[\theta_x],$$
$$-\rho\varepsilon_\theta V[\theta_x] = -[q_x], \tag{1.9}$$

or, writing in matrix form,

$$\begin{pmatrix} -\tau V & K \\ 1 & -\rho\varepsilon_\theta V \end{pmatrix} \begin{pmatrix} [q_x] \\ [\theta_x] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a non-zero solution of this we require

$$\begin{vmatrix} -\tau V & K \\ 1 & -\rho\varepsilon_\theta V \end{vmatrix} = 0$$

and so

$$V^2 = \frac{K(\theta^+)}{\rho\tau\varepsilon_\theta(\theta^+)}, \tag{1.10}$$

where θ^+ denotes the temperature immediately ahead of the wave.

Equation (1.10) shows that in the rigid body a temperature wave moves to the right while an equivalent wave moves to the left with speed V, where

$$V = \sqrt{\frac{K}{\rho \tau \varepsilon_\theta}}.$$ (1.11)

We may analyse the solution behaviour *at the wave* in much greater detail than this. To do this define the wave amplitudes a and b by

$$a(t) = [\theta_x], \quad b(t) = [q_x]$$ (1.12)

and then differentiate each of (1.7) with respect to x,

$$\tau q_{tx} + q_x = -K'(\theta)\theta_x^2 - K(\theta)\theta_{xx},$$

$$\rho \varepsilon_{\theta\theta} \theta_t \theta_x + \rho \varepsilon_\theta \theta_{tx} = -q_{xx},$$ (1.13)

where $K' \equiv \partial K / \partial \theta$.

To simplify the analysis we suppose the region ahead of the wave is in thermal equilibrium, i.e. $\theta =$ constant.

This means

$$\theta^+ = \text{constant}, \quad \theta_x^+ = \theta_t^+ = 0.$$ (1.14)

We take the jumps of (1.13) and use the product relation (1.6), recalling (1.14) to deduce

$$\tau[q_{tx}] + [q_x] = -K'[\theta_x]^2 - K[\theta_{xx}]$$

$$\rho \varepsilon_{\theta\theta}[\theta_t][\theta_x] + \rho \varepsilon_\theta[\theta_{tx}] = -[q_{xx}].$$ (1.15)

Next, we use the Hadamard relation to see that

$$\frac{\delta b}{\delta t} = [q_{xt}] + V[q_{xx}], \quad \frac{\delta a}{\delta t} = [\theta_{tx}] + V[\theta_{xx}]$$

and employ these expressions in (1.15). In this manner one finds

$$\tau\left(\frac{\delta b}{\delta t} - V[q_{xx}]\right) + b = -K'a^2 - K[\theta_{xx}]$$

$$\rho \varepsilon_\theta\left(\frac{\delta a}{\delta t} - V[\theta_{xx}]\right) - \rho \varepsilon_{\theta\theta} V a^2 = -[q_{xx}]$$ (1.16)

From (1.9) we know that

$$\tau V b = K a, \quad \text{or} \quad b = \frac{K}{\tau V} a.$$

Using this we may deduce a satisfies the equations (note V is constant since $\theta^+ \equiv$ constant)

$$\frac{K}{V} \frac{\delta a}{\delta t} - \tau V [q_{xx}] + \frac{K}{\tau V} a = -K' a^2 - K[\theta_{xx}]$$
$$\rho \varepsilon_\theta \frac{\delta a}{\delta t} - \rho \varepsilon_\theta V [\theta_{xx}] - \rho \varepsilon_{\theta\theta} V a^2 = -[q_{xx}] \qquad (1.17)$$

Dividing by K/V, where $K/V \neq 0$ these yield

$$\frac{\delta a}{\delta t} - \frac{\tau V^2}{K} [q_{xx}] + \frac{1}{\tau} a = -\frac{K' V}{K} a^2 - V[\theta_{xx}],$$
$$\frac{\delta a}{\delta t} - V[\theta_{xx}] - \frac{\varepsilon_{\theta\theta}}{\varepsilon_\theta} V a^2 = -\frac{1}{\rho \varepsilon_\theta} [q_{xx}]. \qquad (1.18)$$

Now, add both equations to find

$$2 \frac{\delta a}{\delta t} - [q_{xx}] \left(\frac{\tau V^2}{K} - \frac{1}{\rho \varepsilon_\theta} \right) + \frac{1}{\tau} a + \left(\frac{K'}{K} - \frac{\varepsilon_{\theta\theta}}{\varepsilon_\theta} \right) V a^2 = 0. \qquad (1.19)$$

Thanks to the wavespeed relation (1.10), the coefficient of $[q_{xx}]$ is zero. This yields the following equation for the wave amplitude a,

$$\frac{\delta a}{\delta t} + \frac{1}{2\tau} a + \zeta a^2 = 0, \qquad (1.20)$$

where the constant ζ is given by

$$\zeta = \frac{1}{2} \left(\frac{K' V}{K} - \frac{\varepsilon_{\theta\theta} V}{\varepsilon_\theta} \right)$$
$$= \frac{1}{2} \sqrt{\frac{K(\theta^+)}{\rho \tau \varepsilon_\theta(\theta^+)}} \left(\frac{K_\theta(\theta^+)}{K(\theta^+)} - \frac{\varepsilon_{\theta\theta}(\theta^+)}{\varepsilon_\theta(\theta^+)} \right). \qquad (1.21)$$

To solve (1.20) we put $u = 1/a$ and then find

$$\frac{\delta u}{\delta t} - \frac{u}{2\tau} - \zeta = 0.$$

Using the integrating factor

$$\frac{\delta}{\delta t} (e^{-t/2\tau} u) = \zeta e^{-t/2\tau}.$$

We integrate this to find after rearrangement

$$a(t) = \frac{a(0)}{\exp(t/2\tau) + 2\tau\zeta\{\exp(t/2\tau) - 1\}a(0)}. \tag{1.22}$$

When $a(0) > 0$ this shows $a(t) \to 0$ as t increases. However, if $a(0) < 0$ we have $|a(t)|$ blows up in a finite time. This is associated with thermal shock formation, cf. the calculations in elasticity in Fu and Scott [55]. From (1.22) the blow-up time is when

$$e^{t/2\tau} = |a(0)|\, 2\tau\zeta\left(\exp(t/2\tau) - 1\right) \tag{1.23}$$

and thus we find the blow-up time is

$$T^* = 2\tau \log\left(\frac{2\tau\zeta\,|a(0)|}{2\tau\zeta\,|a(0)| - 1}\right). \tag{1.24}$$

1.2 Outline of thesis

In Chapter 2 we study the problem of thermal convection in a layer of fluid heated from below, but when the heat flux law is one of Cattaneo type, as introduced in Section 1.1. Since we solve this problem for two rigid surfaces we are faced with the numerical solution of an eigenvalue problem for a system of differential equations. Thus, in Chapter 2 we also introduce the D^2 Chebyshev tau numerical method.

Chapter 3 investigates various qualitative properties for a model for second sound derived by Green & Laws [58]. The model of Green & Laws introduces a generalized temperature ϕ which depends on θ and $\dot\theta$, where θ is the usual temperature. We extend this work in Chapter 4 to a model of Batra [13] who generalizes the Green & Laws ideas by allowing θ, $\dot\theta$ and $\ddot\theta$ to be variables in the constitutive theory. In Chapter 4 we investigate uniqueness, instability, and wave motion in the Batra theory.

Chapter 5 studies a more recent model of heat transport in a rigid body, namely, the model of Green & Naghdi [60]. In this chapter we give a detailed analysis of acceleration waves in the Green & Naghdi [60] theory.

In the final chapter, Chapter 6, we investigate an extension of the model in Chapter 5. This is to the case of the thermodynamics of Green & Naghdi [60], but when the theory is extended to cover an inviscid fluid. The extension is due

to Quintanilla & Straughan [155]. We consider a different constitutive theory from Quintanilla & Straughan [155], and study the development of an acceleration wave.

Chapter 2

Thermal convection with the Cattaneo model

Our goal in this section is to solve a thermal convection problem. Firstly, we introduce a numerical method.

2.1 D^2 Chebyshev tau method

To describe the Chebyshev tau technique we follow Dongarra et al. [41] and consider a simple example. Consider the equation and boundary conditions,

$$Lu \equiv u'' + \lambda u = 0, \qquad x \in (-1, 1),$$
$$u(-1) = u(1) = 0, \tag{2.1}$$

where the differential operator L is defined as indicated.

Now write u as a finite series of Chebyshev polynomials

$$u(x) = \sum_{k=0}^{N+2} u_k T_k(x). \tag{2.2}$$

The idea is that (2.2) represent truncations of an infinite series. Due to the truncation, the tau method argues that rather than solving (2.1) one instead solves the equation

$$Lu = \tau_1 T_{N+1} + \tau_2 T_{N+2} \tag{2.3}$$

where τ_1, τ_2 are tau coefficients which may be used to measure the error associated with the truncation of (2.1).

To reduce (2.1) to a finite-dimensional problem the inner product with T_i is taken of (2.3) in the weighted $L^2(-1, 1)$ space with inner product

$$(f, g) = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1 - x^2}} \, dx$$

and associated norm $\|\cdot\|$. The Chebyshev polynomials are orthogonal in this space, and then from (2.3) we obtain $(N + 1)$ equations

$$(Lu, T_i) = 0 \qquad i = 0, 1, \ldots, N. \tag{2.4}$$

There are two further conditions which arise from (2.3),

$$(Lu, T_{N+j}) = \tau_j \|T_{N+j}\|^2, \qquad j = 1, 2,$$

and these may effectively be used to calculate the τ's. The two remaining conditions are found from the boundary conditions, which since $T_n(\pm 1) = (\pm 1)^n$, yield

$$\sum_{n=0}^{N+2} (-1)^n u_n = 0, \qquad \sum_{n=0}^{N+2} u_n = 0. \tag{2.5}$$

Equations (2.4) and (2.5) yield a linear system of $(N + 3)$ equations for the $(N + 3)$ unknowns u_i, $i = 0, \ldots, N + 2$.

The derivative of a Chebyshev polynomial is a linear combination of lower order Chebyshev polynomials, in fact

$$T_n' = 2n \sum_{k=1}^{n-1} T_k, \qquad n \text{ even},$$

$$T_n' = 2n \sum_{k=2}^{n-1} T_k + nT_0, \qquad n \text{ odd}. \tag{2.6}$$

Then (2.4) becomes

$$u_i^{(2)} + \lambda u_i = 0, \qquad i = 0, \ldots, N, \tag{2.7}$$

where the coefficients $u_i^{(2)}$ are given by

$$u_i^{(2)} = \frac{1}{c_i} \sum_{\substack{p=i+2 \\ p+i \text{ even}}}^{p=N+2} p(p^2 - i^2)u_p, \tag{2.8}$$

with the numbers c_i being defined by $c_0 = 2, c_i = 1, i = 1, 2, \dots$. (Actually, (2.8) is really a truncation to the $N + 2$th polynomial of an infinite expansion.) Equations (2.7) and (2.5) represent a matrix equation

$$A\mathbf{x} = -\lambda B\mathbf{x}, \tag{2.9}$$

with $\mathbf{x} = (u_0, \dots, u_{N+2})^T$. However, the B matrix is inevitably singular due to the way the boundary condition rows are added to A. Indeed, the last two lines of B are composed of zeros, while the upper left $(N + 1) \times (N + 1)$ part is simply the identity.

To clarify this point, we observe

$$
\begin{aligned}
u' &= \sum_{s=0}^{N+2} u_s T_s'(x) \\
&= \sum_{s=0}^{N+2} u_s \left(\sum_{r=0}^{N+2} D_{r,s} T_r \right) \\
&= \sum_{r=0}^{N+2} \left(\sum_{s=0}^{N+2} D_{r,s} u_s \right) T_r
\end{aligned}
$$

and so we may make the identification

$$u_r^{(1)} = \sum_{s=0}^{N+2} D_{r,s} u_s.$$

Similarly,

$$u'' = \sum_{r=0}^{N+2} \left(\sum_{s=0}^{N+2} D_{r,s} u_s^{(1)} \right) T_r$$

and, therefore,

$$
\begin{aligned}
u_r^{(2)} &= \sum_{s=0}^{N+2} D_{r,s} u_s^{(1)} \\
&= \sum_{s=0}^{N+2} D_{r,s} \sum_{k=0}^{N+2} D_{s,k} u_k \\
&= \sum_{s=0}^{N+2} \sum_{k=0}^{N+2} D_{r,s} D_{s,k} u_k .
\end{aligned}
$$

This allows us to introduce the differentiation matrix D, and second differentiation

matrix D^2 which are shown to have components

$$D_{0,2j-1} = 2j - 1, \quad j \geq 1,$$

$$D_{i,i+2j-1} = 2(i + 2j - 1), \quad i \geq 1, j \geq 1,$$

$$D_{0,2j}^2 = \frac{1}{2}(2j)^3, \quad j \geq 1,$$

$$D_{i,i+2j}^2 = (i + 2j)4j(i + j), \quad i \geq 1, j \geq 1, \tag{2.10}$$

or

$$D = \begin{pmatrix} 0 & 1 & 0 & 3 & 0 & 5 & 0 & 7 & 0 & 9 & \cdots \\ 0 & 0 & 4 & 0 & 8 & 0 & 12 & 0 & 16 & 0 & \cdots \\ 0 & 0 & 0 & 6 & 0 & 10 & 0 & 14 & 0 & 18 & \cdots \\ 0 & 0 & 0 & 0 & 8 & 0 & 12 & 0 & 16 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 14 & 0 & 18 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$D^2 = \begin{pmatrix} 0 & 0 & 4 & 0 & 32 & 0 & 108 & \cdots \\ 0 & 0 & 0 & 24 & 0 & 120 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 48 & 0 & 192 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

where we observe $D^2 = D \cdot D$ in the sense of matrix multiplication. These matrices are started at $(0,0)$ and truncated at column $N + 2$.

The $N+1$ and $N+2$ rows of the matrix A are replaced by the boundary conditions (2.5) while the same rows of B are replaced by zeros.

The resulting matrix eigenvalue problem is solved by using the QZ algorithm in the NAG library.

2.2 Hydrodynamic stability eigenvalue problems.

To begin our discussion of hydrodynamic stability eigenvalue problems we shall consider the Orr-Sommerfeld equation

$$(D^2 - a^2)^2\phi = iaRe(U - c)(D^2 - a^2)\phi - iaReU''\phi, \quad z \in (-1, 1), \tag{2.11}$$

see Drazin & Reid [42], equation (25.12), where $D = d/dz$, Re, a and c are Reynolds number, wavenumber, and eigenvalue (growth rate), respectively, and ϕ is the amplitude of the stream function. For Poiseuille flow $U = 1 - z^2$, whereas for Couette flow $U = z$. Equation (2.11) is to be solved subject to the boundary conditions

$$\phi = D\phi = 0, \qquad z = \pm 1. \tag{2.12}$$

In Poiseuille flow the basic flow is driven by a pressure gradient in the x-direction whereas Couette flow is driven by the upper boundary being sheared relative to the lower one. The latter is known as shear flow but the whole class of such flows is known as parallel flow.

Equation (2.11) governs the two-dimensional stability problem for parallel flow where Squire's theorem is employed to reduce the three-dimensional problem to a two-dimensional one. This is standard knowledge in the fluid dynamics literature, cf. Drazin & Reid [42]. The function ϕ is related to the stream function ψ by

$$\psi = \phi(z)e^{ia(x-ct)}. \tag{2.13}$$

System (2.11), (2.12) has an infinite number of eigenvalues and associated eigenfunctions. Since the real part of the temporal growth rate in (2.13) is $e^{c_i t}$, $c = c_r + ic_i$, the eigenvalue which has largest imaginary part is the most dangerous in a linear instability analysis. The component in (2.13) of the solution associated with an eigenvalue is referred to as a mode and the one with largest imaginary part is known as the dominant, or leading, mode (eigenvalue).

2.3 D^2 Chebyshev tau method, Orr-Sommerfeld equation

A D^2 method writes (2.11) as two equations

$$L_1(\phi, \chi) \equiv (D^2 - a^2)\phi - \chi = 0,$$
$$L_2(\phi, \chi) \equiv (D^2 - a^2)\chi - iaRe(U - c)\chi + iaReU''\phi = 0. \tag{2.14}$$

We solve exactly the equations

$$L_1(\phi, \chi) = \tau_1 T_{N+1} + \tau_2 T_{N+2},$$

$$L_2(\phi, \chi) = \tau_3 T_{N+1} + \tau_4 T_{N+2}, \tag{2.15}$$

by writing

$$\phi = \sum_{i=0}^{N+2} \phi_i T_i(z), \qquad \chi = \sum_{i=0}^{N+2} \chi_i T_i(z),$$

and then by multiplying each of (2.15) in turn by T_i, $i = 0, \ldots, N$. This yields $2(N + 1)$ equations for the coefficients ϕ_i, χ_i. The equations obtained by taking the inner product of (2.15) with T_{N+1}, T_{N+2} yield equations for the tau coefficients. The difficulty with the above approach, as pointed out by McFadden [113], p. 232, is that the boundary conditions are all on ϕ_i and none are on χ_i.

A D^2-method for (2.11), (2.12) appropriate to Poiseuille flow eventually solves an equation like (2.14) where

$$\mathbf{x} = (\phi_0, \ldots, \phi_{N+2}, \chi_0, \ldots, \chi_{N+2})^T,$$

with

$$A_r = \begin{pmatrix} D^2 - a^2 I & -I \\ BC1 & 0 \ldots 0 \\ BC2 & 0 \ldots 0 \\ \\ 0 & D^2 - a^2 I \\ BC3 & 0 \ldots 0 \\ BC4 & 0 \ldots 0 \end{pmatrix}, \qquad A_i = \begin{pmatrix} 0 & 0 \\ 0 \ldots 0 & 0 \ldots 0 \\ 0 \ldots 0 & 0 \ldots 0 \\ \\ -2aReI & aRe(P - I) \\ 0 \ldots 0 & 0 \ldots 0 \\ 0 \ldots 0 & 0 \ldots 0 \end{pmatrix}$$

and

$$B_r = 0, \qquad B_i = \begin{pmatrix} 0 & 0 \\ \\ 0 & -aReI \\ & 0 \ldots 0 \\ & 0 \ldots 0 \end{pmatrix}$$

where P is the Chebyshev matrix representing z^2, $A = A_r + iA_i$, and $B = B_r + iB_i$. (P is the matrix obtained by writing $z^2 = \frac{1}{2}(1 + T_2(z))$, and then taking the inner product $(T_i, z^2\phi)$.)

The rows $BC1, \ldots, BC4$ refer to the boundary conditions on ϕ_n and for the Orr-Sommerfeld problem. The four discrete boundary conditions are obtained from the boundary conditions (2.12). Since $T_n'(\pm 1) = (\pm 1)^{n+1} n^2$, these are

$$\sum_{i=0}^{N+2}(-1)^i \phi_i = \sum_{i=0}^{N+2} \phi_i = \sum_{i=0}^{N+2}(-1)^{i+1} i^2 \phi_i = \sum_{i=0}^{N+2} i^2 \phi_i = 0. \qquad (2.16)$$

Due to the way the terms split in the discretization of (2.11) when $U = 1 - z^2$ it is then better to write (2.16) as

$$\sum_{\substack{i=0 \\ i \text{ even}}}^{i=N+1} \phi_i = 0, \qquad \sum_{\substack{i=1 \\ i \text{ odd}}}^{i=N+2} \phi_i = 0, \qquad \sum_{\substack{i=1 \\ i \text{ odd}}}^{i=N+2} i^2 \phi_i = 0, \qquad \sum_{\substack{i=2 \\ i \text{ even}}}^{i=N+1} i^2 \phi_i = 0. \qquad (2.17)$$

2.4 Thermal convection with the Cattaneo law

We use the Cattaneo law as in Straughan & Franchi [173]. The equations for fluid motion consist of the balance of mass, balance of linear momentum, balance of energy, and the Cattaneo law [19]. These equations are found in Straughan & Franchi [173]. The balance of linear momentum is,

$$v_{i,t} + v_j v_{i,j} = -\frac{1}{\rho} p_{,i} + \alpha g k_i T + \nu \Delta v_i \qquad (2.18)$$

where v_i, p, T are the velocity, pressure and temperature fields, Δ denotes the Laplace operator in three - dimensions, ρ is the constant density, ν is the kinematic viscosity, g is the gravity, $\mathbf{k} = (0, 0, 1)$ and α is the thermal expansion coefficient. The balance of mass equation is,

$$v_{i,i} = 0. \qquad (2.19)$$

The balance of energy is,

$$\rho c_p (T_t + v_i T_{,i}) = -q_{i,i} \qquad (2.20)$$

where q_i is the heat flux and c_p is the specific heat at constant pressure. The invariant form of the Cattaneo law adopted by Straughan & Franchi [173] is,

$$\tau(q_{i,t} + v_j q_{i,j} - \frac{1}{2} q_j v_{i,j} + \frac{1}{2} q_j v_{j,i}) = -q_i - \kappa T_{,i} , \qquad (2.21)$$

κ being the thermal conductivity.

The fluid occupies the layer $(x, y) \in \mathbb{R}^2$, $z \in (0, d)$ and equations (2.18) - (2.21) hold in the domain $\mathbb{R}^2 \times (0, d) \times \{t > 0\}$. The boundary conditions are

$$v_i = 0 \quad \text{on} \quad z = 0, d,$$
$$T = T_L, \quad z = 0, \qquad T = T_U, \quad z = d, \tag{2.22}$$

with $T_L > T_U$. The steady solution to (2.18) - (2.21) of interest is

$$\bar{v}_i \equiv 0, \qquad \bar{T} = -\beta z + T_L, \qquad \bar{\mathbf{q}} = (0, 0, \kappa\beta),$$

$$\bar{p} = -\frac{\alpha g \beta z^2}{2} + \alpha g \rho T_L z + p_0, \quad \text{where } p_0 \text{ is a reference pressure.} \tag{2.23}$$

Here β is the temperature gradient,

$$\beta = \frac{T_L - T_U}{d}.$$

$T = T_U$

$z = d$

Layers of fluid

$z = 0$

$T = T_L$

$T_L > T_U$
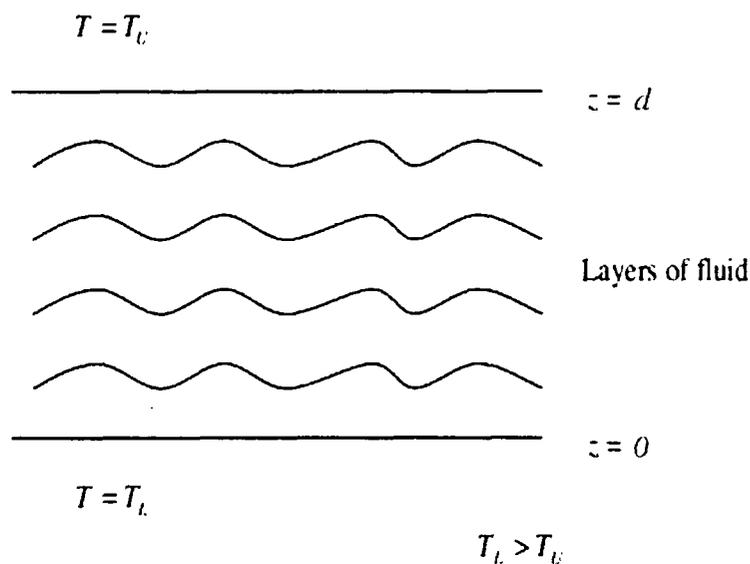
Figure 2.1: Diagram illustrating fluid heated from below.

Instability of solution (2.23) is studied by introducing perturbations (u_i, θ, π, q_i) such that $v_i = \bar{v}_i + u_i$, $T = \bar{T} + \theta$, $p = \bar{p} + \pi$, $q_i = \bar{q}_i + q_i$. Then, from equations (2.18) - (2.21) we derive the linearized equations governing (u_i, θ, π, q_i) as, see Straughan

& Franchi [173],

$$u_{i,t} = -\frac{1}{\rho}\pi_{,i} + \alpha g k_i \theta + \nu \Delta u_i,$$

$$u_{i,i} = 0,$$

$$\rho c_p \theta_t = \beta \rho c_p w - q_{i,i},$$

$$\tau(q_{i,t} - \frac{1}{2}u_{i,z}\kappa\beta + \frac{1}{2}\kappa\beta w_{,i}) = -q_i - \kappa\theta_{,i},$$

(2.24)

where $w = u_3$. Equations (2.24) are non-dimensionalized as in Straughan & Franchi [173]. We need the Prandtl number, Pr, Cattaneo number, C, and Rayleigh number, $Ra = R^2$, which are

$$Pr = \frac{\rho\nu U}{d}, \qquad C = \frac{\tau(\kappa/\rho c_p)}{2d^2}, \qquad R = \sqrt{\frac{\alpha g d^4 \beta}{\nu(\kappa/\rho c_p)}}.$$

The non-dimensional linearized equations (2.24) are

$$u_{i,t} = -\pi_{,i} + R k_i \theta + \Delta u_i,$$

$$u_{i,i} = 0,$$

$$Pr\theta_t = Rw - q_{i,i},$$

$$2CPrq_{i,t} = -q_i + CR(u_{i,z} - w_{,i}) - \theta_{,i}.$$

(2.25)

To study instability we seek an exponential time dependence like

$$u_i(\mathbf{x},t) = e^{\sigma t}u_i(\mathbf{x}), \quad \theta(\mathbf{x},t) = e^{\sigma t}\theta(\mathbf{x}), \quad q_i(\mathbf{x},t) = e^{\sigma t}q_i(\mathbf{x}), \quad \pi(\mathbf{x},t) = e^{\sigma t}\pi(\mathbf{x}),$$

Equations (2.25) yield

$$\sigma u_i = -\pi_{,i} + R k_i \theta + \Delta u_i,$$

$$u_{i,i} = 0,$$

$$\sigma Pr\theta = Rw - q_{i,i},$$

$$2\sigma CPrq_i = -q_i + CR(u_{i,z} - w_{,i}) - \theta_{,i}.$$

(2.26)

Eliminate π from (2.26)$_1$ and put $Q = q_{i,i}$, where " $_{,i}$" means derivative with respect to x_i. Thus we solve the equations

$$\sigma\Delta w = R\Delta^*\theta + \Delta^2 w$$

$$\sigma Pr\theta = Rw - Q$$

(2.27)

$$2\sigma CPrQ = -Q - \Delta\theta - CR\Delta w,$$

where $\Delta^* = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian.

We now use the D^2 Chebyshev tau numerical method to solve equations (2.27). The fixed surface boundary conditions are

$$w = w_z = \theta = 0, \qquad z = 0, 1. \tag{2.28}$$

Equation $(2.27)_1$ is fourth order and hence we introduce the variable χ by $\chi = \Delta w$. We let f be a plane tiling periodic function so that $f_{xx} + f_{yy} = -a^2 f$ where a is a wavenumber. Next write w, χ, θ and Q in the form

$$w = W(z)f(x,y), \quad \chi = \chi(z)f(x,y), \quad \theta = \Theta(z)f(x,y), \quad Q = Q(z)f(x,y).$$

We now solve (2.27) as

$$\begin{aligned}
(D^2 - a^2)W - \chi &= 0, \\
(D^2 - a^2)\chi - Ra^2\Theta &= \sigma\chi, \\
(D^2 - a^2)\Theta + Q + CR\chi &= -2\sigma CPrQ, \\
Q - RW &= -\sigma Pr\Theta.
\end{aligned} \tag{2.29}$$

The functions W, χ, Θ and Q are expanded in terms of Chebyshev polynomials, for N odd,

$$W(z) = \sum_{n=0}^{N} w_n T_n(z), \qquad \chi(z) = \sum_{n=0}^{N} \chi_n T_n(z),$$

$$\Theta(z) = \sum_{n=0}^{N} \Theta_n T_n(z), \qquad Q(z) = \sum_{n=0}^{N} Q_n T_n(z).$$

To use the boundary conditions (2.28) we note that

$$T_n(\pm 1) = (\pm 1)^n, \qquad T_n'(\pm 1) = (\pm 1)^{n-1} n^2.$$

Then the boundary conditions (2.28) become,

$$\sum_{\substack{i=0 \\ i \text{ even}}}^{i=N-1} w_i = 0, \qquad \sum_{\substack{i=1 \\ i \text{ odd}}}^{i=N} w_i = 0, \tag{2.30}$$

$$\sum_{\substack{i=2 \\ i \text{ even}}}^{i=N-1} i^2 w_i = 0, \qquad \sum_{\substack{i=1 \\ i \text{ odd}}}^{i=N} i^2 w_i = 0, \tag{2.31}$$

$$\sum_{\substack{i=0 \\ i \text{ even}}}^{i=N-1} \Theta_i = 0, \qquad \sum_{\substack{i=1 \\ i \text{ odd}}}^{i=N} \Theta_i = 0. \tag{2.32}$$

The Chebyshev tau D^2 method now reduces to solving the matrix system $A\mathbf{x} = \sigma B\mathbf{x}$. Here the $(N+1) \times (N+1)$ matrices A and B are given by

$$
A = \begin{pmatrix}
4D^2 - a^2I & -I & 0 & 0 \\
BC1 & 0\ldots0 & 0\ldots0 & 0\ldots0 \\
BC2 & 0\ldots0 & 0\ldots0 & 0\ldots0 \\
\\
0 & 4D^2 - a^2I & -Ra^2I & 0 \\
BC3 & 0\ldots0 & 0\ldots0 & 0\ldots0 \\
BC4 & 0\ldots0 & 0\ldots0 & 0\ldots0 \\
\\
0 & CRI & 4D^2 - a^2I & I \\
0\ldots0 & 0\ldots0 & BC5 & 0\ldots0 \\
0\ldots0 & 0\ldots0 & BC6 & 0\ldots0 \\
\\
-RI & 0 & 0 & I
\end{pmatrix}
$$

$$
B = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0\ldots0 & 0\ldots0 & 0\ldots0 & 0\ldots0 \\
0\ldots0 & 0\ldots0 & 0\ldots0 & 0\ldots0 \\
\\
0 & I & 0 & 0 \\
0\ldots0 & 0\ldots0 & 0\ldots0 & 0\ldots0 \\
0\ldots0 & 0\ldots0 & 0\ldots0 & 0\ldots0 \\
\\
0 & 0 & 0 & -2CPrI \\
0\ldots0 & 0\ldots0 & 0\ldots0 & 0\ldots0 \\
0\ldots0 & 0\ldots0 & 0\ldots0 & 0\ldots0 \\
\\
0 & 0 & -PrI & 0
\end{pmatrix}
$$

where $\mathbf{x} = (w_0, \ldots, w_N, \chi_0, \ldots, \chi_N, \Theta_0, \ldots, \Theta_N, Q_0, \ldots, Q_N)$. The boundary conditions $BC1, BC2, BC3, BC4, BC5, BC6$ refer respectively to (2.30), (2.31), (2.32). The matrix system is solved by the QZ algorithm used as given in the NAG library,

cf. Dongarra *et al.* [41]. Note that in the last block row of A and B there are no boundary conditions. This is due to the fact that $(2.29)_4$ is an identity and not a differential equation.

