Gravity in spacetimes with cosmological constants

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Abstract

This thesis is composed of two parts: gravity in the spacetime with a negative/positive cosmological constant. The first part, which is the negative case, devotes to constructing the IIB supergravity dual solution in AdS/CFT correspondence for $\mathcal{N} = (1,0)$ and $\mathcal{N} = (1/2,0)$ non-anticommutative deformed super Yang-Mills theory. The non-anticommutativity is realised on N D3-branes in certain constant self-dual RR 5-form background fields. These background fields can be sourced by a set of additional D3-branes intersecting the N D3's. By taking the near horizon limit to the brane configurations, the supergravity solutions are obtained. The mapping between the bulk scalar fields and the boundary operators for $\mathcal{N} = (1,0)$ case is investigated, and it is found that the spectrum of a particular class of the BPS operators is not deformed by the non-anticommutativity. The second part is for the positive cosmological constant case. In this part, a black fusiform solution with a positive cosmological const in $d = 5, \mathcal{N} = 4$ de Sitter supergravity is constructed. The solution is obtained via the braneworld Kaluza-Klein reduction ansatz, and preserves half of the de Sitter supersymmetry. It is static, with the gravitational contraction being balanced by the cosmological repulsion. The black fusiform has an event horizon and a cosmological horizon, and is asymptotically non-de Sitter. The horizons are of *an interval* $\times S^2$ topology, and contain singularities at the opposite ends due to the nature of the reduction ansatz. Despite the singularities, the solution exhibits some physically properties compatible with that of the regular asymptotically de Sitter spacetimes. The entropy and mass observe the N-bound

proposal and the maximal mass conjecture respectively. It also carries a charge which contributes to the 1st law of black hole mechanics.

Declaration

The work in this thesis is based on research carried out at the Center for Particle Theory, the Department of Mathematical Sciences, Durham University, UK. The write-up of this thesis is solely my own work. No part of this thesis has been submitted elsewhere for any other degree or qualification.

The composition of this thesis is based on the published papers [1] and [2]. [1] is a collaboration work of my supervisor Prof. Chong-Sun Chu and myself. [2] is collaborated by Prof. Chu, myself, and Dr. Douglas J Smith. Chapter 1, 4, 5, Section 7.2, Chapter 8, and Appendix C contain original work. Chapter 2, 3, 6, Section 7.1, and Appendix A, B are summaries of previous research, for which I claim no originality.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

Quantum description of gravity has been under quest for a long period. String theories appear to be the promising candidates, but we are still far from fully understanding them. In 1997, a duality was proposed by Maldacena [3], which conjectures that the string theory in $(d+1)$-dimensional anti-de Sitter background corresponds to the conformal field theory living on the d-dimensional boundary. The case of most interest is the duality between the string theory in $AdS_5 \times S^5$ to the boundary $\mathcal{N}=4$ super Yang-Mills theory at large N limit, due to the relevance to the physics of our low energy world. The duality relates the bulk theory at weak (strong) coupling to the strongly (weakly) coupled regime of the boundary gauge theory. Our knowledge so far allows us to examine this correspondence explicitly at the supergravity limit of the bulk theory, and in some non-perturbative aspect such as the instanton test. Nevertheless, Maldacena conjecture provides an approach to understand the gauge theory at strong coupling as well as the quantum gravity. The correspondence is realised by a one-to-one mapping between quantities on each side, in particular by identifying the generating functional of the CFT correlators with the extremised classical AdS supergravity action where the bulk solution is subject to certain asymptotic boundary condition [4, 5].

The boundary field theory such as $\mathcal{N}=4$ super Yang-Mills is supersymmetric and conformal. However the low energy dynamics for the elementary particles such as QCD has none of these invariance. To obtain a boundary theory compatible with QCD, supersymmetry and the conformal invariance must be broken in some way. Moreover, QCD exhibits asymptotic freedom and confinement which cannot be observed in pure $\mathcal{N} = 4$ SYM.

There are many scenarios to break supersymmetry. A recently studied one among those is the non-anticommutative deformations to the supersymmetric gauge theory. In the superspace language, such deformation is expressed by deforming the anticommutation relations of the spinor variables parametrising the invariant sub-superspace. Such deformation gives rise to extra terms in the Lagrangian via the star product and the corresponding fraction of supersymmetry is broken. Non-anticommutativity induces non-commutativity, where the commutation relation of the spacetime coordinates are also deformed. Non-commutative field theory has been well-studied previously and the string origin is also known. Non-commutative spacetime on the D-brane arises from the open string in the background NS-NS 2-form field $B_{\mu\nu}$ along the brane worldvolume direction [6, 7]. This facilitates the construction of the supergravity dual for the non-commutative field theory [8, 9]. The string origin of the non-anticommutativity is also understood: the constant selfdual or anti-selfdual background graviphoton field strength (which belongs to the RR sector) induces the deformed superspace on the D3-brane. The supergravity dual for the non-anticommutative gauge theory is however not yet explored before. One of the goals of this thesis is to construct the supergravity dual of $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1/2, 0)$ super Yang-Mills theory.

To break the conformal invariance of the boundary field theory, a natural way is to deform the action by local operators with scaling dimension $\Delta < 4$. Such deformation is called relevant because it is strong in IR and weak in UV, allowing one to deal with the field theory by flowing from a UV fixed point of CFT to the IR where the theory is non-conformal[1].

To break both conformal invariance and supersymmetry at low energy, one can consider the boundary field theory at finite temperature. The Euclidean time is identified by the period $2\pi R$, where the temperature is expressed by $(2\pi R)^{-1}$. By taking the bosons periodic and the fermions anti-periodic, the bosons receive different mass

[1]See e.g. [10] for an pedagogical account of the relevant, irrelevant, and marginal deformations.

corrections due to the finite temperature compared to the fermions, and therefore the supersymmetry is broken. The effective theory has a non-vanishing beta function since some modes which cancel their counterparts to produce the zero beta function are missing. The theory becomes non-conformal. Maldacena conjecture relates the finite temperature configuration on the field theory side to the thermodynamics of black holes in AdS supergravity. For instance, thermal $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions corresponds to the 5-dimensional Schwarzschild-anti-de Sitter black hole background, and the Hawking-Page like transition between the AdS space with a periodic Euclidean time and the Schwarzschild-AdS black hole is mapped to the confinement-deconfinement phase transition in the thermal boundary theory [11].[2]

For the spacetime of more than 4 dimensions without a cosmological constant, besides the black holes, there exist asymptotically flat black ring solutions [109,111,112] in pure Einstein theory or supergravity. Such solutions have horizons of $S^2 \times S^1$ topology. It is rotating in order to balance the gravitational collapse. A neutral black ring and black hole may carry the same mass and angular momentum, which implies that the uniqueness theorem derived from 4-dimensional black holes cannot be generalised to higher dimensions. Although the research on the asymptotically flat black ring is fruitful, the asymptotically anti-de Sitter or de Sitter black rings are yet to be discovered. Since the first law and a Smarr relation are satisfied on the horizon of the asymptotically flat black ring, one expects that the thermodynamical analysis also applies to the AdS black ring, and if such solution does exist, it will also correspond to some finite temperature configuration of the conformal field theory on the AdS boundary. It will be a very interesting question to find out how the topological information of the bulk geometry is encoded in the objects of the boundary field theory. Since the energy density due to the negative cosmological constant doesn't have a counter-balance effect to the gravitational contraction, in principle the rotation is necessary for an AdS black ring, while it is possible for a static dS black ring to exist since the cosmological constant is positive.

[2]For a comprehensive review on this topic, see e.g [12].

Aiming to find the AdS/dS black rings, and inspired by Emparan and Reall's construction of the asymptotically flat solution, we employ the braneworld Kaluza-Klein supergravity reduction ansatz [135] to oxidise the 4-dimensional C-metrics with cosmological constants. We find that the $dS_4 \subset dS_5$ ansatz gives rise to a $\Lambda > 0$ "black fusiform" solution with topology of *an interval* $\times S^2$, whose geometry looks like a pinched black ring, with the curvature singularities located at the two opposite ends of the cosmological horizon. It is asymptotically non-de Sitter and preserves half of $\mathcal{N} = 4$ supersymmetry of 5-dimensional gauged de Sitter supergravity. The metric is supported by a 2-form and two 3-form fields, and the charge associated with the former contributes in the first law of black hole machenics, analogous to the asymptotically flat dipole black ring case in [111]. This solution exhibits physical properties known for the regular spacetime with a positive cosmological constant, such as conforming to the entropic N-bound [153] and implying generalisation of the maximal mass conjecture [146].

This thesis is composed of two parts. The first part devotes to the construction of the supergravity dual for the non-anticommutative deformed supersymmetric gauge theories. We start in Chapter 2 by reviewing the ingredients needed for the AdS/CFT correspondence, including coincident/intersecting D-branes and $\mathcal{N} = 4$ super Yang-Mills theory, as well as the basics of the Maldacena conjecture. The supergravity dual for the non-commutative gauge theory is also introduced as a particular example of AdS/CFT, where the prescription of taking the near horizon limit will be helpful for our construction. Chapter 3 reviews the non-anticommutative deformations to the supersymmetric gauge theories and their string theory realisation. The supergravity dual for $\mathcal{N} = (1,0)$ and $\mathcal{N} = (1/2, 0)$ super Yang-Mills theory, which involves 4 and 8 intersecting D3-branes respectively, are presented in Chapter 4 and 5. The scalar field-operator correspondence are inspected for the $\mathcal{N} = (1,0)$ case, and we will show that the spectrum of the class of boundary operators dual to the "S-mode" remains undeformed.

The second part of this thesis is on constructing the black fusiform with a positive cosmological constant. This part begins by reviewing how the Emparan and Reall's black ring is constructed via the conventional Kaluza-Klein ansatz and the dilaton

C-metric, as well as summarising the properties of the asymptotically flat neutral and dipole black rings in Chapter 6. Chapter 7 introduces the braneworld Kaluza-Klein supergravity reduction, followed by the main result of the black fusiform with a positive cosmological constant. Some interesting aspects of our solution, including the physical properties and the classical instability near the singularities, are discussed in Chapter 8. The results from recent research on the quest of the AdS/dS black rings are also summarised.

Part I

Gravity in the Spacetime with a Negative Cosmological Constant: the AdS/CFT Correspondence for the Non-anticommutative Deformed Super Yang-Mills Theory

Chapter 2

Introduction to AdS/CFT Correspondence

In this chapter we briefly review the key points of AdS/CFT correspondence, which claims that string theory in $AdS_5\times$ is dual to $\mathcal{N}=4$ super Yang-Mills on the boundary. This is a specific equivalence between a theory containing gravity on the AdS background to a conformal gauge theory without gravity, where the correspondence is provided by mappings between objects in AdS supergravity and the counter parts in SYM.

We start by introducing the ingredients of this duality. AdS_5 spacetime arises from the near horizon geometry of D3-branes. In Section 2.1 the coincident and intersecting D-brane solutions are reviewed, as the intersecting brane solutions will play an important role in our research. The topic of preserved supersymmetry associated with D-branes is also included. The low-energy dynamics of open strings on the D3-brane worldvolume is $\mathcal{N}=4$ super Yang-Mills theory, which is summarised in Section 2.2. AdS/CFT correspondence is covered in Section 2.3. Finally in Section 2.4, we present the generalised gauge theory/string theory duality for the noncommutative deformed gauge theory. Later in Chapter 4 and 5, we will apply similar proposal of taking the near horizon limit used in Section 2.4 in constructing the supergravity dual.

2.1 D-Branes

2.1.1 Coincident D3-branes

In string theories, D-branes are extended objects which carry Ramond-Ramond charges [13] and which are defined by mixed boundary conditions of the open strings in the target space. The end-points of the open strings attach to D-branes in the transverse directions to the brane (i.e. Dirichlet boundary conditions), while are free to move in the worldvolume directions (i.e. Neumann boundary conditions) [14]. The general solutions of coincident N Dp-branes can be found in e.g. [15]. In IIA theory Dp-branes exist for p =even, while in IIB p =odd. Here we present the N D3-brane solution [16][1]:

$$
\begin{aligned}
ds^2 &= H^{-1/2}(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + H^{1/2}(dr^2 + r^2 d\Omega_5^2) \ , \\
H &= 1 + \frac{R^4}{r^4} \ , \qquad\qquad R^4 := 4\pi g_s \alpha'^2 N \ , \\
F_5 &= (1 + *)\, dH^{-1} \wedge \epsilon_{3+1} \ ,
\end{aligned}
\tag{2.1.1}
$$

where ϵ_{3+1} is the volume form on the brane worldvolume and $*$ denotes the Hodge dual. α' is related to the string scale l_s by $\alpha' = l_s^2$. The selfdual RR 5-form field is sourced by the N D3-branes. H is sometimes called the harmonic function, as the field equation of F_5, for which H is the solution, is a Laplace equation.

The metric has Poincaré symmetry on the brane worldvolume and $SO(6)$ invariance in the transverse space. There is a horizon at $r = 0$ which coincides with the singularity. R is of the dimension of length and defines a scale for the D3-brane metric. At infinity where $r >> R$, $H \to 1$ and the spacetime is asymptotically flat. In the near horizon region where $r << R$, $H \sim R^4/r^4$ and the spacetime geometry appears as $AdS_5 \times S^5$:

$$
ds^2 = \frac{r^2}{R^2}(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2 \ , \tag{2.1.2}
$$

where here R defines the radius of both S^5 and AdS_5. The RR 5-form flux is on S^5 as well as AdS_5.

[1] In Section 2.1 and 2.2, we use the mostly-plus Lorentzian signature $(-, +, +, ..., +)$.

The D3-brane solution is supersymmetric, as it preserves 16 out of 32 spacetime supersymmetries in IIB background. This can be illustrated in terms of the spinor projection condition arising from worldvolume Kappa-symmetry of the brane probe [24]. A brane probe in this case is a brane placed in the fixed target space background generated by the same type of branes, such that the backreaction of the probe can be neglected, and no extra supersymmetry is broken by it[2].

The supersymmetric Dp-brane action is invariant under the following kappa transformation [22–24],

$$\delta_\kappa \Theta = \frac{1}{2}(1+\Gamma)\kappa \,, \tag{2.1.3}$$

where Θ denotes the spacetime spinor and $\kappa(\sigma)$ is a local fermionic parameter on the brane worldvolume. Γ is the traceless product structure on the brane with $\Gamma^2 = 1$, given below. This transformation rule arises from cancelling the variations of the Born-Infeld term and of the Wess-Zumino term in the D-brane action. (2.1.3) implies, due to overall supersymmetry arising from the combination of κ-transformations $\delta_\kappa\Theta$ and the spacetime supersymmetry transformations $\delta_\epsilon\Theta = \epsilon$, that half of the spacetime fermionic degrees of freedom are projected out by $(1-\Gamma)/2$; i.e.

$$\Gamma\epsilon = \epsilon \tag{2.1.4}$$

is the condition satisfied by the unbroken supersymmetry. As ϵ is a 32-component Majorana-Weyl spinor in the IIB background, this condition preserves 16 out of 32 supersymmetries for the D-brane probe.

In general, the expressions for Γ depend on the embedding of the worldvolume in the target space, Born-Infeld 2-form field strength, and the pullback of background NS-NS 2-form B-field to the worldvolume. Consider the general case where the Dp-brane with coordinates $\xi^{\mu=0,\dots p}$ is embedded in the target space $X^{M=0,\dots,9}$ with the metric G_{MN}. For the brane probe with vanishing worldvolume fields, Γ is expressed

[2]For the unbroken supersymmetry under "heavy" D-branes, one needs to take into account the coupling of the brane to the background and thus the Killing spinor equation, while kappa-symmetry is a worldvolume description. In [24], it is argued that two methods are compatible.

by

$$\Gamma = \begin{cases} (\Gamma_{11})^{\frac{p}{2}+1}\gamma_{(p+1)} & \text{for IIA} , \\ (\sigma_3)^{\frac{p+1}{2}}(i\sigma_2)\otimes\gamma_{(p+1)} & \text{for IIB} , \end{cases} \tag{2.1.5}$$

where $\sigma_{1,2,3}$ are the Pauli matrices, and

$$\gamma_{(p+1)} := \frac{1}{(p+1)!\sqrt{|g|}}\epsilon^{\mu_0\cdots\mu_p}\gamma_{\mu_0\cdots\mu_p} . \tag{2.1.6}$$

$\epsilon^{\mu_0\cdots\mu_p}$ is the Levi-Civita tensor density with values ± 1, and $\gamma_{\mu_0\ldots\mu_p}$ is the totally antisymmetric product of γ_μ, the pullback of 10-dimensional gamma matrices Γ_M:

$$\gamma_\mu = \partial_\mu X^M \Gamma_M , \qquad \{\Gamma_M, \Gamma_N\} = 2\, G_{MN} . \tag{2.1.7}$$

Here $g_{\mu\nu}$ is the pullback of G_{MN}: $g_{\mu\nu} = \partial_\mu X^M \partial_\nu X^N G_{MN}$. It is easy to check that[3]

$$\gamma^2_{(p+1)} = (-1)^{\frac{(p-1)(p+2)}{2}} . \tag{2.1.8}$$

In terms of 16-component spinors, the projection conditions (2.1.4) associated with Dp-branes read

$$\begin{cases} \Gamma\,\epsilon_L = \epsilon_R & \text{for IIA} , \\ \gamma_{(p+1)}\,\epsilon_2 = \epsilon_1 & \text{for IIB} , \end{cases} \tag{2.1.9}$$

where in the IIB case the 32-component spinor is decomposed into the 16-component ones by

$$\epsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \epsilon_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \epsilon_2 . \tag{2.1.10}$$

If we are in a coordinate system of (2.1.1), and introduce the tangent space gamma matrices $\hat{\Gamma}_{\underline{M}}$ via the vielbein $e_M{}^{\underline{M}}$,

$$\Gamma_M = e_M{}^{\underline{M}}\hat{\Gamma}_{\underline{M}} , \qquad \{\hat{\Gamma}_{\underline{M}}, \hat{\Gamma}_{\underline{N}}\} = 2\eta_{\underline{MN}} , \qquad G_{MN} = e_M{}^{\underline{M}} e_N{}^{\underline{N}}\eta_{\underline{MN}} , \tag{2.1.11}$$

then $\gamma_{(p+1)}$ for the projection condition in IIB is expressed in terms of an antisymmetrised product of $\hat{\Gamma}_{\underline{M}}$ by:

$$\gamma_{(p+1)} = \hat{\Gamma}_{[0}\cdots\hat{\Gamma}_{p]} := \hat{\Gamma}_{0\ldots p} , \tag{2.1.12}$$

and the projection condition (2.1.9) becomes

$$\hat{\Gamma}_{0\ldots p}\,\epsilon_2 = \epsilon_1 . \tag{2.1.13}$$

[3](2.1.8) is for the Lorentzian signature. For the Euclidean signature, the RHS is multiplied by an extra -1.

2.1.2 Intersecting D-branes

We will construct the supergravity dual for the non-anticommutative deformed super Yang-Mills theories in terms of intersecting D3-branes in Chapter 4 and 5. It is helpful to review some of the important aspects here. See [15] [17] for more details of the intersecting branes.

In the previous subsection we see that the solutions of a single set of N coincident D3-branes are characterised by harmonic functions of the transverse coordinates. In general (if the solution exists) one can superpose two or more D-brane solutions into orthogonal intersections via a simple method, the *harmonic function rule* [18–20], which states that, for the intersecting brane solution, the metric components are given by the products of the corresponding components from each constituent brane while the field strength components are given by summing up the contributions from each brane. The rule is first implied by the study of various intersections of M2- and M5-branes in 11-dimensional supergravity [17], and applies to many cases of intersecting branes in IIA and IIB via compactification from 11 dimensions as well as S-duality, which transforms NS branes to RR branes, and T-duality, which relates different RR branes [20, 21]. For example, intersecting M2- and M5-branes in 11 dimensions, denoted $M2\perp M5$, can be compactified into $D2\perp D4$ in 10 dimensions, and then T-dualising a relative transverse dimensions within D4-brane gives rise to $D3\perp D3$.

In the following we give the example of how the harmonic function rule applies to the case of two orthogonal intersecting D3-branes, denoted by $3\perp 3$. Suppose two D3-branes intersect in the following way[4]:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
$D3_1$	•	•	•	•						
$D3_2$	•	•			•	•				

(2.1.14)

[4]The locations of the branes in the common transverse dimensions are the same in order for them to intersect.

The corresponding solution is

$$\begin{aligned} ds^2 &= H_1^{-\frac{1}{2}} H_2^{-\frac{1}{2}}(-dx_0^2+dx_1^2) + H_1^{-\frac{1}{2}} H_2^{\frac{1}{2}}(dx_2^2+dx_3^2) \\ &\quad + H_1^{\frac{1}{2}} H_2^{-\frac{1}{2}}(dx_4^2+dx_5^2) + H_1^{\frac{1}{2}} H_2^{\frac{1}{2}}(\sum_{m=6}^{9} dx_m^2) \ , \\ F_5 &= (1+*)\, dH_1^{-1} \wedge dx^{0,1,2,3} + (1+*)\, dH_2^{-1} \wedge dx^{0,1,4,5} \ , \end{aligned}$$

where $dx^{0,1,2,3}$ is the volume form for D3_1-brane. $H_{1,2}$ are the harmonic functions for D$3_{1,2}$-branes. By setting any of the two harmonic functions to 1, the intersecting D3-brane solution reduce back to ordinary D3-branes. Note that the two branes are smeared in the relative transverse directions, i.e. $H_1 = H_1(x_6, x_7, x_8, x_9)$ and $H_2 = H_2(x_6, x_7, x_8, x_9)$. Localised intersecting branes are subject to extra constraints [21].

The result for $D3 \perp D3$ above can be generalised to $3\perp3\perp3$ or $3\perp3\perp3\perp3$ etc, see Chapter 4.

The intersecting brane solution is not necessarily supersymmetric. Consider that a Dp-brane A intersects a Dq-brane B such that there are r_A relative transverse dimensions on brane A and r_B relative transverse dimensions on brane B. The preserved supersymmetry satisfies the projection conditions arising from both D-branes,

$$\Gamma_A \epsilon = \epsilon \ , \qquad \Gamma_B \epsilon = \epsilon \ , \tag{2.1.15}$$

where $\Gamma_{A,B}$ denote the worldvolume product structures associated with brane A, B. For nontrivial cases where $\Gamma_A \neq \Gamma_B$, Γ_A and Γ_B either commute or anti-commute with each other, and the intersecting brane solution is [15]

$$\begin{cases} supersymmetric & \text{for } r_A + r_B = 0 \bmod 4, \quad [\Gamma_A, \Gamma_B] = 0 \ , \\ nonsupersymmetric & \text{for } r_A + r_B = 2 \bmod 4, \quad \{\Gamma_A, \Gamma_B\} = 0 \ , \end{cases} \tag{2.1.16}$$

where in the $\Gamma_A \Gamma_B = -\Gamma_B \Gamma_A$ case all supersymmetry is broken since it leads to $\epsilon = -\epsilon$. For commuting Γ_A and Γ_B, the condition (2.1.15) is equivalent to

$$\Gamma_A \epsilon = \epsilon \ , \qquad \Gamma_A \Gamma_B \epsilon = \epsilon \ . \tag{2.1.17}$$

This implies that the intersecting branes A, B preserve 1/4 of 32 supersymmetries, since Γ_A and Γ_B are both traceless and squared to 1.

For a configuration of n multi-intersecting D-branes, the condition for the solution to be supersymmetric is that all pairs of Γ from each brane commute. In general, if n projection conditions give rise to m independent ones, $\frac{1}{2^m}$ is the fraction of supersymmetry which is preserved.

2.2 $\mathcal{N}=4$ super Yang-Mills theory

D-branes are dynamical objects and the fluctuations on the branes correspond to the states of the open strings ending on them. The lowest excitation state is a $U(1)$ massless gauge field. From the worldvolume point of view, the open string dynamics on the coincident N D3-brane contains a $\mathcal{N}=4$ supersymmetric gauge multiplet and heavy modes with masses $\sim 1/\sqrt{\alpha'}$. At low energy limit where $\alpha' \to 0$, the heavy modes becomes infinitely heavy and the effective dynamics becomes (massless) $d=4$, $\mathcal{N}=4$, $SU(N)$ super Yang-Mills theory. The field content includes a gauge field A_μ, 4 gauginos $\psi_{\alpha A}$ with $A=1,...,4$, and 6 real scalars ϕ^a with $a=1,...,6$. The Lagrangian and more details can be found in e.g. [47] [25].

The $\mathcal{N}=4$ super Yang-Mills theory is superconformally invariant. The superconformal symmetry is conveniently described by $SU(2,2|4)$ group [34], whose subgroups include conformal symmetry, Poincaré supersymmetry, and $SU(4) \sim SO(6)$ R-symmetry. Many properties of $\mathcal{N}=4$ SYM are understood through these symmetries. There is also a discrete global $SL(2,\mathbf{Z})$ invariance.

The conformal nature of $\mathcal{N}=4$ super Yang-Mills is revealed by the fact that there is no mass-dimensionful parameters in the Lagrangian, as the coupling constants are of dimension 0. As a result the scaling dimension of the Lagrangian $\Delta(\mathcal{L})=4$, and the action is scaling invariant. In principle, the d-dimensional ($d \neq 2$) conformal group is spanned by the generators for Lorentz transformations $L_{\mu\nu}$, translations P_μ, dilations D, and the special conformal transformations K_μ, and is isomorphic to the rotation group $SO(d,2)$. For $d=4$ $\mathcal{N}=4$ SYM, the conformal group is $SO(4,2)$, which matches the AdS_5 isometry in the AdS/CFT correspondence. The superconformal group is spanned by 16 Poincaré supercharges $Q, \bar{Q}$, 16 conformal supercharges $S, \bar{S}$, and 15 R-symmetry generators, as well as the bosonic conformal generators. The complete superconformal algebra in 4 dimensions for general $\mathcal{N}$ is given in [32] [33].

A remarkable feature of the conformal symmetry is that it is not anomalous, and no renormalisation scale is introduced at the quantum level. The beta function $\beta(g) = 0$ for all orders of quantum perturbative expansion and thus $\mathcal{N} = 4$ SYM theory is finite [27–30].

Due to lack of a renormalization scale in CFT, there is no quantities such as asymptotic S-matrix. The observables of interest are the correlation functions of gauge invariant operators. Since the conformal group includes Lorentz group as a subgroup, the symmetry imposes more restrictions to the correlators in $\mathcal{N} = 4$ SYM. As a result the 2-point and 3-point functions among scalar operators in CFT's must be of the forms [26]:

$$\langle\, \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \rangle \;\sim\; \frac{\delta_{\Delta_1,\Delta_2}}{|x_1 - x_2|^{2\Delta_1}}\,, \tag{2.2.1}$$

$$\langle\, \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3)\, \rangle \;\sim\; \frac{1}{|x_1 - x_2|^{\Delta-2\Delta_3}|x_2 - x_3|^{\Delta-2\Delta_1}|x_1 - x_3|^{\Delta-2\Delta_2}} \tag{2.2.2}$$

where $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ and $\Delta_{i=1,2,3}$ are the scale dimensions of $\mathcal{O}_{i=1,2,3}$. In the following we briefly summarise the SYM gauge invariant operators $\mathcal{O}$ that are of interest in the context of AdS/CFT.

The operator multiplets in $\mathcal{N} = 4$ SYM are characterised by the scale dimensions Δ which are the eigenvalues of the operator D, and the Lorentz representations. A superconformal multiplet is composed of operators whose scale dimensions Δ are raised and lowered by Q and S by 1/2 respectively, as a result of $[D, Q^A] = -\frac{i}{2}Q^A$ and $[D, S_A] = \frac{i}{2}S_A$, so that the operators in the representation have distinct helicities. The superconformal primary operator is defined as the operator with the lowest scale dimension, annihilated by S. Other operators in the multiplet are the superconformal descendants, derived from the superconformal primary by applying Q and $\bar{Q}$'s.[5] The highest spin in the multiplet is 4. This is the generic representation and is called the *long multiplet*.

[5]On the other hand, a conformal multiplet is composed of operators whose Δ are raised and lowered by 1 by P_μ and K_μ, due to the algebraic relations $[D, P_\mu] = -iP_\mu$ and $[D, K_\mu] = iK_\mu$. The conformal primary operator is the operator annihilated by K_μ. Unitarity of the CFT requires

The superconformal representation of interest in $\mathcal{N} = 4$ SYM is the *short* (or *1/2 BPS*) *multiplet* in which the highest spin is 2, as the chiral primary is annihilated by 8 out of 16 Poincaré supercharges besides by S. Since the corresponding $\{Q, S\}$ also annihilate the chiral primary operators, by the superconformal algebra, their scale dimensions are uniquely determined by the R-symmetry representation and thus are protected from quantum corrections. No anomalous dimension is developed for the chiral primaries [35] [31]. The short multiplet is important in $\mathcal{N} = 4$ SYM because it corresponds to the graviton multiplet in AdS_5 supergravity in AdS/CFT correspondence.

The supersymmetry transformations of SYM fields [47] implies the chiral primary operators in $\mathcal{N} = 4$ SYM must be combinations of the scalar fields with symmetrised $SO(6)$ indices. The simplest gauge invariant 1/2 BPS operator is given by the single trace operator, which transforms as a rank n totally symmetric $SO(6)$ tensor and carries the Dynkin label $[0, n, 0]$ of $SO(6)$ representation:

$$\mathcal{O}_n := \mathrm{tr}\left(\phi^{(a_1} \cdots \phi^{a_n)}\right) \ , \tag{2.2.3}$$

where the trace is taken over $SU(N)$ gauge group in adjoint representation. The most general 1/2 BPS operators are multi-trace operators, which are products of the single trace ones. In AdS/CFT correspondence, the single trace operators in SYM are interpreted as being dual to the single particle states on AdS spacetime [36], and the multiple trace operators are dual to the multi-particle bound-states in supergravity.[6]

Finally we close the review by summarise the $SL(2, \mathbf{Z})$ symmetry of $\mathcal{N} = 4$ SYM. The complex coupling of the theory is given by

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \ , \tag{2.2.4}$$

where g is the Yang-Mills coupling constant and θ is a parameter (the instanton

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the scale dimensions of the local operators to be positive and bounded from below. For example, $\Delta \geq \frac{d-2}{2}$ for the scalar fields [31]. A superconformal primary is also a conformal primary, but not vice versa.

[6]More details for $SU(2, 2|4)$ representations and multiplet shortening of $\mathcal{N} = 4$ SYM and $AdS_5 \times S^5$ can be found in [35] [37–41].

angle) in the Lagrangian. The theory is invariant under general $SL(2, \mathbf{Z})$ transformations,

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \qquad a, b, c, d \in \mathbf{Z} \quad \text{and} \quad ad - bc = 1 , \tag{2.2.5}$$

which can be reduce to two particular transformations, $\tau \longrightarrow \tau + 1$ and $\tau \longrightarrow -\frac{1}{\tau}$. The former is obtained by $\theta \longrightarrow \theta + 2\pi$. The latter is Montonen-Olive duality [42], an electromagnetic or strong-week duality.

2.3 AdS/CFT correspondence

2.3.1 The Idea

AdS/CFT correspondence was first proposed by Maldacena [3] as a conjecture which states that string theory or M theory on the $AdS_m \times M_{D-m}$ background with $D = 10$ or 11 is dual to the conformal field theory on the boundary of AdS_m. The example of most interest is the duality of IIB supergravity on $AdS_5 \times S^5$ to $\mathcal{N} = 4$ super Yang-Mills theory living on the AdS_5 boundary. The review in this section will focus on this particular duality. AdS/CFT correspondence proposes a mathematical equivalence between a theory containing gravity to a gauge theory without gravity. It is a specific realisation of the underlying holographic principle suggested by 't Hooft [43–45], which claims the phenomena inside a $(d + 1)$-dimensional volume can be described by the degrees of freedom on its d-dimensional boundary as a holographic projection image.

The idea of AdS/CFT correspondence comes from the following observation for N coincident D3-branes in the asymptotically flat background at low energy limit. From the brane worldvolume perspective, heavy modes with masses $\sim 1/\sqrt{\alpha'}$ on the brane disappear, and also the close strings decouple from the open strings, so the dynamics reduces to $\mathcal{N} = 4$ SYM on the brane. From the supergravity point of view, at low energy the dynamics reduces to close strings in the near horizon region of D3-branes, which is $AdS_5 \times S^5$ [10]. Therefore it's natural to identify the two description as being dual to each other and depicting the same physics.

On the supergravity side, the low energy limit is obtained by taken $\alpha' \longrightarrow 0$, which means the string scale is taken to 0 and all massive string modes decouple. At this limit only the supergravity in $AdS_5 \times S^5$ survives. As α' is of mass dimension -2, all dimensionless parameters should be kept fixed while taking the limit:

$$\alpha' \to 0 \ , \qquad U = \frac{r}{\alpha'} = \text{fixed} \ , \tag{2.3.1}$$

and the near horizon metric of D3-branes in (2.1.1) becomes

$$\frac{ds^2}{\alpha'} = \frac{U^2}{\sqrt{\lambda}}(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + \sqrt{\lambda}\,\frac{dU^2}{U^2} + \sqrt{\lambda}\,d\Omega_5^2 \ , \tag{2.3.2}$$

where

$$\lambda := \frac{R^4}{\alpha'^2} = \frac{R^4}{l_s^4} = 4\pi g_s N \tag{2.3.3}$$

is the 't Hooft coupling which is also kept fixed. Maldacena's original proposal is for large N, where the supergravity solution is trustable. This metric together with the rescaled RR 5-form flux define the supergravity dual solution for $SU(N)$ $\mathcal{N}=4$ super Yang-Mills theory on the boundary.

AdS/CFT correspondence relates two theories such that, at large N limit, when one is weakly coupled, the other is strongly coupled. This can be observed as follows [10]. The perturbative description of super Yang-Mills theory is reliable in the regime

$$\lambda = g_{YM}^2 N = \frac{R^4}{l_s^4} << 1 \ . \tag{2.3.4}$$

On the other hand, the supergravity description is good for the string theory when the AdS radius R is much greater than the string scale,

$$\lambda = \frac{R^4}{l_s^4} = g_s N >> 1 \ . \tag{2.3.5}$$

This complementarity poses difficulty on testing the Maldacena conjecture, but it sheds new light on understanding the gauge theories at strong coupling as well as quantum aspect of gravity. Maldacena conjecture inspired fruitful research during the past decade. Detailed reviews of this topic and recent developments are given in [10, 46–50].

2.3.2 The correspondence

The equivalence of AdS supergravity and CFT is provided by precise maps between quantities on both sides, including the global symmetries, and supergravity fields vs. CFT operators. Moreover the expressions of CFT correlators are derived by means of the propagators in AdS supergravity.

The global symmetry for $\mathcal{N}=4$ SYM and supergravity on $AdS_5 \times S^5$ are both $SU(2,2|4)$: for the bosonic part, the global symmetry $SO(4,2)\times SO(6)$ of $\mathcal{N}=4$ SYM is precisely the isometry group of $AdS_5\times S^5$; in the fermionic part, there are 16 Poincaré plus 16 conformal supercharges in $\mathcal{N}=4$ SYM, which matches the number of 32 spacetime supercharges on $AdS_5 \times S^5$. Moreover, Montonen-Olive $SL(2,\mathbf{Z})$ invariance on CFT side also finds its counter part in IIB string theory on AdS side.

There is also a map between the irreducible representations of $SU(2,2|4)$ on both sides. In AdS_5 supergravity, the physical fields arise from KK compactification on S^5, since in the supergravity regime $R >> l_s$, the string excitations of masses $\sim 1/l_s$ are far heavier than the KK states of masses $\sim 1/R$ and are eliminated in the low energy description. The highest spin of KK states is 2, so they fall in the short representations of $SU(2,2|4)$, which correspond to the 1/2 BPS (or, short) multiplets in $\mathcal{N}=4$ super Yang-Mills theory. For completeness we quote the correspondence between the $SU(2,2|4)$ representations of both sides from [47] in Table 2.1. The mapping of the descendant fields in the $SU(2,2|4)$ spectrum on both sides can be found in [51] [52].

The AdS mass of KK fields of various spins arising from compactification on S^5 are summarised in [47] [53]. The mass of scalar fields is given by

$$m_\Delta^2 = \Delta(\Delta-4)/R^2 \ , \tag{2.3.6}$$

where Δ is the integer associated with the S^5 harmonics, and is identified with the scale dimension of the CFT scalar operators via AdS/CFT correspondence.

Before reviewing the AdS description of CFT correlation functions, we first clarify the boundary structure of AdS_{d+1} metric and of the bulk fields. With Euclidean signature, the AdS_{d+1} metric can be expressed by

$$ds^2 = \frac{1}{z^2}\left(d\vec{x}^2 + dz^2\right) \ , \tag{2.3.7}$$

where $\vec{x}$ denotes x^μ with $\mu = 0, ..., d-1$. This metric can be derived from the Euclidean version of (2.3.2) by substituting

$$z = \frac{\sqrt{\lambda}}{U} \ , \tag{2.3.8}$$

IIB string theory	$\mathcal{N}=4$ super-Yang-Mills
Supergravity Excitations 1/2 BPS, spin ≤ 2	Chiral primary + descendants $\mathcal{O}_2 = \mathrm{tr}\phi^{(a}\phi^{b)}$ + descendants
Supergravity KK fields 1/2 BPS, spin ≤ 2	Chiral primary + descendants $\mathcal{O}_\Delta = \mathrm{tr}\phi^{(a_1}\cdots\phi^{a_\Delta)}$ + descendants
Massive string modes non-chiral, long multiplets	Non-Chiral operators, dimensions $\sim \lambda^{1/4}$ e.g. Konishi $\mathrm{tr}\phi^a\phi^b$
n Multiparticle states	Products of operators at distinct points $\mathcal{O}_{\Delta_1}(x_1)\cdots\mathcal{O}_{\Delta_n}(x_n)$
Bound states	Multi-trace operators $\mathcal{O}_{\Delta_1}(x)\cdots\mathcal{O}_{\Delta_n}(x)$

Table 2.1: Mapping of string/supergravity fields and SYM operators

and then rescaling the *AdS* radius to 1 so that the metric is independent of the 't Hooft coupling.

