Gauging Generalised Symmetries : Group-Theoretic Higher Fusion Categories and Higher Representation Theory

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A Thesis presented for the degree of
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Abstract: In this thesis we will construct novel non-invertible symmetries by gauging finite invertible symmetries, discuss the various formalisms for performing this construction using one and two dimensions as motivating examples, and investigate what the most general notion of gauging is in three dimensions. While unitary fusion categorical symmetries are now well under control for describing symmetries of oriented unitary quantum field theories in two dimensions, the same cannot be said for unitary fusion 2-categorical symmetries in three dimensions. It is our hope that by conjecturing that all such underlying fusion 2-categories are group-theoretic, that we can systematically construct all examples and describe how they should lift to unitary fusion 2-categories.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification.

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Theoretical Physics, for all its glory, is not an easy field to make a mark in, and I could never have made it to this point in my career without all of you to support me.

Finally we arrive at the end, where I have saved the best for last; the other half not only to myself, but also to whom this work is dedicated.

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Freedom is the freedom to say that two plus two make four. If that is granted, all else follows. - Winston

— from *1984* by G. Orwell

Dedicated to

My Mum, My Dad

and

Dr Gaia De Angelis

Contents

Chapter 1

Introduction

The role of symmetries in physics is deeply woven into our understanding of the field; appearing in all fields ranging from classical mechanics and general relativity, to particle and condensed matter physics, often providing deep insight into those theories. From a classical perspective they are responsible for conservation laws via Noether's theorem, and the Poincaré symmetry group forms the basis of our understanding of space-time. In the context of quantum field theory these insights include spontaneous symmetry breaking, the Goldstone theorem and the Higgs mechanism, famously leading to the recent discovery of the Higgs boson, but also responsible for our understanding of Pion physics and various quasiparticles appearing in condensed matter systems. Global symmetries in quantum field theories can also carry additional data in the form of anomalies; 't Hooft anomalies in particular are invariants of the renormalisation group flow and as such are matched in the UV and IR limits of a theory; such calculations form a subset of the small number of truly non-perturbative calculations one can perform in a renormalisable QFT.

Indeed the since the inception of generalised symmetries in [3], there has been cause for much excitement as we have observed a precipitation of novel forms of symmetry, including higher-form and non-invertible symmetries, providing powerful new tools and insights as well as generalisations to those we mentioned that already exist generating new and exciting results [4–17]. From a mathematical viewpoint, these symmetries herald the beginning of a new era, one in which the tools and techniques of category theory have become not only necessary, but commonplace, to those intending to study the structure of symmetries in QFT.

One way to construct novel examples of generalised symmetries in two dimensions is to gauge a finite invertible symmetry [18–21]. More recent work, including that which we exposit in this thesis, has shown that these constructions admit categorical lifts to three dimensions [1, 2, 22, 23]. The aim of this thesis is to collect and review these constructions in various low dimensions and further demonstrate a remarkable fact that in three dimensions, a very large class of generalised symmetries (in truth almost all of them) can be constructed through a generalisation of gauge theory.

The role of topological defects, and more generally topological quantum field theories, cannot be understated in the study of symmetries. It is possible to derive the category-theoretic structure needed to describe symmetries by simply considering the calculus of topological defects; their fusion, addition, and the possible topological manipulations thereof. In this way we say that, in two dimensions for example, that the topological extended operators that describe symmetries form a *graphical calculus* for unitary (multi-)fusion categories [24, 25]. Another exciting view point is that the symmetry of a d-dimensional QFT can be captured entirely by the data of $(d + 1)$-dimensional TQFT called the symmetry TFT [26]. Understanding these TQFTs and their boundaries has lead to many new invaluable tools for studying symmetries [27–34], including some that appear frequently in this work.

In this first chapter we shall introduce some basic concepts concerning generalised symmetries using quantum mechanics first as a baseline example. Then we will move on to symmetries in two-dimensional QFTs, introducing some core concepts from category theory and setting them in the context of non-invertible symmetries, followed by a canonical example of non-invertible symmetry that appears in the two-dimensional critical Ising model. Finally we will conclude this introduction by reviewing some basic features of topological quantum field theories in low dimensions, which will appear frequently in the work to come.