2.5 Numerical results

Straughan and Franchi [173] found that for two free surfaces, the following asymptotic formula,

$$Ra = \frac{27\pi^4}{4}\left(1 + \frac{3}{2}C\pi^2 + O(C^2)\right).$$

We present numerical results for two *fixed* surfaces below. Again, we find Ra to be increasing in C.

Table 2.1: Critical Rayleigh numbers

C	Ra	a
0	1707.765	3.12
10^{-4}	1711.180	3.12
2×10^{-4}	1714.596	3.11
4×10^{-4}	1721.475	3.11
6×10^{-4}	1728.402	3.10
8×10^{-4}	1735.369	3.10
10^{-3}	1742.393	3.09
2×10^{-3}	1778.194	3.07
4×10^{-3}	1853.544	3.03
6×10^{-3}	1934.276	2.98
8×10^{-3}	2020.868	2.94
10^{-2}	2113.893	2.89

The results in this table we found by fixing a^2 and solving for σ. The numerical routine seeks that value of Ra for which $Re\sigma = 0$. Then we find

$$\min_{a^2} Ra(a^2).$$

It was found that $\sigma \in \mathbb{R}$ throughout the table. From the table we see that Ra satisfies an approximately linear relationship in C. For small C values we have

$$Ra \simeq 1707.765 + 34150C \tag{2.33}$$

so the slope is approximately 3.5 bigger than that in the free surface case.

Chapter 3

Green and Laws model

3.1 Derivation of the model

In this section we will summarize the derivation of the model which appeared in Green and Laws [58].

In the article of Green & Laws [58], they proposed the entropy production inequality for the whole body b and for the material volumes p, by employing the specific Helmholtz free energy and energy equation together with other standard balance equations. Moreover, they make the constitutive assumptions that ψ, η, ϕ, q_i are functions of $\theta, \dot{\theta}, \theta_{,i}$. Furthermore, they treat both the external volume and surface supplies of entropy on an equal footing and retain a non-zero r. The balance laws for a single phase continuum that are used in Green & Laws [58] are

$$\dot{\rho} + \rho v_{i,i} = 0 \tag{3.1}$$

$$t_{ki,k} + \rho F_i = \rho \dot{v}_i, \tag{3.2}$$

$$\rho r - q_{i,i} + t_{ik} d_{ik} - \rho \dot{\varepsilon} = 0, \tag{3.3}$$

where $\rho, v_i, t_{ki}, F_i, r, q_i$ and ε, are, respectively the mass density, the velocity, the Cauchy stress tensor, the specific body force, the specific heat supply, the heat flux and the specific internal energy. The stress tensor t_{ik} is symmetric, so $t_{ik} = t_{ki}$, and d_{ik} is the symmetric part of the velocity gradient, $d_{ik} = \frac{1}{2}(v_{i,k} + v_{k,i})$. Also a superposed dot denotes material time differentiation.

23

In 1967, Muller [121] proposed an entropy production inequality of the form

$$\frac{d}{dt}\int_p \rho\eta dv - \int_p \frac{\rho r}{\theta}dv + \int_{\partial p} k_i n_i da \geq 0 \tag{3.4}$$

for every material volume p, η is the specific entropy, θ is the absolute temperature ($\theta > 0$), k_i is an entropy flux vector, and n_i is the outward unit normal to the boundary surface, ∂p, of p.

The equation of motion (3.2) and the energy equation (3.3) are balanced by suitable choices of body forces F_i and heat supply r.

Green & Laws [59] suggest that since (3.4) holds for every material volume p, it also holds for the whole body b, therefore they assume that the only external volume supply of entropy is defined in a particular way in terms of externally supplied rate of production of heat r, namely r/θ. Similarly the only external supply of entropy over the boundary ∂b of the continuum is that defined in the same way in terms of the rate of supply of heat. Then they suggest that the entropy flux vector in (3.4) be restricted by the condition

$$k_i n_i = \frac{q_i n_i}{\theta} \quad \text{on } \partial b. \tag{3.5}$$

Otherwise the external supply of entropy over ∂b is of different character from the external volume supply of entropy.

In 1971, Muller [123], [122] considered solutions of (3.1) to (3.3) when $F_i = 0$, $r = 0$. Green & Laws [58] assume that associated with the heat supply r there is an entropy supply r/ϕ where $\phi > 0$ and $\phi = \theta$ in equilibrium. The function $\phi = \phi(\theta, \dot{\theta})$ is a generalised temperature.

For the whole body b, the entropy production inequality is

$$\frac{d}{dt}\int_b \rho\eta dv - \int_b \frac{\rho r}{\phi}dv + \int_{\partial b} \frac{q_i n_i}{\phi}dx \geq 0, \tag{3.6}$$

and for all material volumes p, the entropy production inequality is

$$\frac{d}{dt}\int_p \rho\eta dv - \int_p \frac{\rho r}{\phi}dv + \int_{\partial p} \frac{q_i n_i}{\phi}dx \geq 0. \tag{3.7}$$

Green & Laws [58] define the Helmholtz free energy $\psi = \varepsilon - \eta\phi$, and consider a stationary rigid heat conductor. Moreover, they use the constitutive assumptions

that ψ, η, ϕ, q_i are functions of $\theta, \dot{\theta}, \theta_{,i}$, and exploit the Helmholtz free energy ψ together with the constitutive assumptions into (3.7) to obtain

$$\frac{\partial \phi}{\partial \theta_{,i}} = 0 \text{ (which yields } \phi = \phi(\theta, \dot{\theta})),$$

$$\frac{\partial \phi}{\partial \dot{\theta}} \neq 0, \frac{\partial \psi}{\partial \theta_{,i}} \neq 0. \tag{3.8}$$

For a rigid solid which conducts heat according to a Fourier law

$$q_i = -\kappa_{ij}(\theta, \dot{\theta})\theta_{,j}, \tag{3.9}$$

leads to κ_{ij} is symmetric.

Define equilibrium to be when $\dot{\theta} = 0$, $\theta_{,i} = 0$, and let $\phi|_E = \theta$, hence $(\partial \phi/\partial \theta)\big|_E = 1$. Then Green & Laws [58] show

$$\eta|_E = -\frac{\partial \psi}{\partial \theta}\bigg|_E, \quad q_i|_E = 0, \quad \frac{\partial \psi}{\partial \theta_{,i}}\bigg|_E = 0,$$

$$\text{and } -\frac{\partial q_i}{\partial \theta_{,j}}\bigg|_E \text{ is a positive semi-definite tensor.} \tag{3.10}$$

By employing (3.7) and the Helmholtz free energy: $\psi = \varepsilon - \eta\phi$, then the equation (3.3) becomes

$$\rho r - q_{i,i} - \rho(\dot{\psi} + \eta\dot{\phi} + \phi\dot{\eta}) = 0, \tag{3.11}$$

and the linearized version of (3.11) is

$$\rho r - \frac{\partial q_i}{\partial \dot{\theta}}\bigg|_E \dot{\theta}_{,i} - \frac{\partial q_i}{\partial \theta_{,k}}\bigg|_E \theta_{,ik} - \rho\left(\phi\frac{\partial \eta}{\partial \theta}\right)\bigg|_E \dot{\theta}$$

$$- \rho\left(\phi\frac{\partial \eta}{\partial \dot{\theta}}\right)\bigg|_E \ddot{\theta} - \rho\left(\phi\frac{\partial \eta}{\partial \theta_{,i}}\right)\bigg|_E \dot{\theta}_{,i} = 0. \tag{3.12}$$

Thus, (3.12) is capable of predicting a finite speed of heat propagation. We now study the properties of a solution to equation (3.12) in some detail.

Throughout the remainder of the thesis Ω will denote a domain in $\mathbb{R}^3$. If Ω has a boundary this will be denoted by Γ.

3.2 Uniqueness for the Green and Laws model with k_i constant

In this section $\Omega \subset \mathbb{R}^3$ is a bounded domain with boundary Γ. Now, discuss equation (3.12) and take the specific heat supply $r = 0$ then this equation becomes,

$$\alpha\ddot{\theta} + \beta\dot{\theta} + \xi_{ik}\theta_{,ik} + k_i\dot{\theta}_{,i} = 0, \tag{3.13}$$

where the coefficients are

$$\alpha = \rho\left(\phi\frac{\partial\eta}{\partial\ddot{\theta}}\right)\bigg|_E,$$

$$\beta = \rho\left(\phi\frac{\partial\eta}{\partial\theta}\right)\bigg|_E,$$

$$\xi_{ik} = \frac{\partial q_i}{\partial\theta_{,k}}\bigg|_E,$$

$$k_i = \frac{\partial q_i}{\partial\dot{\theta}}\bigg|_E + \rho\left(\phi\frac{\partial\eta}{\partial\theta_{,i}}\right)\bigg|_E.$$

In this section α, β, ξ_{ik} and k_i are constants, we assume ξ_{ik} to be symmetric, and equation (3.13) holds on the domain Ω, $t > 0$.

The boundary and initial conditions are

Boundary Condition $\qquad\qquad \theta = f(x, t)$ on Γ, $\qquad\qquad$ (3.14)

Initial Conditions $\qquad\qquad \theta(x, 0) = g(x),\ \theta_t(x, 0) = h(x),\ x \in \Omega.$ (3.15)

Let P denote the boundary-initial value problem given by (3.13)-(3.15). To prove that the solution is unique, assume that there are 2 solutions θ_1 and θ_2 which satisfy the equation (3.13) and the boundary-initial conditions (3.14) and (3.15). Then let $w = \theta_1 - \theta_2$. From (3.13) we find w satisfies

$$\alpha\ddot{w} + \beta\dot{w} + \xi_{ik}w_{,ki} + k_i\dot{w}_{,i} = 0, \text{ where } x \in \Omega,\ t > 0, \tag{3.16}$$

with the boundary condition

$$w = 0 \quad \text{on} \quad \Gamma, \tag{3.17}$$

and the initial conditions

$$w(x, 0) = 0,\ x \in \Omega, \tag{3.18}$$

$$w_t(x, 0) = 0, \ x \in \Omega. \tag{3.19}$$

To demonstrate uniqueness we multiply (3.16) by $\dot{w}$ and integrate over Ω, to obtain

$$\alpha \int_\Omega \dot{w}\ddot{w}dx + \beta \int_\Omega \dot{w}^2 dx + \int_\Omega \xi_{ik}w_{,ik}\dot{w}dx + \int_\Omega k_i\dot{w}_{,i}\dot{w}dx = 0, \tag{3.20}$$

or equivalently in the form,

$$\alpha(\dot{w}, \ddot{w}) + \beta(\dot{w}, \dot{w}) + (\xi_{ik}w_{,ki}, \dot{w}) + (k_i\dot{w}_{,i}, \dot{w}) = 0, \tag{3.21}$$

where $(\cdot, \cdot)$ denotes the inner product on $L^2(\Omega)$. Note that

$$\alpha(\dot{w}, \ddot{w}) = \alpha \int_\Omega \dot{w}\ddot{w}dx = \frac{\alpha}{2}\int_\Omega \frac{d}{dt}(\dot{w})^2 dx = \frac{\alpha}{2}\frac{d}{dt}\|\dot{w}\|^2. \tag{3.22}$$

The second term of (3.21) is

$$\beta(\dot{w}, \dot{w}) = \beta \int_\Omega \dot{w}^2 dx = \beta\|\dot{w}\|^2, \tag{3.23}$$

and for the k_i term we have

$$\begin{aligned}
(k_i\dot{w}_{,i}, \dot{w}) &= \int_\Omega k_i\dot{w}_{,i}\dot{w}dx \\
&= \int_\Omega k_i \frac{\partial^2 w}{\partial x_i \partial t}\frac{\partial w}{\partial t}dx \\
&= \int_\Omega \frac{k_i}{2}\frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial t}\right)^2 dx, \text{by using the chain rule} \\
&= \frac{1}{2}\int_\Omega \frac{\partial}{\partial x_i}\left[k_i\left(\frac{\partial w}{\partial t}\right)^2\right]dx, \text{since } k_i \text{ is a constant vector} \\
&= \frac{1}{2}\int_\Gamma n_i k_i(\frac{\partial w}{\partial t})^2 ds, \text{by using the divergence theorem} \\
&= 0. \tag{3.24}
\end{aligned}$$

For the ξ_{ik} term,

$$\begin{aligned}
(\xi_{ik}w_{,ki}, \dot{w}) &= \int_\Omega \xi_{ik}w_{,ki}\dot{w}dx \\
&= \int_\Omega \xi_{ik}\dot{w}(\frac{\partial}{\partial x_k}w_{,i})dx \\
&= \int_\Omega \frac{\partial}{\partial x_k}(\xi_{ik}\dot{w}w_{,i})dx - \int_\Omega w_{,i}\frac{\partial}{\partial x_k}(\xi_{ik}\dot{w})dx \\
&= \int_\Gamma n_k\xi_{ik}w_t w_{,i}ds - \int_\Omega \xi_{ik}w_{,i}w_{,kt}dx,
\end{aligned}$$

where we have integrated by parts and used the divergence theorem. The first term is zero since $w = 0$ on Γ, so that

$$
\begin{aligned}
(\xi_{ik} w_{,ki}, \dot{w}) &= -\int_\Omega \xi_{ik} w_{,i} w_{,kt} dx, \\
&= -\frac{1}{2} \frac{d}{dt} \int_\Omega \xi_{ik} w_{,i} w_{,k} dx.
\end{aligned} \tag{3.25}
$$

Employing (3.22) to (3.25) in (3.21), equation (3.21) becomes

$$
\frac{\alpha}{2} \frac{d}{dt} \|\dot{w}\|^2 + \beta \|\dot{w}\|^2 - \frac{1}{2} \frac{d}{dt} \int_\Omega \xi_{ik} w_{,i} w_{,k} dx = 0, \tag{3.26}
$$

or

$$
\frac{\alpha}{2} \frac{d}{dt} \int_\Omega \dot{w}^2 dx + \beta \int_\Omega \dot{w}^2 dx - \frac{1}{2} \frac{d}{dt} \int_\Omega \xi_{ik} w_{,i} w_{,k} dx = 0. \tag{3.27}
$$

Integrate equation (3.26) from 0 to t,

$$
\frac{1}{2} \int_\Omega \left(\alpha \dot{w}^2 - \xi_{ik} w_{,i} w_{,k} \right) dx + \beta \int_0^t \int_\Omega \dot{w}^2 dx = 0. \tag{3.28}
$$

We now require $\alpha > 0$, $\beta \geq 0$, ξ_{ik} to be a negative semi-definite tensor, i.e. $\xi_{ik} \xi_i \xi_k \leq 0$, $\forall \xi_i$, that is

$$
\frac{\partial q_i}{\partial \theta_{,k}} \bigg|_E \xi_i \xi_k \leq 0, \ \forall \xi_i.
$$

Then from equation (3.28) we obtain

$$
0 \leq \frac{1}{2} \int_\Omega \left(\alpha \dot{w}^2 - \xi_{ik} w_{,i} w_{,k} \right) dx \leq 0,
$$

and so

$$
0 \leq \alpha \int_\Omega \dot{w}^2 dx - \int_\Omega \xi_{ik} w_{,i} w_{,k} dx \leq 0.
$$

Since $\alpha \int_\Omega \dot{w}^2 dx \geq 0$ and $-\int_\Omega \xi_{ik} w_{,i} w_{,k} dx \geq 0$, then

$$
0 \leq \alpha \int_\Omega \dot{w}^2 dx \leq 0. \tag{3.29}
$$

Therefore

$$
\int_\Omega \dot{w}^2 dx = 0, \tag{3.30}
$$

so $\dot{w} \equiv 0$, this leads to $w = 0$, hence $\theta_1 = \theta_2$ which yields uniqueness.

3.3 Uniqueness for Green and Laws model with $k_i = k_i(x)$

Now we use equation (3.13) but allow k_i to depend on the spatial variable, i.e. $k_i = k_i(x)$. Let θ_1 and θ_2 be solutions to the boundary-initial value problem defined by equations (3.13),(3.14) and (3.15), where θ_1 and θ_2 each satisfy (3.14) and (3.15) for the same data functions f, g and h.

Let $w = \theta_1 - \theta_2$. Then from equations (3.13),(3.14) and (3.15) we find w satisfies the boundary-initial value problem (3.16), (3.17) and (3.18). We now, prove uniqueness when $k_i = k_i(x)$.

Multiply (3.16) by $\dot{w}$ and integrate over the domain Ω. Thus

$$\alpha \int_\Omega \dot{w}\ddot{w}dx + \beta \int_\Omega (\dot{w})^2 dx + \int_\Omega \xi_{ik}w_{,ki}\dot{w}dx + \int_\Omega k_i\dot{w}_{,i}\dot{w}dx = 0. \tag{3.31}$$

As in section (2.2) we find

$$\alpha \int_\Omega \dot{w}\ddot{w}dx = \frac{\alpha}{2}\frac{d}{dt}\|\dot{w}\|^2, \tag{3.32}$$

and

$$\int_\Omega \xi_{ik}w_{,ki}\dot{w}dx = -\frac{1}{2}\frac{d}{dt}\int_\Omega \xi_{ik}w_{,i}w_{,k}dx \tag{3.33}$$

For the term in k_i we note

$$\int_\Omega k_i\dot{w}_{,i}\dot{w}dx = \int_\Omega \frac{k_i}{2}\frac{\partial}{\partial x_i}(\dot{w})^2 dx \tag{3.34}$$

and upon integration by parts

$$\begin{aligned}
\int_\Omega k_i\dot{w}_{,i}\dot{w}dx &= \frac{1}{2}\int_\Omega \frac{\partial}{\partial x_i}(k_i\dot{w}^2)dx - \int_\Omega \frac{k_{i,i}}{2}\dot{w}^2 dx \\
&= \frac{1}{2}\int_\Gamma n_i k_i \dot{w}^2 ds - \frac{1}{2}\int_\Omega k_{i,i}\dot{w}^2 dx \\
&= -\frac{1}{2}\int_\Omega k_{i,i}\dot{w}^2 dx, \tag{3.35}
\end{aligned}$$

because $\dot{w} = 0$ on Γ.

Then, employing equation (3.32)-(3.35) in equation (3.31) we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\alpha\|\dot{w}\|^2 - \int_\Omega \xi_{ik}w_{,i}w_{,k}dx\right) + \int_\Omega \left(\beta - \frac{k_{i,i}}{2}\right)\dot{w}^2 dx = 0. \tag{3.36}$$

We now require that

$$\beta - \frac{1}{2}k_{i,i}(\boldsymbol{x}) \geq 0, \quad \forall \boldsymbol{x} \in \Omega. \tag{3.37}$$

Then we discard the $\beta - k_{i,i}/2$ term in (3.36) to find

$$\frac{d}{dt}\left(\frac{\alpha}{2}\|\dot{w}\|^2 - \int_\Omega \frac{\xi_{ik}}{2}w_{,i}w_{,k}dx\right) \leq 0. \tag{3.38}$$

Integrate this from 0 to t to find

$$\frac{\alpha}{2}\|\dot{w}\|^2 - \frac{1}{2}\int_\Omega \xi_{ik}w_{,i}w_{,k}dx \leq 0, \tag{3.39}$$

since $\dot{w}(\boldsymbol{x},0) = 0$ and $w(\boldsymbol{x},0) = 0$. Since $\xi_{ik}\xi_i\xi_k \leq 0$ inequality (3.39) leads to

$$0 \leq \|\dot{w}(t)\|^2 \leq 0 \tag{3.40}$$

Therefore

$$\|\dot{w}(t)\| \equiv 0. \tag{3.41}$$

Thus

$$\dot{w} \equiv 0 \quad \text{in } \Omega, \forall t \tag{3.42}$$

and so

$$w(\boldsymbol{x},t) = 0. \tag{3.43}$$

Hence the solution θ to the boundary-initial value problem (3.13),(3.14) and (3.15) is unique.

3.4   Continuous dependence of the thermal energy

In this section we suppose $k_i = k_i(\boldsymbol{x})$.

We now suppose θ satisfies the boundary and initial conditions

$$\theta = 0 \text{ on } \Gamma,$$

$$\theta(\boldsymbol{x},0) = \theta_0(\boldsymbol{x}), \boldsymbol{x} \in \Omega,$$

$$\dot{\theta}(\boldsymbol{x},0) = \varphi_0(\boldsymbol{x}), \boldsymbol{x} \in \Omega.$$

Then multiply equation (3.13) by $\dot{\theta}$ and integrate over Ω, we obtain,

$$\alpha(\dot{\theta},\ddot{\theta}) + \beta(\dot{\theta},\dot{\theta}) + (\xi_{ik}\theta_{,ki},\dot{\theta}) + (k_i(\boldsymbol{x})\dot{\theta}_{,i},\dot{\theta}) = 0. \tag{3.44}$$

We note,

$$
\begin{aligned}
(\xi_{ik}\theta_{,ik}, \dot\theta) &= \int_\Omega \xi_{ik}\theta_{,ik}\dot\theta dx \\
&= -\frac{d}{dt}\frac{1}{2}\int_\Omega \xi_{ik}\theta_{,i}\theta_{,k}dx,
\end{aligned}
\tag{3.45}
$$

as in Section (3.3) we find

$$
(k_i(\boldsymbol{x})\dot\theta_{,i}, \dot\theta) = -\frac{1}{2}\int_\Omega k_{i,i}\dot\theta^2 dx.
\tag{3.46}
$$

Employing (3.45) and (3.46) into (3.44), we obtain

$$
\frac{\alpha}{2}\frac{d}{dt}\int_\Omega \dot\theta^2 dx + \beta\int_\Omega \dot\theta^2 dx - \frac{d}{dt}\frac{1}{2}\int_\Omega \xi_{ik}\theta_{,i}\theta_{,k}dx - \frac{1}{2}\int_\Omega k_{i,i}\dot\theta^2 dx.
\tag{3.47}
$$

Rearranging equation (3.47), we obtain

$$
\frac{d}{dt}\int_\Omega \left(\frac{\alpha}{2}\dot\theta^2 - \frac{1}{2}\xi_{ik}\theta_{,i}\theta_{,k}\right)dx + \int_\Omega \left(\beta - \frac{k_{i,i}}{2}\right)\dot\theta^2 dx = 0.
\tag{3.48}
$$

Suppose now

$$
\beta - \frac{k_{i,i}}{2} \geq \beta_0 > 0.
\tag{3.49}
$$

Then from equation (3.48) we have

$$
\frac{d}{dt}\int_\Omega \left(\frac{\alpha}{2}\dot\theta^2 - \frac{1}{2}\xi_{ik}\theta_{,i}\theta_{,k}\right)dx + \beta_0\int_\Omega \dot\theta^2 dx \leq 0.
\tag{3.50}
$$

Integrate from 0 to t over Ω, we obtain

$$
\begin{aligned}
&\int_\Omega \frac{\alpha}{2}\dot\theta^2 dx - \frac{1}{2}\int_\Omega \xi_{ik}\theta_{,i}\theta_{,k}dx + \beta_0\int_0^t\int_\Omega \dot\theta^2 dx d\eta \\
&\leq \frac{\alpha}{2}\int_\Omega \varphi_0^2 dx - \frac{1}{2}\int_\Omega \xi_{ik}\theta_{,i}^0\theta_{,k}^0 dx.
\end{aligned}
\tag{3.51}
$$

Suppose

$$
\xi_{ik}\eta_i\eta_k \leq 0,
\tag{3.52}
$$

then

$$
-\int_\Omega \xi_{ik}\theta_{,i}\theta_{,k}dx \geq 0.
\tag{3.53}
$$

Therefore, from equality (3.51) we have

$$
\frac{\alpha}{2}\int_\Omega \dot\theta^2 dx + \beta_0\int_0^t\int_\Omega \dot\theta^2 dx d\eta \leq D_0,
\tag{3.54}
$$

where D_0 is the data term,

$$D_0 = \frac{\alpha}{2} \int_\Omega \varphi_0^2 dx - \frac{1}{2} \int_\Omega \xi_{ik} \theta_{,i}^0 \theta_{,k}^0 dx + \beta_0 \int_0^t \int_\Omega \varphi_0{}^2 dx d\eta. \qquad (3.55)$$

Define now

$$F(t) = \int_0^t \int_\Omega \dot\theta^2 dx d\eta, \text{ thus } F'(t) = \int_\Omega \dot\theta^2 dx. \qquad (3.56)$$

Then the equality (3.54) may be rearranged as

$$\frac{\alpha}{2} F'(t) + \beta_0 F(t) \le D_0, \qquad (3.57)$$

or

$$F'(t) + \frac{2\beta_0}{\alpha} F(t) \le \frac{2}{\alpha} D_0. \qquad (3.58)$$

Multiply (3.58) by $\exp(\frac{2\beta_0 t}{\alpha})$, then we have

$$\frac{d}{dt}\left(e^{\frac{2\beta_0 t}{\alpha}} F \right) \le \frac{2}{\alpha} e^{\frac{2\beta_0}{\alpha} t} D_0.$$

Then integrating

$$\begin{aligned}
e^{\frac{2\beta_0}{\alpha} t} F(t) - F(0) &\le \int_0^t \frac{2}{\alpha} D_0 e^{\frac{2\beta_0}{\alpha} s} ds \\
&= \left(\frac{2}{\alpha} \right) \left(\frac{\alpha}{2\beta_0} \right) D_0 e^{\frac{2\beta_0}{\alpha} s} \bigg|_0^t \\
&= \frac{D_0}{\beta_0} \left(e^{\frac{2\beta_0}{\alpha} t} - 1 \right),
\end{aligned}$$

rearranging to obtain,

$$e^{\frac{2\beta_0}{\alpha} t} F(t) \le F(0) + \frac{D_0}{\beta_0} \left(e^{\frac{2\beta_0}{\alpha} t} - 1 \right).$$

So

$$F(t) \le \frac{D_0}{\beta_0} \left(1 - e^{-\frac{2\beta_0 t}{\alpha}} \right) \qquad (3.59)$$

Inequality (3.59) shows continous dependence of θ on the initial data in the $F(t)$ measure.

3.5 Uniqueness and continuous dependence for a related model

An equation very similar to (3.13) may be taken from the work of Payne & Song [134]. These authors study a model for thermoelasticity which stems from the thermodynamic treatment of Green & Laws [58]. By fixing the displacement, u_i, in

equation (1.11) of Payne & Song [134] so that $\dot{u}_{i,j} = 0$, one arrives at the equation for the temperature field θ of form

$$h\ddot{\theta} + d\dot{\theta} - b_i\dot{\theta}_{,i} - (b_i\dot{\theta})_{,i} = (k_{ik}\theta_{,k})_{,i}. \tag{3.60}$$

Clearly equation (3.60) is very like our equation (3.13) but has the extra term $-(b_i\dot{\theta})_{,i}$.

In this section we establish uniqueness and continuous dependence on the initial data for a solution to (3.60). Hence, let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary Γ and suppose (3.60) is defined on the space-time domain $\Omega \times (0, T)$ for some $T > 0$. The boundary conditions are

$$\theta(\boldsymbol{x}, t) = f(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Gamma, \tag{3.61}$$

and the initial conditions are

$$\theta(\boldsymbol{x}, 0) = g(\boldsymbol{x}), \quad \theta_t(\boldsymbol{x}, 0) = \hat{h}(\boldsymbol{x}), \tag{3.62}$$

where $\boldsymbol{x} \in \Omega$.

The functions h, d may depend on $\boldsymbol{x}$ but $h > 0, d > 0, b_i = b_i(\boldsymbol{x})$ and $k_{ij} = k_{ij}(\boldsymbol{x})$, but $k_{ij}\xi_i\xi_j \geq 0, \forall \boldsymbol{\xi}$.

To establish uniqueness and continuous dependence we let θ_1 and θ_2 be solutions to (3.60) and (3.61) for the same data function f. Let θ_1 satisfy (3.62) for $g = g_1, \hat{h} = h_1$ and let θ_2 satisfy (3.62) for $g = g_2, \hat{h} = h_2$. Define $w = \theta_1 - \theta_2, G = g_1 - g_2, H = h_1 - h_2$, then from (3.60)-(3.62) we find w satisfies the boundary-initial value problem

$$h\ddot{w} + d\dot{w} - b_i\dot{w}_{,i} - (b_i\dot{w})_{,i} = (k_{ij}w_{,j})_{,i}, \text{ in } \Omega \times (0, T), \tag{3.63}$$

$$w(\boldsymbol{x}, t) = 0, \ \boldsymbol{x} \in \Gamma, \tag{3.64}$$

$$w(\boldsymbol{x}, 0) = G(\boldsymbol{x}), \ w_t(\boldsymbol{x}, 0) = H(\boldsymbol{x}), \ \boldsymbol{x} \in \Omega. \tag{3.65}$$

Multiply equation (3.63) by $\dot{w}$ and integrate over Ω to find

$$\frac{d}{dt}\frac{1}{2}\int_\Omega h\dot{w}^2 dx + \int_\Omega d\dot{w}^2 dx + \frac{d}{dt}\frac{1}{2}\int_\Omega k_{ij}w_{,i}w_{,j} dx$$
$$= \int_\Omega b_i\dot{w}\dot{w}_{,i} dx + \int_\Omega (b_i\dot{w})_{,i}\dot{w} dx, \tag{3.66}$$

where we have integrated by parts on the k_{ij} term and used the boundary condition. To handle the right hand side we use the chain rule and integrate by parts to see that

$$\int_\Omega b_i \dot{w} \dot{w}_{,i} dx + \int_\Omega (b_i \dot{w})_{,i} \dot{w} dx$$
$$= \frac{1}{2} \int_\Omega b_i (\dot{w}^2)_{,i} dx + \int_\Omega b_i \dot{w}_{,i} \dot{w} dx + \int_\Omega b_{i,i} \dot{w}^2 dx$$
$$= \frac{1}{2} \int_\Omega b_i (\dot{w}^2)_{,i} dx + \frac{1}{2} \int_\Omega b_i (\dot{w}^2)_{,i} dx + \int_\Omega b_{i,i} \dot{w}^2 dx$$
$$= \int_\Omega b_i (\dot{w}^2)_{,i} dx + \int_\Omega b_{i,i} \dot{w}^2 dx$$
$$= \int_\Omega \frac{\partial}{\partial x_i} (b_i \dot{w}^2) dx$$
$$= \int_\Gamma b_i n_i \dot{w}^2 dS = 0, \tag{3.67}$$

since $\dot{w} = 0$ on Γ. Thus, equation (3.66) becomes

$$\frac{d}{dt} \left(\frac{1}{2} \int_\Omega h \dot{w}^2 dx + \frac{1}{2} \int_\Omega k_{ij} w_{,i} w_{,j} dx \right) + \int_\Omega d \dot{w}^2 dx = 0. \tag{3.68}$$

Since $d > 0$ we drop the last term to obtain

$$\frac{d}{dt} \left(\int_\Omega h \dot{w}^2 dx + \int_\Omega k_{ij} w_{,i} w_{,j} dx \right) \leq 0. \tag{3.69}$$

This equation is integrated from 0 to t and we find

$$\int_\Omega h[\dot{w}(x,t)]^2 dx + \int_\Omega k_{ij} w_{,i}(x,t) w_{,j}(x,t) dx \leq \int_\Omega h H^2 dx + \int_\Omega k_{ij} G_{,i} G_{,j} dx. \tag{3.70}$$

Inequality (3.70) yields continuous dependence on the initial data g and $\hat{h}$ in the measure

$$E(t) = \int_\Omega h[\dot{w}(\boldsymbol{x},t)]^2 dx + \int_\Omega k_{ij} w_{,i}(\boldsymbol{x},t) w_{,j}(\boldsymbol{x},t) dx. \tag{3.71}$$

If we additionally know k_{ij} is positive-definite, i.e.

$$k_{ij} \xi_i \xi_j \geq k_0 |\boldsymbol{\xi}|^2, \quad k_0 > 0, \tag{3.72}$$

then since $h > 0$ we may deduce from (3.70) with the aid of Poincaré's inequality that

$$k_0 \lambda_1 \int_\Omega [w(\boldsymbol{x},t)]^2 dx \leq \int_\Omega h H^2 dx + \int_\Omega k_{ij} G_{,i} G_{,j} dx, \tag{3.73}$$

where $\lambda_1 > 0$ is the constant in Poincaré's inequality.

