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The Dimensions of the Smooth Representations of $GL_n(\mathfrak{o})$

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Jackson

A Thesis presented for the degree of
Doctor of Philosophy



Department of Mathematical Sciences
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United Kingdom
September 2025

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Let \mathfrak{o} be the valuation ring of a non-Archimedean local field with finite residue field and quotients $\mathfrak{o}_r = \mathfrak{o}/\mathfrak{p}^r$. We consider the problem of constructing the irreducible complex representations of $GL_n(\mathfrak{o}_r)$. This is known to be a wild problem, however we prove certain new results about the dimensions of the representations of $GL_n(\mathfrak{o}_r)$.

Our first approach involves studying automorphism groups of finite \mathfrak{o} -modules, whose representations can be used to build those of $GL_n(\mathfrak{o}_2)$. We give a complete description of the dimensions of the representations of $\text{Aut}_{\mathfrak{o}}(\mathfrak{o}_\ell \oplus \mathfrak{o}_1 \oplus \cdots \oplus \mathfrak{o}_1)$ and show that they can be written as polynomials (independent of the choice of \mathfrak{o}) in $q = |\mathfrak{o}_1|$.

We also give a polynomial result for $GL_n(\mathfrak{o}_r)$, namely that subject to the characteristic of \mathfrak{o}_1 being large enough (possibly depending on n and r), there exists

a finite set of polynomials $R \subseteq \mathbb{Q}[x]$, such that when evaluated at $q = |\mathfrak{o}_1|$, all of the dimensions of the irreducible representations of $\mathrm{GL}_n(\mathfrak{o}_r)$ appear in the set $\{m(|\mathfrak{o}_1|) \mid m \in R\}$. Our proof also holds when GL_n is replaced by an arbitrary smooth affine group scheme over \mathbb{Z} .

Finally, we show that if \mathfrak{o} and \mathfrak{o}' are valuation rings as before with $|\mathfrak{o}_1| = |\mathfrak{o}'_1|$, then $\mathbb{C}[\mathrm{GL}_n(\mathfrak{o}_r)] \cong \mathbb{C}[\mathrm{GL}_n(\mathfrak{o}'_r)]$, provided the characteristic of \mathfrak{o}_1 is large enough. This generalises a result of Hadas, who proved the statement when \mathfrak{o} and \mathfrak{o}' are restricted to be unramified extensions of \mathbb{Q}_p if they have characteristic zero.

Declaration

The work in this thesis is based on research carried out at the Department of Mathematical Sciences, Durham University, United Kingdom. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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List of Symbols

- \mathbb{F}_q The finite field with q elements, p. 1
- \mathbb{Q}_l The field of l -adic numbers, p. 1
- R^\times The group of units of the ring R , p. 7
- p, p_K The residue characteristic of a valuation ring \mathfrak{o}_K , p. 1
- q, q_K The residue cardinality of a valuation ring \mathfrak{o}_K , p. 1
- \overline{K} The algebraic closure of a field K , p. 1
- $\mathfrak{o}, \mathfrak{o}_K$ The valuation ring of a non-Archimedean local field K , p. 1
- \mathfrak{o}_r The quotient $\mathfrak{o}/\mathfrak{p}^r$, p. 1
- $\mathfrak{p}, \mathfrak{p}_r$ The non-zero maximal ideal of \mathfrak{o} (resp. \mathfrak{o}_r), p. 1
- ϖ, ϖ_K A uniformiser of a non-Archimedean local field K , p. 1
- $K^{(d)}$ The degree d unramified extension of K , p. 5
- $\mathbb{C}[G]$ The group algebra of G over \mathbb{C} , p. 2
- $\text{Irr}(G)$ The set of irreducible \mathbb{C} -representations of G up to isomorphism, p. 1
- $\text{dimirr}(G)$ The set of dimensions of irreducible representations of G , p. 2
- $\text{Irr}(G \mid \psi)$ The set of irreducible representations of G lying above ψ , p. 11
- $\text{Stab}_G(\psi)$ The stabiliser in G of $\psi \in \text{Irr}(N)$, p. 11
- $\mathcal{R}_G(\mathcal{D})$ The representation zeta polynomial of G , p. 3

$\mathcal{R}_{G \psi}(\mathcal{D})$	The representation zeta polynomial of G above ψ , p. 13
$\text{Ind}_H^G W$	The representation of G induced from W , $\text{Ind}_H^G W = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$, p. 12
$\text{Res}_H^G V, V _H$	The representation of H restricted from V , p. 11
ζ_G	The representation zeta function of G , p. 2
K^i	The i th congruence kernel of $\text{GL}_n(\mathfrak{o}_r)$, $K^i = 1 + \mathfrak{p}^i M_n(\mathfrak{o}_r)$, p. 14
ψ_β	The representation of K^i given by $\psi_\beta(1 + \varpi^i x) = \psi(\text{tr}(\hat{\beta}x))$, p. 15
$G_{\lambda, \mathfrak{o}}$	The automorphism group of the \mathfrak{o} -module of type λ , p. 3
$\text{PGL}(V)$	The projective general linear group on V , p. 7
$\hat{\psi}$	A strong extension of $\psi \in \text{Irr}(N)$ to its stabiliser, p. 11
$B^2(G, \mathbb{C}^\times)$	The group of 2-coboundaries on G with coefficients in the trivial G -module \mathbb{C}^\times , p. 89
$Z^2(G, \mathbb{C}^\times)$	The group of 2-cocycles on G with coefficients in the trivial G -module \mathbb{C}^\times , p. 89
$H^2(G, \mathbb{C}^\times)$	The second cohomology group of G with coefficients in the trivial G -module \mathbb{C}^\times , p. 89
$\mathbb{C}^\alpha[G]$	The twisted group algebra of G corresponding to the cocycle α , p. 8
$\text{Irr}_\alpha(G)$	The set of irreducible projective \mathbb{C} -representations of G with associated cohomology class $[\alpha] \in H^2(G, \mathbb{C}^\times)$, p. 8
μ_{p^∞}	The group of p -power roots of unity in \mathbb{C} , p. 12
$H_c^i(\mathbf{X}, \overline{\mathbb{Q}}_l)$	The i th l -adic cohomology group of \mathbf{X} , p. 51
R_w^θ	The Deligne-Lusztig character corresponding to (w, θ) , p. 52
Gr_{r-1}°	The Greenberg functor $(\text{Sch}/\mathfrak{o}_r) \rightarrow (\text{Sch}/\mathfrak{o}_1)$, p. 65
$R_u(G)$	The unipotent radical of G , p. 59
μ_K	The normalised Haar measure on K , p. 71
$\mathcal{L}_\mathfrak{o}$	The Denef-Pas language with coefficients in \mathfrak{o} , p. 72
$h[m, n, r]$	The trivial definable assignment $K^m \times k_K^n \times \mathbb{Z}^r$, p. 73
$\mathcal{C}(S)$	The ring of constructible motivic functions on S , p. 74
$I_S C(X)$	The ring of S -integrable constructible motivic functions on X , p. 74
s_K	The specialisation map associated to the field K , p. 75

Dedication

To my parents and to my fiancée, who have given me love and support throughout the writing of this thesis.

CHAPTER 1

Introduction

Let \mathfrak{o} be the valuation ring of a non-Archimedean local field with maximal ideal \mathfrak{p} , finite residue field $\mathfrak{o}/\mathfrak{p}$ of order q and characteristic p . We fix a uniformiser ϖ of \mathfrak{o} and integers $n, r \geq 1$ and write $\mathfrak{o}_r = \mathfrak{o}/\mathfrak{p}^r$. The main motivating problem for this thesis will be that of finding the dimensions of the representations of $\mathrm{GL}_n(\mathfrak{o}_r)$. Throughout, we write $\mathrm{Irr}(G)$ to mean the isomorphism classes of irreducible representations of a finite group G (over an algebraically closed field of characteristic zero, e.g. \mathbb{C} or $\overline{\mathbb{Q}_l}$). By abuse of notation, we write $\rho \in \mathrm{Irr}(G)$ to mean either that ρ is an irreducible representation of G or ρ is an irreducible character of G .

There are various levels of detail in which we can understand the irreducible \mathbb{C} -representations of a finite group G . The most complete is to describe them as vector spaces with an explicit G -action. A weaker requirement is to give the character table of G ; knowledge of a character does not give one an explicit description of the representation affording this character. In fact, after Green constructed the characters of $\mathrm{GL}_n(\mathbb{F}_q)$ in [23], it was over twenty years before the classification of the representations of $\mathrm{GL}_n(\mathbb{F}_q)$ was completed by Deligne and Lusztig [15]. This required a new approach using ℓ -adic cohomology, which we will use in Chapter 4. In this thesis, we will generally aim to list the dimensions of the representations, which is a weaker

requirement than finding the character table.¹ However, by the Artin-Wedderburn theorem applied to $\mathbb{C}[G]$, the dimensions of the irreducible representations determine the group algebra up to isomorphism. Further, comparing dimensions in the decomposition of $\mathbb{C}[G]$ into simple modules yields $|G| = \sum_{\rho \in \text{Irr}(G)} (\dim \rho)^2$.

The dimensions of the representations of p -adic groups have been studied extensively. If G is a topological group which is *representation rigid*, that is, has a finite number of smooth irreducible representations of each dimension, define the representation zeta function

$$\zeta_G(s) = \sum_{\rho \in \text{Irr}(G)} \frac{1}{(\dim \rho)^s} = \sum_{n=1}^{\infty} \frac{r_n(G)}{n^s},$$

where $\text{Irr}(G)$ denotes the smooth representations of G and $r_n(G)$ is the number of smooth representations of G of dimension n . Jaikin-Zapirain [37] showed that if G is compact p -adic analytic and FAb,² and either $p > 2$ or G is uniform pro-2, then there exist $n_1, \dots, n_k \in \mathbb{N}$ and $f_1, \dots, f_k \in \mathbb{Q}(x)$ such that $\zeta_G(s) = \sum_{i=1}^k n_i^{-s} f_i(p^{-s})$. Stasinski and Zordan [58] give a single proof which works for all p and all compact, FAb, p -adic analytic G . Avni, Klopsch Onn and Voll have also proven results about the abscissa of convergence of $\zeta_G(s)$ [5, 6, 7] and polynomial representation growth [4, 8] of $r_n(G)$.

Some of the first results about the representations of $\text{GL}_n(\mathfrak{o}_r)$ for $r \geq 2$ were found by Shintani [53], Nagorny [47] and Nobs [49]. Later, Hill [30, 31, 32, 33] rediscovered Shintani's Clifford theoretic approach and constructed certain classes of representations, including so-called *regular* representations. Further constructions for GL_2 were found by Stasinski [56] and Onn [50], and for GL_3 by Onn, Prasad and Singla [51]. The complete construction of the regular representations of $\text{GL}_n(\mathfrak{o}_r)$ was found for $p \neq 2$ by Krakovski, Onn and Singla [43], and independently by Stasinski and Stevens [57] (for arbitrary p).

We write $\dim_{\text{irr}}(G) = \{\dim \rho \mid \rho \in \text{Irr}(G)\}$. Hadas [26] showed the following:

¹If two groups G_1 and G_2 have the same character table, then there is a dimension-preserving bijection $\text{Irr}(G_1) \leftrightarrow \text{Irr}(G_2)$. However, the converse is not true: both $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ have four irreducible representations, all of dimension one, but different character tables.

²“FAb” means that if H is a finite index subgroup of G , then $H/[H, H]$ is finite.

Theorem 1. [26, Theorem 1.1.5] *Let G be a smooth affine group scheme of finite type over \mathbb{Z} . There exist $N > 0$ and $C > 0$ such that for every $p > C$ and every finite field \mathbb{F} of characteristic p , $|\dim_{\text{irr}}(G(\mathbb{F}))| \leq N$.*

We extend this result to obtain a more qualitative statement about the dimensions of the irreducible representations of $G(\mathfrak{o}_r)$, namely that they are given by evaluating some finite set of polynomials. More precisely:

Theorem 2 (Polynomial Theorem for large p). *Let G be a smooth affine group scheme of finite type over \mathbb{Z} and $r \geq 1$ an integer. There exists $C > 0$ such that for every prime $p > C$, there exists a finite set of polynomials $R \subseteq \mathbb{Q}[x]$ such that for every non-Archimedean local field K with valuation ring \mathfrak{o} and finite residue field of characteristic p ,*

$$\dim_{\text{irr}}(G(\mathfrak{o}_r)) \subseteq \{m(|\mathfrak{o}_1|) \mid m \in R\}.$$

Definition 3. [50, §1.1] Let $\lambda = (\ell_1, \dots, \ell_n)$ be a partition, and \mathfrak{o} a valuation ring as before. Define the (finite) \mathfrak{o} -module $M_{\lambda, \mathfrak{o}} = \mathfrak{o}_{\ell_1} \oplus \dots \oplus \mathfrak{o}_{\ell_n}$, and let $G_{\lambda, \mathfrak{o}}$ be its automorphism group, $G_{\lambda, \mathfrak{o}} = \text{Aut}_{\mathfrak{o}}(M_{\lambda, \mathfrak{o}})$.

Note that if $\ell_1 = \dots = \ell_n = r$, then $M_{\lambda, \mathfrak{o}} = \mathfrak{o}_r^n$ and $G_{\lambda, \mathfrak{o}} = \text{GL}_n(\mathfrak{o}_r)$.

In [50, Conjectures 1.3, 1.5], Onn conjectured that the dimensions and frequencies of the representations of $G_{\lambda, \mathfrak{o}}$ can be written as polynomials in $|\mathfrak{o}_1|$. There is room for interpretation about the precise meaning of the original statements, therefore we suggest a new formulation below (see [36, Conjecture 1]).

Definition 4. The **representation zeta polynomial** of a finite group G is

$$\mathcal{R}_G(\mathcal{D}) = \sum_{\rho \in \text{Irr}(G)} \mathcal{D}^{\dim \rho} \in \mathbb{Z}[\mathcal{D}].$$

Conjecture 5. *Let \mathfrak{D} denote the set of rings which are the valuation ring of a non-Archimedean local field with finite residue field, up to isomorphism. Let $n \geq 1$ and a partition λ of n be given. There exist $k \geq 1$, polynomials*

$$d_1(x), \dots, d_k(x) \in \mathbb{Z}[x] \setminus \{0\}, m_1(x), \dots, m_k(x) \in \mathbb{Q}[x] \setminus \{0\},$$

such that for all $\mathfrak{o} \in \mathfrak{D}$,

$$\mathcal{R}_{G_{\lambda, \mathfrak{o}}}(\mathcal{D}) = \sum_{i=1}^k m_i(|\mathfrak{o}_1|) \mathcal{D}^{d_i(|\mathfrak{o}_1|)}.$$

The requirement $m_i \neq 0$ implies that k is unique and the d_i and m_i are determined uniquely up to ordering. We must have $d_i \neq 0$, since an irreducible representation cannot have dimension zero.

The statement for $\lambda = 1^n$ follows from Green’s construction for GL_n over finite fields [23]. It has also been shown for λ of length two by Onn [50]. Singla [55] proved the statement for $\lambda = (2, 1, 1)$ and thus deduced it for $\lambda = 2^3$ and 2^4 . The statement was also shown for $\lambda = (\ell, 1^n)$ by the author [36], and the proof is given in Section 3.1.

Conjecture 5 does not hold if GL_n is replaced by an arbitrary smooth affine group scheme. If GL_n is replaced by SL_2 , then the d_i no longer have integer coefficients; $\mathrm{SL}_2(\mathbb{F}_q)$ has representations of dimension $\frac{1}{2}(q-1)$ and $\frac{1}{2}(q+1)$ when q is odd. Even if we allow $d_i \in \mathbb{Q}[x]$ in Conjecture 5, the statement also does not hold for SL_3 , as it was shown by Simpson and Frame [54] that for $\mathrm{SL}_3(\mathbb{F}_q)$, some of the functions d_i and m_i are given by different polynomials depending on the congruence class of q modulo 3. This behaviour is referred to as being *polynomial on residue classes*, or *PORC* for short. The terminology is due to Higman, and is discussed in Section A.6.1. Moreover, Halasi and Pálffy [27] constructed an example of a so-called “pattern group” G such that the number of conjugacy classes of $G(\mathbb{F}_q)$ is not given by a polynomial in q . However, the question of whether the number of conjugacy classes of the unitriangular group $U_n(\mathbb{F}_q)$ is given by a polynomial in q is still open; this was conjectured by Higman [28].

There are situations in which one must assume that the residue characteristic of \mathfrak{o} is “large enough” (e.g. in [26]), therefore another reasonable conjecture would be to fix a prime p , and replace \mathfrak{D} in Conjecture 5 with the set of all valuation rings of residue characteristic p . This statement would follow immediately from Conjecture 5.

Onn [50] also conjectured the following:

Conjecture 6. *Let λ be a partition. If \mathfrak{o} and \mathfrak{o}' are valuation rings of non-Archimedean local fields with $|\mathfrak{o}_1| = |\mathfrak{o}'_1|$, then there is an isomorphism of algebras*

$$\mathbb{C}[G_{\lambda, \mathfrak{o}}] \cong \mathbb{C}[G_{\lambda, \mathfrak{o}'}].$$

Since this isomorphism is equivalent to there being a dimension-preserving bijection $\text{Irr}(G_{\lambda, \mathfrak{o}}) \leftrightarrow \text{Irr}(G_{\lambda, \mathfrak{o}'})$, Conjecture 5 implies Conjecture 6.

Singla [55] showed that there exists a canonical dimension-preserving bijection between $\text{Irr}(\text{GL}_n(\mathfrak{o}_2))$ and $\text{Irr}(\text{GL}_n(\mathfrak{o}'_2))$, thus proving Conjecture 6 for $\lambda = 2^n$. Furthermore, Hadas [26, Theorem 1.1.4] showed the following:

Theorem 7. *Let G be a smooth affine group scheme of finite type over \mathbb{Z} . Then for every $r \in \mathbb{N}$, there exists $C > 0$ such that for every prime $p > C$ and every $d \in \mathbb{N}$,*

$$\mathbb{C}[G(\mathbb{F}_{p^d}[[t]]/(t^r))] \cong \mathbb{C}[G(\mathfrak{o}_{\mathbb{Q}_p^{(d)}}/\mathfrak{p}_{\mathbb{Q}_p^{(d)}}^r)],$$

where $\mathbb{Q}_p^{(d)}$ is the unique unramified extension of \mathbb{Q}_p of degree d .

Thus, Conjecture 6 holds for $\lambda = r^n$ for all r and n , provided that $p = \text{char } \mathfrak{o}_1$ is large (possibly depending on n and r), and \mathfrak{o} and \mathfrak{o}' are valuation rings of finite unramified extensions of \mathbb{Q}_p in the zero characteristic case. In fact, we will show in Chapter 5 that this can be generalised to the following statement:

Theorem 8. *Let G be a smooth group scheme of finite type over \mathbb{Z} and $r \in \mathbb{N}$. Then there exists $C > 0$ such that for every prime p greater than C , and for every non-Archimedean local field K with finite residue field of characteristic p , we have*

$$\mathbb{C}[G(\mathfrak{o}_K/\mathfrak{p}_K^r)] \cong \mathbb{C}[G(\mathbb{F}_q[t]/(t^r))].$$

The structure of the thesis is as follows: In Chapter 2, we outline the approach to studying the representations of $\text{GL}_n(\mathfrak{o}_r)$ directly using Clifford theory. In Chapter 3, we adapt a method due to Onn (see [55, Proposition 7.7]) to construct the representations of $G_{(\ell, 1^n), \mathfrak{o}}$, prove Conjecture 5 in this case, and give some results on Schur multipliers of $\text{GL}_n(\mathfrak{o}_r)$. In Chapter 4, we prove Theorem 2. This uses an

argument due to Geck [19] for the reductive case, and an argument due to Hadas [26, Theorem 4.0.2] for the general case. A previous attempt at a proof of Theorem 2 is recorded in Section A.6.2. In Chapter 5, we prove Theorem 8, using Hadas' proof of Theorem 7 and an argument using motivic integration. Our main reference for this is Cluckers and Loeser [13]. Appendix A also contains background material on local fields, category theory, group cohomology, the theory of schemes and first-order mathematical logic.

2.1 Projective Representations

The material in this section can be found in, for example, Chapter 1 of [39]. The necessary prerequisites from group cohomology are given in Appendix A.3. In this section, we will only consider cohomology groups $H^2(G, \mathbb{C}^\times)$ where G acts trivially on \mathbb{C}^\times .

Given a \mathbb{C} -vector space V , the **projective general linear group** of V is $\mathrm{PGL}(V) = \mathrm{GL}(V)/\{\lambda I \mid \lambda \in \mathbb{C}\}$.

Definition 9. Let G be a group. A **projective representation** of G (over \mathbb{C}) is a \mathbb{C} -vector space V along with a group homomorphism $G \rightarrow \mathrm{PGL}(V)$.

In order to avoid confusion, we may call a homomorphism $G \rightarrow \mathrm{GL}(V)$ a **linear representation** of G .

Let $\varphi : G \rightarrow \mathrm{PGL}(V)$ be a projective representation. For each $\varphi(g)$, choose a coset representative $\tilde{\varphi}(g) \in \mathrm{GL}(V)$ of $\varphi(g)$. Define the map $\alpha : G \times G \rightarrow \mathbb{C}^\times$ to be such that for all $g_1, g_2 \in G$,

$$\tilde{\varphi}(g_1)\tilde{\varphi}(g_2) = \alpha(g_1, g_2)\tilde{\varphi}(g_1g_2).$$

Then one can show that $\alpha \in Z^2(G, \mathbb{C}^\times)$, that is, the cocycle condition

$$\alpha(g_1g_2, g_3)\alpha(g_1, g_2) = \alpha(g_1, g_2g_3)\alpha(g_2, g_3) \quad (2.1)$$

holds for all $g_1, g_2, g_3 \in G$, and if α_1, α_2 are obtained from any two systems of coset representatives $\tilde{\varphi}_1, \tilde{\varphi}_2$, then $\alpha_1\alpha_2^{-1} \in B^2(G, \mathbb{C}^\times)$, that is, there exists a map $\mu : G \rightarrow \mathbb{C}^\times$ such that for all $g_1, g_2 \in G$, $\alpha_1(g_1, g_2)\alpha_2(g_1, g_2)^{-1} = \mu(g_1)\mu(g_2)\mu(g_1g_2)^{-1}$, or equivalently, $[\alpha_1] = [\alpha_2] \in H^2(G, \mathbb{C}^\times)$. Therefore, every projective representation defines a unique cohomology class in $H^2(G, \mathbb{C}^\times)$. If α is any representative of this cohomology class, we say φ is an α -**representation** of G . In analogy with identifying linear representations of G with $\mathbb{C}[G]$ -modules, we identify α -representations with modules over an algebra defined as follows:

Definition 10. Let $\alpha \in Z^2(G, \mathbb{C}^\times)$. Define the **twisted group algebra** $\mathbb{C}^\alpha[G]$ to be the \mathbb{C} -vector space with basis $\{\bar{g} \mid g \in G\}$ and multiplication given on the basis by $\bar{g}_1 \cdot \bar{g}_2 = \alpha(g_1, g_2)\overline{g_1g_2}$.

Definition 11. If V and W are α -representations of a group G , then they are **isomorphic** if they are isomorphic as $\mathbb{C}^\alpha[G]$ -modules.¹

Definition 12. If V is an α -representation of G , then it is **irreducible** if it has no $\mathbb{C}^\alpha[G]$ -submodules other than 0 and V . The set of irreducible α -representations of G up to isomorphism is written $\text{Irr}_\alpha(G)$.

Many of the concepts in the representation theory of finite groups have analogues for projective representations, including those of conjugacy and class functions. In each case, the correct definition requires a “twist” by a cocycle $\alpha \in Z^2(G, \mathbb{C}^\times)$. In order to understand the projective representations of a group, we only need to understand those corresponding to representatives of the cohomology classes in $H^2(G, \mathbb{C}^\times)$:

Proposition 13. *Let G be a finite group and $\alpha_1, \alpha_2 \in Z^2(G, \mathbb{C}^\times)$. Then $\mathbb{C}^{\alpha_1}[G] \cong \mathbb{C}^{\alpha_2}[G]$ if and only if $[\alpha_1] = [\alpha_2]$.*

¹This is equivalent to the condition of being **linearly equivalent**, see [39, Chapter 1, Theorem 3.4].

Every cohomology class of $H^2(G, \mathbb{C}^\times)$ contains a cocycle α such that $\alpha(1, 1) = 1$. Indeed, given $\alpha \in Z^2(G, \mathbb{C}^\times)$, choose any $\mu : G \rightarrow \mathbb{C}^\times$ such that $\mu(1) = \alpha(1, 1)$, and let $\alpha'(g_1, g_2) = \alpha(g_1, g_2)\mu(g_1)^{-1}\mu(g_2)^{-1}\mu(g_1g_2)$, for then $[\alpha'] = [\alpha]$ and $\alpha'(1, 1) = 1$. We shall assume that cocycles have this property for the remainder of this section. This condition implies that $\alpha(1, g) = \alpha(g, 1) = 1$ for all $g \in G$ by the cocycle condition (Equation 2.1).

Definition 14. Let $\varphi : G \rightarrow \text{PGL}(V)$ be a projective representation and fix a lift $\tilde{\varphi} : G \rightarrow \text{GL}(V)$ with normalised cocycle $\alpha \in Z^2(G, \mathbb{C}^\times)$. Define the **projective character** of φ to be the function $\chi_\varphi(g) = \text{tr}(\varphi(g))$. We also call the character of a projective representation with associated cocycle α an **α -character**.

By [39, Chapter 1, Proposition 6.2(i)], projective characters satisfy an analogous condition to that of a class function, that is, for $g, x \in G$, we have the transformation law

$$\chi_\varphi(g^{-1}xg) = \frac{\alpha(g, g^{-1}xg)}{\alpha(x, g)}\chi_\varphi(x).$$

Definition 15. Let $g \in G$ and $\alpha \in Z^2(G, \mathbb{C}^\times)$. Then g is **α -regular** if for every $g' \in C_G(g)$, $\alpha(g, g') = \alpha(g', g)$.

One can show the following (see [38, Chapter 2, Lemma 6.1] and [39, Chapter 1, Proposition 6.2(ii)]):

Proposition 16. *If $g \in G$ is α -regular, then so are all of the conjugates of g . If χ is an α -character of G and $g \in G$ is not α -regular, then $\chi(g) = 0$.*

Projective characters obey analogous orthogonality relations to the case of linear representations:

Definition 17. Let $G_{\alpha\text{-reg}} = \{g_j \mid 1 \leq j \leq n\}$ be a set of representatives of the α -regular conjugacy classes of G and let $\text{Irr}_\alpha(G) = \{\chi_i \mid 1 \leq i \leq m\}$. The **α -character table of G** is the $m \times n$ matrix whose (i, j) th entry is $\chi_i(g_j)$.

Note that the α -character table depends on the choice of representatives of the α -regular conjugacy classes. Nonetheless, we can state:

Proposition 18. *The α -character table of G is square, i.e. $m = n$. Further, the α -character table obeys the orthogonality relations:*

$$\frac{1}{|G|} \sum_{g \in G_{\alpha\text{-reg}}} \alpha(g, g^{-1})^{-1} \chi_i(g^{-1}) \chi_j(g) = \delta_{ij}; \quad (2.2)$$

$$\frac{1}{|C_G(g_i)|} \sum_{k=1}^n \alpha(g_j, g_j^{-1})^{-1} \chi_k(g_i) \chi_k(g_j^{-1}) = \delta_{ij}. \quad (2.3)$$

Proof. The proofs can be found in [39, Chapter 1, Theorems 3.6 and 11.2]. Alternatively, observe that $\mathbb{C}^\alpha[G]$ is given the structure of a *symmetric algebra* by the map $\tau : \mathbb{C}^\alpha[G] \rightarrow \mathbb{C}$, defined by $\tau(\sum_{g \in G} a_g \bar{g}) = a_1$. That is, τ is linear, and the bilinear form $\langle h, h' \rangle = \tau(hh')$ is symmetric and non-degenerate. Symmetric algebras obey similar orthogonality relations as for characters of groups [21, Corollary 7.2.4]; the orthogonality relation 2.2 follows from this more general fact. Following [39], the character $\chi_i^*(g) = \alpha(g, g^{-1})^{-1} \chi_i(g^{-1})$ is the character of the contragredient of the projective representation affording χ , with associated cohomology class $[\alpha]^{-1}$. Therefore, $\chi_i^* \chi_j$ has trivial cohomology class and is thus a character of a linear representation of G . This implies that $\chi_i^* \chi_j$ is a class function. Writing m_k for the size of the conjugacy class of g_k , we have

$$|G| \delta_{ij} = \sum_{g \in G_{\alpha\text{-reg}}} \chi_i^*(g) \chi_j(g) = \sum_{k=1}^n m_k \chi_i^*(g_k) \chi_j(g_k).$$

These equations can be written in matrix form:

$$|G| I_n = \left(\chi_i^*(g_j) \right)_{i,j} \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \left(\chi_j(g_i) \right)_{i,j}.$$

Since a left inverse of a matrix is also a right inverse and vice versa, we have

$$|G|I_n = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \left(\chi_j(g_i) \right)_{i,j} \left(\chi_i^*(g_j) \right)_{i,j}.$$

Comparing (i, j) th entries,

$$|G|\delta_{ij} = \sum_{k=1}^n m_k \chi_k(g_i) \chi_k^*(g_j).$$

Dividing through by $|G|$ and observing that $|G| = m_i |C_G(g_i)|$ gives Equation 2.3. \square

2.2 Clifford theory

In order to construct representations of a group G , we shall fix a normal subgroup N of G and consider the action of G on $\text{Irr}(N)$ by conjugation; given $g \in G, \psi \in \text{Irr}(N)$, we define ${}^g\psi \in \text{Irr}(N)$ by

$${}^g\psi(n) = \psi(g^{-1}ng).$$

Given $\psi \in \text{Irr}(N)$, we write $\text{Irr}(G \mid \psi) = \{\rho \in \text{Irr}(G) \mid \langle \psi, \text{Res}_N^G \rho \rangle \neq 0\}$ and $\text{Stab}_G(\psi)$ for the stabiliser in G of ψ . We say that $\psi \in \text{Irr}(N)$ is **invariant in** $T \leq G$ if $T \leq \text{Stab}_G(\psi)$.

Definition 19 (Strong Extension). Let $\psi \in \text{Irr}(N)$ be invariant in $T \leq G$. A **strong extension** of ψ to T is a projective representation $\hat{\psi}$ of T such that for all $n \in N, g \in T$, the following hold:

1. $\psi(n) = \hat{\psi}(n)$,
2. $\hat{\psi}(n)\hat{\psi}(g) = \hat{\psi}(ng)$,
3. $\hat{\psi}(g)\hat{\psi}(n) = \hat{\psi}(gn)$.

The terminology is due to Stasinski and Zordan [58]. By means of a projective extension, ψ defines a cohomology class as follows:

Proposition 20. [34, Theorem 11.7] Suppose $\psi \in \text{Irr}(N)$ is invariant in $T \leq G$. Then ψ has a strong extension to an irreducible projective representation $\hat{\psi}$ of $\text{Stab}_G(\psi)$. Any cocycle associated to $\hat{\psi}$ is constant on cosets of N in $\text{Stab}_G(\psi)$, so descends to a cocycle $\alpha \in Z^2(\text{Stab}_G(\psi)/N, \mathbb{C}^\times)$. This defines a unique cohomology class $[\alpha] \in H^2(\text{Stab}_G(\psi)/N, \mathbb{C}^\times)$. Further, ψ extends to a linear representation of $\text{Stab}_G(\psi)$ if and only if $[\alpha] = 1$.

Proposition 21. [1, Lemma 9.0.2] If N is a p -group and $H^2(\text{Stab}_G(\psi)/N, \mu_{p^\infty}) = 1$, then ψ extends to a linear representation of $\text{Stab}_G(\psi)$.

It would therefore be sufficient to understand, for each orbit representative $\psi \in \text{Irr}(N)$, the cohomology class $[\alpha] \in H^2(\text{Stab}_G(\psi)/N, \mathbb{C}^\times)$ associated to ψ , and the set $\text{Irr}_{\alpha^{-1}}(\text{Stab}_G(\psi)/N)$. However, this is not known in general, and in Chapter 4 we will arrange for $H^2(\text{Stab}_G(\psi)/N, \mathbb{C}^\times)$ to be trivial, so that we only need to consider linear representations of $\text{Stab}_G(\psi)/N$.