The *AdS* spacetime is described by the upper-half region $z \geq 0$ in (2.3.7). The boundary locates at $z = 0$, where the metric is singular, and $z \to \infty$, where the spacetime is of infinitely small size. Similarly, some field modes also diverge on the boundary $z = 0$. The boundary quantities can be defined by cancelling the divergence by multiplying to the bulk quantities a function f, which is positive in the bulk and has a simple zero on the boundary, to an appropriate power. This is because there is a conformal structure on the *AdS* boundary, so f is not unique. Take the *AdS* metric as an example, one can multiply, e.g. $f^2 = z^2$, to the bulk metric, giving rise to the boundary metric $\widetilde{ds}^2 = f^2 ds^2$ for $z \to 0$. The boundary conformal structure is revealed in the freedom to replace f with $e^w f$, such that $\widetilde{ds}^2 \to e^{2w}\widetilde{ds}^2$ still defines the boundary metric.

For the scalar fields of dimension Δ in the *AdS* bulk, analysis shows that there are two independent asymptotic solutions to the field equation $(-\nabla^2 + m^2)\,\phi = 0$ with $m^2 = \Delta\,(\Delta - d)$:

$$\phi\,(\vec{x}, z \to 0) \;\sim\; z^{\Delta} \quad \text{or} \quad z^{d-\Delta}\,. \tag{2.3.9}$$

We have $\Delta > d$ if we focus on the modes with $m^2 > 0$ here[7], where z^Δ vanishes while $z^{d-\Delta}$ diverges at $z \to 0$, i.e. z^Δ mode is normalisable and $z^{d-\Delta}$ mode is non-normalisable in AdS supergravity. The former is interpreted to relate to the vev's of bulk excitations, while the latter describes the coupling of external sources (on the boundary) to the bulk fields. The boundary value ϕ_0 of ϕ can be defined by

$$\phi_0 = f^{\Delta-d}\phi = z^{\Delta-d}\phi \ , \tag{2.3.10}$$

which also has the same conformal freedom described above.

Having the asymptotic behavior of AdS bulk field, we now turn to the supergravity description of the operator correlation functions in $\mathcal{N} = 4$ SYM. The mapping is given by identifying the generating functional of the CFT correlators with the extremum of the AdS supergravity action:

$$-\log \langle e^{\int_{\partial(AdS)} \phi_0 \mathcal{O}_\Delta} \rangle_{CFT} = S_{AdS}[\phi(\vec{x}, z)|_{z\to 0}] \ , \tag{2.3.11}$$

where the extremum is obtained by evaluating S_{AdS} with a unique classical solution ϕ to the field equation, with the boundary value $\phi_0(\vec{x})$ [5] [4]. According to this prescription, ϕ_0 is the external source to the field ϕ propagating in the AdS bulk, and also the source coupling to the operator $\mathcal{O}_\Delta$ in the CFT on the boundary. As the operator $\mathcal{O}_\Delta$ is of scale dimension Δ, the coupled ϕ_0 should be of scale dimension $d-\Delta$. In the following we'll derive the CFT correlators following Witten's approach [5].

The solution ϕ with source ϕ_0 on the boundary can be obtained by solving the field equation in terms of the Green's function $K(z, \vec{x}; 0, \vec{x}')$,

$$\Big(- z^{d+1}\,\partial_z(z^{-d+1}\partial_z) - z^2\partial_\mu\partial_\mu + m^2\Big)\ K = 0 \ , \tag{2.3.12}$$

where the boundary-bulk propagator K is

$$K(z, \vec{x}; 0, \vec{x}') = c\left(\frac{z}{z^2 + |\vec{x} - \vec{x}'|^2}\right)^{\Delta} \ , \tag{2.3.13}$$

[7] There exist $m^2 < 0$ modes in SUGRA which describe tachyons, and also in CFT the corresponding relevant perturbations. However these modes do not cause instabilities as long as m^2 is bounded from below, $m^2 \geq -d^2/4$, since the kinetic energy of the scalar fields in the AdS bulk cannot vanish due to the boundary condition at infinity [3–5]. Here we focus on $m^2 > 0$ modes.

with $c = \pi^{-d/2}\Gamma(\Delta)/\Gamma(\Delta - \frac{d}{2})$ the normalisation factor. The bulk field ϕ is given in terms of the boundary value $\phi_0(z = 0)$ by

$$\phi(z, \vec{x}) = \int d^4x' K(z, \vec{x}; 0, \vec{x}')\, \phi_0(\vec{x}') = cz^{d-\Delta} \int d^4x' \frac{z^{2\Delta-d}\, \phi_0(\vec{x}')}{(z^2 + |\vec{x} - \vec{x}'|^2)^\Delta}\,, \quad (2.3.14)$$

where $\int d\vec{x}' \frac{z^{2\Delta-d}}{(z^2+|\vec{x}-\vec{x}'|^2)^\Delta}$ is independent of z and the integrand is proportional to $\delta(\vec{x} - \vec{x}')$ at $z \longrightarrow 0$. (2.3.14) shows that ϕ indeed has the required asymptotic behavior as in (2.3.10).

The on-shell AdS action is obtained by evaluating it with the boundary-bulk propagator K,

$$S(\phi) = \frac{1}{2} \int dv \left(|d\phi|^2 + m^2\phi^2\right) = \frac{c\Delta}{2} \int d\vec{x} d\vec{x}'\, \frac{\phi_0(\vec{x})\phi_0(\vec{x}')}{|\vec{x} - \vec{x}'|^{2\Delta}}\,, \quad (2.3.15)$$

which is identified as the generating functional in boundary CFT. (2.3.15) gives rise to the correct form (2.2.1) of the 2-point correlators in the CFT.

For the massive p-form C on AdS with dimension Δ_p, the mass becomes $m^2 = (\Delta_p + p)(\Delta_p + p - d)$. The boundary field C_0 is expressed as $C_0 = f^{d-p-\Delta_p}C$, and C_0 is of scale dimension $d - \Delta_p$. The $p = 0$ case reduces to the scalar field discussed above.

The 3-point functions are obtained by using the boundary-bulk propagators and integrating out the bulk interaction point in the tree-level Witten diagram [5]. Suppose $\Delta_{1,2,3}$ are the scale dimensions for the chiral primary operators $\mathcal{O}_{\Delta_{1,2,3}}$ coupled to $(\phi_0)_{1,2,3}$, the expression for 3-point correlator is

$$\begin{aligned}&\langle \mathcal{O}_{\Delta_1}(\vec{x}_1)\mathcal{O}_{\Delta_2}(\vec{x}_2)\mathcal{O}_{\Delta_3}(\vec{x}_3)\rangle \\ &= c_3 \int \frac{dz' d\vec{x}'}{z'^5} K_{\Delta_1}(z', \vec{x}'; \vec{x}_1) K_{\Delta_2}(z', \vec{x}'; \vec{x}_2) K_{\Delta_3}(z', \vec{x}'; \vec{x}_3) \\ &= \frac{c_3\, a}{|\vec{x}_1 - \vec{x}_2|^{\Delta_{(3)}-2\Delta_3} |\vec{x}_2 - \vec{x}_3|^{\Delta_{(3)}-2\Delta_1} |\vec{x}_1 - \vec{x}_3|^{\Delta_{(3)}-2\Delta_2}}\,. \end{aligned} \quad (2.3.16)$$

where $\Delta_{(3)} = \Delta_1 + \Delta_2 + \Delta_3$, $\vec{x}_1, \vec{x}_2, \vec{x}_3$ are locations of the sources on the boundary, and c_3, a are some normalisation constants[8]. This expression is of the form of the 3-point CFT correlators in (2.2.2).

The general n-point functions for $n > 3$ are more complicated to calculate as there are contributions from more than one Witten diagrams. In some cases the

[8] Details of calculations can be found in [47].

operator product expansion can be used to simplify the calculations. Discussions on general n-point correlators can also be found in [47].

2.4 AdS/CFT correspondence of noncommutative gauge theory

In the preceding section we reviewed the correspondence between supergravity on $AdS_5 \times S^5$ and a CFT on the boundary. As it is believed that the correspondence is based on the more fundamental holographic principle, the duality is expected to apply in general to supergravity in certain background and the gauge theory on its boundary. For example, finite temperature breaks supersymmetry and the conformal invariance of 4-dimensional super Yang-Mills theory, and it can be described by supergravity in AdS_5 background with a blackhole [11]. This is more applicable to our real world in which the standard model doesn't exhibit conformal invariance.

In the section, we review an example of the generalised gauge theory/supergravity correspondence: noncommutative Yang-Mills theory and its supergravity dual.

2.4.1 Noncommutative gauge theory

In this section we will focus on the Euclidean spacetime $\mathbf{R}^d$ labelled by coordinates x^μ. The noncommutative field theory is obtained by introducing the following noncommutative relation to the spacetime "coordinates":

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} , \tag{2.4.1}$$

where $\theta^{\mu\nu}$ are antisymmetric parameters[9]. This relation is a deformation from the ordinary commutation relation with $\theta^{\mu\nu} = 0$, and thus the physics becomes nonlocal.

The noncommutative algebra of x^μ induces a modification in the multiplication law for the functions over the noncommutative manifold, called the *star product* or

[9]In the Minkowskian case, the quantum field theory is acausal [54] as a consequence of noncommutative time. Moreover, for theories with IR/UV mixing [55], unitarity breaks down for noncommutative time.

Moyal product:

$$\begin{aligned} f(x) * g(x) &= e^{\frac{i}{2}\theta^{\mu\nu}\partial_{a^\mu}\partial_{b^\nu}} f(x+a)\, g(x+b)\Big|_{a=0,b=0} = f(x)\, e^{\left(\frac{i}{2}\overleftarrow{\partial_\mu}\,\theta^{\mu\nu}\,\overrightarrow{\partial_\nu}\right)} g(x) \\ &= f\,g + \frac{i}{2}\,\theta^{\mu\nu}\,\partial_\mu f\;\partial_\nu g + \mathcal{O}(\theta^2)\,, \end{aligned} \tag{2.4.2}$$

where $f(x)$ and $g(x)$ are regarded as functions over ordinary $\mathbf{R}^d$. This expression can be derived by expanding a function in terms of the plane wave basis $e^{ik\cdot x}$, and making use of the fact that the product between $e^{ik\cdot x}$ becomes $e^{ik\cdot x}\, e^{ik'\cdot x} = e^{-\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu}\, e^{i(k+k')\cdot x}$ to the first order of θ by applying the Baker-Campbell-Hausdorff formula under the noncommutative algebra (2.4.1). The star product is associative, and reduces to the ordinary product at the commutative limit $\theta^{ij} = 0$. Moreover

$$\int d^d x\; f(x) * g(x) = \int d^d x\; f(x)\, g(x)\,, \tag{2.4.3}$$

supposing that the appropriate boundary conditions are imposed so that the integration of the total derivative vanishes. Therefore the quadratic terms in the action for noncommutative scalar fields remains the same as for the ordinary case, but the interaction terms proportional to ϕ^n for $n \geq 3$ deviate due to noncommutativity, although the integral of the $*$-product of more than two functions is invariant under cyclic permutations.

The star product rule in (2.4.2) applies while $f(x)$ and $g(x)$ are $N \times N$ matrix-valued functions, and it follows that $\int d^d x\; \mathrm{tr}\, f(x) * g(x) = \int d^d x\; \mathrm{tr}\, g(x) * f(x)$. The expressions for the field strength and the gauge transformations of $U(N)$ Yang-Mills theory are also modified by the $*$-structure:

$$F_{\mu\nu} = \partial_{[\mu}A_{\nu]} - i\,[A_\mu, A_\nu] \quad\longrightarrow\quad \partial_{[\mu}A_{\nu]} - i(A_\mu * A_\nu - A_\nu * A_\mu) \tag{2.4.4}$$

$$\delta A_\mu = \partial_\mu \epsilon + i\,[\epsilon, A_\mu] \quad\longrightarrow\quad \partial_\mu \epsilon + i(\epsilon * A_\mu - A_\mu * \epsilon) \tag{2.4.5}$$

$$\delta F_{\mu\nu} = i\,[\epsilon, F_{\mu\nu}] \quad\longrightarrow\quad i(\epsilon * F_{\mu\nu} - F_{\mu\nu} * \epsilon)\,, \tag{2.4.6}$$

where ϵ is the infinitesimal parameter, and the RHS are the expressions for the noncommutative case. The $N = 1$ example is given explicitly in [56]. Comprehensive accounts for the noncommutative gauge theory and how to construct gauge invariant observables can be found in [57–59]. The gauge invariant operators are not local due to noncommutativity.

The noncommutative field theories exhibit some unusual properties compared to the ordinary theories. A remarkable one is the IR/UV mixing [60]. Take $d = 4$ ϕ^4 theory as an example, for the two-point functions the UV divergence in the integral of the internal propagator is related to the divergence occurring at the low external momentum $p \to 0$, due to the contribution of the non-planar diagram. As a result, the phenomena at high energy scale mixes with physics at the low energy regime. Moreover, Lorentz invariance is broken due to the preferred noncommutative directions in the algebra (2.4.1).

Noncommutative phenomena arise naturally in string theory. The noncommutativity (2.4.1) is realised by open stings with a background NS-NS 2-form $B_{\mu\nu}$ [6] along the D-brane worldvolume where the open strings end. The existence of the background B-field modifies the boundary conditions of the open strings with which the standard canonical quantisation isn't compatible. Upon quantisation, the spacetime coordinates of the open string end-points corresponding to the D-brane worldvolume turn out to be non-commutative, giving rise to the commutator of the form in (2.4.1), with

$$\theta^{\mu\nu} = \pm 2\pi\alpha'(M^{-1})^{\mu\lambda}\mathcal{F}_{\lambda}{}^{\nu} \ , \tag{2.4.7}$$

where the $\pm$ signs correspond to the two end-points, and $\mathcal{F} := B - F$ is the gauge invariant combination of B-field and the Born-Infeld field F on the D-brane. M^{-1} is the inverse of $M_{\mu\nu} = \eta_{\mu\nu} - \mathcal{F}_{\mu}{}^{\lambda}\mathcal{F}_{\lambda\nu}$ with $\eta_{\mu\nu}$ the brane worldvolume components of the background metric. Since the open string end-points are attached to D-branes, it implies noncommutative spacetime arises in the D-brane worldvolume. In the open string theory with a background electric B-field, the acausal behavior of the field theory due to the noncommutative Minkowskian time is cancelled by stringy effects [54].

It is shown in (2.4.7) that the background B-field in the string theory is incorporated in the parameter θ of the worldvolume noncommutative field theory. In fact, the worldvolume dynamics described by the Born-Infeld Lagrangian from the closed string point of view, i.e. with the metric g "seen" by the closed string, the gauge field F, the pullback of B, and the closed string coupling g_s, is equivalent to the description from the open string point of view, i.e. with the metric G seen by the open string, the worldvolume gauge field endowed with the star product in the Born-Infield action, and the open string coupling G_s [7].

2.4.2 Supergravity dual

In the following we will then review the supergravity description dual to the non-commutative gauge theory, following the approach in [8]. (See also [9].)

First we formulate the D3-brane solution in a constant B-field background with nonvanishing components $\hat{B}_{01} = \tan\theta_1$ and $\hat{B}_{23} = \tan\theta_2$ in string frame. Here θ_1 (θ_2) is the angle by which x_0 and x_1 (x_2 and x_3) axes are rotated after $\hat{B}_{01}$ ($\hat{B}_{23}$) is turned on. In the Euclidean target space, the solution is given as follows:

$$\begin{aligned}
ds^2 &= (\sqrt{f})^{-1}\left[h_1(dx_0^2+dx_1^2)+h_2(dx_2^2+dx_3^2)\right]+\sqrt{f}\left[dr^2+r^2d\Omega_5^2\right] , \\
f &= 1+\frac{\alpha'^2R^4}{r^4} , \qquad h_{1,2}^{-1} = f^{-1}\sin^2\theta_{1,2}+\cos^2\theta_{1,2} , \\
B_{01} &= f^{-1}h_1\tan\theta_1 , \qquad B_{23} = f^{-1}h_2\tan\theta_2 , \\
e^{2\phi} &= g^2h_1h_2 , \qquad \chi = \frac{i}{g}f^{-1}\sin\theta_1\sin\theta_2 , \\
A_{01} &= \frac{i}{g}f^{-1}h_1\sin\theta_2\cos\theta_1 , \qquad A_{23} = \frac{i}{g}f^{-1}h_2\sin\theta_1\cos\theta_2 , \\
F_{0123r} &= \frac{i}{g}(h_1h_2\partial_r f^{-1})\cos\theta_1\cos\theta_2 ,
\end{aligned} \tag{2.4.8}$$

where R is defined by $(\cos\theta_1\cos\theta_2)R^4 = 4\pi gN$, and $g = e^{\phi_\infty}$. The solution is derived by T-dualising a D3-brane (in $\hat{B}_{01}, \hat{B}_{23}$ background) along x_3-direction into a D2-brane on a tilted torus, and then T-dualising along x_3 again back to a $D3$-brane. A T-duality transformation inverts the background matrix $E = G + B$ where G is the target space metric [61] [62]. Together with the transformation rules for the RR-fields given in [64], the solution above is obtained. It can be analytically continued to the Minkowskian version by $x_0 \to ix_0$, $\theta' \to i\theta'$ so that the RR-fields become real. The solution preserves 16 supercharges.

As $r \to \infty$, $f, h_1, h_2 \to 1$, so the metric is asymptotically flat, with $B_{01}^{\infty} = \tan\theta_1$, $B_{23}^{\infty} = \tan\theta_2$ while $F_{(3)}, F_{(5)} \to 0$. At the near horizon region, $f \to \alpha'^2R^4/r^4$ and $h_{1,2} \to 1/\cos^2\theta_{1,2}$, which implies that $B_{01}, B_{23} \to 0$ as r reduces to 0 naively, and the solution tends to the conventional $AdS_5 \times S^5$ case.

To obtain the dual supergravity solution for the non-commutative field theory, the decoupling limit is again taken by

$$\alpha' \to 0 \ . \tag{2.4.9}$$

Note that if we only rescale r by r/α' in the meantime, the result is the same as ordinary $AdS_5 \times S^5$ case. Moreover, the noncommutativity scale is given by $\Delta^2 = \alpha' B$ due to $[x_\mu, x_\nu] \sim \Delta^2$; to obtain a finite Δ in the decoupling limit, B^∞-field must be rescaled to infinity. The variables and B-field in (2.4.8) are rescaled as follows for the low energy limit:

$$\begin{aligned}
\alpha' B^\infty_{01} &= \alpha' \tan\theta_1 = \tilde{b}_1 = \text{fixed} \ , \qquad \alpha' B^\infty_{23} = \alpha' \tan\theta_2 = \tilde{b}_2 = \text{fixed} \\
\frac{\tilde{b}_1}{\alpha'} x_{0,1} &= \tilde{x}_{0,1} = \text{fixed} \ , \qquad \frac{\tilde{b}_2}{\alpha'} x_{2,3} = \tilde{x}_{2,3} = \text{fixed} \ , \\
\frac{r}{\alpha' R^2} &= u = \text{fixed} \ , \qquad \frac{\tilde{b}_1 \tilde{b}_2}{\alpha'^2} = \hat{g} = \text{fixed} \ .
\end{aligned} \tag{2.4.10}$$

As a result,

$$\begin{aligned}
ds^2 &= \alpha' R^2 \left\{ u^2 \left(\hat{h}_1 (dx_0^2 + dx_1^2) + \hat{h}_2 (dx_2^2 + dx_3^2) + \frac{du^2}{u^2} + d\Omega_5^2 \right) \right\} \ , \\
e^{2\phi} &= \hat{g}^2 \hat{h}_1 \hat{h}_2 \ , \qquad \chi = i \frac{\tilde{b}_1 \tilde{b}_2}{\hat{g}} R^4 u^4 \ , \\
\hat{h}_1 &= \frac{1}{1 + a_1^4 u^4} \ , \qquad \hat{h}_2 = \frac{1}{1 + a_2^4 u^4} \ , \qquad a_{1,2}^2 = \tilde{b}_{1,2} R^2 \ , \\
B_{01} &= \alpha' R^2 \frac{a_1^2 u^4}{1 + a_1^4 u^4} \ , \qquad B_{23} = \alpha' R^2 \frac{a_2^2 u^4}{1 + a_2^4 u^4} \ , \\
A_{01} &= i\alpha' \frac{\tilde{b}_2}{\hat{g}} \hat{h}_1 R^4 u^4 \ , \qquad A_{23} = i\alpha' \frac{\tilde{b}_1}{\hat{g}} \hat{h}_2 R^4 u^4 \ , \\
F_{0123r} &= i\alpha'^2 \frac{\hat{h}_1 \hat{h}_2}{\hat{g}} \partial_u (R^4 u^4) \ .
\end{aligned} \tag{2.4.11}$$

are proposed as the supergravity dual of the noncommutative gauge theory.

Lorentz invariance is broken for $\theta_1 \neq \theta_2$ in this solution. For small u, $\hat{h}_1, \hat{h}_2 \to 1$, so the metric reduces to $AdS_5 \times S^5$ with $B_{01,23} \sim \alpha' \tilde{b}_{1,2} R^4 u^4$. Since $u \to 0$ correspond to IR regime in the field theory, it agrees with the fact that noncommutative YM theory becomes ordinary at large scale. At infinity, $\hat{h}_{1,2} \to (\tilde{b}_{1,2}^2 R^4 u^4)^{-1}$, and the metric appears again as $AdS_5 \times S^5$ (in a different coordinate) with $B_{01,23} \sim \alpha' / \tilde{b}_{1,2}$. The boundary is taken at infinity although the curvature invariants are finite there, since the fluctuations near the boundary are suppressed by $e^\phi \to 0$ as $u \to \infty$.

The correlation function on the field theory side is again given by evaluating the supergravity action with the boundary value of the supergravity field φ,

$$\log \langle e^{\int d^4k\, \varphi_0(k)\, \mathcal{O}(k)} \rangle = -S_{\rm SUGRA}(\varphi_0(k)) \;, \tag{2.4.12}$$

where $\varphi_0(k)$ is the boundary value of $\varphi(k,u)$ for $u \to \infty$. Here the prescription works more naturally in the momentum space, due to the nonlocality arising from noncommutativity of x_μ.

An explicit attempt to verify the field theory correlators derived from the supergravity field is carried out for $\varphi = g^{00}\gamma_{01}$ in [8], where γ_{01} is the graviton fluctuation in AdS and corresponds to the Fourier transform of the energy-momentum operator T_{01} in Yang-Mills theory. The supergravity equation of motion for φ is

$$\frac{1}{u^5}\partial_u u^5 \partial_u \varphi - k^2(\frac{1}{u^4} + a^4)\varphi = 0 \;, \tag{2.4.13}$$

and the relevant asymptotic solution is $\varphi(k,u) \sim e^{ka^2u}/u^{5/2}$. The actual solution is solved in terms of Mathieu functions. Therefore it is convenient to impose the boundary condition

$$\varphi(k, u \to \infty) = \frac{e^{ka^2u}}{(ka^2u)^{5/2}}\, \varphi_0(k)\Big|_{u\to\infty} \;. \tag{2.4.14}$$

Evaluation of the supergravity action produces UV divergent terms, and are subtracted from the integral. The correlation function then obtained is

$$\langle \mathcal{O}(k)\mathcal{O}(k')\rangle \sim \delta(k+k')\frac{B(k)}{(ka^2)^4} \;, \tag{2.4.15}$$

where $B(k)$ is some function of k. Analysis shows that it has correct IR behavior $\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle \sim k^4 \log k$. However the momentum-dependent renormalisation imposes ambiguity on the behavior of the correlator at UV, $\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle \sim e^{-|k|}$, which is different from the case of ordinary field theory.

Chapter 3

Non-anticommutative Super Yang-Mills Theory

In Section 2.4 we have considered the noncommutative gauge theory which arises from deforming the commuting spacetime coordinates x^μ. As supersymmetry is incorporated in superstring theories, it is natural to generalise such concept to the fermionic part of superspace [65–69]. The simplest model is to introduce a Clifford algebra to the chiral Grassmann coordinates on the Euclidean $\mathcal{N} = 1$ superspace. This implies that half of the supersymmetry is broken, and gives rise to a particular class of deformed supersymmetric gauge theories. Such deformations can be generalised to extended supersymmetry. The fermionic non-anticommutativity leads to noncommutativity in bosonic spacetime.

Non-anticommutative supersymmetric gauge theories are interesting because their supersymmetry is broken in a non-trivial way. Moreover their renormalisability is not destroyed by the deformations. The non-anticommutative SYM Lagrangian contains additional terms in positive powers of the deformation parameter C (see (3.1.15) for example). Since C is of mass dimension -1, if it is regarded as the coupling constant, by power counting the theories appear non-renormalisable. But interestingly, they are in fact renormalisable. This is because these deformed theories are formulated in Euclidean superspace in which the chiral and antichiral Grassmann coordinates are not related by complex conjugation, and the hermiticity of the theories is lost since the deformation occurs to the chiral θ but not to antichiral

$\bar{\theta}$. Due to lack of hermiticity, insertion of certain vertex into a convergent loop is not accompanied by the hermitian conjugate of the vertex to make the diagram divergent, so the power counting scheme fails to apply [72]. Analysis shows that the non-anticommutative $\mathcal{N} = 1/2$ Wess-Zumino Model is multiplicatively renormalisable [70, 71], and the renormalisability of the deformed super Yang-Mills theories can be achieved by shifting the component fields [72–76].

The non-anticommutativity can be realised in string theory. The deformed superspace on D3-branes arises from turning on the constant selfdual graviphoton background field strength which belongs to the RR sector of the closed string excitations. The effects of $\mathcal{N} = 1$ open strings extending in 4-dimensions in such background was first study by Ooguri and Vafa [77], and shortly after by Seiberg [69]. These graviphoton fields produce vanishing target-space energy-momentum tensors and thus generate no backreaction. The Lagrangians of non-anticommutative deformed super Yang-Mills theories on D3-branes are derived via computing the open string scattering amplitudes in the constant graviphoton backgrounds using the RNS formalism [80–82]. For this construction, in general the RR fields are represented by the vertex operators in the $(-1/2, -1/2)$ superghost picture of the form

$$V_{\mathcal{F}}(z, \bar{z}) \sim \tilde{S}^T \mathcal{C} \mathcal{F} S e^{-\phi/2} e^{\tilde{\phi}/2} , \tag{3.0.1}$$

where S and $\tilde{S}$ are left-moving and right-moving spinor fields on the closed strings, $\mathcal{F}$ is the background RR field, $\mathcal{C}$ is the charge conjugation matrix, ϕ is the chiral boson in superghost bosonisation [79], and $z, \bar{z}$ are the closed string worldsheet coordinates. Since the $SO(10)$ spacetime isometry is broken into $SO(4) \times$ (internal isometry) due to the presence of D3-branes, the 10-dimensional spinor fields $S, \tilde{S}$ in (3.0.1) also need to be decomposed. Their decompositions are determined by the choices of internal manifold compactification.

In this chapter we review the non-anticommutative deformations to $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superspace, and the deformed super Yang-Mills theories realised on these superspaces. As the supergravity dual of noncommutative SYM had been constructed in [8, 9], it is interesting to find out the supergravity dual of non-anticommutative super Yang-Mills theory and the consequences of the duality, which has never been done before. We will deal with this topic in Chapter 4 and 5.

3.1 Non-anticommutative $\mathcal{N}=1$ super Yang-Mills theory

The discussions of deformed supersymmetry will be carried out in Euclidean spacetime. In Euclidean superspace, the spacetime symmetry group is $SO(4)$, with the universal covering spin group decomposed into $Spin(4) = SU(2)_L \times SU(2)_R$. The fermionic variables θ and $\bar{\theta}$ are no longer complex conjugates to each other and transform independently under $SU(2)_L$ and $SU(2)_R$, so that $\mathcal{N}=1$ supersymmetry is also referred to as $\mathcal{N} = (\frac{1}{2}, \frac{1}{2})$. Deformations to the Grassmann coordinates such as (3.1.2) are possible only in Euclidean spacetime. The (pseudo-)conjugation[1] of the spinor coordinates is instead defined by

$$(\theta^\alpha)^* = \varepsilon_{\alpha\beta}\theta^\beta , \qquad (\bar{\theta}^{\dot{\alpha}})^* = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}} , \tag{3.1.1}$$

where $\varepsilon_{12} = -\varepsilon^{12} = \varepsilon_{\dot{1}\dot{2}} = -\varepsilon^{\dot{1}\dot{2}} = 1$. Thus the bosonic and fermionic coordinates of Euclidean $\mathcal{N} = (1/2, 1/2)$ superspace describe the real manifold $\mathbf{R}^{4|4}$.

The non-anticommutative deformed $\mathcal{N}=1$ superspace arises from deforming the anticommutation relation of θ^α while keeping other (anti-)commutators involving $\bar{\theta}^{\dot{\alpha}}$ unchanged,

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta} , \quad \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \{\bar{\theta}^{\dot{\alpha}}, \theta^\beta\} = [\bar{\theta}^{\dot{\alpha}}, x^\mu] = 0 , \tag{3.1.2}$$

where $C^{\alpha\beta}$ is a symmetric parameter with respect to α, β, and $\mu = (0, 1, 2, 3)$. In order to preserve the chiral representations of $\mathcal{N}=1$ supersymmetry, it is convenient to use the chiral coordinates[2]

$$y^\mu = x^\mu + i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} , \tag{3.1.3}$$

and impose the commutativity conditions on y^μ [69],

$$[y^\mu, y^\nu] = [y^\mu, \theta^\alpha] = [y^\mu, \bar{\theta}^{\dot{\alpha}}] = 0 , \tag{3.1.4}$$

[1]The pseudo-conjugation operation is defined as being squared to -1 on fermionic objects and to +1 on bosonic objects. The conjugation squares to 1 on any objects.

[2]We follow the supersymmetry notations of Wess and Bagger [83] here except that the spacetime signature is Euclidean. We will switch to the other convention shortly.

one finds that the spacetime coordinates become noncommutative:

$$[x^\mu, x^\nu] = \bar\theta\bar\theta\, C^{\mu\nu} , \qquad [x^\mu, \theta^\alpha] = iC^{\alpha\beta}\sigma^\mu_{\beta\dot\alpha}\bar\theta^{\dot\alpha} , \tag{3.1.5}$$

where

$$C^{\mu\nu} = C^{\alpha\beta}\varepsilon_{\beta\gamma}(\sigma^{\mu\nu})_\alpha{}^\gamma , \qquad C^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\gamma}(\sigma^{\mu\nu})_\gamma{}^\beta C_{\mu\nu} . \tag{3.1.6}$$

It is clear that as C is switched off, the algebras above reduce back to the standard ones. Lorentz symmetry is broken in (3.1.5), due to non-anticommutativity of (3.1.2). Analogous to the non-commutative case discussed in Section 2.4, the non-anticommutative deformation also induces the star product for functions of $(y, \theta, \bar\theta)$, i.e. the superfields $f(y, \theta, \bar\theta)$ and $g(y, \theta, \bar\theta)$:

$$\begin{aligned} f * g \;&:=\; f\, e^{-\frac{C^{\alpha\beta}}{2}\overleftarrow{\partial_\alpha}\overrightarrow{\partial_\beta}}\, g \;=\; f\, e^P\, g \\ &=\; f\left(1 + P + \frac{1}{2}P^2\right) g . \end{aligned} \tag{3.1.7}$$

where ∂_α denotes $\frac{\partial}{\partial\theta^\alpha}$ with the "partial" derivative taken at fixed y and $\bar\theta$, and

$$P := -\frac{1}{2}\overleftarrow{\partial_\alpha} C^{\alpha\beta}\overrightarrow{\partial_\beta} = \overleftarrow{Q_\alpha} C^{\alpha\beta}\overrightarrow{Q_\beta} . \tag{3.1.8}$$

Here $\Omega\overleftarrow{\partial_\alpha} := \pm\frac{\partial}{\partial\theta^\alpha}\Omega$, with $+$ sign for bosonic Ω and $-$ for fermionic Ω. The integral of (3.1.7) over the superspace coördinates is undeformed from the ordinary product case (analogous to (2.4.3)). The star product of more than three superfields in the integral is invariant under cyclic permutations of the superfields.

The star product generator P satisfies $P^3 = 0$ and is called nilpotent [84]. The star product structure is preserved by the supercharges Q given below, but not invariant under $\bar Q$, as $[P, Q] = 0 \neq [P, \bar Q]$. Since only Q is a symmetry in the non-anticommutative deformed superspace, half of $\mathcal{N} = (\frac{1}{2}, \frac{1}{2})$ supersymmetry is broken and the superspace is called denotes as $\mathcal{N} = (\frac{1}{2}, 0)$.

The supercharges in (3.1.8) and the covariant derivatives in terms of the chiral coordinates are given by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial y^\mu} , \qquad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} , \tag{3.1.9}$$

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} , \qquad \bar Q_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + 2i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial y^\mu} . \tag{3.1.10}$$

One can check that the algebras of the covariant derivatives and supercharges are thus

$$\{D_\alpha, Q_\beta\} = \{\bar{D}_{\dot\alpha}, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot\beta}\} = \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = \{Q_\alpha, Q_\beta\} = 0 \,,$$
$$\{\bar{Q}_{\dot\alpha}, Q_\beta\} = 2i\sigma^\mu_{\beta\dot\alpha}\frac{\partial}{\partial y^\mu}\,, \qquad \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = -4C^{\alpha\beta}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\frac{\partial^2}{\partial y^\mu \partial y^\nu}\,. \qquad (3.1.11)$$

Only the $\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\}$ term is deformed by $C^{\alpha\beta}$. It is clear here that $\bar{Q}$ is not a supersymmetric generator under the non-anticommutative deformation because of explicit dependence on θ^α which satisfy the Clifford algebra in (3.1.2).

The pure $\mathcal{N}=1$ super Yang-Mills theory contains a vector multiplet, which is described by a vector superfield in the superspace formalism. To realise the deformed $\mathcal{N}=1$ SYM on $\mathcal{N}=(1/2,0)$ superspace[3], one uses the definitions for the chiral and antichiral field strength superfield in the standard form, $W_\alpha = \frac{-1}{4}\bar{D}\bar{D}e^{-V}D_\alpha e^V$ and $\overline{W}_{\dot\alpha} = \frac{1}{4}DDe^V\bar{D}_{\dot\alpha}e^{-V}$, but replace the ordinary products by the $*$-product. The gauge theory Lagrangian becomes $\int d^2\theta \,\mathrm{tr}\, W^\alpha * W_\alpha + \int d^2\bar\theta \,\mathrm{tr}\, \overline{W}_{\dot\alpha} * \overline{W}^{\dot\alpha}$, which takes exactly the same form as in the ordinary case since the star product is the same as the standard product in the integral. The superfield gauge transformations are defined by $e^V \to e^{-i\bar\Lambda} * e^V * e^{i\Lambda}$, $W_\alpha \to e^{-i\Lambda} * W_\alpha * e^{i\Lambda}$, and $\overline{W}_{\dot\alpha} \to e^{-i\bar\Lambda} * \overline{W}_{\dot\alpha} * e^{i\bar\Lambda}$. In order for the component fields to have the canonical gauge transformation properties, the vector superfield in the Wess-Zumino gauge is chosen to be [69]

$$V(y,\theta,\bar\theta) = V(C=0) - i\,\frac{1}{4}\,\bar\theta\bar\theta\theta^\alpha\varepsilon_{\alpha\beta}C^{\beta\gamma}\sigma^\mu_{\gamma\dot\alpha}\left\{\bar\psi^{\dot\alpha}, A_\mu\right\}\,, \qquad (3.1.12)$$

where the gaugino ψ contains deformation from C,

$$\psi \to \psi + \frac{1}{4}\,\varepsilon_{\alpha\beta}C^{\beta\gamma}\sigma^\mu_{\gamma\dot\alpha}\left\{\bar\psi^{\dot\alpha}, A_\mu\right\}\,. \qquad (3.1.13)$$

Note that, despite the gauge transformations are canonical, with the deformation (3.1.12), the supersymmetry transformation of the gaugino becomes non-canonical, modified by an extra term in C. The field strength superfields arising from V are

[3]Non-anticommutative $\mathcal{N}=1$ SYM with matter can be found in [86].

deformed by

$$\begin{aligned} W_\alpha &= W_\alpha(C=0) + \epsilon_{\alpha\beta}C^{\beta\gamma}\theta_\gamma\bar{\psi}\bar{\psi} , \qquad (3.1.14) \\ \overline{W}_{\dot\alpha} &= \overline{W}_{\dot\alpha}(C=0) \\ & -\bar\theta\bar\theta\left(\frac{C^{\mu\nu}}{2}\{F_{\mu\nu},\bar\psi_{\dot\alpha}\} + C^{\mu\nu}\{A_\nu, D_\mu\bar\psi_{\dot\alpha} - \frac{i}{4}[A_\mu,\bar\psi_{\dot\alpha}]\} + \frac{i}{16}|C|^2\{\bar\psi\bar\psi,\bar\psi_{\dot\alpha}\}\right) , \end{aligned}$$

where $|C|^2 = C^{\mu\nu}C_{\mu\nu} = 4\det C$ and the covariant derivative $D_\mu\bar\psi_{\dot\alpha} = \partial_\mu\bar\psi_{\dot\alpha} + \frac{i}{2}[A_\mu,\bar\psi_{\dot\alpha}]$.

