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Near-critical interfaces and massive Schramm-Loewner evolutions

LEONIE BABETTE MARIE PAPON

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Near-critical interfaces and massive Schramm-Loewner evolutions

Léonie Papon

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Abstract

In this thesis, we study examples of near-critical interfaces in planar statistical mechanics models. These interfaces are defined on a lattice and we are interested in constructing their scaling limit as the meshsize of the lattice tends to 0. We show that in our examples, the limiting curves are described by massive SLE, as conjectured by Makarov and Smirnov [MS10].

In the first chapter, we look at a model called the massive harmonic explorer. This model was proposed by Makarov and Smirnov as a near-critical, or massive, perturbation of the harmonic explorer [MS10]. They argued that the scaling limit of the massive harmonic explorer in a bounded domain is a massive version of chordal SLE_4, called massive SLE_4, which is conformally covariant and absolutely continuous with respect to chordal SLE_4. We provide a full and rigorous proof of this statement. Moreover, we show that a massive SLE_4 curve can be coupled with a massive Gaussian free field as its level line, when the field has appropriate boundary conditions.

In the second chapter, we instigate further the above coupling between a massive GFF and a massive SLE_4 curve. We give an alternative proof of this coupling in which the coupling is constructed by reweighting the law of the standard GFF-SLE_4 coupling. As a consequence of this construction, we derive an explicit expression for the Radon-Nikodym derivative of the law of massive SLE_4 with respect to the law of SLE_4. We then prove that by reweighting the law of the coupling GFF-CLE_4 in a similar way, one obtains a coupling between a massive GFF and a random countable collection of simple loops, that we call massive CLE_4. As in the massless setting, these loops can be seen as the level lines of the massive GFF. Using this coupling, we show that massive CLE_4 can in turn be coupled to a massive Brownian loop soup of intensity $1/2$ in such way that the loops of massive CLE_4 correspond to the outer boundaries of the outermost clusters of loops in the loop soup. This proves a conjecture of Camia [Cam13]. Moreover, this relation between massive CLE_4 and the massive Brownian loop soup of intensity $1/2$ yields an explicit expression for the Radon-Nikodym derivative of the law of massive CLE_4 with respect to that of CLE_4. Finally, we note that, as the law of the massive GFF, the laws of massive SLE_4 and massive CLE_4 are conformally covariant.

In the third chapter, we consider the interface separating $+1$ and -1 spins in the critical planar Ising model with Dobrushin boundary conditions perturbed by an external magnetic field. We prove that this interface has a scaling limit. This result holds when the Ising model is defined on a bounded and simply connected subgraph of $\delta\mathbb{Z}^2$, with $\delta > 0$. We show that if the scaling of the external field is of order $\delta^{15/8}$, then, as $\delta \to 0$, the interface converges in law to a random curve whose law is conformally covariant and absolutely continuous with respect to SLE_3. This limiting law is a massive version of SLE_3 in the sense of Makarov and Smirnov [MS10] and we give an explicit expression for its Radon-Nikodym derivative with respect to SLE_3. We also prove that if the scaling of the external field is of order $\delta^{15/8}g_1(\delta)$ with $g_1(\delta) \to 0$, then the interface converges in law to SLE_3. In contrast, we show that if the scaling of the external field is of order $\delta^{15/8}g_2(\delta)$ with $g_2(\delta) \to \infty$, then the interface degenerates to a boundary arc.

Statement of Copyright

Declaration

I declare that this thesis has been composed solely by myself and that it has not been submitted, in whole or in part, in any previous application for a degree. Except where stated otherwise by reference or acknowledgment, the work presented is entirely my own.

Parts of this work have been published or posted as a preprint in

- Massive SLE_4 and the scaling limit of the massive harmonic explorer, *Annales de l'Institut Henri Poincaré*;

- Massive SLE_4, massive CLE_4 and the massive planar GFF, arXiv:2312.11180;

- Interface scaling limit for the critical planar Ising model perturbed by a magnetic field, arXiv:2411.16452.

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À Sandrine, François, Félicie et Cléophée.

Contents

Chapter 1

Introduction

Planar statistical mechanics models are a class of models which aim at capturing the behavior of a large system of interacting components. These models often depend on a set of parameters, such as the temperature and one of the early achievements of statistical mechanics was to prove that in many cases, these models undergo a phase transition: their macroscopic behavior is strongly influenced by the value of their parameters and drastically changes when these parameters assume a specific value, which is called the critical value or critical point.

These models are usually defined on a graph, typically a subset of the square grid $\delta\mathbb{Z}^d$, for some $\delta > 0$ and where d is the dimension. They give rise to probability measures on the vertices and/or edges of the graph: edges may be open or closed, vertices may be assigned a real, complex, or vector-valued random variable. Many questions in statistical mechanics can therefore be formulated within a probabilistic framework and the macroscopic properties of the models can in turn be thought of as properties of the corresponding probability measures. A question that has fascinated both physicists and mathematicians for more than 50 years is the existence of a scaling limit for these measures when the meshsize δ of the square grid $\delta\mathbb{Z}^d$ is sent to 0. This question, and its answer, is central in quantum field theory and turns out to harness the role played by the parameters of the model.

In two dimensions, at the critical point, physicists have observed that the model, or equivalently the corresponding probability measure, should acquire many symmetries in the scaling limit $\delta \to 0$: many functionals of the limiting measure, such as limits of correlation functions, should be conformally invariant. Moreover, at the lattice-level, these models satisfy an important property, called a domain Markov property. Informally, this can be phrased as follows. If the model is defined in a box $\Delta \subset \delta\mathbb{Z}^2$ and one looks at the system in a smaller box $\Lambda \subset \Delta$, then the state of the system in Λ depends on the state of the system outside Λ only through the state of the system on the boundary of Λ. This can be recast in terms of conditional expectations under the corresponding probability measure. Importantly, this domain Markov property is predicted to be conserved when taking the scaling limit $\delta \to 0$: the limiting measure, if it exists, should also satisfy a domain Markov property. This prediction, together with the above mentioned limiting conformal invariance property, marked the beginning of conformal field theory (CFT) [Pol70; BPZ84]. Indeed, conformal invariance and the domain Markov property impose strong constraints on what the limiting measures can be and this was used by physicists to classify the measures that can arise from scaling limits of planar statistical mechanics models at their critical point [FF84; FF90]. However, mathematically, many questions remain open about this classification and the very existence of limiting measures, even though a lot of progress was made in the last 25 years, at least in some specific cases, see [GKR24] for a review.

Instead of looking at correlation functions as in the approach developed in CFT, another way to study planar statistical mechanics models in the scaling limit is via their graphical representation. Indeed, many of these models can be described in terms of a set of interfaces: for example, interfaces separating $+1$ and -1 spins in a magnet or interfaces separating open and closed edges in a percolation model. If one believes physicists' prediction that these models are in some sense conformally invariant in the scaling limit, these interfaces should converge as $\delta \to 0$ to random curves in the complex plane that are conformally invariant in law. Moreover, at the lattice-level, when appropriate boundary conditions are imposed to the model, an interface joining two boundary points is observed and in the scaling limit, the law of this boundary-to-boundary interface should

become conformally invariant as well. The Markov property of the model should also manifest itself in the law of the limiting interface. More precisely, if one traces a portion of the interface starting from the boundary of the domain, then the law of the part of the interface that remains to be discovered should be the same as that of the original interface, but in the domain slit by the portion of the interface that was traced. This conjectural description of the scaling limit of the law of critical boundary-to-boundary interfaces puzzled mathematicians during more than twenty years until the far-reaching ideas of Oded Schramm sparked a revolution in the mathematical understanding of these conjectures.

Schramm introduced a one-parameter family of laws on random curves, now called Schramm-Loewner evolutions (SLE) [Sch00]. These random curves live on simply connected subsets of the complex plane $\mathbb{C}$ and join two boundary points. They are characterized by two properties. First, they are conformally invariant in law: if $\phi : D \to \tilde{D}$ is a conformal map between two simply connected domains $D, \tilde{D} \subset \mathbb{C}$ and γ has the law of an SLE curve in D, then $\phi(\gamma)$ has the law of an SLE curve in $\tilde{D}$. Second, SLE curves satisfy the following domain Markov property. If γ is an SLE curve in a domain $D \subset \mathbb{C}$ and $t \in (0, \infty)$, then, conditionally on $\gamma([0, t])$, $\gamma([t, \infty))$ has the law of an SLE curve in the unbounded component of the slit domain $D \setminus \gamma([0, t])$. The family of SLE curves is indexed by a non-negative parameter κ, which can be shown to quantify the roughness of the curve. We will give a more precise definition of SLE curves in Section 2.1.

Since SLE_κ curves are characterized by conformal invariance in law and a domain Markov property, they are the natural candidates to be the scaling limits of the above boundary-to-boundary interfaces observed in planar statistical mechanics models with appropriate boundary conditions, when these models are at their critical point. This was first established in the case of loop-erased random walk: the scaling limit of loop-erased random walk is SLE_2 [LSW04]. Via Wilson's algorithm, one can sample a uniform spanning tree using loop-erased random walk trajectories and this connection yielded a proof of the convergence of the so-called Peano curve of the uniform spanning tree to SLE_8 [LSW04]. It was then proved that the interface of critical site percolation with free-wired boundary conditions on the hexagonal lattice converges to SLE_6 [Smi01]. The proof relied on new tools in discrete complex analysis and these tools were subsequently used to prove that the interface of the critical Ising model with Dobrushin boundary conditions converges to SLE_3 while that of the critical FK-Ising model with free-wired boundary conditions converges to $\mathrm{SLE}_{16/3}$ [Che+14]. With different techniques, convergence of the interface of the discrete Gaussian free field to SLE_4 was also established [SS09].

Convergence of the critical interface to an SLE_κ curve can be used to derive many properties of the underlying statistical mechanics model at its critical point, for example its so-called critical exponents [LSW02; SW01; Wu18b; Wu18a], which govern its macroscopic behavior. However, this provides an accurate description of the model at its critical point only, whereas in statistical mechanics, understanding the phase transition itself is also of great importance to gain insights on the behavior of the model. In this regard, typical questions are about the thermodynamical properties of the model near the critical point. As the parameters of the model approach their critical values, these thermodynamical properties, such as the average magnetization in a magnet or the density of the largest cluster in a percolation model, start to exhibit the same behavior as at criticality. One possible way to investigate this phenomena is via the study of the so-called near-critical regime. In this regime, the parameters of the model are not set at their critical value but instead converge to their critical value when the meshsize of the lattice is sent to 0. To be in the near-critical regime, the rate of this convergence must be appropriately chosen.

Physicists, most notably Fisher, Kadanoff and Wilson, understood in the mid-sixties and seventies that in many models, different thermodynamical quantities can be related to each other as the parameters approach their critical values: this is the so-called scaling (and hyperscaling) relations [Fis99]. Studying the near-critical regime of these models is a possible path toward establishing these relations rigorously, as first achieved by Kesten for two-dimensional percolation on the square lattice [Kes87], see also [Nol08]. For some other two-dimensional percolation models, on the mathematical side, these scaling and hyperscaling relations are starting to be better understood [DM22] but a satisfying picture of their near-critical regime is still lacking.

In the near-critical regime, the description of planar statistical mechanics models in terms of a set of interfaces is still valid. However, much less is known about the properties of these interfaces. As in the critical regime, a natural question is whether they have a scaling limit or not, and if so how to describe it. In the critical regime, the properties of the scaling limit of the interfaces are often useful to establish properties of the model, such as the critical exponents mentioned above.

Therefore, one may hope that constructing and studying the scaling limit of these near-critical interfaces is a promising way to understand the behavior of planar models in the near-critical regime.

At the lattice-level, a model in the near-critical regime can be seen as a perturbation of the model in the critical regime. The near-critical interfaces can in turn be seen as a perturbation of the critical ones, in the sense that their law is absolutely continuous with respect to that of the critical interfaces. This absolute continuity may not pertain to the scaling limit but it can be expected that the limiting near-critical interface, if it exists, is still related in some way to the critical one. Moreover, one can observe that the near-critical boundary-to-boundary interface satisfies the same type of domain Markov property as the critical boundary-to-boundary interface. These two observations – absolute continuity at the lattice-level and the domain Markov property – led Makarov and Smirnov to raise the following question [MS10]: can SLE curves be in some sense "perturbed" to describe the scaling limits of near-critical boundary-to-boundary interfaces? In the continuum, the laws of these limiting near-critical interfaces should depend on a parameter, called the mass, which should stem from the near-critical perturbation applied to the model on the lattice. In view of this, Makarov and Smirnov gave the name of massive SLE to these "perturbations" of SLE that should describe the (conjectural) limiting laws of boundary-to-boundary near-critical interfaces.

One main difference between massive SLE and SLE is that the law of massive SLE is not expected to be conformally invariant. Indeed, in the scaling limit, the near-critical perturbation performed at the lattice-level breaks the conformal invariance of the model which is observed in the critical regime. As a consequence, the laws of the limiting near-critical interfaces should be conformally covariant, and not invariant: when mapping one domain to another via a conformal map, the image of the curve via the conformal map would have the same law but only provided that the mass is changed appropriately, in a way that depends on the choice of conformal map.

Moreover, the near-critical perturbation introduces a finite correlation length in the model. Informally, this means that observables depending on parts of the system at different and far away locations are roughly decorrelated. However, correlations between observables depending on parts of the system at close enough locations are roughly the same as in the critical regime, at least when stability below the correlation length holds (which is often expected to be the case). At the level of the interfaces, this should imply that "local" properties, such as the Hausdorff dimension of the curve, should be the same as those of the critical interfaces while global properties may be different. If these differences are not too striking, one expects the limiting law of the near-critical interface to be absolutely continuous with respect to that of the corresponding SLE_κ whereas if these differences are too important, the law of the limiting near-critical interface would be singular with respect to that of the corresponding SLE_κ. When looking at the driving function of the limiting near-critical interface, this should be encoded as follows. The driving function would satisfy an SDE given by $\sqrt{\kappa}$ times a Brownian motion plus a drift term, where we recall that $\sqrt{\kappa}$ times a Brownian motion is the driving function of SLE_κ. The regularity of the drift would then determine whether or not the law of the corresponding massive SLE_κ is absolutely continuous with respect to that of SLE_κ. However, at this stage of development of the theory of massive SLE, making this discussion more precise is out of reach.

As explained above, SLE_κ curves can be characterized by their conformal invariance in law and a Markovian property. Moreover, for a fixed value of κ, SLE_κ can often be characterized as the only law on random curves for which a given observable is a martingale. These two types of characterization are often very useful to prove that a discrete interface in the critical regime converges to an SLE_κ curve in the scaling limit. In the near-critical regime, such characterizations are often lacking, which makes establishing scaling-limit-type results more difficult. Nevertheless, convergence of near-critical interfaces has been established in a few cases. For example, massive loop-erased random walk has been shown to converge to massive SLE_2 [CW21; BH22]. Some progress has also been made towards establishing convergence of the interface of the near-critical FK-Ising model [Par22a] and of near-critical percolation [NW09b; GPS18], although in these two cases proving uniqueness of the limiting law remains a challenge.

In this thesis, we will construct and investigate two other massive SLE_κ curves. In the first chapter, we will look at the case $\kappa = 4$ and study a massive version of the harmonic explorer that was originally introduced by Makarov and Smirnov [MS10]. We will prove that the corresponding near-critical interface has a scaling limit, called massive SLE_4. In the second chapter, we will see that massive SLE_4 has a rich interplay with the massive planar Gaussian free field: when the field

has appropriate boundary conditions, massive SLE$_4$ can be coupled to the field as its level line. Based on this, we will construct a loop analogue of massive SLE$_4$, called massive CLE$_4$, and unveil its connections to the massive Gaussian free field but also to a massive version of the Brownian loop soup. Finally, in the third chapter, we will consider the case $\kappa = 3$ and describe the scaling limit of the interface separating $+1$ and -1 spins in the critical planar Ising model with Dobrushin boundary conditions perturbed by an external magnetic field. In this case, we will also look at the sub-critical and super-critical regimes and prove that in both cases, in the scaling limit, the interface degenerates, either to the critical one or to a boundary arc.

Let us now give a brief overview of the main results of each chapter. We inform the reader that each of these chapters can be read independently from the others. We provide some background material on SLE curves and convergence of discrete interfaces to continuous curves in Chapter 2.

1.1 Massive SLE$_4$ and the scaling limit of the massive harmonic explorer

In this chapter, we are interested in an example of near-critical interface called the massive harmonic explorer. This model of random lattice path was originally introduced by Makarov and Smirnov [MS10] as a near-critical perturbation of the harmonic explorer studied by Schramm and Sheffield [SS05]. Let us first give the construction of the harmonic explorer. We will then see how to perturb it to construct the massive harmonic explorer.

The harmonic explorer is a random discrete path defined on the hexagonal lattice. To construct it, one considers a subset Ω_δ of the triangular lattice $\delta \mathbb{T}$ with meshsize $\delta > 0$ together with two marked points a_δ and b_δ on the boundary of Ω_δ. The vertices on the clockwise oriented boundary arc $(a_\delta b_\delta)$ are assigned the sign $+$ while the vertices on the counter-clockwise oriented boundary arc $(a_\delta b_\delta)$ are assigned the sign $-$, a_δ and b_δ being assigned an arbitrary sign. The path starts in the middle of the edge joining a_δ to the vertex on the boundary of Ω_δ with opposite sign. This singles out a vertex v of Ω_δ which is linked by an edge to a_δ and to this other boundary vertex. Let h_δ be the unique discrete harmonic function in Ω_δ with boundary conditions $+1/2$ on the boundary vertices with sign $+$ and $-1/2$ on the boundary vertices with sign $-$. Then, with probability $1/2 + h_\delta(v)$, the path turns right, that is follows the edges of the hexagonal lattice linking its starting point to the middle of the edge of Ω_δ on its left, and the vertex v is assigned the sign $+$. With complementary probability, the path turns left and in this case, the vertex v is assigned the sign $-$. In both cases, the vertex v becomes a boundary vertex and this defines a new graph, with its associated discrete harmonic function $h_{\delta,1}$. One can then repeat the above procedure, with respect to the harmonic function corresponding to the new graph, to continue tracing the path. This gives rise to a sequence of discrete harmonic functions $(h_{\delta,n})_n$ corresponding to the sequence of graphs obtained while constructing the path. The procedure terminates when the path reaches the edge linking b_δ to a boundary vertex with opposite sign. See Figure 1.1 for a dual perspective on the hexagonal lattice.

To define a near-critical, or massive, perturbation of this model, which we call the massive harmonic explorer as in [MS10], we assign a weight $1 - cm^2\delta^2$ to each edge of the graph. Here, $m^2 > 0$ is a strictly positive constant and $c > 0$ is a constant depending on the lattice, but not on δ. We let h_δ^m be the unique discrete massive harmonic function in Ω_δ with boundary conditions $1/2$ on the boundary vertices with sign $+$ and $-1/2$ on the boundary vertices with sign $-$. The path is then constructed by following the same procedure as above, except that we now consider the function h_δ^m instead of h_δ, thus obtaining a sequence of discrete massive harmonic functions $(h_{\delta,n}^m)_n$. See again Figure 1.1 for a dual perspective on the hexagonal lattice.

The harmonic explorer is known to converge to chordal SLE$_4$ in an appropriate topology [SS05]. Makarov and Smirnov provided arguments in [MS10] to support their assertion that the massive harmonic explorer in turn converges to a massive version of chordal SLE$_4$, called massive SLE$_4$, which is absolutely continuous with respect to SLE$_4$. This was further investigated in an unpublished manuscript [Sha]; however, a conclusive argument was not reached. The main result of this chapter is a fully detailed and rigorous proof of the statement of Makarov and Smirnov, and can informally be written as follows.

Theorem 1.1.1. *Let $\Omega \subset \mathbb{C}$ be a bounded, open and simply connected domain with two marked boundary points a and b. Let $(\Omega_\delta, a_\delta, b_\delta)_\delta$ be discrete approximations of (Ω, a, b), where for each $\delta > 0$, Ω_δ is a subset of the triangular lattice $\delta \mathbb{T}$. Let $m > 0$. Then, as $\delta \to 0$, the scaling limit*
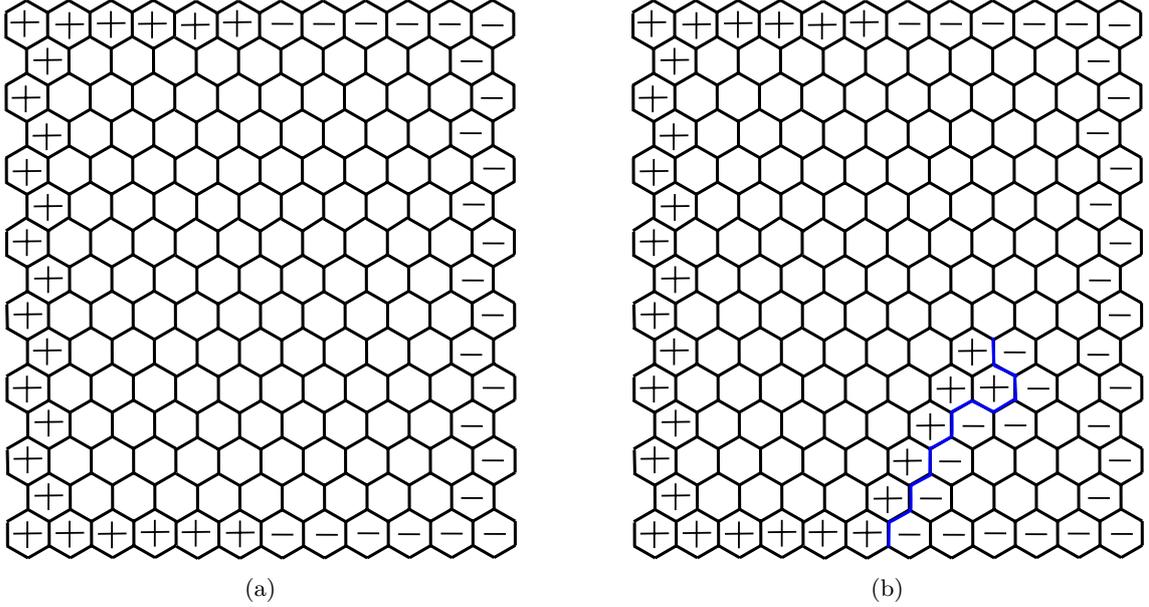
Figure 1.1: (a) Initial configuration: the hexagons on the right-hand side have sign − while those on the left hand-side have sign +. (b) The blue path is a possible path followed by the first steps of the massive harmonic explorer. The hexagons on the right, respectively left, of the path have been assigned the sign −, respectively +.

of the massive harmonic explorer from a_δ to b_δ in Ω_δ is a random curve γ whose law is that of massive SLE_4 with mass m in Ω from a to b.

The precise assumptions on $(\Omega_\delta, a_\delta, b_\delta)_\delta$ and $(\Omega; a, b)$ together with the topologies in which weak convergence is obtained will be described in Chapter 2. Let us now say a word about massive SLE_4. Let $\phi : \Omega \to \mathbb{H}$ be a conformal map from Ω to the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ such that $\phi(a) = 0$ and $\phi(b) = \infty$. A curve γ in Ω from a to b is said to have the law of massive SLE_4 with mass m in Ω from a to b if $\phi(\gamma)$, when parametrized by half-plane capacity, is a chordal stochastic Loewner evolution whose driving function satisfies the SDE

$$dW_t = 2dB_t - 2\pi\left(\int_{\Omega_t} m^2 P_t^m(w) h_t(w) dw\right) dt, \quad W_0 = 0, \tag{1.1}$$

where $(B_t, t \geq 0)$ is a standard one-dimensional Brownian motion. Above, Ω_t is defined as $\Omega \setminus K_t$, with K_t being the hull generated by $\phi(\gamma)([0, t])$, P_t^m is related to the massive Poisson kernel with mass m in $\Omega_t = \Omega \setminus K_t$ evaluated at the growth point $\gamma(t)$ and h_t is the unique harmonic function in Ω_t with boundary conditions $-1/2$ on the counterclockwise oriented boundary arc (ab) and the right side of $\gamma([0, t])$ and $1/2$ on the clockwise oriented boundary arc (ab) and the left side of $\gamma([0, t])$.

All the quantities appearing in the SDE (1.1) will be defined precisely in Section 3.4. We will see in Section 3.5.1 that this SDE has a unique weak solution whose law is absolutely continuous with respect to $(2B_t, t \geq 0)$. This implies that massive SLE_4 with mass m in Ω from a to b is absolutely continuous with respect to SLE_4 in Ω from a to b.

We will show Theorem 1.1.1 under weaker assumptions on the mass m: we will establish the result for a space-dependent mass $m : \Omega \to \mathbb{R}_+$ and its appropriate discretizations $(m_\delta : \Omega_\delta \to \mathbb{R}_+)_\delta$, provided that the function m is continuous and bounded. Defining massive SLE_4 with space-dependent mass also enables us to show that massive SLE_4 is conformally covariant, in a sense made precise in Section 3.5.1.

SLE_4 has a rich interplay with the planar continuum Gaussian free field (GFF). The prime example of this is the existence of a level line coupling between an SLE_4 curve and a GFF with appropriate boundary conditions [Dub09; MS16]. One may wonder whether the massive version of SLE_4 defined via the SDE (1.1) can be coupled in the same way to a massive GFF. The answer to this question turns out to be positive. Let $\Omega \subset \mathbb{C}$ be a bounded, open and simply connected domain and let $m > 0$. The massive GFF in Ω with mass m and Dirichlet boundary conditions is

the centered Gaussian process Γ^m indexed by smooth and compactly supported functions whose covariance is, for f and g two such functions,

$$\mathbb{E}[(\Gamma^m, f)(\Gamma^m, g)] = \int_\Omega f(x) G^m_\Omega(x, y) g(y) dy dx.$$

Above, G^m_Ω is the massive Green function in Ω with mass m and Dirichlet boundary conditions, that is G^m_Ω is the inverse in the sense of the distributions of the operator $-\Delta + m^2$ in Ω with Dirichlet boundary conditions. As the GFF, the massive GFF is not defined pointwise but is only a generalized function. One can also impose boundary conditions to a masive GFF. More precisely, for a function $f : \partial\Omega \to \mathbb{R}$ with finitely many discontinuity points, we say that a massive GFF Γ^m in Ω with mass m has boundary conditions f if Γ^m has the same law as $\Gamma^m_0 + \phi^m_f$, where Γ^m_0 is a massive GFF in Ω with mass m and Dirichlet boundary conditions and ϕ^m_f is the massive harmonic extension of f in Ω. The existence of a coupling between a massive GFF with appropriate boundary conditions and a massive SLE$_4$ curve then reads as follows.

Theorem 1.1.2. *Set $\lambda := \sqrt{\pi/8}$. Let $\Omega \subset \mathbb{C}$ be a bounded, open and simply connected domain and let $a, b \in \partial\Omega$. Denote by $\partial\Omega^+$, respectively $\partial\Omega^-$, the clockwise, respectively counterclockwise, oriented boundary arc (ab). Let $m > 0$. Then there exists a coupling (Γ^m, γ) where Γ^m is a massive GFF in Ω with mass m and boundary conditions $-\lambda$ on $\partial\Omega^-$ and λ on $\partial\Omega^+$ and γ is a massive SLE$_4$ with mass m in Ω from a to b. In this coupling, for any stopping time τ for the filtration generated by γ, conditionally on $\gamma([0, \tau])$,*

$$\Gamma^m = \Gamma^m_\tau + \phi^m_\tau$$

where Γ^m_τ is a massive GFF in $\Omega \setminus \gamma([0, \tau])$ with mass m and Dirichlet boundary conditions and ϕ^m_τ is the massive harmonic function in $\Omega \setminus \gamma([0, \tau])$ with boundary conditions $-\lambda$ on $\partial\Omega^-$ and the right side of $\gamma([0, \tau])$ and λ on $\partial\Omega^+$ and the left side of $\gamma([0, \tau])$. Moreover, Γ^m_τ and ϕ^m_τ are independent.

The existence of such a coupling was already observed in the physics literature [BBC09] assuming absolute continuity of massive SLE$_4$ with respect to SLE$_4$ in the upper half-plane. We emphasize that here, Theorem 1.1.2 is only stated in bounded domains. Its proof, given in Section 3.5, is analogous to that of the existence of a coupling between a GFF and an SLE$_4$ curve. We will actually establish the result in the case of a space-dependent mass $m : \Omega \to \mathbb{R}_+$, provided that m is a bounded and continuous function. Conformal covariance of the massive GFF and of massive SLE$_4$ can then be used to extend this result to unbounded domains with appropriate space-dependent masses, that is masses which are inherited from a bounded domain via conformal mapping, see Section 3.5 for details.

Outline of the proof of Theorem 1.1.1

Let us say a few words about the proof of Theorem 1.1.1. Its strategy can be decomposed into three main steps. The first one is to show tightness of the sequence of massive harmonic explorer paths $(\gamma_\delta)_\delta$ in an appropriate topology. One natural approach would be to show that the massive harmonic explorer is absolutely continuous with respect to the harmonic explorer and that the Radon-Nikodym derivative is well-behaved in the limit $\delta \to 0$. However, it is unclear whether absolute continuity holds at the level of the discrete curves and we therefore adopt a different approach relying on [KS17] and [Kar23]. Thanks to these results, to prove tightness of $(\gamma_\delta)_\delta$, it suffices to show a suitable bound on the probability that the massive harmonic explorer crosses an annulus intersecting the boundary of Ω_δ. This is what we will establish in Section 3.2.

Tightness of the sequence $(\gamma_\delta)_\delta$ then implies the existence of subsequential limits. Characterizing these subsequential limits thus obtained is the aim of the next step of the proof. This step is split into two. First, we will exhibit and study a discrete martingale for the filtration generated by the massive harmonic explorer. This martingale will be given by for fixed $\delta > 0$, the sequence of discrete massive harmonic functions $(h^m_{\delta,n})_n$, see Section 3.1.2 for details. We will prove that the continuum limit as $\delta \to 0$ of h^m_δ is the unique massive harmonic function in Ω with mass m and boundary conditions $1/2$ on the clockwise oriented boundary arc (ab) and $-1/2$ on the counter-clockwise oriented boundary arc (ab). This result will in fact be shown for each $h^m_{\delta,n}$ under precise assumptions on the convergence of the domain at time n to a continumm domain. These assumptions will hold thanks to the tightness of the sequence $(\gamma_\delta)_\delta$ proved in the previous step.

Convergence of these discrete massive harmonic functions is established in Section 3.3 by adapting some of the arguments of [CS11] to the massive setting.

This convergence result suggests the following characterization of the limiting law of any subsequential limit of $(\gamma_\delta)_\delta$: this law is the unique law on non-self-intersecting curves in Ω from a to b such that the massive harmonic function h_t^m with mass m and boundary conditions $1/2$ on the clockwise oriented boundary arc (ab) and the left side of $\gamma([0,t])$ and $-1/2$ on the counterclockwise oriented boundary arc (ab) and the right side of $\gamma([0,t])$ is a martingale for the filtration generated by γ. We will indeed established uniqueness of such a law and show that massive SLE$_4$ in Ω from a to b satisfies the above martingale property. This will thus identify uniquely the law of any subsequential limit of $(\gamma_\delta)_\delta$. Observe that this characterization of massive SLE$_4$ is reminiscent of the characterization of SLE$_4$ by the martingale property of a certain (massless) harmonic function [Dub09; MS16]. In the massive case, the proof follows the same strategy but involves some technicalities due to the presence of a mass. It is given in Section 3.4.

1.2 Massive SLE$_4$, massive CLE$_4$ and the massive planar GFF

In Chapter 3, we showed that a massive SLE$_4$ curve can be coupled with a massive Gaussian free field with appropriate boundary conditions as its level line. This massive SLE$_4$ curve was defined via its driving function and the proof of Theorem 1.1.2 relied on the explicit expression for this function. However, in a simply connected domain $D \subset \mathbb{C}$ and under suitable conditions on the mass, the massive GFF with mass m is absolutely continuous with respect to the GFF with Radon-Nikodym derivative given by

$$\frac{d\mathbb{P}_{\text{mGFF}}}{d\mathbb{P}_{\text{GFF}}}(h) = \frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \int_D m^2(z) : h^2(z) : dz \right) \tag{1.2}$$

where $\mathcal{Z}$ is a normalization constant. In (1.2), $: h^2 :$ is the Wick square of the GFF, which is a renormalized version of the square of the GFF.

In view of (1.2), it is natural to wonder whether the law of the coupling massive GFF-massive SLE$_4$ can be obtained by reweighting the law of the coupling GFF-SLE$_4$. This idea, albeit under a different form, was already suggested in the physics literature [BBC09, Section 4.6] and the first main result of Chapter 4 rigorously formalizes it.

In everything that follows, we set $\lambda := \sqrt{\pi/8}$, and if $D \subset \mathbb{C}$ is an open, bounded and simply connected domain, with $a, b \in \partial D$ (in the sense of prime ends), we denote by ∂D^+, respectively ∂D^-, the clockwise-oriented, respectively counterclockwise-oriented, boundary arc (ab). We also define $F^{(D,a,b)} : \partial D \to \mathbb{R}$ to be equal to λ on ∂D^+ and $-\lambda$ on ∂D^-.

Theorem 1.2.1. *Let D, a, b be as above, and let $\phi : D \to \mathbb{R}$ be the unique harmonic function in D with boundary values $F^{(D,a,b)}$. Denote by $\mathbb{P}$ the law of the coupling $(h + \phi, \gamma)$ between a GFF $h + \phi$ with boundary conditions ϕ in D and an SLE$_4$ curve γ in D from a to b. Let $m : D \to \mathbb{R}_+$ be of the form $|\varphi'|\hat{m} \circ \varphi$, where $\varphi : D \to \hat{D}$ is a conformal isomorphism, $\hat{D} \subset \mathbb{C}$ is a bounded and simply connected domain and $\hat{m} : \hat{D} \to \mathbb{R}_+$ is a bounded and continuous function. Define a new probability measure $\tilde{\mathbb{P}}$ by*

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}((h + \phi, \gamma)) := \frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \int_D m^2(z) : (h + \phi)^2(z) : dz \right) \tag{1.3}$$

where $\mathcal{Z}$ is a normalization constant. Then, under $\tilde{\mathbb{P}}$,

1. *the marginal law of $h + \phi$ is that of a massive GFF in D with mass m and boundary conditions $F^{(D,a,b)}$;*
2. *the marginal law of γ is that of a massive SLE$_4$ curve with mass m in D from a to b, as defined in Section 4.1.1.*

Moreover, let $t \in [0, \infty)$. Then, under $\tilde{\mathbb{P}}$, conditionally on $\gamma([0,t])$, $h + \phi = h_t + \phi_t$ where $h_t + \phi_t$ has the law of a massive GFF in $D \setminus \gamma([0,t])$ with mass m and with boundary conditions $F^{(D \setminus \gamma[0,t], \gamma(t), b)}$. The same result holds at $t = \infty$; with $h_t + \phi_t$ being a sum of independent massive GFFs with mass m and boundary conditions $\pm\lambda$ on either side of the curve.

Theorem 1.2.1 implies that the law of massive SLE_4 is absolutely continuous with respect to that of ordinary SLE_4, and that in the massive GFF-massive SLE_4 coupling (i.e. the law of $(h+\phi,\gamma)$ under $\tilde{\mathbb{P}}$), the curve is a measurable function of the field.[1] Observe also that by conformal invariance of SLE_4 and conformal covariance of the random variable on the right-hand side of (1.3), we can deduce from Theorem 1.2.1 that massive SLE_4 is conformally covariant, in a sense that will be made precise below.

Finally, Theorem 1.2.1 combined with Le Jan's isomorphism theorem [Jan08; Jan10] yield the following expression for the Radon-Nikodym derivative of the law $\mathbb{P}_{\mathrm{mSLE}_4}^{(D,a,b)}$ of massive SLE_4 with respect to the law $\mathbb{P}_{\mathrm{SLE}_4}^{(D,a,b)}$ of SLE_4.

Corollary 1.2.2. *Let $D \subset \mathbb{C}$, $a,b \in \partial D$ and $m : D \to \mathbb{R}_+$ be as in Theorem 1.2.1. Then, for $t \geq 0$,*

$$\left.\frac{\mathrm{d}\mathbb{P}_{\mathrm{mSLE}_4}^{(D,a,b)}}{\mathrm{d}\mathbb{P}_{\mathrm{SLE}_4}^{(D,a,b)}}(\gamma)\right|_{\sigma(\gamma(s),s\leq t)} = \frac{1}{\mathcal{Z}}\exp\left(\frac{1}{2}\mu_t\big(e^{-\langle \ell,m^2\rangle} + \langle \ell,m^2\rangle - 1\big) - \frac{1}{2}\int_{D_t} m^2(z)\phi_t(z)\phi_t^m(z)dz\right)$$

$$\times \exp\left(\int_{D_t}\frac{m^2(z)}{4\pi}\log\frac{\mathrm{CR}(z,\partial D)}{\mathrm{CR}(z,\partial D_t)}dz\right)$$

where $\mathcal{Z}$ is as in the definition (1.3) of $\tilde{\mathbb{P}}$ in Theorem 1.2.1. Above, $D_t := D \setminus \gamma([0,t])$ and ϕ_t, respectively ϕ_t^m, is the harmonic, respectively massive harmonic, function in D_t with boundary conditions $F^{(D_t,\gamma(t),b)}$.

Here μ_t denotes the Brownian loop measure in D_t, and for a loop ℓ, $\langle \ell,m^2\rangle := \int_0^{\tau(\ell)} m^2(\ell(t))dt$, where $\tau(\ell)$ is the lifetime of ℓ. For $z \in D$, $\mathrm{CR}(z,\partial D)$, respectively $\mathrm{CR}(z,\partial D_t)$ is the conformal radius of z in D, respectively D_t.

SLE_κ curves for $\kappa \in (8/3,8)$ have loop versions, called conformal loop ensembles (CLE). CLE_κ are a one-parameter family of random countable collections of planar loops characterized by their conformal invariance and a certain Markovian property [SW12]. There also exists a coupling between a CLE_4 and a GFF with Dirichlet boundary conditions [MS11; ASW19] in which the loops of CLE_4 are in a certain sense, similarly to SLE_4, level lines of the GFF. One may wonder whether a massive version of this coupling exists. To date, no massive version of CLE_4, or of CLE_κ for other values of κ, has been defined. However, using the same ideas as those motivating Theorem 1.2.1, we can construct a coupling between a massive GFF and a random collection of planar loops.

Theorem 1.2.3. *Let $D \subset \mathbb{C}$ and $m : D \to \mathbb{R}_+$ be as in Theorem 1.2.1. Denote by $\mathbb{P}$ the law of the coupling (h,Γ) between a GFF h in D with Dirichlet boundary conditions and a CLE_4 Γ in D. Define a new probability measure $\tilde{\mathbb{P}}$ by*

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}((h,\Gamma)) := \frac{1}{\mathcal{Z}}\exp\left(-\frac{1}{2}\int_D m^2(z):h^2(z):dz\right) \tag{1.4}$$

where $\mathcal{Z}$ is a normalization constant. Then, under $\tilde{\mathbb{P}}$, the marginal law of h is that of a massive GFF in D with mass m and with Dirichlet boundary conditions. Moreover, under $\tilde{\mathbb{P}}$, conditionally on Γ,

$$h = \sum_j h_j + \xi_j$$

where the sum runs over the loops $(L_j)_j$ of Γ and

1. *for each j, $h_j + \xi_j$ has the law of a massive GFF in the interior of the loop L_j with mass m and boundary conditions ξ_j;*
2. *the fields $(h_j + \xi_j)_j$ are independent;*
3. *the random variables $(\xi_j)_j$ are independent and $\tilde{\mathbb{P}}(\xi_j = -2\lambda|\Gamma) = \tilde{\mathbb{P}}(\xi_j = 2\lambda|\Gamma) = 1/2 \ \forall j$;*
4. *the random variables $(\xi_j)_j$ are measurable with respect to the fields $(h_j + \xi_j)_j$.*

Definition 1.2.4. We define massive CLE_4 to be the law of the collection of loops Γ under $\tilde{\mathbb{P}}$, and we denote this law by $\mathbb{P}_{\mathrm{mCLE}_4}^D$.

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[1] If this measurability had already been proven in the previous chapter, then Theorem 1.2.1 would follow from the results of this chapter. However, this fact was not straightforward to prove in the set-up of the previous chapter.

Again it follows from the above theorem and definition that the law of massive CLE$_4$ is absolutely continuous with respect to the law of CLE$_4$ in D. In particular, massive CLE$_4$ defines a countable collection of planar loops in D that are almost surely simple and almost surely do not touch each other or the boundary of D. Moreover, we can deduce from Theorem 1.2.3 that the law of massive CLE$_4$ is conformally covariant.

Corollary 1.2.5. *Let $D \subset \mathbb{C}$ and $m : D \to \mathbb{R}_+$ be as in Theorem 1.2.3. Let $\tilde{\varphi} : D \to \tilde{D}$ be a conformal map. If Γ has the law of massive CLE$_4$ in D with mass m, then $\tilde{\varphi}(\Gamma)$ has the law of massive CLE$_4$ in $\tilde{D}$ with mass given by, for $w \in \tilde{D}$, $\tilde{m}^2(w) = |(\tilde{\varphi}^{-1})'(w)|^2 m^2(\tilde{\varphi}^{-1}(w))$.*

Observe also that the coupling of Theorem 1.2.3 can be iterated in the interiors of the loops to obtain a loop decomposition of the massive GFF similar to that of the GFF [ASW19]. However, in this decomposition, the loops are not identically distributed: this is because massive CLE$_4$ is only conformally covariant and not conformally invariant.

As in the case of massive SLE$_4$, a consequence of Theorem 1.2.3 is an explicit expression for the Radon-Nikodym derivative of $\mathbb{P}^D_{\mathrm{mCLE}_4}$ with respect to the law $\mathbb{P}^D_{\mathrm{CLE}_4}$ of CLE$_4$ in D.

Corollary 1.2.6. *Let $D \subset \mathbb{C}$ and $m : D \to \mathbb{R}_+$ be as in Theorem 1.2.3. Then*

$$\frac{\mathrm{d}\mathbb{P}^D_{\mathrm{mCLE}_4}}{\mathrm{d}\mathbb{P}^D_{\mathrm{CLE}_4}}(\Gamma) = \frac{1}{\mathcal{Z}} \prod_j \exp\left(\frac{1}{2}\mu_j\big(e^{-\langle \ell, m^2 \rangle} + \langle \ell, m^2 \rangle - 1\big) - 2\lambda^2 \int_{\mathrm{Int}(L_j)} m^2(z) H_j^{(m)}(z, L_j) dz\right)$$

$$\times \exp\left(\int_{\mathrm{Int}(L_j)} \frac{m^2(z)}{4\pi} \log \frac{\mathrm{CR}(z, \partial D)}{\mathrm{CR}(z, L_j)} dz\right)$$

where $\mathcal{Z}$ is as in the definition (1.4) of $\tilde{\mathbb{P}}$ in Theorem 1.2.3 and where the product runs over the loops $(L_j)_j$ of Γ. For each j, μ_j is the Brownian loop measure in the interior $\mathrm{Int}(L_j)$ of the loop L_j while, for $z \in \mathrm{Int}(L_j)$, $H_j^{(m)}(z, L_j)$ is the massive harmonic measure of L_j seen from z.

Remark 1.2.7. In the massless case, the partition function of SLE$_\kappa$ can be related to the ζ-regularized determinant of the Laplacian [Dub09; PW24]. This is also the case for massive SLE$_4$: using the computations of [BBC09, Appendix A], one can relate the exponential of the loop measure term of Corollary 1.2.2, respectively Corollary 1.2.6 to $\big(\frac{\det_\zeta(-\Delta_t + m^2)}{\det_\zeta(-\Delta_t)}\big)^{-1/2}$, respectively $\big(\frac{\det_\zeta(-\Delta_j + m^2)}{\det_\zeta(-\Delta_j)}\big)^{-1/2}$. Here, $\det_\zeta$ denotes the ζ-regularized determinant and Δ_t, respectively Δ_j, is the Laplacian in D_t, respectively the interior of the loop L_j, with Dirichlet boundary conditions.

Although Corollary 1.2.6 gives an explicit expression for the Radon-Nikodym derivative of $\mathbb{P}^D_{\mathrm{mCLE}_4}$ with respect to $\mathbb{P}^D_{\mathrm{CLE}_4}$, this does not provide a direct construction of massive CLE$_4$. Such a construction exists in the massless case [SW12]: if one first samples a Brownian loop soup with intensity $1/2$ in D and keeps only the outer boundaries of the outermost clusters of loops of the loop soup, then one obtains a collection of loops in D that has the law of CLE$_4$ in D. In [Cam13], Camia introduced a massive version of the Brownian loop soup and conjectured that the outer boundaries of its outermost clusters at intensity $1/2$ are distributed as the level lines of the massive GFF. Thanks to Theorem 1.2.3 and to the coupling of [QW19, Proposition 5], we can prove this conjecture and give an explicit construction of massive CLE$_4$.

Theorem 1.2.8. *Let $D \subset \mathbb{C}$ and $m : D \to \mathbb{R}_+$ be as in Theorem 1.2.1. Then a massive CLE$_4$ Γ in D with mass m and a massive Brownian loop soup $\mathcal{L}$ in D with mass m and intensity $1/2$ can be coupled together in such way that the outer boundaries of the outermost clusters of $\mathcal{L}$ are the loops of Γ.*

Outline of the proof of Theorem 1.2.1 and Theorem 1.2.3

The proof of Theorem 1.2.1, given in Section 4.2, is composed of two main steps. Note that by definition, under $\tilde{\mathbb{P}}$ as defined via (1.3), the marginal law of $h + \phi$ is that of a massive GFF in D with mass m and boundary conditions $F^{(D,a,b)}$. In the first step, we identify the conditional law under $\tilde{\mathbb{P}}$ of $h + \phi$ given $\gamma([0,t])$, for $t \geq 0$. This involves some technical arguments but the main idea is simple. Let f be a smooth function with compact support and let $t \geq 0$. It suffices to compute the characteristic function of $(h + \phi, f)$ given $\gamma([0,t])$ under $\tilde{\mathbb{P}}$. If the field $h + \phi$ were a

pointwise defined function and thus no renormalization were needed to define its square, we would have that

$$\tilde{\mathbb{E}}[\exp(i(h+\phi,f))|\gamma([0,t])] = \frac{1}{\mathcal{Z}_t}\mathbb{E}[\exp(i(h+\phi,f))\exp(-\frac{1}{2}\int_D m^2(z)(h+\phi)^2(z)dz)|\gamma([0,t])]$$
$$= \frac{1}{\mathcal{Z}_t}\mathbb{E}[\exp(i(h_t+\phi_t,f))\exp(-\frac{1}{2}\int_{D_t} m^2(z)(h_t+\phi_t)^2(z)dz)|\gamma([0,t])],$$
(1.5)

using the fact that under $\mathbb{P}$, $h+\phi$ and γ are coupled in such way that, conditionally on $\gamma([0,t])$, $h+\phi = h_t+\phi_t$ where $h_t+\phi_t$ is a GFF in $D_t := D\setminus\gamma([0,t])$ with boundary conditions $F^{(D_t,\gamma(t),b)}$. This coupling will be recalled more precisely in Section 4.1.3. Similarly, we would have that

$$\mathcal{Z}_t = \mathbb{E}[\exp(-\frac{1}{2}\int_D m^2(z)(h+\phi)^2(z)dz)|\gamma([0,t])] = \mathbb{E}[\exp(-\frac{1}{2}\int_{D_t} m^2(z)(h_t+\phi_t)^2(z)dz)|\gamma([0,t])].$$
(1.6)

Now, observe that, since conditionally on $\gamma([0,t])$, $h_t+\phi_t$ is a GFF in D_t with boundary conditions $F^{D_t,\gamma(t),b}$, according to (1.2), the random variable

$$\frac{1}{\mathcal{Z}_t}\exp\left(-\frac{1}{2}\int_{D_t} m^2(z)(h_t+\phi_t)^2(z)dz\right)$$

is the Radon-Nikodym derivative of the law of the massive GFF in D_t with mass m and boundary conditions $F^{(D_t,\gamma(t),b)}$, with respect to the law of the GFF in D_t with the same boundary boundary conditions. Going back to (1.5), this would complete the proof.

Of course, these computations are only formal since h is not a pointwise defined function. Nevertheless, when introducing the renormalization of h^2 mentioned in the discussion below (1.2), they can be made rigorous. The equalities (1.5) and (1.6) actually hold true when considering the renormalized squares : $(h+\phi)^2$: and : $(h_t+\phi_t)^2$: of $h+\phi$ and $h_t+\phi_t$, but up to a correction term due to boundary effects. This term is the same in both equalities and thus is cancelled out.

The second step of the proof of Theorem 1.2.1 is devoted to the identification of the marginal law of γ under $\tilde{\mathbb{P}}$. This relies on a martingale characterization of massive SLE_4 shown in the previous chapter and on the fact that under $\tilde{\mathbb{P}}$, $h+\phi$ and γ are coupled as described in the statement of Theorem 1.2.1. This fact is used to express the martingale characterizing massive SLE_4 as an observable of the field $h+\phi$ conditioned on $\gamma([0,t])$, for $t \geq 0$. This is possible thanks to the Markov property of the massive GFF and the explicit expression for its covariance.

The strategy for the proof of Theorem 1.2.3 is the same as that of Theorem 1.2.1, except that we must deal with more technicalities. This is due to the fact that under $\mathbb{P}$ and when regularizing the GFF h to define its square, conditionally on Γ, the regularized GFFs $(h_{j,\epsilon})_j$ obtained via the decomposition of h are not supported in the interior of the loops of Γ. This creates correlations and we must ensure that they disappear when we remove the regularization. To identify the conditional law of the variables $(\xi_j)_j$ given Γ under $\tilde{\mathbb{P}}$, we simply use the symmetry of the Wick square of the GFF and of the conditional laws of $(h_j)_j$ and $(\xi_j)_j$ given Γ under $\mathbb{P}$.

Finally, to prove Theorem 1.2.8, we first compute the Radon-Nikodym derivative of the law of the massive Brownian loop soup with respect to that of the Brownian loop soup. We then use this Radon-Nikodym derivative to reweight the coupling of [QW19] between a CLE_4, a GFF with Dirichlet boundary conditions and a Brownian loop soup with intensity $1/2$ in a similar way as in Theorem 1.2.3. We show that we thus obtain a coupling between a massive CLE_4, a massive GFF with Dirichlet boundary conditions and a massive Brownian loop soup with intensity $1/2$, which yields Theorem 1.2.8.

We will prove Theorem 1.2.1, Theorem 1.2.3 and Theorem 1.2.8 under the assumptions that the domain D is bounded and that the mass $m : D \to \mathbb{R}_+$ is a bounded and continuous function. Conformal covariance then extends the results to the stated generality. However, our set of allowable masses is probably not the optimal one. This will be discussed in Section 4.1.5.

1.3 Interface scaling limit for the critical planar Ising model perturbed by a magnetic field

In this chapter, we look at another example of interface in a near-critical regime: the interface separating $+1$ and -1 spins in the planar Ising model at critical temperature with Dobrushin

boundary conditions perturbed by an external magnetic field. When there is no external field, this interface is known to converge to an SLE_3 curve [Che+14]. According to [MS10], in the presence of an external field, one should expect the limiting interface to be a massive version of SLE_3, in the sense that its law is obtained by "perturbing" the law of SLE_3. The main result of this chapter makes this precise. We will give an exact definition of the Ising model perturbed by an external magnetic field and of its interface in the core of the chapter but for now, let us simply state the result.

Theorem 1.3.1. *Let $h : \Omega \to \mathbb{R}_+$ be a bounded and Lipschitz function. Let $\Omega \subset \mathbb{C}$ be a bounded and simply connected domain with two marked boundary points $a, b \in \partial\Omega$. Let $(\Omega_\delta; a_\delta, b_\delta)_\delta$ be discrete approximations of $(\Omega; a, b)$ where for each $\delta > 0$, Ω_δ is a subgraph of $\delta\mathbb{Z}^2$. Consider the interface γ_δ of the critical Ising model in Ω_δ with Dobrushin boundary conditions and external magnetic field $H(x, \delta) = C_\sigma^{-1} h(x)\delta^{15/8}$, where $C_\sigma > 0$ is a constant that will be defined below and corresponds to the fact that we are working on the square lattice. Then, as $\delta \to 0$, γ_δ converges in law to a massive version of SLE_3 in Ω from a to b, with law denoted by $\mathbb{P}_h^{(\Omega,a,b)}$.*

The Radon-Nikodym derivative of $\mathbb{P}_h^{(\Omega,a,b)}$ with respect to the law $\mathbb{P}_{SLE_3}^{(\Omega,a,b)}$ of SLE_3 in Ω from a to b is given by, for any $t \geq 0$,

$$\left.\frac{d\mathbb{P}_h^{(\Omega,a,b)}}{d\mathbb{P}_{SLE_3}^{(\Omega,a,b)}}(\gamma)\right|_{\sigma(\gamma(s):0\leq s\leq t)} = \frac{1}{\mathcal{Z}_h(\Omega)}\sum_{k\geq 0}\frac{1}{k!}\int_{\Omega_t^k} h(z_1)\ldots h(z_k) f_t^{(k)}(z_1,\ldots,z_k)\prod_{j=1}^k dz_j \qquad (1.7)$$

where $\mathcal{Z}_h(\Omega)$ is a normalization constant. Above, we have set $\Omega_t = \Omega \setminus \gamma([0,t])$ and the functions $f_t^{(k)} : \Omega_t^k \to \mathbb{R}$ are explicit.

The precise assumptions on $(\Omega_\delta; a_\delta, b_\delta)_\delta$ and $(\Omega; a, b)$ together with the topologies in which weak convergence is obtained will be detailed in Section 5.1. Above, the constant C_σ and the functions $(f_t^{(k)})_{k\geq 1}$ are such that the k-point spin correlations of the critical Ising model in the discrete slit domain $\Omega_{\delta,t}$ with Dobrushin boundary conditions converge to $C_\sigma^k f_t^{(k)}$ as $\delta \to 0$, see [CHI21] and Section 5.1.3. Rescaling the external field by C_σ^{-1} guarantees that the limiting law $\mathbb{P}_h^{(\Omega,a,b)}$ does not depend on the fact that the sequence $(\gamma_\delta)_\delta$ is defined on the square lattice. In this way, one can expect $\mathbb{P}_h^{(\Omega,a,b)}$ to be universal. Note that rescaling the strength of the perturbation by a lattice-dependent constant is common in the near-critical regime, as for example in the case of the massive harmonic explorer.

Since $\mathbb{P}_h^{(\Omega,a,b)}$ is obtained as the scaling limit of an interface in a near-critical model, it is not conformally invariant, but conformally covariant, as shown below. Not surprisingly given the nature of the near-critical perturbation, the behavior of $\mathbb{P}_h^{(\Omega,a,b)}$ under conformal maps is the same as that of the continuum magnetization field of the critical Ising model constructed in [CGN15].

Proposition 1.3.2. *The law $\mathbb{P}_h^{(\Omega,a,b)}$ is conformally covariant in the following sense. If γ has law $\mathbb{P}_h^{(\Omega,a,b)}$ and $\psi : \Omega \to \tilde\Omega$ is a conformal map, then $\psi(\gamma)$ has law $\mathbb{P}_{\tilde h}^{(\tilde\Omega,\psi(a),\psi(b))}$ where $\tilde h$ is given by, for $w \in \tilde\Omega$,*

$$\tilde h(w) = |(\psi^{-1})'(w)|^{\frac{15}{8}} h(\psi^{-1}(w)).$$

Theorem 1.3.1 describes the scaling limit of the interface when the scaling of the external magnetic field is the near-critical one. A natural question that then arises is to give a description of the scaling limit of the interface when the scaling of the magnetic field is not the near-critical one. The first case to consider is when the scaling of the field goes to 0 faster than $\delta^{15/8}$ as $\delta \to 0$. In this case, we show that the scaling limit of the interface is the same as in the critical regime. The topologies in which this convergence holds are the same as in Theorem 1.3.1 and will be described in Section 5.1.

Proposition 1.3.3. *Let $g_1 : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $g_1(\delta) \to 0$ as $\delta \to 0$. Assume that the function h, the domain $(\Omega; a, b)$ and its approximations $(\Omega_\delta; a_\delta, b_\delta)_\delta$ are as in Theorem 1.3.1. Consider the interface γ_δ of the critical Ising model in Ω_δ with Dobrushin boundary conditions and external magnetic field $H(x, \delta) = h(x)\delta^{15/8}g_1(\delta)$. Then, as $\delta \to 0$, γ_δ converges in law to an SLE_3 in Ω from a to b.*

The second case to consider is that of an external magnetic field whose strength is larger than $\delta^{15/8}$. In this case, as in near-critical percolation away from the near-critical window [NW09b], we prove that the interface degenerates to a boundary arc.

Proposition 1.3.4. *Let $g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $g_2(\delta) \to \infty$ as $\delta \to 0$. Assume that $(\Omega; a, b)$ and its discrete approximations $(\Omega_\delta; a_\delta, b_\delta)_\delta$ are as in Theorem 1.3.1. Consider the interface γ_δ of the critical Ising model in Ω_δ with Dobrushin boundary conditions and external magnetic field $H(\delta) = h\delta^{15/8} g_2(\delta)$, for some $h > 0$. Then, as $\delta \to 0$, γ_δ converges in law to $\partial\Omega^-$. Here, $\partial\Omega^-$ is the boundary arc of $\partial\Omega$ where -1 boundary conditions are imposed.*

Observe that Proposition 1.3.4 in fact covers the case $h < 0$. Indeed, by spin flip (broken) symmetry, using Proposition 1.3.4 but with the opposite boundary conditions for the Ising model, one can easily see that when $h < 0$, γ_δ converges in law to $\partial\Omega^+$, where $\partial\Omega^+$ is the boundary arc of $\partial\Omega$ where $+1$ boundary conditions are imposed.

In view of Proposition 1.3.4, one may wonder if in the continuum, the measure $\mathbb{P}_h^{(\Omega,a,b)}$ of Theorem 1.3.1 also degenerates to a measure supported on $\partial\Omega^-$ as $h \to \infty$. In this direction, we prove that the $\mathbb{P}_h^{(\Omega,a,b)}$-probability that the curve exits an arbitrarily small neighborhood of $\partial\Omega^-$ vanishes as $h \to \infty$.

Proposition 1.3.5. *Let $\Omega \subset \mathbb{C}$ be a bounded and simply connected domain with two marked boundary points $a, b \in \partial\Omega$. Let $\eta > 0$ and define $\Omega^-(\eta) := \{z \in \Omega : \text{dist}(z, \partial\Omega^-) \leq \eta\}$. Then*

$$\lim_{h \to \infty} \mathbb{P}_h^{(\Omega,a,b)}[\gamma \cap (\Omega \setminus \Omega^-(\eta)) \neq \emptyset] = 0.$$

For the same reasons as those explained below Proposition 1.3.4 and using the convergence result of Theorem 1.3.1, it can be deduced from Proposition 1.3.5 that the $\mathbb{P}_h^{(\Omega,a,b)}$-probability that the curve exits an arbitrarily small neighborhood of $\partial\Omega^+$ vanishes as $h \to -\infty$.

Let us conclude this section of the introduction by a few remarks. Contrary to most results establishing convergence of a discrete interface to an SLE_κ curve, the proof of Theorem 1.3.1 does not rely on a martingale characterization of $\mathbb{P}_h^{(\Omega,a,b)}$. As such, it leaves open the question of how to characterize the law of $\mathbb{P}_h^{(\Omega,a,b)}$: the characterization of the limiting interface is often an issue in near-critical regimes. We note that [MS10] gives a conjecture on which observable could provide a characterizing martingale for $\mathbb{P}_h^{(\Omega,a,b)}$ but even if this was established, it would probably not be very useful to prove scaling-limit type results. Indeed, the discrete version of this observable would satisfy a discrete boundary value problem for which convergence of the discrete solution to its continuum analogue is not known.

Let us also mention that Theorem 1.3.1 and its proof suggest that it should be possible to construct a coupling between the continuum magnetization field of the critical Ising model with Dobrushin boundary conditions and an SLE_3 curve, in a spirit similar to the GFF-SLE_4 coupling [SS13; Dub09]. As in the case of the massive GFF studied in the previous chapter, this coupling could be extended to the near-critical regime. These questions will be investigated in future work. Such a coupling may turn out to be useful to compare the arm exponents in the near-critical regime (if they exist) with those in the critical regime, which is a question raised in [MS10].

Besides, we note that from Theorem 1.3.1, given the topologies in which convergence is obtained, we know that the driving function of the curve γ under $\mathbb{P}_h^{(\Omega,a,b)}$ exists, is continuous, and its law is absolutely continuous with respect to that of the driving function of γ under $\mathbb{P}_{\text{SLE}_3}^{(\Omega,a,b)}$. If continuity in t of the random variable on the right-hand side of (1.7) in Theorem 1.3.1 was established, one could deduce an explicit expression for the driving function of γ under $\mathbb{P}_h^{(\Omega,a,b)}$. This driving function would have the form of $\sqrt{3}$ times a Brownian motion plus a drift term, where we recall that $\sqrt{3}$ times a Brownian motion is the driving function of γ under $\mathbb{P}_{\text{SLE}_3}^{(\Omega,a,b)}$.

Finally, perturbing the critical Ising model by an external field is not the only way to move away from the critical regime. Another way is to perturb the temperature and look at the Ising model with inverse temperature $\beta = \beta_c + m\delta$, where β_c is the critical inverse temperature and $m \in \mathbb{R}$ [Par22b; CIM23]. Proving convergence of the interface in this setting is more challenging: this stems from the fact that the energy field of the critical Ising model cannot be constructed as a random field in the continuum, whereas the magnetization field can be [CGN15; FM17].

Let us now briefly discuss the proofs of Theorem 1.3.1, Proposition 1.3.3, Proposition 1.3.4 and Proposition 1.3.5.

Outline of the proofs

The proofs of Theorem 1.3.1 and Proposition 1.3.3, given respectively in Section 5.2 and Section 5.3.1, follow the same strategy. Let us denote by $\mathbb{P}_{\delta,h}^\pm$, respectively $\mathbb{P}_\delta^\pm$, the law of γ_δ when $h \neq 0$,

respectively $h \equiv 0$. The first step of the proofs of Theorem 1.3.1 and Proposition 1.3.3 is to show tightness of $(\gamma_\delta)_\delta$ under $(\mathbb{P}^\pm_{\delta,h})_\delta$. For this, we observe that for each $\delta > 0$, the law of γ_δ under $\mathbb{P}^\pm_{\delta,h}$ has a Radon-Nikodym derivative $F_\delta(\gamma_\delta)$ with respect to the law of γ_δ under $\mathbb{P}^\pm_\delta$. Moreover, $(\gamma_\delta)_\delta$ under $(\mathbb{P}^\pm_\delta)_\delta$ is tight in an appropriate topology by [Che+14]. In view of this, to establish tightness of $(\gamma_\delta)_\delta$ under $(\mathbb{P}^\pm_{\delta,h})_\delta$, it therefore suffices to prove that for any $p \in [1, \infty)$, $\sup_{\delta>0} \mathbb{E}^\pm_\delta[|F_\delta(\gamma_\delta)|^p] < \infty$, where $\mathbb{E}^\pm_\delta$ denotes the expectation with respect to $\mathbb{P}^\pm_\delta$. This is what we achieve in the first part of the proofs of Theorem 1.3.1 and Proposition 1.3.3.

The second step of the proofs of Theorem 1.3.1 and Proposition 1.3.3 is to prove a characterization of the limiting laws obtained along weakly convergent subsequences of $(\gamma_\delta)_\delta$ under $(\mathbb{P}^\pm_{\delta,h})_\delta$. Let $(\gamma_n)_n$ be such a weakly convergent subsequence under $(\mathbb{P}^\pm_{n,h})_n$. To characterize its limiting law, we fist use the fact that $(\gamma_n)_n$ under $(\mathbb{P}^\pm_n)_n$ converges in law to $\mathbb{P}^{(\Omega,a,b)}_{\mathrm{SLE}_3}$ [Che+14]. By Skorokhod representation theorem, we can therefore define on a common probability space $(S, \mathcal{F}, \mathbb{P})$ a sequence $(\gamma_n)_n$ and a random curve γ such that for each n, γ_n has law $\mathbb{P}^\pm_n$, γ has law $\mathbb{P}^{(\Omega,a,b)}_{\mathrm{SLE}_3}$ and $\mathbb{P}$-almost surely, $\gamma_n \to \gamma$. The Radon-Nikodym derivatives $(F_n(\gamma_n))_n$ of $(\mathbb{P}^\pm_{n,h})_n$ with respect to $(\mathbb{P}^\pm_n)_n$ are then all defined on $(S, \mathcal{F}, \mathbb{P})$. In the setting of Proposition 1.3.3, when the scaling of the external field is $\delta_n^{15/8} g_1(\delta_n)$, we characterize the limiting law of weakly convergent subsequences under $(\mathbb{P}^\pm_{n,h})_n$ by showing that the discrete Radon-Nikodym derivatives $(F_n(\gamma_n))_n$ converge to 1 in $L^2(\mathbb{P})$. This yields that the limiting law is $\mathbb{P}^{(\Omega,a,b)}_{\mathrm{SLE}_3}$, as claimed. In the setting of Theorem 1.3.1, when the scaling of the external field is $\delta_n^{15/8}$, the characterization of the limiting laws of weakly convergent subsequences $(\mathbb{P}^\pm_{n,h})_n$ is established by proving that $(F_n(\gamma_n))_n$ has a subsequence that converges in $L^1(\mathbb{P})$ to an explicit random variable. To show this convergence, a crucial result is the convergence of the k-point spin correlations of the critical Ising model obtained in [CHI21]. This ultimately allows us to express the limit of $(F_n(\gamma_n))_n$ along a subsequence in terms of explicit functions of the domain slit by the limiting curve. This is similar in spirit to the construction of the continuum planar Ising magnetization field [CGN15]. However, in our setting, more work is needed as we must also take into account the contribution of points that arbitrarily close to the boundary. Moreover, since the underlying domain is typically Ω slit by a portion of a rough curve, we cannot assume smoothness of the boundary. Another important result used in the proof of the convergence of $F_n(\gamma_n)$ is the computation of the two-arm exponent of the critical Ising model [Wu18a]. This is instrumental to show that a quantity that can be seen as the magnetization of the discrete interface vanishes in the limit $n \to \infty$. This also provides a useful bound on the growth of the size of the interface, which is necessary to control what happens near the boundary of the domain.

The proof of Proposition 1.3.4, given in Section 5.3.2, is more probabilistic in nature. The heuristic behind this result is that spins tend to align with the magnetic field and that the scaling $\delta^{\frac{15}{8}} g_2(\delta)$ makes this phenomena dominate in the limit $\delta \to 0$. To implement this idea, we rely on the Edwards-Sokal coupling for the Ising model with an external magnetic field, which was investigated in [CV16; CJN20b; CJN20a]. For this, the first step is to change the Dobrushin boundary conditions into $+1$ boundary conditions using spatial-mixing-type arguments which build on what happens at criticality [CDH16]. This change of boundary conditions then enables us to use the Edwards-Sokal coupling with an external field to show that, for arbitrary $\eta > 0$, the probability that there is a path of $+1$ spins staying within distance η from $\partial\Omega^-_\delta$ and joining the η-neighborhoods of a_δ and b_δ on $\partial\Omega^+_\delta$ converges to 1 as $\delta \to 0$. To show this, an important result is the joint convergence of the lattice FK-Ising cluster boundaries to $\mathrm{CLE}_{16/3}$ and of the discrete FK-Ising cluster area measures to $\mathrm{CME}_{16/3}$ [CCK19]. Having such a path of $+1$ spins rules out that a path with only -1 spins on one of its sides reaches distance η from $\partial\Omega^-_\delta$ as $\delta \to 0$, which proves Proposition 1.3.4.

Proposition 1.3.5 is shown in Section 5.3.3. The proof relies on both Theorem 1.3.1 and Proposition 1.3.4. In particular, for the argument to work, we use results obtained in the discrete setting and, interestingly, we do not know how to obtain Proposition 1.3.5 directly in the continuum.

Chapter 2

Background on Loewner chains and convergence of discrete curves

2.1 Loewner chains and SLE

Set $\mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$ and let $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ be a non-self-crossing curve targeting ∞ and such that $\gamma(0) = 0$. For $t \geq 0$, let K_t be the hull generated by $\gamma([0, t])$, that is $\mathbb{H} \setminus K_t$ is the unbounded connected component of $\mathbb{H} \setminus \gamma([0, t])$. In the case where $\gamma([0, t])$ is non-self-touching, K_t is simply given by $\gamma([0, t])$. For each $t \geq 0$, it is easy to see that there exists a unique conformal $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ satisfying the normalization $g_t(\infty) = \infty$ and such that $\lim_{z \to \infty}(g_t(z) - z) = 0$. It can then be proved that g_t satisfies the asymptotic

$$g_t(z) = z + \frac{a_1(t)}{z} + O(|z|^{-2}) \quad \text{as } |z| \to \infty.$$

The coefficient $a_1(t)$ is equal to $\mathrm{hcap}(K_t)$, the half-plane capacity of K_t, which, roughly speaking, is a measure of the size of K_t seen from ∞. Moreover, one can show that $a_1(0) = 0$ and that $t \mapsto a_1(t)$ is continuous and strictly increasing. Therefore, the curve γ can be reparametrized in such a way that at each time t, $a_1(t) = 2t$. γ is then said to be parameterized by half-plane capacity.

In this time-reparametrization and with the normalization of g_t just described, it is known that there exists a unique real-valued function $t \mapsto W_t$, called the driving function, such that the following equation, called the Loewner equation, is satisfied:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z, \quad \text{for all } z \in \mathbb{H} \setminus K_t. \tag{2.1}$$

Indeed, it can be shown that g_t extends continuously to $\gamma(t)$ and setting $W_t = g_t(\gamma(t))$ yields the above equation, see e.g. [Law05, Chapter 4] and [Kem17, Chapter 4].

Conversely, given a continuous and real-valued function $t \mapsto W_t$, one can construct a locally growing family of hulls $(K_t)_t$ by solving the equation (2.1). Under additional assumptions on the function $t \mapsto W_t$, the family of hulls obtained using (2.1) is generated by a curve, in the sense explained above.

Schramm-Loewner evolutions, or SLE for short, are random Loewner chains introduced by Schramm [Sch00]. For $\kappa \geq 0$, SLE_κ is the Loewner chain obtained by considering the Loewner equation (2.1) with driving function $W_t = \sqrt{\kappa} B_t$, where $(B_t, t \geq 0)$ is a standard one-dimensional Brownian motion. As such, SLE_κ is defined in $\mathbb{H}$ but, thanks to the conformal invariance of the Loewner equation, SLE_κ can be defined in any simply connected domain $\Omega \subset \mathbb{C}$ with two marked boundary points $a, b \in \partial\Omega$ by considering a conformal map $\phi : \Omega \to \mathbb{H}$ with $\phi(a) = 0$ and $\phi(b) = \infty$ and taking the image of SLE_κ in $\mathbb{H}$ by ϕ^{-1}. In particular, SLE_κ is conformally invariant and it turns out that this conformal invariance property together with a certain domain Markov property characterize the family $(\mathrm{SLE}_\kappa, \kappa \geq 0)$.

2.2 Topology and convergence on the space of curve

Following [KS17], the space of curves that we will consider is a subspace of the space of continuous mappings from $[0, 1]$ to $\mathbb{C}$ modulo reparametrization. More precisely, let

$$\mathcal{C}' := \left\{ f \in \mathcal{C}([0,1], \mathbb{C}) : \begin{array}{l} \text{either } f \text{ is not constant on any subinterval of } [0,1] \\ \text{or } f \text{ is constant on } [0,1] \end{array} \right\}$$

and let $f_1, f_2 \in \mathcal{C}'$ be equivalent if there exists an increasing homeomorphism $\psi : [0,1] \to [0,1]$ with $f_2 = f_1 \circ \psi$. We denote by $[f]$ the equivalence class of f under this equivalence relation and set

$$X(\mathbb{C}) := \{[f] : f \in \mathcal{C}'\}.$$

$X(\mathbb{C})$ is called the space of curves. We turn $X(\mathbb{C})$ into a metric space by equipping it with the metric

$$d_X(f, g) := \inf\{\|f_0 - g_0\|_\infty : f_0 \in [f], g_0 \in [g]\}.$$

$(X(\mathbb{C}), d_X)$ is a separable and complete metric space, but it is not compact. Given $\Omega \subsetneq \mathbb{C}$ with $\partial\Omega \neq \emptyset$ and two marked boundary points a and b, we define the space of simple curves from a to b in Ω as

$$X_{\text{simple}}(\Omega, a, b) := \{[f] : f \in \mathcal{C}', f((0,1)) \subset \Omega, f(0) = a, f(1) = b, f \text{ injective}\}.$$

We then let $X_0(\Omega, a, b)$ be the closure of $X_{\text{simple}}(\Omega, a, b)$ in $X(\mathbb{C})$ with respect to the metric d_X. Curves in $X_0(\Omega, a, b)$ run from a to b, may touch $\partial\Omega$ elsewhere than at their endpoints, may touch themselves and have multiple points but they can have no transversal self-crossings. Notice that if $(\mathbb{P}_n)_n$ is a sequence of probability measures supported on $X_{\text{simple}}(\Omega, a, b)$ that converges weakly to a probability measure $\mathbb{P}^*$, then a priori $\mathbb{P}^*$ is supported on $X_0(\Omega, a, b)$.

Assume that we have a family of random curves $(\gamma_\delta)_\delta$ distributed according to some family $(\mathbb{P}_\delta)_\delta$ such that for each $\delta > 0$, γ_δ is an element of $X_0(\hat{\Omega}_\delta; a_\delta, b_\delta)$. Here, for each $\delta > 0$, $(\hat{\Omega}_\delta; a_\delta, b_\delta)_\delta$ is a suitable polygonal domain associated to $(\Omega_\delta; a_\delta, b_\delta)$ that approximates $(\Omega; a, b)$, see Section 5.1.1 for the precise definition and assumptions. The curves $(\gamma_\delta)_\delta$ can be made to be supported on the same space $X_0(\Omega; a, b)$ by uniformization. More precisely, let $\phi : \Omega \to \mathbb{H}$ be a conformal map such that $\phi(a) = 0$ and $\phi(b) = \infty$ and similarly, for $\delta > 0$, let $\phi_\delta : \hat{\Omega}_\delta \to \mathbb{H}$ be a conformal map such that $\phi_\delta(a_\delta) = 0$ and $\phi_\delta(b_\delta) = \infty$. Setting $\gamma_\delta^{\mathbb{H}} := \phi_\delta(\gamma_\delta)$, the family $(\gamma_\delta^{\mathbb{H}})_\delta$ is supported on $X_0(\mathbb{H}; 0, \infty)$. Moreover, for each $\delta > 0$, when parametrized by half-plane capacity, $\gamma_\delta^{\mathbb{H}}$ has an associated driving function W_δ. In this setting, there are two different topological spaces in which we may wish to show tightness of $(\gamma_\delta)_\delta$:

(T.1) the space of curves $X_0(\mathbb{H}, 0, \infty)$ equipped with the metric d_X;

(T.2) the metrizable space of continuous functions from $[0, \infty)$ to $\mathbb{H}$ with the topology of uniform convergence on compact subsets of $[0, \infty)$.

Tightness could also be established for the driving functions $(W_\delta)_\delta$, in which case the topological space to consider is

(T.3) the metrizable space of continuous functions from $[0, \infty)$ to $\mathbb{R}$ with the topology of uniform convergence on compact subsets of $[0, \infty)$.

It turns out that, by [KS17, Corollary 1.7], weak convergence in one of the topologies (T.1)–(T.3) implies weak convergence in the other two and that the limits agree, in the sense that the limiting random curve is driven by the limiting driving function.

Another natural topological space in which to establish tightness of $(\gamma_\delta)_\delta$ is

(T.4) the space of curves $X(\mathbb{C})$ equipped with the metric d_X.

Under the assumption that $(\hat{\Omega}_\delta; a_\delta, b_\delta)_\delta$ converges in the Carathéodory sense to $(\Omega; a, b)$ and that $(a_\delta)_\delta$ and $(b_\delta)_\delta$ are close approximations of the degenerate prime ends a and b, [Kar23, Theorem 4.2] guarantees that weak convergence of $(\gamma_\delta)_\delta$ in the topology (T.4) implies weak convergence in the other three topologies (T.1)–(T.3), see the discussion below [Kar23, Theorem 4.2]. Moreover, if γ denotes the limit in the topology (T.4) and $\gamma^{\mathbb{H}}$ denotes the limit in the topology (T.1), then γ has the same law as $\phi^{-1}(\gamma^{\mathbb{H}})$. We note that [Kar23, Theorem 4.2] also guarantees that γ is supported on $\overline{\Omega}$.

Chapter 3

Massive SLE$_4$ and the scaling limit of the massive harmonic explorer

3.1 Setup

3.1.1 Assumptions on the domain

We consider an open, bounded and simply connected subset Ω of the complex plane $\mathbb{C}$ such that $0 \in \Omega$. We fix two marked boundary points $a, b \in \partial\Omega$ and we assume that both a and b are degenerate prime ends of Ω. That is, if $f : \mathbb{D} \to \Omega$ is a conformal map from the unit disc $\mathbb{D}$ to Ω and $\hat{f}$ is a bijective mapping from $\partial\mathbb{D}$ to $\partial\Omega$ induced by f, see [Pom92, Theorem 2.15], then f can be extended continuously at $z_a, z_b \in \partial\mathbb{D}$ by taking radial limits, where $z_a \in \partial\mathbb{D}$, respectively $z_b \in \partial\mathbb{D}$, is the preimage of a, respectively b, by $\hat{f}$. Here, the radial limit of f at $\zeta \in \partial\mathbb{D}$ is defined as $\lim_{\epsilon \to 0} f \circ P_\epsilon(\zeta)$, whenever this limit exists and where for $\epsilon > 0$ and $z \in \mathbb{C}$, $P_\epsilon(z) = (z/|z|) \min\{1 - \epsilon, |z|\}$. For a more detailed discussion on degenerate prime ends, the reader can consult [Pom92, Section 2.5]. These assumptions on Ω and the boundary points a and b will in particular allow us to use [Kar23, Theorem 4.2].