We summarise the results we require from Clifford theory. Their proofs can be found in [34, Theorems 6.2, 6.11, 6.17].

Proposition 22. Let G be a finite group and N a normal subgroup of G . Then the following statements hold:

1. for every $\rho \in \text{Irr}(G)$, there exists an integer $e \geq 1$ and a G -orbit $\Omega \subseteq \text{Irr}(N)$ such that

$$\text{Res}_N^G \rho = \left(\bigoplus_{\psi \in \Omega} \psi \right)^e;$$

2. for every $\psi \in \text{Irr}(N)$, there exists a bijection

$$\text{Irr}(\text{Stab}_G(\psi) \mid \psi) \rightarrow \text{Irr}(G \mid \psi)$$

given by induction,

$$\theta \mapsto \text{Ind}_{\text{Stab}_G(\psi)}^G \theta;$$

3. writing $\hat{\psi}$ for a strong extension of ψ to $\text{Stab}_G(\psi)$ and $[\alpha] \in H^2(\text{Stab}_G(\psi)/N, \mathbb{C}^\times)$

for the cohomology class defined by $\hat{\psi}$, there exists a bijection

$$\text{Irr}_{\alpha^{-1}}(\text{Stab}_G(\psi)/N) \rightarrow \text{Irr}(\text{Stab}_G(\psi) \mid \psi)$$

given by

$$\bar{\pi} \mapsto \pi \otimes \hat{\psi},$$

where π is obtained from $\bar{\pi}$ by composition with the natural map $\text{Stab}_G(\psi) \rightarrow \text{Stab}_G(\psi)/N$.

Remark 23. A representation is called **homogeneous** if it is the direct sum of isomorphic irreducible representations. If G is finite with normal subgroup N and $\rho \in \text{Irr}(G)$, then by part 1 of Proposition 22, $\text{Res}_N^G \rho$ is homogeneous precisely when the orbit $\Omega = \{\psi\}$ has size 1. Equivalently, ψ is invariant in G , i.e. $\text{Stab}_G(\psi) = G$.

Definition 24. Given a normal subgroup N of G and $\psi \in \text{Irr}(N)$, the **representation zeta polynomial above ψ** is

$$\mathcal{R}_{G|\psi}(\mathcal{D}) = \sum_{\rho \in \text{Irr}(G|\psi)} \mathcal{D}^{\dim \rho}.$$

Remark 25. By Proposition 22, if X is any set of representatives of the G -orbits on $\text{Irr}(N)$, then

$$\mathcal{R}_G(\mathcal{D}) = \sum_{\psi \in X} \mathcal{R}_{G|\psi}(\mathcal{D}).$$

Also by Proposition 22,

$$\mathcal{R}_{G|\psi}(\mathcal{D}) = \mathcal{R}_{\text{Stab}_G(\psi)|\psi}(\mathcal{D}^{[G:\text{Stab}_G(\psi)]}).$$

We shall also need to describe the representations of a semidirect product where the normal subgroup is abelian:

Proposition 26. [52, Proposition 25] Let $G = N \rtimes H$, where N is abelian. Let ψ_i be representatives of the orbits of H on $\text{Irr}(N)$. For $\rho \in \text{Irr}(\text{Stab}_H(\psi_i))$, define

$\tilde{\rho} \in \text{Irr}(N \rtimes \text{Stab}_H(\psi_i))$ by composition with the projection

$$N \rtimes \text{Stab}_H(\psi_i) \twoheadrightarrow \text{Stab}_H(\psi_i).$$

Define $\tilde{\psi}_i \in \text{Irr}(N \rtimes \text{Stab}_H(\psi_i))$ by $\tilde{\psi}_i(nh) = \psi_i(n)$. The irreducible representations of G are precisely

$$\text{Ind}_{N \rtimes \text{Stab}_H(\psi_i)}^G(\tilde{\psi}_i \otimes \tilde{\rho}),$$

and further, these are all distinct for distinct choices of pairs (i, ρ) .

Note that this is a special case of Proposition 22, where the stabiliser also splits as a semidirect product, $\text{Stab}_G(\psi_i) = N \rtimes \text{Stab}_H(\psi_i)$. In order to find $\mathcal{R}_G(\mathcal{D})$ in this case, it is sufficient to find representatives of the H -orbits on $\text{Irr}(N)$ and the representation zeta polynomial of each of the stabilisers of the orbit representatives.

2.3 Application to the Groups $\text{GL}_n(\mathfrak{o}_r)$

Fix an integer $r \geq 2$ throughout. The canonical ring surjection $\mathfrak{o}_r \twoheadrightarrow \mathfrak{o}_{r-1}$ induces a surjective group homomorphism $\eta : \text{GL}_n(\mathfrak{o}_r) \twoheadrightarrow \text{GL}_n(\mathfrak{o}_{r-1})$. We define the $(r-1)$ th congruence subgroup of $\text{GL}_n(\mathfrak{o}_r)$ to be

$$K = \ker \eta = \{1 + \varpi^{r-1}x \mid x \in M_n(\mathfrak{o}_r)\}.$$

We see that K is abelian; indeed for all $x, y \in M_n(\mathfrak{o}_r)$,

$$\begin{aligned} (1 + \varpi^{r-1}x)(1 + \varpi^{r-1}y) &= 1 + \varpi^{r-1}(x + y) + \varpi^{2r-2}xy \\ &= 1 + \varpi^{r-1}(x + y), \end{aligned}$$

since $\varpi^{2r-2} = 0$. We shall use K as the chosen normal subgroup and apply Clifford theory. One may define the i th congruence subgroup to be $\ker(\text{GL}_n(\mathfrak{o}_r) \twoheadrightarrow \text{GL}_n(\mathfrak{o}_i))$, however we will only consider the case $i = r-1$ in this thesis.

Proposition 27. *There is a canonical group isomorphism $M_n(\mathfrak{o}_1) \rightarrow K$ given by $\bar{x} \mapsto 1 + \varpi^{r-1}x$ for $x \in M_n(\mathfrak{o}_r)$.*

Proof. The map is well-defined, since if $x_1, x_2 \in M_n(\mathfrak{o}_r)$ are such that $\overline{x_1} = \overline{x_2}$ (i.e. $x_1 \equiv x_2 \pmod{\varpi}$), there exists $y \in M_n(\mathfrak{o}_r)$ such that $x_1 = x_2 + \varpi y$. Then $1 + \varpi^{r-1}x_1 = 1 + \varpi^{r-1}(x_2 + \varpi y) = 1 + \varpi^{r-1}x_2$, since $\varpi^r = 0$ in \mathfrak{o}_r .

The map is a group homomorphism, indeed for $x, y \in M_n(\mathfrak{o}_r)$, $\overline{x + y} = \overline{x} + \overline{y} \mapsto 1 + \varpi^{r-1}(x + y) = (1 + \varpi^{r-1}x)(1 + \varpi^{r-1}y)$. The last equality holds because $\varpi^{2r-2} = 0$ in \mathfrak{o}_r .

We claim the map is injective; indeed suppose that $1 + \varpi^{r-1}x = 1$ so that $\varpi^{r-1}x = 0$ in \mathfrak{o}_r . If \hat{x} is any lift of x to \mathfrak{o} , then we have $\varpi^{r-1}\hat{x} \in \mathfrak{p}^r$, thus $r \leq v(\varpi^{r-1}\hat{x}) = v(\varpi^{r-1}) + v(\hat{x}) = (r-1) + v(\hat{x})$. We conclude that $1 \leq v(\hat{x})$, so $\hat{x} \in \mathfrak{p}$ and reducing modulo ϖ gives $\overline{x} = 0 \in \mathfrak{o}_1$.

The map is surjective, since every element of K has the form $1 + \varpi^{r-1}x$ for some $x \in \mathfrak{o}_r$; such an element has a preimage \overline{x} . \square

Fix a non-trivial character $\psi : \mathfrak{o}_1 \rightarrow \mathbb{C}^\times$. We parameterise the irreducible representations of K by means of ψ . For $\beta \in M_n(\mathfrak{o}_1)$, define $\psi_\beta : K \rightarrow \mathbb{C}^\times$ by

$$\psi_\beta(1 + \varpi^{r-1}x) = \psi(\text{tr}(\beta\overline{x}))$$

for $x \in M_n(\mathfrak{o}_r)$, where \overline{x} denotes reduction modulo ϖ . This definition does not depend on the representative x chosen, indeed if $x_1, x_2 \in M_n(\mathfrak{o}_r)$ are such that $1 + \varpi^{r-1}x_1 = 1 + \varpi^{r-1}x_2$, then $x_1 \equiv x_2 \pmod{\varpi}$ and so $\overline{x_1} = \overline{x_2}$.

We can describe the characters of \mathfrak{o}_1 in terms of ψ :

Proposition 28. *There is a group isomorphism*

$$M_n(\mathfrak{o}_1) \rightarrow \text{Irr}(K)$$

given by $\beta \mapsto \psi_\beta$.

Proof. We first show that the map is a group homomorphism. For $\beta_1, \beta_2 \in M_n(\mathfrak{o}_1)$

and $g \in K$, we have

$$\begin{aligned}
\psi_{\beta_1+\beta_2}(1 + \varpi^{r-1}x) &= \psi(\text{tr}((\beta_1 + \beta_2)\bar{x})) \\
&= \psi(\text{tr}(\beta_1\bar{x}) + \text{tr}(\beta_2\bar{x})) \\
&= \psi(\text{tr}(\beta_1\bar{x}))\psi(\text{tr}(\beta_2\bar{x})) \\
&= \psi_{\beta_1}(1 + \varpi^{r-1}x)\psi_{\beta_2}(1 + \varpi^{r-1}x).
\end{aligned}$$

Therefore, $\psi_{\beta_1+\beta_2} = \psi_{\beta_1}\psi_{\beta_2}$ and $\beta \mapsto \psi_\beta$ is a group homomorphism.

In order to show that the map is injective, suppose that $\beta \in M_n(\mathfrak{o}_1)$ is such that $\psi_\beta = 0$. Then for all $x \in M_n(\mathfrak{o}_r)$,

$$0 = \psi_\beta(1 + \varpi^{r-1}x) = \psi(\text{tr}(\beta x)).$$

Let $1 \leq i, j \leq n$ be given, and choose $x = aE_{ij}$ (the matrix with $a \in \mathfrak{o}_1$ in the (i, j) th entry and 0 in all other entries). Then the j th column of βx is equal to a multiplied by the i th column of β , therefore $\text{tr}(\beta x) = a\beta_{ij}$. Since $\psi(a\beta_{ij}) = 0$ for all $a \in \mathfrak{o}_1$, we must have $\beta_{ij} = 0$. Since this holds for all i and j , we must have $\beta = 0$.

We show that the map is surjective. Let $\rho \in \text{Irr}(K)$ be given. Via the isomorphism $\varphi : M_n(\mathfrak{o}_1) \rightarrow K$, $\bar{x} \mapsto 1 + \varpi^{r-1}x$ of Proposition 27, ρ corresponds to a unique $\rho' = \rho \circ \varphi \in \text{Irr}(M_n(\mathfrak{o}_1))$. Since, as an abelian group, $M_n(\mathfrak{o}_1)$ is the direct product of n^2 copies of \mathfrak{o}_1 , there exist characters $\rho_{ij} \in \text{Irr}(\mathfrak{o}_1)$ such that for all $\bar{x} \in M_n(\mathfrak{o}_1)$, $\rho'(\bar{x}) = \prod_{i,j} \rho_{ij}(\bar{x}_{ij})$. Since the characters of \mathfrak{o}_1 are precisely $x \mapsto \psi(ax)$ for $a \in \mathfrak{o}_1$, choose a_{ij} such that $\rho_{ij} = \psi(a_{ij} \cdot -)$. Now define $\beta = (a_{ij})_{ij}^T$. One has $\text{tr}(\beta\bar{x}) = \sum_i \sum_j a_{ij}\bar{x}_{ij}$, therefore

$$\begin{aligned}
\psi_\beta(1 + \varpi^{r-1}x) &= \psi(\text{tr}(\beta\bar{x})) \\
&= \prod_i \prod_j \psi(a_{ij}\bar{x}_{ij}) \\
&= \prod_i \prod_j \rho_{ij}(\bar{x}_{ij}) \\
&= \rho'(\bar{x}) \\
&= \rho(1 + \varpi^{r-1}x).
\end{aligned}$$

We have found $\beta \in M_n(\mathfrak{o}_1)$ such that $\rho = \psi_\beta$, therefore the map $\beta \mapsto \psi_\beta$ is surjective. \square

By Propositions 27 and 28, we have exhibited a non-canonical isomorphism between K and $\text{Irr}(K)$ via ψ .

Further, the action of $\text{GL}_n(\mathfrak{o}_r)$ on $\text{Irr}(K)$ is given by ${}^g\psi_\beta = \psi_{\bar{g}\beta\bar{g}^{-1}}$, where \bar{g} is the image of g in $M_n(\mathfrak{o}_1)$. Indeed, for all $x \in M_n(\mathfrak{o}_r)$,

$$\begin{aligned} {}^g\psi_\beta(1 + \varpi^{r-1}x) &= \psi_\beta(g^{-1}(1 + \varpi^{r-1}x)g) \\ &= \psi_\beta(1 + \varpi^{r-1}g^{-1}xg) \\ &= \psi(\text{tr}(\beta\bar{g}^{-1}\bar{x} \cdot \bar{g})) \\ &= \psi(\text{tr}(\bar{g} \cdot \beta\bar{g}^{-1}\bar{x})) \\ &= \psi_{\bar{g}\beta\bar{g}^{-1}}(1 + \varpi^{r-1}x). \end{aligned}$$

Therefore, when choosing representatives of the orbits, we can assume a canonical form for β , and we call $\rho \in \text{Irr}(\text{GL}_n(\mathfrak{o}_r) \mid \psi_\beta)$ semisimple, nilpotent, etc. if β is semisimple, nilpotent, etc. respectively.

In order to apply Clifford theory, first observe that $\text{Irr}(\text{GL}_n(\mathfrak{o}_r) \mid \psi_\beta)$ is in bijection with $\text{Irr}(\text{Stab}_{\text{GL}_n(\mathfrak{o}_r)}(\psi_\beta) \mid \psi_\beta)$. We claim that it is sufficient to consider β which is nilpotent; for the proof, we follow Hill [30].

Definition 29. Let $\bar{s} \in M_n(\mathfrak{o}_1)$ be semisimple. For $\beta \in M_n(\mathfrak{o}_1)$, write $\beta \sim_{\text{s.s.}} \bar{s}$ if the semisimple part of the Jordan-Chevalley decomposition of β is conjugate to \bar{s} .

Define the set

$$\mathcal{C}_{\text{GL}_n(\mathfrak{o}_r)}(\bar{s}) = \{\rho \in \text{Irr}(\text{GL}_n(\mathfrak{o}_r)) \mid \langle \psi_\beta, \rho|_{K^{r-1}} \rangle \neq 0 \text{ for some } \beta \in M_n(\mathfrak{o}_1) \text{ s.t. } \beta \sim_{\text{s.s.}} \bar{s}\}.$$

In [30], $\mathcal{C}_{\text{GL}_n(\mathfrak{o}_r)}(\bar{s})$ is called the “geometric conjugacy class” of \bar{s} , however we reserve this terminology for the notion of geometric conjugacy introduced in Section 4.1. The following result is proved in the discussion preceding Theorem 2.13 in [30]:

Lemma 30. *For each semisimple $\bar{s} \in M_n(\mathfrak{o}_1)$, there exists a lift $s \in M_n(\mathfrak{o}_r)$ with the property that there exist unramified extensions $\mathfrak{o}^{(j)}$ of \mathfrak{o} such that*

$$C_{\mathrm{GL}_n(\mathfrak{o}_r)}(s) \cong \prod_{j=1}^t \mathrm{GL}_{m_j}(\mathfrak{o}^{(j)}/(\varpi^r)),$$

where $\sum_{j=1}^t m_j = n$, and $t = 1$ if and only if $\bar{s} = aI$ for some $a \in \mathfrak{o}_1$. The $\mathfrak{o}^{(j)}$ and m_j depend only on the decomposition of $\bar{V} = \mathfrak{o}_1^n$ into \bar{s} -invariant subspaces $\bar{V}_1, \dots, \bar{V}_t$.

In order to illustrate the ideas behind the proof of this claim, we give a worked example.

Example 31. Put $\mathfrak{o} = \mathbb{Z}_5$ and

$$\bar{s} = \begin{pmatrix} 1 & & & & \\ & 0 & -1 & & \\ & 1 & 1 & & \\ & & & 0 & -1 \\ & & & 1 & 1 \end{pmatrix} \in M_5(\mathbb{F}_5).$$

Note that \bar{s} is the direct sum of (1) and two copies of the companion matrix of the irreducible polynomial $x^2 - x + 1 \in \mathbb{F}_5[x]$. First, we decompose $\bar{V} = \mathbb{F}_5^5$ into \bar{s} -invariant subspaces:

$$\bar{V} = \langle e_1 \rangle_{\mathbb{F}_5} \oplus \langle e_2, e_3 \rangle_{\mathbb{F}_5} \oplus \langle e_4, e_5 \rangle_{\mathbb{F}_5} = \bar{V}_1 \oplus \bar{V}_2 \oplus \bar{V}_3.$$

The minimal polynomials of \bar{s} restricted to \bar{V}_1 , \bar{V}_2 and \bar{V}_3 are $x - 1$, $x^2 - x + 1$ and $x^2 - x + 1$ respectively. Define the following extensions of \mathbb{Z}_5 :

$$\begin{aligned} \mathbb{Z}_5^{(1)} &= \mathbb{Z}_5[x]/(x - 1) \cong \mathbb{Z}_5, \\ \mathbb{Z}_5^{(2)} &= \mathbb{Z}_5^{(3)} = \mathbb{Z}_5[x]/(x^2 - x + 1). \end{aligned}$$

Since $x^2 - x + 1$ is irreducible modulo 5, by [52, Chapter 1, §6, Corollary 2], $\mathbb{Z}_5^{(2)}$ is a degree 2 unramified extension of \mathbb{Z}_5 , which is also a free \mathbb{Z}_5 -module of rank 2. Write $\theta_1 = x \bmod x - 1 \in \mathbb{Z}_5^{(1)}$ and $\theta_2 = \theta_3 = x \bmod x^2 - x + 1 \in \mathbb{Z}_5^{(2)}$.

For this example, we put $r = 2$. For $1 \leq j \leq 3$, define $\mu_j : \mathbb{Z}_5^{(j)}/(25) \rightarrow \mathbb{Z}_5^{(j)}/(25)$ by $y \mapsto \tilde{\theta}_j y$, where the tilde denotes reduction modulo 25. Then μ_1 is the trivial map on $\mathbb{Z}/25\mathbb{Z}$, while μ_2 is the \mathbb{Z}_5 -module endomorphism of $\mathbb{Z}_5^{(2)}/(25) = (\mathbb{Z}/25\mathbb{Z})[x]/(x^2 - x + 1) =: R$, a free $\mathbb{Z}/25\mathbb{Z}$ -module of rank 2. Under this isomorphism, the multiplication action μ_2 of $\tilde{\theta}_2$ on $\mathbb{Z}_5^{(2)}/(25)$ corresponds to the action of $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ on $(\mathbb{Z}/25\mathbb{Z})^2$. Suppose $f : R^2 \rightarrow R^2$ is a $\mathbb{Z}/25\mathbb{Z}$ -module homomorphism, and let $(\mu_2, \mu_2) : R^2 \rightarrow R^2$ be given by $(\mu_2, \mu_2)(r_1, r_2) = (\tilde{\theta}_2 r_1, \tilde{\theta}_2 r_2)$. Then f is an R -module homomorphism if and only if $f \circ (\mu_2, \mu_2) = (\mu_2, \mu_2) \circ f$, since R is generated by $\mathbb{Z}/25\mathbb{Z}$ and $\tilde{\theta}_2$. Therefore,

$$\mathrm{M}_2(R) \cong \mathrm{End}_R(R^2) \cong C_{\mathrm{M}_4(\mathbb{Z}/25\mathbb{Z})} \begin{pmatrix} 0 & -1 & & \\ & 1 & 1 & \\ & & & 0 & -1 \\ & & & 1 & 1 \end{pmatrix}.$$

Now put

$$s = \begin{pmatrix} 1 & & & & \\ & 0 & -1 & & \\ & 1 & 1 & & \\ & & & 0 & -1 \\ & & & 1 & 1 \end{pmatrix} \in \mathrm{M}_5(\mathbb{Z}/25\mathbb{Z});$$

this is a lift of \bar{s} to a matrix over $\mathbb{Z}/25\mathbb{Z}$. Since the centraliser of a block diagonal matrix is the direct product of the centralisers of the blocks,

$$\begin{aligned} C_{\mathrm{M}_3(\mathbb{Z}/25\mathbb{Z})}(s) &= C_{\mathrm{M}_1(\mathbb{Z}/25\mathbb{Z})}(\mathbf{1}) \times C_{\mathrm{M}_4(\mathbb{Z}/25\mathbb{Z})} \begin{pmatrix} 0 & -1 & & \\ & 1 & 1 & \\ & & & 0 & -1 \\ & & & 1 & 1 \end{pmatrix} \\ &\cong \mathbb{Z}/25\mathbb{Z} \times \mathrm{M}_2(\mathbb{Z}_5^{(2)}/(25)). \end{aligned}$$

Taking units, we get

$$C_{\mathrm{GL}_3(\mathbb{Z}/25\mathbb{Z})}(s) \cong \mathrm{GL}_1(\mathbb{Z}/25\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}_5^{(2)}/(25)).$$

Thus, in the case where $\mathfrak{o} = \mathbb{Z}_5$ and $r = 2$, we have found a lift of \bar{s} to a matrix over \mathfrak{o}_r whose centraliser is a product of $\mathrm{GL}_{n_j}(\mathfrak{o}^{(j)}/\varpi^r)$ where $\mathfrak{o}^{(j)}$ are unramified extensions of \mathfrak{o} , depending only on the factorisation of the characteristic polynomial of \bar{s} .

Note that our choice of lift s was not unique; we could have chosen any lift of $x^2 - x + 1 \in \mathbb{F}_5[x]$ to $\mathbb{Z}_5[x]$ in the definition of $\mathbb{Z}_5^{(2)}$ and $\mathbb{Z}_5^{(3)}$. Consequently, the 2×2 diagonal blocks of s can be chosen to be the companion matrices of arbitrary lifts of $x^2 - x + 1$ to $(\mathbb{Z}/25\mathbb{Z})[x]$.

We have the following Jordan decomposition for representations [30, Theorem 2.13]:

Proposition 32. *Let $s \in M_n(\mathfrak{o}_r)$, and suppose that its reduction $\bar{s} \in M_n(\mathfrak{o}_1)$ is semisimple. There is a bijection*

$$\mathcal{C}_{\mathrm{GL}_n(\mathfrak{o}_r)}(\bar{s}) \rightarrow \mathcal{C}_{\mathrm{GL}_n(\mathfrak{o}_r)(s)}(0); \quad \rho \mapsto \rho_{\mathrm{nil}},$$

where we interpret the set $\mathcal{C}_{\mathrm{GL}_n(\mathfrak{o}_r)(s)}(0)$ to consist of tensor products of representations in each $\mathcal{C}_{\mathrm{GL}_{m_j}(\mathfrak{o}^{(j)}/(\varpi^r))}(0)$. Further, for all $\rho \in \mathcal{C}_{\mathrm{GL}_n(\mathfrak{o}_r)}(\bar{s})$,

$$\dim \rho = (\dim \rho_{\mathrm{nil}}) \cdot \frac{|\mathrm{GL}_n(\mathbb{F}_q)|}{\prod_j |\mathrm{GL}_{m_j}(\mathbb{F}_{q^{\lambda_j}})|} q^{(r-2) \binom{n^2 - \sum_{j=1}^r \lambda_j m_j^2}{2}},$$

where λ_j is the degree of the extension $\mathfrak{o}^{(j)}/\mathfrak{o}$.

In the case where $r = 2$ and $i = 1$, ψ_β has an extension $\hat{\psi}_\beta \in \mathrm{Irr}(\mathrm{Stab}_{\mathrm{GL}_n(\mathfrak{o}_2)}(\psi_\beta))$ [55, Proposition 2.2]. This extension is not canonical in the sense that the proof relies on a choice of extension of a character $1 + \varpi \mathfrak{o}_2 \rightarrow \mathbb{C}^\times$ to \mathfrak{o}_2^\times . By Proposition 22, the representations of the stabiliser containing ψ_β are given by

$$\mathrm{Irr}(\mathrm{Stab}(\psi_\beta) \mid \psi_\beta) = \{\hat{\psi}_\beta \otimes \theta \mid \bar{\theta} \in \mathrm{Irr}(\mathrm{Stab}(\psi_\beta)/K^1)\}.$$

By [32, Proposition 2.3(2)], we have $\mathrm{Stab}(\psi_\beta)/K^1 \cong C_{\mathrm{GL}_n(\mathfrak{o}_1)}(\beta)$, so the representations of $\mathrm{GL}_n(\mathfrak{o}_2)$ are determined by those of $C_{\mathrm{GL}_n(\mathfrak{o}_1)}(\beta)$, for $\beta \in M_n(\mathfrak{o}_1)$. When β

is nilpotent, $C_{\mathrm{GL}_n(\mathfrak{o}_1)}(\beta)$ is isomorphic to the group of \mathfrak{o} -automorphisms of a finite \mathfrak{o} -module. The following definition is due to Onn [50].

Definition 33. Let $\lambda = (\ell_1, \dots, \ell_n)$ be a partition and \mathfrak{o} the ring of integers of a non-Archimedean local field with finite residue field. Define $G_{\lambda, \mathfrak{o}}$ to be the automorphism group of the \mathfrak{o} -module $\mathfrak{o}_{\ell_1} \oplus \dots \oplus \mathfrak{o}_{\ell_n}$:

$$G_{\lambda, \mathfrak{o}} = \mathrm{Aut}_{\mathfrak{o}}(\mathfrak{o}_{\ell_1} \oplus \dots \oplus \mathfrak{o}_{\ell_n}).$$

By choosing the basis $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ of $\mathfrak{o}_{\ell_1} \oplus \dots \oplus \mathfrak{o}_{\ell_n}$, we can make the identification

$$G_{\lambda, \mathfrak{o}} = \left(\mathrm{Hom}_{\mathfrak{o}}(\mathfrak{o}_{\ell_j}, \mathfrak{o}_{\ell_i}) \right)_{i,j}^{\times},$$

where

$$\mathrm{Hom}_{\mathfrak{o}}(\mathfrak{o}_{\ell_j}, \mathfrak{o}_{\ell_i}) \cong \begin{cases} \mathfrak{o}_{\ell_j} & \text{if } i \geq j \\ \mathfrak{p}_{\ell_i}^{\ell_i - \ell_j} & \text{if } i < j \end{cases}$$

given by $f \mapsto f(1)$. Multiplication of group elements in the matrix form is done by usual matrix multiplication, taking lifts or reductions as appropriate. Indeed,

$$f_{ik} \circ g_{kj}(1) = f_{ik}(g_{kj}(1)) = \widehat{g_{kj}(1)} \cdot f_{ik}(1),$$

where the hat denotes a lift to \mathfrak{o} . Since \mathfrak{o} acts via its quotient modulo \mathfrak{p}^{ℓ_i} on \mathfrak{o}_{ℓ_i} , $\widehat{g_{kj}(1)} \cdot f_{ik}(1)$ is to be interpreted as the product in \mathfrak{o}_{ℓ_i} of $f_{ik}(1)$ and either the reduction or an arbitrary lift of $g_{kj}(1)$ to \mathfrak{o}_{ℓ_i} .

Proposition 34. [55, Proposition 4.11] *Let λ be a partition of n and β a nilpotent Jordan matrix of type λ . Then there is an isomorphism of groups*

$$C_{\mathrm{GL}_n(\mathfrak{o}_1)}(\beta) \cong G_{\lambda, \mathfrak{o}_1[[t]]}.$$

Proof. We sketch the construction of the isomorphism given in [55]. Write $\lambda = (\ell_1, \dots, \ell_n)$. The centraliser $C_{\mathrm{M}_n(\mathfrak{o}_1)}(\beta)$ consists of *block upper Toeplitz matrices* $(T_{ij})_{1 \leq i, j \leq n}$ of type λ , where $T_{ij} \in \mathrm{M}_{\ell_i \times \ell_j}(\mathfrak{o}_1)$. That is, T_{ii} has the form of an

upper Toeplitz matrix:

$$\begin{pmatrix} a_0 & a_1 & \dots & a_{\ell_j-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & a_0 & a_1 \\ 0 & \dots & 0 & a_0 \end{pmatrix},$$

and T_{ij} takes the form

$$T_{ij} = \begin{pmatrix} 0 & \tilde{T}_{ij} \end{pmatrix}$$

where $i > j$ and \tilde{T}_{ij} is an upper Toeplitz matrix, or

$$T_{ij} = \begin{pmatrix} \tilde{T}_{ij} \\ 0 \end{pmatrix}$$

where $i < j$ and \tilde{T}_{ij} is an upper Toeplitz matrix. The centraliser $C_{\mathrm{GL}_n(\mathfrak{o}_1)}(\beta)$ is the group of invertible block upper Toeplitz matrices of type λ over \mathfrak{o}_1 .

Let $i > j$. Let $f_{ij} \in \mathrm{Hom}_{\mathfrak{o}_1[[t]]}(\mathfrak{o}_1[t]/t^{\ell_i}, \mathfrak{o}_1[t]/t^{\ell_j})$ and write $f(1) = a_0 + a_1 t + \dots + a_{\ell_j-1} t^{\ell_j-1}$ with $a_i \in \mathfrak{o}_1$. Define

$$T(f_{ij}) = \begin{pmatrix} 0 & \dots & 0 & a_0 & a_1 & \dots & a_{\ell_j-1} \\ \vdots & & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \vdots & & & a_0 & a_1 \\ 0 & \dots & 0 & 0 & \dots & 0 & a_0 \end{pmatrix}.$$

Similarly, replace the zero columns to the left with zero rows on the bottom of the Toeplitz matrix in the case $i < j$. Writing an element $f \in G_{\lambda, \mathfrak{o}_1[[t]]}$ in matrix form (f_{ij}) with $f_{ij} \in \mathrm{Hom}_{\mathfrak{o}_1[[t]]}(\mathfrak{o}_1[t]/t^{\ell_i}, \mathfrak{o}_1[t]/t^{\ell_j})$, map f to the matrix $(T(f_{ij}))_{i,j}$. This defines an isomorphism from $G_{\lambda, \mathfrak{o}_1[[t]]}$ to the group of invertible block upper Toeplitz matrices of type λ over \mathfrak{o}_1 , as required. \square

Therefore, if β is nilpotent, then $\mathrm{Irr}(\mathrm{GL}_n(\mathfrak{o}_2) \mid \psi_\beta)$ is determined by $\mathrm{Irr}(G_{\lambda, \mathfrak{o}_1[[t]]})$, where β has Jordan canonical form of type λ .

In fact, we show that $\mathcal{R}_{\mathrm{GL}_n(\mathfrak{o}_2)}(\mathcal{D})$ can be obtained from $\mathcal{R}_{G_{\lambda, \mathfrak{o}'_1[[t]]}}(\mathcal{D})$ for certain choices of λ and \mathfrak{o}' . The following result extends existing work of Singla [55, §7.1], the difference being that we only assume knowledge about centralisers of nilpotent

matrices (which are isomorphic to some $G_{\lambda, \mathfrak{o}}$ by Proposition 34). A similar result was also obtained where \mathfrak{o} has positive characteristic by Aubert, Onn, Prasad and Stasinski [3, Theorem 6.2].

Proposition 35. *The representation zeta polynomial of $\mathrm{GL}_n(\mathfrak{o}_2)$ is determined by*

1. $\mathcal{R}_{G_{\lambda, \mathfrak{o}_1[[t]]}}(\mathcal{D})$ for all partitions λ of n ,
2. $\mathcal{R}_{G_{\lambda, \mathfrak{o}'_1[[t]]}}(\mathcal{D})$ for all partitions λ of natural numbers less than n and all finite unramified extensions $\mathfrak{o}'/\mathfrak{o}$ of degree at most n .

If the $G_{\lambda, \mathfrak{o}_1[[t]]}$ and $G_{\lambda, \mathfrak{o}'_1[[t]]}$ have the polynomial property as in Conjecture 5, then the same is true for $\mathrm{GL}_n(\mathfrak{o}_2)$.