Substituting W_α and $\overline{W}_{\dot\alpha}$ into the Lagrangian $\mathrm{tr}\,(W^\alpha W_\alpha|_{\theta\theta} + \overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}|_{\bar\theta\bar\theta})$, one finds the modification to the ordinary $\mathcal{N}=1$ SYM Lagrangian by the following extra terms due to C-deformation:

$$\begin{aligned} \int d^2\theta\ \mathrm{tr}\,W^\alpha W_\alpha(C\neq 0) &= -iC^{\mu\nu}\,\mathrm{tr}\,F_{\mu\nu}\bar\psi\bar\psi + \frac{|C|^2}{4}\,\mathrm{tr}(\bar\psi\bar\psi)^2 , \qquad (3.1.15) \\ \int d^2\bar\theta\ \mathrm{tr}\,\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}(C\neq 0) &= -iC^{\mu\nu}\,\mathrm{tr}\,F_{\mu\nu}\bar\psi\bar\psi + \frac{|C|^2}{4}\,\mathrm{tr}(\bar\psi\bar\psi)^2 + \text{total derivative} , \end{aligned}$$

where the total derivative in $\mathrm{tr}\,\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}$ can be dropped for the integration of the action.

As for the Wess-Zumino Lagrangian, the star product of the chiral superfields Φ is straightforward. The antichiral superfield $\bar\Phi$ depends on $(\bar y^\mu,\bar\theta)$ where $\bar y^\mu = y^\mu - 2i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}$. The star product of the antichiral superfields is carried out by making use of $[\bar y^\mu,\bar y^\nu] = 4\bar\theta\bar\theta C^{\mu\nu}$ such that the bi-differential operator $P = 2\bar\theta\bar\theta C^{\mu\nu}\overleftarrow{\frac{\partial}{\partial\bar y^\mu}}\overrightarrow{\frac{\partial}{\partial\bar y^\nu}}$. As a result, the Wess-Zumino Lagrangian is modified by an additional term $-\frac{g}{3}\det C\,F^3$ (where F is the auxiliary field in chiral superfield expansion), contributed from the deformation in $\Phi*\Phi*\Phi|_{\theta\theta}$. $\mathcal{L}_{WZ}$ remains Lorentz invariant despite the deformation (3.1.2) is not.

The renormalisability of non-anticommutative $\mathcal{N}=1$ super Yang-Mills theory to all orders of perturbation is proven in [72, 73] by making use of the non-hermiticity argument. In [74], it is shown by explicit calculations that the multiplicative renormalisability at one-loop level can be achieved by the complement of a shift in the gaugino field ψ.

3.1.1 String theory construction

In string theory, $\mathcal{N}=(1/2,0)$ superspace arises from open strings in a constant graviphoton background. The graviphoton is the RR 5-form field wrapped around the 3-cycle on the internal Calabi-Yau manifold so that the components are along the D3-brane worldvolume. It is shown in [69] by using the Berkovits formalism [85] that, for type II open strings extended in 4-dimensions and ending on D-branes in the background of constant selfdual $\mathcal{F}^{\alpha\beta}$, the deformation to $\mathcal{N}=1$ superspace $\{\theta^\alpha,\theta^\beta\}=C^{\alpha\beta}\sim\mathcal{F}^{\alpha\beta}$ is obtained by imposing appropriate boundary conditions for the worldsheet fermionic coordinates and by analysing the two-point correlator $\langle\,\theta^\alpha\theta^\beta\,\rangle$ on the boundary.

(3.1.2) implies the deformation parameter $C^{\alpha\beta}$ is of length dimension $[L]^{+1}$. The dimensions $\mathcal{F}^{\alpha\beta}$ may be of different dimension, and therefore it needs to be dimensionally fixed and scaled. It is convenient to choose $\mathcal{F}^{\alpha\beta}$ of canonical dimension $[L]^{-2}$ [81, 82] such that[4]

$$(2\pi\alpha')^{\frac{3}{2}}\mathcal{F}^{\alpha\beta}=C^{\alpha\beta}=\{\theta^\alpha,\theta^\beta\}\ . \tag{3.1.16}$$

We will use this rescaling in our discussions.

Non-anticommutative deformed $\mathcal{N}=1$ SYM is derived in [80] as the low energy dynamics on D3-branes compactified on the internal orbifold $\mathbf{R}^6/(\mathbf{Z}_2\times\mathbf{Z}_2)$ in a constant graviphoton background, using the RNS formalism[5]. For scattering of open strings, each asymptotic string state is described by a vertex operator on the boundary of a disk representing the open string worldsheet. Besides the vertex operators of A_μ,ψ and $\bar\psi$, a selfdual auxiliary field $H_{\mu\nu}$ is introduced to decouple the quartic interactions of gauge fields into cubic ones. By summing up all inequivalent

[4]Note that the dimension of $\mathcal{F}$ chosen in this chapter is different from that of the ordinary RR gauge fields in the supergravity action

$$S=(2\kappa_{10}^2)^{-1}\int d^{10}x\,\sqrt{g}\,(R-\frac{1}{2}F^2)\ ,$$

where $[F]=[L]^{-1}$. The choice of the power of α' depends on the normalisation of α' in the action.

[5]In this subsection, the convention for the 2-component gamma matrices is temporarily switched to that in (3.3.3).

color-ordered scattering amplitudes among certain combination of vertex operators, the corresponding interaction term in the Lagrangian is obtained.

The constant background graviphoton with a definite duality[6] is represented by an RR vertex operator $V_{\mathcal{F}}$ at zero momentum, which can be inserted into the interior of the disk to evaluate the mixed open and closed string scattering amplitude. For D3-branes compactified on $\mathbf{R}^6/(\mathbf{Z}_2\times\mathbf{Z}_2)$, the spacetime is decomposed into $\mathbf{R}^4\times(\mathbf{R}^6/(\mathbf{Z}_2\times\mathbf{Z}_2))$. The 10-dimensional spinor field S^{λ} is thus decomposed into $(S_\alpha S^{(-)}, S^{\dot\alpha}S^{(+)})$, where $S_\alpha, S^{\dot\alpha}$ are 4-dimensional spinors with $\alpha,\dot\alpha=(1,2)$, and $S^{(+)}, S^{(-)}$ are 6-dimensional spinors with weights $(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2})$ and $(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$ respectively. The graviphoton vertex operator in the $(-1/2,-1/2)$ superghost picture is given by

$$V_{\mathcal{F}}^{(-1/2,-1/2)}(z,\bar z)=(2\pi\alpha')^{3/2}\,\mathcal{F}^{\alpha\beta}S_\alpha S^{(-)}e^{-\phi/2}(z)\,\tilde S_\beta \tilde S^{(-)}e^{-\tilde\phi/2}(\bar z)\ , \tag{3.1.17}$$

where $\mathcal{F}^{\alpha\beta}$ is selfdual.

The calculation of the disk amplitude with insertion of $V_{\mathcal{F}}$ is facilitated by the boundary condition that the left-moving and right-moving spin fields on the closed string are identified on the disk,

$$S_\alpha(z)S^{(-)}(z)=\tilde S_\alpha(\bar z)\tilde S^{(-)}(\bar z)\Big|_{z=\bar z}\ . \tag{3.1.18}$$

As a result, $\tilde S_\alpha \tilde S^{(-)}$ in $V_{\mathcal{F}}$ is replaced by $S_\alpha S^{(-)}$ while evaluating the mixed open/closed string scattering amplitudes. The fact of two copies of $S^{(-)}$ being in $V_{\mathcal{F}}$ rules out vanishing scattering amplitudes among $V_{\mathcal{F}}$ and the open string vertex operators $V_A, V_\psi, V_{\bar\psi}, V_H$. By considering all nonvanishing amplitudes, summing over inequivalent color orders, and integrating out the auxiliary field $H_{\mu\nu}$, one recovers the non-anticommutative $\mathcal{N}=1$ SYM Lagrangian which agrees with the RHS of (3.1.15). The details of calculation can be found in [80].

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[6] A generic 2-form $\mathcal{F}_{\mu\nu}$ can be decomposed into the selfdual part $\mathcal{F}^+_{\mu\nu}$ and the anti-selfdual part $\mathcal{F}^-_{\mu\nu}$: $\mathcal{F}_{\mu\nu}=\mathcal{F}^+_{\mu\nu}+\mathcal{F}^-_{\mu\nu}$ where $\mathcal{F}^\pm_{\mu\nu}=\pm\frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\mathcal{F}^{\rho\lambda}$. The selfdual component $\mathcal{F}^{\alpha\beta}=\frac{1}{2}\mathcal{F}^+_{\mu\nu}(\sigma^{\mu\nu})^{\alpha\beta}$ gives rise to the deformation $\{\theta^\alpha,\theta^\beta\}=C^{\alpha\beta}$, while the anti-selfdual component $\mathcal{F}^{\dot\alpha\dot\beta}=\frac{1}{2}\mathcal{F}^-_{\mu\nu}(\bar\sigma^{\mu\nu})^{\dot\alpha\dot\beta}$ is responsible for $\{\bar\theta^{\dot\alpha},\bar\theta^{\dot\beta}\}=C^{\dot\alpha\dot\beta}$. To break half of $\mathcal{N}=1$ supersymmetry, the graviphoton field is either selfdual or anti-selfdual. The calculation in [80] is given for anti-selfdual $\mathcal{F}^{\dot\alpha\dot\beta}$. For consistency of our discussions here, we will use selfdual $\mathcal{F}^{\alpha\beta}$.

3.2 Deformed $\mathcal{N} = (1,1)$ SUSY and non-anticommutative SYM

The 4-dimensional Euclidean $\mathcal{N} = (1,1)$ superspace in the chiral basis is parametrised by $(y^\mu, \theta_k^\alpha, \bar{\theta}^{\dot{\alpha}k})$, where $k = (1,2)$ is the $SU(2)$ R-symmetry indices and $y^\mu = x^\mu + i\theta_k^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}k}$. Two different (pseudo-)conjugations are compatible with the reality conditions for the superspace $\mathbf{R}^{4|8}$ [84]:

$$\widetilde{\theta_k^\alpha} = \varepsilon^{kj}\varepsilon_{\alpha\beta}\theta_j^\beta \ , \qquad \widetilde{\bar{\theta}^{\dot{\alpha}k}} = -\varepsilon_{kj}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}j} \qquad \widetilde{y^\mu} = y^\mu \ , \qquad \widetilde{fg} = \tilde{g}\tilde{f} \ , \tag{3.2.1}$$

$$(\theta_k^\alpha)^* = \varepsilon_{\alpha\beta}\theta_k^\beta \ , \qquad (\bar{\theta}^{\dot{\alpha}k})^* = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}k} \ , \qquad (y^\mu)^* = y^\mu \ , \qquad (fg)^* = g^* f^* \ . \tag{3.2.2}$$

$\tilde{}$ is the standard conjugation, which preserves the irreducible representations of $SU(2)_L \times SU(2)_R$ and the $SU(2)$ R-symmetry group, but since $\tilde{}$ relates θ_1^α to θ_2^β, the $\mathcal{N} = (1/2, 1/2)$ superspace cannot be contained in $\mathcal{N} = (1,1)$ one as a real subspace. On the other hand, the pseudo-conjugation * is compatible with enclosing $\mathcal{N} = (1/2, 1/2)$ as a subspace of $\mathcal{N} = (1,1)$ superspace as well as the action of $Spin(4)$ group, but the R-symmetry preserved by * is $SL(2,\mathbf{R})$ instead of $SU(2)$. Only the $U(1)$ factor of $SU(2)$ is preserved by *.

The non-anticommutative deformation to $\mathcal{N} = (1,1)$ superspace is obtained by introducing the following Clifford algebra:

$$\{\theta_i^\alpha, \theta_j^\beta\} = C_{ij}^{\alpha\beta} \ , \tag{3.2.3}$$

with the (anti-)commutators involving $\bar{\theta}^{\alpha j}$ and y^μ remaining vanished. $C_{ij}^{\alpha\beta} = C_{ji}^{\beta\alpha}$ is some constant parameter of $[L]^{+1}$. The spacetime commutativity is also broken by (3.2.3) [90]. The two choices of conjugations lead to different conditions for the deformation parameter[7]:

$$\widetilde{C_{ij}^{\alpha\beta}} = C^{ij}_{\alpha\beta} \ , \qquad (C_{ij}^{\alpha\beta})^* = C_{\alpha\beta ij} \ . \tag{3.2.4}$$

[7]The difference between the two (pseudo-)conjugations can be illustrated here: by specifying * as the conjugation for $\mathbf{R}^{4|8}$, if $C_{11}^{\alpha\beta}$ is the only nonvanishing component, (3.2.3) reduces back to (3.1.2). Only $\bar{Q}_{\dot{\alpha}1}$ is broken and the preserved supersymmetry becomes $\mathcal{N} = (1, 1/2)$. Such degenerate deformation is not possible for taking the conjugation $\tilde{}$, since nonvanishing $C_{11}^{\alpha\beta}$ implies nonzero $C_{22}^{\alpha\beta}$, and also the bi-differential operator obeys $\widetilde{P} = P$; as a result the unbroken susy is $\mathcal{N} = (1,0)$.

Due to the deformation, $SU(2)$ R-symmetry is generically broken, but the $SU(2)_R$ part is unchanged. Such deformation gives rise to the star product for superfields, where the bi-differential operator is given by

$$P := -\frac{1}{2}\overleftarrow{\partial^i_\alpha}\, C^{\alpha\beta}_{ij}\overrightarrow{\partial^j_\beta} = -\frac{1}{2}\overleftarrow{Q^i_\alpha}\, C^{\alpha\beta}_{ij}\overrightarrow{Q^j_\beta}\,, \tag{3.2.5}$$

and subject to the identity $P^5 = 0$. Here ∂^i_α stands for $\frac{\partial}{\partial\theta^\alpha_i}$, etc.. Chirality of the superfields is preserved by the operator P. Generically P breaks $\mathcal{N}=(1,1)$ supersymmetry to $\mathcal{N}=(1,0)$ due to nonvanishing $[P,\bar{Q}_i]$, but for some particular $C^{\alpha\beta}_{ij}$ (such as that in the footnote), the preserved supersymmetry is enhanced to $\mathcal{N}=(1,1/2)$. The star product of two superfields is expanded explicitly by

$$f * g = f\, e^P g = fg + f\,(1+\sum_{n=1}^{4}\frac{1}{n!}P^n)\, g\,. \tag{3.2.6}$$

Analogous to $\mathcal{N}=(1/2,0)$ case, here the supercharges in (3.2.5) and the covariant derivatives are expressed explicitly in terms of the chiral basis by

$$Q^i_\alpha = \frac{\partial}{\partial\theta^\alpha_i}\,,\qquad \bar{Q}_{\dot{\alpha}i} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}i}} + 2i\theta^\alpha_i\,\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^\mu}, \tag{3.2.7}$$

$$D^i_\alpha = \frac{\partial}{\partial\theta^\alpha_i} + 2i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}i}\frac{\partial}{\partial y^\mu}\,,\qquad \bar{D}_{\dot{\alpha}i} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}i}}\,. \tag{3.2.8}$$

The deformation parameter $C^{\alpha\beta}_{ij}$ can be decomposed into the trace and the traceless parts of the $SU(2)_L$ spinor indices α,β and the $SU(2)$ R-symmetry indices i,j [84] [88]:

$$C^{\alpha\beta}_{ij} = \varepsilon^{\alpha\beta}\varepsilon_{ij}I + C^{(\alpha\beta)}_{(ij)}\,, \tag{3.2.9}$$

where I is some constant. The first term gives rise to the *singlet* deformation [88–92] and the second term to the *non-singlet* deformation [90] [93–96].

For the singlet deformation $C^{\alpha\beta}_{ij} = \varepsilon^{\alpha\beta}\varepsilon_{ij}I$, the bi-differential operator becomes

$$P_{\text{singlet}} = -\frac{1}{2}\overleftarrow{Q^i_\alpha}\, I\, \overrightarrow{Q^\alpha_i}\,. \tag{3.2.10}$$

P_{singlet} preserves $SO(4)$ symmetry and $SU(2)$ R-symmetry, but the supercharges $\bar{Q}_i$ are broken by P_{singlet}. The remaining supersymmetry for the singlet deformation is thus $\mathcal{N}=(1,0)$.

The non-singlet deformation arising from $C^{\alpha\beta}_{ij} = C^{(\alpha\beta)}_{(ij)}$ generically breaks the $SO(4)$ symmetry and $SU(2)$ R-symmetry. The preserved supersymmetry is $\mathcal{N}=$

$(1,0)$. An exception occurs for a particular factorisable decomposition of the deformation parameter:

$$C^{\alpha\beta}_{ij} = C^{\alpha\beta} b_{ij} \tag{3.2.11}$$

with $\det b = b^2 = \varepsilon^{ik}\varepsilon^{jl} b_{ij} b_{kl} = 0$. In this case b_{ij} is a matrix of rank 1, and allows the degenerate deformation

$$b_{11} \neq 0\ , \qquad b_{12} = b_{22} = 0\ . \tag{3.2.12}$$

So that the corresponding bi-differential operator of the star product

$$P_{\text{deg}} = -\frac{1}{2} C^{12}_{11} (\overleftarrow{Q}^1_1 \overrightarrow{Q}^1_2 + \overleftarrow{Q}^1_2 \overrightarrow{Q}^1_1) \tag{3.2.13}$$

breaks only $\bar{Q}_{i=1}$ supercharges, and the preserved supersymmetry is enhanced to $\mathcal{N} = (1, 1/2)$ [95,96]. As previously mentioned, this could happen only when taking $\bullet$ as the conjugation in the Euclidean $\mathcal{N} = (1,1)$ superspace.

3.2.1 Harmonic superspace approach and deformed $\mathcal{N} = 2$ SYM

The standard $\mathcal{N} = (1,1)$ superspace $(x^\mu, \theta_i, \bar{\theta}^i)$ is isomorphic to the coset space $\left\{\frac{\mathcal{N}=2 \text{ super-Poincaré group}}{SO(4)}\right\}$. Since the supercharges and the bosonic translation generator P_μ form $SU(2)$ R-symmetry doublets and singlet respectively, $\mathcal{N} = (1,1)$ superspace can be enlarged by enclosing the $SU(2)$ structure realised on the coset space $SU(2)/U(1) \sim S^2$. Consequently the harmonic superspace [97,98] is obtained by introducing extra harmonic variables $u^\pm_i$ on S^2 to the original superspace coordinates $(x^\mu, \theta_i, \bar{\theta}^i)$. With these extra variables u, there exists an invariant subspace for the harmonic superspace parametrised by half of the original fermionic variables (not in the chiral sense as in $\mathcal{N} = (1/2, 1/2)$), on which the superfields can be defined with definite harmonic Grassmann analyticity. This is not possible for the standard $\mathcal{N} = (1,1)$ superspace if the $SU(2)$ automorphism is to be preserved. With the harmonic superspace formalism, construction of unconstraint off-shell $\mathcal{N} = 2$ manifestly supersymmetric theories of Minkowskian signature becomes possible. Such construction is not available for the $\mathcal{N} = 2$ chiral superspace, because the chirality constraints imposed on superfields turn out to be the mass shell conditions [99].

The harmonic variables $u_i^\pm$ form a $SU(2)$ matrix

$$\begin{pmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^- \end{pmatrix} \tag{3.2.14}$$

with the $U(1)$ phase of each component being identified but the $U(1)$ charge being preserved, and satisfy the constraints

$$\begin{aligned} u^{i+}u_i^- &= 1 , \qquad u^{\pm i} = \varepsilon^{ij} u_j^\pm , \\ u^{+i}u_i^+ &= u^{-i}u_i^- = 0 . \end{aligned} \tag{3.2.15}$$

They are regarded as spin 1/2 object as they can be obtained from the Pauli matrices τ^1 and τ^2 [99]. With these harmonic variables, a function F of $u_i^\pm$ on S^2 carries a definite $U(1)$ charge q, denoted by $F^{(q)}$.

$u_i^\pm$ serve as a mapping between $SU(2)$ and $U(1)$ groups and convert the R-symmetry indices into U(1) indices. By contracting $\theta^i, \bar\theta^i$ with $u_i^\pm$, the analytic basis $(x_A, \theta^\pm, \bar\theta^\pm, u^\pm)$ for the harmonic superspace is constructed, such that[8]

$$\begin{aligned} x_A^\mu &:= x^\mu - i(\theta^-\sigma^\mu\bar\theta^+ + \theta^+\sigma^\mu\bar\theta^-) \qquad (3.2.16) \\ \theta_\alpha^\pm &:= u_i^\pm\theta_\alpha^i , \qquad \bar\theta_{\dot\alpha}^\pm := u_i^\pm\bar\theta_{\dot\alpha}^i , \end{aligned}$$

and the fermionic variables carry $U(1)$ charges $q = \pm 1$. By generalising the pseudo-conjugation (3.2.1) (as it preserves the $SU(2)$ R-symmetry structure) to the harmonics

$$\widetilde{u^{\pm i}} = u_i^\pm , \tag{3.2.17}$$

and combining with the standard conjugation (3.2.1) on $\theta_i, \bar\theta^i$, one obtains the pseudo-conjugations to the analytic variables

$$\widetilde{\theta^{\pm\alpha}} = \varepsilon_{\alpha\beta}\theta^{\pm\beta} , \qquad \widetilde{\bar\theta^{\pm\dot\alpha}} = \varepsilon_{\dot\alpha\dot\beta}\bar\theta^{\pm\dot\beta} , \qquad \widetilde{x_A^\mu} = x_A^\mu , \tag{3.2.18}$$

[8]The supersymmetric transformations of the analytic variables are also projected by the harmonics,

$$\begin{aligned} \delta_\epsilon x_A^\mu &= -2i(\epsilon^-\sigma^\mu\bar\theta^+ + \theta^+\sigma^\mu\bar\epsilon^-) , \\ \delta_\epsilon\theta_\alpha^\pm &= u_i^\pm\epsilon_\alpha^i := \epsilon_\alpha^\pm , \qquad \delta_\epsilon\bar\theta_{\dot\alpha}^\pm = u_i^\pm\bar\epsilon_{\dot\alpha}^i := \bar\epsilon_{\dot\alpha}^\pm , \end{aligned}$$

due to the fact that $u_i^\pm$ is supersymmetry invariant. $u_i^\pm$ themselves transform under $SU(2)$ R-symmetry by $u^\pm \to U_i^j u_j^\pm e^{\pm i\,(\text{phase})}$, where U_i^j is an $SU(2)$ matrix and the exponential factor is an induced $U(1)$ transformation.

in which objects with positive and negative $U(1)$ charges are not mixed.

The supercharges and the supercovariant derivatives are also constructed by analytic projections,

$$Q^{\pm}_{\alpha} = u^{\pm}_i Q^i_{\alpha}, \qquad \bar{Q}^{\pm}_{\dot{\alpha}} = u^{\pm}_i \bar{Q}^i_{\dot{\alpha}}, \qquad D^{\pm}_{\alpha} = u^{\pm}_i D^i_{\alpha}, \qquad \bar{D}^{\pm}_{\dot{\alpha}} = u^{\pm}_i \bar{D}^i_{\dot{\alpha}} . \tag{3.2.19}$$

Their explicit expressions in the analytic basis are

$$\begin{aligned}
Q^{+}_{\alpha} &= \frac{\partial}{\partial\theta^{-\alpha}} - 2i\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{+\dot{\alpha}}\frac{\partial}{\partial x^{\mu}_{A}}, & Q^{-}_{\alpha} &= -\frac{\partial}{\partial\theta^{+\alpha}}, \\
\bar{Q}^{+}_{\dot{\alpha}} &= \frac{\partial}{\partial\bar{\theta}^{-\dot{\alpha}}} + 2i\theta^{+\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^{\mu}_{A}}, & \bar{Q}^{-}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{+\dot{\alpha}}}, \\
D^{+}_{\alpha} &= \frac{\partial}{\partial\theta^{-\alpha}}, & D^{-}_{\alpha} &= -\frac{\partial}{\partial\theta^{+\alpha}} + 2i\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{-\dot{\alpha}}\frac{\partial}{\partial x^{\mu}_{A}}, \\
\bar{D}^{+}_{\dot{\alpha}} &= \frac{\partial}{\partial\bar{\theta}^{-\dot{\alpha}}}, & \bar{D}^{-}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{+\dot{\alpha}}} - 2i\theta^{-\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^{\mu}_{A}},
\end{aligned} \tag{3.2.20}$$

which satisfy the algebra

$$\{Q^{+}_{\alpha}, \bar{Q}^{-}_{\dot{\alpha}}\} = -\{Q^{-}_{\alpha}, \bar{Q}^{+}_{\dot{\alpha}}\} = -\{D^{+}_{\alpha}, \bar{D}^{-}_{\dot{\alpha}}\} = \{D^{-}_{\alpha}, \bar{D}^{+}_{\dot{\alpha}}\} = 2i\sigma^{\mu}_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^{\mu}_{A}} . \tag{3.2.21}$$

Note that $D^{+}_{\alpha}, \bar{D}^{+}_{\dot{\alpha}}$ form a closed subalgebra by $\{D^{+}_{\alpha}, D^{+}_{\beta}\} = \{D^{+}_{\alpha}, \bar{D}^{+}_{\dot{\alpha}}\} = \{\bar{D}^{+}_{\dot{\alpha}}, \bar{D}^{+}_{\dot{\beta}}\} = 0$, and so do $D^{-}_{\alpha}, \bar{D}^{-}_{\dot{\alpha}}$.

One can construct the superfield depending only on the invariant subspace $(x_A, \theta^+, \bar{\theta}^+, u^{\pm})$ by imposing the covariant conditions $D^{+}_{\alpha}\Phi = \bar{D}^{+}_{\dot{\alpha}}\Phi = 0$. The superfield surviving such constraint is called the analytic superfield, $\Phi = \Phi(x_A, \theta^+, \bar{\theta}^+, u^{\pm})$.

An analytic superfield contains infinitely many component fields by expansion. This can be obtained as follows. The analytic superfield $\Phi^{(q)}$ with $U(1)$ charge q can be expanded in terms of θ^+ and $\bar{\theta}^+$ up to $(\theta^+)^2(\bar{\theta}^+)^2$, where the component fields carry $U(1)$ charges ranging from q to $q-4$. Each θ^+- or $\bar{\theta}^+$-component, say $f^{(q)}(x_A, u)$ with $q > 0$, is subject to the harmonic expansion [99]

$$f^{(q)} = \sum_{n=0}^{\infty} f^{(i_1 \cdots i_{n+q} j_1 \cdots j_n)} u^{+}_{i_1} \cdots u^{+}_{i_{n+q}} u^{-}_{j_1} \cdots u^{-}_{j_n} . \tag{3.2.22}$$

Such harmonic expansion is supported by the fact that the symmetrised products $u^{+}_{(i} \cdots u^{+}_{j} u^{-}_{l} \cdots u^{-}_{m)}$ of the harmonics $u^{\pm}_i$ form a complete set of spherical harmonics on S^2, in terms of which the square integrable $f^{(q)}$ can be expanded. The expansion gives rise to an infinite number of u-independent components $f^{(i_1 \cdots i_{n+q} j_1 \cdots j_n)}$, among which there are finite physical fields and the remaining are the auxiliary fields or pure gauge degrees of freedom. Analogous procedures also applies to the $q < 0$ cases.

The charge q of a function is obtained by

$$D^0 F^{(q)} = qF^{(q)} \ , \tag{3.2.23}$$

where D^0 is one of the harmonic derivatives defined in the chiral basis $(x^\mu, \theta_i, \bar\theta^i, u^\pm)$:

$$D^{\pm\pm} = u^{\pm i}\frac{\partial}{\partial u^{\mp i}} \ , \qquad D^0 = [D^{++}, D^{--}] = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}} \ . \tag{3.2.24}$$

In the analytic basis, their explicit expressions are

$$\begin{aligned} D^{\pm\pm} &= \partial^{\pm\pm} - 2i\theta^\pm\sigma^\mu\bar\theta^\pm\frac{\partial}{\partial x^\mu_A} + \theta^{\pm\alpha}\frac{\partial}{\partial\theta^{\mp\alpha}} + \bar\theta^{\pm\dot\alpha}\frac{\partial}{\partial\bar\theta^{\mp\dot\alpha}} \ , \\ D^0 &= u^+_i\frac{\partial}{\partial u^+_i} - u^-_i\frac{\partial}{\partial u^-_i} + \theta^{+\alpha}\frac{\partial}{\partial\theta^{+\alpha}} + \bar\theta^{+\dot\alpha}\frac{\partial}{\partial\bar\theta^{+\dot\alpha}} - \theta^{-\alpha}\frac{\partial}{\partial\theta^{-\alpha}} - \bar\theta^{-\dot\alpha}\frac{\partial}{\partial\bar\theta^{-\dot\alpha}} \ , \end{aligned} \tag{3.2.25}$$

where $\partial^{\pm\pm} = u^{\pm i}\frac{\partial}{\partial u^{\mp i}}$, and D^{++}, D^{--}, D^0 satisfy a $su(2)$ algebra.

The deformation (3.2.5) can also be realised by analytic harmonic construction. By substituting $Q^i_\alpha = u^{+i}Q^-_\alpha - u^{-i}Q^+_\alpha$ into (3.2.5), together with the projections $u^{+i}u^{+j}C^{\alpha\beta}_{ij} = C^{++\alpha\beta}$ etc., one finds the corresponding operator P is expression by $P = \frac{-1}{2}\left(\overleftarrow{Q^-_\alpha}\, C^{++\alpha\beta}\overrightarrow{Q^-_\beta} - \overleftarrow{Q^-_\alpha}\, C^{+-\alpha\beta}\overrightarrow{Q^+_\beta} - \overleftarrow{Q^+_\alpha}\, C^{-+\alpha\beta}\overrightarrow{Q^-_\beta} + \overleftarrow{Q^+_\alpha}\, C^{--\alpha\beta}\overrightarrow{Q^+_\beta}\right)$. The non-(anti)commutative deformations become [90] [84]

$$\begin{aligned} \{\theta^{\pm\alpha}, \theta^{\pm\beta}\} &= C^{\pm\pm\alpha\beta} \ , & \{\theta^{\pm\alpha}, \theta^{\mp\beta}\} &= C^{\pm\mp\alpha\beta} \ , \\ [x^\mu_A, x^\nu_A] &= 4C^{--\mu\nu}(\bar\theta^+)^2 \ , & [x^\mu_A, \theta^{\pm\alpha}] &= -2iC^{-\pm\beta\alpha}(\sigma^\mu\bar\theta^+)_\beta \ . \end{aligned} \tag{3.2.26}$$

Since $[D^+_\alpha, P] = [\bar D^+_{\dot\alpha}, P] = 0$, the star-product preserves the Grassmann analyticity. For the singlet deformation (3.2.10), the bi-differential operator is simplified to

$$P_{\text{singlet}} = -i(\sigma^\mu\bar\theta^+)^\alpha I\left(\overleftarrow{\frac{\partial}{\partial x^\mu_A}}\overrightarrow{\frac{\partial}{\partial\theta^{+\alpha}}} - \overleftarrow{\frac{\partial}{\partial\theta^{+\alpha}}}\overrightarrow{\frac{\partial}{\partial x^\mu_A}}\right) \tag{3.2.27}$$

and becomes nilpotent $P^3 = 0$.

The field content of pure $\mathcal{N} = 2$ SYM is a gauge multiplet, which is described in the harmonic superspace formalism by an analytic gauge superfield V^{++} with $U(1)$ charge $q = +2$. In the non-abelian theory the gauge superfield takes value in $U(N)$

gauge group, $V^{++}=V^{++M}T^M$ with T^M the Lie algebra generators. The action of the non-commutative deformed $\mathcal{N}=2$ SYM action is given in terms of an infinite series of the star products of V^{++} [84],

$$S_V=\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\,\mathrm{tr}\int d^{12}\underline{z}\,du_1\ldots du_n\frac{V^{++}(z,u_1)*V^{++}(z,u_2)*\ldots*V^{++}(z,u_n)}{(u_1^+u_2^+)(u_2^+u_3^+)\ldots(u_n^+u_1^+)}\,, \tag{3.2.28}$$

with $\underline{z}=(x,\theta_i^\alpha,\bar{\theta}_i^{\dot{\alpha}})$. The action takes the same form as that of the ordinary $\mathcal{N}=2$ SYM [100], except that the star product is used.[9] [10] The deformed gauge transformation for V^{++} is given by

$$\delta_\Lambda V^{++}=D^{++}\Lambda+[V^{++}\,\overset{*}{,}\,\Lambda]\,, \tag{3.2.32}$$

where Λ is an analytic superfield parameter. With the presence of the star product, this formula is valid even for the $U(1)$ gauge group, although in the ordinary case the commutator doesn't appear in the gauge transformation.

By expanding the superfields and integrating out θ and u variables, one obtain the action in terms of component fields. Although the expansion of V^{++} give rise to infinite number of component fields as described previously, one can make use of the gauge freedom of Λ to eliminate all the auxiliary fields and retain finite number of physical fields. For example, the gauge superfield in Wess-Zumino gauge is expressed

[9]Another way to obtain the non-anticommutative $\mathcal{N}=2$ SYM action is via the integral over the left chiral coordinates $\int\frac{1}{4}\mathrm{tr}\int d^4y d^4\theta\, W^2$, where the gauge superfield strength $W,\bar{W}$ requires to define the superfield V^{--}. See e.g. [101] for a summary. The action (3.2.28) is more favorable at the quantum level as it is an expansion into infinite numbers of vertices (with star products) of V^{++}.

[10]The harmonic integrals are defined by [99]

$$\int du\, f^{(q)}(u) = 0 \quad \text{if } q\neq 0\,, \tag{3.2.29}$$

$$\int du\, 1 = 1\,,\qquad \int du\, u^+_{(i_1}\ldots u^+_{i_n}u^-_{j_1}\ldots u^-_{j_m)}=0\,, \tag{3.2.30}$$

and the integrations over analytic Grassmann variables are [101]

$$\int d^2\theta^+d^2\theta^-\,(\theta^+)^2(\theta^-)^2=1\,,\quad \int d^4\theta^-\,(\theta^+)^2(\bar{\theta}^+)^2=1\,,\quad \int d^4\theta^+d^4\theta^-\,(\theta^+)^2(\bar{\theta}^+)^2(\theta^-)^2(\bar{\theta}^-)^2=1\,. \tag{3.2.31}$$

by [84]

$$\begin{aligned} V_{WZ}^{++} &= (\theta^+)^2\bar{\phi} + (\bar{\theta}^+)^2\phi + 2(\theta^+\sigma^\mu\bar{\theta}^+)A_\mu + 4(\bar{\theta}^+)^2\theta^+\psi^i u_i^- \\ &\quad +4(\theta^+)^2\bar{\theta}^+\bar{\psi}^i u_i^- + 3(\theta^+)^2(\bar{\theta}^+)^2\mathcal{D}^{ij}u_i^- u_j^- \ . \end{aligned} \tag{3.2.33}$$

The component fields are all functions of x_A and form an off-shell gauge multiplet of $\mathcal{N}=(1,1)$ supersymmetry.

The gauge transformations of the component fields following (3.2.32) turn out to be deformed by C, but they can be reduced to the canonical forms by performing component field redefinitions. The supersymmetry transformations (which are compensated by gauge transformations to preserve the WZ gauge) are also C-deformed, but again can be switched back to the canonical ones via field redefinitions. For the singlet deformations to SYM with $U(1)$ gauge group [89, 91], there exist field redefinitions under which both the gauge and supersymmetry transformations of the component fields are canonical. The invariant action of singlet-deformed $\mathcal{N}=(1,0)$ SYM is found by such constructions. For generic non-single deformations, one can restore the canonical gauge transformations by the component field redefinitions, by which, however, the supersymmetry transformations still remain deformed by C. This resembles the situation reviewed in Section 3.1 for case of non-anticommutative $\mathcal{N}=1$ SYM [69]. The example of the non-singlet deformed $\mathcal{N}=2$ $U(1)$ theory is elaborated in [90, 93].

The quantum divergence of the gauge multiplet with singlet deformations can be eliminated by a C-related shift of the scalar field ϕ. Therefore the corresponding deformed $\mathcal{N}=2$ SYM is renormalisable [76].

3.2.2 Harmonic superspace construction of non-anticommutative $\mathcal{N}=4$ SYM

So far there is no available superspace formalism for unconstrained manifest $\mathcal{N}=4$ supersymmetry. Non-anticommutative deformation of $\mathcal{N}=4$ super Yang-Mills theory can be realised in deformed $\mathcal{N}=(1/2,1/2)$ superspace in terms of a vector superfield and three chiral superfields, or in deformed harmonic superspace in terms of a gauge and a hypermultiplet superfield. The gauge multiplet is represented by an analytic V^{++} in the harmonic superspace formalism. The hypermultiplet can either be described by a complex analytic superfield q^+ with $+1$ $U(1)$ charge, or by a $U(1)$ neutral real analytic superfield ω. Since the two models are related by a duality [99], we adopt the q^+ description here.

The hypermultiplet is in adjoint representations of $U(N)$, and the corresponding superfield is $U(N)$-valued, $q^+ = q^{+M}T^M$. Under the non-anticommutative deformation, the free action of q^+ is undeformed, and is given by an integral over the analytic subspace

$$S_q^{\text{free}} = -\text{tr} \int d^4x_A d^4\theta^- du \ \tilde{q}^+ D^{++} q^+ \ , \tag{3.2.34}$$

where $\tilde{q}^+$ is the conjugate to q^+. However the interaction terms are deformed. Therefore, in adjoint representation, the hypermultiplet superfields follow the gauge transformations

$$\delta_\Lambda \tilde{q}^+ = [\tilde{q}^+ \overset{*}{,} \Lambda] \ , \qquad \delta_\Lambda q^+ = [q^+ \overset{*}{,} \Lambda] \ . \tag{3.2.35}$$

and in the presence of the gauge superfield, $D^{++}q^+$ is replaced by

$$D^{++}q^+ \to D^{++}q^+ + [V^{++} \overset{*}{,} q^+] \ . \tag{3.2.36}$$

The coupling of q^+ to V^{++} is given by

$$S_q = -\text{tr} \int d^4x_A d^4\theta^- du \ \tilde{q}^+ \left(D^{++}q^+ + [V^{++} \overset{*}{,} q^+] \right) \ . \tag{3.2.37}$$

Together with the gauge superfield action in (3.2.28), the action of the non-anticommutative deformed $\mathcal{N} = 4$ super Yang-Mills theory is expressed in terms of $\mathcal{N} = 2$ superfields by

$$S_{\mathcal{N}=4SYM} = S_V + S_q \ . \tag{3.2.38}$$

For generic non-anticommutative deformations, the supersymmetry for such theory is manifest $\mathcal{N} = (1, 0)$. At the limit of vanishing deformation constant $C = 0$, the action (3.2.38) has an on-shell $\mathcal{N} = (1, 1)$ supersymmetry and a manifest $\mathcal{N} = (1, 1)$, together forming $\mathcal{N} = (2, 2)$ supersymmetry.