We assume that $(\Omega_\delta)_\delta$ is a sequence of graphs approximating Ω in a sense that we will now explain. For each $\delta > 0$, Ω_δ is a simply connected subgraph of the triangular lattice $\delta\mathbb{T}$, so that every edge of Ω_δ has length δ. We denote by $V(\Omega_\delta)$ the set of vertices of Ω_δ and define the boundary $\partial\Omega_\delta$ of Ω_δ as

$$\partial\Omega_\delta := \{w \in \delta\mathbb{T} \setminus V(\Omega_\delta) : \text{there exists } v \in V(\Omega_\delta) \text{ such that } v \sim w\}$$

where $v \sim w$ means that there is an edge of $\delta\mathbb{T}$ connecting v and w. With this definition of $\partial\Omega_\delta$, it is legitimate to set $\text{Int}(\Omega_\delta) := V(\Omega_\delta)$.

We associate to each Ω_δ an open and simply connected polygonal domain $\hat{\Omega}_\delta \subset \mathbb{C}$ by taking the union of all open hexagons with side length δ centered at vertices in $V(\Omega_\delta)$. We assume that there exists $\bar{R} > 0$ such that for any $\delta > 0$, $\hat{\Omega}_\delta \subset B(0, \bar{R})$ and $\Omega \subset B(0, \bar{R})$. The marked boundary points a and b of $\partial\Omega$ are then approximated by sequences $(a_\delta)_\delta$ and $(b_\delta)_\delta$ where, for each $\delta > 0$, a_δ and b_δ belong to $\partial\hat{\Omega}_\delta$. We assume that for all $\delta > 0$, 0 belongs to $\hat{\Omega}_\delta$, which is necessary to apply [Kar23, Theorem 4.2]. More importantly, we assume that the approximations $(\hat{\Omega}_\delta; a_\delta, b_\delta)_\delta$ converge to $(\Omega; a, b)$ in the Carathéodory sense. That is,

- each inner point $z \in \Omega$ belongs to $\hat{\Omega}_\delta$ for δ small enough;

- for every boundary point $\zeta \in \partial\Omega$, there exists a sequence $(\zeta_\delta)_\delta$ such that $\zeta_\delta \to \zeta$ as $\delta \to 0$, where, for each $\delta > 0$, $\zeta_\delta \in \partial\hat{\Omega}_\delta$.

This can be rephrased in terms of conformal maps. Let $\psi : \Omega \to \mathbb{D}$ be a conformal map such that $\psi(a) = 1$ and $\psi(0) = 0$. Similarly, for each $\delta > 0$, let $\psi_\delta : \hat{\Omega}_\delta \to \mathbb{D}$ be a conformal map such that $\psi_\delta(a_\delta) = 1$ and $\psi_\delta(0) = 0$. Then, by [Pom92, Theorem 1.8], the Carathéodory convergence of $(\hat{\Omega}_\delta; a_\delta, b_\delta)_\delta$ to $(\Omega; a, b)$ is equivalent to

$$\psi_\delta \to \psi \quad \text{uniformly on compact subsets of } \Omega \text{ and}$$

$$\psi_\delta^{-1} \to \psi^{-1} \quad \text{uniformly on compact subsets of } \mathbb{D}.$$

Furthermore, we assume that a_δ, respectively b_δ, is a close approximation of a, respectively b, as defined by Karrila in [Kar23, Section 4.3]. Let us recall what this means. To lighten the notations, we identify the prime ends a_δ and a with their corresponding radial limit points. For $r > 0$, let S_r be the arc of $\partial B(a, r) \cap \Omega$ disconnecting in Ω the prime end a from 0 and that is closest to a. In other words, S_r is the last arc from the (possibly countable) collection $\partial B(a, r) \cap \Omega$ of arcs that a path running from 0 to a inside Ω must cross. Such an arc exists by [Kar23, Lemma A.1] and approximation by radial limits. a_δ is then said to be a close approximation of a if

- $a_\delta \to a$ as $\delta \to 0$; and

- for each r small enough and for all sufficiently (depending on r) small δ, the boundary point a_δ of $\hat{\Omega}_\delta$ is connected to the midpoint of S_r inside $\hat{\Omega}_\delta \cap B(a, r)$.

3.1.2 Definition of the discrete model

The massive harmonic explorer is a massive version of the harmonic explorer studied by Schramm and Sheffield in [SS05]. Let $m : \Omega \to \mathbb{R}_+$ be a continuous function bounded above by some constant $\overline{m} > 0$. For each $\delta > 0$, we assign a weight to the edges of the graph Ω_δ as follows: if z and w belong to $V(\Omega_\delta)$ and are connected by an edge in Ω_δ, then the oriented edge (zw) from z to w has weight $1 - m_d^2(z)\delta^2$, where we have set, for $v \in V(\Omega_\delta)$,

$$m_d^2(v) := \frac{cm^2(v)}{6\tan(\theta)}. \tag{3.1}$$

Above, the constant $c > 0$ is such that the faces of the hexagonal lattice dual to $\delta\mathbb{T}$ have area $A_\delta = c\delta^2$, that is $c = 9\sqrt{3}/8$. θ is defined as [CS11, Figure 1.B], that is $\theta = \pi/6$ for the triangular lattice. If $v \notin \Omega \cap \hat{\Omega}_\delta$, we set $m_d^2(v) = 0$. Notice that the weight of an edge depends on its orientation: the oriented edge (wz) in Ω_δ has weight $1 - m_d^2(w)\delta^2$. Observe also that for any $v \in V(\Omega_\delta)$,

$$m_d^2(v) \leq \frac{c\overline{m}^2}{6\tan(\theta)} =: \overline{m}_d^2.$$

For each $\delta > 0$ such that $\delta < \overline{m}_d^{-1}$, these edge weights naturally give rise to a massive random walk X^m on Ω_δ. More precisely, if the walk X^m is at a vertex $v \in \Omega_\delta$, then at the next step, it jumps to one of its neighbors with probability $1 - m_d^2(v)\delta^2$ or is killed with probability $m_d^2(v)\delta^2$. The jumps to its neighbors are uniform, that is if w is connected to v by an edge, then the walk has probability $(1 - m_d^2(v)\delta^2)/6$ to jump to w at its next step. We denote by $\tau^\star$ the killing time of the walk and by $\tau_{\partial\Omega_\delta}$ the hitting time of $\partial\Omega_\delta$ by the walk, when it started in $\text{Int}(\Omega_\delta)$. Accordingly, we denote by $\mathbb{P}_w^{(m)}(\tau^\star \leq \tau_{\partial\Omega_\delta})$ the probability that the walk is killed before reaching the boundary $\partial\Omega_\delta$ when it started at $w \in \text{Int}(\Omega_\delta)$. Such a massive random walk is intimately connected to discrete massive harmonic functions with mass m, in the same way as random walk is connected to discrete harmonic functions. Discrete massive harmonic functions are defined as follows: a function $h : \Omega_\delta \to \mathbb{R}$ is said to be discrete massive harmonic with mass m if for any $v \in \text{Int}(\Omega_\delta)$,

$$h(v) = \frac{1 - m_d^2(v)\delta^2}{6} \sum_{v \sim w} h(w). \tag{3.2}$$

As in the non-massive case, one can also define the massive harmonic measure $H^{(m)}(\cdot, E_1)$ of a subset E_1 of $\partial\Omega_\delta$. This is the unique discrete massive harmonic function with mass m in Ω_δ and boundary value 1 on E_1 and 0 on $\partial\Omega_\delta \setminus E_1$. For $v \in \text{Int}(\Omega_\delta)$, $H^{(m)}(v, E_1)$ can be interpreted as the probability that a massive random walk with mass m started at v is not killed before leaving Ω_δ and exits Ω_δ through E_1. Observe that $H^{(m)}(v, \partial\Omega_\delta) = 1 - \mathbb{P}_v^{(m)}(\tau^\star \leq \tau_{\partial\Omega_\delta})$.

We can now define the massive harmonic explorer. For $0 < \delta < \overline{m}_d^{-1}$, the massive harmonic explorer with mass m is a random path γ_δ on the dual of Ω_δ defined as follows. On $\partial\Omega$, we assign the sign $+$ to the clockwise oriented boundary arc (ab), denoted $\partial\Omega^+$, and the sign $-$ to the counter-clockwise oriented boundary arc (ab), denoted $\partial\Omega^-$. Correspondingly, for each $\delta > 0$, the boundary $\partial\hat{\Omega}_\delta$ of $\hat{\Omega}_\delta$ is split into two parts: the clockwise oriented boundary arc $(a_\delta b_\delta)$ has sign $+$ while the counter-clockwise oriented boundary arc $(a_\delta b_\delta)$ has sign $-$. This naturally defines a partition of $\partial\Omega_\delta$ into two sets of vertices: the vertices of $\partial\Omega_\delta$ that belong to the clockwise oriented boundary arc $(a_\delta b_\delta)$ have sign $+$ while the vertices that belong to counter-clockwise oriented boundary arc

$(a_\delta b_\delta)$ have sign $-$. The vertices a_δ and b_δ, in case they belong to $\partial\Omega_\delta$, are assigned an arbitrary sign. We denote the set of vertices with sign $+$, respectively $-$, by $\partial\Omega_\delta^+$, respectively $\partial\Omega_\delta^-$.

Using this partition of $\partial\Omega_\delta$, we then let h_δ^m be the unique discrete massive harmonic function with mass m in Ω_δ and boundary values $-1/2$ on $\partial\Omega_\delta^-$ and $1/2$ on $\partial\Omega_\delta^+$. Let m_0 be the middle point of the unique edge e_0 connecting two vertices of opposite sign to which a_δ belongs; m_0 is the starting point of the massive harmonic explorer γ_δ. The edge e_0 bounds a triangle f_0 of Ω_δ and we denote by v_1 the vertex of $\Omega_\delta \cup \partial\Omega_\delta$ that is not an endpoint of e_0. The explorer then turns right with probability $h_\delta^m(v_1) + 1/2$, that is, with probability $h_\delta^m(v_1) + 1/2$, γ_δ traces the broken line from m_0 to the center of f_0 and then from the center of f_0 to the middle point m_1 of the right edge of f_0. In that case, v_1 becomes a boundary vertex with sign $+$ and we set $\Omega_{\delta,1} := \Omega_\delta \setminus \{v_1\}$, $\partial\Omega_{\delta,1}^+ := \partial\Omega_\delta^+ \cup \{v_1\}$ and $\partial\Omega_{\delta,1}^- := \partial\Omega_\delta^-$. With complementary probability, the explorer turns left, that is γ_δ traces the broken line from m_0 to the center of f_0 and then from the center of f_0 to the middle point m_1 of the left edge of f_0. In that case, v_1 becomes a boundary vertex with sign $-$ and we set $\Omega_{\delta,1} := \Omega_\delta \setminus \{v_1\}$, $\partial\Omega_{\delta,1}^- := \partial\Omega_\delta^- \cup \{v_1\}$ and $\partial\Omega_{\delta,1}^+ := \partial\Omega_\delta^+$. In both cases, on $\Omega_{\delta,1}$, we let $h_{\delta,1}^m$ be the unique discrete massive harmonic function with mass m and boundary value $-1/2$ on $\partial\Omega_{\delta,1}^-$ and $1/2$ on $\partial\Omega_{\delta,1}^+$. For the second step, we repeat the same procedure but with respect to the vertex v_2, defined analogously to v_1, and using the function $h_{\delta,1}^m$. We continue until the path γ_δ hits the edge connecting two vertices of opposite sign to which b_δ belongs, which almost surely happens in finite time. We update the graphs and their boundary at each step to obtain random sequences $(\Omega_{\delta,n})_n$, $(\partial\Omega_{\delta,n}^+)_n$ and $(\partial\Omega_{\delta,n}^-)_n$ and the corresponding sequence of discrete massive harmonic functions $(h_{\delta,n}^m)_n$. See again Figure 1.1 for a dual perspective on the hexagonal lattice.

This discrete model is well-defined, in the sense that each step of the massive harmonic explorer is chosen according to a probability measure. Indeed, notice that almost surely, for any $n \in \mathbb{N}$ and any $w \in \text{Int}(\Omega_{\delta,n}) \cup \partial\Omega_{\delta,n}$,

$$h_{\delta,n-1}^m(w) = \frac{1}{2}\left(H_{\delta,n-1}^{(m)}(w) - \tilde{H}_{\delta,n-1}^{(m)}(w) \right) \tag{3.3}$$

where $H_{\delta,n-1}^{(m)}(w)$, respectively $\tilde{H}_{\delta,n-1}^{(m)}(w)$, is the discrete massive harmonic measure with mass m of $\partial\Omega_{\delta,n-1}^+$, respectively $\partial\Omega_{\delta,n-1}^-$, seen from w. This equality is a consequence of the uniqueness of discrete massive harmonic functions with prescribed boundary conditions: the functions on each side of the equality (3.3) are discrete massive harmonic with mass m and boundary values $1/2$ on $\partial\Omega_{\delta,n-1}^+$ and $-1/2$ on $\partial\Omega_{\delta,n-1}^-$. This shows that almost surely, for any $n \in \mathbb{N}$,

$$h_{\delta,n-1}^m(v_n) + \frac{1}{2} = \frac{1}{2}\left(H_{\delta,n-1}^{(m)}(v_n) - \tilde{H}_{\delta,n-1}^{(m)}(v_n) \right) + \frac{1}{2}$$

and the right-hand side almost surely takes values in $[0,1]$ as both $H_{\delta,n-1}^{(m)}(v_n)$ and $\tilde{H}_{\delta,n-1}^{(m)}(v_n)$ almost surely take values in $[0,1]$.

Observe also that the equality (3.3) implies that this model has the following symmetry: almost surely, for all $n \in \mathbb{N}$ and all $v \in \text{Int}(\Omega_\delta) \cup \partial\Omega_\delta$,

$$1 - (h_{\delta,n}^m(v) + 1/2) = \tilde{h}_{\delta,n}^m(v) + 1/2 \tag{3.4}$$

where $\tilde{h}_{\delta,n}^m$ is the unique discrete massive harmonic function with mass m in $\Omega_{\delta,n}$ and boundary value $-1/2$ on $\partial\Omega_{\delta,n}^+$ and $1/2$ on $\partial\Omega_{\delta,n}^-$.

Throughout the text, for $0 < \delta < \overline{m}_d^{-1}$, we denote by $\mathbb{P}_\delta^{(\Omega,a,b,m)}$ the probability measure on paths on the dual of Ω_δ induced by the massive harmonic explorer from a to b in Ω with mass m. $\mathbb{E}_\delta^{(\Omega,a,b,m)}$ denotes the corresponding expectation. For ease of notations, when $\delta > 0$ and it is clear from the context that the path γ_δ is distributed according to $\mathbb{P}_\delta^{(\Omega,a,b,m)}$, we simply write γ instead of γ_δ. Notice also that the family of probability measures $(\mathbb{P}_\delta^{(\Omega,a,b,m)})_\delta$ implicitly depends on the sequence $(\hat{\Omega}_\delta, a_\delta, b_\delta)_\delta$ approximating (Ω, a, b).

With these notations, observe that for $0 < \delta < \overline{m}_d^{-1}$, $\mathbb{P}_\delta^{(\Omega,a,b,m)}$ satisfies the following domain Markov property. Let τ be a stopping time for the filtration generated by γ. Then, almost surely,

$$\mathbb{P}_\delta^{(\Omega,a,b,m)}(\cdot \,|\, \gamma([0,\tau])) = \mathbb{P}_\delta^{(\Omega_\tau, \gamma(\tau), b, m)}(\cdot) \tag{3.5}$$

where, with a slight abuse of notation, $\mathbb{P}_\delta^{(\Omega_\tau, \gamma(\tau), b, m)}$ denotes the probability measure induced on paths by the massive harmonic explorer with mass m from $\gamma(\tau)$ to b_δ in $\Omega_\tau = \Omega_{\delta,\tau}$ and exploration starting with respect to $h_{\delta,\tau}^m(v_{\tau+1}) + 1/2$. That is, conditionally on $\gamma([0,\tau])$, at step $\tau+1$, the explorer turns right with probability $h_{\delta,\tau}^m(v_{\tau+1}) + 1/2$ and, with complementary probability, it turns left and the rest of the path is traced following the same procedure as the one described above.

Finally, as in the case of the non-massive harmonic explorer, for each $v \in \mathrm{Int}(\Omega_\delta) \cup \partial\Omega_\delta$, we notice that for each δ, under $\mathbb{P}_\delta^{(\Omega,a,b,m)}$, the process $(h_{\delta,n}^m(v), n \geq 0)$ is a martingale with respect to the filtration $(\mathcal{F}_{\delta,n})_n$ generated γ_δ. That is, for $n \in \mathbb{N}$, $\mathcal{F}_{\delta,n} = \sigma(\gamma_\delta([0,n]))$. We record this fact in the proposition below.

Proposition 3.1.1. *Let $\delta > 0$. In the above setting and using the same notations, for each $v \in \mathrm{Int}(\Omega_\delta) \cup \partial\Omega_\delta$, the process $(h_{\delta,n}^m(v), n \geq 0)$ is a martingale with respect to the filtration $(\mathcal{F}_{\delta,n})_n$.*

Proof. Let $\delta > 0$ and fix $n \in \mathbb{N}$. We first consider the case $v = v_{n+1}$. Then, almost surely,

$$
\begin{aligned}
\mathbb{E}_\delta^{(\Omega,a,b,m)}[h_{\delta,n+1}^m(v_{n+1})|\mathcal{F}_{\delta,n}] &= \frac{1}{2} \times \mathbb{P}_\delta^{(\Omega,a,b,m)}(h_{\delta,n+1}^m(v_{n+1}) = 1/2 \,|\, \mathcal{F}_{\delta,n}) \\
&\quad - \frac{1}{2} \times (1 - \mathbb{P}_\delta^{(\Omega,a,b,m)}(h_{\delta,n+1}^m(v_{n+1}) = 1/2 \,|\, \mathcal{F}_{\delta,n})) \\
&= \frac{1}{2} \times \left(h_{\delta,n}^m(v_{n+1}) + \frac{1}{2}\right) + \frac{-1}{2} \times \left(\frac{1}{2} - h_{\delta,n}^m(v_{n+1})\right) \\
&= h_{\delta,n}^m(v_{n+1}).
\end{aligned}
$$

$h_{\delta,n}^m$ is also the discrete massive harmonic extension of its restriction to $\partial\Omega_{\delta,n+1}$ and similarly for $h_{\delta,n+1}^m$. Since taking the massive harmonic extension of a function defined on the boundary of a domain is a linear operation and almost surely $\mathbb{E}_\delta^{(\Omega,a,b,m)}[h_{\delta,n+1}^m(v)|\mathcal{F}_{\delta,n}] = h_{\delta,n}(v)$ for all $v \in \partial\Omega_{\delta,n+1}$, the same relation holds almost surely for every $v \in \mathrm{Int}(\Omega_{\delta,n+1})$. Thus, for each $v \in \mathrm{Int}(\Omega_\delta) \cup \partial\Omega_\delta$, $(h_{\delta,n}^m(v), n \geq 0)$ is a martingale. $\square$

In view of Proposition 3.1.1, we call the functions $(h_{\delta,n}^m)_n$ the martingale observables of the massive harmonic explorer.

3.2 Tightness

In this section, we establish tightness of the sequence of massive harmonic explorer paths $(\gamma_\delta)_\delta$, where for each $\delta > 0$, γ_δ is distributed according to $\mathbb{P}_\delta^{(\Omega,a,b,m)}$. Tightness is shown in the topologies (T.1)–(T.3), using the approach laid out in [KS17]. This approach applies to families of probability measures supported on a metric space of curves, whose construction is explained in Section 2.2. Under a condition on the probability of a certain crossing event, tightness of a sequence of random curves supported on this space then follows from [KS17, Theorem 1.7]. This theorem is phrased in terms of Loewner chains, that we introduced in Section 2.1. [KS17, Theorem 1.7] is then discussed in Section 3.2.1, where we also explain the condition on the probability of the crossing event and describe the topologies in which tightness is obtained. Finally, in Section 3.2.2, we show that the condition on the probability of the crossing event is satisfied by the sequence $(\gamma_\delta)_\delta$ distributed according to $(\mathbb{P}_\delta^{(\Omega,a,b,m)})_\delta$, thus proving tightness of $(\gamma_\delta)_\delta$ in the aforementioned topologies.

3.2.1 Annulus crossing estimate and tightness

To show tightness, we rely on the framework developed by Kemppainen and Smirnov in [KS17]. According to these results, tightness of the sequence of massive harmonic explorers $(\gamma_\delta)_\delta$ can be established in the topologies (T.1)–(T.3) if, under $\mathbb{P}_\delta^{(\Omega,a,b,m)}$, an appropriate and uniform in δ upper bound on the probability of a certain crossing event holds. In our case, we will prove that under $\mathbb{P}_\delta^{(\Omega,a,b,m)}$, the probability that the massive harmonic explorer γ crosses a so-called avoidable annulus of modulus r/R, defined just below, decays geometrically in r/R, with constants independent of δ. Essentially, we will show that the family $(\mathbb{P}_\delta^{(\Omega,a,b,m)})_{\delta < \overline{m}_d^{-1}}$ satisfies Condition G.3 of [KS17]. Let us now describe in more detail this condition and its consequences regarding tightness of the family $(\mathbb{P}_\delta^{(\Omega,a,b,m)})_{\delta < \overline{m}_d^{-1}}$.

Let $z_0 \in \mathbb{C}$ and $0 < r \le R$ and denote by $A(z_0, r, R)$ the annulus centered at z_0 with inner radius r and outer radius R, that is

$$A(z_0, r, R) := \{z \in \mathbb{C} : r < |z_0 - z| < R\}.$$

Let τ be a stopping time for γ and set $\Omega_\tau := \Omega \setminus \gamma([0, \tau])$. The avoidable component A_τ^Ω of an annulus $A(z_0, r, R)$ at time τ in Ω is defined as follows. If $\partial B(z_0, r) \cap \partial \Omega_\tau = \emptyset$, then $A_\tau^\Omega = \emptyset$. Otherwise,

$$A_\tau^\Omega := \left\{ z \in \Omega_\tau \cap A(z_0, r, R) : \begin{array}{l} \text{the connected component of } z \text{ in } \Omega_\tau \cap A(z_0, r, R) \\ \text{does not disconnect } \gamma(\tau) \text{ from } b \text{ in } \Omega_\tau \end{array} \right\}.$$

If an annulus $A(z_0, r, R)$ is such that $A_\tau^\Omega \neq \emptyset$, we say that $A(z_0, r, R)$ is an avoidable annulus at time τ. Furthermore, if $\gamma([\tau, 1])$ crosses $A(z_0, r, R)$ in one of the connected components of A_τ^Ω, γ is said to make an unforced crossing of $A(z_0, r, R)$ in Ω_τ. The reader may wish to look at Figure 3.1 for an example of A_τ.

For a family $(\mathbb{P}_n)_n$ of probability measures supported on $X_0(\Omega, a, b)$, Condition G.3 of [KS17] then reads as follows. Here, curves are parametrized from 0 to 1.

Condition G.3 The family $(\mathbb{P}_n)_n$ is said to satisfy a geometric power-law bound on an unforced crossing if there exist $K > 0$ and $\Delta > 0$ such that for any n, for any stopping time $0 \le \tau \le 1$ and for any annulus $A = A(z_0, r, R)$ where $0 < r \le R$,

$$\mathbb{P}_n(\gamma([\tau, 1]) \text{ makes a crossing of } A \text{ which is contained in } A_\tau^\Omega \,|\, \gamma([0, \tau])) \le K \left(\frac{r}{R} \right)^\Delta.$$

By [KS17, Theorem 1.7], verifying this condition for the family $(\mathbb{P}_n)_n$ allows one to establish tightness of the sequence of curves $(\gamma_n)_n$ distributed according to $(\mathbb{P}_n)_n$ in the topologies (T.1)–(T.3) (see below for the details in our setting).

We wish to apply this result to the family of massive harmonic explorers under $\mathbb{P}_\delta^{(\Omega, a, b, m)}$. However, note that under this law, $\gamma = \gamma_\delta$ is an element of $X_0(\hat{\Omega}_\delta, a_\delta, b_\delta)$ for each δ, whereas Condition G.3 is stated for a family of probability measures all supported on the same $X_0(\Omega, a, b)$. Thus we first need to uniformize the picture. To this end, we let $\phi : \Omega \to \mathbb{H}$ be a conformal map such that $\phi(a) = 0$ and $\phi(b) = \infty$ and similarly, for $\delta > 0$, let $\phi_\delta : \hat{\Omega}_\delta \to \mathbb{H}$ be a conformal map such that $\phi_\delta(a_\delta) = 0$ and $\phi_\delta(b_\delta) = \infty$. For $\delta > 0$, we denote by W_δ the driving function of $\gamma_\delta^{\mathbb{H}} := \phi_\delta(\gamma_\delta)$, when the curve is parametrized by half-plane capacity.

In this setting [KS17, Theorem 1.7] yields the following. Suppose that the laws of $(\gamma_\delta^{\mathbb{H}})_{0 < \delta < \overline{m}_\delta^{-1}}$ satisfy Condition G.3, (where γ_δ has law $(\mathbb{P}_\delta^{(\Omega, a, b, m)})$ for each δ, and ϕ_δ is as above). Then $(\gamma_\delta^{\mathbb{H}})_\delta$ is tight in the topologies (T.1)–(T.2) and $(W_\delta)_\delta$ is tight in the topology (T.3).

Moreover, under the assumption that $(\hat{\Omega}_\delta; a_\delta, b_\delta)_\delta$ converges in the Carathéodory sense to $(\Omega; a, b)$, that (a_δ) and $(b_\delta)_\delta$ are close approximations of the degenerate prime ends a and b, [KS17, Corollary 1.8] together with [Kar23, Theorem 4.2] imply the following stronger result.

Theorem 3.2.1. *Assume that a family of measures $(\mathbb{P}_n)_n$ satisfies Condition 3.2.1. Then, the family $(\gamma_n, \gamma_n^{\mathbb{H}})_n$ is tight in the topologies ((T.1)) and ((T.4)). Furthermore, any subsequential limit $(\gamma, \gamma^{\mathbb{H}})$ almost surely satisfies $\gamma = \phi^{-1}(\gamma^{\mathbb{H}})$ where ϕ^{-1} is extended by radial limits.*

3.2.2 Proof of the annulus crossing estimate

As explained above, to establish tightness of the massive harmonic explorers $(\gamma_\delta)_\delta$ in the topologies ((T.1)) – ((T.3)), we show that the family $(\mathbb{P}_\delta^{(\Omega, a, b, m)})_\delta$ satisfies Condition G.3. This is the content of the following proposition.

Proposition 3.2.2. *There exist constants $K, \alpha > 0$ such that for any $0 < \delta < \overline{m}_d^{-1}$, for any stopping time τ and any annulus $A = A(z_0, r, R)$,*

$$\mathbb{P}_\delta^{(\Omega, a, b, m)}(\gamma([\tau, 1]) \text{ makes a crossing of } A \text{ which is contained in } A_\tau^{\hat{\Omega}_\delta} \,|\, \gamma([0, \tau])) \le K \left(\frac{r}{R} \right)^\alpha.$$

This implies in particular that the family $(\mathbb{P}_\delta^{(\Omega, a, b, m)})_{\delta < \overline{m}_d^{-1}}$ satisfies Condition G.3.

The proof of Proposition 3.2.2 relies on a martingale argument similar to that used in the proof of [SS05, Proposition 6.3]. Our martingale is a sum of two terms. One of them is the massive version of the martingale used in the proof of [SS05, Proposition 6.3] or, in other words, the total mass of massive random-walk excursions from a well-chosen set of boundary vertices to boundary vertices with sign $+$. However, this term by itself is not a martingale because in our setting, the massive harmonic measure of $\partial\Omega_{\delta,n}$ seen from a vertex $v \in \Omega_{\delta,n}$ is not a martingale. To compensate the drift that arises from it, we must add another term, which is the second term in our martingale. This term accounts for the probability that excursions get killed before leaving the domain. To control the first term, using simple inequalities for the massive harmonic measure and up to some minor modifications, one can argue as in the proof of [SS05, Propostion 6.3]. However, controlling the second term requires new ideas but makes use of several lemmas proved in [SS05].

We remark that it was already observed in [KS17, Section 4.4] that the proof of [SS05, Proposition 6.3] could be used to deduce Proposition 3.2.2 when $m \equiv 0$, that is to show that the harmonic explorer satisfies Condition G.3. However, in [KS17, Section 4.4], the argument did not take into account a certain type of geometric configuration for $A_\tau^{\hat\Omega_\delta}$, corresponding to the collection $\mathcal{A}_2$ in our proof of Proposition 3.2.2, thus making the proof incomplete. This gap is filled below.

Before turning to the proof of Proposition 3.2.2, we recall [SS05, Lemma 6.1] and its corollary [SS05, Corollary 6.2] as we will repeatedly use them. In order to do so, we need to introduce the discrete excursion measures, which are the discrete analogues of the Brownian excursion measures. For $\delta > 0$, let $G_\delta \subset \delta\mathbb{T}$ be a graph with boundary ∂G_δ. For an oriented edge (vw) of $\delta\mathbb{T}$, we denote by $\mathrm{rev}(vw)$ the same edge with reverse orientation. We let $E = E(G_\delta)$ denote the set of edges of G_δ whose inital vertex is in ∂G_δ and whose terminal vertex is in $\mathrm{Int}(G_\delta)$. Let $E_1 \subset E$ and $E_2 \subset \mathrm{rev}(E)$. For every $v \in \partial G_\delta$, let X^v be a simple random walk on $\delta\mathbb{T}$ that starts at v and is stopped at the first time $n \geq 1$ such that $X_n^v \notin G_\delta$. Let ν^v denote the restriction of the law of X^v to those walks that use an edge of E_1 as their first step and an edge of E_2 as their last step. Finally, define

$$\nu_{(G_\delta, E_1, E_2)} := \sum_{v \in \partial G_\delta} \nu^v.$$

$\nu_{(G_\delta, E_1, E_2)}$ is called the discrete excursion measure from E_1 to E_2 in G_δ: this is a measure on paths starting with an edge of E_1 and ending with an edge of E_2 and staying in G_δ in between. When $E_2 = \mathrm{rev}(E)$, we simply write $\nu_{(G_\delta, E_1, E_2)} = \nu_{(G_\delta, E_1)}$. The first result about the measure $\nu_{(G_\delta, E_1)}$ that will be instrumental in the proof of Proposition 3.2.2 is a relation between the expected number of visits to a vertex $x \in \mathrm{Int}(G_\delta)$ under $\nu_{(G_\delta, E_1)}$ and the probability that a walk started from x exits G_δ using an edge of $\mathrm{rev}(E_1)$. This is [SS05, Lemma 6.1].

Lemma 3.2.3. *Let G_δ be as above and let $E_1 \subset E$. Fix $x \in \mathrm{Int}(G_\delta)$ and for a path ω, let $n_x(\omega)$ be the number of times ω visits x. Then*

$$\int n_x(\omega) \nu_{(G_\delta, E_1)}(d\omega) = H_{G_\delta}(x, \mathrm{rev}(E_1))$$

where $H_{G_\delta}(x, \mathrm{rev}(E_1))$ is the probability that a simple random walk started at x will first exit G_δ via an edge in $\mathrm{rev}(E_1)$. In particular, $\int n_x(\omega) \nu_{(G_\delta, E)}(d\omega) = 1$.

This lemma can be used to estimate the $\nu_{(G_\delta, E_1)}$-mass of paths that visits a ball, provided the ball is sufficiently far away from ∂G_δ. This is stated as [SS05, Corollary 6.2] and since this result will be useful in the proof of Proposition 3.2.2, let us recall it.

Lemma 3.2.4. *Let G_δ and E_1 be as above and let $x \in \mathrm{Int}(G_\delta)$. Denote by $rad_x(G_\delta)$ the Euclidean distance between x and the boundary of G_δ. Assume that the boundary of G_δ is connected. Let B be the ball centered at x whose radius is $\frac{1}{2}rad_x(G_\delta)$ and let Γ_B be the set of paths that visit B. Then*

$$c^{-1} H_{G_\delta}(x, \mathrm{rev}(E_1)) < \nu_{(G_\delta, E_1)}(\Gamma_B) < c H_{G_\delta}(x, \mathrm{rev}(E_1))$$

for some absolute constant $c > 0$.

The last fact that will be useful in the course of the proof of Proposition 3.2.2 is the following simple inequality between discrete massive and massless harmonic measures.

Lemma 3.2.5. *Let $G_\delta \subset \delta\mathbb{T}$ be a finite graph with boundary ∂G_δ and let $\tilde{\partial}$ be a subset of ∂G_δ. For $w \in \mathrm{Int}(G_\delta)$, denote by $H_\delta^{(0)}(w)$, respectively $H_\delta^{(m)}(w)$, the massless, respectively the massive, discrete harmonic measure of $\tilde{\partial}$ seen from w. Then, for any $w \in \mathrm{Int}(G_\delta)$,*

$$H_\delta^{(m)}(w) \leq H_\delta^{(0)}(w).$$

Proof. Let $w \in \mathrm{Int}(G_\delta)$. By definition,

$$H_\delta^{(m)}(w) = \mathbb{P}_{\delta,w}^{(m)}(X_{\tau_{\partial G_\delta}} \in \tilde{\partial})$$

where under $\mathbb{P}_{\delta,w}^{(m)}$, X is a simple massive random walk on $\delta\mathbb{T}$ started at w and $\tau_{\partial G_\delta}$ denotes its first hitting time of ∂G_δ. Therefore, we have that

$$H_\delta^{(m)}(w) = \sum_{\omega: w \to \tilde{\partial}} \prod_{j=1}^{\tau(\omega)} \frac{1 - m^2(\omega_j)\delta^2}{6}$$

where the sum is over the set of paths ω on $\mathbb{T}_\delta$ starting at w and ending at a vertex of $\tilde{\partial}$. For such a path ω, ω_j denotes the jth vertex it visits and $\tau(\omega)$ is its length. The above equality then yields that

$$\begin{aligned}
H_\delta^{(m)}(w) &\leq \sum_{\omega: w \to \tilde{\partial}} \prod_{j=1}^{\tau(\omega)} \frac{1}{6} \\
&= \mathbb{P}_{\delta,w}^{(0)}(X_{\tau_{\partial G_\delta}} \in \tilde{\partial}) \\
&= H_\delta^{(0)}(w)
\end{aligned}$$

where under $\mathbb{P}_{\delta,w}^{(0)}$, X is a simple (massless) random walk started at w. $\qquad\square$

With these lemmas in hand, let us now turn to the proof of Proposition 3.2.2. We first prove the following proposition, which is a special case of Proposition 3.2.2 when the stopping time τ is almost surely equal to 0. Thanks to the Markov property (3.5) of the massive harmonic explorer, the proof of Proposition 3.2.2 will follow the same strategy as that of the proof of this proposition, and we find it easier to first explain the arguments for the time $\tau = 0$ and then show how to adapt them to the case of a general stopping time.

Proposition 3.2.6. *There exist constants $K, \alpha > 0$ such that for any $0 < \delta < \overline{m}_d^{-1}$ and any annulus $A = A(z_0, r, R)$,*

$$\mathbb{P}_\delta^{(\Omega, a, b, m)}(\gamma \text{ makes a crossing of } A \text{ which is contained in } A^{\hat{\Omega}_\delta}) \leq K\left(\frac{r}{R}\right)^\alpha. \tag{3.6}$$

Proof. Fix $0 < \delta < \overline{m}_d^{-1}$. For clarity, as δ is fixed, we write A^Ω for $A^{\hat{\Omega}_\delta}$. A^Ω is a collection $(A_j^\Omega)_j$ of connected components of $A(z_0, r, R) \cap \hat{\Omega}_\delta$. We are going to split it into two disjoint sub-collections $\mathcal{A}_1^\Omega$ and $\mathcal{A}_2^\Omega$ of connected components. These collections correspond to two different geometric configurations for the intersection between $\hat{\Omega}_\delta$ and the annulus $A(z_0, r, R)$. We will then upper bound the probability of a crossing of a component of $\mathcal{A}_1^\Omega$ and that of a crossing of a component of $\mathcal{A}_2^\Omega$ separately. In both cases, the upper bound is established using a martingale argument. Indeed, we will see that thanks to the optional stopping theorem, upper bounding the probability of a crossing in $\mathcal{A}_1^\Omega$ or that of a crossing in $\mathcal{A}_2^\Omega$ amounts to upper bound a certain martingale at time 0 and lower bound it at a well-chosen stopping time. However, because of the different geometric configurations reflected in the collections $\mathcal{A}_1^\Omega$ and $\mathcal{A}_2^\Omega$, we cannot use the same martingale in both cases and this is why we must distinguish between these two cases.

Let us now define the splitting of A^Ω into two disjoint sub-collections $\mathcal{A}_1^\Omega := (A_{1,j}^\Omega)_j$ and $\mathcal{A}_2^\Omega := (A_{2,j}^\Omega)_j$ as mentioned above. The collection $\mathcal{A}_1^\Omega$ is such the following holds. $A \subset A^\Omega$ is an element of $\mathcal{A}_1^\Omega$ if and only if γ must first intersect $\partial B(z_0, R)$ to cross A. In turn, the collection $\mathcal{A}_2^\Omega$ is made of those components of A^Ω that do not satisfy this property. In other words, A_j^Ω belongs to $\mathcal{A}_2^\Omega$ if and only if γ must first intersect $\partial B(z_0, r)$ to cross A_j^Ω. See Figure 3.1 for an illustration.
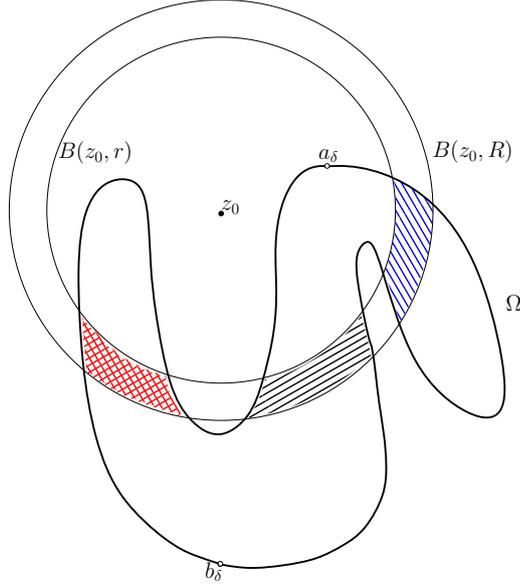
Figure 3.1: Example of a domain Ω together with an annulus $A(z_0, r, R)$. Here, the collection $\mathcal{A}_1$ consists of the connected component of $A(z_0, r, R) \cap \Omega$ shaded with red cross-hatch while the collection $\mathcal{A}_2$ consists of the connected component of $A(z_0, r, R) \cap \Omega$ with blue lines. The connected component with black lines is an unavoidable component of $\Omega \cap A(z_0, r, R)$.

Observe that $\mathcal{A}_1^\Omega$ or $\mathcal{A}_2^\Omega$ may be empty. Given this splitting, we define two collections of connected components of $\Omega \setminus A^\Omega$

$$B_1^{av} := \left\{ D \subset B(z_0, r) \cap \hat{\Omega}_\delta : \ D \text{ connected such that } \exists A_{1,j}^\Omega \in \mathcal{A}_1^\Omega \text{ such that } \partial D \cap \partial A_{1,j}^\Omega \neq \emptyset \right\}$$

$$B_2^{av} := \left\{ D \subset \hat{\Omega}_\delta \setminus B(z_0, R) : \ D \text{ connected such that } \exists A_{2,j}^\Omega \in \mathcal{A}_2^\Omega \text{ such that } \partial D \cap \partial A_{2,j}^\Omega \neq \emptyset \right\}.$$

Notice that B_1^{av}, respectively B_2^{av}, is chosen such that if γ makes a crossing of $A_{1,j}^\Omega$, respectively of $A_{2,j}^\Omega$, for some j, then there exists $n \in \mathbb{N}$ such that $\gamma(n) \in B_1^{av}$, respectively $\gamma(n) \in B_2^{av}$. B_1^{av} and B_2^{av} depend on δ but since δ is fixed and our estimates will be uniform in δ, we do not mention explicitly this dependency in the notation.

Let $\mathcal{Q}$ be the event that there exists $j \in \mathbb{N}$ such that $v_j \in B_1^{av} \cup B_2^{av}$ and $\gamma([0, j])$ contains a crossing of A^Ω. On $\mathcal{Q}$, denote by σ the least such j. Observe that $\mathcal{Q}$ can be decomposed as a disjoint union $\mathcal{Q} = \mathcal{Q}_1 \sqcup \mathcal{Q}_2$, where $\mathcal{Q}_1 := \mathcal{Q} \cap \{v_\sigma \in B_1^{av}\}$ and $\mathcal{Q}_2 := \mathcal{Q} \cap \{v_\sigma \in B_2^{av}\}$. Therefore, we see that to bound $\mathbb{P}_\delta^{(\Omega, a, b, m)}(\mathcal{Q})$, it is enough to bound the probabilities $\mathbb{P}_\delta^{(\Omega, a, b, m)}(\mathcal{Q}_1)$ and $\mathbb{P}_\delta^{(\Omega, a, b, m)}(\mathcal{Q}_2)$ separately. We claim that the following bound on $\mathbb{P}_\delta^{(\Omega, a, b, m)}(\mathcal{Q}_1)$ holds.

Claim 3.2.7. *There exist universal constants $K_1, \alpha_1 > 0$ such that*

$$\mathbb{P}_\delta^{(\Omega, a, b, m)}(\mathcal{Q}_1) \leq K_1 \left(\frac{r}{R} \right)^{\alpha_1}. \tag{3.7}$$

As for $\mathbb{P}_\delta^{(\Omega, a, b, m)}(\mathcal{Q}_2)$, we claim that it satisfies a similar bound.

Claim 3.2.8. *There exist universal constants $K_2, \alpha_2 > 0$ such that*

$$\mathbb{P}_\delta^{(\Omega, a, b, m)}(\mathcal{Q}_2) \leq K_2 \left(\frac{r}{R} \right)^{\alpha_2}. \tag{3.8}$$

Claim 3.2.7 and Claim 3.2.8 together imply that the inequality (3.6) holds with $K = 2 \max(K_1, K_2)$ and $\alpha = \min(\alpha_1, \alpha_2)$, thus establishing Proposition 3.2.6. Let us now turn to the proof of these claims. We start by showing Claim 3.2.7.

Proof of Claim 3.2.7. Set $\bar{D} = 2\bar{R}$, where we recall that $\bar{R} > 0$ is such that for any $\delta > 0$, $\hat{\Omega}_\delta \subset B(0, \bar{R})$ and $\Omega \subset B(0, \bar{R})$. In particular, for any $\delta > 0$, $\text{diam}(\hat{\Omega}_\delta) \leq \bar{D}$. Let us assume that (z_0, r, R) is in the following subset of $\mathbb{C} \times \mathbb{R}_+ \times \mathbb{R}_+$

$$\mathcal{G}_1 := \{(z_0, r, R) : \text{dist}(z_0, \partial\hat{\Omega}_\delta) \leq \bar{D}, \, r \leq \frac{1}{100}\bar{D}, \, R \leq 2\bar{D}, \, 0 < 10r \leq R\}.$$

In this case, we will see that there exist universal constants $\tilde{K}_1, c_1, \hat{\alpha}_1 > 0$ such that

$$\mathbb{P}_\delta^{(\Omega, a, b, m)}(\mathcal{Q}_1) \leq \frac{2\tilde{K}_1}{c_1}\left(\frac{r}{R}\right)^{\hat{\alpha}_1} + \frac{2^{1+\hat{\alpha}_1/2}\tilde{K}_1}{c_1}\left(\frac{r}{R}\right)^{\hat{\alpha}_1/2} + 2\frac{\bar{m}_d^2\bar{D}^2}{c_1}\frac{r}{R}. \tag{3.9}$$

Therefore, for $(z_0, r, R) \in \mathcal{G}_1$, the inequality (3.7) is satisfied with

$$C_1 := K_1 = \frac{2\tilde{K}_1}{c_1} + \frac{2^{1+\hat{\alpha}_1/2}\tilde{K}_1}{c_1} + \frac{2\overline{m}_d^2}{c_1}\bar{D}^2, \quad \text{and} \quad \alpha_1 = \frac{\hat{\alpha}_1}{2} \wedge 1. \tag{3.10}$$

If $(z_0, r, R) \notin \mathcal{G}_1$, then notice that either there are no crossings of $A(z_0, r, R)$ that stay in $\hat{\Omega}_\delta$ or the ratio r/R is greater than or equal to a deterministic constant $c > 0$ (independent of δ). Indeed:

1. If $\text{dist}(z_0, \partial\hat{\Omega}_\delta) \geq \bar{D}$, then, in order to have a crossing of $A(z_0, r, R)$ that stays in $\hat{\Omega}_\delta$, one must have $r \geq \text{dist}(z_0, \partial\hat{\Omega}_\delta)$ and $R \leq \text{dist}(z_0, \partial\hat{\Omega}_\delta) + \text{diam}(\hat{\Omega}_\delta)$. This implies that

$$\frac{R}{r} \leq \frac{\text{dist}(z_0, \partial\hat{\Omega}_\delta) + \text{diam}(\hat{\Omega}_\delta)}{\text{dist}(z_0, \partial\hat{\Omega}_\delta)} \leq 1 + \frac{\bar{D}}{\bar{D}} = 2.$$

Therefore, the ratio r/R is lower bounded by $1/2$ in that case.

2. If $10r \geq R$, then the ratio r/R is lower bounded by $1/10$.

3. If $\text{dist}(z_0, \partial\hat{\Omega}_\delta) \leq \bar{D}$ and $R \geq 2\bar{D}$, then there are no crossings of $A(z_0, r, R)$ that stay in $\hat{\Omega}_\delta$.

4. If $\text{dist}(z_0, \partial\hat{\Omega}_\delta) \leq \bar{D}$ and $r \geq (1/100)\bar{D}$, then in order to have a crossing of $A(z_0, r, R)$, one must have $R \leq \text{dist}(z_0, \partial\hat{\Omega}_\delta) + \text{diam}(\hat{\Omega}_\delta)$ and, in this case, $\text{dist}(z_0, \partial\hat{\Omega}_\delta) + \text{diam}(\hat{\Omega}_\delta) \leq 2\bar{D}$. This implies that the ratio r/R is lower bounded by $1/200$.

We thus see that if $(z_0, r, R) \notin \mathcal{G}_1$, then either there are no crossings of $A(z_0, r, R)$ that stay in $\hat{\Omega}_\delta$ or the ratio r/R is lower bounded by $1/200$. Taking α_1 as in (3.10) and $C_{b1} > 0$ such that $C_{b1} \times (1/200)^{\alpha_1} = 1$, we trivially obtain that for $(z_0, r, R) \notin \mathcal{G}_1$,

$$\mathbb{P}_\delta^{(\Omega, a, b, m)}(\mathcal{Q}_1) \leq C_{b1}\left(\frac{r}{R}\right)^{\alpha_1}$$

since the right-hand side is greater than 1. Setting $K_1 = \max(C_1, C_{b1})$ where C_1 is the constant found in (3.10), we then obtain the inequality (3.7) for any $(z_0, r, R) \in \mathbb{C} \times \mathbb{R}_+ \times \mathbb{R}_+$ with $r \leq R$.

Let us now turn to the proof of the inequality (3.9). Let (z_0, r, R) be in $\mathcal{G}_1$ and consider the annulus $A(z_0, r, R)$. By definition of the collection A^Ω, each connected component of $B(z_0, 3r) \cap \Omega_\delta$ which intersects some $A_{1,j}^\Omega$ for some j has boundary entirely in $\partial\Omega_\delta^-$ or $\partial\Omega_\delta^+$. Recall that on the event $\mathcal{Q}_1$, the stopping time σ is defined as the least j such that $v_j \in B_1^{av}$ and $\gamma([0, j])$ contains a crossing of A^Ω. On the event $\mathcal{Q}_1$, let S be the connected component of $B(z, 3r) \cap \Omega_\delta$ intersecting $\gamma([\sigma - 1, \sigma])$ and let $\mathcal{Q}_1^- \subset \mathcal{Q}_1$, respectively $\mathcal{Q}_1^+ \subset \mathcal{Q}_1$, be the event that $\partial S \subset \partial\Omega_\delta^-$, respectively $\partial S \subset \partial\Omega_\delta^+$. Then $\mathbb{P}_\delta^{(\Omega, a, b, m)}(\mathcal{Q}_1) = \mathbb{P}_\delta^{(\Omega, a, b, m)}(\mathcal{Q}_1^-) + \mathbb{P}_\delta^{(\Omega, a, b, m)}(\mathcal{Q}_1^+)$ and the inequality (3.9) will hold if we can show that there exist constants $\tilde{K}_1, c_1, \hat{\alpha}_1 > 0$ such that

$$\mathbb{P}_\delta^{(\Omega, a, b, m)}(\mathcal{Q}_1^\pm) \leq \frac{\tilde{K}_1}{c_1}\left(\frac{r}{R}\right)^{\hat{\alpha}_1} + \frac{2^{\hat{\alpha}_1}\tilde{K}_1}{c_1}\left(\frac{r}{R}\right)^{\hat{\alpha}_1/2} + \frac{\overline{m}_d^2\bar{D}^2}{c_1}\frac{r}{R}.$$

Let us prove this inequality for $\mathbb{P}_\delta^{(\Omega, a, b, m)}(\mathcal{Q}_1^-)$. By symmetry (see the discussion around (3.4)), the proof for $\mathbb{P}_\delta^{(\Omega, a, b, m)}(\mathcal{Q}_1^+)$ is virtually the same. Let E_- denote the set of directed edges in $E = E(\Omega_\delta)$ whose initial vertex is in $\partial\Omega_\delta^- \cap B(z_0, 3r)$ and is disconnected from b in Ω_δ by a

connected component of $\mathcal{A}_1^\Omega$. Denote by V_- the set of initial vertices of the edges in E_-. Then, by Lemma 3.1.1, the process

$$\|\tilde{\nu}_{\delta,n}^{(m)}\| := \sum_{v \in V_-} \frac{1}{6} \sum_{v \sim w} \left(h_{\delta,n}^m(w) + \frac{1}{2} \right), \quad n \geq 0,$$

is a non-negative martingale for the filtration $(\mathcal{F}_{\delta,n})_n$. Therefore, by the optional stopping theorem,

$$\|\tilde{\nu}_{\delta,0}^{(m)}\| = \mathbb{E}_\delta^{(\Omega,a,b,m)}\big[\|\tilde{\nu}_{\delta,\sigma}^{(m)}\|\big] \geq \mathbb{E}_\delta^{(\Omega,a,b,m)}\big[\mathbb{I}_{\mathcal{Q}_1^-}\|\tilde{\nu}_{\delta,\sigma}^{(m)}\|\big]. \tag{3.11}$$

We thus see that in order to bound $\mathbb{P}_\delta^{(\Omega,a,b,m)}(\mathcal{Q}_1^-)$, it is enough to exhibit an appropriate upper bound for $\|\tilde{\nu}_{\delta,0}^{(m)}\|$ and to show that $\|\tilde{\nu}_{\delta,\sigma}^{(m)}\|$ is bounded away from 0 by a universal constant on the event $\mathcal{Q}_1^-$. Let us first focus on $\|\tilde{\nu}_{\delta,0}^{(m)}\|$.

Observe that almost surely, for any $n \in \mathbb{N}$ and any $w \in \text{Int}(\Omega_{\delta,n}) \cup \partial\Omega_{\delta,n}$,

$$h_{\delta,n}^m(w) + \frac{1}{2} = H_{\delta,n}^{(m)}(w) + \frac{1}{2}\mathbb{P}_w^{(m)}(\tau^\star \leq \tau_{\partial\Omega_{\delta,n}})$$

where, as explained in Section 3.1.2, $H_{\delta,n}^{(m)}(w)$ is the discrete massive harmonic measure of $\partial\Omega_{\delta,n}^+$ seen from w. By Lemma 3.2.5, we then have that, almost surely, for any $n \in \mathbb{N}$,

$$\|\tilde{\nu}_{\delta,n}^{(m)}\| \leq \sum_{v \in V_-} \frac{1}{6} \sum_{v \sim w} \left(H_{\delta,n}^{(0)}(w) + \frac{1}{2}\mathbb{P}_w^{(\overline{m}_d)}(\tau^\star \leq \tau_{\partial\Omega_{\delta,n}}) \right),$$

where $H_{\delta,n}^{(0)}(w)$ is the discrete (massless) harmonic measure of $\partial\Omega_{\delta,n}^+$ seen from w. Taking $n = 0$, this yields that

$$\|\tilde{\nu}_{\delta,0}^{(m)}\| \leq \sum_{v \in V_-} \frac{1}{6} \sum_{v \sim w} \left(H_{\delta,0}^{(0)}(w) + \frac{1}{2}\mathbb{P}_w^{(\overline{m}_d)}(\tau^\star \leq \tau_{\partial\Omega_{\delta,0}}) \right). \tag{3.12}$$

Our upper bound on $\|\tilde{\nu}_{\delta,0}^{(m)}\|$ that will allow us to establish the inequality (3.9) is a consequence of the following two lemmas.

Lemma 3.2.9. *There exist universal constants $\tilde{K}_1, \hat{\alpha}_1 > 0$ such that, for any $(z_0, r, R) \in \mathcal{G}_1$,*

$$\sum_{v \in V_-} \frac{1}{6} \sum_{v \sim w} H_{\delta,0}^{(0)}(w) \leq \tilde{K}_1 \left(\frac{r}{R} \right)^{\hat{\alpha}_1}. \tag{3.13}$$

Lemma 3.2.9 can be derived using the same arguments as those used in the first part of the proof of [SS05, Proposition 6.3], since we assume that $10r \leq R$ for $(z_0, r, R) \in \mathcal{G}_1$. The next lemma, which controls the term in $\|\tilde{\nu}_{\delta,0}^{(m)}\|$ arising from the killing, will require a bit more work.

Lemma 3.2.10. *For the same constants $\tilde{K}_1$ and $\hat{\alpha}_1$ as in Lemma 3.2.9 and still assuming that $(z_0, r, R) \in \mathcal{G}_1$,*

$$\sum_{v \in V_-} \frac{1}{6} \sum_{v \sim w} \frac{1}{2}\mathbb{P}_w^{(\overline{m}_d)}(\tau^\star \leq \tau_{\partial\Omega_{\delta,0}}) \leq 2^{\hat{\alpha}_1/2}\tilde{K}_1 \left(\frac{r}{R} \right)^{\hat{\alpha}_1/2} + \overline{m}_d^2\bar{D}^2 \times \frac{r}{R}.$$

We postpone the proof of Lemma 3.2.10 to the end and show how to proceed from here. Observe that, using the inequality (3.12), Lemma 3.2.9 and Lemma 3.2.10 yield that

$$\|\tilde{\nu}_{\delta,0}^{(m)}\| \leq \tilde{K}_1 \left(\frac{r}{R} \right)^{\hat{\alpha}_1} + 2^{\hat{\alpha}_1/2}\tilde{K}_1 \left(\frac{r}{R} \right)^{\hat{\alpha}_1/2} + \overline{m}_d^2\bar{D}^2 \frac{r}{R} \tag{3.14}$$

where $\tilde{K}_1, \hat{\alpha}_1 > 0$ are universal constants.

Let us now exhibit a lower bound for $\|\tilde{\nu}_{\delta,\sigma}^{(m)}\|$ on the event $\mathcal{Q}_1^-$. For $n \geq 0$ and $w \in \text{Int}(\Omega_{\delta,n})$, denote by $\tilde{H}_{\delta,n}^{(m)}(w)$, respectively $\tilde{H}_{\delta,n}^{(0)}(w)$, the discrete massive, respectively massless, harmonic measure of $\partial\Omega_{\delta,n}^-$ seen from w. Almost surely, it holds that, for any $n \in \mathbb{N}$ and $w \in \text{Int}(\Omega_{\delta,n})$,

$$H_{\delta,n}^{(m)}(w) + \tilde{H}_{\delta,n}^{(m)}(w) + \mathbb{P}_w^{(m)}(\tau^\star \leq \tau_{\partial\Omega_{\delta,n}}) = H_{\delta,n}^{(0)}(w) + \tilde{H}_{\delta,n}^{(0)}(w).$$

It follows that almost surely, for any $n \in \mathbb{N}$ and $w \in \mathrm{Int}(\Omega_{\delta,n})$,

$$H_{\delta,n}^{(m)}(w) + \mathbb{P}_w^{(m)}(\tau^\star \leq \tau_{\partial\Omega_{\delta,n}}) \geq H_{\delta,n}^{(0)}(w) \tag{3.15}$$

since almost surely, for any $n \in \mathbb{N}$ and $w \in \Omega_{\delta,n}$, $\tilde{H}_{\delta,n}^{(0)}(w) - \tilde{H}_{\delta,n}^{(m)}(w) \geq 0$. Therefore, almost surely,

$$
\begin{aligned}
\|\tilde{\nu}_{\delta,\sigma}^{(m)}\| &= \sum_{v \in V_-} \frac{1}{6} \sum_{v \sim w} \left(H_{\delta,\sigma}^{(m)}(w) + \frac{1}{2}\mathbb{P}_w^{(m)}(\tau^\star \leq \tau_{\partial\Omega_{\delta,\sigma}}) \right) \\
&\geq \sum_{v \in V_-} \frac{1}{6} \sum_{v \sim w} \left(\frac{1}{2}H_{\delta,\sigma}^{(m)}(w) + \frac{1}{2}\mathbb{P}_w^{(m)}(\tau^\star \leq \tau_{\partial\Omega_{\delta,\sigma}}) \right) \\
&\geq \sum_{v \in V_-} \frac{1}{6} \sum_{v \sim w} \frac{1}{2}H_{\delta,\sigma}^{(0)}(w) \tag{3.16}
\end{aligned}
$$

where the inequality (3.16) follows from the inequality (3.15) by multiplying both sides by $1/2$. The first inequality simply uses the fact that for any $n \in \mathbb{N}$ and any $w \in \mathrm{Int}(\Omega_{\delta,n})$, $H_{\delta,n}^{(m)}(w)$ is non-negative. The second part of the proof of [SS05, Proposition 6.3] shows that

$$\mathbb{E}_\delta^{(\Omega,a,b,m)}\left[\mathbb{I}_{\mathcal{Q}_1^-}\left(\sum_{v \in V_-} \frac{1}{6} \sum_{v \sim w} \frac{1}{2}H_{\sigma,\delta}^{(0)}(w) \right) \right] \geq c_1 \mathbb{P}_\delta^{(\Omega,a,b,m)}(\mathcal{Q}_1^-)$$

where $c_1 > 0$ is a universal constant. This inequality together with (3.14) and the optional stopping theorem argument explained in (3.11) yield that

$$c_1 \mathbb{P}_\delta^{(\Omega,a,b,m)}(\mathcal{Q}_1^-) \leq \tilde{K}_1\left(\frac{r}{R}\right)^{\hat{\alpha}_1} + 2^{\hat{\alpha}_1/2}\tilde{K}_1\left(\frac{r}{R}\right)^{\hat{\alpha}_1/2} + \overline{m}_d^2 \bar{D}^2 \frac{r}{R},$$

which, as explained above, implies the inequality (3.9). $\qquad\square$

To complete the proof of Claim 3.2.7, we must prove the auxiliary lemma that we used along the way.

Proof of Lemma 3.2.10. Using the same notations as in the proof of Claim 3.2.7, we want to upper bound the quantity

$$\|K_{\delta,0}^{(\overline{m}_d)}\| := \sum_{v \in V_-} \frac{1}{6} \sum_{v \sim w} \mathbb{P}_w^{(\overline{m}_d)}(\tau^\star \leq \tau_{\partial\Omega_\delta}).$$

Let us express this quantity in terms of the integral of a functional with respect to the excursion measure $\nu_{(\Omega_\delta, E_-, E_\delta)}$, where E_δ is the set of edges in Ω_δ whose endpoint is in $\partial\Omega_\delta$. We have that

$$
\begin{aligned}
\|K_{\delta,0}^{(\overline{m}_d)}\| &= \sum_{v \in V_-} \frac{1}{6} \sum_{v \sim w} \mathbb{E}_w^{(0)}\left[\overline{m}_d^2 \delta^2 \sum_{k=0}^{\tau_{\partial\Omega_\delta}} (1 - \overline{m}_d^2\delta^2)^k \right] \\
&= \sum_{v \in V_-} \frac{1}{6} \sum_{v \sim w} \sum_{\omega: w \to \partial\Omega_\delta} \overline{m}_d^2 \delta^2 \left[\sum_{k=0}^{|\omega|} (1 - \overline{m}_d^2\delta^2)^k \right] \mathbb{P}_\delta^{(0)}(\omega)
\end{aligned}
$$

where $\mathbb{P}_\delta^{(0)}(\omega)$ denotes the probability that a (non-massive) random walk on $\delta\mathbb{T}$ traces the path ω, $\mathbb{E}_w^{(0)}$ denotes the expectation with respect to (non-massive) random walk started at w and where, for $w \in \Omega_\delta$, we write $\omega: w \to \partial\Omega_\delta$ to indicate that ω is a path from w to $\partial\Omega_\delta$ in Ω_δ. We continue with this expansion on paths to obtain that

$$
\begin{aligned}
\|K_{\delta,0}^{(\overline{m}_d)}\| &= \sum_{v \in V_-} \sum_{\omega: v \to \partial\Omega_\delta} \overline{m}_d^2 \delta^2 \left[\sum_{k=1}^{|\omega|} (1 - \overline{m}_d^2\delta^2)^k \right] \frac{1}{6}\mathbb{P}_\delta^{(0)}(\omega_{|\geq 1}) \\
&= \sum_{\omega: E_- \to \partial\Omega_\delta} \overline{m}_d^2 \delta^2 \left[\sum_{k=1}^{|\omega|} (1 - \overline{m}_d^2\delta^2)^k \right] \mathbb{P}_\delta^{(0)}(\omega) \\
&= \int \overline{m}_d^2 \delta^2 \left[\sum_{k=1}^{|\omega|} (1 - \overline{m}_d^2\delta^2)^k \right] d\nu_{(\Omega_\delta, E_-, E_\delta)}(\omega)
\end{aligned}
$$

where $\omega : E_- \to \partial\Omega_\delta$ is the set of path starting with an edge of E_- and ending with an edge of E_δ and $\omega_{|\geq 1}$ denotes the path ω minus its first edge. The above representation of $\|K_{\delta,0}^{(\overline{m}_d)}\|$ is useful as the discrete excursion measure $\nu_{(\Omega_\delta, E_-, E_\delta)}$ is well-understood. To obtain a bound on this integral with respect to this excursion measure, we are going to split the set of excursions into two disjoint sets: the set of excursions that remain in a well-chosen ball $B(z_0, \tilde{r})$ and the set of excursions which exit this ball. The radius $\tilde{r}$ of this ball is going to be chosen such that the total mass of excursions that exit in $B(z_0, \tilde{r})$ can be well-controlled while excursions that stay in $B(z_0, \tilde{r})$ are not long enough to have a macroscopic probability to be killed. To find the appropriate radius $\tilde{r}$, it is more convenient to first rescale Ω. So, let us now rescale $\hat\Omega_\delta$, and thus Ω_δ, by $(r\bar{D})^{-1}$ and denote by $\Omega_\delta(r)$ the rescaled version of Ω_δ. $\Omega_\delta(r)$ is a piece of the triangular lattice with meshsize $\tilde\delta := \delta(r\bar{D})^{-1}$. As we want the killing probabilities to agree on Ω_δ and $\Omega_\delta(r)$, we must choose the mass $\tilde{m}^2 = \tilde{m}^2(r)$ on $\Omega_\delta(r)$ such that

$$\tilde{m}^2 \frac{\delta^2}{r^2\bar{D}^2} = \overline{m}_d^2\delta^2, \quad \text{that is} \quad \tilde{m}^2 = \overline{m}_d^2 r^2 \bar{D}^2.$$

Denote by $E_-(r)$, respectively $E_\delta(r)$, the image of E_-, respectively E_δ, after the rescaling. Notice that for any path ω starting in E_- and ending in $E_\delta(r)$, $\nu_{(\Omega_\delta, E_-, E_\delta)}(\omega) = \nu_{(\Omega_\delta(r), E_-(r), E_\delta(r))}(\omega_r)$, where ω_r is the rescaled version of ω. Therefore, we have that

$$\|K_{\delta,0}^{(\overline{m}_d)}\| = \int \tilde{m}^2 \tilde\delta^2 \left[\sum_{k=1}^{|\omega|} (1 - \tilde{m}^2 \tilde\delta^2)^k \right] d\nu_{(\Omega_\delta(r), E_-(r), E_\delta(r))}(\omega).$$

We now use a kind of restriction property of $\nu_{(\Omega_\delta(r), \mathrm{rev}(E_\delta(r)), E_\delta(r))}$ to write

$$\|K_{\delta,0}^{(\overline{m}_d)}\| = \int_{(\omega_0\omega_1) \in E_-(r),\, e(\omega) \in E_\delta(r)} \tilde{m}^2 \tilde\delta^2 \left[\sum_{k=1}^{|\omega|} (1 - \tilde{m}^2 \tilde\delta^2)^k \right] d\nu_{(\Omega_\delta(r), \mathrm{rev}(E_\delta(r)), E_\delta(r))}(\omega)$$

where for a path ω, $(\omega_0\omega_1)$ is the first edge traversed by ω and $e(\omega)$ is the last one. Denote by z_0' the image of z_0 by the rescaling $(r\bar{D})^{-1}$ and define the following sets of paths

$$\mathcal{P}_{ext}(r) = \left\{ \omega : (\omega_0\omega_1) \in E_-(r),\, e(\omega) \in E_\delta(r),\, \exists k \text{ such that } \omega_k \notin B\left(z_0', \frac{1}{\sqrt{r\bar{D}}}\right) \right\}$$

$$\mathcal{P}_{in}(r) = \left\{ \omega : (\omega_0\omega_1) \in E_-(r),\, e(\omega) \in E_\delta(r),\, \forall k\, \omega_k \in B\left(z_0', \frac{1}{\sqrt{r\bar{D}}}\right) \right\}.$$

We then have that

$$\|K_{\delta,0}^{(\overline{m}_d)}\| = \tilde{m}^2 \tilde\delta^2 \int_{\mathcal{P}_{ext}(r)} \left[\sum_{k=1}^{|\omega|} (1 - \tilde{m}^2 \tilde\delta^2)^k \right] d\nu_{(\Omega_\delta(r), \mathrm{rev}(E_\delta(r)), E_\delta(r))}(\omega)$$

$$+ \tilde{m}^2 \tilde\delta^2 \int_{\mathcal{P}_{in}(r)} \left[\sum_{k=1}^{|\omega|} (1 - \tilde{m}^2 \tilde\delta^2)^k \right] d\nu_{(\Omega_\delta(r), \mathrm{rev}(E_\delta(r)), E_\delta(r))}(\omega)$$

$$\leq \nu_{(\Omega_\delta(r), \mathrm{rev}(E_\delta(r)), E_\delta(r))}\left(\mathcal{P}_{ext}(r)\right) + \tilde{m}^2 \tilde\delta^2 \int_{\mathcal{P}_{in}(r)} |\omega| d\nu_{(\Omega_\delta(r), \mathrm{rev}(E_\delta(r)), E_\delta(r))}(\omega). \quad (3.17)$$

Using this upper bound, we are going to show that,

$$\|K_{\delta,0}^{\overline{m}_d}\| \leq 2^{\hat\alpha_1/2} \tilde{K}_1 \left(\frac{r}{R}\right)^{\hat\alpha_1/2} + 2\overline{m}_d^2 \bar{D}^2 \frac{r}{R},$$

where $\tilde{K}_1$ and $\hat\alpha_1$ are as in (3.13). First, since for $(z_0, r, R) \in \mathcal{G}_1$, $10/\bar{D} \leq 1/\sqrt{r\bar{D}}$, the same arguments as those used in the first part of the proof of [SS05, Proposition 6.3] show that

$$\nu_{(\Omega_\delta(r), \mathrm{rev}(E_\delta(r)), E_\delta(r))}\left(\mathcal{P}_{ext}(r)\right) \leq \tilde{K}_1 \left(\frac{1/\bar{D}}{1/\sqrt{r\bar{D}}}\right)^{\hat\alpha_1} = \tilde{K}_1 \left(\frac{r}{\bar{D}}\right)^{\hat\alpha_1/2}, \quad (3.18)$$

where $\tilde{K}_1$ and $\hat\alpha_1$ are as in (3.13). Since for $(z_0, r, R) \in \mathcal{G}_1$, the ratio $r/\bar{D}$ is upper bounded by $2r/R$ because $R \leq 2\bar{D}$, it thus only remains to upper bound the second term in the sum on the

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right-hand side of (3.17). For convenience, let us set $B_\delta(r) = B(z, (r\bar{D})^{-1/2}) \cap \delta\mathbb{T}$. We first use the Fubini-Tonelli theorem to write

$$\tilde{m}^2\tilde{\delta}^2 \int_{\mathcal{P}_{in}(r)} |\omega| d\nu_{(\Omega_\delta(r),\mathrm{rev}(E_\delta(r)),E_\delta)(r)}(\omega) = \tilde{m}^2\tilde{\delta}^2 \int_{\mathcal{P}_{in}(r)} \left[\sum_{x \in B_\delta(r)} n_x(\omega) \right] d\nu_{(\Omega_\delta(r),\mathrm{rev}(E_\delta(r)),E_\delta(r))}(\omega)$$

$$= \tilde{m}^2\tilde{\delta}^2 \sum_{x \in B_\delta(r)} \int_{\mathcal{P}_{in}(r)} n_x(\omega) d\nu_{(\Omega_\delta(r),\mathrm{rev}(E_\delta(r)),E_\delta(r))}(\omega).$$

By Lemma 3.2.3, for any $x \in \Omega_\delta$,

$$\int n_x(\omega) d\nu_{(\Omega_\delta(r),\mathrm{rev}(E_\delta(r)),E_\delta)(r)}(\omega) = 1.$$

This yields that

$$\tilde{m}^2\tilde{\delta}^2 \int_{\mathcal{P}_{in}(r)} |\omega| d\nu_{(\Omega_\delta(r),\mathrm{rev}(E_\delta(r)),E_\delta)(r)}(\omega) \leq \tilde{m}^2\tilde{\delta}^2 \sum_{x \in B_\delta(r)} 1$$

$$= \frac{\tilde{m}^2}{r\bar{D}}\tilde{\delta}^2\tilde{\delta}^{-2}.$$

Using that $\tilde{m}^2 = \overline{m}_d^2 r^2 \bar{D}^2$ and the upper bound (3.18), we thus obtain that

$$\|K_{\delta,0}^{(\overline{m}_d)}\| \leq \tilde{K}_1 \left(\frac{r}{\bar{D}} \right)^{\hat{\alpha}_1/2} + \overline{m}_d^2 \bar{D} r.$$

We observe that for $(z_0, r, R) \in \mathcal{G}_1$, the ratio r/R is lower bounded by $r/2\bar{D}$. Hence, it follows from the inequality above that

$$\|K_0^{(\overline{m}_d)}\| \leq \tilde{K}_1 \left(\frac{r}{\bar{D}} \right)^{\hat{\alpha}_1/2} + \overline{m}_d^2 \bar{D}^2 \times \frac{r}{\bar{D}}$$

$$\leq 2^{\hat{\alpha}_1/2} \tilde{K}_1 \left(\frac{r}{R} \right)^{\hat{\alpha}_1/2} + 2\overline{m}_2^2 \bar{D}^2 \times \frac{r}{R},$$

which, after multiplying both sides by $1/2$, is exactly the claim of Lemma 3.2.10. $\qquad\square$

We now turn to the proof of Claim 3.2.8.

Proof of Claim 3.2.8. As in the proof of Claim 3.2.7, we set $\bar{D} = 2\bar{R}$. Let us assume that (z_0, r, R) is in the following subset of $\mathbb{C} \times \mathbb{R}_+ \times \mathbb{R}_+$

$$\mathcal{G}_2 := \{(z_0, r, R) : \mathrm{dist}(z_0, \partial\hat{\Omega}_\delta) \leq \bar{D}, \ R \leq 2\bar{D}, \ 0 < 10^4 r \leq R\}.$$

In this case, we will see that there exist universal constants $\tilde{K}_2, c_2, \hat{\alpha}_2 > 0$ such that

$$\mathbb{P}_\delta^{(\Omega,a,b,m)}(\mathcal{Q}_2) \leq \frac{2\tilde{K}_2}{c_2} \left(\frac{r}{R} \right)^{\hat{\alpha}_2} + \frac{2\tilde{K}_2}{c_2} \left(\frac{r}{R} \right)^{\hat{\alpha}_2/4} + 2\frac{\overline{m}_d^2 \bar{D}^2}{c_2} \sqrt{\frac{r}{R}}. \tag{3.19}$$

Therefore, for $(z_0, r, R) \in \mathcal{G}_2$, the inequality (3.8) is satisfied with

$$C_2 := K_2 = \frac{4\tilde{K}_2}{c_2} + \frac{4\overline{m}_d^2}{c_2}\bar{D}^2, \quad \text{and} \quad \alpha_2 = \frac{\hat{\alpha}_2}{4} \wedge \frac{1}{2}. \tag{3.20}$$

If $(z_0, r, R) \notin \mathcal{G}_2$, then notice that, as explained above for the set $\mathcal{G}_1$, either there are no crossings of A^Ω that stay in $\hat{\Omega}_\delta$ or the ratio r/R is lower bounded by 10^{-4}. Taking α_2 as in (3.20) and $C_{b2} > 0$ such that $C_{b2} \times (10^{-4})^{\alpha_2} = 1$, we trivially obtain that for $(z_0, r, R) \notin \mathcal{G}_2$,

$$\mathbb{P}_\delta^{(\Omega,a,b,m)}(\mathcal{Q}_2) \leq C_{b2} \left(\frac{r}{R} \right)^{\alpha_2}$$

since the right-hand side is greater than 1. Setting $K_2 = \max(C_2, C_{b2})$ where C_2 is the constant found in (3.20), we then obtain the inequality (3.8) for any $(z_0, r, R) \in \mathbb{C} \times \mathbb{R}_+ \times \mathbb{R}_+$ with $r \leq R$.