Inequality (3.73) demonstrates continuous dependence on the initial data g and $\hat{h}$, in the $L^2(\Omega)$ measure of w, and hence θ.

Uniqueness follows immediately from (3.73) when (3.72) holds, for in that case $G \equiv 0, H \equiv 0$ and so from (3.73)

$$0 \le \int_\Omega w^2 dx \le 0 \tag{3.74}$$

which shows $w \equiv 0 \ \forall (x, t) \in \Omega \times (0, T)$. Thus $\theta_1 = \theta_2$ and hence uniqueness.

When k_{ij} is simply positive, i.e. $k_{ij}\xi_i\xi_j \ge 0$, then uniqueness follows from (3.70). In that case we see

$$0 \le \int_\Omega h\dot{w}^2 dx \le 0 \tag{3.75}$$

and since $h > 0$ we must have $\dot{w} \equiv 0$ in $\Omega \times (0, T)$. It then follows $w \equiv 0$ and so $\theta_1 \equiv \theta_2$ in $\Omega \times (0, T)$, hence uniqueness.

3.6   An instability result for a solution to (3.13)

We return now to equation (3.13) which we recollect together with the boundary and initial data as

$$\alpha\ddot{\theta} + \beta\dot{\theta} + \xi_{ik}\theta_{,ik} + k_i\dot{\theta}_{,i} = 0, \text{ in } \Omega \times (0, \infty), \tag{3.76}$$

$$\theta(x, t) = 0, \ x \in \Gamma, \tag{3.77}$$

$$\theta(x, 0) = g(x), \theta_t(x, 0) = \hat{h}(x), \ x \in \Omega. \tag{3.78}$$

In this case we assume α, β are constants but allow k_i to depend on x.

Our aim is to see whether the continuous dependence (stability) result of Section 3.4 may be negated if k_i is sufficiently large and ξ_{ij} is not negative. Thus, multiply (3.76) by $\dot{\theta}$ and integrate over Ω. After calculations like those of Section 3.4 we may show that

$$\frac{d}{dt}\frac{\alpha}{2}\int_\Omega \dot{\theta}^2 dx - \int_\Omega \left(\frac{k_{i,i}}{2} - \beta\right)\dot{\theta}^2 dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \xi_{ij}\theta_{,i}\theta_{,j} dx. \tag{3.79}$$

Suppose now that

$$k_{i,i}(x) \geq k_1 > 0 \tag{3.80}$$

and

$$k_1 > 2\beta. \tag{3.81}$$

We also assume that

$$\xi_{ij}\xi_i\xi_j \geq 0, \quad \forall \xi_i. \tag{3.82}$$

Under conditions (3.80)-(3.82) one now shows from (3.79) that

$$\int_\Omega \dot{\theta}^2 dx \geq \left(\frac{k_1 - 2\beta}{\alpha}\right) \int_0^t \int_\Omega \dot{\theta}^2 dx ds + L_1. \tag{3.83}$$

Put $\lambda = (k_1 - 2\beta)/\alpha > 0$. Define the function $K(t)$ by

$$K(t) = \int_0^t \int_\Omega \dot{\theta}^2 dx ds, \tag{3.84}$$

where

$$L_1 = \int_\Omega \hat{h}^2 dx - \frac{1}{\alpha} \int_\Omega \xi_{ij} g_{,i} g_{,j} dx. \tag{3.85}$$

Then inequality (3.83) is equivalent to

$$K' \geq \lambda K + L_1. \tag{3.86}$$

Using an integrating factor one finds

$$\frac{d}{dt}\left(e^{-\lambda t} K\right) \geq L_1 e^{-\lambda t} \tag{3.87}$$

and upon integration from 0 to t one obtains

$$K(t) \geq \frac{L_1}{\lambda}(e^{\lambda t} - 1) \tag{3.88}$$

or

$$\int_0^t \int_\Omega \dot{\theta}^2(x, t) dx ds \geq \left[\int_\Omega \hat{h}^2(x) dx - \frac{1}{\alpha} \int_\Omega \xi_{ij} g_{,i} g_{,j} dx\right]\left(\frac{e^{\lambda t} - 1}{\lambda}\right). \tag{3.89}$$

This inequality clearly shows $\int_\Omega \dot{\theta}^2 dx$ grows exponentially fast in time if conditions (3.80) - (3.82) hold and $\int_\Omega \hat{h}^2(x) dx > \frac{1}{\alpha} \int_\Omega \xi_{ij} g_{,i} g_{,j} dx$. This in turn demonstrates instability of a solution to (3.76)-(3.78). Thus, the conditions imposed in Sections 3.3 and 3.4 are important in ensuring a well posed boundary initial value problem.

3.7 Uniqueness on an unbounded spatial domain

In this section we let Ω be a domain in $\mathbb{R}^3$ exterior to a bounded set $\Omega_0 \subset \mathbb{R}^3$. The interior boundary of Ω is $\partial\Omega$. Suppose without loss of generality the origin $0 \in \Omega_0$. Define Ω_R to be the domain $B_R \backslash \Omega_0$ where B_R is the ball of radius R with R so large that Ω_0 does not intersect B_R. The boundary of Ω_R, i.e. the spherical surface of radius R, is denoted by Γ_R. The smallest R which intersects with $\partial\Omega$ is denoted by R_0.
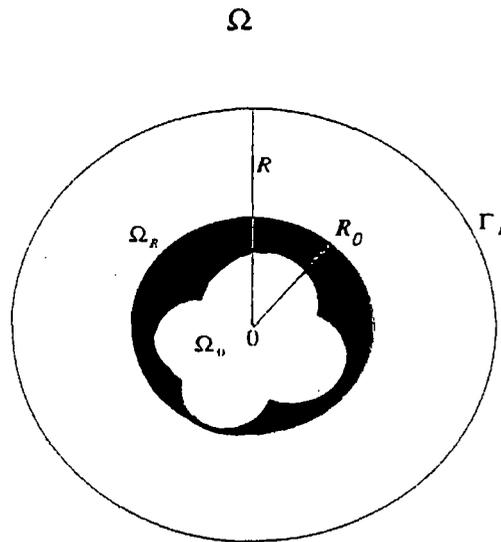
Figure 3.1: Diagram illustrating unbounded spatial domain

We wish to establish a uniqueness result for a solution to equation (3.13) but now on the unbounded domain Ω. Our aim is to establish uniqueness by **not** assuming the solution θ decays as $R \to \infty$. To achieve this we employ a method due to Graffi [56].

Let us recollect the boundary-initial value problem for (3.13), i.e.

$$\alpha\ddot{\theta} + \beta\dot{\theta} + k_i(x)\theta_{,i} + (k_{ij}\theta_{,j})_{,i} = 0, \text{ in } \Omega \times (0, T), \tag{3.90}$$

$$\theta(x, t) = \theta_1(x), \text{ on } \partial\Omega, \tag{3.91}$$

$$\theta(x, 0) = g(x), \ \theta_t(x, 0) = \hat{h}(x), \ x \in \Omega. \tag{3.92}$$

To establish uniqueness via the Graffi method we assume θ_1 and θ_2 are solutions to (3.90)-(3.92) for the same data functions θ_1, g and h. Let $w = \theta_1 - \theta_2$. Then w satisfies the boundary-initial value problem,

$$\alpha \ddot{w} + \beta \dot{w} + k_i(x)\dot{w}_{,i} + (k_{ij}w_{,j})_{,i} = 0, \ x \in \Omega, \ t \in (0,T), \tag{3.93}$$

$$w(x,t) = 0, \ x \in \partial\Omega, \tag{3.94}$$

$$w(x,0) = 0, \ w_t(x,0) = 0, \ x \in \Omega. \tag{3.95}$$

Multiply equation (3.93) by $\dot{w}$ and integrate over Ω_R for R fixed. We find

$$\frac{d}{dt}\int_{\Omega_R}\left(\frac{\alpha}{2}\dot{w}^2 - \frac{k_{ij}}{2}w_{,i}w_{,j}\right)dx + \int_{\Gamma_R}k_{ij}w_{,i}\dot{w}n_j dS$$
$$+ \int_{\Omega_R}\beta\dot{w}^2 dx + \int_{\Omega_R}\frac{k_i}{2}(\dot{w}^2)_{,i}dx = 0, \tag{3.96}$$

where we have integrated by parts on the k_{ij} term. For the last term

$$\int_{\Omega_R}\frac{k_i}{2}(\dot{w}^2)_{,i}dx = \int_{\Gamma_R}\frac{k_i n_i}{2}\dot{w}^2 dS - \int_{\Omega_R}\frac{k_{i,i}}{2}\dot{w}^2 dx. \tag{3.97}$$

Combining (3.97) with (3.96) we see that

$$\frac{d}{dt}\int_{\Omega_R}\left(\frac{\alpha}{2}\dot{w}^2 - \frac{1}{2}k_{ij}w_{,i}w_{,j}\right)dx + \int_{\Omega_R}\left(\beta - \frac{k_{i,i}}{2}\right)\dot{w}^2 dx$$
$$= -\int_{\Gamma_R}\left(k_{ij}n_j\dot{w}w_{,i} + \frac{k_i n_i}{2}\dot{w}^2\right)dS. \tag{3.98}$$

Integrate equation (3.98) twice over the time region $(0,T)$,

$$\int_0^T\int_{\Omega_R}\left(\frac{\alpha}{2}\dot{w}^2 - \frac{1}{2}k_{ij}w_{,i}w_{,j}\right)dxd\eta + \int_0^T\int_0^s\int_{\Omega_R}\left(\beta - \frac{k_{i,i}}{2}\right)\dot{w}^2 dxd\mu d\eta$$
$$= -\int_0^T\int_0^s\int_{\Gamma_R}\left(k_{ij}n_j\dot{w}w_{,i} + \frac{k_i n_i}{2}\dot{w}^2\right)dSd\mu d\eta. \tag{3.99}$$

We now require

$$\beta - \frac{k_{i,i}}{2} \geq 0. \tag{3.100}$$

Then the term $\beta - \frac{k_{i,i}}{2}$ may be dropped, hence we obtain

$$\int_0^T\int_{\Omega_R}\left(\frac{\alpha}{2}\dot{w}^2 - \frac{1}{2}k_{ij}w_{,i}w_{,j}\right)dxd\eta$$
$$\leq -\int_0^T\int_0^s\int_{\Gamma_R}\left(k_{ij}n_j\dot{w}w_{,i} + \frac{k_i n_i}{2}\dot{w}^2\right)dSd\mu d\eta. \tag{3.101}$$

We now suppose k_{ij} is a bounded, negative-definite tensor, so that

$$-k_{ij}\xi_i\xi_j \geq \alpha_0|\boldsymbol{\xi}|^2, \quad \alpha_0 > 0. \tag{3.102}$$

By employing (3.102) in (3.101), we then obtain

$$\int_0^T \int_{\Omega_R} \left(\frac{\alpha}{2}\dot{w}^2 + \frac{1}{2}\alpha_0 w_{,i}w_{,i}\right) dxd\eta$$

$$\leq -\int_0^T \int_0^s \int_{\Gamma_R} \left(k_{ij}n_j\dot{w}w_{,i} + \frac{k_i n_i}{2}\dot{w}^2\right) dSd\mu d\eta. \tag{3.103}$$

Suppose $|k_{ij}| \leq K$, $|k_i| \leq k_1$, $k_1 > 0$, then we obtain

$$\frac{1}{2}\int_0^T \int_{\Omega_R} \left(\frac{\alpha}{2}\dot{w}^2 + \alpha_0 w_{,i}w_{,i}\right) dxd\eta$$

$$\leq \int_0^T \int_0^s \int_{\Gamma_R} \left(K|\dot{w}||w_{,i}| + \frac{k_1}{2}\dot{w}^2\right) dSd\mu d\eta. \tag{3.104}$$

We then use the arithmetic-geometric mean inequality on the right to see that

$$\frac{1}{2}\int_0^T \int_{\Omega_R} \left(\frac{\alpha}{2}\dot{w}^2 + \alpha_0 w_{,i}w_{,i}\right) dxd\eta$$

$$\leq \int_0^T \int_0^s \int_{\Gamma_R} \left\{K\left(\frac{|\dot{w}|^2}{2} + \frac{|w_{,i}|^2}{2}\right) + \frac{k_1}{2}\dot{w}^2\right\} dSd\mu d\eta. \tag{3.105}$$

or,

$$\frac{1}{2}\int_0^T \int_{\Omega_R} \left(\frac{\alpha}{2}\dot{w}^2 + \alpha_0 w_{,i}w_{,i}\right) dxd\eta$$

$$\leq \int_0^T \int_0^s \int_{\Gamma_R} \left(\frac{K + k_1}{2}|\dot{w}|^2 + \frac{|\nabla w|^2}{2}\right) dSd\mu d\eta. \tag{3.106}$$

Hence, we obtain

$$\frac{1}{2}\int_0^T \int_{\Omega_R} \left(\alpha\dot{w}^2 + \alpha_0 w_{,i}w_{,i}\right) dxd\eta \leq \frac{(K + k_1)T}{2}\int_0^T \int_{\Gamma_R} \left(|\dot{w}|^2 + |\nabla w|^2\right) dSd\eta. \tag{3.107}$$

Let $a_1 = \min\{\alpha, \alpha_0\}$, then

$$\frac{a_1}{2}\int_0^T \int_{\Omega_R} \left(\dot{w}^2 + |\nabla w|^2\right) dxd\eta$$

$$\leq \frac{1}{2}\int_0^T \int_{\Omega_R} \left(\alpha\dot{w}^2 + \alpha_0|\nabla w|^2\right) dxd\eta$$

$$\leq \frac{(K + k_1)T}{2}\int_0^T \int_{\Gamma_R} \left(|\dot{w}|^2 + |\nabla w|^2\right) dSd\eta,$$

or

$$a_1 \int_0^T \int_{\Omega_R} \left(\dot{w}^2 + |\nabla w|^2 \right) dx d\eta \leq (K + k_1) T \int_0^T \int_{\Gamma_R} \left(|\dot{w}|^2 + |\nabla w|^2 \right) dS d\eta. \quad (3.108)$$

Define now

$$F(R) = \int_0^T \int_{\Omega_R} (\dot{w}^2 + |\nabla w|^2) dx d\eta, \quad (3.109)$$

and put $\lambda = a_1/(K + k_1)T$. From inequality (3.108) one shows

$$\lambda F \leq \frac{dF}{dR}. \quad (3.110)$$

Thus,

$$\frac{d}{dR} [F \exp(-\lambda R)] \geq 0.$$

Integrate this from $0 < R_0$ to R to find

$$F(R) \geq F(R_0) \exp[\lambda(R - R_0)]. \quad (3.111)$$

Suppose now that our class of solutions is such that

$$|\dot{w}|, \ |\nabla w| \leq e^{\xi R} \qquad \text{for some } \xi > 0. \quad (3.112)$$

Then, since $F(R_0) = F_0$ is a constant (3.111) shows that

$$F_0 \exp[\lambda(R - R_0)] \leq F(R) \leq A e^{2\xi R}, \quad (3.113)$$

some $A > 0$. If we pick T small enough then (3.113) yields a contradiction, i.e. for

$$\lambda > 2\xi,$$

or

$$T < \frac{a_1}{2\xi(K + k_1)}.$$

Thus, $w \equiv 0$ on $\Omega \times (0, T)$ and uniqueness follows.

However the bound (3.112) is independent of T and so we may repeat this argument on $(T, 2T)$, and so on to establish uniqueness on $(0, T)$.

N.B. The original Graffi method, see Graffi [56], was developed for the Navier-Stokes equations and for the equations of compressible flow. The extension of the Graffi method to hyperbolic equations of linear elasticity was due to Straughan [167].

3.8   A non-standard problem for equation (3.13)

Payne & Schaefer [127] began a study into non-standard problems for a class of partial differential equations. They argued that rather than prescribe initial conditions on the function, θ, say, one prescribes a combination of the solution at $t = 0$ and at $t = T$, for some time $T > 0$. Such a class of problem may be employed to obtain bounds for the solution when the problem is improperly posed. The class of non-standard problems studied by Payne & Schaefer [127] has proved to be a very fruitful area of research as may be witnessed by the extensions of Ames & Payne [3], Payne et al. [128, 129], Ames et al. [4, 5], and Quintanilla & Straughan [153, 154].

To state the class of non-standard problem we are interested in, suppose in (3.13) we change α to m, β to d, ξ_{ij} to $-\alpha_{ij}$ and for $m, d > 0$ but constant we have

$$m\ddot{\theta} + d\dot{\theta} + k_i(x)\dot{\theta}_{,i} - (\alpha_{ij}\theta_{,j})_{,i} = 0. \tag{3.114}$$

Equation (3.114) holds in $\Omega \times (0, T)$ where $\Omega \in \mathbb{R}^3$ is bounded. On the boundary of Ω, Γ, we have

$$\theta(x, t) = 0, \qquad x \in \Gamma. \tag{3.115}$$

For constants α, β, the "initial" conditions are replaced by

$$\alpha\theta(x, 0) + \theta(x, T) = g,$$

$$\beta\theta_t(x, 0) + \theta_t(x, T) = h, \quad x \in \Omega. \tag{3.116}$$

Our goal is to establish that a solution to (3.114)-(3.116) depends continuously in an appropriate manner on the data functions g and h. To achieve this we begin with a general estimate which follows by multiplying (3.114) by $\dot{\theta}$ and integrating over Ω. After integration by parts and use of the boundary condition (3.115) we may arrive at

$$\frac{m}{2}\|\dot{\theta}\|^2 + \int_0^t \int_\Omega \left(d - \frac{k_{i,i}}{2}\right)\dot{\theta}^2 dx d\eta + \frac{1}{2}\int_\Omega \alpha_{ij}\theta_{,i}(t)\theta_{,j}(t)dx$$

$$= \frac{m}{2}\|\dot{\theta}(0)\|^2 + \frac{1}{2}\int_\Omega \alpha_{ij}\theta_{,i}(0)\theta_{,j}(0)dx, \tag{3.117}$$

where here and throughout this section we have suppressed the dependence on x in θ.

Observe that (3.117) does not by itself yield a continuous dependence estimate for $\theta(t)$ since the right hand side consists of unknown functions $\dot{\theta}(0), \nabla\theta(0)$. We do not know $\theta(\boldsymbol{x}, 0), \dot{\theta}(\boldsymbol{x}, 0)$ we only know the data functions $g(\boldsymbol{x})$ and $h(\boldsymbol{x})$.

We suppose throughout this section that

$$d - \frac{k_{i,i}}{2} \geq 0 \quad \forall \boldsymbol{x} \in \Omega.$$

$$|\alpha|, |\beta| > 1.$$

We firstly consider the case $|\alpha|, |\beta| > 1$. Thus, we evaluate (3.117) for $t = T$ and drop the second term on the left of (3.117). This yields

$$m\|\dot{\theta}(T)\|^2 + \int_\Omega \alpha_{ij}\theta_{,j}(T)\theta_{,i}(T)dx$$
$$\leq m\|\dot{\theta}(0)\|^2 + \int_\Omega \alpha_{ij}\theta_{,j}(0)\theta_{,i}(0)dx. \tag{3.118}$$

Now, use equation (3.116) in the left hand side of (3.118) to find

$$m(h - \beta\dot{\theta}(0), h - \beta\dot{\theta}(0)) + \int_\Omega \alpha_{ij}[g_{,j} - \alpha\theta_{,j}(0)][g_{,i} - \alpha\theta_{,i}(0)]dx$$
$$\leq m\|\dot{\theta}(0)\|^2 + \int_\Omega \alpha_{ij}\theta_{,j}(0)\theta_{,i}(0)dx, \tag{3.119}$$

where $(.,.)$ denotes the inner product on $L^2(\Omega)$.

Note now that

$$(h - \beta\theta_t(0), h - \beta\theta_t(0)) = \|h\|^2 + \beta^2\|\theta_t(0)\|^2 - 2\beta(h, \theta_t(0))$$
$$\geq \|h\|^2 + \beta^2\|\theta_t(0)\|^2 - \frac{\beta^2}{\varepsilon}\|\theta_t(0)\|^2 - \varepsilon\|h\|^2, \tag{3.120}$$

for $\varepsilon > 0$, where we have used the arithmetic-geometric mean inequality. For $\varepsilon_1 > 0$ we similarly establish

$$\int_\Omega \alpha_{ij}[g_{,j} - \alpha\theta_{,j}(0)][g_{,i} - \alpha\theta_{,i}(0)]dx$$
$$= \int_\Omega \alpha_{ij}g_{,j}g_{,i}dx + \alpha^2\int_\Omega \alpha_{ij}\theta_{,j}(0)\theta_{,i}(0)dx - 2\alpha\int_\Omega \alpha_{ij}g_{,i}\theta_{,j}(0)dx,$$
$$\geq \int_\Omega \alpha_{ij}g_{,i}g_{,j}dx + \alpha^2\int_\Omega \alpha_{ij}\theta_{,i}(0)\theta_{,j}(0)dx$$
$$- \frac{\alpha^2}{\varepsilon_1}\int_\Omega \alpha_{ij}\theta_{,i}(0)\theta_{,j}(0)dx - \varepsilon_1\int_\Omega \alpha_{ij}g_{,i}g_{,j}dx. \tag{3.121}$$

Combining (3.120) and (3.121) in (3.119) we thus obtain

$$
m\beta^2 \left(1 - \frac{1}{\varepsilon}\right) \|\theta_t(0)\|^2 + \alpha^2 \left(1 - \frac{1}{\varepsilon_1}\right) \int_\Omega \alpha_{ij}\theta_{,i}(0)\theta_{,j}(0)dx
$$
$$
+ m(1 - \varepsilon)\|h\|^2 + (1 - \varepsilon_1) \int_\Omega \alpha_{ij}g_{,i}g_{,j}dx
$$
$$
\leq m\|\theta_t(0)\|^2 + \int_\Omega \alpha_{ij}\theta_{,i}(0)\theta_{,j}(0)dx. \tag{3.122}
$$

Since $|\beta| > 1, |\alpha| > 1$ we choose

$$
\varepsilon = \frac{2\beta^2}{\beta^2 - 1} > 1 \qquad \text{and} \qquad \varepsilon_1 = \frac{2\alpha^2}{\alpha^2 - 1} > 1.
$$

These choices in (3.122) yield the inequality

$$
m\left(\frac{\beta^2 - 1}{2}\right) \|\theta_t(0)\|^2 + \left(\frac{\alpha^2 - 1}{2}\right) \int_\Omega \alpha_{ij}\theta_{,i}(0)\theta_{,j}(0)dx
$$
$$
\leq m\left(\frac{\beta^2 + 1}{\beta^2 - 1}\right) \|h\|^2 + \left(\frac{\alpha^2 + 1}{\alpha^2 - 1}\right) \int_\Omega \alpha_{ij}g_{,i}g_{,j}dx. \tag{3.123}
$$

Put $\xi = \min\{|\alpha|, |\beta|\}$ and then from (3.123) we may obtain

$$
m\|\theta_t(0)\|^2 + \int_\Omega \alpha_{ij}\theta_{,i}(0)\theta_{,j}(0)dx
$$
$$
\leq k_1\|h\|^2 + k_2 \int_\Omega \alpha_{ij}g_{,i}g_{,j}dx, \tag{3.124}
$$

where

$$
k_1 = m\left(\frac{\beta^2 + 1}{\beta^2 - 1}\right)\left(\frac{2}{\xi^2 - 1}\right) > 0, \quad k_2 = \left(\frac{\alpha^2 + 1}{\alpha^2 - 1}\right)\left(\frac{2}{\xi^2 - 1}\right) > 0.
$$

Finally, employ the bound (3.124) in (3.117) to find

$$
\frac{m}{2}\|\dot{\theta}(t)\|^2 + \frac{1}{2}\int_\Omega \alpha_{ij}\theta_{,i}(t)\theta_{,j}(t)dx + \int_0^t \int_\Omega \left(d - \frac{k_{i,i}}{2}\right)\dot{\theta}^2 dx d\eta
$$
$$
\leq \frac{k_1}{2}\|h\|^2 + \frac{k_2}{2}\int_\Omega \alpha_{ij}g_{,i}g_{,j}dx. \tag{3.125}
$$

Inequality (3.125) holds for any $t \in [0, T]$ and thus yields our desired estimate of how the solution θ depends continuously on changes in the data functions $g(x)$ and $h(x)$.

The case $|\alpha|, |\beta| < 1$

In this situation we follow the method of Payne & Schaefer [127], p.88.

Let us define the bilinear form $(A., .)$ by

$$(A\theta(t), \theta(t)) = \int_\Omega \alpha_{ij}\theta_{,i}(t)\theta_{,j}(t)dx. \tag{3.126}$$

Then we follow the procedure leading to (3.117) but now integrate from t to T(fixed) rather than from 0 to t.

Recall $d - k_{i,i}/2 \geq 0$. Then put

$$E(t) = m\|\theta_t(t)\|^2 + (A\theta(t), \theta(t)). \tag{3.127}$$

One obtains

$$m\|\theta_t(t)\|^2 + (A\theta(t), \theta(t))$$
$$= \int_t^T \int_\Omega (2d - k_{i,i})|\theta_\eta|^2 dxd\eta + m\|\theta_t(T)\|^2 + (A\theta(T), \theta(T)). \tag{3.128}$$

We cannot discard the $2d - k_{i,i}$ term on the right of (3.128) since it is non-negative. However, we suppose

$$\max_\Omega[2d - k_{i,i}(x)] \leq a, \tag{3.129}$$

for a constant. Then, from (3.128)

$$E(t) \leq a \int_t^T \|\theta_\eta\|^2 d\eta + E(T) \tag{3.130}$$

and since (Ax, x) is a positive form,

$$E(t) \leq a \int_t^T E(\eta)d\eta + E(T). \tag{3.131}$$

Now put

$$P(t) = \int_t^T E(\eta)d\eta. \tag{3.132}$$

Then (3.131) may be rewritten as

$$-P'(t) \leq aP(t) + E(T). \tag{3.133}$$

Since T is fixed $E(T)$ is constant. Hence from (3.133)

$$\frac{d}{dt}\left(e^{at}P\right) + e^{at}E(T) \geq 0. \tag{3.134}$$

Upon integration from t to T we find

$$\frac{E(T)}{a}(e^{aT} - e^{at}) \geq e^{at}P(t). \tag{3.135}$$

Thus

$$P(t) \le \frac{E(T)}{a} \left(e^{a(T-t)} - 1 \right). \tag{3.136}$$

Now put this in (3.133) to obtain

$$-P'(t) \le E(T) \left(e^{a(T-t)} - 1 \right) + E(T) = E(T)e^{a(T-t)} \tag{3.137}$$

Now recall definition (3.132) so that $P'(t) = -E(t)$, hence

$$E(t) \le E(T)e^{a(T-t)}, \qquad t \in [0, T]. \tag{3.138}$$

Evaluate this for $t = 0$,

$$m\|\theta_t(0)\|^2 + (A\theta(0), \theta(0)) \le [m\|\theta_t(T)\|^2 + (A\theta(T), \theta(T))]e^{aT}. \tag{3.139}$$

Using the data relations

$$\alpha\theta(0) + \theta(T) = g, \quad \beta\theta_t(0) + \theta_t(T) = h, \tag{3.140}$$

we then find

$$m \left\| \frac{h}{\beta} - \frac{\theta_t(T)}{\beta} \right\|^2 + \left(A\left(\frac{g}{\alpha} - \frac{\theta(T)}{\alpha} \right), \frac{g}{\alpha} - \frac{\theta(T)}{\alpha} \right)$$
$$\le m\|\theta_t(T)\|^2 e^{aT} + (A\theta(T), \theta(T))e^{aT}. \tag{3.141}$$

The left hand side is expanded out to see that

$$\frac{m}{\beta^2}\|h\|^2 + \frac{m}{\beta^2}\|\theta_t(T)\|^2 - \frac{2m}{\beta^2}(h, \theta_t(T))$$
$$+ \frac{1}{\alpha^2}(Ag, g) + \frac{1}{\alpha^2}(A\theta(T), \theta(T)) - \frac{2}{\alpha^2}(Ag, \theta(T))$$
$$\le m\|\theta_t(T)\|^2 e^{aT} + (A\theta(T), \theta(T))e^{aT}. \tag{3.142}$$

Next, use the arithmetic-geometric mean inequality to obtain

$$-\frac{2m}{\beta^2}(h, \theta_t(T)) \ge -\frac{m\delta_1}{\beta^2}\|\theta_t(T)\|^2 - \frac{m}{\delta_1\beta^2}\|h\|^2 \tag{3.143}$$

$$-\frac{2}{\alpha^2}(Ag, \theta(T)) \ge -\frac{\delta_2}{\alpha^2}(A\theta(T), \theta(T)) - \frac{1}{\alpha^2\delta_2}(Ag, g), \tag{3.144}$$

where $\delta_1, \delta_2 > 0$ are constants.