Before giving the proof, we prove a short lemma:

Lemma 36. *Let $a \in \mathrm{M}_n(\mathfrak{o}_1)$. Then ψ_{aI_n} extends to $\mathrm{GL}_n(\mathfrak{o}_r)$.*

Proof. We have $\psi_{aI_n}(1 + \varpi^{r-1}x) = \psi(\mathrm{tr}(a\bar{x}))$. Note also that $\det(1 + \varpi^{r-1}x) = 1 + \varpi^{r-1} \mathrm{tr}(x) \in \mathfrak{o}_r^\times$, therefore $\overline{\det}(1 + \varpi^{r-1}x) = 1 \in \mathfrak{o}_1^\times$, where $\overline{\det} : K \rightarrow \mathfrak{o}_1^\times$ denotes the composition of $\det : K \rightarrow \mathfrak{o}_r^\times$ with the canonical map $\mathfrak{o}_r^\times \rightarrow \mathfrak{o}_1^\times$. We see that ψ_{aI_n} is trivial on $\ker(\overline{\det} : K \rightarrow \mathfrak{o}_1^\times)$, therefore by the Fundamental Homomorphism Theorem, there exists a homomorphism $\chi_a : \mathfrak{o}_1^\times \rightarrow \mathbb{C}^\times$ that makes the diagram

$$\begin{array}{ccc} K & \xrightarrow{\overline{\det}} & \mathfrak{o}_1^\times \\ & \searrow \psi_{aI_n} & \vdots \chi_a \\ & & \mathbb{C}^\times \end{array}$$

commute. Then $\hat{\psi}_{aI_n} = \chi_a \circ (\overline{\det} : \mathrm{GL}_n(\mathfrak{o}_r) \rightarrow \mathfrak{o}_1^\times)$ extends ψ_{aI_n} . □

Proof of Proposition 35. We give an argument due to Hill [30]. Let $\bar{s} \in \mathrm{M}_n(\mathfrak{o}_1)$ be semisimple. By Lemma 30, for every semisimple $\bar{s} \in \mathrm{M}_n(\mathfrak{o}_1)$, there is a lift $s \in \mathrm{M}_n(\mathfrak{o}_2)$ with the property that there exist unramified extensions $\mathfrak{o}^{(j)}/\mathfrak{o}$ of degree at most n , and depending only on \bar{s} , such that

$$C_{\mathrm{GL}_n(\mathfrak{o}_2)}(s) \cong \prod_{j=1}^t \mathrm{GL}_{m_j}(\mathfrak{o}^{(j)}/(\varpi^2)),$$

where $\sum_{j=1}^t m_j = n$, and $t = 1$ if and only if $\bar{s} = aI$ for some $a \in \mathfrak{o}_1$. By Theorem 2.13 of [30], there is a bijection

$$\mathcal{C}_{\mathrm{GL}_n(\mathfrak{o}_2)}(\bar{s}) \rightarrow \mathcal{C}_{\mathrm{GL}_n(\mathfrak{o}_2)(s)}(0); \quad \rho \mapsto \rho_{\mathrm{nil}},$$

where

$$\mathcal{C}_{\mathrm{GL}_n(\mathfrak{o}_2)(s)}(0) = \{\sigma_1 \otimes \cdots \otimes \sigma_t \mid \sigma_j \in \mathcal{C}_{\mathrm{GL}_{m_j}(\mathfrak{o}^{(j)}/(\varpi^2))}(0)\}.$$

By Proposition 32, for all $\rho \in \mathcal{C}_{\mathrm{GL}_n(\mathfrak{o}_2)}(\bar{s})$, $\dim \rho / \dim \rho_{\mathrm{nil}}$ is a polynomial in $q = |\mathfrak{o}_1|$ which depends only on the decomposition of \mathfrak{o}_1^n into \bar{s} -invariant subspaces and is independent of ρ . This allows us to work inductively; if we know $\mathcal{C}_{\mathrm{GL}_m(\mathfrak{o}'_2)}(0)$ for all $m < n$ and every unramified extension $\mathfrak{o}'/\mathfrak{o}$ of degree at most n , then we can construct $\mathcal{C}_{\mathrm{GL}_n(\mathfrak{o}_2)}(\bar{s})$ where \bar{s} is not a scalar matrix. On the other hand, if $\rho \in \mathrm{Irr}(\mathrm{GL}_n(\mathfrak{o}_2) \mid \psi_{aI+N})$ where N is nilpotent and $a \in \mathfrak{o}_1$, then by Lemma 36, ψ_{-aI} extends to $\hat{\psi}_{-aI} \in \mathrm{Irr}(\mathrm{GL}_n(\mathfrak{o}_2))$, and $\rho \otimes \hat{\psi}_{-aI} \in \mathrm{Irr}(\mathrm{GL}_n(\mathfrak{o}_2) \mid \psi_N)$ has the same dimension as ρ .

This completes the construction of $\mathcal{C}_{\mathrm{GL}_n(\mathfrak{o}_2)}(\bar{s})$ for all semisimple \bar{s} , given knowledge of $\mathcal{C}_{\mathrm{GL}_n(\mathfrak{o}_2)}(0)$ and $\mathcal{C}_{\mathrm{GL}_m(\mathfrak{o}'_2)}(0)$ for all $m < n$ and unramified extensions $\mathfrak{o}'/\mathfrak{o}$ of degree at most n . Since for each m , the dimensions of the representations in $\mathcal{C}_{\mathrm{GL}_m(\mathfrak{o}_2)}(0)$ are determined by $\mathcal{R}_{G_{\lambda, \mathfrak{o}}}(\mathcal{D})$ for all partitions λ of m , the result follows. \square

The Representations of $G_{\lambda, \mathfrak{o}}$

3.1 The Representations of $G_{(\ell, 1^n), \mathfrak{o}}$

We construct the representations of $G_{(\ell, 1^n), \mathfrak{o}}$; these results can be found in [36, Propositions 3 and 4, Theorem 5]. This work was done independently of, and extends, the existing construction of Crisp, Meir and Onn [14, §7.2] which was stated for positive characteristic. However, we consider the arbitrary characteristic case and give a formula for the representation zeta polynomial of $G_{(\ell, 1^n), \mathfrak{o}}$. This gives an affirmative answer to Conjecture 5 for all partitions of the form $(\ell, 1^n)$.

3.1.1 The Representation Zeta Polynomial of $G_{(\ell, 1^n), \mathfrak{o}}$

In order to describe $\mathcal{R}_{G_{(\ell, 1^n), \mathfrak{o}}}(\mathcal{D})$, we need to introduce two families of groups whose representation zeta polynomials we derive:

Proposition 37. *The representation zeta polynomial of*

$$P_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{pmatrix} \cong \mathfrak{o}_1^{n-1} \rtimes G_{(1^{n-1}), \mathfrak{o}}$$

is given inductively by $\mathcal{R}_{P_1}(\mathcal{D}) = \mathcal{D}$ and for $n \geq 2$,

$$\mathcal{R}_{P_n}(\mathcal{D}) = \mathcal{R}_{P_{n-1}}(\mathcal{D}^{q^{n-1}-1}) + \mathcal{R}_{\mathrm{GL}_{n-1}(\mathfrak{o}_1)}(\mathcal{D}).$$

The entries marked $*$ are from the finite field \mathfrak{o}_1 of order q , therefore P_n depends only on $q = |\mathfrak{o}_1|$. However, to avoid cumbersome notation, we do not write q as a subscript.

Proposition 38. *The representation zeta polynomial of*

$$T_n = \begin{pmatrix} 1 & * & * & \dots & * \\ 0 & 1 & 0 & \dots & 0 \\ 0 & * & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \dots & * \end{pmatrix} \cong \mathfrak{o}_1^{n-1} \rtimes (\mathfrak{o}_1^{n-2} \rtimes G_{(1^{n-2}), \mathfrak{o}})$$

is given inductively by $\mathcal{R}_{T_2}(\mathcal{D}) = q\mathcal{D}$ and for $n \geq 3$,

$$\mathcal{R}_{T_n}(\mathcal{D}) = (q-1)\mathcal{R}_{\mathrm{GL}_{n-2}(\mathfrak{o}_1)}(\mathcal{D}^{q^{n-2}}) + \mathcal{R}_{T_{n-1}}(\mathcal{D}^{q^{n-2}-1}) + \mathcal{R}_{P_{n-1}}(\mathcal{D}).$$

Knowing $\mathcal{R}_{P_n}(\mathcal{D})$ and $\mathcal{R}_{T_n}(\mathcal{D})$, one can calculate $\mathcal{R}_{G_{(\ell, 1^n), \mathfrak{o}}}(\mathcal{D})$ as follows:

Theorem 39. *The representation zeta polynomial of $G_{(\ell, 1^n), \mathfrak{o}}$ is given by*

$$\mathcal{R}_{G_{(\ell, 1^n), \mathfrak{o}}}(\mathcal{D}) = q^{\ell-2}(q-1)^2 \mathcal{R}_{\mathrm{GL}_n(\mathfrak{o}_1)}(\mathcal{D}^{q^n}) + q^{\ell-2}(q-1) \mathcal{R}_{((\mathfrak{o}_1^\ell \times \mathfrak{o}_1^n) \rtimes \mathrm{GL}_n(\mathfrak{o}_1))}(\mathcal{D}),$$

where

$$\begin{aligned} \mathcal{R}_{(\mathfrak{o}_1^n \times \mathfrak{o}_1^n) \rtimes \mathrm{GL}_n(\mathfrak{o}_1)}(\mathcal{D}) &= \mathcal{R}_{\mathrm{GL}_n(\mathfrak{o}_1)}(\mathcal{D}) + 2\mathcal{R}_{P_n}(\mathcal{D}^{q^n-1}) + \mathcal{R}_{T_n}(\mathcal{D}^{(q^n-1)(q^{n-1}-1)}) \\ &\quad + (q-1)\mathcal{R}_{\mathrm{GL}_{n-1}(\mathfrak{o}_1)}(\mathcal{D}^{q^{n-1}(q^n-1)}). \end{aligned}$$

In particular, Conjecture 5 holds for $\lambda = (\ell, 1^n)$.

Using the method of proof of [55, Lemma 7.7] (an argument attributed to U. Onn), we derive the representation zeta polynomial of $G_{(\ell, 1^n), \mathfrak{o}}$, thus proving Theorem 39.

Proposition 40. [10] *Let G be a finite group and N a normal subgroup of G such that $V = G/N$ is an elementary abelian p -group, regarded as an \mathbb{F}_p -vector space. Let $\chi \in \mathrm{Irr}(N)$ be invariant in G and suppose that the bilinear form*

$$\langle \cdot, \cdot \rangle_\chi : V \times V \rightarrow \mu_p(\mathbb{C}) \cong \mathbb{F}_p; \quad \langle g_1 N, g_2 N \rangle = \chi([g_1, g_2])$$

is non-degenerate. Then there exists a unique $\rho_\chi \in \mathrm{Irr}(G)$ such that $\langle \chi, \rho_\chi|_N \rangle \neq 0$. Moreover, $\rho_\chi|_N = e\chi$ for some $e \geq 1$, and $\dim \rho_\chi = [G : N]^{\frac{1}{2}}$.

Outline of the proof. Choose a maximal isotropic subspace J/N for the bilinear form. The representation ρ_χ is obtained by extending χ to J and then inducing to G . One can show that ρ_χ does not depend on the choice of J or the extension of χ . \square

Lemma 41. [55, Lemma 5.4] *Let G be a finite group, $N \triangleleft G$ and $M \leq G$ such that $G = NM$. Let ψ_1, ψ_2 be one-dimensional representations of N and M respectively such that ψ_1 is invariant in M and $\psi_1|_{M \cap N} = \psi_2|_{M \cap N}$. Then $\psi_1 \psi_2 \in \mathrm{Irr}(G)$ defined by $\psi_1 \psi_2(nm) = \psi_1(n)\psi_2(m)$ is the unique one-dimensional representation of G extending both ψ_1 and ψ_2 .*

Proof of Theorem 39. We can use Proposition 40 to construct representations of $G_{(\ell, 1^n), \mathfrak{o}}$, following the method in [55, Lemma 7.7]. Denote by \mathfrak{p}_ℓ the image of \mathfrak{p} in \mathfrak{o}_ℓ . Fix a

non-trivial character $\psi : \mathfrak{o}_1 \rightarrow \mathbb{C}^\times$ and define the following groups:

$$G = G_{(\ell, 1^n), \mathfrak{o}} = \begin{pmatrix} \mathfrak{o}_\ell^\times & \mathfrak{p}_\ell^{\ell-1} & \cdots & \mathfrak{p}_\ell^{\ell-1} \\ \mathfrak{o}_1 & & & \\ \vdots & & \text{GL}_n(\mathfrak{o}_1) & \\ \mathfrak{o}_1 & & & \end{pmatrix}, H = \begin{pmatrix} 1 + \mathfrak{p}_\ell^{\ell-1} & \mathfrak{p}_\ell^{\ell-1} & \cdots & \mathfrak{p}_\ell^{\ell-1} \\ \mathfrak{o}_1 & 1 & & \\ \vdots & & \ddots & \\ \mathfrak{o}_1 & & & 1 \end{pmatrix}.$$

Then H is a normal subgroup of G with centre

$$Z(H) = \begin{pmatrix} 1 + \mathfrak{p}_\ell^{\ell-1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \cong \mathfrak{o}_1,$$

such that $H/Z(H)$ is an elementary abelian p -group. The non-trivial irreducible representations of $Z(H)$ are parameterised for $z \in Z(H), z \neq 1$ by

$$\psi_z \begin{pmatrix} 1 + \varpi^{\ell-1}v & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \psi(\overline{xv}),$$

where the bar denotes reduction modulo ϖ , and

$$z = \begin{pmatrix} 1 + \varpi^{\ell-1}x & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

We claim that the alternating bilinear form on $H/Z(H)$ given by

$$\langle \overline{h_1}, \overline{h_2} \rangle_{\psi_z} = \psi_z([h_1, h_2])$$

is non-degenerate.

To prove this claim, we choose two arbitrary elements $h_1, h_2 \in H$:

$$h_1 = \begin{pmatrix} 1 + \varpi^{\ell-1}u & \varpi^{\ell-1}v \\ w & 1 \end{pmatrix}, h_2 = \begin{pmatrix} 1 + \varpi^{\ell-1}u' & \varpi^{\ell-1}v' \\ w' & 1 \end{pmatrix},$$

where $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)^T$ (and similar notation for v' and w').

We compute the commutator $[h_1, h_2]$ explicitly:

$$h_1 = \begin{pmatrix} 1 + \varpi^{\ell-1}u & \varpi^{\ell-1}v \\ w & 1 \end{pmatrix}, \quad h_1^{-1} = \begin{pmatrix} 1 + \varpi^{\ell-1}(-u + \sum v_i \tilde{w}_i) & -\varpi^{\ell-1}v \\ -w & 1 \end{pmatrix},$$

$$\begin{aligned} h_1 h_2 &= \begin{pmatrix} 1 + \varpi^{\ell-1}u & \varpi^{\ell-1}v \\ w & 1 \end{pmatrix} \begin{pmatrix} 1 + \varpi^{\ell-1}u' & \varpi^{\ell-1}v' \\ w' & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 + \varpi^{\ell-1}(u + u' + v\tilde{w}') & \varpi^{\ell-1}(v + v') \\ w + w' & 1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} h_1^{-1} h_2^{-1} &= \begin{pmatrix} 1 + \varpi^{\ell-1}(-u + v\tilde{w}) & -\varpi^{\ell-1}v \\ -w & 1 \end{pmatrix} \begin{pmatrix} 1 + \varpi^{\ell-1}(-u' + v'\tilde{w}') & -\varpi^{\ell-1}v' \\ -w' & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 + \varpi^{\ell-1}(-(u + u') + v\tilde{w} + v'\tilde{w}' + v\tilde{w}') & -\varpi^{\ell-1}(v + v') \\ -(w + w') & 1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} [h_1, h_2] &= \begin{pmatrix} 1 + \varpi^{\ell-1}(v\tilde{w} + v'\tilde{w}' + 2v\tilde{w}' - (v + v')(\tilde{w} + \tilde{w}')) & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 + \varpi^{\ell-1}(v\tilde{w}' - v'\tilde{w}) \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 + \varpi^{\ell-1} \sum_{i=1}^n (v_i w'_i - v'_i w_i) \\ 1 \end{pmatrix}. \end{aligned}$$

Suppose $\langle \overline{h_1}, \overline{h_2} \rangle_{\psi_z} = 0$ for all h_2 . Then

$$\psi(x(\overline{v}w' - \overline{v}'w)) = 1$$

for all v' and w' . Choose $v' = 0$ and $w' = (w'_1, 0, \dots, 0)^T$ to get $\psi(x\overline{v}_1 w'_1) = 0$ for all w'_1 . Since $x \neq 0$ and ψ is non-trivial, we must have $\overline{v}_1 = 0$, i.e. $v_1 \equiv 0 \pmod{\varpi}$. Similarly, all other $v_i \equiv 0 \pmod{\varpi}$ and $w_i = 0$. Therefore, $h_1 \in Z(H)$, so $\overline{h_1} = 1$. We conclude that $\langle \cdot, \cdot \rangle_{\psi_z}$ is non-degenerate, as claimed.

Applying Proposition 40, there are $q - 1$ pairwise inequivalent irreducible representations of H lying over the $q - 1$ distinct non-trivial characters of $Z(H)$, namely ρ_χ for each non-trivial character $\chi : Z(H) \rightarrow \mathbb{C}^\times$. We see that $Z(H) \leq Z(G)$, indeed suppose that

$$h = \begin{pmatrix} 1 + \varpi^{\ell-1}u & \\ & 1 \end{pmatrix} \in Z(H), \quad g = \begin{pmatrix} u' & \varpi^{\ell-1}v' \\ w' & D' \end{pmatrix} \in G.$$

Then

$$hg = \begin{pmatrix} u' + \varpi^{\ell-1}uu' & \varpi^{\ell-1}v' \\ w' & D' \end{pmatrix} = gh.$$

Since $g \in G$ was arbitrary, $Z(H) \leq Z(G)$. In particular, $Z(H)$ is normal in G . For $g \in G, h \in Z(H)$,

$${}^g\rho_\chi(h) = \rho_\chi(g^{-1}hg) = \rho_\chi(h),$$

therefore $\text{Res}_{Z(H)}^H {}^g\rho_\chi = \text{Res}_{Z(H)}^H \rho_\chi = \chi$. Thus, ${}^g\rho_\chi = \rho_\chi$ and ρ_χ is invariant in G .

In the following discussion, fix a non-trivial character $\chi \in \text{Irr}(Z(H))$. The subgroup

$$\overline{H^{\text{iso}}} = \begin{pmatrix} 1 & & & \\ \mathfrak{o}_1 & 1 & & \\ \vdots & & \ddots & \\ \mathfrak{o}_1 & & & 1 \end{pmatrix} \leq G/Z(H)$$

is a maximal isotropic subspace for $\langle \cdot, \cdot \rangle_{\psi_z}$. Consider the inverse image

$$H^{\text{iso}} = \begin{pmatrix} 1 + \mathfrak{p}_\ell^{\ell-1} & & & \\ \mathfrak{o}_1 & 1 & & \\ \vdots & & \ddots & \\ \mathfrak{o}_1 & & & 1 \end{pmatrix}$$

and define the character $\chi^{\text{iso}} \in \text{Irr}(H^{\text{iso}})$ by

$$\chi^{\text{iso}} \begin{pmatrix} u & \\ w & 1 \end{pmatrix} = \chi(u).$$

Choose an extension $\tilde{\chi}$ of χ to $\mathfrak{o}_\ell^\times \times \text{GL}_n(\mathfrak{o}_1) =: M$. Define $G^{\text{iso}} = H^{\text{iso}}M$ so that

$$G^{\text{iso}} = \begin{pmatrix} \mathfrak{o}_\ell^\times & 0 & \dots & 0 \\ \mathfrak{o}_1 & & & \\ \vdots & & \text{GL}_n(\mathfrak{o}_1) & \\ \mathfrak{o}_1 & & & \end{pmatrix}.$$

Observe that H^{iso} is normal in G^{iso} . Further, we show that χ^{iso} is invariant in M , and that $\chi^{\text{iso}}|_{H^{\text{iso}} \cap M} = \tilde{\chi}|_{H^{\text{iso}} \cap M}$ (for then we can apply Lemma 41).

Let $m = \begin{pmatrix} y & \\ & D \end{pmatrix} \in M$. Then

$$\begin{aligned} {}^m\chi^{\text{iso}} \begin{pmatrix} u & \\ w & 1 \end{pmatrix} &= \chi^{\text{iso}} \left(\begin{pmatrix} y & \\ & D \end{pmatrix} \begin{pmatrix} u & \\ w & 1 \end{pmatrix} \begin{pmatrix} y^{-1} & \\ & D^{-1} \end{pmatrix} \right) \\ &= \chi^{\text{iso}} \begin{pmatrix} u & \\ \bar{y}^{-1}Dw & 1 \end{pmatrix} \\ &= \chi(u). \end{aligned}$$

Therefore, χ^{iso} is invariant in M . Finally, $H^{\text{iso}} \cap M = Z(H)$, therefore

$$\chi^{\text{iso}}|_{H^{\text{iso}} \cap M} = \chi = \tilde{\chi}|_{H^{\text{iso}} \cap M}.$$

Thus, by Lemma 41, we get a unique one-dimensional representation $\chi^{\text{iso}} \tilde{\chi} : G^{\text{iso}} \rightarrow \mathbb{C}^\times$ extending both χ^{iso} and $\tilde{\chi}$.

We have $\dim \rho_\chi = [H : Z(H)]^{\frac{1}{2}} = q^n$. Define $\rho_{\chi^{\text{iso}}} = \text{Ind}_{G^{\text{iso}}}^G \chi^{\text{iso}} \tilde{\chi}$ and note also that $\dim \rho_{\chi^{\text{iso}}} = q^n$. We claim that $\rho_{\chi^{\text{iso}}}|_H$ contains ρ_χ .

One possible proof of this claim is by a similar argument as in [55, Lemma 7.7]. Since $\rho_{\chi^{\text{iso}}}$ contains χ , $\rho_{\chi^{\text{iso}}}|_H$ also contains χ . Since ρ_χ is the unique irreducible representation of H containing χ , $\rho_{\chi^{\text{iso}}}|_H$ contains ρ_χ .

Alternatively, by the Mackey formula,

$$\rho_{\chi^{\text{iso}}}|_H = \text{Res}_H^G \text{Ind}_{G^{\text{iso}}}^G \chi^{\text{iso}} \tilde{\chi} = \bigoplus_{s \in H \backslash G / G^{\text{iso}}} \text{Ind}_{s G^{\text{iso}} \cap H}^H \text{Res}_{s G^{\text{iso}} \cap H}^{s G^{\text{iso}}} (\chi^{\text{iso}} \tilde{\chi}).$$

Choosing a set of double coset representatives which contains 1, the summand corresponding to $s = 1$ is

$$\begin{aligned} \text{Ind}_{H^{\text{iso}}}^H \text{Res}_{H^{\text{iso}}}^{G^{\text{iso}}} (\rho) &= \text{Ind}_{H^{\text{iso}}}^H \text{Res}_{H^{\text{iso}}}^{G^{\text{iso}}} (\chi^{\text{iso}} \tilde{\chi}) \\ &= \text{Ind}_{H^{\text{iso}}}^H \chi^{\text{iso}} \\ &= \rho_\chi, \end{aligned}$$

by the construction of ρ_χ (see the proof of Proposition 40). Therefore, $\rho_{\chi^{\text{iso}}}|_H$ contains ρ_χ .

Furthermore, $\rho_{\chi^{\text{iso}}}$ and ρ_χ have equal dimension, therefore $\rho_{\chi^{\text{iso}}}|_H = \rho_\chi$, that is, ρ_χ extends to $\rho_{\chi^{\text{iso}}} \in \text{Irr}(G)$.

By Proposition 22,

$$\text{Irr}(G \mid \rho_\chi) = \{\rho_{\chi^{\text{iso}}} \otimes \phi \mid \phi \in \text{Irr}(G/H)\},$$

where $G/H \cong \mathfrak{o}_{\ell-1}^\times \times \text{GL}_n(\mathfrak{o}_1)$. Therefore,

$$\begin{aligned} \sum_{\substack{\chi \in \text{Irr}(Z(H)) \\ \chi \neq 1}} \mathcal{R}_{G|\rho_\chi}(\mathcal{D}) &= (q-1) \mathcal{R}_{\mathfrak{o}_{\ell-1}^\times \times \text{GL}_n(\mathfrak{o}_1)}(\mathcal{D}^{q^n}) \\ &= q^{\ell-2} (q-1)^2 \mathcal{R}_{\text{GL}_n(\mathfrak{o}_1)}(\mathcal{D}^{q^n}). \end{aligned}$$

This gives an expression for the representation zeta polynomial of $G_{(\ell,1^n),\mathfrak{o}}$:

$$\mathcal{R}_{G_{(\ell,1^n),\mathfrak{o}}}(\mathcal{D}) = q^{\ell-2}(q-1)^2 \mathcal{R}_{\mathrm{GL}_n(\mathfrak{o}_1)}(\mathcal{D}^{q^n}) + \mathcal{R}_{G/Z(H)}(\mathcal{D}).$$

The quotient $G/Z(H)$ splits as the direct product

$$G/Z(H) = \left\{ \begin{pmatrix} u & & & \\ & \bar{u}I_n & & \\ & & & \\ & & & \end{pmatrix} \middle| u \in \mathfrak{o}_{\ell-1}^\times \right\} \times \begin{pmatrix} 1 & \mathfrak{p}_\ell^{\ell-1} & \dots & \mathfrak{p}_\ell^{\ell-1} \\ \mathfrak{o}_1 & & & \\ \vdots & & \mathrm{GL}_n(\mathfrak{o}_1) & \\ \mathfrak{o}_1 & & & \end{pmatrix}$$

which we can write as $\mathfrak{o}_{\ell-1}^\times \times ((\mathfrak{o}_1^n \times \mathfrak{o}_1^n) \rtimes \mathrm{GL}_n(\mathfrak{o}_1))$, therefore

$$\mathcal{R}_{G_{(\ell,1^n),\mathfrak{o}}}(\mathcal{D}) = q^{\ell-2}(q-1)^2 \mathcal{R}_{\mathrm{GL}_n(\mathfrak{o}_1)}(\mathcal{D}^{q^n}) + q^{\ell-2}(q-1) \mathcal{R}_{((\mathfrak{o}_1^n \times \mathfrak{o}_1^n) \rtimes \mathrm{GL}_n(\mathfrak{o}_1))}(\mathcal{D}).$$

The final task is to find $\mathcal{R}_{(\mathfrak{o}_1^n \times \mathfrak{o}_1^n) \rtimes \mathrm{GL}_n(\mathfrak{o}_1)}(\mathcal{D})$. The action in the semidirect product is the conjugation action

$$\begin{pmatrix} 1 & & & \\ & \mathrm{GL}_n(\mathfrak{o}_1) & & \end{pmatrix} \curvearrowright \begin{pmatrix} 1 & \mathfrak{p}_\ell^{\ell-1} & \dots & \mathfrak{p}_\ell^{\ell-1} \\ \mathfrak{o}_1 & 1 & & \\ \vdots & & \ddots & \\ \mathfrak{o}_1 & & & 1 \end{pmatrix} \cong \mathfrak{o}_1^n \times \mathfrak{o}_1^n$$

given by

$$\begin{pmatrix} 1 & \\ & D \end{pmatrix} \begin{pmatrix} 1 & \varpi^{\ell-1}v^T \\ w & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & D^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \varpi^{\ell-1}v^T D^{-1} \\ Dw & 1 \end{pmatrix}.$$

As in [55], this can be identified with the action $\mathrm{GL}_n(\mathfrak{o}_1) \curvearrowright (\mathfrak{o}_1^n \times \mathfrak{o}_1^n)$ given by

$$g^{-1}(\hat{v}, \hat{w}) = (D^{-1}\hat{v}, \hat{w}D), \text{ where } g = \begin{pmatrix} 1 & \\ & D \end{pmatrix}.$$

The orbits and stabilisers are given in Table 3.1, from which the representation zeta polynomial of $(\mathfrak{o}_1^n \times \mathfrak{o}_1^n) \rtimes \mathrm{GL}_n(\mathfrak{o}_1)$ can now be deduced as in the statement of

Orbit representative χ	$\text{Stab}_{\text{GL}_n(\mathfrak{o}_1)}(\chi)$	$[\text{GL}_n(\mathfrak{o}_1) : \text{Stab}_{\text{GL}_n(\mathfrak{o}_1)}(\chi)]$
$\left[\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, (0 \ \dots \ 0) \right]$	$\text{GL}_n(\mathfrak{o}_1)$	1
$\left[\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, (1 \ 0 \ \dots \ 0) \right]$	$\begin{pmatrix} 1 & 0 \\ \mathfrak{o}_1^{n-1} & \text{GL}_{n-1}(\mathfrak{o}_1) \end{pmatrix} \cong P_n$	$q^n - 1$
$\left[\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, (0 \ \dots \ 0) \right]$	$\begin{pmatrix} 1 & \mathfrak{o}_1^{n-1} \\ 0 & \text{GL}_{n-1}(\mathfrak{o}_1) \end{pmatrix} \cong P_n$	$q^n - 1$
$\left[\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, (0 \ 1 \ 0 \ \dots \ 0) \right]$	T_n	$(q^n - 1)(q^{n-1} - 1)$
$\left[\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, (u \ 0 \ \dots \ 0) \right]; u \in \mathfrak{o}_1^\times$	$\begin{pmatrix} 1 & \\ & \text{GL}_{n-1}(\mathfrak{o}_1) \end{pmatrix}$	$q^{n-1}(q^n - 1)$

Table 3.1: The action of $\text{GL}_n(\mathfrak{o}_1)$ on $\mathfrak{o}_1^n \times \mathfrak{o}_1^n$

Theorem 39. □

Example 42. One can check by substituting $\ell = 2, n = 2$ that the expression obtained for $\mathcal{R}_{G_{(2,1,1),\mathfrak{o}}}(\mathcal{D})$ agrees with [55, Lemma 7.7]. Moreover, for $\ell = 3, n = 2$ one can calculate

$$\begin{aligned}
\mathcal{R}_{G_{(3,1,1),\mathfrak{o}}}(\mathcal{D}) &= q(q-1)^2\mathcal{D} + \frac{1}{2}q^2(q-1)^2\mathcal{D}^{q-1} + q(q-1)^2\mathcal{D}^q \\
&\quad + \frac{1}{2}q(q-1)^2(q-2)\mathcal{D}^{q+1} + 2q(q-1)^2\mathcal{D}^{(q-1)(q+1)} \\
&\quad + q(q-1)^3\mathcal{D}^{q^2} + q(q-1)(q+2)\mathcal{D}^{(q-1)^2(q+1)} \\
&\quad + \frac{1}{2}q^2(q-1)^3\mathcal{D}^{q^2(q-1)} + q(q-1)^3\mathcal{D}^{q(q-1)(q+1)} \\
&\quad + q(q-1)^3\mathcal{D}^{q^3} + \frac{1}{2}q(q-1)^3(q-2)\mathcal{D}^{q^2(q+1)}.
\end{aligned}$$

Further, $\mathcal{R}_{G_{(2,1,1,1),\mathfrak{o}}}(\mathcal{D})$ has 24 terms, with the polynomials giving the dimensions of degree at most 6. We omit the expression due to its length.

3.1.2 Proofs of the Auxiliary Results

In this section, we prove Propositions 37 and 38. We fix the following notation throughout the subsequent proofs:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_{n-2} \end{pmatrix}, \mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_{n-1} \end{pmatrix}, Y = \begin{pmatrix} y_{11} & \cdots & y_{1,n-2} \\ \vdots & \ddots & \vdots \\ y_{n-2,1} & \cdots & y_{n-2,n-2} \end{pmatrix}.$$

For $i \in \mathfrak{o}_1$, let σ_i be the additive character of \mathfrak{o}_1 defined by $\sigma_i(x) = \zeta_p^{\text{tr}(ix)}$, where ζ_p is a fixed primitive p th root of unity in \mathbb{C} and tr denotes the absolute trace $\text{tr}_{\mathfrak{o}_1/\mathbb{F}_p}$.