Let us now turn to the proof of the inequality (3.19). Let (z_0, r, R) be in $\mathcal{G}_2$ and consider the annulus $A(z_0, r, R)$. By definition of A^Ω, the boundary arcs of the connected components in $\mathcal{A}_2^\Omega$ that are also arcs of $\partial\Omega_\delta$ are either all contained in $\partial\Omega_\delta^-$ or all contained in $\partial\Omega_\delta^+$. Let us define the martingale that will plays the role of the martingale $(\|\tilde{\nu}_{\delta,n}^{(m)}\|, n \geq 0)$ that we used to estimate $\mathbb{P}_\delta^{(\Omega,a,b,m)}(\mathcal{Q}_1)$. We start by rescaling $\delta\mathbb{T}$ by $1/R$ using the map $f_R : z \mapsto z/R$. We denote by z_0' the image of z_0 by f_R. The image of the ball $B(z_0, r)$, respectively $B(z_0, R)$, under f_R is $B(z_0', r/R)$, respectively $B(z_0', 1)$. We also denote by $\tilde{\Omega}_\delta$ the image of Ω_δ by f_R; this is a piece of the triangular lattice with meshsize $\delta_a := \delta R^{-1}$. For the killing probabilities to agree on $\tilde{\Omega}_\delta$ and Ω_δ, we must choose the mass m_a^2 on $\tilde{\Omega}_\delta$ such that

$$m_a^2 \frac{\delta^2}{R^2} = \overline{m}_d^2 \delta^2, \quad \text{that is} \quad m_a^2 = \overline{m}_d^2 R^2.$$

Observe that we have the inclusions

$$B\left(z_0', \frac{r}{R}\right) \subset B\left(z_0', \frac{1}{3}\sqrt{\frac{r}{R}}\right) \subset B\left(z_0', \sqrt{\frac{r}{R}}\right) \subset B(z_0', 1)$$

where the first inclusion holds because for $(z_0, r, R) \in \mathcal{G}_2$, $10r \leq R$. Moreover, the boundary of a connected component D of $A(z_0', (1/3)\sqrt{r/R}, \sqrt{r/R}) \cap f_R(\mathcal{A}_2^\Omega)$ is made of four arcs. One of them is an arc of $\partial B(z_0', (1/3)\sqrt{r/R})$ and its opposite boundary arc is an arc of $\partial B(z_0', \sqrt{r/R})$. The two other boundary arcs, denoted $b_1(D)$ and $b_2(D)$, that are opposite to each other are either two boundary arcs of $\partial\tilde{\Omega}_\delta^-$ or of $\partial\tilde{\Omega}_\delta^+$. On the event $\mathcal{Q}_2$, let S be the connected component of $A(z_0', (1/3)\sqrt{r/R}, \sqrt{r/R}) \cap f_R(\mathcal{A}_2^\Omega)$ crossed by $\tilde{\gamma}_\delta([0, \sigma])$, where $\tilde{\gamma}_\delta$ denotes the rescaled version of γ_δ. Note that the rescaling f_R does not affect the value of σ. Let $\mathcal{Q}_2^-$, respectively $\mathcal{Q}_2^+$, be the event that $b_1(S), b_2(S) \subset \partial\tilde{\Omega}_\delta^-$, respectively $b_1(S), b_2(S) \subset \partial\tilde{\Omega}_\delta^+$. Notice that we have

$$\mathbb{P}_\delta^{(\Omega,a,b,m)}(\mathcal{Q}_2) = \mathbb{P}_\delta^{(\Omega,a,b,m)}(f_R^{-1}(\mathcal{Q}_2^-)) + \mathbb{P}_\delta^{(\Omega,a,b,m)}(f_R^{-1}(\mathcal{Q}_2^+))$$

and the inequality (3.19) will hold if we can show that there exist universal constants $\tilde{K}_2, c_2, \hat{\alpha}_2 > 0$ such that

$$\mathbb{P}_\delta^{(\Omega,a,b,m)}(f_R^{-1}(\mathcal{Q}_2^\pm)) \leq \frac{\tilde{K}_2}{c_2}\left(\frac{r}{R}\right)^{\hat{\alpha}_2/2} + \frac{\tilde{K}_2}{c_2}\left(\frac{r}{R}\right)^{\hat{\alpha}_2/4} + 2\frac{\overline{m}_d^2 \bar{D}^2}{c_2}\left(\frac{r}{R}\right)^{1/2}.$$

Let us prove this inequality for $\mathbb{P}_\delta^{(\Omega,a,b,m)}(f_R^{-1}(\mathcal{Q}_2^-))$. By symmetry (see the discussion around (3.4)), the proof for $\mathbb{P}_\delta^{(\Omega,a,b,m)}(f_R^{-1}(\mathcal{Q}_2^+))$ is virtually the same. Let $\tilde{E}_-$ denote the set of directed edges in $\tilde{E} = \tilde{E}(\tilde{\Omega}_\delta)$ whose initial vertex is in $\partial\tilde{\Omega}_\delta^- \cap A(z_0', (1/3)\sqrt{r/R}, \sqrt{r/R})$ and on the boundary of $f_R(\mathcal{A}_2^\Omega)$. Denote $\tilde{V}_-$ the set of initial vertices of the edges in $\tilde{E}_-$. Then, by Lemma 3.1.1, the process

$$\|\mathcal{V}_{\delta,n}^{(m)}\| := \sum_{v \in f_R^{-1}(\tilde{V}_-)} \frac{1}{6}\sum_{v \sim w}\left(h_{\delta,n}^{(m)}(w) + \frac{1}{2}\right), \quad n \geq 0,$$

is a non-negative martingale for the filtration $(\mathcal{F}_{\delta,n})_n$. Therefore, by the optional stopping theorem,

$$\|\mathcal{V}_{\delta,0}^{(m)}\| = \mathbb{E}_\delta^{(\Omega,a,b,m)}\left[\|\mathcal{V}_{\delta,\sigma}^{(m)}\|\right] \geq \mathbb{E}_\delta^{(\Omega,a,b,m)}\left[\mathbb{I}_{f_R^{-1}(\mathcal{Q}_2^-)}\|\mathcal{V}_{\delta,\sigma}^{(m)}\|\right].$$

From here, we proceed as in the first case, that is we exhibit an upper bound for $\|\mathcal{V}_{\delta,0}^{(m)}\|$ and a lower bound for $\|\mathcal{V}_{\delta,\sigma}^{(m)}\|$ on the event $f_R^{-1}(\mathcal{Q}_2^-)$. Observe that, almost surely, for any $n \in \mathbb{N}$,

$$\sum_{v \in f_R^{-1}(\tilde{V}_-)} \frac{1}{6}\sum_{v \sim w}\left(h_{\delta,n}^m(w) + \frac{1}{2}\right) \leq \sum_{v \in f_R^{-1}(\tilde{V}_-)} \frac{1}{6}\sum_{v \sim w}\left(H_{\delta,n}^{(0)}(w) + \frac{1}{2}\mathbb{P}_w^{(\overline{m}_d)}(\tau^\star \leq \tau_{\partial\Omega_{\delta,n}})\right)$$

$$= \sum_{v \in \tilde{V}_-} \frac{1}{6}\sum_{v \sim w}\left(H_{\delta,n,R}^{(0)}(w) + \frac{1}{2}\mathbb{P}_w^{(m_a)}(\tau^\star \leq \tau_{\partial\tilde{\Omega}_{\delta,n}})\right),$$

where $H_{\delta,n,R}^{(0)}$ denote the (non-massive) harmonic measure of $\partial\tilde{\Omega}_{\delta,n}^+$. This implies in particular that

$$\|\mathcal{V}_{\delta,0}^{(m)}\| \leq \sum_{v \in \tilde{V}_-} \frac{1}{6}\sum_{v \sim w} H_{\delta,0,R}^{(0)}(w) + \sum_{v \in \tilde{V}_-} \frac{1}{6}\sum_{v \sim w} \frac{1}{2}\mathbb{P}_w^{(m_a)}(\tau^\star \leq \tau_{\partial\tilde{\Omega}_\delta}). \tag{3.21}$$

Our upper bound on $\|\mathcal{V}_{\delta,0}^{(m)}\|$ that will ultimately allow us to establish the inequality (3.19) is a consequence of the following two lemmas.

Lemma 3.2.11. *There exist universal constants $\tilde{K}_2, \hat{\alpha}_2 > 0$ such that, for any $(z_0, r, R) \in \mathcal{G}_2$,*

$$\sum_{v \in \tilde{V}_-} \frac{1}{6} \sum_{v \sim w} H_{\delta,0,R}^{(0)}(w) \leq \tilde{K}_2 \left(\frac{r/R}{\sqrt{r/R}} \right)^{\hat{\alpha}_2} = \tilde{K}_2 \left(\frac{r}{R} \right)^{\hat{\alpha}_2/2}.$$

The proof of Lemma 3.2.11 follows the same lines as that of the first part of the proof of [SS05, Proposition 6.3] but for the sake of completeness, we will sketch the necessary modifications below. The next lemma controls the term in $\|\mathcal{V}_{\delta,0}^{(m)}\|$ arising from the killing and we will prove it using the same strategy as that used to show Lemma 3.2.10.

Lemma 3.2.12. *For the same constants K_2 and $\hat{\alpha}_2$ as in Lemma 3.2.11 and still assuming that $(z_0, r, R) \in \mathcal{G}_2$,*

$$\sum_{v \in \tilde{V}_-} \frac{1}{6} \sum_{v \sim w} \frac{1}{2} \mathbb{P}_w^{(m_a)}(\tau^\star \leq \tau_{\partial \tilde{\Omega}_\delta}) \leq \tilde{K}_2 \left(\frac{r}{R} \right)^{\hat{\alpha}_2/4} + \frac{1}{2} \times 4 \overline{m}_d^2 \bar{D}^2 \left(\frac{r}{R} \right)^{1/2}.$$

We postpone the proof of Lemma 3.2.12 to the end and show how to proceed from here. Observe that combined together, Lemma 3.2.11 and Lemma 3.2.12 yield that

$$\|\mathcal{V}_{\delta,0}^{(m)}\| \leq \tilde{K}_2 \left(\frac{r}{R} \right)^{\hat{\alpha}_2/2} + \tilde{K}_2 \left(\frac{r}{R} \right)^{\hat{\alpha}_2/4} + \frac{1}{2} \times 4 \overline{m}_d^2 \bar{D}^2 \left(\frac{r}{R} \right)^{1/2}$$

where $\tilde{K}_2, \hat{\alpha}_2 > 0$ are universal constants.

It now remains to exhibit a lower bound for $\|\mathcal{V}_{\delta,\sigma}^{(m)}\|$ on the event $f_R^{-1}(\mathcal{Q}_2^-)$. As in (3.16), we first observe that, using Lemma 3.2.5, almost surely,

$$\|\mathcal{V}_{\delta,\sigma}^{(m)}\| \geq \sum_{v \in f_R^{-1}(\tilde{V}_-)} \frac{1}{6} \sum_{w \sim v} \frac{1}{2} H_{\delta,\sigma}^{(0)}(w).$$

To lower bound the right-hand side on the event $f_R^{-1}(\mathcal{Q}_2^-)$, we can use the same reasoning as in the proof of [SS05, Propostion 6.3], using the annulus $B(z_0', \sqrt{r/R}) \setminus B(z_0', (1/3)\sqrt{r/R})$ and the arc $\partial B(z_0', (1/2)\sqrt{r/R})$. Since we have the same scaling relations, we obtain the same lower bound as in [SS05, Propostion 6.3] and therefore, there exists a universal constant $c_2 > 0$ such that

$$\mathbb{E}_\delta^{(\Omega,a,b,m)} \left[\mathbb{I}_{f_R^{-1}(\mathcal{Q}_2^-)} \left(\sum_{v \in f_R^{-1}(\tilde{V}_-)} \frac{1}{6} \sum_{w \sim v} \frac{1}{2} H_{\delta,\sigma}^{(0)}(w) \right) \right] \geq c_2 \mathbb{P}_\delta^{(\Omega,a,b,m)}(f_R^{-1}(\mathcal{Q}_2^-)).$$

Putting everything together, we have thus shown that

$$c_2 \mathbb{P}_\delta^{(\Omega,a,b,m)}(f_R^{-1}(\mathcal{Q}_2^-)) \leq \tilde{K}_2 \left(\frac{r}{R} \right)^{\hat{\alpha}_2/2} + \tilde{K}_2 \left(\frac{r}{R} \right)^{\hat{\alpha}_2/4} + 2 \overline{m}_d^2 \bar{D}^2 \left(\frac{r}{R} \right)^{1/2}$$

which as explained above implies the inequality (3.19). This also concludes the proof of the claim, conditionally on Lemma 3.2.11 and Lemma 3.2.12. □

To complete the proof of Claim 3.2.8, we must prove the two auxiliary lemmas that we used along the way. Let us start with the proof of Lemma 3.2.11. As it is very similar to the first part of the proof of [SS05, Proposition 6.3], we only briefly explain how to adapt the arguments.

Proof of Lemma 3.2.11. We use here the same notations as in the proof of Proposition 3.2.6. Lemma 3.2.11 can be shown by applying the same arguments as in the first part of the proof of [SS05, Proposition 6.3] with respect to the balls $B(z_0', (1/3)\sqrt{r/R})$, $B(z_0', (1/6)\sqrt{r/R})$ and $B(z_0', r/R)$. Indeed, excursions starting from a vertex in V_- and ending with an edge of $\partial \Omega_{\delta,0}^+$ must exit the ball $B(z_0', r/R)$. Moreover, observe that the estimates of Lemma 3.2.3 and Lemma

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3.2.4 do not depend on the meshsize of the graph, and therefore the rescaling f_R is harmless. This yields that there exist universal constants $\tilde{K}_2, \hat{\alpha}_2 > 0$ such that

$$\sum_{v \in \tilde{V}_-} \frac{1}{6} \sum_{v \sim w} H_{\delta,0,R}^{(0)}(w) \leq \tilde{K}_2 \left(\frac{r/R}{\sqrt{r/R}} \right)^{\hat{\alpha}_2} = \tilde{K}_2 \left(\frac{r}{R} \right)^{\hat{\alpha}_2/2}, \tag{3.22}$$

which is the statement of Lemma 3.2.11. $\qquad\square$

Let us finally establish Lemma 3.2.12.

Proof of Lemma 3.2.12. As the proof is similar to that of Lemma 3.2.10, we will be somewhat brief. Using the same notations as in the proof of Claim 3.2.8, we rescale $\tilde{\Omega}_\delta$ by $1/\sqrt{(r/R)}$, which yields the rescaled mass

$$m_b^2 := \overline{m}_d^2 R r.$$

We define the sets $\mathcal{P}_{\mathrm{in}}(r)$ and $\mathcal{P}_{\mathrm{ext}}(r)$ in the same way as before, but with respect to the ball $B(z_0'', (r/R)^{-1/4})$ instead, where z_0'' denotes the image of z_0' after the rescaling. Notice once again that the conclusions of Lemma 3.2.3 hold for any graph, independently of its meshsize. When considering excursions in $\mathcal{P}_{\mathrm{in}}(r)$, we thus obtain a term of the form

$$m_b^2 \times \left(\frac{R}{r} \right)^{1/2}.$$

Using the fact that for $(z_0, r, R) \in \mathcal{G}_2$, $R \leq 2\bar{D}$, we then get that

$$\begin{aligned} m_b^2 \times \left(\frac{R}{r} \right)^{1/2} &= \overline{m}_d^2 R r \left(\frac{R}{r} \right)^{1/2} \\ &= \overline{m}_d^2 \left(\frac{r}{2\bar{D}} \right)^{1/2} \times \sqrt{2} R^{3/2} \bar{D}^{1/2} \\ &\leq 4 \overline{m}_d^2 \bar{D}^2 \left(\frac{r}{R} \right)^{1/2}. \end{aligned}$$

On the other hand, it follows from Lemma 3.2.3 and Lemma 3.2.4 that the term arising from excursions in $\mathcal{P}_{\mathrm{ext}}(r)$ is upper bounded by

$$\tilde{K}_2 \left(\frac{1}{(r/R)^{-1/4}} \right)^{\hat{\alpha}_2} = \tilde{K}_2 \left(\frac{r}{R} \right)^{\hat{\alpha}_2/4},$$

since $10 \leq (r/R)^{-1/4}$ for $(z_0, r, R) \in \mathcal{G}_2$. Therefore, we obtain that

$$\sum_{v \in \tilde{V}_-} \frac{1}{6} \sum_{v \sim w} \frac{1}{2} \mathbb{P}_w^{(m_a)}(\tau^\star \leq \tau_{\partial \tilde{\Omega}_\delta}) \leq \tilde{K}_2 \left(\frac{r}{R} \right)^{\hat{\alpha}_2/4} + \frac{1}{2} \times 4 \overline{m}_d^2 \bar{D}^2 \left(\frac{r}{R} \right)^{1/2},$$

which is exactly the inequality claimed in the statement of Lemma 3.2.12. $\qquad\square$

The proof of Proposition 3.2.6 is now complete. $\qquad\square$

We now turn to the proof of Proposition 3.2.2.

Proof of Proposition 3.2.2. Let us explain how to adapt the arguments of the proof of Proposition 3.2.6 to show an estimate similar to (3.6) for the conditional probabilities $\mathbb{P}_\delta^{(\Omega,a,b,m)}(\cdot \,|\, \gamma([0,\tau]))$, as required by Condition G.3. To this end, fix $0 < \delta < \overline{m}_d^{-1}$ and let τ be a stopping time for the filtration generated by γ. Let $A(z_0, r, R)$ be an annulus. As before, we divide the connected components of A_τ^Ω into two sub-collections $\mathcal{A}_{\tau,1}^\Omega$ and $\mathcal{A}_{\tau,2}^\Omega$ that are defined in the same way as $\mathcal{A}_1^\Omega$ and $\mathcal{A}_2^\Omega$ but with respect to Ω_τ, $\gamma(\tau)$ and b instead of Ω, a and b. Notice that in both cases, we can use the same sets $\mathcal{G}_1$ and $\mathcal{G}_2$ as above. Moreover, in both cases, we can also define the same events and processes as for the time $\tau = 0$ estimates, except that now everything is conditioned on $\gamma([0,\tau])$. More precisely, conditionally on $\gamma([0,\tau])$, the event $\mathcal{Q}$ is defined as for the time 0 estimate but with respect to A_τ^Ω. Conditionally on $\gamma([0,\tau])$, on $\mathcal{Q}$, the stopping time σ is then defined as

the least j such that $v_{\tau+j} \in B_{1,\tau}^{av} \cup B_{2,\tau}^{av}$ and $\gamma([\tau, \tau+j])$ contains a crossing of A_τ^Ω. Conditionally on $\gamma([0, \tau])$, the events $\mathcal{Q}_1^\pm$ and $f_R^-(\mathcal{Q}_2^\pm)$ are then defined as above but with respect to $\mathcal{A}_{\tau,1}^\Omega$, $\mathcal{A}_{\tau,2}^\Omega$ and Ω_τ. The argument based on the optional stopping theorem is replaced by the observation that the martingale property of $(h_{\delta,n}^m(w), n \geq 0)$ for any w together with the domain Markov property (3.5) imply that, almost surely,

$$
\begin{aligned}
\|\tilde{\nu}_{\delta,\tau}^{(m)}\| &= \mathbb{E}_\delta^{(\Omega,a,b,m)}[\|\tilde{\nu}_{\delta,\tau}^{(m)}\| \,|\, \gamma([0,\tau])] \\
&= \mathbb{E}_\delta^{(\Omega,a,b,m)}[\|\tilde{\nu}_{\delta,\tau+\sigma}^{(m)}\| \,|\, \gamma([0,\tau])] \\
&\geq \mathbb{E}_\delta^{(\Omega,a,b,m)}[\mathbb{I}_{\mathcal{Q}_1^-}\|\tilde{\nu}_{\delta,\tau+\sigma}^{(m)}\| \,|\, \gamma([0,\tau])].
\end{aligned}
$$

Similarly, we have that, almost surely,

$$
\begin{aligned}
\|\mathcal{V}_{\delta,\tau}^{(m)}\| &= \mathbb{E}_\delta^{(\Omega,a,b,m)}[\|\mathcal{V}_{\delta,\tau}^{(m)}\| \,|\, \gamma([0,\tau])] \\
&= \mathbb{E}_\delta^{(\Omega,a,b,m)}[\|\mathcal{V}_{\delta,\tau+\sigma}^{(m)}\| \,|\, \gamma([0,\tau])] \\
&\geq \mathbb{E}_\delta^{(\Omega,a,b,m)}[\mathbb{I}_{f_R^{-1}(\mathcal{Q}_2^-)}\|\mathcal{V}_{\delta,\tau+\sigma}^{(m)}\| \,|\, \gamma([0,\tau])].
\end{aligned}
$$

Now observe that the strategy used to upper bound $\|\mathcal{V}_{\delta,0}^{(m)}\|$ and $\|\tilde{\nu}_{\delta,0}^{(m)}\|$ that we used to prove Claim 3.2.7 and Claim 3.2.8 can be apply to obtain an almost sure upper bound on $\|\mathcal{V}_{\delta,\tau}^{(m)}\|$ and $\|\tilde{\nu}_{\delta,\tau}^{(m)}\|$ conditionally on $\gamma_\delta([0, \tau])$, without making any change in the proof. In particular, the upper bounds do not depend on $\text{diam}(\Omega_\tau)$ but only on $\bar{D}$. Similarly, the exact same arguments as in the time $\tau = 0$ case yield the same almost sure lower bound as in the case $\tau = 0$ on $\|\tilde{\nu}_{\delta,\tau+\sigma}^{(m)}\|$ on the event $\mathcal{Q}_1^-$ conditionally on $\gamma([0, \tau])$. The analogous statement holds for $\|\mathcal{V}_{\delta,\tau+\sigma}^{(m)}\|$ on the event $f_R^{-1}(\mathcal{Q}_2^-)$ conditionally on $\gamma([0, \tau])$. This crucially implies that the constants in the upper and lower bounds do not depend on τ and are the same as in the case $\tau = 0$. We can therefore conclude that for the same K and α as in the Proposition 3.2.6, almost surely,

$$
\mathbb{P}_\delta^{(\Omega,a,b,m)}(\gamma([\tau, 1]) \text{ makes a crossing of } A \text{ which is contained in } A_\tau^{\hat{\Omega}_\delta} \,|\, \gamma([0,\tau])) \leq K \left(\frac{r}{R} \right)^\alpha.
$$

$\square$

3.3 Scaling limit of the martingale observable

In this section, we study the scaling limit of the martingale observable $(h_{\delta,n}^m, n \geq 0)$. The results of Section 3.2 show that the sequence $(\gamma_\delta^\mathbb{H})_\delta$ is tight in the topologies $((\text{T.1})) - ((\text{T.3}))$, which implies that $(\gamma_\delta^\mathbb{H})_\delta$ converges along subsequences in these topologies. By [KS17, Corollary 1.8] and [Kar23, Theorem 4.2], if $(\gamma_{\delta_k}^\mathbb{H})_k$ is such a convergent subsequence, then $(\gamma_{\delta_k})_k$ in turn converges weakly in $X(\mathbb{C})$ equipped with the metric d_X to a random curve that is almost surely supported on $\bar{\Omega}$ and has the same law as $\phi^{-1}(\gamma^\mathbb{H})$, provided that for each δ_k, γ_{δ_k} is parametrized by the half-plane capacity of $\gamma_{\delta_k}^\mathbb{H}$. Here, assuming that $(\hat{\Omega}_{\delta_k}, a_{\delta_k}, b_{\delta_k})$ converges in the Carathéodry topology to (Ω, a, b), we show that the corresponding subsequence of massive harmonic functions $(h_{\delta_k}^m)_{\delta_k}$ converges pointwise to the continuous massive harmonic function with the same boundary conditions. After suitable reparametrization, we will prove that this in fact holds almost surely for the time-dependent subsequences $(h_{\delta_k,t(\delta_k)}^m)_{\delta_k}$ in the domains $(\Omega_{\delta_k,t(\delta_k)})_{\delta_k}$. This is established under the assumptions that the time-reparametrized (random) sequence $(\hat{\Omega}_{\delta_k,t(\delta_k)}, a_{t(\delta_k)}, b_{\delta_k})_k$ almost surely converges in the Carathéodory topology to (Ω_t, a_t, b) where $\Omega_t := \Omega \setminus K_t$, with K_t being the hull generated at time t by the limit γ of $(\gamma_{\delta_k})_{\delta_k}$ and $a_t = \gamma(t)$. This will be the key to characterize in Section 3.4 the subsequential limits obtained as a consequence of the tightness of $(\gamma_\delta^\mathbb{H})_\delta$ established in the previous section.

To study the scaling limit of the martingale observable, we choose to use the framework developed by Chelkak and Smirnov in [CS11]. For $\delta > 0$, we define the Laplacian Δ_δ on Ω_δ by, for $f : \Omega_\delta \to \mathbb{R}$ and $z \in \text{Int}(\Omega_\delta)$,

$$
\Delta_\delta f(z) = \frac{1}{A_\delta} \sum_{z \sim w} \tan(\theta)(f(w) - f(z)),
$$

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where as above $A_\delta = c\delta^2$ is the area of a face of the graph dual to Ω_δ and θ is defined as in the definition (3.1) of m_d^2. Let $f : \Omega_\delta \to \mathbb{R}$ be a discrete massive harmonic function with mass m in Ω_δ. It follows from the definition of a discrete massive harmonic function given in (3.2) that f satisfies, for all $z \in \text{Int}(\Omega_\delta)$,

$$\sum_{z \sim w} \frac{1 - m_d^2(z)\delta^2}{6}(f(z) - f(w)) + m_d^2(z)\delta^2 f(z) = 0.$$

Multiplying both sides by $6\tan(\theta)/A_\delta$, this is equivalent to

$$-\Delta_\delta f(z) + m^2(z)f(z) + A_\delta \frac{m^2(z)}{6\tan(\theta)}\Delta_\delta f(z) = 0. \tag{3.23}$$

The following lemma gives a continuity estimate for discrete massive harmonic functions. This estimate will be useful to show precompactness of the family $(h_\delta^m)_\delta$ in the proof of Proposition 3.3.2.

Lemma 3.3.1. *Let $m : \overline{\Omega} \to \mathbb{R}_+$ be a continuous function bounded by some constant $\overline{m} > 0$ in $\overline{\Omega}$. Let $\delta > 0$ be such that $\delta < \overline{m}_d^{-1}$. There exist constants $C, \beta > 0$ depending on $\overline{m}$ such that the following holds. Let H be a positive massive discrete harmonic function with mass m defined in $B(z, 2r) \cap \delta\mathbb{T}$ with $0 < r \le \overline{m}^{-1}$. Then, for any $w_1, w_2 \in B(z, 2r) \cap \delta\mathbb{T}$,*

$$|H(w_1) - H(w_2)| \le C\left(\frac{|w_1 - w_2|}{r}\right)^\beta \max_{w \in B(z,2r) \cap \delta\mathbb{T}} H(w).$$

Proof. The proof is almost identical to that of [BH22, Lemma 3.4], where this lemma is established for constant mass. Indeed, the proof relies on an estimate on the probability that a massive random walk makes a non-contractible loop in an annulus of modulus $1/2$ before exiting it or being killed, when the walk started not too far from the center of the annulus. Therefore, in our setting, we can lower bound this probability by that of the same event taking place when the walk instead has probability $\overline{m}_d^2\delta^2$ to be killed at each step. This latter probability is exactly the probability that is analyzed in the proof of [BH22, Lemma 3.4]. $\square$

Recall that for $\delta > 0$, $\phi_\delta : \hat{\Omega}_\delta \to \mathbb{H}$ is a conformal map such that $\phi_\delta(a_\delta) = 0$ and $\phi_\delta(b_\delta) = \infty$. Recall also that, for $\delta > 0$, we denote by $\gamma_\delta^{\mathbb{H}}$ the curve $\phi_\delta(\gamma_\delta)$. To establish convergence of the martingale observable, it will be more convenient to parametrize the discrete curves $(\gamma_\delta)_\delta$ by the half-plane capacity of their conformal images $(\gamma_\delta^{\mathbb{H}})_\delta$. Indeed, the curves $(\gamma_\delta^{\mathbb{H}})_\delta$ can be considered as continuous curves in $\mathbb{H}$. As such, they can be canonically parametrized by half-plane capacity, as explained in Section 2.1. For $n \ge 0$, let $\mathcal{F}_{\delta,n}$ be the σ-algebra generated by $\gamma_\delta([0,n])$ and, for $t \ge 0$ and $\delta > 0$, let us define the following stopping time for the filtration $(\mathcal{F}_{\delta,n})_n$

$$t(\delta) := \inf\{n \in \mathbb{N} : \text{hcap}(\phi_\delta(\gamma_\delta[0,n])) \ge t\}.$$

We then let $\Omega_{t(\delta)}$ be the connected component of $\Omega_\delta \setminus \gamma_\delta([0, t(\delta)])$ which contains both $a_{t(\delta)}$ and b_δ on its boundary, where $a_{t(\delta)}$ is the last vertex added to $\partial\Omega_{t(\delta)}$.

By the results of Section 3.2, the sequence $(\gamma_\delta^{\mathbb{H}})_\delta$ is tight in the topologies ((T.1))–((T.3)) and each subsequence $(\gamma_{\delta_k})_k$ has a further subsequence which weakly converges in $X(\mathbb{C})$ with the metric d_X to a curve γ supported on $\overline{\Omega}$ (γ may depend on the subsequence at this stage), provided that the curves $(\gamma_{\delta_k})_k$ are parametrized by the half-plane capacitiy of their conformal images $(\gamma_{\delta_k}^{\mathbb{H}})_{\delta_k}$. Moreover, γ has the same law as $\phi^{-1}(\gamma^{\mathbb{H}})$. The space of continuous functions on $[0, \infty)$ is a separable and metrizable space and therefore, by Skorokhod representation theorem, we can suppose that for each weakly convergent subsequence of $(\gamma_{\delta_k})_k$, we also have $\gamma_{\delta_k} \to \gamma$ almost surely. In particular, we can assume that, almost surely, in the Carathéodory sense,

$$(\hat{\Omega}_{t(k)}; a_{t(k)}, b_{\delta_k}) \to (\Omega_t; a_t, b) \quad \text{as } \delta \to 0,$$

where we have set $t(k) := t(\delta_k)$ and Ω_t is the connected component of $\Omega \setminus \gamma([0, t])$ which contains $a_t : \gamma(t)$ and b.

Proposition 3.3.2. *Let $t \ge 0$. Let $(\delta_k)_k$ be a subsequence such that the sequence $(\gamma_{\delta_k}^{\mathbb{H}})_k$ converges weakly in the topologies ((T.1)) – ((T.3)) to a random curve $\gamma^{\mathbb{H}}$, and thus $(\gamma_{\delta_k})_k$ also converges*

weakly to a random curve γ in $X(\mathbb{C})$ equipped with the metric d_X. Then the sequence of discrete massive harmonic functions $(h^m_{\delta_k, t(\delta_k)})_k$ almost surely converges pointwise in $\overline{\Omega_t}$ to the massive harmonic function $h^m_t : \Omega_t \to \mathbb{R}$ solving the Dirichlet problem

$$\begin{cases} [-\Delta + m^2(z)]h^m_t(z) = 0, & z \in \Omega_t \\ h^m_t{}_{|\partial\Omega_t^+} = \frac{1}{2}, & h^m_t{}_{|\partial\Omega_t^-} = \frac{-1}{2}. \end{cases}$$

Proposition 3.3.2 will be a consequence of a general deterministic result. To state this result, let us introduce a few notations, which mimic the setting of Proposition 3.3.2. Let Λ be an open, bounded and simply connected subset of $\mathbb{C}$ and let $a, b \in \partial\Lambda$. We denote by $\partial\Lambda^+$, respectively $\partial\Lambda^-$, the clockwise boundary arc from a to b, respectively counterclockwise boundary arc from a to b. We then approximate Λ by a sequence of graphs $(\Lambda_\delta)_\delta$ where for each $\delta > 0$, Λ_δ is a portion of $\delta\mathbb{T}$. We define $\partial\Lambda_\delta$ as in Section 3.1.1. Recall also that $\hat{\Lambda}_\delta$ denotes the open and simply connected polygonal domain obtained from Λ_δ by taking the union of all open hexagons with side length δ centered at vertices of Λ_δ. As in Section 3.1.1, we obtain two sequences $(a_\delta)_\delta$ and $(b_\delta)_\delta$ that approximate the boundary points a and b. We separate $\partial\Lambda_\delta$ into two subsets, $\partial\Lambda_\delta^+$ and $\partial\Lambda_\delta^-$, where $\partial\Lambda_\delta^-$, respectively $\partial\Lambda_\delta^-$ are defined in a similar fashion as $\partial\Omega_\delta^+$ and $\partial\Omega_\delta^-$. Finally, we let $m : \overline{\Lambda} \to \mathbb{R}_+$ be a continuous function which is bounded by some constant $\overline{m} > 0$. For $\delta < \overline{m}_d^{-1}$, where m_d is defined as in (3.1), we denote by h^m_δ the discrete massive harmonic function in Λ_δ with boundary value $1/2$ on $\partial\Lambda_\delta^+$ and $-1/2$ on $\partial\Lambda_\delta^-$.

Lemma 3.3.3. *In the above setting, assume that $(\hat{\Lambda}_\delta; a_\delta, b_\delta)$ converges to $(\Lambda; a, b)$ in the Carathéodory sense. Then the sequence of functions $(h^m_\delta)_\delta$ converges pointwise in $\overline{\Lambda}$ to the massive harmonic function $h^m : \overline{\Lambda} \to \mathbb{R}$ solving the Dirichlet problem*

$$\begin{cases} [-\Delta + m^2(z)]h^m(z) = 0, & z \in \Lambda \\ h^m_{|\partial\Lambda^+} = \frac{1}{2}, & h^m_{|\partial\Lambda^-} = \frac{-1}{2}. \end{cases}$$

Proof. We first observe that, for any $\delta < \overline{m}_d^{-1}$ and any $z \in \text{Int}(\Lambda_\delta) \cup \partial\Lambda_\delta$,

$$|h^m_\delta(z)| \leq \frac{1}{2},$$

that is the sequence $(h^m_\delta)_{\delta < \overline{m}_d^{-1}}$ is uniformly bounded. $(h^m_\delta)_{\delta < \overline{m}_d^{-1}}$ is also equicontinuous. Indeed, we have, for any $z \in \text{Int}(\Lambda_\delta)$,

$$h^m_\delta(z) = \frac{1}{2}\left(H^{(m)}_\delta(z) - \tilde{H}^{(m)}_\delta(z) \right),$$

where $H^{(m)}_\delta(z)$, respectively $\tilde{H}^{(m)}_\delta(z)$, is the discrete massive harmonic measure of $\partial\Lambda_\delta^+$, respectively $\partial\Lambda_\delta^-$, seen from z. Therefore, for any $\delta < \overline{m}_d^{-1}$, applying Lemma 3.3.1 to both $H^{(m)}_\delta$ and $\tilde{H}^{(m)}_\delta$, we can see that for any $z \in \Lambda_\delta$ and any $w_1, w_2 \in B(z, 2r) \cap \delta\mathbb{T}$ with $r \leq \overline{m}_d^{-1} \wedge \text{dist}(z, \partial\Lambda_\delta)$,

$$\begin{aligned} |h^m_\delta(w_1) - h^m_\delta(w_2)| &\leq \frac{1}{2}\left(|H^{(m)}_\delta(w_1) - H^{(m)}_\delta(w_2)| + |\tilde{H}^{(m)}_\delta(w_1) - \tilde{H}^{(m)}_\delta(w_2)| \right) \\ &\leq \frac{C}{2}\left(\frac{|w_1 - w_2|}{r} \right)^\beta \\ &\quad \times \left(\max_{w \in B(z,2r) \cap \delta\mathbb{T}} H^{(m)}_\delta(w) + \max_{w \in B(z,2r) \cap \delta\mathbb{T}} \tilde{H}^{(m)}_\delta(w) \right) \\ &\leq C\left(\frac{|w_1 - w_2|}{r} \right)^\beta. \end{aligned}$$

By the Arzela-Ascoli theorem, uniform boundedness and equicontinuity of the sequence $(h^m_\delta)_{\delta < \overline{m}_d^{-1}}$ implies that there exists a function $h^m : \Lambda \to \mathbb{R}$ and a subsequence $(h^m_{\delta_k})_k$ such that $(h^m_{\delta_k})_k$ converges uniformly on compact subsets of Λ to h^m. Let us show that h^m is the massive harmonic function with mass m in Λ and boundary conditions $-1/2$ on $\partial\Lambda^-$ and $1/2$ on $\partial\Lambda^+$.

We first prove that h^m is massive harmonic with mass m in Λ. Let $\varphi : \Lambda \to \mathbb{R}$ be a smooth and compactly supported function on Λ. We then have

$$\int_\Lambda h^m(z)(-\Delta\varphi(z) + m^2(z)\varphi(z))dz = \lim_{\delta = \delta_k \to 0} \sum_{z \in \text{Int}(\Lambda_\delta)} A_\delta(h^m)^\delta(z)(-(\Delta\varphi)^\delta(z) + m^2(z)\varphi^\delta(z))$$

where, as in [CS11, Section 2], for $f : \Lambda \to \mathbb{R}$ and $\delta > 0$, $f^\delta : \Lambda_\delta \to \mathbb{R}$ is defined as the restriction of f to Λ_δ. Using [CS11, Lemma 2.2], we get that

$$\lim_{\delta = \delta_k \to 0} \sum_{z \in \mathrm{Int}(\Lambda_\delta)} A_\delta (h^m)^\delta(z)(-(\Delta\varphi)^\delta(z) + m^2(z)\varphi^\delta(z))$$

$$= \lim_{\delta = \delta_k \to 0} \sum_{z \in \mathrm{Int}(\Lambda_\delta)} A_\delta h_\delta^m(z)(-\Delta_\delta \varphi^\delta(z) + m^2(z)\varphi^\delta(z)).$$

We also have that

$$\lim_{\delta = \delta_k \to 0} A_\delta \sum_{z \in \mathrm{Int}(\Lambda_\delta)} A_\delta h_\delta^m(z) \frac{m^2(z)}{6\tan(\theta)} \Delta_\delta \varphi^\delta(z) = 0$$

since $A_\delta \to 0$ as $\delta \to 0$ and

$$\lim_{\delta = \delta_k \to 0} \sum_{z \in \mathrm{Int}(\Lambda_\delta)} A_\delta h_\delta^m(z) \frac{m^2(z)}{6\tan(\theta)} \Delta_\delta \varphi^\delta(z) = \int_\Lambda h^m(z) \frac{m^2(z)}{6\tan(\theta)} \Delta\varphi(z) dz.$$

Therefore, we have that

$$\int_\Lambda h^m(z)(-\Delta\varphi(z) + m^2(z)\varphi(z)) dz = \lim_{\delta = \delta_k \to 0} \sum_{z \in \mathrm{Int}(\Lambda_\delta)} A_\delta h_\delta^m(z)(-\Delta_\delta \varphi^\delta(z) + m^2(z)\varphi^\delta(z))$$

$$+ A_\delta \sum_{z \in \mathrm{Int}(\Lambda_\delta)} A_\delta h_\delta^m(z) \frac{m^2(z)}{6\tan(\theta)} \Delta_\delta \varphi^\delta(z).$$

By discrete integration by parts, this implies that

$$\int_\Lambda h^m(z)(-\Delta\varphi(z) + m^2(z)\varphi(z)) = \lim_{\delta = \delta_k \to 0} \sum_{z \in \mathrm{Int}(\Lambda_\delta)} A_\delta \varphi^\delta(z) \Bigg[-\Delta_\delta h_\delta^m(z) + m^2(z) h_\delta^m(z)$$

$$+ A_\delta \frac{m^2(z)}{6\tan(\theta)} \Delta_\delta h_\delta^m(z) \Bigg].$$

Since h_δ^m is discrete massive harmonic with mass m, by (3.23), the right-hand side is equal to 0 and thus,

$$\int_\Lambda h^m(z)(-\Delta\varphi(z) + m^2(z)\varphi(z)) dz = 0.$$

Therefore, h^m is weakly massive harmonic with mass m in Λ. But this implies that h^m is in fact massive harmonic with mass m in Λ.

We now want to show that h^m is equal to the massive harmonic function with mass m and boundary conditions $1/2$ on $\partial\Lambda^+$ and $-1/2$ on $\partial\Lambda^-$. Recall that, for any $0 < \delta < \overline{m}_d^{-1}$ and any $z \in \mathrm{Int}(\Lambda_\delta) \cup \partial\Lambda_\delta$,

$$h_\delta^m(z) = \frac{1}{2}\left(H_\delta^{(m)}(z) - \tilde{H}_\delta^{(m)}(z) \right). \tag{3.24}$$

The same reasoning as above shows that there exist subsequences $(H_{\delta_q}^{(m)})_q$ and $(\tilde{H}_{\delta_r}^{(m)})_r$ and functions $H^m : \Lambda \to \mathbb{R}$ and $\tilde{H}^m : \Lambda \to \mathbb{R}$ such that $(H_{\delta_q}^m)_q$, respectively $(\tilde{H}_{\delta_r}^m)_r$, converges uniformly on compact subsets of Λ to H^m, respectively $\tilde{H}^m$. Moreover, H^m and $\tilde{H}^m$ are both massive harmonic with mass m in Λ. Let us show that H^m, respectively $\tilde{H}^m$, is in fact the massive harmonic measure of $\partial\Lambda^+$, respectively $\partial\Lambda^-$. As the proof is virtually the same for both H^m and $\tilde{H}^m$, we only detail the arguments for H^m.

Observe that by the weak Beurling estimate for (massless) harmonic measure, see e.g. [CS11, Proposition 2.11], we have, for any $z \in \mathrm{Int}(\Lambda_{\delta_q})$,

$$0 \le H_{\delta_q}^{(m)}(z) \le \mathrm{const}\left(\frac{\mathrm{dist}(z, \partial\Lambda_{\delta_q})}{\mathrm{dist}(z, \partial\Lambda_{\delta_q}^+)} \right)^{\hat{\beta}}$$

where the constant and $\hat{\beta} > 0$ are independent of δ_q. Passing to the limit $\delta_q \to 0$, we obtain that, for any $z \in \Lambda$,

$$0 \le H^{(m)}(z) \le C\left(\frac{\mathrm{dist}(z, \partial\Lambda)}{\mathrm{dist}(z, \partial\Lambda^+)} \right)^{\hat{\beta}}.$$

We can thus conclude that, for $z \in \partial\Lambda^-$, $H^m(z) = 0$. Now, recall that, for any $q \in \mathbb{N}$ and $z \in \mathrm{Int}(\Lambda_{\delta_q})$,

$$\tilde{H}^{(m)}_{\delta_q} = 1 - H^{(m)}_{\delta_q}(z) - \mathbb{P}^{(m)}_z(\tau^\star \leq \tau_{\partial\Lambda_{\delta_q}}). \tag{3.25}$$

Once again, by the weak Beurling estimate for (massless) harmonic measure, we have that, for any $z \in \mathrm{Int}(\Lambda_{\delta_q})$,

$$0 \leq \tilde{H}^{(m)}_{\delta_q}(z) \leq \mathrm{const}\left(\frac{\mathrm{dist}(z, \partial\Lambda_{\delta_q})}{\mathrm{dist}(z, \partial\Lambda^-_{\delta_q})}\right)^{\hat{\beta}}. \tag{3.26}$$

Moreover, if $(z_{\delta_q})_q$ is a sequence of points such that for each $q \in \mathbb{N}$, $z_{\delta_q} \in \delta_q \mathbb{T}$ and $z_{\delta_q} \to z$, then

$$\liminf_{\delta=\delta_q \to 0} \mathbb{P}^{(m)}_{z_\delta}(\tau^\star \leq \tau_{\partial\Lambda_{\delta_q}}) \leq 1 - \mathbb{I}_{z \notin \Lambda}.$$

Taking the $\limsup$ as $\delta_q \to 0$ on both sides of the equality (3.25), this yields that

$$1 - H^m(z) - (1 - \mathbb{I}_{z \notin \Lambda}) \leq 1 - H^m(z) - \liminf_{\delta=\delta_q \to 0} \mathbb{P}^z_\delta(\tau^\star \leq \tau_{\partial\Lambda_\delta}) \leq C\left(\frac{\mathrm{dist}(z, \partial\Lambda)}{\mathrm{dist}(z, \partial\Lambda^-)}\right)^{\hat{\beta}}$$

which implies that

$$\limsup_{z \to \partial\Lambda^+} 1 - H^m(z) \leq 0.$$

On the other hand, since for any $\delta < \overline{m}_d^{-1}$ and $z \in \mathrm{Int}(\Lambda_\delta) \cup \partial\Lambda_\delta$, $H^m_\delta(z) \leq 1$, we have that

$$0 \leq \limsup_{z \to \partial\Lambda^+} \lim_{\delta=\delta_q \to 0} 1 - H^{(m)}_\delta(z) = \limsup_{z \to \partial\Lambda^+} 1 - H^m(z).$$

Therefore, we obtain that $\lim_{z \to \partial\Lambda^+} 1 - H^m(z) = 0$, which yields that H^m is equal to 1 on $\partial\Lambda^+$.

The above arguments show that in fact the whole sequences $(H^{(m)}_\delta)_\delta$ and $(\tilde{H}^{(m)}_\delta)_\delta$ converge pointwise to the massive harmonic measure of $\partial\Lambda^+$ and $\partial\Lambda^-$, respectively. Recalling the decomposition (3.24) of h^m_δ, we can thus conclude that $(h^m_\delta)_\delta$ converges pointwise along any subsequence, and thus converges pointwise, to the function

$$z \in \Lambda \mapsto \frac{1}{2}\left(H^m(z) - \tilde{H}^m(z)\right).$$

This function is massive harmonic with mass m in Λ and has boundary conditions $1/2$ on $\partial\Lambda^+$ and $-1/2$ on $\partial\Lambda^-$. By uniqueness of such massive harmonic functions, we obtain that the limit h^m of h^m_δ is indeed solution to the Dirichlet problem of the statement of the lemma. $\qquad\square$

3.4 Characterization of the limiting continuum curve

Recall that we have a random sequence $(\gamma_\delta)_\delta$ of curves where for each $\delta > 0$, γ_δ is distributed according to $\mathbb{P}^{(\Omega,a,b,m)}_\delta$. For each $\delta > 0$, we also have a conformal map $\phi_\delta: \hat{\Omega}_\delta \to \mathbb{H}$ such that $\phi_\delta(a_\delta) = 0$ and $\phi_\delta(b_\delta) = \infty$ and we denote by $\gamma^{\mathbb{H}}_\delta$ the curve $\phi_\delta(\gamma_\delta)$. We have shown in Section 3.2 that the sequence $(\gamma^{\mathbb{H}}_\delta)_\delta$ is tight in the topologies ((T.1)) – ((T.3)), which implies that $(\gamma^{\mathbb{H}}_\delta)_\delta$ converges weakly along subsequences in these topologies. If $(\gamma^{\mathbb{H}}_{\delta_k})_k$ is such a convergent subsequence, then its limit $\gamma^{\mathbb{H}}$ is a random non-self crossing curve in $\mathbb{H}$ whose time evolution can therefore be described by the Loewner equation, see (2.1). Moreover, in this case, $(\gamma_{\delta_k})_k$ converges weakly in $X(\mathbb{C})$ equipped with the metric d_X to a random curve that is almost surely supported on $\overline{\Omega}$ and has the same law as $\phi^{-1}(\gamma^{\mathbb{H}})$, provided that for each δ_k, γ_{δ_k} is parametrized by the half-plane capacity of $\gamma^{\mathbb{H}}_{\delta_k}$. Our goal here is to characterize the limits of such subsequences, or equivalently the Loewner chain describing their time evolution: we are going to show that this limiting Loewner chain is characterized by the martingale property of a certain massive harmonic function. Before stating precisely the result, in Section 3.4.1, we introduce a few notations and recall how to express massive harmonic functions in terms of their harmonic counterparts. The characterization of the limiting Loewner chain is then stated in Section 3.4.2 and Section 3.4.3 is devoted to its proof. In Section 3.4.4, we reformulate Theorem 1.1.1 and show how to prove it by combining the results of Section 3.2.2, Section 3.3 and Section 3.4.2.

3.4.1 Massive harmonic functions and massive Poisson kernels

Let $\Lambda \subset \mathbb{C}$ be a bounded, open and simply connected domain. The Laplace operator $-\Delta$ in Λ with Dirichlet boundary conditions has a unique Green function G_Λ, which is defined as its inverse in the sense of distributions, that is for $z \in \Lambda$, $-\Delta G_\Lambda(z, \cdot) = \delta_z(\cdot)$. In what follows, we will be interested in quantities related to the massive Laplace operator $-\Delta + m^2$ in Λ with Dirichlet boundary conditions. This operator acts on a function $f \in \mathcal{C}_c^\infty(\Lambda)$ as $[-\Delta + m^2(z)]f(z) = -\Delta f(z) + m^2(z)f(z)$. It also has a unique Green function G_Λ^m defined as its inverse in the sense of distributions, that is, for $z \in \Lambda$, $[-\Delta + m^2(\cdot)]G_\Lambda^m(z, \cdot) = \delta_z(\cdot)$. Existence and uniqueness of this Green function follows from the fact that $-\Delta + m^2(\cdot)$ is a uniformly elliptic operator, see for example [WGH63]. We call G_Λ^m the massive Green function (with mass m in Λ). Since for any $z \in \Lambda$, $-\Delta G_\Lambda^m(z, \cdot) = \delta_z(\cdot) - m^2(\cdot)G_\Lambda^m(z, \cdot)$ (in the sense of distributions), G_Λ^m is related to G_Λ as follows: for $z, w \in \Lambda$,

$$G_\Lambda^m(z, w) = G_\Lambda(z, w) - \int_\Lambda m^2(y)G_\Lambda(z, y)G_\Lambda^m(w, y)dy. \tag{3.27}$$

Indeed, one can check that the right-hand side of this equality is the inverse of $-\Delta + m^2$ in the sense of distributions, which thus establishes (3.27). Moreover, G_Λ^m is conformally covariant in the following sense. Let $\phi : \Lambda \to \tilde{\Lambda}$ be a conformal map and set, for $y \in \tilde{\Lambda}$, $\tilde{m}^2(y) = |(\phi^{-1})'(y)|^2 m^2(\phi^{-1}(y))$. Then, for any $z, w \in \Lambda$,

$$G_\Lambda^m(z, w) = G_{\tilde{\Lambda}}^{\tilde{m}}(\phi(z), \phi(w)). \tag{3.28}$$

This equality is a consequence of the conformal covariance of the two-dimensional massive (killed) Brownian motion. Indeed, as in the case of standard Brownian motion, if $A \subset \Lambda$ is an open set, then

$$\int_A G_\Lambda^m(z, w)dw = \mathbb{E}_z^{(m)}\left[\mathbb{I}_{\tau^\star > \tau_\Lambda} \int_0^{\tau_\Lambda} \mathbb{I}_A(B_t)dt\right] \tag{3.29}$$

where under $\mathbb{E}_z^{(m)}$, B has the law of a massive Brownian motion with mass m started at $z \in \Lambda$, τ_Λ is its first exit time of Λ and $\tau^\star$ is its killing time. This equality can be deduced from [FOT11, Theorem 4.2.6] and [FOT11, Section 6.1] by observing that in the terminology of Dirichlet forms, in the bilinear form

$$\mathcal{E}(u, v) = \int_D \nabla u(z) \nabla v(z)dz + \int_D m^2(z)u(z)v(z)dz$$

associated to the generator $-\Delta + m^2(\cdot)$ of massive Brownian motion, $m^2(z)dz$ is a killing measure. This allows to connect massive Brownian motion and Brownian motion as claimed in (3.29).

Using the massive Green function G_Λ^m, one can express massive harmonic functions in Λ in terms of their harmonic counterparts. More precisely, let $f : \partial\Lambda \to \mathbb{R}$ be a piecewise smooth function with finitely many discontinuity points. Let h be the unique harmonic extension of f in Λ. Let h^m be the unique massive harmonic extension of f in Λ, that is h^m is the unique solution to the boundary value problem

$$\begin{cases} [-\Delta + m^2(z)]u(z) = 0 & \text{in } \Lambda \\ u = f & \text{on } \partial\Lambda. \end{cases}$$

Then, it is easy to see that, for any $z \in \Lambda$,

$$h^m(z) = h(z) - \int_\Lambda m^2(w)h(w)G_\Lambda^m(z, w)dw \tag{3.30}$$

Indeed, this follows from the facts that $[-\Delta + m^2(z)]h(z) = m^2(z)h(z)$ for $z \in \Lambda$, $h(z) = f(z)$ for $z \in \partial\Lambda$ and that, by definition of G_Λ^m, the function

$$\zeta^m : z \mapsto \int_\Lambda m^2(w)h(w)G_\Lambda^m(z, w)dw$$

is the unique solution to the boundary value problem

$$\begin{cases} [-\Delta + m^2(z)]\zeta^m(z) = m^2(z)h(z) & \text{in } \Lambda \\ \zeta^m = 0 & \text{on } \partial\Lambda. \end{cases}$$

Note that h^m can also be rewritten in the form

$$h^m(z) = h(z) - \int_\Lambda m^2(w)h^m(w)G_\Lambda(z,w)dw. \tag{3.31}$$

Indeed, we have that, using the relation between G_Λ and G_Λ^m and Fubini's theorem (Λ is bounded by assumption),

$$\begin{aligned}
\int_\Lambda m^2(w)G_\Lambda^m(z,w)h(w)dw &= \int_\Lambda m^2(w)\left[G_\Lambda(z,w) - \int_\Lambda m^2(y)G_\Lambda(z,y)G_\Lambda^m(y,w)dy\right]h(w)dw \\
&= \int_\Lambda m^2(w)G_\Lambda(z,w)h(w)dw \\
&\quad - \int_{\Lambda\times\Lambda} m^2(y)m^2(w)G_\Lambda(z,y)G_\Lambda^m(y,w)h(w)dwdy \\
&= \int_\Lambda m^2(y)G_\Lambda(z,y)h(y)dy \\
&\quad - \int_\Lambda m^2(y)G_\Lambda(z,y)\int_\Lambda m^2(w)G_\Lambda^m(y,w)h(w)dw \\
&= \int_\Lambda m^2(y)G_\Lambda(z,y)\left[h(y) - \int_\Lambda m^2(w)G_\Lambda^m(y,w)h(w)dw\right]dy \\
&= \int_\Lambda m^2(y)G_\Lambda(z,y)h^m(y)dy.
\end{aligned}$$

We will also need a massive object related to the massive Poisson kernel in Λ. As we will use it only at one given point on the boundary, we find it convenient to introduce it as follows. Assume that a and b are two marked boundary points of $\partial\Lambda$. Let $\phi_\Lambda : \Lambda \to \mathbb{H}$ be a conformal map such that $\phi_\Lambda(a) = 0$ and $\phi_\Lambda(b) = \infty$. For $z \in \Lambda$, set

$$P_\Lambda(z) := \frac{1}{\pi}\Im\left(\frac{-1}{\phi_\Lambda(z)}\right). \tag{3.32}$$

Then $P_\Lambda(z) = P_\mathbb{H}(\phi_\Lambda(z))$ is the bulk-to-boundary Poisson kernel in $\mathbb{H}$ evaluated at the bulk point $\phi_\Lambda(z)$ and at the boundary point 0, i.e. $P_\Lambda(z)$ is the density at 0 of the harmonic measure of $\mathbb{R}$ seen from $\phi_\Lambda(z)$. Notice that $P_\Lambda(z)$ depends on the boundary points a and b but for conciseness, we do not mention explicitly this dependency in the notation. The massive version of P_Λ is then defined by, for $z \in \Lambda$,

$$P_\Lambda^m(z) := P_\Lambda(z) - \int_\Lambda m^2(w)P_\Lambda(w)G_\Lambda^m(z,w)dw. \tag{3.33}$$

Finiteness of the above integral is shown in [CW21, Equation (4.6)]. Observe that by making the change of variable $u = \phi_\Lambda(w)$ in this integral and using the conformal covariance property of the massive Green function given by (3.28), we have that

$$P_\Lambda^m(z) = P_\Lambda(z) - \int_\mathbb{H} \tilde{m}^2(w)\frac{1}{\pi}\Im\left(\frac{-1}{w}\right)G_\mathbb{H}^{\tilde{m}}(\phi_\Lambda(z),w)dw.$$

where $\tilde{m}^2(w) = |(\phi_\Lambda^{-1})'(w)|^2 m^2(\phi_\Lambda^{-1}(w))$. We can thus see that $P_\Lambda^m(z) = P_\mathbb{H}^{\tilde{m}}(\phi_\Lambda(z))$ is the massive bulk-to-boundary Poisson kernel in $\mathbb{H}$ with mass $\tilde{m}$ evaluated at the bulk point $\phi_\Lambda(z)$ and at the boundary point 0, i.e. $P_\Lambda^m(z)$ is the density at 0 of the massive harmonic measure with mass $\tilde{m}$ of $\mathbb{R}$ seen from $\phi_\Lambda(z)$. Moreover, using conformal covariance of the Poisson kernel and that of its massive counterpart (for which the mass also changes under conformal maps), one can see that

$$\frac{P_\Lambda^m(z)}{P_\Lambda(z)} = \frac{\mathfrak{P}_\Lambda^m(z)}{\mathfrak{P}_\Lambda(z)} \tag{3.34}$$

where $\mathfrak{P}_\Lambda^m(z)$, respectively $\mathfrak{P}_\Lambda(z)$, is the massive, respectively massless, bulk-to-boundary Poisson kernel in Λ evaluated at the bulk point z and at the boundary point a. In other words, $\mathfrak{P}_\Lambda^m(z)$, respectively $\mathfrak{P}_\Lambda(z)$, is the density at a of the masseless, respectively massive, harmonic measure of $\partial\Lambda$ seen from z. Here, we consider ratios as $P_\Lambda(z)$ and $\mathfrak{P}_\Lambda(z)$ are related by the multiplicative factor $|\phi_\Lambda'(a)|$ and similarly for $P_\Lambda^m(z)$ and $\mathfrak{P}_\Lambda^m(z)$. This requires that the conformal map ϕ extends as a differentiable function at a, which is not necessarily the case. But the above ratios are nevertheless always well-defined.

3.4.2 Martingale characterization of massive SLE$_4$

Let us now state our characterization result. Although we have in mind its application to the characterization of the scaling limit of the massive harmonic explorer, this result holds under fairly general assumptions, that we now describe. Recall the assumptions made on the domain Ω and the boundary points $a, b \in \partial\Omega$ in Subsection 3.1.1. In this setting, as in Subsection 3.1.2, we divide the boundary of Ω into two parts, $\partial\Omega^+$ and $\partial\Omega^-$, which are the clockwise, respectively counterclockwise, oriented boundary arcs between a and b. Let $\phi : \Omega \to \mathbb{H}$ be a conformal map such that $\phi(a) = 0$ and $\phi(b) = \infty$. As before, we also let $m : \Omega \to \mathbb{R}_+$ be a continuous function bounded by some constant $\overline{m} > 0$. Assume that $(\gamma(t), t \geq 0)$ is a random non-self-crossing curve in $\overline{\Omega}$ with $\gamma(0) = a$ and $\gamma(\infty) = b$. Let $(\phi(\gamma(t)), t \geq 0)$ be its image in $\mathbb{H}$. This is a non-self crossing curve in $\mathbb{H}$ starting at 0 and targeting ∞. We assume that $(\phi(\gamma(t)), t \geq 0)$ is parametrized by half-plane capacity, or else reparametrize it. For $t \geq 0$, we denote by K_t the hull generated by $\phi(\gamma)([0,t])$. $(K_t, t \geq 0)$ is a random locally growing family of hulls generated by a curve and therefore, as explained in Section 2.1, its growth can be described using the Loewner equation. In other words, from the family $(K_t, t \geq 0)$, we can construct a random Loewner chain $(g_t, t \geq 0)$ whose time-evolution is described by a random driving function $(W_t, t \geq 0)$ and the Loewner equation (2.1). We set $\Omega_t := \phi^{-1}(\mathbb{H} \setminus K_t)$ and denote by $\partial\Omega_t^+$, respectively $\partial\Omega_t^-$, the clockwise, respectively counter-clockwise, oriented boundary arc of Ω_t from $\gamma(t)$ to b. For $z \in \Omega$, we also define the (possibly infinite) stopping time

$$\tau_z := \inf\{t \geq 0 : |g_t(\phi(z)) - W_t| = 0\}.$$

τ_z corresponds to the time at which $\phi(z)$ is swallowed by the hulls $(K_t, t \geq 0)$ and with this definition, for $t \geq 0$, $K_t = \{w \in \overline{\mathbb{H}} : \tau_{\phi^{-1}(w)} \leq t\}$.

In what follows, we are going to consider the time-evolution of the massive Green function and of P_Ω^m under the Loewner maps $(f_t)_t$, where for $t \geq 0$, $f_t := g_t - W_t$. In view of this, we introduce the following notations. We denote by G_t^m the massive Green function with mass m in Ω_t, defined as in the discussion around (3.27). We also define, for $t \geq 0$ and $z \in \Omega_t$,

$$P_t^m(z) := \frac{1}{\pi}\Im\left(\frac{-1}{f_t(\phi(z))}\right) - \int_{\Omega_t} m^2(w)\frac{1}{\pi}\Im\left(\frac{-1}{f_t(\phi(w))}\right)G_t^m(z,w)dw. \tag{3.35}$$

Remark that in the notations of Section 3.4.1, $P_t^m(z) = P_{\Omega_t}^m(z)$ and, as already mentioned there (notice that $f_t \circ \phi$ satisfies the assumptions made on the map denoted ϕ_Λ in (3.33)), the integral on the right-hand side of the above equality is well-defined. Setting $P_t(z) := \frac{1}{\pi}\Im(\frac{-1}{f_t(\phi(z))})$, the ratio $P_t^m(z)/P_t(z)$ can be given the same interpretation as in (3.34), with the Poisson kernels being evaluated at the bulk point z and at the boundary point $\gamma(t)$, the tip of the curve. Our characterization result then reads as follows.

Proposition 3.4.1. *Suppose that $\Omega, a, b, (\gamma(t), t \geq 0)$ and m are as described in the previous two paragraphs. For each $t \geq 0$, let $h_t^m : \Omega_t \to \mathbb{R}$ be the massive harmonic function in Ω_t with mass m and boundary conditions $-1/2$ on $\partial\Omega_t^-$ and $1/2$ on $\partial\Omega_t^+$ and assume that $(h_t^m(z), t \leq \tau_z)$ is a martingale for all $z \in \Omega$. Let h_t be the massless harmonic function in Ω_t with the same boundary conditions as h_t^m and recall the definition of $P_t^m(z)$ given in (3.35). Then γ is distributed as a massive SLE$_4$ curve from a to b in Ω, that is the driving function $(W_t, t \geq 0)$ of $\phi(\gamma)$ in $\mathbb{H}$ is given by, for $t \geq 0$,*

$$W_t = 2B_t - 2\pi \int_0^t \int_{\Omega_s} m^2(w)P_s^m(w)h_s(w)dwds \tag{3.36}$$

where $(B_t, t \geq 0)$ is a standard one-dimensional Brownian motion.

In the course of the proof of Proposition 3.4.1, we will repeatedly use the following massive version of the Hadamard formula.

Lemma 3.4.2. *Under the same assumptions on $\Omega, a, b, (\gamma(t), t \geq 0)$ and m as in Proposition 3.4.1, for each $z, w \in \Omega$, the function $G_t^m(z,w)$ is differentiable in t, until the first time that either $z \notin \Omega_t$ or $w \notin \Omega_t$. Its differential is given by*

$$\partial_t G_t^m(z,w) = -2\pi P_t^m(z)P_t^m(w),$$

where $P_t^m(z)$ and $P_t^m(w)$ are given by (3.35).

Proof. When the mass m is constant, the result is shown in [CW21, Lemma 4.7]. The arguments can be straightforwardly adapted to the case of a bounded and continuous mass function. $\qquad\square$

Before turning to the proof of Proposition 3.4.1, we state a preliminary lemma which will allow us to control the ration $P_t^m(w)/P_t(w)$ uniformly in t and w.

Lemma 3.4.3. *Let $R > 0, \overline{m} > 0$ be such that $\Omega \subset B(0, R)$, and $m^2 \leq \overline{m}^2$. Then almost surely, for any $t \geq 0$ and any $w \in \Omega_t$,*

$$\frac{P_t^m(w)}{P_t(w)} \geq \exp(-c_0 \overline{m}^2 R^2)$$

where $c_0 > 0$ is an absolute constant.

Proof. We first observe that, almost surely, for any $t \geq 0$ and any $w \in \Omega_t$,

$$\frac{P_t^m(w)}{P_t(w)} \geq \frac{P_t^{\overline{m}}(w)}{P_t(w)}.$$

This follows from the fact that $P_t^m(w)$ is a non-negative massive harmonic function in Ω_t with mass $m \leq \overline{m}$ while $P_t^{\overline{m}}$ is a non-negative massive harmonic function in Ω_t with $\overline{m}$ and both have the same boundary values (in the distributional sense). One can then use [CW21, Equation (4.10)] to obtain the lower bound

$$\frac{P_t^{\overline{m}}(w)}{P_t(w)} \geq \exp(-c_0 \overline{m}^2 R^2)$$

where $c_0 > 0$ is an absolute constant. In [CW21], this inequality is first shown for the discrete counterpart of the ratio $P_t^{\overline{m}}(w)/P_t(w)$ on the square grid of meshsize $\delta > 0$ and, since the lower bound is uniform in $\delta > 0$, the inequality in the continuum follows from convergence of the discrete ratio to the continuum one. This convergence holds provided that the discrete domains converge to Ω_t in the Carathéodory topology. We emphasize that the proof in [CW21] thus does not rely on the fact that the dynamics is that of the massive loop-erased random walk. $\qquad\square$

With this lemma in hand, let us now turn to the proof of Proposition 3.4.1.

3.4.3 Proof of Proposition 3.4.1

We prove Proposition 3.4.1 through a sequence of claims, that we now state and will prove in turn.

Claim 3.4.4. *The driving function $(W_t, t \geq 0)$ of $\phi(\gamma)$ is a semi-martingale. It can therefore be decomposed as $W_t = M_t + V_t$ where $(M_t, t \geq 0)$ is a local martingale and $(V_t, t \geq 0)$ is a process with bounded variations.*

In view of Claim 3.4.4, in order to prove Proposition 3.4.1, we must identify the local martingale $(M_t, t \geq 0)$ and the process $(V_t, t \geq 0)$. To do this, we rely on the assumption that for each $z \in \Omega$, the process $(h_t^m(z), 0 \leq t \leq \tau_z)$ is a martingale. Indeed, by computing its Ito derivative and using its martingale property, we will obtain equations satisfied by the process $(V_t, t \geq 0)$ and the quadratic variation $(\langle M \rangle_t, t \geq 0)$ of $(M_t, t \geq 0)$ that will uniquely determine $(V_t, t \geq 0)$ and $(M_t, t \geq 0)$. Let us first compute the Ito derivative of $(h_t^m(z), 0 \leq t \leq \tau_z)$.

Claim 3.4.5. *For each $z \in \Omega$, the process*

$$Q_t^m(z) = \frac{1}{\pi} \Im \left(\frac{-1}{(g_t(\phi(z)) - W_t)^2} \right) - \int_{\Omega_t} m^2(w) G_t^m(z, w) \frac{1}{\pi} \Im \left(\frac{-1}{(g_t(\phi(w)) - W_t)^2} \right) dw, \quad t \leq \tau_z,$$

is well defined, and the process $(h_t^m(z), t \leq \tau_z)$ satisfies the SDE

$$dh_t^m(z) = P_t^m(z) dM_t + P_t^m(z) dV_t + \frac{1}{2} Q_t^m(z) d\langle M \rangle_t - 2Q_t^m(z) dt$$

$$+ 2\pi P_t^m(z) \int_{\Omega_t} m^2(w) P_t^m(w) h_t(w) dw dt, \quad t \leq \tau_z. \tag{3.37}$$

Since by assumption, for any $z \in \Omega$, $(h_t^m(z), 0 \le t \le \tau_z)$ is a martingale, we can deduce from Claim 3.4.5 that, almost surely, for any $t \ge 0$ and $z \in \Omega_t$,

$$\int_0^t P_s^m(z) \left[dV_s + 2\pi \int_{\Omega_s} m^2(w) P_s^m(w) h_s^m(w) dw ds \right] + \int_0^t Q_s^m(z) \left[\frac{1}{2} d\langle M \rangle_t - 2dt \right] = 0. \quad (3.38)$$

To identify $(V_t, t \ge 0)$ and $(\langle M \rangle_t, t \ge 0)$, we will use this equality evaluated at a well-chosen sequence of points and then take a limit. The next claim establishes the existence of this (subsequential) limit.

Claim 3.4.6. *Set*

$$\tilde{V}_t = V_t + 2\pi \int_0^t \int_{\Omega_s} m^2(w) P_s^m(w) h_s(w) dw ds \quad and \quad A_t = \frac{1}{2}\langle M \rangle_t - 2t. \quad (3.39)$$

Fix $t > 0$ and consider the sequence of points $(z_n)_n = (\phi^{-1}(in))_n$. Then, almost surely, there exists a subsequence $(n(k))_k$ such that

$$\mathcal{P}_s^m(b) := \lim_{k \to \infty} \frac{P_s^m(z_{n(k)})}{P_s(z_{n(k)})} \quad and \quad \mathcal{Q}_s^m(b) := \lim_{n \to \infty} \frac{Q_s^m(z_{n(k)})}{P_s^m(z_{n(k)})} \quad (3.40)$$

exist, and moreover,

$$\lim_{k \to \infty} \int_0^t n(k) P_s^m(z_{n(k)}) d\tilde{V}_s + \int_0^t n(k) Q_s^m(z_{n(k)}) dA_s = \frac{1}{\pi} \int_0^t \mathcal{P}_s^m(b) d\tilde{V}_s + \frac{1}{\pi} \int_0^t \mathcal{P}_s^m(b) \mathcal{Q}_s^m(b) dA_s. \quad (3.41)$$

Finally, based on Claim 3.4.6, we will be able to identify $(M_t, t \ge 0)$ and $(V_t, t \ge 0)$.

Claim 3.4.7. *$(M_s)_{s \ge 0}$ has the law of 2 times a standard Brownian motion, and with probability one,*

$$V_s = -2\pi \int_0^s \int_{\Omega_r} m^2(w) P_r^m(w) h_r(w) dw dr$$

for all $s \ge 0$.

Proof of Proposition 3.4.1. This follows immediately from Claims 3.4.4 and 3.4.7. $\qquad \square$

Proof of Claim 3.4.4. For $t \ge 0$, let us first relate the massive harmonic function h_t^m to the massless harmonic function h_t. As explained in (3.30), for $z \in \Omega_t$, we have

$$h_t^m(z) = h_t(z) - \int_{\Omega_t} m^2(w) G_t^m(z, w) h_t(w) dw \quad (3.42)$$

where h_t can be written explicitly as

$$h_t(z) = \frac{1}{\pi} \arg(g_t(\phi(z)) - W_t) - \frac{1}{2} = \frac{1}{\pi} \arg(f_t(\phi(z))) - \frac{1}{2}.$$

Using the representation (3.31) of massive harmonic functions, we have the equality

$$h_t^m(z) = h_t(z) - \int_{\Omega_t} m^2(w) G_t(z, w) h_t^m(w) dw$$

from which we deduce that

$$h_t(z) = h_t^m(z) + \int_{\Omega_t} m^2(w) G_t(z, w) h_t^m(w) dw.$$

Moreover, since G_t is almost surely equal to zero outside $\Omega_t \subset \Omega$, this yields that for $t \ge 0$ and $z \in \Omega_t$,

$$h_t(z) = h_t^m(z) + \int_{\Omega} m^2(w) G_t(z, w) h_t^m(w) dw. \quad (3.43)$$

By the (non-massive) Hadamard formula, almost surely, for any $w \in \Omega_t$, $\partial_t G_t(z, w) = -2\pi P_t(z) P_t(w)$ and therefore, almost surely, for any $w \in \Omega_t$, the function $s \mapsto G_s(z, w)$ is decreasing on $[0, t]$.

Since $(h_s^m(w), 0 \leq s \leq t)$ is a martingale by assumption, we deduce from this that, for any $w \in \Omega_t$, $(G_s(z, w)h_s^m(w), 0 \leq s \leq t)$ is a semi-martingale. This implies that the process

$$\left(\int_\Omega m^2(w) G_t(z, w) h_t^m(w) dw, \ 0 \leq t \leq \tau_z \right)$$

is a semi-martingale as well. Again, since $(h_t^m(z), 0 \leq t \leq \tau_z)$ is a martingale by assumption, the equality (3.43) then shows that for each $z \in \Omega_t$, $(h_t(z), 0 \leq t \leq \tau_z)$ is a semi-martingale. Now, writing $f_t(z) = X_t(z) + iY_t(z)$, we have

$$Y_t(z) - Y_0 = \int_0^t \Im\left(\frac{2}{f_s(z)} \right) ds \quad \text{and}$$

$$X_t(z) - X_0 = \frac{Y_t(z)}{\tan(\pi(h_t(z) + 1/2))}.$$

The process $(Y_t(z), t \geq 0)$ has bounded variations since $Y_t(z) = \Im(f_t(z)) = \Im(g_t(z))$. As we have just established that $(h_t(z), 0 \leq t \leq \tau_z)$ is a semi-martingale, this in turn implies that the process $(X_t(z), 0 \leq t \leq \tau_z)$ is a semi-martingale. Writing

$$W_t = -X_t(z) + X_0 + \int_0^t \Re\left(\frac{2}{f_s(z)} \right) ds,$$

then shows that $(W_t, t \geq 0)$ is a semi-martingale. $\qquad \square$

Proof of Claim 3.4.7 given Claims 3.4.5 and 3.4.6. Let us deduce from Claim 3.4.7 and Claim 3.4.5 that the processes $(\tilde{V}_s, s \leq t)$ and $(A_s, s \leq t)$ defined in (3.39) are both 0. We will then explain why this yields Claim 3.4.7. From the equality (3.38), we obtain that almost surely, for any $k \in \mathbb{N}$,

$$\int_0^t n(k) P_s^m(z_{n(k)}) d\tilde{V}_s + \int_0^t n(k) Q_s^m(z_{n(k)}) dA_s = 0,$$

where $(n(k))_k$ is the subsequence obtained in Claim 3.4.6. Together with (3.41), this implies that, almost surely,

$$\frac{1}{\pi} \int_0^t \mathcal{P}_s^m(b) \left[d\tilde{V}_s + \mathcal{Q}_s^m(b) dA_s \right] = 0. \tag{3.44}$$

The process $(\mathcal{P}_s^m(b), s \leq t)$ is almost surely strictly positive on $[0, t]$ due to Lemma 3.4.3. Therefore, we obtain from (3.44) that, almost surely, for all $r_1, r_2 \in [0, t]$ and any measurable function $f : [0, t] \to \mathbb{R}$,

$$\int_{r_1}^{r_2} f(s) d\tilde{V}_s = - \int_{r_1}^{r_2} f(s) \mathcal{Q}_s^m(b) dA_s. \tag{3.45}$$

This equality applied to the function $f(s) = P_s^m(z)$ for some $z \in \Omega_t$ together with (3.38) yields that for any $z \in \Omega_t$, almost surely, for all $r_1, r_2 \in [0, t]$,

$$\int_{r_1}^{r_2} P_s^m(z) \mathcal{Q}_s^m(b) dA_s = \int_{r_1}^{r_2} Q_s^m(z) dA_s.$$

This implies that $A_s = 0$ for all $s \in [0, t]$. Indeed, using the fact that, almost surely, for any $z \in \Omega_s$, $P_s^m(z)$ is strictly positive and rewriting the above equality as

$$\int_{r_1}^{r_2} P_s^m(z) \left(\mathcal{Q}_s^m(b) - \frac{Q_s^m(z)}{P_s^m(z)} \right) dA_s = 0,$$

we obtain that for any $z \in \Omega_t$, almost surely, for any $0 \leq r_1 < r_2 \leq t$,

$$\int_{r_1}^{r_2} \frac{Q_s^m(z)}{P_s^m(z)} dA_s = \int_{r_1}^{r_2} \mathcal{Q}_s^m(b) dA_s.$$

As this equality holds for any $z \in \Omega_t$, we can deduce from this that for any $z, w \in \Omega_t$, almost surely, for any $0 \leq r_1 < r_2 \leq t$,

$$\int_{r_1}^{r_2} \frac{Q_s^m(z)}{P_s^m(z)} dA_s = \int_{r_1}^{r_2} \frac{Q_s^m(w)}{P_s^m(w)} dA_s. \tag{3.46}$$

Almost surely, for any $0 \leq s \leq t$, the function $(z, w) \mapsto Q_s^m(z)/P_s^m(z) - Q_s^m(w)/P_s^m(w)$ is not identically 0 in Ω_t^2. Indeed, if this function were identically 0, then, since P_s^m is almost surely strictly positive in Ω_t, this would imply that Q_s^m almost surely has constant sign in Ω_t whereas it massless version, $Q_s : z \mapsto (1/\pi)\Im(-1/f_t(\phi(z))^2)$, does not (as it can be seen by considering the points $\phi^{-1}(i + n)$ and $\phi^{-1}(i + n)$ which almost surely belong to Ω_t for n large enough). Moreover, since for any $(z, w) \in \Omega_t^2$, the function $s \mapsto Q_s^m(z)/P_s^m(z) - Q_s^m(w)/P_s^m(w)$ is almost surely continuous, if $Q_s^m(z)/P_s^m(z) - Q_s^m(w)/P_s^m(w) > 0$, then there almost surely exists $\epsilon > 0$ such that this quantity remains strictly positive on $[s, s + \epsilon]$. Taking $s = 0$ and $(z, w) \in \Omega_t^2$ such that $Q_0^m(z)/P_0^m(z) - Q_0^m(w)/P_0^m(w) > 0$, we deduce from this and the equality (3.46) that there almost surely exists $\epsilon > 0$ such that

$$\int_0^\epsilon \frac{Q_s^m(z)}{P_s^m(z)} - \frac{Q_s^m(w)}{P_s^m(w)} dA_s = 0.$$

Since $Q_s^m(z)/P_s^m(z) - Q_s^m(w)/P_s^m(w) > 0$ for any $s \in [0, \epsilon]$, this yields that, almost surely, $A_s = 0$ on $[0, \epsilon]$. We can use a similar argument to show that, almost surely, for any $s \in [0, t]$, $A_s = 0$. Indeed, assume by contradiction that there exists $s \in [0, t]$ such that $A_s \neq 0$. Choose $(z, w) \in \Omega_t^2$ such that $Q_s^m(z)/P_s^m(z) - Q_s^m(w)/P_s^m(w) > 0$. By continuity, there almost surely exists $\epsilon > 0$ such that $Q_r^m(z)/P_r^m(z) - Q_r^m(w)/P_r^m(w) > 0$ for any $r \in [s - \epsilon, s + \epsilon]$. The equality (3.46) then yields that almost surely $A_r = 0$ on $[s - \epsilon, s + \epsilon]$, which is a contradiction.