Upon insertion of (3.143), (3.144) into (3.142) and rearranging one finds

$$m\left(\frac{1-\delta_1}{\beta^2} - e^{aT} \right)\|\theta_t(T)\|^2 + \left(\frac{1-\delta_2}{\alpha^2} - e^{aT} \right)(A\theta(T), \theta(T))$$
$$\le \frac{m}{\beta^2}\left(\frac{1}{\delta_1} - 1 \right)\|h\|^2 + \frac{1}{\alpha^2}\left(\frac{1}{\delta_2} - 1 \right)(Ag, g), \tag{3.145}$$

where we assume $0 < \{\delta_1, \delta_2\} < 1$. Pick $\delta_1 = \delta_2 = \frac{1}{2}$. Then

$$m\left(\frac{1}{2\beta^2} - e^{aT}\right)\|\theta_t(T)\|^2 + \left(\frac{1}{2\alpha^2} - e^{aT}\right)(A\theta(T), \theta(T))$$

$$\leq \frac{m\|h\|^2}{\beta^2} + \frac{1}{\alpha^2}(Ag, g). \tag{3.146}$$

Suppose now $|\alpha|, |\beta|$ are such that

$$\frac{1}{2\beta^2} - e^{aT} \geq \mu_1 > 0,$$

$$\frac{1}{2\alpha^2} - e^{aT} \geq \mu_2 > 0.$$

This requires $|\alpha|, |\beta| < e^{-aT/2}/\sqrt{2}$ and is a restriction on the data coefficients α and β. Let $\xi_1 = \min[\mu_1, \mu_2]$ and then from (3.146) we find

$$E(T) \leq \frac{\|h\|^2}{\xi_1\beta^2} + \frac{1}{\xi_1\alpha^2}(Ag, g). \tag{3.147}$$

Now, we use (3.147) in inequality (3.138) to find

$$E(t) \leq \left[\frac{\|h\|^2}{\xi_1\beta^2} + \frac{1}{\xi_1\alpha^2}(Ag, g)\right] \exp[a(T - t)], \tag{3.148}$$

for $0 \leq t \leq T$.

Inequality (3.148) is our bound for $\theta(t)$ in terms of the data functions g and h and the data constants α and β. Note, however, that α and β must in the case of $|\alpha|$, $|\beta| < 1$ be restricted.

When $\alpha > 1$, $\beta < 1$, or $\alpha < 1$, $\beta > 1$, Payne & Schaefer [127] show that non-uniqueness or non-existence of a solution is possible for an equation with $d = 0$, $k_i = 0$, i.e. $m\ddot{\theta} + (k_{ij}\theta_{,j})_{,i} = 0$.

We expect similar undesirable behaviour for the more complicated equation (3.13).

Chapter 4

Batra Model

4.1 Model and uniqueness?

In 1975, Batra [13] studied heat conduction and wave propagation in non-simple rigid solids for which the constitutive quantities at a point depend upon the present value of temperature and of all its derivatives up to second order at that point. Batra's theory is also developed in Batra [12].

He considered the balance of the internal energy

$$\dot{\varepsilon} + q_{i,i} - r = 0, \tag{4.1}$$

where ε is the internal energy density, q is the heat flux measured per unit surface area and r is the supply density of the internal energy. In that paper he used $\dot{f}$ for the time derivative of a function f of $\boldsymbol{x}$, t, $f_{,i}$ for the partial derivative of f with respect to x_i, where $\boldsymbol{x}$ is the position of a material particle in rectangular Cartesian coordinates. Moreover, he assumes that a material point $\boldsymbol{x}$ and at time t, the variables q, ε, η and ϕ are smooth functions of $\theta, \dot{\theta}, \boldsymbol{g}, \ddot{\theta}, \dot{\boldsymbol{g}}, \boldsymbol{G}$ and he denotes the corresponding functions by a superimposed caret i.e.

$$\varepsilon(x,t) = \hat{\varepsilon}(\theta, \dot{\theta}, \boldsymbol{g}, \ddot{\theta}, \dot{\boldsymbol{g}}, \boldsymbol{G}, x), \tag{4.2}$$

where θ is the empirical temperature, and $\boldsymbol{g}$, $\boldsymbol{G}$ denote the temperature gradient and second gradient, i.e.

$$g_i \equiv \theta_{,i} \quad \text{and} \quad G_{ij} \equiv \theta_{,ij}. \tag{4.3}$$

47

He notes that isotropic heat conductors, $\hat{\varepsilon}, \hat{\eta}, \hat{q}$ and $\hat{\phi}$ are isotropic functions of their variables. Furthermore, he derives for the balance of the internal energy in the linear case when heat supply $r = 0$ as

$$C_1\dot{\theta} + C_2\ddot{\theta} + C_3\dddot{\theta} + (C_4 + K_4)\dot{\theta}_{,ii} = K_1\theta_{,ii}, \qquad (4.4)$$

where

$$C_1 = \frac{\partial \varepsilon}{\partial \theta}\bigg|_E, \quad C_2 = \frac{\partial \varepsilon}{\partial \dot{\theta}}\bigg|_E$$

$$C_3 = \frac{\partial \varepsilon}{\partial \ddot{\theta}}\bigg|_E, \quad C_4 = \frac{\partial \varepsilon}{\partial I_1}\bigg|_E,$$

$$K_1 = -Q_1|_E, \quad K_4 = Q_4|_E, \quad I_1 = G_{ii}, \qquad (4.5)$$

where Q_1 and Q_4 are the coefficients of g_i and $\dot{g}_i$ in the expression for q_i.

He shows that the equation (4.4) gives C_3 and K_4 have the same sign. Moreover, he assumes that $\theta, \dot{\theta}, \boldsymbol{g}, \ddot{\theta}, \dot{\boldsymbol{g}}$, and $\boldsymbol{G}$ are continuous across a singular surface, from which it follows that the jumps $[\theta], [\dot{\theta}]$ and $[\theta_{,i}]$ across the wave vanish. He therefore obtains the solutions for the wavespeeds in the linear case as

$$u_i n_i = 0, \quad u_i n_i = \pm\sqrt{-\frac{K_4 + C_4}{C_3}}. \qquad (4.6)$$

He also establishes a uniqueness theorem for (4.4) by assuming that

$$C_1 > 0, K_1 > 0, C_2 \leq 0, C_3 < 0, C_4 + K_4 \leq 0, \qquad (4.7)$$

and the initial and boundary conditions are

$$\theta = \dot{\theta} = \ddot{\theta} = 0 \quad \text{on } \Omega \text{ at } t = 0,$$

$$\theta = 0 \text{ on } \partial_1\Omega \times (0, T), \quad \frac{\partial \theta}{\partial n} = 0 \text{ on } \partial_2\Omega \times (0, T), \qquad (4.8)$$

where

$$\partial\Omega = \partial_1\Omega \cup \partial_2\Omega, \quad \partial_1\Omega \cap \partial_2\Omega = \emptyset, \qquad (4.9)$$

Here Ω is the region occupied by the body and $\partial\Omega$ is its boundary. The proof of the uniqueness theorem in Batra [13] is wrong, as we now show. Recall the equation (4.4) with $M = C_4 + K_4$ as

$$C_1\dot{\theta} + C_2\ddot{\theta} + C_3\dddot{\theta} + M\Delta\dot{\theta} = K_1\Delta\theta. \qquad (4.10)$$

To prove uniqueness, Batra multiplies (4.10) by $\dot{\theta}$ and integrates over Ω, to obtain

$$C_1(\dot{\theta}, \dot{\theta}) + C_2(\ddot{\theta}, \dot{\theta}) + C_3(\dddot{\theta}, \dot{\theta}) + M(\dot{\theta}, \Delta\dot{\theta}) = K_1(\Delta\theta, \dot{\theta}). \qquad (4.11)$$

Note that,

$$(\dot{\theta}, \dot{\theta}) = \int_{\Omega} (\dot{\theta})^2 dx = \|\dot{\theta}\|^2. \qquad (4.12)$$

For the second term on the left hand side we find

$$(\ddot{\theta}, \dot{\theta}) = \int_{\Omega} \dot{\theta}\ddot{\theta} dx = \frac{1}{2}\frac{d}{dt}\int_{\Omega} (\dot{\theta})^2 dx = \frac{1}{2}\frac{d}{dt}\|\dot{\theta}\|^2. \qquad (4.13)$$

For the last term on the left hand side we find

$$
\begin{aligned}
(\dot{\theta}, \Delta\dot{\theta}) &= \int_{\Omega} \dot{\theta}\frac{\partial^2 \dot{\theta}}{\partial x_i \partial x_i} dx \\
&= \int_{\Omega} \frac{\partial}{\partial x_i}\left(\dot{\theta}\frac{\partial \dot{\theta}}{\partial x_i}\right) dx - \int_{\Omega} \left(\frac{\partial \dot{\theta}}{\partial x_i}\frac{\partial \dot{\theta}}{\partial x_i}\right) dx \\
&= \int_{\Gamma} n_i \dot{\theta}\frac{\partial \dot{\theta}}{\partial x_i} ds - \int_{\Omega} (\nabla\dot{\theta})^2 dx \\
&= -\int_{\Omega} (\nabla\dot{\theta})^2 dx = -\|\nabla\dot{\theta}\|^2.
\end{aligned}
\qquad (4.14)
$$

For the term on the right hand side one finds

$$
\begin{aligned}
(\Delta\theta, \dot{\theta}) &= \int_{\Omega} \dot{\theta}\frac{\partial^2 \theta}{\partial x_i \partial x_i} dx \\
&= \int_{\Omega} \frac{\partial}{\partial x_i}\left(\dot{\theta}\frac{\partial \theta}{\partial x_i}\right) dx - \int_{\Omega} \frac{\partial \theta}{\partial x_i}\frac{\partial \dot{\theta}}{\partial x_i} dx \\
&= \int_{\Gamma} \dot{\theta}\frac{\partial \theta}{\partial x_i} n_i ds - \int_{\Omega} \nabla\theta\nabla\dot{\theta} dx \\
&= -\frac{1}{2}\frac{d}{dt}\int_{\Omega} (\nabla\theta)^2 dx = -\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2.
\end{aligned}
\qquad (4.15)
$$

Note, now that

$$\frac{d}{dt}(\dot{\theta}, \ddot{\theta}) = (\dot{\theta}, \dddot{\theta}) + (\ddot{\theta}, \ddot{\theta}),$$

therefore

$$
\begin{aligned}
(\dot{\theta}, \dddot{\theta}) &= \frac{d}{dt}(\dot{\theta}, \ddot{\theta}) - (\ddot{\theta}, \ddot{\theta}) \\
&= \frac{1}{2}\frac{d^2}{dt^2}\int_{\Omega} (\dot{\theta})^2 dx - \int_{\Omega} (\ddot{\theta})^2 dx \\
&= \frac{1}{2}\frac{d^2}{dt^2}\|\dot{\theta}\|^2 - \|\ddot{\theta}\|^2.
\end{aligned}
\qquad (4.16)
$$

Therefore (4.11) becomes,

$$C_1\|\dot\theta\|^2 + \frac{C_2}{2}\frac{d}{dt}\|\dot\theta\|^2 + \frac{C_3}{2}\frac{d^2}{dt^2}\|\dot\theta\|^2 - C_3\|\ddot\theta\|^2 - M\|\nabla\dot\theta\|^2 = -\frac{K_1}{2}\frac{d}{dt}\|\nabla\theta\|^2, \quad (4.17)$$

or

$$\frac{d}{dt}\left(\frac{C_2}{2}\|\dot\theta\|^2 + \frac{K_1}{2}\|\nabla\theta\|^2\right) + \frac{d^2}{dt^2}\frac{C_3}{2}\|\dot\theta\|^2 = C_3\|\ddot\theta\|^2 + M\|\nabla\dot\theta\|^2 - C_1\|\dot\theta\|^2. \quad (4.18)$$

Under the conditions of Batra's theorem, $M \leq 0, C_3 \leq 0, C_1 \geq 0$, so drop the right hand-side of (4.18), and integrate from 0 to t, to obtain

$$\frac{C_2}{2}\|\dot\theta\|^2 + \frac{K_1}{2}\|\nabla\theta\|^2 + \frac{d}{dt}\frac{C_3}{2}\|\dot\theta\|^2 \leq 0. \quad (4.19)$$

Now suppose $K_1 \geq 0$, then with $K(t) = \frac{1}{2}\|\dot\theta\|^2$, inequality (4.19) yields

$$C_2 K + C_3 \dot K \leq 0. \quad (4.20)$$

If $C_3 \neq 0$, then inequality (4.20) may be rewritten

$$\frac{C_2 K}{C_3} + \dot K \geq 0. \quad (4.21)$$

Define $\lambda = C_2/C_3$, thus

$$\dot K + \lambda K \geq 0, \quad (4.22)$$

then multiply by the integrating factor $e^{\lambda t}$ and then one finds

$$e^{\lambda t}(\dot K + \lambda K) \geq 0, \quad (4.23)$$

or

$$\frac{d}{dt}(e^{\lambda t}K) \geq 0. \quad (4.24)$$

Then integrate from 0 to t, to find

$$e^{\lambda t}K(t) - K(0) \geq 0$$

$$K(t) \geq e^{-\lambda t}K(0). \quad (4.25)$$

Uniqueness does not follow from inequality (4.25) as claimed by Batra [13].

We now investigate the Batra theory further. We are able to show that under certain conditions one achieves exponential growth in a suitable solution measure, or possibly even finite time blow-up.

4.2 Exponential growth

Recall the Batra model of heat conduction in homogeneous, isotropic, non-simple rigid heat conduction,

$$C_1 \dot{\theta} + C_2 \ddot{\theta} + C_3 \dddot{\theta} + M \dot{\theta}_{,ii} = K_1 \theta_{,ii}. \tag{4.26}$$

The boundary and initial conditions are

$$\text{Boundary Condition} \qquad \theta(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \Gamma, \tag{4.27}$$

$$\text{Initial Conditions} \qquad \theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}),$$

$$\frac{\partial \theta}{\partial t}(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}),$$

$$\frac{\partial^2 \theta}{\partial t^2}(\boldsymbol{x}, 0) = a_0(\boldsymbol{x}). \tag{4.28}$$

To establish exponential growth of a solution, we multiply (4.26) by θ and integrate over Ω, then we obtain

$$C_1(\theta, \dot{\theta}) + C_2(\theta, \ddot{\theta}) + C_3(\theta, \dddot{\theta}) + M(\theta, \dot{\theta}_{,ii}) = K_1(\theta, \theta_{,ii}). \tag{4.29}$$

Note now that

$$(\theta, \dot{\theta}) = \int_\Omega \theta \dot{\theta} dx = \frac{1}{2} \frac{d}{dt} \int_\Omega \theta^2 dx = \frac{1}{2} \frac{d}{dt} \|\theta\|^2. \tag{4.30}$$

For the term of M we have

$$
\begin{aligned}
(\theta, \dot{\theta}_{,ii}) &= \int_\Omega \theta \left(\frac{\partial^2 \dot{\theta}}{\partial x_i \partial x_i} \right) dx \\
&= \int_\Omega \frac{\partial}{\partial x_i} \left(\theta \frac{\partial \dot{\theta}}{\partial x_i} \right) dx - \int_\Omega \frac{\partial \dot{\theta}}{\partial x_i} \frac{\partial \theta}{\partial x_i} dx \\
&= \int_\Gamma n_i \theta \frac{\partial \dot{\theta}}{\partial x_i} ds - \int_\Omega \frac{\partial \dot{\theta}}{\partial x_i} \frac{\partial \theta}{\partial x} dx \\
&= - \int_\Omega \frac{\partial \dot{\theta}}{\partial x_i} \frac{\partial \theta}{\partial x} dx \\
&= -\frac{1}{2} \frac{d}{dt} \int_\Omega (\nabla \theta)^2 dx = -\frac{1}{2} \frac{d}{dt} \|\nabla \theta\|^2.
\end{aligned}
\tag{4.31}
$$

For the term on the right hand side we obtain

$$
\begin{aligned}
(\theta, \theta_{,ii}) &= \int_\Omega \theta \frac{\partial^2 \theta}{\partial x_i \partial x_i} dx \\
&= \int_\Omega \frac{\partial}{\partial x_i}\left(\theta \frac{\partial \theta}{\partial x_i}\right) dx - \int_\Omega \frac{\partial \theta}{\partial x_i}\frac{\partial \theta}{\partial x_i} dx \\
&= \int_\Gamma n_i \theta \frac{\partial \theta}{\partial x_i} ds - \int_\Omega \left(\frac{\partial \theta}{\partial x_i}\right)^2 dx \\
&= -\int_\Omega \left(\frac{\partial \theta}{\partial x_i}\right)^2 dx = -\|\nabla\theta\|^2.
\end{aligned}
\tag{4.32}
$$

Now, put (4.30), (4.31) and (4.32) in (4.29) to obtain,

$$
\frac{1}{2}C_1 \frac{d}{dt}\|\theta\|^2 + C_2(\theta, \ddot\theta) + C_3(\theta, \dddot\theta) - \frac{M}{2}\frac{d}{dt}\|\nabla\theta\|^2 = -K_1\|\nabla\theta\|^2.
\tag{4.33}
$$

Define $\mathscr{F} = (\theta, \ddot\theta)$, thus

$$
\begin{aligned}
\frac{d\mathscr{F}}{dt} &= \frac{d}{dt}\int_\Omega \theta\ddot\theta dx = \int_\Omega \frac{d}{dt}(\theta\ddot\theta) dx \\
&= \int_\Omega \theta\dddot\theta\, dx + \int_\Omega \ddot\theta\dot\theta dx \\
&= (\theta, \dddot\theta) + (\dot\theta, \ddot\theta) = (\theta, \dddot\theta) + \frac{1}{2}\frac{d}{dt}\int_\Omega (\dot\theta)^2 dx \\
&= (\theta, \dddot\theta) + \frac{1}{2}\frac{d}{dt}\|\dot\theta\|^2,
\end{aligned}
\tag{4.34}
$$

therefore

$$
(\theta, \dddot\theta) = \frac{d\mathscr{F}}{dt} - \frac{1}{2}\frac{d}{dt}\|\dot\theta\|^2.
\tag{4.35}
$$

Since $\mathscr{F} = (\theta, \ddot\theta)$, put (4.35) in (4.33), and we find

$$
\frac{1}{2}C_1 \frac{d}{dt}\|\theta\|^2 + C_2\mathscr{F} + C_3\frac{d\mathscr{F}}{dt} - \frac{C_3}{2}\frac{d}{dt}\|\dot\theta\|^2 - \frac{M}{2}\frac{d}{dt}\|\nabla\theta\|^2 + K_1\|\nabla\theta\|^2 = 0.
\tag{4.36}
$$

Then, rearrange to obtain,

$$
C_3\frac{d\mathscr{F}}{dt} + C_2\mathscr{F} + \frac{d}{dt}\left(\frac{C_1}{2}\|\theta\|^2 - \frac{C_3}{2}\|\dot\theta\|^2 - \frac{M}{2}\|\nabla\theta\|^2\right) + K_1\|\nabla\theta\|^2 = 0.
\tag{4.37}
$$

Divide (4.37) by $C_3 \neq 0$, thus (4.37) becomes

$$
\frac{d\mathscr{F}}{dt} + \frac{C_2}{C_3}\mathscr{F} + \frac{d}{dt}\left(\frac{C_1}{2C_3}\|\theta\|^2 - \frac{1}{2}\|\dot\theta\|^2 - \frac{M}{2C_3}\|\nabla\theta\|^2\right) + \frac{K_1}{C_3}\|\nabla\theta\|^2 = 0.
\tag{4.38}
$$

Put $\mu = -C_2/C_3$, then equation (4.38) may be rewritten,

$$
\frac{d\mathscr{F}}{dt} - \mu\mathscr{F} = \frac{d}{dt}G - \frac{K_1}{C_3}\|\nabla\theta\|^2,
\tag{4.39}
$$

where $G(t) = \frac{\|\dot{\theta}\|^2}{2} + \frac{M}{2C_3}\|\nabla\theta\|^2 - \frac{C_1}{2C_3}\|\theta\|^2$.

Let $\xi = -\frac{K_1}{C_3}$, then (4.39) is

$$\frac{d\mathscr{F}}{dt} - \mu\mathscr{F} = \frac{d}{dt}G + \xi\|\nabla\theta\|^2. \tag{4.40}$$

Use an integrating factor

$$\frac{d}{dt}\left(e^{-\mu t}\mathscr{F}\right) = e^{-\mu t}\frac{dG}{dt} + e^{-\mu t}\xi\|\nabla\theta\|^2, \tag{4.41}$$

but

$$\frac{d}{dt}\left(e^{-\mu t}G\right) = e^{-\mu t}\frac{dG}{dt} - \mu e^{-\mu t}G, \tag{4.42}$$

then

$$\left(e^{-\mu t}\right)\frac{dG}{dt} = \frac{d}{dt}\left(e^{-\mu t}G\right) + \mu e^{-\mu t}G. \tag{4.43}$$

Put (4.43) in (4.41), therefore

$$\begin{aligned}\frac{d}{dt}\left(e^{-\mu t}\mathscr{F}\right) &= \frac{d}{dt}\left(e^{-\mu t}G\right) + \mu e^{-\mu t}G + e^{-\mu t}\xi\|\nabla\theta\|^2 \\ &= \frac{d}{dt}\left(e^{-\mu t}G\right) + e^{-\mu t}\left(\mu G + \xi\|\nabla\theta\|^2\right),\end{aligned} \tag{4.44}$$

and upon integration from 0 to t we find

$$\left(e^{-\mu s}\mathscr{F}\right)\Big|_0^t = \left(e^{-\mu s}G\right)\Big|_0^t + \int_o^t e^{-\mu s}\left(\mu G + \xi\|\nabla\theta\|^2\right)ds, \tag{4.45}$$

or

$$e^{-\mu t}\mathscr{F} - \mathscr{F}(0) = e^{-\mu t}G - G(0) + \int_0^t e^{-\mu s}\left(\mu G + \xi\|\nabla\theta\|^2\right)ds, \tag{4.46}$$

or, rearranging

$$e^{-\mu t}\mathscr{F} = e^{-\mu t}G + \int_0^t e^{-\mu s}\left(\mu G + \xi\|\nabla\theta\|^2\right)ds + \mathscr{F}(0) - G(0). \tag{4.47}$$

Let now $Q(t) = \mu G + \xi\|\nabla\theta\|^2$, then (4.47) may be wriiten

$$e^{-\mu t}\mathscr{F} = e^{-\mu t}G + \int_0^t e^{-\mu s}Q(s)ds + \mathscr{F}(0) - G(0). \tag{4.48}$$

Divide (4.48) by $e^{-\mu t}$, to obtain

$$\mathscr{F}(t) = G(t) + \int_0^t e^{\mu(t-s)}Q(s)ds + e^{\mu t}(\mathscr{F}(0) - G(0)). \tag{4.49}$$

Now recall that

$$\mathcal{F}(t) = (\theta, \ddot{\theta}) = \frac{d}{dt}(\theta, \dot{\theta}) - \|\dot{\theta}\|^2$$

$$= \frac{d^2}{dt^2}\frac{1}{2}\|\theta\|^2 - \|\dot{\theta}\|^2. \tag{4.50}$$

In addition we define

$$K(0) = \mathcal{F}(0) - G(0). \tag{4.51}$$

Next, employ (4.50) and (4.51) in (4.49) to see that

$$\frac{d^2}{dt^2}\frac{1}{2}\|\theta\|^2 - \|\dot{\theta}\|^2 = G(t) + \int_0^t e^{\mu(t-s)}Q(s)ds + e^{\mu t}K(0). \tag{4.52}$$

Define now $F(t) = \|\theta\|^2$, then from (4.52) we have

$$\frac{1}{2}F'' - \|\dot{\theta}\|^2 = G(t) + \int_0^t e^{\mu(t-s)}Q(s)ds + e^{\mu t}K(0), \tag{4.53}$$

or taking a term to the right hand side

$$\frac{F''}{2} = \|\dot{\theta}\|^2 + G(t) + \int_0^t e^{\mu(t-s)}Q(s)ds + e^{\mu t}K(0), \tag{4.54}$$

and then multiplying by 2 we arrive at

$$F'' = 2\|\dot{\theta}\|^2 + 2G(t) + 2\int_0^t e^{\mu(t-s)}Q(s)ds + 2e^{\mu t}K(0). \tag{4.55}$$

Next, substitute for $G(t)$ in (4.55), to obtain

$$F'' = 2\|\dot{\theta}\|^2 + \|\dot{\theta}\|^2 + \frac{M}{C_3}\|\nabla\theta\|^2 - \frac{C_1}{C_3}\|\theta\|^2 + 2\int_0^t e^{\mu(t-s)}Q(s)ds + 2e^{\mu t}K(0), \tag{4.56}$$

or,

$$F'' = 3\|\dot{\theta}\|^2 + \frac{M}{C_3}\|\nabla\theta\|^2 - \frac{C_1}{C_3}\|\theta\|^2 + 2\int_0^t e^{\mu(t-s)}Q(s)ds + 2e^{\mu t}K(0). \tag{4.57}$$

If $M/C_3 > 0, -C_1/C_3 > 0, Q(s) \geq 0, \forall s \geq 0$, thus (4.57) becomes

$$F'' \geq 2e^{\mu t}K(0). \tag{4.58}$$

Recall $Q(t) = \mu G(t) + \xi\|\nabla\theta\|^2$

$$= \frac{\mu}{2}\|\dot{\theta}\|^2 + \frac{\mu M}{2C_3}\|\nabla\theta\|^2 - \frac{\mu C_1}{2C_3}\|\theta\|^2 + \xi\|\nabla\theta\|^2 \tag{4.59}$$

where $\mu = -C_2/C_3$ and $\xi = -K_1/C_3$. Thus, if we suppose $\mu > 0$ and $\xi > 0$, then $Q(t) \geq 0$, as required.

Inequality (4.58) is the fundamental inequality from which exponential growth follows.

Now, integrate inequality (4.58) from 0 to t, thus

$$F'(s)|_0^t \geq \frac{2K(0)}{\mu}(e^{\mu s})|_0^t, \tag{4.60}$$

or

$$F'(t) \geq \frac{2K(0)}{\mu}(e^{\mu t} - 1) + F'(0). \tag{4.61}$$

Further, integrate (4.61) from 0 to t, to find

$$
\begin{aligned}
F(t) - F(0) &\geq \frac{2K(0)}{\mu}\int_0^t (e^{\mu s} - 1)ds + \int_0^t F'(0)ds \\
&= \frac{2K(0)}{\mu}\left(\frac{e^{\mu t}}{\mu} - t - \frac{1}{\mu}\right) + \int_0^t 2(\theta,\dot\theta)_0 ds \\
&= \frac{2K(0)}{\mu}\left(\frac{e^{\mu t}}{\mu} - t - \frac{1}{\mu}\right) + 2(\theta_0, v_0)t.
\end{aligned}
\tag{4.62}
$$

Rearrange inequality (4.62) to obtain

$$
\begin{aligned}
F(t) &\geq F(0) + \frac{2K(0)}{\mu}\left(\frac{e^{\mu t}}{\mu} - t - \frac{1}{\mu}\right) + 2(\theta,\dot\theta)_0 t \\
&= F(0) + \left(\frac{2}{\mu^2}(e^{\mu t} - 1) - \frac{2t}{\mu}\right)K(0) + 2(\theta,\dot\theta)_0 t \\
&= \|\theta_0\|^2 + \left(\frac{2}{\mu^2}(e^{\mu t} - 1) - \frac{2t}{\mu}\right)(\mathscr{F}(0) - G(0)) + 2(\theta,\dot\theta)_0 t.
\end{aligned}
$$

Therefore

$$\|\theta(t)\|^2 \geq \|\theta_0\|^2 + \left(\frac{2}{\mu^2}(e^{\mu t} - 1) - \frac{2t}{\mu}\right)(\mathscr{F}(0) - G(0)) + 2(\theta,\dot\theta)_0 t, \tag{4.63}$$

Provided $\mathscr{F}(0) - G(0) > 0$, since $\mu > 0$, inequality (4.63) leads to exponential growth of $\|\theta(t)\|^2$.

4.3 Explosive instability in a Batra model

We now investigate a generalization of equation (4.26) for which we are able to establish that a solution ceases to exist in a finite time. A similar result for another

third order in time equation was established by Quintanilla and Straughan [151]. Recall that with $M = K_4 + C_4$ and $K = K_1$, equation (4.26) is

$$C_1 \dot{\theta} + C_2 \ddot{\theta} + C_3 \dddot{\theta} + M \Delta \dot{\theta} = K \Delta \theta. \tag{4.64}$$

We here suppose that K is a function of temperature. This is realistic because in general the thermal conductivity does depend on temperature. Thus, we consider the following generalization to equation (4.64),

$$C_3 \dddot{\theta} + C_2 \ddot{\theta} + C_1 \dot{\theta} + M \Delta \dot{\theta} = \nabla(K(\theta) \nabla \theta). \tag{4.65}$$

For our instability result we require either

$$C_3 > 0, \ C_2 < 0, \ C_1 < 0, \ M > 0 \text{ and } K < 0, \tag{4.66}$$

or

$$C_3 < 0, \ C_2 > 0, \ C_1 > 0, \ M < 0 \text{ and } K > 0. \tag{4.67}$$

It is easy to see (4.66) and (4.67) are equivalent. Suppose now (4.66) holds. Then divide by C_3 and we find θ satisfies the equation

$$\dddot{\theta} - \alpha \ddot{\theta} - \alpha_2 \dot{\theta} = -\beta \Delta \dot{\theta} - \nabla(f(\theta) \nabla \theta), \tag{4.68}$$

where

$$\alpha = \left| \frac{C_2}{C_3} \right|, \quad \alpha_2 = \left| \frac{C_1}{C_3} \right|, \quad \beta = \left| \frac{M}{C_3} \right|, \tag{4.69}$$

and we have chosen

$$f(\theta) = -K(\theta) = 1 + a\theta^\varepsilon, \tag{4.70}$$

for a, ε positive constants. We could easily have $f = k_0 + a\theta^\varepsilon$ for $k_0 > 0$ and our proof will carry through.