After expanding the hypermultiplet superfield in analytic Grassmann and harmonic variables, the infinite auxiliary fields can be eliminated by making use of the non-dynamical equations of motion derived from the superfield equation of motion for q^+. The action in terms of component fields is obtained by using such q^+ together with V^{++}_{WZ} in (3.2.38) and integrating over θ and u variables. Again, one can restore the canonical gauge transformations by redefinitions of the component fields, but supersymmetry transformations for the redefined fields remain deformed [92]. The singlet-deformed non-anticommutative $\mathcal{N}=4$ SYM is also renormalisable, as the divergence in the Feynman diagrams can be removed by a shift of the scalar field ϕ in the gauge multiplet [76].

3.3 Non-anticommutative $\mathcal{N}=4$ SYM in string theory

Similar to the previously discussed cases, non-anticommutative deformed $\mathcal{N}=(2,2)$ supersymmetry arises from the deformation

$$\{\theta^\alpha_A, \theta^\beta_B\} = C^{\alpha\beta}_{AB} , \tag{3.3.1}$$

where $(y^\mu, \theta^\alpha_A, \bar{\theta}^{\dot{\alpha}A})$ are the chiral coordinates parametrising the $\mathcal{N}=(2,2)$ superspace. $\alpha, \dot{\alpha} = (1,2)$ are the spinor indices and $A=(1,2,3,4)$ are the $SU(4)$ R-symmetry indices. The deformation parameter satisfies $C^{\alpha\beta}_{AB} = C^{\beta\alpha}_{BA}$. The deformation (3.3.1) can also be decomposed into the singlet part $C^{\alpha\beta}_{AB} = \varepsilon^{\alpha\beta}\varepsilon_{AB} I$ and the non-singlet part $C^{\alpha\beta}_{AB} = C^{(\alpha\beta)}_{(AB)}$, as in (3.2.9) for the $\mathcal{N}=(1,1)$ case.

Non-anticommutative deformed $\mathcal{N}=4$ SYM is obtained as the low energy effective theory on D3-branes in [82], by means of the string scattering amplitudes in the constant RR field background $\mathcal{F}^{\alpha\beta AB}$ using RNS formalism[11]. $\mathcal{F}^{\alpha\beta AB}$ is of canonical length dimension -2 and scaled by

$$(2\pi\alpha')^{\frac{3}{2}}\mathcal{F}^{\alpha\beta AB} := C^{\alpha\beta AB} , \tag{3.3.2}$$

such that at the field theory limit $\alpha' \to 0$, $\mathcal{F}^{\alpha\beta AB}$ is scaled to infinity to keep $C^{\alpha\beta AB}$ fixed. For N D3-branes in IIB background, the 10-dimensional Euclidean Lorentz symmetry is decomposed into $SO(4)\times SO(6)$. The 10-dimensional spin field S^λ is decomposed accordingly into $(S_\alpha S_A, S^{\dot{\alpha}} S^A)$, where $S_\alpha, S^{\dot{\alpha}}$ and S_A, S^A

[11] Same approach for non-anticommutative deformed $\mathcal{N}=2$ SYM can be found in [81].

are 4- and 6-dimensional Weyl spinors respectively. The 6-dimensional spinors are in (anti-) fundamental $SU(4)$ representations. By the same decomposition, the 10-dimensional gamma matrices are decomposed into the 4-dimensional part $\sigma^\mu, \bar{\sigma}^\mu$ with $\mu = 0, 1, 2, 3$ and the 6-dimensional part $\Sigma^a, \bar{\Sigma}^a$ with $a = 4, ..., 9$. For convenience, from here on we use a supersymmetry notation appeared in [82] (summarised in Appendix B) different from Wess and Bagger's. The 2-component gamma matrices are defined by

$$\sigma^\mu = (i\tau^1, i\tau^2, i\tau^3, \mathbf{1}) , \qquad \bar{\sigma}^\mu = (-i\tau^1, -i\tau^2, -i\tau^3, \mathbf{1}) , \tag{3.3.3}$$

where τ^1, τ^2, τ^3 are Pauli matrices, and $\Sigma^a, \bar{\Sigma}^a$ are defined by

$$\begin{aligned} \Sigma^a &= (\ \eta^3, -i\bar{\eta}^3, \ \ \eta^2, -i\bar{\eta}^2, \ \ \eta^1, i\bar{\eta}^1) , \\ \bar{\Sigma}^a &= (-\eta^3, -i\bar{\eta}^3, -\eta^2, -i\bar{\eta}^2, -\eta^1, i\bar{\eta}^1) , \end{aligned} \tag{3.3.4}$$

where $\eta^1, \eta^2, \eta^3, \bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3$ are the 't Hooft symbols whose properties are reviewed in Appendix A and matrix expressions are given explicitly in (B.2.2).

The massless open string excitations are gauge boson A_μ, 4 gauginos $\psi_{\alpha A}$, and 6 scalars ϕ^a with $a = 1, ..., 6$. Each field is represented by a vertex operator. The auxiliary fields also introduced to decouple the quartic interactions into cubic ones in the open string sector. As for the closed string fields, with the spinor field decompositions mentioned above, the constant RR background is described by the vertex operator in $(-\frac{1}{2}, -\frac{1}{2})$ picture,

$$V_{\mathcal{F}}^{(-1/2,-1/2)}(z, \bar{z}) = (2\pi\alpha')^{3/2} \mathcal{F}^{\alpha\beta AB} S_\alpha S_A e^{-\phi/2}(z) \tilde{S}_\beta \tilde{S}_B e^{-\tilde{\phi}/2}(\bar{z}) , \tag{3.3.5}$$

where the RR field $\mathcal{F}^{\alpha\beta AB}$ is a product the 4-dimensional part $f^{\alpha\beta}$ and the 6-dimensional part g^{AB}. Both $f^{\alpha\beta}$ and g^{AB} can be decomposed into the symmetric and the anti-symmetric parts:

$$f^{\alpha\beta} = f^{[\alpha\beta]} + f^{(\alpha\beta)} = f \epsilon^{\alpha\beta} + f_{\mu\nu} (\sigma^{\mu\nu})^{\alpha\beta} , \tag{3.3.6}$$

$$g^{AB} = g^{[AB]} + g^{(AB)} = g_a (\Sigma^a)^{AB} + g_{abc} (\Sigma^{abc})^{AB} , \tag{3.3.7}$$

in which ( ) denotes symmetrised indices and [ ] anti-symmetrised indices. $\sigma^{\mu\nu}$ and Σ^{abc} are defined by

$$\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) , \qquad \Sigma^{abc} := \Sigma^{[a} \bar{\Sigma}^b \Sigma^{c]} , \tag{3.3.8}$$

and are self-dual in 4 dimensions and 6 dimensions respectively:

$$\sigma^{\mu\nu} = \frac{1}{2!}\epsilon^{\mu\nu\rho\lambda}\sigma^{\rho\lambda} , \qquad \Sigma^{abc} = \frac{i}{3!}\epsilon^{abcdef}\Sigma^{def} . \tag{3.3.9}$$

In (3.3.6) and (3.3.7), $f^{\alpha\beta}$ is composed of a singlet (anti-symm.) and a selfdual 2-form (symm.), while g^{AB} contains a vector (anti-symm.) and a selfdual 3-form (symm.). Consequently, $\mathcal{F}^{\alpha\beta AB}$ is decomposed into components of (A,A), (A,S), (S,A), (S,S) types:

$$\begin{aligned}
&\mathcal{F}^{\alpha\beta AB} \\
&= f g_a \epsilon^{\alpha\beta}(\Sigma^a)^{AB} + f g_{abc}\epsilon^{\alpha\beta}(\Sigma^{abc})^{AB} + f_{\mu\nu} g_a (\sigma^{\mu\nu})^{\alpha\beta}(\Sigma^a)^{AB} + f_{\mu\nu} g_{abc}(\sigma^{\mu\nu})^{\alpha\beta}(\Sigma^{abc})^{AB}, \\
&= \quad \mathcal{F}^{[\alpha\beta][AB]} \quad + \quad \mathcal{F}^{[\alpha\beta](AB)} \quad + \quad \mathcal{F}^{(\alpha\beta)[AB]} \quad + \quad \mathcal{F}^{(\alpha\beta)(AB)} \\
&= \quad (\mathrm{A},\mathrm{A}) \quad + \quad (\mathrm{A},\mathrm{S}) \quad + \quad (\mathrm{S},\mathrm{A}) \quad + \quad (\mathrm{S},\mathrm{S}) .
\end{aligned} \tag{3.3.10}$$

The (A,A)-component is an RR 1-form. The constant (A,A)-type background corresponds to the singlet deformation of harmonic superspace [89] which has back-reactions to the RR background [81]. The (A,S)- and (S,A)-type components are RR 3-forms, giving rise to non-trivial deformation to $\mathcal{N}=2$ and $\mathcal{N}=4$ SYM which cannot be realised by deformed superspaces [103] [102]. In particular, the (A,S)-type deformation (with a different scaling by $(2\pi\alpha')^{1/2}\mathcal{F}^{[\alpha\beta](AB)}=$ fixed) is interpreted in terms of $\mathcal{N}=1$ superfields as the mass deformation of super Yang-Mills theory [102]. The (S,S) type is an RR 5-form carrying the tensor structure of the Clifford algebra in (3.1.2), (3.2.3) and (3.3.1), and is regarded as being responsible for the generic non-singlet deformations to superspaces, provided that the RR 5-form is scaled according to (3.3.2)

Since $\sigma^{\mu\nu}$ and Σ^{abc} are 4- and 6-dimensionally selfdual, consequently the constant (S,S)-type RR 5-form $F_{\mu\nu abc} = f_{\mu\nu}g_{abc}$ satisfies the "double self-duality" condition:

$$F_{\mu\nu abc} = \frac{1}{2!}\epsilon_{\mu\nu\rho\lambda}F_{\rho\lambda abc} , \qquad F_{\mu\nu abc} = \frac{-i}{3!}\epsilon_{abcdef}F_{\mu\nu def} . \tag{3.3.11}$$

An example of the minimal constant RR-flux configuration under these conditions is

$$F_{01456} = F_{23456} = -iF_{01789} = -iF_{23789} = \text{constant} , \tag{3.3.12}$$

In next chapter, we are going to construct the supergravity background associated with such selfdual RR 5-forms.

The Lagrangian for non-anticommutative deformed $\mathcal{N}=4$ super Yang-Mills theory constructed in [82] up to first order in C by means of the open string scattering amplitudes with insertion of one RR vertex operator. The bosonic terms of C^1 order in the Lagrangian is interpreted as the Chern-Simons coupling of D3-branes to the constant RR 5-form, which can be realised by $\mathcal{N}=4$ SYM on $\mathcal{N}=(1/2,0)$ superspace [104, 105].

We close this section by making a few comments regarding the deformed $\mathcal{N}=2$ supersymmetric gauge theory. The discussions on the string theory construction of non-anticommutative $\mathcal{N}=4$ SYM is a generalisation from the $\mathcal{N}=2$ case. The non-anticommutative $\mathcal{N}=2$ SYM is realised on D3-branes compactified on $\mathbf{C}^2/\mathbf{Z}_2$ in the constant graviphoton background $\mathcal{F}^{\alpha\beta ij}$ with $i=1,2$. The spacetime symmetry is broken into $SO(10)=SO(4)\times SO(2)\times SU(2)$ by the compactification, and the spinor field on the close string is decomposed into $S^\lambda \rightarrow (S^\alpha S^{(-)} S^i, S^{\dot\alpha} S^{(+)} S^i)$. The non-singlet deformation to $\mathcal{N}=2$ is also induced by the (S,S)-type RR background $\mathcal{F}^{(\alpha\beta)(ij)}$ [89] [81]. In the case where only $\mathcal{F}^{(\alpha\beta)(11)} \sim C^{\alpha\beta}$ is non-zero, the Lagrangian is precisely the $\mathcal{N}=2$ SYM defined on $\mathcal{N}=(1/2,0)$ superspace. For the abelian gauge theory, this deformed Lagrangian can be identified with that of $\mathcal{N}=2$ constructed on the harmonic superspace subject to $\{\theta^{\alpha i},\theta^{\beta j}\}=C^{\alpha\beta}b^{ij}$ non-singlet deformation with $\det b=0$ [81]. The singlet deformation arise from the (A,A)-type background $\mathcal{F}^{[\alpha\beta][ij]}$ which in $\mathcal{N}=2$ case is an RR scalar [89].

Chapter 4

The Supergravity Dual to $\mathcal{N} = (1,0)$ Super Yang-Mills Theory

In this chapter we will present the supergravity dual for the non-anticommutative deformed $\mathcal{N} = 4$ super Yang-Mills theory with $\mathcal{N} = (1,0)$ supersymmetry.

The action for the deformed SYM on the D3-branes can be expressed in terms of a gauge and a hypermultiplet superfield in the deformed $\mathcal{N} = (1,1)$ harmonic superspace. Such deformation is introduced to the harmonic superspace by turning on a rank two (A, B)-indexed part of the (S,S) background graviphoton field $\mathcal{F}^{\alpha\beta AB} = \mathcal{F}^{(\alpha\beta)(AB)}$ in the target space.

To construct the supergravity dual, we start by proposing an RR 5-form configuration which satisfies the double self-duality conditions in (3.3.11) and gives rise to the required $\mathcal{F}$. The 5-form field strength is generated by a set of intersecting D3-branes which preserves the same fraction of supersymmetries as in $\mathcal{N} = (1,0)$ super Yang-Mills. We will see that the metric is complex due to the complex nature of the RR 5-forms. The supergravity dual is obtained by taking the near horizon limit of the N D3's, where the radial coordinate transverse to AdS_5 and the background RR 5-forms are both rescaled by α'. Finally, we show that the spectrum of the field theory operators dual to the bulk scalar field modes which are independent of the "deformed S^5" is undeformed by non-anticommutativity.

The results in Chapter 4 and Chapter 5 can be found in [2].

4.1 RR-flux configuration

As reviewed in the previous chapter, from the string theory point of view, the non-anticommutative deformation to the SU(N) super Yang-Mills theory arises from the nonvanishing constant RR 5-form background fields on the N D3-branes where the SYM lives. The RR 5-form satisfies the "double self-duality" condition (3.3.11):

$$F_{\mu\nu abc} = \frac{1}{2!}\epsilon_{\mu\nu\rho\lambda}F_{\rho\lambda abc} \ , \qquad F_{\mu\nu abc} = \frac{-i}{3!}\epsilon_{abcdef}F_{\mu\nu def} \ .$$

The supersymmetric gauge theory with $\mathcal{N} = (1,0)$ supersymmetry can be realised on the D3-branes by turning on the following background field configuration on the branes which satisfies (3.3.11):

$$\begin{aligned} F_{01456} = F_{23456} = -iF_{23789} = -iF_{01789} = 2c \ , \\ F_{01786} = F_{23786} = -iF_{23459} = -iF_{01459} = 2c \ , \end{aligned} \qquad (4.1.1)$$

where c is a real constant. Note that each line in (4.1.1) is a minimal configuration for the double self-duality condition. The supergravity background which gives rise to this RR-field configuration will be presented in the next section; it will be clear then why a single set of minimal configuration is not sufficient for our construction of the supergravity dual for $\mathcal{N} = (1,0)$ SYM.

Given the field components in (4.1.1), the (S,S) part $\mathcal{F}^{(\alpha\beta)(AB)}$ of (3.3.10) is expressed by

$$\begin{aligned} \mathcal{F}^{\alpha\beta AB} = F_{\mu\nu abc}(\sigma^{\mu\nu})^{\alpha\beta}(\Sigma^{abc})^{AB} &= 48\,c\,(\sigma^{01})^{\alpha\beta}(\Sigma^{456} + i\Sigma^{459})^{AB} \\ &= 48\,c\,i(\tau^3)^{\alpha\beta}M^{AB} \ , \end{aligned} \qquad (4.1.2)$$

where the convention for the gamma matrices σ^μ in (3.3.3) is used, and $M^{AB} := \Sigma^{456} + i\Sigma^{459}$. Half of $\mathcal{N} = (1,1)$ supersymmetry is broken by turning on a 2×2 sub-block of the (A, B)-indexed 4×4 part of $\mathcal{F}$ explicitly, and this can be achieved by taking the following basis of Σ^a from that given in (3.3.4):

$$\Sigma^{6,9,4,5,7,8} = \Sigma^{4,5,6,7,8,9}_{(3.3.4)} \ , \qquad (4.1.3)$$

such that

$$M^{AB} = \Sigma^{456} + i\Sigma^{459} = 2i \begin{pmatrix} \tau_1 & 0 \\ 0 & 0 \end{pmatrix} := M_{(1,0)} \ . \qquad (4.1.4)$$

The identification of Σ^a in (4.1.3) gives rise to an explicit rank two 4×4 matrix $M_{(1,0)}$. Identifications of Σ^a other than (4.1.3) produce different matrices $M^{AB} = M_1$ which can be related to $M_{(1,0)}$ by the bi-unitary transformations[1]

$$M_1 = V^T M_{(1,0)} V \ , \tag{4.1.5}$$

where V, V^T are unitary but $V^T V \neq 1$. The vertex operator (3.3.5) with $M^{AB} = M_1$ for $\mathcal{F}$ is equivalent to the vertex operator with $M^{AB} = M_{(1,0)}$ and the spin fields under the transformations

$$S \to VS \ , \quad \tilde{S} \to V\tilde{S} \ . \tag{4.1.6}$$

For instance, the identification

$$\Sigma^{6,9,4,5,7,8} \doteq \Sigma^{4,8,6,7,5,9}_{(3.3.4)} \tag{4.1.7}$$

gives rise to a matrix M_1 relating to $M_{(1,0)}$ by the bi-unitary transformation under

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} \tau^1 & -\tau^2 \\ -\tau^2 & \tau^1 \end{pmatrix} \tag{4.1.8}$$

where

$$V^T V = -V V^T = -i \begin{pmatrix} 0 & \tau^3 \\ \tau^3 & 0 \end{pmatrix} \neq 1 \ . \tag{4.1.9}$$

With $M_{(1,0)}$, we show that the RR 5-form components in (4.1.1) constitute the graviphoton field $F^{\alpha\beta AB}$ which takes the factorised form (3.2.11) of non-singlet deformation with $\det b \neq 0$ for $\mathcal{N} = (1, 1)$ supersymmetry.

Note that in the flat Euclidean background spacetime, the constant RR 5-forms (4.1.1) can be accommodated in the target space without altering the background geometry, since the energy-momentum tensor generated by the fields in (4.1.1) vanishes:

$$T_{MN} = F_{MM_1M_2M_3M_4} F_N{}^{M_1M_2M_3M_4} - \frac{1}{10} g_{MN} F^2 = 0 \ . \tag{4.1.10}$$

In the presence of N D3-branes, however, the background becomes $AdS_5 \times S^5$ at the near horizon limit and is no longer flat, so the energy momentum tensor doesn't vanish and the geometry is expected to deviate from $AdS_5 \times S^5$.

[1]The supergravity configuration which gives rise to $M_{(1,0)}$ involving a particular choice of the basis for Σ^a is given in Appendix C.

4.2 The supergravity dual

Intersecting brane configuration

Let the coordinates $(x^0, ..., x^9)$ denote the 10-dimensional target space. We propose that the RR 5-form field strength in (4.1.1) is sourced by the following setup of intersecting D3-branes, where "•" indicates the worldvolume dimensions of each brane, and "×" denotes the transverse dimensions in which each brane is localized:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
N D3	•	•	•	•	×	×	×	×	×	×
$D3_1$	•	•			•	•	×			×
$D3_2$	•	•					×	•	•	×
$D3_3$			•	•	•	•	×			×
$D3_4$			•	•			×	•	•	×

(4.2.1)

The N D3-branes where the super Yang-Mills lives are intersected by 4 additional D3's. The former are localized in all of the 6 transverse dimensions, while the latter are localised in (x^6, x^9) only. Each of the additional D3-branes intersects the N D3-branes in two common longitudinal dimensions, in (x_0, x_1) for $D3_{1,2}$ and (x_2, x_3) for $D3_{3,4}$. Additionally, $D3_1$, $D3_3$ intersect in (x^4, x^5) and $D3_2$, $D3_4$ intersect in (x^7, x^8). It is assumed that all branes locate at the same position in their transverse dimensions in order to intersect.

The partial localisation allows the branes $D3_1 \cdots D3_4$ to generate the constant RR 5-form "background" fields of (4.1.1) which induces non-anticommutative deformation to the $\mathcal{N} = (1,1)$ superspace and break half of supersymmetry on the N D3-branes.

Preserved supersymmetry

To construct the supergravity dual of $\mathcal{N} = (1,0)$ non-anticommutative SYM out of the brane configuration in (4.2.1), these branes at the near horizon limit of the N D3's should preserve the same fraction of supersymmetry as that of the deformed SYM. In the following we show that this is indeed the case.

As reviewed in Section 2.1, in general a D-brane breaks the spacetime supersymmetry by half. In our supergravity background, the two Majorana-Weyl spacetime spinors ϵ_1 and ϵ_2 are related by the projection condition associated with the N D3-branes

$$\hat{\Gamma}_{0123}\,\epsilon_1 = \epsilon_2 \ , \tag{4.2.2}$$

and those due to $D3_1 \cdots D3_4$:

$$\hat{\Gamma}_{0145}\,\epsilon_1 = \epsilon_2 \ , \quad \hat{\Gamma}_{0178}\,\epsilon_1 = \epsilon_2 \ , \quad \hat{\Gamma}_{2345}\,\epsilon_1 = \epsilon_2 \ , \quad \hat{\Gamma}_{2378}\,\epsilon_1 = \epsilon_2 \ , \tag{4.2.3}$$

where $\hat{\Gamma}_a$'s are the 10-dimensional tangent space gamma matrices and $\hat{\Gamma}_{0123} = \hat{\Gamma}_0\hat{\Gamma}_1\hat{\Gamma}_2\hat{\Gamma}_3$, etc.. The conditions in (4.2.3) are not totally independent and can reduce to

$$\hat{\Gamma}_{0123}\,\epsilon_1 = -\epsilon_1 \ , \quad \hat{\Gamma}_{4578}\,\epsilon_1 = -\epsilon_1 \ , \quad \hat{\Gamma}_{2378}\,\epsilon_1 = \epsilon_2 \ . \tag{4.2.4}$$

(4.2.2) and (4.2.4) combine into 3 independent projection conditions on ϵ_1. ϵ_2 can be derived from ϵ_1 by (4.2.2). Therefore, the brane configuration (4.2.1) in general preserves 1/16 of supersymmetry, i.e. 2 supersymmetries.

At the near horizon limit of a D3-brane, the geometry becomes $AdS_5 \times S^5$, so the projection condition associated with the brane is lifted. As a result, the constraint (4.2.2) is lifted at the near horizon limit of the N D3-branes, while only (4.2.4) remains valid. The third condition in (4.2.4) relates ϵ_2 to ϵ_1, and the first two conditions reduce 16 ϵ_1 components down to 4. Moreover, $\hat{\Gamma}_{0123}\,\epsilon_1 = -\epsilon_1$ implies that ϵ_1 is chiral both in the 4-dimensional and 6-dimensinal sense. So the preserved supersymmetry can be denoted by

$$\epsilon^{\alpha A} : \qquad \alpha = 1, 2; \quad A = 1, 2 \ , \tag{4.2.5}$$

which agrees with that of $\mathcal{N} = (1, 0)$ non-anticommutative super Yang-Mills theory.

To summarise, the existence of the 4 additional intersecting branes generates the exact background RR field components of (4.1.1) for the N D3-branes as well as preserves the correct fraction of supersymmetry corresponding to that of the non-anticommutative deformed supersymmetric gauge theory on the N D3's.

Metric and fields

Let's denote the harmonic function associated with N D3-branes H and those with $D3_{1,2,3,4}$-branes $H_{extra} := \{H_1, H_2, H_3, H_4\}$ respectively. The harmonic functions in general depends on the transverse dimensions in which the brane is localised.

According to the "harmonic function rule" reviewed in Chapter 2, the metric describing the intersecting branes (4.2.1) is

$$\begin{aligned} ds^2 &= \sqrt{\frac{H_3H_4}{HH_1H_2}}(dx_0^2+dx_1^2)+\sqrt{\frac{H_1H_2}{HH_3H_4}}(dx_2^2+dx_3^2)+\sqrt{\frac{HH_2H_4}{H_1H_3}}(dx_4^2+dx_5^2) \\ &+\sqrt{\frac{HH_1H_3}{H_2H_4}}(dx_7^2+dx_8^2)+\sqrt{HH_1H_2H_3H_4}(dx_6^2+dx_9^2) \ , \end{aligned} \tag{4.2.6}$$

The self-dual RR 5-form fields sourced by the N D3-branes is

$$F_0 := d(\frac{1}{H})dx^{0123} + \text{dual} \ , \tag{4.2.7}$$

and those sourced by the additional set of intersecting branes are,

$$F_1 := d(\frac{1}{H_1})dx^{0145} + d(\frac{1}{H_2})dx^{0178} + d(\frac{1}{H_3})dx^{2345} + d(\frac{1}{H_4})dx^{2378} + \text{dual} \ . \tag{4.2.8}$$

The wedge products are implicit in the expressions above. The overall 5-form fields generated by the branes in (4.2.1) is $F = F_0 + F_1$.

As the constant c in (4.1.1) is switched off, the correspondence is between the undeformed $\mathcal{N} = 4$ SYM and the near horizon limit of the N D3-branes which are fully localised in the transverse space. Therefore, it is natural to choose H as a function of all transverse dimensions, $H := H(x_a)$ where $a = 4, ..., 9$. The harmonic functions H_{extra} are chosen to depend on (x_6, x_9) only, in order for the $D3_1 \cdots D3_4$-branes to produce exact 5-form components as in (4.1.1):

Source	$D3_1$	$D3_2$	$D3_3$	$D3_4$
Components	F_{01456}, F_{01459}	F_{01786}, F_{01789}	F_{23456}, F_{23459}	F_{23786}, F_{23789}
	F_{23789}, F_{23786}	F_{23459}, F_{23456}	F_{01789}, F_{01786}	F_{01459}, F_{01456}

(4.2.9)

In this table, the first row consists of the electric part in (4.2.8), and the second row corresponds to the dual part in (4.2.8).

Comparing (4.2.8) and (4.1.1), we obtain the following differential equations:

$$\partial_6\left(\frac{1}{H_{1,2,3,4}}\right) = c \ , \quad \partial_9\left(\frac{1}{H_{1,2,3,4}}\right) = ic \ . \tag{4.2.10}$$

To produce such a complex structure, we complexify (x_6, x_9) coordinates by

$$z := x_6 + ix_9 \ , \quad \bar{z} := x_6 - ix_9 \ , \tag{4.2.11}$$

and it is necessary that the harmonic functions H_{extra} depend on z only[2]:

$$H_{extra}(z) = \{H_1(z), H_2(z), H_3(z), H_4(z)\} \ . \tag{4.2.12}$$

The harmonic functions

The field equations arising from the partially localised intersecting brane configuration (4.2.1) are

$$\begin{aligned}
(H_1 H_3 \partial_i^2 + H_2 H_4 \partial_m^2 + \partial_p^2) H &= 0 \ , & (4.2.13)\\
\partial_p \left(\frac{H_1^2}{H_4^2} \partial_p (\frac{1}{H_1}) \right) &= 0 \ , & (4.2.14)\\
\partial_p \left(\frac{H_2^2}{H_3^2} \partial_p (\frac{1}{H_2}) \right) &= 0 \ , & (4.2.15)\\
\partial_p \left(\frac{H_3^2}{H_2^2} \partial_p (\frac{1}{H_3}) \right) &= 0 \ , & (4.2.16)\\
\partial_p \left(\frac{H_4^2}{H_1^2} \partial_p (\frac{1}{H_4}) \right) &= 0 \ , & (4.2.17)
\end{aligned}$$

where the abbreviation $i = 4,5$,$m = 7,8$, $p = 6,9$ are used and the Einstein summation convention applies to ∂_i^2, etc.

By complexifying the coordinates (x_6, x_9) such that $\partial_p^2 = \partial_6^2 + \partial_9^2 = 4\,\partial_z \partial_{\bar{z}}$, the equations (4.2.14)-(4.2.17) are trivially satisfied since H_{extra} are holomorphic functions. H_{extra} can be obtained by solving the complexified version of (4.2.10),

$$\partial_z \left(\frac{1}{H_{extra}} \right) = c \ , \quad \partial_{\bar{z}} \left(\frac{1}{H_{extra}} \right) = 0 \ , \tag{4.2.18}$$

[2]Suppose a general complex function f depends on z and $\bar{z}$: $f = f(z, \bar{z})$. To comply with the complex structure in (4.1.1), f satisfies the simultaneous differential equations

$$\begin{aligned}
\partial_6 f(z, \bar{z}) &= \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} = c \ , \\
\partial_9 f(x, \bar{z}) &= i\frac{\partial f}{\partial z} - i\frac{\partial f}{\partial \bar{z}} = ic \ ,
\end{aligned}$$

which result in $\frac{\partial f}{\partial \bar{z}} = 0$ and thus f is a holomorphic function: $f = f(z)$.

under the constraint that the charges associated with F_1 are well defined. Since $D3_1 \cdots D3_4$-branes are eventually localised in one dimension z, this implies that $*F_1$ should be finite at $|z| = \infty$. Moreover, H_{extra} satisfy the requirement that $H_{extra} \longrightarrow 1$ as $c \longrightarrow 0$. The unique solutions are given by

$$H_1 = H_2 = H_3 = H_4 = \frac{1}{1+cz} . \tag{4.2.19}$$

With these given solutions, the dual part of F_1 in (4.2.8) can be easily obtained. In fact, each field component in (4.1.1) contains equal electric and magnetic contributions from two branes respectively. For example, $F_{01456} = \partial_6(1/H_1) - i\partial_9(1/H_4) = 2c$, in which the first term is sourced by $D3_1$ and the second term is sourced by $D3_4$, with overall $F_{01456} = 2c$.

Using the expressions for H_{extra} given above, the equation (4.2.13) becomes

$$\left(\partial_i^2 + \partial_m^2 + \frac{4}{H_1^2}\partial_z\partial_{\bar{z}}\right) H = 0 , \tag{4.2.20}$$

To solve this differential equation, it's convenient to introduce a new variable w,

$$w := \int H_1^2 dz = \frac{z}{1+cz} , \tag{4.2.21}$$

where $w = 0$ as $z = 0$, and $z = w$ as c is turned off. Then (4.2.20) transforms into

$$(\partial_i^2 + \partial_m^2 + 4\partial_w\partial_{\bar{z}})H = 0 , \tag{4.2.22}$$

which is reminiscent of the six-dimensional Laplace equation, apart from that the variables w and $\bar{z}$ are not complex conjugate to each other. With the aid of the standard technique of solving Laplace equations, we obtain

$$H = 1 + \frac{R^4}{\rho^4} , \tag{4.2.23}$$

with

$$\rho^2 := x_i^2 + x_m^2 + w\bar{z} = x_i^2 + x_m^2 + \frac{z\bar{z}}{1+cz} = x_i^2 + x_m^2 + H_1 z\bar{z} , \tag{4.2.24}$$

$$R^4 := 4\pi\alpha'^2 g_s N = \alpha'^2\lambda . \tag{4.2.25}$$

Note that the small c expansion of H_1 is $H_1 = 1 - cz + c^2z^2 - \cdots$, and ρ^2 reduces to $r^2 = x_a^2$ at the limit of $c = 0$, such that H in (4.2.23) reduces to the standard harmonic function for $D3$-branes.

Substituting H back to (4.2.7), one obtains the 5-form fields sourced by the N D3-branes. The F_0 components are complex due to nonvanishing c, but they become real when c is switched off, as in the case of only N D3-branes without the presence of the intersecting ones.

With H and H_1 provided, the metric for the intersecting brane configuration can be written down,

$$ds^2 = \frac{1}{\sqrt{H}}(dx_0^2+dx_1^2+dx_2^2+dx_3^2)+\sqrt{H}\Big(\frac{dzd\bar{z}}{(1+cz)^2}+dx_4^2+dx_5^2+dx_7^2+dx_8^2\Big) \,. \quad (4.2.26)$$

This is of the form of the D3-brane metric, except that the harmonic function H and $g_{z\bar{z}}$ are modified by c. Comparing with the standard $AdS_5 \times S^5$ case, the effect of turning on c is to replace $z \to w$ in ρ^2 and $dzd\bar{z} \to dwd\bar{z}$ in the metric.

It appears that the metric (4.2.26) is singular at $z = -1/c$. In fact this singularity is at infinite proper distance, which can be shown by taking the trajectory of z along the real axis in the calculation of the proper distance. Also, all F_0 components sourced by N D3-branes vanish at $z = -1/c$.

The Euclidean Lorentz symmetry is broken in the non-anticommutative deformed gauge theories due to non-commutativity of the spacetime. This is reflected in the fact that the background RR 5-forms (4.1.1) break the $SO(4)$ invariance although the metric (4.2.26) is $SO(4)$ symmetric.

Supergravity dual for $\mathcal{N} = (1,0)$ super Yang-Mills

In the near horizon region of the N $D3$-branes where $\rho << R$ in (4.2.26), the harmonic function $H \sim R^4/\rho^4$ and the geometry becomes

$$ds^2 = \frac{\rho^2}{R^2}dx_\mu^2 + \frac{R^2}{\rho^2}d\rho^2 + R^2 ds_{M_5}^2 \,, \quad (4.2.27)$$

which describes the geometry of $AdS_5 \times M_5$, where M_5 is the manifold deformed from S^5 due to the presence of c, and $ds_{M_5}^2$ is independent of ρ.

The decoupling limit is taken by $\alpha' \to 0$, with

$$\tilde{x}^a := \frac{x^a}{\alpha'}, \quad \tilde{z} := z/\alpha' \,, \quad (4.2.28)$$

being fixed. Note that in the supergravity dual for the non-commutative field theory reviewed in Section 2.4, besides taking $\alpha' \to 0$ and rescaling r (where r is the radial coordinate) by fixing r/α', it is also necessary to rescale the B-field at infinity with $\alpha' B^\infty$ fixed in order to maintain the non-commutative effect at the near horizon limit. The same reason applies here due to the Clifford algebra (3.3.1). Therefore we also rescale c by fixing

$$\tilde{c} := \alpha' c = \alpha' F_{01456} \ , \tag{4.2.29}$$

such that

$$U^2 := \left(\frac{\rho}{\alpha'}\right)^2 = \tilde{x}_4^2 + \tilde{x}_5^2 + \tilde{x}_7^2 + \tilde{x}_8^2 + \frac{\tilde{z}\bar{\tilde{z}}}{1+\tilde{c}\tilde{z}} \ . \tag{4.2.30}$$

The metric for the intersecting brane configuration (4.2.1) at the near horizon limit reads

$$\frac{ds^2}{\alpha'} = \frac{U^2}{\sqrt{\lambda}} dx_\mu^2 + \frac{\sqrt{\lambda}}{U^2}\Big(\frac{d\tilde{z}d\bar{\tilde{z}}}{(1+\tilde{c}\tilde{z})^2} + d\tilde{x}_4^2 + d\tilde{x}_5^2 + d\tilde{x}_7^2 + d\tilde{x}_8^2\Big) \ . \tag{4.2.31}$$

The RR field strengths are

$$\frac{F_0}{\alpha'^2} = d(\frac{U^4}{\lambda}) dx^{0123} + \text{dual} \tag{4.2.32}$$

and

$$\begin{aligned} \frac{F_1}{\alpha'^2} = \tilde{c}\ & (dx^{01}d\tilde{x}^{456} + i dx^{01} d\tilde{x}^{789} + dx^{23} d\tilde{x}^{456} + i dx^{23} d\tilde{x}^{789} \\ & + i dx^{01} d\tilde{x}^{459} + dx^{01} d\tilde{x}^{786} + i dx^{23} d\tilde{x}^{459} + dx^{23} d\tilde{x}^{786}) \\ & + \text{dual} \ . \end{aligned} \tag{4.2.33}$$

(4.2.31)-(4.2.33) constitute the supergravity dual for $\mathcal{N} = (1,0)$ non-anticommutative supersymmetric Yang-Mills theory.

One can tell from (4.2.31) that in the near horizon limit, the geometry is no longer $AdS_5 \times S^5$. The S^5 part is deformed by c and thus the $SO(6)$ symmetry is broken. The RR 5-forms are not $SO(6)$ symmetric either. This corresponds to the broken $\mathcal{N} = 4$ supersymmetry on the field theory side.

4.3 The scalar field-operator correspondence

In the standard AdS/CFT duality, the spectrum of the bulk scalar fields is mapped to the spectrum of the corresponding field theory operators on the boundary [4] [5]. In this section we check the correspondence between the boundary operators in

$\mathcal{N} = (1,0)$ super Yang-Mills and the bulk scalar fields which don't depend on M_5 in the supergravity dual. We will demonstrate that the spectrum of this class of operators on the field theory side is not modified.