From the equality (3.45), we then conclude that $\tilde{V}_s = 0$ for all $s \in [0, t]$ as well. In view of the definitions of $(\tilde{V}_s, 0 \leq s \leq t)$ and $(A_s, 0 \leq s \leq t)$ given in (3.39), this yields that, almost surely, for $s \in [0, t]$,

$$V_s = -2\pi \int_0^s \int_{\Omega_r} m^2(w) P_r^m(w) h_r(w) dw dr \quad \text{and}$$

$$\langle M \rangle_s = 4s.$$

Since $M_0 = 0$ and $(M_t, t \geq 0)$ is a continuous process, by Lévy's characterization of Brownian motion, this implies that $M_t = 2B_t$, where $(B_t, t \geq 0)$ is a standard one-dimensional Brownian motion. Therefore, we can conclude that

$$W_t = 2B_t - 2\pi \int_0^t \int_{\Omega_s} m^2(w) P_s^m(w) h_s(w) dw ds$$

which is the statement of Claim 3.4.7. $\qquad \square$

Proof of Claim 3.4.6 given Claim 3.4.5. We first observe that, since Ω is a bounded domain, there exists $C > 0$ such that for any $n \in \mathbb{N}$, $|z_n| \leq C$. Therefore, the sequence $(z_n)_n$ is bounded in $(\mathbb{C}, |\cdot|)$. This implies that there exists a subsequence $(z_{n(k)})_k$ along which $(z_n)_n$ converges. We denote by b^* the limit of this subsequence. Let $s \in [0, t]$ and let us now show that the limits in (3.40) almost surely exist. The first step is to show that the the sequence $(P_s^m(z_{n(k)}/P_s(z_{n(k)})))_k$ almost surely converges. By [CW21, Lemma 4.4], almost surely,

$$P_s^m(z_{n(k)}) = P_s(z_{n(k)}) - \int_{\Omega_s} m^2(w) P_s^m(w) G_s(z_{n(k)}, w) dw.$$

We note that [CW21, Lemma 4.4] is proved for constant mass but the proof can easily be adapted to the case where the mass is a bounded function. The above equality yields that, almost surely,

$$\frac{P_s^m(z_{n(k)})}{P_s(z_{n(k)})} = 1 - \int_{\Omega_s} m^2(w) P_s^m(w) \frac{G_s(z_{n(k)}, w)}{P_s(z_{n(k)})} dw.$$

Therefore, to complete the first step, it suffices to show that

$$\lim_{k \to \infty} \int_{\Omega_s} m^2(w) P_s^m(w) \frac{G_s(z_{n(k)}, w)}{P_s(z_{n(k)})} dw \tag{3.47}$$

almost surely exists. To this end, observe that, by definition of $(z_{n(k)})$ and conformal invariance of the Green function, we have that, almost surely,

$$\frac{G_s(z_{n(k)}, w)}{P_s(z_{n(k)})} = \pi G_{\mathbb{H}_s}(\phi(z_{n(k)}), \phi(w)) \Im\left(-\frac{1}{f_s(\phi(z_{n(k)}))}\right)^{-1}$$

$$= \pi G_{\mathbb{H}_s}(in(k), \phi(w)) \Im\left(-\frac{1}{f_s(in(k))}\right)^{-1}$$

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where, as before, $\mathbb{H}_s := \phi(\Omega \setminus \gamma([0, s]))$. We can now argue as in [CW21, Section 3.2] to show that the limit as $k \to \infty$ of the left-hand side exists. Indeed, the right-hand side of the above equality is a ratio of two non-negative harmonic functions that have Dirichlet boundary conditions at the boundary point $\infty = \lim_{k\to\infty} in(k)$. This implies that, almost surely,

$$\frac{G_s(b^*, w)}{P_s(b^*)} := \lim_{k\to\infty} \frac{G_s(z_{n(k)}, w)}{P_s(z_{n(k)})}, \tag{3.48}$$

see [CW21, Section 3.2] for the detailed arguments. To show that the limit (3.47) exists, we must now explain why one can exchange the limit as $k \to \infty$ and the integration over Ω_s. As discussed below [CW21, Equation (4.12)], this simply follows from the almost sure uniform estimate

$$m^2(w)P_s^m(w)G_s(z_{n(k)}, w) \le \overline{m}^2 P_s(w)G_s(z_{n(k)}, w) \le \overline{m}^2 P_s(z_{n(k)})G_s(z_{n(k)}, w) + \overline{m}^2 C P_s(z_{n(k)})$$

where $C > 0$ is a universal constant, see [CW21, Equation (4.6)], which shows that, almost surely, for any $k \in \mathbb{N}$ and any $w \in \Omega_s$

$$m^2(w)P_s^m(w)\frac{G_s(z_{n(k)}, w)}{P_s(z_{n(k)})} \le \overline{m}^2 G_s(z_{n(k)}, w) + \overline{m}^2 C.$$

Moreover, $\max_{z\in B(b^*,r)} \int_{B(b^*,r)} m^2(w)G_s(z, w)dw$ almost surely converges to 0 as $r \to 0$, where $B(b^*, r)$ is the ball of radius r centered at b^*. This allows us to neglect the contribution from points close to b^*, where the Green function cannot be uniformly bounded. Thus, by domianted convergence, we obtain that, almost surely,

$$\lim_{k\to\infty} \frac{P_s^m(z_{n(k)})}{P_s(z_{n(k)})} = 1 - \int_{\Omega_s} m^2(w)P_s^m(w)\frac{G_s(b^*, w)}{P_s(b^*)}dw =: \mathcal{P}_s^m(b). \tag{3.49}$$

The second step is to show that $(Q_s^m(z_{n(k)})/P_s^m(z_{n(k)}))_k$ almost surely converges as $k \to \infty$. By definition, we have that, almost surely,

$$\frac{Q_s^m(z_{n(k)})}{P_s^m(z_{n(k)})} = \frac{Q_s(z_{n(k)})}{P_s^m(z_{n(k)})} - \int_{\Omega_s} m^2(w)Q_s(w)\frac{G_s^m(z_{n(k)}, w)}{P_s^m(z_{n(k)})}dw.$$

The ratio $Q_s(z_{n(k)})/P_s^m(z_{n(k)})$ almost surely converges to 0 as $k \to \infty$. Indeed, we have that, almost surely,

$$\frac{|Q_s(z_{n(k)})|}{P_s^m(z_{n(k)})} \le \exp(c_0 \overline{m}^2 R^2)\frac{|Q_s(z_{n(k)})|}{P_s(z_{n(k)})}$$

and the ratio of the right-hand side almost surely converges to 0. This follows from the definition of $(z_n)_n$ and the behavior of $f_s(z)$ for $|z| \to \infty$: as $k \to \infty$, almost surely,

$$P_s(z_{n(k)}) = \frac{1}{\pi}\Im\left(\frac{-1}{f_s(in(k))}\right) \sim \frac{1}{\pi n(k)} \quad \text{and} \quad |Q_s(z_{n(k)})| = \frac{1}{\pi}\left|\Im\left(\frac{-1}{f_s(in(k))^2}\right)\right| \sim \frac{1}{\pi n(k)^2}.$$

To complete the second step, we need to establish that

$$\lim_{k\to\infty} \int_{\Omega_s} m^2(w)Q_s(w)\frac{G_s^m(z_{n(k)}, w)}{P_s^m(z_{n(k)})}dw$$

almost surely exists. Using the same type of arguments as in the proof of [CW21, Lemma 4.4] and the fact that almost surely, $\int_{\Omega_s} P_s(w)dw < \infty$, one can show that, almost surely, for each k,

$$\int_{\Omega_s} m^2(w)Q_s(w)\frac{G_s^m(z_{n(k)}, w)}{P_s^m(z_{n(k)})}dw = \int_{\Omega_s} m^2(w)Q_s^m(w)\frac{G_s(z_{n(k)}, w)}{P_s^m(z_{n(k)})}dw.$$

Let us now show that $(G_s(z_{n(k)}, w)/P_s^m(z_{n(k)}))_k$ almost surely converges as $k \to \infty$ for any $w \in \Omega_s$. For this, we simply observe that, almost surely, for any $k \in \mathbb{N}$ and any $w \in \Omega_s$,

$$\frac{G_s(z_{n(k)}, w)}{P_s^m(z_{n(k)})} = \frac{G_s(z_{n(k)}, w)}{P_s(z_{n(k)})} \times \frac{P_s(z_{n(k)})}{P_s^m(z_{n(k)})}. \tag{3.50}$$

It follows from (3.48) and (3.49) that the two ratios on the above right-hand side almost surely converge. Indeed, note that the convergent sequence $(P_s^m(z_{n(k)})/P_s(z_{n(k)}))_k$ is almost surely bounded below by $\exp(-c_0\overline{m}^2R^2)$ and therefore, its inverse also almost surely converges.

Finally, to show that the limit

$$\lim_{k\to\infty}\int_{\Omega_s}m^2(w)Q_s^m(w)\frac{G_s(z_{n(k)},w)}{P_s^m(z_{n(k)})}dw$$

almost surely exists, we must justify why the limit and the integration over Ω_s can be exchanged. This can be achieved using (3.50), [CW21, Equation (4.7)] and the same arguments as above. We leave the details to the reader. We thus obtain that, almost surely,

$$\lim_{k\to\infty}\frac{Q_s^m(z_{n(k)})}{P_s^m(z_{n(k)})}=-\mathcal{P}_s^m(b)^{-1}\int_{\Omega_s}m^2(w)Q_s^m(w)\frac{G_s(b^*,w)}{P_s(b^*)}dw=:\mathcal{Q}_s^m(b).$$

The third and last step of the proof of the lemma is to show that the limit on the right-hand side of (3.41) exists and is such that the equality (3.41) holds. To establish this, we use the dominated convergence theorem. We first establish that, almost surely,

$$\lim_{k\to\infty}\int_0^t n(k)P_s^m(z_{n(k)})d\tilde{V}_s=\frac{1}{\pi}\int_0^t \mathcal{P}_s^m(b)d\tilde{V}_s$$

where $\mathcal{P}_s^m(b)$ is given by (3.40) and $(\tilde{V}_s, s\geq 0)$ is as defined in (3.39). The process $(\tilde{V}_s, 0\leq s\leq t)$ is a process of bounded variations. It can thus be decomposed as $\tilde{V}_s=\tilde{V}_s^+-\tilde{V}_s^-$ where $\mu^+([0,s))=\tilde{V}_s^+$ and $\mu^-([0,s))=\tilde{V}_s^-$ are non-negative measures. We first observe that, almost surely,

$$\lim_{n\to\infty}nP_s(z_n)=\frac{1}{\pi} \tag{3.51}$$

and the convergence is almost surely uniform on the interval $[0,t]$. Let $\epsilon>0$. The previous observation implies that there almost surely exists $K_0\in\mathbb{N}$ such that almost surely, for any $k\geq K_0$ and all $s\in[0,t]$, $n(k)P_s(z_{n(k)})\leq 1/\pi+\epsilon$. Therefore, almost surely, for any $k\geq K_0$ and any $s\in[0,t]$,

$$n(k)P_s^m(z_{n(k)})=n(k)P_s(z_{n(k)})\frac{P_s^m(z_{n(k)})}{P_s(z_{n(k)})}\leq\frac{1}{\pi}+\epsilon$$

where in the last inequality, we also used the fact that, almost surely, for any $s\in[0,t]$ and $w\in\Omega_t$, $P_s^m(w)\leq P_s(w)$. The right-hand side of the above inequality is integrable with respect to $d\tilde{V}_s^+$ and $d\tilde{V}_s^-$. Therefore, by the dominated convergence theorem, almost surely,

$$\lim_{k\to\infty}\int_0^t n(k)P_s^m(z_{n(k)})d\tilde{V}_s^+=\frac{1}{\pi}\int_0^t \mathcal{P}_s^m(b)d\tilde{V}_s^+\quad\text{and}$$

$$\lim_{k\to\infty}\int_0^t n(k)P_s^m(z_n(k))d\tilde{V}_s^-=\frac{1}{\pi}\int_0^t \mathcal{P}_s^m(b)d\tilde{V}_s^-,$$

which shows that, almost surely,

$$\lim_{k\to\infty}\int_0^t n(k)P_s^m(z_{n(k)})d\tilde{V}_s=\frac{1}{\pi}\int_0^t \mathcal{P}_s^m(b)d\tilde{V}_s.$$

Above, note that we used the decomposition $n(k)P_s^m(z_{n(k)})=n(k)P_s(z_{n(k)})\times(P_s^m(z_{n(k)})/P_s(z_{n(k)}))$ together with the fact that these two factors almost surely converge to $1/\pi$ and $\mathcal{P}_s^m(b)$ respectively. Let us now show that, almost surely,

$$\lim_{k\to\infty}\int_0^t n(k)Q_s^m(z_{n(k)})dA_s=\frac{1}{\pi}\int_0^t \mathcal{Q}_s^m(b)\mathcal{P}_s^m(b)dA_s$$

where $\mathcal{Q}_s^m(b)$ is given by (3.40) and $(A_s, s\geq 0)$ is as defined in (3.39). As before, we decompose the process $(A_s, 0\leq s\leq t)$ as $A_s=A_s^+-A_s^-$ in order to apply the dominated convergence theorem with respect to dA_s^+ and dA_s^-. We first observe that, almost surely,

$$\lim_{n\to\infty}nQ_s(z_n)=0$$

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and the convergence is almost surely uniform on the interval $[0, t]$. Let $\epsilon > 0$. The previous observation, together with the uniform convergence (3.51), implies that there almost surely exists $K_1 \in \mathbb{N}$ such that almost surely for any $k \geq K_1$ and any $s \in [0, t]$, $n(k)|Q_s(z_{n(k)})| \leq \epsilon$ and $n(k)P_s(z_{n(k)}) \leq 1/\pi + \epsilon$. Furthermore, the function $s \mapsto \int_{\Omega_s} P_s(w)dw$ is almost surely continuous on $[0, t]$ and therefore has an almost sure maximum $M(t)$ on $[0, t]$. This maximum is almost surely non-negative since P_s is almost surely non-negative. As above, using [CW21, Equation (4.7)], we then obtain that, almost surely, for any $k \geq K_1$ and any $s \in [0, t]$,

$$
n(k)|Q_s^m(z_{n(k)})| \leq n(k)|Q_s(z_{n(k)})| + n(k)\int_{\Omega_s} m^2(w)|Q_s(w)|G_s^m(z_{n(k)}, w)dw
$$

$$
\leq \epsilon + n(k)C\overline{m}^2 \int_{\Omega_s} P_s(z_{n(k)})P_s(w)dw + C\overline{m}^2 \int_{\Omega_s} n(k)|Q_s(z_{n(k)})|G_s(z_{n(k)}, w)dw
$$

$$
+ C\overline{m}^2 n(k)|Q_s(z_{n(k)})|\mathrm{vol}(\Omega_s)
$$

$$
\leq \epsilon + C\overline{m}^2(\frac{1}{\pi} + \epsilon)\int_{\Omega_s} P_s(w)dw + C\overline{m}^2\epsilon \int_{\Omega_s} G_s(z_{n(k)}, w)dw + C\overline{m}^2\epsilon\,\mathrm{vol}(\Omega_s)
$$

$$
\leq \epsilon + C\overline{m}^2(\frac{1}{\pi} + \epsilon)M(t) + C\tilde{C}\overline{m}^2\epsilon\,\mathrm{diam}(\Omega)^2 + C\overline{m}^2\epsilon\,\mathrm{vol}(\Omega)
$$

where $C, \tilde{C} > 0$ are (non-random) constants. The right-hand side of this inequality is integrable on $[0, t]$ with respect to dA_s^+ and dA_s^-. Therefore, by the dominated convergence theorem, almost surely,

$$
\lim_{k \to \infty} \int_0^t n(k)Q_s^m(z_{n(k)})dA_s^+ = \frac{1}{\pi}\int_0^t \mathcal{Q}_s^m(b)\mathcal{P}_s^m(b)dA_s^+ \quad \text{and}
$$

$$
\lim_{k \to \infty} \int_0^t n(k)Q_s^m(z_{n(k)})dA_s^- = \frac{1}{\pi}\int_0^t \mathcal{Q}_s^m(b)\mathcal{P}_s^m(b)dA_s^-
$$

which shows that, almost surely,

$$
\lim_{k \to \infty} \int_0^t n(k)Q_s^m(z_{n(k)})dA_s = \frac{1}{\pi}\int_0^t \mathcal{P}_s^m(b)\mathcal{Q}_s^m(b)dA_s.
$$

Above, we have also used the decomposition

$$
n(k)Q_s^m(z_{n(k)}) = n(k)P_s(z_{n(k)}) \times \frac{P_s^m(z_{n(k)})}{P_s(z_{n(k)})} \times \frac{Q_s^m(z_{n(k)})}{P_s^m(z_{n(k)})}
$$

and the fact that the three factors in this product almost surely converge to $1/\pi$, $\mathcal{P}_s^m(b)$ and $\mathcal{Q}_s^m(b)$, respectively, as $k \to \infty$. $\qquad\square$

Proof of Claim 3.4.5. Recall the equality (3.42) relating $h_t^m(z)$ to $h_t(z)$ for $z \in \Omega$ and $t \leq \tau_z$. In view of this equality, a natural strategy to compute the Ito derivative of $h_t^m(z)$ would be to first apply Ito's lemma to the product $G_t^m(z, w)h_t(w)$ and then use the stochastic Fubini theorem to switch the stochastic integral and the integral over Ω. However, as we have a priori no information of the local martingale $(M_t, t \geq 0)$, checking that the conditions of the stochastic Fubini theorem hold is not possible. We therefore follow a different strategy. Note that if absolute continuity of the limiting curve with respect to SLE_4 could be established from the discrete, then one may be able to use the stochastic Fubini theorem to compute the Ito derivative of h_t^m, as in the case of massive loop-erased random walk and massive SLE_2 [CW21].

Let us first use the equality (3.42) to express h_t^m in terms of the bulk-to-boundary Poisson kernel $P_{\mathbb{H}} : \mathbb{R} \times \mathbb{H} \to \mathbb{R}_+$ in $\mathbb{H}$ given by $P_{\mathbb{H}}(x, z) = (1/\pi)\Im(-1/(z - x))$. As explained around (3.32), $P_{\mathbb{H}}(x, z)$ is the density at x of the harmonic measure of $\mathbb{R}$ seen from z and, with the notation of (3.32), $P_{\mathbb{H}}(0, z) = P_{\mathbb{H}}(z)$. Using the fact that

$$
\frac{1}{\pi}\arg(f_t(z)) = \int_{-\infty}^0 P_{\mathbb{H}}(x, f_t(\phi(z)))dx,
$$

we have that, for $z \in \Omega$ and $t \leq \tau_z$,

$$
h_t^m(z) = \int_{-\infty}^0 P_{\mathbb{H}}(x, f_t(\phi(z)))dx - \frac{1}{2} - \int_{\Omega_t} m^2(w)G_t^m(z, w)\left[\int_{-\infty}^0 P_{\mathbb{H}}(x, f_t(\phi(w)))dx - \frac{1}{2}\right]dw.
$$

By Fubini-Tonelli theorem, since the function $(w, x) \mapsto m(w)^2 G_t^m(z, w) P_{\mathbb{H}}(x, f_t(\phi(w)))$ is non-negative, we can switch the integral over Ω_t and $(-\infty, 0)$. This yields that

$$\int_{\Omega_t} m^2(w) G_t^m(z, w) \int_{-\infty}^0 P_{\mathbb{H}}(x, f_t(\phi(w))) dx dw = \int_{-\infty}^0 \int_{\Omega_t} m^2(w) G_t^m(z, w) P_{\mathbb{H}}(x, f_t(\phi(w))) dw dx.$$

Notice that the integral $\int_{\Omega_t} m(w)^2 G_t^m(z, w) P_{\mathbb{H}}(x, f_t(\phi(w))) dw$ is finite since the only divergence is at $z = w$ where the integrand is bounded from above by a multiple of the (massless) Green function $G_0(z, w)$. Therefore, we obtain that

$$h_t^m(z) = \int_{-\infty}^0 \left[P_{\mathbb{H}}(x, f_t(\phi(z))) - \int_{\Omega_t} m^2(w) G_t^m(z, w) P_{\mathbb{H}}(x, f_t(\phi(w))) dw \right] dx \qquad (3.52)$$

$$- \frac{1}{2} + \frac{1}{2} \int_{\Omega_t} m^2(w) G_t^m(z, w) dw. \qquad (3.53)$$

We now make the change of variable $u = g_t(\phi(w))$ in the first integral over Ω_t. By conformal covariance of the massive Green function stated in (3.28), we have, for any $z, w \in \Omega_t$,

$$G_t^m(z, w) = G_{\mathbb{H}}^{m_t}(g_t(\phi(z)), g_t(\phi(w)))$$

where $G_{\mathbb{H}}^{m_t}$ is the massive Green function in $\mathbb{H}$ with mass m_t given by, for $u \in \mathbb{H}$,

$$m_t(u)^2 = |((g_t \circ \phi)^{-1})'(u)|^2 m((g_t \circ \phi)^{-1}(u))^2. \qquad (3.54)$$

Going back to the equality (3.52) for $h_t^m(z)$, we see that the changes of variables $u = g_t(\phi(w))$ and $v = x + W_t$ in the integral (3.52) yield that

$$h_t^m(z) = \int_{-\infty}^{W_t} P_{\mathbb{H}}^{m_t}(v, g_t(\phi(z))) dv \qquad (3.55)$$

$$- \frac{1}{2} + \frac{1}{2} \int_{\Omega_t} m^2(w) G_t^m(z, w) dw. \qquad (3.56)$$

where we have set for $z \in \Omega_t$ and $x \in \mathbb{R}$,

$$P_{\mathbb{H}}^{m_t}(x, g_t(\phi(z))) := P_{\mathbb{H}}(x, g_t(\phi(z))) - \int_{\mathbb{H}} m_t^2(w) G_{\mathbb{H}}^{m_t}(g_t(\phi(z)), w) P_{\mathbb{H}}(x, w) dw. \qquad (3.57)$$

Observe that by using the same changes of variable as above in the definition of P_t^m, we obtain that

$$P_t^m(z) = P_{\mathbb{H}}(W_t, g_t(\phi(z))) - \int_{\mathbb{H}} m_t^2(w) G_{\mathbb{H}}^{m_t}(g_t(\phi(z)), w) P_{\mathbb{H}}(W_t, w) dw.$$

Next, we want to compute the Itô derivative of $h_t^m(z)$ using the expression (3.55) and Itô's lemma. Let us first write the result and then explain how each term arises. The Itô derivative of $h_t^m(z)$ reads (we will justify the appearance of each term below):

$$dh_t^m(z) = P_t(z) dM_t - \int_{\Omega_t} m^2(w) P_t(w) G_t^m(z, w) dw dM_t \qquad (3.58)$$

$$+ P_t(z) dV_t - \int_{\Omega_t} m^2(w) P_t(w) G_t^m(z, w) dw dV_t \qquad (3.59)$$

$$+ \frac{1}{2} Q_t(z) d\langle M \rangle_s - \frac{1}{2} \int_{\Omega_t} m^2(w) Q_t(w) G_t^m(z, w) dw d\langle M \rangle_s \qquad (3.60)$$

$$- 2 Q_t(z) dt - 2 \int_{\Omega_t} m^2(w) Q_t(w) G_t^m(z, w) dw dt \qquad (3.61)$$

$$+ 2\pi P_t(z) \int_{\Omega_t} m^2(w) P_t^m(w) h_t(w) dw dt. \qquad (3.62)$$

where we have set, for $z \in \Omega_t$,

$$Q_t(z) = \frac{1}{\pi} \Im \left(\frac{-1}{(g_t(\phi(z)) - W_t)^2} \right).$$

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Let us start by explaining where the term (3.58) comes from. Using (3.57), we can write

$$\int_{-\infty}^{W_t} P_{\mathbb{H}}^{m_t}(x, g_t(\phi(z)))dx = \int_{-\infty}^{W_t} P_{\mathbb{H}}(x, g_t(\phi(z)))dx \qquad (3.63)$$

$$- \int_{-\infty}^{W_t} \int_{\mathbb{H}} m_t^2(w)G_{\mathbb{H}}^{m_t}(g_t(\phi(z)), w)P_{\mathbb{H}}(x, w)dwdx. \qquad (3.64)$$

Observe that the function

$$y \in \mathbb{R} \mapsto \int_{-\infty}^{y} P_{\mathbb{H}}^{m_t}(x, g_t(\phi(z)))dx$$

is differentiable and its derivative at $y \in \mathbb{R}$ is

$$P_{\mathbb{H}}^{m_t}(y, g_t(\phi(z))).$$

To evaluate this derivative at $y = W_t$ in order to compute the term depending on dM_t in the stochastic derivative of h_t^m, we must be slightly careful since both W_t and the integrand in (3.63)–(3.64) depend on t. Quite straightforwardly, the term (3.63) gives rise to the term $P_t(z)dM_t$, while the term (3.64) gives rise to the other term depending on dM_t in (3.58). Indeed, observe that from the expression of m_t given in (3.54), no terms depending on dM_t arise from m_t. Similarly, the derivative of $G_{\mathbb{H}}^{m_t}(g_t(\phi(z)), w)$ in (3.64) does not yield any term depending on dM_t. Indeed, by the massive Hadamard formula of Lemma 3.4.2, we have that

$$\partial_t G_{\mathbb{H}}^{m_t}(u, v) = \partial_t G_t^m((g_t \circ \phi)^{-1}(u), (g_t \circ \phi)^{-1}(v)) - 2\pi P_t^m(((g_t \circ \phi)^{-1}(u))P_t^m(((g_t \circ \phi)^{-1}(v))dt$$

Therefore, going back to (3.63)–(3.64), we can conclude that the term depending on dM_t in the Itô derivative of $h_t^m(z)$ is

$$P_{\mathbb{H}}^{m_t}(W_t, g_t(\phi(z)))dM_t = P_{\mathbb{H}}(W_t, g_t(\phi(z)))dM_t - \left[\int_{\mathbb{H}} m_t^2(w)G_{\mathbb{H}}^{m_t}(g_t(\phi(z)), w)P_{\mathbb{H}}(W_t, w)dw\right]dM_t$$

which can also be rewritten as

$$P_t(z)dM_t - \left[\int_{\Omega_t} m^2(w)G_t^m(z, w)P_t(w)dw\right]dM_t = P_t^m(z)dM_t.$$

The term (3.59) in the stochastic derivative of h_t^m arises for exactly the same reasons as the term (3.58). Let us now explain where the quadratic variation term (3.60) comes from. To compute it, we see that we must take the derivative of the function

$$x \in \mathbb{R} \mapsto \Im\left(\frac{-1}{g_t(\phi(z)) - x}\right) - \int_{\Omega_t} m^2(w)G_t^m(z, w)\Im\left(\frac{-1}{g_t(\phi(w)) - x}\right)dw \qquad (3.65)$$

and evaluate it at $x = W_t$. The above function is indeed differentiable and its derivative at $x \in \mathbb{R}$ is

$$\Im\left(\frac{-1}{(g_t(\phi(z)) - x)^2}\right) - \int_{\Omega_t} m^2(w)G_t^m(z, w)\Im\left(\frac{-1}{(g_t(\phi(w)) - x)^2}\right)dw \qquad (3.66)$$

Indeed, the integral

$$\int_{\Omega_t} m^2(w)G_t^m(z, w)\left|\Im\left(\frac{-1}{(g_t(\phi(w)) - x)^2}\right)\right|dw$$

is finite since the only divergence is at $z = w$ where the integral is bounded from above by a multiple of the Green function $G_0(z, w)$. One can thus differentiate under the integral sign, which yields the expression (3.66) for the derivative of the function (3.65). Once again, to evaluate the derivative of this function at $x = W_t$, we must be careful since both W_t and the integrand depend on t. However, for the same reasons as above, no term depending on $d\langle M \rangle_t$ arise from the integrand and therefore, we obtain that the quadratic variation term in the Itô derivative of h_t^m is

$$\frac{1}{2}\left[\Im\left(\frac{-1}{(g_t(\phi(z)) - W_t)^2}\right) - \int_{\Omega_t} m^2(w)G_t^m(z, w)\Im\left(\frac{-1}{(g_t(\phi(w)) - W_t)^2}\right)dw\right]d\langle M \rangle_t,$$

which, using the definition of Q_t, is exactly the term (3.60). Let us now turn to the terms (3.61) and (3.62). These terms come from the time-derivative of g_t, whose expression is given by the

Loewner equation, and of G_t^m, which can be computed using the massive Hadamard formula of Lemma 3.4.2. Writing

$$h_t^m(z) = \frac{1}{\pi}\arg(g_t(\phi(z)) - W_t) - \frac{1}{2} - \int_{\Omega_t} m^2(w)G_t(z,w)\left(\frac{1}{\pi}\arg(g_t(\phi(w)) - W_t) - \frac{1}{2}\right)dw,$$

we can see that the first summand in the term (3.61) simply comes from the first term in the above expression for $h_t^m(z)$. As for the second summand in (3.61) and the term (3.62), we observe that by the Loewner equation and the massive Hadamard formula of Lemma 3.4.2, we have that, for $z, w \in \Omega_t$,

$$\partial_t\big(G_t^m(z,w)h_t(w)\big) = -2\pi P_t^m(z)P_t^m(w)h_t(w) + G_t^m(z,w)Q_t(w).$$

Moreover, the integral $\int_{\Omega_t} P_t^m(w)h_t(w)dw$ is well-defined since $|h_t(w)|$ is bounded from above by $\frac{1}{2}$ and the integral $\int_{\Omega_t} P_t^m(w)dw$ is finite by [CW21, Corollary 4.6(i)], see also [CW21, Remark 4.3]. We have also seen above that the integral $\int_{\Omega_t} G_t^m(z,w)Q_t(w)dw$ is well-defined. Therefore, using the fact that $G_t^m(z,w) = 0$ for $w \notin \Omega_t$, we obtain that

$$\partial_t\left(\int_{\Omega_t} m^2(w)G_t^m(z,w)h_t(w)dw\right) = \partial_t\left(\int_{\Omega} m^2(w)G_t^m(z,w)h_t(w)dw\right)$$

$$= -2\pi P_t^m(z)\int_{\Omega_t} m^2(w)P_t^m(w)h_t(w)dw$$

$$+ \int_{\Omega_t} m^2(w)G_t^m(z,w)Q_t(w)dw,$$

which exactly corresponds to the second summand of (3.61) and the term (3.62). Moreover, inspecting the above arguments, one can see that no other terms arise in the Ito derivative of $h_t^m(z)$. Therefore, from the definition of $Q_t^m(z)$ in the statement of Claim 3.4.5, we see that we have obtained the desired SDE for $h_t^m(z)$ and the proof of Claim 3.4.5 is complete. $\qquad\square$

3.4.4   Reformulation and proof of Theorem 1.1.1

To conclude, let us now give a rigorous formulation of Theorem 1.1.1 and show how to combine the results of the previous sections to prove it.

Theorem 3.4.8. *Let $(\Omega_\delta, a_\delta, b_\delta)_\delta$ be a sequence of subgraphs of $\delta\mathbb{T}$ with two marked boundary points a_δ and b_δ. Assign a sign to boundary vertices of Ω_δ in the fashion described in Section 3.1.2. Assume that $(\hat{\Omega}_\delta, a_\delta, b_\delta)$ converges in the Carathéodory topology to (Ω, a, b), where $\Omega \subset \mathbb{C}$ and $a, b \in \partial\Omega$ satisfy the assumptions of Section 3.1.1. Let $m : \Omega \to \mathbb{R}_+$ be a bounded and continuous function. For $\delta > 0$, assign a mass to each edge of Ω_δ in the manner described in Section 3.1.2. Let $(\gamma_\delta)_\delta$ be a sequence of random paths distributed according to $(\mathbb{P}_\delta^{(\Omega,a,b,m)})_\delta$.*

Then, in the notations of Section 3.2.1, the sequence $(\gamma_\delta^{\mathbb{H}})_\delta$ converges weakly to a random curve $\gamma^{\mathbb{H}}$ in the topologies $((\mathrm{T}.1)) - ((\mathrm{T}.3))$. The driving function of $\gamma^{\mathbb{H}}$ parametrized by half-plane capacity satisfies the SDE

$$dW_t = 2dB_t - 2\pi\left(\int_{\Omega_t} m^2(w)P_t^m(w)h_t(w)dw\right)dt, \quad W_0 = 0, \qquad (3.67)$$

where $(B_t, t \geq 0)$ is a standard one-dimensional Brownian motion and where P_t^m and h_t have been defined in Section 3.4.2. This SDE has a unique weak solution whose law is absolutely continuous with respect to that of $(2B_t, t \geq 0)$.

Moreover, provided that for each $\delta > 0$, γ_δ is parametrized by the half-plane capacity of $\gamma_\delta^{\mathbb{H}}$, the sequence $(\gamma_\delta)_\delta$ converges weakly in $X(\mathbb{C})$ equipped with the metric d_X to a random curve γ that is almost surely supported on $\overline{\Omega}$ and has the same law as $\phi_\Omega^{-1}(\gamma^{\mathbb{H}})$. This implies in particular that γ is absolutely continuous with respect to SLE_4 in Ω from a to b.

Proof. By the results of Section 3.2.2, the sequence $(\gamma_\delta^{\mathbb{H}})_\delta$ is tight in the topologies $((\mathrm{T}.1)) - ((\mathrm{T}.3))$. Provided that for each $\delta > 0$, γ_δ is parametrized by the half-plane capacity of $\gamma_\delta^{\mathbb{H}}$, this implies that $(\gamma_\delta)_\delta$ is tight as well, in the space $X(\mathbb{C})$ equipped with the metric d_X. Let $(\gamma_{\delta_k})_k$ be a convergent subsequence and denote by γ its weak limit. By Proposition 3.3.2, for each $t \geq 0$, the corresponding subsequence of discrete massive harmonic functions $(h_{\delta_k, t(\delta_k)}^m)_k$ almost surely converges pointwise

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to h_t^m. Moreover, for each δ_k and any $v \in \mathrm{Int}(\Omega_{\delta_k}) \cup \partial\Omega_{\delta_k}$, $(h_{\delta_k,n}^m(v), n \geq 0)$ is a martingale for the filtration $(\mathcal{F}_{\delta_k,n})_n$. Therefore, we obtain that for any $z \in \Omega$, $(h_t^m(z), 0 \leq t \leq \tau_z)$ is a martingale for the filtration generated by γ. To conclude, we use the characterization result of Section 3.4.2. Indeed, since for any subsequential limit γ of $(\gamma_\delta)_\delta$ and any $z \in \Omega$, $(h_t^m(z), 0 \leq t \leq \tau_z)$ is a martingale, Proposition 3.4.1 implies that the driving function of all subsequential limits of $(\gamma_\delta^{\mathbb{H}})_\delta$ satisfies the SDE (3.67). The fact that this SDE has a unique weak solution whose law is absolutely continuous with respect to $(2B_t, t \geq 0)$ is shown below in Lemma 3.5.1. The last part of the statement of Theorem 3.4.8 is a consequence of Theorem 3.2.1 stated in Section 3.2.1. $\square$

3.5 Massive Gaussian free field and massive SLE$_4$: level line coupling

3.5.1 Absolute continuity of massive SLE$_4$ with respect to SLE$_4$ and conformal covariance of massive SLE$_4$

Absolute continuity of the massive harmonic explorer with respect to the (non-massive) harmonic explorer is not straightforward to see at the discrete level, which explains why establishing tightness is more involved than in the case of massive loop-erased random walk [CW21]. However, in the continuum, absolute continuity of massive SLE$_4$ with respect to SLE$_4$ is easily shown, as pointed out by Makarov and Smirnov in [MS10]. Here, we prove this fact for space-dependent mass, following the sketch of proof given in [MS10, Section 3.2] when the mass is constant. This implies in particular that massive SLE$_4$ shares many geometric properties of SLE$_4$.

Lemma 3.5.1. *Let $\Omega \subset \mathbb{C}$ be a bounded, open and simply connected domain with two marked boundary points $a, b \in \partial\Omega$. Let $\alpha > 0$. With the same notations as in Section 3.4.2, there exists a unique weak solution to the stochastic differential equation*

$$dW_t = 2dB_t - F_t^m dt, \quad W_0 = 0,$$

where, for $t \geq 0$,

$$F_t^m := \alpha \int_{\Omega_t} m^2(w) P_t^m(w) h_t(w) dw.$$

The law of this solution is absolutely continuous with respect to that of $(2B_t, t \geq 0)$. This implies that the massive SLE$_4$ Loewner chain $(f_t^m)_t$ with mass m from a to b in Ω as defined in (3.36) is absolutely continuous with respect to the SLE$_4$ Loewner chain $(f_t)_t$ from a to b in Ω. In particular, the hulls of massive SLE$_4$ are almost surely generated by a simple continuous curve γ of Hausdorff dimension $3/2$. Moreover, γ almost surely reaches its target point b and does not intersect $\partial\Omega$, except at its endpoints.

Proof. We adapt the strategy outlined in [MS10], taking into account the fact that here the mass is position-dependent and therefore, a priori, $G^m(z, w) \neq G^m(w, z)$. To show existence, uniqueness and absolute continuity with respect to $(2B_t, t \geq 0)$ of the solution to the above SDE, we apply Novikov's criterion to the drift term F_t^m. Since the mass function m and the harmonic function h_t are both almost surely bounded for any $t \geq 0$, we have that, almost surely, for any $t \geq 0$,

$$|F_t^m| \leq \alpha M \int_{\Omega_t} P_t(z) dz,$$

for some (non-random) constant $M > 0$ depending on $\overline{m}^2$. We recall that the integral on the right-hand side is almost surely finite by [CW21, Corollary 4.6]. Using the Fubini-Tonelli theorem and the massive Hadamard formula of Lemma 3.4.2, the above inequality yields that, almost surely,

$$\int_0^\infty |F_t^m|^2 dt \leq \alpha M^2 \int_0^\infty \int_{\Omega_t \times \Omega_t} P_t^m(z) P_t^m(w) dz dw dt$$

$$= \alpha M^2 \int_{\Omega \times \Omega} \int_0^\infty \mathbb{I}_{z \in \Omega_t} \mathbb{I}_{w \in \Omega_t} P_t^m(z) P_t^m(w) dz dw$$

$$= (-2\pi) \alpha M^2 \int_{\Omega \times \Omega} \int_0^\infty \mathbb{I}_{z \in \Omega_t} \mathbb{I}_{w \in \Omega_t} \partial_t G_t^m(z, w) dz dw dt$$

$$= \tilde{M} \int_{\Omega \times \Omega} G_\Omega^m(z,w) dz dw - \int_{\Omega_\infty \times \Omega_\infty} G_\infty^m(z,w) dz dw$$

$$\leq \tilde{M} \int_{\Omega \times \Omega} G_\Omega^m(z,w) dz dw$$

$$\leq \tilde{M} \int_{\Omega \times \Omega} G_\Omega(z,w) dz dw.$$

Since Ω is bounded, the last (non-random) integral on the right-hand side is finite. Therefore, we obtain that

$$\mathbb{E}\left[\exp\left(\frac{1}{2} \int_0^\infty |F_t|^2 dt \right) \right] \leq \exp(C)$$

for some constant $C > 0$ depending on α, m and Ω. This shows that Novikov's criterion holds and therefore that there exists a unique weak solution to the SDE

$$dW_t = 2dB_t - F_t^m dt, \quad W_0 = 0,$$

which is absolutely continuous with respect to $(2B_t, t \geq 0)$. The rest of the statement of Lemma 3.5.1 follows from the corresponding properties of SLE_4, see e.g. [Kem17, Chapter 5]. $\qquad\square$

The next lemma shows that massive SLE_4 is conformally covariant. Note that as a consequence of this result, one can extend the definition of massive SLE_4 and its absolute continuity with respect to SLE_4 to the case of unbounded domains provided that the mass is inherited from a bounded domain via conformal mapping.

Lemma 3.5.2. *Let $\Omega \subset \mathbb{C}$ be a bounded, open and simply connected domain with two marked boundary points $a, b \in \partial\Omega$. Let $\varphi : \Omega \to \tilde{\Omega}$ be a conformal map such that $\varphi(a) = \tilde{a}$ and $\varphi(b) = \tilde{b}$, where $\tilde{a}, \tilde{b} \in \partial\tilde{\Omega}$. Let $m : \Omega \to \mathbb{R}_+$ be a bounded and continuous function. If γ has the law of a massive SLE_4 curve in Ω from a to b with mass m, then $\varphi(\gamma)$ has the law of a massive SLE_4 in $\tilde{\Omega}$ from $\tilde{a}$ to $\tilde{b}$ with mass $\tilde{m} : \tilde{\Omega} \to \mathbb{R}_+$ given by, for $\tilde{w} \in \tilde{\Omega}$,*

$$\tilde{m}^2(\tilde{w}) = |(\varphi^{-1})'(\tilde{w})|^2 m^2(\varphi^{-1}(\tilde{w})). \tag{3.68}$$

In other words, massive SLE_4 is conformally covariant.

Proof. This simply follows from a change of variable in the drift term of massive SLE_4 in Ω from a to b with mass m. Recall that ϕ is a conformal map from Ω to $\mathbb{H}$ such that $\phi(a) = 0$ and $\phi(b) = \infty$. We have that

$$P_t^m(z) = \frac{1}{\pi} \Im\left(\frac{-1}{f_t(\phi(z))} \right) - \int_{\Omega_t} m^2(w) G_t^m(z,w) \frac{1}{\pi} \Im\left(\frac{-1}{f_t(\phi(w))} \right) dw.$$

Setting $z = \varphi^{-1}(\tilde{z})$ and $w = \varphi^{-1}(\tilde{w})$ and using the conformal covariance of the massive Green function (see (3.28)), we then obtain that

$$P_t^m(z) = \frac{1}{\pi} \Im\left(\frac{-1}{f_t((\phi \circ \varphi^{-1})(\tilde{z}))} \right) - \int_{\tilde{\Omega}_t} \tilde{m}^2(\tilde{w}) \tilde{G}_t^{\tilde{m}}(\tilde{z}, \tilde{w}) \frac{1}{\pi} \Im\left(\frac{-1}{f_t((\phi \circ \varphi^{-1})(\tilde{w}))} \right) d\tilde{w}$$

where $\tilde{m}$ is as in (3.68) and $\tilde{G}_t^{\tilde{m}}$ denotes the massive Green function with mass $\tilde{m}$ in $\tilde{\Omega}_t = \varphi(\Omega_t)$. In the notation of Section 3.4.1, the right-hand side of the above equality is equal to $P_{\tilde{\Omega}_t}^{\tilde{m}}(\tilde{z})$ since the map $\phi \circ \varphi^{-1}$ is a conformal map from $\tilde{\Omega}$ to $\mathbb{H}$ such that $(\phi \circ \varphi^{-1})(\tilde{a}) = 0$ and $(\phi \circ \varphi^{-1})(\tilde{b}) = \infty$. By conformal invariance of harmonic functions, this then yields that

$$\int_{\Omega_t} m^2(w) P_t^m(w) h_t(w) dw = \int_{\tilde{\Omega}_t} \tilde{m}^2(\tilde{w}) P_{\tilde{\Omega}_t}^{\tilde{m}}(\tilde{w}) \tilde{h}_t(\tilde{w}) d\tilde{w} \tag{3.69}$$

where $\tilde{h}_t$ is the harmonic function in $\tilde{\Omega}_t = \tilde{\Omega} \setminus \varphi(\gamma([0,t]))$ with boundary values $-1/2$ on the counter-clockwise oriented boundary arc $(\tilde{a}\tilde{b})$ and the right side of $\varphi(\gamma([0,t]))$ and $+1/2$ on the clockwise oriented boundary arc $(\tilde{a}\tilde{b})$ and the left side of $\varphi(\gamma([0,t]))$. The right-hand side of (3.69) is exactly the drift term in the driving function of $\phi \circ \varphi^{-1}(\tilde{\gamma})$ if $\tilde{\gamma}$ has the law of massive SLE_4 in $\tilde{\Omega}$ from $\tilde{a}$ to $\tilde{b}$ with mass $\tilde{m}$. This thus shows that $\varphi(\gamma)$ has indeed the law of massive SLE_4 in $\tilde{\Omega}$ from $\tilde{a}$ to $\tilde{b}$ with mass $\tilde{m}$. $\qquad\square$

3.5.2 Coupling of the massive Gaussian free field and massive SLE$_4$

In this section, we show the existence of a coupling between a massive GFF and a massive SLE$_4$ curve stated in the introduction as Theorem 1.1.2. This result is shown for space-dependent mass, so let us first define the massive GFF in this case. The definition is very similar to that in the constant mass case.

Let $\Omega \subset \mathbb{C}$ be an open, bounded and simply connected domain and let $a, b \in \partial\Omega$ be two boundary points. Fix a conformal map $\phi : \Omega \to \mathbb{H}$ such that $\phi(a) = 0$ and $\phi(b) = \infty$. Let $m : \Omega \to \mathbb{R}_+$ be a continuous function bounded by some constant $\overline{m} > 0$ in Ω. A massive GFF Γ in Ω with mass m and Dirichlet boundary conditions is a centered Gaussian process indexed by $\mathcal{C}_c^\infty(\Omega)$ with covariance given by, for $f, g \in \mathcal{C}_c^\infty(\Omega)$,

$$\mathbb{E}[(\Gamma, f)(\Gamma, g)] = \int_{\Omega \times \Omega} f(z) G_\Omega^m(z, w) g(w) dz dw.$$

Above, as before, G_Ω^m is the massive Green function in Ω with mass m or, in other words, this is the inverse in the sense of distributions of the operator $-\Delta + m^2$ with Dirichlet boundary conditions in Ω. The massive GFF Γ is absolutely continuous with respect to the massless GFF, with Radon-Nikodym derivative

$$\frac{1}{Z} \exp\left(-\frac{1}{2} \int_\Omega m^2(z) : \Gamma^0(z)^2 : dz \right)$$

where $: \Gamma^0(z)^2 :$ denotes the Wick-ordered square of the massless GFF Γ^0 and Z is a normalization constant chosen so that the expectation of this random variable is one. Finiteness of the exponential term is established in [LRV17, Lemma 3.5] when the mass m is constant but the proof can easily be adapted to the case of a non-constant mass by viewing $m^2(z)dz$ as the volume form $\exp(\log(m^2(z)))dz$.

With these definitions in hand, we can now state the existence of a coupling between a massive GFF and a massive SLE$_4$ curve. In the statement below, we stress that the domain Ω satisfies the assumptions introduced at the beginning of this subsection, so that in particular Ω is bounded.

Lemma 3.5.3. *Set $\lambda = \sqrt{\pi/8}$. There exists a coupling (Γ, γ) where Γ is a massive GFF in Ω with mass m and Dirichlet boundary conditions and γ is a massive SLE$_4$ curve with mass m from a to b in Ω. This coupling is such that the following Markov property holds. Assume that τ is an almost surely finite stopping time for the filtration generated by γ and define the following massive harmonic function*

$$\eta_\tau^m : z \in \Omega_\tau \mapsto \frac{1}{\sqrt{2\pi}} arg(f_\tau(\phi(z))) - \lambda - \int_{\Omega_\tau} m^2(w) G_\tau^m(z, w) \left(\frac{1}{\sqrt{2\pi}} arg(f_\tau(\phi(w))) - \lambda \right) dw.$$

In other words, η_τ^m is the unique massive harmonic function with mass m in $\Omega \setminus \gamma([0, \tau])$ that has boundary conditions $-\lambda$ on the counter-clockwise oriented boundary arc (ab) and the right side of $\gamma([0, \tau])$ and λ on the clockwise oriented boundary arc (ab) and the left side of $\gamma([0, \tau])$. Then, given $\gamma([0, \tau])$, the conditional law of $\Gamma + \eta_0^m$ restricted to Ω_τ is that of the sum of a massive GFF in Ω_τ with mass m and Dirichlet boundary conditions plus the function η_τ^m.

Let us point out that the drift term in the definition of massive SLE$_4$ satisfies the equality

$$2\pi \int_{\Omega_t} m^2(w) P_t^m(w) h_t(w) \, dw = 2\sqrt{2\pi} \int_{\Omega_t} m^2(w) P_t^m(w) \left(\frac{1}{\sqrt{2\pi}} arg(f_t(\phi(w))) - \lambda \right) dw.$$

In other words, the drift of massive SLE$_4$ can be expressed in terms of the harmonic function

$$z \in \Omega_t \mapsto \left(\frac{1}{\sqrt{2\pi}} arg(f_t(\phi(z))) - \lambda \right).$$

This function corresponds to the harmonic function appearing in the coupling between a (massless) GFF in Ω and a SLE$_4$ curve in Ω from a to b. The function η_t^m of Lemma 3.5.3 is its massive version.

Moreover, we also note that by conformal covariance of massive SLE$_4$, see Lemma 3.5.2, and of the massive GFF (which follows from the conformal covariance of the massive Green function, see (3.28)), Lemma 3.5.3 can be extended to unbounded domains with appropriate mass functions. These mass functions are of the form (3.68), that is are inherited from a bounded domain via conformal mapping.

Proof of Lemma 3.5.3. The proof goes along the same lines as in the massless case, see e.g. [WW17, Proposition 2.2.7]. Let η be the continuous harmonic function in Ω with boundary conditions $-\lambda$ on the boundary arc (ab) oriented clockwise and λ on the boundary arc (ba) oriented clockwise. More explicitly, for $z \in \Omega$,

$$\eta(z) = \frac{1}{\sqrt{2\pi}}\arg(\phi(z)) - \lambda.$$

For $t \geq 0$ and z such that $\tau_z > t$, set

$$\eta_t(z) = \frac{1}{\sqrt{2\pi}}\arg(f_t(\phi(z))) - \lambda.$$

Let η^m be the massive harmonic function in Ω with the same boundary values as η, that is

$$\eta^m = \eta(z) - \int_\Omega m^2(w)\eta(w)G_\Omega^m(z,w)dw.$$

For $t \geq 0$, let η_t^m be the massive harmonic function in Ω_t with boundary values $-\lambda$ on the left side of $\gamma([0,t])$ and the clockwise-oriented boundary arc (ab) and λ on the right side of $\gamma([0,t])$ and the clockwise-oriented boundary arc (ba). That is, for $z \in \Omega$ such that $\tau_z > t$,

$$\eta_t^m(z) = \eta_t(z) - \int_{\Omega_t} m^2(w)\eta_t(w)G_t^m(z,w)dw.$$

Fix $z \in \Omega$ and let us show that $t \mapsto \eta_t^m(z)$ is a continuous martingale until the possibly infinite stopping time τ_z. Indeed, by the computations done in the proof of Claim 3.4.5, $\eta_t^m(z)$ satisfies the SDE

$$d\eta_t^m(z) = \sqrt{\frac{\pi}{2}}P_t^m(z)2dB_t = \sqrt{2\pi}P_t^m(z)dB_t. \tag{3.70}$$

Therefore, $t \mapsto \eta_t^m(z)$ is a local martingale. But since $t \mapsto \eta_t^m(z)$ is almost surely bounded uniformly over t by λ, this is in fact a continuous martingale.

Next, let us show that for $z, w \in \Omega$, $t \mapsto \eta_t^m(z)\eta_t^m(w) + G_t^m(z,w)$ is a continuous martingale until the first time that either $\tau_z \leq t$ or $\tau_w \leq t$, This essentially follows from the massive Hadamard formula of Lemma 3.4.2 which, together with the SDE (3.70), implies that

$$d\langle\eta^m(z),\eta^m(w)\rangle_t = -\partial_t G_t^m(z,w).$$

Therefore, $t \mapsto \eta_t^m(z)\eta_t^m(w) + G_t^m(z,w)$ is a local martingale until the first time that either $\tau_z \leq t$ or $\tau_w \leq t$, Moreover, $\eta_t^m(w)$ and $\eta_t^m(w)$ are continuous and uniformly bounded over z, w by λ and G_t^m is non-increasing in t. Thus, $t \mapsto \eta_t^m(z)\eta_t^m(w) + G_t^m(z,w)$ is a continuous martingale until the first time that either $\tau_z \leq t$ or $\tau_w \leq t$,

Now, let $\varphi \in \mathcal{C}_c^\infty(\Omega)$ and define, for $t \geq 0$,

$$E_t^m(\varphi) = \int_{\Omega_t \times \Omega_t} \varphi(z)G_t^m(z,w)\varphi(w)dzdw.$$

We want to show that (η_t^m, φ) is a continuous martingale with quadratic variation

$$d\langle(\eta^m,\varphi)\rangle_t = -dE_t^m(\varphi). \tag{3.71}$$

Since $\eta_t^m(z)$ is a continuous martingale and is bounded uniformly over in Ω by λ, by Fubini's theorem, the integral (η_t^m, φ) is also a bounded and continuous martingale. To show that its quadratic variation is given by (3.71), it suffices to show that $(\eta_t^m, \varphi)^2 + E_t^m(\varphi)$ is a martingale. We observe that

$$(\eta_t^m,\varphi)^2 + E_t^m(\varphi) = \int_{\Omega_t \times \Omega_t} \varphi(z)\varphi(w)[\eta_t^m(z)\eta_t^m(w) + G_t^m(z,w)]dzdw.$$

We have already shown that $\eta_t^m(z)\eta_t^m(w) + G_t^m(z,w)$ is a continuous martingale and that $\eta_t^m(z)$, $\eta_t^m(w)$ are bounded uniformly over z, w. Moreover, $G_t^m(z,w)$ is non-negative and non-increasing in t. Therefore, we can apply Fubini's theorem, which yields that $(\eta_t^m, \varphi)^2 + E_t^m(\varphi)$ is a continuous martingale.

It now remains to construct a coupling that satisfies the domain Markov property. For $z \in \Omega$, define

$$\eta_\infty^m := \lim_{t \to \infty} \eta_t^m(z).$$

This limit exists almost surely for fixed z since $\eta_t^m(z)$ is a bounded martingale. For $z, w \in \Omega$ and $\varphi \in \mathcal{C}_c^\infty(\Omega)$ non-negative, define also

$$G_\infty^m(z, w) := \lim_{t \to \infty} G_t^m(z, w), \quad E_\infty^m(\varphi) := \lim_{t \to \infty} E_t^m(\varphi).$$

These limits exist because $G_t^m(z, w)$ and $E_t^m(\varphi)$ are both non-negative. Let h be a massive GFF in Ω_∞ with mass m and boundary conditions given by $\eta_\infty^m - \eta_0^m$. Then, for any $\varphi \in \mathcal{C}_c^\infty(\Omega)$ which is non-negative and any $\mu \geq 0$, we have

$$
\begin{aligned}
\mathbb{E}[\exp(-\mu(\Gamma, \varphi))] &= \mathbb{E}[\mathbb{E}[\exp(-\mu(\Gamma, \varphi))|K_\infty]] \\
&= \mathbb{E}\left[\exp\left(-(\eta_\infty^m - \eta_0^m, \varphi) - \frac{\mu^2}{2} E_\infty^m(\varphi) \right) \right] \\
&= \exp\left(-\frac{\mu^2}{2} E_0^m(\varphi) \right) \mathbb{E}\left[\exp\left(-(\eta_\infty^m - \eta_0^m, \varphi) - \frac{\mu^2}{2}(E_\infty^m(\varphi) - E_0^m(\varphi)) \right) \right] \\
&= \exp\left(-\frac{\mu^2}{2} E_0^m(\varphi) \right).
\end{aligned}
$$

The last equality holds because (η_t^m, φ) is a continuous and bounded martingale with mean (η_0^m, φ) and quadratic variation $E_0^m(\varphi) - E_t^m(\varphi)$. Finally, the coupling (Γ, γ) satisfies the domain Markov property since for any function $\varphi \in \mathcal{C}_c^\infty(\Omega)$, the conditional law of $(\Gamma + \eta_{0|\Omega_\tau}^m, \varphi)$ given $\gamma([0, \tau])$ is that of a Gaussian random variable with mean (η_τ^m, φ) and variance $E_\tau^m(\varphi)$. $\qquad \square$

Remark 3.5.4. We observe that using exactly the same arguments as in the proof of Proposition 3.5.3, one can show that a massive GFF with appropriate boundary conditions can be coupled to a massive version of $\mathrm{SLE}_4(\rho)$, where the drift is exactly the same as that of massive SLE_4 except that the harmonic function h_t has different boundary conditions.

Chapter 4

Massive SLE$_4$, massive CLE$_4$ and the massive planar GFF

4.1 Background

4.1.1 Reminder on massive SLE$_4$

Massive SLE$_4$ is a massive version of SLE$_4$ originally introduced in [MS10]. There are many possible ways to define massive versions of SLE$_4$ but we will use the name massive SLE$_4$ for this particular one, as in [MS10].

When mapped to $\mathbb{H}$ via a conformal map, a random curve whose law is massive SLE$_4$ can be described via its driving function as follows. Let $D \subset \mathbb{C}$ be a bounded, open and simply connected domain and let $a, b \in \partial D$. Denote by ∂D^+, respectively ∂D^-, the clockwise, respectively counterclockwise, oriented boundary arc from a to b. Let $m : D \to \mathbb{R}_+$ be a bounded and continuous function. Let $\varphi : D \to \mathbb{H}$ be a conformal map such that $\varphi(a) = 0$ and $\varphi(b) = \infty$. Then, a random curve γ has the law of massive SLE$_4$ in D from a to b with mass m if the driving function $(W_t, t \geq 0)$ of $\varphi(\gamma)$, when parametrized by half-plane capacity, satisfies the SDE

$$dW_t = 2dB_t - 2\pi \left(\int_{D_t} m^2(z) P_t^m(z) h_t(z) dz \right) dt, \quad W_0 = 0, \tag{4.1}$$

where $(B_t, t \geq 0)$ is a one-dimensional standard Brownian motion. Above, for $z \in D_t$, we have set

$$P_t^m(z) := \frac{1}{\pi} \Im \left(\frac{-1}{g_t(\varphi(z)) - W_t} \right) - \int_{D_t} m^2(w) \frac{1}{\pi} \Im \left(\frac{-1}{g_t(\varphi(w)) - W_t} \right) G_t^m(z, w) dw$$

where G_t^m is the massive Green function with mass m in D_t. In (4.1), the function h_t is the unique harmonic function with boundary values $1/2$ on ∂D^- and the left side of $\gamma([0,t])$ and $-1/2$ on ∂D^- and the right side of $\gamma([0,t])$. It was shown in Chapter 3 that the SDE (4.1) has a unique weak solution whose law is absolutely continuous with respect to that of $(2B_t, t \geq 0)$. This implies that the law of massive SLE$_4$ in D from a to b with mass m is absolutely continuous with respect to that of SLE$_4$ in D from a to b.

Moreover, using the conformal covariance of the drift term in the SDE (4.1), one can extend the definition of massive SLE$_4$ to pairs $(\tilde{D}, \tilde{m})$ where $\tilde{D} \subset \mathbb{C}$ is the image under a conformal map φ of a bounded, open and simply connected domain $D \subset \mathbb{C}$ and, for $w \in \tilde{D}$, $\tilde{m}(w)^2 = |(\varphi^{-1})'(w)|^2 m(\varphi^{-1}(w))^2$, where $m : D \to \mathbb{R}$ is a bounded and continuous function.

4.1.2 Conformal loop ensembles

Conformal loop ensembles (CLE) are a family of probability distributions on countable ensembles of non-nested loops (closed curves) in open and simply connected domains of the complex plane [She09; SW12]. As in [SW12, Section 2.1], we define a simple loop in the plane to be the image of the unit circle under a continuous and injective map. With this definition, a loop ℓ with time-duration $\tau(\ell)$ is equivalent to the loop $\tilde{\ell}$ with time-duration $\tau(\tilde{\ell})$ if there exists a bijective map $\psi : [0, \tau(\ell)] \to [0, \tau(\tilde{\ell})]$ such that for any $t \in [0, \tau(\ell)]$, $\ell(\psi(t)) = \tilde{\ell}(t)$. This defines a space of

loops, that we denote by $\text{Loop}(\mathbb{C})$, and we endow it with the σ-field Σ generated by all the events of the form $\{O \subset \text{Int}(\ell)\}$, where O spans the set of open subsets of $\mathbb{C}$ and for a loop ℓ, $\text{Int}(\ell)$ denotes its interior. Note that $\text{Loop}(\mathbb{C})$ can also be turned into a metric space, for example by equipping it with the distance induced by the supremum norm. A countable collection $\Gamma = (\ell_j)_{j \in J}$ of loops in $\text{Loop}(\mathbb{C})$ can be identified with the point-measure $\mu_\Gamma = \sum_{j \in J} \delta_{\ell_j}$. The space of countable collections of loops is then naturally equipped with the σ-field generated by the sets $\{\Gamma : \mu_\Gamma(A) = k\}$ where $A \in \Sigma$ and $k \geq 0$.

CLE_κ are random countable collections of loops indexed by a parameter $\kappa \in (8/3, 8)$. CLE_κ is connected to SLE_κ via the so-called branching tree construction [She09]. The geometry of the loops of a CLE_κ in an open and simply connected domain D depends on the value of κ: when $\kappa \in (8/3, 4]$, these loops are almost surely simple loops that do not intersect each other or ∂D; on the contrary, when $\kappa \in (4, 8)$, they are almost surely non-simple but non-self-crossing and they may touch (but not cross) ∂D and each other. Another important property of CLE_κ is their conformal invariance in law: if $\varphi : D \to \tilde{D}$ is a conformal map between two open and simply connected domains of $\mathbb{C}$ and Γ is a CLE_κ in D, then $\varphi(\Gamma)$ has the law of a CLE_κ in $\tilde{D}$.

4.1.3 Level lines of the Gaussian free field

In this section, unless mentioned otherwise, we assume that $D \subset \mathbb{C}$ is an open and simply connected domain. We also set $\lambda := \sqrt{\pi/8}$.

Definition of the GFF

The Gaussian free field (GFF) in D with Dirichlet boundary conditions is a centered Gaussian process h indexed by smooth functions with compact support in D whose covariance is given by, for f and g two such functions,

$$\mathbb{E}[(h, f)(h, g)] = \int_{D \times D} f(z) G_D(z, w) g(w) dz dw.$$

Above, G_D is the Green function in D: this is the inverse (in the sense of distributions) of the operator $-\Delta$ in D with Dirichlet boundary conditions. As the Green function blows up on the diagonal, the GFF is not defined pointwise but is instead a generalized function.

One can also define a GFF with specified boundary conditions. Let $f : \partial D \to \mathbb{R}$ be a bounded function and let ϕ_f be its unique harmonic extension in D. That is, ϕ_f is the unique function $u : D \to \mathbb{R}$ such that $-\Delta u(z) = 0$ for $z \in D$ and $u(z) = f(z)$ for $z \in \partial D$. Then, a GFF in D is said to have boundary conditions f if it has the same law as $h + \phi_f$ where h has the law of a GFF in D with Dirichlet boundary conditions.

Coupling between an SLE$_4$ curve and a GFF

Let $a, b \in \partial D$. Denote by ∂D^+, respectively ∂D^-, the clockwise-oriented, respectively counterclockwise-oriented, boundary arc (ab). Let $\phi : D \to \mathbb{R}$ be the unique harmonic function in D with boundary conditions λ on ∂D^+ and $-\lambda$ on ∂D^-. Then, there exists a coupling $(h + \phi, \gamma)$ between a GFF $h + \phi$ in D with boundary conditions ϕ and an SLE$_4$ curve in D from a to b [Dub09; MS16]. In this coupling, for any $t \geq 0$, conditionally on $\gamma([0, t])$, $h + \phi = h_t + \phi_t$ where h_t is a GFF with Dirichlet boundary conditions in $D_t := D \setminus \gamma([0, t])$ and ϕ_t is the unique harmonic function in D_t with boundary conditions λ on ∂D^+ and the left side of $\gamma([0, t])$ and $-\lambda$ on ∂D^- and the right side of $\gamma([0, t])$.

When $t = \infty$, since $\gamma([0, \infty))$ is almost surely a simple curve which does not intersect the boundary of D except at its endpoints, $D \setminus \gamma([0, \infty))$ is almost surely composed of two simply connected components. Denote by D_1, respectively D_2, the component on the left, respectively right, of $\gamma([0, \infty))$. Then, conditionally on $\gamma([0, \infty))$, $h = h_1 + \phi_1 + h_2 + \phi_2$ where [1] h_1 and h_2 are independent GFFs with Dirichlet boundary conditions in D_1 and D_2 and ϕ_1, respectively ϕ_2 is almost surely equal to λ, respectively 0, in D_1 and to 0, respectively $-\lambda$, in D_2.

These decompositions for $t < \infty$ and $t = \infty$ explain why in this coupling, the curve γ is sometimes called the level line of the field $h + \phi$. In this coupling, it can also be shown that the curve γ is in fact measurable with respect to the GFF h, see [MS16].

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[1] We will never use the decompositions at time $t < \infty$ and $t = \infty$ simultaneously, therefore no confusion should arise from the subscript notations used for these two decomposition.

Coupling between an CLE$_4$ and a GFF

There exists a coupling (h, Γ) between a GFF h in D with Dirichlet boundary conditions and a CLE$_4$ Γ in D [ASW19]. In this coupling, conditionally on Γ, $h = \sum_j h_j + \xi_j$ where the sum runs over the loops $(L_j)_j$ of Γ and

- $(h_j)_j$ are independent GFFs with Dirichlet boundary conditions in the interiors of $(L_j)_j$;
- $(\xi_j)_j$ are independent random variables with $\mathbb{P}(\xi_j = 2\lambda|\Gamma) = \mathbb{P}(\xi_j = -2\lambda|\Gamma) = 1/2 \; \forall j$;
- $(h_j)_j$ and $(\xi_j)_j$ are independent.

In this coupling, it can also be shown that the CLE$_4$ Γ, the fields $(h_j)_j$ and the labels $(\xi_j)_j$ are in fact measurable with respect to the GFF h [MS11; ASW19].

4.1.4   The Brownian loop measure and Le Jan's isomorphism theorem

In this section, unless mentioned otherwise, we assume that $D \subset \mathbb{C}$ is an open, bounded and simply connected domain and that $m : D \to \mathbb{R}_+$ is a bounded and continuous function.

The Brownian loop measure and the Brownian loop soup

Let us denote by $\mathbb{P}_t^{z,w}$ the bridge probability from z to w of length t for a two-dimensional standard Brownian motion $(B_s, s \geq 0)$. We also let $\tau_{\partial D}$ denote the first hitting time of ∂D by $(B_s, s \geq 0)$. The rooted Brownian loop measure [LW04] $\mu_{R,D}$ in D is defined as

$$\mu_{R,D}(\cdot) := \int_0^\infty \int_D \frac{1}{t} \frac{1}{2\pi t} \mathbb{P}_t^{z,z}(\cdot, \tau_{\partial D} > t) dz dt.$$

This is a measure on rooted loops. One can define an equivalence relation between such loops as follows. For ℓ a rooted loop, denote by $\tau(\ell)$ its lifetime. Then, for $u \in \mathbb{R}$, $\theta_u \ell : t \mapsto \ell(u+t \mod \tau(\ell))$ is again a loop. We then say that two loops ℓ and $\tilde{\ell}$ are equivalent if there exists $u \in \mathbb{R}$ such that $\ell = \theta_u \tilde{\ell}$. The quotient of the rooted loop measure $\mu_{R,D}$ under this equivalence relation is the Brownian loop measure μ_D in D. One important property of this measure is its conformal invariance: if $f : \Omega \to D$ is a conformal map, then $f \circ \mu_\Omega = \mu_D$.

For $\alpha > 0$, one can then sample a Poisson point process with intensity $\alpha \mu_D$ which gives rise to a countable collection of unrooted loops in D. This process is called a Brownian loop soup in D with intensity α [LW04]. It can be shown that for $\kappa(8/3, 4]$, a CLE$_\kappa$ can be constructed out a Brownian loop soup with the correct intensity [SW12]. In particular, a CLE$_4$ Γ can be coupled to a Brownian loop soup $\mathcal{L}$ with intensity $1/2$ in such way that the outer boundaries of the outermost clusters of loops of $\mathcal{L}$ are the loops of Γ. Here, we say that two loops ℓ and $\tilde{\ell}$ belong to the same cluster of loops of $\mathcal{L}$ if there exists a sequence $(\ell_k)_{k=0}^n$ of loops in $\mathcal{L}$ such that $\ell_0 = \ell$, $\ell_n = \tilde{\ell}$ and for any k, $\ell_k \cap \ell_{k+1} \neq \emptyset$.

An important observable of a Brownian loop soup $\mathcal{L}$ in D with intensity $\alpha > 0$ is its renormalized occupation-time field $: L :$. For $z \in D$, $: L(z) :$ should be thought of as the total amount of time that loops in $\mathcal{L}$ spend at z. However, since this quantity is in fact almost surely infinite, $: L :$ is constructed via renormalization as follows. Let $\epsilon > 0$ and set, for a bounded function $f : D \to \mathbb{R}$,

$$\int_D : L_\epsilon(z) : f(z) dz := \sum_{\ell \in \mathcal{L}: \tau(\ell) \geq \epsilon} \int_0^{\tau(\ell)} f(\ell(t)) dt - \alpha \mu_D \left(\mathbb{I}_{\tau(\ell) \geq \epsilon} \int_0^{\tau(\ell)} f(\ell(t)) dt \right)$$

where we note that $\alpha \mu_D(\mathbb{I}_{\tau(\ell) \geq \epsilon} \int_0^{\tau(\ell)} f(\ell(t)) dt) = \mathbb{E}[\sum_{\ell \in \mathcal{L}: \tau(\ell) \geq \epsilon} \int_0^{\tau(\ell)} f(\ell(t)) dt]$. Then, by [Jan08, Section 10.2], as ϵ tends to 0, $\langle : L_\epsilon(\mathcal{L}) :, f \rangle$ converges in $L^2(\mathbb{P})$ and almost surely to a random variable that we denote by $\int_D f(z) : L(z) : dz$. One can then make sense of $: L :$ as a random field indexed by bounded functions in D and this field is what we call the renormalized ocupation-time field of $\mathcal{L}$.

Wick square of the Gaussian free field

Since the GFF is only a generalized function, its square is a priori ill-defined and must be constructed via renormalization, as we now explain. Let us first introduce the circle-average approximation of the GFF. Let h be a GFF in D with Dirichlet boundary conditions. For $z \in D$ and

$\epsilon > 0$, denote by $B(z, \epsilon)$ the ball of radius ϵ centered at z and let ρ_ϵ^z be the uniform measure on $\partial B(z, \epsilon)$ seen from z. We define the random variable

$$h_\epsilon(z) := (h, \rho_\epsilon^z).$$

Even though ρ_ϵ^z is not a smooth function, this random variable is always well-defined and the process $(h_\epsilon(z), z \in D, \epsilon > 0)$ is in fact jointly Hölder continuous in (z, ϵ), see [WP21, Section 3.3.4].

As the expectation of $h_\epsilon(z)^2$ blows up as ϵ tends to 0, a divergent counterterm must be introduced to define the square of h. This counterterm is chosen such that the limit field, denoted by $: h^2 :$, is a centered field. Here, we will not need to construct the field $: h^2 :$ itself. Instead, we must only be able to make sense of the random variable $(: h^2 :, m^2)$, or more generally of $(: (h + \phi)^2 :, m^2)$, where ϕ will correspond to the boundary conditions of h and m^2 will be the mass of the massive GFF. This is achieved thanks to the following lemma. Below, for $\phi : D \to \mathbb{R}$ a bounded harmonic function, $z \in D$ and $\epsilon > 0$, we use the notation

$$: (h_\epsilon(z) + \phi_\epsilon(z))^2 : \overset{\text{def}}{=} h_\epsilon(z)^2 + 2h_\epsilon(z)\phi_\epsilon(z) + \phi_\epsilon(z)^2 - \mathbb{E}[h_\epsilon(z)^2] \tag{4.2}$$

where $\phi_\epsilon(z)$ the circle average approximation of $\phi(z)$, that is

$$\phi_\epsilon(z) := \int \phi(w)\rho_\epsilon^z(dw). \tag{4.3}$$

Notice that if $B(z, \epsilon) \subset D$, then, by harmonicity, $\phi_\epsilon(z) = \phi(z)$. We also denote by $\dim_H(A)$ the Hausdorff dimension of the set A.

Lemma 4.1.1. *Let $m : D \to \mathbb{R}_+$ be a bounded and continuous function. Let h be a GFF with Dirichlet boundary conditions in D and $\phi : D \to \mathbb{R}$ be a harmonic function bounded by some constant $b_\phi > 0$. Then*

$$\lim_{\epsilon \to 0} \int_D m^2(z) : (h_\epsilon(z) + \phi_\epsilon(z))^2 : dz = \int_D m^2(z) : h(z)^2 : dz + 2(h, m^2\phi) + \int_D m^2(z)\phi(z)^2 dz$$

where the limit is in $\mathrm{L}^2(\mathbb{P})$. Above, $(h, m^2\phi)$ denotes the GFF h tested against the function $z \mapsto m^2(z)\phi(z)$ and

$$\int_D m^2(z) : h(z)^2 : dz = \lim_{\epsilon \to 0} \int_D m^2(z) : h_\epsilon(z)^2 : dz$$

where the limit is in $\mathrm{L}^2(\mathbb{P})$. Moreover, for any $b \in (0, 2 - \dim_H(\partial D))$, there exists $C > 0$ such that, for any $0 < \epsilon < 1/2$,

$$\left\| \int_D m^2(z) : (h + \phi)^2(z) : dz - \int_D m^2(z) : (h_\epsilon(z) + \phi_\epsilon(z))^2 : dz \right\|_{\mathrm{L}^2(\mathbb{P})} \leq C\epsilon^b. \tag{4.4}$$

The proof of Lemma 4.1.1 is given in Appendix 6.1. The assumption that ϕ is a harmonic function is not really needed but it simplifies the proof and will always be satisfied in the cases in which we will be interested. Let us also stress that in the above statement, the notation $\int_D m^2(z) : h(z)^2 : dz$ is only formal since $: h^2 :$ is not defined pointwise.

For convenience, as in (4.4), we will sometimes use the formal notation

$$\int_D m^2(z) : (h + \phi)^2(z) : dz := \int_D m^2(z) : h(z)^2 : dz + 2(h, m^2\phi) + \int_D m^2(z)\phi(z)^2 dz. \tag{4.5}$$

Also, if the function $m : D \to \mathbb{R}_+$ is of the form $|\varphi'|(\hat{m} \circ \varphi)$ where $\varphi : D \to \hat{D}$ is a conformal isomorphism, $\hat{D} \subset \mathbb{C}$ is an open, bounded and simply connected domain and $\hat{m} : \hat{D} \to \mathbb{R}_+$ is a bounded and continuous function, we set

$$\int_D m^2(z) : (h + \phi)^2(z) : dz = \int_{\hat{D}} \hat{m}^2(w) : (\hat{h} + \hat{\phi})^2(w) : dw$$

with $h + \phi = (\hat{h} + \hat{\phi}) \circ \varphi^{-1}$.

Le Jan's isomorphism theorem

The Brownian loop soup in D is related to the GFF in D with Dirichlet boundary conditions via Le Jan's isomorphism theorem [Jan08; Jan10].

Proposition 4.1.2 (Le Jan's isomorphism)**.** *The renormalized occupation-time field* $: L :$ *of a Brownian loop soup in D with intensity $1/2$ has the same law as half the Wick square $1/2 : h^2 :$ of a GFF h in D with Dirichlet boundary conditions.*

We also recall the following result. It is stated as Theorem 8 in [Jan08] under the additional assumption that the function m is smooth and has compact support in D. But the proof in fact extends to the case where m is a bounded function and D a bounded domain.

Proposition 4.1.3. *For a bounded and continuous function $m : D \to \mathbb{R}_+$ and a loop $\ell : [0, \tau(\ell)] \to D$, set $\langle \ell, m^2 \rangle := \int_0^{\tau(\ell)} m^2(\ell(t))dt$. Then*

$$\mathbb{E}_{\mathrm{GFF}}\left[\exp\left(-\frac{1}{2} \int_D m^2(z) : h^2(z) : dz \right) \right] = \exp\left(\frac{1}{2} \mu_D(e^{-\langle \ell, m^2 \rangle} + \langle \ell, m^2 \rangle - 1) \right) < \infty$$

where under $\mathbb{E}_{\mathrm{GFF}}$, h has the law of a GFF in D with Dirichlet boundary conditions.

Coupling between a GFF, a CLE$_4$ and a Brownian loop soup

There also exists a coupling between a GFF, a CLE$_4$ and a Brownian loop soup with intensity $1/2$ such that all the relations described above hold [QW19].

Proposition 4.1.4 (Qian-Werner)**.** *One can couple a Brownian loop soup $\mathcal{L}$ in D with intensity $1/2$, a CLE$_4$ Γ in D and a GFF h in D with Dirichlet boundary conditions in such a way that:*

- *the loops of Γ are the level lines of h as in Section 4.1.3;*
- *the loops of Γ are the outer boundaries of the outermost clusters of $\mathcal{L}$;*
- *the renormalized occupation-time field $: L :$ of $\mathcal{L}$ is exactly $\frac{1}{2} : h^2 :$.*

4.1.5 The massive Gaussian free field

In this section, unless mentioned otherwise, we assume that $D \subset \mathbb{C}$ is an open, bounded and simply connected domain and that $m : D \to \mathbb{R}_+$ is a bounded and continuous function.

Definition of the massive GFF

The massive GFF in D with mass m and Dirichlet boundary conditions is a centered Gaussian process h indexed by smooth functions with compact support in D whose covariance is given by, for f and g two such functions,

$$\mathbb{E}[(h, f)(h, g)] = \int_{D \times D} f(z) G_D^m(z, w) g(w) dz dw.$$

Above, G_D^m is the massive Green function in D with mass m: this is the inverse (in the sense of distributions) of the operator $-\Delta + m^2$ in D with Dirichlet boundary conditions. Like the GFF, the massive GFF is not defined pointwise but is instead a random generalized function.

From a statistical mechanics point of view, the massive GFF can be seen as a GFF "perturbed by a mass". The presence of this mass breaks the conformal invariance of the field, which is one of the main features of the GFF: the massive GFF is not conformally invariant but only conformally covariant, in the following sense. Let $\varphi : D \to \tilde{D}$ be conformal map. If h is a massive GFF in D with mass m and Dirichlet boundary conditions, then the pushforward of h via φ is a massive GFF in $\tilde{D}$ with Dirichlet boundary conditions and mass $\tilde{m}$ given by, for $z \in \tilde{D}$,

$$\tilde{m}^2(z) = |(\varphi^{-1})'(z)|^2 m^2(\varphi^{-1}(z)). \tag{4.6}$$

This simply follows from the conformal covariance of the massive Green function G_D^m.

One can also define a massive GFF with specified boundary conditions. Let $f : \partial D \to \mathbb{R}$ be a bounded function and let ϕ_f^m be its unique massive harmonic extension in D. That is, ϕ_f^m is

the unique function $u : D \to \mathbb{R}$ such that $(-\Delta + m^2(z))u(z) = 0$ for $z \in D$ and $u(z) = f(z)$ for $z \in \partial D$. Then, a massive GFF in D with mass m is said to have boundary conditions f if it has the same law as $h + \phi_f^m$ where h has the law of a massive GFF in D with mass m and Dirichlet boundary conditions. Note that since massive harmonic functions are conformally covariant, with the mass changing as in (4.6) when conformally mapping the domain to another one, a massive GFF with specified boundary conditions is also conformally covariant, in the same sense as for a massive GFF with Dirichlet boundary conditions.