Proof of Proposition 37. Note $P_n = \tilde{N}_n \rtimes \tilde{H}_n$, where

$$\tilde{N}_n = \left\{ \begin{pmatrix} 1 & 0 \\ \mathbf{v} & I \end{pmatrix} \mid \mathbf{v} \in \mathfrak{o}_1^{n-1} \right\}, \tilde{H}_n = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix} \mid Y \in \text{GL}_{n-1}(\mathfrak{o}_1) \right\}.$$

Parameterise the irreducible representations of \tilde{N}_n :

$$\begin{aligned} \omega_{i_1, \dots, i_{n-1}} \begin{pmatrix} 1 & 0 \\ \mathbf{v} & I \end{pmatrix} &= \sigma_{i_1}(v_1) \cdots \sigma_{i_{n-1}}(v_{n-1}) \\ &= \zeta_p^{\text{tr}(i_1 v_1 + \cdots + i_{n-1} v_{n-1})}. \end{aligned}$$

The conjugate of an element of \tilde{N}_n by an element of \tilde{H}_n takes the following form:

$$\begin{pmatrix} 1 & 0 \\ Y\mathbf{v} & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathbf{v} & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y & I \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ Y\mathbf{v} & I \end{pmatrix}.$$

Writing $h = \begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix}$,

$$\begin{aligned} \omega_{i_1, \dots, i_{n-1}}^h \begin{pmatrix} 1 & 0 \\ \mathbf{v} & I \end{pmatrix} &= \omega_{i_1, \dots, i_{n-1}} \begin{pmatrix} 1 & 0 \\ Y\mathbf{v} & I \end{pmatrix} \\ &= \zeta_p^{\text{tr}(\theta(\mathbf{v}))}, \end{aligned}$$

where

$$\begin{aligned}
\theta(\mathbf{v}) &= i_1(y_{11}v_1 + \cdots + y_{1,n-1}v_{n-1}) \\
&+ \dots \\
&+ i_{n-1}(y_{n-1,1}v_1 + \cdots + y_{n-1,n-1}v_{n-1}) \\
&= (i_1y_{11} + \cdots + i_{n-1}y_{n-1,1})v_1 \\
&+ \dots \\
&+ (i_1y_{1,n-1} + \cdots + i_{n-1}y_{n-1,n-1})v_{n-1}.
\end{aligned}$$

Therefore, the right action of \tilde{H}_n on $\text{Irr}(\tilde{N}_n)$ is given explicitly:

$$\omega_{i_1, \dots, i_{n-1}}^h = \omega_{(i_1y_{11} + \cdots + i_{n-1}y_{n-1,1}), \dots, (i_1y_{1,n-1} + \cdots + i_{n-1}y_{n-1,n-1})}.$$

The index transformation is given by the matrix Y , which is non-singular of free choice, and since every non-zero vector can be mapped to any other by multiplication by a non-singular matrix, there are two orbits, $\{\omega_{i_1, \dots, i_{n-1}} | i_k \neq 0 \text{ for some } k\}$ and $\{\omega_{0, \dots, 0}\}$. We find equations for the stabilisers:

$$\begin{aligned}
\omega_{i_1, \dots, i_{n-1}}^h = \omega_{i_1, \dots, i_{n-1}} &\iff \text{for all } \mathbf{v} \in \mathfrak{o}_1^{n-1}, \zeta_p^{\text{tr}(\theta(\mathbf{v}))} = \zeta_p^{\text{tr}(i_1v_1 + \cdots + i_{n-1}v_{n-1})} \\
&\iff \text{for all } \mathbf{v} \in \mathfrak{o}_1^{n-1}, \zeta_p^{\text{tr}(\theta'(\mathbf{v}))} = 1,
\end{aligned}$$

where

$$\begin{aligned}
\theta'(\mathbf{v}) &= \theta(\mathbf{v}) - (i_1v_1 + \cdots + i_{n-1}v_{n-1}) \\
&= (i_1(y_{11} - 1) + \cdots + i_{n-1}y_{n-1,1})v_1 \\
&+ \dots \\
&+ (i_1y_{1,n-1} + \cdots + i_{n-1}(y_{n-1,n-1} - 1))v_{n-1}
\end{aligned}$$

Setting all but one of the v_j to zero in turn, we obtain the equations

$$\begin{aligned} i_1(y_{11} - 1) + \cdots + i_{n-1}y_{n-1,1} &= 0 \\ &\vdots \\ i_1y_{1,n-1} + \cdots + i_{n-1}(y_{n-1,n-1} - 1) &= 0. \end{aligned}$$

Choose the representative $\omega_{1,0,\dots,0}$ of the non-trivial orbit:

$$\begin{aligned} \text{Stab}_{\tilde{H}_n}(\omega_{1,0,\dots,0}) &= \left\{ \begin{pmatrix} 1 & & & & \\ & 1 & 0 & \cdots & 0 \\ & y_{21} & y_{22} & \cdots & y_{2,n-1} \\ & \vdots & \vdots & \ddots & \vdots \\ & y_{n-1,1} & y_{n-1,2} & \cdots & y_{n-1,n-1} \end{pmatrix} \middle| \begin{array}{l} y_{jk} \in \mathfrak{o}_1, \\ \det \neq 0 \end{array} \right\} \\ &\cong P_{n-1}, \end{aligned}$$

with index

$$\frac{|\text{GL}_{n-1}(\mathfrak{o}_1)|}{|\mathfrak{o}_1^{n-2} \rtimes \text{GL}_{n-2}(\mathfrak{o}_1)|} = q^{n-1} - 1$$

in \tilde{H}_n . In the case of $n = 2$, this is the trivial group. Applying Proposition 26, the representation zeta polynomial above $\omega_{1,0,\dots,0}$ is

$$\mathcal{R}_{P_n|\omega_{1,0,\dots,0}}(\mathcal{D}) = \mathcal{R}_{P_{n-1}}(\mathcal{D}^{q^{n-1}-1}).$$

In the case of the trivial orbit, $\text{Stab}_{\tilde{H}_n}(\omega_{0,\dots,0}) = \tilde{H}_n \cong \text{GL}_{n-1}(\mathfrak{o}_1)$, and

$$\mathcal{R}_{P_n|\omega_{0,\dots,0}}(\mathcal{D}) = \mathcal{R}_{\text{GL}_{n-1}(\mathfrak{o}_1)}(\mathcal{D}).$$

We can deduce the following formula for the representation zeta polynomial of P_n for $n \geq 2$:

$$\mathcal{R}_{P_n}(\mathcal{D}) = \mathcal{R}_{P_{n-1}}(\mathcal{D}^{q^{n-1}-1}) + \mathcal{R}_{\text{GL}_{n-1}(\mathfrak{o}_1)}(\mathcal{D}).$$

Note that $P_1 = 1$, therefore $\mathcal{R}_{P_1}(\mathcal{D}) = \mathcal{D}$. □

Proof of Proposition 38. First note that

$$T_2 = \left\{ \begin{pmatrix} 1 & w_1 \\ 0 & 1 \end{pmatrix} \middle| w_1 \in \mathfrak{o}_1 \right\},$$

has representation zeta polynomial $q\mathcal{D}$, as claimed. From now on, let $n \geq 3$. Label the characters of N_n by

$$\chi_{i_1, \dots, i_{n-1}} \begin{pmatrix} 1 & \mathbf{w}^T \\ 0 & I \end{pmatrix} = \sigma_{i_1}(w_1) \sigma_{i_2}(w_2) \dots \sigma_{i_{n-1}}(w_{n-1}).$$

Let $Z = \begin{pmatrix} 1 & 0 \\ \mathbf{v} & Y \end{pmatrix}$, and $h = \begin{pmatrix} 1 & 0 \\ 0 & Z \end{pmatrix}$. The following formula for the conjugate holds:

$$\begin{pmatrix} 1 & 0 \\ 0 & Z \end{pmatrix}^{-1} \begin{pmatrix} 1 & \mathbf{w}^T \\ 0 & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Z \end{pmatrix} = \begin{pmatrix} I & \mathbf{w}^T Z \\ 0 & 1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} {}^h \chi_{i_1, \dots, i_{n-1}} \begin{pmatrix} 1 & \mathbf{w}^T \\ 0 & I \end{pmatrix} &= \chi_{i_1, \dots, i_{n-1}} \begin{pmatrix} I & \mathbf{w}^T Z \\ 0 & 1 \end{pmatrix} \\ &= \zeta_p^{\text{tr}(\eta(\mathbf{w}))}, \end{aligned}$$

where

$$\begin{aligned} \eta(\mathbf{w}) &= i_1(w_1 + v_1 w_2 + \dots + v_{n-2} w_{n-1}) \\ &\quad + i_2(y_{11} w_2 + \dots + y_{n-2,1} w_{n-1}) \\ &\quad + \dots \\ &\quad + i_{n-1}(y_{1,n-2} w_2 + \dots + y_{n-2,n-2} w_{n-1}) \\ &= i_1 w_1 + (i_1 v_1 + i_2 y_{11} + \dots + i_{n-1} y_{1,n-2}) w_2 \\ &\quad + \dots \\ &\quad + (i_1 v_{n-2} + i_2 y_{n-2,1} + \dots + i_{n-1} y_{n-2,n-2}) w_{n-1}. \end{aligned}$$

Therefore, there is a left action of H_n on $\text{Irr}(N_n)$ given explicitly by the index transformation

$${}^h\chi_{i_1, \dots, i_{n-1}} = \chi_{i_1, (i_1 v_1 + i_2 y_{11} + \dots + i_{n-1} y_{1, n-2}), \dots, (i_1 v_{n-2} + i_2 y_{n-2, 1} + \dots + i_{n-1} y_{n-2, n-2})}.$$

For each $i_1 \neq 0$, choosing Y and then taking a free choice of v_1, \dots, v_{n-2} gives an orbit $\chi_{i_1, *, \dots, *} = \{\chi_{i_1, \dots, i_{n-1}} \mid i_2, \dots, i_{n-1} \in \mathfrak{o}_1\}$, which we shall refer to as type (a). There are $q - 1$ such orbits, one for each i_1 .

If $i_1 = 0$, the index transformation is given by the matrix Y^T , which is invertible of free choice, giving an orbit

$$\{\chi_{0, i_2, \dots, i_{n-1}} \mid i_k \neq 0 \text{ for some } 2 \leq k \leq n - 1\},$$

which we call type (b). The remaining orbit is the trivial one, $\{\chi_{0, \dots, 0}\}$, which we call type (c).

We deduce equations for the stabilisers:

$$\begin{aligned} {}^h\chi_{i_1, \dots, i_{n-1}} = \chi_{i_1, \dots, i_{n-1}} &\iff \text{for all } \mathbf{w} \in \mathfrak{o}_1^{n-1}, \zeta_p^{\text{tr}(\eta(\mathbf{w}))} = \zeta_p^{\text{tr}(i_1 w_1 + \dots + i_{n-1} w_{n-1})} \\ &\iff \text{for all } \mathbf{w} \in \mathfrak{o}_1^{n-1}, \zeta_p^{\text{tr}(\eta'(\mathbf{w}))} = 1, \end{aligned}$$

where

$$\begin{aligned} \eta'(\mathbf{w}) &= \eta(\mathbf{w}) - (i_1 w_1 + \dots + i_{n-1} w_{n-1}) \\ &= (i_1 v_1 + i_2 (y_{11} - 1) + \dots + i_{n-1} y_{1, n-2}) w_2 \\ &\quad + \dots \\ &\quad + (i_1 v_{n-2} + i_2 y_{n-2, 1} + \dots + i_{n-1} (y_{n-2, n-2} - 1)) w_{n-1}. \end{aligned}$$

Setting all but one of w_2, \dots, w_{n-1} to zero in turn, we obtain the following system

of equations:

$$\begin{aligned}
i_1 v_1 + i_2 (y_{11} - 1) + \cdots + i_{n-1} y_{1,n-2} &= 0 \\
&\vdots \\
i_1 v_{n-2} + i_2 y_{n-2,1} \cdots + i_{n-1} (y_{n-2,n-2} - 1) &= 0
\end{aligned}$$

We find explicit forms for the stabilisers of irreducible representations of N_n inside H_n .

Type (a) ($q - 1$ orbits)

For $i_1 \neq 0, i_2 = \cdots = i_{n-1} = 0$, the system becomes $i_1 v_1 = \cdots = i_1 v_{n-2} = 0$, therefore $v_1 = \cdots = v_{n-2} = 0$, and

$$\text{Stab}_H(\chi_{i_1,0,\dots,0}) = \left\{ \left(\begin{array}{c|c} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & \\ \hline & Y \end{array} \right) \middle| Y \in \text{GL}_{n-2}(\mathfrak{o}_1) \right\},$$

with index q^{n-2} . For each $i_1 \neq 0$,

$$\mathcal{R}_{T_n|\chi_{i_1,0,\dots,0}}(\mathcal{D}) = \mathcal{R}_{\text{GL}_{n-2}(\mathfrak{o}_1)}(\mathcal{D}^{q^{n-2}}),$$

therefore

$$\sum_{i_1 \neq 0} \mathcal{R}_{T_n|\chi_{i_1,0,\dots,0}}(\mathcal{D}) = (q - 1) \mathcal{R}_{\text{GL}_{n-2}(\mathfrak{o}_1)}(\mathcal{D}^{q^{n-2}}).$$

Type (b)

For the non-trivial orbit with $i_1 = 0$, we choose $i_2 = 1$ and all other $i_k = 0$. The

system becomes $y_{11} - 1 = y_{21} = \dots = y_{n-2,1} = 0$, therefore the stabiliser is

$$\text{Stab}_{H_n}(\chi_{0,1,0,\dots,0}) = \left\{ \left(\begin{array}{cccccc} 1 & & & & & \\ & 1 & & & & \\ v_1 & 1 & y_{12} & \dots & y_{1,n-2} & \\ v_2 & 0 & y_{22} & \dots & y_{2,n-2} & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ v_{n-2} & 0 & y_{n-2,2} & \dots & y_{n-2,n-2} & \end{array} \right) \middle| \begin{array}{l} v_i, y_{jk} \in \mathfrak{o}_1, \\ \det \neq 0 \end{array} \right\},$$

with index $q^{n-2} - 1$. There is an isomorphism $\text{Stab}_{H_n}(\chi_{0,1,0,\dots,0}) \cong T_{n-1}$ which can be seen by conjugating by the permutation matrix corresponding to the transposition (2, 3). By induction, we can assume that the representations of this are known, and deal with the base case of $n = 3$:

$$\text{Stab}_{H_3}(\chi_{01}) \cong T_2 = \left\{ \left(\begin{array}{cc} 1 & w_1 \\ & 1 \end{array} \right) \middle| w_1 \in \mathfrak{o}_1 \right\},$$

which has representation zeta polynomial

$$\mathcal{R}_{T_2}(\mathcal{D}) = q\mathcal{D}.$$

For general n , we have

$$\mathcal{R}_{T_n|\chi_{0,1,0,\dots,0}}(\mathcal{D}) = \mathcal{R}_{T_{n-1}}(\mathcal{D}^{q^{n-2}-1}).$$

Type (c)

The group H_n is isomorphic to P_{n-1} . Consider the case of the trivial representation,

$$\text{Stab}_{H_n}(\chi_{0,\dots,0}) = H_n.$$

The contribution to the representation zeta polynomial of T_n is

$$\mathcal{R}_{T_n|\chi_{0,\dots,0}}(\mathcal{D}) = \mathcal{R}_{P_n}(\mathcal{D}),$$

which we know by Proposition 37.

Collecting the representation zeta polynomials above the representatives and adding them (see Remark 25) will give the representation zeta polynomial of T_n as claimed. \square

3.2 The Representations of $G_{(2,2,1),\mathfrak{o}}$

The smallest value of n for which the dimensions of the irreducible representations of $G_{\lambda,\mathfrak{o}}$ are not known to be polynomial for some partition λ of n is $n = 5$. This is settled in the affirmative for the Jordan canonical forms of type: (5) (abelian), (1, 1, 1, 1, 1) by Green [23], (4, 1) and (3, 2) by Onn [50], and (2, 1, 1, 1) and (3, 1, 1) by Theorem 39. The question is still open in the case of (2, 2, 1). Based on computer calculations for $q \leq 5$, we conjecture that for all \mathfrak{o} :

$$\begin{aligned} \mathcal{R}_{G_{(2,2,1),\mathfrak{o}}}(\mathcal{D}) &= (q-1)^2\mathcal{D} + \frac{1}{2}q(q-1)^2\mathcal{D}^{q-1} + (q-1)^2\mathcal{D}^q \\ &\quad + \frac{1}{2}(q-1)^2(q-2)\mathcal{D}^{q+1} + 2(q-1)^2\mathcal{D}^{(q-1)(q+1)} \\ &\quad + (q-1)^3\mathcal{D}^{q^2} + (q-1)(q+2)\mathcal{D}^{(q-1)^2(q+1)} \\ &\quad + \frac{1}{2}q(q-1)^3\mathcal{D}^{q^2(q-1)} + (q-1)^2(2q-1)\mathcal{D}^{q(q-1)(q+1)} \\ &\quad + (q-1)^3\mathcal{D}^{q^3} + \frac{1}{2}(q-1)^3(3q-4)\mathcal{D}^{q^2(q+1)} \\ &\quad + q(q-1)(q+1)\mathcal{D}^{(q-1)^2(q+1)} + \frac{1}{2}q(q-1)^3(q+1)\mathcal{D}^{q^3(q-1)} \\ &\quad + (q-1)^3(2q+1)\mathcal{D}^{q^2(q-1)(q+1)} + \frac{1}{2}(q-1)^4(q-2)\mathcal{D}^{q^3(q+1)}. \end{aligned}$$

If this is true, then Conjecture 5 will be proven for $\mathrm{GL}_5(\mathfrak{o}_2)$ (by Proposition 35).

There are some differences which complicate matters in this case. First, if we define

$$H = \begin{pmatrix} 1 + \mathfrak{p}M_n(\mathfrak{o}_2) & \mathfrak{p}_2^2 \\ \mathfrak{o}_1^2 & 1 \end{pmatrix},$$

then H is normal in G with centre $Z(H) = \mathrm{diag}(1 + \mathfrak{p}M_n(\mathfrak{o}_2), 1) \cong M_2(\mathfrak{o}_1)$. In this case, $Z(H)$ is normal in G , but $Z(H) \not\subseteq Z(G)$, so the argument that every irreducible representation of H lying above a non-trivial character of $Z(H)$ extends

to G does not work here.

Nonetheless, we are able to construct the representations of H , and we give a formula for $\mathcal{R}_H(\mathcal{D})$:

Proposition 43. $\mathcal{R}_H(\mathcal{D}) = q^4\mathcal{D} + q^2(q-1)(q+1)^2\mathcal{D}^q + q(q-1)^2(q+1)\mathcal{D}^{q^2}$.

We use the ‘‘Heisenberg lifting’’ method as before, however the bilinear forms $\langle \cdot, \cdot \rangle_{\psi_z}$ obtained in this case may be degenerate. We therefore need the following generalisation of Proposition 40 (see [57, Corollary 3.3]):

Proposition 44. *Let H be a finite group, Z a normal subgroup of H such that H/Z is elementary abelian, and $\chi \in \text{Irr}(Z)$, $\chi \neq 1$ invariant in H . Define an alternating bilinear form on H/Z by*

$$\langle \overline{h_1}, \overline{h_2} \rangle_{\chi} = \chi([h_1, h_2]).$$

Let R_{χ} be the inverse image of the radical of $\langle \cdot, \cdot \rangle_{\chi}$ under the natural map $H \mapsto H/Z$. Then χ has exactly $[R_{\chi} : Z]$ many extensions to R_{χ} , and there is a unique irreducible representation of H lying above each extension. In the case where Z is a central subgroup of H , the representation zeta polynomial of H is given by

$$\mathcal{R}_H(\mathcal{D}) = \mathcal{R}_{H/Z}(\mathcal{D}) + \sum_{\chi \neq 1} [R_{\chi} : Z] \mathcal{D}^{[H:R_{\chi}]^{\frac{1}{2}}}.$$

Outline of the proof. The existence and uniqueness of an irreducible representation of H lying above each extension to R_{χ} can be deduced by taking quotients by R_{χ} and applying Proposition 40. □

As before, there is a canonical isomorphism $M_2(\mathfrak{o}_1) \rightarrow Z(H)$ given by $\overline{u} \mapsto 1 + \varpi u$ for $u \in M_2(\mathfrak{o}_2)$, and a non-canonical isomorphism $M_2(\mathfrak{o}_1) \rightarrow \text{Irr}(Z)$ given by fixing a non-trivial character $\psi : \mathfrak{o}_1 \rightarrow \mathbb{C}^{\times}$ and mapping $u \in M_2(\mathfrak{o}_1)$ to $\psi_u \in \text{Irr}(Z)$ given by

$$\psi_u \left(\begin{pmatrix} 1 + \varpi v & \\ & 1 \end{pmatrix} \right) = \psi(\text{tr}(uv)).$$

Throughout, write

$$h_1 = \begin{pmatrix} 1 + \varpi x & \varpi v \\ w & 1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 1 + \varpi x' & \varpi v' \\ w' & 1 \end{pmatrix}.$$

Given $u \in M_2(\mathfrak{o}_1)$, define a bilinear form $\langle \cdot, \cdot \rangle_u$ on $H/Z(H)$ by

$$\langle \overline{h_1}, \overline{h_2} \rangle_u = \psi_u([h_1, h_2]) = \psi(\text{tr}(u(vw' - v'w))).$$

Write $\overline{R}_u = \text{Rad}\langle \cdot, \cdot \rangle_u$, and R_u for the inverse image of \overline{R}_u under the map $H \rightarrow H/Z(H)$. It can be shown that $\overline{R_{gug^{-1}}} = g\overline{R}_ug^{-1}$, therefore it suffices to find \overline{R}_u for representatives u of the $\text{GL}_2(\mathfrak{o}_1)$ -similarity classes in $M_2(\mathfrak{o}_1)$.

Proposition 45. *The following is a list of representatives of the $\text{GL}_2(\mathfrak{o}_1)$ -similarity classes in $M_2(\mathfrak{o}_1)$ along with their sizes and a description of R_u :*

1. (scalar) $u = aI_2$, orbit size 1. If $a \neq 0$, then $R_u = Z$.

2. (split regular semisimple) $u = \begin{pmatrix} a & \\ & d \end{pmatrix}$, $a \neq d$, orbit size $q(q+1)$.

If $d = 0$, then

$$R_u = \begin{pmatrix} 1 + \mathfrak{p}M_n(\mathfrak{o}_2) & 0 \\ & \mathfrak{p}_2 \\ 0 & \mathfrak{o}_1 & 1 \end{pmatrix}.$$

If $a, d \neq 0$, then $R_u = Z$.

3. (nonsplit regular semisimple) $u = \begin{pmatrix} 0 & 1 \\ -\Delta & \tau \end{pmatrix}$, where $x^2 - \tau x + \Delta \in \mathfrak{o}_1[x]$ is irreducible, orbit size $q(q-1)$. In this case, $R_u = Z$.

4. (split regular non-semisimple) $u = \begin{pmatrix} a & 1 \\ & a \end{pmatrix}$, $a \neq 0$, orbit size $(q-1)(q+1)$.

Then $R_u = Z$. If $a = 0$, then

$$R_u = \begin{pmatrix} 1 + \mathfrak{p}M_n(\mathfrak{o}_2) & \mathfrak{p}_2 \\ & 0 \\ 0 & \mathfrak{o}_1 & 1 \end{pmatrix}.$$

Proof. Note that

$$vw' - v'w = \begin{pmatrix} v_1w'_1 - v'_1w_1 & v_1w'_2 - v'_1w_2 \\ v_2w'_1 - v'_2w_1 & v_2w'_2 - v'_2w_2 \end{pmatrix},$$

and that $h_1 \in R_u$ if and only if for all $h_2 \in H$, $\langle \overline{h_1}, \overline{h_2} \rangle = 1$, or equivalently, for all $v', w', \psi(\text{tr}(u(vw' - v'w))) = 1$. We show (2), since the other cases are proven similarly. Suppose that $u = \begin{pmatrix} a \\ d \end{pmatrix}$ and that for all $v', w', \psi(\text{tr}(u(vw' - v'w))) = 1$, that is,

$$\psi(a(v_1w'_1 - v'_1w_1) + d(v_2w'_2 - v'_2w_2)) = 1. \quad (3.1)$$

Suppose first that $a \neq 0, d = 0$. Put $v'_1 = 0$ to obtain $\psi(av_1w'_1) = 1$ for all $w'_1 \in \mathfrak{o}_1$. This implies that $av_1 = 0$, therefore $v_1 = 0$. Similarly, $w_1 = 0$. Conversely, if $v_1 = w_1 = 0$, then Equation 3.1 is satisfied. Therefore,

$$R_u = \begin{pmatrix} 1 + \mathfrak{p}M_n(\mathfrak{o}_2) & 0 \\ & \mathfrak{p}_2 \\ 0 & \mathfrak{o}_1 & 1 \end{pmatrix}.$$

If $d \neq 0$, similar reasoning gives $v_1 = v_2 = w_1 = w_2 = 0$, so $R_u = Z$. \square

We collect the preceding information to deduce the final expression for the representation zeta polynomial of H .

$$\begin{aligned} \mathcal{R}_H(\mathcal{D}) &= \mathcal{R}_{H/Z}(\mathcal{D}) + \sum_{u \in M_2(\mathfrak{o}_1) \setminus \text{GL}_2(\mathfrak{o}_1), u \neq 0} |\mathcal{O}_{\text{GL}_2(\mathfrak{o}_1)}(u)| [R_u : Z] \mathcal{D}^{[H:R_u]^{\frac{1}{2}}} \\ &= q^4 \mathcal{D} + (q-1) \mathcal{D}^{q^2} + \frac{1}{2}(q-1)(q-2) \cdot q(q+1) \mathcal{D}^{q^2} + (q-1) \cdot q(q+1) \cdot q^2 \mathcal{D}^q \\ &\quad + \frac{1}{2}q(q-1) \cdot q(q-1) \cdot \mathcal{D}^{q^2} + (q-1) \cdot (q-1)(q+1) \mathcal{D}^{q^2} + (q-1)(q+1) \cdot q^2 \mathcal{D}^q \\ &= q^4 \mathcal{D} + q^2(q-1)(q+1)^2 \mathcal{D}^q + q(q-1)^2(q+1) \mathcal{D}^{q^2}. \end{aligned}$$

3.3 Schur Multipliers

We give results about the Schur multipliers $H^2(\mathrm{GL}_n(\mathfrak{o}_r), \mathbb{C}^\times)$. For brevity, we write $G_r = \mathrm{GL}_n(\mathfrak{o}_r)$.

Theorem 46. *There is an infinite chain of embeddings of abelian groups*

$$H^2(G_{r-1}, \mathbb{C}^\times) \hookrightarrow H^2(G_r, \mathbb{C}^\times); r > 1.$$

Proof. Consider the inflation-restriction exact sequence [22, Proposition 3.3.14]:

$$\begin{aligned} 0 \rightarrow \mathrm{Hom}(G_{r-1}, \mathbb{C}^\times) &\xrightarrow{\mathrm{Inf}^1} \mathrm{Hom}(G_r, \mathbb{C}^\times) \xrightarrow{\mathrm{Res}^1} \mathrm{Hom}(K, \mathbb{C}^\times)^{G_{r-1}} \\ &\xrightarrow{\mathrm{Tra}^{1,2}} H^2(G_{r-1}, \mathbb{C}^\times) \xrightarrow{\mathrm{Inf}^2} H^2(G_r, \mathbb{C}^\times) \end{aligned}$$

(since the action of G on \mathbb{C}^\times is trivial, the first cohomology groups are simply $\mathrm{Hom}(G, \mathbb{C}^\times)$). By exactness, it suffices to show that the restriction map Res^1 is surjective. Indeed, we have

$$\begin{aligned} \mathrm{Hom}(K, \mathbb{C}^\times)^{G_{r-1}} &= \{\psi_\beta \mid \beta \in M_n(\mathfrak{o}_{r-i}), \forall g \in G_{r-1}, \psi_\beta = \psi_{\bar{g}\beta\bar{g}^{-1}}\} \\ &= \{\psi_\beta \mid \beta \in M_n(\mathfrak{o}_1), \forall g \in G_{r-1}, \beta = \bar{g}\beta\bar{g}^{-1}\} \\ &= \{\psi_{aI_n} \mid a \in \mathfrak{o}_1\}, \end{aligned}$$

and by Lemma 36, every ψ_{aI_n} extends to a character $G_r \rightarrow \mathbb{C}^\times$; this is precisely the statement that Res^1 is surjective. Thus, Inf^2 is injective. \square

Theorem 47. [38, Chapter 16, Theorem 3.7] *Let $n \geq 2$ and q a prime power. Then $H^2(\mathrm{GL}_n(\mathbb{F}_q), \mathbb{C}^\times) = \mathbb{Z}/2$ if $(n, q) \in \{(2, 4), (3, 2), (4, 2)\}$, and trivial otherwise.*

In fact, [38] gives two possibilities, 1 and $(\mathbb{Z}/4)^2$, for $H^2(\mathrm{GL}_3(\mathbb{F}_4), \mathbb{C}^\times)$, however computation in GAP showed that $H^2(\mathrm{GL}_3(\mathbb{F}_4), \mathbb{C}^\times)$ is trivial.

Theorem 48. [40, Corollary 2.1.15] *Let G be finite and N a normal p -subgroup of G . Then the p' -parts of $H^2(G, \mathbb{C}^\times)$ and $H^2(G/N, \mathbb{C}^\times)$ are isomorphic.*

Given a finite abelian group G , there exist unique groups H_p and $H_{p'}$ such that

$G \cong H_p \times H_{p'}$, $|H_p|$ is a power of p and p does not divide $|H_{p'}|$. We call H_p the p -part of G and $H_{p'}$ the p' -part of G .

Theorem 49. *For all $n \geq 2$ and all $r \geq 1$, the p' -part of $H^2(\mathrm{GL}_n(\mathfrak{o}_r), \mathbb{C}^\times)$ is trivial.*

Proof. Put $N = K^1 = 1 + \mathfrak{pM}_n(\mathfrak{o}_r)$ in Theorem 48 to conclude that the p' -parts of $H^2(\mathrm{GL}_n(\mathfrak{o}_r), \mathbb{C}^\times)$ and $H^2(\mathrm{GL}_n(\mathfrak{o}_1), \mathbb{C}^\times)$ are isomorphic. By the list given in Theorem 47, the p' -part of $H^2(\mathrm{GL}_n(\mathfrak{o}_1), \mathbb{C}^\times)$ is trivial. \square

Given n and r , and two rings \mathfrak{o} and \mathfrak{o}' with equal residue cardinality, we may have $|H^2(\mathrm{GL}_n(\mathfrak{o}_r), \mathbb{C}^\times)| \neq |H^2(\mathrm{GL}_n(\mathfrak{o}'_r), \mathbb{C}^\times)|$. For example, we have checked using GAP that $H^2(\mathrm{GL}_3(\mathbb{Z}/27\mathbb{Z}), \mathbb{C}^\times) = 1$, but $H^2(\mathrm{GL}_3(\mathbb{F}_3[t]/(t^3)), \mathbb{C}^\times) = (\mathbb{Z}/3)^2$. This appears as an obstruction when one attempts to formulate Conjecture 5 for projective representations in general; one would need a canonical bijection between $H^2(\mathrm{GL}_n(\mathfrak{o}_r), \mathbb{C}^\times)$ and $H^2(\mathrm{GL}_n(\mathfrak{o}'_r), \mathbb{C}^\times)$, however the example shows that no bijection exists.

Polynomial Results for Smooth Affine Group Schemes

In this chapter, we prove Theorem 2. In order to do this, we prove Theorem 51, which is a polynomial analogue of [26, Theorem 4.0.2]. In Section 4.1, we also employ an argument due to Geck [19] to deal with the reductive case. The new results in this chapter can be found in [35]. For background material about schemes, see Appendix A.4. For an introduction to affine group schemes, see Appendix A.4.1, and for affine varieties and Frobenius maps, see Appendix A.4.2.

Tao [59] defined the notion of complexity of a variety over an algebraically closed field. We will use a notion of Hadas [26, Definition 3.1.1] which is valid for varieties over fields which may not be algebraically closed.