For simplicity, we will drop the $\tilde{}$ notation for the decoupling limit and denote the rescaled radial coordinate U in (4.2.30) by ρ in this section, and use the standard expression

$$ds^2 = \frac{\rho^2}{\sqrt{\lambda}}\, dx_\mu^2 + \frac{\sqrt{\lambda}}{\rho^2}\left(\frac{dzd\bar{z}}{(1+cz)^2} + dx_i^2 + dx_m^2\right) \tag{4.3.1}$$

for the near horizon metric by assuming the rescaling of ds^2, ρ, x^a, z, and c with α'. Following Witten's approach introduced in Section 2.3 to derive the two-point function for the boundary theory operators from the bulk theory, we first evaluate the minimised supergravity action

$$I_{SUGRA} = \int d^{10}x\, \frac{1}{2}\sqrt{g}\left(\partial_M\phi\, \partial^M\phi + m^2\phi\right) . \tag{4.3.2}$$

The calculation involves identifying the conformal boundary of the spacetime in order to insert the boundary-to-bulk propagators. In (4.2.23)-(4.2.26), the metric of our supergravity dual is formally the same as that of $AdS_5 \times S^5$, but with z replaced by another variable $w = z/(1+cz)$ which is not complex conjugate to $\bar{z}$. The "radial coordinate" $\rho^2 = x_i^2 + x_m^2 + w\bar{z}$ becomes complex and is subtle in physical implications. Since the non-anticommutative field theory is a deformation from $\mathcal{N} = 4$ super Yang-Mills, the supergravity background (4.3.1) is also regarded as a deformation from $AdS_5 \times S^5$ geometry such that $\rho^2 = x_i^2 + x_m^2 + z\bar{z} \to r^2$ as c is turned off. Hence we will evaluate (4.3.2) in the original radial coordinate r with the standard notion of the boundary at $r \to \infty$ where also $\rho \to \infty$, and investigate the effect of the deformation on the boundary theory.

The scalar field ϕ which minimises (4.3.2) is obtained by solving the 10-dimensional full equation of motion

$$\left\{\frac{\lambda}{\rho^4}\, \partial_\mu^2 + \partial_i^2 + \partial_m^2 + 4\, \partial_w\partial_{\bar{z}} - m^2\frac{1}{\rho^2}\right\}\phi = 0 , \tag{4.3.3}$$

where m^2 is the square of 10-dimensional mass, and $\sqrt{\lambda}$ in the metric (4.3.1) has been rescaled to 1 to get rid the 't Hooft coupling. The solution which is independent of M_5 and vanishes at $\rho \to \infty$, is

$$K = \frac{\xi^\Delta}{(x^{\mu 2} + \xi^2)^\Delta} , \tag{4.3.4}$$

where $\xi = \frac{1}{\rho}$, and

$$\Delta = 2 + \sqrt{4 + m^2} . \tag{4.3.5}$$

In this case, m is also the AdS_5 mass. By (4.3.4) one can obtain the boundary-bulk propagator $\frac{\xi^{2\Delta-4}}{(x^{\mu 2}+\xi^2)^\Delta}$, since it is a delta function supported at $x^\mu = 0$ as $\xi \to 0$. The bulk field expressed in terms of the boundary-bulk propagator and the source on the boundary is given by

$$\begin{aligned} \phi(\xi, \vec{x}) &= \xi^{4-\Delta} \int_{\partial(AdS_5)} d^4\vec{x}\,' \frac{\xi^{2\Delta-4}}{((\vec{x} - \vec{x}\,')^2 + \xi^2)^\Delta} \phi_0(\vec{x}\,') \\ &= \xi^{4-\Delta} \phi_0(\vec{x}) . \end{aligned} \tag{4.3.6}$$

In this expression, ϕ_0 is of conformal weight $4-\Delta$. The field theory operators coupled to ϕ_0 are of conformal dimensions Δ. The relation of Δ and m^2 given in (4.3.5) is not deformed from that in (2.3.6) of the standard AdS/CFT correspondence for $\mathcal{N} = 4$ SYM .

By making use of the equation of motion and Stokes theorem, (4.3.2) becomes

$$I_{SUGRA} = \frac{1}{2} \int d^4x \, d^5\Omega \, (\sqrt{g} \, \phi \, \partial^r \phi)_{r\to\infty} . \tag{4.3.7}$$

To write down the 5-dimensional "angular" part explicitly, we use the following standard parameterisation of S^5:

$$\begin{aligned} x^4 + ix^5 &= r \sin\alpha \cos\beta \, e^{i\phi_2} , \\ x^7 + ix^8 &= r \sin\alpha \sin\beta \, e^{i\phi_3} , \\ x^6 + ix^9 &= r \cos\alpha \, e^{i\phi_1} , \end{aligned} \tag{4.3.8}$$

such that

$$\rho^2 = \frac{\lambda}{\xi^2} = r^2 \sin^2\alpha + \frac{r^2 \cos^2\alpha}{1 + c\, r \cos\alpha \, e^{i\phi_1}} . \tag{4.3.9}$$

When the angles are fixed, $\xi \to 0$ as $r \to \infty$. The asymptotic behaviour of ρ^2 as $r \to \infty$ varies at $\alpha = 0$ and $\alpha \neq 0$. When $\alpha \neq 0$, $\rho^2 \to r^2 \sin^2\alpha$ at large r while $\rho^2 \to r/(ce^{i\phi_1})$ at $\alpha = 0$.

The explicit expression of the 6-dimensional part metric is

$$\begin{aligned} ds^2_{(6d)} = \frac{\sqrt{\lambda}}{\rho^2} & \Big\{(\sin^2\alpha + H_1^2\cos^2\alpha)\,dr^2 + 2r\sin\alpha\cos\alpha(1-H_1^2)\,drd\alpha \\ & +r^2(H_1^2\sin^2\alpha+\cos^2\alpha)\,d\alpha^2 + r^2\sin^2\alpha\,d\beta^2 + H_1^2r^2\cos^2\alpha\,d\phi_1^2 \\ & +r^2\sin^2\alpha\cos^2\beta\,d\phi_2^2 + r^2\sin^2\alpha\sin^2\beta\,d\phi_3^2\Big\}\,. \qquad (4.3.10) \end{aligned}$$

So we obtain the expressions for the ingredients in (4.3.7):

$$\sqrt{g} = \frac{1}{\sqrt{\lambda}}\,H_1^2\,\xi^2\,r^5|\sin^3\alpha\cos\alpha\sin\beta\cos\beta|\,, \qquad (4.3.11)$$

$$\phi(\xi\to 0,\vec{x}) = \xi^{4-\Delta}\,\phi_0(\vec{x})\,, \qquad (4.3.12)$$

$$\begin{aligned} \partial^r\phi_{(\xi\to 0)} &= g^{rr}\partial_r\phi + g^{r\alpha}\partial_\alpha\phi \\ &= -\Delta\,\xi^{\Delta-1}\left\{\left(\sin^2\alpha + \frac{\cos^2\alpha}{H_1^2}\right)\partial_r\rho + \frac{1}{r}\,\sin\alpha\,\cos\alpha\left(\frac{1}{H_1^2}-1\right)\partial_\alpha\rho\right\}\times \\ & \int d^4\vec{x}'\,\frac{\phi_0(\vec{x}')}{|\vec{x}-\vec{x}'|^{2\Delta}}\,. \qquad (4.3.13) \end{aligned}$$

With (4.3.11), (4.3.12) and (4.3.13), the on-shell supergravity action can be evaluated. Following the prescription summarised in Section 2.3, it serves as the generating functional from which the two-point function of the boundary field theory operators dual to the bulk scalar fields of (4.3.6) is derived,

$$<\mathcal{O}(\vec{x})\mathcal{O}(\vec{x}')> = \frac{\delta I}{\delta\phi_0(\vec{x})\delta\phi_0(\vec{x}')} = \frac{C_1}{|\vec{x}-\vec{x}'|^{2\Delta}}\,, \qquad (4.3.14)$$

where

$$\begin{aligned} C_1 = & -\frac{\Delta}{2}\lambda^2\int d^5\Omega\,|\sin^3\alpha\cos\alpha\sin\beta\cos\beta|\,\times \\ & \left\{\frac{r^5}{\rho^5}\left(H_1^2\sin^2\alpha+\cos^2\alpha)\,\partial_r\rho + \frac{1}{r}\sin\alpha\cos\alpha(1-H_1^2)\,\partial_\alpha\rho\right)\right\}_{r\to\infty} \qquad (4.3.15) \end{aligned}$$

It is difficult to evaluate the integral of C_1 since ρ behaves differently at $\alpha = 0$ and $\alpha \neq 0$ as $r \to \infty$. For $\alpha \neq 0$, the integrand in (4.3.15) tends to $2|\frac{\cos^3\alpha\sin\beta\cos\beta}{\sin\alpha}|$ as $r \to \infty$, while at $\alpha = 0$ the integrand $\sim c^2r^2$ at large r. However (4.3.14) shows that the operator $\mathcal{O}$ is indeed of conformal dimension Δ.

(4.3.5) and (4.3.14) indicate that the spectrum of the boundary field theory operators corresponding to the bulk field modes independent of M_5 is not deformed by non-anticommutativity. This fulfills our expectation, since the bulk fields we investigate have no dependence on M_5 whose geometry deviates from S^5 due to c, as a result of Maldacena's conjecture, the spectrum of operators dual to this class of fields remains undeformed.

Due to the nature of the deformation occurred to the supergravity background, we predict that similar result applies to the boundary operators dual to other bulk fields. In particular, since the chiral operators are built from scalar canonical fields [47], it is expected that their conformal dimensions are also undeformed.

Nevertheless there are two possible ways to deform the operator spectrum [2]. One is realised by those Green's function solutions which depend on M_5 to the full Klein-Gordon equation (4.3.3). The AdS_5 scalar field mass spectrum is expected to be modified by c, and thus the spectrum of the corresponding field theory operators will also be modified accordingly. It requires more investigation to understand how the spectrum is deformed by non-anticommutativity. The other way may be via the modification of the full string theory mass spectrum by c, which gives rise to the dependence of the 10-dimensional m^2 on c.

Chapter 5

The Supergravity Dual to $\mathcal{N} = (\frac{1}{2}, 0)$ Super Yang-Mills Theory

In the previous chapter we have presented the supergravity dual for $\mathcal{N} = (1,0)$ non-anticommutative super Yang-Mills theory. In this chapter we turn to construct the supergravity dual for the deformed SYM with $\mathcal{N} = (\frac{1}{2}, 0)$ supersymmetry.

The description of Euclidean $\mathcal{N} = 4$ super Yang-Mills theory in the $\mathcal{N} = (\frac{1}{2}, \frac{1}{2})$ superspace language is well established. In the context of string theory, the non-anticommutative deformation to such superspace on the D3-brane is introduced by switching on a rank one (A, B)-indexed part of the (S, S) background graviphoton field $\mathcal{F}^{\alpha\beta AB} = \mathcal{F}^{(\alpha\beta)(AB)}$. Half of supersymmetry is broken by non-anticommutativity. The outcome is the super Yang-Mills with $\mathcal{N} = (\frac{1}{2}, 0)$ supersymmetry.

To construct the supergravity dual for $\mathcal{N} = (\frac{1}{2}, 0)$ SYM, we follow the approaches as in Chapter 4. First, a configuration of constant background RR 5-form fields which exhibits double self-duality (3.3.11) and gives rise to the aforementioned $\mathcal{F}^{\alpha\beta AB}$ is given. Then the supergravity background, involving N coincident D3-branes and 8 additional D3-branes which intersect the N D3's and source the background RR flux, is constructed. The intersecting branes preserve the same fraction of supersymmetry as that of the $\mathcal{N} = (\frac{1}{2}, 0)$ SYM. The supergravity dual of the deformed SYM is obtained by taking the near horizon limit of the N D3-branes, where, besides the transverse coordinates to the N D3-branes, the RR 5-forms are also rescaled by α'.

5.1 RR-flux configuration

To break half of supersymmetry in $\mathcal{N} = (1/2, 1/2)$ superspace by non-anticommutative deformation, we propose that the following components of the background RR 5-form field strength are turned on on the N coincident D3 branes,

$$\begin{aligned} F_{01456} &= -iF_{01789} = F_{23456} = -iF_{23789} = 2k, \\ F_{01786} &= -iF_{01459} = F_{23786} = -iF_{23459} = 2k, \\ F_{01476} &= iF_{01589} = F_{23476} = iF_{23589} = 2ik, \\ F_{01586} &= iF_{01479} = F_{23586} = iF_{23479} = -2ik, \end{aligned} \tag{5.1.1}$$

where k is a real constant. These RR fields satisfy the double self-duality condition in (3.3.11). With the 4-dimensional gamma matrices given by (3.3.3) and the 6-dimensional ones denoted by $\Sigma^{4,\dots,9}$, the background graviphoton field $\mathcal{F}^{\alpha\beta AB} = \mathcal{F}^{(\alpha\beta)(AB)}$ is expressed explicitly by

$$\mathcal{F}^{\alpha\beta AB} = 48ik(\tau^3)^{\alpha\beta}M^{AB}, \tag{5.1.2}$$

where in this case $M^{AB} = \Sigma^{456} + i\Sigma^{459} + i(\Sigma^{476} + i\Sigma^{479})$. The explicit expression of $F^{\alpha\beta AB}$ which breaks half of supersymmetry of $\mathcal{N} = (1/2, 1/2)$ superspace has a 1×1 sub-block of M^{AB} turned on[1], e.g.

$$M_{(\frac{1}{2},0)} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.1.3}$$

In general, M^{AB} in (5.1.2) doesn't take the form of $M_{(\frac{1}{2},0)}$, but they are related by a bi-unitary transformation as for the $\mathcal{N} = (1,0)$ case. For example, if we choose the basis for the 6-dimensional gamma matrices as in (4.1.3),

$$(\Sigma)^{6,9,4,5,7,8} = \Sigma^{4,5,6,7,8,9}_{(3.3.4)},$$

[1]It can be checked that all $[\Sigma^{abc} + i\Sigma^{abd} + i(\Sigma^{aec} + i\Sigma^{aed})]$ are of rank 1.

then $M^{AB} = M_2$ is expressed by

$$M_2^{AB} = \Sigma^{456}+i\Sigma^{459}+i(\Sigma^{476}+i\Sigma^{479}) = 2i \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 4iU^T M_{(\frac{1}{2},0)} U \ , \quad (5.1.4)$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix} \quad (5.1.5)$$

The explicit expression of U depends on the choice of the basis for Σ^a. As in the $\mathcal{N} = (1,0)$ case, the vertex operator (3.3.5) for $\mathcal{F}^{\alpha\beta AB}$ with components given by (5.1.1) and $M^{AB} = M_2$ is equivalent to the vertex operator with $M^{AB} = M_{(\frac{1}{2},0)}$ and the spin fields under the transformations

$$S \to US \ , \quad \tilde{S} \to U\tilde{S} \ . \quad (5.1.6)$$

Therefore, the background graviphoton field with components (5.1.1) and M^{AB} of rank 1 gives rise to non-anticommutative deformation to the $\mathcal{N} = (1/2, 1/2)$ superspace on $D3$-branes.

5.2 The supergravity dual

In this section, we present the supergravity dual solution for $\mathcal{N} = (1/2, 0)$ super Yang-Mills theory. First, we construct an intersecting D3-brane configuration which sources the RR 5-form fields in (5.1.1):

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
N D3	•	•	•	•	×	×	×	×	×	×
$D3_1$	•	•			•	•	×			×
$D3_2$	•	•					×	•	•	×
$D3_3$			•	•	•	•	×			×
$D3_4$			•	•			×	•	•	×
$D3_5$	•	•			•		×	•		×
$D3_6$	•	•				•	×		•	×
$D3_7$			•	•	•		×	•		×
$D3_8$			•	•		•	×		•	×

, (5.2.1)

where • indicates worldvolume dimensions of the branes, and × indicates the (partial) localisations. The N D3-branes where the super Yang-Mills lives are intersected by 8 additional D3-branes. The former are chosen to localise in all of their transverse dimensions, as in the $\mathcal{N} = (1,0)$ case, while the latter are localised in their commonly transverse space (x_6, x_9) only. Each of the additional branes intersects the N D3-branes in two common dimensions, either (x_0, x_1) or (x_2, x_3).

The partial localisation arranged in (5.2.1) allows the 8 additional intersecting branes to generate no more 5-form field components than those given in (5.1.1). Note that the components in the first two rows of (5.1.1) are sourced by $D3_1 \cdots D3_4$-branes and the last two rows are sourced by $D3_5 \cdots D3_8$.

Preserved supersymmetry

We then check that the intersecting brane configuration at the N D3's near horizon limit preserve the same fraction of spacetime supersymmetry as the number of supersymmetries in $\mathcal{N} = (1/2, 0)$ SYM.

The N D3-branes give rise to a constraint which relates the two 10-dimensional spacetime spinors ϵ_1 and ϵ_2,

$$\hat{\Gamma}_{0123}\, \epsilon_1 = \epsilon_2 \ , \tag{5.2.2}$$

while the conditions associated with $D3_1 \cdots D3_8$-branes are

$$\begin{aligned} &\hat{\Gamma}_{0145}\,\epsilon_1 = \epsilon_2\ , \quad \hat{\Gamma}_{0178}\,\epsilon_1 = \epsilon_2\ , \quad \hat{\Gamma}_{2345}\,\epsilon_1 = \epsilon_2\ , \quad \hat{\Gamma}_{2378}\,\epsilon_1 = \epsilon_2\ , \\ &\hat{\Gamma}_{0147}\,\epsilon_1 = \epsilon_2\ , \quad \hat{\Gamma}_{0185}\,\epsilon_1 = \epsilon_2\ , \quad \hat{\Gamma}_{2347}\,\epsilon_1 = \epsilon_2\ , \quad \hat{\Gamma}_{2385}\,\epsilon_1 = \epsilon_2\ , \end{aligned} \tag{5.2.3}$$

where $\hat{\Gamma}$ denote the 10-dimensional tangent-space gamma matrices. At the near horizon limit of the N D3-branes, (5.2.2) is lifted. The conditions in (5.2.3) are not completely independent. They can reduce to

$$\hat{\Gamma}_{0123}\,\epsilon_1 = -\epsilon_1\ , \quad \hat{\Gamma}_{4578}\,\epsilon_1 = -\epsilon_1\ , \quad \hat{\Gamma}_{48}\,\epsilon_1 = \epsilon_1\ , \quad \hat{\Gamma}_{2378}\,\epsilon_1 = \epsilon_2\ . \tag{5.2.4}$$

The first three constraints preserve $\frac{1}{8}$ of the ϵ_1 components, while the last one fixes all components of ϵ_2 by ϵ_1. So the number of unbroken supersymmetries is 2, (i.e. 1/16 of supersymmetry is preserved,) and they can be expressed as

$$\epsilon^{\alpha}, \qquad \alpha = 1, 2\ , \tag{5.2.5}$$

since ϵ_1 is chiral in 4 dimensions due to $\hat{\Gamma}_{0123}\,\epsilon_1 = -\epsilon_1$. Therefore the supersymmetry preserved by the intersecting branes in (5.2.1) agrees with that in the $\mathcal{N} = (\frac{1}{2}, 0)$ super Yang-Mills theory.

Metric and fields

Suppose the harmonic function associated with N D3-branes is denoted by H and those with $D3_1 \cdots D3_8$-branes are

$$H_{extra} := \{H_1, ..., H_8\}$$

respectively. The metric which describes the intersecting brane configuration (5.2.1) is

$$\begin{aligned} ds^2 \;=\; & \sqrt{\frac{H_3H_4H_7H_8}{HH_1H_2H_5H_6}}(dx_0^2 + dx_1^2) + \sqrt{\frac{H_1H_2H_5H_6}{HH_3H_4H_7H_8}}(dx_2^2 + dx_3^2) \\ & + \sqrt{\frac{HH_2H_4H_6H_8}{H_1H_3H_5H_7}}dx_4^2 + \sqrt{\frac{HH_2H_4H_5H_7}{H_1H_3H_6H_8}}dx_5^2 \\ & + \sqrt{\frac{HH_1H_3H_6H_8}{H_2H_4H_5H_7}}dx_7^2 + \sqrt{\frac{HH_1H_3H_5H_7}{H_2H_4H_6H_8}}dx_8^2 \\ & + \sqrt{HH_1H_2H_3H_4H_5H_6H_7H_8}(dx_6^2 + dx_9^2)\ . \end{aligned} \tag{5.2.6}$$

The metric is supported by the overall 5-form field strength generated by those branes,

$$F = F_0 + F_1 \ . \tag{5.2.7}$$

where F_0 is sourced by the N D3's and F_1 by the additional $D3_1 \cdots D3_8$:

$$\begin{aligned} F_0 &= d(\frac{1}{H})dx^{0123} + \text{dual}, \ , & (5.2.8) \\ F_1 &= d(\frac{1}{H_1})dx^{0145} + d(\frac{1}{H_2})dx^{0178} + d(\frac{1}{H_3})dx^{2345} + d(\frac{1}{H_4})dx^{2378} \\ &\quad + d(\frac{1}{H_5})dx^{0147} + d(\frac{1}{H_6})dx^{0158} + d(\frac{1}{H_7})dx^{2347} + d(\frac{1}{H_8})dx^{2358} \\ &\quad + \text{dual} \ . & (5.2.9) \end{aligned}$$

Note that the first line of F_1 contains fields generated by $D3_{1,2,3,4}$ and the second line by $D3_{5,6,7,8}$.

It is natural to choose H to depend on $x_a := (x_4, ..., x_9)$ in (5.2.8). In order that (5.2.9) gives rise to the exact RR 5-form components given in (5.1.1), the harmonic functions H_{extra} are chosen to depend on (x_6, x_9) only. This is why we choose the partial localisation in (5.2.1) for these branes.

Comparing the components in (5.2.9) and (5.1.1), the following differential equations are obtained,

$$\begin{aligned} \partial_6\left(\frac{1}{H_{1,2,3,4}}\right) &= k \ , & \partial_9\left(\frac{1}{H_{1,2,3,4}}\right) &= ik \ , \\ \partial_6\left(\frac{1}{H_{5,7}}\right) &= ik \ , & \partial_6\left(\frac{1}{H_{6,8}}\right) &= -ik \ , \\ \partial_9\left(\frac{1}{H_{5,7}}\right) &= -k \ , & \partial_9\left(\frac{1}{H_{6,8}}\right) &= k \ . \end{aligned} \tag{5.2.10}$$

Due to the same reason explained in Chapter 4, in order to satisfy (5.2.10), H_{extra} can only depend on $z := x_6 + ix_9$:

$$H_{extra}(z) := \{H_1(z), ..., H_8(z)\} \ . \tag{5.2.11}$$

Solving for the harmonic functions

The field equation associated with the N D3-branes is

$$\left(H_1H_3(H_5H_7\partial_4^2 + H_6H_8\partial_5^2) + H_2H_4(H_5H_7\partial_7^2 + H_6H_8\partial_8^2) + \partial_p^2\right)H = 0 \tag{5.2.12}$$

and those for $D3_{1,\ldots,8}$ are

$$\begin{aligned}
\partial_p(\frac{H_5H_6}{H_7H_8}\frac{H_1^2}{H_4^2}\partial_p(\frac{1}{H_1})) &= 0\,, & (5.2.13)\\
\partial_p(\frac{H_5H_6}{H_7H_8}\frac{H_2^2}{H_3^2}\partial_p(\frac{1}{H_2})) &= 0\,, & (5.2.14)\\
\partial_p(\frac{H_7H_8}{H_5H_6}\frac{H_3^2}{H_2^2}\partial_p(\frac{1}{H_3})) &= 0\,, & (5.2.15)\\
\partial_p(\frac{H_7H_8}{H_5H_6}\frac{H_4^2}{H_1^2}\partial_p(\frac{1}{H_4})) &= 0\,, & (5.2.16)\\
\partial_p(\frac{H_1H_2}{H_3H_4}\frac{H_5^2}{H_8^2}\partial_p(\frac{1}{H_5})) &= 0\,, & (5.2.17)\\
\partial_p(\frac{H_1H_2}{H_3H_4}\frac{H_6^2}{H_7^2}\partial_p(\frac{1}{H_6})) &= 0\,, & (5.2.18)\\
\partial_p(\frac{H_3H_4}{H_1H_2}\frac{H_7^2}{H_6^2}\partial_p(\frac{1}{H_7})) &= 0\,, & (5.2.19)\\
\partial_p(\frac{H_3H_4}{H_1H_2}\frac{H_8^2}{H_5^2}\partial_p(\frac{1}{H_8})) &= 0\,, & (5.2.20)
\end{aligned}$$

where $p = 6,9$. By complexifying the coordinates (x_6, x_9) such that $\partial_p^2 = 4\partial_z\partial_{\bar{z}}$, equations (5.2.13) - (5.2.20) are trivially satisfied since H_{extra} are holomorphic functions. We can solve H_{extra} by the complexifed version of equation (5.2.10):

$$\begin{aligned}
\partial_z\left(\frac{1}{H_{1,2,3,4}}\right) &= k\,,\\
\partial_z\left(\frac{1}{H_{5,7}}\right) &= ik\,,\quad \partial_z\left(\frac{1}{H_{6,8}}\right) = -ik\,,\\
\partial_{\bar{z}}\left(\frac{1}{H_{extra}}\right) &= 0\,.
\end{aligned} \qquad (5.2.21)$$

Taking into account that F_1 should be finite at infinity in order for the charges of $D3_{1,\ldots,8}$-branes to be well defined, and requiring $H_{extra} = 1$ when k is turned off, we obtain the unique solution:

$$\begin{aligned}
H_1 = H_2 = H_3 = H_4 &= \frac{1}{1+kz},\\
H_5 = H_7 &= \frac{1}{1+ikz}\\
H_6 = H_8 &= \frac{1}{1-ikz}.
\end{aligned} \qquad (5.2.22)$$

With these H_{extra}, it's is clear now each component of the doubly self-dual RR 5-forms in (5.1.1) contains equal contribution k from two D3-branes in the configuration (5.2.1). For example, F_{01456} is sourced by $D3_1$ and $D3_4$, with $(F_{01456})_{D3_1} = \partial_6(1/H_1) = k$ and $(F_{01456})_{D3_4} = -i\partial_9(1/H_4) = k$ such that F_{01794} amounts to $2k$; F_{01476} is generated by $D3_5$ and $D3_8$, etc..

In $(z, \bar{z})$ coordinates, the equation (5.2.12) is simplified to

$$(A\,\partial_i^2 + \frac{1}{A}\,\partial_m^2 + \frac{4}{H_1^2 H_5 H_6}\,\partial_z\partial_{\bar{z}})H = 0. \tag{5.2.23}$$

where $i = 4, 7$ and $m = 5, 8$. The function $A(z)$ is defined by

$$A := \frac{H_5}{H_6} = \frac{1 - ikz}{1 + ikz}. \tag{5.2.24}$$

To solve H, we apply the technique used in Chapter 4 by introducing the variable u,

$$u := \int H_1^2 H_5 H_6\, dz = \frac{1}{4k} \ln \frac{(1 + kz)^2}{1 + k^2 z^2} + \frac{z}{2(1 + kz)} \tag{5.2.25}$$

such that $u = 0$ when $z = 0$ or $k = 0$. The equation (5.2.23) for H is expressed in $(u, \bar{z})$ by

$$\left(A\partial_i^2 + \frac{1}{A}\partial_m^2 + 4\,\partial_u\partial_{\bar{z}}\right) H = 0\ . \tag{5.2.26}$$

It can be solved by the ansatz

$$H = 1 + \frac{R^4}{\rho^4}, \tag{5.2.27}$$

where

$$\rho^2 = B_1(u)x_i^2 + B_2(u)x_m^2 + C(u)\bar{z} \tag{5.2.28}$$

and $R^4 := 4\pi\alpha'^2 g_s N = \alpha'^2\lambda$. Substituting the ansatz for H and ρ^2 into (5.2.23), finding the solution for H is equivalent to solving

$$-2AB_1 + \frac{B_2}{A} + \frac{1}{H_1^2 H_5 H_6}\left(\frac{dC}{dz} - 3C\frac{\frac{dB_1}{dz}}{B_1}\right) = 0\ , \tag{5.2.29}$$

$$AB_1 - \frac{2B_2}{A} + \frac{1}{H_1^2 H_5 H_6}\left(\frac{dC}{dz} - 3C\frac{\frac{dB_2}{dz}}{B_2}\right) = 0\ , \tag{5.2.30}$$

$$AB_1 + \frac{B_2}{A} - \frac{2\frac{dC}{dz}}{H_1^2 H_5 H_6} = 0\ . \tag{5.2.31}$$

By changing the variable from z to $u(z)$ such that $\frac{du}{dz} = H_1^2 H_5 H_6$, (5.2.29)-(5.2.31) reduces to

$$(B_1B_2)' = 0, \quad (5.2.32)$$
$$(\frac{C}{B_1})' = A, \quad (5.2.33)$$
$$(\frac{C}{B_2})' = \frac{1}{A}, \quad (5.2.34)$$

where $'$ refers to differentiation by u. Equation (5.2.32) suggest B_1B_2 is a constant. Since at $k = 0$, ρ^2 in (5.2.28) reduces to the undeformed radial coordinate $r^2 = x_i^2 + x_m^2 + z\bar{z}$, so the integration constant is 1, and

$$B_1 = 1/B_2\ . \quad (5.2.35)$$

Then by (5.2.33) and (5.2.34) we can solve

$$CB_1 = \int \frac{1}{A}\frac{du}{dz}dz := N(z) \quad (5.2.36)$$
$$= \frac{1}{4k}\Big[(1-i)\ln\frac{(1+kz)^2}{1+k^2z^2} + 2(1+i)\tan^{-1}(kz) - 2(1+i)\frac{k^2z^2}{(1+kz)(i+kz)}\Big].$$
$$\frac{C}{B_1} = \int A\frac{du}{dz}dz := D(z) \quad (5.2.37)$$
$$= \frac{1}{4k}\Big[(1+i)\ln\frac{(1+kz)^2}{1+k^2z^2} + 2(1-i)\tan^{-1}(kz) - 2(1-i)\frac{k^2z^2}{(1+kz)(-i+kz)}\Big],$$

and hence

$$B_1(u(z)) = \frac{1}{B_2(u(z))} = \sqrt{\frac{N(z)}{D(z)}}\ , \quad (5.2.38)$$
$$C(u(z)) = \sqrt{N(z)D(z)}\ . \quad (5.2.39)$$

Substituting B_1, B_2, C into the ansatz (5.2.28) and (5.2.27), one obtains the solution for H. The small k expansions of B_1, B_2 and C are

$$B_1 = 1 + ikz + O(k^2z^2), \quad (5.2.40)$$
$$B_2 = 1 - ikz + O(k^2z^2), \quad (5.2.41)$$
$$C = z(1 - kz + O(k^2z^2)). \quad (5.2.42)$$

As $k \to 0$, these functions reduce to the undeformed ones $B_1 = B_2 = 1$, $C = z$.

With the harmonic functions H and H_{extra} provided, the metric for the intersecting branes now can be written down explicitly,

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{H}}dx_\mu^2 + \sqrt{H}\left(\frac{dz d\bar{z}}{(1+kz)^2(1+k^2z^2)} + \frac{1}{A}(dx_4^2+dx_7^2) + A(dx_5^2+dx_8^2)\right) \\ &= \frac{1}{\sqrt{H}}dx_\mu^2 + \sqrt{H}\left(4\,du\,d\bar{z} + \frac{1}{A}(dx_4^2+dx_7^2) + A(dx_5^2+dx_8^2)\right) . \end{aligned} \tag{5.2.43}$$

As $k = 0$, this metric reduces to the standard D3-branes one.

It appears that the metric (5.2.43) is singular at $z = -\frac{1}{k}, \pm\frac{i}{k}$. One can carry out calculation similar to that below (4.2.26) and show that $z = -\frac{1}{k}$ is at infinite proper distance in the metric (5.2.43), while the singularities at $z = \pm\frac{i}{k}$ are at finite proper distance. Therefore the metric (5.2.43) is singular.

Near horizon limit

We follow similar approach to take the near horizon limit of the N D3-branes for the supergravity solution as in Chapter 4. The limit is obtained by taking $\alpha' \to 0$, and rescaling x^a and k by α' such that

$$\tilde{x}^a := \frac{x^a}{\alpha'}, \quad \tilde{z} := \frac{z}{\alpha'}, \quad \tilde{\rho} := \frac{\rho}{\alpha'}, \tag{5.2.44}$$

$$\tilde{k} := \alpha' k , \tag{5.2.45}$$

are fixed, where $\tilde{k}\tilde{z} = kz$. At this limit the rescaled "radial coordinate" is given by

$$\tilde{\rho}^2 := \left(\frac{\rho}{\alpha'}\right)^2 = B_1\tilde{x}_i^2 + B_2\tilde{x}_m^2 + \frac{C}{\alpha'}\bar{\tilde{z}} , \tag{5.2.46}$$

and also

$$\tilde{A} := \frac{1 - i\tilde{k}\tilde{z}}{1 + i\tilde{k}\tilde{z}} = A . \tag{5.2.47}$$

Then the metric at the near horizon limit of the N D3-branes reads

$$\frac{ds^2}{\alpha'} = \frac{\tilde{\rho}^2}{\sqrt{\lambda}}\,dx_\mu^2 + \frac{\sqrt{\lambda}}{\tilde{\rho}^2}\left(\frac{d\tilde{z}d\bar{\tilde{z}}}{(1+\tilde{k}\tilde{z})^2(1+\tilde{k}^2\tilde{z}^2)} + \frac{1}{\tilde{A}}(d\tilde{x}_4^2 + d\tilde{x}_7^2) + \tilde{A}(d\tilde{x}_5^2 + d\tilde{x}_8^2)\right) . \tag{5.2.48}$$

When k is turned off, this metric describes the standard $AdS_5 \times S^5$ geometry. On the other hand, the RR 5-form fields are given by

$$\frac{F_0}{\alpha'^2} = d(\frac{\tilde{\rho}^4}{\lambda})dx^{0123} + \text{dual} , \tag{5.2.49}$$

$$\begin{aligned}\frac{F_1}{\alpha'^2} = \ & \tilde{k}\ \{dx^{01}d\tilde{x}^{456} + idx^{01}d\tilde{x}^{789} + dx^{23}d\tilde{x}^{456} + idx^{23}d\tilde{x}^{789} \\ & +dx^{01}d\tilde{x}^{786} + idx^{01}d\tilde{x}^{459} + dx^{23}d\tilde{x}^{786} + idx^{23}d\tilde{x}^{459} \\ & +idx^{01}d\tilde{x}^{476} + dx^{01}d\tilde{x}^{589} + idx^{23}d\tilde{x}^{476} + dx^{23}d\tilde{x}^{589} \\ & -idx^{01}d\tilde{x}^{586} - dx^{01}d\tilde{x}^{479} - idx^{23}d\tilde{x}^{586} - dx^{23}d\tilde{x}^{479}\} \\ & + \text{dual}\,. \end{aligned} \tag{5.2.50}$$

At the near horizon limit, the metric and the RR 5-form fields are both well defined, and we claim that (5.2.48), (5.2.49), and (5.2.50) constitute the supergravity dual for the $\mathcal{N} = (1/2, 0)$ super Yang-Mills theory. The existence of the constant RR 5-forms breaks the $SO(4)$ symmetry on the worldvolume of the N D3-branes, despite the full metric is still $SO(4)$ invariant. This corresponds to the fact that the Euclidean Lorentz symmetry is broken in the non-anticommutative deformed gauge theories. Non-anticommutativity also breaks the $\mathcal{N} = 4$ supersymmetry, which is reflected in the broken $SO(6)$ symmetry both in the metric and the RR 5-form field strength.

As in the field theory operator - bulk field correspondence for the $\mathcal{N} = (1,0)$ case, we also expect that, for the $\mathcal{N} = (1/2, 0)$ case, the spectrum of the boundary field theory operators corresponding to the bulk field modes which are independent of the "deformed S^5" is not altered by k. We omit the calculation here due to its complication, but the approach is similar to that given in Section 4.3.

Part II

Gravity in the Spacetime with a Positive Cosmological Constant: an Asymptotically Non-de Sitter Black Fusiform Solution

Chapter 6

Review on Asymptotically Flat Black Rings

The black holes in 4-dimensional Einstein gravity are "simple" objects because they are highly constrained. For a stationary vacuum black hole in the asymptotically flat background, topological censorship claims that the event horizon must be of spherical topology [106], and the state of the black hole is fully dictated by its mass and spin according to the uniqueness theorem [107].

In five dimensions, the black object family contains more members. Besides the Myers-Perry solution [108] which is the direct generalisation of 4d Kerr black holes, there is a distinct class of solutions called the *black ring*, first discovered by Emparan and Reall [109] not long ago. The vacuum black ring is neutral and has a horizon of $S^1 \times S^2$ topology. It is rotating along an axis in order to balance its self-gravitation. The solutions with the same mass and angular momentum may correspond to the black hole or black rings, and hence the uniqueness is violated. The ring solution is later generalised to those with conserved electric charges [110] and with dipole charge [111]. The latter exhibits continuous non-uniqueness due to the presence of the non-conserved dipole charge, and the dipole term also contributes in the 1st law of thermodynamics [111] [116], which is different from the conventional black hole cases. The supersymmetric black rings are constructed in [112–115]. A comprehensive review of this topic is given in [117].

In this chapter we review some interesting properties of asymptotically flat black ring solutions. In order to present a self-content review for our research, we focus only on the non-supersymmetric solutions in this chapter.

6.1 Neutral rotating black rings

The asymptotically flat neutral rotating black ring is a vacuum solution to the 5-dimensional pure Einstein equation [118],

$$\begin{aligned} ds^2 &= -\frac{F(x)}{F(y)}\left[dt + R\sqrt{\lambda\nu}\,(1+y)\,d\psi\right]^2 \qquad (6.1.1)\\ &+\frac{R^2}{(x-y)^2}\left[-F(x)\left(G(y)d\psi^2 + \frac{F(y)}{G(y)}dy^2\right) + F(y)^2\left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\varphi^2\right)\right] , \end{aligned}$$

where

$$F(\xi) = 1 - \lambda\xi \ , \qquad G(\xi) = (1-\xi^2)(1-\nu\xi) \, . \qquad (6.1.2)$$

Compared to the original solution in [109], $G(\xi)$ in this expression takes a modified form and the roots are more tractable. In this solution there are two dimensionless parameters ν, λ, and one length scale R which basically indicates the ring radius at the limit of large thin ring. This solution also contains a branch equivalent to the Myers-Perry black hole, depicted by $\lambda = 1$. For the black ring case, λ is related to ν. The parameter ν is related to the shape of the black object in both cases[1].