Absolute continuity with respect to the GFF

The law $\mathbb{P}_{\mathrm{mGFF}}^D$ of the massive GFF in D with mass m and Dirichlet boundary conditions is absolutely continuous with respect to the law $\mathbb{P}_{\mathrm{GFF}}^D$ of the GFF in D with Dirichlet boundary conditions. The corresponding Radon-Nikodym derivative has an explicit expression: it is given by

$$\frac{d\mathbb{P}_{\mathrm{mGFF}}^D}{d\mathbb{P}_{\mathrm{GFF}}^D}(h) = \frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \int_D m^2(z) : h^2(z) : dz \right) \tag{4.7}$$

where $\mathcal{Z}$ is a normalization constant and $\int_D m^2(z) : h^2(z) : dz$ is as defined in Section 4.1.4. See [ALS20, Lemma 3.10] for a detailed proof of this fact. Let us nevertheless point out that the fact that the random variable on the right-hand side is $\sigma(h)$-measurable and almost strictly positive can be justified using Lemma 6.1.2 (see Lemma 6.1.4 in Appendix 6.1). Moreover, by Proposition 4.1.3, this random variable is in $L^1(\mathbb{P}_{\mathrm{GFF}}^D)$, so $\mathcal{Z}$ is finite.

Note also that, by Proposition 4.1.3 applied with $f = m^2$,

$$\mathcal{Z} = \mathbb{E}_{\mathrm{GFF}}^D\left[\exp\left(-\frac{1}{2} \int_D m^2(z) : h^2(z) : dz \right) \right] = \exp\left(\frac{1}{2}\mu_D\left(e^{-\langle \ell, m^2 \rangle} + \langle \ell, m^2 \rangle - 1 \right) \right). \tag{4.8}$$

Similarly, for $f : \partial D \to \mathbb{R}$ a bounded function, the law of the massive GFF in D with mass m and boundary conditions f is absolutely continuous with respect to the law $\mathbb{P}_{\mathrm{GFF}}^{D,f}$ of the GFF in D with boundary conditions f. Indeed, if we define

$$\frac{d\mathbb{P}_{\mathrm{mGFF}}^{D,f}}{d\mathbb{P}_{\mathrm{GFF}}^{D,f}}(h + \phi_f) = \frac{1}{\mathcal{Z}_f} \exp\left(-\frac{1}{2} \int_D m^2(z) : (h + \phi_f)^2(z) : dz \right) \tag{4.9}$$

where $\mathcal{Z}_f$ is a normalization constant, then the law of $h + \phi_f$ under $\mathbb{P}_{\mathrm{mGFF}}^{D,f}$ is that of a massive GFF in D with mass m and boundary conditions f. The fact that the random variable on the right-hand side of (4.9) is almost surely strictly positive and measurable with respect to $h + \phi$ follows from Lemma 4.1.1 and Lemma 6.1.4 in Appendix 6.1.

To prove that under $\mathbb{P}_{\mathrm{mGFF}}^{D,f}$ defined via (4.9), $h + \phi_f$ has the claimed law, one can compute the Laplace transform of $h + \phi_f$ under $\mathbb{P}_{\mathrm{mGFF}}^{D,f}$. Although this computation is rather straightforward, let us detail it as we could not find any reference in the existing literature.

Proof of (4.9). Let g be a smooth function with compact support in D. Then, we have that

$$\mathbb{E}_{\mathrm{mGFF}}^{D,f}\left[\exp((h + \phi_f, g)) \right] = \frac{1}{\mathcal{Z}_f} \mathbb{E}_{\mathrm{GFF}}^D\left[\exp\left((h + \phi_f, g) - \frac{1}{2} \int_D m^2(z) : (h + \phi_f)^2(z) : dz \right) \right]$$

$$= \frac{1}{\mathcal{Z}_f} \exp\left(\int_D \phi_f(z)g(z) - \frac{1}{2}m^2(z)\phi_f(z)^2 dz \right)$$

$$\times \mathbb{E}_{\mathrm{GFF}}^D\left[\exp\left((h, g - m^2\phi_f) - \frac{1}{2} \int_D m^2(z) : h^2(z) : dz \right) \right]$$

$$= \frac{\mathcal{Z}}{\mathcal{Z}_f} \exp\left(\int_D \phi_f(z)g(z) - \frac{1}{2}m^2(z)\phi_f(z)^2 dz \right) \mathbb{E}_{\mathrm{mGFF}}^D\left[\exp((h, g - m^2\phi_f)) \right]. \tag{4.10}$$

The fact that the two integrals above are finite follows from the fact that D is a bounded domain and that ϕ_f and m are bounded functions in D. Similarly, we have that

$$\mathcal{Z}_f = \mathcal{Z} \exp\left(-\frac{1}{2} \int_D m^2(z)\phi_f(z)^2 dz \right) \mathbb{E}_{\mathrm{mGFF}}^D\left[\exp((h, m^2\phi_f)) \right]. \tag{4.11}$$

Going back to (4.10), this yields that

$$\mathbb{E}_{\mathrm{mGFF}}^{D,f}\big[\exp((h+\phi_f,g))\big]$$
$$= \exp\left(\int_D g(z)\left(\phi_f(z) - \int_D m^2(w)\phi_f(w)G_D^m(z,w)dw\right)dz + \frac{1}{2}\int_{D\times D} g(z)G_D^m(z,w)g(w)dzdw\right)$$
$$= \exp\left(\int_D g(z)\phi_f^m(z)dz + \frac{1}{2}\int_{D\times D} g(z)G_D^m(z,w)g(w)dzdw\right)$$

where the last equality uses Fubini theorem. This is exactly the Laplace transform of $(h+\phi_f^m,g)$ when h has the law of a massive GFF in D with mass m and Dirichlet boundary conditions. $\qquad\square$

For later reference, let us record the computation of the normalization constant $\mathcal{Z}_f$ in the following lemma.

Lemma 4.1.5. *Let $D \subset \mathbb{C}$ be an open, bounded and simply connected domain. Let $\phi : D \to \mathbb{R}$ be a bounded harmonic function in D and let $m : D \to \mathbb{R}_+$ be a bounded and continuous function. Then*

$$\mathcal{Z}_f = \exp\left(-\frac{1}{2}\int_D m^2(z)\phi_f(z)\phi_f^m(z)dz + \frac{1}{2}\mu_D\big(e^{-\langle \ell,m^2\rangle} + \langle \ell,m^2\rangle - 1\big)\right)$$

where the integral is finite.

Proof. By (4.11) and using the explicit expression (4.8) for $\mathcal{Z}$, we have that

$$\mathcal{Z}_f = \exp\left(-\frac{1}{2}\int_D m^2(z)\phi_f(z)^2 dz + \frac{1}{2}\mu_D\big(e^{-\langle \ell,m^2\rangle} + \langle \ell,m^2\rangle - 1\big)\right)\mathbb{E}_{\mathrm{mGFF}}^D\big[\exp((h,m^2\phi_f))\big]$$

where

$$\mathbb{E}_{\mathrm{mGFF}}^D\big[\exp((h,m^2\phi_f))\big] = \exp\left(\frac{1}{2}\int_{D\times D} m^2(z)\phi_f(z)G_D^m(z,w)m^2(w)\phi_f(w)dwdz\right)$$
$$= \exp\left(-\frac{1}{2}\int_D m^2(z)\phi_f(z)\phi_f^m(z)dz + \frac{1}{2}\int_D m^2(z)\phi_f(z)^2 dz\right). \quad (4.12)$$

Above, we used Fubini theorem whose application can easily be justified. Indeed, ϕ_f is a bounded function in D and the massive Green function G_D^m is integrable in $D \times D$ since D is a bounded domain. This yields the expression of Lemma 4.1.5 for $\mathcal{Z}_f$. The finiteness of the two integrals in (4.12) simply follows from the fact that m^2, ϕ_f and ϕ_f^m are bounded functions and that D is a bounded domain. $\qquad\square$

Note that the definition of the massive GFF and the changes of measure (4.7) and (4.9) can be extended to unbounded domains, provided that in such a domain D, the mass function $m : D \to \mathbb{R}_+$ is of the form $|\varphi'|(\hat{m}\circ\varphi)$ where $\varphi : D \to \hat{D}$ is a conformal isomorphism, $\hat{D}$ is a bounded domain and $\hat{m} : \hat{D} \to \mathbb{R}_+$ is a bounded and continuous function.

Modes of convergence of the Radon-Nikodym derivative of the massive GFF with respect to the GFF

In this section, we collect some technical results on the mode of convergence as $\epsilon \to 0$ of the random variables

$$\exp\left(-\frac{1}{2}\int_D m^2(z)\big(h_\epsilon(z)^2 - \mathbb{E}[h_\epsilon(z)^2]\big)dz\right) \quad \text{and}$$
$$\exp\left(-\frac{1}{2}\int_D m^2(z)\big((h_\epsilon(z)+\phi_\epsilon(z))^2 - \mathbb{E}[h_\epsilon(z)^2]\big)dz\right)$$

where h is a GFF with Dirichlet boundary conditions in D and $\phi : D \to \mathbb{R}$ is a bounded harmonic function. These random variables approximate the Radon-Nikodym derivatives (4.7) and (4.9). Their convergence in an $\mathrm{L}^p(\mathbb{P})$ sense, which is shown below, will be useful in the proof of Theorem 1.2.1 and Theorem 1.2.3. Indeed, there, we will need to take conditional expectations of the Radon-Nikodym derivatives (4.7) and (4.9), which will be well behaved since conditioning is a contraction in L^p.

In the statement below, for $\epsilon > 0$, the random variable $\int_D m^2(z) : (h_\epsilon(z)+\phi_\epsilon(z))^2 : dz$ is defined as in (4.2) and we use the notation (4.5) for the random variable $\int_D m^2(z) : (h+\phi)^2(z) : dz$.

Lemma 4.1.6. *Let $D \subset \mathbb{C}$ be a bounded, open and simply connected domain and let $m : D \to \mathbb{R}_+$ be a bounded and continuous function. Let h be a GFF with Dirichlet boundary conditions in D and let $\phi : D \to \mathbb{R}$ be a harmonic function bounded by some constant $b_\phi > 0$. Then, for any $p \in [1, \infty)$,*

$$\lim_{\epsilon \to 0} \exp\left(-\frac{1}{2} \int_D m^2(z) : (h_\epsilon(z) + \phi_\epsilon(z))^2 : dz \right) = \exp\left(-\frac{1}{2} \int_D m^2(z) : (h + \phi)^2(z) : dz \right)$$

where the limit is in $\mathrm{L}^p(\mathbb{P})$.

The proof of Lemma 4.1.6 is given in Appendix 6.1. This is for the proof of this lemma that the rate of convergence (4.4) obtained in Lemma 4.1.1 is important.

Note that by taking $\phi \equiv 0$ in Lemma 4.1.6, we obtain that the random variables approximating the Radon-Nikodym derivative (4.7) converge in $\mathrm{L}^p(\mathbb{P})$, for any $p \in [1, \infty)$.

Assumptions on the mass and on the domain of definition of the massive GFF

So far, in our discussion about the massive GFF, the domain $D \subset \mathbb{C}$ and the mass $m : D \to \mathbb{R}_+$ have always been chosen to satisfy the assumptions of Theorem 1.2.1 and Theorem 1.2.3. However, these assumptions are not optimal, as we now explain.

In Theorem 1.2.1, we will show that a massive GFF can be coupled to a random curve which has the law of massive SLE$_4$. Massive SLE$_4$ is in fact well-defined and absolutely continuous with respect to SLE$_4$ provided that the domain D is bounded and that the mass $m : D \to \mathbb{R}_+$ satisfies

$$\int_D m^2(z) dz < \infty \quad \text{and} \quad \int_{D \times D} m^2(z) m^2(w) G_D(z, w) dz dw < \infty, \tag{4.13}$$

where G_D is the Green function in D. For the massive GFF to be absolutely continuous with respect to the GFF with Radon-Nikodym derivative (4.7), the mass m must also be such that

$$\int_{D \times D} m^2(z) m^2(w) G_D(z, w)^2 dz dw < \infty. \tag{4.14}$$

Indeed, the condition (4.14) ensures that the random variable $\int_D m^2(z) : h(z)^2 : dz$ can be constructed as the limit in $L^2(\mathbb{P})$ of $\int_D m^2(z) : h_\epsilon(z)^2 : dz$ as $\epsilon \to 0$. This is because if this condition is not satisfied, then the variance of $\int_D m^2(z) : h_\epsilon(z)^2 : dz$ blows up as $\epsilon \to 0$.

Observe that the conditions (4.13) and (4.14) on m are conformally invariant in the following sense. If $\varphi : D \to \tilde{D}$ is a conformal map and that m satisfies (4.13) and (4.14), then the mass $\tilde{m} : \tilde{D} \to \mathbb{R}$ defined via, for $w \in \tilde{D}$, $\tilde{m}(w)^2 = |(\varphi^{-1})'(w)|^2 m(\varphi^{-1}(w))^2$ also satisfies (4.13) and (4.14). However, in their current version, the proofs of Theorem 1.2.1 and Theorem 1.2.3 require stronger assumptions on the domain $D \subset \mathbb{C}$ and the function $m : D \to \mathbb{R}_+$. These assumptions are helpful to prove Lemma 4.1.6.

We remark that the question of what happens when, for example, $D = \mathbb{H}$ and the mass m^2 is a constant $m^2 > 0$ is interesting. The massive GFF in $\mathbb{H}$ with mass m^2 does exist, since the massive Green function in $\mathbb{H}$ with mass m^2 is symmetric and positive-definite. However, this field is only locally absolutely continuous with respect to the GFF in $\mathbb{H}$ and thus we cannot define its level lines via a change of measure similar to (1.3) or (1.4).

4.2 Massive SLE$_4$ via change of measure

In this section, we prove Theorem 1.2.1 and its corollary, Corollary 1.2.2. The proof of Theorem 1.2.1 is given in Section 4.2.1. There, we first identify the conditional law of $h + \phi$ given $\gamma([0, t])$ under $\tilde{\mathbb{P}}$ and then use the knowledge of this conditional law to show that the marginal law of γ under $\tilde{\mathbb{P}}$ is that of massive SLE$_4$. We will explain how to use these results to show Theorem 1.2.1. In the proof, we will assume that the domain D is bounded and that the mass $m : D \to \mathbb{R}_+$ is a bounded and continuous function. Conformal covariance then enables us to extend the results to pairs (D, m) satisfying the more general assumptions of Theorem 1.2.1.

Finally, Section 4.2.2 is devoted to the proof of Corollary 1.2.2. Conformal covariance of massive SLE$_4$ is also established there.

4.2.1 A level line of the massive GFF

Let us first observe that the change of measure (1.3) of Theorem 1.2.1 is a valid change of measure. Indeed, this is just the change of measure (4.9), which extends to a change of measure on $(h+\phi, \gamma)$ since γ is measurable with respect to h under $\mathbb{P}_{\mathrm{GFF}}$. This implies in particular that the marginal law of $h + \phi$ under $\tilde{\mathbb{P}}$ is that of a massive GFF in D with mass m and boundary conditions λ on ∂D^+ and $-\lambda$ on ∂D^-.

To prove Theorem 1.2.1, we must now understand what happens to the field $h + \phi$ under $\tilde{\mathbb{P}}$ when conditioning on $\gamma([0,t])$, for $t \geq 0$. Denote by $\mathcal{S}'(\mathbb{R}^2)$ the space of Schwartz distributions on $\mathbb{R}^2$ and observe that the field $h + \phi$ can be seen as a random element of $\mathcal{S}'(\mathbb{R}^2)$ by extending it to 0 outside of D. By definition of $\tilde{\mathbb{P}}$, the conditional law of $h + \phi$ under $\tilde{\mathbb{P}}$ given γ is such that, for $t \geq 0$ and $F : \mathcal{S}'(\mathbb{R}^2) \to \mathbb{R}$ a bounded and continuous function, almost surely,

$$\tilde{\mathbb{E}}[F(h+\phi) \,|\, \gamma([0,t])] = \frac{1}{\mathcal{Z}_t} \mathbb{E}\left[F(h+\phi) \exp\left(-\frac{1}{2} \int_D m^2(z) : (h+\phi)^2(z) : dz \right) \,|\, \gamma([0,t]) \right] \quad (4.15)$$

where we have set

$$\mathcal{Z}_t := \mathbb{E}\left[\exp\left(-\frac{1}{2} \int_D m^2(z) : (h+\phi)^2(z) : dz \right) \,|\, \gamma([0,t]) \right]. \quad (4.16)$$

Observe that in (4.15), the random partition function $\mathcal{Z}_t$ is almost surely strictly positive: this follows from the fact that if X is an almost surely strictly positive random variable and $\mathcal{F}$ is a σ-algebra, then $\mathbb{E}[X|\mathcal{F}]$ is almost surely strictly positive. This implies in particular that the right-hand side of (4.15) is well-defined.

Since under $\mathbb{P}$, conditionally on $\gamma([0,t])$, $h + \phi = h_t + \phi_t$ where h_t and ϕ_t are as described in Section 4.1.3, one expects the random partition function $\mathcal{Z}_t$, or more generally weighted conditional expectations of the form

$$\mathbb{E}\left[\exp(i(h,f)) \exp\left(-\frac{1}{2} \int_D m^2(z) : (h+\phi)^2(z) : dz \right) \,|\, \gamma([0,t]) \right] \quad (4.17)$$

where $f : D \to \mathbb{R}$ is a bounded and continuous function, to have an expression in terms of the field h_t and the harmonic function ϕ_t. As a first step toward the identification of the conditional law of $h + \phi$ given $\gamma([0,t])$ under $\tilde{\mathbb{P}}$, let us show that this is indeed the case.

The weighted conditional expectation

Here and in the rest of Section 4.2, if $\gamma : [0,\infty) \to D$ is a simple and continuous curve in D that does not touch the boundary of D except at its endpoints, we set, for $t \geq 0$, $D_t := D \setminus \gamma([0,t])$. We also denote by ∂D_t^+, respectively ∂D_t^-, the boundary arc of D_t formed by the union of ∂D^+ and the left side of $\gamma([0,t])$, respectively the union of ∂D^- and the right side of $\gamma([0,t])$. For $t \geq 0$, we also let ϕ_t, respectively ϕ_t^m, be the unique harmonic function, respectively massive harmonic function with mass m, with boundary conditions λ on ∂D_t^+ and $-\lambda$ on ∂D_t^-.

Observe that by absolute continuity of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$, under $\tilde{\mathbb{P}}$, the marginal law of γ is such that γ is almost surely a simple and continuous curve that does not touch the boundary of D except at its endpoints. Indeed, under $\mathbb{P}$, the marginal law of γ is that of SLE_4 in D from a to b, so that these properties are satisfied by γ under $\mathbb{P}$.

Let us now state and prove how to express weighted conditional expectations of the form (4.17) in terms of the field h_t and the harmonic function ϕ_t. We first start with the case $t \in (0, \infty)$.

Proposition 4.2.1. *Under the same assumptions as in Theorem 1.2.1, let $f : D \to \mathbb{R}$ be a bounded and continuous function. Then, for any $t \in (0,\infty)$, almost surely,*

$$\mathbb{E}\left[\exp(i(h+\phi,f)) \exp\left(-\frac{1}{2} \int_D m^2(z) : (h+\phi)^2(z) : dz \right) \,|\, \gamma([0,t]) \right]$$

$$= \mathbb{E}\left[\exp(i(h_t+\phi_t,f)) \exp\left(-\frac{1}{2} \int_{D_t} m^2(z) : (h_t+\phi_t)^2(z) : dz \right) |\, \gamma([0,t]) \right]$$

$$\times \exp\left(\int_{D_t} \frac{m^2(z)}{4\pi} \log \frac{\mathrm{CR}(z, \partial D)}{\mathrm{CR}(z, \partial D_t)} dz \right)$$

where $\mathrm{CR}(z, \partial D)$, respectively $\mathrm{CR}(z, \partial D_t)$, is the conformal radius of z in D, respectively D_t. Above, the random variable $\int_{D_t} m^2(z) : (h_t + \phi_t)^2(z) : dz$ is as defined via the notation (4.5).

Proof. The proof is similar to that of Proposition 4.3.6, once the claim below is proved. Details are left to the reader.

Claim 4.2.2. *Almost surely,*

$$\lim_{\epsilon \to 0} \exp\left(\frac{1}{2}\int_{D_t} m^2(z)\big(\mathbb{E}[h_\epsilon(z)^2] - \mathbb{E}_t[h_{t,\epsilon}(z)^2]\big)dz\right) = \exp\left(\int_{D_t} \frac{m^2(z)}{4\pi}\log\frac{\mathrm{CR}(z,\partial D)}{\mathrm{CR}(z,\partial D_t)}dz\right).$$

$\square$

Proof of Claim 4.2.2. Let $\epsilon > 0$ and set $D_{t,\epsilon} := \{z \in D_t : \mathrm{dist}(z, \gamma([0,t])) > \epsilon\}$. Observe that, almost surely, if $z \in D_{t,\epsilon}$ is such that $\mathrm{dist}(z, \partial D) > \epsilon$, then

$$\mathbb{E}[h_\epsilon(z)^2] - \mathbb{E}_t[h_{t,\epsilon}(z)^2] = \frac{1}{2\pi}\log\frac{\mathrm{CR}(z,\partial D)}{\mathrm{CR}(z,\partial D_t)}. \tag{4.18}$$

On the other hand, it is easy to see that there exists $K > 0$ such that for any $\epsilon > 0$, almost surely, for any $z \in D_{t,\epsilon}$ with $\mathrm{dist}(z, \partial D) \leq \epsilon$,

$$0 \leq \mathbb{E}[h_\epsilon(z)^2] - \mathbb{E}_t[h_{t,\epsilon}(z)^2] \leq \mathbb{E}[h_\epsilon(z)^2] \leq K. \tag{4.19}$$

Since almost surely for any $z \in D_t$, $\mathrm{CR}(z,\partial D) \geq \mathrm{CR}(z,\partial D_t)$ and since the function m is bounded, we obtain from (4.18) and (4.19) that there exists $C > 0$ such that, almost surely, for any $z \in D_{t,\epsilon}$,

$$m^2(z)|\mathbb{E}[h_\epsilon(z)^2] - \mathbb{E}_t[h_{t,\epsilon}(z)^2]| \leq C\left(\log\frac{\mathrm{CR}(z,\partial D)}{\mathrm{CR}(z,\partial D_t)} + 1\right). \tag{4.20}$$

Observe also that, almost surely,

$$\lim_{\epsilon \to 0} m^2(z)(\mathbb{E}[h_\epsilon(z)^2] - \mathbb{E}_t[h_{t,\epsilon}(z)^2])\mathbb{I}_{D_{t,\epsilon}}(z) = \frac{m^2(z)}{2\pi}\log\frac{\mathrm{CR}(z,\partial D)}{\mathrm{CR}(z,\partial D_t)}$$

pointwise in D_t. Moreover, the function $z \mapsto \log(\mathrm{CR}(z,\partial D)/\mathrm{CR}(z,\partial D_t))$ is almost surely integrable in D_t. Therefore, using the dominated convergence theorem with the bound (4.20), we obtain that, almost surely,

$$\lim_{\epsilon \to 0} \int_{D_{t,\epsilon}} m^2(z)(\mathbb{E}[h_\epsilon(z)^2] - \mathbb{E}_t[h_{t,\epsilon}(z)^2])dz = \int_{D_t} \frac{m^2(z)}{2\pi}\log\frac{\mathrm{CR}(z,\partial D)}{\mathrm{CR}(z,\partial D_t)}dz.$$

To prove Claim 4.2.2, it remains to show that, almost surely,

$$\lim_{\epsilon \to 0} \int_{D_t \setminus D_{t,\epsilon}} m^2(z)(\mathbb{E}[h_\epsilon(z)^2] - \mathbb{E}_t[h_{t,\epsilon}(z)^2])dz = 0. \tag{4.21}$$

For this, we simply observe that the estimate (6.1) shows that there exists $C > 0$ such that for any $0 < \epsilon < 1/2$ and any $z \in D_t$, $0 \leq \mathbb{E}[h_\epsilon(z)^2] - \mathbb{E}_t[h_{t,\epsilon}(z)^2] \leq \sup_{z \in D} \mathbb{E}[h_\epsilon(z)^2] \leq C\log\epsilon^{-1}$. Together with the boundedness of m, this yields that, almost surely,

$$\left|\int_{D_t \setminus D_{t,\epsilon}} m^2(z)(\mathbb{E}[h_\epsilon(z)^2] - \mathbb{E}_t[h_{t,\epsilon}(z)^2])dz\right| \leq \overline{m}^2 C(\log\epsilon^{-1})\mathrm{Area}(D_t \setminus D_{t,\epsilon}). \tag{4.22}$$

Since $\gamma([0,t])$ almost surely has Hausdorff dimension $3/2$ [Bef08], we have that, almost surely, $\mathrm{Area}(D_t \setminus D_{t,\epsilon}) = O(\epsilon^{\frac{1}{2}-})$. Therefore, the inequality (4.22) implies the almost sure convergence (4.21), which completes the proof of the claim. $\square$

Let us now give an expression for weighted conditional expectations of the form (4.26) when $t = \infty$. In this case, $D_\infty := D \setminus \gamma([0,\infty))$ is composed of two simply connected components. Recall that we denote by D_1, respectively D_2, the component on the left, respectively right, of $\gamma([0,\infty))$. Moreover, under $\mathbb{P}$, conditionally on $\gamma([0,\infty])$, $h + \phi = h_1 + \phi_1 + h_2 + \phi_2$, where h_1, h_2, ϕ_1 and ϕ_2 are as described in Section 4.1.3.

Proposition 4.2.3. *Under the same assumptions as in Theorem 1.2.1, let $f : D \to \mathbb{R}$ be a bounded and continuous function. Then, almost surely,*

$$
\mathbb{E}\left[\exp(i(h+\phi,f)) \exp\left(-\frac{1}{2} \int_D m^2(z) : (h+\phi)^2(z) : dz \right) | \gamma([0,\infty)) \right]
$$

$$
= \mathbb{E}\left[\exp(i(h_1+\phi_1,f)) \exp\left(-\frac{1}{2} \int_{D_1} m^2(z) : (h_1+\phi_1)^2(z) : dz \right) | \gamma([0,\infty)) \right]
$$

$$
\times \mathbb{E}\left[\exp(i(h_2+\phi_2,f)) \exp\left(-\frac{1}{2} \int_{D_2} m^2(z) : (h_2+\phi_2)^2(z) : dz \right) | \gamma([0,\infty)) \right]
$$

$$
\times \exp\left(\int_{D_1} \frac{m^2(z)}{4\pi} \log \frac{\mathrm{CR}(z,\partial D)}{\mathrm{CR}(z,\partial D_1)} dz + \int_{D_2} \frac{m^2(z)}{4\pi} \log \frac{\mathrm{CR}(z,\partial D)}{\mathrm{CR}(z,\partial D_2)} \right)
$$

where $\mathrm{CR}(z,\partial D)$, *respectively* $\mathrm{CR}(z,\partial D_j)$, $j=1,2$, *is the conformal radius of z in D, respectively D_j.*

Proof. This is left to the reader, since the proof is identical to but simpler than the proof of Proposition 4.3.1, or alternatively may be obtained by taking t to ∞ in Proposition 4.2.1. $\square$

The conditional law of $h + \phi$ under $\tilde{\mathbb{P}}$

With Proposition 4.2.1 in hands, let us now identify the conditional law of $h + \phi$ given $\gamma([0,t])$ under $\tilde{\mathbb{P}}$, for $t \geq 0$. In this section, we keep the same notations as those adopted in Section 4.2.1 for the slit domains $(D_t)_{t\geq0}$, their boundary arcs $(\partial D_t^+)_{t\geq0}$ and $(\partial D_t^-)_{t\geq0}$ and the functions $(\phi_t)_{t\geq0}$ and $(\phi_t^m)_{t\geq0}$.

Proposition 4.2.4. *With the same assumptions as in Theorem 1.2.1, let $t \geq 0$. Then, under $\tilde{\mathbb{P}}$, conditionally on $\gamma([0,t])$, $h + \phi = h_t + \phi_t$ where $h_t + \phi_t$ has the law of a massive GFF in D_t with mass m and with boundary conditions λ on ∂D_t^+ and $-\lambda$ on ∂D_t^-. In other words, under $\tilde{\mathbb{P}}$, the conditional law of $h + \phi$ given $\gamma([0,t])$ is that of $h_t^m + \phi_t^m$ where h_t^m has the law of a massive GFF in D_t with mass m and with Dirichlet boundary conditions and ϕ_t^m is the massive harmonic function in D_t with boundary values λ on ∂D_t^+ and $-\lambda$ on ∂D_t^-.*

Proof. This is left to the reader, since, using Proposition 4.2.1, the proof is identical to but simpler than the proof of Proposition 4.3.6. $\square$

Let us now establish the conditional law of $h + \phi$ under $\tilde{\mathbb{P}}$ conditionally on $\gamma([0,\infty))$. Below, the domains D_j and the functions ϕ_j for $j = 1, 2$ are as described in Section 4.1.3.

Proposition 4.2.5. *With the same assumptions as in Theorem 1.2.1, under $\tilde{\mathbb{P}}$, conditionally on $\gamma([0,\infty))$, $h + \phi = h_1 + \phi_1 + h_2 + \phi_2$ where for $j = 1, 2$, $h_j + \phi_j$ has the law of a massive GFF in D_j with mass m and boundary conditions ϕ_j. Moreover, under $\tilde{\mathbb{P}}$, conditionally on $\gamma([0,\infty))$, $h_1 + \phi_1$ and $h_2 + \phi_2$ are independent.*

Proof. This is left to the reader, since, using Proposition 4.2.3, the proof is identical to but simpler than the proof of Proposition 4.3.6. $\square$

The marginal law of γ under $\tilde{\mathbb{P}}$

To conclude the proof of Theorem 1.2.1, we must identify the marginal of γ under $\tilde{\mathbb{P}}$.

Proposition 4.2.6. *With the same assumptions as in Theorem 1.2.1, under $\tilde{\mathbb{P}}$, the marginal law of γ is that of massive SLE_4 with mass m in D from a to b.*

The proof of Proposition 4.2.6 relies on the martingale characterization of massive SLE_4 established in Section 3.4 in Chapter 3. We recall it just below in a slightly weaker version than in Chapter 3, in the sense that the statement actually holds if the curve is only non-self-touching, and not necessarily simple. However, since this stronger assumption is satisfied here (by absolute continuity of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$), this does not really matter.

Proposition 4.2.7. *Let $D \subset \mathbb{C}$ be an open, bounded and simply connected domain and let $a, b \in \partial D$. Let $\gamma : [0, \infty) \to D$ be a random simple curve from a to b which does not hit ∂D except at its endpoints and denote by $(\mathcal{F}_t)_t$ the filtration generated by γ. Let $m : D \to \mathbb{R}_+$ be a bounded and continuous function. For $z \in D$, define the stopping time τ_z for $(\mathcal{F}_t)_t$ by $\tau_z := \inf\{t \geq 0 : z \in \gamma([0, t])\}$. For $t \geq 0$, denote also by H_t^m the unique massive harmonic function in $D_t := D \setminus \gamma([0, t])$ with boundary conditions $1/2$ on ∂D_t^+ and $-1/2$ on ∂D_t^-. Assume that, for any $z \in D$, $(H_t^m(z), 0 \leq t \leq \tau_z)$ is a martingale with respect to the filtration $(\mathcal{F}_t)_t$. Then, when appropriately time-parametrized, γ has the law of a massive SLE_4 curve in D from a to b with mass m.*

Proof of Proposition 4.2.6. To show Proposition 4.2.6, we wish to use the martingale characterization of massive SLE_4 given by Proposition 4.2.7. Observe first that since $\tilde{\mathbb{P}}$ is absolutely continuous with respect to $\mathbb{P}$, under $\tilde{\mathbb{P}}$, the marginal law of γ is such that γ is almost surely a simple and continuous curve that almost surely does not touch ∂D except at its endpoints. Thus, the marginal law of γ under $\tilde{\mathbb{P}}$ is such that γ is a random curve satisfying the first part of the assumptions of Proposition 4.2.7. To apply this proposition, it remains show that for any $z \in D$, $(H_t^m(z), 0 \leq t \leq \tau_z)$ is a martingale under $\tilde{\mathbb{P}}$, where H_t^m is defined as in Proposition 4.2.7. This will be established by using the coupling given by Proposition 4.2.4.

For $z \in D$ and $\epsilon > 0$, set $T_\epsilon(z) := \inf\{t \geq 0 : \operatorname{dist}(z, \gamma([0, t])) \leq \epsilon\}$. $T_\epsilon(z)$ is a stopping time for the filtration generated by γ. We denote by $\rho_{z,\epsilon}^m$ the massive harmonic measure with mass m of $\partial B(z, \epsilon)$ seen from z. That is, for any $\tilde{\partial} \subset \partial B(z, \epsilon)$,

$$\int_{\tilde{\partial}} \rho_{z,\epsilon}^m(dw) = \mathbb{E}_z^{(m)}[\mathbb{1}_{\tau^* > \tau_\epsilon} \mathbb{1}_{B_{\tau_\epsilon} \in \tilde{\partial}}]$$

where under $\mathbb{E}_z^{(m)}$, B has the law of a massive (killed) Brownian motion with mass (killing rate) m started at z, τ^* denotes its killing time and τ_ϵ its first hitting time of $\partial B(z, \epsilon)$. Moreover, by definition, if a function $f : D \to \mathbb{R}$ is a bounded massive harmonic function in D, then for any $\epsilon > 0$ and any $z \in D$ such that $\operatorname{dist}(z, \partial D) > \epsilon$, $\int f(w) \rho_{z,\epsilon}^m(dw) = f(z)$. Going back to the setting of Proposition 4.2.6, observe that, almost surely, for any $t \geq 0$ and any $z \in D_t$,

$$H_t^m(z) = \frac{1}{2\lambda} \phi_t^m(z).$$

Indeed, the functions on each side of the equality are massive harmonic in D_t with the same boundary values and therefore, they are equal. Now, let $z \in D$ and $\epsilon > 0$. To show that $(H_t^m(z), 0 \leq t \leq \tau_z)$ is a martingale under $\tilde{\mathbb{P}}$, we are going to look at the "massive circle average" of the massive GFF $h + \phi$. This will allow us to express $H_t^m(z)$ as an observable of the field $h + \phi$ conditioned on $\gamma([0, t])$ under $\tilde{\mathbb{P}}$. This massive circle average is defined in a similar way as the circle average of the GFF (see Section 4.1.4): this is the random variable $(h + \phi, \rho_{z,\epsilon}^m)$. Even though $\rho_{z,\epsilon}^m$ is not a smooth function, this is a well-defined random variable for the same reasons as in the case of the circle average of the GFF.

We can then deduce from the above discussion and Proposition 4.2.4 that, almost surely,

$$\tilde{\mathbb{E}}\big[(h + \phi, \rho_{z,\epsilon}^m)|\gamma([0, t])\big]\mathbb{1}_{t < T_\epsilon(z)} = \tilde{\mathbb{E}}\big[(h_t + \phi_t, \rho_{z,\epsilon}^m)|\gamma([0, t])\big]\mathbb{1}_{t < T_\epsilon(z)}$$
$$= (\phi_t^m, \rho_{z,\epsilon}^m)\mathbb{1}_{t < T_\epsilon(z)} = \phi_t^m(z)\mathbb{1}_{t < T_\epsilon(z)}.$$

The second equality follows from Proposition 4.2.4 which shows that under $\tilde{\mathbb{P}}$, conditionally on $\gamma([0, t])$, $h + \phi = h_t^m + \phi_t^m$ where h_t^m has the law of a massive GFF in D_t with mass m and Dirichlet boundary conditions. The third equality uses the massive harmonicity of ϕ_t^m. This implies that, almost surely,

$$H_t^m(z)\mathbb{1}_{\{t < T_\epsilon(z)\}} = \frac{1}{2\lambda}\tilde{\mathbb{E}}\big[(h + \phi, \rho_{z,\epsilon}^m)|\gamma([0, t])\big]\mathbb{1}_{\{t < T_\epsilon(z)\}}.$$

Taking $\epsilon \to 0$, it follows that $(H_t^m(z), 0 \leq t \leq \tau_z)$ is a martingale with respect to $(\mathcal{F}_t)_t$ under $\tilde{\mathbb{P}}$, where $(\mathcal{F}_t)_t$ is the filtration generated by γ. By Proposition 4.2.7, this shows that under $\tilde{\mathbb{P}}$, the marginal law of γ is that of a massive SLE_4 with mass m in D from a to b. $\qquad\square$

We can now conclude the proof of Theorem 1.2.1.

Proof of Theorem 1.2.1. $\tilde{\mathbb{P}}$ is a well-defined probability measure which is absolutely continuous with respect to $\mathbb{P}$. Item (1) follows by definition of $\tilde{\mathbb{P}}$ while Proposition 4.2.6 shows Item (2). Proposition 4.2.4 establishes the correct conditional law of $h + \phi$ given $\gamma([0, t])$, for $t \in (0, \infty)$ under $\tilde{\mathbb{P}}$. Finally, Proposition 4.2.5 shows the part of the statement about the conditional law of $h + \phi$ given $\gamma([0, \infty))$ under $\tilde{\mathbb{P}}$. $\square$

4.2.2 Some properties of massive SLE$_4$

We now use Theorem 1.2.1 to show two properties of massive SLE$_4$, stated in Corollary 1.2.2 and Lemma 4.2.8.

Proof of Corollary 1.2.2. By definition of $\tilde{\mathbb{P}}$, we have that, for $t \geq 0$,

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}((h + \phi, \gamma))\Big|_{\sigma(\gamma(s), s \leq t)} = \frac{1}{\mathcal{Z}}\mathbb{E}\left[\exp\left(-\frac{1}{2}\int_D m^2(z) : (h + \phi)^2(z) : dz\right)\Big|\gamma([0, t])\right] = \frac{\mathcal{Z}_t}{\mathcal{Z}}$$

where $\mathcal{Z}$ is as in the statement of Theorem 1.2.1. In other words, for any random variable Y which is $\sigma(\gamma(s), s \leq t)$-measurable, $\tilde{\mathbb{E}}[Y] = \mathbb{E}[\mathcal{Z}_t/\mathcal{Z}]$. Since under $\mathbb{P}$, the marginal law of γ is that of SLE$_4$ in D from a to b while, by Theorem 1.2.1, under $\tilde{\mathbb{P}}$, the marginal law of γ is that of massive SLE$_4$ in D from a to b with mass m, this implies that, almost surely,

$$\frac{\mathrm{d}\mathbb{P}_{\mathrm{mSLE}_4}^{(D,a,b)}}{\mathrm{d}\mathbb{P}_{\mathrm{SLE}_4}^{(D,a,b)}}\Big|_{\sigma(\gamma(s), s \leq t)} = \frac{\mathcal{Z}_t}{\mathcal{Z}}. \tag{4.23}$$

Proposition 4.2.1 applied with $f \equiv 0$ and Lemma 4.1.5 used in the domain D_t yield that, almost surely,

$$\mathcal{Z}_t = \exp\left(-\frac{1}{2}\int_{D_t} m^2(z)\phi_t(z)\phi_t^m(z)dz + \int_{D_t}\frac{m^2(z)}{4\pi}\log\frac{\mathrm{CR}(z, \partial D)}{\mathrm{CR}(z, \partial D_t)}dz\right)$$
$$\exp\left(\frac{1}{2}\mu_t(e^{-\langle \ell, m^2\rangle} + \langle \ell, m^2\rangle - 1)\right).$$

The statement of the corollary then follows from this equality and (4.23). $\square$

Lemma 4.2.8. *Under the same assumptions as in Theorem 1.2.1 on D and m, let $\varphi : D \to \tilde{D}$ be a conformal map. If a curve γ has the law of massive SLE$_4$ in D from a to b with mass m, then $\varphi(\gamma)$ has the law of massive SLE$_4$ in $\tilde{D}$ from $\varphi(a)$ to $\varphi(b)$ with mass m given by, for $w \in \tilde{D}$, $\tilde{m}^2(w) = |(\varphi^{-1})'(w)|^2 m^2(\varphi^{-1}(w))$.*

Proof. This follows from the conformal invariance of $\mathbb{P}_{\mathrm{SLE}_4}^{(D,a,b)}$ and the conformal covariance of each term appearing in the Radon-Nikodym derivative $\mathbb{P}_{\mathrm{mSLE}_4}^{(D,a,b)}/\mathbb{P}_{\mathrm{SLE}_4}^{(D,a,b)}$ given by Corollary 1.2.2. $\square$

4.3 Massive CLE$_4$ via change of measure

In this section, we prove Theorem 1.2.3 and its corollaries, Corollary 1.2.6 and Corollary 1.2.5. Theorem 1.2.3 is established in Section 4.3.1. There, we first identify the conditional law under $\tilde{\mathbb{P}}$ of the random variables $(\xi_j)_j$ given Γ. Then, we turn to the identification of the conditional law under $\tilde{\mathbb{P}}$ of the fields $(h_j + \xi_j)_j$ given Γ. We will explain how to use these results to prove Theorem 1.2.3. In the proof, we will assume that the domain D is bounded and that the mass $m : D \to \mathbb{R}_+$ is a bounded and continuous function. Conformal covariance then enables us to extend the results to pairs (D, m) satisfying the more general assumptions of Theorem 1.2.3. Finally, in Section 4.3.2, we establish Corollary 1.2.6 and Corollary 1.2.5.

4.3.1 Level lines of the massive GFF

Let us first observe that the change of measure (1.4) of Theorem 1.2.3 is well-defined. Indeed, this is just the change of measure (4.7), which extends to a change of measure on (h, Γ) since Γ is measurable with respect to h under $\mathbb{P}_{\mathrm{GFF}}$. This implies in particular that the marginal law of h under $\tilde{\mathbb{P}}$ is that of a massive GFF in D with mass m and Dirichlet boundary conditions.

The next step to prove Theorem 1.2.3 is to understand how the field h behaves under $\tilde{\mathbb{P}}$ when conditioning on Γ. By definition of $\tilde{\mathbb{P}}$ via the change of measure (1.4), the conditional law of h under $\tilde{\mathbb{P}}$ given Γ is such that, for $F : \mathcal{S}'(\mathbb{R}^2) \to \mathbb{R}$ a bounded and continuous function, almost surely,

$$\tilde{\mathbb{E}}[F(h)|\Gamma] = \frac{1}{\mathcal{Z}(\Gamma)} \mathbb{E}\left[F(h) \exp\left(-\frac{1}{2} \int_D m^2(z) : h^2(z) : dz \right) | \Gamma \right] \tag{4.24}$$

where we have set

$$\mathcal{Z}(\Gamma) := \mathbb{E}\left[\exp\left(-\frac{1}{2} \int_D m^2(z) : h^2(z) : dz \right) | \Gamma \right]. \tag{4.25}$$

Note that the random partition function $\mathcal{Z}(\Gamma)$ is almost surely strictly positive: this again follows from the fact that if X is an almost surely strictly positive random variable and $\mathcal{F}$ is a σ-algebra, then $\mathbb{E}[X|\mathcal{F}]$ is almost surely strictly positive. This implies that the right-hand side of (4.24) is well-defined.

Since under $\mathbb{P}$, conditionally on Γ, $h = \sum_j h_j + \xi_j$ where $(h_j)_j$ and $(\xi_j)_j$ are as described in Section 4.1.3, the random partition function $\mathcal{Z}(\Gamma)$, or more generally weighted conditional expectations of the form

$$\mathbb{E}\left[\exp(i(h,f)) \exp\left(-\frac{1}{2} \int_D m^2(z) : h^2(z) : dz \right) | \Gamma \right] \tag{4.26}$$

where $f : D \to \mathbb{R}$ is a bounded and continuous function, should have an expression in terms of the fields $(h_j)_j$ and the labels $(\xi_j)_j$. As a first step toward the identification of the conditional law of h given Γ under $\tilde{\mathbb{P}}$, let us establish such an expression.

The weighted conditional expectation

Observe that by absolute continuity of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$, under $\tilde{\mathbb{P}}$, the marginal law of Γ is such that Γ is almost surely a countable collection of planar simple loops which do not touch each other or ∂D. Indeed, under $\mathbb{P}$, the marginal law of Γ is that of a CLE_4 in D, so that these properties are satisfied by Γ under $\mathbb{P}$.

With these preliminary remarks made, let us now show how to express the weighted conditional expectations of the form (4.26) in terms of the fields $(h_j)_j$ and the labels $(\xi_j)_j$.

Proposition 4.3.1. *Under the same assumptions as in Theorem 1.2.3 and with the same notations, let $f : D \to \mathbb{R}$ be a bounded and continuous function. Then, almost surely,*

$$\mathbb{E}\left[\exp(i(h,f)) \exp\left(-\frac{1}{2} \int_D m^2(z) : h^2(z) : dz \right) | \Gamma \right]$$

$$= \prod_j \mathbb{E}\left[\exp(i(h_j + \xi_j, f)) \exp\left(-\frac{1}{2} \int_{\mathrm{Int}(L_j)} m^2(z) : (h_j + \xi_j)^2(z) : dz \right) | \Gamma \right]$$

$$\times \exp\left(\int_{\mathrm{Int}(L_j)} \frac{m^2(z)}{4\pi} \log \frac{\mathrm{CR}(z, \partial D)}{\mathrm{CR}(z, L_j)} dz \right)$$

where the product runs over the loops $(L_j)_j$ of Γ and $\mathrm{CR}(z, \partial D)$, respectively $\mathrm{CR}(z, L_j)$, denotes the conformal radius of z in D, respectively in $\mathrm{Int}(L_j)$. Above, $(h_j)_j$ and $(\xi_j)_j$ are as described in Section 4.1.3 and the random variables $(\int_{\mathrm{Int}(L_j)} m^2(z) : (h_j + \xi_j)^2(z) : dz)_j$ are as defined via (4.5).

Proof. Let $f : D \to \mathbb{R}$ be a bounded and continuous function. By Lemma 4.1.6 and since almost surely $|\exp(i(h,f))| \le 1$, we have that

$$\exp(i(h,f)) \exp\left(-\frac{1}{2} \int_D m^2(z) : h(z)^2 : dz \right)$$

$$= \lim_{\epsilon \to 0} \exp(i(h,f)) \exp\left(-\frac{1}{2} \int_D m^2(z) (h_\epsilon(z)^2 - \mathbb{E}[h_\epsilon(z)^2]) dz \right)$$

where the limit is in $\mathrm{L}^1(\mathbb{P})$. This yields that, almost surely,

$$\mathbb{E}\left[\exp(i(h,f)) \exp\left(-\frac{1}{2} \int_D m^2(z) : h(z)^2 : dz \right) | \Gamma \right]$$

$$= \lim_{\epsilon \to 0} \mathbb{E}\left[\exp(i(h,f)) \exp\left(-\frac{1}{2} \int_D m^2(z) (h_\epsilon(z)^2 - \mathbb{E}[h_\epsilon(z)^2]) dz \right) | \Gamma \right]$$

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where again the limit is in $L^1(\mathbb{P})$. Using the fact that, as explained in Section 4.1.3, under $\mathbb{P}$, conditionally on Γ, $h = \sum_j h_j + \xi_j$ with $(h_j)_j$ and $(\xi_j)_j$ as described there, we obtain that, almost surely,

$$
\mathbb{E}\left[\exp(i(h, f)) \exp\left(-\frac{1}{2} \int_D m^2(z) : h(z)^2 : dz\right) \Big| \Gamma\right]
$$
$$
= \lim_{\epsilon \to 0} \mathbb{E}\left[\exp(i(\sum_j h_j + \xi_j, f))\right.
$$
$$
\exp\left(-\frac{1}{2} \int_D m^2(z)\left(\left(\sum_j h_{j,\epsilon}(z) + \xi_{j,\epsilon}(z)\right)^2 - \sum_j \mathbb{E}[h_{j,\epsilon}(z)^2 | \Gamma]\right) dz\right) \Big| \Gamma\right] \quad (4.27)
$$
$$
\times \exp\left(\frac{1}{2} \int_D m^2(z)\left(\mathbb{E}[h_\epsilon(z)^2] - \sum_j \mathbb{E}[h_{j,\epsilon}(z)^2 | \Gamma]\right) dz\right),
$$

where the limit is in $L^1(\mathbb{P})$. Above, for $\epsilon > 0$, $z \in D$ and any j, we have set

$$
\xi_{j,\epsilon}(z) := \xi_j(\mathbb{I}_{\mathrm{Int}(L_j)}, \rho_\epsilon^z) = \frac{\xi_j}{2\pi\epsilon} \mathcal{L}\big(\partial B(z, \epsilon) \cap \mathrm{Int}(L_j)\big) \quad (4.28)
$$

where $\mathcal{L}$ denotes the length with respect to the arc-length measure. As before, for any j, $h_{j,\epsilon}(z)$ stands for the random variable (h_j, ρ_ϵ^z). Now, for $\epsilon > 0$, let us define

$$
I_\epsilon := -\frac{1}{2} \int_D m^2(z)\left(\left(\sum_j h_{j,\epsilon}(z) + \xi_{j,\epsilon}(z)\right)^2 - \sum_j \mathbb{E}[h_{j,\epsilon}(z)^2 | \Gamma]\right) dz,
$$
$$
J_\epsilon := \frac{1}{2} \int_D m^2(z)\left(\mathbb{E}[h_\epsilon(z)^2] - \sum_j \mathbb{E}[h_{j,\epsilon}(z)^2 | \Gamma]\right) dz.
$$

To prove Proposition 4.3.1, we must control the limit of I_ϵ and J_ϵ as ϵ tends to 0. We claim the following. Below, we denote by $\mathbb{P}_\Gamma$ the (random) conditional law induced by $\mathbb{E}_\Gamma[\cdot] := \mathbb{E}[\cdot | \Gamma]$.

Claim 4.3.2. *For $\mathbb{P}$-almost every Γ,*

$$
\lim_{\epsilon \to 0} I_\epsilon = -\frac{1}{2} \sum_j \int_{\mathrm{Int}(L_j)} m^2(z) : (h_j + \xi_j)^2(z) : dz
$$

where the limit is in $L^2(\mathbb{P}_\Gamma)$.

Claim 4.3.3. *Almost surely,*

$$
\lim_{\epsilon \to 0} J_\epsilon = \sum_j \int_{\mathrm{Int}(L_j)} \frac{m^2(z)}{4\pi} \log \frac{\mathrm{CR}(z, \partial D)}{\mathrm{CR}(z, L_j)} dz
$$

where the sum on the right-hand side is almost surely finite.

We postpone the proof of these claims to the end and show how to prove Proposition 4.3.1 based on them. On the one hand, we know that $e^{I_\epsilon} e^{J_\epsilon}$ converges to $\exp(-\frac{1}{2} \int_D m^2(z) : h(z)^2 : dz)$ in $\mathbb{P}$-probabilty as ϵ tends to 0. On the other hand, for any fixed $\delta > 0$ and for $\mathbb{P}$-almost every Γ, $\mathbb{P}_\Gamma(|e^{I_\epsilon} - e^{I_0}| > \delta)$ converges to 0 as ϵ tends to 0, where I_0 denotes the limiting random variable in Claim 4.3.2. By dominated convergence, this implies that e^{I_ϵ} converges to e^{I_0} in $\mathbb{P}$-probability as ϵ tends to 0. Together with Claim 4.3.3, this in turn yields that $e^{I_\epsilon} e^{J_\epsilon}$ converges to $e^{I_0} e^{J_0}$ in $\mathbb{P}$-probability as ϵ tends to 0, where J_0 is the limiting random variable in Claim 4.3.3. Combining this with our first observation, we obtain that, $\mathbb{P}$-almost surely, $\exp\left(-\frac{1}{2} \int_D m^2(z) : h(z)^2 : dz\right) = e^{I_0} e^{J_0}$. Proposition 4.3.1 then follows from this and the independence of $(h_j)_j$ and $(\xi_j)_j$ under $\mathbb{P}_\Gamma$. $\qquad\square$

It now remains to prove Claim 4.3.2 and Claim 4.3.3.

Proof of Claim 4.3.2. We first observe that, almost surely,

$$I_\epsilon = -\frac{1}{2}\left(\int_D m^2(z)\sum_j (h_{j,\epsilon}(z)^2 - \mathbb{E}[h_{j,\epsilon}(z)^2|\Gamma])dz + 2\int_D m^2(z)\sum_j h_{j,\epsilon}(z)\xi_{j,\epsilon}(z)dz\right.$$

$$+ \int_D m^2(z)\sum_j \xi_{j,\epsilon}(z)^2 dz + 2\int_D m^2(z)\sum_j h_{j,\epsilon}(z)\sum_{k:k\neq j} h_{k,\epsilon}(z)dz$$

$$\left.+ 2\int_D m^2(z)\sum_j h_{j,\epsilon}(z)\sum_{k:k\neq j}\xi_{j,\epsilon}(z)dz + \int_D m^2(z)\sum_j\sum_{k:k\neq j}\xi_{j,\epsilon}(z)\xi_{k,\epsilon}(z)dz\right).$$

Conditionally on Γ, the first term should give rise to a collection of Wick squares of the GFFs $(h_j)_j$ in the interiors of the loops $(L_j)_j$ while the second term should yield a collection of GFFs tested against the function $z \mapsto m^2(z)\xi_j$ inside each loop. The third term should in turn give rise to a collection of random variables $(\xi_j^2(m^2, \mathbb{I}_{\text{Int}(L_j)}))_j$, one for each loop. Moreover, intuitively, the three last terms should all converge to 0 since in the limit as ϵ tends to 0, we should obtain random functions/distributions supported on the interior of different loops. Making these rigorous is the content of the following claim.

Claim 4.3.4. *For $\mathbb{P}$-almost every Γ,*

$$\lim_{\epsilon\to 0}\int_D m^2(z)\left(\sum_j h_{j,\epsilon}(z)^2 - \mathbb{E}_\Gamma[h_{j,\epsilon}(z)^2]\right)dz = \sum_j (: h_j^2 :, m^2) \tag{4.29}$$

$$\lim_{\epsilon\to 0}\int_D m^2(z)\sum_j h_{j,\epsilon}(z)\xi_{j,\epsilon}(z)dz = \sum_j \xi_j(h_j, m^2) \tag{4.30}$$

$$\lim_{\epsilon\to 0}\int_D m^2(z)\sum_j h_{j,\epsilon}(z)\sum_{k:k\neq j}\xi_{k,\epsilon}(z)dz = 0 \tag{4.31}$$

$$\lim_{\epsilon\to 0}\int_D m^2(z)\sum_j h_{j,\epsilon}(z)\sum_{k:k\neq j}h_{k,\epsilon}(z)dz = 0 \tag{4.32}$$

$$\lim_{\epsilon\to 0}\int_D m^2(z)\sum_j \xi_{j,\epsilon}(z)^2 dz = \sum_j \xi_j^2\int_{\text{Int}(L_j)} m^2(z)dz \tag{4.33}$$

$$\lim_{\epsilon\to 0}\int_D m^2(z)\sum_j \xi_{j,\epsilon}(z)\sum_{k:k\neq j}\xi_{k,\epsilon}(z)dz = 0 \tag{4.34}$$

where the limits are in $L^2(\mathbb{P}_\Gamma)$.

Claim 4.3.2 straightforwardly follows from this claim (recall the notation (4.5)). □

Proof of Claim 4.3.4. Let us start with (4.29). Let $\epsilon > 0$. Using that Γ almost surely has Lebesgue measure 0 and the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we observe that, almost surely,

$$\int_D m^2(z)\left(\sum_j h_{j,\epsilon}(z)^2 - \mathbb{E}_\Gamma[h_{j,\epsilon}(z)^2]\right)dz$$

$$\leq 2\mathbb{E}_\Gamma\left[\left(\sum_j\int_{\text{Int}(L_j)} m^2(z)\left(h_{j,\epsilon}(z)^2 - \mathbb{E}_\Gamma[h_{j,\epsilon}(z)^2]\right)dz - \sum_j (: h_j^2 :, m^2)\right)^2\right]$$

$$+ 2\mathbb{E}_\Gamma\left[\left(\int_D m^2(z)\sum_{k:k\neq j(z)} h_{k,\epsilon}(z)^2 - \mathbb{E}_\Gamma[h_{k,\epsilon}(z)^2]dz\right)^2\right]$$

$$=: 2E_{1,\epsilon}(\Gamma) + 2E_{2,\epsilon}(\Gamma) \tag{4.35}$$

where for $z \in D$, $j(z)$ denotes the index of the loop of Γ surrounding z, which exists for almost every z. Let us show that the conditional expectations $E_{1,\epsilon}(\Gamma)$ and $E_{2,\epsilon}(\Gamma)$ almost surely converge to 0 as ϵ tends to 0. For $E_{1,\epsilon}(\Gamma)$, we have that, almost surely,

$$E_{1,\epsilon}(\Gamma) = \sum_j \mathbb{E}_\Gamma\left[\left(\int_{\text{Int}(L_j)} m^2(z)\left(h_{j,\epsilon}(z)^2 - \mathbb{E}_\Gamma[h_{j,\epsilon}(z)^2]\right)dz - (: h_j^2 :, m^2)\right)^2\right] \tag{4.36}$$

since the cross-terms vanish due to the conditional independence of the fields $(h_j)_j$. Now, observe that, almost surely, for any j,

$$\lim_{\epsilon \to 0} \mathbb{E}_\Gamma \left[\left(\int_{\text{Int}(L_j)} m^2(z) \left(h_{j,\epsilon}(z)^2 - \mathbb{E}_\Gamma[h_{j,\epsilon}(z)^2] \right) dz - (: h_j^2 :, m^2) \right)^2 \right] = 0.$$

This follows from the fact that, under $\mathbb{P}_\Gamma$, h_j is a GFF with Dirichlet boundary conditions in $\text{Int}(L_j)$ and Lemma 4.1.1. So, by the dominated convergence theorem, to prove that $E_{1,\epsilon}(\Gamma)$ almost surely converges to 0 as $\epsilon \to 0$, it suffices to show that a uniform bound in ϵ on the conditional expectations in (4.36) which is summable. Using Lemma 6.1.3 for the fields $(h_j)_j$ under $\mathbb{P}_\Gamma$, we have that almost surely, for any j,

$$\mathbb{E}_\Gamma \left[\left(\int_{\text{Int}(L_j)} m^2(z) \left(h_{j,\epsilon}(z)^2 - \mathbb{E}_\Gamma[h_{j,\epsilon}(z)^2] \right) dz - (: h_j^2 :, m^2) \right)^2 \right]$$

$$\leq 2 \int_{\text{Int}(L_j) \times \text{Int}(L_j)} m^2(z) m^2(w) \tilde{G}_{j,\epsilon}(z,w) dz dw + 4 \int_{\text{Int}(L_j) \times \text{Int}(L_j)} m^2(z) m^2(w) G_j(z,w)^2 dz dw$$

where $\tilde{G}_{j,\epsilon}$ denotes the covariance of the field $h_{j,\epsilon}^2 - \mathbb{E}_\Gamma[h_{j,\epsilon}(\cdot)^2]$ under $\mathbb{P}_\Gamma$ and G_j is the Green function in $\text{Int}(L_j)$. One can easily check that, almost surely, for any $z, w \in D$, $\tilde{G}_{j,\epsilon}(z,w) = 2\mathbb{E}[h_{j,\epsilon}(z) h_{j,\epsilon}(w)]^2$. This implies that, almost surely, for any $z, w \in D$, $\tilde{G}_{j,\epsilon}(z,w) \leq 2G_j(z,w)^2 + O(1) \leq 2G_D(z,w)^2 + O(1)$, where the implied constant is deterministic and independent of j. We deduce from this that, almost surely,

$$\sum_j \mathbb{E}_\Gamma \left[\left(\int_{\text{Int}(L_j)} m^2(z) \left(h_{j,\epsilon}(z)^2 - \mathbb{E}_\Gamma[h_{j,\epsilon}(z)^2] \right) dz - (: h_j^2 :, m^2) \right)^2 \right]$$

$$\leq 8 \int_{D \times D} m^2(z) m^2(w) (2G_D(z,w)^2 + O(1)) dz dw. \tag{4.37}$$

The final inequality follows from the fact that $G_j(z,w) = 0$ if $j(z) \neq j(w)$ and $G_D(z,w)^2 \geq G_j(z,w)^2$ if $j(z) = j(w)$. We also used the fact that, almost surely, $\sum_j \text{Area}(\text{Int}(L_j))^2 \leq \text{Area}(D)^2$. As explained above, this shows that $E_{1,\epsilon}(\Gamma)$ almost surely converges to 0 as ϵ tends to 0. Let us now show that $E_{2,\epsilon}(\Gamma)$ also almost surely converges to 0. For $\epsilon > 0$, define the set

$$G_\epsilon := \{ z \in D : \text{dist}(z, L(z)) \leq \epsilon \} \tag{4.38}$$

where for $z \in D$, $L(z)$ denotes the loop of Γ surrounding z, which almost surely exists. Note that by [SSW09; NW09a], $\text{Area}(G_\epsilon) = O(\epsilon^{\frac{1}{8}-})$. We then have that, almost surely,

$$E_{2,\epsilon}(\Gamma) = \mathbb{E}_\Gamma \left[\left(\int_{G_\epsilon} m^2(z) \sum_{k : k \neq j(z)} h_{k,\epsilon}(z)^2 - \mathbb{E}_\Gamma[h_{k,\epsilon}(z)^2] dz \right)^2 \right].$$

Indeed, for $z \in D \setminus G_\epsilon$, the integrand inside the conditional expectation is almost surely equal to 0 as the fields $(h_k)_{k \neq j(z)}$ are almost surely equal to 0 in $\text{Int}(L(z))$. Note also that the set of points not surrounded by any loop of Γ almost surely has Lebesgue measure 0, so we can ignore it when integrating. Pursuing our computations, we then have that, almost surely,

$$E_{2,\epsilon}(\Gamma) = \int_{G_\epsilon \times G_\epsilon} m^2(z) m^2(w) \sum_{\substack{k : k \neq j(z) \\ p : p \neq j(w)}} \mathbb{E}_\Gamma \left[(h_{k,\epsilon}(z)^2 - \mathbb{E}_\Gamma[h_{k,\epsilon}(z)^2])(h_{p,\epsilon}(w)^2 - \mathbb{E}_\Gamma[h_{p,\epsilon}(w)^2]) \right] dz dw,$$

where we apply Fubini theorem term by term. In the sum above, by conditional independence of h_k and h_p given Γ, only the terms $k = p$ contribute. Since conditionally on Γ, for each k, h_k is a GFF with Dirichlet boundary conditions in $\text{Int}(L_k)$, this yields that, almost surely,

$$E_{2,\epsilon}(\Gamma) = \int_{G_\epsilon \times G_\epsilon} m^2(z) m^2(w) \sum_{k : k \neq j(z), k \neq j(w)} \tilde{G}_{k,\epsilon}(z,w) dz dw$$

where, as before, for each k, $\tilde{G}_{k,\epsilon}$ denotes the covariance function of the field $h_{k,\epsilon}^2 - \mathbb{E}_\Gamma[h_{k,\epsilon}(\cdot)^2]$ under $\mathbb{P}_\Gamma$. For the same reasons as those in the first part of the proof, we get that, almost surely,

$$E_{2,\epsilon}(\Gamma) \leq \int_{G_\epsilon \times G_\epsilon} m^2(z)m^2(w)(2G_D(z,w)^2 + O(1))dzdw.$$

Since the function $z, w \mapsto G_D(z,w)^2$ is integrable in $D \times D$ and $\mathrm{Area}(G_\epsilon)$ almost surely converges to 0 as ϵ tends to 0, the right-hand side of the above inequality almost surely converges to 0 as ϵ tends to 0. Going back to (4.35) and combining this with the first part of the proof, we deduce the equality (4.29).

We now prove (4.30). Let $\epsilon > 0$. Since Γ almost surely has Lebesgue measure 0, we then have that, almost surely,

$$\mathbb{E}_\Gamma\left[\left(\int_D m^2(z)\sum_j h_{j,\epsilon}(z)\xi_{j,\epsilon}(z)dz - \sum_j \xi_j(h_j, m^2)\right)^2\right]$$

$$\leq 2\mathbb{E}_\Gamma\left[\left(\sum_j \int_{\mathrm{Int}(L_j)} m^2(z)h_{j,\epsilon}(z)\xi_{j,\epsilon}(z)dz - \sum_j \xi_j(h_j, m^2)\right)^2\right]$$

$$+ 2\mathbb{E}_\Gamma\left[\left(\sum_j \int_{\mathrm{Int}(L_j)} m^2(z)\sum_{k:k\neq j} h_{k,\epsilon}(z)\xi_{k,\epsilon}(z)dz\right)^2\right] \tag{4.39}$$

$$=: \tilde{E}_{1,\epsilon}(\Gamma) + \tilde{E}_{2,\epsilon}(\Gamma) \tag{4.40}$$

Let us now show that $\tilde{E}_{1,\epsilon}(\Gamma)$ and $\tilde{E}_{2,\epsilon}(\Gamma)$ almost surely converges to 0 as $\epsilon \to 0$. For $\tilde{E}_{1,\epsilon}(\Gamma)$, by conditional independence of $(h_j)_j$ and $(\xi_j)_j$ given Γ, we have that, almost surely,

$$\tilde{E}_{1,\epsilon}(\Gamma) = \sum_j \mathbb{E}_\Gamma\left[\left(\int_{\mathrm{Int}(L_j)} m^2(z)h_{j,\epsilon}(z)\xi_{j,\epsilon}(z)dz - \xi_j(h_j, m^2)\right)^2\right]. \tag{4.41}$$

Observe that for any j, almost surely,

$$\lim_{\epsilon \to 0} \mathbb{E}_\Gamma\left[\left(\int_{\mathrm{Int}(L_j)} m^2(z)h_{j,\epsilon}(z)\xi_{j,\epsilon}(z)dz - \xi_j(h_j, m^2)\right)^2\right] = 0.$$

This follows from the fact that under $\mathbb{P}$, conditionally on Γ, h_j is a GFF with Dirichlet boundary conditions in the interior $\mathrm{Int}(L_j)$ of the loop L_j. Therefore, by the dominated convergence theorem, to prove that $\tilde{E}_{1,\epsilon}(\Gamma)$ almost surely converges to 0 as $\epsilon \to 0$, it suffices to show that a uniform bound in ϵ on the conditional expectations in (4.41) which is summable. To this end, observe that, almost surely,

$$\mathbb{E}_\Gamma\left[\left(\int_{\mathrm{Int}(L_j)} m^2(z)h_{j,\epsilon}(z)\xi_{j,\epsilon}(z)dz - \xi_j(h_j, m^2)\right)^2\right]$$

$$\leq 8\lambda^2 \int_{\mathrm{Int}(L_j)} m^2(z)m^2(w)G_{j,\epsilon}(z,w)dzdw + 8\lambda^2 \int_{\mathrm{Int}(L_j)} m^2(z)m^2(w)G_j(z,w)dzdw \tag{4.42}$$

where $G_{j,\epsilon}$ denotes the covariance of the field $h_{j,\epsilon}$ when conditioning on Γ (recall that under $\mathbb{P}$, conditionally on Γ, h_j is a GFF in $\mathrm{Int}(L_j)$ with Dirichlet boundary conditions) and G_j is the Green function in $\mathrm{Int}(L_j)$. To obtain this inequality, we also used the fact that under $\mathbb{P}$, almost surely, for any j, $\xi_j^2 = 4\lambda^2$, which implies that for any $z \in D$, almost surely, $\xi_{j,\epsilon}^2(z) \leq 4\lambda^2$. Moreover, almost surely, for any $z, w \in D$, we have that $G_{j,\epsilon}(z,w) \leq G_j(z,w) + O(1) \leq G_D(z,w) + O(1)$, where the implied constant is deterministic and independent of j. This yields that, almost surely,

$$\sum_j \mathbb{E}_\Gamma\left[\left(\int_{\mathrm{Int}(L_j)} m^2(z)h_{j,\epsilon}(z)\xi_{j,\epsilon}(z)dz - \xi_j(h_j, m^2)\right)^2\right]$$

$$\leq 16\lambda^2 \int_{D \times D} m^2(z)m^2(w)(G_D(z,w) + O(1))dzdw,$$

since, almost surely, $G_j(z,w) = 0$ if $j(z) \neq j(w)$ and $G_D(z,w) \geq G_j(z,w)$ if $j(z) = j(w)$. This establishes the convergence to 0 as $\epsilon \to 0$ of $\tilde{E}_{1,\epsilon}(\Gamma)$. Let us now turn to $\tilde{E}_{2,\epsilon}(\Gamma)$. With G_ϵ as in (4.38), observe that, almost surely,

$$\tilde{E}_{2,\epsilon}(\Gamma) = \mathbb{E}_\Gamma\left[\left(\int_{G_\epsilon} m^2(z) \sum_{j:z\notin\mathrm{Int}(L_j)} h_{j,\epsilon}(z)\xi_{j,\epsilon}(z)dz\right)^2\right].$$

The last equality follows from the fact that if $z \in D\backslash G_\epsilon$, then almost surely $\sum_{j:z\notin\mathrm{Int}(L_j)} h_{j,\epsilon}(z)\xi_{j,\epsilon}(z) = 0$. This is because h_j is almost surely 0 outside of $\mathrm{Int}(L_j)$ and $\xi_{j,\epsilon}(z)$ is almost surely 0 if $\mathrm{dist}(z, \mathrm{Int}(L_j)) > \epsilon$. Using the conditional independence of $(h_j)_j$ and $(\xi_j)_j$ together with the fact that conditionally on Γ, these random variables are centered, it then follows that, almost surely,

$$\tilde{E}_{2,\epsilon}(\Gamma) = \int_{G_\epsilon \times G_\epsilon} m^2(z)m^2(w) \sum_{j:z,w\notin\mathrm{Int}(L_j)} \mathbb{E}_\Gamma[h_{j,\epsilon}(z)h_{j,\epsilon}(w)]\mathbb{E}_\Gamma[\xi_{j,\epsilon}(z)\xi_{j,\epsilon}(w)]dzdw.$$

Notice that almost surely, for any $z, w \in D$, $\mathbb{E}_\Gamma[\xi_{j,\epsilon}(z)\xi_{j,\epsilon}(w)] \leq 4\lambda^2$. This implies that, almost surely,

$$\tilde{E}_{2,\epsilon}(\Gamma) \leq 4\lambda^2\overline{m}^4 \int_{G_\epsilon \times G_\epsilon} \sum_{j:z,w\notin\mathrm{Int}(L_j)} \mathbb{E}_\Gamma[h_{j,\epsilon}(z)h_{j,\epsilon}(w)]dzdw. \tag{4.43}$$

Moreover, almost surely, for any $z, w \in D$,

$$\sum_{j:z,w\notin\mathrm{Int}(L_j)} \mathbb{E}_\Gamma[h_{j,\epsilon}(z)h_{j,\epsilon}(w)] \leq \mathbb{E}[h_\epsilon(z)h_\epsilon(w)] \leq O(-\log\epsilon), \tag{4.44}$$

where the right-most inequality follows from Cauchy-Schwarz inequality and the estimate (6.1). The inequality (4.44) applied to the right-hand side of (4.43) yields that, almost surely, $\tilde{E}_{2,\epsilon}(\Gamma) \leq O(-\log\epsilon)\mathrm{Area}(G_\epsilon)^2 = O(-\epsilon^{\frac{1}{4}-}\log\epsilon)$, which completes the proof of the equality (4.30).

We now show (4.31). Let $\epsilon > 0$. For G_ϵ defined as in (4.38), notice that conditionally on Γ, if $z \in D\backslash G_\epsilon$, then, almost surely, $\sum_{j\neq k} h_{j,\epsilon}(z)\xi_{k,\epsilon}(z) = 0$. Indeed, since we are summing over indices $j \neq k$, conditionally on Γ, for $z \in D \backslash G_\epsilon$, if $z \in \mathrm{Int}(L_j)$, then $\sum_{k:k\neq j} \xi_{k,\epsilon}(z)$ and $\sum_{k:k\neq j} h_{k,\epsilon}(z)$ are almost surely equal to 0. Therefore, it follows that, almost surely,

$$\mathbb{E}_\Gamma\left[\left(\int_D m^2(z) \sum_{j\neq k} h_{j,\epsilon}(z)\xi_{k,\epsilon}(z)dz\right)^2\right] \leq C \int_{G_\epsilon \times G_\epsilon} \sum_{j\neq k} \mathbb{E}_\Gamma[h_{j,\epsilon}(z)h_{j,\epsilon}(w)]\mathbb{E}_\Gamma[\xi_{k,\epsilon}(z)\xi_{k,\epsilon}(w)]dzdw$$

$$\tag{4.45}$$

where $C > 0$ is a (non-random) constant depending only on $\overline{m}^2$ and where we used the fact that the fields $(h_j)_j$ and the labels $(\xi_j)_j$ are conditionally independent given Γ and that all these random variables have mean 0 conditionally on Γ. Observe that, for the same reasons as in (4.44), for any $z, w \in G_\epsilon$, almost surely, $\sum_j \mathbb{E}_\Gamma[h_{j,\epsilon}(z)h_{j,\epsilon}(w)] \leq \mathbb{E}[h_\epsilon(z)h_\epsilon(w)] = O(-\log\epsilon)$ with constant independent of z and w. Moreover, we have that, almost surely,

$$\mathbb{E}_\Gamma[\xi_{k,\epsilon}(z)\xi_{k,\epsilon}(w)] = 4\lambda^2 \mathbb{E}_\Gamma[(\mathbb{I}_{\mathrm{Int}(L_k)}, \rho_\epsilon^z)(\mathbb{I}_{\mathrm{Int}(L_k)}, \rho_\epsilon^w)]$$

$$= \frac{4\lambda^2}{4\pi^2\epsilon^2}\mathcal{L}(\partial B(z,\epsilon) \cap \mathrm{Int}(L_k))\mathcal{L}(\partial B(w,\epsilon) \cap \mathrm{Int}(L_k)), \tag{4.46}$$

where as in (4.28), $\mathcal{L}$ denote the arc-length measure. Observe also that almost surely, for any k and any w, $(1/2\pi\epsilon)\,\mathrm{Length}(\partial B(w,\epsilon) \cap \mathrm{Int}(L_k)) \leq 1$. Moreover, it is easy to see that, since the loops of Γ are almost surely disjoint, we have that, almost surely, $\frac{1}{2\pi\epsilon}\sum_k \mathcal{L}(\partial B(z,\epsilon) \cap \mathrm{Int}(L_k)) \leq 1$. From (4.46) and these facts, we therefore obtain that, almost surely,

$$\sum_k \mathbb{E}_\Gamma[\xi_{k,\epsilon}(z)\xi_{k,\epsilon}(w)] = \frac{4\lambda^2}{4\pi^2\epsilon^2}\sum_k \mathcal{L}(\partial B(z,\epsilon) \cap \mathrm{Int}(L_k))\mathcal{L}(\partial B(w,\epsilon) \cap \mathrm{Int}(L_k))$$

$$\leq \frac{4\lambda^2}{2\pi\epsilon}\sum_k \mathcal{L}(\partial B(z,\epsilon) \cap \mathrm{Int}(L_k)) \leq 4\lambda^2.$$

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In view of (4.45), this yields that, almost surely,

$$\mathbb{E}_\Gamma\Big[\big(\int_D m^2(z)\sum_{j\neq k}h_{j,\epsilon}(z)\xi_{k,\epsilon}(z)dz\big)^2\Big] \leq O(-\epsilon^{\frac{1}{4}-}\log\epsilon),$$

which shows the equality (4.31).

We now turn to (4.32). Let $\epsilon > 0$. With G_ϵ as in (4.38), observe that if $z \in D \setminus G_\epsilon$, then $\sum_j h_{j,\epsilon}(z)\sum_{k:k\neq j}h_{k,\epsilon}(z) = 0$ almost surely. This is because for any j, the field h_j is 0 outside $\mathrm{Int}(L_j)$. Therefore, almost surely,

$$\mathbb{E}_\Gamma\Big[\Big(\int_D m^2(z)\sum_j h_{j,\epsilon}(z)\sum_{k:k\neq j}h_{k,\epsilon}(z)dz\Big)^2\Big]$$

$$= \int_{G_\epsilon\times G_\epsilon} m^2(z)m^2(w)\mathbb{E}_\Gamma\Big[\Big(\sum_j h_{j,\epsilon}(z)\sum_{k:k\neq j}h_{k,\epsilon}(z)\Big)\Big(\sum_j h_{j,\epsilon}(w)\sum_{k:k\neq j}h_{k,\epsilon}(w)\Big)\Big]dzdw. \quad (4.47)$$

Observe that since the fields $(h_j)_j$ are centered and conditionally independent given Γ, almost surely,

$$\mathbb{E}_\Gamma\Big[\Big(\sum_j h_{j,\epsilon}(z)\sum_{k:k\neq j}h_{k,\epsilon}(z)\Big)\Big(\sum_j h_{j,\epsilon}(w)\sum_{k:k\neq j}h_{k,\epsilon}(w)\Big)\Big]$$

$$= \sum_{j\neq k}\mathbb{E}_\Gamma[h_{j,\epsilon}(z)h_{j,\epsilon}(w)]\mathbb{E}_\Gamma[h_{k,\epsilon}(z)h_{k,\epsilon}(w)].$$

Under $\mathbb{P}_\Gamma$, for each j, h_j is a GFF in $\mathrm{Int}(L_j)$ with Dirichlet boundary conditions and therefore, for the same reasons as in (4.44), we have that, almost surely, for any j, $\sum_{k:k\neq j}\mathbb{E}_\Gamma[h_{k,\epsilon}(z)h_{k,\epsilon}(w)] \leq \mathbb{E}[h_\epsilon(z)h_\epsilon(w)] \leq O(-\log\epsilon)$. This yields that, almost surely,

$$\mathbb{E}_\Gamma\Big[\Big(\int_D m^2(z)\sum_j h_{j,\epsilon}(z)\sum_{k:k\neq j}h_{k,\epsilon}(z)dz\Big)^2\Big] \leq O(\log(\epsilon)^2)\mathrm{Area}(G_\epsilon)^2 = O(-\epsilon^{\frac{1}{4}-}(\log\epsilon)^2). \quad (4.48)$$

By the inequality (4.48), this completes the proof of the equality (4.32).