Suppose $\theta \geq 0$, an assumption which is realistic given θ is temperature. Then θ satisfies equation (4.68) on the domain $\Omega \times (0, T)$ together with the boundary condition

$$\theta(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \Gamma, \tag{4.71}$$

and the initial conditions

$$\theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}), \quad \frac{\partial \theta}{\partial t}(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}),$$
$$\frac{\partial^2 \theta}{\partial t^2}(\boldsymbol{x}, 0) = a_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{4.72}$$

We define the functions $G(t)$ and $K(t)$ to be

$$G(t) = \|\theta(t)\|^2, \quad K(t) = \|\dot\theta\|^2. \tag{4.73}$$

Then, we multiply equation (4.68) by θ and integrate over Ω to find

$$(\theta, \dddot\theta) - \alpha(\theta, \ddot\theta) - \alpha_2(\theta, \dot\theta) = -\beta(\theta, \Delta\dot\theta) - \int_\Omega \theta\nabla(f\nabla\theta)dx. \tag{4.74}$$

Next, integrate by parts and use the boundary condition (4.71) to establish the following chain of results.

$$-(\theta, \dot\theta) = -\frac{d}{dt}\frac{1}{2}\|\theta\|^2 = -\frac{1}{2}G', \tag{4.75}$$

and

$$-(\theta, \Delta\dot\theta) = (\nabla\theta, \nabla\dot\theta) = \frac{d}{dt}\frac{1}{2}\|\nabla\theta\|^2, \tag{4.76}$$

and

$$-(\theta, \nabla(f\nabla\theta)) = \int_\Omega f|\nabla\theta|^2 dx = (f\theta_{,i}, \theta_{,i}), \tag{4.77}$$

and

$$\begin{aligned}
-(\theta, \ddot\theta) &= -\frac{d}{dt}(\theta, \dot\theta) + \|\dot\theta\|^2 \\
&= -\frac{1}{2}\frac{d^2}{dt^2}\|\theta\|^2 + \|\dot\theta\|^2 \\
&= -\frac{1}{2}G'' + K,
\end{aligned} \tag{4.78}$$

and

$$\begin{aligned}
(\theta, \dddot\theta) &= \frac{d}{dt}(\theta, \ddot\theta) - (\dot\theta, \ddot\theta) \\
&= \frac{d}{dt}\left[\frac{d}{dt}(\theta, \dot\theta) - \|\dot\theta\|^2\right] - \frac{1}{2}\frac{d}{dt}\|\dot\theta\|^2 \\
&= \frac{1}{2}\frac{d^3}{dt^3}\|\theta\|^2 - \frac{3}{2}\frac{d}{dt}\|\dot\theta\|^2 \\
&= \frac{1}{2}G''' - \frac{3}{2}K'.
\end{aligned} \tag{4.79}$$

Using relations (4.75)-(4.79) we rewrite equation (4.74) as

$$\frac{1}{2}G''' - \frac{3}{2}K' - \frac{\alpha}{2}G'' + \alpha K - \frac{\alpha_2}{2}G' = \frac{\beta}{2}\frac{d}{dt}\|\nabla\theta\|^2 + (f\theta_{,i}, \theta_{,i}).$$

We multiply by 2 and rearrange this equation to obtain

$$G''' - \alpha G'' = 3K' - 2\alpha K + \frac{d}{dt}(\alpha_2 G + \beta\|\nabla\theta\|^2) + 2(f\theta_{,i}, \theta_{,i}). \qquad (4.80)$$

Now, multiply by the integrating factor $\exp(-\alpha t)$ and write $-2\alpha K = -3\alpha K + \alpha K$, and then from (4.80) one sees that

$$\frac{d}{dt}[\exp(-\alpha t)G''] = 3\frac{d}{dt}[\exp(-\alpha t)K] + \alpha K e^{-\alpha t} + \frac{d}{dt}[e^{-\alpha t}(\alpha_2 G + \beta\|\nabla\theta\|^2)]$$

$$+ \alpha \exp(-\alpha t)(\alpha_2 G + \beta\|\nabla\theta\|^2) + 2\exp(-\alpha t)(f\theta_{,i}, \theta_{,i}). \qquad (4.81)$$

We integrate this equation from 0 to t and multiply the result by $\exp(\alpha t)$ to arrive at

$$G''(t) = 3K + \alpha_2 G + \beta\|\nabla\theta\|^2$$

$$+ \alpha \int_0^t e^{\alpha(t-s)}(K + \alpha_2 G + \beta\|\nabla\theta\|^2)ds$$

$$+ 2\int_0^t e^{\alpha(t-s)}(f\theta_{,i}, \theta_{,i})ds + K_0 e^{\alpha t}, \qquad (4.82)$$

where the initial data term K_0 given by

$$K_0 = G''(0) - 3K(0) - \alpha_2 G(0) - \beta\|\nabla\theta_0\|^2. \qquad (4.83)$$

Recalling the definitions of G and K in (4.73) and the initial data in (4.72), we find

$$K_0 = 2(\theta_0, a_0) - \|v_0\|^2 - \alpha_2\|\theta_0\|^2 - \beta\|\nabla\theta_0\|^2. \qquad (4.84)$$

We henceforth require

$$K_0 \geq 0. \qquad (4.85)$$

This condition can always be achieved and requires the initial temperature and temperature acceleration to be sufficiently different that (4.85) holds.

Next, we return to (4.82). Since α_2, β, α and K_0 are all non-negative we may deduce from (4.82) that

$$G'' \geq 2\int_0^t \exp[\alpha(t-s)](f\theta_{,i}, \theta_{,i})ds. \qquad (4.86)$$

Recall $f = 1 + a\theta^\varepsilon$. Then

$$(f\theta_{,i}, \theta_{,i}) = \int_\Omega |\nabla\theta|^2 dx + a\int_\Omega \theta^\varepsilon \theta_{,i}\theta_{,i}dx$$

$$\geq a\int_\Omega \theta^\varepsilon \theta_{,i}\theta_{,i}dx. \qquad (4.87)$$

Now, define

$$F(t) = \int_0^t G(s)ds. \tag{4.88}$$

Combine (4.87) and (4.88) in inequality (4.86) to find

$$F''' \geq 2a \int_0^t e^{\alpha(t-s)} \int_\Omega \theta^\varepsilon \theta_{,i}\theta_{,i}dxds. \tag{4.89}$$

To handle the right hand side of (4.89) we write

$$\begin{aligned}
a \int_\Omega \theta^\varepsilon \theta_{,i}\theta_{,i}dx &= \frac{a}{(1+\varepsilon/2)^2} \int_\Omega (\theta^{1+\varepsilon/2})_{,i}(\theta^{1+\varepsilon/2})_{,i}dx \\
&\geq \frac{\lambda_1 a}{(1+\varepsilon/2)^2} \int_\Omega (\theta^{1+\varepsilon/2})^2 dx \\
&= \frac{\lambda_1 a}{(1+\varepsilon/2)^2} \int_\Omega \theta^{2+\varepsilon}dx, \tag{4.90}
\end{aligned}$$

where $\lambda_1 > 0$ is the constant in Poincaré's inequality $\|\nabla\phi\|^2 \geq \lambda_1\|\phi\|^2$ for functions ϕ which vanish on Ω. Next, use Hölder's inequality

$$\int_\Omega \theta^2 dx \leq (\int_\Omega dx)^{1/q}(\int_\Omega \theta^{2p}dx)^{1/p}, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{4.91}$$

Pick

$$p = \frac{2+\varepsilon}{2} \quad \text{and then} \quad q = \frac{2+\varepsilon}{\varepsilon}.$$

Thus, with m denoting the volume of Ω we find

$$\int_\Omega \theta^2 dx \leq m^{\varepsilon/(2+\varepsilon)}\left(\int_\Omega \theta^{2+\varepsilon}dx\right)^{2/(2+\varepsilon)},$$

or rearranging,

$$\int_\Omega \theta^{2+\varepsilon}dx \geq \frac{1}{m^{\varepsilon/2}}\left(\int_\Omega \theta^2 dx\right)^{(2+\varepsilon)/2} = \frac{1}{m^{\varepsilon/2}}\|\theta\|^{2+\varepsilon}. \tag{4.92}$$

Thus, combining (4.92) and (4.90) one sees that

$$a \int_\Omega \theta^\varepsilon \theta_{,i}\theta_{,i}dx \geq \frac{a\lambda_1}{(1+\varepsilon/2)^2 m^{\varepsilon/2}}\|\theta\|^{2+\varepsilon}. \tag{4.93}$$

Now use (4.93) in inequality (4.89) to find

$$F''' \geq \hat{k} \int_0^t e^{\alpha(t-s)}\|\theta(s)\|^{2+\varepsilon}ds, \tag{4.94}$$

where

$$\hat{k} = \frac{8a\lambda_1}{(2+\varepsilon)^2 m^{\varepsilon/2}}. \tag{4.95}$$

We must now estimate the right hand side of inequality (4.94) in terms of F. To do this we again employ Hölder's inequality as follows, where $\xi > 0$ is a constant to be chosen,

$$
\int_0^t \|\theta\|^2 ds = \int_0^t e^{\alpha\xi(t-s)} \|\theta\|^2 e^{-\alpha\xi(t-s)} ds
$$
$$
\leq \left(\int_0^t e^{-\alpha\xi q(t-s)} ds \right)^{1/q} \left(\int_0^t e^{\alpha\xi p(t-s)} \|\theta\|^{2p} ds \right)^{1/p}. \qquad (4.96)
$$

Pick

$$
p = \frac{2+\varepsilon}{2}, \quad \text{whence} \quad q = \frac{2+\varepsilon}{\varepsilon}
$$

and from the inequality above one derives

$$
\int_0^t \|\theta\|^2 ds \leq \left(\int_0^t \exp\left[-\alpha\xi(t-s)\frac{(2+\varepsilon)}{\varepsilon} \right] ds \right)^{\varepsilon/(2+\varepsilon)}
$$
$$
\times \left(\int_0^t \exp\left[\alpha\xi(t-s)\left(\frac{2+\varepsilon}{2} \right) \right] \|\theta\|^{2+\varepsilon} ds \right)^{2/(2+\varepsilon)}. \qquad (4.97)
$$

Now, choose $\xi = 2/(2+\varepsilon)$, then

$$
\int_0^t \|\theta\|^2 ds \leq \left[\int_0^t e^{\alpha(t-s)} \|\theta\|^{2+\varepsilon} ds \right]^{2/(2+\varepsilon)} \left(\int_0^t \exp\left[-\alpha\frac{2}{\varepsilon}(t-s) \right] ds \right)^{\varepsilon/(2+\varepsilon)}
$$
$$
= \left[\int_0^t e^{\alpha(t-s)} \|\theta\|^{2+\varepsilon} ds \right]^{2/(2+\varepsilon)} \left(e^{-2\alpha t/\varepsilon} \int_0^t e^{2\alpha s/\varepsilon} ds \right)^{\varepsilon/(2+\varepsilon)}
$$
$$
= \left(\frac{\varepsilon}{2\alpha} \right)^{\varepsilon/(2+\varepsilon)} \left[1 - \exp\left(\frac{-2\alpha t}{\varepsilon} \right) \right]^{\varepsilon/(2+\varepsilon)} \left(\int_0^t e^{\alpha(t-s)} \|\theta\|^{2+\varepsilon} ds \right)^{2/(2+\varepsilon)}.
$$
$$
\qquad (4.98)
$$

We now bound the second term on the right by 1 to find

$$
\int_0^t \|\theta\|^2 ds \leq \left(\frac{\varepsilon}{2\alpha} \right)^{\varepsilon/(2+\varepsilon)} \left[\int_0^t e^{\alpha(t-s)} \|\theta\|^{2+\varepsilon} ds \right]^{2/(2+\varepsilon)}. \qquad (4.99)
$$

Thus, rearranging (4.99) yields

$$
\int_0^t e^{\alpha(t-s)} \|\theta(s)\|^{2+\varepsilon} ds \geq \left(\frac{2\alpha}{\varepsilon} \right)^{\varepsilon/2} F^{(2+\varepsilon)/2}. \qquad (4.100)
$$

Finally, use of (4.100) in inequality (4.94) shows

$$
F''' \geq k F^{1+\varepsilon/2}, \quad \text{where} \quad k = \hat{k}(2\alpha/\varepsilon)^{\varepsilon/2}. \qquad (4.101)
$$

To show (4.101) leads to global non existence of a solution suppose $\theta_0 \neq 0$, then $F'(0) = \|\theta_0\|^2 > 0$. Multiply (4.101) by F' to show

$$
(F'F'')' \geq (F'')^2 + \left(\frac{2k}{4+\varepsilon} \right) \frac{d}{dt} F^{2+\varepsilon/2}. \qquad (4.102)
$$

Integrate this inequality from 0 to t to obtain

$$F'F'' \geq F'(0)F''(0) + \left(\frac{2k}{4+\varepsilon}\right)F^{2+\varepsilon/2}. \tag{4.103}$$

Now multiply this inequality by F', integrate from 0 to t and multiply the result by 3. One finds

$$(F')^3 \geq [F'(0)]^3 + 3F'(0)F''(0)F(t) + \gamma F^{3+\varepsilon/2}, \tag{4.104}$$

where the constant γ is given by

$$\gamma = \frac{12k}{(4+\varepsilon)(6+\varepsilon)}.$$

The proof now proceeds by contradiction. Suppose θ exists for all time $t > 0$. Then

$$F' \geq (\alpha_1 + \beta_1 F + \gamma F^{3+\varepsilon/2})^{1/3},$$

$$\alpha_1 = [F'(0)]^3, \quad \beta_1 = 3F'(0)F''(0), \text{ and so}$$

$$\int_{F(0)}^{F(t)} \frac{dF}{(\alpha_1 + \beta_1 F + \gamma F^{3+\varepsilon/2})^{1/3}} \geq \int_0^t dt. \tag{4.105}$$

The left hand side is bounded above by the analogous integral from $F(0)$ to ∞ and then

$$t \leq T_u, \tag{4.106}$$

where

$$T_u = \int_0^\infty \frac{dF}{(\alpha_1 + \beta_1 F + \gamma F^{3+\varepsilon/2})^{1/3}}, \tag{4.107}$$

where we have noted $F(0) = 0$.

Since $T_u < \infty$, inequality (4.105) shows t must be bounded which contradicts the fact that θ exists for all time $t > 0$. In fact, θ cannot exist globally beyond the value $t = T_u$.

The expected behaviour of θ is that it blows-up in a finite time $T \geq T_u$.

4.4 Thermal discontinuity waves in the Batra theory

Batra [13] found the wavespeed of a discontinuity wave in the linear version of his theory. We now analyse discontinuity waves in his model, but for the fully nonlinear

theory. A further study of thermal waves in a rigid heat conductor is contained in Chapter 5.

Recall the complete equation of Batra [13], with the heat supply $r = 0$,

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{4.108}$$

In this section we restrict attention to a one-dimensional discontinuity wave. Thus, ε and q are functions of the variables

$$\theta, \ \dot{\theta}, \ \ddot{\theta}, \ \theta_x, \ \dot{\theta}_x, \ \theta_{xx} \tag{4.109}$$

Using (4.109) in (4.108) and expanding we see that

$$\varepsilon_\theta \dot{\theta} + \varepsilon_{\dot{\theta}} \ddot{\theta} + \varepsilon_{\ddot{\theta}} \dddot{\theta} + \varepsilon_{\theta_x} \dot{\theta}_x + \varepsilon_{\dot{\theta}_x} \ddot{\theta}_x + \varepsilon_{\theta_{xx}} \dot{\theta}_{xx}$$
$$+ q_\theta \theta_x + q_{\dot{\theta}} \dot{\theta}_x + q_{\ddot{\theta}} \ddot{\theta}_x + q_{\theta_x} \theta_{xx} + q_{\dot{\theta}_x} \dot{\theta}_{xx} + q_{\theta_{xx}} \theta_{xxx} = 0. \tag{4.110}$$

A discontinuity wave for (4.110) is a singular surface, analogous to an acceleration wave, S, across which $\dddot{\theta}, \dot{\theta}_x, \theta_{xx}$ and θ_{xxx} possess a finite discontinuity. The higher derivatives may also be discontinuous across S. However, $\theta \in C^2$ in both x and t. As stated in the introduction the jump of a quantity f is

$$[f] = f^- - f^+$$

where

$$f^\pm = \lim_{x \to S^\pm} f(x, t).$$

Taking the jump of equation (4.110) and recalling the regularity properties of S, we obtain

$$\varepsilon_{\ddot{\theta}}[\dddot{\theta}] + \varepsilon_{\dot{\theta}_x}[\dot{\theta}_x] + \varepsilon_{\theta_{xx}}[\dot{\theta}_{xx}] + q_{\ddot{\theta}}[\ddot{\theta}_x] + q_{\dot{\theta}_x}[\dot{\theta}_{xx}] + q_{\theta_{xx}}[\theta_{xxx}] = 0. \tag{4.111}$$

The Hadamard relation is

$$\frac{\delta[f]}{\delta t} = [f_t] + V[f_x], \tag{4.112}$$

where $\delta/\delta t$ is the derivative of a function at the wave S, as seen by an observer on the wave, and V is the speed of the wave moving in the x-direction. Since $\theta \in C^2(\mathbb{R} \times \mathbb{R})$ we find the chain of relations

$$\frac{\delta}{\delta t}[\ddot{\theta}] = 0 = [\dddot{\theta}] + V[\ddot{\theta}_x], \tag{4.113}$$

and

$$\frac{\delta}{\delta t}[\dot{\theta}_x] = 0 = [\ddot{\theta}_x] + V[\dot{\theta}_{xx}], \tag{4.114}$$

and

$$\frac{\delta}{\delta t}[\theta_{xx}] = 0 = [\dot{\theta}_{xx}] + V[\theta_{xxx}]. \tag{4.115}$$

From (4.113)-(4.115) we derive

$$[\dddot{\theta}] = -V[\ddot{\theta}_x] = V^2[\dot{\theta}_{xx}] = -V^3[\theta_{xxx}], \tag{4.116}$$

$$[\ddot{\theta}_x] = -V[\dot{\theta}_{xx}] = V^2[\theta_{xxx}], \tag{4.117}$$

and

$$[\dot{\theta}_{xx}] = -V[\theta_{xxx}]. \tag{4.118}$$

Next, use (4.116)-(4.118) in equation (4.111) to find

$$[\theta_{xxx}]\{-V^3\varepsilon_{\ddot{\theta}} + V^2\varepsilon_{\dot{\theta}_x} - V\varepsilon_{\theta_{xx}}\} + [\theta_{xxx}]\{V^2 q_{\ddot{\theta}} - V q_{\dot{\theta}_x} + q_{\theta_{xx}}\} = 0. \tag{4.119}$$

We require that $[\theta_{xxx}] \neq 0$, otherwise we do not have a discontinuity wave, and then equation (4.119) yields the following equation for the wavespeed V,

$$\varepsilon_{\ddot{\theta}} V^3 - V^2(\varepsilon_{\dot{\theta}_x} + q_{\ddot{\theta}}) + V(\varepsilon_{\theta_{xx}} + q_{\dot{\theta}_x}) - q_{\theta_{xx}} = 0. \tag{4.120}$$

This is thus a cubic equation for V. In general we shall find three solutions for V. To make a comparison with the work of Batra [13] (who only studied discontinuity waves in the linear theory) we suppose the wave, S, is moving into an equilibrium region for which $\theta^+ \equiv$ constant and so $\theta_x^+ = 0, \theta_{xx}^+ = 0, \dot{\theta}_x^+ = 0, \ddot{\theta}^+ = 0$. In this case

$$\left.\frac{\partial q}{\partial \ddot{\theta}}\right|_E = 0, \quad \left.\frac{\partial \varepsilon}{\partial \dot{\theta}_x}\right|_E = 0, \quad \left.\frac{\partial q}{\partial \theta_{xx}}\right|_E = 0. \tag{4.121}$$

[Equations (4.121) follow since by the representation we must have for q_i and ε in the three dimensional theory, terms like $\partial q_i/\partial \ddot{\theta}$ will involve $\theta_{,i}$, etc, and so $\partial q_i/\partial \ddot{\theta}|_E = 0$]. For a wave moving into equilibrium, equation (4.120) reduces to

$$V\{\varepsilon_{\ddot{\theta}} V^2 + \varepsilon_{\theta_{xx}} + q_{\dot{\theta}_x}\} = 0. \tag{4.122}$$

Equation (4.122) has solution

$$V = 0, \quad V = \pm\sqrt{-\frac{q_{\dot{\theta}_x} + \varepsilon_{\theta_{xx}}}{\varepsilon_{\ddot{\theta}}}}, \tag{4.123}$$

Solutions (4.123) correspond to those of Batra [13], his equation (4.7) when we adopt a linear theory and the appropriate expressions for q and ε. Of course, our equations (4.123) are valid for a general nonlinear theory. The only solutions of interest are

$$V = \pm\sqrt{-\frac{q_{\dot{\theta}_x} + \varepsilon_{\theta_{xx}}}{\varepsilon_{\ddot{\theta}}}}. \tag{4.124}$$

Since physically we must have $V^2 > 0$, this imposes the restriction on the derivatives of q and ε, in the sense that we must have

$$\frac{\varepsilon_{\theta_{xx}} + q_{\dot{\theta}_x}}{\varepsilon_{\ddot{\theta}}} < 0. \tag{4.125}$$

Equation (4.125) imposes a restriction on the possible forms of nonlinear functions ε and q may have.

Chapter 5

Thermal waves in a rigid heat conductor

This work is published in Jaisaardsuetrong and Straughan [81]. There has been a huge amount of interest in theories which allow heat to propagate as a wave. For example, extensive reviews of Chandrasekharaiah [22] and Hetnarsky & Ignaczak [69] study several thermal wave theories coupled to the equations of elasticity. There are many theories which allow heat to propagate as a thermal wave and among these we quote the theories of Green & Laws [58] where a generalised temperature $\phi(T, \dot{T})$, T being absolute temperature, is introduced, the two temperature theory of Chen & Gurtin [25], the history-dependent theory of Gurtin & Pipkin [67], the dual phase lag theory of Tzou [183], the τ theory of Cattaneo [19], and its extension to thermoelasticity by Lord & Shulman [109], and the internal variable theory used by Caviglia et al. [20]. We note that the theory of [9] in *the linear case* reduces to that of Lord & Shulman [109], as connectly observed by Chandrasekharaiah [22].

A more recent theory which allows heat to propagate as thermal wave is introduced by Green & Naghdi [60] whose thermodynamics employs *an entropy balance equation*, rather than an entropy inequality, and a thermal displacement variable

$$\alpha(\mathbf{x}, t) = \int_{t_0}^{t} T(\mathbf{x}, s) ds. \qquad (5.1)$$

In equation (5.1) the temperature field T is defined by Green & Naghdi [60] to be an empirical temperature. This temperature is, however, equivalent to the absolute

temperature as shown by Green & Naghdi [[64], pp. 338-340]. The theory of Green & Naghdi [60] has been extended to thermoelasticity and many applications are reviewed in [1] and [2], mainly in a linear setting. The theory of Green & Naghdi [60] relies on a variable θ which they call a temperature, although it is actually a function of α and T, so that $\theta = \theta(\alpha, T) = \theta(\alpha, \dot{\alpha})$. While there may appear some formal similarity with the Green & Laws [58] theory where $\phi = \phi(T, \dot{T})$ the two theories are very different. In fact, most applications to date of the Green & Naghdi [60] theory assume a particular form and work with α rather than θ. In this work the goal is to analyse the full nonlinear theory of Green & Naghdi [60], the theory they refer to as type II, for a rigid body, and we keep $\theta = \theta(\alpha, T)$. We have not seen any study like this before. (It is worth pointing out that both type I and type III heat conduction in a rigid body lead to nonlinear theories, see Green & Naghdi [60], but wave motion in these theories is very different.) We maintain that which of the many heat propagation theories will be useful in a mundane situation will depend on an extensive and rigorous analysis of each theory and the aim in this chapter is in this line.

There are many recent applications of the various theories outlined above, to a variety of problems, mostly in a linear context. While these are too numerous to list we do draw attention to the paper of Jordan & Puri [94] where they make a very useful comparison of the Lord & Shulman [109] model and the Green & Laws [58] model with that of classical Fourier theory. Christov & Jordan [26] analyse the Cattaneo theory as do Franchi & Straughan [51], who also dwell on the fact that a constant relaxation time τ in the Cattaneo theory is not consistent with thermodynamics. Puri & Jordan [139] and Quintanilla & Straughan [152] study wave motion in thermoelastic bodies of type III. The Tzou [183](third order) theory is studied from various angles by Quintanilla & Racke [147], Quintanilla [144] , Jou and Criado-Sancho[19], and Serdyukov[20], while Quintanilla & Straughan [151] show that finite time blow up is possible with third order theory when the thermal conductivity is a nonlinear function of temperature, as it often is in practice. Puri & Jordan [140] analyse the two temperature theory of Gurtin & Pipkin [25] in some depth. Many other recent references may be found in these chapter.

The goal is to perform a complete analysis of acceleration wave motion in a Green & Naghdi [60] rigid solid of type II. By a rigid body we follow the definition of Green & Naghdi [[60], p.177] and such a body is one for which the distance between any two particles remains unchanged whatever the motion maybe. We keep the equations of [10] in their full generality, full nonlinearity, and we completely determine the wavespeed and the amplitude of the wave as a function of time. While the mathematics behind acceleration waves, cf. Chen [24], has been around for some time, the fact that its use reveals very useful understanding of the physics of a problem justifies its use. In this regard we cite the recent papers of Jordan & Puri [92, 93], Jordan [86], Christov *et al.* [27], Fu & Scott [55], Quintanilla & Straughan [152] where acceleration waves are employed, and many other articles dealing with acceleration waves are cited there.

5.1 Basic equations and thermodynamic restrictions

The fundamental equation of Green & Naghdi [60] is the balance of entropy, equation (7.20a), which is

$$\rho\dot{\eta} = \rho\xi + \rho s - p_{i,i}, \tag{5.2}$$

where ρ, η, ξ, s and p_i are, respectively, density, entropy, internal rate of production of entropy per unit mass, external rate of production of entropy per unit mass, and the entropy flux vector. Standard notation is used throughout, so a superposed dot denotes $\partial/\partial t$, subscript comma i denotes $\partial/\partial x_i$, and repeated indices denote summation from 1 to 3.

Fundamental theory are a temperature function $\theta = \theta(T, \alpha)$ such that $\theta > 0$ and $\partial\theta/\partial T > 0$. In addition, Green & Naghdi [[60], p185], show how the energy balance equation may be exploited to deduce restrictions on the thermodynamic variables which may be interpreted as defining η, p_i and ξ in terms of a Helmholtz free energy function $\psi = \psi(\theta, \beta_i)$. The relevant relations follow from Green & Naghdi [60],

equation (8.15), and are

$$\eta = -\frac{\partial \psi}{\partial T}\Big/\frac{\partial \theta}{\partial T}, \tag{5.3}$$

$$p_i = -\rho\frac{\partial \psi}{\partial \beta_i}\Big/\frac{\partial \theta}{\partial T}, \text{ where } \beta_i = \alpha_{,i}, T = \dot{\alpha} \tag{5.4}$$

$$\xi = 2\Lambda\psi_\Lambda\frac{\theta_\alpha}{\theta\theta_T} + \frac{T}{\theta}\left(\frac{\psi_T\theta_\alpha}{\theta_T} - \psi_\alpha\right), \tag{5.5}$$

$$\Lambda = \alpha_{,i}\alpha_{,i} = \beta_i\beta_i, \tag{5.6}$$

and we write e.g. $\psi_\Lambda = \partial\psi/\partial\Lambda$.

For our analysis there is no less in generality in setting the external supply of entropy to be zero and so we henceforth assume $s = 0$. Then, using (5.3)-(5.6), equation (5.2) may be written as

$$-\frac{\partial A}{\partial t} = \xi + \frac{\partial \mu_i}{\partial x_i}, \tag{5.7}$$

where we have defined

$$A(\alpha, \dot{\alpha}, \alpha_{,i}) = A = \frac{\psi_T}{\theta_T} \quad \text{and} \quad \mu_i = \frac{\psi_{\beta_i}}{\theta_T}, \tag{5.8}$$

The above equations represent the general theory of heat flow in a type II rigid body of Green & Naghdi [60]. However, we may wish to consider the following special cases,

$$\psi = c(\theta - \theta\ln\theta) + \frac{k}{2}\beta_i\beta_i \tag{5.9}$$

and

$$\psi = c(\theta - \theta\ln\theta) + \frac{k}{2}\beta_i\beta_i, \quad \theta = a + bT \tag{5.10}$$

c, k, a, b positive constants, where c and k denote the specific heat and heat conductivity. Case (5.10) is considered by Green & Naghdi [60]. Note that, in these cases the theory is still nonlinear. Moreover, from equation $(8.15)_4$ of Green & Naghdi [60], for case (10), $\xi = 0$.

While we concentrate on a rigid heat conductor of type II, we point out that Green & Naghdi [[60], pp.183-188] also define such a heat conductor in type I or type III cases. Very briefly, the key difference is the list of independent constitutive variables. This list comprises T and $T_{,i}$ for type I , T, α and $\alpha_{,i}$ for type II, and $T, \alpha, T_{,i}$ and $\alpha_{,i}$ for type III. Type II leads to a very different equation to both type I and III, the latter types both containing substantial damping.