Suppose \mathbb{F} is a field and V is an affine \mathbb{F} -variety (i.e. a reduced affine \mathbb{F} -scheme of finite type over \mathbb{F}) equipped with an \mathbb{F} -embedding (i.e. a closed immersion of \mathbb{F} -schemes) $\iota : V \rightarrow \mathbb{A}_{\mathbb{F}}^n$; this corresponds to a surjective \mathbb{F} -algebra homomorphism $\pi : \mathbb{F}[x_1, \dots, x_n] \rightarrow R$, where $V = \text{Spec } R$. Consider the morphism obtained after extending scalars to $\overline{\mathbb{F}}$: $\iota_{\overline{\mathbb{F}}} : V_{\overline{\mathbb{F}}} \rightarrow \mathbb{A}_{\overline{\mathbb{F}}}^n$. This corresponds to an $\overline{\mathbb{F}}$ -algebra homomorphism $\pi_{\overline{\mathbb{F}}} : \overline{\mathbb{F}}[x_1, \dots, x_n] \rightarrow R \otimes_{\mathbb{F}} \overline{\mathbb{F}}$. Since the functor $- \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is right exact, $\pi_{\overline{\mathbb{F}}}$ is also surjective, and $\iota_{\overline{\mathbb{F}}}$ is a closed immersion.

Definition 50. The **complexity** of V is the smallest integer $M \geq n$ such that $V_{\overline{\mathbb{F}}}$

is the zero set of at most M polynomials, all of degree at most M . That is, $M \geq n$ is minimal such that there exists a generating set of $\ker \pi_{\overline{\mathbb{F}}}$ containing at most M polynomials in $\overline{\mathbb{F}}[x_1, \dots, x_n]$, all of degree at most M .

The complexity of a variety depends on the embedding. However, this does not matter since we will work with closed subgroups of GL_n , which are varieties already equipped with an embedding.

Following [26], let \mathbb{F} be a finite field and define $\mathcal{A}(n', M_{\mathrm{dim}}, M_{\mathrm{cmp}}, \mathbb{F})$ to be the set of all reduced group schemes over \mathbb{F} with dimension at most M_{dim} and complexity at most M_{cmp} , equipped with a closed immersion into $(\mathrm{GL}_n)_{\mathbb{F}}$. If we say that an affine group scheme G over \mathbb{F} is connected, reductive or unipotent, we take this to mean that the extension of scalars $G \times_{\mathbb{F}} \overline{\mathbb{F}}$ is connected, reductive or unipotent, respectively.

If G is a group scheme over \mathfrak{o}_r , we can apply the Greenberg functor (see Section 4.3.1) to obtain an algebraic group over \mathfrak{o}_1 , therefore we aim to prove the statement for algebraic groups over finite fields. It was shown in [26, Theorem 4.0.2] that there exists a bound, uniform in $|\mathbb{F}|$, on the cardinality of the set

$$\bigcup_{G \in \mathcal{A}(n', M_{\mathrm{dim}}, M_{\mathrm{cmp}}, \mathbb{F})} \mathrm{dimirr}(G(\mathbb{F})).$$

The following result states further that the dimensions of the representations of $G(\mathbb{F})$ are given by evaluating finitely many polynomials at $|\mathbb{F}|$. It is the main step in proving Theorem 2.

Theorem 51. [35, Theorem 28] *For all $n', M_{\mathrm{dim}}, M_{\mathrm{cmp}} \in \mathbb{N}$, there exists $C > 0$ such that for every prime $p > C$, there exists a finite set of polynomials $R_{n', M_{\mathrm{dim}}, M_{\mathrm{cmp}}} \subseteq \mathbb{Q}[x]$ such that for every finite field \mathbb{F} of characteristic p ,*

$$\bigcup_{G \in \mathcal{A}(n', M_{\mathrm{dim}}, M_{\mathrm{cmp}}, \mathbb{F})} \mathrm{dimirr}(G(\mathbb{F})) \subseteq \{m(|\mathbb{F}|) \mid m \in R_{n', M_{\mathrm{dim}}, M_{\mathrm{cmp}}}\}. \quad (4.1)$$

We give a preliminary outline of the proof; the full proof can be found in Section 4.3. The proof follows that of [26, Theorem 4.0.2], with the difference that we employ the argument of Section 4.1 to deal with the reductive case, and we argue

at each step that the dimensions of the representations constructed are in fact given by finitely many polynomials.

Outline of the proof of Theorem 51. We consider groups $G \in \mathcal{A}(n', M_{\dim}, M_{\text{cmp}}, \mathbb{F})$ and reduce to the case where G is connected, and then work by induction on M_{\dim} . The dimension zero case is now trivial, since the only connected group of dimension zero is the trivial group. For higher dimensions, we deal with the reductive case by Section 4.1, and the unipotent case by the Kirillov orbit method (see [26, Corollary 4.3.2]).

If G is neither reductive nor unipotent, let the unipotent radical be U . Given $\rho \in \text{Irr}(U(\mathbb{F}))$, it was shown in [26, Lemma 4.3.4] that subject to the assumption that $\text{char } \mathbb{F}$ is large, $\text{Stab}_{G(\mathbb{F})}(\rho) = K(\mathbb{F})$ for some closed subgroup K of G with bounded complexity. This leads to two cases:

1. If $K = G$, we will arrange for p to be large enough so that $H^2(G(\mathbb{F})/U(\mathbb{F}), \mathbb{C}^\times) = 1$. In this case, ρ extends to $\tilde{\rho} \in \text{Irr}(G(\mathbb{F}))$ and we can apply Proposition 22 to conclude that $\text{Irr}(G(\mathbb{F}) \mid \rho) = \{\tilde{\rho} \otimes \varphi \mid \varphi \in \text{Irr}((G(\mathbb{F})/U(\mathbb{F}))\}$. Since $G(\mathbb{F})/U(\mathbb{F}) = (G/U)(\mathbb{F})$ is the \mathbb{F} -points of a reductive group, this case has already been covered.
2. If $K \neq G$, then $\dim K < \dim G$, since G is connected. Since K comes with a complexity bound, we can apply the inductive hypothesis to K (that is, the dimension of the irreducible representations of $K(\mathbb{F})$ are given by finitely many evaluated at $|\mathbb{F}|$). In Lemma 74, we show that the cardinalities $|G(\mathbb{F})|$ of groups G with bounded dimension are given by finitely many polynomials. Hence, we show that the quantity $[G(\mathbb{F}) : K(\mathbb{F})]$ is given by finitely many polynomials. The dimension of a representation induced from the stabiliser $K(\mathbb{F})$ is then given by a product with the index $[G(\mathbb{F}) : K(\mathbb{F})]$, which is therefore also given by finitely many polynomials.

□

4.1 Finite Groups of Lie Type

Our aim in this section is to show that reductive groups have the polynomial property of Theorem 2, that is:

Theorem 52. *Let q be a power of a prime and G be a connected reductive linear algebraic group over \mathbb{F}_q . There exists a finite set $R \subseteq \mathbb{Q}[x]$ such that for every $d \geq 1$,*

$$\dim_{\text{irr}}(G(\mathbb{F}_{q^d})) \subseteq \{m(q^d) \mid m \in R\}.$$

This result follows from the classification given by Lusztig [45, 46], however we give an argument due to Geck [19, Remark 4.2].

4.1.1 Deligne-Lusztig Theory

It can be shown that if F is a Frobenius map on a connected reductive group \mathbf{G} over $\overline{\mathbb{F}_p}$, the fixed points \mathbf{G}^F form a finite group, known as a **finite group of Lie type**. We will be interested in the representations of these groups. A partition of the representations was done by Deligne and Lusztig [15], which we outline:

Let \mathbf{T}_0 be an F -stable maximal torus (i.e. $F(\mathbf{T}_0) \subseteq \mathbf{T}_0$), contained in an F -stable Borel subgroup \mathbf{B}_0 (such a torus is called **maximally split**).

Definition 53. For $w \in \mathbf{W}$, define

$$\mathbf{T}_0[w] = \{t \in \mathbf{T}_0 \mid F(t) = \dot{w}^{-1}t\dot{w}\},$$

where \dot{w} is a lift of w to $N_{\mathbf{G}}(\mathbf{T}_0)$; this is independent of the choice of representative, since \mathbf{T}_0 is normal in $N_{\mathbf{G}}(\mathbf{T}_0)$. Where we need to make reference to the Frobenius map F , we write $\mathbf{T}_0[w, F]$.

Following [20], if G is a group acting by algebraic automorphisms on a variety \mathbf{X} , then for each i , $H_c^i(\mathbf{X}, \overline{\mathbb{Q}_l})$ is a $\overline{\mathbb{Q}_l}[G]$ -module, where each $g \in G$ acts via $(g^*)^{-1}$.¹

¹Each g induces an automorphism $\mathbf{X} \rightarrow \mathbf{X}$, and any morphism $f : \mathbf{X} \rightarrow \mathbf{X}'$ induces a linear map $f^* : H_c^i(\mathbf{X}', \overline{\mathbb{Q}_l}) \rightarrow H_c^i(\mathbf{X}, \overline{\mathbb{Q}_l})$ by functoriality. Therefore, g induces a linear automorphism g^* of $H_c^i(\mathbf{X}, \overline{\mathbb{Q}_l})$.

The **Lefschetz number** of g on \mathbf{X} is defined as

$$\mathfrak{L}(g, \mathbf{X}) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{tr}(g, H_c^i(\mathbf{X}, \overline{\mathbb{Q}}_l)).$$

Define the set $\mathfrak{X} = \{(w, \theta) \mid w \in \mathbf{W}, \theta \in \operatorname{Irr}(\mathbf{T}_0[w])\}$. Given $(w, \theta) \in \mathfrak{X}$, one defines the **Deligne-Lusztig character** R_w^θ associated to the pair (w, θ) as follows: let \dot{w} be a lift of w to $N_{\mathbf{G}}(\mathbf{T}_0)$. Denote by $\mathcal{L} : \mathbf{G} \rightarrow \mathbf{G}$, $\mathcal{L}(g) = g^{-1}F(g)$ the Lang-Steinberg map. Then $\mathbf{G}^F \times \mathbf{T}_0[w]$ acts on the variety $\mathcal{L}^{-1}(\dot{w}\mathbf{U}_0)$ by $(g, t) \cdot x = gxt^{-1}$. This induces an action of $\overline{\mathbb{Q}}_l[\mathbf{G}^F \times \mathbf{T}_0[w]]$ on $H_c^*(\mathcal{L}^{-1}(\dot{w}\mathbf{U}_0), \overline{\mathbb{Q}}_l)$. We set:

$$R_w^\theta(g) = \frac{1}{|\mathbf{T}_0[w]|} \sum_{t \in \mathbf{T}_0[w]} \mathfrak{L}((g, t), \mathcal{L}^{-1}(\dot{w}\mathbf{U}_0)) \theta(t).$$

One can show that R_w^θ is a virtual character of \mathbf{G}^F , that is, a \mathbb{Z} -linear combination of characters of \mathbf{G}^F .

4.1.2 Polynomial Property for Finite Groups of Lie Type

The objective of this subsection is to prove the following:

Proposition 54. *For all $M_{\dim} > 0$, there exists a finite set $T_{\operatorname{red}, M_{\dim}} \subseteq \mathbb{Q}[x]$ such that for every prime p and every finite field \mathbb{F} of characteristic p ,*

$$\bigcup_{\substack{\dim G \leq M_{\dim} \\ G \text{ connected, reductive,} \\ \text{defined over } \mathbb{F}}} \operatorname{dimirr}(G(\mathbb{F})) \subseteq \{m(|\mathbb{F}|) \mid m \in T_{\operatorname{red}, M_{\dim}}\}.$$

The proof follows an argument due to Geck [19], which we outline here.

Proposition 55 (Inner product formula). [20, Example 2.3.22] *Let $R_w^\theta, R_{w'}^{\theta'}$ be two Deligne-Lusztig characters. Then \mathbf{W} acts on \mathfrak{X} by $x \cdot (w, \theta) = (xwF(x)^{-1}, {}^x\theta)$, and*

$$\langle R_w^\theta, R_{w'}^{\theta'} \rangle = |\{x \in \mathbf{W} \mid x \cdot (w, \theta) = (w', \theta')\}|.$$

In particular, $0 \leq \langle R_w^\theta, R_{w'}^{\theta'} \rangle \leq |\mathbf{W}|$.

Proposition 56. [20, Example 2.3.23] The regular character of \mathbf{G}^F can be written as

$$\chi_{\text{reg}} = \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \sum_{\theta \in \text{Irr}(\mathbf{T}_0[w])} R_w^\theta(1) R_w^\theta.$$

Corollary 57. [11, Corollary 7.5.8] Every $\rho \in \text{Irr}(\mathbf{G}^F)$ occurs in some Deligne-Lusztig character R_w^θ .

Proof. Write

$$\begin{aligned} \dim \rho &= \langle \rho, \chi_{\text{reg}} \rangle \\ &= \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \sum_{\theta \in \text{Irr}(\mathbf{T}_0[w])} R_w^\theta(1) \langle R_w^\theta, \rho \rangle. \end{aligned}$$

Then one of the terms in the sum must be non-zero, in particular $\langle R_w^\theta, \rho \rangle \neq 0$ for some R_w^θ . \square

We refer to [20, §2.4] for the following definitions:

Definition 58 (Norm map). Given $w \in \mathbf{W}$ and $F : \mathbf{T}_0 \rightarrow \mathbf{T}_0$, define $wF : \mathbf{T}_0 \rightarrow \mathbf{T}_0$ by $(wF)(t_0) = \dot{w}F(t_0)\dot{w}^{-1}$. We say $d \geq 1$ is **admissible** if it is a multiple of the order of the induced Frobenius map $F : \mathbf{W} \rightarrow \mathbf{W}$ and $(wF)^d = F^d$ on \mathbf{T}_0 for all $w \in \mathbf{W}$. The d th norm map is

$$N_{F^d/F} : \mathbf{T}_0 \rightarrow \mathbf{T}_0; \quad t \mapsto tF(t) \dots F^{d-1}(t).$$

We define $N_d^{(w)} = N_{(wF)^d/wF} : \mathbf{T}_0 \rightarrow \mathbf{T}_0$.

Since, by [20, Lemma 2.3.1], $N_d^{(w)}(\mathbf{T}_0^{F^d}) = \mathbf{T}^{wF} = \mathbf{T}_0[w]$, we can consider $N_d^{(w)}$ as a surjective map $\mathbf{T}_0^{F^d} \twoheadrightarrow \mathbf{T}_0[w]$.

Note also that $N_{\mathbf{G}}(\mathbf{T}_0)^{F^d}$ acts on $\text{Irr}(\mathbf{T}_0^{F^d})$ by conjugation. Since $\mathbf{T}_0^{F^d}$ is abelian, it acts trivially and the action descends to an action of $N_{\mathbf{G}}(\mathbf{T}_0)^{F^d}/\mathbf{T}_0^{F^d} = \mathbf{W}^{F^d} = \mathbf{W}$ on $\text{Irr}(\mathbf{T}_0^{F^d})$.

Definition 59. Let $w_1, w_2 \in \mathbf{W}$ and $\theta_1 \in \text{Irr}(\mathbf{T}_0[w_1]), \theta_2 \in \text{Irr}(\mathbf{T}_0[w_2])$. Then (w_1, θ_1) and (w_2, θ_2) are **geometrically conjugate** if there exists an admissible

$d \geq 1$ and $y \in \mathbf{W}$ such that $\theta_2 \circ N_d^{(w_2)} = y(\theta_1 \circ N_d^{(w_1)})$, or equivalently, $\theta_1 \circ N_d^{(w_1)}$ and $\theta_2 \circ N_d^{(w_2)}$ are in the same orbit of the action of \mathbf{W} on $\text{Irr}(\mathbf{T}_0^{F^d})$.

Note that w_1 and w_2 do not have to be conjugate in \mathbf{W} in order for (w_1, θ_1) to be geometrically conjugate to (w_2, θ_2) ; indeed, these pairs are necessarily geometrically conjugate when θ_1 and θ_2 are the trivial characters of $\mathbf{T}_0[w_1]$ and $\mathbf{T}_0[w_2]$ respectively.

We say that two virtual representations, R_1, R_2 of a group G are **disjoint** if for all $\rho \in \text{Irr}(G)$, either $\langle R_1, \rho \rangle = 0$ or $\langle R_2, \rho \rangle = 0$. The following important result relates geometric conjugacy with disjointness of Deligne-Lusztig characters.

Proposition 60 (Exclusion theorem). *[20, Corollary 2.4.24] Let $w_1, w_2 \in \mathbf{W}$ and $\theta_1 \in \text{Irr}(\mathbf{T}_0[w_1]), \theta_2 \in \text{Irr}(\mathbf{T}_0[w_2])$ be such that (w_1, θ_1) and (w_2, θ_2) are not geometrically conjugate. Then $R_{w_1}^{\theta_1}$ and $R_{w_2}^{\theta_2}$ are disjoint.*

Proposition 61. *Let ρ be an irreducible representation of \mathbf{G}^F , occurring in some Deligne-Lusztig character, R_w^θ . Then*

$$|\langle R_w^\theta, \rho \rangle| \leq |\mathbf{W}|^{\frac{1}{2}}.$$

Proof. Write R_w^θ as a \mathbb{Z} -linear combination of irreducible representations:

$$R_w^\theta = e_1 \rho_1 + \cdots + e_k \rho_k,$$

where $\rho_1 = \rho$. Then

$$\langle R_w^\theta, R_w^\theta \rangle = e_1^2 + \cdots + e_k^2.$$

By Proposition 55, the inner product of two Deligne-Lusztig characters is bounded above by $|\mathbf{W}|$. Therefore, we have $\langle R_w^\theta, \rho \rangle^2 = e_1^2 \leq |\mathbf{W}|$, and the result follows. \square

We record the following useful result:

Lemma 62. *[11, p.33] Let p be a prime and \mathbf{G} an algebraic group over $\overline{\mathbb{F}}_p$ with Frobenius map F , and \mathbf{H} a connected F -stable closed subgroup. Then there is a canonical bijection $\mathbf{G}^F/\mathbf{H}^F \leftrightarrow (\mathbf{G}/\mathbf{H})^F$.*

To prove this, we require the following theorem of Lang and Steinberg (see, for example, [11]):

Theorem 63. *Let \mathbf{H} be a connected algebraic group over $\overline{\mathbb{F}}_p$ with Frobenius map F . Then the map $\mathbf{H} \rightarrow \mathbf{H}; h \mapsto h^{-1}F(h)$ is surjective.*

Proof of Lemma 62. The map $\mathbf{G}^F/\mathbf{H}^F \rightarrow (\mathbf{G}/\mathbf{H})^F$ given by $g\mathbf{H}^F \rightarrow g\mathbf{H}$ is well-defined and injective. Let $g\mathbf{H} \in (\mathbf{G}/\mathbf{H})^F$. Here, $F(g)\mathbf{H} = F(g\mathbf{H}) = g\mathbf{H}$, therefore $g^{-1}F(g) \in \mathbf{H}$. By Theorem 63, there exists $h \in \mathbf{H}$ such that $g^{-1}F(g) = h^{-1}F(h)$, so $F(gh^{-1}) = gh^{-1}$, that is, $gh^{-1} \in \mathbf{G}^F$. Therefore, $gh^{-1}\mathbf{H}^F \mapsto gh^{-1}\mathbf{H} = g\mathbf{H}$. \square

Proposition 64. *[20, Lemma 2.3.5] Let $w \in \mathbf{W}$ and $\theta_1 \in \text{Irr}(\mathbf{T}_0[w])$. There are at most $|\mathbf{W}|$ choices of $\theta_2 \in \text{Irr}(\mathbf{T}_0[w])$ for which (w, θ_2) is geometrically conjugate to (w, θ_1) .*

Proof. Let $d \geq 1$ be admissible as in Definition 59 and fix $w \in \mathbf{W}$. Since the map $N_d^{(w)} : \mathbf{T}_0^{F^d} \rightarrow \mathbf{T}_0[w]$ is surjective, the map $\text{Irr}(\mathbf{T}_0[w]) \rightarrow \text{Irr}(\mathbf{T}_0^{F^d})$ given by $\theta \mapsto \theta \circ N_d^{(w)}$ is injective (this follows from the left-exactness of the functor $\text{Irr}(-) = \text{Hom}(-, \mathbb{C}^\times)$ from abelian groups to sets). Since the size of each \mathbf{W} -orbit on $\text{Irr}(\mathbf{T}_0^{F^d})$ has size at most $|\mathbf{W}|$, and $\theta \circ N_d^{(w)}$ are distinct for distinct θ , there are at most $|\mathbf{W}|$ choices of θ for which $\theta \circ N_d^{(w)}$ lie in the same orbit. Equivalently, there are at most $|\mathbf{W}|$ choices of θ for which (w, θ) lie in the same geometric conjugacy class. \square

Before continuing with the discussion, we state the following useful result:

Lemma 65. *[61] Suppose $f, g \in \mathbb{Q}[x]$ and that $\frac{f(a)}{g(a)} \in \mathbb{Z}$ for infinitely many $a \in \mathbb{Z}$. Then $g \mid f$ in $\mathbb{Q}[x]$.*

Proof. Applying Euclidean division, there exist $q, r \in \mathbb{Q}[x]$ such that $f = qg + r$ and $\deg r < \deg g$.

We are given that

$$\frac{f(a)}{g(a)} = q(a) + \frac{r(a)}{g(a)} \in \mathbb{Z}$$

for infinitely many $a \in \mathbb{Z}$. Suppose that $n \in \mathbb{Z}$ is such that $nq \in \mathbb{Z}[x]$. This implies

$$\frac{nr(a)}{g(a)} = \frac{nf(a)}{g(a)} - nq(a) \in \mathbb{Z}$$

for infinitely many $a \in \mathbb{Z}$. Since $\deg r < \deg g$, $\frac{nr(x)}{g(x)}$ tends to 0 as $x \rightarrow \pm\infty$. In particular, $\frac{nr(x)}{g(x)}$ can only attain non-zero integer values in some bounded interval

$[-K, K]$, so there are still infinitely many $a \in \mathbb{Z}$ such that $|a| > K$ and $\frac{nr(a)}{g(a)} \in \mathbb{Z}$. In particular, for such a , $r(a) = 0$. Since r has infinitely many roots, we have $r = 0$ and the result follows. \square

For a connected reductive group \mathbf{G} with Frobenius map F , one can assign a positive real number q as in [11]. In this thesis, this will be equal to the order of the finite field over which \mathbf{G} is defined (see Appendix A.4, Definition 141 and Proposition 142 for details).

Proposition 66 (Deligne-Lusztig character degrees are polynomial). *Let \mathbf{G} be connected and reductive with Frobenius map F and F -stable maximally split torus \mathbf{T}_0 . For every $w \in \mathbf{W}$, there exists $f_w \in \mathbb{Q}[x]$ such that for all $d \geq 1$ and all $\theta \in \text{Irr}(\mathbf{T}_0[w, F^d])$, $R_w^\theta(1) = f_w(q^d)$, where R_w^θ is a virtual representation of \mathbf{G}^{F^d} .*

Proof. The following argument is due to Geck [19, Remark 4.2]. Denote by $n_{p'}$ the p' -part of the integer n , that is, the unique integer m coprime to p such that $n = p^i m$ for some integer i . By [11, Theorem 7.5.1], $R_w^\theta(1) = \pm[\mathbf{G}^F : \mathbf{T}_0[w]]_{p'}$, where the sign depends only on \mathbf{G} and w . By [11, §2.9], we can write

$$|\mathbf{G}^F| = |(\mathbf{Z}^\circ)^F| q^N \prod_J (q^{|J|} - 1) \sum_{w \in \mathbf{W}^F} q^{l(w)},$$

where the product runs over orbits J of simple roots of \mathbf{G} , $l(w)$ is the minimal length of an expression of w as a product of the Coxeter generators of \mathbf{W} , and \mathbf{Z}° is the connected component of the centre of \mathbf{G} .

By [11, Proposition 3.3.8],

$$|\mathbf{T}_0[w]| = |(\mathbf{Z}^\circ)^F| \chi(q)$$

where χ is a polynomial depending only on F, w and the semisimple algebraic group \mathbf{G}' such that $\mathbf{G} = \mathbf{G}'\mathbf{Z}^\circ$. Since p does not divide $|\mathbf{T}_0[w]|$ [11, p.83], we see that

$$[\mathbf{G}^F : \mathbf{T}_0[w]]_{p'} = \frac{\prod_J (q^{|J|} - 1) \sum_{w \in \mathbf{W}^F} q^{l(w)}}{\chi(q)},$$

which is a polynomial in q by Lemma 65. \square

Suppose that $\rho \in \text{Irr}(\mathbf{G}^F)$. Then writing the regular character of \mathbf{G}^F in terms of Deligne-Lusztig characters by Proposition 56, we obtain

$$\begin{aligned} \dim \rho &= \langle \chi_{\text{reg}}, \rho \rangle \\ &= \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \sum_{\theta \in \text{Irr}(\mathbf{T}_0[w])} R_w^\theta(1) \langle R_w^\theta, \rho \rangle \\ &= \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \sum_{\theta \in \text{Irr}(\mathbf{T}_0[w])} \langle R_w^\theta, \rho \rangle f_w(q). \end{aligned}$$

By Proposition 64, the inner sum has at most $|\mathbf{W}|$ non-zero terms, and for each θ , we have $|\langle R_w^\theta, \rho \rangle| \leq |\mathbf{W}|^{\frac{1}{2}}$. Therefore, the finite set of polynomials

$$R = \left\{ \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} a_w f_w \mid a_w \in \mathbb{Z}, |a_w| \leq |\mathbf{W}|^{\frac{3}{2}} \right\}$$

will give all the dimensions of the irreducible representations of \mathbf{G}^F . This completes the proof of Theorem 52.

Proof of Proposition 54. Even though the polynomials f_w in the proof of Proposition 66 depend on the Frobenius map, for a collection of connected reductive groups with bounded dimension, there are finitely many root systems (and hence finitely many possibilities for the Weyl group and the sizes of the orbits of simple roots). Since χ is a polynomial dividing the numerator in the expression for $[\mathbf{G}^F : \mathbf{T}_0[w]]_p$ (as polynomials, by Lemma 65), we get finitely many options for the polynomial f_w overall. The parameter q in Definition 141 is equal to $|\mathbb{F}|$; see the proof of Proposition 142 (Appendix A.4.2). \square

We shall also require a similar polynomial result for the cardinalities of \mathbb{F} -points of reductive groups:

Lemma 67. *Let M_{dim} be a non-negative integer and p a prime. There exists a finite set of polynomials $R_{\text{card,red}} \subseteq \mathbb{Q}[x]$ such that for every reductive group G defined over a finite field \mathbb{F} of characteristic p with dimension at most M_{dim} ,*

$$|G(\mathbb{F})| \in \{m(|\mathbb{F}|) \mid m \in R_{\text{card,red}}\}.$$

Proof. By the formula for the F -points of a reductive group \mathbf{G} ,

$$|\mathbf{G}^F| = |(\mathbf{Z}^\circ)^F| q^N \prod_J (q^{|J|} - 1) \sum_{w \in \mathbf{W}^F} q^{l(w)}. \quad (4.2)$$

Since \mathbf{Z}° is a torus with dimension at most M_{\dim} , we have

$$|(\mathbf{Z}^\circ)^F| \in \{(q-1)^i \mid 1 \leq i \leq M_{\dim}\}.$$

Since $N = |\Phi^+|$ is the number of positive roots, $N \leq M_{\dim}$. Since J runs over orbits of the Frobenius action on the simple roots, we must have $|J| \leq M_{\dim}$, so there are at most M_{\dim} factors, each of degree at most M_{\dim} . Therefore,

$$\prod_J (q^{|J|} - 1) \in \{(q^{j_1} - 1) \dots (q^{j_l} - 1) \mid l \leq M_{\dim}, j_k \leq M_{\dim} \text{ for all } k\}.$$

Since for $w \in \mathbf{W}^F$, its length is at most the number of positive roots $|\Phi^+|$ (see e.g. [11, Proposition 2.2.11]), we have $l(w) \leq M_{\dim}$ also. Since the Weyl group is determined by the root system of \mathbf{G} , and there are finitely many options for the root system of a reductive group of bounded dimension, there exists $C_{\text{Weyl}}(M_{\dim}) > 0$, depending only on M_{\dim} , such that $|\mathbf{W}| \leq C_{\text{Weyl}}(M_{\dim})$. Therefore,

$$\sum_{w \in \mathbf{W}^F} q^{l(w)} \in \left\{ \sum_{i=1}^{C_{\text{Weyl}}(M_{\dim})} q^{a_i} \mid a_i \leq M_{\dim} \right\}.$$

Putting all of the above polynomials together, define

$$R_{\text{card,red}} = \left\{ (x-1)^{i_1} x^{i_2} (x^{j_1} - 1) \dots (x^{j_l} - 1) \sum_{i=1}^{C_{\text{Weyl}}(M_{\dim})} x^{a_i} \mid i_1, i_2, l, j_k, a_i \leq M_{\dim} \right\}.$$

By its construction, the set $R_{\text{card,red}} \subseteq \mathbb{Q}[x]$ and has the required property. \square

4.2 The Kirillov Orbit Method and its Consequences

In [42], Kirillov gave a complete description of the irreducible representations of a simply connected nilpotent Lie group; they are in bijection with the coadjoint orbits on the Lie algebra. We give the analogue of this bijection in our setting of algebraic groups, and give corollaries regarding the stabilisers of representations of $U(\mathbb{F})$, with a view to applying Clifford theory. For the following statement of the orbit method, see [26, Theorem 4.3.1].

Theorem 68 (The Kirillov Orbit Method). *Let $n \geq 1$ be an integer, \mathbb{F} a finite field of characteristic $p > n$, \mathfrak{g} a nilpotent Lie subalgebra of $M_n(\overline{\mathbb{F}})$ defined over \mathbb{F} and $G = \exp(\mathfrak{g})$. Fix a nontrivial character $\varphi : \mathbb{F} \rightarrow \mathbb{C}^\times$. Then there is a bijection*

$$\begin{aligned} \mathfrak{g}(\mathbb{F})^*/G(\mathbb{F}) &\rightarrow \text{Irr}(G(\mathbb{F})) \\ \Omega &\mapsto \chi_\Omega, \end{aligned}$$

where $G(\mathbb{F})$ has the coadjoint action on $\mathfrak{g}(\mathbb{F})^*$ given by $(g \cdot \lambda)(x) = \lambda(g^{-1}xg)$, and

$$\chi_\Omega(g) = \frac{1}{\sqrt{|\Omega|}} \sum_{\omega \in \Omega} \varphi(\omega(\log(g))).$$

The size of each orbit Ω is an even power of $|\mathbb{F}|$, and $\dim \chi_\Omega = \sqrt{|\Omega|}$.

The following results are corollaries of Theorem 68; see [26] for the proofs.

Lemma 69. [26, Corollary 4.3.2] *For all $n', M \geq 1$, there exists $C_{\text{uni}}(n', M) \geq 1$ such that for every finite field \mathbb{F} of characteristic greater than $C_{\text{uni}}(n', M)$, and every unipotent subgroup U of $(\text{GL}_{n'})_{\overline{\mathbb{F}}}$ which is defined over \mathbb{F} and has complexity at most M ,*

$$\{\dim \rho \mid \rho \in \text{Irr}(U(\mathbb{F}))\} \subseteq \{|\mathbb{F}|^k \mid 0 \leq k \leq (n')^2\}.$$

Lemma 70. [26, Lemma 4.3.4] *Let $n', M \in \mathbb{N}$. There exist $C_{\text{nonhomog}}(n', M) > 0$ and $C_{\text{stab}}(n', M) > 0$ such that for every prime $p > C_{\text{stab}}(n', M)$, every finite field \mathbb{F} of characteristic p , every closed connected subgroup G of $(\text{GL}_{n'})_{\overline{\mathbb{F}}}$ defined over \mathbb{F} and of complexity at most M , and every $\chi \in \text{Irr}(R_u(G)(\mathbb{F}))$, there exists a closed*

subgroup K of G , defined over \mathbb{F} and of complexity at most $C_{\text{nonhomog}}(n', M)$, such that $\text{Stab}_{G(\mathbb{F})}(\chi) = K(\mathbb{F})$.

4.3 Proof of the Polynomial Result

We prove Theorem 51, and deduce Theorem 2 in Section 4.3.2. First, we need to deal with the homogeneous case:

Lemma 71. [26, Lemma 4.2.1] *For every $m \in \mathbb{N}$, there exists $C_{H^2}(m) > 0$ such that for every prime $p > C_{H^2}(m)$, every finite field \mathbb{F} of characteristic p and every connected reductive G defined over \mathbb{F} of dimension at most m , $H^2(G(\mathbb{F}), \mu_{p^\infty}) = 1$.*

Lemma 72. [26, Corollary 4.2.6] *Let $p > C_{H^2}(1 + \dim G)$ and G a connected linear algebraic group defined over a finite field \mathbb{F} of characteristic p with unipotent radical U . Then every $\rho \in \text{Irr}(G(\mathbb{F}))$ whose restriction to $U(\mathbb{F})$ is homogeneous has the form $\rho = \pi_1 \otimes \pi_2$, where π_1 is the pullback of a representation of $(G/U)(\mathbb{F})$ to $G(\mathbb{F})$, and π_2 is the extension of a representation of $U(\mathbb{F})$ to $G(\mathbb{F})$.*

We also need a result (Lemma 74) which shows that the cardinality of \mathbb{F} -points of algebraic groups with bounded dimension are given by finitely many polynomials. It is a generalisation of the reductive case proved earlier (Lemma 67).