The geometry and physics of the black rings are explored in [109]. The roots for the functions F and G are $\xi_1 = \lambda^{-1}$ and respectively $\xi_2 = -1, \xi_3 = +1, \xi_4 = \nu^{-1}$, as displayed in Figure 6.1. (It will be clear shortly why $\xi_3 < \xi_1 < \xi_4$.) To ensure that the roots of G are distinct and the metric is real, ν is restricted to the range[2]

$$0 \leq \nu < 1 \, . \qquad (6.1.3)$$

The ranges for (x, y) coordinates (with analytic extension in y) are

$$-1 \leq x \leq 1 \ , \qquad -1 \leq \frac{1}{y} < \lambda \, . \qquad (6.1.4)$$

[1]By setting ν to 0 and leaving λ as a free parameter ($\lambda \neq 1$), (6.1.1) reduces to the static black ring solution with a conical singularity in [119].

[2]For $-1 < \nu \leq 0$ and $\lambda \to -\lambda$, the metric is still real, and G, F becomes the mirror reflections along the $\xi = 0$ axis of the original functions in (6.1.2). Therefore we only need to consider the case in (6.1.3).

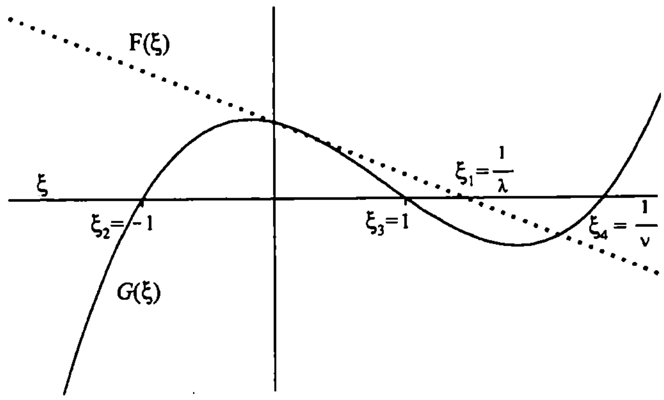

Figure 6.1: The roots of the functions F, G. $F(\xi)$ has one root $\xi_1 = \lambda^{-1}$ and $G(\xi)$ has three roots $\xi_2 = -1, \xi_3 = 1, \xi_4 = \nu^{-1}$. They satisfy $1 < \lambda^{-1} < \nu^{-1}$.

To maintain the correct signature, the metric (6.1.1) is for $-1 \leq x \leq 1$, $-\infty < y \leq -1$, but y coordinate can be analytically extended to $\nu^{-1} < y < \infty$, which is the ergoregion, by the transformation $Y = y^{-1}$ which gives rise to a regular metric at $Y = 0$. $|y| = \infty$ is the ergosurface, as the Killing vector $\partial/\partial t$ becomes spacelike upon crossing $Y = 0$ from the right. The event horizon locates at $y = \nu^{-1} = \xi_4$, where the metric can be demonstrated to be regular by changing the coordinates to

$$d\chi = -d\psi + \frac{\sqrt{-F(y)}}{G(y)}dy\ , \qquad dv = dt + R\sqrt{\lambda\nu}\,(1+y)\frac{\sqrt{-F(y)}}{G(y)}dy\ . \tag{6.1.5}$$

y is then extended to $y < \nu^{-1}$, and the signature of g_{yy} is flipped to minus inside the horizon with all other metric components being positive. The spacelike singularity is reached at $y = \lambda^{-1} = \xi_1$.

For $\lambda \neq 1$, there exists conical singularities at $x = \pm 1$ and $y = -1$, where $g_{\varphi\varphi}$ and $g_{\psi\psi}$ vanish respectively. These conical singularities are removed by identifying the periods of φ and ψ with

$$\Delta\psi|_{y=-1} = \Delta\varphi|_{x=-1} = 2\pi\frac{\sqrt{1+\lambda}}{1+\nu}\ , \tag{6.1.6}$$

$$\Delta\varphi|_{x=+1} = 2\pi\frac{\sqrt{1-\lambda}}{1-\nu}\ . \tag{6.1.7}$$

For a rotation-gravitation balanced black ring, the periods of φ at $x = \pm 1$ must be equal, $\Delta\varphi|_{x=-1} = \Delta\varphi|_{x=+1}$, which gives rise to a constraint on λ,

$$\lambda = \frac{2\nu}{1+\nu^2}\ , \qquad 0 \leq \nu < \lambda < 1\ , \tag{6.1.8}$$

as Figure 6.1 displays. Then (x, φ) parametrise a surface of topology S^2 with the two poles located at $x = \pm 1$, and ψ parametrises a circle S^1. The solution (6.1.1) has the horizon of topology $S^1 \times S^2$ and thus it is a black ring.

For $\lambda = 1$, $g_{\varphi\varphi}$ no longer vanishes at $x = 1$ and no conical singularity exists there. In this case (x, φ, ψ) parametrise a hypersurface of topology S^3, and the solution describes a 5-dimensional black hole. Such solution can be reduced to the ordinary Myers-Perry black hole metric rotating along ψ axis [109] [117]. To summarise,

$$\lambda = \begin{cases} \dfrac{2\nu}{1+\nu} & \text{(black ring)} \\ 1 & \text{(black hole)} \ . \end{cases} \tag{6.1.9}$$

The asymptotical infinity of the solution locates at $x = y = -1$ for both the black ring and black hole cases. This can be demonstrated via the transformations

$$\tilde{\psi} = \left(\frac{2\pi}{\Delta\varphi}\right)\psi = \frac{1+\nu}{\sqrt{1+\lambda}}\psi \ , \qquad \tilde{\varphi} = \left(\frac{2\pi}{\Delta\varphi}\right)\varphi = \frac{1+\nu}{\sqrt{1+\lambda}}\varphi \ , \tag{6.1.10}$$

$$\zeta = \tilde{R}\frac{\sqrt{-1-y}}{x-y} \ , \qquad \eta = \tilde{R}\frac{\sqrt{1+x}}{x-y} \ , \qquad \tilde{R} = \frac{\sqrt{2}(1+\lambda)}{\sqrt{1+\nu}}R \ , \tag{6.1.11}$$

such that the metric in the new coordinates is asymptotically flat,

$$ds^2 \sim -dt^2 + d\zeta^2 + \zeta^2 d\tilde{\psi}^2 + d\eta^2 + \eta^2 d\tilde{\varphi}^2 \tag{6.1.12}$$

The ADM mass and angular momentum of the black rings are

$$M = \frac{3\pi R^2}{4G}\frac{\lambda(\lambda+1)}{\nu+1} \ , \qquad J = \frac{\pi R^3}{2G}\frac{\sqrt{\lambda\nu}(\lambda+1)^{5/2}}{(\nu+1)^2} \ , \tag{6.1.13}$$

where J describes the rotation around S^1.[3] In 4-dimensional gravity, the uniqueness theorem states that the black hole is complete characterised by the mass and the angular momentum, but such physical property doesn't generalise to 5 dimensions. In order to show this, it is convenient to define the dimensionless spin j by

$$j^2 := \frac{27\pi}{32G}\frac{J^2}{M^3} = \begin{cases} \dfrac{(1+\nu)^3}{8\nu} & \text{(black ring)} \\ \dfrac{2\nu}{\nu+1} & \text{(black hole)} \ , \end{cases} \tag{6.1.14}$$

[3]The black ring with two independent angular momenta, which is the most general solution in 5 dimensions, can be found in [120].

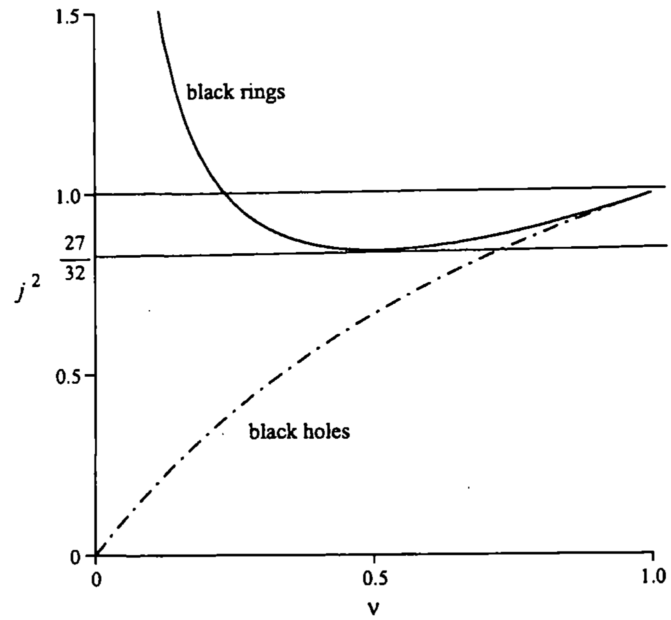

Figure 6.2: The dimensionless spin j^2 of the black ring and black hole against ν. There is a minimum of $j^2_{BR} = 27/32$ at $\nu = 1/2$, while j^2_{BH} increases monotonically from 0 to 1 over the range of ν. This figure shows that for j in $27/32 < j^2 < 1$, there exist solutions of spherical and ring topology. Non-uniqueness occurs to the black ring for $\sqrt{5} - 2 \leq \nu < 1$ and to the black hole for $27/37 \leq \nu < 1$. This figure is similar to the one in [110], but for the metric with the factorisation of (6.1.1) and (6.1.2).

for which (6.1.9) is used to distinguish the two cases. This is equivalent to discussing the angular momentum under the fixed mass M. As shown in Figure 6.1, j^2_{BR} is bounded from below by 27/32 while j^2_{BH} is bounded from above by 1. For $27/32 \leq j^2 < 1$, the "black solution" is not unique in terms of topology: solutions with identical mass and spin may correspond to a black hole ($\lambda = 1$) or two black rings ($\lambda \neq 1$) with $\nu > 1/2$ (fat ring), $\nu < 1/2$ (thin ring) as $j^2 \neq 27/32$, while may correspond to a black hole and a black ring as $j^2 = 27/32$.

For the case of black rings, the Killing vector ω which is null on the horizon can be expressed in terms of the coordinates $(v, \tilde{\chi})$ of (6.1.5),

$$\omega = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \tilde{\psi}} = \frac{\partial}{\partial v} + \Omega_H \frac{\partial}{\partial \tilde{\chi}} \,. \qquad (6.1.15)$$

where Ω_H is the angular velocity at which the event horizon rotate around the ψ

axis,

$$\Omega_H = \frac{1}{R}\sqrt{\frac{\nu}{\lambda(1+\lambda)}} \,. \tag{6.1.16}$$

The event horizon area and the surface gravity κ which is derived from the fact of ω being null $\nabla^\mu \omega^2 = -2\,\kappa\,\omega^\mu$ are

$$A_H = 8\pi^2 R^3 \frac{\lambda^{1/2}(1+\lambda)(\lambda-\nu)^{3/2}}{(1+\nu)(1-\nu^2)} \,, \qquad \kappa = \frac{1}{2R}\frac{1-\nu}{\sqrt{\lambda}\sqrt{\lambda-\nu}} \,. \tag{6.1.17}$$

The entropy and temperature are defined as usual by $S = A/4$ and $T = \kappa/2\pi$. A Smarr relation can be derived immediately,

$$M = \frac{3}{2}\left(\frac{\kappa A_H}{8\pi G} + \Omega_H J\right) \,. \tag{6.1.18}$$

The first law of black hole mechanics [121]

$$dM = \frac{\kappa}{8\pi G}\, dA_H + \Omega_H\, dJ \tag{6.1.19}$$

is also valid in terms of perturbations between two neighboring stable solutions of black rings.

The solutions of different topologies with the same mass and angular momentum in fact correspond to different entropy. In order to show this, it is convenient to define the dimensionless area a_H [117],

$$a_H := \frac{3}{16}\sqrt{\frac{3}{\pi}}\frac{A_H}{(GM)^{3/2}} = \begin{cases} 2\sqrt{\nu\,(1-\nu)} & \text{(black ring)} \\ 2\sqrt{2}\sqrt{\dfrac{1-\nu}{1+\nu}} & \text{(black hole)} \,. \end{cases} \tag{6.1.20}$$

The solutions with the same j^2, the one with higher entropy is thermodynamically preferred. Using the expression of j^2 given in (6.1.14), it is found that the solution with the largest horizon area is the Myers-Perry black hole for $j^2 < 0.8874$ and the black ring for $j^2 > 0.8874$. Therefore, a phase transition from the black hole to the black ring may happen when the angular momentum increases over $j_c^2 = 0.8874$ [109].

Small ν parameter corresponds to a thin black ring, as the inner radius of S^1 at $x = 1$ and the outer radius at $x = -1$ tend to the same limit at $\nu \to 0$. For $\lambda = 1$ case, $\nu \to 0$ describes a non-rotating Myers-Perry black hole. If ν, λ are both set to zero initially, the metric (6.1.1) reduces to

$$ds^2 = -dt^2 + \frac{R^2}{(x-y)^2}\left[\frac{dy^2}{y^2-1} + (y^2-1)d\psi^2 + \frac{dx^2}{1-x^2} + (1-x^2)d\varphi^2\right] \,, \tag{6.1.21}$$

which is exactly the flat spacetime metric given in [117], and can be transformed into the manifest flat form (6.1.12) by transformations similar to (6.1.11) [119]. However, if we keep λR and νR fixed while taking the limit $\lambda, \nu \to 0$, (6.1.1) describes a boosted black string, and R is interpreted as the black ring radius in this limit [118].[4] As ν increases and becomes close to 1, the black ring is flattened along the plane of rotation and is called the fat ring. If $\nu = 1$, the roots $\xi_1 = \xi_4 = \xi_3 = 1$, and the black ring has a naked singularity. For the case of black hole, the shape also becomes flattened along the rotation plane for $\nu \to 1$.

6.2 Construction of the neutral rotating black rings

The asymptotically flat black ring solutions can be derived systematically by the solitonic solution-generating technique [122] and by the inverse scattering method [123]. In our review on construction of the black ring, however, we follow the historic line by means of the Kaluza-Klein reduction. In the next chapter we will construct the black ring with a positive cosmological constant using the approach based on dimensional reduction in supergravity.

The schematic construction of the asymptotically flat black ring is to generalise the charged C-metric to dilaton gravity, and then oxidise to 5 dimensions using the Kaluza-Klein ansatz, followed by double Wick rotation and redefinition of the parameters. The details are given below.

6.2.1 C-metric

The most general electrically charged, asymptotically flat C-metric in 4-dimensions takes the form [124, 125] :

$$\begin{aligned} ds^2 &= \frac{1}{A^2(x-y)^2}\left[G(y)\,dt^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + G(x)\,d\varphi^2\right] , \qquad (6.2.1) \\ A &= (qy)\,dt , \end{aligned}$$

[4] This is compatible with conceptual construction of the black ring by rotating up a circular black string which is obtained by a 4-dimensional Schwarzschild black hole times a periodically identified flat extra dimension.

where q is the charge. $q = 0$ correspond to the vacuum case. Considering the mass m of the C-metric, the structure function G is given by [125][5]

$$G(\xi) = (1-\xi^2)(1+2mA\xi+q^2A^2\xi^2) , \tag{6.2.5}$$

This is a solution to the 4-dimensional Einstein-Maxwell theory. The C-metric describes a pair of black holes uniformly accelerating in the opposite directions. The acceleration, characterised by the parameter A, arises from the external force due to each black hole being pulled by a semi-infinite string (or equivalently, pushed by a strut connected the two black holes) represented by the conical singularity in the metric. $G(\xi)$ has 4 distinct roots $\xi = (-r_-A)^{-1}, (-r_+A)^{-1}, \pm 1$, where

$$r_\pm = m \pm \sqrt{m^2-q^2} , \tag{6.2.6}$$

provided that $m > q$. Moreover, it is assumed that $0 \leq r_-A \leq r_+A < 1$.

6.2.2 Dilaton C-metric

The charged C-metric can be generalised to the Einstein-Maxwell dilaton theory in 4 dimensions:

$$S = \int d^4x\sqrt{-g}\left(R - 2(\nabla\phi)^2 - e^{-2a\phi}F^2\right) , \tag{6.2.7}$$

[5] The most general form of $G(\xi)$ for the charged C-metric is a quartic function,

$$\tilde{G}(\tilde{\xi}) = a_0 + a_1\tilde{\xi} + a_2\tilde{\xi}^2 + a_3\tilde{\xi}^3 + a_4\tilde{\xi}^4 , \tag{6.2.2}$$

where the equations of motion require $a_4 = q^2A^2$. The metric still takes the same form under the rescaling

$$\tilde{x} = Bc_0x + c_1 , \quad \tilde{y} = Bc_0y + c_1 , \quad \tilde{t} = \frac{c_0}{B}t , \quad \tilde{\varphi} = \frac{c_0}{B}\varphi . \tag{6.2.3}$$
$$\tilde{A} = \frac{A}{B} , \quad \tilde{G}(\tilde{\xi}) = B^2G(\xi) .$$

The coefficients in G are different from those in $\tilde{G}$. One can make use of these transformations to fix two coefficients out of four in $G(\xi)$, leaving two coefficients free. Besides (6.2.5), another widely used form of G is [124]:

$$G(\xi) = 1 - \xi^2 - 2mA\xi^3 - q^2A^2\xi^4 . \tag{6.2.4}$$

which gives rise to the equations of motion

$$\begin{aligned}
\nabla_\mu \left(e^{-2a\phi} F^{\mu\nu}\right) &= 0 \,, \\
\nabla^2 \phi + \frac{a}{2} e^{-2a\phi} F^2 &= 0 \,, \\
2\nabla_\mu \phi \nabla_\nu \phi + 2e^{-2a\phi} F_{\mu\rho} F_\nu{}^\rho - \frac{1}{2} g_{\mu\nu} e^{-2a\phi} F^2 &= R_{\mu\nu} \,.
\end{aligned} \tag{6.2.8}$$

The action includes several theories. The $a = 0$ case is standard Einstein-Maxwell gravity. $a = 1$ corresponds to the low energy dynamics of string theory. The $a = \sqrt{3}$ case describes the Kaluza-Klein reduction from 5-dimensional pure Einstein theory with the following ansatz,

$$ds_5^2 = e^{-4\phi/\sqrt{3}} (dx^5 + A_\mu dx^\mu)^2 + e^{2\phi/\sqrt{3}} ds_4^2 \,, \tag{6.2.9}$$

where x^5 is the extra dimension. We take $a = \sqrt{3}$ from here on.

The equations of motion are invariant under the electromagnetic duality transformation

$$\tilde{F}_{\mu\nu} = \frac{1}{2} e^{-2a\phi} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \qquad \tilde{\phi} = -\phi \,. \tag{6.2.10}$$

There exists electric and magnetic dilaton C-metric solutions for (6.2.7). The magnetically charged dilaton C-metric is given explicitly in [126]:

$$\begin{aligned}
ds^2 &= \frac{1}{A^2(x-y)^2} \sqrt{\frac{F(x)}{F(y)}} \left[F(x) \left(G(y) dt^2 - \frac{F(y)}{G(y)} dy^2 \right) + \right. \\
&\qquad \left. F(y)^2 \left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\varphi^2 \right) \right] \,, \\
& F(\xi) = (1 + r_- A \xi) \,, \\
& G(\xi) = (1 - \xi^2)(1 + r_+ A \xi) \,, \\
e^{-2\sqrt{3}\phi} &= \left(\frac{F(y)}{F(x)} \right)^{3/2} \,, \qquad A_\varphi = qx \,,
\end{aligned} \tag{6.2.11}$$

where $r_\pm$ follow (6.2.6) such that $q = 0$ (i.e. $r_- = 0$) corresponds to the neutral case.[6]

The electric dilaton C-metric is obtained by applying the duality transformations (6.2.10) to (6.2.11), which leaves the metric unchanged but transforms the dilaton

[6] By factorising $\bar{F}(\xi) = F(\xi)^{3/2}$ and $\bar{G}(\xi) = G(\xi) F(\xi)^{-1/2}$, the metric (6.2.11) takes the form

and the gauge potential to

$$e^{-2\sqrt{3}\phi} = \left(\frac{F(x)}{F(y)}\right)^{3/2} , \qquad A_t = qy . \tag{6.2.13}$$

6.2.3 The black ring

By oxidising the electric dilaton C-metric to 5 dimensions using the KK ansatz (6.2.9), one obtains the solution which describes two accelerating black holes or KK bubbles [127]:

$$\begin{aligned} ds^2 &= \frac{F(x)}{F(y)}\left(dx^5 + qy\,dt\right)^2 + \frac{1}{A^2(x-y)^2}\times \\ &\left[F(x)\left(G(y)dt^2 - \frac{F(y)}{G(y)}dy^2\right) + F(y)^2\left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\varphi^2\right)\right] , \end{aligned} \tag{6.2.14}$$

where F and G are given in (6.2.11).

Then, by double Wick rotations

$$t \longrightarrow i\psi , \qquad x^5 \longrightarrow it , \tag{6.2.15}$$

the metric assumes the primordial form of the black ring. However, since the neutral rotating black ring is a vacuum solution to 5-dimensional pure Einstein theory, there is no notion for the "electromagnetic charge" q in the solution. Moreover, the mass m is a 4-dimensional quantity and requires redefinition. These amount to redefining the parameters r_+, r_- by

$$r_-A = -\lambda , \qquad r_+A = -\nu , \qquad q = \sqrt{r_+r_-} = \frac{\sqrt{\lambda\nu}}{A} = R\sqrt{\lambda\nu} , \tag{6.2.16}$$

---

of the dilaton C-metric

$$\begin{aligned} ds^2 &= \frac{1}{A^2(x-y)^2}\left(\bar{F}(x)\left(\bar{G}(y)dt^2 - \frac{dy^2}{\bar{G}(y)}\right) + \bar{F}(y)\left(\frac{dx^2}{\bar{G}(x)} + \bar{G}(x)d\varphi^2\right)\right) ,\\ e^{-2\sqrt{3}\phi} &= \frac{\bar{F}(y)}{\bar{F}(x)} . \end{aligned} \tag{6.2.12}$$

The powers in $\bar{F}$ and $\bar{G}$ shown above is for $a = \sqrt{3}$; in general they are related to the value of a. As $a = 0$, the dilaton decouples and $\bar{F} = 1$, $\bar{G} = $ (6.2.5). The metric (6.2.11)(6.2.12) reduce to the standard charged C-metric in (6.2.1). Note that we still use the same choice of the coefficient function as in [125] for the dilaton C-metric.

where the 4-dimensional acceleration parameter A has been replaced with the 5-dimensional length scale R by $R = A^{-1}$. If one uses the gauge field $A_t = q\,(y+1)$ instead of $A_t = qy$ in (6.2.13) (where the field strength isn't altered) to derive the uniformly accelerating KK bubbles solution (6.2.14), by all those manipulations presented above, one eventually construct the asymptotically flat, neutral rotating black ring solution given in (6.1.1).

6.3 Dipole black ring solutions

The neutral black ring/black-hole solution introduced in Section 6.1 exhibits three-fold non-uniqueness. There are black rings with conserved electric charge [110], which can be heuristically regarded as a circular black string coupled naturally to 2-form potential in 5 dimensions. The conserved charge doesn't contribute to non-uniqueness. The rotating electric charges induce magnetic dipoles which depend on the charges and the rotation, so the dipoles have no contribution to non-uniqueness either. If we consider instead a black ring magnetically coupled to a 1-form potential, the net charge measured from infinity vanishes, but there are non-zero local charges (which are not conserved) on the S^1 circle, and those at the diametrically opposite ends of the circle form a dipole. As a result, the black rings with the same mass and spin carry infinitely many distinct values of dipole charges which label continuous violation of uniqueness. Such black ring solutions are naturally in the context of string theory (e.g. 5-dimensional dilatonic Einstein-Maxwell theory) or M-theory. In this section we present an example of the dipole black rings embedded in M-theory.

Compared to the neutral solution in Section 6.1, the dipole black ring contains additional functions H_i which incorporate new parameters μ_i with $i = 1, 2, 3$ [117] [111]:

$$
\begin{aligned}
ds^2 \;=\; & -\frac{F(y)}{F(x)}\frac{H(x)}{H(y)}\left(dt + CR\,\frac{1+y}{F(y)}\,d\psi\right)^2 + \frac{R^2}{(x-y)^2}\,F(x)H(x)H(y)^2 \times \\
& \left[-\frac{G(y)}{F(y)H(y)^3}\,d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)H(x)^3}\,d\varphi^2\right] \;, \quad (6.3.1)
\end{aligned}
$$

where the functions F, G and H are

$$\begin{aligned} F(\xi) &= 1+\lambda\xi\,, & G(\xi) &= (1-\xi^2)(1+\nu\xi)\,, \\ H(\xi) &= \left(H_1(\xi)\,H_2(\xi)\,H_3(\xi)\right)^{1/3}, & H_i(\xi) &= 1-\mu_i\xi\,. \end{aligned} \tag{6.3.2}$$

and the parameter C is a function of ν and λ,

$$C=\sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}\,. \tag{6.3.3}$$

Note that we use slightly different coordinates in this section compared to the neutral black ring in (6.1.1), although the notation is the same. (See equation (6.3.11) and the transformations that follow for explanation.)

This dipole black ring is a solution to 5-dimensional $U(1)^3$ supergravity

$$S=\frac{1}{16\pi G}\int\left\{R*\mathbb{1}-\sum_{i=1}^{3}\frac{1}{2(X^i)^2}(dX^i\wedge *dX^i-F^i\wedge *F^i)-\frac{C_{ijk}}{6}\,F^i\wedge F^j\wedge A^k\right\}\,, \tag{6.3.4}$$

where $C_{ijk}=1$ for $(i,j,k)=$permutation of $(1,2,3)$ and $C_{ijk}=0$ otherwise. The 2-form field strength $F^i=dA^i$. The 5d supergravity is obtained from the dimensional reduction of 11-dimensional supergravity with the ansatz

$$\begin{aligned} ds_{11}^2 &= ds_5^2+X^1(dz_1^2+dz_2^2)+X^2(dz_3^2+dz_4^2)+X^3(dz_5^2+dz_6^2)\,, \\ A_{(3)} &= A^1_{(1)}\wedge dz_1\wedge dz_2+A^2_{(1)}\wedge dz_3\wedge dz_4+A^3_{(1)}\wedge dz_5\wedge dz_6\,, \end{aligned} \tag{6.3.5}$$

in which the 5-dimensional metric ds_5^2, magnetic 1-form gauge fields $A^i_{(1)}$ and scalars X^i are independent of $(z_1,\ldots,z_6)$ which parametrise the T^6 internal space[7]. This ansatz is interpreted as triple intersections of M5-branes [128] [18], as the 1-form magnetic gauge potential A^i and the scalars X^i are given by

$$A^i=C_i\,R\,\frac{1+x}{H_i(x)}\,d\varphi\,, \qquad X^i=\frac{H(x)H_i(y)}{H(y)H_i(x)}\,, \tag{6.3.6}$$

with

$$C_i=\sqrt{\mu_i(\mu_i+\nu)\frac{1-\mu_i}{1+\mu_i}}\,. \tag{6.3.7}$$

Thus the dipole black ring solution arises from the intersecting M5-branes, with four worldvolume dimensions of each brane wrapping 4-cycles of T^6 and the remaining

[7]It is assumed that $X^1X^2X^3=1$ in order for the T^6 to have a constant volume.

spacelike direction winding around S^1 of the black ring, and the rotation derives from momentum flux along the intersection.

The conserved electric charges in 5 dimensions is defined by

$$Q_i = \frac{1}{16\pi G} \int_{S^3} \frac{1}{(X^i)^2} * F^i \,, \tag{6.3.8}$$

where S^3 is taken at asymptotic infinity. The topology of the black ring also makes it possible to define the local, non-conserved charges (or dipole charges) by considering an S^2 surface enclosing a point on S^1 at constant (t, y):

$$q_i = \frac{1}{2\pi} \int_{S^2} F^i \,, \tag{6.3.9}$$

where an extra factor of 2 has been included on the RHS by taking into account the normalisation of the Lagrangian. q_i are non-conserved as they can be changed by local processes which leave the fields at infinity unaltered, as the cycle the ring wraps is topologically trivial.

The dimensionless parameters λ, ν, μ_i are subject to the conditions

$$0 < \nu \leq \lambda < 1 \,, \qquad 0 \leq \mu_i < 1 \,, \tag{6.3.10}$$

where $\mu_1 = \mu_2 = \mu_3 = 0$ correspond to the neutral rotating black ring in Section 6.1.[8] R again corresponds to the radius of thin ring as R is large. The coordinates (x, y) take values in

$$-1 \leq x \leq 1 \,, \qquad -1 \leq \frac{1}{y} < \min(\mu_1, \mu_2, \mu_3) \,. \tag{6.3.12}$$

The analysis of the geometry of the dipole black ring solution is similar to that in Section 6.1. Asymptotic infinity locates at $(x = -1, y = -1)$, while $y = -1/\lambda$

[8] In the neutral limit, the metric reads

$$\begin{aligned} ds^2 \;&=\; -\frac{F(y)}{F(x)} \left(dt + CR\,\frac{1+y}{F(y)}\, d\psi \right)^2 \\ &\quad + \frac{R^2}{(x-y)^2}\, F(x) \left[-\frac{G(y)}{F(y)} d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\varphi^2 \right] \,, \end{aligned} \tag{6.3.11}$$

which can be transformed into the form of (6.1.1) by $x = \frac{x'-\lambda'}{1-\lambda' x'}$, $y = \frac{y'-\lambda'}{1-\lambda' y'}$, $(\psi, \varphi) = \frac{1-\lambda'\nu'}{\sqrt{1-\lambda'^2}}(\psi', \varphi')$, $\nu = \frac{\lambda'-\nu'}{1-\lambda'\nu'}$, $\lambda = \lambda'$ [117], where the primed coordinates and parameters represent those used in (6.1.1).

represents the ergosurface. There are two regular horizons in this solution: the outer one is at $y = -1/\nu$ and the inner one at $|y| = \infty$. y can be analytically extended to $y < \infty$, until reaching the curvature singularity at $y = 1/\mu_{min}$ where $\mu_{min} = \min(\mu_1, \mu_2, \mu_3)$. In the neutral limit, the inner horizon becomes singular.

The conical singularities are present at $y = -1$ and $x = -1$, and can be removed by identifying the periods of φ and ψ by

$$\Delta\varphi(x=-1) = \Delta\psi(y=-1) = 2\pi\frac{\sqrt{1-\lambda}}{1-\nu}\prod_{i=1}^{3}\sqrt{1+\mu_i}\,. \tag{6.3.13}$$

For the balanced black ring, the conical singularity at $x = +1$ is also removed by $\Delta\varphi(x = +1) = \Delta\varphi(x = -1)$, which gives rise to a constraint on the parameters λ, ν, μ_i:

$$\frac{1-\lambda}{1+\lambda}\prod_{i=1}^{3}\frac{1+\mu_i}{1-\mu_i} = \left(\frac{1-\nu}{1+\nu}\right)^2 . \tag{6.3.14}$$

The mass, angular momentum, outer horizon area and dipole charges of the solution are given by

$$M = \frac{3\pi R^2}{4G}\frac{1}{1-\nu}\left(\lambda + \sum_{i=1}^{3}\frac{\mu_i(1-\lambda)}{3(1+\mu_i)}\right)\prod_{i=1}^{3}(1+\mu_i)\,, \tag{6.3.15}$$

$$J = \frac{\pi R^3}{2G}\frac{\sqrt{\lambda(\lambda-\nu)(1+\lambda)}}{(1-\nu)^2}\prod_{i=1}^{3}(1+\mu_i)^{3/2}\,, \tag{6.3.16}$$

$$A_H = 8\pi^2 R^3\frac{\sqrt{\lambda(1-\lambda^2)}}{(1-\nu)^2(1+\nu)}\prod_{i=1}^{3}(1+\mu_i)\sqrt{(\mu_i+\nu)}\,, \tag{6.3.17}$$

$$q_i = 2R\,\frac{\sqrt{\mu_i(\mu_i+\nu)(1-\lambda)}}{(1-\nu)\sqrt{1-\mu_i^2}}\prod_{j=1}^{3}\sqrt{1+\mu_j}\,. \tag{6.3.18}$$

The condition (6.3.14) implies that, if ν (i.e. the outer horizon position) is fixed, upon turning on the dipole charges, the ergosurface shifts outwards, and the mass also increases. The dimensionless spin j^2, defined as in (6.1.14), is bounded both from below and from above as the dipole charges deviate from zero. For fixed mass and spin, the surface area decreases for both the thin ring and fat ring as q_i increase. The maxima of the dipole charges can be found as the black ring becomes extremal, i.e. $\nu = 0$ such that the outer and inter horizons overlap [111].

The temperature and angular velocity on the outer horizon are

$$T = \frac{1}{4\pi R}\frac{\nu(1+\nu)}{\prod_{i=1}^{3}(\mu_i+\nu)^{1/2}}\sqrt{\frac{1-\lambda}{\lambda(1+\lambda)}}\,, \tag{6.3.19}$$

$$\Omega_H = \frac{1}{R}\frac{1}{\prod_{i=1}^{3}(1+\mu_i)^{1/2}}\sqrt{\frac{\lambda-\nu}{\lambda(1+\lambda)}}\,. \tag{6.3.20}$$

If the conserved charge vanishes, using the physical quantities given above, one derives the Smarr relation which incorporates a new term contributed by the dipole charges,

$$M = \frac{3}{2}\left(\frac{T}{4G}A_H + \Omega_H J\right) + \frac{1}{2}\,\Phi_i q_i\,, \tag{6.3.21}$$

where Φ_i correspond to the dipole potentials defined by the difference of the dual electric 2-forms $B^i = C_i R\frac{1+y}{2H_i(y)}\,dt\wedge d\psi$ (up to a gauge freedom) at infinity and at the outer horizon,

$$\begin{aligned}\Phi_i &= \frac{\pi}{2G}\left(B^i|_{\text{infinity}} - B^i|_{\text{horizon}}\right)\\ &= \frac{\pi R}{4G}\sqrt{\frac{\mu_i(1-\mu_i)(1-\lambda)}{(1+\mu_i)(\mu_i+\nu)}}\prod_{j=1}^{3}\sqrt{1+\mu_j}\,.\end{aligned} \tag{6.3.22}$$

Hence the first law for dipole black ring reads

$$dM = \frac{T}{4G}\,dA_H + \Omega_H\,dJ + \Phi_i\,dq_i\,. \tag{6.3.23}$$

Note that the last term is from non-conserved charges, which doesn't seem compatible with the known form of the first law of black hole mechanics in which only the conserved charges that can be obtained by surface integral at infinity are involved. This puzzle is resolved by Copsey and Horowitz [129]. The previous derivation of the first law by Sudarsky and Ward [130] assumes that the 2-form potential $B_{\mu\nu}$ is globally well-defined and non-singular all over the space outside and on the horizon. Consistency of physics also requires that $B_{t\psi}$ must vanish both at the rotation axis and on the horizon. These however contradict the properties of the dipole ring solution that the dipole charge is nonzero and the potential $B_{\mu\nu}$ is invariant under symmetries of $\partial/\partial t$ and $\partial/\partial\psi$. Since the coordinate (x, y) break down near the rotation axis $y = -1$ (and thus the asymptotic infinity), the manifestly asymptotically flat metric is obtained by transforming (x, y) to new coordinates in which $B_{t\psi}$ is set to zero at $y = -1$. In the new coordinates, however, $B_{t\psi}$ diverge on the horizon. Consequently a new surface term arises in the first law, and (6.3.23) is reproduced precisely.

Chapter 7

Black Fusiform with a Positive Cosmological Constant

In the previous chapter we reviewed various properties of the black ring solutions. Those black rings are placed in the background without a cosmological constant. No satisfactory black ring solution with a cosmological constant is constructed yet, except in $AdS_3 \times S^3$ [131]. There will be very interesting applications in (A)dS/CFT correspondence if such solutions in 5 dimensional AdS space can be found, so that one can investigate how the physics in the bulk with non-spherical horizons is described by the dual field theory on the boundary.

In order to construct a 5-dimensional black ring solution with a cosmological constant from 4-dimensional ones as the way Emparan and Reall's black ring is constructed, certain dimensional reduction ansatz applicable to spacetime with a non-vanishing cosmological constant must be used. Attempting to construct an asymptotically AdS/dS black ring, in this chapter we oxidise a 4 dimensional C-metric with $\Lambda > 0$ [132] using the braneworld Kaluza-Klein supergravity reduction ansatz of $dS_4 \subset dS_5$ [135] which preserves half of $\mathcal{N} = 4$ supersymmetry in 5-dimensional de Sitter supergravity. The outcome is a solution of *an interval* $\times S^2$ topology, with curvature sigularities at the two ends of the interval. The interval is labelled by $z \in [-\pi/2k, \pi/2k]$, where k is related to the 5-dimensional cosmological constant. We call this solution a *black fusiform*, as the size of S^2 is largest in the middle of the interval ($z = 0$) and shrinks toward the tips ($z = \pm\pi/2k$) where the singularities reside. As two singularities sit on the opposite sides of the cosmological horizon, the black fusiform solution may be interpreted as a pinched-off black ring solution.

This black fusiform is static and asymptotically non-de Sitter despite a positive cosmological constant is present. It has two horizons of *an interval* $\times S^2$ topology. It is supported by a magnetic 2-form field and two 3-form fields, and the charge associated with the former contributes in the 1st law of thermodynamics.

This chapter is started by reviewing the braneworld Kaluza-Klein supergravity reduction, followed by construction of the black fusiform solution using the reduction ansatz. Despite the singularities, our solution still exhibits some interesting physical properties. Their implications will be discussed in the next chapter.