5.2 Acceleration waves

Define an acceleration wave for the theory of section 5.1 to be a two-dimensional surface, Σ, in $\mathbb{R}^3$, across which $\ddot{\alpha}(\mathbf{x}, \mathbf{t}), \dot{\alpha}_{,i}$, and $\alpha_{,ij}$ suffer a finite discontinuity, but $\alpha \in C^1$ everywhere. The jump, $[f]$, of a function f, across Σ is defined by

$$[f] = f^- - f^+, \tag{5.11}$$

where - and + refer to the limits of f as Σ is approached from the region into which the wave is advancing, or through which it has just passed. The jump is assumed to be even along the wave surface, cf. Chen [24], so that $[f]$ is a function only of t. By expanding (5.7) one may arrive at the following equation

$$-\left(\frac{\partial A}{\partial T}\dot{T} + \frac{\partial A}{\partial \alpha}\dot{\alpha} + \frac{\partial A}{\partial \beta_i}\dot{\beta}_i\right) = \xi + \left(\frac{\partial \mu_i}{\partial T}T_{,i} + \frac{\partial \mu_i}{\partial \alpha}\alpha_{,i} + \frac{\partial \mu_i}{\partial \beta_j}\beta_{j,i}\right). \tag{5.12}$$

After taking the jumps, and since $\alpha \in C^1$ everywhere, then $\dot{\alpha} = 0$ and $\alpha_{,i} = 0$, moreover ξ is continuous then $[\xi] = 0$, hence

$$-\left(\frac{\partial A}{\partial T}[\dot{T}] + \frac{\partial A}{\partial \beta_i}[\dot{\beta}_i]\right) = \frac{\partial \mu_i}{\partial T}[T_{,i}] + \frac{\partial \mu_i}{\partial \beta_j}[\beta_{j,i}]. \tag{5.13}$$

or equivalently,

$$-\frac{\partial A}{\partial T}[\ddot{\alpha}] - \frac{\partial A}{\partial \beta_i}[\dot{\alpha}_{,i}] = \frac{\partial \mu_i}{\partial T}[\dot{\alpha}_{,i}] + \frac{\partial \mu_i}{\partial \beta_j}[\alpha_{,ji}]. \tag{5.14}$$

Using equation (4.15) of Chen [24], then

$$0 = \frac{\delta}{\delta t}[\dot{\alpha}] = [\ddot{\alpha}] + Vn_i[\dot{\alpha}_{,i}], \tag{5.15}$$

$$0 = \frac{\delta}{\delta t}[\alpha_{,i}] = [\dot{\alpha}_{,i}] + Vn_j[\alpha_{,ji}], \tag{5.16}$$

from (5.16) one obtains

$$[\dot{\alpha}_{,i}] = -Vn_j[\alpha_{,ji}], \tag{5.17}$$

and from (5.15) one finds

$$[\ddot{\alpha}] = -Vn_i[\dot{\alpha}_{,i}]. \tag{5.18}$$

Now put (5.17) into (5.18) then

$$[\ddot{\alpha}] = V^2 n_i n_j[\alpha_{,ji}]. \tag{5.19}$$

Recall that

$$\frac{\partial \mu_i}{\partial \beta_j} = \frac{\partial}{\partial \beta_j}\left(\frac{\psi_{\beta_i}}{\theta_T}\right) \text{ since } \theta = \theta(\alpha, \dot\alpha) \text{ then}$$

$$\frac{\partial \mu_i}{\partial \beta_j} = \frac{\psi_{\beta_i \beta_j}}{\theta_T}. \tag{5.20}$$

Use equation (4.14) in Chen [24], obtains

$$[\alpha_{,ij}] = n_i n_j [n^r n^s \alpha_{,rs}], \tag{5.21}$$

then

$$[\dot\alpha_{,i}] = -V n_j n_i n_j [n^r n^s \alpha_{,rs}]$$

$$= -V n_i [n^r n^s \alpha_{,rs}] \tag{5.22}$$

therefore,

$$\frac{\partial \mu_i}{\partial \beta_j}[\alpha_{,ji}] = \frac{\partial \mu_i}{\partial \beta_j} n_i n_j [n^r n^s \alpha_{,rs}]. \tag{5.23}$$

By employing (5.17) and (5.21) in the second term on the left hand side of (5.14) we find

$$-\frac{\partial A}{\partial \beta_i}[\dot\alpha_{,i}] = \frac{\partial A}{\partial \beta_i} V n_k [\alpha_{,ki}] = \frac{\partial A}{\partial \beta_i} V n_i [n^k n^j \alpha_{,kj}]. \tag{5.24}$$

Put (5.19), (5.23) and (5.24) into (5.14), then (5.14) becomes

$$- A_T V^2 [n^i n^j \alpha_{,ji}] + n_i \frac{\partial A}{\partial \beta_i} V [n^r n^s \alpha_{,rs}]$$

$$= -\frac{\partial \mu_i}{\partial T} V n_i [n^p n^q \alpha_{,pq}] + \frac{\partial \mu_i}{\partial \beta_j} n_i n_j [n^p n^q \alpha_{,pq}]. \tag{5.25}$$

Since we assume the amplitude is non-zero, therefore in (5.21) we have $[n^p n^q \alpha_{,pq}] \neq 0$, and hence (5.25) becomes

$$A_T V^2 - V\left(n_i \frac{\partial \mu_i}{\partial T} + n_i \frac{\partial A}{\partial \beta_i}\right) + n_i n_j \frac{\partial \mu_i}{\partial \beta_j} = 0. \tag{5.26}$$

We now wish to develop (5.26) and to do this we recall the form of A, μ_i and Λ from (5.6) and (5.8);

$$A = \frac{\psi_T}{\theta_T}, \quad \mu_i = \frac{\psi_{\beta_i}}{\theta_T}, \text{ and } \Lambda = \beta_i \beta_i,$$

then,

$$A_T = \frac{\psi_{TT}}{\theta_T} - \frac{\psi_T \theta_{TT}}{\theta_T{}^2}. \tag{5.27}$$

Therefore

$$\frac{\partial \mu_i}{\partial T} = \frac{\partial}{\partial T}\left(\frac{\psi_{\beta_i}}{\theta_T}\right) = \frac{\psi_{\beta_i T}}{\theta_T} - \frac{\psi_{\beta_i}\theta_{TT}}{\theta_T{}^2} \quad,\text{where } \psi_{\beta_i} = \frac{\partial \psi}{\partial \Lambda}\frac{\partial \Lambda}{\partial \beta_i} = 2\beta_i\psi_\Lambda,$$

$$\text{then } \frac{\partial \mu_i}{\partial T} = \frac{2\beta_i\psi_{\Lambda T}}{\theta_T} - \frac{2\beta_i\psi_\Lambda\theta_{TT}}{\theta_T{}^2}. \tag{5.28}$$

Now we must calculate A_T, $\partial A/\partial\beta_i$, $\partial\mu_i/\partial T$ and $\partial\mu_i/\partial\beta_j$ in terms of the Helmholtz free energy ψ.

$$\frac{\partial A}{\partial \beta_i} = \frac{\partial A}{\partial \Lambda}\frac{\partial \Lambda}{\partial \beta_i} = \frac{\partial A}{\partial \Lambda}(2\beta_i) = \frac{\partial}{\partial \Lambda}\left(\frac{\psi_T}{\theta_T}\right)(2\beta_i) = \frac{1}{\theta_T}(\psi_{T\Lambda})(2\beta_i) = \frac{2\beta_i}{\theta_T}\psi_{T\Lambda}, \tag{5.29}$$

then derive the last term of (5.26) obtains

$$\frac{\partial \mu_i}{\partial \beta_j} = \frac{\partial}{\partial \beta_j}\left(\frac{\psi_{\beta_i}}{\theta_T}\right) = \frac{\partial}{\partial \beta_j}\left(\frac{2\beta_i\psi_\Lambda}{\theta_T}\right) = \frac{1}{\theta_T}\frac{\partial}{\partial \beta_j}(2\beta_i\psi_\Lambda)$$

$$= \frac{2}{\theta_T}(\delta_{ij}\psi_\Lambda + \beta_i\psi_{\Lambda\Lambda}(2\beta_j)) = \frac{2\delta_{ij}\psi_\Lambda}{\theta_T} + \frac{4\beta_i\beta_j\psi_{\Lambda\Lambda}}{\theta_T},$$

therefore

$$n_i n_j\frac{\partial \mu_i}{\partial \beta_j} = \frac{2\psi_\Lambda}{\theta_T} + 4(\beta_i n_i)^2\frac{\psi_{\Lambda\Lambda}}{\theta_T}. \tag{5.30}$$

Finally, put (5.27), (5.28), (5.29) and (5.30) into (5.26), then the wavespeed equation is

$$V^2\left(\frac{\psi_{TT}}{\theta_T} - \frac{\psi_T\theta_{TT}}{\theta_T{}^2}\right) - 2n_i\beta_i\left\{\frac{2\psi_{T\Lambda}}{\theta_T} - \frac{\psi_\Lambda\theta_{TT}}{\theta_T{}^2}\right\}V$$

$$+ \frac{2\psi_\Lambda}{\theta_T} + 4(\beta_i n_i)^2\frac{\psi_{\Lambda\Lambda}}{\theta_T} = 0. \tag{5.31}$$

This means there are wave moving in each direction with speed V given by the roots of (5.31).

For the special theory (5.9), we can find wavespeed as follows, since

$$\psi = c(\theta - \theta\ln\theta) + \frac{1}{2}k\beta_i\beta_i, \qquad \theta = \theta(T,\alpha)$$

$$\psi_T = \psi_\theta \cdot \theta_T \qquad \psi_{TT} = \psi_\theta\theta_{TT} + \psi_{\theta\theta}\theta_T{}^2$$

$$\psi_\Lambda = \frac{k}{2} \qquad \psi_{\Lambda\Lambda} = 0 \qquad \psi_{T\Lambda} = 0$$

$$\psi_\theta = c(1 - \ln\theta - 1) = -c\ln\theta \qquad \psi_{\theta\theta} = -\frac{c}{\theta}$$

put these equations into (5.31), then the wavespeed equation reduces to

$$V^2\left(\frac{\psi_\theta \theta_{TT} + \theta_T{}^2 \psi_{\theta\theta}}{\theta_T} - \frac{\psi_\theta \theta_{TT}}{\theta_T}\right) - 2n_i\beta_i\left(-\frac{k}{2}\frac{\theta_{TT}}{\theta_T{}^2}V\right) + \frac{k}{\theta_T}, \qquad (5.32)$$

rearrange to obtains

$$V^2(\theta_T \psi_{\theta\theta}) + kn_i\beta_i\frac{\theta_{TT}}{\theta_T{}^2}V + \frac{k}{\theta_T} = 0,$$

or equivalently,

$$-\frac{c\theta_T V^2}{\theta} + kn_i\beta_i\frac{\theta_{TT}}{\theta_T^2}V + \frac{k}{\theta_T} = 0. \qquad (5.33)$$

For the case (5.10), as follows,

$$\psi = c(\theta - \theta\ln\theta) + \frac{k}{2}\beta_i\beta_i, \qquad \theta = a + bT,$$

equation (5.31) reduces further by

$$\theta_T = b, \quad \theta_{TT} = 0,$$

then the wave speed theory for case (5.10) is

$$-\frac{cb}{\theta}V^2 + \frac{k}{b} = 0, \quad \text{or } V^2 = \frac{k\theta}{b^2 c} \qquad (5.34)$$

from which it is clearly seen there are waves moving in opposite directions with speeds $V = \pm b^{-1}\sqrt{k\theta/c}$.

5.3   Homogeneous region

We can calculate the wave amplitude

$$a(t) = [\ddot{\alpha}], \qquad (5.35)$$

exactly, even in the general case studied in section 5.2. However, the key physics is perhaps easier to see by considering a wave moving into a homogeneous region for which

$$\alpha_{,i} = \beta_i \equiv 0. \qquad (5.36)$$

We henceforth restrict attention to this situation.

For the wavespeed we find equation (5.31) reduces to

$$V^2 = 2\psi_\Lambda \left/ \left(\frac{\psi_T \theta_{TT}}{\theta_T} - \psi_{TT} \right) \right. .$$

(5.37)

Under the same conditions, equation (5.33) becomes

$$V^2 = \frac{k\theta}{c\theta_T{}^2},$$

(5.38)

while equation (5.34) remains the same.

5.4 Amplitude behaviour

To calculate the wave amplitude $a(t) = [\ddot{\alpha}]$ one expand (5.7),

$$-(A_T \ddot{\alpha} + A_\alpha \dot{\alpha} + A_\Lambda \dot{\Lambda}) = \xi(\alpha, T, \Lambda) + \frac{\partial \mu_i}{\partial T} \dot{\alpha}_{,i} + \frac{\partial \mu_i}{\partial \alpha} \alpha_{,i} + \frac{\partial \mu_i}{\partial \beta_j} \beta_{j,i}.$$

Then differentiate with respect to t,

$$- (A_T \dddot{\alpha} + \ddot{\alpha}\{A_{TT} \ddot{\alpha} + A_{T\alpha} \dot{\alpha} + A_{T\Lambda} \dot{\Lambda}\}) - (A_\alpha \ddot{\alpha} + \dot{\alpha}\{A_{\alpha T} \ddot{\alpha} + A_{\alpha\alpha} \dot{\alpha} + A_{\alpha\Lambda} \dot{\Lambda}\})$$

$$- (A_\Lambda \ddot{\Lambda} + \dot{\Lambda}\{A_{\Lambda T} \ddot{\alpha} + A_{\Lambda\alpha} \dot{\alpha} + A_{\Lambda\Lambda} \dot{\Lambda}\}) = \xi_\alpha \dot{\alpha} + \xi_T \ddot{\alpha} + \xi_\Lambda \dot{\Lambda}$$

$$+ \frac{\partial \mu_i}{\partial T} \ddot{\alpha}_{,i} + \dot{\alpha}_{,i} \left\{ \frac{\partial^2 \mu_i}{\partial\alpha\partial T} \dot{\alpha} + \frac{\partial^2 \mu_i}{\partial T^2} \ddot{\alpha} + \frac{\partial^2 \mu_i}{\partial\beta_j\partial T} \dot{\beta}_j \right\}$$

$$+ \frac{\partial \mu_i}{\partial \alpha} \dot{\alpha}_{,i} + \alpha_{,i} \left\{ \frac{\partial^2 \mu_i}{\partial\alpha\partial T} \ddot{\alpha} + \frac{\partial^2 \mu_i}{\partial\alpha^2} \dot{\alpha} + \frac{\partial^2 \mu_i}{\partial\beta_j\partial\alpha} \dot{\beta}_j \right\}$$

$$+ \frac{\partial \mu_i}{\partial \beta_j} \dot{\beta}_{j,i} + \beta_{j,i} \left\{ \frac{\partial^2 \mu_i}{\partial\alpha\partial\beta_j} \dot{\alpha} + \frac{\partial^2 \mu_i}{\partial T\partial\beta_j} \ddot{\alpha} + \frac{\partial^2 \mu_i}{\partial\beta_k\partial\beta_j} \dot{\beta}_k \right\}.$$

(5.39)

Recalling that

$$\Lambda = \beta_i \beta_i = \alpha_{,i} \alpha_{,i}$$

$$\dot{\Lambda} = 2\beta_i \dot{\beta}_i = 2\alpha_{,i} \dot{\alpha}_{,i}$$

$$\ddot{\Lambda} = 2\{\alpha_{,i} \ddot{\alpha}_{,i} + \dot{\alpha}_{,i} \dot{\alpha}_{,i}\},$$

then (5.39) becomes,

$$- (A_{TT}\ddot{\alpha}^2 + A_{T\alpha}\ddot{\alpha}\dot{\alpha} + A_{T\Lambda}\ddot{\alpha}(2\alpha_{,i}\dot{\alpha}_{,i}) + A_T\dddot{\alpha}) - (A_{\alpha T}\dot{\alpha}\ddot{\alpha} + A_{\alpha\alpha}\dot{\alpha}\dot{\alpha} + A_{\alpha\Lambda}2\dot{\alpha}\alpha_{,i}\dot{\alpha}_{,i} + A_\alpha\ddot{\alpha})$$

$$- A_{\Lambda T}\ddot{\alpha}(2\alpha_{,i}\dot{\alpha}_{,i}) + A_{\Lambda\alpha}\dot{\alpha}(2\alpha_{,i}\dot{\alpha}_{,i}) + A_{\Lambda\Lambda}(4\alpha_{,i}\dot{\alpha}_{,i}\alpha_{,k}\dot{\alpha}_{,k}) + A_\Lambda 2\{\alpha_{,i}\ddot{\alpha}_{,i} + \dot{\alpha}_{,i}\dot{\alpha}_{,i}\}$$

$$= \xi_\alpha\dot{\alpha} + \xi_T\ddot{\alpha} + \xi_\Lambda(2\alpha_{,i}\dot{\alpha}_{,i})$$

$$+ \frac{\partial\mu_i}{\partial T}\ddot{\alpha}_{,i} + \frac{\partial^2\mu_i}{\partial\alpha\partial T}\dot{\alpha}_{,i}\dot{\alpha} + \frac{\partial^2\mu_i}{\partial T^2}\dot{\alpha}_{,i}\ddot{\alpha} + \frac{\partial^2\mu_i}{\partial\beta_j\partial T}\dot{\alpha}_{,i}\dot{\beta}_j$$

$$+ \frac{\partial\mu_i}{\partial\alpha}\ddot{\alpha}_{,i} + \frac{\partial^2\mu_i}{\partial\alpha\partial T}\alpha_{,i}\ddot{\alpha} + \frac{\partial^2\mu_i}{\partial\alpha^2}\alpha_{,i}\dot{\alpha} + \frac{\partial^2\mu_i}{\partial\beta_j\partial\alpha}\alpha_{,i}\dot{\beta}_j$$

$$+ \frac{\partial\mu_i}{\partial\beta_j}\dot{\beta}_{j,i} + \frac{\partial^2\mu_i}{\partial\alpha\partial\beta_j}\alpha_{,ij}\dot{\alpha} + \frac{\partial^2\mu_i}{\partial T\partial\beta_j}\alpha_{,ij}\ddot{\alpha} + \frac{\partial^2\mu_i}{\partial\beta_k\partial\beta_j}\alpha_{,ij}\dot{\beta}_{,k}.$$

Take the jumps of the resulting equation with take into account that the wave is moving into a homogeneous region, $\alpha_{,i} = 0$ and since $\alpha \in C^1$, then this becomes,

$$- (A_{TT}[\ddot{\alpha}^2] + A_{T\alpha}[\ddot{\alpha}]\dot{\alpha} + A_T[\dddot{\alpha}]) - (A_{\alpha T}\dot{\alpha}[\ddot{\alpha}] + A_\alpha[\ddot{\alpha}]) + 2A_\Lambda[\dot{\alpha}_{,i}\dot{\alpha}_{,i}]$$

$$= \xi_T[\ddot{\alpha}] + \frac{\partial\mu_i}{\partial T}[\ddot{\alpha}_{,i}] + \frac{\partial^2\mu_i}{\partial\alpha\partial T}[\dot{\alpha}_{,i}]\dot{\alpha} + \frac{\partial^2\mu_i}{\partial T^2}[\dot{\alpha}_{,i}\ddot{\alpha}] + \frac{\partial^2\mu_i}{\partial\beta_j\partial T}[\dot{\alpha}_{,i}\dot{\beta}_j] + \frac{\partial\mu_i}{\partial\alpha}[\ddot{\alpha}_{,i}]$$

$$+ \frac{\partial\mu_i}{\partial\beta_j}[\dot{\alpha}_{,ij}] + \frac{\partial^2\mu_i}{\partial\alpha\partial\alpha_{,j}}[\alpha_{,ij}]\dot{\alpha} + \frac{\partial^2\mu_i}{\partial T\partial\beta_j}[\alpha_{,ij}\ddot{\alpha}] + \frac{\partial^2\mu_i}{\partial\beta_k\partial\beta_j}[\alpha_{,ij}\dot{\beta}_k].$$

Recall

$$\mu_i = \frac{\psi_{\beta_i}}{\theta_T} = \frac{1}{\theta_T}\psi_\Lambda\frac{\partial\Lambda}{\partial\beta_i} = \frac{\psi_\Lambda}{\theta_T}2\beta_i,$$

Define "$|_E$" denotes the homogeneous region, therefore in the homogeneous region we have

$$\mu_{i\alpha T}|_E = 2\beta_i\left(\frac{\psi_\Lambda}{\theta_T}\right)_{\alpha T} = 0, \qquad (5.40)$$

also,

$$\mu_{iTT}|_E = 2\alpha_{,i}\left(\frac{\psi_\Lambda}{\theta_T}\right)_{TT} = 0. \qquad (5.41)$$

Furthermore,

$$\mu_{iT\beta_j}|_E = \frac{\partial}{\partial\beta_j}\left\{2\beta_i\left(\frac{\psi_\Lambda}{\theta_T}\right)_T\right\} = 2\delta_{ij}\left(\frac{\psi_\Lambda}{\theta_T}\right)_T + 2\beta_j\left(\frac{\psi_\Lambda}{\theta_T}\right)_{T\Lambda}2\beta_j$$

$$= 2\delta_{ij}\left(\frac{\psi_\Lambda}{\theta_T}\right)_T, \qquad (5.42)$$

and so,

$$\frac{\partial\mu_i}{\partial T}|_E = 2\beta_i\left(\frac{\psi_\Lambda}{\theta_T}\right)_T = 0, \qquad (5.43)$$

similarly,

$$\frac{\partial^2 \mu_i}{\partial \alpha \partial \beta_j}|_E = \frac{\partial}{\partial \beta_j}\left\{2\beta_i\left(\frac{\psi_\Lambda}{\theta_T}\right)_\alpha\right\} = 2\delta_{ij}\left(\frac{\psi_\Lambda}{\theta_T}\right)_\alpha + 4\beta_i\beta_j\left(\frac{\psi_\Lambda}{\theta_T}\right)_{\alpha\Lambda}$$

$$= 2\delta_{ij}\left(\frac{\psi_\Lambda}{\theta_T}\right)_\alpha, \tag{5.44}$$

and,

$$\frac{\partial \mu_i}{\partial \alpha}|_E = 2\beta_i\left(\frac{\psi_\Lambda}{\theta_T}\right)_\alpha = 0. \tag{5.45}$$

Moreover,

$$\frac{\partial^2 \mu_i}{\partial \beta_k \partial \beta_j}\bigg|_E = \frac{\partial^2}{\partial \beta_k \partial \beta_j}\left\{2\beta_i\frac{\psi_\Lambda}{\theta_T}\right\} = \frac{\partial}{\partial \beta_k}\left\{2\delta_{ij}\left(\frac{\psi_\Lambda}{\theta_T}\right) + 4\beta_i\beta_j\left(\frac{\psi_\Lambda}{\theta_T}\right)_\Lambda\right\}$$

$$= 2\delta_{ij}\left(\frac{\psi_\Lambda}{\theta_T}\right)_\Lambda 2\alpha_{,i}\delta_{ik} + 2\frac{\psi_\Lambda}{\theta_T}\frac{\partial}{\partial \beta_k}\delta_{ij}$$

$$+ 4\beta_i\beta_j\left(\frac{\psi_\Lambda}{\theta_T}\right)_{\Lambda\Lambda} 2\alpha_{,i}\delta_{ik} + 4\left(\frac{\psi_\Lambda}{\theta_T}\right)_\Lambda \frac{\partial}{\partial \beta_k}(\beta_i\beta_j)$$

$$= 4\left(\frac{\psi_\Lambda}{\theta_T}\right)_\Lambda (\beta_i\delta_{jk} + \beta_j\delta_{ik}) = 0. \tag{5.46}$$

Also,

$$\frac{\partial \mu_i}{\partial \beta_j}\bigg|_E = 2\delta_{ij}\left(\frac{\psi_\Lambda}{\theta_T}\right) + 4\left(\frac{\psi_\Lambda}{\theta_T}\right)_\Lambda \beta_i\beta_j = 2\delta_{ij}\left(\frac{\psi_\Lambda}{\theta_T}\right).$$

Hence the amplitude equation becomes,

$$- A_{TT}[\ddot{\alpha}]^2 - 2A_{T\alpha}T[\ddot{\alpha}] - A_T[\ddot{\alpha}] - A_\alpha[\ddot{\alpha}] - 2A_\Lambda[\dot{\alpha}_{,i}\dot{\alpha}_{,i}]$$

$$= \xi_T[\ddot{\alpha}] + 2\left(\frac{\psi_\Lambda}{\theta_T}\right)_T\left\{[\dot{\alpha}_{,i}\dot{\alpha}_{,i}] + [\ddot{\alpha}\alpha_{,ii}]\right\}$$

$$+ 2T\left(\frac{\psi_\Lambda}{\theta_T}\right)_\alpha [\alpha_{,ii}] + 2\frac{\psi_\Lambda}{\theta_T}[\dot{\alpha}_{,ii}]. \tag{5.47}$$

The Hadamard relation or condition of kinematical compatibility, see Chen [24], is

$$\frac{\delta}{\delta t}[f] = [f_t] + V n_i[f_{,i}] \tag{5.48}$$

where $\delta/\delta t$ is the derivative of a function on the wave as seen by the observer on the wave. We now use (5.48) and the wave speed equation to remove the $[\ddot{\alpha}]$ and $[\dot{\alpha}_{,ii}]$ terms, therefore

$$\frac{\delta}{\delta t}[\ddot{\alpha}] = [\dddot{\alpha}] + V[n^i\ddot{\alpha}_{,i}], \tag{5.49}$$

$$\frac{\delta}{\delta t}[\dot{\alpha}_{,i}] = [\ddot{\alpha}_{,i}] + V n_j[\dot{\alpha}_{,ij}], \tag{5.50}$$

$$0 = \frac{\delta}{\delta t}[\dot{\alpha}] = [\ddot{\alpha}] + V[n^r \dot{\alpha}_{,r}], \tag{5.51}$$

$$0 = \frac{\delta}{\delta t}[\alpha_{,r}] = [\dot{\alpha}_{,r}] + V[n^j \alpha_{,rj}], \tag{5.52}$$

then (5.51) is

$$[\ddot{\alpha}] = -V[n^r \dot{\alpha}_{,r}] = V^2[n^r n^j \dot{\alpha}_{,rj}], \tag{5.53}$$

and

$$0 = \frac{\delta}{\delta t}[\alpha_{,i}] = [\dot{\alpha}_{,i}] + V[n^k \alpha_{,ik}], \tag{5.54}$$

then

$$[\dot{\alpha}_{,i}] = -V[n^k \alpha_{,ik}], \tag{5.55}$$

$$[n_i \dot{\alpha}_{,i}] = -V n_i n_j [\alpha_{,ij}]. \tag{5.56}$$

Therefore

$$a(t) = [\ddot{\alpha}] = -V[n^i \dot{\alpha}_{,i}] \tag{5.57}$$

or

$$a = V^2 n_i n_j [\alpha_{,ij}]. \tag{5.58}$$

Hence

$$\frac{\delta a}{\delta t} = [\ddot{\alpha}] + V[\ddot{\alpha}_{,i} n^i]. \tag{5.59}$$

Seeking for the term $[\ddot{\alpha}_{,i} n^i]$ in the equation (5.59).

So use equation (4.15) of Chen [24] to obtains

$$\frac{\delta}{\delta t}[n_i \dot{\alpha}_{,i}] = [\ddot{\alpha}_{,i} n^i] + V n_i n_j [\dot{\alpha}_{,ij}], \tag{5.60}$$

but from (5.57) we have $[n_i \dot{\alpha}_{,i}] = -a/V$ then

$$-\frac{\delta}{\delta t}(\frac{a}{V}) = [\ddot{\alpha}_{,i} n^i] + V n_i n_j [\dot{\alpha}_{,ij}].$$

Hence

$$[\ddot{\alpha}_{,i} n^i] = -\frac{\delta}{\delta t}(\frac{a}{V}) - V n_i n_j [\dot{\alpha}_{,ij}]. \tag{5.61}$$

Put (5.61) into (5.59), therefore

$$\frac{\delta a}{\delta t} = [\ddot{\alpha}] + V\left\{ \frac{\delta}{\delta t}(-\frac{a}{V}) - V n_i n_j [\dot{\alpha}_{,ij}] \right\}, \tag{5.62}$$

since

$$V\frac{\delta}{\delta t}\left(-\frac{a}{V}\right) = -\frac{\delta a}{\delta t} + \frac{a}{V}\frac{\delta V}{\delta t},$$

then (5.62) becomes

$$\frac{\delta a}{\delta t} = [\ddot{\alpha}] - \frac{\delta a}{\delta t} + \frac{a}{V}\frac{\delta a}{\delta t} - V^2 n_i n_j[\dot{\alpha}_{,ij}]. \tag{5.63}$$

Therefore

$$V^2 n_i n_j[\dot{\alpha}_{,ij}] = -2\frac{\delta a}{\delta t} + [\ddot{\alpha}] + \frac{a}{V}\frac{\delta V}{\delta t}$$

$$V^2[\dot{\alpha}_{,ii}] = -2\frac{\delta a}{\delta t} + [\ddot{\alpha}] + \frac{a}{V}\frac{\delta V}{\delta t}. \tag{5.64}$$

Now, we can obtain the term $[\dot{\alpha}_{,ii}]$ as follows

$$[\dot{\alpha}_{,ii}] = \frac{1}{V^2}\left\{-2\frac{\delta a}{\delta t} + [\ddot{\alpha}] + \frac{a}{V}\frac{\delta V}{\delta t}\right\}. \tag{5.65}$$

Put (5.65) into the last term of the amplitude equation (5.47), therefore,

$$-A_{TT}[\ddot{\alpha}]^2 - 2A_{T\alpha}T[\ddot{\alpha}] - A_T[\ddot{\alpha}] - A_\alpha[\ddot{\alpha}] - 2A_\Lambda[\dot{\alpha}_{,i}\dot{\alpha}_{,i}]$$

$$= \xi_T[\ddot{\alpha}] + 2\left(\frac{\psi_\Lambda}{\theta_T}\right)_T\left\{[\dot{\alpha}_{,i}\dot{\alpha}_{,i}] + [\ddot{\alpha}\alpha_{,ii}]\right\}$$

$$+ 2T\left(\frac{\psi_\Lambda}{\theta_T}\right)_\alpha[\alpha_{,ii}] + 2\frac{\psi_\Lambda}{\theta_T V^2}\left\{-2\frac{\delta a}{\delta t} + [\ddot{\alpha}] + \frac{a}{V}\frac{\delta V}{\delta t}\right\},$$

By rearranging,

$$-A_{TT}[\ddot{\alpha}]^2 - 2A_{T\alpha}T[\ddot{\alpha}] - A_\alpha[\ddot{\alpha}] - 2A_\Lambda[\dot{\alpha}_{,i}\dot{\alpha}_{,i}] - A_T[\ddot{\alpha}] - 2\frac{\psi_\Lambda}{V^2\theta_T}[\ddot{\alpha}]$$

$$= \xi_T[\ddot{\alpha}] + 2\left(\frac{\psi_\Lambda}{\theta_T}\right)_T\left\{[\dot{\alpha}_{,i}\dot{\alpha}_{,i}] + [\ddot{\alpha}\alpha_{,ii}]\right\} + 2T\left(\frac{\psi_\Lambda}{\theta_T}\right)_\alpha[\alpha_{,ii}]$$

$$+ 2\frac{\psi_\Lambda}{\theta_T V^2}\left\{-2\frac{\delta a}{\delta t} + \frac{a}{V}\frac{\delta V}{\delta t}\right\}, \tag{5.66}$$

or simplicity,

$$-A_{TT}[\ddot{\alpha}]^2 - 2A_{T\alpha}T[\ddot{\alpha}] - A_\alpha[\ddot{\alpha}] - 2A_\Lambda[\dot{\alpha}_{,i}\dot{\alpha}_{,i}] + (-A_T - \frac{2\psi_\Lambda}{\theta_T V^2})[\ddot{\alpha}]$$

$$= \xi_T[\ddot{\alpha}] + 2\left(\frac{\psi_\Lambda}{\theta_T}\right)_T\left\{[\dot{\alpha}_{,i}\dot{\alpha}_{,i}] + [\ddot{\alpha}\alpha_{,ii}]\right\} + 2T\left(\frac{\psi_\Lambda}{\theta_T}\right)_\alpha[\alpha_{,ii}]$$

$$+ 2\frac{\psi_\Lambda}{\theta_T V^2}\left\{-2\frac{\delta a}{\delta t} + \frac{a}{V}\frac{\delta V}{\delta t}\right\}, \tag{5.67}$$

Recall that the wavespeed relation in homogeneous region (5.37) is

$$V^2 = 2\psi_\Lambda \bigg/ \left(\frac{\psi_T \theta_{TT}}{\theta_T} - \psi_{TT} \right),$$

$$\text{or } V^2 A_T + 2\frac{\psi_\Lambda}{\theta_T} = 0,$$

$$\text{or } A_T + 2\frac{\psi_\Lambda}{V^2 \theta_T} = 0, \tag{5.68}$$

therefore this term has been vanished in (5.67), hence the amplitude equation becomes

$$- A_{TT}a^2 - 2A_{T\alpha}Ta - A_\alpha a - 2A_\Lambda[\dot{\alpha}_{,i}\dot{\alpha}_{,i}] = \xi_T a + 2\Big(\frac{\psi_\Lambda}{\theta_T}\Big)_T \Big([\dot{\alpha}_{,i}\dot{\alpha}_{,i}] + [\ddot{\alpha}\alpha_{,ii}]\Big)$$

$$+ 2T\left(\frac{\psi_\Lambda}{\theta_T}\right)_\alpha [\alpha_{,ii}] + 2\frac{\psi_\Lambda}{V^2\theta_T}\left(-2\frac{\delta a}{\delta t} + \frac{a}{V}\frac{\delta V}{\delta t} \right). \tag{5.69}$$

Since the formula for the jump of a product that has been discussed by Lindsay & Straughan [104]: $[fg] = f^+[g] + g^+[f] + [f][g]$, therefore

$$[\dot{\alpha}_{,i}\dot{\alpha}_{,i}] = 2\dot{\alpha}_{,i}^+[\dot{\alpha}_{,i}] + [\dot{\alpha}_{,i}][\dot{\alpha}_{,i}] = [\dot{\alpha}_{,i}][\dot{\alpha}_{,i}], \tag{5.70}$$

the term $2\dot{\alpha}_{,i}^+[\dot{\alpha}_{,i}]$ of (5.70) vanished since the wave is moving ahead of the region, $\dot{\alpha}_{,i}^+ = 0$.