Definition 73. [26, Lemma 4.5.1] Let $n', M_{\text{dim}}, M_{\text{cmp}}$ be given, and \mathbb{F} a finite field. Define

$$GC(n', M_{\text{dim}}, M_{\text{cmp}}, \mathbb{F}) = \{|G(\mathbb{F})| \mid G \in \mathcal{A}(n', M_{\text{dim}}, M_{\text{cmp}}, \mathbb{F})\}$$

The following result is an analogue of [26, Lemma 4.5.1], with polynomials in place of an upper bound on $|GC(n', M_{\text{dim}}, M_{\text{cmp}}, \mathbb{F})|$.

Lemma 74. *Let $n', M_{\text{dim}}, M_{\text{cmp}}$ be given. There exists $C_{\text{card}}(n', M_{\text{dim}}, M_{\text{cmp}})$ such that for all $p > C_{\text{card}}(n', M_{\text{dim}}, M_{\text{cmp}})$, there exists a finite set $R_{\text{card}, n', M_{\text{dim}}, M_{\text{cmp}}} \subseteq \mathbb{Q}[x]$ such that for every finite field \mathbb{F} of characteristic p , we have*

$$GC(n', M_{\text{dim}}, M_{\text{cmp}}, \mathbb{F}) \subseteq \{m(|\mathbb{F}|) \mid m \in R_{\text{card}}\}.$$

Proof. Let $G \in \mathcal{A}(n', M_{\dim}, M_{\text{cmp}}, \mathbb{F})$. Let U denote the unipotent radical of G° . By Lemma 62, we have

$$|G(\mathbb{F})| = |(G/G^\circ)(\mathbb{F})| \cdot |(G^\circ/U)(\mathbb{F})| \cdot |U(\mathbb{F})|.$$

By [26, Lemma 3.1.2], there exists $C_{\text{irred}}(M_{\text{cmp}})$, depending only on M_{cmp} , such that G has at most $C_{\text{irred}}(M_{\text{cmp}})$ geometrically irreducible components. Thus, $|(G/G^\circ)(\mathbb{F})| \leq C_{\text{irred}}(M_{\text{cmp}})$.

Since G°/U is reductive, by Lemma 67 there exists $R_{\text{card,red}} \subseteq \mathbb{Q}[x]$ finite such that for all finite fields \mathbb{F} ,

$$|(G^\circ/U)(\mathbb{F})| \in \{m(|\mathbb{F}|) \mid m \in R_{\text{card,red}}\}.$$

Since U is unipotent, $|U(\mathbb{F})|$ is a power of $|\mathbb{F}|$, and by the dimension bound, $|U(\mathbb{F})| \in \{1, |\mathbb{F}|, \dots, |\mathbb{F}|^{M_{\dim}}\}$ (see e.g. [18, Theorem 4.2.4]). By the above construction, the set

$$R_{\text{card}} = \{am(x)x^b \mid 1 \leq a \leq C_{\text{irred}}(M_{\text{cmp}}), m \in R_{\text{card,red}}, 1 \leq b \leq M_{\dim}\}$$

satisfies the required properties. □

Proof of Theorem 51. By [59, Lemma 11], there exists $C_{\text{ic}}(M_{\text{cmp}}) > 0$ (assume it is an integer) such that for every finite field \mathbb{F} and every $G \in \mathcal{A}(n', M_{\dim}, M_{\text{cmp}}, \mathbb{F})$, we have

$$[G(\mathbb{F}) : G^\circ(\mathbb{F})] = |(G/G^\circ)(\mathbb{F})| \leq C_{\text{ic}}(M_{\text{cmp}}),$$

and that the dimensions of the representations of $G(\mathbb{F})$ are sums of dimensions of representations of $G^\circ(\mathbb{F})$ with at most $C_{\text{ic}}(M_{\text{cmp}})$ terms. Therefore, if the result holds for connected groups, it holds for G , so we assume that G is connected.

We argue by induction on M_{\dim} . For the base case, $M_{\dim} = 0$, the only connected zero-dimensional closed subgroup of $(\text{GL}_{n'})_{\overline{\mathbb{F}}}$ is the trivial group, so we choose $R_{n',0,M_{\text{cmp}}} = \{1\}$. We proceed with the inductive step.

Step 1: Reduce to those G which are neither reductive nor unipotent.

We assume for induction that for all $M'_{\text{cmp}} \in \mathbb{N}$, there exists a finite set of polynomials $R_{n', M_{\text{dim}}-1, M'_{\text{cmp}}} \subseteq \mathbb{Q}[x]$ such that for every \mathbb{F} of characteristic p ,

$$\bigcup_{G \in \mathcal{A}(n', M_{\text{dim}}-1, M'_{\text{cmp}}, \mathbb{F})} \dim_{\text{irr}}(G(\mathbb{F})) \subseteq \{m(|\mathbb{F}|) \mid m \in R_{n', M_{\text{dim}}-1, M'_{\text{cmp}}}\}.$$

We need to show that there exists a finite set $R_{n', M_{\text{dim}}, M_{\text{cmp}}} \subseteq \mathbb{Q}[x]$ such that

$$\bigcup_{G \in \mathcal{A}(n', M_{\text{dim}}, M_{\text{cmp}}, \mathbb{F})} \dim_{\text{irr}}(G(\mathbb{F})) \subseteq \{m(|\mathbb{F}|) \mid m \in R_{n', M_{\text{dim}}, M_{\text{cmp}}}\}.$$

Consider the case where G is reductive. By Proposition 54, there exists a finite set $T_{\text{red}, M_{\text{dim}}} \subseteq \mathbb{Q}[x]$ such that for every prime p and every finite field \mathbb{F} of characteristic p ,

$$\bigcup_{\substack{G \in \mathcal{A}(n', M_{\text{dim}}, M_{\text{cmp}}, \mathbb{F}) \\ G \text{ reductive}}} \dim_{\text{irr}}(G(\mathbb{F})) \subseteq \{m(|\mathbb{F}|) \mid m \in T_{\text{red}, M_{\text{dim}}}\}.$$

Consider the case where G is unipotent. We may assume $p > C_{\text{uni}}(n', M_{\text{cmp}})$, and by Lemma 69, there exists a finite set $T_{\text{uni}, M_{\text{dim}}} \subseteq \mathbb{Q}[x]$ and $C_{\text{uni}} > 0$ such that for every finite field \mathbb{F} of characteristic p ,

$$\bigcup_{\substack{G \in \mathcal{A}(n', M_{\text{dim}}, M_{\text{cmp}}, \mathbb{F}) \\ G \text{ unipotent}}} \dim_{\text{irr}}(G(\mathbb{F})) \subseteq \{m(|\mathbb{F}|) \mid m \in T_{\text{uni}, M_{\text{dim}}}\}.$$

In fact, Lemma 69 implies that we can choose $T_{\text{uni}, M_{\text{dim}}} = \{1, x, \dots, x^{(n')^2}\}$.

Step 2: Homogeneous and non-homogeneous restrictions to $U(\mathbb{F})$

Let $U = R_u(G)$ and suppose now that $U \neq 1, G$ and $\rho \in \text{Irr}(G(\mathbb{F}))$. In order to apply the results of Section 4.2 onwards, we need to assume that p is large enough relative to the dimension and complexity of the groups in question. We bound p below by functions of n' , M_{dim} and M_{cmp} , which is allowable since the quantifiers for n' , M_{dim} and M_{cmp} come before the lower bound for p in Theorem 51. We split the proof into two cases, depending on whether or not $\text{Res}_{U(\mathbb{F})}^{G(\mathbb{F})} \rho$ is homogeneous.

Our assumptions on p will allow us to conclude that $\dim \rho$ is either:

- (a) a product of dimensions of irreducible representations of $U(\mathbb{F})$ and $(G/U)(\mathbb{F})$

(the homogeneous case), or

- (b) a product of $[G(\mathbb{F}) : K(\mathbb{F})]$ and the dimension of a representation of $K(\mathbb{F})$, where K is a proper closed subgroup of G defined over \mathbb{F} , with bounded complexity (the non-homogeneous case).

Step 2(a): The homogeneous case

If $\text{Res}_{U(\mathbb{F})}^{G(\mathbb{F})} \rho$ is homogeneous, we may assume that $p > C_{H^2}(1 + M_{\dim})$. By Lemma 72, ρ is a tensor product of an extension of a representation of $U(\mathbb{F})$ with a pullback of a representation of $G(\mathbb{F})/U(\mathbb{F}) = (G/U)(\mathbb{F})$. Define

$$T_{\text{homog}, M_{\dim}} = \{m_1(x)m_2(x) \mid m_1 \in T_{\text{uni}, M_{\dim}}, m_2 \in T_{\text{red}, M_{\dim}}\}.$$

Then since U is unipotent and G/U is reductive, for every finite field \mathbb{F} of characteristic p ,

$$\begin{aligned} & \bigcup_{\substack{G \in \mathcal{A}(n', M_{\dim}, M_{\text{cmp}}, \mathbb{F}) \\ R_u(G) \neq 1, G}} \{\dim \rho \mid \rho \in \text{Irr}(G(\mathbb{F})), \text{Res}_{U(\mathbb{F})}^{G(\mathbb{F})} \rho \text{ is homogeneous}\} \\ & \subseteq \{m(|\mathbb{F}|) \mid m \in T_{\text{homog}, M_{\dim}}\}. \end{aligned}$$

Step 2(b): The non-homogeneous case

We claim that there is a finite set $T_{\text{nonhomog}, M_{\dim}} \subseteq \mathbb{Q}[x]$ such that for every finite field \mathbb{F} of characteristic p ,

$$\begin{aligned} & \bigcup_{\substack{G \in \mathcal{A}(n', M_{\dim}, M_{\text{cmp}}, \mathbb{F}) \\ R_u(G) \neq 1, G}} \{\dim \rho \mid \rho \in \text{Irr}(G(\mathbb{F})), \text{Res}_{U(\mathbb{F})}^{G(\mathbb{F})} \rho \text{ is non-homogeneous}\} \\ & \subseteq \{m(|\mathbb{F}|) \mid m \in T_{\text{nonhomog}, M_{\dim}}\}. \end{aligned}$$

If $\text{Res}_{U(\mathbb{F})}^{G(\mathbb{F})} \rho$ is not homogeneous, let ψ be an irreducible constituent of $\text{Res}_{U(\mathbb{F})}^{G(\mathbb{F})} \rho$ and assume that $p > C_{\text{stab}}(n', M_{\text{cmp}})$. By Lemma 70, there exists a proper closed subgroup $K < G$, defined over \mathbb{F} , with $K \in \mathcal{A}(n', M_{\dim} - 1, C_{\text{nonhomog}}(n', M_{\text{cmp}}), \mathbb{F})$ such that $\text{Stab}_{G(\mathbb{F})}(\psi) = K(\mathbb{F})$. Hence, by part (2) of Proposition 22, there exists

$\theta \in \text{Irr}(K(\mathbb{F}) \mid \psi)$ such that $\rho = \text{Ind}_{K(\mathbb{F})}^{G(\mathbb{F})} \theta$. Comparing dimensions,

$$\dim \rho = [G(\mathbb{F}) : K(\mathbb{F})] \dim \theta.$$

Assume that $p > C_{\text{card}}(n', M_{\text{dim}}, M_{\text{cmp}})$ and $p > C_{\text{card}}(n', M_{\text{dim}-1}, C_{\text{nonhomog}}(n', M_{\text{cmp}}))$.

By Lemma 74, for every finite field \mathbb{F} of characteristic p ,

$$[G(\mathbb{F}) : K(\mathbb{F})] = \frac{m_1(|\mathbb{F}|)}{m_2(|\mathbb{F}|)},$$

where $m_1 \in R_{\text{card}, n', M_{\text{dim}}, M_{\text{cmp}}}$ and $m_2 \in R_{\text{card}, n', M_{\text{dim}-1}, C_{\text{nonhomog}}(n', M_{\text{cmp}})}$. Therefore, we obtain a finite set of rational functions

$$\left\{ \frac{m_1}{m_2} \mid \begin{array}{l} m_1 \in R_{\text{card}, n', M_{\text{dim}}, M_{\text{cmp}}}, \\ m_2 \in R_{\text{card}, n', M_{\text{dim}-1}, C_{\text{nonhomog}}(n', M_{\text{cmp}})} \end{array} \right\} \subseteq \mathbb{Q}(x).$$

If there are infinitely many \mathbb{F} such that $m_1(|\mathbb{F}|)/m_2(|\mathbb{F}|) = [G(\mathbb{F}) : K(\mathbb{F})]$ for some pair (G, K) appearing in the process, then m_1/m_2 is a polynomial by Lemma 65. If there are only finitely many \mathbb{F} such that $m_1(|\mathbb{F}|)/m_2(|\mathbb{F}|) = [G(\mathbb{F}) : K(\mathbb{F})]$ for some (G, K) , then replace m_1/m_2 with the values $[G(\mathbb{F}) : K(\mathbb{F})]$ for all such \mathbb{F} ; this is a finite set of integers. We obtain a finite set of polynomials $T_{\text{index}, M_{\text{dim}}} \subseteq \mathbb{Q}[x]$ with the property that for all \mathbb{F} of characteristic p ,

$$[G(\mathbb{F}) : K(\mathbb{F})] \in \{m(|\mathbb{F}|) \mid m \in T_{\text{index}, M_{\text{dim}}}\}.$$

By the equation $\dim \rho = [G(\mathbb{F}) : K(\mathbb{F})] \dim \theta$, the finite set

$$T_{\text{nonhomog}, M_{\text{dim}}} = \{m_3(x)m_4(x) \mid m_3 \in T_{\text{index}, M_{\text{dim}}}, m_4 \in T_{M_{\text{dim}-1}, C_{\text{nonhomog}}(n', M_{\text{cmp}})}\}$$

has the property that for all finite fields \mathbb{F} of characteristic p , all $G \in \mathcal{A}(n', M_{\text{dim}}, M_{\text{cmp}}, \mathbb{F})$ and all $\rho \in \text{Irr}(G(\mathbb{F}))$, we have $\dim \rho \in \{m(|\mathbb{F}|) \mid m \in T_{\text{nonhomog}, M_{\text{dim}}}\}$, as claimed.

Step 3: Conclusion

Since we already have defined $R_{n',0,M_{\text{cmp}}}$, define $R_{n',M_{\text{dim}},M_{\text{cmp}}}$ inductively by:

$$R_{n',M_{\text{dim}},M_{\text{cmp}}} := R_{n',M_{\text{dim}}-1,M_{\text{cmp}}} \cup T_{\text{red},M_{\text{dim}}} \cup T_{\text{uni},M_{\text{dim}}} \cup T_{\text{homog},M_{\text{dim}}} \cup T_{\text{nonhomog},M_{\text{dim}}}.$$

By the construction in the inductive step, $R_{n',M_{\text{dim}},M_{\text{cmp}}}$ has the required properties in the statement of the theorem. \square

4.3.1 The Greenberg Functor

Throughout this section, let \mathfrak{o} be the ring of integers of a non-Archimedean local field with finite residue field k , and \mathfrak{o}' an unramified extension of \mathfrak{o} with residue field k' .

In [24], Greenberg constructed a functor, Gr_{r-1}° , mapping schemes of finite type over \mathfrak{o}_r to schemes of finite type over k , with the property that

$$\text{Gr}_{r-1}^{\circ}(X)(k) = X(\mathfrak{o}_r).$$

Proposition 75. *If X is smooth and affine of finite type over \mathfrak{o}_r , then $\text{Gr}_{r-1}^{\circ}(X)$ is smooth and affine of finite type over k . Furthermore, Gr_{r-1}° preserves fibre products, so if X is a group scheme, then $\text{Gr}_{r-1}^{\circ}(X)$ is also naturally a group scheme over k .*

Proof. This result is the content of Corollary 1 of Proposition 4 and Corollary 4 of Proposition 6 in [24], and [25, Corollary 1]. \square

We also require the following result of Nicaise and Sebag [48, Proposition 3.3]:

Theorem 76. *Let X be a scheme of finite type over \mathfrak{o}_r . Observe that \mathfrak{o}'_r is an \mathfrak{o}_r -algebra via the natural map $\mathfrak{o}_r \rightarrow \mathfrak{o}'_r$; this induces a isomorphism of k' -schemes:*

$$\text{Gr}_{r-1}^{\circ}(X) \times_k k' \cong \text{Gr}_{r-1}^{\circ'}(X \times_{\mathfrak{o}_r} \mathfrak{o}'_r).$$

This result is stated only for finite extensions $\mathfrak{o}'/\mathfrak{o}$ in [48], however the proof does not make use of the assumption that the extension is finite.

4.3.2 Proof of Theorem 2

We can now prove a polynomial result for unramified extensions of a fixed base field:

Corollary 77. *Let G be a smooth affine group scheme over \mathbb{Z} and $r \geq 1$ be an integer. There exists $C > 0$ such that for every $p > C$ and every non-Archimedean local field K with finite residue field of characteristic p , there exists a finite set of polynomials $R \subseteq \mathbb{Q}[x]$ such that for every \mathfrak{o} which is the valuation ring of some finite unramified extension of K ,*

$$\dim_{\text{irr}}(G(\mathfrak{o}_r)) \subseteq \{m(|\mathfrak{o}_1|) \mid m \in R\}.$$

Proof. Let $\mathcal{G} = \text{Gr}_{r-1}^{\mathfrak{o}}(G \times_{\mathbb{Z}} \mathfrak{o}_r)$, an algebraic group over \mathbb{F}_q , where $q = |\mathfrak{o}_1|$. Then $G(\mathfrak{o}_r) \cong \mathcal{G}(\mathbb{F}_q)$. By Theorem 76, all such \mathcal{G} are given by extension of scalars from $\mathcal{G}_0 := \text{Gr}_{r-1}^{\mathfrak{o}_K}(G)$. Therefore, all such \mathcal{G} have the same extension of scalars to an algebraic group over $\overline{\mathbb{F}_p}$, which is contained in some $\mathcal{A}(n', d, c, \mathbb{F}_q)$. The result now follows from Theorem 51. \square

Note that the Greenberg functor does not preserve the property of group schemes being reductive, which is why we have to treat the non-reductive case in Theorem 51.

We would like to extend Corollary 77 to all non-Archimedean local fields K with finite residue field. It is already known that such K must be isomorphic as a topological field to either $\mathbb{F}_q((t))$ or a finite extension of \mathbb{Q}_p . As it turns out, we can ignore the case where K has ramification index greater than or equal to r over \mathbb{Q}_p .

Proposition 78. *Let K be a finite extension of \mathbb{Q}_p or $\mathbb{F}_p((t))$ with residue cardinality q and ramification index $e \geq r$. Then $\mathfrak{o}_K/\mathfrak{p}_K^r \cong \mathbb{F}_q[[t]]/(t^r)$.*

Proof. Let L be the maximal unramified extension of \mathbb{Q}_p contained in K . Write $\mathfrak{p}_L, \mathfrak{p}_K$ for the primes in L and K respectively. There are natural maps

$$\mathbb{F}_q = \mathfrak{o}_L/\mathfrak{p}_L \rightarrow \mathfrak{o}_K/\mathfrak{p}_K^e \twoheadrightarrow \mathfrak{o}_K/\mathfrak{p}_K^r.$$

The first map is well-defined because $\mathfrak{p}_K^e = \mathfrak{p}_L \mathfrak{o}_K \supseteq \mathfrak{p}_L$, and the second natural map is well-defined because $e \geq r$. Therefore, $\mathfrak{o}_K/\mathfrak{p}_K^r$ is naturally an \mathbb{F}_q -algebra, and we

can extend to a map

$$\mathbb{F}_q[[t]] \rightarrow \mathfrak{o}_K/\mathfrak{p}_K^r$$

by mapping $t \mapsto \varpi$. This latter map is surjective with kernel (t^r) . \square

We are now ready to prove the main result of this chapter.

Proof of Theorem 2. For the moment, fix a finite extension K/\mathbb{Q}_p of ramification index $e < r$ and residue degree f . Since the lower bound for p (C in the statement of the theorem) is allowed to depend on r , we can assume $p \geq r$ so that p does not divide e , and K is tamely ramified over \mathbb{Q}_p . Denote by K^{unr} the maximal unramified extension of K , which has residue field $\overline{\mathbb{F}_p}$ (see Proposition 106). By Corollary 104 (Appendix A.1), we can write $L = K^{\text{unr}} = \mathbb{Q}_p^{\text{unr}}(\sqrt[e]{p})$ and $\mathcal{G}_{\mathfrak{o}_K, r} = \text{Gr}_{r-1}^{\mathfrak{o}_K}(G_{\mathfrak{o}_K/\mathfrak{p}_K^r})$, where $G_{\mathfrak{o}_K/\mathfrak{p}_K^r} = G \times_{\mathbb{Z}} \mathfrak{o}_K/\mathfrak{p}_K^r$. By Theorem 76 applied to the extension $\mathfrak{o}_L/\mathfrak{o}_K$, we get

$$\begin{aligned} \mathcal{G}_{\mathfrak{o}_K, r} \times_{\mathbb{F}_p, f} \overline{\mathbb{F}_p} &\cong \text{Gr}_{r-1}^{\mathfrak{o}_L}(G_{\mathfrak{o}_K/\mathfrak{p}_K^r} \times_{\mathfrak{o}_K/\mathfrak{p}_K^r} \mathfrak{o}_L/\mathfrak{p}_L^r) \\ &\cong \text{Gr}_{r-1}^{\mathfrak{o}_L}(G_{\mathfrak{o}_L/\mathfrak{p}_L^r}), \end{aligned}$$

which does not depend on K . Therefore, if K_1 and K_2 are extensions of \mathbb{Q}_p with ramification index e , then $\mathcal{G}_{\mathfrak{o}_{K_1}, r}$ and $\mathcal{G}_{\mathfrak{o}_{K_2}, r}$ have the same extension of scalars \mathcal{H} to the residue field of $K_1^{\text{unr}} = K_2^{\text{unr}} = \mathbb{Q}_p^{\text{unr}}(\sqrt[e]{p})$. We can now repeat the argument of Corollary 77: since \mathcal{H} is contained in some $\mathcal{A}(n', d, c, \mathbb{F}_q)$, there exists $C_e > 0$ such that for all $p > C_e$, there exists a finite set $R_e \subseteq \mathbb{Q}[x]$ such that for every \mathfrak{o} which is the valuation ring of an extension of \mathbb{Q}_p of ramification index e ,

$$\dim_{\text{irr}}(G(\mathfrak{o}_r)) \subseteq \{m(|\mathfrak{o}_1|) \mid m \in R_e\}.$$

Now apply Corollary 77 in the positive characteristic case to get $C_{\text{fun}} > 0$ such that for all $p > C_{\text{fun}}$, there exists a finite set $R_{\text{fun}} \subseteq \mathbb{Q}[x]$ such that for every power q of p ,

$$\dim_{\text{irr}}(G(\mathbb{F}_q[[t]]/(t^r))) \subseteq \{m(q) \mid m \in R_{\text{fun}}\}.$$

Choose $C = \max(C_1, \dots, C_{r-1}, C_{\text{fun}})$ and let

$$R = R_{\text{fun}} \cup \bigcup_{e=1}^{r-1} R_e$$

for the result. □

Application of Motivic Integration to $\mathrm{GL}_n(\mathfrak{o}_r)$

5.1 The Weak Onn Conjecture for Large p

Throughout, fix integers $n, r \geq 1$. For a local field K and $d \geq 1$, denote by $K^{(d)}$ the unique unramified extension of K with degree d .

Let G be a smooth affine group scheme of finite type over \mathbb{Z} . In [26, Theorem 1.1.4], Hadas showed that for all $r \geq 1$, there exists $C > 0$ such that for all prime $p > C$ and every power q of p , there is an isomorphism of group algebras $\mathbb{C}[G(\mathfrak{o}_{\mathbb{Q}_p^{(d)}}/\mathfrak{p}_{\mathbb{Q}_p^{(d)}}^r)] \cong \mathbb{C}[G(\mathbb{F}_{p^d}[[t]]/t^r)]$. An essential part of the proof is the result [2, Proposition 3.0.2] that for every scheme X of finite type over \mathbb{Z} and every integer $r \geq 1$, there exists $C > 0$ such that for every prime $p > C$ and every power q of p ,

$$|X(\mathfrak{o}_{\mathbb{Q}_p^{(d)}}/\mathfrak{p}_{\mathbb{Q}_p^{(d)}}^r)| = |X(\mathbb{F}_q[[t]]/(t^r))|.$$

The proof follows from results of [12, §9]; we will explain why this is the case in Section 5.2. The same argument generalises to obtain the following result:

Theorem 79. *Let X be a scheme of finite type over \mathbb{Z} and $r \geq 1$ an integer. There exists $C > 0$ such that for every prime $p > C$, and every non-Archimedean local*

field K with finite residue field and residue characteristic p ,

$$|X(\mathfrak{o}_K/\mathfrak{p}_K^r)| = |X(\mathbb{F}_q[[t]]/(t^r))|.$$

This result is “folklore” for experts. Before proving Theorem 79, we summarise the argument in [26] which shows that Theorem 79 implies Theorem 8.

The following result is due to Frobenius (see e.g. [44, Lemma 3.1(ii)]):

Proposition 80. *Let G be a finite group and m an integer. The number of solutions to the equation $[x_1, y_1] \cdots [x_m, y_m] = 1$ in G is $|G|^{2m-1} \zeta_G(2m-2)$, where ζ_G is the representation zeta function of G .*

We need to determine when the representation zeta functions of two finite groups are equal. It turns out that we only need to check whether they agree on a finite (but sufficiently large) number of points.

Lemma 81. [26, Corollary 5.0.2] *Let G_1, G_2 be finite groups, and k an integer such that $|\dim_{\text{irr}}(G_1)| \leq k$. Then if $\zeta_{G_1}(s) = \zeta_{G_2}(s)$ for $2k+1$ distinct values of s , then $\zeta_{G_1} = \zeta_{G_2}$.*

Lemma 82. (cf. [26, Lemma 5.0.3]) *Let G be a smooth affine group scheme of finite type over \mathbb{Z} , $r \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{-1\}$. Then there exists $C > 0$ such that for every prime $p > C$, every p -power q and every non-Archimedean local field K with residue cardinality q , $\zeta_{G(\mathbb{F}_q[[t]]/(t^r))}(2m) = \zeta_{G(\mathfrak{o}_K/\mathfrak{p}_K^r)}(2m)$.*

Proof. For $m = -1$, the statement is equivalent to Theorem 79 for $X = G$. For $m \geq 1$, Theorem 80 gives

$$\zeta_{G(\mathfrak{o}_r)}(2m) = |G(\mathfrak{o}_r)|^{1-2m} |\{(g_1, \dots, g_m, h_1, \dots, h_m) \in G(\mathfrak{o}_r)^{2m} \mid [g_1, h_1] \cdots [g_m, h_m] = 1\}|,$$

where $\mathfrak{o} = \mathfrak{o}_K$ or $\mathbb{F}_q[[t]]$. Therefore, if we define

$$X(R) = \{(g_1, \dots, g_m, h_1, \dots, h_m) \in G(R) \mid [g_1, h_1] \cdots [g_m, h_m] = 1\},$$

it is sufficient to show that $|X(\mathfrak{o}_K/\mathfrak{p}_K^r)| = |X(\mathbb{F}_q[[t]]/(t^r))|$. In fact, X is an affine

scheme of finite type. Indeed, X is representable by

$$\mathbb{Z}[x_1^\pm, \dots, x_m^\pm, y_1^\pm, \dots, y_m^\pm] / (x_1 y_1 x_1^{-1} y_1^{-1} \dots x_m y_m x_m^{-1} y_m^{-1} - 1),$$

therefore $|X(\mathfrak{o}_K/\mathfrak{p}_K^r)| = |X(\mathbb{F}_q[[t]]/(t^r))|$ by Theorem 79. The result follows. \square

Note that on its own, Lemma 82 does not guarantee that we have equality of zeta functions for large enough p . Indeed, $|\dim_{\text{irr}}(G(\mathbb{F}_q[[t]]/(t^r)))|$ could grow without bound as q varies among powers of p . However, by Theorem 1, this is not the case:

Proof that Theorem 79 implies Theorem 8. The same argument as in the proof of [26, Theorem 1.1.4] applies. We have a group isomorphism $\text{Gr}_{r-1}^{\mathfrak{o}_K}(G)(\mathbb{F}_q) \cong G(\mathbb{F}_q[[t]]/(t^r))$. By Theorem 1, there exist $C, N > 0$ such that for every $p > C$ and every p -power q , $|\dim_{\text{irr}}(G(\mathbb{F}_q[[t]]/(t^r)))| < N$. By Lemma 82, the zeta functions $\zeta_{G(\mathbb{F}_q[[t]]/(t^r))}$ and $\zeta_{\mathfrak{o}_K/\mathfrak{p}_K^r}$ agree on at least $2N + 1$ points, so are equal by Lemma 81. \square

5.2 Motivic Integration

We turn our attention to proving Theorem 79. Suppose X is given by the polynomial equations $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$, and let K be a non-Archimedean local field with valuation ring \mathfrak{o} and finite residue field. From now on, fix an integer $r \geq 1$. We want to show that $|X(\mathfrak{o}_r)|$ depends only on q when p is large enough. The first step of the proof is to relate $|X(\mathfrak{o}_r)|$ to the measure of a subset in K^n as follows: define

$$\begin{aligned} A_r(K) &= \{x \in \mathfrak{o}_K^n \mid f_i(x) \equiv 0 \pmod{\mathfrak{p}_K^r} \text{ for all } i\} \\ &= \{x \in \mathfrak{o}_K^n \mid \text{ord}(f_i(x)) \geq r \text{ for all } i\}, \end{aligned}$$

where ord denotes the discrete valuation on K .

Proposition 83. *We have $|X(\mathfrak{o}_K/\mathfrak{p}_K^r)| = q^{rn} \mu_K(A_r(K))$, where μ_K is the Haar measure on K normalised so that $\mu_K(\mathfrak{o}_K) = 1$.*

Proof. The Haar measure is translation-invariant, therefore the fibres of the map $\pi : \mathfrak{o}_K^n \rightarrow (\mathfrak{o}_K/\mathfrak{p}_K^r)^n$ have constant measure q^{-rn} . Regarding $X(\mathfrak{o}_K/\mathfrak{p}_K^r)$ as a sub-

set of $(\mathfrak{o}_K/\mathfrak{p}_K^r)^n$, one has $A_r(K) = \pi^{-1}(X(\mathfrak{o}_K/\mathfrak{p}_K^r))$, and is a union of fibres of π . Therefore, we get $\mu_K(A_r(K)) = q^{-rn}|X(\mathfrak{o}_K/\mathfrak{p}_K^r)|$. \square

In order to show that $|X(\mathfrak{o}_r)|$ depends only on q , it is sufficient to show that the measure of the set $\mu_K(A_r(K))$ depends only on q . However, the difficulty arises from having to compare the Haar measures μ_K for different choices of K . We will outline a concept due to Denef and Loeser [16], known as the **motivic measure**, which allows us to assign a “size” (albeit not an element of \mathbb{R}) to the assignment $K \mapsto A_r(K)$. This motivic measure will specialise to the Haar measure on each local field K , for large residue characteristic (Theorem 92). This will allow us to compare measures on different non-Archimedean local fields, in particular enabling a comparison between the zero and positive characteristic cases.

Definition 84. Let L be a number field with ring of integers \mathfrak{o} . Write $\mathcal{C}_\mathfrak{o}$ for the collection of local fields which are either p -adic completions of the finite extensions of L , or those of positive characteristic equipped with an \mathfrak{o} -algebra structure. Let $\mathcal{C}_{\mathfrak{o},N}$ be the collection of all the fields in $\mathcal{C}_\mathfrak{o}$ whose residue characteristic is greater than N .