We now prove (4.33). Observe first that since $D \setminus \Gamma$ almost surely has Lebesgue measure 0, we have that, almost surely

$$\mathbb{E}_\Gamma\Big[\Big(\int_D m^2(z)\sum_j \xi_{j,\epsilon}(z)^2 dz - \sum_j \xi_j^2\int_{\mathrm{Int}(L_j)} m^2(z)dz\Big)^2\Big]$$

$$= \mathbb{E}_\Gamma\Big[\Big(\int_D m^2(z)\sum_j \xi_{j,\epsilon}(z)^2 dz - \int_D 4\lambda^2 m^2(z)dz\Big)^2\Big],$$

where we have also used the fact that under $\mathbb{P}_\Gamma$, for any j, $\xi_j^2 = 4\lambda^2$. Moreover, under $\mathbb{P}_\Gamma$, for $z \in D \setminus G_\epsilon$, where G_ϵ is defined as in (4.38), $\xi_{j,\epsilon}(z)^2 = 0$ unless $L_j = L(z)$, in which case $\xi_{j,\epsilon}(z)^2 = 4\lambda^2$. Here, as in the proof of Claim 4.30, for $z \in D$, we have denoted by $L(z)$ the loop of Γ that surrounds z. Together Fubini-Tonelli theorem, Hölder's inequality and the boundedness of m, this implies that, almost surely,

$$\mathbb{E}_\Gamma\Big[\Big(\int_D m^2(z)\sum_j \xi_{j,\epsilon}(z)^2 dz - \sum_j \xi_j^2\int_{\mathrm{Int}(L_j)} m^2(z)dz\Big)^2\Big]$$

$$\leq \overline{m}^4\mathrm{Area}(D)\mathbb{E}_\Gamma\Big[\int_{G_\epsilon} 2\Big(\sum_j \xi_{j,\epsilon}(z)^2\Big)^2 + 32\lambda^4 dz\Big]$$

$$= \overline{m}^4\mathrm{Area}(D)\int_{G_\epsilon} 2\mathbb{E}_\Gamma\Big[\Big(\sum_j \xi_{j,\epsilon}(z)^2\Big)^2\Big] + 32\lambda^4 dz. \quad (4.49)$$

Notice that almost surely, for any $z \in D$, $\xi_{j,\epsilon}(z) = \xi_j(\mathbb{I}_{\mathrm{Int}(L_j)}, \rho_\epsilon^z) = \frac{\xi_j}{2\pi\epsilon}\mathcal{L}(\partial B(z,\epsilon)\cap\mathrm{Int}(L_j))$, where as in (4.28), $\mathcal{L}$ denotes the length with respect to the arc-length measure. Moreover, as the loops are almost surely disjoint, we have that, almost surely, for any $z \in D$,

$$\frac{1}{2\pi\epsilon}\sum_j \mathcal{L}(\partial B(z,\epsilon)\cap\mathrm{Int}(L_j)) \leq 1. \quad (4.50)$$

This inequality implies that, almost surely, for any $z \in G_\epsilon$, $\mathbb{E}_\Gamma\left[\left(\sum_j \xi_{j,\epsilon}(z)^2\right)^2\right] \leq 16\lambda^4$. Using this in (4.49) yields that, almost surely,

$$\mathbb{E}_\Gamma\left[\left(\int_D m^2(z)\sum_j \xi_{j,\epsilon}(z)^2 dz - \sum_j \xi_j^2 \int_{\mathrm{Int}(L_j)} m^2(z)dz\right)^2\right] \leq C\mathrm{Area}(G_\epsilon) = O(\epsilon^{\frac{1}{8}-})$$

where $C > 0$ is a deterministic constant depending only on D, $\overline{m}^2$ and λ^2. This inequality implies (4.33).

Let us now prove (4.34). Let $\epsilon > 0$. Observe that, under $\mathbb{P}_\Gamma$, for $j \neq k$ and $z \in D \setminus G_\epsilon$, where G_ϵ is defined as in (4.38), $\xi_{j,\epsilon}(z)\xi_{k,\epsilon}(z) = 0$. This implies that, almost surely,

$$\mathbb{E}_\Gamma\left[\left(\int_D m^2(z)\sum_j \xi_{j,\epsilon}(z)\sum_{k:k\neq j}\xi_{j,\epsilon}(z)dz\right)^2\right] = \mathbb{E}_\Gamma\left[\left(\int_{G_\epsilon} m^2(z)\sum_j \xi_{j,\epsilon}(z)\sum_{k:k\neq j}\xi_{j,\epsilon}(z)dz\right)^2\right].$$

Using Hölder inequality, the boundedness of m and Fubini-Tonelli theorem, we then obtain that, almost surely,

$$\mathbb{E}_\Gamma\left[\left(\int_D m^2(z)\sum_j \xi_{j,\epsilon}(z)\sum_{k:k\neq j}\xi_{j,\epsilon}(z)dz\right)^2\right] \leq \overline{m}^4\mathrm{Area}(D)\int_{G_\epsilon}\mathbb{E}_\Gamma\left[\left(\sum_j \xi_{j,\epsilon}(z)\sum_{k:k\neq j}\xi_{j,\epsilon}(z)\right)^2\right]dz \tag{4.51}$$

Next, observe that, by the inequality (4.50) in the proof of Claim 4.33, almost surely, for any $z \in G_\epsilon$,

$$\mathbb{E}_\Gamma\left[\left(\sum_{j,k}\xi_{j,\epsilon}(z)\xi_{k,\epsilon}(z)\right)^2\right] \leq \mathbb{E}_\Gamma\left[\left(\sum_j 2\lambda\frac{|\xi_j|}{2\pi\epsilon}\mathcal{L}(\partial B(z,\epsilon)\cap\mathrm{Int}(L_j))\right)^2\right] \leq (4\lambda^2)^2.$$

Using this in (4.51) yields that, almost surely,

$$\mathbb{E}_\Gamma\left[\left(\int_D m^2(z)\sum_j \xi_{j,\epsilon}(z)\sum_{k:k\neq j}\xi_{j,\epsilon}(z)dz\right)^2\right] \leq 16\lambda^4\overline{m}^4\mathrm{Area}(D)\mathrm{Area}(G_\epsilon) = O(\epsilon^{\frac{1}{8}-}),$$

which establishes the equality (4.34). $\qquad\qquad\square$

Let us now establish Claim 4.3.3.

Proof of Claim 4.3.3. Let $\epsilon > 0$. We first observe that if $z \in D$ is such that $\mathrm{dist}(z,L(z)) > \epsilon$ where as in the proof of Claim 4.30, $L(z)$ denotes the loop of Γ surrounding z, then, almost surely,

$$\mathbb{E}[h_\epsilon(z)^2] - \sum_j \mathbb{E}_\Gamma[h_{j,\epsilon}(z)^2] = \mathbb{E}[h_\epsilon(z)^2] - \mathbb{E}_\Gamma[h_{j(z),\epsilon}(z)^2] = \frac{1}{2\pi}\log\frac{\mathrm{CR}(z,\partial D)}{\mathrm{CR}(z,L(z))},$$

where $h_{j(z)}$ denotes the field in the loop $L(z)$. This equality follows from the fact that under $\mathbb{P}$, conditionally on Γ, for any $j \neq j(z)$, $h_{j,\epsilon}(z) = 0$ almost surely since for $j \neq j(z)$, h_j is almost surely equal to 0 in $\mathrm{Int}(L(z))$. For $\epsilon > 0$, define the set G_ϵ as in (4.38). Since the set $\{z \in D : z \text{ is not surrounded by a loop of } \Gamma\}$ almost surely has Lebesgue measure 0, we then have that, almost surely,

$$\int_D m^2(z)(\mathbb{E}[h_\epsilon(z)^2] - \sum_j \mathbb{E}[h_{j,\epsilon}(z)^2|\Gamma])dz$$
$$= \int_{D\setminus G_\epsilon}\frac{m^2(z)}{2\pi}\log\frac{\mathrm{CR}(z,\partial D)}{\mathrm{CR}(z,L(z))}dz + \int_{G_\epsilon} m^2(z)(\mathbb{E}[h_\epsilon(z)^2] - \sum_j \mathbb{E}_\Gamma[h_{j,\epsilon}(z)^2])dz. \tag{4.52}$$

The second term in this sum almost surely converges to 0 as ϵ tends to 0. Indeed, almost surely,

$$0 \leq \int_{G_\epsilon} m^2(z)(\mathbb{E}[h_\epsilon(z)^2] - \sum_j \mathbb{E}_\Gamma[h_{j,\epsilon}(z)^2])dz \leq \overline{m}^2\int_{G_\epsilon}\mathbb{E}[h_\epsilon(z)^2]dz. \tag{4.53}$$

Above, the inequality on the left-hand side simply follows from the fact that, almost surely, for any $z \in D$, $\mathbb{E}[h_\epsilon(z)^2] - \sum_j \mathbb{E}_\Gamma[h_{j,\epsilon}(z)^2] \geq 0$. Moreover, the estimate (6.1) shows that there exists $C > 0$ such that, for any $z \in D$, $\mathbb{E}[h_\epsilon(z)^2] \leq C \log(\epsilon^{-1})$. Using this and the boundedness of m in (4.53) yields that, almost surely,

$$0 \leq \int_{G_\epsilon} m^2(z)(\mathbb{E}[h_\epsilon(z)^2] - \sum_j \mathbb{E}_\Gamma[h_{j,\epsilon}(z)^2])dz \leq O(-\log \epsilon)\mathrm{Area}(G_\epsilon) = O(-\epsilon^{\frac{1}{8}-}\log \epsilon),$$

This shows that the second term in the sum (4.52) indeed almost surely converges to 0. For the first term in the sum (4.52), by the monotone convergence theorem, almost surely,

$$\lim_{\epsilon \to 0} \int_{D \setminus G_\epsilon} \frac{m^2(z)}{2\pi} \log \frac{\mathrm{CR}(z, \partial D)}{\mathrm{CR}(z, L(z))} dz = \int_D \frac{m^2(z)}{2\pi} \log \frac{\mathrm{CR}(z, \partial D)}{\mathrm{CR}(z, L(z))} dz.$$

The random variable on the right-hand side is almost surely finite. Indeed, by [ASW19, Proposition 20], for each $z \in D$, $\log \mathrm{CR}(z, \partial D) - \log \mathrm{CR}(z, L(z))$ has the law of $\tau_{-\pi,\pi}$, the first hitting time of $\{-\pi, \pi\}$ by a standard one-dimensional Brownian motion started at 0. Moreover, for any $z \in D$, $\log \mathrm{CR}(z, \partial D) - \log \mathrm{CR}(z, L(z))$ is almost surely non-negative. Therefore, denoting by $T_{-\pi,\pi}$ the expectation of $\tau_{-\pi,\pi}$ and using Fubini-Tonelli theorem, we have that

$$0 \leq \mathbb{E}\left[\int_D \frac{m^2(z)}{2\pi} \log \frac{\mathrm{CR}(z, \partial D)}{\mathrm{CR}(z, L(z))} dz \right] = \int_D \frac{m^2(z)}{2\pi} T_{-\pi,\pi} dz < \infty.$$

This implies that, almost surely,

$$\int_D \frac{m^2(z)}{2\pi} \log \frac{\mathrm{CR}(z, \partial D)}{\mathrm{CR}(z, L(z))} dz < \infty.$$

Since $D \setminus \Gamma$ almost surely has Lebesgue measure 0 and since the loops of Γ are almost surely disjoint, the above integral over D can be written as a sum of integrals over the interiors of the loops of Γ. This yields that, almost surely,

$$\int_D \frac{m^2(z)}{2\pi} \log \frac{\mathrm{CR}(z, \partial D)}{\mathrm{CR}(z, L(z))} dz = \sum_j \int_{\mathrm{Int}(L_j)} \frac{m^2(z)}{2\pi} \log \frac{\mathrm{CR}(z, \partial D)}{\mathrm{CR}(z, L_j)} dz.$$

Going back to (4.52), this completes the proof of the claim. □

The conditional law of $(\xi_j)_j$ under $\tilde{\mathbb{P}}$

Let us now show that under $\tilde{\mathbb{P}}$, conditionally on Γ, the random variables $(\xi_j)_j$ are independent and Rademacher distributed on $\{-2\lambda, 2\lambda\}$.

Proposition 4.3.5. *With the same assumptions as in Theorem 1.2.3, under $\tilde{\mathbb{P}}$, conditionally on Γ, the random variables $(\xi_j)_j$ are independent and almost surely, for any j, $\tilde{\mathbb{P}}(\xi_j = -2\lambda|\Gamma) = \tilde{\mathbb{P}}(\xi_j = 2\lambda|\Gamma) = \frac{1}{2}$.*

Proof. Observe that under $\mathbb{P}$, almost surely, for any j, ξ_j takes values in $\{-2\lambda, 2\lambda\}$. Since $\tilde{\mathbb{P}}$ is absolutely continuous with respect to $\mathbb{P}$, this also holds under $\tilde{\mathbb{P}}$. To identify the law of $(\xi_j)_j$ given Γ under $\tilde{\mathbb{P}}$, it thus suffices to show that for any $a \in \mathbb{R}$, almost surely,

$$\tilde{\mathbb{E}}[e^{ia\xi_j}|\Gamma] = \tilde{\mathbb{E}}[e^{-ia\xi_j}|\Gamma]. \tag{4.54}$$

Indeed, it follows from this equality and the fact that almost surely $\xi_j \in \{-2\lambda, 2\lambda\}$ that, almost surely, $\tilde{\mathbb{P}}(\xi_j = -2\lambda|\Gamma) = \tilde{\mathbb{P}}(\xi_j = 2\lambda|\Gamma) = \frac{1}{2}$. Thus, let $a \in \mathbb{R}$. To establish (4.54), we are going to make use of the symmetry properties of the Wick square and of the conditional laws of $(h_j)_j$ and $(\xi_j)_j$ conditionally on Γ under $\mathbb{P}$. We first observe that, almost surely,

$$\tilde{\mathbb{E}}[e^{-ia\xi_j}|\Gamma] = \frac{1}{\mathcal{Z}(\Gamma)} \mathbb{E}\left[e^{-ia\xi_j} \exp\left(-\frac{1}{2} \int_D m^2(z) :h^2(z): dz \right)|\Gamma \right]$$

where $\mathcal{Z}(\Gamma)$ is defined in (4.25). Proposition 4.3.1 applied with $f \equiv 0$ shows that, almost surely,

$$\mathcal{Z}(\Gamma) = \prod_j \mathcal{Z}_j(\Gamma) \exp\left(\int_{\text{Int}(L_j)} \frac{m^2(z)}{4\pi} \log \frac{\text{CR}(z, \partial D)}{\text{CR}(z, L_j)} dz\right) \tag{4.55}$$

where the random variables $(\mathcal{Z}_j(\Gamma))_j$ are defined as

$$\mathcal{Z}_j(\Gamma) := \mathbb{E}\left[\exp\left(-\frac{1}{2}\int_{\text{Int}(L_j)} m^2(z) : (h_j + \xi_j)^2(z) : dz\right) \Big| \Gamma\right]. \tag{4.56}$$

On the other hand, one can argue as in the proof of Proposition 4.3.1 to show that, almost surely,

$$\mathbb{E}\left[e^{-ia\xi_j} \exp\left(-\frac{1}{2}\int_D m^2(z) : h^2(z) : dz | \Gamma\right]\right.$$
$$= \lim_{\epsilon \to 0} \mathbb{E}\left[e^{-ia\xi_j} \exp\left(-\frac{1}{2}\int_D m^2(z) : h_\epsilon^2(z) : dz\right) | \Gamma\right]$$
$$= \exp\left(\sum_j \int_{\text{Int}(L_j)} \frac{m^2(z)}{4\pi} \log \frac{\text{CR}(z, \partial D)}{\text{CR}(z, L_j)} dz\right) \tag{4.57}$$
$$\times \mathbb{E}\left[e^{-ia\xi_j} \exp\left(-\frac{1}{2}\int_{\text{Int}(L_j)} m^2(z) : (h + \xi_j)^2(z) : dz\right) | \Gamma\right] \prod_{k: k \neq j} \mathcal{Z}_j(\Gamma).$$

This yields that, almost surely,

$$\tilde{\mathbb{E}}[e^{-ia\xi_j} | \Gamma] = \frac{1}{\mathcal{Z}_j(\Gamma)} \mathbb{E}\left[e^{-ia\xi_j} \exp\left(-\frac{1}{2}\int_D m^2(z) : (h_j + \xi_j)^2(z) : dz\right) | \Gamma\right]. \tag{4.58}$$

Since under $\mathbb{P}$, conditionally on Γ, ξ_j has the same law as $-\xi_j$, we then obtain that, almost surely,

$$\tilde{\mathbb{E}}[e^{-ia\xi_j} | \Gamma] = \frac{1}{\mathcal{Z}_j(\Gamma)} \mathbb{E}\left[e^{ia\xi_j} \exp\left(-\frac{1}{2}\int_D m^2(z) : (h_j - \xi_j)^2(z) : dz\right) | \Gamma\right].$$

It is also easy to see that if h is a GFF with Dirichlet boundary conditions and $c \in \mathbb{R}$, then $\int_D m^2(z) : (h - c)^2(z) : dz \overset{\text{law}}{=} \int_D m^2(z) : (-h + c)^2(z) : dz$. This simply follows from the fact that $-h$ has the same law as h and that we can write

$$\int_D m^2(z) : (h - c)^2(z) : dz = \lim_{\epsilon \to 0} \int_D m^2(z)((h_\epsilon(z) - c)^2 - \mathbb{E}[h_\epsilon(z)^2])dz$$
$$= \lim_{\epsilon \to 0} \int_D m^2(z)((-h_\epsilon(z) + c)^2 - \mathbb{E}[h_\epsilon(z)^2])dz$$
$$= \int_D m^2(z) : (-h + c)^2(z) : dz$$

where the limit is in $L^2(\mathbb{P})$. Therefore, we have that, almost surely,

$$\tilde{\mathbb{E}}[e^{-ia\xi_j} | \Gamma] = \frac{1}{\mathcal{Z}_j(\Gamma)} \mathbb{E}\left[e^{ia\xi_j} \exp\left(-\frac{1}{2}\int_D m^2(z) : (-h_j + \xi_j)^2(z) : dz\right) | \Gamma\right].$$

Moreover, under $\tilde{\mathbb{P}}$, conditionally on Γ, h_j has the same law as $-h_j$. This implies that, almost surely,

$$\tilde{\mathbb{E}}[e^{-ia\xi_j} | \Gamma] = \frac{1}{\mathcal{Z}_j(\Gamma)} \mathbb{E}\left[e^{ia\xi_j} \exp\left(-\frac{1}{2}\int_D m^2(z) : (h_j + \xi_j)^2(z) : dz\right) | \Gamma\right]. \tag{4.59}$$

On the other hand, the same arguments as those used to prove (4.58) show that, almost surely,

$$\tilde{\mathbb{E}}[e^{ia\xi_j} | \Gamma] = \frac{1}{\mathcal{Z}_j(\Gamma)} \mathbb{E}\left[e^{ia\xi_j} \exp\left(-\frac{1}{2}\int_D m^2(z) : (h_j + \xi_j)^2(z) : dz\right) | \Gamma\right]. \tag{4.60}$$

We can deduce from this equality and (4.59) that (4.54) holds, thus identifying the conditional law of ξ_j given Γ under $\tilde{\mathbb{P}}$. Let us now show that under $\tilde{\mathbb{P}}$, the random variables $(\xi_j)_j$ are conditionally

independent given Γ. To this end, conditionally on Γ, let $(a_j)_j$ be a collection of real numbers, one for each loop. We then have that, almost surely,

$$\tilde{\mathbb{E}}\big[\prod_j \exp(ia_j\xi_j)|\Gamma\big] = \frac{1}{\mathcal{Z}(\Gamma)}\mathbb{E}\bigg[\Big(\prod_j \exp(ia_j\xi_j)\Big)\exp\Big(-\frac{1}{2}\int_D m^2(z):h^2(z):dz\Big)|\Gamma\bigg]$$

where $\mathcal{Z}(\Gamma)$ is defined as in (4.25). Arguing as in the proof of Proposition 4.3.1 and using the conditional independence of $(\xi_j)_j$ given Γ under $\mathbb{P}$, we obtain that, almost surely,

$$\tilde{\mathbb{E}}\big[\prod_j \exp(ia_j\xi_j)|\Gamma\big] = \prod_j \frac{1}{\mathcal{Z}_j(\Gamma)}\mathbb{E}\bigg[e^{ia_j\xi_j}\exp\Big(-\frac{1}{2}\int_{\mathrm{Int}(L_j)} m^2(z):(h+\xi_j)^2(z):dz\Big)|\Gamma\bigg].$$

Together with (4.60), this implies that, almost surely, $\tilde{\mathbb{E}}\big[\prod_j \exp(ia_j\xi_j)|\Gamma\big] = \prod_j \tilde{\mathbb{E}}\big[\exp(ia_j\xi_j)|\Gamma\big]$, which shows that under $\tilde{\mathbb{P}}$, the random variables (ξ_j) are conditionally independent given Γ. $\qquad\square$

The conditional law of h under $\tilde{\mathbb{P}}$

Let us now show that under $\tilde{\mathbb{P}}$, conditionally on Γ, the field h can be decomposed as claimed in Theorem 1.2.3. We will then establish Theorem 1.2.3.

Proposition 4.3.6. *With the same assumptions as in Theorem 1.2.3, under $\tilde{\mathbb{P}}$, conditionally on Γ, $h = \sum_j h_j + \xi_j$ where the sum runs over the loops $(L_j)_j$ of Γ and the fields $(h_j + \xi_j)_j$ are independent fields whose law is, for each j, that of a massive GFF with mass m and boundary conditions ξ_j in the interior of the loop L_j.*

Proof. To establish the conditional law of h given Γ under $\tilde{\mathbb{P}}$, we are going to compute its characteristic function. To this end, let f be a smooth function with compact support in D. We have that, almost surely,

$$\tilde{\mathbb{E}}\big[\exp(i(h,f))|\Gamma\big] = \frac{1}{\mathcal{Z}(\Gamma)}\mathbb{E}\bigg[\exp(i(h,f))\exp\Big(-\frac{1}{2}\int_D m^2(z):h^2(z):dz\Big)|\Gamma\bigg], \qquad (4.61)$$

where $\mathcal{Z}(\Gamma)$ is defined in (4.25). By Proposition 4.3.1, almost surely,

$$\mathbb{E}\bigg[\exp(i(h,f))\exp\Big(-\frac{1}{2}\int_D m^2(z):h^2(z):dz\Big)|\Gamma\bigg]$$

$$= \prod_j \mathbb{E}\bigg[\exp(i(h_j+\xi_j,f))\exp\Big(-\frac{1}{2}\int_{\mathrm{Int}(L_j)} m^2(z):(h_j+\xi_j)^2(z):dz\Big)|\Gamma\bigg] \qquad (4.62)$$

$$\times \exp\bigg(\int_{\mathrm{Int}(L_j)} \frac{m^2(z)}{4\pi}\log\frac{\mathrm{CR}(z,\partial D)}{\mathrm{CR}(z,L_j)}dz\bigg).$$

Using the product decomposition (4.55), we deduce from this that, almost surely,

$$\tilde{\mathbb{E}}\big[\exp(i(h,f))|\Gamma\big]$$

$$= \prod_j \frac{1}{\mathcal{Z}_j(\Gamma)}\mathbb{E}\bigg[\exp(i(h_j+\xi_j,f))\exp\Big(-\frac{1}{2}\int_{\mathrm{Int}(L_j)} m^2(z):(h_j+\xi_j)^2(z):dz\Big)|\Gamma\bigg].$$

with $(\mathcal{Z}_j)_j$ defined as in (4.56). Now, observe that by (4.9), conditionally on Γ, for any j, $\mathcal{Z}_j(\Gamma)^{-1}\exp\big(-\frac{1}{2}\int_{\mathrm{Int}(L_j)} m^2(z):(h_j+\xi_j)^2(z):dz\big)$ is almost surely equal to the Radon-Nikodym derivative of the massive GFF with mass m and boundary conditions ξ_j in $\mathrm{Int}(L_j)$ with respect to the GFF with boundary conditions ξ_j in $\mathrm{Int}(L_j)$. Going back to (4.61), this implies that under $\tilde{\mathbb{P}}$, conditionally on Γ, $h = \sum_j h_j + \xi_j$ where $h_j + \xi_j$ is a massive GFF in $\mathrm{Int}(L_j)$ with boundary conditions ξ_j.

Let us now establish the conditional independence of the fields $(h_j + \xi_j)_j$ given Γ under $\tilde{\mathbb{P}}$. To this end, under $\tilde{\mathbb{P}}$, conditionally on Γ, let $(f_j)_j$ be a collection of smooth functions such that for each j, f_j has compact support in $\mathrm{Int}(L_j)$. We then have that, almost surely,

$$\tilde{\mathbb{E}}\big[\prod_j \exp(i(h_j+\xi_j,f_j))|\Gamma\big]$$

$$= \frac{1}{\mathcal{Z}(\Gamma)}\mathbb{E}\bigg[\Big(\prod_j \exp(i(h_j+\xi_j,f_j))\Big)\exp\Big(-\frac{1}{2}\int_D m^2(z):h^2(z):dz\Big)|\Gamma\bigg]$$

where $\mathcal{Z}(\Gamma)$ is as in (4.25). Since, almost surely, $|\prod_j \exp(i(h_j + \xi_j, f_j))| \leq 1$, for the same reasons as those explained in the first part of the proof, we have that, almost surely,

$$\tilde{\mathbb{E}}\big[\prod_j e^{i(h_j + \xi_j, f_j)}|\Gamma\big] = \prod_j \frac{1}{\mathcal{Z}_j(\Gamma)} \mathbb{E}\Big[e^{i(h_j + \xi_j, f_j)} \exp\Big(-\frac{1}{2}\int_{\text{Int}(L_j)} m^2(z) : (h_j + \xi_j)^2(z) : dz\Big)|\Gamma\Big].$$

$$(4.63)$$

On the other hand, using once again the same reasoning as in the first part of the proof, it is easy to see that almost surely, for any j,

$$\tilde{\mathbb{E}}\big[\exp(i(h_j + \xi_j, f_j))|\Gamma\big]$$
$$= \frac{1}{\mathcal{Z}_j(\Gamma)}\mathbb{E}\Big[\exp(i(h_j + \xi_j, f_j)) \exp\Big(-\frac{1}{2}\int_{\text{Int}(L_j)} m^2(z) : (h_j + \xi_j)^2(z) : dz\Big)|\Gamma\Big]$$

which, by (4.63), implies that under $\tilde{\mathbb{P}}$, conditionally on Γ, the fields $(h_j + \xi_j)_j$ are independent. $\quad\square$

We can now conclude the proof of Theorem 1.2.3.

Proof of Theorem 1.2.3. $\tilde{\mathbb{P}}$ defined via the Radon-Nikodym derivative (1.4) is a well-defined probability measure which is absolutely continuous with respect to $\mathbb{P}$. Moreover, by definition, the marginal law under $\tilde{\mathbb{P}}$ of h is that of a massive GFF in D with mass m and Dirichlet boundary conditions. Proposition 4.3.6 and Proposition 4.3.5 show the first three items of the statement of Theorem 1.2.3. The last item follows from the fact that under $\mathbb{P}$, conditionally on Γ, for each j, ξ_j is almost surely measurable with respect to $h_j + \xi_j$. Since $\tilde{\mathbb{P}}$ is absolutely continuous with respect to $\mathbb{P}$, the same holds true under $\tilde{\mathbb{P}}$. This gives the joint law of $(h_j + \xi_j, \xi_j)$ under $\tilde{\mathbb{P}}$ conditionally on Γ. This also concludes the proof of Theorem 1.2.3. $\quad\square$

4.3.2 Some properties of massive CLE$_4$

We show below two properties of massive CLE$_4$, stated in Corollaries 1.2.6 and 1.2.5, that can be established thanks to Theorem 1.2.3.

Proof of Corollary 1.2.6. The proof is very similar to that of Corollary 1.2.2, using that under $\mathbb{P}$, conditionally on Γ, $(h_j)_j$ and (ξ_j) are independent. We leave the details to the reader. $\quad\square$

Proof of Corollary 1.2.5. The proof of this corollary follows the same strategy as that of Lemma 4.2.8 and we leave the details to the reader. $\quad\square$

4.4 Massive CLE$_4$, the massive Brownian loop soup and the massive GFF

In this section, we prove Theorem 1.2.8. In Section 4.4.1, we first define the massive Brownian loop soup and show that its law is absolutely continuous with respect to that of the Brownian loop soup. Then, in Section 4.4.2, we use this result together with Theorem 1.2.3 to deduce Theorem 1.2.8.

4.4.1 The massive Brownian loop soup

The massive Brownian loop soup is a massive version of the Brownian loop soup introduced by Camia in [Cam13; Cam17]. To define it, one first defines a massive version $\mu_D^{(m)}$ of the Brownian loop measure, called the massive Brownian loop measure, via the following change of measure:

$$\mu_D^{(m)}(d\ell) := \exp\Big(-\int_0^{\tau(\ell)} m^2(\ell(t))dt\Big)\mu_D(d\ell).$$

Here, we assume that the domain D and the mass function m satify the assumptions of Theorem 1.2.8 and we recall that μ_D was defined in Section 4.1.4. The measure $\mu_D^{(m)}$ is then a measure on unrooted loops in D. The presence of a mass breaks the conformal invariance of the measure:

$\mu_D^{(m)}$ is conformally covariant, in the following sense. If $f : D \to \Omega$ is a conformal map, then $\mu_\Omega^{(\tilde{m})} = f \circ \mu_D^{(m)}$ where, for $z \in \Omega$, $\tilde{m}^2(z) = |(f^{-1})'(z)|^2 m^2(f^{-1}(z))$.

The massive Brownian loop soup in D with mass m and intensity $\alpha > 0$ is then defined analogously to the Brownian loop soup in D with intensity α (see Section 4.1.4): this is the Poisson point process with intensity measure $\alpha \mu_D^{(m)}$. The massive Brownian loop soup is actually absolutely continuous with respect to the Brownian loop soup and the corresponding Radon-Nikodym can be explicitly computed, as shown below.

Lemma 4.4.1. *Let $D \subset \mathbb{C}$ and $m : D \to \mathbb{R}_+$ be as in Theorem 1.2.8. Let $\alpha > 0$ and denote by $\mathbb{P}_\alpha$, respectively $\mathbb{P}_\alpha^{(m)}$, the law of the Brownian loop soup, respectively massive Brownian loop soup with mass m, in D with intensity α. Then*

$$\frac{\mathrm{d}\mathbb{P}_\alpha^{(m)}}{\mathrm{d}\mathbb{P}_\alpha}(\mathcal{L}) = \frac{1}{\mathcal{Z}} \exp\left(-\int_D m^2(z) : L(z) : dz\right) \tag{4.64}$$

where under $\mathbb{P}_\alpha$, $: L :$ is the renormalized occupation-time field of $\mathcal{L}$ (see Section 4.1.4).

Proof. The proof goes along the same lines as that of the "if" part of the main theorem of [Tak90]. For convenience of the reader, we give the details in Appendix 6.2. □

4.4.2 Construction of massive CLE$_4$ via a massive Brownian loop soup

We now turn to the proof of Theorem 1.2.8. This theorem is in fact a consequence of the following result, which we show below.

Proposition 4.4.2. *Let $D \subset \mathbb{C}$ and $m : D \to \mathbb{R}_+$ be as in Theorem 1.2.8. Denote by $\mathbb{P}$ the law of the coupling $(h, \mathcal{L}, \Gamma)$ of Proposition 4.1.4 between a GFF h in D with Dirichlet boundary conditions, a Brownian loop soup $\mathcal{L}$ in D with intensity $1/2$ and a CLE$_4$ Γ in D. Define a new probability measure $\tilde{\mathbb{P}}$ via*

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}((h, \mathcal{L}, \Gamma)) := \frac{1}{\mathcal{Z}} \exp\left(-\int_D m^2(z) : L(z) : dz\right) \tag{4.65}$$

where, under $\mathbb{P}$, $\int_D m^2(z) : L(z) : dz$ is the renormalized occupation-time field of $\mathcal{L}$ tested against the function m^2 and $\mathcal{Z}$ is a normalization constant. Then, under $\tilde{\mathbb{P}}$,

1. *the marginal law of h is that of a massive GFF in D with mass m and Dirichlet boundary conditions;*
2. *the marginal law of Γ is that of a massive CLE$_4$ in D with mass m;*
3. *the loops of Γ are the level lines of h as in Theorem 1.2.3;*
4. *the marginal law of $\mathcal{L}$ is that of a massive Brownian loop soup in D with mass m and intensity $1/2$;*
5. *the loops of Γ are almost surely equal to the outer boundaries of the outermost clusters of $\mathcal{L}$.*

Proof. As noted in the proof of Lemma 4.4.1, the change of measure (4.65) is well-defined. Items (1), (2) and (3) are consequences of the definition of $\tilde{\mathbb{P}}$ and Theorem 1.2.3. Indeed, under $\mathbb{P}$, $\frac{1}{2} \int_D m^2(z) : h(z)^2 : dz = \int_D m^2(z) : L(z) : dz$ and therefore, by Theorem 1.2.3, (h, Γ) has the required joint law under $\tilde{\mathbb{P}}$. Item (4) is in turn a direct consequence of Lemma 4.4.1. Finally, Item (5) follows by absolute continuity, since the equality holds almost surely under $\mathbb{P}$. □

Chapter 5

Interface scaling limit for the critical planar Ising model perturbed by a magnetic field

5.1 Setting and background

We detail here the assumptions under which Theorem 1.3.1, Proposition 1.3.3 and Proposition 1.3.4 are proved: the assumptions on $(\Omega_\delta; a_\delta, b_\delta)_\delta$ and $(\Omega; a, b)$ are made precise in Section 5.1.1 and the topologies in which weak convergence is established are described in Section 2.2. Section 5.1.2 and Section 5.1.3 provide some background on the Ising model.

5.1.1 Assumptions on the domain

We consider a bounded, open and simply connected subset Ω of the complex plane $\mathbb{C}$ such that $0 \in \Omega$. We fix two marked boundary points $a, b \in \partial\Omega$ and we assume that both a and b are degenerate prime ends of Ω, see [Pom92, Section 2.5] for a definition. These assumptions on Ω and the boundary points a and b will in particular allow us to use [Kar23, Theorem 4.2].

Regarding the assumptions on the boundary $\partial\Omega$ of Ω, Theorem 1.3.1 and Proposition 1.3.3 will be shown assuming that the Hausdorff dimension of $\partial\Omega$ is strictly smaller that $\frac{7}{4}$. However, the proof of Proposition 1.3.4 requires $\partial\Omega$ to be more regular and we will actually take it to be smooth, for reasons that we explain in the proof of Claim 5.3.3.

We assume that $(\Omega_\delta)_\delta$ is a sequence of graphs approximating Ω in a sense that we will now explain. For each $\delta > 0$, Ω_δ is a simply connected subgraph of the square lattice $\delta\mathbb{Z}^2$, so that every edge of Ω_δ has length δ. We denote by $V(\Omega_\delta)$ the set of vertices of Ω_δ and define the boundary $\partial\Omega_\delta$ of Ω_δ as

$$\partial\Omega_\delta := \{w \in \delta\mathbb{Z}^2 \setminus V(\Omega_\delta) : \text{there exists } v \in V(\Omega_\delta) \text{ such that } v \sim w\}$$

where $v \sim w$ means that there is an edge of $\delta\mathbb{Z}^2$ connecting v and w. With this definition of $\partial\Omega_\delta$, it is legitimate to set $\text{Int}(\Omega_\delta) := V(\Omega_\delta)$. We also let $E(\Omega_\delta)$ denote the set of edges of $\delta\mathbb{Z}^2$ with at least one endpoint in $\text{Int}(\Omega_\delta)$.

We associate to each Ω_δ an open and simply connected polygonal domain $\hat{\Omega}_\delta \subset \mathbb{C}$ by taking the union of all open squares with side length δ centered at vertices in $V(\Omega_\delta)$. We assume that for any $\delta > 0$, 0 belongs to $\hat{\Omega}_\delta$ and that there exists $\bar{R} > 0$ such that for any $\delta > 0$, $\hat{\Omega}_\delta \subset B(0, \bar{R})$ and $\Omega \subset B(0, \bar{R})$. These assumptions are necessary to apply [Kar23, Theorem 4.2]. The marked boundary points a and b of $\partial\Omega$ are then approximated by sequences $(a_\delta)_\delta$ and $(b_\delta)_\delta$ where, for each $\delta > 0$, a_δ and b_δ belong to $\partial\hat{\Omega}_\delta$.

The sequence $(\hat{\Omega}_\delta; a_\delta, b_\delta)_\delta$ is assumed to converge to $(\Omega; a, b)$ in the Carathéodory topology. That is,

- each inner point $z \in \Omega$ belongs to $\hat{\Omega}_\delta$ for δ small enough;

- for every boundary point $\zeta \in \partial\Omega$, there exists a sequence $(\zeta_\delta)_\delta$ such that $\zeta_\delta \to \zeta$ as $\delta \to 0$, where, for each $\delta > 0$, $\zeta_\delta \in \partial\hat{\Omega}_\delta$.

This can be rephrased in terms of conformal maps. Let $\psi : \Omega \to \mathbb{D}$ be a conformal map such that $\psi(a) = 1$ and $\psi(0) = 0$. Similarly, for each $\delta > 0$, let $\psi_\delta : \hat{\Omega}_\delta \to \mathbb{D}$ be a conformal map such that $\psi_\delta(a_\delta) = 1$ and $\psi_\delta(0) = 0$. Then, by [Pom92, Theorem 1.8], the Carathéodory convergence of $(\hat{\Omega}_\delta; a_\delta, b_\delta)_\delta$ to $(\Omega; a, b)$ is equivalent to

$$\psi_\delta \to \psi \quad \text{uniformly on compact subsets of } \Omega \text{ and}$$

$$\psi_\delta^{-1} \to \psi^{-1} \quad \text{uniformly on compact subsets of } \mathbb{D}.$$

Furthermore, we suppose that a_δ, respectively b_δ, is a close approximation of a, respectively b, as defined by Karrila in [Kar23, Section 4.3]. Let us recall what this means. To lighten the notations, we identify the prime ends a_δ and a with their corresponding radial limit points. For $r > 0$, let S_r be the arc of $\partial B(a, r) \cap \Omega$ disconnecting in Ω the prime end a from 0 and that is closest to a. In other words, S_r is the last arc from the (possibly countable) collection $\partial B(a, r) \cap \Omega$ of arcs that a path running from 0 to a inside Ω must cross. Such an arc exists by [Kar23, Lemma A.1] and approximation by radial limits. a_δ is then said to be a close approximation of a if

- $a_\delta \to a$ as $\delta \to 0$; and

- for each r small enough and for all sufficiently (depending on r) small δ, the boundary point a_δ of $\hat{\Omega}_\delta$ is connected to the midpoint of S_r inside $\hat{\Omega}_\delta \cap B(a, r)$.

In what follows, we will need to split the boundary $\partial \Omega_\delta$ of Ω_δ into two set of vertices, depending on their position with respect to a_δ and b_δ. We therefore denote by $\partial \Omega_\delta^-$, respectively $\partial \Omega_\delta^+$, the vertices of $\partial \Omega_\delta$ that are traversed when following the vertices of $\partial \Omega_\delta$ from a_δ to b_δ counterclockwise, respectively clockwise.

5.1.2 The Ising model

For $\delta > 0$, let $(\Omega_\delta; a_\delta, b_\delta)$ be as above. The Ising model on Ω_δ is a probability measure on spin configurations $\sigma : V(\Omega_\delta) \cup \partial \Omega_\delta \to \{-1, 1\}^{|V(\Omega_\delta) \cup \partial \Omega_\delta|}$. In our setting, since the boundary $\partial \Omega_\delta$ of Ω_δ is non-empty, boundary conditions can be imposed to specify the measure. Two different choices of boundary conditions will be of interest here. The first one is that of Dobrushin boundary conditions: the spins of the vertices of $\partial \Omega_\delta^-$ are all -1 while the spins of the vertices of $\partial \Omega_\delta^+$ are all $+1$. The probability of an Ising spin configuration σ in Ω_δ with Dobrushin boundary conditions is given by

$$\mathbb{P}_{\Omega_\delta, \beta}^{\pm}(\sigma) = \frac{1}{\mathcal{Z}_\beta} \exp \left(\beta \sum_{x \sim y} \sigma_x \sigma_y \right) \mathbb{I}_{\sigma_{|\partial \Omega_\delta^+} = +1} \mathbb{I}_{\sigma_{|\partial \Omega_\delta^-} = -1} \tag{5.1}$$

where $\beta > 0$ is a parameter called the inverse temperature. Above, the sum is over nearest neighbor vertices $x \sim y$ of $\delta \mathbb{Z}^2$ such that at least one of them belong to $V(\Omega_\delta)$.

The other set of boundary conditions that we will consider is that of $+1$ boundary conditions and -1 boundary conditions. The probability $\mathbb{P}_{\Omega_\delta, \beta}^{+}(\sigma)$, respectively $\mathbb{P}_{\Omega_\delta, \beta}^{-}(\sigma)$, of an Ising spin configuration σ in Ω_δ with $+1$, respectively -1, boundary conditions is given by replacing $\mathbb{I}_{\sigma_{|\partial \Omega_\delta^+} = +1} \mathbb{I}_{\sigma_{|\partial \Omega_\delta^-} = -1}$ by $\mathbb{I}_{\sigma_{|\partial \Omega_\delta} = +1}$, respectively $\mathbb{I}_{\sigma_{|\partial \Omega_\delta} = -1}$ in (5.1). In the course of the proof of Proposition 1.3.4, we will also use the Ising model with free boundary conditions: in this case, the spins of the vertices on $\partial \Omega_\delta$ is not prescribed.

The parameter β plays an important role in the macroscopic behavior of the Ising model on Ω_δ. Indeed, there exists a critical value $0 < \beta_c < \infty$ at which the model undergoes a phase transition. On the square lattice, it has been shown that $\beta_c = \frac{1}{2} \log(\sqrt{2} + 1)$ [FV17]. Moreover, at $\beta = \beta_c$, as $\delta \to 0$, the Ising model on the square lattice exhibits conformal invariance properties and this has been the topic of extensive research in the past twenty years, see [Che18; Dum22] for a survey. Here, we will always take $\beta = \beta_c$ and we set $\mathbb{P}_{\Omega_\delta}^{\pm} := \mathbb{P}_{\Omega_\delta, \beta_c}^{\pm}$ and $\mathbb{P}_{\Omega_\delta}^{+} := \mathbb{P}_{\Omega_\delta, \beta_c}^{+}$ and call the corresponding Ising model the critical Ising model.

In what follows, we will consider the critical Ising model perturbed by an external magnetic field. For $H : V(\Omega_\delta) \to \mathbb{R}$ a bounded function, when we impose Dobrushin boundary conditions, this is a probability measure denoted $\mathbb{P}_{\Omega_\delta, H}^{\pm}$ on spin configurations $\sigma : V(\Omega_\delta) \cup \partial \Omega_\delta \to \{-1, +1\}^{|V(\Omega_\delta) \cup \partial \Omega_\delta|}$ that can be defined via its Radon-Nikodym derivative with respect to $\mathbb{P}_{\Omega_\delta}^{\pm}$ as follows:

$$\frac{d\mathbb{P}_{\Omega_\delta, H}^{\pm}}{d\mathbb{P}_{\Omega_\delta}^{\pm}}(\sigma) = \frac{1}{\mathcal{Z}_H} \exp \left(\sum_{x \in V(\Omega_\delta)} H_x \sigma_x \right)$$

where $\mathcal{Z}_H$ is a normalization constant. We can also impose $+1$ boundary conditions to the model, in which case the corresponding probability measure $\mathbb{P}^+_{\Omega_\delta, H}$ has Radon-Nikodym derivative with respect to $\mathbb{P}^+_{\Omega_\delta}$ given by

$$\frac{d\mathbb{P}^+_{\Omega_\delta, H}}{d\mathbb{P}^+_{\Omega_\delta}}(\sigma) = \frac{1}{\mathcal{Z}^+_H} \exp\left(\sum_{x \in V(\Omega_\delta)} H_x \sigma_x \right).$$

Below, we will typically choose functions $H : V(\Omega_\delta) \to \mathbb{R}$ which depend on the meshsize $\delta > 0$ of Ω_δ. Moreover, if $h : \Omega \to \mathbb{R}$, we define the function $h_\delta : V(\Omega_\delta) \to \mathbb{R}$ as the restriction of h to $V(\Omega_\delta)$, that is, for $x \in V(\Omega_\delta)$, $h_\delta(x) = h(x)$.

An important property of the Ising model at arbitrary inverse temperature $\beta > 0$ perturbed by an external magnetic field H is the FKG property [FV17, Theorem 3.21]. We introduce the partial order $\leq$ on spin configurations defined by $\sigma \leq \sigma'$ if and only if $\sigma_x \leq \sigma'_x$ for any $x \in V(\Omega_\delta) \cup \partial\Omega_\delta$. A function $f : \{-1, 1\}^{V(\Omega_\delta) \cup \partial\Omega_\delta} \to \mathbb{R}$ is said to be increasing if $f(\sigma) \leq f(\sigma')$ whenever $\sigma \leq \sigma'$. Similarly, we say that an event E is increasing if the indicator function $\mathbb{I}_E$ is increasing.

Proposition 5.1.1 (FKG inequality). *Let f, g be two increasing functions and $H : V(\Omega_\delta) \to \mathbb{R}$. Then, for any boundary conditions ξ, $\mathbb{E}^\xi_{\Omega_\delta, \beta, H}[fg] \geq \mathbb{E}^\xi_{\Omega_\delta, \beta, H}[f]\mathbb{E}^\xi_{\Omega_\delta, \beta, H}[g]$.*

As a consequence of the FKG inequality, we have the following comparison between the Ising model perturbed by a non-negative external field and the Ising model with no external field.

Lemma 5.1.2. *Let f be an increasing function and let $H : V(\Omega_\delta) \to \mathbb{R}_+$. Then, $\mathbb{E}^+_{\Omega_\delta, \beta, H}[f] \geq \mathbb{E}^+_{\Omega_\delta, \beta}[f]$.*

Ising measures with different boundary conditions can also be compared with one another. The following inequality will be useful, see [Dum13, Theorem 7.6] for a proof.

Proposition 5.1.3. *Let f be an increasing function. Then, $\mathbb{E}^+_{\Omega_\delta, \beta}[f] \geq \mathbb{E}^{\text{free}}_{\Omega_\delta, \beta}[f]$.*

The discrete interface of the Ising model

Under $\mathbb{P}^\pm_{\Omega_\delta}$ and $\mathbb{P}^\pm_{\Omega_\delta, H}$, in each spin configuration, there is an interface γ_δ separating $+1$ and -1 spins: this is a discrete path running from a_δ to b_δ such that a vertex adjacent to it has spin $+1$ if it is on its left and -1 if it is on its right. Note that this interface is a priori not unique. In what follows, we always choose the leftmost such interface but this choice is irrelevant: any other reasonable way of prescribing how to turn when encountering a face surrounded by four alternating spins would yield the same scaling limit as $\delta \to 0$. As in [Che+14], for technical reasons, we draw γ_δ on the auxiliary square-octagon lattice, with octagons corresponding to vertices of Ω_δ. This for example prevents γ_δ from having self-intersection points.

The Edwards-Sokal coupling

The Edwards-Sokal coupling provides a coupling between an edge configuration on the graph $G_\delta = (V(\Omega_\delta), E_\delta)$, sampled from the FK-Ising measure, and an Ising configuration on Ω_δ (with no external field). As this coupling will be a useful tool in what follows, we recall it below.

We call a function $\omega : E(\Omega_\delta) \to \{0, 1\}^{|E(\Omega_\delta)|}$ an edge configuration. We say that an edge e is open in ω if $\omega(e) = 1$. Otherwise, e is said to be closed. We identify ω with the subgraph whose vertex set is $V(\Omega_\delta)$ and whose edge set is $\{e \in E(\Omega_\delta) : \omega(e) = 1\}$. A cluster of ω is a connected component of this subgraph. One can also impose boundary conditions on $\partial\Omega_\delta$: this is a partition ξ of $\partial\Omega_\delta$ and two vertices are wired together if they are in the same element of the partition. When $\xi = \text{free}$, no vertices of $\partial\Omega_\delta$ are wired together and when $\xi = \text{wired}$, all boundary vertices are wired together.

With these notations, the FK-Ising percolation measure with parameter $p \in [0, 1]$ is defined as follows: for ξ a set of boundary conditions and ω an edge configuration,

$$\mathbb{P}^{\text{FK}, \xi}_{\delta, p}(\omega) = \frac{1}{\mathcal{Z}_p} 2^{k(\omega^\xi)} \prod_{e \in E(\Omega_\delta)} p^{\omega(e)}(1-p)^{1-\omega(e)}$$

where ω^ξ is the edge configuration obtained by identifying wired boundary vertices and $k(\omega^\xi)$ the number of clusters of ω^ξ. Above, $\mathcal{Z}_p$ is a normalization constant chosen such that $\mathbb{P}^{\text{FK}, \xi}_{\delta, p}$ is a probability measure.

The measure $\mathbb{P}^{\mathrm{FK},\xi}_{\delta,p}$ satisfies an FKG property similar to that of the Ising model and we will use it on several occasions. To compare edge configurations, we introduce a partial order $\leq$ defined as follows: for ω,ω' two edge configurations, $\omega \leq \omega'$ if and only if $\omega(e) \leq \omega'(e)$ for any $e \in E(\Omega_\delta)$. A function $f : \{0,1\}^{E(\Omega_\delta)} \to \mathbb{R}$ is then said to be increasing if $f(\omega) \leq f(\omega')$ whenever $\omega \leq \omega'$. Similarly, an event E is said to be increasing if the indicator function $\mathbb{I}_E$ is increasing. The FKG property of the FK-Ising model is the following inequality [Dum13, Theorem 4.14].

Proposition 5.1.4. *Fix $p \in [0,1]$. Let f and g be two increasing functions and let ξ be a set of boundary conditions. Then $\mathbb{E}^{\mathrm{FK},\xi}_{\delta,p}[fg] \geq \mathbb{E}^{\mathrm{FK},\xi}_{\delta,p}[f]\mathbb{E}^{\mathrm{FK},\xi}_{\delta,p}[g]$.*

This proposition yields the following comparison between boundary conditions, see [Dum13, Corollary 4.19] for a proof.

Proposition 5.1.5. *Fix $p \in [0,1]$. Let $\xi \leq \psi$ be two sets of boundary conditions and let f be an increasing function. Then, $\mathbb{E}^{\mathrm{FK},\psi}_{\delta,p}[f] \geq \mathbb{E}^{\mathrm{FK},\xi}_{\delta,p}[f]$.*

We can now describe the Edwards-Sokal coupling for the Ising model in Ω_δ. Fix $p \in [0,1]$ and let $\xi \in \{\mathrm{wired, free}\}$. Sample first an edge configuration ω on G_δ according to $\mathbb{P}^{\mathrm{FK},\xi}_{\delta,p}$. Then, assign independently to each cluster of ω not connected to $\partial\Omega_\delta$ a spin $+1$ or -1 with probability $1/2$. If $\xi = \mathrm{free}$, proceed in a similar way for clusters connected to $\partial\Omega_\delta$ and if $\xi = \mathrm{wired}$, assign the spin $+1$ to the cluster connected to $\partial\Omega_\delta$. Finally, for each vertex $x \in V(\Omega_\delta)$, define σ_x to be the spin of the cluster of x in ω.

Proposition 5.1.6 (Edwards-Sokal coupling [ES88])**.** *The spin configuration σ constructed above is distributed according to the Ising measure with inverse temperature $\beta = -\frac{1}{2}\log(1-p)$ and boundary conditions ξ, where $\xi \in \{\mathrm{wired, free}\}$.*

Below, when $\beta = \beta_c$, we simply write $\mathbb{P}^{\mathrm{FK},\xi}_{\delta}$ instead of $\mathbb{P}^{\mathrm{FK},\xi}_{\delta,p_c}$, where p_c is given by Proposition 5.1.6.

The critical Ising interface

A conformal invariance property of the critical Ising model emerges when looking at the scaling limit of the interface γ_δ under $\mathbb{P}^{\pm}_{\Omega_\delta}$ as $\delta \to 0$. Indeed, as a consequence of the next theorem, its limiting law is conformally invariant.

Theorem 5.1.7 ([Che+14])**.** *Under the above assumptions on $(\Omega; a,b)$ and $(\Omega_\delta, a_\delta, b_\delta)_\delta$, as $\delta \to 0$, $(\gamma_\delta)_\delta$ under $(\mathbb{P}^{\pm}_\delta)_\delta$ converges weakly in the topologies (T.1)–(T.4) to SLE_3 in Ω from a to b.*

Theorem 5.1.7 will be the starting point of the proof of Theorem 1.3.1.

5.1.3 Spin correlations at $\beta = \beta_c$

The crucial input for the proof of Theorem 1.3.1 is the existence of a scaling limit for the k-point spin correlations of the critical Ising model, when the boundary conditions are either $+1$, -1 or Dobrushin. This result, established in [CHI21], holds under no assumption on the smoothness of the boundary of the underlying domain Ω. This fact is essential as we will typically apply this result when the underlying domain is a domain slit by a portion of an SLE_3 curve.

Theorem 5.1.8 ([CHI21])**.** *Let $\Omega \subset \mathbb{C}$ be a bounded and simply connected domain and $a,b \in \partial\Omega$ be two prime ends of $\partial\Omega$. Let $(\Omega_\delta; a_\delta, b_\delta)_\delta$ be discrete approximations of $(\Omega; a,b)$ satisfying the assumptions of Section 5.1.1. Then, for any $\eta > 0$ and any $k \geq 1$, we have that*

$$\delta^{-\frac{k}{8}}\mathbb{E}^{\pm}_\delta[\sigma_{x_1}\ldots\sigma_{x_k}] = C^k_\sigma f^{(\pm,k)}_\Omega(x_1,\ldots,x_k) + o(1)$$

$$\delta^{-\frac{k}{8}}\mathbb{E}^{+}_\delta[\sigma_{x_1}\ldots\sigma_{x_k}] = C^k_\sigma f^{(+,k)}_\Omega(x_1,\ldots,x_k) + o(1)$$

$$\delta^{-\frac{k}{8}}\mathbb{E}^{-}_\delta[\sigma_{x_1}\ldots\sigma_{x_k}] = C^k_\sigma f^{(-,k)}_\Omega(x_1,\ldots,x_k) + o(1)$$

as $\delta \to 0$, uniformly over all $x_1,\ldots,x_k \in \Omega$ at distance at least η from $\partial\Omega$ and from each other. Here, $C_\sigma > 0$ is a constant that corresponds to the fact that we are working on the square lattice. The functions $f^{(\pm,k)}_\Omega$, $f^{(+,k)}_\Omega$ and $f^{(-,k)}_\Omega$ are explicit and satisfy the following conformal covariance

property. If $\psi : \Omega \to \tilde{\Omega}$ is a conformal map, then for any $k \geq 1$ and any $x_1, \ldots, x_k \in \Omega$, with $\xi \in \{\pm, -, +\}$,

$$f_\Omega^{(\xi,k)}(x_1, \ldots, x_k) = \prod_{j=1}^k |\psi'(x_j)|^{1/8} f_{\tilde{\Omega}}^{(\xi,k)}(\psi(x_1), \ldots, \psi(x_k)).$$

Remark 5.1.9. In [CHI21], Theorem 5.1.8 is established under the assumption that Ω_δ is a subset of the rotated square lattice $\sqrt{2}\delta e^{i\frac{\pi}{4}} \mathbb{Z}^2$ with meshsize $\sqrt{2}\delta$. The difference of meshsize does not change anything while rotating $\delta \mathbb{Z}^2$ only affects the value of the constant C_σ. The functions $(f_\Omega^{(\xi,k)})_{k \geq 1}$, $\xi \in \{\pm, -, +\}$, remain the same.

Remark 5.1.10. Let us briefly explain the scaling by $\delta^{-\frac{k}{8}}$ that appears in Theorem 5.1.8. Let $\Lambda_\delta = [-L, L]^2 \cap \delta\mathbb{Z}^2$. It is known since the work of Onsager [Ons44] that there exist $c_1, c_2 > 0$ such that

$$\frac{c_1 \delta^{\frac{1}{8}}}{L^{1/8}} \leq \mathbb{E}_{\Lambda_\delta}^+[\sigma_0] \leq \frac{c_2 \delta^{\frac{1}{8}}}{L^{1/8}} \tag{5.2}$$

where σ_0 is the spin at the origin. One can see from this inequality that in order to obtain a non-trivial scaling limit for $\mathbb{E}_{\Lambda_\delta}^+[\sigma_0]$ as $\delta \to 0$, one should rescale this expectation by $\delta^{-\frac{1}{8}}$. For k-spin correlation functions $\mathbb{E}_{\Lambda_\delta}^+[\sigma_{x_1} \ldots \sigma_{x_k}]$, one way to understand why the correct scaling is $\delta^{-\frac{k}{8}}$ is to use the Edwards-Sokal coupling described in Section 5.3.2. This yields that

$$\mathbb{E}_{\Lambda_\delta}^+[\sigma_{x_1} \ldots \sigma_{x_k}] \leq \prod_{j=1}^k \mathbb{P}_{\Lambda_\delta(x_j)}^{FK,\text{wired}}[x_j \longleftrightarrow \partial \Lambda_\delta(x_j)]$$

where for each j, $\Lambda_\delta(x_j) = (x_j + [-d_j, d_j]) \cap \delta\mathbb{Z}^2$ where $d_j = \frac{1}{2} \min_{p:p \neq j} |x_p - x_j| \wedge \text{dist}(x_j, \partial\Lambda)$. $\{x_j \longleftrightarrow \partial\Lambda_\delta(x_j)\}$ is the event that there is an open path from x_j to the boundary of $\Lambda_\delta(x_j)$. Moreover, the Edwards-Sokal coupling shows that for each j,

$$\mathbb{P}_{\Lambda_\delta(x_j)}^{FK,\text{wired}}[x_j \longleftrightarrow \partial\Lambda_\delta(x_j)] = \mathbb{E}_{\Lambda_\delta(x_j)}^+[\sigma_{x_j}].$$

Using the upper bound (5.2), we thus obtain that

$$\mathbb{E}_{\Lambda_\delta}^+[\sigma_{x_1} \ldots \sigma_{x_k}] \leq c_2^k \delta^{\frac{k}{8}} \prod_{j=1}^k d_j^{-\frac{1}{8}}.$$

This suggests that $\delta^{-\frac{k}{8}}$ is indeed the correct scaling to obtain a non-trivial limit for $\mathbb{E}_{\Lambda_\delta}^+[\sigma_{x_1} \ldots \sigma_{x_k}]$ as $\delta \to 0$. When the boundary conditions are $\pm$, the k-spin correlations functions also scale like $\delta^{\frac{k}{8}}$. When the points $x_1, \ldots, x_k$ are in the bulk, this is can be seen using a mixing argument which shows that the boundary conditions do not affect the scaling of the spin correlations.

Spin correlations of the critical Ising model with $+1$ boundary conditions are often easier to work with than those of the critical Ising model with Dobrushin boundary conditions. The following lemma bounds the latter correlations by the former and this bound will enable us to get sufficient control on the moments of $\sum_x \sigma_x$.

Lemma 5.1.11. *Let $k \geq 1$. For any $x_1, \ldots, x_k \in \Omega_\delta$, $|\mathbb{E}_\delta^\pm[\prod_{j=1}^k \sigma_{x_j}]| \leq \mathbb{E}_\delta^+[\prod_{j=1}^k \sigma_{x_j}]$.*

Proof. The proof relies on the Edwards-Sokal coupling of Proposition 5.1.6. Let us first examine the case when the Ising model in Ω_δ has $+$ boundary conditions. Let $k \geq 1$. We identify the vertices on the boundary of Ω_δ with a single vertex b_+. For $x_1, \ldots, x_k \in \Omega_\delta$, an FK-Ising percolation configuration on Ω_δ induces a graph on $\{x_1, \ldots, x_k, b_+\}$, where two vertices x_i and x_j are connected via an edge if there is a direct open path between them and a vertex x_i is connected to b_+ if there is a direct open path between x_i and the boundary of Ω_δ. Let us denote by $A_{x_1, \ldots, x_k}$ the event that the graph on $\{x_1, \ldots, x_k, b_+\}$ induced by the FK-Ising percolation configuration is such that each cluster contains an even number of vertices, except possibly the cluster containing b_+. Then, using the Edwards-Sokal coupling, it is easy to see that

$$\mathbb{E}_\delta^+[\prod_{j=1}^k \sigma_{x_j}] = \mathbb{P}_\delta^{FK,\text{wired}}(A_{x_1, \ldots, x_k})$$

Note also that $A_{x_1,\ldots,x_k}$ is an increasing event.

The Edwards-Sokal coupling can also be used in the case of Dobrushin boundary conditions. The coupling is as with $+$ boundary conditions except that the vertices of $\partial\Omega_\delta^+$ are wired together, the vertices of $\partial\Omega_\delta^-$ are wired together but $\partial\Omega_\delta^+$ is not wired to $\partial\Omega_\delta^-$. Moreover, the FK-Ising percolation configuration is in that case conditioned on the event $\{\partial\Omega_\delta^+ \not\leftrightarrow \partial\Omega_\delta^-\}$ that there are no clusters containing both a vertex of $\partial\Omega_\delta^+$ and a vertex of $\partial\Omega_\delta^-$. We denote by $\mathbb{P}_\delta^{2-w}(\cdot|\partial\Omega_\delta^+ \not\leftrightarrow \partial\Omega_\delta^-)$ the corresponding probability measure. We identify the vertices on $\partial\Omega_\delta^+$, respectively $\partial\Omega_\delta^-$, with a single vertex b_+, respectively b_-. As before, for $x_1,\ldots,x_k \in \Omega_\delta$, such an FK-Ising percolation configuration on Ω_δ induces a graph on $\{x_1,\ldots,x_k,b_+,b_-\}$, where two vertices x_i and x_j are connected via an edge if there is a direct open path between them and a vertex x_i is connected to b_+, respectively b_-, if there is a direct open path between x_i and $\partial\Omega_\delta^+$, respectively $\partial\Omega_\delta^-$. Let us denote by $A_{x_1,\ldots,x_k}^{(\mathrm{even})}$, respectively $A_{x_1,\ldots,x_k}^{(\mathrm{odd})}$, the event that the graph induced on $\{x_1,\ldots,x_k,b_+,b_-\}$ by the FK-Ising percolation configuration is such that each cluster not connected to b_+ or b_- contains an even number of vertices and that the cluster of b_- contains an even, respectively odd, number of vertices. No restrictions are imposed on the cluster connected to b_+. Then, it is easy to see that

$$\mathbb{E}_\delta^\pm[\prod_{j=1}^k \sigma_{x_j}] = \mathbb{P}_\delta^{2-w}(A_{x_1,\ldots,x_k}^{(\mathrm{even})}|\partial\Omega_\delta^+ \not\leftrightarrow \partial\Omega_\delta^-) - \mathbb{P}_\delta^{2-w}(A_{x_1,\ldots,x_k}^{(\mathrm{odd})}|\partial\Omega_\delta^+ \not\leftrightarrow \partial\Omega_\delta^-).$$

This yields that $|\mathbb{E}_\delta^\pm[\prod_{j=1}^k \sigma_{x_j}]| \leq \mathbb{P}_\delta^{2-w}(A_{x_1,\ldots,x_k}^{(\mathrm{even})} \cup A_{x_1,\ldots,x_k}^{(\mathrm{odd})}|\partial\Omega_\delta^+ \not\leftrightarrow \partial\Omega_\delta^-)$. Now, since $A_{x_1,\ldots,x_k}^{(\mathrm{even})} \cup A_{x_1,\ldots,x_k}^{(\mathrm{odd})}$ is an increasing event and $\{\partial\Omega_\delta^+ \not\leftrightarrow \partial\Omega_\delta^-\}$ is a decreasing event, we have that, by the FKG inequality (Proposition 5.1.4),

$$\begin{aligned}
\mathbb{P}_\delta^{2-w}(A_{x_1,\ldots,x_k}^{(\mathrm{even})} \cup A_{x_1,\ldots,x_k}^{(\mathrm{odd})}|\partial\Omega_\delta^+ \not\leftrightarrow \partial\Omega_\delta^-) &= \frac{\mathbb{P}_\delta^{2-w}((A_{x_1,\ldots,x_k}^{(\mathrm{even})} \cup A_{x_1,\ldots,x_k}^{(\mathrm{odd})}) \cap \{\partial\Omega_\delta^+ \not\leftrightarrow \partial\Omega_\delta^-\})}{\mathbb{P}_\delta^{2-w}(\partial\Omega_\delta^+ \not\leftrightarrow \partial\Omega_\delta^-)} \\
&\leq \frac{\mathbb{P}_\delta^{2-w}(A_{x_1,\ldots,x_k}^{(\mathrm{even})} \cup A_{x_1,\ldots,x_k}^{(\mathrm{odd})})\mathbb{P}_\delta^{2-w}(\partial\Omega_\delta^+ \not\leftrightarrow \partial\Omega_\delta^-)}{\mathbb{P}_\delta^{2-w}(\partial\Omega_\delta^+ \not\leftrightarrow \partial\Omega_\delta^-)} \\
&= \mathbb{P}_\delta^{2-w}(A_{x_1,\ldots,x_k}^{(\mathrm{even})} \cup A_{x_1,\ldots,x_k}^{(\mathrm{odd})}).
\end{aligned}$$

Moreover, by Proposition 5.1.5, $\mathbb{P}_\delta^{2-w}(A_{x_1,\ldots,x_k}^{(\mathrm{even})} \cup A_{x_1,\ldots,x_k}^{(\mathrm{odd})}) \leq \mathbb{P}_\delta^{\mathrm{FK,wired}}(A_{x_1,\ldots,x_k}^{(\mathrm{even})} \cup A_{x_1,\ldots,x_k}^{(\mathrm{odd})})$. Since when $\partial\Omega_\delta^+$ and $\partial\Omega_\delta^-$ are wired together, $A_{x_1,\ldots,x_k}^{(\mathrm{even})} \cup A_{x_1,\ldots,x_k}^{(\mathrm{odd})}$ and $A_{x_1,\ldots,x_k}$ are the same events, this implies that $|\mathbb{E}_\delta^\pm[\prod_{j=1}^k \sigma_{x_j}]| \leq \mathbb{P}_\delta^{\mathrm{FK,wired}}(A_{x_1,\cdots,A_{x_k}})$, from which we deduce the lemma. $\square$

The next lemma provides a bound on k-point spin correlations of the critical Ising model with $+1$ boundary conditions. It is certainly well-known but for completeness, we prove it below.

Lemma 5.1.12. *There exists a constant $C > 0$ such that for any $k \geq 1$ and any $x_1,\ldots,x_k \in \Omega_\delta$,*

$$\mathbb{E}_\delta^+[\prod_{j=1}^k \sigma_{x_j}] \leq C^k \delta^{\frac{k}{8}} \exp\left(\frac{1}{8}\sum_{j=1}^k \log \frac{1}{\frac{1}{2}L(x_j) \wedge d(x_j,\partial\Omega_\delta)}\right)$$

where for each j, $L(x_j) = \min_{p:p\neq j}|x_p - x_j|$. Above, for $x \in \Omega_\delta$, $d(x,\partial\Omega_\delta)$ is the Euclidean distance between x and $\partial\Omega_\delta$.

Proof. The proof of this lemma relies on the Edwards-Sokal coupling and the Markov property of FK-Ising percolation. We note that similar ideas appear in the proof of [FM17, Proposition 3.10]. Let $k \geq 1$ and let $x_1,\ldots,x_k \in \Omega_\delta$. For each $j \in \{1,\ldots,k\}$, set $L(x_j) := \frac{1}{2}\min_{p:p\neq j}|x_j - x_p|$ and $d(x_j) := d(x_j,\partial\Omega_\delta)$. We then define $\Lambda(x_j)$ to be the Euclidean square of side-length $\frac{1}{\sqrt{2}} \times (d(x_j) \wedge L(x_j))$ centered at x_j. Observe that $\Lambda(x_j) \subset \Omega$. Using that $\mathbb{E}_\delta^+[\sigma_{x_1}\ldots\sigma_{x_k}]$ can expressed in terms of connection events in a FK-Ising percolation configuration (see the proof of Lemma 5.1.11) and the Markov property of FK-Ising percolation, we obtain that

$$\mathbb{E}_\delta^+[\sigma_{x_1}\ldots\sigma_{x_k}] \leq \prod_{j=1}^k \mathbb{P}_{\Lambda_\delta(x_j)}^{\mathrm{FK,wired}}(x_j \leftrightarrow \partial\Lambda(x_j)),$$

where $\Lambda_\delta(x_j) := \Lambda(x_j) \cap \delta\mathbb{Z}^2$. By [DHN11, Lemma 5.4], there exists a constant $C > 0$ such that for any j, $\mathbb{P}_{\Lambda(x_j)}^w(x_j \leftrightarrow \partial\Lambda(x_j)) \leq C\delta^{1/8}(d(x_j) \wedge L(x_j))^{-1/8}$. This concludes the proof of the lemma. $\square$

5.2 Scaling limit of the near-critical interface: proof of Theorem 1.3.1

In this section, we prove Theorem 1.3.1. We give the proof for $H_x = C_\sigma^{-1} h \delta^{15/8}$ where $h \in \mathbb{R}$ and C_σ is the constant of Theorem 5.1.8. The extension to space-varying external fields can easily be obtained, as long as the function $h : \Omega \to \mathbb{R}$ is bounded and Lipschitz. Indeed, this last property guarantees that discrete sums approximate continuum integrals in a sufficiently nice way, see [CS11], which makes the generalization of the proof below straightforward.

To lighten notations, we set $\mathbb{P}_\delta^\pm := \mathbb{P}_{\Omega_\delta}^\pm$ and $\mathbb{P}_{\delta,h}^\pm := \mathbb{P}_{\Omega_{\delta,H}}^\pm$. Similarly, when $(\delta_n)_n$ is a sequence converging to 0, we set $\Omega_n := \Omega_{\delta_n}$, $\mathbb{P}_n^\pm := \mathbb{P}_{\Omega_{\delta_n}}^\pm$ and $\mathbb{P}_{n,h}^\pm := \mathbb{P}_{\Omega_{\delta_n,H}}^\pm$. We will also often write $x \in \Omega_\delta$ instead of $x \in \mathrm{Int}(\Omega_\delta)$.

Remark 5.2.1. Let us give some intuition on why $\delta^{\frac{15}{8}}$ is indeed the near-critical scaling. If we want to be in the near-critical regime, we should choose α such that $\delta^\alpha \mathbb{E}_\delta^\pm [\sum_{x \in D_\delta} \sigma_x]$ has a non-trivial limit when $\delta \to 0$. The sum should be interpreted as a discrete integral and therefore, it should be rescaled by δ^2. Moreover, recall from Theorem 5.1.8 that for $x \in \Omega_\delta$, $\mathbb{E}_\delta^\pm[\sigma_x]$ is of order $\delta^{\frac{1}{8}}$. Thus, ignoring the effect of the boundary, we should choose $\alpha = 2 - \frac{1}{8} = \frac{15}{8}$ to obtain a non-trivial limit for $\delta^\alpha \mathbb{E}_\delta^\pm[\sum_{x \in D_\delta} \sigma_x]$. A similar reasoning can be used to show that for any $k \in \mathbb{N}$, $\mathbb{E}_\delta^\pm[(\sum_x \sigma_x)^k]$ should be rescaled by $\delta^{\frac{15k}{8}}$ in order to have a non-trivial limit as $\delta \to 0$.

Proof of Theorem 1.3.1. The proof consists of two main steps. First, in Proposition 5.2.2, we establish tightness of $(\gamma_\delta)_\delta$ under $(\mathbb{P}_{\delta,h}^\pm)_\delta$ in the topology (T.4). Then, in Proposition 5.2.3, we prove a characterization of the limiting law of any weakly convergent subsequence obtained as a consequence of the tightness of $(\gamma_\delta)_\delta$ under $(\mathbb{P}_{\delta,h}^\pm)_\delta$. This yields convergence in law of $(\gamma_\delta)_\delta$ under $(\mathbb{P}_{\delta,h}^\pm)_\delta$ in the topology (T.4). As explained in Section 2.2, this also implies tightness and convergence of $(\gamma_\delta)_\delta$ under $(\mathbb{P}_{\delta,h}^\pm)_\delta$ in the topologies (T.1)–(T.3) and the limiting laws in each topology agree.

Both to prove tightness and characterize the limiting law, we are going to use the fact that for each $\delta > 0$, the law of γ_δ under $\mathbb{P}_{\delta,h}^\pm$ is absolutely continuous with respect to that of γ_δ under $\mathbb{P}_\delta^\pm$ with Radon-Nikodym given by

$$F_\delta(\gamma_\delta) := \frac{d\mathbb{P}_{\delta,h}^\pm}{d\mathbb{P}_\delta^\pm}(\gamma_\delta) = \frac{1}{\mathcal{Z}_\delta(h)} \mathbb{E}_\delta^\pm \left[\exp\left(C_\sigma^{-1} h \delta^{15/8} \sum_{x \in \Omega_\delta} \sigma_x \right) \Big| \gamma_\delta \right]. \tag{5.3}$$

Tightness of $(\gamma_\delta)_\delta$ under $(\mathbb{P}_{\delta,h}^\pm)_\delta$ in the topology (T.4) in fact follows from the uniform boundedness of $(F_\delta(\gamma_\delta))_\delta$ in L^p ($p \geq 1$), which is established in the proposition below.

Proposition 5.2.2 (tightness)**.** *For any $p \geq 1$, $\sup_{\delta > 0} \mathbb{E}_\delta^\pm[F_\delta(\gamma_\delta)^p] < \infty$. In particular, since $(\gamma_\delta)_\delta$ under $(\mathbb{P}_\delta^\pm)_\delta$ is tight in the topology (T.4), $(\gamma_\delta)_\delta$ under $(\mathbb{P}_{\delta,h}^\pm)_\delta$ is also tight in the topology (T.4).*

We postpone the proof of Proposition 5.2.2 to Section 5.2.1 and now state our characterization of the limiting law of weakly convergent subsequences.

Proposition 5.2.3 (characterization of the limiting law)**.** *There exists a unique law $\mathbb{P}_h^{(\Omega,a,b)}$ such that the following holds. The Radon-Nikodym derivative of $\mathbb{P}_h^{(\Omega,a,b)}$ with respect to $\mathbb{P}_{\mathrm{SLE}_3}^{(\Omega,a,b)}$ is given by*

$$\frac{d\mathbb{P}_h^{(\Omega,a,b)}}{d\mathbb{P}_{\mathrm{SLE}_3}^{(\Omega,a,b)}}(\gamma) = \frac{1}{\mathcal{Z}_h(\Omega)} \left(\sum_{k \geq 0} \frac{h^k}{k!} \int_{\Omega_L^k} f_L^{(+,k)}(z_1, \ldots, z_k) \prod_{j=1}^k dz_j \right)$$

$$\times \left(\sum_{k \geq 0} \frac{h^k}{k!} \int_{\Omega_R^k} f_R^{(-,k)}(z_1, \ldots, z_k) \prod_{j=1}^k dz_j \right) \tag{5.4}$$

and, for any sequence $(\delta_n)_n$ such that $(\mathbb{P}_{n,h}^\pm)_n$ converges weakly in the topology (T.4), there exists a further subsequence $(\delta_{n_k})_k$ such that $\mathbb{P}_{n_k,h}^\pm \to \mathbb{P}_h^{(\Omega,a,b)}$ weakly as $k \to \infty$ in the topology (T.4). Above, Ω_R, respectively Ω_L, denotes the connected component of $\Omega \setminus \gamma$ on the right, respectively left, of γ and the functions $f_L^{(+,k)} := f_{\Omega_L}^{(+,k)}$ and $f_R^{(-,k)} := f_{\Omega_R}^{(-,k)}$ are as in Theorem 5.1.8.

We will give the proof of Proposition 5.2.3 in Section 5.2.2. In view of Proposition 5.2.2 and Proposition 5.2.3, the convergence part of Theorem 1.3.1 follows from Prokhorov's theorem. It remains to prove the equality (1.7) for the Radon-Nikodym derivative of $\mathbb{P}_h^{(\Omega,a,b)}$ with respect to $\mathbb{P}_{\mathrm{SLE}_3}^{(\Omega,a,b)}$ when restricted to $\sigma(\gamma(s) : 0 \leq s \leq t)$. This is the content of the next proposition, whose proof is postponed to Section 5.2.2. This proposition concludes the proof of Theorem 1.3.1.

Proposition 5.2.4. *Let $t \geq 0$. The Radon-Nikodym derivative of $\mathbb{P}_h^{(\Omega,a,b)}$ with respect to $\mathbb{P}_{\mathrm{SLE}_3}^{(\Omega,a,b)}$ restricted to the σ-field $\sigma(\gamma(s) : 0 \leq s \leq t)$ is given by*

$$\frac{\mathrm{d}\mathbb{P}_h^{(\Omega,a,b)}}{\mathrm{d}\mathbb{P}_{\mathrm{SLE}_3}^{(\Omega,a,b)}}(\gamma)\Bigg|_{\sigma(\gamma(s):0\leq s\leq t)} = \frac{1}{\mathcal{Z}_h(\Omega)} \sum_{k\geq 0} \frac{h^k}{k!} \int_{\Omega_t^k} f_t^{(\pm,k)}(z_1,\ldots,z_k) \prod_{j=1}^k dz_j \tag{5.5}$$

where we have set $\Omega_t := \Omega \setminus \gamma([0,t])$ and the functions $f_t^{(\pm,k)} := f_{\Omega_t}^{(\pm,k)}$ are as in Theorem 5.1.8.