In the latter chapters 2, 3, and 4 we will study the formalisms used to describe the gauging of finite invertible symmetries to produce group-theoretic symmetries in one, two and three dimensions, respectively. These chapters all share a similar structure, starting first with a discussion of the general categorical structure of symmetries including invertible symmetries, then moving to the construction of non-invertible symmetries through gauging invertible ones following [1, 2], concluding with a discussion of various formalisms that seek to generalise that process.

1.1 Symmetry in Quantum Mechanics

To introduce some of the core ideas of generalised symmetries, we shall start with the simple example of a one-dimensional QFT, or quantum mechanics. Such quantum mechanical systems are characterised by a Hilbert space $\mathcal{H}$, together with a Hermitian linear map $H : \mathcal{H} \to \mathcal{H}$ called the Hamiltonian.

We define a Hilbert space in the usual fashion as a complex vector space $\mathcal{H}$, equipped with an inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ satisfying

- Conjugate symmetry:
$$\langle x, y \rangle = \langle y, x \rangle^* \tag{1.1.1}$$
for all $x, y \in \mathcal{H}$.

- Linearity in the second argument:
$$\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle \tag{1.1.2}$$
for all $x, y, z \in \mathcal{H}$ and $\lambda \in \mathbb{C}$.

- positive definiteness:
$$\langle x, x \rangle \geq 0 \tag{1.1.3}$$
for all $x \in \mathcal{H}$, with equality holding if and only if $x = 0$.

- Completeness:
$$\lim_{m,n \to \infty} |\sum_{i=m}^{n} x_i|^2 = 0 \implies \exists!\ S = \sum_{i=0}^{\infty} x_i \in \mathcal{H} \tag{1.1.4}$$
where $|x|^2 = \langle x, x \rangle$, for every converging infinite set $\{x_i \in \mathcal{H}\}_i$.

Given two Hilbert spaces $\mathcal{H}, \mathcal{H}'$, we define homomorphisms $f \in \mathrm{Hom}(\mathcal{H}, \mathcal{H}')$ as bounded linear maps [1]
$$f : \mathcal{H} \to \mathcal{H}' \,. \tag{1.1.5}$$

[1] That is, linear maps with finite norm. In the literature sometimes the stronger condition of short is taken, where the norm is at most one.

Given a single Hilbert space $\mathcal{H}$, the set of automorphisms $\mathrm{Aut}(\mathcal{H})$ has the structure of a group, while the full space of endomorphisms $\mathrm{End}(\mathcal{H})$, together with transpose complex conjugation on linear maps, has the structure of a C*-algebra.

We define a C*-algebra as a complex associative (Banach) algebra $\mathcal{A}$, equipped with a norm $\|\cdot\| : \mathcal{A} \to \mathbb{C}$ and anti-linear involution $* : \mathcal{A} \to \mathcal{A}$ such that

$$\|x^* x\| = \|x^*\| \, \|x\| \qquad\qquad \|x^* x\| = \|x\|^2 \,, \tag{1.1.6}$$

for all $x \in \mathcal{A}$. For our purposes however, and for intuition, we can adopt a simpler definition: the Gelfand-Naimark theorem states that *every* C*-algebra can be expressed as the algebra of bounded linear endomorphisms of some appropriate choice of Hilbert space.

The Hamiltonian H generates a one-parameter family of automorphisms called time-translation operators

$$U(t) = \exp(-iHt) : \mathcal{H} \to \mathcal{H} \,, \tag{1.1.7}$$

for finite times $t \in \mathbb{R}$. Hermicity of the Hamiltonian $H = H^*$ implies that these automorphisms are bounded and unitarity:

$$U(t)^{-1} = U(-t) = U(t)^* \,. \tag{1.1.8}$$

Given a quantum mechanical theory with Hilbert space $\mathcal{H}$, we identify normalised vectors $x, y \in \mathcal{H}$ with physical states, and their inner products $\langle x, y \rangle$ with amplitudes. We define the cross-section to be

$$P(x \to y) = \frac{\langle x, y \rangle \langle y, x \rangle}{\langle x, x \rangle \langle y, y \rangle} \,. \tag{1.1.9}$$