5.2.1 The Denef-Pas Language

Definition 85. The Denef-Pas language is the 3-sorted language

$$\mathcal{L}_{\text{DP}} = (\mathcal{L}_{\text{Val}}, \mathcal{L}_{\text{Res}}, \mathcal{L}_{\text{Pres}}, \text{ord}, \overline{\text{ac}}),$$

where $\mathcal{L}_{\text{Val}}, \mathcal{L}_{\text{Res}}$ are both the language of rings, $\mathcal{L}_{\text{Pres}}$ is the Presburger language (totally ordered abelian groups with 0, 1 and congruence modulo n , \equiv_n , for each $n \geq 1$), ord for the valuation map on a valued field, and $\overline{\text{ac}}$ for the angular component map

$$\overline{\text{ac}}(x) = x\varpi^{-\text{ord}(x)} \bmod \varpi.$$

Denote by $\mathcal{L}_\mathfrak{o}$ the language \mathcal{L}_{DP} with coefficients in $\mathfrak{o}[[t]]$, that is, with one constant symbol for each element of $\mathfrak{o}[[t]]$. If K is in $\mathcal{C}_\mathfrak{o}$, then a choice of uniformiser ϖ_K of K

gives K an $\mathfrak{o}[[t]]$ -algebra structure by the map $\lambda_{\mathfrak{o},K} : t \mapsto \varpi_K$. In an $\mathcal{L}_{\mathfrak{o}}$ -structure, we interpret an element of $\mathfrak{o}[[t]]$ as its image under $\lambda_{\mathfrak{o},K}$.

Definition 86. Let K be a non-Archimedean local field with finite residue field k_K and φ an \mathcal{L}_{DP} -formula with m, n, r free variables in the respective sorts. For every field extension K/k , let $h_{\varphi}(K)$ be the subset of $K^m \times k_K^n \times \mathbb{Z}^r$ defined by φ . Any assignment of a definable set to every extension of k in this manner is called a **definable assignment**. There is a trivial case, the definable assignment given by $h[m, n, r](K) = K^m \times k_K^n \times \mathbb{Z}^r$. A **point** of a definable assignment S is a pair (x_0, K) , where K is a field extension of k and $x_0 \in S(K)$.

The category of definable assignments over k , with morphisms given by definable maps, is denoted Def_k . Given S in Def_k , we write $\text{Def}_S = \text{Def}_k/S$. An important subcategory of Def_S is RDef_S , the category of definable subassignments of $S \times h[0, n, 0]$. Where we need to make reference to the language $\mathcal{L}_{\mathfrak{o}}$, we write $\text{RDef}_S(\mathcal{L}_{\mathfrak{o}})$.

5.2.2 Constructible Motivic Functions

Definition 87. Let \mathbb{L} be a formal symbol and

$$\mathbb{A} = \mathbb{Z} \left[\mathbb{L}, \mathbb{L}^{-1}, \left(\frac{1}{1 - \mathbb{L}^{-i}} \right)_{i>0} \right].$$

For a definable assignment S in Def_k , the ring of **Presburger functions on S** is the subring $\mathcal{P}(S)$ of functions from the set of points of S to \mathbb{A} generated by constant functions, definable functions $S \rightarrow \mathbb{Z}$ and \mathbb{L}^{β} , where $\beta : S \rightarrow \mathbb{Z}$ is definable.

Let $\mathcal{P}^0(S)$ be the subring of $\mathcal{P}(S)$ generated by \mathbb{L} and the indicator functions 1_Y , for definable subassignments Y of S .

Example 88. Let $\Lambda = h[0, 0, 0]$ be the point. Then a definable function $\Lambda \rightarrow \mathbb{Z}$ can be viewed as an element of \mathbb{Z} , therefore $\mathcal{P}(\Lambda) = \mathbb{A}$ and $\mathcal{P}^0(\Lambda) = \mathbb{Z}[\mathbb{L}]$.

Definition 89. [13, §5.1.2] The Grothendieck ring $K_0(\text{RDef}_S)$ is the free abelian group on symbols $[Y]$, for Y in RDef_S , modulo the relations $[Y \rightarrow S] = [Y' \rightarrow S]$ if $Y \rightarrow S$ is isomorphic to $Y' \rightarrow S$, and

$$[Y \cup Y' \rightarrow S] = [Y \rightarrow S] + [Y' \rightarrow S] - [Y \cap Y' \rightarrow S]$$

(the inclusion-exclusion principle). The multiplication on $K_0(\mathrm{RDef}_S)$ is given by the fibre product, $[Y \rightarrow S][Y' \rightarrow S] = [Y \times_S Y' \rightarrow S]$.

Definition 90. Let S be in Def_k . The ring of **motivic functions on S** is

$$\mathcal{C}(S) = K_0(\mathrm{RDef}_S) \otimes_{\mathcal{P}^0(S)} \mathcal{P}(S),$$

where $K_0(\mathrm{RDef}_S)$ is given a $\mathcal{P}^0(S)$ -algebra structure via the unique map $\mathcal{P}^0(S) \rightarrow K_0(\mathrm{RDef}_S)$ given by $1_Y \mapsto [Y]$ and $\mathbb{L} \mapsto [h[0, 1, 0]]$.

In [12], the dimension of a definable assignment is defined (this is an analogue of the definition of the dimension of a subset of K^n as the dimension of its Zariski closure). Defining $\mathcal{C}^{\leq d}(S)$ to be the ideal of $\mathcal{C}(S)$ generated by 1_Z for definable subassignments Z of S with $\dim Z \leq d$, put

$$C(S) = \bigoplus_{d=0}^{\infty} \mathcal{C}^{\leq d}(S) / \mathcal{C}^{\leq d-1}(S),$$

where $\mathcal{C}^{-1}(S) = 0$. We think of $C(S)$ as identifying functions which differ on a set of strictly smaller dimension as their supports.

5.2.3 Motivic Integration and Specialisation

Let $f : X \rightarrow Y$ be a morphism in Def_S . Cluckers and Loeser [13] define the set of S -integrable constructible motivic functions $I_S C(X)$ on a definable assignment X and define a pushforward map $f_! : I_S C(X) \rightarrow I_S C(Y)$. When $Y = \Lambda$ is the point, $f_!$ is the operation of motivic integration (see [12, p. 1015]). The fact we will need for our application is that the indicator function 1_{A_r} is Λ -integrable (as is the indicator function of any definable assignment), and its integral is interpreted as the motivic measure of A_r .

We need to define the specialisation of an arbitrary motivic function on Λ . Recall from Example 88 that $\mathcal{C}(\Lambda) = K_0(\mathrm{RDef}_\Lambda(\mathcal{L}_0)) \otimes_{\mathbb{Z}[\mathbb{L}]} \mathbb{A}$. Therefore, it is sufficient to specify the values of the specialisation map on \mathbb{L} and on $[Y]$, where Y is in $\mathrm{RDef}_\Lambda(\mathcal{L}_0)$.

Definition 91. [12, §9.1] The specialisation map associated to the point Λ and a local field K is the ring homomorphism $s_K : \mathcal{C}(\Lambda) \rightarrow \mathbb{Q}$ is given by $\mathbb{L} \mapsto q_K$ and $[Y] \mapsto |Y(k_K)|$, where Y is in $\text{RDef}_\Lambda(\mathcal{L}_\circ)$. We write $\varphi_K = s_K(\varphi)$.

The critical observation here is that if K_1 and K_2 are local fields with equal residue cardinality, then $s_{K_1} = s_{K_2}$.

Theorem 92 (Specialisation Principle). [12, Theorem 9.1.4] *Let S be in $\text{Def}(\mathcal{L}_\circ)$ and φ in $\text{I}_\Lambda\text{C}(S, \mathcal{L}_\circ)$. Then there exists N such that for all K in $\mathcal{C}_{\circ, N}$ and every choice of uniformiser ϖ_K of \mathfrak{o}_K ,*

$$(\mu(\varphi))_K = \int \varphi_K d\mu_K,$$

where $\mu = f_! : \text{I}_\Lambda\text{C}(S) \rightarrow \text{I}_\Lambda\text{C}(\Lambda)$ and $f : S \rightarrow \Lambda$ is the map to the final object of $\text{Def}(\mathcal{L}_\circ)$.

Recall that our aim is to show that $\mu_K(A_r(K))$ depends only on q . Putting $S = A_r$ and $\varphi = 1_{A_r}$, we get $(\mu(A_r))_K = \mu_K(A_r(K))$. Since $\mu(A_r)$ is in $C(\Lambda)$, its specialisation to K depends only on q . This completes the proof of Theorem 79.

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A.1 Local Fields

The results in this section can be found in [52].

Definition 93. A **discrete valuation** on a field F is a map $\text{ord} : F \rightarrow \mathbb{Z} \cup \{\infty\}$ such that for all $x, y \in F$:

1. $\text{ord}(x) = \infty$ if and only if $x = 0$;
2. $\text{ord}(xy) = \text{ord}(x) + \text{ord}(y)$;
3. $\text{ord}(x + y) \geq \min\{\text{ord}(x), \text{ord}(y)\}$.

A **non-Archimedean local field** is a field with a discrete valuation which satisfies the strong triangle inequality, $|x + y| \leq \max\{|x|, |y|\}$.

Example 94. Let p be prime. Every $\alpha \in \mathbb{Q}$ can be written uniquely as $p^n \frac{a}{b}$, where $a, b \in \mathbb{Z}$ are not divisible by p . If we define $\text{ord}_p(\alpha) = n$, then ord_p is a discrete valuation on \mathbb{Q} , the **p -adic valuation**.

Example 95. Let q be a prime power. Define $\mathbb{F}_q((t))$ to be the field of **formal Laurent series** over \mathbb{F}_q :

$$\mathbb{F}_q((t)) = \left\{ \sum_{i=N}^{\infty} a_i t^i \mid N \in \mathbb{Z}, a_i \in \mathbb{F}_q \right\}.$$

Take $f = \sum_{i=N}^{\infty} a_i t^i$ with $a_N \neq 0$ and define $\text{ord}_t(f) = N$. Also defining $\text{ord}_t(0) = \infty$, one can check that ord_t is a discrete valuation.

Definition 96. Let F be a field with a discrete valuation. The **valuation ring** of F is the set $\mathfrak{o}_F = \{x \in F \mid \text{ord}(x) \geq 0\}$.

One can show that \mathfrak{o}_F is a subring of F . Further, $\mathfrak{p}_F = \{x \in \mathfrak{o}_F \mid \text{ord}(x) \geq 1\}$ is the unique maximal ideal of \mathfrak{o}_F and $\mathfrak{o}_F^\times = \{x \in \mathfrak{o}_F \mid \text{ord}(x) = 0\}$.

Definition 97. Let F be a local field. The **residue field** of F is $k_F = \mathfrak{o}_F/\mathfrak{p}_F$. The **residue characteristic** of F is $p_F = \text{char } k_F$.

We assume from now on that the residue field of F is finite. The maximal ideal \mathfrak{p} is principal, and any generator of \mathfrak{p} is called a **uniformiser** of F . The ideals of \mathfrak{o}_F are precisely the powers of \mathfrak{p} ; in particular, \mathfrak{o}_F is a Dedekind domain (every ideal can be expressed uniquely as a product of prime ideals).

Example 98. The residue fields of \mathbb{Q}_p and $\mathbb{F}_q((t))$ are $k_{\mathbb{Q}_p} = \mathbb{F}_p$ and $k_{\mathbb{F}_q((t))} = \mathbb{F}_q$.

Definition 99. Let E/F be a finite extension of local fields and Let ϖ_F be a uniformiser of F . Consider the ideal $\varpi_F \mathfrak{o}_E$ of \mathfrak{o}_E . Then there exists an integer $e \geq 1$ such that $\varpi_F \mathfrak{o}_E = (\mathfrak{p}_E)^e$. Then e is called the **ramification index** of the extension E/F . The **residue degree** of E/F is $(k_E : k_F)$.

Definition 100. Let E/F be a finite extension of local fields with ramification index e and residue degree f . Then E/F is called **unramified** if $e = 1$, **totally ramified** if $e = n$, and **tamely ramified** if p_F does not divide e .

With abuse of notation, we may write the degree of $\mathfrak{o}'/\mathfrak{o}$ to mean the degree of the extension F'/F .

Proposition 101. *Suppose the extension F'/F is finite of degree n and ramification index e . Then k' is a finite extension of k of degree f such that $n = ef$.*

Proposition 102. *Let F be a non-Archimedean local field with finite residue field and residue characteristic p . Then F is isomorphic to either:*

- *a finite extension of \mathbb{Q}_p , if $\text{char } F = 0$, or:*
- *$\mathbb{F}_q((t))$ for some power q of p , if $\text{char } F > 0$.*

The following is a form of Hensel's lemma:

Lemma 103. *Let \mathfrak{o} be the ring of integers of a non-Archimedean local field and $f \in \mathfrak{o}[x]$. If there exists $a \in \mathfrak{o}$ such that $f(a) \equiv 0 \pmod{\mathfrak{p}}$ and $f'(a) \not\equiv 0 \pmod{\mathfrak{p}}$, then there exists $b \in \mathfrak{o}$ such that $b \equiv a \pmod{\mathfrak{p}}$ and $f(b) = 0$.*

As an example of an application, we describe the totally tamely ramified extensions when the residue field is algebraically closed:

Corollary 104. *Suppose that \mathfrak{o} is the ring of integers of F , \mathfrak{o}_1 is algebraically closed and $e \geq 1$ is not divisible by p . Then if E/F is a totally ramified extension of degree e , then $E = F(\sqrt[e]{\varpi_F})$.*

Proof. Consider the polynomial $f(x) = x^e - \varpi_F \in \mathfrak{o}_E[x]$. Since the residue field of E is algebraically closed, choose a (non-zero) root of $f \pmod{\mathfrak{p}_E}$, say $\bar{a} \in \mathfrak{o}_E/\mathfrak{p}_E$. Then $\overline{f'(a)} = e\bar{a}^{e-1} \neq 0 \in \mathfrak{o}_E/\mathfrak{p}_E$, since $e \neq 0$ in the residue field. By Hensel's lemma, there exists $b \in \mathfrak{o}_E$ such that lifts \bar{a} and $f(b) = b^e - \varpi_F = 0$. \square

Proposition 105. *Let F be a non-Archimedean local field with finite residue field. Then F has a unique finite unramified extension of each degree.*

Proposition 106 (Maximal unramified extension). *Let F be a non-Archimedean local field and \overline{F} the algebraic closure of F . There is a unique subextension F^{unr} of \overline{F}/F , called the **maximal unramified extension** of F , such that F^{unr}/F is unramified and for every unramified extension F'/F with $F' \subseteq \overline{F}$, one has $F' \subseteq F^{\text{unr}}$. If F has finite residue field of characteristic p , then F^{unr} has residue field $\overline{\mathbb{F}_p}$.*

A.2 Concepts from Category Theory

Throughout, we denote by (Set) the category of sets, (Grp) the category of groups, (CRing) the category of commutative rings, and (k – Alg) the category of commutative k -algebras (where k is any commutative ring).

Definition 107. Let \mathcal{C}, \mathcal{I} be categories and $F : \mathcal{I} \rightarrow \mathcal{C}$ a functor. A **cone to** F is an object C of \mathcal{C} , together with morphisms $\phi_I : C \rightarrow F(I)$ for each object I of \mathcal{I} . We write C to denote the cone where the morphisms are understood. If $(C_1, \{\phi_{1,I}\}_{I \in \mathcal{I}})$ and $(C_2, \{\phi_{2,I}\}_{I \in \mathcal{I}})$ are cones to F , a **morphism** of cones from C_1 to C_2 is a \mathcal{C} -morphism $\psi \in \text{Hom}_{\mathcal{C}}(C_1, C_2)$ such that for all $I \in \mathcal{I}$, $\phi_{2,I} \circ \psi = \phi_{1,I}$. A **limit** of F is a cone L to F such that for every cone C to F , there is a unique morphism of cones $C \rightarrow L$. We write $L = \varprojlim F$.

The limit of a functor (if it exists) is unique up to isomorphism. In fact, the cones to F form a category, and the limit is the final object in the category of cones to F . We think of \mathcal{I} as an indexing category, so that the functor F can be viewed as some collection of objects in \mathcal{C} , possibly with morphisms between them.

We can also dualise the definition of limit to define **cocone** and **colimit**.

Example 108. If \mathcal{I} is a category with no morphisms, then the limit of F is called a **product**. If, further, \mathcal{I} has no objects, then the limit of F is the final object of \mathcal{C} .

Definition 109. Let X be an object of \mathcal{C} and $Y \rightarrow X$ and $Z \rightarrow X$ be morphisms. The limit of the diagram $Y \rightarrow X \leftarrow Z$, if it exists, is called the **fibre product** and is denoted $Y \times_X Z$.

Example 110. In the category of commutative rings, the diagram

$$\cdots \rightarrow \mathbb{Z}/p^3\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$$

has a limit, the p -adic integers \mathbb{Z}_p . Putting the discrete topology on each $\mathbb{Z}/p^i\mathbb{Z}$ and taking the limit of the same diagram in the category of topological commutative rings, \mathbb{Z}_p inherits a topology, the p -adic topology. A basis for this topology is given by open sets of the form $\{x \in \mathbb{Z}_p \mid x \equiv a \pmod{p^i}\}$. These are precisely the open

balls in the p -adic absolute value, since $x \equiv a \pmod{p^i}$ if and only if $|x - a|_p \leq p^{-i}$, or equivalently, $|x - a|_p < p^{-(i-1)}$.

More generally, if \mathfrak{o} is the valuation ring of a non-Archimedean local field, we have $\mathfrak{o} = \varprojlim_i \mathfrak{o}/\mathfrak{p}^i$ as topological rings. A similar fact is true for general linear groups, $\mathrm{GL}_n(\mathfrak{o}) = \varprojlim_i \mathrm{GL}_n(\mathfrak{o}/\mathfrak{p}^i)$ as topological groups, where each $\mathrm{GL}_n(\mathfrak{o}/\mathfrak{p}^i)$ has the discrete topology.

Definition 111. A group is called **profinite** if it is the limit of a diagram of discrete finite groups in the category of topological groups. Profinite commutative rings are defined similarly.

As in Example 110, the group $\mathrm{GL}_n(\mathfrak{o})$ is profinite. The ring \mathbb{Z}_p also has the property of being **pro- p** , i.e. the limit of finite discrete commutative rings of p -power order.

Example 112. The coproduct in the category of abelian groups is the direct sum. For example, $\mathbb{Z}/2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/6$. However, the coproduct in the category of groups is the free product, $\mathbb{Z}/2 * \mathbb{Z}/3 = \langle x, y \mid x^2 = y^3 = 1 \rangle$, which in this case is an infinite non-abelian group. This illustrates the fact that limits and colimits may be extremely different in a subcategory.

Definition 113. Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be functors. A **natural transformation** from F to G is a collection $\{\Phi_X : F(X) \rightarrow G(X)\}_{X \in \mathcal{C}}$ of morphisms in \mathcal{D} such that for every morphism $X \rightarrow Y$ of \mathcal{C} , the square

$$\begin{array}{ccc} F(X) & \xrightarrow{\Phi_X} & G(X) \\ F(f) \downarrow & & \downarrow G(f) \\ F(Y) & \xrightarrow{\Phi_Y} & G(Y) \end{array}$$

commutes. A **natural isomorphism** from F to G is a natural transformation $\{\Phi_X\}_{X \in \mathcal{C}}$ such that each Φ_X is an isomorphism. In this case, we say that F and G are **naturally isomorphic**, and write $F \cong G$.

Definition 114. Let \mathcal{C} be a category. The **opposite category** of \mathcal{C} is the category $\mathcal{C}^{\mathrm{op}}$ such that $\mathrm{Obj} \mathcal{C}^{\mathrm{op}} = \mathrm{Obj} \mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$. Composition of morphisms in $\mathcal{C}^{\mathrm{op}}$ is given by composition of morphisms in \mathcal{C} in reverse order.

Definition 115. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is an **equivalence of categories** if there exists a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ such that $F \circ G \cong \text{id}_{\mathcal{D}}$ and $G \circ F \cong \text{id}_{\mathcal{C}}$. If $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ is an equivalence of categories, then F is said to be a **duality** between \mathcal{C} and \mathcal{D} .

Definition 116. Let \mathcal{C}, \mathcal{D} be categories and $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be functors. Then F is **left adjoint** to G if, for every $X \in \mathcal{C}, Y \in \mathcal{D}$, there exists a bijection $\text{Hom}_{\mathcal{D}}(F(X), Y) \rightarrow \text{Hom}_{\mathcal{C}}(X, G(Y))$ which is natural in X and Y , i.e. such that for all morphisms $X_1 \rightarrow X_2$ of \mathcal{C} and all objects Y of \mathcal{D} , the square

$$\begin{array}{ccc} \text{Hom}_{\mathcal{D}}(F(X_2), Y) & \longrightarrow & \text{Hom}_{\mathcal{C}}(X_2, G(Y)) \\ \downarrow & & \downarrow \\ \text{Hom}_{\mathcal{D}}(F(X_1), Y) & \longrightarrow & \text{Hom}_{\mathcal{C}}(X_1, G(Y)) \end{array}$$

commutes, and for all morphisms $Y_1 \rightarrow Y_2$ in \mathcal{D} and all objects X of \mathcal{C} , the square

$$\begin{array}{ccc} \text{Hom}_{\mathcal{D}}(F(X), Y_1) & \longrightarrow & \text{Hom}_{\mathcal{C}}(X, G(Y_1)) \\ \downarrow & & \downarrow \\ \text{Hom}_{\mathcal{D}}(F(X), Y_2) & \longrightarrow & \text{Hom}_{\mathcal{C}}(X, G(Y_2)) \end{array}$$

commutes.

Theorem 117 (Frobenius Reciprocity). *Given a finite group G and a subgroup H , the functor Ind_H^G is left adjoint to Res_H^G . Consequently, given representations ψ of H and ρ of G , $\langle \text{Ind}_H^G \psi, \rho \rangle = \langle \psi, \text{Res}_H^G \rho \rangle$.*

Lemma 118 (Yoneda Lemma). *Let A be an object of a category \mathcal{C} and let F be a functor $\mathcal{C} \rightarrow (\text{Set})$. There is a one-to-one correspondence between natural transformations of functors from $\text{Hom}(A, -)$ to F and elements of $F(A)$. In particular, if B is an object of \mathcal{C} and $F = \text{Hom}(B, -)$, then morphisms from $\text{Hom}(A, -)$ to $\text{Hom}(B, -)$ are in one-to-one correspondence with morphisms $B \rightarrow A$.*

The Yoneda Lemma is a fundamental result about functors which are naturally isomorphic to $\text{Hom}(A, -)$ for some object A ; such a functor is said to be **representable** by A .

Definition 119. Let \mathcal{C} be a category and X an object of \mathcal{C} . Define \mathcal{C}/X to be the category whose objects are morphisms $Y \rightarrow X$, for each object Y of \mathcal{C} , and morphisms given by commuting triangles, that is, a \mathcal{C}/X -morphism from $Y \rightarrow X$ to $Z \rightarrow X$ is a \mathcal{C} -morphism $Y \rightarrow Z$ such that the triangle

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow & & \swarrow \\ & & X \end{array}$$

commutes.

A.3 Group Cohomology

We outline the construction of group cohomology (see [22, Chapter 3]). Let G be a group and A any abelian group. Take any projective resolution of the trivial $\mathbb{Z}[G]$ -module \mathbb{Z}

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0.$$

Consider the cochain complex

$$C^\bullet : 0 \rightarrow \text{Hom}(P_0, A) \xrightarrow{d_1^*} \text{Hom}(P_1, A) \xrightarrow{d_2^*} \text{Hom}(P_2, A) \xrightarrow{d_3^*} \dots$$

and define the i th cohomology group

$$H^i(G, A) := H^i(C^\bullet) = \ker d_{i+1}^* / \text{im } d_i^*.$$

One can show that this construction does not depend on the choice of projective resolution. Therefore, for the purposes of computation, we can choose the free resolution where $P_i = \mathbb{Z}[G^i]$. In order to define the maps, let $s_j^i : \mathbb{Z}[G^{i+1}] \rightarrow \mathbb{Z}[G^i]$ be given on the basis by $s_j^i(g_0, \dots, g_i) = (g_0, \dots, \hat{g}_j, \dots, g_i)$ where the hat denotes omission of the term, and

$$d_i : \mathbb{Z}[G^{i+1}] \rightarrow \mathbb{Z}[G^i]$$

is given by $d_i = \sum_{j=0}^i s_j^i$. Applying the functor $\text{Hom}_{\mathbb{Z}[G]}(-, A)$, consider the induced cochain complex

$$0 \rightarrow \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \xrightarrow{d_1^*} \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^2], A) \xrightarrow{d_2^*} \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^3], A) \xrightarrow{d_3^*} \dots$$

For $g_1, \dots, g_i \in G$, define the element $[g_1, \dots, g_i] = (1, g_1, g_1g_2, \dots, g_1 \dots g_i) \in G^{i+1}$. One can show that $\{[g_1, \dots, g_i] \mid g_j \in G\}$ is a $\mathbb{Z}[G]$ -basis for $\mathbb{Z}[G^{i+1}]$. Therefore, any $\mathbb{Z}[G]$ -module homomorphism $\mathbb{Z}[G^{i+1}] \rightarrow A$ can be regarded as a function $G^i \rightarrow A$. One can show that the maps d_i are given on the basis by

$$d_i([g_1, \dots, g_i]) = g_1[g_2, \dots, g_i] + \sum_{j=1}^{i-1} (-1)^j [g_1, \dots, g_j g_{j+1}, \dots, g_i] + (-1)^i [g_1, \dots, g_{i-1}].$$

Therefore, given $\varphi \in \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^i], A)$, viewed as a map $\varphi : G^{i-1} \rightarrow A$, one has

$$\begin{aligned} d_i^*(\varphi)(g_1, \dots, g_i) &= g_1\varphi(g_2, \dots, g_i) + \sum_{j=1}^{i-1} (-1)^j \varphi(g_1, \dots, g_j g_{j+1}, \dots, g_i) \\ &\quad + (-1)^i \varphi(g_1, \dots, g_{i-1}). \end{aligned}$$

Definition 120. Let $i \geq 1$ be an integer. We define $Z^i(G, A) = \ker d_{i+1}^*$, $B^i(G, A) = \text{im } d_i^*$ and $H^i(G, A) = Z^i(G, A)/B^i(G, A)$. Elements of $Z^i(G, A)$ are called ***i -cocycles***, elements of $B^i(G, A)$ are called ***i -coboundaries***, and elements of $H^i(G, A)$ are called ***i -cohomology classes***. We call $H^i(G, A)$ the ***i th cohomology group of G with coefficients in A*** .

We will be interested in the second cohomology group, that is, $\ker d_3^*/\text{im } d_2^*$. One can compute explicitly that

$$\begin{aligned} Z^2(G, A) &= \{\varphi : G^2 \rightarrow A \mid g_1\varphi(g_2, g_3) - \varphi(g_1g_2, g_3) + \varphi(g_1, g_2g_3) - \varphi(g_1, g_2) = 0\}, \\ B^2(G, A) &= \{\varphi : G^2 \rightarrow A \mid \exists \mu : G \rightarrow A, \varphi(g_1, g_2) = g_1\mu(g_2) - \mu(g_1g_2) + \mu(g_1)\}. \end{aligned}$$

Definition 121. The **Schur multiplier** of a group G is the second cohomology group $H^2(G, \mathbb{C}^\times)$, where G acts trivially on \mathbb{C}^\times .

As can be seen in Proposition 20, a criterion for a representation to have an

extension to its stabiliser is given in terms of a cohomology class in a Schur multiplier.

A.4 Generalities on Schemes

Classical algebraic geometry is concerned with varieties, that is, curves or surfaces in affine space, \mathbb{A}^n which are the solution sets to some set of polynomial equations. However, this approach has some shortcomings. The most serious in our applications is the fact that one must work over a field in the classical setting, however we would like to consider sets of solutions of polynomial equations over the local ring \mathfrak{o}_r . The material in this section can be found in, for example, Eisenbud and Harris [17].

We introduce the notion of a sheaf, an abstraction of the notion of functions on a space. More precisely, a sheaf assigns to every open set U of a topological space X a set $\mathcal{F}(U)$, which we think of as being the set of functions on U .

Definition 122. Let X be a topological space. Denote by $\text{Top}(X)$ the category whose objects are the open subsets of X and whose morphisms are given by the inclusion of open sets. A **presheaf of sets** on X is a functor $\mathcal{F} : \text{Top}(X)^{\text{op}} \rightarrow (\text{Set})$. Let \mathcal{F}, \mathcal{G} be presheaves on X . A **morphism** of presheaves $\mathcal{F} \rightarrow \mathcal{G}$ is a natural transformation of functors $\mathcal{F} \rightarrow \mathcal{G}$.

Equivalently, a presheaf assigns to each open set U a set $\mathcal{F}(U)$, and for every inclusion of open sets $U \subseteq V$, a map $\text{res}_U^V : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$, called the **restriction map** from V to U . The restriction maps satisfy $\text{res}_U^U = \text{id}_{\mathcal{F}(U)}$ and for every inclusion $U \subseteq V \subseteq W$, $\text{res}_U^W = \text{res}_U^V \circ \text{res}_V^W$. Elements $s \in \mathcal{F}(U)$ are called **sections** of \mathcal{F} . Given an inclusion of open sets $U \subseteq V$ and $s \in \mathcal{F}(V)$, we also write $s|_U = \text{res}_U^V(s)$.

We also define a **presheaf of rings** to be a functor $\text{Top}(X)^{\text{op}} \rightarrow (\text{CRing})$. Presheaves of abelian groups, modules etc. are defined similarly.

In order to specify a presheaf on X , it is sufficient to choose a basis \mathcal{B} of the topology on X and define $\mathcal{F}(U)$ for every $U \in \mathcal{B}$, along with the restriction maps res_U^V for $U, V \in \mathcal{B}$. One then defines, for arbitrary open sets $U \subseteq X$,

$$\mathcal{F}(U) = \varprojlim_{\substack{V \subseteq U \\ V \in \mathcal{B}}} \mathcal{F}(V)$$

Functions on a space also satisfy the property that if functions s_1 on U_1 and s_2 on U_2 restrict to the same function on $U_1 \cap U_2$, then there is a unique function s on $U_1 \cup U_2$ which restricts to s_1 on U_1 and s_2 on U_2 . This motivates the following definition:

Definition 123. A **sheaf** on a topological space X is a presheaf \mathcal{F} satisfying the following condition, called the **sheaf axiom**: for every open $U \subseteq X$ and every open cover $\{U_i\}_{i \in I}$ of U (that is, $\cup_{i \in I} U_i = U$), if for each $i \in I$, $s_i \in \mathcal{F}(U_i)$ is a section with the property that for every $i, j \in I$, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists a unique section $s \in \mathcal{F}(U)$ such that for all $i \in I$, $s|_{U_i} = s_i$. If \mathcal{F} and \mathcal{G} are sheaves on X , then a **morphism** of sheaves $\mathcal{F} \rightarrow \mathcal{G}$ is the same as a morphism of presheaves $\mathcal{F} \rightarrow \mathcal{G}$, i.e. a natural transformation of functors.

Definition 124. Let \mathcal{F} be a presheaf on a topological space X and $x \in X$. The **stalk** of \mathcal{F} at x , denoted \mathcal{F}_x , is the set

$$\{(U, s) \mid U \ni x \text{ open}, s \in \mathcal{F}(U)\}$$

modulo the relation $(U_1, s_1) \sim (U_2, s_2)$ if and only if there exists an open set $V \subseteq U_1 \cap U_2$ such that $s_1|_V = s_2|_V$. equivalence class $[(U, s)]$ is called the **germ** of s at x and is denoted s_x .

Alternatively, a more categorical description of the stalk at x is given by

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U).$$

One can show that these two definitions are equivalent; moreover, the limit can be taken over basic open sets U and this will not affect the outcome.

A morphism of sheaves $\mathcal{F} \rightarrow \mathcal{G}$ induces a map of stalks $\mathcal{F}_x \rightarrow \mathcal{G}_x$ as follows: for each $x \in X$ and every open set $U \ni x$, one has a composition of maps $\mathcal{F}(U) \rightarrow \mathcal{G}(U) \rightarrow \mathcal{G}_x$. Since $\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$, by the defining property of the limit and the fact that $\mathcal{F} \rightarrow \mathcal{G}$ is a natural transformation of functors, there is a unique map $\mathcal{F}_x \rightarrow \mathcal{G}_x$ making every diagram

$$\begin{array}{ccc}
\mathcal{F}(U) & & \\
\downarrow & \searrow & \\
\mathcal{F}_x & \longrightarrow & \mathcal{G}_x
\end{array}$$

commute. More explicitly, let $[(U, s)] \in \mathcal{F}_x$. Write t for the image of s under the map $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$. One maps $[(U, s)]$ to $[(U, t)] \in \mathcal{G}_x$, defining the map $\mathcal{F}_x \rightarrow \mathcal{G}_x$.