7.1 Braneworld Kaluza-Klein supergravity reduction

7.1.1 Overview

In the previous chapter, we reviewed that the first asymptotically flat black ring solution is constructed by means of KK reduction (6.2.9) based on a factorisable geometry. For a spacetime with nonvanishing cosmological constant, there is a dimensional reduction ansatz proposed by Randall and Sundrum for the famous braneworld model [133],

$$d\hat{s}_5^2 = dz^2 + e^{-2k|z|}\, ds_4^2\ , \qquad (k > 0)\ , \tag{7.1.1}$$

which basically describes a 4-dimensional Minkowskian spacetime (i.e. a 3-brane) embedded in a 5-dimensional anti-de Sitter spacetime. It was demonstrated later in [134] that by this reduction ansatz, together with its extension to other fields, $\mathcal{N} = 2, D = 4$ ungauged Einstein-Maxwell supergravity can be consistently embedded in $\mathcal{N} = 4, D = 5$ gauged supergravity.

The Randall-Sundrum ansatz is based on warped geometry,

$$d\hat{s}_{D+1}^2 = dz^2 + f(z)^2\, ds_D^2\ , \tag{7.1.2}$$

where the higher dimensional quantities are hatted while the lower dimensional ones are unhatted. ds_D^2 is Lorentzian and z denotes the "extra" dimension. $f(z)$ is the warp factor. The $(D+1)$-dimensional Ricci tensor is expressed in terms of the metric components and the lower dimensional quantities by

$$\hat{R}_{zz} = -D\frac{f''}{f}, \qquad \hat{R}_{\mu z} = 0,$$
$$\hat{R}_{\mu\nu} = \left(R_{\mu\nu} + (D-1)(f''f - f'^2)\, g_{\mu\nu}\right) - D\,\frac{f''}{f}\,\hat{g}_{\mu\nu}\,, \tag{7.1.3}$$

where $\mu, \nu = 0, \ldots, D-1$. If we consider the braneworld embedding of D-dimensional Einstein spacetime in a $(D+1)$-dimensional Einstein manifold, namely $R_{\mu\nu} = \pm(D-1)k^2 g_{\mu\nu}$ and $\hat{R}_{MN} = \pm Dk^2 \hat{g}_{MN}$, the differential equations of f can be read out from (7.1.3) for various embeddings:

$$\frac{f''}{f} = \begin{cases} +k^2\,, & AdS_{D+1} \quad (a) \\ -k^2\,, & dS_{D+1} \quad (b) \end{cases} \tag{7.1.4}$$

and also

$$f''f - (f')^2 = \begin{cases} +k^2\,, & AdS_D \quad (a) \\ -k^2\,. & dS_D \quad (b) \end{cases} \tag{7.1.5}$$

The independent solutions of f are e^{kz}, e^{-kz} (resp. e^{ikz}, e^{-ikz}) if the $(D+1)$-dimensional spacetime is AdS (resp. dS), and f must also satisfy (a) or (b) of (7.1.5). If the warp factor f is required to be real, the braneworld embedding of $AdS_D \subset AdS_{D+1}$, $dS_D \subset AdS_{D+1}$, and $dS_D \subset dS_{D+1}$ are thus obtained [135],

$$AdS_D \subset AdS_{D+1}: \quad f = \cosh(kz), \quad \hat{R}_{zz} = -Dk^2\,, \tag{7.1.6}$$
$$\hat{R}_{\mu\nu} = [R_{\mu\nu} + (D-1)k^2 g_{\mu\nu}] - Dk^2 \hat{g}_{\mu\nu}\,,$$

$$dS_D \subset AdS_{D+1}: \quad f = \sinh(kz), \quad \hat{R}_{zz} = -Dk^2\,, \tag{7.1.7}$$
$$\hat{R}_{\mu\nu} = [R_{\mu\nu} - (D-1)k^2 g_{\mu\nu}] - Dk^2 \hat{g}_{\mu\nu}\,,$$

$$dS_D \subset dS_{D+1}: \quad f = \cos(kz), \quad \hat{R}_{zz} = Dk^2\,, \tag{7.1.8}$$
$$\hat{R}_{\mu\nu} = [R_{\mu\nu} - (D-1)k^2 g_{\mu\nu}] + Dk^2 \hat{g}_{\mu\nu}\,.$$

Together with the reduction ansatz for other fields, it is shown in [135] that $AdS_4 \subset AdS_5$ and $dS_4 \subset AdS_5$ give rise to consistent reductions from gauged $\mathcal{N} = 4$ AdS_5

supergravity to gauged $\mathcal{N}=2$ AdS_4 and dS_4 supergravity respectively, while $dS_4 \subset dS_5$ embeds gauged $\mathcal{N}=2$ dS_4 supergravity in gauged $\mathcal{N}=4$ dS_5 one. They are called the braneworld Kaluza-Klein reductions. In this section we will review the $dS_4 \subset dS_5$ case in detail, as this ansatz will be used later to construct our solution. This particular reduction scheme is chosen because it is the only ansatz with compact warp factor for $z \in \mathbf{R}$, and thus allows the 5-dimensional solution to be interpreted as a pinched black ring. This will be clear later.

7.1.2 $dS_4 \subset dS_5$ case

The bosonic part of gauged $\mathcal{N}=4, D=5$ de Sitter supergravity, which contains the metric, a dilaton field ϕ, three $SU(2)$ Yang-Mills potentials $\hat{A}^i_{(1)}$ $(i=1,2,3)$, a $U(1)$ gauge potential $\hat{B}_{(1)}$, and two 2-form potentials $\hat{A}^\alpha_{(2)}$ $(\alpha=1,2)$ transforming as a charged doublet under the $U(1)$, is described by the Lagrangian [136]

$$\begin{aligned} \mathcal{L}_5 \;=\;& \hat{R}\,\hat{*}\mathbb{1} - \frac{1}{2}\,\hat{*}d\phi\wedge d\phi - \frac{1}{2}\,X^4\,\hat{*}\hat{G}_{(2)}\wedge\hat{G}_{(2)} + \frac{1}{2}\,X^{-2}\,\hat{*}\hat{F}^i_{(2)}\wedge\hat{F}^i_{(2)} \\ & + \frac{1}{2g}\,\varepsilon_{\alpha\beta}\,\hat{A}^\alpha_{(2)}\wedge d\hat{A}^\beta_{(2)} - \frac{1}{2}\,\theta_{(\alpha)}\,(X^{-2}\,\hat{*}\hat{A}^\alpha_{(2)}\wedge\hat{A}^\alpha_{(2)} - \hat{A}^\alpha_{(2)}\wedge\hat{A}^\alpha_{(2)}\wedge\hat{B}_{(1)}) \\ & + \frac{1}{2}\,\hat{F}^i_{(2)}\wedge\hat{F}^i_{(2)}\wedge\hat{B}_{(1)} - 4g^2\,(X^2+2X^{-1})\,\hat{*}\mathbb{1}\,, \end{aligned} \tag{7.1.9}$$

where $\hat{*}$ denotes the 5-dimensional Hodge dual, $X=e^{-\frac{1}{\sqrt{6}}\phi}$, $\hat{F}^i_{(2)} = d\hat{A}^i_{(1)} - \frac{1}{\sqrt{2}}\,g\,\varepsilon^{ijk}\,\hat{A}^j_{(1)}\wedge \hat{A}^k_{(1)}$, $\hat{G}_{(2)} = d\hat{B}_{(1)}$, and $\theta_{(\alpha=1)}=-1$, $\theta_{(\alpha=2)}=1$. The Einstein summation convention is applied over the indices i and α, and g is the gauge coupling constant. This theory is obtained by performing Kaluza-Klein reduction to type IIB* theory on the hyperbolic 5-space H^5, where the IIB* supergravity arises from T-dualising type IIA on a timelike circle, or directly from the ordinary IIB by replacing all Ramond-Ramond fields with $\Psi \to i\Psi$, such that the RR fields have the "wrong-sign" kinetic terms [137,138]. Consequently, the Lagrangian (7.1.9) can be obtained from that of gauged $\mathcal{N}=4, D=5$ AdS supergravity by carrying out the analytic continuations

$$g \to ig\,, \qquad \hat{A}^1_{(2)} \to i\hat{A}^1_{(2)}\,, \qquad \hat{A}^i_{(1)} \to i\,\hat{A}^i_{(1)}\,, \tag{7.1.10}$$

and thus the $\hat{A}^i_{(1)}$ kinetic terms, the $\hat{A}^1_{(2)}$ interaction to itself, and the g^2 coupling term in (7.1.9) carry opposite signs compared to those in the AdS supergravity

Lagrangian. Moreover, in the definition of $\hat{F}^i_{(2)}$, the sign of $\frac{1}{\sqrt{2}}\, g\, \varepsilon^{ijk}\, \hat{A}^j_{(1)} \wedge \hat{A}^k_{(1)}$ term is flipped.

The Lagrangian (7.1.9) gives rise to the following equations of motion:

$$\begin{aligned}
d(X^{-1}\hat{*}dX) &= \frac{1}{3}X^4\hat{*}\hat{G}_{(2)}\wedge\hat{G}_{(2)} - \frac{1}{6}X^{-2}(-\hat{*}\hat{F}^i_{(2)}\wedge\hat{F}^i_{(2)} + \theta_{(\alpha)}\hat{*}\hat{A}^\alpha_{(2)}\wedge\hat{A}^\alpha_{(2)}) \\
&\quad +\frac{4}{3}g^2(X^2 - X^{-1})\hat{*}1\!\!1\,, \\
d(X^4\hat{*}\hat{G}_{(2)}) &= -\frac{1}{2}\,\theta_{(\alpha)}\hat{A}^\alpha_{(2)}\wedge\hat{A}^\alpha_{(2)} + \frac{1}{2}\,\hat{F}^i_{(2)}\wedge\hat{F}^i_{(2)}\,, \\
d(X^{-2}\hat{*}\hat{F}^i_{(2)}) &= \sqrt{2}\,gX^{-2}\varepsilon^{ijk}\,\hat{*}\hat{F}^j_{(2)}\wedge\hat{A}^k_{(1)} - \hat{F}^i_{(2)}\wedge\hat{G}_{(2)}\,, \qquad (7.1.11)\\
X^2\hat{*}\hat{F}^\alpha_{(3)} &= g\,\hat{A}^\alpha_{(2)}\,, \\
\hat{R}_{MN} &= \frac{3}{X^2}\,\partial_M X\,\partial_N X + \frac{4}{3}g^2(X^2 + \frac{2}{X})\,\hat{g}_{MN} + \frac{X^4}{2}\left[\hat{G}_M{}^P\hat{G}_{NP} - \frac{\hat{g}_{MN}}{6}\,(\hat{G}_{(2)})^2\right] \\
&\quad + \frac{1}{2X^2}\left[-\hat{F}^i_M{}^P\hat{F}^i_{NP} + \frac{\hat{g}_{MN}}{6}\,(\hat{F}^i_{(2)})^2\right] + \frac{\theta_{(\alpha)}}{2X^2}\left[\hat{A}^\alpha_M{}^P\hat{A}^\alpha_{NP} - \frac{\hat{g}_{MN}}{6}\,(\hat{A}^\alpha_{(2)})^2\right],
\end{aligned}$$

where $\hat{F}^\alpha_{(3)} = d\hat{A}^\alpha_{(2)} + g\,\hat{B}_{(1)}\wedge\hat{A}^\alpha_{(2)}$. Here M, N denotes the dS_5 indices. The consistent reduction to gauged $\mathcal{N}=2$, $D=4$ dS supergravity is given by taking $g = k$ together with the reduction ansatz

$$\begin{aligned}
d\hat{s}_5{}^2 &= dz^2 + \cos^2(kz)\,ds_4{}^2, \\
\hat{A}^1_{(2)} &= -\frac{1}{\sqrt{2}}\cos(kz)\, *F_{(2)}, \qquad \hat{A}^2_{(2)} = -\frac{1}{\sqrt{2}}\sin(kz)\,F_{(2)}, \qquad (7.1.12)\\
\hat{A}^1_{(1)} &= \frac{1}{\sqrt{2}}A_{(1)}\,,
\end{aligned}$$

with all other bosonic fields in dS_5 supergravity being set to zero. $*$ denotes the Hodge dual in 4-dimensional spacetime, and $F_{(2)}$ is the Maxwell field strength in dS_4, $F_{(2)} = dA_{(1)}$. The coordinate z takes values in the interval

$$z \in [-\frac{\pi}{2k}, \frac{\pi}{2k}]\,, \qquad (7.1.13)$$

i.e. $\Delta z = \pi/k$. These ansatz imply that the 4-dimensional $U(1)$ gauge group is matched with that of $\hat{A}^\alpha_{(2)}$ and the $U(1)$ subgroup of $\hat{A}^i_{(1)}$ in 5-dimensions, i.e. the photon field in dS_4 is derived from $\hat{A}_{(2)}$ and $\hat{A}^1_{(1)}$ in dS_5 supergravity, which is unconventional compared to the standard Kaluza-Klein scenario. Then the 4-dimensional equations of motion reduced from dS_5 supergravity ones are

$$\begin{aligned}
R_{\mu\nu} &= -\frac{1}{2}\,(F_{\nu\lambda}\,F_\mu{}^\lambda - \frac{1}{4}\,F^2_{(2)}\,g_{\mu\nu}) + 3k^2 g_{\mu\nu}\,, \qquad (7.1.14)\\
d(*F_{(2)}) &= 0\,,
\end{aligned}$$

which are exactly the bosonic equations of motion for the gauged $\mathcal{N} = 2, D = 4$ de Sitter supergravity. Note that the Maxwell terms carry opposite signs compared to those in the standard Einstein-Maxwell theory, which is the feature of de Sitter supergravity.

The reduction of the fermionic part works similarly; the details of the $AdS_4 \subset AdS_5$ case can be found in [135]. By performing the analytic continuation in (7.1.10), the ansatz for $AdS_4 \subset AdS_5$ are obtained. $N = 4$ dS_5 supergravity contains spin-$\frac{1}{2}$ fields $\hat{\chi}_p$ and spin-$\frac{3}{2}$ fields ψ_{Mp}, where p is the 4-dimensional $USp(4)$ index, $p = \{\alpha = (1,2)\} \cup \{i = (3,4,5)\}$.[1] The reduction ansatz involves setting $\hat{\chi}_\alpha$, $\hat{\psi}_{zp} = 0$ to zero, and

$$\begin{aligned} \hat{\epsilon}_p &= \left\{\cos\left(\frac{kz}{2}\right) + \sin\left(\frac{kz}{2}\right)\gamma_5\Gamma_{12}\right\}\epsilon_p \\ \hat{\psi}_{\mu p} &= \left\{\cos\left(\frac{kz}{2}\right) + \sin\left(\frac{kz}{2}\right)\gamma_5\Gamma_{12}\right\}\psi_{\mu p}\ , \end{aligned} \tag{7.1.15}$$

where $\hat{\epsilon}$ and ϵ are the SUSY transformation parameters, and ψ_μ satisfies the 4-dimensional gravitino transformation rule. Both ϵ and ψ_μ are constrained by the component-halving condition,

$$\gamma_5\Gamma_{13}\,\epsilon = \epsilon\ , \qquad \gamma_5\Gamma_{13}\,\psi = \psi\ . \tag{7.1.16}$$

Therefore the full $\mathcal{N} = 2, D = 4$ de Sitter supergravity is recovered via the braneworld KK reduction.

7.1.3 $D = 4$ and $D = 5$ cosmological constants

In the braneworld embedding (7.1.2), the $(D+1)$-dimensional cosmological constant is given by $\Lambda_{D+1} = \frac{D(D-1)}{2}k^2$. As Λ_{D+1} is fixed, Λ_D can be arbitrarily rescaled to the required value by shifting z. The case of $AdS_4 \subset AdS_5$ was elaborated in [135]. For $dS_4 \subset dS_5$, the rescaling of Λ_4 is achieved by $z \to z + ic$, together with

$$f = \cos kz \to f = \lambda e^{ikz} + e^{-ikz}\ , \qquad g_{\mu\nu} \to 4\lambda g_{\mu\nu}\ , \qquad A_\mu \to 2\sqrt{\lambda}A_\mu\ , \tag{7.1.17}$$

[1] i is the same as the $SU(2)$ triplet index of $\hat{A}^i_{(1)}$. Here i is taken as $(3,4,5)$ just for the convenience of calculation.

where $\lambda = e^{-2kc}$ and c is a real constant. The shifted f still satisfies the differential equations (7.1.4)(b) and (7.1.5)(b), so the embedding is still $dS_4 \subset dS_5$. For the vacuum case, the dS_4 Einstein equation becomes $R_{\mu\nu} = 3k^2 g_{\mu\nu} \rightarrow R_{\mu\nu} = 12\lambda k^2 g_{\mu\nu}$, where Λ_4 is rescaled, at the cost that the warp factor f becomes complex. The supergravity reduction ansatz becomes

$$\begin{aligned} d\hat{s}_5{}^2 &= dz^2 + (\lambda e^{ikz} + e^{-ikz})^2 (kz)\, ds_4{}^2 \,, \qquad (7.1.18) \\ \hat{A}^1_{(2)} &= -\frac{1}{\sqrt{2}}(\lambda e^{ikz} + e^{-ikz})\, *F_{(2)} \,, \qquad \hat{A}^2_{(2)} = -\frac{1}{\sqrt{2}}(\lambda e^{ikz} + e^{-ikz})\, F_{(2)} \,, \\ \hat{A}^1_{(1)} &= \sqrt{2}\lambda\, A_{(1)} \,, \end{aligned}$$

and the 4-dimensional equations of motion is given simply by replacing the $3k^2 g_{\mu\nu}$ term in (7.1.14) with $12\lambda k^2 g_{\mu\nu}$.

7.2 Black fusiform with a positive cosmological constant

To construct the black fusiform solution with the topology of S^2 times an interval in the background with a positive cosmological constant, we start from the charged $\Lambda > 0$ C-metric [132, 139], which carries a positive Λ and is a solution to the dS_4 supergravity equations of motion in (7.1.14),

$$\begin{aligned} ds_4{}^2 &= \frac{1}{A^2(x-y)^2}\left[G(y)\, dt^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{\tilde{G}(x)} + \tilde{G}(x) d\varphi^2\right], \\ A_\varphi &= qx + c_0 \,. \end{aligned} \qquad (7.2.1)$$

The coefficient functions $G(\xi)$ and $\tilde{G}(\xi)$ are quartic,

$$\begin{aligned} G(\xi) &= q^2 A^2 \xi^4 + a_3 \xi^3 + a_2 \xi^2 + a_1 \xi + a_0 \,, \\ \tilde{G}(\xi) &= G(\xi) - k^2/A^2 \,, \end{aligned} \qquad (7.2.2)$$

where c_0, $a_{0,1,2,3}$ and $A > 0$ are constant parameters. This metric describes a pair of uniformly accelerating charged black holes in the background with cosmological constant $\Lambda = 3k^2$, where A denotes the acceleration. (7.2.1) is called the "de Sitter" C-metric in [132] because the curvature invariants are asymptotically de Sitter, but the metric itself in fact is not maximally symmetric at asymptotic infinity (which we'll see shortly). Therefore the "de Sitter" C-metric is a misnomer.

By substituting the $\Lambda > 0$ C-metric into the $dS_4 \subset dS_5$ braneworld KK supergravity reduction ansatz (7.1.12), a solution to $D = 5$ de Sitter supergravity can be constructed [1],

$$\begin{aligned} d\hat{s}_5{}^2 &= dz^2 + \frac{\cos^2(kz)}{A^2(x-y)^2}\left[G(y)\,dt^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{\tilde{G}(x)} + \tilde{G}(x)d\varphi^2\right] , \\ \hat{A}^1_{ty} &= -\frac{1}{\sqrt{2}}\cos(kz)\,(*F)_{ty} , \qquad \hat{A}^2_{x\varphi} = -\frac{1}{\sqrt{2}}\sin(kz)\,F_{x\varphi} , \\ \hat{A}^1_{\varphi} &= \frac{1}{\sqrt{2}}A_{\varphi} . \end{aligned} \tag{7.2.3}$$

Suppose the roots $\xi_{1,2,3,4}$ for $G(\xi)$ and $\tilde{\xi}_{1,2,3,4}$ for $\tilde{G}(\xi)$ are distinct and are arranged such that $\xi_1 < \xi_2 < \xi_3 < \xi_4$ and $\tilde{\xi}_1 < \tilde{\xi}_2 < \tilde{\xi}_3 < \tilde{\xi}_4$. (See Fig. 7.2.) We choose the functions $G, \tilde{G}$ to be even so that $a_3 = a_1 = 0$. (See footnote [3].) By rescaling the coordinates, the remaining coefficients can be set to $a_0 = 1 = -a_2$, such that $G(\xi)$ and $\tilde{G}(\xi)$ are fixed to the form

$$G(y) = 1 - y^2 + q^2A^2y^4 = q^2A^2(y^2 - \xi_3^2)(y^2 - \xi_4^2) , \tag{7.2.4}$$

$$\tilde{G}(x) = (1 - k^2/A^2) - x^2 + q^2A^2x^4 = q^2A^2(x^2 - \tilde{\xi}_3^2)(x^2 - \tilde{\xi}_4^2) , \tag{7.2.5}$$

where $\xi_{1,2} = -\xi_{4,3}$, $\tilde{\xi}_{1,2} = -\tilde{\xi}_{4,3}$. It is also assumed that

$$k^2 < A^2 , \qquad 0 \le q \le \frac{1}{2A} \tag{7.2.6}$$

in order for the roots to be real. The explicit expressions for the roots are

$$\xi_{3,4}^2 = \frac{1 \mp \sqrt{1 - 4q^2A^2}}{2q^2A^2} , \qquad \tilde{\xi}_{3,4}^2 = \frac{1 \mp \sqrt{1 - 4q^2A^2(1 - k^2/A^2)}}{2q^2A^2} , \tag{7.2.7}$$

where the "$-$" and "$+$" signs corresponds to $\xi_3, \tilde{\xi}_3$ and $\xi_4, \tilde{\xi}_4$ respectively.

For $q = 0$, G and $\tilde{G}$ becomes quadratic, and the metric (7.2.1) is called the massless C-metric [132] in terms of the choice of coefficients in (6.2.4). It carries a positive cosmological constant and takes the form of the maximally symmetric spacetime metric, i.e

$$K_{\mu\nu\lambda\sigma} := R_{\mu\nu\lambda\sigma} - k^2(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda}) = 0 . \tag{7.2.8}$$

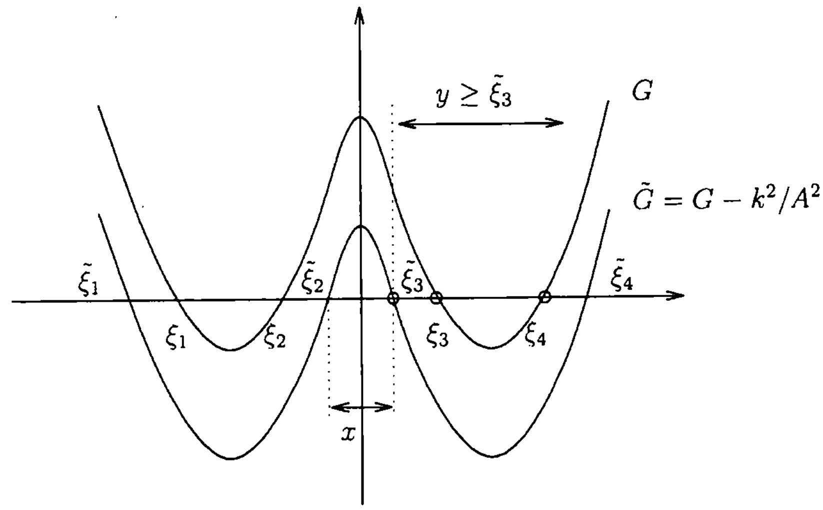

Figure 7.1: The even functions $G(\xi)$ and $\tilde{G}(\xi)$. The roots are arranged as $\tilde{\xi}_1 < \xi_1 < \xi_2 < \tilde{\xi}_2 < \tilde{\xi}_3 < \xi_3 < \xi_4 < \tilde{\xi}_4$, and $\xi_{1,2} = -\xi_{4,3}$, $\tilde{\xi}_{1,2} = -\tilde{\xi}_{4,3}$. The coordinate range is $\tilde{\xi}_2 \leq x \leq \tilde{\xi}_3$ for x and $\tilde{\xi}_3 \leq y < \infty$ for y.

Now G and $\tilde{G}$ respectively have two roots $\xi_{+,-} = \pm 1$ and $\tilde{\xi}_{+,-} = \pm\sqrt{1 - k^2/A^2}$. The coordinate range is $-1 \leq x \leq 1$ and $-x \leq y < \infty$. There is a cosmological horizon at $y = \tilde{\xi}_+$, and a Rindler-like acceleration horizon also at $y = \tilde{\xi}_+$. Therefore the spacetime is not exactly the same as the pure de Sitter space. This is analogous to the difference between the Minkowskian space and the Rindler spacetime such that they are only locally the same. One can perform the coordinate transformations $\tau = \sqrt{1 + A^2/k^2}A^{-1}t$, $\rho = \sqrt{1 + A^2/k^2}(Ay)^{-1}$ and finds that only when $A = 0$ also, (7.2.1) recovers the pure de Sitter metric. The massless dS C-metric with $q = 0$ (and thus the corresponding 5-dimensional solution in (7.2.3)) is locally de Sitter due to the symmetry condition (7.2.8), and can be regarded as the background upon which the general $\Lambda > 0$ solutions arise by turning on q.[2]

For general $q \neq 0$, the 5-dimensional metric in (7.2.3)) with $G, \tilde{G}$ given in (7.2.4) and (7.2.5) has horizons of *an interval* $\times$ S^2 topology. It carries a cosmological constant $\hat{\Lambda} = 6k^2$ but not asymptotically de Sitter. We justify this claim in the

[2]The massless $\Lambda > 0$ C-metrics with general q can be transformed into "dS" Reissner-Nordström solutions by the same changes of coordinates mentioned in this paragraph [132].

following.

First, we specify the range of coordinates as $\tilde{\xi}_2 \leq x \leq \tilde{\xi}_3$, $\tilde{\xi}_3 \leq y < \infty$, and so that $\tilde{G}(x) \geq 0$. (See Figure (7.2).) There are conical singularities at $x = \tilde{\xi}_2$ and $x = \tilde{\xi}_3$ where $g_{\varphi\varphi} = 0$, unless the periods of φ at these positions are identified with

$$\Delta\varphi \Big|_{x=\tilde{\xi}_2} = \frac{4\pi}{|G'(\tilde{\xi}_2)|} , \qquad \Delta\varphi \Big|_{x=\tilde{\xi}_3} = \frac{4\pi}{|G'(\tilde{\xi}_3)|} , \tag{7.2.9}$$

where the to periods are automatically equal dual to the choice that G are even[3],

$$\Delta\varphi = \frac{2\pi}{q^2 A^2 |\tilde{\xi}_3(\tilde{\xi}_4^2 - \tilde{\xi}_3^2)|} . \tag{7.2.10}$$

Then (x, φ) parametrise a hypersurface of S^2 topology, with two poles at $x = \tilde{\xi}_3$ and $x = \tilde{\xi}_2 = -\tilde{\xi}_3$. The coordinate z which appears in the warp factor $\cos^2(kz)$ takes values in $-\pi/2k \leq z \leq \pi/2k$ with

$$\Delta z = \frac{\pi}{k} . \tag{7.2.11}$$

Therefore the constant (t, y) slice is of *an interval* $\times$ S^2 topology. We call this solution a *black fusiform*, since the S^2 size is largest at $z = 0$ and reduces to zero towards the singularities at $z = \pm\pi/2k$.

The asymptotic infinity is at $x \to y = \tilde{\xi}_3$. The "radial" coordinate is given by [132]

$$r = \frac{1}{A(y - x)} , \tag{7.2.12}$$

where $0 < r < \infty$ and $r \to \infty$ as $x \to y = \tilde{\xi}_3$. The curvature invariants of the 5-dimensional metric in terms of r are expressed by

$$\begin{aligned} \hat{R} &= 20k^2 , \\ \hat{R}_{MN}\hat{R}^{MN} &= 80k^4 + \frac{4q^4}{\cos^4(kz)r^8} , \\ \hat{R}_{MNPQ}\hat{R}^{MNPQ} &= 40k^4 + \frac{192q^4A^2x^2 + 96q^2a_3x + 12a_3^2/A^2}{\cos^4(kz)r^6} \\ &\quad + \frac{192q^4Ax + 48q^2a_3/A}{\cos^4(kz)r^7} + \frac{56q^4}{\cos^4(kz)r^8} . \end{aligned} \tag{7.2.13}$$

[3] In fact, by starting from $G(\xi)$ with 4 arbitrary distinct roots, the condition $\Delta\varphi\,(x = \tilde{\xi}_2) = \Delta\varphi\,(x = \tilde{\xi}_3)$ demands G must be even.

It is straight forward to conclude that the curvature invariants tend to those of pure dS_5 as $r \to \infty$, i.e. $\hat{R} \to 20k^2$, $\hat{R}_{MN}\hat{R}^{MN} \to 80k^4$, and $\hat{R}_{MNPQ}\hat{R}^{MNPQ} \to 40k^4$. Moreover,

$$\begin{aligned} \frac{\hat{R}_{zz}}{\hat{g}_{zz}} &= 4k^2 , \\ \frac{\hat{R}_{tt}}{\hat{g}_{tt}} &= \frac{\hat{R}_{yy}}{\hat{g}_{yy}} = 4k^2 + \frac{q^2}{r^4\cos^2(kz)} , \\ \frac{\hat{R}_{xx}}{\hat{g}_{xx}} &= \frac{\hat{R}_{\varphi\varphi}}{\hat{g}_{\varphi\varphi}} = 4k^2 - \frac{q^2}{r^4\cos^2(kz)} . \end{aligned} \tag{7.2.14}$$

As $r \to \infty$, they amount to the Einstein equation with a cosmological constant $\hat{\Lambda} = 6k^2$.

It is tempting to infer that the 5-dimensional spacetime is asymptotically de Sitter, but in fact it is not, since the spacetime is not maximally symmetric. This can be verified via the tensor

$$\hat{K}_{MNPQ} := \hat{R}_{MNPQ} - k^2(\hat{g}_{MP}\,\hat{g}_{NQ} - \hat{g}_{MQ}\,\hat{g}_{NP}) , \tag{7.2.15}$$

where the de Sitter metric should satisfy $\hat{K}_{MNPQ} = 0$ as in (7.2.8). It is found that not all components of $\hat{K}_{MNPQ}$ vanish, for example, $\hat{K}_{x\varphi x\varphi} = \cos(kz)^2(4A\,q^2\,x\,r + a_3\,r/A + q^2)$. Therefore our solution is *not* asymptotic de Sitter. Consequently, we have a black fusiform with a positive cosmological constant.

The curvature invariants in (7.2.13) imply that the metric is singular at $z = \pm\pi/2k$, where the warp factor vanishes and the 4-dimensional hypersurface shrinks to zero size. For the non-supersymmetric solution, the spacetime near the singularities at $z = \pm\pi/2k$ is highly curved and thus unstable. We will discuss the classical instability and the possible outcome of the instability in the next chapter.

The metric in (7.2.3) has the Killing vectors $\partial/\partial t$ and $\partial/\partial\varphi$. The coordinate y takes value in $\tilde{\xi}_3 \le y < \infty$, but the coordinate system breaks down at $y = \xi_3, \xi_4$, where $G(y) = 0$. In fact the spacetime is regular at these positions which are also the locations of the Killing horizons. This can be demonstrated by introducing the new coordinate v,

$$dv = dt + \frac{dy}{G(y)} . \tag{7.2.16}$$

such that the metric (7.2.3) is transformed into

$$d\hat{s}_5{}^2 = dz^2 + \frac{\cos^2(kz)}{A^2(x-y)^2}\left[G(y)\,dv^2 - 2dvdy + \frac{dx^2}{\tilde{G}(x)} + \tilde{G}(x)d\varphi^2\right] . \quad (7.2.17)$$

which is explicitly regular at $y = \xi_3, \xi_4$. Now y can be analytically continued beyond ξ_3 and ξ_4. The Killing vector

$$\eta = \frac{\partial}{\partial v} \quad (7.2.18)$$

becomes null at $y = y_h := \xi_3, \xi_4$ and therefore they are Killing horizons. The areas of the horizons are

$$\begin{aligned} A_{h\pm} &= \frac{2\pi^2}{kq^2A^4}\cdot\frac{1}{(y_h^2-\tilde{\xi}_3^2)(\tilde{\xi}_4^2-\tilde{\xi}_3^2)} \\ &= \frac{4\pi^2q^2/k}{\sqrt{1-4q^2A^2(1-k^2/A^2)}}\frac{1}{\sqrt{1-4q^2A^2(1-k^2/A^2)}\pm\sqrt{1-4q^2A^2}} , \end{aligned} \quad (7.2.19)$$

where (7.2.7) is used in the second line, and "$-$" and "$+$" are for quantities at $y = \xi_3$ and $y = \xi_4$ respectively. We conclude that $y = \xi_3$ is the outer (cosmological) horizon and $y = \xi_4$ is the inner (event) horizon, since

$$A_{h-}(y_h = \xi_3) > A_{h+}(y_h = \xi_4) > 0 , \quad (7.2.20)$$

i.e. the horizon at ξ_3 has larger r and larger area. Originally the y range is $\xi_3 < y < \xi_4$ as $G(y) < 0$, so that the metric has the correct signature. As y is analytically continued to $\tilde{\xi}_3 < y < \xi_3$, g_{tt} becomes spacelike and g_{yy} timelike. This is similar to the pure de Sitter space where the timelike Killing vector becomes spacelike outside the cosmological horizon. y can also be analytically continued to $y > \xi_4$, where the signature of the Killing vectors are again flipped, and reaches the curvature singularity at $y = \infty$.

The surface gravity are given by

$$\kappa_\pm = \frac{|G'(y_h)|}{2} = q^2A^2y_h(\xi_4^2-\xi_3^2) = \frac{\sqrt{1-4q^2A^2}\sqrt{1\pm\sqrt{1-4q^2A^2}}}{\sqrt{2}qA} , \quad (7.2.21)$$

which are finite despite the singularities at $z = \pm\pi/2k$ on the horizon. We have $\kappa_+ > \kappa_-$, where κ_+ and κ_- are associated with the inner / outer horizon. The temperature and the entropy associated with the event / cosmological horizons are defined to be proportional to the surface gravity and the horizon area,

$$T_\pm = \frac{\kappa_\pm}{2\pi} , \qquad S_\pm = \frac{A_{h\pm}}{4} \quad (7.2.22)$$

The temperatures are different on the two horizons, which implies that the black fusiform is not in thermal equilibrium. This reminds us the case of general charged non-extremal Kerr-de Sitter black holes [152].

For the class of known solutions with horizons which allows Euclidean continuations and requires the Euclidean time τ to be periodic with a period $\tau = \beta$, it is found that the temperature associated with their horizons can be given by $T = \beta^{-1}$ [140, 141]. The temperature for our solution can also be derived this way. By Wick rotating the metric (7.2.3) with $t = -i\tau$ and then removing the conical singularities of the (y, τ) part at y_h by identifying the period of τ with

$$\Delta\tau = \frac{4\pi}{|G'(y_h)|} = \frac{2\pi}{\kappa_\pm}\,, \tag{7.2.23}$$

the expressions for the temperature are reproduced. This is also why we choose the normalisation of the Killing vector in (7.2.18).

By (7.2.22) and (7.2.19), one finds that the entropy for the outer horizon is greater than that for the inner horizon, and the total entropy is the sum of both, as the case of the de Sitter Schwarzschild black hole, since both horizons are present in the real Euclidean geometry and contribute in the derivation of $S = \beta(\partial/\partial\beta - 1)\,I$, where I is the Euclidean action.

The specific heats for both horizons are given by

$$C_\pm := T\frac{\partial S_\pm}{\partial T} = \pm\frac{\pi^2}{k}\frac{q^2\sqrt{1-4q^2A^2}}{(1-4q^2A^2(1-k^2/A^2))^{3/2}}\frac{1}{\left(-1+\frac{1}{2}(1-4q^2A^2\pm\sqrt{1-4q^2A^2})\right)}\,. \tag{7.2.24}$$

C_- (for the cosmological horizon) is positive and C_+ (for the event horizon) is negative, which implies the cosmological/event horizon is thermally stable/unstable. This is similar to the case of a de Sitter Schwarzschild black hole.