$\square$

Remark 5.2.5. Let us note that a priori, Proposition 5.2.4 could be derived from Proposition 5.2.3 by computing the conditional expectation of the random variable on the right-hand side of (5.4) given $\sigma(\gamma(s) : 0 \leq s \leq t)$. However, it is not clear how to do this. Instead, to establish Proposition 5.2.4, we obtain the right-hand side of (5.5) by computing the scaling limit of the corresponding discrete quantity. This thus makes the statement of Theorem 1.3.1 stronger than the one we would obtain by only proving Proposition 5.2.3.

5.2.1 Tightness of $(\mathbb{P}_\delta^\pm)_\delta$: proof of Proposition 5.2.2

In this section, we prove Proposition 5.2.2. Recall the definition (5.3) of $F_\delta(\gamma_\delta)$.

Proof of Proposition 5.2.2. We give the proof in the case where $h : \Omega \to \mathbb{R}$ is a bounded and Lipchitz function: compared to the case when h is a constant, this requires a few modifications that we prefer to detail. We first show that for any $p \in [1,\infty)$, $\sup_\delta \mathbb{E}_\delta^\pm[F_\delta(\gamma_\delta)^p] < \infty$. Fix $p \in [1,\infty)$ and let $\delta > 0$. Set

$$S_\delta(\gamma_\delta) := \mathbb{E}_\delta^\pm\left[\exp(\delta^{15/8} \sum_{x\in\Omega_\delta} \tilde{h}_\delta(x)\sigma_x)|\gamma_\delta\right]$$

with $\tilde{h}_\delta(x) := C_\sigma^{-1} h_\delta(x)$. We are going to obtain uniform bounds on $\mathcal{Z}_\delta(h)^{-p}$ and $\mathbb{E}_\delta^\pm[S_\delta(\gamma_\delta)^p]$ separately. For $\mathbb{E}_\delta^\pm[S_\delta(\gamma_\delta)^p]$, using first Jensen's inequality for the conditional expectation and then Hölder's inequality, we have that

$$\mathbb{E}_\delta^\pm[S_\delta(\gamma_\delta)^p] \leq \mathbb{E}_\delta^\pm\left[\exp\left(p\delta^{\frac{15}{8}} \sum_{x\in\Omega_\delta} \tilde{h}_\delta(x)\sigma_x\right)\right]$$

$$= \mathbb{E}_\delta^\pm\left[\exp\left(p\delta^{\frac{15}{8}} \sum_{x\in\Omega_\delta} (\tilde{h}_\delta(x) - |\tilde{h}_\delta(x)|)\sigma_x\right) \exp\left(p\delta^{\frac{15}{8}} \sum_{x\in\Omega_\delta} |\tilde{h}_\delta(x)|\sigma_x\right)\right]$$

$$\leq \mathbb{E}_\delta^\pm\left[\exp\left(2p\delta^{\frac{15}{8}} \sum_{x\in\Omega_\delta} (\tilde{h}_\delta(x) - |\tilde{h}_\delta(x)|)\sigma_x\right)\right]^{1/2} \mathbb{E}_\delta^\pm\left[\exp\left(2p\delta^{\frac{15}{8}} \sum_{x\in\Omega_\delta} |\tilde{h}_\delta(x)|\sigma_x\right)\right]^{1/2}. \tag{5.6}$$

Setting $M = C_\sigma^{-1} \max_{x\in\Omega} |h(x)|$ and applying Proposition 5.1.3 with the increasing function $\sigma \mapsto \sum_x |\tilde{h}_\delta(x)|\sigma_x$, we obtain that

$$\mathbb{E}_\delta^\pm\left[\exp\left(2p\delta^{\frac{15}{8}} \sum_{x\in\Omega_\delta} |\tilde{h}_\delta(x)|\sigma_x\right)\right] \leq \mathbb{E}_\delta^+\left[\exp\left(2p\delta^{\frac{15}{8}} \sum_{x\in\Omega_\delta} |\tilde{h}_\delta(x)|\sigma_x\right)\right]$$

$$\leq \mathbb{E}_\delta^+\left[\exp\left(2pM\delta^{\frac{15}{8}} \sum_{x\in\Omega_\delta} \sigma_x\right)\right] \tag{5.7}$$

where the last inequality follows by expanding the exponential and noticing that for any $k \geq 1$, $\mathbb{E}_\delta^+[\prod_{j \leq k} \sigma_j] \geq 0$. The proof of [CGN15, Proposition 3.5] shows that

$$\log \mathbb{E}_\delta^+ \left[\exp\left(2pM\delta^{\frac{15}{8}} \sum_{x \in \Omega_\delta} \sigma_x \right) \right] \leq 2pM\delta^{\frac{15}{8}} \mathbb{E}_\delta^+ \left[\sum_{x \in \Omega_\delta} \sigma_x \right] + 2p^2M^2\delta^{\frac{15}{4}} \mathbb{E}_\delta^+ \left[\left(\sum_{x \in \Omega_\delta} \sigma_x - \mathbb{E}_\delta^+ \left[\sum_{x \in \Omega_\delta} \sigma_x \right] \right)^2 \right].$$
(5.8)

Since both $\delta^{\frac{15}{8}} \mathbb{E}_\delta^+[\sum_x \sigma_x]$ and $\delta^{\frac{15}{4}} \mathbb{E}_\delta^+[(\sum_x \sigma_x)^2]$ converge as $\delta \to 0$, see e.g. [CGN15] or Lemma 5.2.18 below, it follows from the above inequality that there exist $\delta_0 > 0$ and $c_1, c_2 > 0$ (depending only on Ω) such that for any $0 < \delta < \delta_0$,

$$\mathbb{E}_\delta^+ \left[\exp\left(2pM\delta^{\frac{15}{8}} \sum_{x \in \Omega_\delta} \sigma_x \right) \right] \leq \exp(2pMc_1 + 2p^2M^2c_2).$$

On the other hand, using the spin flip symmetry of the Ising model, we have that

$$\mathbb{E}_\delta^\pm \left[\exp\left(2p\delta^{\frac{15}{8}} \sum_{x \in \Omega_\delta} (\tilde{h}_\delta(x) - |\tilde{h}_\delta(x)|)\sigma_x \right) \right] = \mathbb{E}_\delta^\mp \left[\exp\left(2p\delta^{\frac{15}{8}} \sum_{x \in \Omega_\delta} (|\tilde{h}_\delta(x)| - \tilde{h}_\delta(x))\sigma_x \right) \right]$$

where under $\mathbb{E}_\delta^\mp$, the Ising spin configuration σ in Ω_δ has -1, respectively $+1$, boundary conditions on $\partial\Omega_\delta^+$, respectively $\partial\Omega_\delta^-$. Note that the function $x \mapsto |\tilde{h}_\delta(x)| - \tilde{h}_\delta(x)$ is non-negative and bounded by $2M$. Therefore, we can proceed as in (5.7) and (5.8) to obtain that, with δ_0, c_1 and c_2 as above, for any $0 < \delta < \delta_0$,

$$\mathbb{E}_\delta^\mp \left[\exp\left(2p\delta^{\frac{15}{8}} \sum_{x \in \Omega_\delta} (\tilde{h}_\delta(x) - |\tilde{h}_\delta(x)|)\sigma_x \right) \right] \leq \exp(4pMc_1 + 8p^2M^2c_2).$$

Going back to the inequality (5.6), this yields that for any $0 < \delta < \delta_0$,

$$\mathbb{E}_\delta^\pm [S_\delta(\gamma_\delta)^p] \leq \exp(3pMc_1 + 5p^2M^2c_2).$$

It now remains to obtain a lower bound on $\mathcal{Z}_\delta(h)$. For this, we simply observe that Jensen's inequality implies that $\mathcal{Z}_\delta(h) \geq \exp(\delta^{\frac{15}{8}} \mathbb{E}_\delta^\pm[\sum_x \tilde{h}_\delta(x)\sigma_x])$. Moreover, the results of Section 5.2.2 show that $\delta^{\frac{15}{8}} \mathbb{E}_\delta^\pm[\sum_x \tilde{h}_\delta(x)\sigma_x]$ converges to a finite limit as $\delta \to 0$. Note that this limit may be negative as the model has Dobrushin boundary conditions and h does not have a constant sign. We obtain from this that there exist $\delta_1 > 0$ and $c > 0$ such that for any $0 < \delta < \delta_1$, $\mathcal{Z}_\delta(h) \geq \exp(-c)$, which concludes the proof of the uniform boundedness of $\mathbb{E}_\delta[F_\delta(\gamma_\delta)^p]$. Tightness of $(\gamma_\delta)_\delta$ under $(\mathbb{P}_{\delta,h}^\pm)_\delta$ in the topology (T.4) now follows from this uniform boundedness and tightness of $(\gamma_\delta)_\delta$ under $(\mathbb{P}_\delta^\pm)_\delta$. Indeed, using Cauchy-Schwarz inequality, for any $K \subset X(\mathbb{C})$, $\mathbb{P}_{\delta,h}^\pm(K)$ can be bounded by a constant, that is uniform in δ, times $\mathbb{P}_\delta^\pm(K)^{1/2}$. This concludes the proof of Proposition 5.2.2. $\qquad\qquad\square$

5.2.2 Characterization of the limiting law: proof of Propositions 5.2.3 and 5.2.4

In this section, we prove Proposition 5.2.3 and Proposition 5.2.4. We start with Proposition 5.2.3.

Proof of Proposition 5.2.3. Let $(\delta_n)_n$ be a sequence such that $(\mathbb{P}_{n,h}^\pm)_n$ converges weakly as $n \to \infty$ in the topology (T.4). Note that such a sequence exists by Proposition 5.2.2. By Theorem 5.1.7, we also know that $(\mathbb{P}_n^\pm)_n$ converges weakly in the topologies (T.1)–(T.4) to $\mathbb{P}_{\mathrm{SLE}_3}^{(\Omega,a,b)}$. The first step of the proof is to use Skorokhod representation theorem to define a sequence $(\gamma_n)_n$ and a random curve γ on the same probability space $(S, \mathcal{F}, \mathbb{P})$ in such a way that for each n, γ_n has law $\mathbb{P}_n^\pm$, γ has law $\mathbb{P}_{\mathrm{SLE}_3}^{(\Omega,a,b)}$ and $\mathbb{P}$-almost surely, $\gamma_n \to \gamma$ as $n \to \infty$ in the topologies (T.1)–(T.4).

Now, the random variables $(F_n(\gamma_n))_n$ defined via (5.3) (with $\delta = \delta_n$) can all be defined on $(S, \mathcal{F}, \mathbb{P})$. To prove Proposition 5.2.3, we are going to show that there exists a subsequence $(\delta_m)_m \subset (\delta_n)_n$ such that $F_m(\gamma_m)$ converges in $L^1(\mathbb{P})$ to the random variable of the right-hand side of (5.4). As Proposition 5.2.2 applied with $p = 2$ shows that $(F_n(\gamma_n))_n$ is uniformly bounded in $L^2(\mathbb{P})$, it actually suffices to show that there exists a subsequence $(\delta_m)_m \subset (\delta_n)_n$ such that $F_m(\gamma_m)$ converges $\mathbb{P}$-almost surely to the random variable on the right-hand side of (5.4). For this, we first

observe that by the Markov property of the Ising model and the definition of γ_n, for any $n \geq 1$, $F_n(\gamma_n)$ can be decomposed as

$$F_n(\gamma_n) = \frac{1}{\mathcal{Z}_n(h)} \exp\left(C_\sigma^{-1} h \delta_n^{\frac{15}{8}} \left(|V_L(\gamma_n)| - |V_R(\gamma_n)|\right)\right)$$

$$\times \mathbb{E}_{n,L}^+ \left[\exp\left(C_\sigma^{-1} h \delta_n^{15/8} \sum_{x \in \Omega_{n,L}} \sigma_x\right)\right] \mathbb{E}_{n,R}^- \left[\exp\left(C_\sigma^{-1} h \delta_n^{15/8} \sum_{x \in \Omega_{n,R}} \sigma_x\right)\right] \qquad (5.9)$$

where $\Omega_{n,L}$, respectively $\Omega_{n,R}$, is the connected component of $\Omega_n \setminus V(\gamma_n)$ on the left, respectively right, of γ_n and, conditionally on γ_n, $\mathbb{E}_{n,L}^+$, respectively $\mathbb{E}_{n,R}^-$, denotes the expectation with respect to the critical Ising model with $+$, respectively $-$, boundary conditions in $\Omega_{n,L}$, respectively $\Omega_{n,R}$. Here, $V(\gamma_n)$ is the set of vertices adjacent to γ_n, where a vertex $x \in \Omega_n$ is said to be adjacent to γ_n if γ_n traverses an edge of the octagon whose center is x. $V_L(\gamma_n)$, respectively $V_R(\gamma_n)$, denotes the set of vertices that are adjacent to γ_n and on its left, respectively right. For $j \in \{L, R\}$, $|V_j(\gamma_n)|$ is the number of vertices in $V_j(\gamma_n)$. Note that the $\mathbb{P}$-almost sure convergence of γ_n to γ in the topologies (T.1)–(T.4) implies that almost surely, $\hat{\Omega}_{n,L} \to \Omega_L$ and $\hat{\Omega}_{n,R} \to \Omega_R$ in the Carathéodory topology, where Ω_L, respectively Ω_R, is the connected component of $\Omega \setminus \gamma$ on the left, respectively right, of γ.

Expanding the two exponentials in (5.9), the existence of a subsequence along which almost sure convergence of $F_n(\gamma_n)$ holds is a consequence of the following lemmas. Their proofs are given in Section 5.2.2 and Section 5.2.2.

Lemma 5.2.6. *There exists a subsequence $(\delta_m)_m \subset (\delta_n)_n$ such that, almost surely, as $m \to \infty$, $\delta_m^{\frac{15}{8}}\left(|V_L(\gamma_m)| - |V_R(\gamma_m)|\right) \to 0$.*

Lemma 5.2.7. *For $(\delta_m)_m$ as in Lemma 5.2.6, almost surely, for $\xi \in \{+, -\}$ with $j(\xi) = L$ if $\xi = +$ and $j(\xi) = R$ if $\xi = -1$,*

$$\lim_{m \to \infty} \sum_{k=0}^\infty \frac{C_\sigma^{-k} h^k}{k!} \delta_m^{\frac{15k}{8}} \mathbb{E}_{m,j(\xi)}^\xi \left[\left(\sum_{x \in \Omega_{m,j(\xi)}} \sigma_x\right)^k\right] = \sum_{k=0}^\infty \frac{h^k}{k!} \int_{\Omega_{j(\xi)}^k} f_{j(\xi)}^{(\xi,k)}(x_1, \ldots, x_k) \prod_{j=1}^k dx_j < \infty$$

$$(5.10)$$

where the functions $f_{j(\xi)}^{(\xi,k)} := f_{\Omega_{j(\xi)}}^{(\xi,k)}$ are as in Theorem 5.1.8.

Proposition 5.2.3 directly follows from Lemma 5.2.6 and Lemma 5.2.7. $\qquad \square$

Let us now turn to the proof of Proposition 5.2.4; this follows a very similar strategy to the proof of Proposition 5.2.3.

Proof of Proposition 5.2.4. As in the proof of Proposition 5.2.3 and with $(\delta_n)_n$ as there, we start by coupling $(\gamma_n)_n$ under $(\mathbb{P}_n^\pm)_n$ and γ under $\mathbb{P}_{\mathrm{SLE}_3}^{(\Omega,a,b)}$ on a common probability space $(S, \mathcal{F}, \mathbb{P})$. We then parametrize the curves $(\gamma_n)_n$ by the half-plane capacity of their conformal images $(\gamma_n^\mathbb{H})_n$, where for each n, $\gamma_n^\mathbb{H}$ is defined as in Section 2.2. For $n \geq 1$, denote by $\mathcal{F}_{n,m}$ the σ-algebra generated by $\gamma_n([0,m])$. Here, for $m \in \mathbb{N}$, $\gamma_n([0,m])$ is the curve obtained by following γ_n for m steps, where a step corresponds to γ_n traversing edges of the square-octagon lattice to join two edges of $\delta_n \mathbb{Z}^2$ that surrounds the same face of $\delta_n \mathbb{Z}^2$. If m is larger than the total number of steps taken by γ_n, we set $\gamma_n([0,m]) := \gamma_n$. For $n \geq 1$ and $t \geq 0$, we then define the following stopping time for the filtration $(\mathcal{F}_{n,m})_m$:

$$t_n := \inf\{m \geq 0 : \mathrm{hcap}(\phi_n(\gamma_n([0,m]))) \geq t\}.$$

Accordingly, we let $\Omega_{n,t}$ be the discrete domain whose vertex set is $V(\Omega_n) \setminus V(\gamma_n; t)$ and whose boundary $\partial \Omega_{n,t}$ is split into two subarcs $\partial \Omega_{n,t}^+ := \partial \Omega_{n,t}^+ \cup V_L(\gamma_n; t)$ and $\partial \Omega_{n,t}^+ := \partial \Omega_{n,t}^+ \cup V_R(\gamma_n; t)$. Here, $V(\gamma_n; t)$ is the set of vertices adjacent to $\gamma_n([0, t_n])$. $V_L(\gamma_n; t)$, respectively $V_R(\gamma_n; t)$, denotes the set of vertices that are adjacent to $\gamma_n([0, t_n])$ and on its left, respectively right.

Since $\mathbb{P}$-almost surely, $\gamma_n \to \gamma$, we have that, $\mathbb{P}$-almost surely, in the Carathéodory topology,

$$(\hat{\Omega}_{n,t}; a_{n,t}, b_n) \to (\Omega_t; a_t, b) \quad \text{as } n \to \infty$$

where $a_{n,t}$ denotes the tip of $\gamma_n([0, t_n])$ and Ω_t is the connected component of $\Omega \setminus \gamma([0, t])$ whose boundary contains both $a_t := \gamma(t)$ and b.

Observe now that, for $t \geq 0$ and $n \geq 1$, the Radon-Nikodym of the law of γ_n under $\mathbb{P}_{n,h}^{\pm}$ with respect to that of γ_n under $\mathbb{P}_n^{\pm}$ restricted to $\mathcal{F}_{n,t_n}$ is given by

$$F_n(\gamma_n; t) := \left. \frac{d\mathbb{P}_{n,h}^{\pm}}{d\mathbb{P}_n^{\pm}}(\gamma_n) \right|_{\sigma(\gamma_n(s):0 \leq s \leq t_n)} = \frac{1}{\mathcal{Z}_n(h)} \mathbb{E}_n^{\pm} \left[\exp\left(C_\sigma^{-1} h \delta_n^{\frac{15}{8}} \sum_{x \in \Omega_n} \sigma_x \right) \middle| \gamma_n([0, t_n]) \right]. \quad (5.11)$$

The family $(F_n(\gamma_n; t), t \geq 0)_n$ is well-defined on $(S, \mathcal{F}, \mathbb{P})$. To prove Proposition 5.2.4, we are going to show that there exists a subsequence $(\delta_m)_m \subset (\delta_n)_n$ such that for each $t \geq 0$, $F_m(\gamma_m; t)$ converges in $L^1(\mathbb{P})$ to the random variable on the right-hand side of (5.5). As $(\mathbb{P}_{m,h}^{\pm})_m$ converges weakly to $\mathbb{P}_h^{(\Omega, a, b)}$ by Proposition 5.2.2 and Proposition 5.2.3, this is indeed sufficient to establish Proposition 5.2.4.

We first observe that by the Markov property of the Ising model and the definition of γ_n, almost surely, for any $t \geq 0$ and any $n \geq 1$,

$$F_n(\gamma_n; t) = \frac{1}{\mathcal{Z}_n(h)} \exp\left(C_\sigma^{-1} h \delta_n^{\frac{15}{8}} (|V_L(\gamma_n; t)| - |V_R(\gamma_n; t)|) \right) \mathbb{E}_{n,t}^{\pm} \left[\exp\left(C_\sigma^{-1} h \delta_n^{15/8} \sum_{x \in \Omega_{n,t}} \sigma_x \right) \right]$$
$$(5.12)$$

where conditionally on $\gamma_n([0, t_n])$, $\mathbb{E}_{n,t}^{\pm}[\cdot]$ denotes the expectation with respect to the critical Ising model in $\Omega_{n,t}$ with $+$, respectively $-$, boundary conditions on $\partial\Omega_{n,t}^+$, respectively $\partial\Omega_{n,t}^-$. Above, for $j \in \{L, R\}$, $|V_j(\gamma_n; t)|$ is the number of vertices in $V_j(\gamma_n; t)$. Proposition 5.2.2 applied with $p = 2$ and combined with Jensen's inequality shows that $(F_n(\gamma_n; t), t \geq 0)_n$ is uniformly bounded in t and n in $L^2(\mathbb{P})$. To prove Proposition 5.2.4, it therefore suffices to show that there exists a subsequence $(\delta_m)_m \subset (\delta_n)_n$ such that for each $t \geq 0$, $F_m(\gamma_m; t)$ almost surely converges to the right-hand side of (5.5) as $m \to \infty$. Expanding the exponential in (5.12), this follows from the two lemmas below, which are analogues of Lemma 5.2.6 and Lemma 5.2.7 and whose proofs are also postponed to Section 5.2.2 and Section 5.2.2.

Lemma 5.2.8. *Let $(\delta_m)_m \subset (\delta_n)_n$ be as in Lemma 5.2.6. Then, almost surely, for any $t \geq 0$, as $m \to \infty$, $\delta_m^{\frac{15}{8}} (|V_L(\gamma_m; t)| - |V_R(\gamma_m; t)|) \to 0$.*

Lemma 5.2.9. *Let $t \in (0, \infty)$ and let $(\delta_m)_m$ be as Lemma 5.2.8. Then, almost surely,*

$$\lim_{m \to \infty} \sum_{k=0}^{\infty} \frac{C_\sigma^{-k} h^k}{k!} \delta_m^{\frac{15k}{8}} \mathbb{E}_{m,t}^{\pm} \left[\left(\sum_{x \in \Omega_{m,t}} \sigma_x \right)^k \right] = \sum_{k=0}^{\infty} \frac{h^k}{k!} \int_{\Omega_t^k} f_t^{(\pm, k)}(x_1, \ldots, x_k) \prod_{j=1}^{k} dx_j < \infty \quad (5.13)$$

where the functions $f_t^{(\pm, k)} := f_{\Omega_t}^{(\pm, k)}$ are as in Theorem 5.1.8.

These two lemmas conclude the proof of Proposition 5.2.4. $\qquad\square$

Vanishing of the magnetization of the interface

In this section, we prove Lemma 5.2.6 and Lemma 5.2.8. Note that these lemmas do not follow from the $\mathbb{P}$-almost sure convergence of $\hat{\Omega}_{n,t}$ to Ω_t in the Carathéodory topology. Indeed, this topology is not strong enough to guarantee convergence of the Minkowski dimension of $\partial\hat{\Omega}_{n,t}$ to that of $\partial\Omega_t$. Instead, Lemma 5.2.6 and Lemma 5.2.8 will be proved thanks to the following lemma, which provides an almost sure estimate on the size of $|V(\gamma_n)|$ along a subsequence, where $|V(\gamma_n)|$ is the number of vertices adjacent to γ_n.

Lemma 5.2.10. *Let $\beta > 0$. There exists a subsequence $(\delta_m)_m \subset (\delta_n)_n$ such that, almost surely, $\delta_m^{\frac{11}{8} + \beta} |V(\gamma_m)| \to 0$ as $m \to \infty$.*

Lemma 5.2.6 and Lemma 5.2.6 now straightforwardly follow from Lemma 5.2.10. Below, we only give the proof of Lemma 5.2.8, as that of Lemma 5.2.6 is almost identical.

Proof of Lemma 5.2.8. Observe that, almost surely, for any $t \geq 0$ and any $n \geq 1$, $-|V(\gamma_n; t)| \leq |V_L(\gamma_n; t)| - |V_R(\gamma_n; t)| \leq |V(\gamma_n; t)|$, where $|V(\gamma_n; t)|$ denotes the number of vertices in $V(\gamma_n; t)$. Moreover, almost surely, for any $n \geq 1$, $t \mapsto |V(\gamma_n; t)|$ is an increasing function. Therefore, almost surely, for any $t \geq 0$ and any $n \geq 1$, $-|V(\gamma_n)| \leq |V_L(\gamma_n; t)| - |V_R(\gamma_n; t)| \leq |V(\gamma_n)|$. The statement then follows directly from Lemma 5.2.10. $\qquad\square$

Let us now turn to the proof of Lemma 5.2.10. For $L > 0$ and $v \in \mathbb{C}$, we set $\Lambda_{L,\delta}(v) = \Lambda_L(v) \cap \delta\mathbb{Z}^2$, where $\Lambda_L(v)$ is the Euclidean square of side length $2L$ centered at v. For $0 < r < R$ and $v \in \Omega_\delta$, we say that there is an arm of sign $+$, respectively $-$, in the annulus $\Lambda_{R,\delta}(v) \setminus \Lambda_{r,\delta}(v)$ if there is a nearest neighbor path joining $\partial\Lambda_{r,\delta}(v)$ to $\partial\Lambda_{R,v}$ along which all vertices have spin $+1$, respectively -1. For $0 < r < R$ and $v \in \Omega_\delta$, we denote by $A_2(v; r, R)$ the event that there exist two arms of opposite signs crossing the annulus $\Lambda_{R,\delta}(v) \setminus \Lambda_{r,\delta}(v)$. We then define

$$\alpha_\delta(v; r, R) := \mathbb{P}_\delta^\pm(A_2(v; r, R)), \quad \alpha_{L,\delta}(v; r, R) := \mathbb{P}_{\Lambda_{L,\delta}(v)}^\pm(A_2(v; r, R))$$

where the boundary conditions on $\partial\Lambda_{L,\delta}(v)$ under $\mathbb{P}_{\Lambda_{L,\delta}(v)}^\pm$ are -1 on the bottom side of $\Lambda_{L,\delta}(v)$ and $+1$ elsewhere (this choice is somewhat arbitrary but convenient).

Proof of Lemma 5.2.10. The strategy of the proof is to upper bound $\mathbb{E}_n^\pm[|V(\gamma_n)|]$ and show that, once multiplied by $\delta_n^{\frac{11}{8} + \beta}$, this expectation decays fast enough to apply Borel-Cantelli lemma to get almost sure convergence along a subsequence $(\delta_m)_m$. Let $\beta > 0$. Let $\eta > 0$ and set $A_n(\eta) := \{x \in \Omega_n : d(x, \partial\Omega) < \eta\}$. We have that, for any n,

$$\mathbb{E}_n^\pm[|V(\gamma_n)|] = \sum_{x \in \Omega_n} \mathbb{P}_n^\pm(x \in V(\gamma_n)) \leq \sum_{x \in \Omega_n} \alpha_n(x; \delta_n, d(x, \partial\Omega_n)).$$

For $x \in \Omega_n$, set $L(x) := \frac{1}{2}d(x, \partial\Omega)$ and note that $\Lambda_{L(x),\delta_n}(x) \subset \Omega_n$. By the Markov property of the Ising model, we have that

$$\alpha_\delta(x; \delta_n, d(x, \partial\Omega_n)) \leq \alpha_n(x; \delta_n, \frac{1}{4}d(x, \partial\Omega_n)) = \mathbb{E}_n^\pm[\mathbb{P}_{\Lambda_{L(x),\delta_n}(x)}^{\sigma_{|\partial\Lambda_{L(x)}(x)}}(A_2(x; \delta_n, \frac{1}{2}L(x)))]$$

where $\mathbb{P}_{\Lambda_{L(x),\delta_n}(x)}^{\sigma_{|\partial\Lambda_{L(x)}(x)}}$ denotes the critical Ising measure in $\Lambda_{L(x),\delta_n}(x)$ with boundary conditions $\sigma_{|\partial\Lambda_{L(x)}(x)}$. Now, [Wu18a, Corrolary 5.2] guarantees that, when $\beta = \beta_c$, changing the boundary conditions does not change by more than a multiplicative constant the probability of events depending on edges at macroscopic distance from the boundary. This implies that, almost surely,

$$\mathbb{P}_{\Lambda_{L(x)}(x)}^{\sigma_{|\partial\Lambda_{L(x)}(x)}}(A_2(x; \delta_n, \frac{1}{2}L(x))) \leq 2\alpha_{L(x),\delta_n}(x; \delta_n, \frac{1}{2}L(x))) = 2\alpha_{L(x),\delta_n}(0; \delta_n, \frac{1}{2}L(x)))$$

where the last equality uses translation invariance of the critical Ising model. Moreover, the proof of [Wu18a, Theorem 1.2] shows that, as $\delta_n \to 0$, uniformly in $x \in \Omega \setminus A(\eta)$,

$$\alpha_{L(x),\delta_n}(0; \delta_n, \frac{1}{2}L(x))) \lesssim (\delta_n/L(x))^{5/8 - \tilde\beta},$$

for arbitrary $\tilde\beta > 0$, see e.g. [Wu18a, Equation (5.2)]. Choose $\tilde\beta = \beta/2$. We thus obtain that there exist $C_\eta, \delta_0 > 0$ such that for any $0 < \delta < \delta_0$,

$$\sum_{x \in \Omega_n \setminus A_n(\eta)} \mathbb{P}_n^\pm(x \in V(\gamma_n)) \leq C_\eta \eta^{-\frac{7}{8} + \frac{\beta}{2}} \delta_n^{-2 - \frac{\beta}{2} + \frac{5}{8}}.$$

To estimate $\mathbb{E}_n^\pm[|V(\gamma_n)|]$, it now remains to deal with the sum over vertices in $A_n(\eta)$. The argument below goes along the same lines as the proof of [GPS13, Theorem 5.5], except that we must make sure that we can get estimates in δ_n on the probability of a two-arm event that are uniform in space, even when points are close to the boundary of the domain. We divide $A_n(\eta)$ into $\eta^{-d_H(\partial\Omega)}$ squares of side length η, where $d_H(\partial\Omega)$ denotes the Hausdorff dimension of $\partial\Omega$. Among these squares, one touches a, call it s_a, and another touches b, call it s_b. For $j \geq 0$, we define the set $P_j(a) := \{x \in s_a : 2^j \delta_n \leq d(x, a) \leq 2^{j+1}\delta_n\}$ and $P_j(b)$ is defined analogously with a replaced by b. We then have that, for any $j \geq 0$,

$$\sum_{x \in P_j(a)} \mathbb{P}_n^\pm(x \in V(\gamma_n)) \leq \sum_{x \in P_j(a)} \alpha_{\Omega_n}(x; \delta_n, 2^j \delta_n).$$

By the Markov property of the Ising model and using again [Wu18a, Corollary 5.2], we have that there exist $n_0 \geq 1$ and an absolute constant $C > 0$ such that, for any $n \geq n_0$ and any $j \geq 0$,

$$\alpha_n(x; \delta_n, 2^j \delta_n) \leq C\alpha_{L_{j+1},\delta_n}(x; \delta_n, 2^j \delta_n),$$

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where we have set $L_{j+1} = 2^{j+1}\delta_n$. Rescaling by $1/L_{j+1}$, so that the meshsize becomes $2^{-(j+1)}$, and translating the domain, we then obtain that, for any $j \geq 0$,

$$\alpha_{L_{j+1},\delta_n}(x; \delta_n, 2^j\delta_n) = \alpha_{1,2^{-(j+1)}}(x; 2^{-(j+1)}, \tfrac{1}{2}) = \alpha_{1,2^{-(j+1)}}(0; 2^{-(j+1)}, \tfrac{1}{2}).$$

We note that the event $A_2(0; 2^{-(j+1)}, \tfrac{1}{2})$ depends only on edges in the square $\Lambda_{\frac{1}{2},2^{-(j+1)}}(0)$, which is at macroscopic distance from $\partial\Lambda_{1,2^{-(j+1)}}(0)$. As above, by [Wu18a], there exists $K > 0$ such that for any $j \geq 0$,

$$\alpha_{1,2^{-(j+1)}}(0; 2^{-(j+1)}, \tfrac{1}{2}) \leq K 2^{-(\frac{5}{8} - \frac{\beta}{2})j}.$$

The same reasoning applies for $x \in P_j(b)$. We thus have that there exist $n_0 \geq 1$ and a constant $\tilde{K} > 0$ such that for any $n \geq n_0$,

$$\sum_{x \in s_a \cup s_b} \mathbb{P}_n^{\pm}(x \in V(\gamma_n)) \leq \tilde{K} \sum_{j=0}^{O(\log_2(\frac{\eta}{\delta_n}))} \Big(\sum_{x \in P_j(a)} 2^{-\frac{5j}{8} + \beta j} + \sum_{x \in P_n(b)} 2^{-\frac{5j}{8} + \beta j} \Big)$$

$$\leq \tilde{K} \sum_{j=0}^{O(\log_2(\frac{\eta}{\delta_n}))} 2^{2j+1} 2^{(\beta - \frac{5}{8})j} \lesssim \eta^{\frac{5}{8} - \beta} \delta_n^{-(2+\beta-\frac{5}{8})}.$$

We are now left with controlling the contribution of points in $A_n(\eta)$ that are in squares other than s_a and s_b. This can be done following the strategy of the proof of [GPS13, Theorem 5.5] and we leave the details to the reader. We note that tighter estimates for the contribution of these points could be obtained by using the half-plane two-arm exponents but it is not really necessary here and would pose additional technical issues. Putting all the above estimates together, we obtain that there exist $n_0 \geq 1$ and $C > 0$ such that for any $n \geq n_0$, $\delta_n^{\frac{11}{8}+\beta} \mathbb{E}_n^{\pm}[|V(\gamma_n)|] \leq C\delta_n^{\frac{\beta}{2}}$. As $(\delta_n)_n$ converges to 0, there exists a subsequence $(\delta_m)_m \subset (\delta_n)_n$ such that $\sum_m \delta_m^{\frac{\beta}{2}} < \infty$. Borel-Cantelli lemma then yields the statement of the lemma. □

Controlling the exponential of the magnetization field in the slit domain

In this section, we prove Lemma 5.2.9 and Lemma 5.2.7. Let us start with the proof of Lemma 5.2.9. Our end goal is to apply the dominated convergence theorem. The proof is therefore split into two parts. First, in Lemma 5.2.11, we establish bounds on the terms $\delta_m^{15k/8} \mathbb{E}_{m,t}^{\pm}[(\sum_x \sigma_x)^k]$ on the left-hand side of (5.13) that are uniform in m and tight enough to be summable once multiplied by $h^k/k!$. Then, in Lemma 5.2.12, we prove that, almost surely, for any $k \geq 0$, the k-th term of the sum on the left-hand side of (5.13) almost surely converges as $m \to \infty$ to the k-th term of the sum on the right-hand side of (5.13).

Lemma 5.2.11. *Let $t \in (0, \infty)$. There almost surely exist $m_0 \geq 1$ and a sequence $(C_{k,t})_k$ such that, for any $m \geq m_0$ and any $k \geq 1$,*

$$\sum_{(x_1,\ldots,x_k) \in \Omega_{m,t}} \delta_m^{\frac{15k}{8}} |\mathbb{E}_{m,t}^{\pm}[\sigma_{x_1} \ldots \sigma_{x_k}]| \leq C_{k,t}$$

with $\sum_k \frac{h^k}{k!} C_{k,t} < \infty$.

Lemma 5.2.12. *Let $t \in (0, \infty)$. Almost surely, for any $k \geq 1$,*

$$\lim_{m \to \infty} \sum_{(x_1,\ldots,x_k) \in \Omega_{m,t}^k} C_\sigma^{-k} \delta_m^{\frac{15k}{8}} \mathbb{E}_{m,t}^{\pm}[\sigma_{x_1} \ldots \sigma_{x_k}] = \int_{\Omega_t^k} f_t^{(\pm,k)}(x_1, \ldots, x_k) \prod_{j=1}^k dx_j \qquad (5.14)$$

where the functions $f_t^{(\pm,k)} := f_{\Omega_t}^{(\pm,k)}$ are as in Theorem 5.1.8.

Before proving Lemma 5.2.11 and Lemma 5.2.12, let us give the proof of Lemma 5.2.9.

Proof of Lemma 5.2.9. This follows from the dominated convergence theorem: Lemma 5.2.11 gives us tight enough bounds on $|\delta_m^{\frac{15k}{8}} \mathbb{E}_{m,t}^{\pm}[(\sum_x \sigma_x)^k]|$ while Lemma 5.2.12 shows that, almost surely, for any $k \geq 1$, $\delta_m^{\frac{15k}{8}} \mathbb{E}_{m,t}^{\pm}[(\sum_x \sigma_x)^k]$ converges to the correct quantity. □

We now turn to the proof Lemma 5.2.11. Let us first introduce some notations. Let $\eta > 0$. For $t \geq 0$, $m \geq 1$ and $k \geq 1$, we set

$$G_{m,t}^{(k)}(\eta) := \{(x_1, \ldots, x_k) \in \Omega_{m,t}^k : \forall j, d(x_j, \partial\Omega_t) \geq \eta \text{ and } \forall i \neq j, |x_i - x_j| \geq \eta\}.$$

$G_{m,t}^{(k)}(\eta)$ is the "good" set in which points are well-separated and at macroscopic distance from the boundary. We also set $A_{m,t}(\eta) := \{x \in \Omega_{m,t} : d(x, \partial\Omega_t) < \eta\}$ and for $k \geq 2$,

$$B_{m,t}^{(k)}(\eta) := \{(x_1, \ldots, x_k) \in (\Omega_{m,t} \setminus A_{m,t}(\eta))^k : \exists i \neq j, |x_i - x_j| < \eta\}.$$

Proof of Lemma 5.2.11. To prove the lemma, we are going to control the contribution of points in $G_{m,t}^{(k)}(\eta)$ and in $\Omega_{m,t}^k \setminus G_{m,t}^{(k)}(\eta)$ separately. Let $t \in (0, \infty)$. Let us begin with the sum over $G_{m,t}^{(k)}(\eta)$ for $k \geq 1$. We want to show that there almost surely exists a sequence $(C_{k,t})_k$ and $m_0 \geq 1$ such that, for any $m \geq m_0$, any $\eta > 0$ and any $k \geq 1$,

$$\sum_{(x_1, \ldots, x_k) \in G_{m,t}^{(k)}(\eta)} \delta_m^{\frac{15k}{8}} |\mathbb{E}_{m,t}^{\pm}[\sigma_{x_1} \ldots \sigma_{x_k}]| \leq C_{k,t} \tag{5.15}$$

with $\sum_{k \geq 0} \frac{h^k}{k!} C_{k,t} < \infty$. By Lemma 5.1.11 and Lemma 5.1.12, we have that, almost surely,

$$|\sum_{(x_1, \ldots, x_k) \in G_{m,t}^{(k)}(\eta)} \delta_m^{\frac{15k}{8}} \mathbb{E}_{m,t}^{\pm}[\sigma_{x_1} \ldots \sigma_{x_k}]| \leq \sum_{(x_1, \ldots, x_k) \in G_{m,t}^{(k)}(\eta)} \delta_m^{\frac{15k}{8}} \mathbb{E}_{m,t}^{+}[\sigma_{x_1} \ldots \sigma_{x_k}]$$

$$\leq \sum_{(x_1, \ldots, x_k) \in G_{m,t}^{(k)}(\eta)} C^k \delta_m^{2k} \prod_{j=1}^{k} \left(\frac{1}{2}L(x_j) \wedge d_t(x_j)\right)^{-\frac{1}{8}}$$

$$\leq \sum_{(x_1, \ldots, x_k) \in \Omega_{m,t}^k \setminus D_{m,t}^{(k)}} C^k \delta_m^{2k} \prod_{j=1}^{k} \left(\frac{1}{2}L(x_j) \wedge d_t(x_j)\right)^{-\frac{1}{8}}$$

$$=: I_t^{(k)}(m)$$

where we have set $D_{m,t}^{(k)} := \{(x_1, \ldots, x_k) \in \Omega_{m,t}^k : \exists i \neq j, x_i = x_j\}$ and, for $x \in \Omega_{m,t}$, $d_t(x) := \text{dist}(x, \partial\Omega_{m,t})$. The last inequality follows from the non-negativity of the summand. Set $d_\Omega = 2\text{diam}(\Omega)$. Since almost surely, for any m, $t \mapsto \text{diam}(\Omega_{m,t})$ is a decreasing function, we can assume that almost surely, for any m and any $t \geq 0$, $\text{diam}(\Omega_{m,t}) \leq d_\Omega$. We now rescale $\Omega_{m,t}$ by d_Ω^{-1}. Denote by $\tilde\Omega_{m,t}$ the graph thus obtained and observe that almost surely, for $m \geq 1$, for any $t \geq 0$ and for any $x \in \tilde\Omega_{m,t}$, $d_t(x) \leq 1$. We then obtain that, almost surely, for any $m \geq 1$, any $k \geq 1$ and any $t \geq 0$,

$$I_t^{(k)}(m) \leq C^k \sum_{(x_1, \ldots, x_k) \in \tilde\Omega_{m,t}^k \setminus D_{m,t}^{(k)}} \delta_m^{2k} d_\Omega^{-\frac{k}{8}} C_k(x_1, \ldots, x_k; \frac{1}{8}) \prod_{j=1}^{k} d_t(x_j)^{-\frac{1}{8}} C_k(x_1, \ldots, x_k; \frac{1}{8}) \tag{5.16}$$

where the last inequality uses that almost surely, for any $m \geq 1$ and any $t \geq 0$, $d_t(x) \leq 1$ for $x \in \tilde\Omega_{m,t}$ and where we have set, for $\alpha > 0$,

$$C_k(x_1, \ldots, x_k; \alpha) := \prod_{j=1}^{k} (\frac{1}{2}L(x_j))^{-\alpha}.$$

Fix $\beta \in (0, \frac{1}{100})$. To estimate the right-hand side of (5.16), we use Hölder inequality with $\alpha, \tilde\alpha > 1$ such that $\frac{1}{\alpha} + \frac{1}{\tilde\alpha} = 1$, $\frac{\tilde\alpha}{8} < \frac{5}{8} - \beta$ and $\frac{\alpha}{8} < 1$. This choice of $\tilde\alpha$ will guarantee that Lemma 5.2.10 can be used to estimate the contributions of points near the boundary. We get that, almost surely,

$$I_t^{(k)}(m) \leq C^k \delta_m^{2k} d_\Omega^{-\frac{k}{8}} \left(\sum_{(x_1, \ldots, x_k) \in \tilde\Omega_{m,t}^k} \prod_{j=1}^{k} d(x_j)^{-\frac{\tilde\alpha}{8}}\right)^{1/\tilde\alpha} \left(\sum_{(x_1, \ldots, x_k) \in \tilde\Omega_{m,t}^k \setminus D_{m,t}^{(k)}} C_k(x_1, \ldots, x_k; \frac{\alpha}{8})\right)^{1/\alpha} \tag{5.17}$$

Set $J_t(m, \tilde\alpha) := \sum_{x \in \tilde\Omega_{m,t}} d(x)^{-\tilde\alpha/8}$. We claim the following.

Claim 5.2.13. *Let $\tilde{\alpha} > 1$ be such that $\frac{\tilde{\alpha}}{8} < \frac{5}{8} - \beta$ and let $t \in (0, \infty)$. Then, there almost surely exists $m_0 \geq 1$ such that, for any $m \geq m_0$, $J_t(m, \tilde{\alpha}) \leq C_{t,\tilde{\alpha}} \delta_m^{-2}$, where $C_{t,\tilde{\alpha}}$ is a random constant depending on $\tilde{\alpha}$, t and Ω.*

Claim 5.2.14. *Let $\alpha > 1$ be such that $\alpha/8 < 1$. There exists $c > 0$ such that almost surely, for any $m \geq 1$, any $k \geq 2$ and any $t \geq 0$,*

$$\sum_{(x_1,\ldots,x_k) \in \tilde{\Omega}_{m,t}^k \setminus D_{m,t}^{(k)}} C_k\left(x_1, \ldots, x_k; \frac{\alpha}{8}\right) \leq c^k \delta_m^{-2k} k^{\frac{k\alpha}{16}}.$$

These two claims together with the inequality (5.17) imply that there almost surely exists $m_0 \geq 1$ such that for any $m \geq m_0$ and any $k \geq 1$,

$$I_t^{(k)}(m) \leq d_\Omega^{-\frac{k}{8}} \delta_m^{2k} (C_{t,\tilde{\alpha}} \delta_m^{-2k})^{\frac{1}{\tilde{\alpha}}} c^{\frac{k}{\alpha}} \delta_m^{-\frac{2k}{\alpha}} k^{\frac{k}{16}},$$

which yields the inequality (5.15).

To prove Lemma 5.2.11, it now remains to control the contribution of points in $\Omega_{m,t}^k \setminus G_{m,t}^{(k)}(\eta)$. We are going to show that there almost surely exists $m_0 \geq 1$ such that for any $m \geq m_0$, any $k \geq 1$ and any $\eta > 0$,

$$M_t^{(k)}(m; \eta) := \sum_{(x_1,\ldots,x_k) \in \Omega_{m,t}^k \setminus G_{m,t}^{(k)}(\eta)} \delta_m^{\frac{15k}{8}} |\mathbb{E}_{m,t}^\pm[\sigma_{x_1} \ldots \sigma_{x_k}]| \leq C_{1,t}^k \eta^{\frac{5}{8}-\beta} k^{\frac{k}{16}} + C_{2,t}^k \eta k^{\frac{k}{16}} \quad (5.18)$$

where $C_{1,t}, C_{2,t} > 0$ are random constants depending on t and Ω. Observe in particular that the terms on the above right-hand side are summable in k, once multiplied by $\frac{h^k}{k!}$. We have that, almost surely, for any $k \geq 1$,

$$M_t^{(k)}(m; \eta) = \sum_{(x_1,\ldots,x_k) \in B_{m,t}^{(k)}(\eta)} \delta_m^{\frac{15k}{8}} |\mathbb{E}_{m,t}^\pm[\sigma_{x_1} \ldots \sigma_{x_k}]|$$

$$+ \sum_{r=1}^k \sum_{W_r} \sum_{(x_1,\ldots,x_k) \in \phi_{W_r}(A_{m,t}(\eta)^r \times (\Omega_{m,t} \setminus A_{m,t}(\eta))^{k-r})} \delta_m^{\frac{15k}{8}} |\mathbb{E}_{m,t}^\pm[\sigma_{x_1} \ldots \sigma_{x_k}]|. \quad (5.19)$$

Above, $\sum_{W_r}$ ranges over the different possible choices of r coordinates in $A_{m,t}(\eta)$ and $k-r$ ones in $\Omega_{m,t} \setminus A_{m,t}(\eta)$. Given such a choice W_r, $\phi_{W_r}(A_{m,t}(\eta)^r \times (\Omega_{m,t} \setminus A_{m,t}(\eta))^{k-r})$ is the permutation of $A_{m,t}(\eta)^r \times (\Omega_{m,t} \setminus A_{m,t}(\eta))^{k-r}$ corresponding to this choice. However, it is easy to see that the sum over points in $\phi_{W_r}(A_{m,t}(\eta)^r \times (\Omega_{m,t} \setminus A_{m,t}(\eta))^{k-r})$ does not depend on this choice of permutations. To obtain the inequality (5.18), we thus see that we must upper bound the two sums appearing in the decomposition (5.19) of $M_t^{(k)}(m; \eta)$. These bounds are the content of the following two claims.

Claim 5.2.15. *There almost surely exists m_0 such that for any $m \geq m_0$, any $k \geq 2$ and any $\eta > 0$,*

$$\sum_{(x_1,\ldots,x_k) \in B_{m,t}^{(k)}(\eta)} \delta_m^{\frac{15k}{8}} |\mathbb{E}_{m,t}^\pm[\sigma_{x_1} \ldots \sigma_{x_k}]| \leq C_1^k \eta k^{\frac{k}{16}}$$

where $C_1 = C_1(t, \Omega)$ is a random constant depending on t and Ω.

Claim 5.2.16. *There almost surely exists m_0 such that, for any $m \geq m_0$, any $k \geq r \geq 1$ and any $\eta > 0$,*

$$\sum_{(x_1,\ldots,x_k) \in A_{m,t}(\eta)^r \times (\Omega_{m,t} \setminus A_{m,t}(\eta))^{k-r}} \delta_m^{\frac{15k}{8}} |\mathbb{E}_{m,t}^\pm[\sigma_{x_1} \ldots \sigma_{x_k}]| \leq C_2^k \eta^{\frac{r}{\rho}(\frac{5}{8}-\beta)} k^{\frac{k}{16}}$$

where $C_2 = C_2(t, \Omega)$ is a random constant depending on t and Ω.

Claim 5.2.15 and Claim 5.2.16 yield that there almost surely exists $m_0 \geq 1$ such that, for any $m \geq m_0$, any $k \geq 1$ and any $\eta > 0$,

$$M_t^{(k)}(m; \eta) \leq C^k \eta k^{\frac{k}{16}} + \sum_{r=1}^k |W_r| C^k \eta^{(\frac{5}{8}-\beta)} k^{\frac{k}{16}} \leq C_1^k \eta k^{\frac{k}{16}} + 2^k C_2^k \eta^{\frac{5}{8}-\beta} k^{\frac{k}{16}},$$

which shows the inequality (5.18) and concludes the proof of the lemma, $\qquad \square$

Let us now prove Claim 5.2.13, Claim 5.2.14, Claim 5.2.15 and Claim 5.2.16.

Proof of Claim 5.2.13. Let $\eta > 0$. Let us first estimate the contribution of points in $\tilde{\Omega}_{m,t} \setminus \tilde{A}_{m,t}(\eta)$ to $J_t(m, \tilde{\alpha})$, where $\tilde{A}_{m,t}(\eta) := \{x \in \tilde{\Omega}_{m,t} : \mathrm{dist}(x, \partial\Omega_{m,t}) < d_\Omega \eta\}$. We have that, almost surely, for any $m \geq 1$,

$$\sum_{x \in \tilde{\Omega}_{m,t} \setminus \tilde{A}_{m,t}(\eta)} d_t(x)^{-\frac{\tilde{\alpha}}{8}} \lesssim \eta^{-\frac{\tilde{\alpha}}{8}} \mathrm{vol}(\tilde{\Omega}) \delta_m^{-2}.$$

We now want to estimate the contribution of points in $\tilde{A}_{m,t}(\eta)$ to $J_t(m, \tilde{\alpha})$. Let us write $\tilde{A}_{m,t}(\eta) = \tilde{A}_{m,t}^{(1)}(\eta) \sqcup \tilde{A}_{m,t}^{(2)}(\eta)$ where we have set

$$\tilde{A}_{m,t}^{(1)}(\eta) = \{x \in \tilde{A}_{m,t}(\eta) : \mathrm{dist}(x, \partial\Omega) < \mathrm{dist}(x, \gamma_m([0, t_m]))\}$$
$$\tilde{A}_{m,t}^{(2)}(\eta) = \{x \in \tilde{A}_{m,t}(\eta) : \mathrm{dist}(x, \gamma_m([0, t_m])) \leq \mathrm{dist}(x, \partial\Omega)\}.$$

Observe that almost surely, for any $m \geq 1$ and any $t \geq 0$, $\tilde{A}_{m,t}^{(1)}(\eta) \subset \cup_{k=0}^{K(m)} D_k(m)$ where for $k \geq 0$, $D_k(m) := \{x \in \tilde{\Omega}_m : 2^k \delta_m \leq d_\Omega \, \mathrm{dist}(x, \partial\Omega_m) \leq 2^{k+1} \delta_m\}$ with $K(m) = \log_2(\eta/\delta_m) + 1$. Therefore, using that almost surely, for any $t \geq 0$ and for any $x \in \tilde{A}_{m,t}^{(1)}(\eta)$, $d_t(x) = \mathrm{dist}(x, \partial\Omega_m)$, the contribution of points in $\tilde{A}_{m,t}^{(1)}(\eta)$ is almost surely bounded by

$$\sum_{x \in \tilde{A}_{m,t}^{(1)}(\eta)} \mathrm{dist}(x, \partial\Omega_m)^{-\tilde{\alpha}/8} \leq \sum_{k=0}^{O(\log_2(\eta/\delta_m))} 2^{-\frac{\tilde{\alpha}}{8}k} \delta_m^{-\frac{\tilde{\alpha}}{8}} |D_k(m)|$$
$$\leq \sum_{k=0}^{O(\log_2(\eta/\delta_m))} 2^{\frac{\tilde{\alpha}}{8}k} \delta_m^{-\frac{\tilde{\alpha}}{8}} 2^k \delta_m^{-1} \lesssim \delta_m^{-2} \eta^{1-\frac{\tilde{\alpha}}{8}}. \tag{5.20}$$

Let us now estimate the contributions of points in $\tilde{A}_{m,t}^{(2)}(\eta)$. Observe that almost surely, for any $x \in \tilde{A}_{m,t}^{(2)}(\eta)$ and any $t \geq 0$, $d_t(x) = \mathrm{dist}(x, \gamma_m([0, t_m]))$. Moreover, we have that, almost surely, $\tilde{A}_{m,t}^{(2)}(\eta) \subset \cup_{k=0}^{K(m)} C_k(\gamma_m; \eta, t)$ where $C_k(\gamma_m; \eta, t) := \{x \in \tilde{\Omega}_{m,t} : 2^k \delta_m \leq d_\Omega \, \mathrm{dist}(x, \gamma_m([0, t_m])) \leq 2^{k+1} \delta_m\}$. Therefore, we have that, almost surely, for any $m \geq 1$ and any $t \geq 0$,

$$\sum_{x \in \tilde{A}_{m,t}^{(2)}(\eta)} d_t(x)^{-\frac{\tilde{\alpha}}{8}} \leq \sum_{k=0}^{O(\log_2(\eta/\delta_m))} 2^{-k\frac{\tilde{\alpha}}{8}} \delta_m^{-\frac{\tilde{\alpha}}{8}} |C_k(\gamma_m; \eta, t)| \leq \sum_{k=0}^{O(\log_2(\eta/\delta_m))} 2^{-k\frac{\tilde{\alpha}}{8}} \delta_m^{-\frac{\tilde{\alpha}}{8}} |C_k(\gamma_m; \eta, \infty)|.$$

By Lemma 5.2.6, almost surely $\delta_m^{\frac{11}{8}+\beta} |V(\gamma_m)| \to 0$ as $m \to \infty$. This implies that there almost surely exists $m_0 \geq 1$ such that for any $\eta > 0$, the set $C_k(\gamma_m, \eta, \infty)$ can be covered by $C_\infty \eta^{-\frac{11}{8}-\beta}$ balls of radius η, where C_∞ is a random constant independent of t. Each of these balls contains a number of vertices of order $\eta^2 \delta_m^{-2}$. This yields that almost surely, for any $m \geq m_0$ and any $\eta > 0$,

$$\sum_{x \in \tilde{A}_{m,t}^{(2)}(\eta)} d_t(x)^{-\frac{\tilde{\alpha}}{8}} \leq C_\infty \delta_m^{-2} \sum_{k=0}^{O(\log_2(\eta/\delta_m))} 2^{\frac{\tilde{\alpha}(k+1)}{8}} \eta^{-\frac{\tilde{\alpha}}{8}} \eta^{\frac{5}{8}-\beta} 2^{k(\frac{11}{8}+\beta)} 2^{-2k}$$
$$\leq C_\infty \delta_m^{-2} \eta^{\frac{5}{8}-\beta-\frac{\tilde{\alpha}}{8}} \sum_{k=0}^{\infty} 2^{-(\frac{5}{8}-\beta-\frac{\tilde{\alpha}}{8})}. \tag{5.21}$$

The sum on the above right-hand side is finite since $\frac{\tilde{\alpha}}{8} < \frac{7}{8} - \beta$ by assumption. Putting everything together, we get that there almost surely exists $m_0 \geq 1$ such that for any $m \geq m_0$, any $t \geq 0$ and any $\eta > 0$,

$$J_t(m, \tilde{\alpha}) \lesssim \delta_m^{-2} \eta^{\frac{5}{8}-\beta-\frac{\tilde{\alpha}}{8}} + \delta_m^{-2} \eta^{1-\frac{\tilde{\alpha}}{8}} + \eta^{-\frac{\tilde{\alpha}}{8}} \delta_m^{-2}$$

which yields the claim. $\qquad\square$

Proof of Claim 5.2.14. Since C_k is a nonnegative function and Ω is a bounded domain, there exists $\tilde{d}_\Omega > 1$ such that, almost surely, for any $m \geq 1$, any $t \geq 0$ and any $k \geq 0$,

$$\sum_{(x_1,\ldots,x_k) \in \tilde{\Omega}_{m,t} \setminus D_{m,t}^{(k)}} C_k(x_1, \ldots, x_k; \frac{\alpha}{8}) \leq \sum_{(x_1,\ldots,x_k) \in B_m(0, \tilde{d}_\Omega)^k \setminus D_m^{(k)}(\tilde{d}_\Omega)} C_k(x_1, \ldots, x_k; \frac{\alpha}{8})$$

where $B_m(0, \tilde{d}_\Omega)$ denotes the δ_n-approximation of the Euclidean ball $B(0, \tilde{d}_\Omega)$ and $D_m^{(k)}(\tilde{d}_\Omega) :=$ $\{(x_1, \ldots, x_k) \in B_m(0, \tilde{d}_\Omega)^k : \exists i \neq j, x_i = x_j\}$. By [FM17, Proposition 3.10] and [JSW20, Lemma 3.10], as $\frac{\alpha}{8} < 1$, we obtain that there exists a constant $c > 0$ such that, almost surely, for any $m \geq 1$ and any $k \geq 2$,

$$\sum_{(x_1, \ldots, x_k) \in B_m(0, \tilde{d}_\Omega)^k \setminus D^{(k)}(\tilde{d}_\Omega)} C_k(x_1, \ldots, x_k; \frac{\alpha}{8}) \leq c^k \delta_m^{-2k} k^{\frac{k\alpha}{16}},$$

which yields the claim. We note that the estimate of [JSW20, Lemma 3.10] is obtained in the continuum but can also be derived in the discrete by using the same type of arguments as in the proof of [FM17, Proposition 3.10]. $\qquad\square$

Proof of Claim 5.2.15. By Lemma 5.1.11, we have that, almost surely,

$$F_t^{(k)}(m; \eta) := \delta_m^{\frac{15k}{8}} \sum_{(x_1, \ldots, x_k) \in B_{m,t}^{(k)}(\eta)} |\mathbb{E}_{m,t}^{\pm}[\sigma_{x_1} \ldots \sigma_{x_k}]| \leq \delta_m^{\frac{15k}{8}} \sum_{(x_1, \ldots, x_k) \in B_{m,t}^{(k)}(\eta)} |\mathbb{E}_{m,t}^{+}[\sigma_{x_1} \ldots \sigma_{x_k}]|$$

Using the same arguments as in the second part of the proof of [FM17, Proposition 3.10] together with [JSW20, Lemma 3.10] to estimate the k-dependent constant, we get that there almost surely exists $m_0 \geq 1$ such that for any $m \geq m_0$, any $k \geq 2$ and any $\eta > 0$,

$$\sum_{(x_1, \ldots, x_k) \in B_{m,t}^{(k)}(\eta)} |\mathbb{E}_{m,t}^{+}[\sigma_{x_1} \ldots \sigma_{x_k}]| \leq C_1^k \eta \delta_m^{-\frac{15k}{8}} k^{\frac{k}{16}}$$

where $C_1 = C_1(t, \Omega)$ is a random constant depending on t and Ω. This concludes the proof of the claim. Note that the η-dependence of the above right-hand side could be made to depend on k but these tighter estimates are not necessary here. $\qquad\square$

Proof of Claim 5.2.16. Using the Edwards-Sokal coupling of Proposition 5.1.6 and with the same notations as in the proof of Lemma 5.1.11, we have that $\mathbb{E}_{m,t}^{+}[\sigma_{x_1} \ldots \sigma_{x_k}] = \mathbb{P}_{m,t}^{\mathrm{FK,wired}}(A_{x_1, \ldots, x_k})$. By the Markov property of the FK-Ising model, we then have that, almost surely, for any $m \geq 1$, any $r \geq 1$, any $k \geq r$ and any $(x_1, \ldots, x_k) \in A_{m,t}(\eta)^r \times (\Omega_{m,t} \setminus A_{m,t}(\eta))^{k-r}$,

$$\mathbb{P}_{m,t}^{\mathrm{FK,wired}}(A_{x_1, \ldots, x_k}) \lesssim \mathbb{P}_{m,t}^{\mathrm{FK,wired}}(A_{x_{r+1}, \ldots, x_k}) \prod_{j=1}^{r} \mathbb{P}_{\Lambda_{d_t(x_j), m}(x_j)}^{\mathrm{FK,wired}}(x_j \longleftrightarrow \partial\Lambda_{d_t(x_j), m}(x_j))$$

where for $x \in A_{m,t}(\eta)$, $\Lambda_{d_t(x_j), m}(x_j) \subset \delta_m \mathbb{Z}^2$ is the square centered at x_j with side-length $\frac{1}{2} d_t(x_j)$ and $\mathbb{P}_{\Lambda_{d_t(x_j), m}(x_j)}^{\mathrm{FK,wired}}$ is the FK-Ising measure in $\Lambda_{d_t(x_j), m}(x_j)$ with wired boundary conditions. Here, we recall that $d_t(x_j) = \mathrm{dist}(x, \partial\Omega_{m,t})$. Denoting by $S_{m,t}^{(r,k)}$ the sum in the statement of Claim 5.2.16, this implies that

$$S_{m,t}^{(r,k)} \lesssim \delta_m^{\frac{15k}{8}} \left(\sum_{x \in A_{m,t}(\eta)} \mathbb{P}_{\Lambda_{d_t(x)}(x)}^{\mathrm{FK,wired}}(x_j \longleftrightarrow \partial\Lambda_{d_t(x)}(x)) \right)^r$$

$$\times \sum_{(x_{r+1}, \ldots, x_{k-r}) \in (\Omega_{m,t} \setminus A_{m,t}(\eta))^{k-r}} \mathbb{P}_{m,t}^{\mathrm{FK,wired}}(A_{x_{r+1}, \ldots, x_k}).$$

By [DHN11, Lemma 5.4], there exists $C > 0$ such that, almost surely, for any $m \geq 1$, any $t \geq 0$ and any $x \in A_{m,t}(\eta)$,

$$\mathbb{P}_{\Lambda_{d_t(x), m}(x)}^{\mathrm{FK,wired}}(x_j \longleftrightarrow \partial\Lambda_{d_t(x), m}(x)) \leq C \delta_m^{1/8} d_t(x_j)^{-1/8}.$$

The sum $\sum_{A_{m,t}(\eta)} d_t(x_j)^{-1/8}$ can be estimated as in we did in the proof of Claim 5.2.13, see the discussion leading to the inequalities (5.20) and (5.21). This yields that there almost surely exits m_0 such that, almost surely, for any $m \geq m_0$, any $k \geq r \geq 1$ and any $\eta > 0$,

$$S_{m,t}^{(r,k)} \lesssim \delta_m^{\frac{15(k-r)}{8}} \eta^{r(\frac{5}{8} - \beta)} \sum_{(x_1, \ldots, x_{k-r}) \in (\Omega_{m,t} \setminus A_{m,t}(\eta))^{k-r}} \mathbb{P}_{m,t}^{\mathrm{FK,wired}}(A_{x_1, \ldots, x_{k-r}}).$$

The sum on the above right-hand side can now be upper bounded using Claim 5.2.14 and Claim 5.2.15, which concludes the proof. $\qquad\square$

We now turn to the proof of Lemma 5.2.12. We keep the same definitions for $G_{m,t}^{(k)}(\eta)$, $A_{m,t}(\eta)$ and $B_{m,t}^{(k)}(\eta)$.

Proof of Lemma 5.2.12. We have that, almost surely, for any $m \geq 1$, any $t \geq 0$ and any $k \geq 1$,

$$\sum_{(x_1,\ldots,x_k) \in \Omega_{m,t}^k} C_\sigma^{-k} \delta_m^{\frac{15k}{8}} \mathbb{E}_{m,t}^\pm[\sigma_{x_1} \ldots \sigma_{x_k}] = \sum_{(x_1,\ldots,x_k) \in G_{m,t}^{(k)}(\eta)} C_\sigma^{-k} \delta_m^{\frac{15k}{8}} \mathbb{E}_{m,t}^\pm[\sigma_{x_1} \ldots \sigma_{x_k}]$$
$$+ \sum_{(x_1,\ldots,x_k) \in \Omega_{m,t}^k \setminus G_{m,t}^{(k)}(\eta)} C_\sigma^{-k} \delta_m^{\frac{15k}{8}} \mathbb{E}_{m,t}^\pm[\sigma_{x_1} \ldots \sigma_{x_k}].$$

Since $(\hat{\Omega}_{m,t}; a_{m,t}, b_m)$ almost surely converges to $(\Omega; a, b)$ in the Carathéodory topology, we obtain from Theorem 5.1.8 that, almost surely, for any $k \geq 1$,

$$\lim_{m \to \infty} \sum_{(x_1,\ldots,x_k) \in G_{m,t}^{(k)}(\eta)} C_\sigma^{-k} \delta_m^{\frac{15k}{8}} \mathbb{E}_{m,t}^\pm[\sigma_{x_1} \ldots \sigma_{x_k}] = \int_{G_t^{(k)}(\eta)} f_t^{(k)}(x_1, \ldots, x_k) \prod dx_j \qquad (5.22)$$

where we have set $G_t^{(k)}(\eta) := \{(x_1, \ldots, x_k) \in \Omega_t^k : \forall j, d(x_j, \partial\Omega_t) \geq \eta \text{ and } \forall i \neq j, |x_i - x_j| \geq \eta\}$ and $f_t^{(k)} := f_{\Omega_t}^{(\pm,k)}$. Observe now that the inequality (5.18) shows that the contribution of points in $\Omega_{m,t}^k \setminus G_{m,t}^{(k)}(\eta)$ is uniformly bounded in m by a quantity that converges to 0 as $\eta \to 0$. Together with (5.22), this yields that, almost surely, for any $k \geq 1$,

$$\lim_{m \to \infty} \sum_{(x_1,\ldots,x_k) \in \Omega_{m,t}^k} C_\sigma^{-k} \delta_m^{\frac{15k}{8}} \mathbb{E}_{m,t}^\pm[\sigma_{x_1} \ldots \sigma_{x_k}] = \lim_{\eta \to 0} \int_{G_t^{(k)}(\eta)} f_t^{(k)}(x_1, \ldots, x_k) \prod_{j=1}^k dx_j.$$

Lemma 5.2.12 then follows from the claim below, which shows that in the continuum, the contribution of points in $\Omega_t^k \setminus G_t^{(k)}(\eta)$ to the integral on the right-hand side of (5.14) vanishes as $\eta \to 0$.

Claim 5.2.17. *Almost surely, for any $k \geq 1$ and any $t \in (0, \infty)$,*

$$\lim_{\eta \to 0} \int_{G_t^{(k)}(\eta)} f_t^{(k)}(x_1, \ldots, x_k) \prod_{j=1}^k dx_j = \int_{\Omega_t^k} f_t^{(k)}(x_1, \ldots, x_k) \prod dx_j < \infty.$$

$\square$

We now prove Claim 5.2.17.

Proof of Claim 5.2.17. Let $k \geq 1$. Observe that almost surely, as $\eta \to 0$, $\mathbb{I}_{G_t^{(k)}(\eta)} f_t^{(k)}(x_1, \ldots, x_k) \to f_t^{(k)}(x_1, \ldots, x_k)$ pointwise. Therefore, to show the lemma, it is enough to show that $\mathbb{I}_{G_t^{(k)}(\eta)} f_t^{(k)}(x_1, \ldots, x_k)$ is almost surely uniformly bounded in $L^2(\Omega_t^k)$. Going back to the discrete level and using Lemma 5.1.11 and Theorem 5.1.8, we have that, almost surely,

$$\int_{G_t^{(k)}(\eta)} |f_t^{(k)}(x_1, \ldots, x_k)|^2 \prod dx_j = \lim_{n \to \infty} \sum_{(x_1,\ldots,x_k) \in G_{n,t}^{(k)}(\eta)} \delta_n^{\frac{15k}{8}} C_\sigma^{-2k} |\mathbb{E}_{n,t}^\pm[\sigma_{x_1} \ldots \sigma_{x_k}]|^2$$
$$\leq \lim_{n \to \infty} \sum_{(x_1,\ldots,x_k) \in G_{n,t}^{(k)}(\eta)} \delta_n^{\frac{15k}{8}} C_\sigma^{-2k} |\mathbb{E}_{n,t}^+[\sigma_{x_1} \ldots \sigma_{x_k}]|^2$$
$$= \int_{G_t^{(k)}(\eta)} |f_t^{(+,k)}(x_1, \ldots, x_k)|^2 \prod dx_j$$
$$\leq \int_{\Omega_t^k} |f_t^{(+,k)}(x_1, \ldots, x_k)|^2 \prod dx_j$$

where $f_t^{(+,k)} := f_{\Omega_t}^{(+,k)}$ is as in Theorem 5.1.8. Almost sure finiteness of the last integral follows from Lemma 6.3.1 in Appendix 6.3 together with the fact since γ has law $\mathbb{P}_{\text{SLE}_3}^{(\Omega,a,b)}$, $\partial\Omega_t$ almost surely has Hausdorff dimension 11/8. $\square$

Let us now turn to the proof of Lemma 5.2.7. The proof follows the same strategy as that of the proof of Lemma 5.2.9. Namely, we first show that each term in the series on the left-hand side of (5.10) almost surely converges and then prove estimates on these terms that allow us to use the dominated convergence theorem. These two statements are the content of the two lemmas below. We omit their proof as it is almost identical to that of Lemma 5.2.12 and Lemma 5.2.11.

Lemma 5.2.18. *Let $\xi \in \{-, +\}$. Almost surely, for any $k \geq 1$,*

$$\lim_{m \to \infty} \sum_{(x_1, \ldots, x_k) \in \Omega_{m,j(\xi)}^k} C_\sigma^{-k} \delta_m^{\frac{15k}{8}} \mathbb{E}_{m,j(\xi)}^\xi [\sigma_{x_1} \ldots \sigma_{x_k}] = \int_{\Omega_{j(\xi)}^k} f_{j(\xi)}^{(\xi,k)}(x_1, \ldots, x_k) \prod_{j=1}^k dx_j$$

where $j(\xi)$, $\Omega_{j(\xi)}$ and the functions $f_{j(\xi)}^{(\xi,k)}$ are as in Lemma 5.2.7.

Lemma 5.2.19. *Let $\xi \in \{-, +\}$. Almost surely, there exists m_0 such that, for any $m \geq m_0$ and any $k \geq 1$,*

$$\sum_{(x_1, \ldots, x_k) \in \Omega_{m,j(\xi)}} \delta_m^{\frac{15k}{8}} \mathbb{E}_{m,j(\xi)}^\xi [\sigma_{x_1} \ldots \sigma_{x_k}] \leq C_{k,\infty}^{(\xi)}$$

with $\sum_k \frac{h^k}{k!} C_{k,\infty}^{(\xi)} < \infty$.

Proof of Lemma 5.2.7. Lemma 5.2.7 directly follows from Lemma 5.2.18, Lemma 5.2.19 and dominated convergence. □

5.2.3 Conformal covariance of the limiting interface

In this section, we show that the law of $\mathbb{P}_h^{(\Omega,a,b)}$ is conformally covariant.

Proof of Proposition 1.3.2. This follows from the expression (1.7) for the Radon-Nikodym derivative of $\mathbb{P}_h^{(\Omega,a,b)}$ with respect to $\mathbb{P}_{\text{SLE}_3}^{(\Omega,a,b)}$. Let $\psi : \Omega \to \tilde{\Omega}$ be a conformal map. $\mathbb{P}_{\text{SLE}_3}^{(\Omega,a,b)}$ is conformally invariant while by Theorem 5.1.8, a change of variable yields that, almost surely, for any $k \geq 1$,

$$\int_{\Omega_t^k} [\prod_{j=1}^k h(z_j)] f_t^{(\pm,k)}(z_1, \ldots, z_k) \prod_{j=1}^k dz_j = \int_{\tilde{\Omega}_t^k} [\prod_{j=1}^k \tilde{h}(z_j)] \tilde{f}_t^{(\pm,k)}(z_1, \ldots, z_k) \prod_{j=1}^k dz_j$$

where we have set $\tilde{\Omega}_t = \psi(\Omega_t)$, $\tilde{f}_t^{(\pm,k)} := f_{\tilde{\Omega}_t}^{(\pm,k)}$ and for $z \in \tilde{\Omega}$, $\tilde{h}(z) = |(\psi^{-1})'(z)|^{\frac{15}{8}} h(\psi^{-1}(z))$. Similarly, $\mathcal{Z}_h(\Omega) = \mathcal{Z}_{\tilde{h}}(\tilde{\Omega})$, which concludes the proof. □

5.3 Scaling limit of the interface away from near-criticality

5.3.1 The case of a small magnetic perturbation: proof of Proposition 1.3.3

In this section, we prove Proposition 1.3.3. We give the proof in the case $H_x = h\delta^{\frac{15}{8}} g_1(\delta)$ with $h \in \mathbb{R}$ but the extension to the case of space-varying external fields that are bounded is straightforward. We keep the same notations as in Section 5.2 for $\mathbb{P}_\delta^\pm$, $\mathbb{P}_{\delta,h}^\pm$, $\mathbb{P}_n^\pm$ and $\mathbb{P}_{n,h}^\pm$.

Proof of Proposition 1.3.3. The proof of Proposition 1.3.3 consists of two steps: first showing tightness of $(\gamma_\delta)_\delta$ under $(\mathbb{P}_{\delta,h}^\pm)_\delta$ in the topology (T.4) and then characterizing the law of any subsequential limit of $(\gamma_\delta)_\delta$ under $(\mathbb{P}_{\delta,h}^\pm)_\delta$. For both of these steps, we rely on the observation that for any $\delta > 0$, the law of γ_δ under $\mathbb{P}_{\delta,h}^\pm$ is absolutely continuous with respect to that of γ_δ under $\mathbb{P}_\delta^\pm$ with corresponding Radon-Nikodym derivative given by

$$F_\delta(\gamma_\delta) := \frac{d\mathbb{P}_{\delta,h}^\pm}{d\mathbb{P}_\delta^\pm}(\gamma_\delta) = \frac{1}{\mathcal{Z}_\delta} \mathbb{E}_\delta^\pm \left[\exp \left(h\delta^{\frac{15}{8}} g_1(\delta) \sum_{x \in \Omega_\delta} \sigma_x \right) | \gamma_\delta \right]. \tag{5.23}$$

With this observation, tightness of $(\gamma_\delta)_\delta$ under $(\mathbb{P}_{\delta,h}^\pm)_\delta$ in the topology (T.4) follows from the same arguments as those used in the proof of Proposition 5.2.2. Indeed, using the same inequalities

as in this proof but with $\delta^{15/8}$ replaced by $\delta^{15/8}g_1(\delta)$, it is not hard to see that for any $p \geq 1$, $\sup_{\delta>0}\mathbb{E}_\delta[F_\delta(\gamma_\delta)^p] < \infty$. We leave the details to the reader.

To characterize the law of any subsequential limits of $(\gamma_\delta)_\delta$ under $(\mathbb{P}^\pm_{\delta,h})_\delta$, we proceed as in the proof of Proposition 5.2.3. Let $(\delta_n)_n$ be a sequence such that $\mathbb{P}^\pm_{n,h}$ converges weakly in the topology (T.4). We use Theorem 5.1.7 and Skorokhod representation theorem to define on a common probability space $(S, \mathcal{F}, \mathbb{P})$ a sequence $(\gamma_n)_n$ and a random curve γ such that for any $n \geq 1$, γ_n has law $\mathbb{P}^\pm_n$, γ has law $\mathbb{P}^{(\Omega,a,b)}_{\text{SLE}_3}$ and $\mathbb{P}$-almost surely, $\gamma_n \to \gamma$ in the topologies (T.1)–(T.4). The random variables $(F_n(\gamma_n))_n$ given by (5.23) (with $\delta = \delta_n$) can then all be defined on $(S, \mathcal{F}, \mathbb{P})$. To characterize the limiting law of γ_n under $\mathbb{P}^\pm_{n,h}$, we are going to show that $F_n(\gamma_n)$ converges in $L^1(\mathbb{P})$ to 1. This will imply that the Radon-Nikodym derivative of the law of any subsequential limit of $(\gamma_\delta)_\delta$ under $(\mathbb{P}^\pm_{\delta,h})_\delta$ with respect to $\mathbb{P}^{(\Omega,a,b)}_{\text{SLE}_3}$ is equal to 1, thus proving Proposition 1.3.3.