Definition 125. A morphism of presheaves $f : \mathcal{F} \rightarrow \mathcal{G}$ is **injective** (resp. **surjective**, **bijective**) if for all $x \in X$, the induced map of stalks $\mathcal{F}_x \rightarrow \mathcal{G}_x$ is injective (resp. surjective, bijective).

Definition 126. Let $f : X \rightarrow Y$ be a continuous map of topological spaces and let \mathcal{F} be a presheaf on X . The **pushforward** of \mathcal{F} along f is the presheaf $f_*\mathcal{F}$ on Y , defined by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ for every open set $U \subseteq Y$. The restriction maps are defined by those of \mathcal{F} , since given an inclusion of open sets $U \subseteq V$ in Y , one has $f^{-1}(U) \subseteq f^{-1}(V)$.

One can show that the pushforward of a sheaf is again a sheaf.

Definition 127. A **ringed space** is a topological space X with a sheaf of rings \mathcal{O}_X . A **locally ringed space** is a ringed space (X, \mathcal{O}_X) such that for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring (i.e. has a unique maximal ideal).

There is a natural way to form a locally ringed space from any commutative ring. In what follows, given a commutative ring R and a prime ideal \mathfrak{p} of R , $R_{\mathfrak{p}}$ shall denote the localisation away from \mathfrak{p} , that is, $R[S^{-1}]$ where $S = R - (\mathfrak{p} \cup \{0\})$. Note that $R_{\mathfrak{p}}$ is local with unique maximal ideal which can be identified with \mathfrak{p} .

Definition 128. Let R be a commutative ring. Define the locally ringed space $\text{Spec } R$ as the set of prime ideals of R , with topology and structure sheaf as follows: given $f \in R$, define $V(f) = \{\mathfrak{p} \in \text{Spec } R \mid f \in \mathfrak{p}\}$. The $V(f)$ form the closed sets of a topology on $\text{Spec } R$, the **Zariski topology**. One can show that the sets $D(f) = \{\mathfrak{p} \in \text{Spec } R \mid f \notin \mathfrak{p}\}$ form a basis for the Zariski topology; the $D(f)$ are called **basic open sets**.

Proposition 129. For $f, g \in R$, $D(f) \subseteq D(g)$ if and only if there exists $n \geq 1$ such that $g|f^n$.

It is a general fact about (pre)sheaves that in order to specify a presheaf \mathcal{F} on a topological space X , it is sufficient to specify a basis \mathcal{B} for X , along with the sets $\mathcal{F}(B)$ for $B \in \mathcal{B}$ and a restriction map $\mathcal{F}(B_1) \rightarrow \mathcal{F}(B_2)$ for every inclusion $B_2 \subseteq B_1$ of open sets in \mathcal{B} . If U is an arbitrary open set, then the sections over U are determined uniquely by

$$\mathcal{F}(U) = \varprojlim_i \mathcal{F}(B_i),$$

where $U = \cup_{i \in I} B_i$ is any open cover with $B_i \in \mathcal{B}$. If, further, V is an open set contained in U and $V = \cup_{j \in J} C_j$ is an open cover by basic open sets, then define the restriction map $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$. Furthermore, if \mathcal{F} satisfies the sheaf axiom on open sets in \mathcal{B} , then \mathcal{F} is a sheaf.

Definition 130. The structure sheaf on $\text{Spec } R$ is given on basic open sets by

$$\mathcal{O}_{\text{Spec } R}(D(f)) = R[f^{-1}],$$

where the restriction maps $\mathcal{O}_{\text{Spec } R}(D(g)) \rightarrow \mathcal{O}_{\text{Spec } R}(D(f))$ are the natural maps $R[g^{-1}] \rightarrow R[f^{-1}]$ (recall that by Proposition 129, $f^n = gh$ for some $h \in R$, therefore $ghf^{-n} = 1$ in $R[f^{-1}]$, and $g^{-1} \mapsto hf^{-n}$ under the map).

One can check that $\mathcal{O}_{\text{Spec } R}$ satisfies the sheaf axiom on the basic open sets $D(f)$ and that the stalks are $(\mathcal{O}_{\text{Spec } R})_{\mathfrak{p}} = R_{\mathfrak{p}}$.

One can regard any open subset of a locally ringed space as a locally ringed space itself. It is sufficient to define a notion of the restriction of the structure sheaf to an open subset:

Definition 131. Let X be a topological space, \mathcal{F} a sheaf on X and $U \subseteq X$ an open subset. Define the **restriction** of \mathcal{F} to U to be the sheaf on U (with the subspace topology) defined for open $V \subseteq U$ by

$$\mathcal{F}|_U(V) = \mathcal{F}(V),$$

and the restriction maps are the same as those of \mathcal{F} .

Thus, if (X, \mathcal{O}_X) is a locally ringed space and $U \subseteq X$ is an open subset, then $(U, \mathcal{O}_X|_U)$ is also a locally ringed space.

Definition 132. Let R_1, R_2 be local rings with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2$. A **local homomorphism** from R_1 to R_2 is a ring homomorphism $f : R_1 \rightarrow R_2$ such that $f(\mathfrak{m}_1) \subseteq \mathfrak{m}_2$.

Definition 133. Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be locally ringed spaces. A **morphism** from (X, \mathcal{O}_X) to (Y, \mathcal{O}_Y) is a continuous map $f : X \rightarrow Y$ along with a morphism of sheaves $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ which induces a local homomorphism $f_y^\# : (\mathcal{O}_Y)_y \rightarrow (f_*\mathcal{O}_X)_y$ on stalks. Further, f is an **isomorphism** if $f : X \rightarrow Y$ is a homeomorphism and $f^\#$ is an isomorphism of sheaves.

We are now ready to define schemes.

Definition 134. An **affine scheme** is a locally ringed space X which is isomorphic to $\text{Spec } R$ for some ring R . A **scheme** is a locally ringed space X such that there exists an open cover $\{U_i\}_{i \in I}$ of X and commutative rings R_i such that for all i , there is an isomorphism of locally ringed spaces $U_i \cong \text{Spec } R_i$. A **morphism** of schemes is the same as a morphism of locally ringed spaces between schemes.

Let (Sch) denote the category of schemes, and (AffSch) the full subcategory of affine schemes. If k is a commutative ring, we write $(\text{Sch}/k) = (\text{Sch})/\text{Spec } k$ (see Appendix A.2, Definition 119). Similarly, we write $(\text{AffSch}/k) = (\text{AffSch})/\text{Spec } k$.

Proposition 135. *The functor $\text{Spec} : (k - \text{Alg}) \rightarrow (\text{AffSch}/k)$ is a duality.*

Theorem 136. *The functor $\text{Spec} : (k - \text{Alg}) \rightarrow (\text{AffSch}/k)^{\text{op}}$ is left adjoint to the functor $\Gamma : (\text{AffSch}/k)^{\text{op}} \rightarrow (k - \text{Alg})$ which maps a scheme X over k to its global sections. That is, given a k -algebra R and a scheme X over k , there is a bijection, natural in R and X , between morphisms of schemes $X \rightarrow \text{Spec } R$ and k -algebra homomorphisms $R \rightarrow \mathcal{O}_X(X)$.*

A.4.1 Affine Group Schemes as Functors

This exposition follows that of Waterhouse [60]. As a motivating example, we consider the special linear groups, $\text{SL}_n(R)$. For different choices of the ring R , the prop-

erties of the group $\mathrm{SL}_n(R)$ may vary dramatically, however all the groups $\mathrm{SL}_n(R)$ are related in a way which we shall describe.

Fix a commutative ring k and let R be a k -algebra, that is, we are given a ring homomorphism $k \rightarrow R$. Then choosing a matrix $(a_{ij}) \in \mathrm{SL}_n(R)$ is equivalent to choosing a k -algebra homomorphism $k[\{x_{ij}\}_{i,j=1}^n] \rightarrow R$ which maps x_{ij} to a_{ij} , such that $\det(a_{ij}) = 1$. This condition is equivalent to $\det(x_{ij}) - 1$ being in the kernel of the map $k[\{x_{ij}\}_{i,j=1}^n] \rightarrow R$, that is, a homomorphism $k[\{x_{ij}\}_{i,j=1}^n]/(\det(x_{ij}) - 1) \rightarrow R$. We can now identify SL_n with a functor $\mathrm{Hom}_{k\text{-Alg}}(k[\{x_{ij}\}_{i,j=1}^n]/(\det(x_{ij}) - 1), -)$ from k -algebras to groups. This motivates the following definition:

Definition 137. An **affine scheme over k** is a representable functor $(k - \mathrm{Alg}) \rightarrow (\mathrm{Set})$. An **affine group scheme over k** is a functor $(k - \mathrm{Alg}) \rightarrow (\mathrm{Grp})$ whose composition with the forgetful functor $(\mathrm{Grp}) \rightarrow (\mathrm{Set})$ is representable.

There is a duality from the category of k -algebras to the category of affine schemes over k , given by sending A to $\mathrm{Hom}(A, -)$. In order to determine the object which is dual to an affine group scheme G represented by A , observe that the group operation is a collection of maps $\mathrm{Hom}(A, R) \times \mathrm{Hom}(A, R) \rightarrow \mathrm{Hom}(A, R)$, natural in R . This is equivalent to a natural transformation of functors $\mathrm{Hom}(A \otimes_k A, -) \rightarrow \mathrm{Hom}(A, -)$. By the Yoneda Lemma, this corresponds to a unique **comultiplication** map $\Delta : A \rightarrow A \otimes_k A$. Similarly, the identity of G is a natural transformation $e \rightarrow G$, where $e = \mathrm{Hom}(k, -)$ is the trivial group scheme over k ; this corresponds to the **counit** $\varepsilon : A \rightarrow k$. Finally, the inverse map $G \rightarrow G$ corresponds to the **antipode** $S : A \rightarrow A$. With these extra operations, A becomes a **Hopf algebra**, and in fact, the category of Hopf algebras over k is dual to the category of affine schemes over k .

The axioms for a group can be written in terms of commutative diagrams; these can be reinterpreted in the category of affine group schemes and are equivalent to the requirement that $\mathrm{Hom}(A, R)$ must have a group structure which is natural in R . These give rise to dual diagrams which give conditions on the Hopf algebra operations on A . For example, the associativity axiom for groups corresponds to coassociativity for Hopf algebras, as in the diagram below.

$$\begin{array}{ccc}
G \times G \times G & \xrightarrow{(\mu, \text{id})} & G \times G & & A \otimes A \otimes A & \xleftarrow{(\Delta, \text{id})} & A \otimes A \\
(\text{id}, \mu) \downarrow & & \downarrow \mu & & (\text{id}, \Delta) \uparrow & & \uparrow \Delta \\
G \times G & \xrightarrow{\mu} & G & & A \otimes A & \xleftarrow{\Delta} & A
\end{array}$$

Example 138. Take $G = \text{SL}_n$, for which $A = k[\{x_{ij}\}_{i,j=1}^n]/(\det(x_{ij}) - 1)$. We need a map $\Delta : A \rightarrow A \otimes A$ such that the diagram

$$\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \otimes A \\
x_{ij} \mapsto \sum_k r_{ik}s_{kj} \searrow & & \downarrow \varphi = (x_{ij} \mapsto r_{ij}, x_{ij} \mapsto s_{ij}) \\
& & R
\end{array}$$

commutes. Setting $\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$ works, indeed:

$$\begin{aligned}
\varphi \circ \Delta(x_{ij}) &= \varphi \left(\sum_k x_{ik} \otimes x_{kj} \right) \\
&= \sum_k \varphi(x_{ik} \otimes x_{kj}) \\
&= \sum_k r_{ik}s_{kj}.
\end{aligned}$$

By the Yoneda Lemma, Δ is the only k -algebra homomorphism making the diagram commute for all R .

A.4.2 Affine Algebraic Groups as Varieties

The material below, with the exception of Proposition 142, can be found in [11]. Let k be a field. An **affine algebraic group over k** is an affine group scheme represented by a finitely generated k -algebra which is reduced (i.e. contains no non-zero nilpotent elements). If k is algebraically closed, every affine algebraic group G over k is isomorphic to a closed subgroup of $\text{GL}_n(k)$. In this situation, we identify G with an affine variety (as a subset of k^n) with a group operation $G \times G \rightarrow G$ which is a morphism of varieties.

Definition 139. Let p be a prime and \mathbf{G} be an affine algebraic group over $\overline{\mathbb{F}_p}$. Then \mathbf{G} contains a maximal closed connected unipotent normal subgroup, called

the **unipotent radical**, $R_u(\mathbf{G})$. We say \mathbf{G} is **reductive** if $R_u(\mathbf{G}) = 1$.

An **algebraic torus** is an algebraic group which is isomorphic to \mathbb{G}_m^n , for some $n \in \mathbb{N}$ (so n is the dimension of the torus). Every reductive group contains maximal tori, all of which are conjugate in \mathbf{G} and thus have the same dimension. This common value is called the **rank** of \mathbf{G} .

A **Borel subgroup** of \mathbf{G} is defined as a maximal closed connected solvable subgroup of \mathbf{G} . Since any torus is connected and abelian (which implies it is solvable), every maximal torus is contained in a Borel subgroup. Furthermore, it can be shown that any two Borel subgroups of \mathbf{G} are conjugate. Let \mathbf{T} be a maximal torus in \mathbf{G} , \mathbf{B} a Borel subgroup containing \mathbf{T} , and \mathbf{U} the unipotent radical of \mathbf{B} . It is a general fact that $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}$. We also define the **Weyl group** of \mathbf{G} to be $\mathbf{W} = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. Up to isomorphism, this does not depend on the choice of \mathbf{T} .

Reductive groups have a very tightly controlled structure, as we shall now see. For more details, we refer to Carter [11]. For the remainder of this subsection, \mathbf{G} will denote a connected reductive group.

Example 140. Let $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$. Choose \mathbf{T} to be the set of diagonal matrices in \mathbf{G} and \mathbf{B} to be the set of upper triangular matrices. One can check that \mathbf{U} is the set of matrices in \mathbf{B} whose diagonal entries are all equal to 1, that $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}$, \mathbf{N} is the set of all monomial matrices (those with exactly one non-zero entry in each row and column), and $\mathbf{W} \cong S_n$.

We now define Frobenius maps, which are related to k -structures. This relies on the fact that every affine algebraic group over $\overline{\mathbb{F}}_p$ is linear, i.e. is a closed subgroup of GL_n for some n .

Let q be a power of p . The **standard Frobenius map** on $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$ with parameter q is $F_q : \mathrm{GL}_n(\overline{\mathbb{F}}_p) \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ given by $F_q(a_{ij}) = (a_{ij}^q)$. If \mathbf{G} is a closed subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$, then a standard Frobenius map on \mathbf{G} is the restriction of a standard Frobenius map on $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$. A **Frobenius map** on \mathbf{G} is a map $F : \mathbf{G} \rightarrow \mathbf{G}$ such that some power of F is a standard Frobenius map on \mathbf{G} . One can show that a Frobenius map is a group isomorphism and a morphism of varieties.

Let \mathbf{T} be a maximal torus of \mathbf{G} . The **character lattice** of \mathbf{G} (relative to \mathbf{T}) is the free abelian group $X = \mathrm{Hom}(\mathbf{T}, \mathbb{G}_m) \cong \mathbb{Z}^n$. Let F be a Frobenius map

on \mathbf{G} . Then F induces a homomorphism of abelian groups $X \rightarrow X$ by setting $F(\chi)(t) = \chi(F(t))$ for $t \in \mathbf{T}, \chi \in X$. It can be shown that some power of F , say F^δ , is a scalar multiple of the identity on X . Let $\delta \geq 1$ be minimal with this property. Then define q to be the unique positive real number such that $F^\delta = q^\delta \cdot \text{id}$ on X .

Definition 141. Following Carter [11, §1.18], to any Frobenius map F on an algebraic group \mathbf{G} we may associate a parameter q as follows: let δ be the smallest positive integer such that F^δ is an integer multiple of the identity on the character lattice of \mathbf{G} . Define q to be the unique positive real number such that $F^\delta = q^\delta \cdot \text{id}$ on the character lattice.

We can relate the notions of k -structures (see Chapter 4, Definition ??) and Frobenius maps as follows:

Proposition 142. *Let G be a closed subgroup of $\text{GL}_n(\overline{\mathbb{F}}_q)$ with \mathbb{F}_q -rational structure given by $G = G_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. Then there exists an embedding of G into GL_n as $\overline{\mathbb{F}}_q$ -group schemes (which can be thought of as coordinates x_i on G) such that $G_0(\mathbb{F}_q)$ can be identified with the fixed points of the standard Frobenius map on $\text{GL}_n(\overline{\mathbb{F}}_q)$ restricted to $G(\overline{\mathbb{F}}_q)$. The parameter of Definition 141 associated to G with this rational structure is q .*

Proof. Let $G = G_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. Dually, if A and A_0 are the coordinate algebras of G and G_0 respectively, then $A = A_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. Choose coordinates x_i such that $A_0 = \mathbb{F}_q[x_i]/(f_j)$ for some polynomials $f_j \in \mathbb{F}_q[x_i]$, so $A = \overline{\mathbb{F}}_q[x_i]/(f_j)$. Then $G_0(\mathbb{F}_{q^d}) = \text{Hom}(A_0, \mathbb{F}_{q^d})$ is the set of solutions to f_j in \mathbb{F}_{q^d} , which is the intersection of the fixed points of the standard Frobenius $F_{q^d} : (a_{ij}) \mapsto (a_{ij}^{q^d})$ and the group $G(\overline{\mathbb{F}}_q) \subseteq \mathbb{A}^n(\overline{\mathbb{F}}_q)$. The q is the same as in Definition 141 because $G_0(\mathbb{F}_q)$ is the fixed points of the standard Frobenius F_q on $G(\overline{\mathbb{F}}_q)$ and $F_q = q \cdot \text{id}$ on the character lattice X . Indeed, any split torus $T = \mathbb{G}_m \times \cdots \times \mathbb{G}_m$ can be parameterised by elements (t_1, \dots, t_n) , and the characters are given by $\chi_{i_1, \dots, i_n}(t_1, \dots, t_n) = \prod_k t_k^{i_k}$. Then

$$F_q(\chi_{i_1, \dots, i_n})(t_1, \dots, t_n) = \chi_{i_1, \dots, i_n}(t_1^q, \dots, t_n^q) = \prod_k t_k^{q i_k},$$

so $F_q(\chi_{i_1, \dots, i_n}) = \chi_{q i_1, \dots, q i_n}$. □

A.5 First-Order Logic

We give the basic definitions necessary from first-order logic; for more details, we refer to Kirby [41].

Definition 143. The **basic first-order language** is the collection of countably many variables (usually denoted by x_i or x, y, z, \dots), equality \doteq , the quantifiers \forall, \exists , the logical connectives $\neg, \rightarrow, \vee, \wedge$, the comma and brackets. A **first-order language** is the basic first-order language along with any number of function symbols, relation symbols and constant symbols. Each function and relation symbol is assigned an **arity**, which is a natural number n ; we say that such a symbol is n -**ary** in this case.

Example 144. The language of rings is $\mathcal{L}_{\text{Ring}} = \{+, \cdot, -, 0, 1\}$, where $+, \cdot$ are binary (i.e. 2-ary) function symbols, $-$ is a unary (i.e. 1-ary) function symbol, and $0, 1$ are constant symbols.

Definition 145. Let \mathcal{L} be a first-order language. An \mathcal{L} -**structure** is a collection \mathcal{M} consisting of the following data: a set $M = |\mathcal{M}|$; for each n -ary function symbol f , a map $f^{\mathcal{M}} : M^n \rightarrow M$; for each n -ary relation symbol R , a subset $R^{\mathcal{M}} \subseteq M^n$; and for each constant symbol c , an element $c^{\mathcal{M}} \in M$.

Where no confusion arises, we drop the superscripts and write f in place of $f^{\mathcal{M}}$, R in place of $R^{\mathcal{M}}$ and c in place of $c^{\mathcal{M}}$.

Example 146. Any ring R is an $\mathcal{L}_{\text{Ring}}$ -structure, where $+^R$ is the addition in R , $-^R$ is the map $a \mapsto -a$ on R , and so on.

Given a language \mathcal{L} , one may define what it means for a string of \mathcal{L} -symbols to be an \mathcal{L} -formula. Instead of giving the full definition, it suffices for our purposes to think of an \mathcal{L} -formula with n free variables $\varphi(x_1, \dots, x_n)$ as a statement about each tuple $(a_1, \dots, a_n) \in |\mathcal{M}|^n$, where \mathcal{M} is an \mathcal{L} -structure. For example, if $\mathcal{L} = \mathcal{L}_{\text{Ring}}$ and $\varphi(x)$ is $\exists y(x = y^2)$, then for any ring R and $a \in R$, $\varphi(a)$ is true precisely when a is the square of an element of R .

Definition 147. Let \mathcal{L} be a first-order language, \mathcal{M} an \mathcal{L} -structure and $\varphi(x_1, \dots, x_n)$ an \mathcal{L} -formula. The **set defined by** φ is the set

$$\varphi(\mathcal{M}) = \{(a_1, \dots, a_n) \in M^n \mid \mathcal{M} \models \varphi(a_1, \dots, a_n)\}.$$

A set is called **definable** if it is of the above form, for some \mathcal{L} -structure \mathcal{M} and some \mathcal{L} -formula φ . A function is definable if its graph is definable.

Example 148. If $\mathcal{L} = \mathcal{L}_{\text{Ring}}$, $\mathcal{M} = \mathbb{R}$ and $\varphi(x)$ is $\exists y(x = y^2)$, then $\varphi(\mathcal{M}) = \mathbb{R}_{\geq 0}$.

A first-order language can be **n -sorted**, that is, a language \mathcal{L} equipped with a partition of the variables into n disjoint, countably infinite sets known as **sorts**, and to each function, relation and constant symbol, a list of sorts from which terms can be inputted to the symbol to produce a formula. In order to avoid a cumbersome definition, we give an example of how such a formula can be built in practice:

Example 149. Let \mathcal{L} be the Denef-Pas language of Definition 85, which is a 3-sorted language. The function symbol ord is unary from the \mathcal{L}_{Val} -sort to the $\mathcal{L}_{\text{Pres}}$ -sort, and $\overline{\text{ac}}$ is a unary function symbol from the \mathcal{L}_{Val} -sort to the \mathcal{L}_{Res} -sort. Let the \mathcal{L}_{Val} -variables be denoted x_1, x_2, \dots , the \mathcal{L}_{Res} -variables y_1, y_2, \dots , and the $\mathcal{L}_{\text{Pres}}$ -variables z_1, z_2, \dots . Then $\text{ord}(x_1) = z_1$ and $\overline{\text{ac}}(x_1) = y_1$ are valid formulae, but $\text{ord}(y_1) = x_1$ is not, since ord must take a valued field variable and equal a value group variable.

A.6 Homogeneous Varieties and PORC Functions

It was hoped that in Step 2(b) of the proof of Theorem 51, one could realise the index $[G(\mathbb{F}) : K(\mathbb{F})]$ as an integer closely related to the \mathbb{F} -points of a variety which is homogeneous under a linear algebraic group. By work of Brion and Peyre (Proposition 151), the number of \mathbb{F} -points is *polynomial on residue classes* (or PORC, see Section A.6.1). The aim is to use the PORC property to obtain the required finite set of polynomials. However, there arises a subtle issue with this approach which we outline in Section A.6.2. Whether this issue can be resolved directly in order to prove Theorem 51 by this method remains unclear.

A.6.1 PORC Functions

Given an algebraic group G and an algebraic subgroup K both defined over \mathbb{F}_q for some prime p , we will want to describe the behaviour of $[G(\mathbb{F}_{q^d}) : K(\mathbb{F}_{q^d})]$ as d varies over \mathbb{N} . This does not necessarily define a polynomial function, but one with a weaker property which we will define. Higman [29] defined the notion of a PORC function, however we modify the definition for convenience here.

Definition 150. Let $q \in \mathbb{N}$. A function $f : \{q^d \mid d \in \mathbb{N}\} \rightarrow \mathbb{Z}$ is **q -PORC** (polynomial on residue classes) if there exist $N \in \mathbb{N}$ and polynomials $g_0, \dots, g_{N-1} \in \mathbb{Q}[x]$ such that for all $d_1, d_2 \geq 1$ with $d_1 \equiv d_2 \pmod{N}$, $f(q^{d_1}) = g_{d_2}(q^{d_1})$. We call N the **period** of f and g_0, \dots, g_{N-1} the **constituents** of f . We say f is **q -CORC** (constant on residue classes) if the g_i can be chosen to be constant polynomials.

We will always consider q to be a prime power in our applications. The quotient of a q -PORC function of period N_1 by a q -CORC function of period N_2 is again q -PORC, of period dividing $N_1 N_2$.

Proposition 151. [9, Theorem 1.2, Remark 5.1(ii)] *Let G be an algebraic group defined over a finite field \mathbb{F}_q and let X be a variety with homogeneous G -action. Then the function*

$$q^d \mapsto |X(\mathbb{F}_{q^d})|$$

is a q -PORC function. Its period is bounded above by a constant that depends only on the rank of G .

The statement in [9] states further that the constituents of this q -PORC function have integer coefficients.

The following result states that the values taken by any set of q -PORC functions whose period is bounded above are also taken by some finite set of polynomials.

Lemma 152 (PORC Lemma). *Let $q \in \mathbb{N}$ and $\{f_i\}_{i \in I}$ be a set of q -PORC functions $\{q^d \mid d \geq 1\} \rightarrow \mathbb{Z}$ such that there exists $N > 0$ and for all $i \in I$, the period of f_i is at most N . Suppose further that there exists $C > 0$ such that for all $d \geq 1$, $|\{f_i(q^d) \mid i \in I\}| \leq C$. Then there exists a finite set of polynomials $A \subseteq \mathbb{Q}[x]$ such that for every $i \in I, d \geq 1$, there exists $g \in A$ such that $f_i(q^d) = g(q^d)$.*

Proof. Let N be the least common multiple of the periods of the f_i and fix any residue class m modulo N . Then there exist polynomials $g_{m,i}$, for $i \in I$ such that for all $d \equiv m \pmod{N}$, $f_i(q^d) = g_{m,i}(q^d)$. Let

$$C_m = \max_{d \equiv m \pmod{N}} |\{f_i(q^d) \mid i \in I\}|.$$

Choose C_m distinct polynomials from the set $\{g_{m,i} \mid i \in I\}$, say $A_m = \{g_{m,j} \mid j \in J\}$ where $J \subseteq I$ and $|J| = C_m$. Then there exists $d_m > 0$ such that for all $j_1, j_2 \in J$ with $j_1 \neq j_2$ and all $d \geq d_m$, we have $g_{m,j_1}(q^d) \neq g_{m,j_2}(q^d)$.

Write $d_0 = \max_m d_m$ and $A^+ = \bigcup_{m \in \mathbb{Z}/N\mathbb{Z}} A_m$. Then for all $d \geq d_0$ and all $i \in I$, there exists $g \in A^+$ such that $f_i(q^d) = g(q^d)$. Now put

$$A = A^+ \cup \{f_i(q^d) \mid i \in I, d < d_0\},$$

which is still a finite set by the assumption that for each d , the set $\{f_i(q^d) \mid i \in I\}$ is finite. \square

In the proof of Theorem 51, recalling the Clifford theory setup, we begin with a stabiliser subgroup $K(\mathbb{F})$ and induce representations to $G(\mathbb{F})$. Inducing representations corresponds to multiplication by the index $[G(\mathbb{F}) : K(\mathbb{F})]$, and we show that this is a PORC function in $|\mathbb{F}|$. More precisely:

Lemma 153. *Suppose that p is prime, I is an indexing set, and for each $i \in I$, $G_i \in \mathcal{A}(n', M_{\dim}, M_{\text{cmp}}, \mathbb{F}_q)$ for some p -power q , and K_i is a closed subgroup of G_i with complexity at most $C_{\text{nonhomog}}(n', M_{\text{cmp}})$. Then for each i , $[G_i(\mathbb{F}_{q^j}) : K_i(\mathbb{F}_{q^j})]$ is a q -PORC function, and there exists a finite set of polynomials $T_{\text{index}} \subseteq \mathbb{Q}[x]$ with the property that for all $j \geq 1$,*

$$\{[G_i(\mathbb{F}_{q^j}) : K_i(\mathbb{F}_{q^j})] \mid i \in I\} \subseteq \{m(q^j) \mid m \in T_{\text{index}}\}.$$

Proof. By Proposition 151, the functions f_{G/K° and f_{K/K° , given by $f_{G/K^\circ}(q^j) = |(G/K^\circ)(\mathbb{F}_{q^j})|$ and $f_{K/K^\circ}(q^j) = |(K/K^\circ)(\mathbb{F}_{q^j})|$ respectively, are both q -PORC. Furthermore, f_{K/K° is at most the number of connected components of K , and thus bounded above by $C_{\text{nonhomog}}(n', M_{\text{cmp}})$. Therefore, f_{K/K° must also be q -CORC.

Thus the quotient

$$[G(\mathbb{F}_{q^j}) : K(\mathbb{F}_{q^j})] = \frac{f_{G/K^\circ}(q^j)}{f_{K/K^\circ}(q^j)}$$

is also q -PORC.

Further, we can apply the same argument to show that $f_K(q^j) := |K(\mathbb{F}_{q^j})|$ is q -PORC, therefore $[G(\mathbb{F}_{q^j}) : K(\mathbb{F}_{q^j})]$ is also a q -PORC function.

By [26, Lemma 4.5.1], there exist $C_{GC,G}, C_{GC,K} > 0$ such that for all $j \geq 1$, the sets $A_j := \{|G_i(\mathbb{F}_{q^j})| \mid i \in I\}$ and $B_j := \{|K_i(\mathbb{F}_{q^j})| \mid i \in I\}$ are finite, with $|A_j| \leq C_{GC,G}, |B_j| \leq C_{GC,K}$. Therefore, for all $j \geq 1$,

$$|\{[G_i(\mathbb{F}_{q^j}) : K_i(\mathbb{F}_{q^j})] \mid i \in I\}| \leq C_{GC,G} C_{GC,K} =: C_{GC}.$$

Suppose without loss of generality that this bound is optimal, i.e. there exists j such that $|\{[G_i(\mathbb{F}_{q^j}) : K_i(\mathbb{F}_{q^j})] \mid i \in I\}| = C_{GC}$. By Proposition 151, there exist bounds on the periods of f_G and f_K depending only on the ranks of G and K (both bounded above by M_{\dim}), therefore we can apply Lemma 152 for the result. \square

A.6.2 A Previous Attempt at a Proof of Theorem 51

Recall that the objective of step 2(b) of the proof of Theorem 51 is to find a finite set $T_{\text{nonhomog}, M_{\dim}} \subseteq \mathbb{Q}[x]$ such that for every finite field \mathbb{F} of characteristic p ,

$$\begin{aligned} & \bigcup_{\substack{G \in \mathcal{A}(n', M_{\dim}, M_{\text{cmp}}, \mathbb{F}) \\ R_u(G) \neq 1, G}} \{ \dim \rho \mid \rho \in \text{Irr}(G(\mathbb{F})), \text{Res}_{U(\mathbb{F})}^{G(\mathbb{F})} \rho \text{ is non-homogeneous} \} \\ & \subseteq \{ m(|\mathbb{F}|) \mid m \in T_{\text{nonhomog}, M_{\dim}} \}. \end{aligned}$$

Suppose that $\rho \in \text{Irr}(G(\mathbb{F}))$ has non-homogeneous restriction to $U(\mathbb{F})$, hence $\rho = \text{Ind}_{K(\mathbb{F})}^{G(\mathbb{F})} \psi$, exactly as before.

By the inductive hypothesis,

$$\dim \psi \in \{ m(|\mathbb{F}|) \mid m \in T_{M_{\dim}-1, C_{\text{nonhomog}}(n', M_{\text{cmp}})} \}.$$

Each choice of a pair (G, ρ) , where $G \in \mathcal{A}(n', M_{\dim}, M_{\text{cmp}}, \mathbb{F})$ neither reductive nor

and so on. It remains unclear whether a PORC Lemma-style result holds in this situation, since the proof of the PORC Lemma relied on fixing a congruence class modulo N , which no longer corresponds to an evenly spaced set of rows.