From (5.51) obtains

$$n^r[\dot{\alpha}_{,i}] = -\frac{[\ddot{\alpha}]}{V} = -\frac{a}{V},$$

then,

$$|[\dot{\alpha}_{,i}]| = \frac{|a|}{V} \text{ , hence } [\dot{\alpha}_{,i}][\dot{\alpha}_{,i}] = \frac{a^2}{V^2} = [\dot{\alpha}_{,i}\dot{\alpha}_{,i}],$$

and from (5.58) we have $V^2[\alpha_{,ii}] = a$, therefore $[\alpha_{,ii}] = a/V^2$, moreover

$$[\ddot{\alpha}\alpha_{,ii}] = \ddot{\alpha}^+[\alpha_{,ii}] + \alpha_{,ii}^+[\ddot{\alpha}] + [\ddot{\alpha}][\alpha_{,ii}]$$

$$= [\ddot{\alpha}][\alpha_{,ii}] = (a)\left(\frac{a}{V^2}\right) = \frac{a^2}{V^2}. \tag{5.71}$$

Henceforth (5.69) can be reduced further as follows,

$$- A_{TT}a^2 - 2A_{T\alpha}Ta - A_\alpha a - 2A_\Lambda\frac{a^2}{V^2}$$

$$= \xi_T a + 4\Big(\frac{\psi_\Lambda}{\theta_T}\Big)_T\Big(\frac{a^2}{V^2}\Big) + \frac{2T}{V^2}\left(\frac{\psi_\Lambda}{\theta_T}\right)_\alpha a + \frac{2\psi_\Lambda}{V^2\theta_T}\left(\frac{a}{V}\frac{\delta V}{\delta t} - 2\frac{\delta a}{\delta t} \right). \tag{5.72}$$

Multiply both sides of (5.72) by $V^2\theta_T/2\psi_\Lambda$, then obtains

$$2\frac{\delta a}{\delta t} - \frac{a}{V}\frac{\delta V}{\delta t} - \frac{V^2\theta_T}{2\psi_\Lambda}\Big(A_{TT}a^2 + 2TA_{T\alpha}a + A_\alpha a + 2\frac{A_\Lambda}{V^2}a^2\Big)$$

$$-\frac{V^2\theta_T}{2\psi_\Lambda}\xi_T a - 4\Big(\frac{\psi_\Lambda}{\theta_T}\Big)_T \frac{\theta_T}{2\psi_\Lambda}a^2 - 2T\Big(\frac{\psi_\Lambda}{\theta_T}\Big)_\alpha \frac{\theta_T}{2\psi_\Lambda}a = 0. \tag{5.73}$$

By rearranging (5.73) obtains,

$$2\frac{\delta a}{\delta t} + a\left\{ -\frac{1}{V}\frac{\delta V}{\delta t} - \frac{V^2\theta_T T A_{T\alpha}}{\psi_\Lambda} - \frac{V^2\theta_T A_\alpha}{2\psi_\Lambda} - \frac{V^2\theta_T \xi_T}{2\psi_\Lambda} - \frac{T\theta_T}{\psi_\Lambda}\Big(\frac{\psi_\Lambda}{\theta_T}\Big)_\alpha \right\}$$

$$-\Big(\frac{V^2\theta_T A_{TT}}{2\psi_\Lambda} + \frac{\theta_T A_\Lambda}{\psi_\Lambda} + 2\frac{\theta_T}{\psi_\Lambda}\Big(\frac{\psi_\Lambda}{\theta_T}\Big)_T\Big)a^2 = 0. \tag{5.74}$$

Now the Maxwell relation and the wave speed equation are employed to remove the $[\ddot\alpha]$ and $[\dot\alpha_{,ii}]$ terms and we arrive at the amplitude equation

$$2\frac{\delta a}{\delta t} + \omega a - \zeta a^2 = 0, \tag{5.75}$$

where $\delta/\delta t$ is the derivative of function on the wave as seen by an observer on the wave.

The coefficients ω and ζ are given by

$$\omega = -\frac{1}{V}\frac{\delta V}{\delta t} - \frac{V^2\theta_T T}{\psi_\Lambda}\Big(\frac{\psi_T}{\theta_T}\Big)_{T\alpha} - \frac{V^2\theta_T}{2\psi_\Lambda}\Big(\frac{\psi_T}{\theta_T}\Big)_\alpha - \frac{T\theta_T}{\psi_\Lambda}\Big(\frac{\psi_\Lambda}{\theta_T}\Big)_\alpha$$

$$-\frac{V^2\theta_T}{2\psi_\Lambda}\left\{\Big(2\Lambda\psi_\Lambda\frac{\theta_\alpha}{\theta\theta_T}\Big)_T + \Big(T\frac{\psi_T\theta_\alpha}{\theta\theta_T}\Big)_T - \Big(T\frac{\psi_\alpha}{\theta}\Big)_T\right\}$$

$$=-\frac{1}{V}\frac{\delta V}{\delta t} - \frac{V^2\theta_T T}{\psi_\Lambda}\Big(\frac{\psi_T}{\theta_T}\Big)_{T\alpha} - \frac{V^2\theta_T}{2\psi_\Lambda}\Big(\frac{\psi_T}{\theta_T}\Big)_\alpha - \frac{T\theta_T}{\psi_\Lambda}\Big(\frac{\psi_\Lambda}{\theta_T}\Big)_\alpha$$

$$-\frac{V^2\theta_T}{2\psi_\Lambda}\left\{\Big(T\frac{\psi_T\theta_\alpha}{\theta\theta_T}\Big)_T - \Big(T\frac{\psi_\alpha}{\theta}\Big)_T\right\}, \tag{5.76}$$

$$\zeta = \frac{V^2\theta_T}{2\psi_\Lambda}\Big(\frac{\psi_T}{\theta_T}\Big)_{TT} + \frac{\theta_T}{\psi_\Lambda}\Big(\frac{\psi_T}{\theta_T}\Big)_\Lambda + 2\frac{\theta_T}{\psi_\Lambda}\Big(\frac{\psi_\Lambda}{\theta_T}\Big)_T. \tag{5.77}$$

We observe the effect of ξ is manifest in the last term of (5.76), the one involving the braces:

$$\xi = 2\Lambda\psi_\Lambda\frac{\theta_\alpha}{\theta\theta_T} + \frac{T}{\theta}\Big(\frac{\psi_T\theta_\alpha}{\theta_T} - \psi_\alpha\Big)$$

$$=\frac{T}{\theta}\Big(\frac{\psi_T\theta_\alpha}{\theta_T} - \psi_\alpha\Big) \quad \text{in equilibrium region.} \tag{5.78}$$

The general solution to (5.75) is easily found, cf. Chen [24] as follows.

$$2\frac{\delta a}{\delta t} + \omega a - \zeta a^2 = 0,$$

$$\frac{\delta a}{\delta t} + \alpha a - \beta a^2 = 0, \text{ where } \alpha = \frac{\omega}{2} \text{ and } \beta = \frac{\zeta}{2},$$

$$\frac{\delta a}{\delta t} + \alpha a = \beta a^2,$$

$$\frac{1}{a^2}\frac{\delta a}{\delta t} + \frac{\alpha}{a} = \beta.$$

Let $f = \frac{1}{a}$ then

$$-\frac{\delta f}{\delta t} + \alpha f = \beta,$$

$$\frac{\delta f}{\delta t} - \alpha f = -\beta. \tag{5.79}$$

Use integrating factor $\mu(t) = e^{-\int_0^t \alpha(\tau)d\tau}$, therefore (5.79) becomes

$$\frac{\delta}{\delta t}(\mu f) = -\beta\mu,$$

$$\mu(t)f(t) = -\int_0^t \beta(\tau)\mu(\tau)d\tau + \mu(0)f(0),$$

$$f(t) = -\frac{1}{\mu(t)}\int_0^t \beta(\tau)\mu(\tau)d\tau + \frac{\mu(0)f(0)}{\mu(t)}, \tag{5.80}$$

since

$$f(t) = \frac{1}{a(t)},$$

thus (5.80) becomes

$$a(t) = \frac{\mu(t)}{\mu(0)f(0) - \int_0^t \beta(\tau)\mu(\tau)d\tau},$$

since

$$\mu(0) = e^{-\int_0^0 \alpha(\tau)d\tau} = 1,$$

therefore

$$a(t) = \frac{e^{-\int_0^t \alpha(\tau)d\tau}}{\frac{1}{a(0)} - \int_0^t \beta(\tau)e^{-\int_0^\tau \alpha(s)ds}d\tau}.$$

However, its interpretation is perhaps easier seen by considering the special theories (5.9) and (5.10).

For (5.9), we have

$$\psi = c(\theta - \theta\ln\theta) + \frac{k}{2}\Lambda, \qquad \theta = \theta(T, \alpha),$$

then

$$\psi_\Lambda = \frac{k}{2}$$

$$\psi_T = \psi_\theta \theta_T = c(1 - \ln\theta - 1)\theta_T = -c\ln\theta \cdot \theta_T.$$

$$\psi_\alpha = \psi_\theta \theta_\alpha = c(1 - \ln\theta - 1)\theta_\alpha = -c\ln\theta \cdot \theta_\alpha$$

Hence, in this case,

$$\zeta = \frac{V^2 \theta_T}{k}(-c\ln\theta)_{TT} + \frac{4\theta_T}{k}\left(\frac{k}{2\theta_T}\right)_T = -\frac{V^2\theta_T}{k}c(\ln\theta)_{TT} - 2\frac{\theta_T}{\theta_T^2}\theta_{TT},$$

and from (5.37) we have

$$V^2 = \frac{k\theta}{c}\frac{1}{\theta_T^2},$$

so ζ for special case (5.9) is

$$\zeta = -\frac{\theta}{\theta_T}(\ln\theta)_{TT} - \frac{2\theta_{TT}}{\theta_T} = -\frac{\theta}{\theta_T}\left(\frac{1}{\theta}\cdot\theta_T\right)_T - \frac{2\theta_{TT}}{\theta_T},$$

then,

$$\zeta = \frac{\theta_T}{\theta} - 3\frac{\theta_{TT}}{\theta_T}. \tag{5.81}$$

and,

$$\omega = -\frac{1}{V}\frac{\delta V}{\delta t} - \frac{2\theta_T T V^2}{k}(-c\ln\theta)_{T\alpha} - \frac{V^2\theta_T}{k}(-c\ln\theta)_\alpha$$

$$- \frac{T\theta_T}{k}\left(\frac{k}{\theta_T}\right)_\alpha - \frac{V^2\theta_T}{k}\left\{\Lambda\left(\frac{k\theta_\alpha}{\theta\theta_T}\right)_T + \left(-T\frac{c\ln\theta\cdot\theta_\alpha}{\theta}\right)_T - \left(T\frac{c\ln\theta\cdot\theta_\alpha}{\theta}\right)_T\right\}$$

Put $V^2 = \frac{k\theta}{c}\frac{1}{\theta_T^2}$ then the term ω can be rewritten as

$$\omega = -\frac{1}{V}\frac{\delta V}{\delta t} - \frac{2T\theta_T}{k}\frac{k\theta}{c\theta_T^2}c(-\ln\theta)_{T\alpha} + \frac{c\theta_T}{k}\frac{k\theta}{c\theta_T^2}(\ln\theta)_\alpha - \frac{T\theta_{T\alpha}}{\theta_T}$$

$$- \frac{\theta_T}{k}\left(\frac{k\theta}{c\theta_T^2}\right)\left\{\Lambda k\left(\frac{\theta_\alpha}{\theta\theta_T}\right)_T - c\left(T\frac{\ln\theta\cdot\theta_\alpha}{\theta}\right)_T - c\left(T\frac{\ln\theta\cdot\theta_\alpha}{\theta}\right)_T\right\}$$

$$= -\frac{1}{V}\frac{\delta V}{\delta t} + \frac{2T\theta}{\theta_T}(\ln\theta)_{T\alpha} + \frac{\theta}{\theta_T}(\ln\theta)_\alpha - T\frac{\theta_{T\alpha}}{\theta_T}$$

$$- \frac{\theta\Lambda k}{c\theta_T}\left(\frac{\theta_\alpha}{\theta\theta_T}\right)_T + \frac{2\theta}{\theta_T}\left(T\frac{\ln\theta\cdot\theta_\alpha}{\theta}\right)_T.$$

Since the wave is moving into the homogeneous region, $\alpha_{,i} = 0$, then

$$\omega = -\frac{1}{V}\frac{\delta V}{\delta t} + \frac{2T\theta}{\theta_T}(\ln\theta)_{T\alpha} + \frac{\theta}{\theta_T}(\ln\theta)_\alpha - T\frac{\theta_{T\alpha}}{\theta_T} + \frac{2\theta}{\theta_T}(T\frac{\ln\theta\cdot\theta_\alpha}{\theta})_T. \tag{5.82}$$

The effect of ξ is due to the last term of (5.82).

Finally, in case (5.10), we further assume $T^+=$ constant, i.e. the temperature ahead of the wave is constant, thus

$$\theta = a + bT, \qquad \theta_T = b, \qquad \theta_{TT} = 0$$

$$\text{then } V^2 = \frac{k\theta}{b^2 c} \equiv \text{constant, since } \theta = a + bT \equiv \text{constant,}$$

hence

$$\omega = 0, \quad \zeta = \frac{b}{a + bT}, \tag{5.83}$$

so equation (5.75) becomes

$$\frac{\delta a}{\delta t} - \frac{\zeta}{2}a^2 = 0. \tag{5.84}$$

$$\int \frac{da}{a^2} = \frac{1}{2}\int \zeta dt$$

$$-\frac{1}{a(t)} + \frac{1}{a(0)} = \frac{1}{2}\int_0^t \zeta(s)ds$$

$$\frac{1}{a(t)} = \frac{1}{a(0)} - \frac{1}{2}\int_0^t \zeta(s)ds.$$

Therefore solution is

$$a(t) = \frac{a(0)}{1 - \frac{a(0)}{2}\zeta t}. \tag{5.85}$$

Since $\zeta > 0$, we see that if $a(0) > 0$ then $a(t)$ blows up in a finite time t_0, where t_0 has derived from

$$\frac{a(0)}{2}\int_0^t \zeta(s)ds = 1,$$

$$\int_0^t \zeta(s)ds = \frac{2}{a(0)},$$

$$\int_0^t \frac{b}{a + bT}ds = \frac{2}{a(0)},$$

$$\frac{bt}{a + bT} = \frac{2}{a(0)},$$

$$t = \frac{2(a + bT^+)}{a(0)b} = t_0. \tag{5.86}$$

Thus, t_0 depends on T in the homogeneous region. In the last case we have $\xi = 0$ and it is interesting to note that this may always lead to the thermal shock formation. For a deeper discussion of how an acceleration wave may develop into a shock wave,

see Fu & Scott [55]. To interpret thermal shock formation we note that the aid of Hadamard relation,

$$[\ddot{\alpha}] = -V[\dot{\alpha}_x] = -V[T_x]. \tag{5.87}$$

Since $T^+ = $ constant we may deduce $T_x^+ = 0$ and hence $[\ddot{\alpha}] = -VT_x^-$. The condition $a(0) > 0$ then implies $T_x^- < 0$ and from (5.85) T_x^- increases in absolute value as time increases. Thus, as the wave moves right the temperature gradient increases(negatively) and blow-up is consistent with the thermal shock formation, i.e. a jump in T (or θ).

Chapter 6

Green-Naghdi theory for a fluid.

We now consider a generalization of the theory of chapter 5, that developed by Green and Naghdi [60].

Green and Naghdi [65] extended their theory of [60] to be applicable to a perfect fluid. They, however, adopted a special relation for the Helmholtz free energy of form

$$\psi = \frac{1}{2} m \alpha_{,i} \alpha_{,i} + f(\rho, \theta), \qquad (6.1)$$

where m is a constant. A general theory satisfying the laws of continuum thermodynamics was constructed by Quintanilla and Straughan [155]. Quintanilla and Straughan [155] also developed a fully nonlinear acceleration wave analysis with a very general form for $\psi = \psi(\rho, \theta, \nabla \alpha)$.

In this section we derive the wavespeeds of an acceleration wave when the Helmholtz free energy is more general than (6.1), but less general than that proposed by Quintanilla and Straughan [155]. We assume that

$$\psi = \frac{1}{2} m(\rho, \theta) |\nabla \alpha|^2 + F(\rho, \theta). \qquad (6.2)$$

Here, m and F are allowed to be functions of ρ and θ. Note that again α is a temperature displacement function,

$$\alpha = \int_0^t \theta ds, \qquad (6.3)$$

while ρ is the density.

The equations of Quintanilla and Straughan [155] are those of continuity of mass, the momentum equation, and the energy balance law, namely,

$$\rho_t + v_i \rho_{,i} + \rho v_{i,i} = 0, \tag{6.4}$$

$$\rho(v_{i,t} + v_j v_{i,j}) = -p_{,i} - \frac{1}{2}\frac{\partial}{\partial x_j}\{\rho(\psi_{\alpha,j}\alpha_{,i} + \psi_{\alpha,i}\alpha_{,j})\}, \tag{6.5}$$

$$-\rho(\psi_{\theta,t} + v_i \psi_{\theta,i}) = (\rho\psi_{\alpha,i})_{,i}, \tag{6.6}$$

where the pressure p is given by

$$p = \rho^2 \psi_\rho. \tag{6.7}$$

We note that from (6.2),

$$\psi_\theta = \frac{1}{2}m_\theta|\nabla\alpha|^2 + F_\theta, \tag{6.8}$$

$$\psi_{\alpha,i} = m\alpha_{,i}, \tag{6.9}$$

$$\psi_\rho = \frac{1}{2}m_\rho|\nabla\alpha|^2 + F_\rho, \tag{6.10}$$

$$\psi_{\alpha,j}\alpha_{,i} + \psi_{\alpha,i}\alpha_{,j} = 2m\alpha_{,j}\alpha_{,i}. \tag{6.11}$$

Employing the above relations, equations (6.4)-(6.6) become

$$\rho_t + v_i \rho_{,i} + \rho v_{i,i} = 0,$$

$$\rho(v_{i,t} + v_j v_{i,j}) = -p_{,i} - (\rho m \alpha_{,i}\alpha_{,j})_{,j},$$

$$-\rho\left\{\frac{\partial}{\partial t}\psi_\theta + v_i\frac{\partial}{\partial x_i}\psi_\theta\right\} = (\rho m \alpha_{,i})_{,i}. \tag{6.12}$$

We now analyse the behaviour of an acceleration wave $\mathcal{S}$ for equations (6.12). By an acceleration wave we here mean a surface $\mathcal{S}$ across which $\alpha_{tt}, \alpha_{,ti}, \alpha_{,ij}, v_{i,j}, v_{i,t}, \rho_{,t}, \rho_{,i}$ and higher derivatives posses a finite discontinuity, but $\alpha \in C^1(\mathbb{R}^3)$, $v_i, \rho \in C^0(\mathbb{R}^3)$.

Equations (6.12) are expanded recalling $p = \rho^2\psi_\rho$ and $\psi = \psi(\rho, \theta, \alpha_{,i})$ to find after taking the jump $[\,.\,]$ across $\mathcal{S}$,

$$[\rho_t] + v_i[\rho_{,i}] + \rho[v_{i,i}] = 0, \tag{6.13}$$

$$\rho([v_{i,t}] + v_j[v_{i,j}]) = -p_\theta[\theta_{,i}] - p_\rho[\rho_{,i}] - p_{\alpha,j}[\alpha_{,ji}] - [\rho_{,j}]m\alpha_{,i}\alpha_{,j} - \rho m_\rho[\rho_{,j}]\alpha_{,i}\alpha_{,j}$$
$$- \rho m_\theta[\theta_{,j}]\alpha_{,j}\alpha_{,i} - \rho m\alpha_{,j}[\alpha_{,ij}] - \rho m\alpha_{,i}[\alpha_{,jj}], \tag{6.14}$$

$$- \rho \left\{ \frac{1}{2} m_{\theta\theta}[\theta_t]\alpha_{,k}\alpha_{,k} + \frac{1}{2} m_{\theta\rho}[\rho_t]\alpha_{,k}\alpha_{,k} + m_\theta \alpha_{,i}[\alpha_{,it}] + F_{\theta\theta}[\theta_t] + F_{\theta\rho}[\rho_t] \right\}$$

$$- v_i \rho \left\{ \frac{1}{2} m_{\theta\theta}[\theta_{,i}]\alpha_{,r}\alpha_{,r} + \frac{1}{2} m_{\theta\rho}[\rho_{,i}]\alpha_{,r}\alpha_{,r} + m_\theta \alpha_{,j}[\alpha_{,ji}] + F_{\theta\theta}[\theta_{,i}] + F_{\theta\rho}[\rho_{,i}] \right\}$$

$$= [\rho_{,i}]m\alpha_{,i} + \rho\alpha_{,i}(m_\theta[\theta_{,i}] + m_\rho[\rho_{,i}]) + \rho m[\alpha_{,ii}]. \tag{6.15}$$

We now analyse equations (6.13)-(6.15), but when the accelaration wave $\mathcal{S}$ is moving into an equilibrium region. An equilibrium region is one for which

$$\rho = \rho_0, \ v_i = 0, \ \theta = \theta_0, \ \alpha = \alpha_0.$$

Since in an equilibrium region $\alpha_{,i} \equiv 0$, $v_i = 0$, equations (6.13)-(6.15) reduce to

$$\rho[v_{i,t}] = -p_\theta[\theta_{,i}] - p_\rho[\rho_{,i}], \tag{6.16}$$

$$-\rho F_{\theta\theta}[\theta_t] - \rho F_{\theta\rho}[\rho_t] = \rho m[\alpha_{,ii}], \tag{6.17}$$

$$[\rho_t] + \rho[v_{i,i}] = 0. \tag{6.18}$$

Since $p = \rho^2 \psi_\rho$ we find

$$p_\theta = \frac{\rho^2}{2} m_{\rho\theta}|\nabla\alpha|^2 + \rho^2 F_{\rho\theta}$$

$$p_\rho = |\nabla\alpha|^2 \left\{ \frac{\alpha^2}{2} m_{\rho\rho} + m_\rho\rho \right\} + \rho^2 F_{\rho\rho} + 2\rho F_\rho.$$

In equilibrium, therefore,

$$p_\theta = \rho^2 F_{\theta\rho}, \ \ p_\rho = \rho^2 F_{\rho\rho} + 2\rho F_\rho. \tag{6.19}$$

Define now the wave amplitudes A^i, B and C by

$$A^i = [n_j v_{i,j}], \ \ B = [n^i \rho_{,i}], \ \ C = [n^i n^j \alpha_{,ij}].$$

Use the compatibility relations given in chapter 1 we see

$$[\rho_t] = -VB,$$

$$[v_{i,i}] = A^i n_i,$$

$$[v_{i,t}] = -VA_i,$$

$$[\rho_{,i}] = n_i B,$$

$$[\theta_{,i}] = [\alpha_{,ti}] = -Vn_iC,$$

$$[\theta_t] = [\alpha_{tt}] = V^2 C,$$

$$[\alpha_{,ii}] = C.$$

$$\tag{6.20}$$

Next, employ equations (6.20) in (6.16)-(6.18) and we derive the relations

$$- \rho V A_i = \rho^2 F_{\theta\rho} V n_i C - (\rho^2 F_{\rho\rho} + 2\rho F_\rho) n_i B, \tag{6.21}$$

$$- \rho F_{\theta\theta} V^2 C + \rho F_{\theta\rho} V B = \rho m C. \tag{6.22}$$

$$- V B + \rho A^i n_i = 0, \tag{6.23}$$

For equation (6.23) we see immediately that $\mathcal{S}$ is a longitudinal wave, i.e. $A^i = A n^i$ where $A = [n_i n_j \alpha_{,ij}]$. Equations (6.21)-(6.23) then reduce to a system of three simultaneous equations for the wave amplitudes A, B and C, namely

$$- V A = \rho F_{\theta\rho} V C - (\rho F_{\rho\rho} + 2 F_\rho) B$$

$$- F_{\theta\theta} V^2 C + F_{\theta\rho} V B = m C$$

$$V B - \rho A = 0$$

To require non-zero wave amplitudes A, B, C we require non-vanishing of the determinant

$$\begin{vmatrix} -V & \rho F_{\rho\rho} + 2F_\rho & -\rho F_{\theta\rho} V \\ 0 & V F_{\theta\rho} & -(m + F_{\theta\theta} V^2) \\ -\rho & V & 0 \end{vmatrix} = 0.$$

Expanding this determinant we find

$$-V^4 F_{\theta\theta} + V^2 (\rho^2 F_{\rho\rho} F_{\theta\theta} - m - \rho^2 F_{\theta\rho}^2 + 2\rho F_{\theta\theta} F_\rho) + m\rho^2 F_{\rho\rho} + 2m\rho F_\rho = 0. \tag{6.24}$$

Equation (6.24) is a quadratic equation for V^2. This, in general, allows for a slow wave, speed V_1, and a fast wave, speed V_2. (Note $V_1^2 < V_2^2$, where each fast and slow wave moves in the + and - directions.)

To analyse (6.24) further we may consider a further simplification to the constitutive theory for ψ, equation (6.2). Hence, we suppose

$$\psi = c(\rho)(\theta - \theta \ln \theta) + \frac{1}{2} m(\rho, \theta) \alpha_{,i} \alpha_{,i}. \tag{6.25}$$

Thus, we find

$$F_\theta = -c \ln \theta$$

$$F_{\theta\theta} = -\frac{c(\rho)}{\theta}$$

$$F_\rho = c'(\rho)(\theta - \theta \ln \theta).$$

The solution to the wavespeed equation (6.24) is then

$$
V^2 = -\frac{1}{2F_{\theta\theta}} \Bigg\{ -(\rho^2 F_{\rho\rho} F_{\theta\theta} - m - \rho^2 F_{\theta\rho}^2 + 2\rho F_{\theta\theta} F_\rho)
$$
$$
\pm \big[(\rho^2 F_{\rho\rho} F_{\theta\theta} - m - \rho^2 F_{\theta\rho}^2 + 2\rho F_{\theta\theta} F_\rho)^2 + 4F_{\theta\theta}(m\rho^2 F_{\rho\rho} + 2m\rho F_\rho)\big]^{1/2} \Bigg\}.
$$

If $c = $ constant, so that all the ρ dependence is in $m(\rho, \theta)$, then $F_\rho = 0$,

$$
V^2 = -\frac{1}{2F_{\theta\theta}} \{ m \pm m \} = 0 \quad \text{or} \quad -\frac{m}{F_{\theta\theta}}.
$$

Thus, we find a standing wave, $V = 0$, and one with wavespeed given by

$$
V^2 = -\frac{m}{F_{\theta\theta}} = \frac{m\theta}{c}
$$

6.1 Conclusions

In this thesis we have analysed a variety of models in continuum mechanics which are capable of admitting temperature waves of finite speed (second sound). In particular, in Chapter 1 we saw how an acceleration wave evolves in a rigid body with a heat flux law of Cattaneo type. Chapter 2 investigated the problem of a layer of fluid heated from below when the Cattaneo law is involved. In Chapter 3 we investigated uniqueness, stability and instability together with a class of non - standard problems for a class of rigid bodies using thermodynamics of Green & Laws. Chapter 4 investigated an extension of the Green - Laws model due to Batra. Here we analysed uniqueness, exponential growth, finite time blow - up and discontinuity waves.

In Chapters 5 and 6 we concentrated on a model for a rigid solid and then a fluid when the thermodynamics is based on more recent ideas due to Green and Naghdi.

All of these theories have shown to yield desirable physical properties. Which model is preferable can only be decided by comparision with experimental results.

There are still many open problems. In particular, the theory of type II fluids studied in Chapter 6 is very new. There are many open questions concerning uniqueness, continuous dependence, and other stability issues. Such studies will form part of future work.

Bibliography

[1] J. Alvarez-Ramirez, G. Fernandez-Anaya, F. J. Valdes-Parada, and J. A. Ochoa-Tapia. A high order extension for Cattaneo's diffusion equation. *Physica A - Statistical Mechanics and its Applications*, 368:345–354, 2006.

[2] J. Alvarez-Ramirez, F. J. Valdes-Parada, and J. A. Ochoa-Tapia. A lattice - Boltzmann scheme for Cattaneo's diffusion equation. *Physica A - Statistical Mechanics and its Applications*, 387:1475–1484, 2008.

[3] K. A. Ames and L. E Payne. Asymptotic behavior for two regularizations of the cauchy problem for the backward heat equation. *Mathematical Models and Methods in Applied Sciences*, 8:187–202, 1998.

[4] K. A. Ames, L. E. Payne, and P. W. Schaefer. Energy and pointwise bounds in some non-standard parabolic problems. *Proc. Roy. Soc. Edinburgh A*, 134:1–9, 2004.

[5] K. A. Ames, L. E. Payne, and J. C. Song. On two classes of nonstandard parabolic problems. *Journal of Mathematical Analysis and Applications*, 311:254–267, 2005.

[6] A. M. Anile and V. Romano. Numerical investigation of shock waves in a radiating gas described by a variable Eddington factor. In B. Straughan, R. Greve, H. Ehrentraut, and Y. Wang, editors, *Continuum Mechanics and Applications in Geophysics and the Environment.* Springer, 2001.

[7] S. Bargmann, R. Greve, and P. Steinmann. Simulation of cryovolcanism on Saturns moon Enceladus with the Green - Naghdi theory of thermoelasticity. *Bulletin of Glaciological Research*, 26:23–32, 2008.

[8] S. Bargmann and P. Steinmann. Finite element approaches to non-classical heat conduction in solids. *Comput. Model. Eng. Sci.*, 9:133–150, 2005.

[9] S. Bargmann and P. Steinmann. Theoretical and computational aspects of non-classical thermoelasticity. *Comp. Methods Appl. Mech. Engrg.*, 196:516–527, 2006.

[10] S. Bargmann and P. Steinmann. Classical results for a non-classical theory: Remarks on thermodynamic relations in Green - Naghdi thermo-hyperelasticity. *Cont. Mech. Thermodyn.*, 19:59–66, 2007.

[11] S. Bargmann, P. Steinmann, and P. M. Jordan. On the propagation of second-sound in linear and nonlinear media: Results from Green Naghdi theory. *Physics Letters A,* 372:4418–4424, 2008.

[12] R. C. Batra. A thermodynamic theory of rigid heat conductors. *Arch. Rational Mech. Anal.*, 53:359–365, 1974.

[13] R. C. Batra. On heat conduction and wave propagation in non-simple rigid solids. *Letters in applied and engineering sciences*, 3:97–107, 1975.

[14] P. Brusov, P. Brusov, G. Lawes, C. Lee, A. Matsubara, O. Ishikawa, and P. Majumdar. Novel sound phenomena in superfluid helium in aerogel and other impure superfluids. *Physics Letters A*, 310:311–321, 2003.