To establish the above mentioned convergence, we are going to show that

$$\lim_{n\to\infty}\mathbb{E}^\pm_n[(F_n(\gamma_n)-1)^2] = 0. \tag{5.24}$$

Developing the square and using that $\mathbb{E}^\pm_n[F_n(\gamma_n)] = 1$, to show (5.24), we see that it suffices to prove that $\mathbb{E}^\pm_n[F_n(\gamma_n)^2]$ converges to 1. Let us set $S_n(\gamma_n) := \mathbb{E}^\pm_n[\exp(h\delta_n^{15/8}g_1(\delta_n)\sum_x \sigma_x)|\gamma_n]$. We are first going to show that $\mathbb{E}^\pm_n[S_n(\gamma_n)^2]$ converges to 1 as $n \to \infty$ by exhibiting lower and upper bounds on $\mathbb{E}^\pm_n[S_n(\gamma_n)^2]$ that both converge to 1 as $n \to \infty$. For the upper bound, we can proceed as in the proof of Proposition 5.2.2 (with $p = 2$) to get that there exists $n_0 \geq 1$ and $c_1, c_2 > 0$ such that for any $n \geq n_0$,

$$\mathbb{E}^\pm_n[S_n(\gamma_n)^2] \leq \exp(6Mc_1g_1(\delta_n) + 20M^2c_2g_1(\delta_n)^2) \tag{5.25}$$

where $M := \max_{x\in\Omega}|h(x)|$. Observe that the right-hand side of (5.25) converges to 1 as $n \to \infty$. On the other hand, by Jensen's inequality, $\mathbb{E}^\pm_n[S_n(\gamma_n)^2] \geq \mathcal{Z}_n(h)^2$. The same reasoning as in the proof of Proposition 5.2.2 now shows that there exist $n_1 \geq 1$ and $c > 0$ such that for any $n \geq n_1$, $\mathcal{Z}_n(h) \geq \exp(-cg_1(\delta_n))$. Together with the inequality (5.25), this implies that $\mathbb{E}^\pm_n[S_n(\gamma_n)^2]$ converges to 1 as $n \to \infty$.

To prove (5.24), it remains to show $\mathcal{Z}_n(h)$ converges to 1 as $n \to \infty$. For this, we can once again apply the same reasoning as in the proof of Proposition 5.2.2 (with $p = 1$) to get that for any $n \geq n_0$, $\mathcal{Z}_n(h) \leq \exp(4Mc_1g_1(\delta_n) + 8M^2c_2g_1(\delta_n)^2)$. As we have already shown that for any $n \geq n_1$, $\mathcal{Z}_n(h) \geq \exp(-cg_1(\delta_n))$, this yields that $\mathcal{Z}_n(h) \to 1$ as $n \to \infty$. As explained above, this also completes the proof of Proposition 1.3.3. $\qquad\square$

5.3.2 The case of a large magnetic perturbation: proof of Proposition 1.3.4

The Edwards-Sokal coupling with non-zero magnetic field

For $\delta > 0$, let $(\Omega_\delta; a_\delta, b_\delta)_\delta$ be as in Section 5.1.1. The Edwards-Sokal coopuling of Proposition 5.1.6 can be extended to the case when the Ising model is perturbed by an external magnetic field, at least when the external field is non-negative and when the boundary conditions are chosen to be $+$ or free. This coupling will be instrumental for the proof of Proposition 1.3.4, so let us describe it.

To construct the Edwards-Sokal coupling with an external field, one direct way to proceed is to consider the extended graph obtained from $(V(\Omega_\delta), E(\Omega_\delta))$ by adding a ghost vertex g to the graph and drawing an edge (vg) between each vertex $v \in V(\Omega_\delta)$ and g. We denote the set of such edges $E^*(\Omega_\delta)$ and call a function $\omega : E(\Omega_\delta) \cup E^*(\Omega_\delta) \to \{0,1\}^{|E(\Omega_\delta)|+|E^*(\Omega_\delta)|}$ an edge configuration. We say that the edge e is open (in ω) if $\omega(e) = 1$. Otherwise, e is said to be closed. We identify ω with the subgraph whose vertex set is $V(\Omega_\delta) \cup \{g\}$ and whose edge set is $\{e \in E(\Omega_\delta) \cup E^*(\Omega_\delta) : \omega(e) = 1\}$. A cluster of ω is a connected component of this subgraph.

With these notations, the FK-Ising percolation measure with parameter p and external field H on $(V(\Omega_\delta) \cup g, E(\Omega_\delta) \cup E^*(\Omega_\delta))$ is defined as follows: for $\xi \in \{\text{free}, \text{wired}\}$ and ω an edge configuration,

$$\mathbb{P}^{\text{FK},\xi}_{\delta,p,H}(\omega) = \frac{1}{\mathcal{Z}_{p,H}}2^{k(\omega^\xi)}\prod_{e\in E(\Omega_\delta)}p^{\omega(e)}(1-p)^{1-\omega(e)}\prod_{e^*\in E^*(\Omega_\delta)}(1-e^{-2H})^{\omega(e^*)}(e^{-2H})^{1-\omega(e^*)}$$

where ω^ξ is the configuration obtained by identifying wired vertices and $k(\omega^\xi)$ is the number of clusters of ω^ξ not connected to g. $\mathcal{Z}_{p,H}$ is a normalizing constant chosen such that $\mathbb{P}_{\delta,p,H}^{\mathrm{FK},\xi}$ is a probability measure. Note that when $H \equiv 0$, $\mathbb{P}_{\delta,p,0}^{\mathrm{FK},\xi}$ can be seen as a measure on edge configurations in $(V(\Omega_\delta), E(\Omega_\delta))$ and it is equivalent to the measure $\mathbb{P}_{\delta,p}^\xi$ introduced in Section 5.1.2.

The Edwards-Sokal coupling with external field H goes as follows [CV16; CJN20b; CJN20a]. One first samples an edge configuration ω on $(V(\Omega_\delta) \cup g, E(\Omega_\delta) \cup E^*(\Omega_\delta))$ according to $\mathbb{P}_{\delta,p,H}^{\mathrm{FK},\xi}$ with $p = 1 - e^{-2\beta}$, where β is the inverse temperature of the Ising model. Then, all clusters of ω connected to g, and also the cluster of ω connected to $\partial\Omega_\delta$ if $\xi = \mathrm{wired}$, are assigned the spin $+1$. The spins of the remaining clusters, i.e. the clusters not connected to g, are chosen by tossing independent fair coins, one for each cluster. This gives rise to a spin configuration in Ω_δ, where each vertex has the spin of its cluster. This spin configuration has the same distribution as the Ising model in Ω_δ with inverse temperature β, external field H and boundary conditions ξ.

From now on, we will set $\beta = \beta_c$ and for $H \geq 0$ and $\xi \in \{\mathrm{free}, \mathrm{wired}\}$, we denote by $\mathbb{P}_{\delta,H}^{\mathrm{FK},\xi}$ the corresponding FK-Ising percolation measure with external field H. For the proof of Proposition 1.3.4, it will be useful to have a version of the Edwards-Sokal coupling with external field in which one first samples an edge configuration ω on $(V(\Omega_\delta), E(\Omega_\delta))$ and then, conditionally on ω, samples the set of clusters that are connected to g. This procedure does not sample an edge configuration on $(V(\Omega_\delta) \cup g, E(\Omega_\delta) \cup E^*(\Omega_\delta))$. However, given the edge configuration ω on $(V(\Omega_\delta), E(\Omega_\delta))$ and the set of clusters of ω connected to g, one can still sample the sign of the clusters in such a way that the resulting spin configuration in Ω_δ has the same distribution as a critical Ising spin configuration in Ω_δ with external field H and boundary conditions ξ.

We summarize this alternative procedure in the following proposition, which is [CJN20a, Proposition 1] and [CJN20a, Lemma 4] and appears in an implicit form in [CV16]. We state these results for free boundary conditions as these are the boundary conditions that we will use in the proof of Proposition 1.3.4. Below, for $\omega : E(\Omega_\delta) \to \{0,1\}^{|E(\Omega_\delta)|}$, we denote by $\mathcal{C}_\delta(\omega)$ the set of clusters of ω and for $\mathcal{C} \in \mathcal{C}_\delta(\omega)$, $|\mathcal{C}|$ is the number of vertices in $\mathcal{C}$. The event that a cluster $\mathcal{C} \in \mathcal{C}_\delta(\omega)$ is connected to the ghost vertex g is denoted by $\{\mathcal{C} \longleftrightarrow g\}$.

Proposition 5.3.1. *Let $\hat{\mathbb{P}}_{\delta,H}^{\mathrm{free}}$ denote the law of the Edwards-Sokal coupling with external field $H \geq 0$. For $\omega : E(\Omega_\delta) \to \{0,1\}^{|E(\Omega_\delta)|}$ and $\mathcal{C} \in \mathcal{C}_\delta(\omega)$, we let $\sigma(\mathcal{C})$ denote the spin of $\mathcal{C}$ assigned by the coupling. We then have that*

$$\mathbb{P}_{\delta,H}^{\mathrm{FK,free}}(\mathcal{C} \longleftrightarrow g | \omega) = \tanh(H|\mathcal{C}|), \quad \hat{\mathbb{P}}_{\delta,H}^{\mathrm{free}}(\sigma(\mathcal{C}) = +1 | \omega) = \frac{1}{2} + \frac{1}{2}\tanh(H|\mathcal{C}|).$$

Moreover, conditionally on ω, the events $(\{\mathcal{C} \longleftrightarrow g\})_{\mathcal{C} \in \mathcal{C}_\delta(\omega)}$ are mutually independent. Similarly, conditionally on ω, the events $(\{\sigma(\mathcal{C}) = +1\})_{\mathcal{C} \in \mathcal{C}_\delta(\omega)}$ are mutually independent. The Radon-Nikodym derivative of $\mathbb{Q}_{\delta,H}^{\mathrm{free}}$, the marginal law of $\mathbb{P}_{\delta,H}^{\mathrm{FK,free}}$ on edge configurations $\omega : E(\Omega_\delta) \to \{0,1\}^{|E(\Omega_\delta)|}$, with respect to $\mathbb{P}_\delta^{\mathrm{FK,free}}$ is given by

$$\frac{d\mathbb{Q}_{\delta,H}^{\mathrm{free}}}{d\mathbb{P}_\delta^{\mathrm{FK,free}}}(\omega) = \frac{1}{\mathcal{Z}_{\delta,H}} \prod_{\mathcal{C} \in \mathcal{C}_\delta(\omega)} \cosh(H|\mathcal{C}|)$$

where $\mathcal{Z}_{\delta,H}$ is a normalization constant.

Let us also record two stochastic domination results for the measures $\mathbb{P}_{\delta,H}^{\mathrm{FK,free}}$, $\mathbb{Q}_{\delta,H}^{\mathrm{free}}$ and $\mathbb{P}_\delta^{\mathrm{FK,free}}$. They are established in [CJN20a, Lemma 2] and [CV16, Theorem 6].

Lemma 5.3.2. *For any $H \geq 0$, $\mathbb{P}_{\delta,H}^{\mathrm{FK,free}}$ stochastically dominates $\mathbb{P}_\delta^{\mathrm{FK,free}}$. Similarly, for any $H \geq 0$, $\mathbb{Q}_{\delta,H}^{\mathrm{free}}$ stochastically dominates $\mathbb{P}_\delta^{\mathrm{FK,free}}$.*

Proof of Proposition 1.3.4

In this section, we prove Proposition 1.3.4. We give the proof in the case $h > 0$ but it could easily be adapted to handle the case of a non-constant perturbation $h : x \mapsto h(x)$, as long as h is lower bounded by a positive constant.

Proof of Proposition 1.3.4. Let $\eta > 0$. For $\delta > 0$, we set $\Omega_\delta^-(\eta) := \{x \in \Omega_\delta : d(x, \partial\Omega_\delta^-) \leq \eta\}$. We are going to show that

$$\lim_{\delta \to 0} \mathbb{P}_{\delta,h}^\pm[\gamma_\delta \cap (\Omega_\delta \setminus \Omega_\delta^-(\eta)) \neq \emptyset] = 0. \tag{5.26}$$

The first step is to turn the Dobrushin boundary conditions on $\partial\Omega_\delta$ into $+1$ boundary conditions, that are easier to handle. For this, let us start by briefly recalling the results of [Wu18a, Section 5.1]. A spin configuration in Ω_δ gives rise to an edge configuration in Ω_δ: an edge is open if its two endvertices have the same spin, closed otherwise. Let E be an event measurable with respect to the edge configuration in $\Omega_\delta \setminus \Omega_\delta^-(\eta/2)$. As E is measurable with respect to edges at macroscopic distance from $\partial\Omega_\delta^-$, one can expect that the boundary conditions on $\partial\Omega_\delta^-$ do not really influence the probability of E: this phenomena is called spatial mixing. More precisely, [Wu18a, Section 5.1] shows the following. Define $a(\eta/2)$, respectively $b(\eta/2)$, as the intersection point of $\partial B(a,\eta/2)$, respectively $\partial B(b,\eta/2)$, with $\partial\Omega^+$. If this intersection point is not unique, then choose the first one encountered when tracing $\partial\Omega^+$ starting from a, respectively b. For $\delta > 0$, let $a_\delta(\eta/2)$, respectively $b_\delta(\eta/2)$, be the discrete approximation of $a(\eta/2)$, respectively $b(\eta/2)$. Denote by $\{(b_\delta b_\delta(\eta/2)) \overset{+}{\longleftrightarrow} (a_\delta(\eta/2)a_\delta) \text{ in } \Omega_\delta^-(\eta/2)\}$ the event that the topological rectangle $\Omega_\delta^-(\eta/2)$ is crossed by a path of $+1$ spins staying in $\Omega_\delta^-(\eta/2)$ and going from the boundary arc $(b_\delta b_\delta(\eta/2))$ to the boundary arc $(a_\delta(\eta/2)a_\delta)$. Here, the boundary arcs are oriented counter-clockwise. The results of [Wu18a, Section 5.1] show that if there exists $c > 0$ and $\delta_0 > 0$ such that for any $\delta > 0$,

$$\mathbb{P}_{\delta,h}^{\pm}[(b_\delta b_\delta(\eta/2)) \overset{+}{\longleftrightarrow} (a_\delta(\eta/2)a_\delta) \text{ in } \Omega_\delta^-(\eta/2)] \geq c, \tag{5.27}$$

then there exists $C > 0$ such that for any $0 < \delta < \delta_0$, $\mathbb{P}_{\delta,h}^{\pm}[E] \leq C\mathbb{P}_{\delta,h}^+[E]$.

Below, we will apply this result with the event $E = \mathcal{C}^{\pm}(\eta/2,\eta)$, where $\mathcal{C}^{\pm}(\eta/2,\eta)$ is defined as follows. Define $a(\eta)$, $b(\eta)$, $a_\delta(\eta)$ and $b_\delta(\eta)$ as $a(\eta/2)$, $b(\eta/2)$, $a_\delta(\eta/2)$ and $b_\delta(\eta/2)$ but with $\eta/2$ replaced by η. Consider the topological rectangle $(R_\delta; a_\delta(\eta/2), b_\delta(\eta/2), b_\delta(\eta), b_\delta(\eta))$, where the four marked points are in counterclockwise order and $R_\delta := \Omega_\delta^-(\eta) \setminus \Omega_\delta^-(\eta/2)$. With these notations, the event $\mathcal{C}^{\pm}(\eta/2,\eta)$ is the event that there exists a path $\tilde{\gamma}$ on the dual of Ω_δ going from the boundary arc $(a_\delta(\eta/2)b_\delta(\eta/2))$ to the boundary arc $(a_\delta(\eta)b_\delta(\eta))$ such the following is fulfilled: if $x,y \in \Omega_\delta \cup \partial\Omega_\delta$ are adjacent to $\tilde{\gamma}$, then $\sigma_x = \sigma_y$ if x and y are on the same side of $\tilde{\gamma}$, and $\sigma_x = -\sigma_y$ if x and y are on different sides of $\tilde{\gamma}$. Here, we say that a vertex is adjacent to $\tilde{\gamma}$ if it is the endpoint of a primal edge crossed by $\tilde{\gamma}$. Moreover, as before, the boundary arcs $(a_\delta(\eta/2)b_\delta(\eta/2))$ and $(a_\delta(\eta)b_\delta(\eta))$ are oriented counter-clockwise. Observe now that the event $\mathcal{C}^{\pm}(\eta/2,\eta)$ is measurable with respect to the edge configuration in $\Omega_\delta \setminus \Omega_\delta^-(\eta/2)$ and that

$$\mathbb{P}_{\delta,h}^{\pm}[\gamma_\delta \cap (\Omega_\delta \setminus \Omega_\delta^-(\eta)) \neq \emptyset] \leq \mathbb{P}_{\delta,h}^{\pm}[\mathcal{C}^{\pm}(\eta/2,\eta)]. \tag{5.28}$$

As explained above, to change the boundary conditions and show that there exist $C > 0$ and $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$,

$$\mathbb{P}_{\delta,h}^{\pm}[\mathcal{C}^{\pm}(\eta/2,\eta)] \leq C\mathbb{P}_{\delta,h}^+[\mathcal{C}^{\pm}(\eta/2,\eta)], \tag{5.29}$$

we must show that the inequality (5.27) is satisfied. Here, we recall that under $\mathbb{P}_{\delta,h}^+$, the boundary conditions of the Ising model in Ω_δ are $+1$. Using the Markov property of the Ising model and the fact that $\{(b_\delta b_\delta(\eta/2)) \overset{+}{\longleftrightarrow} (a_\delta(\eta/2)a_\delta) \text{ in } \Omega_\delta^-(\eta/2)\}$ is an increasing event, we have that

$$\mathbb{P}_{\delta,h}^{\pm}[(b_\delta b_\delta(\eta/2)) \overset{+}{\longleftrightarrow} (a_\delta(\eta/2)a_\delta) \text{ in } \Omega_\delta^-(\eta/2)] \geq \mathbb{P}_{\Omega_\delta^-(\eta/2),h}^{\pm}[(b_\delta b_\delta(\eta/2)) \overset{+}{\longleftrightarrow} (a_\delta(\eta/2)a_\delta)]$$

$$\geq \mathbb{P}_{\Omega_\delta^-(\eta/2)}^{\pm}[(b_\delta b_\delta(\eta/2)) \overset{+}{\longleftrightarrow} (a_\delta(\eta/2)a_\delta)]$$

where $\mathbb{P}_{\Omega_\delta^-(\eta/2)}^{\pm}$ is the probability measure on Ising spin configurations in $\Omega_\delta^-(\eta/2)$ with -1 boundary conditions on $\partial\Omega_\delta^-$ and the boundary arc $(b_\delta(\eta/2)a_\delta(\eta/2))$ and $+1$ boundary conditions on the boundary arcs $(a_\delta(\eta/2)a_\delta)$ and $(b_\delta b_\delta(\eta/2))$. By [CDH16, Corollary 1.7], there exist $c = c(\eta) > 0$ and $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$,

$$\mathbb{P}_{\Omega_\delta(\eta/2)}^{\pm}[(b_\delta b_\delta(\eta/2)) \overset{+}{\longleftrightarrow} (a_\delta(\eta/2)a_\delta)] \geq c,$$

which yields (5.29).

With the inequality (5.29) at hands, the next step of the proof is to replace $\mathcal{C}^{\pm}(\eta/2,\eta)$ by an event whose probability is easier to estimate. To this end, define $\{(a_\delta(\eta)a_\delta(\eta/2)) \overset{+}{\longleftrightarrow} (b_\delta(\eta/2)b_\delta(\eta)) \text{ in } R_\delta\}$ as the event that there exits a crossing of R_δ by a path of $+1$ spins going

from the boundary arc $(a_\delta(\eta)a_\delta(\eta/2))$ to the boundary arc $(b_\delta(\eta/2)b_\delta(\eta))$ and staying in R_δ. Observe that if $\{(a_\delta(\eta)a_\delta(\eta/2)) \overset{+}{\longleftrightarrow} (b_\delta(\eta/2)b_\delta(\eta)) \text{ in } R_\delta\}$ happens, then $\mathcal{C}^\pm(\eta/2, \eta)$ cannot happen. Therefore,

$$\mathbb{P}_{\delta,h}^+[\mathcal{C}^\pm(\eta/2,\eta)] \le 1 - \mathbb{P}_{\delta,h}^+\big[(a_\delta(\eta)a_\delta(\eta/2)) \overset{+}{\longleftrightarrow} (b_\delta(\eta/2)b_\delta(\eta)) \text{ in } R_\delta\big]. \tag{5.30}$$

We are now going to show that the lim inf as $\delta \to 0$ of the probability on the right-hand side above is equal to 1. For this, we will use the Edwards-Sokal coupling for the Ising model with an external field, see Section 5.3.2. For $K \in \mathbb{N}$, we let $\mathcal{C}_\delta(K)$ be the event that there exists a sequence $\mathcal{C}_1, \ldots, \mathcal{C}_k$ of open clusters in Ω_δ such that $k \le K$, $|\mathcal{C}_i| \ge \delta^{-15/8}/K$ for each i and one can find a path $\tilde{\gamma}$ staying in R_δ that joins $(a_\delta(\eta)a_\delta(\eta/2))$ to $(b_\delta(\eta/2)b_\delta(\eta))$ such that the edges traversed by $\tilde{\gamma}$ are either open edges of some $\mathcal{C}_i$ or closed edges of the form (xy) with $x \in \mathcal{C}_i$, $y \in \mathcal{C}_j$ and $i \ne j$. Here, we stress that connections are consider in $(V(\Omega_\delta) \cup \partial\Omega_\delta, E(\Omega_\delta))$. We further define the event $\mathcal{C}_\delta(g; K)$ as $\mathcal{C}_\delta(K)$ but we additionally require that all clusters $(\mathcal{C}_i)_i$ visited by the path $\tilde{\gamma}$ are connected to the ghost vertex g.

In the Edwards-Sokal coupling with an external field, on the event $\mathcal{C}_\delta(g; K)$, the clusters visited by $\tilde{\gamma}$ all have spin $+1$. Thus, we have that $\mathcal{C}_\delta(g; K) \subset \{(a_\delta(\eta)a_\delta(\eta/2)) \overset{+}{\longleftrightarrow} (b_\delta(\eta/2)b_\delta(\eta)) \text{ in } R_\delta\}$. We note that on the event $\mathcal{C}_\delta(g; K)$, no restriction is imposed on the edges of $\tilde{\gamma}$ joining the boundary arcs $(a_\delta(\eta)a_\delta(\eta/2))$ and $(b_\delta(\eta/2)b_\delta(\eta))$ to some vertex in some $\mathcal{C}_i$: these edges may be open or closed. This is because under $\mathbb{P}_{\delta,h}^+$, the Ising model has $+1$ boundary conditions and therefore, the spins on $(a_\delta(\eta)a_\delta(\eta/2))$ and $(b_\delta(\eta/2)b_\delta(\eta))$ are $+1$.

Since $\mathcal{C}_\delta(g; K)$ is an increasing event, by Proposition 5.3.1, we obtain that

$$\mathbb{P}_{\delta,h}^+\big[(a_\delta(\eta)a_\delta(\eta/2)) \overset{+}{\longleftrightarrow} (b_\delta(\eta/2)b_\delta(\eta)) \text{ in } R_\delta\big] \ge \hat{\mathbb{P}}_{\delta,H}^{\text{wired}}[\mathcal{C}_\delta(g; K)]$$
$$\ge \hat{\mathbb{P}}_{\delta,H}^{\text{free}}[\mathcal{C}_\delta(g; K)] = \hat{\mathbb{P}}_{\delta,H}^{\text{free}}[\mathcal{C}_\delta(g; K)|\mathcal{C}_\delta(K)]\hat{\mathbb{P}}_{\delta,H}^{\text{free}}[\mathcal{C}_\delta(K)]. \tag{5.31}$$

Using that $\tanh(x) \le 1$ and $x \mapsto \tanh(x)$ is increasing, Proposition 5.3.1 yields that

$$\hat{\mathbb{P}}_{\delta,H}^{\text{free}}[]\mathcal{C}_\delta(g; K)|\mathcal{C}_\delta(K)] = \hat{\mathbb{E}}_{\delta,H}^{\text{free}}\big[\prod_{\mathcal{C}_i \cap \tilde{\gamma} \ne \emptyset} \tanh(H(\delta)|\mathcal{C}_i|)|\mathcal{C}_\delta(K)\big] \ge \left[\tanh\left(H(\delta)\frac{\delta^{-\frac{15}{8}}}{K}\right)\right]^K$$
$$= \left[\tanh\left(\frac{g_2(\delta)}{K}\right)\right]^K. \tag{5.32}$$

On the other hand, by stochastic domination, see Lemma 5.3.2, $\mathbb{P}_{\delta,H}^{\text{free}}[\mathcal{C}_\delta(K)] \ge \mathbb{P}_\delta^{\text{FK,free}}[\mathcal{C}_\delta(K)]$. Since $\mathcal{C}_\delta(K)$ is an increasing event, we then arrive at

$$\mathbb{P}_{\delta,H}^{\text{free}}[\mathcal{C}_\delta(K)] \ge \mathbb{P}_{R_\delta}^{\text{FK,free}}[\mathcal{C}_\delta(K)] \tag{5.33}$$

where $\mathbb{P}_{R_\delta}^{\text{FK,free}}$ is the probability measure on FK-Ising configurations in R_δ with free boundary conditions. We now claim the following.

Claim 5.3.3. *Let $\epsilon > 0$. There exists $K = K(\epsilon) < \infty$ such that $\liminf_{\delta \to 0} \mathbb{P}_{R_\delta}^{\text{FK,free}}[\mathcal{C}_\delta(K)] > 1 - \epsilon$.*

We postpone the proof of Claim 5.3.3 to the end and first explain how to conclude the proof of the proposition. Let $\epsilon > 0$. Claim 5.3.3 together with the inequalities (5.31), (5.32) and (5.33) yield that there exists $K > 0$ such that

$$\liminf_{\delta \to 0} \mathbb{P}_{\delta,h}^+\big[(a_\delta(\eta)a_\delta(\eta/2)) \overset{+}{\longleftrightarrow} (b_\delta(\eta/2)b_\delta(\eta)) \text{ in } R_\delta\big] > (1 - \epsilon)\liminf_{\delta \to 0}\left[\tanh\left(\frac{g_2(\delta)}{K}\right)\right]^K.$$

Since $g_2(\delta) \to \infty$ as $\delta \to 0$, the lim inf on the right-hand side above is equal to 1. By choosing $\epsilon > 0$ small enough, this implies that for any $\rho > 0$, there exists $0 < K < \infty$ such that $\liminf_{\delta \to 0} \mathbb{P}_{\delta,h}^+\big[(a_\delta(\eta)a_\delta(\eta/2)) \overset{+}{\longleftrightarrow} (b_\delta(\eta/2)b_\delta(\eta))\big] > 1 - \rho$. Going back to (5.29) and (5.30), this shows that for any $\rho > 0$,

$$\limsup_{\delta \to 0} \mathbb{P}_{\delta,h}^\pm[\mathcal{C}^\pm(\eta/2, \eta)] < C\rho,$$

which, by (5.28), establishes (5.26) and concludes the proof. $\qquad\square$

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Let us now turn to the proof of Claim 5.3.3.

Proof of Claim 5.3.3. The proof is similar to the proofs of [CJN20a, Lemma 5] and [CJN20a, Lemma 7]. Indeed, under $\mathbb{P}_{R_\delta}^{\text{FK,free}}$, the event $\mathcal{C}_\delta(K)$ is the same as the event $E^a(K, N)$ with $a = \delta$ and $N = 1$ of [CJN20a, Lemma 7]. This is because under $\mathbb{P}_{R_\delta}^{\text{FK,free}}$, on the event $\mathcal{C}_\delta(g; K)$, the path $\tilde{\gamma}$ automatically stays in R_δ. The proofs of [CJN20a, Lemma 5] and [CJN20a, Lemma 7] are given when the domain is a rectangle of a fixed aspect ratio. However, one can observe that it is easy to extend the arguments to the case of an arbitrary bounded and simply connected domain D and its discrete approximation D_δ, as long as one considers boundary arcs of D that have positive harmonic measure seen from a point in the interior of the domain. The only restriction that must be imposed is that ∂D has Minkowski dimension 1: if ∂D is too rough, one may not be able to rule out that an arm event of type 010 takes place near ∂D_δ in the limit $\delta \to 0$. This point is crucial to deduce results for $\mathbb{P}_{R_\delta}^{\text{FK,free}}(\mathcal{C}_\delta(K))$ from results obtained for $\text{CLE}_{16/3}$ and $\text{CME}_{16/3}$ in the continuum. We leave the details to the reader. □

5.3.3 On the $h \to \infty$ limit of $\mathbb{P}_h^{(\Omega,a,b)}$: proof of Proposition 1.3.5

Proof of Proposition 1.3.5. For $h \in \mathbb{R}$ and $\eta > 0$, set $p(h, \eta) := \mathbb{P}_h^{(\Omega,a,b)}[\gamma \cap (\Omega \setminus \Omega^-(\eta)) \neq \emptyset]$. To prove the proposition, we argue by contradiction. Assume that there exists $\eta > 0$ such that $p(h, \eta)$ does not converge to 0 as $h \to \infty$. Then, there exist $\epsilon > 0$ and a sequence $(h_n)_n$ such that $\lim_{n\to\infty} h_n = \infty$ and for any $n \geq 1$, $p(h_n, \eta) \geq \epsilon$. Moreover, with $H(n, \delta) = C_\sigma^{-1} h_n \delta^{\frac{15}{8}}$, Theorem 1.3.1, when considering convergence in the topology (T.4), implies that for any $n \geq 1$,

$$\lim_{\delta \to 0} \mathbb{P}_{\delta, H(n,\delta)}^{\pm}[\gamma_\delta \cap (\Omega_\delta \setminus \Omega_\delta^-(\eta)) \neq \emptyset] = p(h_n, \eta) \geq \epsilon \tag{5.34}$$

where we have set $\Omega_\delta^-(\eta) := \{x \in \Omega_\delta : \text{dist}(x, \partial\Omega_\delta) \leq \eta\}$. Now, let $(\delta_n)_n$ be a sequence converging to 0 such that for any $n \geq 1$,

$$\mathbb{P}_{\delta_n, H(n,\delta_n)}^{\pm}[\gamma_{\delta_n} \cap (\Omega_{\delta_n} \setminus \Omega_{\delta_n}^-(\eta)) \neq \emptyset] \geq \epsilon. \tag{5.35}$$

Note that such a sequence $(\delta_n)_n$ exists by (5.34). As h_n tends to ∞, we can write $h_n = g(\delta_n)$ with $g(\delta_n) \to \infty$ as $n \to \infty$. Therefore, by Proposition 1.3.4, we obtain that

$$\lim_{n \to \infty} \mathbb{P}_{n, g(\delta_n)}^{\pm}[\gamma_{\delta_n} \cap (\Omega_{\delta_n} \setminus \Omega_{\delta_n}^-(\eta)) \neq \emptyset] = 0.$$

This contradicts (5.35) and completes the proof of Proposition 1.3.5. □

Chapter 6

Appendix

In this section, $D \subset \mathbb{C}$ is assumed to be a bounded, open and simply connected domain.

6.1 Proof of Lemma 4.1.1 and Lemma 4.1.6

Below, we will repeatedly make use of the following estimate. Let h be a GFF in D with Dirichlet boundary conditions. Then, by [Ber17, Lemma 3.5], there exists $C > 0$ such that for any $0 < \epsilon < 1/2$,

$$\sup_{z \in D} \mathbb{E}[h_\epsilon(z)^2] \leq C \log(\epsilon^{-1}). \tag{6.1}$$

Let us now turn to the proof of Lemma 4.1.1. To prove this result, we first show the following auxiliary lemma.

Lemma 6.1.1. *Let h be a GFF in D with Dirichlet boundary conditions. Let $0 < \delta < \epsilon < 1/2$. Then, for any $z, w \in D$ such that $\max(\operatorname{dist}(z, \partial D), \operatorname{dist}(w, \partial D)) > \epsilon$ and $|z - w| > 2\epsilon$,*

$$\mathbb{E}[(: h_\epsilon(z)^2 : - : h_\delta(z)^2 :)(: h_\epsilon(w)^2 : - : h_\delta(w)^2 :)] = 0. \tag{6.2}$$

Moreover, there exists $K > 0$ such that for any $0 < \delta < \epsilon < 1/2$ and any $z, w \in D$,

$$\mathbb{E}[(: h_\epsilon(z)^2 : - : h_\delta(z)^2 :)(: h_\epsilon(w)^2 : - : h_\delta(w)^2 :)] \leq K(\log \delta)^2. \tag{6.3}$$

Proof. Let us first prove (6.2). Let $0 < \delta < \epsilon < 1/2$ and let $z, w \in D$ be such that $|z - w| > 2\epsilon$ and assume without loss of generality that $\operatorname{dist}(z, \partial D) > \epsilon$. Applying the Markov property in the ball $B(z, 2\epsilon)$ and using that $(W_r)_{r \geq 0} := (h_{2\epsilon e^{-r}}(z) - h_{2\epsilon}(z))_{r \geq 0}$ is a Brownian motion, we see that for some centered random variables $X, \tilde{X}$ (not independent but independent of W),

$$(: h_\epsilon(z)^2 : - : h_\delta(z)^2 :)(: h_\epsilon(w)^2 : - : h_\delta(w)^2 :) = ((W_t - t^2) - (W_s - s^2) + 2X(W_t - W_s))\tilde{X}.$$

Taking expectation and using independence, we get zero, which completes the proof of (6.2).

Let us now prove the estimate (6.3). Let $0 < \delta < \epsilon < 1/2$ and let $z, w \in D$. Observe first that

$$\mathbb{E}[(: h_\epsilon(z)^2 : - : h_\delta(z)^2 :)^2] \leq 2\mathbb{E}[(: h_\epsilon(z)^2 :)^2] + 2\mathbb{E}[(: h_\delta(z)^2 :)^2]. \tag{6.4}$$

Moreover, since $h_\epsilon(z)$ is a Gaussian random variable, we have that $\mathbb{E}[(: h_\epsilon(z)^2 :)^2] = \mathbb{E}[h_\epsilon(z)^4] - 2\mathbb{E}[h_\epsilon(z)^2]^2 + \mathbb{E}[h_\epsilon(z)^2]^2 = 2\mathbb{E}[h_\epsilon(z)^2]^2$. Going back to (6.4), we deduce from this and the estimate (6.1) that $\mathbb{E}[(: h_\epsilon(z)^2 : - : h_\delta(z)^2 :)] \leq 4C^2 \log(\delta)^2$. The estimate (6.3) then simply follows from this bound and the Cauchy-Schwarz inequality. $\square$

With Lemma 6.1.1 at hands, let us now prove Lemma 4.1.1. We first deal with the convergence in $L^2(\mathbb{P})$ of $\int_D m^2(z) : h_\epsilon(z)^2 : dz$, that is with the case $\phi \equiv 0$.

Lemma 6.1.2. *Let $m : D \to \mathbb{R}_+$ be a bounded and continuous function. Let h be a GFF with Dirichlet boundary conditions in D. Then, as $\epsilon \to 0$, the random variable $\int_D m^2(z) : h_\epsilon(z)^2 : dz$ converges in $\mathrm{L}^2(\mathbb{P})$ to a limit denoted by $\int_D m^2(z) : h(z)^2 : dz$. Moreover, for any $b \in (0, 2 - \dim_H(\partial D))$, there exists $C > 0$ such that for any $0 < \epsilon < 1/2$,*

$$\left\| \int_D m^2(z) : h(z)^2 : dz - \int_D m^2(z) : h_\epsilon(z)^2 : dz \right\|_{\mathrm{L}^2(\mathbb{P})} \leq C\epsilon^b. \tag{6.5}$$

Proof of Lemma 6.1.2. For conciseness, let us set $\alpha := 2 - \dim_H(\partial D)$. We are going to show that the sequence $\left(\int_D m^2(z) : h_\epsilon(z)^2 : dz, \epsilon > 0\right)$ is a Cauchy sequence in $L^2(\mathbb{P})$ and thus has a limit in $L^2(\mathbb{P})$ as $\epsilon \to 0$. To this end, let $0 < \delta < \epsilon < 1/2$. Then, by Fubini-Tonelli theorem, we have that

$$\mathbb{E}\left[\left|\int_D m^2(z) : h_\epsilon(z)^2 : dz - \int_D m^2(z) : h_\delta(z)^2 : dz\right|^2\right]$$
$$= \int_{D \times D} m^2(z) m^2(w) \mathbb{E}[(: h_\epsilon(z)^2 : - : h_\delta(z)^2 :)(: h_\epsilon(w)^2 : - : h_\delta(w)^2 :)] dw dz. \qquad (6.6)$$

Indeed, when we expand the product of the difference of the Wick squares, we obtain a sum of non-negative terms and we can apply Fubini-Tonelli theorem to each of these terms separately. With $D_\epsilon := \{z \in D : \text{dist}(z, \partial D) \geq \epsilon\}$, we decompose the region of integration into $D \times D_\epsilon$, $D_\epsilon \times D$ and $(D \setminus D_\epsilon) \times (D_\epsilon \setminus D_\epsilon)$. For the integral over $D \times D_\epsilon$, Lemma 6.1.1 shows that only the points $z, w \in D \times D_\epsilon$ such that $|z - w| \leq 2\epsilon$ contribute. Using the estimate (6.3) of Lemma 6.1.1, this yields that

$$\int_D m^2(z) \int_{B(z, 2\epsilon) \cap D_\epsilon} m^2(w) \mathbb{E}[(: h_\epsilon(z)^2 : - : h_\delta(z)^2 :)(: h_\epsilon(w)^2 : - : h_\delta(w)^2 :)] dw dz$$
$$\leq \int_D m^2(z) \int_{B(z, 2\epsilon) \cap D_\epsilon} m^2(w) K (\log \delta)^2 dw dz \leq O(\epsilon^2 (\log \delta)^2) \qquad (6.7)$$

where the implied constant depends only on D and m^2. The integral over $D_\epsilon \times D$ can be bounded in a similar way. We are now left to control the integral over $(D \setminus D_\epsilon) \times (D \setminus D_\epsilon)$. To do so, we can simply use the estimate (6.3) of Lemma 6.1.1 to obtain the bound

$$\int_{(D \setminus D_\epsilon) \times (D \setminus D_\epsilon)} m^2(z) m^2(w) \mathbb{E}[(: h_\epsilon(z)^2 : - : h_\delta(z)^2 :)(: h_\epsilon(w)^2 : - : h_\delta(w)^2 :)] dw dz \leq O(\epsilon^{2\alpha} \log(\delta)^2).$$

Going back to (6.6), since $\alpha \in (0, 1]$, this bound together with the bound (6.7) imply that

$$\mathbb{E}\left[\left|\int_D m^2(z) : h_\epsilon(z)^2 : dz - \int_D m^2(z) : h_\delta(z)^2 : dz\right|^2\right] \leq O(\epsilon^{2\alpha} (\log \delta)^2). \qquad (6.8)$$

This shows that $(\int_D m^2(z) : h_\epsilon(z)^2 : dz, \epsilon > 0)$ is a Cauchy sequence in $L^2(\mathbb{P})$ and therefore has a limit in $L^2(\mathbb{P})$, that we denote by $\int_D m^2(z) : h(z)^2 : dz$.

It now remains to prove the claim about the rate of convergence of $\int_D m^2(z) : h_\epsilon(z)^2 : dz$ to $\int_D m^2(z) : h(z)^2 : dz$. Let $0 < \epsilon < 1/2$ and let $k \in \mathbb{N}$ be such that $\epsilon \in [2^{-(k+1)}, 2^{-k})$. For $n \in \mathbb{N}$, set $a_n = 2^{-n}$. Using the triangle inequality and the inequality (6.8), we have that, for some constant $C > 0$,

$$\left\|\int_D m^2(z) : h(z)^2 : dz - \int_D m^2(z) : h_\epsilon(z)^2 : dz\right\|_{L^2(\mathbb{P})}$$
$$\leq \left\|\int_D m^2(z) : h_{a_{k+1}}(z)^2 : dz - \int_D m^2(z) : h_\epsilon(z)^2 : dz\right\|_{L^2(\mathbb{P})}$$
$$+ \sum_{n=k+1}^{\infty} \left\|\int_D m^2(z) : h_{a_{n+1}}(z)^2 : dz - \int_D m^2(z) : h_{a_n}(z)^2 : dz\right\|_{L^2(\mathbb{P})}$$
$$\leq C 2^{-k\alpha}(k+1) + C \sum_{n=k+1}^{\infty} (a_n^{2\alpha}(\log(a_{n+1})^2))^{1/2}.$$

Let $\eta \in (0, \alpha)$. Then, we have that $\sum_{n=k}^{\infty} 2^{-n\alpha}(n+1) \leq 2^{(\alpha-\eta)} \epsilon^{(\alpha-\eta)} \sum_{n=0}^{\infty} 2^{-n\eta}(n+1)$, where the last sum is finite. This concludes the proof of (6.5) and thus of the lemma. $\qquad \square$

Let us now turn to the proof of Lemma 4.1.1.

Proof of Lemma 4.1.1. For conciseness, let us set $\alpha := 2 - \dim_H(\partial D) \in (0, 1]$. Observe first that, almost surely, for any $0 < \epsilon < 1/2$,

$$\int_D m^2(z) : (h_\epsilon(z) + \phi_\epsilon(z))^2 : dz = \int_D m^2(z) : h_\epsilon(z)^2 : dz + \int_D 2m^2(z) h_\epsilon(z) \phi_\epsilon(z) dz + \int_D m^2(z) \phi_\epsilon(z)^2 dz.$$

Therefore, by Lemma 6.1.2, it suffices to prove that

$$\lim_{\epsilon \to 0} \int_D m^2(z) h_\epsilon(z) \phi_\epsilon(z) dz = (h, m^2 \phi) \quad \text{in } L^2(\mathbb{P}) \tag{6.9}$$

$$\lim_{\epsilon \to 0} \int_D m^2(z) \phi_\epsilon(z)^2 dz = \int_D m^2(z) \phi(z)^2 dz. \tag{6.10}$$

Let us start with (6.10). By harmonicity of ϕ and our assumptions on ϕ and α, we have that, for any $b \in (0, \alpha]$,

$$\left| \int_D m^2(z) \phi(z)^2 dz - \int_D m^2(z) \phi_\epsilon(z)^2 dz \right| = \left| \int_{D \setminus D_\epsilon} m^2(z)(\phi(z)^2 - \phi_\epsilon(z)^2) dz \right|$$

$$\leq 2 b_\phi^2 \int_{D \setminus D_\epsilon} m^2(z) dz \leq O(\epsilon^b) \tag{6.11}$$

from which (6.10) follows. Let us now turn to (6.9). We are going to show that, for any $b \in (0, \alpha)$, there exists $C_1 > 0$ such that for any $0 < \epsilon < 1/2$,

$$\left\| 2(h, m^2 \phi) - 2 \int_D m^2(z) h_\epsilon(z) \phi_\epsilon(z) dz \right\|_{L^2(\mathbb{P})} \leq C_1 \epsilon^b. \tag{6.12}$$

Let $0 < \epsilon < 1/2$ and let $k \in \mathbb{N}$ be such that $\epsilon \in [2^{-(k+1)}, 2^{-k})$. As before, for $n \in \mathbb{N}$, set $a_n = 2^{-n}$. By the triangular inequality, we have that

$$\left\| 2(h, m^2 \phi) - 2 \int_D m^2(z) h_\epsilon(z) \phi_\epsilon(z) dz \right\|_{L^2(\mathbb{P})}$$

$$\leq 2 \left\| \int_D m^2(z) h_{a_{k+1}}(z) \phi_{a_{k+1}}(z) dz - \int_D m^2(z) h_\epsilon(z) \phi_\epsilon(z) dz \right\|_{L^2(\mathbb{P})}$$

$$+ 2 \sum_{n=k+1}^{\infty} \left\| \int_D m^2(z) h_{a_{n+1}}(z) \phi_{a_{n+1}}(z) dz - \int_D m^2(z) h_{a_n}(z) \phi_{a_n}(z) dz \right\|_{L^2(\mathbb{P})}. \tag{6.13}$$

To show (6.12), we are going to upper bound each term in the above sum. To this end, let $n \geq k$. We have that

$$\left\| \int_D m^2(z) h_{a_{n+1}}(z) \phi_{a_{n+1}}(z) dz - \int_D m^2(z) h_{a_n}(z) \phi_{a_n}(z) dz \right\|_{L^2(\mathbb{P})}^2$$

$$= \mathbb{E} \left[\int_{D \times D} m^2(z) m^2(w) (h_{a_{n+1}}(z) \phi_{a_{n+1}}(z) - h_{a_n}(z) \phi_{a_n}(z)) \right.$$

$$\left. \times (h_{a_{n+1}}(w) \phi_{a_{n+1}}(w) - h_{a_n}(w) \phi_{a_n}(w)) dw dz \right]$$

Set $D_n := \{ z \in D : \text{dist}(z, \partial D) > a_n \}$ and observe that, by harmonicity of ϕ, for any $z \in D_n$, $\phi_{a_n}(z) = \phi(z) = \phi_{a_{n+1}}(z)$. We decompose the region of integration into $D_n \times D$, $D \times D_n$ and $(D \setminus D_n) \times (D \setminus D_n)$. To bound the integral over $D_n \times D$, if $(z, w) \in D_n \times D$ are such that $|z - w| > 2a_n$, then we can use the Markov property in $B(z, 2a_n)$ and the harmonicity of ϕ to obtain that

$$\mathbb{E}[(h_{a_{n+1}}(z) \phi_{a_{n+1}}(z) - h_{a_n}(z) \phi_{a_n}(z))(h_{a_{n+1}}(w) \phi_{a_{n+1}}(w) - h_{a_n}(w) \phi_{a_n}(w))]$$

$$= \phi(z) \mathbb{E}[(h_{a_{n+1}}(z) - h_{a_n}(z))(h_{a_{n+1}}(w) \phi_{a_{n+1}}(w) - h_{a_n}(w) \phi_{a_n}(w))] = 0.$$

Therefore, only points z, w such that $|z - w| \leq 2a_n$ contribute to the integral over $D_n \times D$. Note also that, by harmonicity, for any $n \in \mathbb{N}$ and any $z \in D$, $|\phi_{a_n}(z)| \leq b_\phi$. Therefore, using the estimate (6.1), Cauchy-Schwarz inequality and the boundedness of m, we obtain that

$$\int_{D_n \times D} m^2(z) m^2(w) \phi(z) \mathbb{E}[(h_{a_{n+1}}(z) - h_{a_n}(z))(h_{a_{n+1}}(w) \phi_{a_{n+1}}(w) - h_{a_n}(w) \phi_{a_n}(w))] dw dz$$

$$\leq O(a_n^2 \log a_{n+1}^{-1}).$$

The integral over $D \times D_n$ can be treated similarly. To control the integral over $(D \setminus D_n) \times (D \setminus D_n)$, we simply use the estimate (6.1), Cauchy-Schwarz inequality, the uniform boundedness of ϕ_{a_n} in n and z and the boundedness of m and D to get that

$$\int_{(D\setminus D_n) \times (D\setminus D_n)} m^2(z) m^2(w) \mathbb{E}[(h_{a_{n+1}}(z)\phi_{a_{n+1}}(z) - h_{a_n}(z)\phi_{\epsilon_p}(z))$$
$$\times (h_{a_{n+1}}(w)\phi_{a_{n+1}}(w) - h_{a_n}(w)\phi_{a_n}(w))]dwdz$$
$$\leq O(a_n^{2\alpha} \log(a_{n+1}^{-1})).$$

Going back to (6.13), we can deduce from the above bounds that for some constant $C > 0$,

$$\left\| 2(h, m^2\phi) - 2 \int_D m^2(z) h_\epsilon(z) \phi_\epsilon(z) dz \right\|_{L^2(\mathbb{P})} \leq 2C 2^{-k\alpha} \sqrt{k+1} + 2C \sum_{n=k+1}^{\infty} a_n^{\alpha} \sqrt{(\log(a_{n+1}^{-1}))}.$$

For the same reasons as those explained in the proof of Lemma 6.1.2, this bound yields the inequality (6.12). Moreover, by the triangular inequality, the statement of the lemma about the rate of convergence in $L^2(\mathbb{P})$ follows from Lemma 6.1.2, (6.12) and (6.11). □

As a corollary, we obtain an explicit expression for the variance of the Wick square of the GFF.

Lemma 6.1.3. *Under the same assumptions as in Lemma 6.1.2,*

$$\mathbb{E}\left[\left(\int_D m^2(z) : h(z)^2 : dz \right)^2 \right] = 2 \int_{D \times D} m^2(z) m^2(w) G_D(z,w)^2 dwdz.$$

Proof. From the $L^2(\mathbb{P})$ convergence result established in Lemma 6.1.2, we have that

$$\mathbb{E}\left[\left(\int_D m^2(z) : h(z)^2 : dz \right)^2 \right] = \lim_{\epsilon \to 0} \mathbb{E}\left[\left(\int_D m^2(z) : h_\epsilon(z)^2 : dz \right)^2 \right]$$
$$= \lim_{\epsilon \to 0} \int_{D \times D} m^2(z) m^2(w) \mathbb{E}[: h_\epsilon(z)^2 :: h_\epsilon(w)^2 :] dwdz$$

where in the last equality, the application of Fubini theorem can be justified as in the proof of Lemma 6.1.2. By [Sim74, Theorem I.3], for $z \in D$, $\mathbb{E}[: h_\epsilon(z)^2 :: h_\epsilon(w)^2 :] = 2\mathbb{E}[h_\epsilon(z)h_\epsilon(w)]^2$. This yields that

$$\mathbb{E}\left[\left(\int_D m^2(z) : h(z)^2 : dz \right)^2 \right] = \lim_{\epsilon \to 0} \int_{D \times D} m^2(z) m^2(w) 2\mathbb{E}[h_\epsilon(z)h_\epsilon(w)]^2 dwdz.$$

Moreover, it is easy to see that the integral on the above right-hand side converges to

$$\int m^2(z) m^2(w) 2 G_D(z,w)^2 dwdz,$$

which completes the proof the lemma. □

Let us now establish Lemma 4.1.6. The idea of proof is similar to [Nel73a; Nel73b], see also [Sim74, Chapter V].

Proof of Lemma 4.1.6. Fix $p \in [1, \infty)$. Observe first that

$$\lim_{\epsilon \to 0} \exp\left(-\frac{p}{2} \int_D m^2(z) : (h_\epsilon(z) + \phi_\epsilon(z))^2 : dz \right) = \exp\left(-\frac{p}{2} \int_D m^2(z) : (h + \phi)^2(z) : dz \right) \quad (6.14)$$

in probability. Indeed, applying Lemma 4.1.1 with the mass $z \mapsto (p/2)m^2(z)$, we obtain that $\frac{p}{2} \int_D m^2(z) : (h_\epsilon(z) + \phi_\epsilon(z))^2 : dz$ converges in $L^2(\mathbb{P})$ to $\frac{p}{2} \int_D m^2(z) : (h + \phi)^2(z) : dz$ as $\epsilon \to 0$. (6.14) then follows from the continuity of $x \mapsto e^{-x}$. To prove the lemma, it is thus enough to show that there exists $M > 0$ such that for any $0 < \epsilon < 1/2$,

$$\mathbb{E}\left[\left(\exp\left(-\frac{p}{2} \int_D m^2(z) : (h_\epsilon(z) + \phi_\epsilon(z))^2 : dz \right) \right)^2 \right] \leq M. \quad (6.15)$$

Indeed, (6.15) implies that $\left(\exp\left(-\frac{p}{2} \int_D m^2(z) : (h_\epsilon(z) + \phi_\epsilon(z))^2 : dz \right), 0 < \epsilon < 1/2 \right)$ is uniformly integrable and therefore that it converges in $L^1(\mathbb{P})$ to its limit in probability, which is given by (6.14). Noting that

$$\mathbb{E}\left[\exp\left(-\frac{p}{2} \int_D m^2(z) : (h_\epsilon(z) + \phi_\epsilon(z))^2 : dz \right) \right] = \mathbb{E}\left[\left(\exp\left(-\frac{1}{2} \int_D m^2(z) (h_\epsilon(z) + \phi_\epsilon(z))^2 : dz \right) \right)^p \right]$$

and similarly for $\exp(-\int_D m^2(z) : (h + \phi)^2(z) : dz)$, we deduce the lemma.

So, let us show the uniform bound (6.15). Let $0 < \epsilon < 1/2$ and for conciseness, set $Y_\epsilon := p \int_D m^2(z) : (h_\epsilon(z) + \phi_\epsilon(z))^2 : dz$ and $Y := p \int_D m^2(z) : (h + \phi)^2(z) : dz$. We first observe that for that almost surely, for any $0 < \epsilon < 1/2$ and any $z \in D$,

$$h_\epsilon(z)^2 + 2h_\epsilon(z)\phi_\epsilon(z) \geq h_\epsilon(z)^2 - 2|h_\epsilon(z)|b_\phi \geq -b_\phi^2.$$

The rightmost inequality is obtained by noting that the minimal value of the polynomial $x \mapsto x^2 - 2|x|b_\phi$ is $-b_\phi^2$ (reached at $x = \pm b_\phi$). It follows from this that, almost surely, for any $\epsilon > 0$,

$$\int_D m^2(z) : (h_\epsilon(z) + \phi_\epsilon(z))^2 : dz \geq -2b_\phi^2 \int_D m^2(z) dz - \int_D m^2(z)\mathbb{E}[h_\epsilon(z)^2] dz.$$

Together with the estimate (6.1), this yields that there exists $K > 0$ such that, almost surely, for any $0 < \epsilon < 1/2$,

$$\int_D m^2(z) : (h_\epsilon(z) + \phi_\epsilon(z))^2 : dz \geq -K \log(\epsilon^{-1}).$$

Above, the constant K depends only on D, m^2 and b_ϕ. Using this almost sure lower bound, we then have that, for some constant $\tilde{C} > 0$,

$$\mathbb{E}\left[\left(\exp\left(-\frac{p}{2} \int_D m^2(z) : (h_\epsilon(z) + \phi_\epsilon(z))^2 : dz \right) \right)^2 \right]$$
$$= \mathbb{E}\left[\exp\left(-(Y_\epsilon - Y) \right) \exp\left(-Y \right) \mathbb{I}_{\{Y_\epsilon - Y > -1\}} \right] + \mathbb{E}\left[\exp\left(-(Y_\epsilon - Y) \right) \exp\left(-Y \right) \mathbb{I}_{\{Y_\epsilon - Y \leq -1\}} \right]$$
$$\leq \exp(1)\mathbb{E}[\exp(-Y)] + \exp\left(\tilde{C}\|m^2\|_{L^1(D)} \log(\epsilon^{-1}) \right) \mathbb{P}(Y_\epsilon - Y \leq -1)$$
$$\leq \exp(1)\mathbb{E}[\exp(-Y)] + \exp\left(\tilde{C}\|m^2\|_{L^1(D)} \log(\epsilon^{-1}) \right) \mathbb{P}(|Y_\epsilon - Y| \geq 1), \tag{6.16}$$

where $\mathbb{E}[\exp(-Y)]$ is finite by [Jan08]. We thus see that to prove the uniform bound (6.15), we must show that the probability $\mathbb{P}(|Y_\epsilon - Y| \geq 1)$ decays fast enough as $\epsilon \to 0$. This will be achieved thanks to Lemma 4.1.1. Indeed, by Markov inequality, for any $q \in [2, \infty)$, $\mathbb{P}(|Y_\epsilon - Y| \geq 1) \leq \mathbb{E}[|Y_\epsilon - Y|^q]$. Moreover, as both Y_ϵ and Y are random variables in $\Gamma_0(\mathcal{H}) \oplus \Gamma_1(\mathcal{H}) \oplus \Gamma_2(\mathcal{H})$ where $\mathcal{H}$ is the Gaussian Hilbert space corresponding to the GFF h, by [Sim74, Theorem I.22], we have that

$$\|Y_\epsilon - Y\|_{L^q(\mathbb{P})} \leq (q - 1)\|Y_\epsilon - Y\|_{L^2(\mathbb{P})}.$$

Together with Lemma 4.1.1, this yields that, for any $b \in (0, \alpha)$, there exists a constant $C > 0$ such that for any $0 < \epsilon < 1/2$,

$$\mathbb{P}(|Y_\epsilon - Y| \geq 1) \leq (q - 1)^q \mathbb{E}[|Y_\epsilon - Y|^2]^{q/2} \leq C^q (q - 1)^q \epsilon^{qb}.$$

We are now going to choose q in a ϵ-dependent way. Namely, set $q = \epsilon^{-b/3}$. Then, $(q - 1)^q \leq q^q = \epsilon^{-qb/3}$. Moreover, there exists $\tilde{\epsilon}_1 > 0$ such that for any $0 < \epsilon < \tilde{\epsilon}_1$, $C^q \leq \epsilon^{-qb/3}$. Therefore, for any $0 < \epsilon < \tilde{\epsilon}_1$,

$$\mathbb{P}(|Y_\epsilon - Y| \geq 1) \leq \epsilon^{-qb/3} \epsilon^{-qb/3} \epsilon^{bq} = \epsilon^{qb/3}.$$

Thanks to our choice of q, it then easy to see that there exists $\tilde{\epsilon}_2 > 0$ such that for any $0 < \epsilon < \tilde{\epsilon}_2$, $\mathbb{P}(|Y_\epsilon - Y| \geq 1) \leq \exp(-\epsilon^{-b/3})$. Going back to (6.16), we obtain that for any $0 < \epsilon < \tilde{\epsilon}_2$,

$$\mathbb{E}[\exp(-Y_\epsilon)] \leq \exp(1)\mathbb{E}[\exp(-Y)] + \exp\left(\tilde{C}\|m^2\|_{L^1(D)} \log(\epsilon^{-1}) \right) \exp(-\epsilon^{-b/3}).$$

Since $b > 0$, the second summand on the right-hand side of the above inequality converges to 0 as $\epsilon \to 0$ and is therefore bounded by some constant $C > 0$ independent of ϵ. This shows that, for any $0 < \epsilon < \tilde{\epsilon}_2$,

$$\mathbb{E}[\exp(-Y_\epsilon)] \leq \exp(1)\mathbb{E}[\exp(-Y)] + C.$$

We deduce from this the uniform bound (6.15), which as explained above, concludes the proof of the lemma. $\qquad\square$

Finally, let us show the almost sure strict positiveness of the random variable on the right-hand side of (4.9). Note that by taking $\phi \equiv 0$, the lemma below implies that the random variable on the right-hand side of (4.7) is also almost surely strictly positive.

Lemma 6.1.4. *Almost surely,*

$$\exp\left(-\frac{1}{2} \int_D m^2(z) : (h+\phi)^2(z) : dz \right) > 0.$$

Proof. This lemma follows from the fact that

$$\mathbb{P}\left(\exp\left(-\frac{1}{2} \int_D m^2(z) : (h+\phi)^2(z) : dz \right) > 0 \right) = \mathbb{P}\left(\frac{1}{2} \int_D m^2(z) : (h+\phi)^2(z) : dz < \infty \right) = 1$$

since the random variable $\int_D m^2(z) : (h+\phi)^2(z) : dz$ is in $L^2(\mathbb{P})$. $\qquad\square$

6.2 Proof of Lemma 4.4.1

Proof of Lemma 4.4.1. Let $\alpha > 0$. As a preliminary remark, observe that the change of measure (4.64) is well-defined. Indeed, $\int_D m^2(z) : L(z) : dz$ is a limit in $L^2(\mathbb{P}_\alpha)$ and is therefore measurable with respect to $\mathcal{L}$ while finiteness of the normalization constant $\mathcal{Z}$ is shown in [Jan08, Theorem 8].

Let us denote by $\mathcal{X}(D)$ the space of unrooted loops contained in D. We turn $\mathcal{X}(D)$ into a metric space by endowing with the metric $d_\mathcal{X}$ induced by the norm, for $\ell \in \mathcal{X}(D)$, $\|\ell\|_\mathcal{X} := \tau(\ell) + \sup_{t \in [0,\tau(\ell)]} |\ell(t)|$ where $\tau(\ell)$ is the lifetime of the loop ℓ. See [Law05, Section 5.1] for more details about these definitions. Let $f : \mathcal{X}(D) \to \mathbb{R}_+$ be a nonnegative function with compact support in $\mathcal{X}(D)$. To prove Lemma 4.4.1, we must show that

$$\mathbb{E}_\alpha^{(m)}[\exp(-\langle \mathcal{L}, f \rangle)] = \exp\left(-\alpha \int (1 - e^{-f(\ell)}) \mu_D^m(d\ell) \right) \tag{6.17}$$

or, in words, that the Laplace transform of $\mathcal{L}$ under $\mathbb{P}_\alpha^{(m)}$ is the same as that of a massive Brownian loop in D with mass m and intensity α. Above, the random variable $\langle \mathcal{L}, f \rangle$ is defined as $\langle \mathcal{L}, f \rangle := \sum_{\ell \in \mathcal{L}} f(\ell)$. For ease of notations, let us set

$$\phi(\ell) := \exp\left(-\int_0^{\tau(\ell)} m^2(\ell(t)) dt \right)$$

and note that $\mu_D^m(d\ell) = \phi(\ell)\mu_D(d\ell)$. Next, we observe that in the integral on the right-hand side of (6.17), the integrand is 0 outside the support of f. We are going to use this to decompose this integral as a sum over well-chosen compact sets of loops. To this end, let us define the sets of loops, for $n \geq 1$, $K_n := \{\ell \in \mathcal{X}(D) : \bar{\ell} \subset \bar{D}, \tau(\ell) \in [2^{-n}, 2^n]\}$. Observe that the set $(K_n)_n$ are such that $K_n \subset \text{Int}(K_{n+1})$ and $\mathcal{X}(D) = \cup_n K_n$. Moreover, we claim the following.

Claim 6.2.1. *The sets $(K_n)_n$ are compact sets of $(\mathcal{X}(D), d_\mathcal{X})$.*

We postpone the proof of Claim 6.2.1 to the end and continue the proof of the equality (6.17). By [Eng89, Exercise 3.8.C], Claim 6.2.1 and the covering property of the sets (K_n) imply that $(\mathcal{X}(D), d_\mathcal{X})$ is hemicompact: for any compact set $K \subset \mathcal{X}(D)$, there exists $n \in \mathbb{N}$ such that $K \subset K_n$, see for example [Wil04, Section 17.I]. Therefore, since f has compact support, for n large enough,

$$\int (1 - e^{-f(\ell)}) \mu_D^m(d\ell) = \int_{K_n} (1 - e^{-f(\ell)}) \phi(\ell) \mu_D(d\ell).$$

Note that this integral is finite since loops in K_n have lifetime at least 2^{-n} (the divergence of the total mass of μ_D is only due to short loops). For $n \geq 1$, let us further set $B_n = \{\ell \in \mathcal{X}(D) : \bar{\ell} \subset \bar{D}, \tau(\ell) \geq 2^{-n}\}$ and observe that $K_n \subset B_n$. Using the fact that f is compactly supported in $K_n \subset B_n$,

$$\exp\left(-\alpha \int_{K_n} (1 - e^{-f(\ell)}) \phi(\ell) \mu_D(d\ell) \right)$$
$$= \exp\left(-\alpha \int_{B_n} (1 - e^{-f(\ell)}) \phi(\ell) \mu_D(d\ell) \right) = \mathbb{E}_\alpha\left[\exp(-\langle \mathcal{L}_{B_n}, f \rangle + Y_n(\mathcal{L})) \right] \tag{6.18}$$

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where for a set B and a function g, $\langle \mathcal{L}_B, g \rangle := \sum_{\ell \in \mathcal{L}: \ell \in B} g(\ell)$ and where we have set, for $n \geq 1$,

$$Y_n(\mathcal{L}) := \langle \mathcal{L}_{B_n}, \log \phi \rangle + \alpha \int_{B_n} (1 - \phi(\ell)) \mu_D(d\ell).$$

Note that in this definition, the integral $\int_{B_n} (1 - \phi(\ell)) \mu_D(d\ell)$ is finite since it is bounded above by $2\mu_D(B_n)$, which is finite because loops in B_n have lifetime at least 2^{-n}. The equality (6.18) implies that to establish (6.17), it suffices to show that the random variable $\exp(-\langle \mathcal{L}_{B_n}, f \rangle + Y_n(\mathcal{L}))$ on the right-hand side of (6.18) converges in $L^1(\mathbb{P}_\alpha)$ as $n \to \infty$ to $\exp(-\langle \mathcal{L}, f \rangle) R(\mathcal{L})$ where $R(\mathcal{L})$ denotes the random variable on the right-hand side of (4.64) (with the normalization constant $\mathcal{Z}$ included).

To this end, observe that $\exp(-\langle \mathcal{L}_{B_n}, f \rangle)$ almost surely converges to $\exp(-\langle \mathcal{L}, f \rangle)$ and moreover, since f is non-negative, almost surely, for any n, $\exp(-\langle \mathcal{L}_{B_n}, f \rangle) \leq 1$. By dominated convergence, this implies that $\exp(-\langle \mathcal{L}_{B_n}, f \rangle)$ converges in $L^1(\mathbb{P}_\alpha)$ to $\exp(-\langle \mathcal{L}, f \rangle)$. Therefore, using once again that, almost surely, for any n, $\exp(-\langle \mathcal{L}_{\tilde{B}_n}, f \rangle) \leq 1$, the following claim then yields convergence of $\exp(-\langle \mathcal{L}_{B_n}, f \rangle + Y_n(\mathcal{L}))$ to $\exp(-\langle \mathcal{L}, f \rangle) R(\mathcal{L})$ in $L^1(\mathbb{P}_\alpha)$, and thus concludes the proof of Lemma 4.4.1.

Claim 6.2.2. *The random variables* $(\exp(Y_n(\mathcal{L})))_n$ *converge in* $L^1(\mathbb{P}_\alpha)$ *to the random variable on the right-hand side of* (4.64).

$\square$

We now turn to the proof of Claim 6.2.1 and Claim 6.2.2.

Proof of Claim 6.2.1. Let $n \geq 1$ and let $(\ell_k)_k$ be a sequence of loops in K_n. To show that K_n is compact in $(\mathcal{X}(D), d_\mathcal{X})$, we are going to show that there exists a subsequence $(k(p))_p$ such that $(\ell_{k(p)})_p$ converges in $(\mathcal{X}(D), d_\mathcal{X})$ to a limit ℓ^* with $\ell^* \in K_n$. To this end, observe that, for any k, since $\ell_k \in K_n$ and ℓ_k is contained in the bounded domain D, $2^{-n} \leq \|\ell_k\|_\mathcal{X} = \tau(\ell_k) + \sup_t |\ell(t)| \leq 2^n + C_D$ where $C_D > 0$ is a finite constant that depends only on D. Therefore, $(\ell_k)_k$ is bounded in $(\mathcal{X}(D), d_\mathcal{X})$, which implies that there exists a subsequence $(k(p))_p$ such $(\ell_{k(p)})_p$ converges in $(\mathcal{X}(D), d_\mathcal{X})$ to a limit ℓ^*. To see that ℓ^* belongs to K_n, observe first that the function $\tau : \ell \mapsto \tau(\ell)$ is continuous in $(\mathcal{X}(D), d_\mathcal{X})$ and therefore, the fact that for any p, $2^{-n} \leq \tau(\ell_{k(p)}) \leq 2^n$ implies that $2^{-n} \leq \tau(\ell^*) \leq 2^n$. Moreover, we have that $\ell^* \subset \overline{D}$ since for any p, $\ell_{k(p)}$ belongs to $\mathcal{X}(D)$ and hence $\overline{\ell}^* \subset \overline{D}$. $\square$

Proof of Claim 6.2.2. Let us set, for $n \geq 1$, $\tilde{Y}_n(\mathcal{L}) := \langle \mathcal{L}_{B_n}, \log \phi \rangle - \alpha \mu_D(\mathbb{I}_{B_n} \log \phi)$. Note that, in this definition, the loop measure term is finite. Indeed, we have that

$$\mu_D(|\log \phi| \mathbb{I}_{B_n}) = \mu_D \left(\mathbb{I}_{\ell \in B_n} \int_0^{\tau(\ell)} m^2(\ell(t)) dt \right) \leq \int_{2^{-n}}^\infty \int_D m^2(z) p_t(z, z) dz dt < \infty \qquad (6.19)$$

as D is bounded (here, p_t is the heat kernel in D with Dirichlet boundary conditions on ∂D). Now, observe that, almost surely, $\exp(Y_n(\mathcal{L})) = \exp(\tilde{Y}_n(\mathcal{L})) \exp \left(\alpha \int_{B_n} (1 - \phi(\ell) + \log \phi(\ell)) \mu_D(d\ell) \right)$ where finiteness of the integral $\int_{B_n} (1 - \phi(\ell) + \log \phi(\ell)) \mu_D(d\ell)$ follows from (6.19) and the fact that, for any ℓ, $|1 - \phi(\ell) + \log \phi(\ell)| \leq 2|\log \phi(\ell)|$. Therefore, to prove the claim, it suffices to show that

$$\lim_{n \to \infty} \exp(\tilde{Y}_n(\mathcal{L})) \exp \left(\alpha \int_{\tilde{B}_n} (1 - \phi(\ell) + \log \phi(\ell)) \mu_D(d\ell) \right)$$
$$= \frac{1}{\mathcal{Z}} \exp \left(- \int_D m^2(z) : L(z) : dz \right) \qquad (6.20)$$

where the limit is in $L^1(\mathbb{P}_\alpha)$. To establish this, we first show that $(\exp(Y_n(\mathcal{L})))_n$ is a bounded martingale in $L^1(\mathbb{P}_\alpha)$, which implies that it has a limit in $L^1(\mathbb{P}_\alpha)$. To identify this limit with the random variable on the right-hand side of (6.20), we then prove that $(\exp(Y_n(\mathcal{L})))_n$ almost surely converges to this random variable.

The fact that $(\exp(Y_n(\mathcal{L})))_n$ is a martingale follows from the Poissonian nature of $\mathcal{L}$, as we now explain. For $n \geq 2$, let us set $B'_1 = B_1$ and $B'_n = B_n \setminus B_{n-1}$. We then have that, for any $n \geq 2$,

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almost surely,

$$
\begin{aligned}
\mathbb{E}_\alpha[&\exp(Y_n(\mathcal{L}))|\exp(Y_{n-1}(\mathcal{L}))] \\
&= \exp(Y_{n-1}(\mathcal{L}))\mathbb{E}_\alpha[\exp(Y_n(\mathcal{L}) - Y_{n-1}(\mathcal{L}))|\exp(Y_{n-1}(\mathcal{L}))] \\
&= \exp(Y_{n-1}(\mathcal{L}))\exp\left(\alpha\int_{B'_n}(1-\phi(\ell))\mu_D(d\ell)\right)\mathbb{E}_\alpha[\exp(\langle\mathcal{L}_{B'_n},\log\phi\rangle)|\exp(Y_{n-1}(\mathcal{L}))] \\
&= \exp(Y_{n-1}(\mathcal{L}))\exp\left(\alpha\int_{B'_n}(1-\phi(\ell))\mu_D(d\ell)\right)\mathbb{E}_\alpha[\exp(\langle\mathcal{L}_{B'_n},\log\phi\rangle)] = \exp(Y_{n-1}(\mathcal{L}))
\end{aligned}
$$

where in the penultimate equality, we used the fact that $\exp(\langle\mathcal{L}_{B'_n},\log\phi\rangle) = \exp(\langle\mathcal{L}_{B_n},\log\phi\rangle - \langle\mathcal{L}_{B_{n-1}},\log\phi\rangle)$ is independent of $\exp(Y_{n-1}(\mathcal{L}))$ since $\mathcal{L}$ is a Poisson point process. This shows that $(\exp(Y_n(\mathcal{L})))_n$ is a martingale. Boundedness in $L^1(\mathbb{P}_\alpha)$ is straightforward to see since for each n, $\exp(Y_n(\mathcal{L}))$ is almost surely a non-negative random variable and $\mathbb{E}_\alpha[\exp(Y_n(\mathcal{L}))] = \exp\left(\alpha\int_{B_n}(1-\phi(\ell))\mu_D(d\ell)\right)\mathbb{E}_\alpha[\exp(\langle\mathcal{L}_{B_n},\log\phi\rangle)] = 1$.

Having established that $(\exp(Y_n(\mathcal{L})))_n$ converges in $L^1(\mathbb{P}_\alpha)$, let us now show that it also converges almost surely to the random variable on the right-hand side of (6.20). To this end, we first apply the dominated convergence theorem to the loop measure term. Observe that, μ_D-almost everywhere, $\lim_{n\to\infty}\mathbb{I}_{B_n}(1-\phi(\ell)+\log\phi(\ell)) = 1-\phi(\ell)+\log\phi(\ell)$. Moreover, there exists a constant $C > 0$ such that, for any loop ℓ with $\phi(\ell) \geq 1/2$, $|1-\phi(\ell)+\log\phi(\ell)| \leq C(\sqrt{\phi(\ell)}-1)^2$. Therefore, setting $E := \{\ell \in \mathcal{X}(D) : \phi(\ell) \leq 1/2\}$, we have that, for any $n \geq 1$, μ_D-almost everywhere, $\mathbb{I}_{B_n\setminus E}|1-\phi(\ell)+\log\phi(\ell)| \leq \mathbb{I}_{B_n\setminus E}C(\sqrt{\phi(\ell)}-1)^2$. Moreover, it is easy to see that for a loop ℓ, $(\sqrt{\phi(\ell)}-1)^2 \leq (\log\phi(\ell))^2/4$ and $\mu_D((\log\phi(\ell))^2) = \mu_D(\langle m^2,\ell\rangle^2) < \infty$ by [Jan08, Lemma 2] since D is a bounded domain. This shows that, for any $n \geq 1$, μ_D-almost everywhere,

$$
\begin{aligned}
\mathbb{I}_{B_n}|1-\phi(\ell)+\log\phi(\ell)| &\leq \mathbb{I}_{B_n\setminus E}C(\sqrt{\phi(\ell)}-1)^2 + \mathbb{I}_E|1-\phi(\ell)+\log\phi(\ell)| \\
&\leq C(\log\phi(\ell))^2 + \mathbb{I}_E|1-\phi(\ell)+\log\phi(\ell)|
\end{aligned}
$$

and the right-hand side is μ_D-integrable (loops in E have lifetime at least $\overline{m}^{-2}\log(2)$). Therefore, by dominated convergence, we obtain that

$$
\lim_{n\to\infty}\exp\left(\alpha\int_{B_n}(1-\phi(\ell)+\log\phi(\ell))\mu_D(d\ell)\right) = \exp(\alpha\mu_D(1-\phi(\ell)+\log\phi(\ell))) = \frac{1}{\mathcal{Z}}
$$

where the rightmost equality follows from [Jan08, Theorem 8]. On the other hand, as recalled in Section 4.1.4, we have that $\tilde{Y}_n(\mathcal{L})$ converges $\mathbb{P}_\alpha$-almost surely to $-\int_D m^2(z) : L(z) : dz$ as $n \to \infty$. Therefore, by continuity of $x \mapsto e^x$, we have that, $\mathbb{P}_\alpha$-almost surely, $\lim_{n\to\infty}\exp(\tilde{Y}_n(\mathcal{L})) = \exp\left(-\int_D m^2(z) : L(z) : dz\right)$. As explained above, this yields (6.20) and concludes the proof of Claim 6.2.2. $\qquad\square$

6.3 Proof of Lemma 6.3.1

Lemma 6.3.1. *Let $D \subset \mathbb{C}$ be a bounded, open and simply connected domain such that ∂D has Hausdorff dimension strictly smaller than $7/4$. Then, for any $k \geq 1$,*

$$
\int_{D^k}|f_D^{(+,k)}(x_1,\ldots,x_k)|^2\prod_{j=1}^k dx_j < \infty
$$

where $f_D^{(+,k)}$ is as in Theorem 5.1.8.

Proof. To show this lemma, we use the explicit expression of $f_D^{(+,k)}$: by [CHI21, Theorem 7.1], we have that, for any $k \geq 1$ and any $x_1,\ldots,x_k \in D^k$,

$$
f_D^{(+,k)}(x_1,\ldots,x_k) = \prod_{j=1}^k \mathrm{CR}(x_j,\partial D)^{-\frac{1}{8}}\left(2^{-\frac{k}{2}}\sum_{\mu\in\{-1,1\}^k}\prod_{1\leq j<m\leq k}\exp\left(\frac{\mu_i\mu_m}{2}G_D(x_j,x_m)\right)\right)^{\frac{1}{2}}
$$

where G_D denotes the Green function in D with Dirichlet boundary conditions. Here, G_D is normalized so that as $y \to x$, $G_D(x,y) \sim -\log(|x-y|)$. Let $\alpha,\tilde{\alpha} > 1$ be such that $\frac{1}{\alpha} + \frac{1}{\tilde{\alpha}} = 1$

and $\frac{\tilde{\alpha}}{4} < 2 - \dim_H(\partial D)$ and $\alpha < 2$, where $\dim_H(\partial D)$ denotes the Hausdorff dimension of ∂D. Applying Hölder inequality with the functions

$$g_1(x_1, \ldots, x_k) = \prod_{j=1}^k \mathrm{CR}(x_j, \partial D)^{-\frac{1}{4}}, \quad g_2(x_1, \ldots, x_k) = \sum_{\mu \in \{-1,1\}^k} \prod_{1 \le j < m \le k} \exp\left(\frac{\mu_i \mu_m}{2} G_D(x_j, x_m)\right),$$

we obtain that for any $k \ge 1$,

$$\int_{D^k} |f_\Omega^{(+,k)}(x_1, \ldots, x_k)|^2 \prod_{j=1}^k dx_j \le \left(\int_D \mathrm{CR}(x, \partial D)^{-\frac{\tilde{\alpha}}{4}} dx\right)^{\frac{k}{\tilde{\alpha}}} \left(\int_{D^k} g_2(x_1, \ldots, x_k)^\alpha \prod_{j=1}^k dx_j\right)^{\frac{1}{\alpha}}.$$

By our assumption on $d_H(\partial D)$ and our choice of $\tilde{\alpha}$, it is easy to see that the first integral on the above right-hand side is indeed finite. On the other hand, since G_D is a non-negative function, we have that

$$\int_{D^k} g_2(x_1, \ldots, x_k)^\alpha \prod_{j=1}^k dx_j \le 2^k \int_{D^k} \prod_{1 \le j < m \le k} \exp\left(\frac{\alpha}{4} G_D(x_j, x_m)\right) \prod_{j=1}^k dx_j.$$

Finiteness of the above integral then follows by combining [JSW20, Proposition 3.9] and [JSW20, Lemma 3.10]. $\square$

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