The black fusiform metric is supported by a magnetic 2-form field $\hat{F}^1_{(2)} = d\hat{A}^1_{(1)}$ and two 3-form fields $\hat{F}^{\alpha=1,2}_{(3)} = d\hat{A}^{\alpha=1,2}_{(2)}$, with the field equations

$$d(\hat{*}\hat{F}^1_{(2)}) = 0\,, \qquad \hat{*}\hat{F}^\alpha_{(3)} = k\hat{A}^\alpha_{(2)} \tag{7.2.25}$$

simplified from (7.1.11). Similar to the case reviewed in Section 6.3, the conserved charge associated with the 2-form $\hat{F}^1_{(2)}$ vanishes. The field equation which takes a non-canonical form for $\hat{F}^{\alpha=1,2}_{(3)}$ doesn't lead to a non-trivial conserved charge either. A charge associated with $\hat{F}^1_{(2)}$ is given by

$$q_e = \frac{\sqrt{2}}{4\pi}\int_{S^2} \hat{F}^1_{(2)} \, , \tag{7.2.26}$$

where the factor $\sqrt{2}$ is inherited from the non-standard normalisation of $\hat{F}^i_{(2)}$ terms in the de Sitter supergravity Lagrangian (7.1.9). The integral is performed over an S^2 enclosing a constant (t, y, z) section on the horizons. With the only nonvanishing component $\hat{F}^1_{x\varphi} = q/\sqrt{2}$ in the solution (7.2.1), the charge q_e is

$$q_e = \frac{q}{\sqrt{1 - 4q^2A^2(1 - k^2/A^2)}} \, . \tag{7.2.27}$$

The dual 2-form potential $\hat{B}_{(2)}$ for $\hat{F}^1_{(2)}$ is given by

$$\sqrt{2}\,\hat{B}_{tz} = qy + c_1 \, , \tag{7.2.28}$$

where c_1 is a constant, and $\hat{B}_{(2)}$ is defined by

$$d\hat{B}_{(2)} = (\hat{*}\hat{F}^1)_{(3)} \, . \tag{7.2.29}$$

It was reviewed in Section 6.3 that for the asymptotically flat dipole black ring, an unusual non-conserved dipole charge term contributes in the first law of black ring mechanics. The explanation provided by Copsey and Horowitz [129] for that case also fits our solution: the dual 2-form potential $\hat{B}_{tz}$ in (7.2.28), which is defined up to gauge transformations, can not be set to zero at the asymptotic infinity and on the horizons simultaneously using a single coordinate patch, and it does not vanish on both horizons either, although the relation $\hat{B}_{tz} = \eta^M B_{MN}(\partial/\partial z)^N$ requires so. If the gauge potential vanishes at infinity, it necessarily diverges on the horizon. Therefore we expect a surface term contributed by q_e and the associated potential ϕ_e to appear in the first law, in a similar way reviewed in Section 6.3. For our later use in the expression of the 1st law, we define the potential here by evaluating $\hat{B}_{t\bar{z}}$ (where $\bar{z} = (\pi/\Delta z)z = kz$) on the horizons

$$\phi_e = \frac{\sqrt{2}\pi}{2}\hat{B}_{t\bar{z}}\Big|_{y=\xi_3,\,\xi_4} \, , \tag{7.2.30}$$

while $\hat{B}_{t\bar{z}}$ is set to zero at infinity, so that

$$\phi_e = \frac{\pi q}{2k}(\xi_{3,4} - \tilde{\xi}_3) \tag{7.2.31}$$

on the two horizons.

Next, we determine the mass of our solution. A well-known approach for the black hole mass is the quasilocal formalism[4] proposed by Brown and York [142]. This approach involves defining a surface stress-energy tensor

$$T^{\mu\nu} := \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{grav}}}{\delta \gamma_{\mu\nu}} , \tag{7.2.32}$$

on the "history of the boundary" ^3B (with the metric $\gamma_{\mu\nu}$) of a given region in the spacetime, where S_{grav} is regarded as a functional of $\gamma_{\mu\nu}$. $T^{\mu\nu}$ typically diverges as the boundary is taken at infinity. To obtain a finite stress-energy tensor, it is necessary to introduce a boundary term in the action to cancel the divergence without altering the bulk equation of motion. In Brown-York proposal, the subtraction is achieved by a counter term arising from embedding the boundary with the same intrinsic metric in some reference spacetime[5]. The energy (i.e. mass) contained in the specified region is given by integrating the energy surface density, obtained by normal projections of the finite $T^{\mu\nu}$ to a family of spacelike two-hypersurfaces B which foliates ^3B, over B. For the asymptotically flat spacetime, the quasilocal energy agrees with the ADM mass [144]. The quasilocal energy for the asymptotically anti-de Sitter spacetime [145] and de Sitter spacetime [146] are derived using different subtraction schemes, in which the counter terms are constructed in terms of the boundary metric and the Einstein tensor of the boundary geometry, with the coefficients fixed by the requirement of cancelling the divergence. The mass of the asymptotically flat black ring is derived under quasilocal formalism in [147].

Unfortunately, for our solution, the prescriptions mentioned above for constructing the counter terms are not successful in cancelling the divergence of $T^{\mu\nu}$ on the boundary at infinity where the asymptotic behaviour is non-de Sitter due to the presence of the nontrivial form fields in the supergravity action. This however doesn't imply that Brown-York approach is wrong, as obtaining a well-defined boundary stress tensor is still possible in the case where the asymptotically non-AdS behavior is caused by a nontrivial dilaton potential [148]. We don't reject the possibility to

[4]A general review on the quasilocal energy and angular momentum can be found in [143].

[5]In general, however, it is not always possible to find such embeddings.

obtain the quasilocal mass via a similar procedure to construct a finite $T^{\mu\nu}$ on the boundary, and thus to verify the 1st law and the Smarr-like relation as in the case of the asymptotically flat black ring. Here however since the satisfying quasilocal approach is not available for us to calculate the mass of our solution, instead of verifying the 1st law by the mass derived in this way, we simply *assume* the validity of the 1st law of thermodynamics and use it to determine the mass of our solution.

In the case of asymptotically de Sitter black hole, there are a cosmological and an event horizon, to each of which a 1st law of thermodynamics associates [149–152], and that for the cosmological horizon involves physics for the whole asymptotically dS spacetime including the black hole itself. Moreover, since the mass is determined from the asymptotic infinity outside the cosmological horizon, to which the relevant first law is related. It is natural to conjecture that given an appropriate definition of the energy, the 1st law for the outer horizon will always be valid, and vice versa. We will thus use it to derive the mass of our solution.

The 1st law associated with the cosmological horizon reads

$$dE = TdS + \phi_e dq_e \ . \tag{7.2.33}$$

dE is an exact differential so that the mass can be obtained by integration. Although there are three parameters in our solution, k, A should be regarded as specifying the background and q as specific to the solution after it is turned on. Therefore the mass is given by

$$\begin{aligned} M \ &:= \ E - E_0 = \int_0^q \left(T\frac{dS}{dq} + \phi_e \frac{dq_e}{dq} \right) dq \\ &= \ \int_0^q \left(T\frac{dS}{dq} + \frac{\pi}{2k}\, q\, \xi_3 \frac{dq_e}{dq} - \frac{\pi}{2k}\, q\, \tilde{\xi}_3 \frac{dq_e}{dq} \right) dq, \end{aligned} \tag{7.2.34}$$

where E_0 stands for the energy of the background as $q = 0$. By substituting in the expressions for T, S, q_e and ξ_3, one finds that the first and second term in the second line cancel each other,

$$T\frac{dS}{dq} = -\frac{\pi}{2\sqrt{2}kA} \frac{\sqrt{1 - \sqrt{1 - 4q^2A^2}}}{(1 - 4q^2A^2(1 - k^2/A^2))^{3/2}} = -\frac{\pi}{2k}\, q\, \xi_3 \frac{dq_e}{dq} \ , \tag{7.2.35}$$

in which $T\frac{dS}{dq}$ is manifestly negative. The mass is obtained by integrating the last

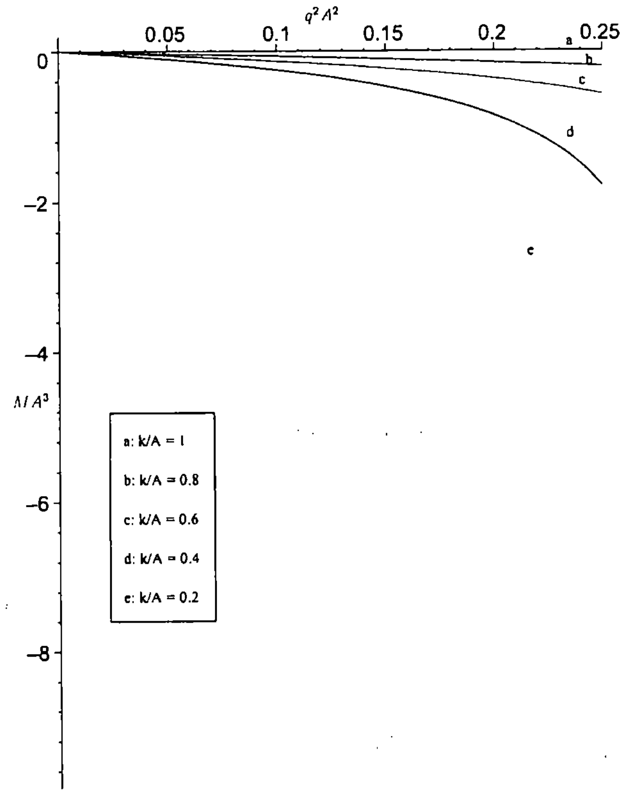

Figure 7.2: M against q in (7.2.36) for various values of k/A. As $k/A = 1$, $M = 0$. For $0 < k/A < 1$, M is a monotonically decreasing function of q. M is undefined at $k/A = 0$ unless $q = 0$.

term in (7.2.34),

$$\begin{aligned} M &= -\frac{\pi}{2k}\int_0^q q\,\tilde{\xi}_3\frac{dq_e}{dq}dq \qquad (7.2.36) \\ &= -\frac{\pi}{4\sqrt{2}\,kA^2\sqrt{1-k^2/A^2}}\left(\frac{\sqrt{1+Y}}{Y}-\sqrt{2}-\frac{1}{2}\ln\frac{(\sqrt{1+Y}+1)(\sqrt{2}-1)}{(\sqrt{1+Y}-1)(\sqrt{2}+1)}\right), \end{aligned}$$

where

$$Y := \sqrt{1-4q^2A^2(1-k^2/A^2)} = \left|\frac{q}{q_e}\right| \leq 1. \qquad (7.2.37)$$

(7.2.36) is the mass gap between the background spacetime ($q = 0$) and the solution with certain q such that $M \leq 0$. It vanishes as $q = 0$ and the minimum occurs at the maximal value of $q^2 = 1/4A^2$. M is a decreasing function against q, as displayed in Figure 7.2. The fact of $M \leq 0$ is reminiscent of that the asymptotically de Sitter Schwarzschild black hole is less massive than the pure de Sitter spacetime.

The implication of this result will be discussed in the next chapter.

Chapter 8

Discussion on the Black Fusiform with a Positive Cosmological Constant

In the previous chapter we have constructed a 5-dimensional static black fusiform in de Sitter supergravity which preserves half of supersymmetry. It is with a positive cosmological constant while asymptotically non-de Sitter. The horizons are singular at $z = \pm\pi/2k$, where the warp factor vanishes and the S^2 part reduces to zero size. The black fusiform solution carries no angular momentum; the gravitational contraction is balanced by the cosmological repulsion due to the positive cosmological constant. The thermodynamic quantities associated with the horizons, such as temperature, entropy and specific heat, are well-defined and finite. The solution is supported by a non-trivial 2-form and two 3-form fields, and the former gives rise to a charge that contributes to the first law of black hole mechanics.

In this chapter, we will discuss some interesting aspects of the $\Lambda > 0$ black fusiform. We first compare the physics arising from our singular, asymptotically non-de Sitter solution with that in asymptotically de Sitter spacetime. We'll see that in spite of the singularity on the horizon, the black fusiform displays properties parallel to the de Sitter case, supporting the entropic N-bound proposal [153] and the maximal mass conjecture [146]. The spacetime near the singularities is highly curved and unstable. We'll discuss the possible outcome of the instability. Finally, we briefly summarise recent progress in constructing the (anti-)de Sitter black rings and point out the future direction.

8.1 Comparison with de Sitter gravity

The entropies associated with the horizons of the black fusiform solution are derived in (7.2.19) and (7.2.22). Focusing on the cosmological horizon, for which a first law is assumed in the previous chapter, the entropy can be expressed by the following expansion as q is small,

$$S_- = \frac{\pi^2}{2k^3} - \frac{\pi^2}{2k}q^2 - \frac{\pi^2}{2k}(4A^2 - 3k^2)q^4 + O(q^6). \tag{8.1.1}$$

The entropy decreases monotonically as q increases, so that

$$S_- \leq \frac{\pi^2}{2k^3} := S_{\text{de Sitter}}, \tag{8.1.2}$$

where $S_{\text{de Sitter}}$ is the entropy of the 5-dimensional locally pure de Sitter space of the same cosmological constant, which can be obtained from (7.2.19).

In [153], Bousso proposed the *N-bound* for the general spacetime with a positive cosmological constant, for which the total entropy (including the matter entropy) is claimed to be bounded by that of the pure de Sitter space with the same cosmological constant. This generalises an earlier proposal by Banks [154] for the asymptotically de Sitter universe. For our asymptotically non-de Sitter singular solution, the total entropy S_T is the sum of those related to the inner horizon and the cosmological horizon, and is equal to the de Sitter entropy obtained above,

$$S_T = S_- + S_+ = S_{\text{de Sitter}}, \tag{8.1.3}$$

such that the N-bound is precisely saturated. This result shows that, even though our black fusiform carries unusual features by construction, e.g. having the singular horizons and containing the fields with the wrong-sign kinetic terms, the N-bound is still respected. Since an entropy bounded from above means that the underlying gravity theory has a finite number of degrees of freedom, we regard this as an implication of the existence of a quantum gravity description for the black fusiform.

Besides the entropy, according to (7.2.36), the mass of the black fusiform reduces monotonically against q, and the maximum appears at $q = 0$. It was observed that the masses of the Schwarzschild-de Sitter [146] and the Kerr-de Sitter [152] black holes are less than that of the pure de Sitter spacetime. This leads to the *maximal mass conjecture* proposed in [146], which states that *any asymptotically de Sitter spacetime which is more massive than de Sitter must contain a cosmological singularity*. As our asymptotically non-de Sitter object has a lower mass compared to the de Sitter spacetime, it may imply a generalised form of the maximal mass conjecture: *In any spacetime with a positive cosmological constant, if the presence of matter causes increase in energy, such a spacetime must contain a cosmological singularity* [1].

In [153], the entropic N-bound is proved for the spherical symmetric causal space associated with an observer in the spacetime with a positive cosmological constant. Proof of the maximal mass conjecture is still not available. There are known examples violating the former [155,156] and the latter [156]. These might due to violation of certain asymptotic energy conditions. More specific energy conditions are needed in order to obtain the complete descriptions and general proofs of both conjectures. Our solution offers an explicit example which supports the N-bound proposal and the maximal mass conjecture, and may help to clarify the appropriate conditions for them.

In de Sitter supergravity, the action contains the wrong-sign kinetic terms, and there exists ghost modes which carries negative energy. The solution may be thus unstable. However, it is argued in [137] that the ghost modes might be merely artifacts arising from the truncation to de Sitter supergravity; once in the full string theory where all the string modes and string corrections are considered, one can expect that the black fusiform becomes a solution to the full string theory (as it is supersymmetric by construction) and free from ghost modes.

8.2 Instability associated with the singularities at $z = \pm\frac{\pi}{2k}$

The black fusiform with a positive cosmological constant in (7.2.3) is singular at $z = \pm\pi/2k$, where the warp factor $\cos^2(kz)$ vanishes and the S^2 factor on the horizon becomes zero size. Despite the spacetime is highly curved near the singularities, we expect the solution to be classically stable since it is protected by the preserved $\mathcal{N} = 2$ supersymmetry [159, 160]. However, we may consider the finite temperature configuration of the solution as in (7.2.23) where all supersymmetry is broken. The non-supersymmetric solution has a classical Gregory-Laflamme instability [158].

The singular horizon appears to be a generic feature of the braneworld black holes. An example of a singularity on the horizon is given in [157], in which the 5-dimensional black string obtained from embedding a Schwarzschild black hole on the brane in the Randall-Sundrum model is singular on the AdS horizon. The solution suffers from the Gregory-Laflamme instability near the singularity, but is stable far away from the AdS horizon. The authors of [157] suggest that the horizon pinches off due to the instability, and the solution evolves into a "black cigar" shape and becomes non-singular.

A neutral uniform black string in $(d+1)$-dimensions has an event horizon of $S^{d-2} \times \mathbf{R}$ topology. Under compactification along the length direction, the horizon becomes $S^{d-2} \times S^1$. For $d \geq 5$, the black string in pure Einstein gravity with a mass below a critical value (called the Gregory-Laflamme mass) is found to be unstable under classical long-wavelength perturbations, and it is proposed that the event horizon eventually pinches off and a series of localised black holes form as an endpoint of the instability. This conjecture arises from the fact that, with the same mass, the entropy of the localised black holes is higher than that of the black string. The evolution involves a topology-changing process, and might violate the Cosmic Censorship Hypothesis as a naked singularity may form during the pinch-off of the black string horizon.

So far the end state of the Gregory-Laflamme instability is still an open question and yet to be concluded. Recent progress in this topic is summarised in the review articles [161] [162] and the references within. It is discovered there exists a time-independent threshold mode of the perturbation which gives rise to a black string non-uniformly distributed along the length direction [163–165]. The uniform black string, non-uniform black string, and the localised black holes together form three branches on the phase diagram characterised by the rescaled dimensionless mass μ and the relative tension n which is defined as the ratio of the total binding energy to the (dimensionful) mass. The pinch-off of the event horizon (if it happens) is in fact the phase transition of the black string to the localised black holes. The Gregory-Laflamme instability also applies to the non-extremal charged black p-branes in supergravity, where the phases of the black branes are obtained from those of the neutral black strings by a combination of the boost and U-duality transformations. The black brane is unstable when the charge is much smaller than the mass.

Another scenario for the instability is proposed by Horowitz and Maeda [166]. They argue that the classical evolution of the black strings wouldn't result in the localised black holes, as the pinch-off cannot happen in finite affine parameter on the horizon. Instead, the solution stabilises in some non-translationally invariant black string, for example the non-uniform black string, or a chain of black holes connected by small necks. The numerical data obtained later confirm the possibility that the classical unstable horizon can pinch off at infinite affine parameter [167–169]. [170] also argues that the infinite affine parameter on the horizon may correspond to a finite advanced time along the past null infinity, and thus the neck size may shrink to zero at finite advanced time for an asymptotic observer.

As discussed in the preceding section that the black fusiform displays properties which may be valid in general for a spacetime with a $\Lambda > 0$, it is possible that an underlying framework exists for the singularities to be resolved so that the solution becomes regular and makes good physical sense. The Gregory-Laflamme instability may lead to the pinch-off of the horizon at the two tips, giving rise to a "black cigar" stretching between two opposite sides of the cosmological horizon. On the other hand, if the Horowitz-Maeda scenario happens, the horizons do not pinch off, while instead a new regular horizon forms a neck surrounding each tip. In particular, if our solution can be rotated up in φ, as the angular momentum becomes large (while below the critical value where the S^2 is destroyed), the Horowitz-Maeda scenario might be more favorable.

8.3 Toward the (anti-)de Sitter black ring solution

It is natural to expect that if a black ring in the background with a negative cosmological constant exists, it should carry certain angular momentum in order to counter-balance the contraction due to gravity and the negative cosmological constant. So far the asymptotically AdS/dS black rings are still not yet found. In [171], research is carried out on classification of the supersymmetric black solutions admitting two rotational symmetries in 5-dimensional gauged supergravity, by classifying their near horizon geometries [172,173]. It turns out that the regular, asymptotically AdS, supersymmetric black ring in gauged supergravity doesn't exist, as the only allowed near horizon geometries are those of the topologically spherical black hole and the "unbalanced" black ring with a conical singularity on the S^2 part[1]. This implies that if the regular supersymmetric black ring does exist, it might have only one rotational symmetry. A simple physical interpretation is provided by [174]: coupling of the black ring to AdS gravity requires nonzero pressure on the ring along S^1 in order to counteract the AdS and self-gravitational contraction, but the pressure is cancelled due to supersymmetry. As a result, external force (which is represented by the conical singularity) is necessary to hold the black ring together. The authors of [171] also regard their singular solution as an evidence supporting the existence of regular non-supersymmetric AdS black rings, since the conical singularity might be resolved by increasing the angular momentum, as in the asymptotically flat case.

In [174], an approximate construction of the asymptotically AdS/dS thin black rings with one angular momentum is carried out in $d + 1 \geq 5$ dimensions. The

[1]The conical singularity can only be removed when the cosmological constant vanishes, and the near horizon geometry becomes $AdS_3 \times S^2$ of the asymptotically flat supersymmetric black ring in [112].

basic idea is as follows. As a rotating thin black ring can be obtained by bending a boosted string into a circular shape, the solution appears as a circular distribution of energy-momentum tensor for an observer far away from the black ring horizon, while it can be approximated near the horizon by a perturbed black string bent into a curve with a large radius of curvature. For the 5-dimensional thin AdS black ring, it is found that the S^1 radius of is determined by the boost parameter in order for the centrifugal force to be in equilibrium with the gravitation, and the angular momentum is bounded from above by the *BPS bound* for the regular AdS solution [175], $J \leq ML_{AdS}$, where J, M, L_{AdS} are the angular momentum, mass, and the AdS cosmological radius. For de Sitter thin black ring in 5 dimensions, the equilibrium condition becomes

$$\sinh^2 \alpha = 1 - 3\left(\frac{R}{L_{dS}}\right)^2 , \tag{8.3.1}$$

where α is the boost parameter and R, L_{dS} are the S^1 radius of the ring and the de Sitter scale respectively. The angular momentum is also bounded from above. The condition (8.3.1) implies that there exists a maximal radius R_{st} such that the black ring becomes static,

$$0 \leq R \leq R_{st} := \frac{L}{\sqrt{3}} , \tag{8.3.2}$$

To summarise, since the discovery of the asymptotically flat black ring in 2001, efforts had been made to construct the black rings in (anti-)de Sitter backgrounds. We only manage to construct a solution with topology of S^2 times an interval which may be interpreted as a pinched black ring. As the discovery of asymptotically flat black ring help us to understand that the uniqueness theorem derived from the 4-dimensional black holes cannot be naively extended to higher dimensions while the black hole mechanics is still valid, we expect that the explicit formulation of the AdS/dS black rings will lead us to better understanding in (anti-)de Sitter physics, and there will also be interesting applications of these solutions to the gauge theory/gravity correspondence.

Appendix A

't Hooft Symbols

This appendix is a summary of the 't Hooft symbols, in order for defining the 6-dimensional gamma matrices used in Chapter 4, 5 and [2]. The 6d gamma matrices will be given in Appendix B.

The 4-dimensional Minkowskian spacetime is $SO(3,1)$ invariant, locally equivalent to $SU(2) \times SU(2)$ [176]. As for the Euclidean case, the symmetry becomes $SO(4)$. Given the Euclidean Lorentz generator $L_{\mu\nu}$ where $\mu, \nu = 1...4$, $L_i = \frac{1}{2}\epsilon_{ijk}L_{jk}$ and $K_i = L_{4i}$ are the rotation generator and the Euclidean "boost" generator respectively, with $i, j = 1, 2, 3$. With the following recombination

$$M_i = \frac{1}{2}(J_i - K_i), \qquad N_i = \frac{1}{2}(J_i + K_i), \tag{A.0.1}$$

M_i and N_i each forms an individual $SU(2)$ algebra. Unlike the Minkowskian case, these two $su(2)$'s are not related to each other by complex conjugation.

The 't Hooft symbols $\eta^i_{\mu\nu}$ and $\bar{\eta}^i_{\mu\nu}$ were first introduced as covariant mappings of $SO(3)$ vectors on (anti-)selfdual $SO(4)$ tensors [177]. They can also be regarded as mappings between $SO(4)$ and $SU(2)$ algebras [178]:

$$M_i = \frac{1}{4}\eta_{i\mu\nu}L_{\mu\nu} \ , \qquad N_i = \frac{1}{4}\bar{\eta}_{i\mu\nu}L_{\mu\nu} \ . \tag{A.0.2}$$

In order to write down the expression for the 't Hooft symbols, we can formulate the Lorentz generators in terms of differential operators, $L_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)$.

Then the eta symbols are conveniently given by

$$\begin{aligned} \eta_{i\mu\nu} &= \bar{\eta}_{i\mu\nu} = \epsilon_{i\mu\nu} & (\mu,\nu = 1,2,3) \\ \eta_{i4\nu} &= -\bar{\eta}_{i4\nu} = -\delta_{i\nu} & (\mu = 4) \\ \eta_{i\mu 4} &= -\bar{\eta}_{i\mu 4} = \delta_{i\mu} & (\nu = 4) \\ \eta_{i44} &= \bar{\eta}_{i44} = 0. \end{aligned} \tag{A.0.3}$$

In brief, η^i and $\bar{\eta}^i$ are related by $\bar{\eta}^i_{\mu\nu} = (-1)^{\delta_{4\mu}+\delta_{4\nu}}\eta^i_{\mu\nu}$.

The expressions of $\eta_{i\mu\nu}$ and $\bar{\eta}_{i\mu\nu}$ should be independent of representations, as it is a mapping between algebras. According to (A.0.3), the 't Hooft symbols are antisymmetric in μ,ν indices, so η^i and $\bar{\eta}^i$ form a basis for general 4×4 antisymmetric real matrices. Moreover, $\eta^i_{\mu\nu}$ is selfdual and $\bar{\eta}^i_{\mu\nu}$ anti-selfdual:

$$\eta^i_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\eta^i_{\rho\lambda}, \qquad \bar{\eta}^i_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\bar{\eta}^i_{\rho\lambda}. \tag{A.0.4}$$

The 't Hooft symbols obey the following identities, which will be useful for our later discussions:

$$\epsilon^{ijk}\eta^j_{\mu\nu}\eta^k_{\rho\lambda} = \delta_{\mu\rho}\eta^i_{\nu\lambda} + \delta_{\nu\lambda}\eta^i_{\mu\rho} - \delta_{\mu\lambda}\eta^i_{\nu\rho} - \delta_{\nu\rho}\eta^i_{\mu\lambda}, \quad \text{(same for } \bar{\eta}\text{'s)} \tag{A.0.5}$$

$$\epsilon^{ijk}\,\eta^i_{\rho\mu}\eta^j_{\mu\nu}\eta^k_{\nu\lambda} = 3!\,\delta_{\rho\lambda} = \epsilon^{ijk}\,\bar{\eta}^i_{\rho\mu}\bar{\eta}^j_{\mu\nu}\bar{\eta}^k_{\nu\lambda}, \tag{A.0.6}$$

$$\eta^i_{\mu\nu}\eta^i_{\rho\lambda} = \delta_{\mu\rho}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\rho} + \epsilon_{\mu\nu\rho\lambda}, \tag{A.0.7}$$

$$\bar{\eta}^i_{\mu\nu}\bar{\eta}^i_{\rho\lambda} = \delta_{\mu\rho}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\rho} - \epsilon_{\mu\nu\rho\lambda}, \tag{A.0.8}$$

$$\eta^i_{\mu\rho}\eta^j_{\mu\lambda} = \delta^{ij}\delta_{\rho\lambda} + \epsilon^{ijk}\eta^k_{\rho\lambda}, \quad \text{(same for } \bar{\eta}\text{'s)} \tag{A.0.9}$$

where (A.0.9) implies

$$\eta^i_{\mu\rho}\eta^i_{\mu\lambda} = 3\delta_{\rho\lambda}, \quad \eta^i_{\mu\nu}\eta^j_{\mu\nu} = 4\delta^{ij}, \quad \eta^i_{\mu\nu}\eta^i_{\mu\nu} = 12. \tag{A.0.10}$$

The (anti-)selfduality conditions (A.0.4) for η^i and $\bar{\eta}^i$ also give rise to the identities

$$\eta^i_{\mu\nu}\bar{\eta}^j_{\mu\lambda} = \eta^i_{\mu\lambda}\bar{\eta}^j_{\mu\nu}, \tag{A.0.11}$$

$$\eta^i_{\mu\nu}\bar{\eta}^j_{\mu\nu} = 0. \tag{A.0.12}$$

We can rewrite the formulas which define the 't Hooft symbols, by applying (A.0.7) and (A.0.8) to (A.0.2),

$$\begin{aligned} \eta^i_{\mu\nu}M^i &= \frac{1}{4}\left(2L_{\mu\nu} + \epsilon_{\mu\nu\rho\lambda}L_{\rho\lambda}\right), \\ \bar{\eta}^i_{\mu\nu}N^i &= \frac{1}{4}\left(2L_{\mu\nu} - \epsilon_{\mu\nu\rho\lambda}L_{\rho\lambda}\right). \end{aligned}$$

Then, if the (anti)-selfdual conditions are imposed on the Lorentz generators, we obtain the alternative expressions for (A.0.2):

$$L_{\mu\nu} = \eta^i_{\mu\nu} M^i \ , \qquad N^i = 0 \qquad \text{for selfdual } L_{\mu\nu}; \tag{A.0.13}$$

$$L_{\mu\nu} = \bar{\eta}^i_{\mu\nu} N^i \ , \qquad M^i = 0 \qquad \text{for anti-selfdual } L_{\mu\nu}. \tag{A.0.14}$$

Appendix B

Gamma matrices

Consider N Euclidean D3-branes in 10-dimsional IIB background. The spacetime symmetry group $SO(10)$ is decomposed into $SO(4) \times SO(6)$, and the 10-dimensional gamma matrices are also decomposed into the 4-dimsional and the 6-dimensional parts:

$$\Gamma^{M}_{(10d)} = \{\gamma^\mu \otimes \mathbf{1},\ \gamma^5 \otimes \Gamma^a\} \ , \tag{B.0.1}$$

where the indices $M = \{0, ..., 9\}$ is for the overall 10-dimensions, $\mu = \{0, ..., 3\}$ indicates the worldvolume directions and $a = \{4, ..., 9\}$ indicates the transverse dimensions. γ^μ and Γ^a are defined by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \Gamma^a = \begin{pmatrix} 0 & \Sigma^a \\ \bar{\Sigma}^a & 0 \end{pmatrix} \tag{B.0.2}$$

and satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$ and $\{\Gamma^a, \Gamma^b\} = 2\delta^{ab}$ respectively.

B.1 4-dimensional part

We follow the convention of [81, 82] in defining the basis of the 4-demsional part gamma matrices in Section 3.3 and Chapters 4 and 5:

$$\begin{aligned} \sigma^\mu &= (i\tau^1, i\tau^2, i\tau^3, \mathbf{1}) \ , \\ \bar{\sigma}^\mu &= (-i\tau^1, -i\tau^2, -i\tau^3, \mathbf{1}) \ , \end{aligned} \tag{B.1.1}$$

where τ_1, τ_2, τ_3 are Pauli matrices,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{B.1.2}$$

The SU(2) Lorentz generators are defined by

$$\sigma_{\mu\nu} = \frac{1}{4}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu) \qquad \bar{\sigma}_{\mu\nu} = \frac{1}{4}(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu) \tag{B.1.3}$$

and satisfy the Lorentz algebra

$$[\sigma_{\mu\nu}, \sigma_{\rho\lambda}] = -(\delta_{\mu\rho}\sigma_{\nu\lambda} + \delta_{\nu\lambda}\sigma_{\mu\rho} - \delta_{\mu\lambda}\sigma_{\nu\rho} - \delta_{\nu\rho}\sigma_{\mu\lambda}). \tag{B.1.4}$$

σ_μ and $\bar{\sigma}_\mu$ themselves obey the Clifford algebra,

$$\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = \bar{\sigma}_\mu \sigma_\nu + \bar{\sigma}_\nu \sigma_\mu = 2\,\delta_{\mu\nu}. \tag{B.1.5}$$

Next we show the (anti-)selfduality of $\sigma_{\mu\nu}(\bar{\sigma}_{\mu\nu})$. By using the identity

$$\gamma^{\mu\nu} := \gamma^{[\mu}\gamma^{\nu]} = \frac{1}{2}\,\epsilon^{\mu\nu\rho\lambda}\gamma^5\gamma^{\rho\lambda}, \tag{B.1.6}$$

where the γ^5 matrix is defined by

$$\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{1}{4!}\,\epsilon^{\mu\nu\rho\lambda}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & -1_{2\times 2} \end{pmatrix} . \tag{B.1.7}$$

It follows directly from (B.1.6) that $\sigma_{\mu\nu}$ is selfdual and $\bar{\sigma}_{\mu\nu}$ anti-selfdual,

$$\sigma_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\sigma_{\rho\lambda} , \qquad \bar{\sigma}_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\bar{\sigma}_{\rho\lambda} . \tag{B.1.8}$$

According to (A.0.13) and (A.0.14), we can express the $SU(2)$ Lorentz generators using the 't Hooft symbols and the Pauli matrices τ^i,

$$\sigma_{\mu\nu} = \frac{i}{2}\eta^i_{\mu\nu}\tau^i, \qquad \bar{\sigma}_{\mu\nu} = \frac{i}{2}\bar{\eta}^i_{\mu\nu}\tau^i. \tag{B.1.9}$$

B.2 6-dimensional part

As Σ^a and $\bar{\Sigma}^a$ are antisymmetric with respect to the SU(4) indices $A, B = 1...4$, it is convenient to define the 6-dementional part of the gamma matrices in terms of the 't Hooft symbols [82]:

$$\begin{aligned} \Sigma^a &= (\ \eta^3, -i\bar{\eta}^3,\ \ \eta^2, -i\bar{\eta}^2,\ \ \eta^1, i\bar{\eta}^1), \\ \bar{\Sigma}^a &= (-\eta^3, -i\bar{\eta}^3, -\eta^2, -i\bar{\eta}^2, -\eta^1, i\bar{\eta}^1), \end{aligned} \tag{B.2.1}$$

where η's and $\bar{\eta}$'s are realized as 4×4 matrices:

$$\eta^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{\eta}^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\eta^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \bar{\eta}^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{B.2.2}$$

$$\eta^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \bar{\eta}^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The identities (A.0.9) and (A.0.11) give rise to the algebras for the eta matrices,

$$\{\eta^i, \eta^j\} = \{\bar{\eta}^i, \bar{\eta}^j\} = -2\,\delta^{ij}\,\mathbf{1}\,, \tag{B.2.3}$$

$$[\eta^i, \bar{\eta}^j] = 0\,, \tag{B.2.4}$$

which allow Σ and $\bar{\Sigma}$ matrices to satisfy the Clifford algebra

$$(\Sigma^a)^{AB}(\bar{\Sigma}^b)_{BC} + (\Sigma^b)^{AB}(\bar{\Sigma}^a)_{BC} = 2\,\delta^{ab}\,\delta^A_C\,. \tag{B.2.5}$$

To show that $\Sigma^{abc} := \Sigma^{[a}\bar{\Sigma}^b\Sigma^{c]}$ is selfdual and $\bar{\Sigma}^{abc} := \bar{\Sigma}^{[a}\Sigma^b\bar{\Sigma}^{c]}$ anti-selfdual. First define

$$\Gamma^{(7)} := i\,\Gamma^4\Gamma^5\Gamma^6\Gamma^7\Gamma^8\Gamma^9 = \frac{i}{6!}\,\epsilon^{abcdef}\,\Gamma^a\Gamma^b\Gamma^c\Gamma^d\Gamma^e\Gamma^f = \begin{pmatrix} \mathbf{1}_{4\times4} & 0 \\ 0 & -\mathbf{1}_{4\times4} \end{pmatrix}, \tag{B.2.6}$$

where $\epsilon^{456789} = 1$. The identities (A.0.6) and (B.2.4) are used in (B.2.6) for $\Sigma^4\bar{\Sigma}^5\Sigma^6\bar{\Sigma}^7\Sigma^8\bar{\Sigma}^9 = -i\,\mathbf{1}$ and $\bar{\Sigma}^4\Sigma^5\bar{\Sigma}^6\Sigma^7\bar{\Sigma}^8\Sigma^9 = i\,\mathbf{1}$. The above equation gives rise to the identity

$$\Gamma^{abc} := \Gamma^{[a}\Gamma^b\Gamma^{c]} = \frac{i}{3!}\,\epsilon^{abcdef}\,\Gamma^{(7)}\,\Gamma^{def}, \tag{B.2.7}$$

which immediately leads to

$$\Sigma^{abc} = \frac{i}{3!}\epsilon^{abcdef}\,\Sigma^{def}\,, \qquad \bar{\Sigma}^{abc} = -\frac{i}{3!}\epsilon^{abcdef}\,\bar{\Sigma}^{def}\,. \tag{B.2.8}$$

Appendix C

A Particular Supergravity Configuration for $\mathcal{N} = (1, 0)$

In this appendix we present the supergravity configuration which gives rise to an explicit rank 2 $M^{AB} = M_{(1,0)}$ in (4.1.4) for the background graviphoton field $\mathcal{F}^{\alpha\beta AB}$ inducing $\mathcal{N} = (1, 0)$ non-anticommutativity.

The doubly self-dual RR 5-forms are given by:

$$\begin{aligned} F_{01674} &= F_{23674} = -iF_{23895} = -iF_{01895} = 2c \ , \\ F_{01894} &= F_{23894} = -iF_{23675} = -iF_{01675} = 2c \ . \end{aligned} \tag{C.0.1}$$

To obtain the expression for $\mathcal{F}^{\alpha\beta AB}$, here we choose the basis for the 4-dimensional gamma matrices σ^{μ} as in (3.3.3), and the 6-dimensional ones Σ^{a} as in (3.3.4),

$$\Sigma^{4,5,6,7,8,9} = \Sigma^{4,5,6,7,8,9}_{(3.3.4)} \ . \tag{C.0.2}$$

Then

$$\begin{aligned} \mathcal{F}^{\alpha\beta AB} = F_{\mu\nu abc}(\sigma^{\mu\nu})^{\alpha\beta}(\Sigma^{abc})^{AB} &= 48\, c\, (\sigma^{01})^{\alpha\beta}(\Sigma^{674} + i\Sigma^{675})^{AB} \\ &= 48\, c\, i(\tau^{3})^{\alpha\beta} M^{AB} \ , \end{aligned} \tag{C.0.3}$$

where

$$M^{AB} = \Sigma^{674} + i\Sigma^{675} = 2i \begin{pmatrix} \tau_1 & 0 \\ 0 & 0 \end{pmatrix} := M_{(1,0)} \ . \tag{C.0.4}$$

Only a 2×2 sub-block of M^{AB} in $\mathcal{F}$ is turned on to deform the $\mathcal{N} = (1, 1)$ superspace.

The intersecting brane configuration to generate the RR 5-form fields in (C.0.1) is given by

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
N D3	•	•	•	•	×	×	×	×	×	×
$D3_1$	•	•			×	×	•	•		
$D3_2$	•	•			×	×			•	•
$D3_3$			•	•	×	×	•	•		
$D3_4$			•	•	×	×			•	•

(C.0.5)

where "•" indicates the worldvolume dimensions and "×" indicates the transverse dimensions in which each brane is localized. The harmonic functions $H_{extra} :=$ $H_{1,2,3,4}$ are functions of (x_4, x_5), and thus precisely the field components in (C.0.1) are turned on. This is just a simple relabelling of the coordinates used in Chapter 4.

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