[15] T. Buishvili, S. Kekutia, O. Tkeshelashvili, and L. Tkeshelashvili. Theory of sound propagation in superfluid - filled porous media. *Physics Letters A*, 300:672–686, 2002.

[16] R. Cai, C. Gou, and H. Li. Algebraically explicit solutions of unsteady 3-D nonlinear non-Fourier (hyperbolic) heat conduction. *Int. J. Thermal Sciences*, 45:893–896, 2006.

[17] M. Carr. *Convection in porous media flows*. PhD thesis, University of Durham, 2003.

[18] M. Carr. Penetrative convection in a superposed porous - medium - fluid layer via internal heating. *J. Fluid Mech.*, 509:305–329, 2004.

[19] C. Cattaneo. Sulla conduzione del calore. *Atti Sem. Mat. Fis. Modena*, 3:83–101, 1948.

[20] G. Caviglia, A. Morro, and B. Straughan. Thermoelasticity at cryogenic temperatures. *Int. J. Nonlinear Mech.*, 27:251–263, 1992.

[21] D. S. Chandrasekharaiah. Hyperbolic thermoelasticity: a review. *Appl. Mech. Rev.*, 39:355–376, 1986.

[22] D. S. Chandrasekharaiah. Hyperbolic thermoelasticity: a review of recent literature. *Appl. Mech. Rev.*, 51:705–729, 1998.

[23] M. H. Chang, F. Chen, and B. Straughan. Instability of Poiseuille flow in a fluid overlying a porous layer. *J. Fluid Mech.*, 564:287–303, 2006.

[24] P. J. Chen. Growth and decay of waves in solids. In S. Flügge and C. Truesdell, editors, *Handbuch der Physik*, volume VIa/3, pages 303–402. Springer, Berlin, 1973.

[25] P. J. Chen and M. E. Gurtin. On a theory of heat conduction involving two temperatures. *ZAMP*, 19:614–627, 1968.

[26] C. I. Christov and P. M. Jordan. Heat conduction paradox involving second sound propagation in moving media. *Physical Review Letters*, 94:154301-1–154301-4, 2005.

[27] I. Christov, P. M. Jordan, and C. I. Christov. Nonlinear acoustic propagation in homentropic perfect gases: a numerical study. *Phys. Letters A*, 353:273–280, 2006.

[28] I. Christov, P. M. Jordan, and C. I. Christov. Modelling weakly nonlinear acoustic wave propagation. *Quart. Jl. Mech. Appl. Math.*, 60:473–495, 2007.

[29] V. Ciancio and R. Quintanilla. Thermodynamics of materials with internal variables in the context of the Green and Naghdi theories. *Balkan J. Geom. Applications*, 12:16–31, 2007.

[30] M. Ciarletta and D. Iesan. *Non-classical elastic solids*. Longman, New York, 1993.

[31] M. Ciarletta and B. Straughan. Poroacoustic acceleration waves. *Proc. Roy. Soc. London A*, 462:3493–3499, 2006.

[32] M. Ciarletta and B. Straughan. Poroacoustic acceleration waves with second sound. *J. Sound Vibration*, 306:725–731, 2007.

[33] M. Ciarletta and B. Straughan. Thermo-poroacoustic acceleration waves in elastic materials with voids. *J. Math. Anal. Appl.*, 333:142–150, 2007.

[34] M. Ciarletta, B. Straughan, and V. Zampoli. Thermo-poroacoustic acceleration waves in elastic materials with voids without energy dissipation. *Int. J. Engng. Sci.*, 45:736–743, 2007.

[35] V. A. Cimmelli and K. Frischmuth. Gradient generalization to the extended thermodynamic approach and diffusive - hyperbolic heat conduction. *Physica B*, 400:257–265, 2007.

[36] B. D. Coleman, M. Fabrizio, and D. R. Owen. On the thermodynamics of second sound in dielectric crystals. *Arch. Rational Mech. Anal.*, 80:135–158, 1982.

[37] B. D. Coleman, M. Fabrizio, and D. R. Owen. Thermodynamics and the constitutive relations for second sound in dielectric crystals. In J. Serrin, editor, *New perspectives in thermodynamics*. Springer, 1986.

[38] B. D. Coleman and D. C. Newman. Implications of a nonlinearity in the theory of second sound in solids. *Phys. Rev. B*, 37:1492–1498, 1988.

[39] C. Curro, G. Valenti, M. Sugiyama, and S. Taniguchi. Propagation of an acceleration wave in layers of isotropic solids at finite temperatures. *Wave Motion*, pages 1–23, 2008. doi:10.1016/j.wavemoti.2008.09.003.

[40] S. De Cicco and M. Diaco. A theory of thermoelastic materials with voids without energy dissipation. *J. Thermal Stresses*, 25:493–503, 2002.

[41] J. J. Dongarra, B. Straughan, and D. W. Walker. Chebyshev tau - QZ algorithm methods for calculating spectra of hydrodynamic stability problems. *Appl. Numer. Math.*, 22:399–435, 1996.

[42] P. G. Drazin and W. H. Reid. *Hydrodynamic Stability*. Cambridge University Press, Cambridge, 1981.

[43] W. Dreyer and H. Struchtrup. Heat pulse experiments revisited. *Continuum Mech. Thermodyn.*, 5:3–50, 1993.

[44] P. Duhamel. A new finite integral transform pair for hyperbolic conduction problems in heterogeneous media. *Int. J. Heat Mass Trans.*, 44:3307–3320, 2001.

[45] V. A. Eremeyev. Acceleration waves in incompressible elastic media. *Doklady Physics*, 50:204–206, 2005.

[46] M. Fabrizio. *Introduzione alla meccanica razionale e ai suoi metodi matematici*. Zanichelli, Bologna, second edition, 1994.

[47] M. Fabrizio, F. Franchi, and B. Straughan. On a model for thermo-poroacoustic waves. *Int. J. Engng. Sci.*, 46:790–798, 2008.

[48] M. Fabrizio, G. Gentili, and D. W. Reynolds. On rigid linear heat conductors with memory. *Int. J. Engng. Sci.*, 36:765–782, 1998.

[49] M. Fabrizio and A. Morro. *Electromagnetism of continuous media*. Oxford University Press, Oxford, 2003.

[50] G. Fichera. Is the Fourier theory of heat propagation paradoxical? *Rend. Circolo Matem. Palermo*, 41:5–28, 1992.

[51] F. Franchi and B. Straughan. Continuous dependence on the relaxation time and modelling, and unbounded growth, in theories of heat conduction with finite propagation speeds. *J. Math. Anal. Appl*, 185:726–746, 1994.

[52] F. Franchi and B. Straughan. Thermal convection at low temperature. *J. Non-Equilibrium Thermodynamics*, 19:368–374, 1994.

[53] Y. B. Fu and N. H. Scott. Acceleration wave propagation in an inhomogeneous heat conducting elastic rod of slowly varying cross section. *J. Thermal Stresses*, 15:253–264, 1988.

[54] Y. B. Fu and N. H. Scott. One-dimensional shock waves in simple materials with memory. *Proc. Roy. Soc. London A*, 428:547–571, 1990.

[55] Y. B. Fu and N. H. Scott. The transistion from acceleration wave to shock wave. *Int. J. Engng. Sci.*, 29:617–624, 1991.

[56] D. Graffi. Opere scelte. *CNR Gruppo Nazionale per la Fisica Matematica*, 1999.

[57] A. E. Green. A note on linear thermoelasticity. *Mathematika*, 19:69–75, 1972.

[58] A. E. Green and N. Laws. On the entropy production inequality. *Arch. Rational Mech. Anal.*, 45:47–53, 1972.

[59] A. E. Green and N. Laws. On the global entropy production inequality. *The Quarterly Journal of Mechanics and Applied Mathematics*, 25:1–11, 1972.

[60] A. E. Green and P. M. Naghdi. A re-examination of the basic postulates of thermomechanics. *Proc. Roy. Soc. London A*, 432:171–194, 1991.

[61] A. E. Green and P. M. Naghdi. On undamped heat waves in an elastic solid. *J. Thermal Stresses*, 15:253–264, 1992.

[62] A. E. Green and P. M. Naghdi. Thermoelasticity without energy-dissipation. *J. Elasticity*, 31:189–208, 1993.

[63] A. E. Green and P. M. Naghdi. A new thermoviscous theory for fluids. *J. Non-Newtonian Fluid Mech.*, 56:289–306, 1995.

[64] A. E. Green and P. M. Naghdi. A unified procedure of construction of theories of deformable media.i. *Proc. Roy. Soc. London A*, 448:335–342, 1995.

[65] A.E. Green and P. M. Naghdi. An extended theory for incompressible viscous fluid flow. *J. Non-Newtonian Fluid Mech.*, 66:233–255, 1996.

[66] T. Gultop. On the propagation of acceleration waves in incompressible hyperelastic solids. *J. Sound Vibration*, 462:409–418, 2006.

[67] M. E. Gurtin and A. C. Pipkin. A general theory of heat conduction with finite wavespeeds. *Arch. Rational Mech. Anal.*, 31:113–126, 1968.

[68] P. Han, D. Tang, and L. Zhou. Numerical analysis of two-dimensional lagging thermal behaviour under short-pulse-laser heating on surface. *Int. J. Engng. Sci.*, 44:1510–1519, 2006.

[69] R. B. Hetnarski and J. Ignaczak. Generalized thermoelasticity. *J. Thermal Stresses*, 22:451–476, 1999.

[70] A. A. Hill. *Convection problems in porous media*. PhD thesis, University of Durham, 2005.

[71] A. A. Hill. Double-diffusive convection in a porous medium with a concentration based internal heat source. *Proc. Roy. Soc. London A*, 461:561–574, 2005.

[72] A. A. Hill and B. Straughan. A Legendre spectral element method for eigenvalues in hydrodynamic stability. *J. Computational Appl. Math.*, 193:363–381, 2005.

[73] A. A. Hill and B. Straughan. Linear and nonlinear stability thresholds for thermal convection in a box. *Math. Meth. Appl. Sci.*, 29:2123–2132, 2006.

[74] A. A. Hill and B. Straughan. Poiseuille flow of a fluid overlying a porous layer. *J. Fluid Mech.*, 603:137–149, 2008.

[75] C. O. Horgan and R. Quintanilla. Spatial behaviour of solutions of the dual - phase - lag heat equation. *Math. Meth. Appl. Sci.*, 28:43–57, 2005.

[76] D. Iesan. On the theory of heat conduction in micromorphic continua. *Int. J. Engng. Sci.*, 40:1859–1878, 2002.

[77] D. Iesan. *Thermoelastic models of continua.* Kluwer, Dordrecht, 2004.

[78] D. Iesan. Thermopiezoelectricity without energy dissipation. *Proc. Roy. Soc. London A*, 464:631–656, 2008.

[79] D. Iesan and L. Nappa. On the theory of heat for micromorphic bodies. *Int. J. Engng. Sci.*, 43:17–32, 2005.

[80] D. Iesan and A. Scalia. Propagation of singular surfaces in thermo-microstretch continua with memory. *Int. J. Engng. Sci.*, 44:845–858, 2006.

[81] J. Jaisaardsuetrong and B. Straughan. Thermal waves in a rigid heat conductor. *Phys. Letters A*, 36:433–436, 2007.

[82] D. L. Johnson, T. J. Plona, and H. Kajima. Probing porous media with first and second sound. II. Acoustic properties of water saturated porous media. *J. Appl. Phys.*, 76:115–125, 1994.

[83] P. M. Jordan. An analytical study of Kuznetsov's equation: diffusive solitons, shock formation, and solution bifurcation. *Phys. Letters A*, 326:77–84, 2004.

[84] P. M. Jordan. Growth and decay of acoustic acceleration waves in Darcy-type porous media. *Proc. Roy. Soc. London A*, 461:2749–2766, 2005.

[85] P. M. Jordan. Growth and decay of shock and acceleration waves in a traffic flow model with relaxation. *Physica D*, 207:220–229, 2005.

[86] P. M. Jordan. Finite amplitude acoustic traveling waves in a fluid that saturates a porous medium: acceleration wave formation. *Phys. Letters A*, 355:216–221, 2006.

[87] P. M. Jordan. Growth, decay and bifurcation of shock amplitudes under the type-II flux law. *Proc. Roy. Soc. London A*, 463:2783–2798, 2007.

[88] P. M. Jordan. On the growth and decay of transverse acceleration waves on a nonlinear, externally damped string. *J. Sound Vibration*, 311:597–607, 2008.

[89] P. M. Jordan and C. I. Christov. A simple finite difference scheme for modelling the finite-time blow-up of acoustic acceleration waves. *J. Sound Vibration*, 281:1207–1216, 2005.

[90] P. M. Jordan and C. Feuillade. On the propagation of harmonic acoustic waves in bubbly liquids. *Int. J. Engng. Sci.*, 42:1119–1128, 2004.

[91] P. M. Jordan and A. Puri. Digital signal propagation in dispersive media. *J. Appl. Phys.*, 85:1273–1282, 1999.

[92] P. M. Jordan and A. Puri. Growth/decay of transverse acceleration waves in nonlinear elastic media. *Phys. Letters A*, 341:427–434, 2005.

[93] P. M. Jordan and A. Puri. A note on traveling wave solutions for a class of nonlinear viscoelastic media. *Phys. Letters A*, 335:150–156, 2005.

[94] P. M. Jordan and P. Puri. Thermal stresses in a spherical shell under three thermoelastic models. *J. Thermal Stresses*, 24:47–70, 2001.

[95] P. M. Jordan and B. Straughan. Acoustic acceleration waves in homentropic Green and Naghdi gases. *Proc. Roy. Soc. London A*, 462:3601–3611, 2006.

[96] D. D. Joseph and L. Preziosi. Heat waves. *Rev. Mod. Phys.*, 61:41–73, 1989.

[97] D. D. Joseph and L. Preziosi. Heat waves: addendum. *Rev. Mod. Phys.*, 62:375–391, 1990.

[98] D. Jou, J. Casas-Vázquez, and G. Lebon. *Extended irreversible thermodynamics*. Springer, Berlin, second edition, 1996.

[99] D. Jou and M. Criado-Sancho. Thermodynamic stability and temperature overshooting in dual-phase-lag heat transfer. *Phys. Letters A*, 248:172–178, 1998.

[100] N. Kameyama and M. Sugiyama. Analysis of acceleration waves in crystalline solids based on a continuum model incorporating microscopic thermal vibration. *Continuum Mech. Thermodyn.*, 8:351–359, 1996.

[101] W. Kaminski. Hyperbolic heat conduction equation for materials with a nonhomogeneous inner structure. *ASME J. Heat Transfer*, 112:555–560, 1990.

[102] C. Lin and L. E. Payne. Continuous dependence of heat flux on spatial geometry for the generalized Maxwell-Cattaneo system. *ZAMP*, 55:575–591, 2004.

[103] H. Lin and A. J. Szeri. Shock formation in the presence of entropy gradients. *J. Fluid Mech.*, 431:161–188, 2001.

[104] K. A. Lindsay and B. Straughan. Temperature waves in a rigid heat conductor. *J. Appl. Math. Phys(ZAMP)*, 27:653–662, 1976.

[105] K. A. Lindsay and B. Straughan. Acceleration waves and second sound in a perfect fluid. *Arch. Rational Mech. Anal.*, 68:53–87, 1978.

[106] K. A. Lindsay and B. Straughan. Propagation of mechanical and temperature acceleration waves in thermoelastic materials. *ZAMP*, 30:477–490, 1979.

[107] D. Linton-Johnson, T. J. Plona, and H. Kajima. Probing porous media with first and second sound. Acoustic properties of water saturated porous media. *J. Appl. Phys.*, 76:115–125, 1994.

[108] J. S. Loh, I. A. Azid, K. N. Seetharamu, and G. A. Quadir. Fast transient thermal analysis of Fourier and non-Fourier heat conduction. *Int. J. Heat Mass Trans.*, 50:4400–4408, 2007.

[109] H. W. Lord and Y. Shulman. A generalized dynamical theory of thermoelasticity. *J. Mech. Phys. Solids*, 15:299–309, 1967.

[110] A. Marasco. On the first-order speeds in any direction of acceleration waves in prestressed second - order isotropic, compressible, and homogeneous materials. *Mathematical and Computer Modelling*, pages 1–17, 2008. doi:10.1016/j.mcm.2008.07.037.

[111] A. Marasco and A. Romano. On the acceleration waves in second - order elastic, isotropic, compressible, and homogeneous materials. *Mathematical and Computer Modelling*, pages 1–15, 2008. doi:10.1016/j.mcm.2008.06.005.

[112] P. M. Mariano and L. Sabatini. Homothermal acceleration waves in multifield theories of continua. *Int. J. Non-Linear Mech.*, 35:963–977, 2000.

[113] G. B. McFadden, B. T. Murray, and R. F. Boisvert. Elimination of spurious eigenvalues in the Chebyshev tau spectral method. *J. Computational Physics*, 91:228–239, 1990.

[114] A. Mentrelli, T. Ruggeri, M. Sugiyama, and N. Zhao. Interaction between a shock and an acceleration wave in a perfect gas for increasing shock strength. *Wave Motion*, 45:498–517, 2008.

[115] R. Metzler and A. Compte. Stochastic foundation of normal and anomalous Cattaneo - type transport. *Physica A*, 268:454–468, 1999.

[116] R. J. Meyer. Ultrasonic drying of saturated porous solids via second sound. http://www.freepatentsonline.com/6376145.html, 2006.

[117] K. Mitra, S. Kumar, A. Vedavarz, and M. K. Moallemi. Experimental evidence of hyperbolic heat conduction in processed meat. *Trans. ASME, J. Heat Transfer*, 117:568–573, 1995.

[118] A. Morro, L. E. Payne, and B. Straughan. Decay, growth, continuous dependence and uniqueness results in generalized heat conduction theories. *Applicable Analysis*, 38:231–243, 1990.

[119] A. Morro and T. Ruggeri. Second sound and internal energy in solids. *Int. J. Nonlinear Mech.*, 22:27–36, 1987.

[120] A. Morro and T. Ruggeri. Non-equilibrium properties of solids obtained from second - sound measurements. *J. Phys. C: Solid State Phys.*, 21:1743–1752, 1988.

[121] I. Muller. On the entropy inequality. *Archive for Rational Mechanics and Analysis*, 26:118–141, 1967.

[122] I. Muller. The coldness, a universal function in thermoelastic bodies. *Archive for Rational Mechanics and Analysis*, 41:319–332, 1971.

[123] I. Muller. Die kaltefunktion, eine universelle funktion in der thermodynamik viskoser warmeleitender flussigkeiten. *Archive for Rational Mechanics and Analysis*, 40:1–36, 1971.

[124] R. W. Ogden. *Non-linear elastic deformations*. Dover, Mineola, New York, 1997.

[125] S. A. Orszag. Accurate solution of the Orr-Sommerfeld stability equation. *J. Fluid Mech.*, 50:689–703, 1971.

[126] M. Ostoja-Starzewski and J. Trebicki. On the growth and decay of acceleration waves in random media. *Proc. Roy. Soc. London A*, 455:2577–2614, 1999.

[127] L. E. Payne and P. W. Schaefer. Energy bounds for some nonstandard problems in partial differential equations. *Journal of Mathematical Analysis and Applications*, 273:75–92, 2002.

[128] L. E. Payne, P. W. Schaefer, and J. C. Song. Improved bounds for some nonstandard problems in generalized heat conduction. *Journal of Mathematical Analysis and Applications*, 298:325–340, 2004.

[129] L. E. Payne, P. W. Schaefer, and J. C. Song. Some nonstandard problems in generalizized heat conduction. *Z. angew. Math. Phys.*, 56:931–941, 2005.

[130] L. E. Payne and J. C. Song. Phragmén - Lindelöf and continuous dependence type results in generalized heat conduction. *ZAMP*, 47:527–538, 1996.

[131] L. E. Payne and J. C. Song. Continuous dependence on initial-time and spatial geometry in generalized heat conduction. *J. Math. Anal. Appl.*, 214:173–190, 1997.

[132] L. E. Payne and J. C. Song. Continuous dependence on initial-time geometry in generalized heat conduction. *Math. Models Meth. Appl. Sci.*, 7:125–138, 1997.

[133] L. E. Payne and J. C. Song. Convergence results for generalized heat conduction as the relaxation time tends to zero. *J. Math. Anal. Appl.*, 256:175–189, 2001.

[134] L. E. Payne and J. C. Song. Growth and decay in generalized thermoelasticity. *Int. J. Engng. Sci.*, 40:385–400, 2002.

[135] L. E. Payne and J. C. Song. Temporal decay bounds in generalized heat conduction. *J. Math. Anal. Appl.*, 294:82–95, 2004.

[136] L. E. Payne and J. C. Song. Improved spatial decay bounds in generalized heat conduction. *ZAMP*, 56:805–820, 2005.

[137] P. Puri and P. M. Jordan. Stokes's first problem for a dipolar fluid with nonclassical heat conduction. *J. Engng. Math.*, 36:219–240, 1999.

[138] P. Puri and P. M. Jordan. Wave structure in Stokes' second problem for a dipolar fluid with nonclassical heat conduction. *Acta Mechanica*, 133:145–160, 1999.

[139] P. Puri and P. M. Jordan. On the propagation of plane waves in type-iii thermoelastic media. *Proc. Roy. Soc. London A*, 460:3203–3221, 2004.

[140] P. Puri and P. M. Jordan. On the propagation of harmonic plane waves under the two-temperature theory. *Int. J. Engng. Sci.*, 44:1113–1126, 2006.

[141] P. Puri and P. K. Kythe. Discontinuities in velocity gradients and temeprature in the Stokes' first problem with nonclassical heat conduction. *Quart. Appl. Math.*, 55:167–176, 1997.

[142] P. Puri and P. K. Kythe. Stokes' first and second problems for Rivlin - Ericksen fluids with nonclassical heat conduction. *J. Heat Transfer, ASME*, 120:44–50, 1998.

[143] R. Quintanilla. Instability and non-existence in the nonlinear theory of thermoelasticity without energy dissipation. *Continuum Mech. Thermodyn.*, 13:121–129, 2001.

[144] R. Quintanilla. Exponential stability in the dual-phase-lag heat conduction theory. *J. Non-Equilibrium Thermodyn.*, 27:217–227, 2002.

[145] R. Quintanilla. On the impossibility of localization in linear thermoelasticity. *Proc. Roy. Soc. London A*, 463:3311–3322, 2007.

[146] R. Quintanilla and R. Racke. Stability in thermoelasticity of type III. *Discrete and Continuous Dynamical Systems B*, 1:383–400, 2003.

[147] R. Quintanilla and R. Racke. A note on stability in dual-phase-lag heat conduction. *Int. J. Heat Mass Transfer*, 49:1209–1213, 2006.

[148] R. Quintanilla and R. Racke. Qualitative aspects in dual-phase-lag heat conduction. *Proc. Roy. Soc. London A*, 463:659–674, 2007.

[149] R. Quintanilla and R. Racke. A note on stability in three-phase lag heat conduction. *Int. J. Heat Mass Transfer*, 51:24–29, 2008.

[150] R. Quintanilla and B. Straughan. Growth and uniqueness in thermoelasticity. *Proc. Roy. Soc. London A*, 456:1419–1429, 2000.

[151] R. Quintanilla and B. Straughan. Explosive instabilities in heat transmission. *Proc. Roy. Soc. London A*, 458:2833–2837, 2002.

[152] R. Quintanilla and B. Straughan. Discontinuity waves in type III thermoelasticity. *Proc. Roy. Soc. London A*, 460:1169–1175, 2004.

[153] R. Quintanilla and B. Straughan. Bounds for some non-standard problems in porous flow and viscous Green-Naghdi fluids. *Proc. Roy. Soc. London A*, 461:3159–3168, 2005.

[154] R. Quintanilla and B Straughan. Energy bounds for some non-standard problems in thermoelasticity. *Proc. Roy. Soc. London A*, 461:1147–1162, 2005.

[155] R. Quintanilla and B. Straughan. Nonlinear waves in a Green-Naghdi dissipationless fluid. *J. Non-Newtonian Fluid Mech.*, 154:207–210, 2008.

[156] A. Rai. Breakdown of acceleration waves in radiative magnetic fluids. *Defence Science Journal*, 53:425–430, 2003.

[157] K. R. Rajagopal and C. Truesdell. *An introduction to the mechanics of fluids.* Birkhäuser, Basel, 1999.

[158] S. Roy, A. S. V. Murthy, and R. B. Kudenati. A numerical method for the hyperbolic - heat conduction equation based on multiple scale technique. *Appl. Numer. Math.*, pages 1–12, 2008. doi:10.1016/j.apnum.2008.09.001.

[159] T. Ruggeri. The binary mixtures of Euler fluids: a unified theory of second sound phenomena. In B. Straughan, R. Greve, H. Ehrentraut, and Y. Wang, editors, *Continuum Mechanics and Applications in Geophysics and the Environment.* Springer, 2001.

[160] T. Ruggeri and M. Sugiyama. Hyperbolicity, convexity and shock waves in one-dimensional crystalline solids. *J. Phys. A: Math. Gen.*, 38:4337–4347, 2005.

[161] L. Sabatini and G. Augusti. Homothermal acceleration waves in nematic liquid crystals. *Int. J. Solids Struct.*, 38:1227–1242, 2001.

[162] A. Saleh and M. Al-Nimr. Variational formulation of hyperbolic heat conduction problems applying Laplace transform technique. *Int. Communications in Heat and Mass Transfer*, 35:204–214, 2008.

[163] T. Sanderson, C. Ume, and J. Jarzynski. Hyperbolic heat conduction effects caused by temporally modulated laser pulses. *Ultrasonics*, 33:423–427, 1995.

[164] S. I. Serdyukov. A new version of extended irreversible thermodynamics and dual-phase-lag model in heat transfer. *Phys. Letters A*, 281:16–20, 2001.

[165] S. I. Serdyukov, N. M. Voskresenskii, V. K. Bel'nov, and I. I. Karpov. Extended irreversible thermodynamics and generation of the dual-phase-lag model in heat transfer. *J. Non-Equilib. Thermodyn.*, 28:207–219, 2003.

[166] I. Shnaid. Thermodynamically consistent description of heat conduction with finite speed of heat propagation. *Int. J. Heat Mass Transfer*, 46:3853–3863, 2003.

[167] B. Straughan. *Qualitative estimates for some equations in contemporary Continuum Mechanics*. PhD thesis, Heriot-Watt University, 1974.

[168] B. Straughan. Porous convection, the Chebyshev tau method, and spurious eigenvalues. In B. Straughan, R. Greve, H. Ehrentraut, and Y. Wang, editors, *Continuum Mechanics and Applications in Geophysics and the Environment*. Springer, 2001.

[169] B. Straughan. *The energy method, stability, and nonlinear convection*, volume 91 of *Appl. Math. Sci.* Springer, New York, second edition, 2004.

[170] B. Straughan. *Stability, and wave motion in porous media*, volume 165 of *Appl. Math. Sci.* Springer, New York, 2008.

[171] B. Straughan. *Heat waves*. Springer, New York, 2009. to be published.

[172] B. Straughan. Nonlinear acceleration waves in porous media. *Mathematics and Computers in Simulation*, pages 1–9, 2009. in the press.

[173] B. Straughan and F. Franchi. Bénard convection and the Cattaneo law of heat conduction. *Proc. Roy. Soc. Edinburgh A*, 96:175–178, 1984.

[174] B. Straughan and D. W. Walker. Anisotropic porous penetrative convection. *Proc. Roy. Soc. London A*, 452:97–115, 1996.

[175] B. Straughan and D. W. Walker. Two very accurate and efficient methods for computing eigenvalues and eigenfunctions in porous convection problems. *J. Computational Phys.*, 127:128–141, 1996.

[176] B. Straughan and D. W. Walker. Multi component diffusion and penetrative convection. *Fluid Dyn. Res.*, 19:77–89, 1997.

[177] S. Su and W. Dai. Comparison of the solutions of a phase-lagging heat transport equation and damped wave equation with a heat source. *Int. J. Heat Mass Transfer*, 49:2793–2801, 2006.

[178] S. Su, W. Dai, P. M. Jordan, and R. E. Mickens. Comparison of the solutions of a phase-lagging heat transport equation and damped wave equation. *Int. J. Heat Mass Transfer*, 48:2233–2241, 2005.

[179] M. Sugiyama. Statistical mechanical study of wave propagation in crystalline solids at finite temperatures. *Current Topics in Acoustics Research*, 1:139–158, 1994.

[180] C. Truesdell and W. Noll. *The non-linear field theories of mechanics*. Springer, Berlin, second edition, 1992.

[181] C. Truesdell and R. A. Toupin. The classical field theories. In S. Flügge, editor, *Handbuch der Physik*, volume III/1, pages 226–793. Springer, 1960.

[182] D. Y. Tzou. The generalized lagging response in small-scale and high-rate heating. *Int. J. Heat Mass Transfer*, 38:3231–3234, 1995.

[183] D. Y. Tzou. A unified approach for heat conduction from macro to microscales. *J. Heat Transfer*, 117:8–16, 1995.

[184] J. J. Vadasz, S. Govender, and P. Vadasz. Heat transfer enhancement in nanofluids suspensions: possible mechanisms and explanations. *Int. J. Heat Mass Transfer*, 48:2673–2683, 2005.

[185] P. Vadasz. Lack of oscillations in dual - phase - lagging heat conduction for a porous slab subject to heat flux and temperature. *Int. J. Heat Mass Transfer*, 48:2822–2828, 2005.

[186] G. Valenti, C. Curro, and M. Sugiyama. Acceleration waves analysed by a new continuum model of solids incorporating microscopic thermal vibrations. *Continuum Mech. Thermodyn.*, 16:185–198, 2004.

[187] A. Vedavarz, K. Mitra, S. Kumar, and M. K. Moallemi. Effect of hyperbolic heat conduction on temperature distribution in laser irradiated tissue with blood perfusion. *Adv. Biological Heat Mass Transfer, ASME HTD*, 231:7–16, 1992.

[188] M. Webber. *Instability of fluid flows, including boundary slip.* PhD thesis, Durham University, 2007.

[189] M. Webber. Instability of thread - annular flow with small characteristic length to three - dimensional disturbances. *Proc. Roy. Soc. London A*, 464:673–690, 2008.

[190] B. Weingartner, V. A. Osinov, and W. Wu. Acceleration wave speeds in a hypoplastic constitutive model. *Int. J. Non-linear Mech.*, 41:991–999, 2006.

[191] B. Weingartner, V. A. Osinov, and W. Wu. Effect of inherent anisotropy on acceleration wave speeds in hypoplasticity. *Int. J. Engng. Sci.*, 46:286–292, 2008.

[192] G. B. Whitham. *Linear and nonlinear waves.* John Wiley and Sons, New York, 1974.

[193] X. Zhang and E. Zuazua. Decay of solutions of the system of thermoelasticity of type III. *Comm. Contemporary Math.*, 5:25–84, 2003.