Physically this captures the probability for a system starting in state $x \in \mathcal{H}$ to be measured in state $y \in \mathcal{H}$. These cross-sections are (by design) independent of the choice of normalisation, meaning that the physical content of the quantum mechanics is actually captured by the *projective* Hilbert space

$$\mathcal{H}\mathbb{P} = \mathcal{H}/\mathbb{C}^\times \,, \tag{1.1.10}$$

and that we should identify those linear operators related by rescaling the normalisation. One definition for a symmetry in quantum mechanics then, is an automorphism $\mathcal{S} \in \mathrm{Aut}(\mathcal{H})$ of the Hilbert space (up to rescaling) that preserves the cross-sections:

$$P(\mathcal{S}x \to \mathcal{S}y) = P(x \to y)\,, \tag{1.1.11}$$

and commutes with the Hamiltonian. However, this restriction to invertible symmetries is too strict. By extending linearly from automorphisms to all endomorphisms we can discuss more general symmetries, whose mathematical description comes in the form of C*-algebras. We should say that in this context, this is not really a "new" idea; algebraic structures appear frequently going all the way back to the Dirac-von Neumann axiomatisation of quantum mechanics. Viewed in the larger context of quantum field theory however, the notion of non-invertible symmetries in general dimension (most notably dimensions greater than two) is certainly more recent.

1.1.1 Non-invertible Symmetry in Quantum Mechanics

We can pinpoint the source of non-invertible symmetries in quantum mechanics as being due to the projectivity of the Hilbert space. Were it not the case that physical states are independent of normalisation, we might have alternatively defined cross-sections by the square modulus of the inner product. In that case, a symmetry $\mathcal{S}$ is non-invertible if there is a non-zero vector $x \neq 0$ in its kernel $\mathcal{S}x = 0$. Our assumption that $\mathcal{S}$ is a symmetry then implies

$$\langle x, x\rangle = \langle \mathcal{S}x, \mathcal{S}x\rangle = 0, , \tag{1.1.12}$$

violating the positive-definiteness assumption $\langle x, x\rangle > 0$ for $x \neq 0$. This shows that without projectivity, there can be no non-invertible symmetry.

To better understand the origin of non-invertible symmetries in quantum mechanics, we will now adopt a more general definition of symmetry that better uses the projectivity of the cross-section. We consider the case of oriented quantum mechanics, that is, a quantum mechanics whose states are all bosonic and whose partition function is defined on a oriented 1-manifold. We define a global symmetry as an endomorphism [2] (up to rescaling) of the Hilbert space $\mathcal{S} \in \mathrm{End}(\mathcal{H})$, such that there exists a (non-zero complex) one-parameter family of automorphisms $\mathcal{S}_\tau \in \mathrm{Aut}(\mathcal{H})$ for $\tau \in \mathbb{C}^\times$ satisfying:

1. Invariance under time-translations:

$$[H, \mathcal{S}_\tau] = 0\,, \tag{1.1.13}$$

   for all $\tau \in \mathbb{C}^\times$.

2. Preservation of the cross-section:

$$P(\mathcal{S}_\tau x \to \mathcal{S}_\tau y) = P(x \to y)\,, \tag{1.1.14}$$

   for all $x, y \in \mathcal{H}$ and $\tau \in \mathbb{C}^\times$.

3. The limit $\tau \to 0$ is well defined:

$$\lim_{\tau \to 0} \mathcal{S}_\tau = \mathcal{S}\,. \tag{1.1.15}$$

This circumvents the need for a symmetry $\mathcal{S}$ to be invertible by recognising that an endomorphism of $\mathcal{H}$ that preserves the cross-section in some appropriately defined limit is still a symmetry. In this way we have extended linearly from the subgroup of automorphisms $\mathrm{Aut}(\mathcal{H})/[H, -]$, to the subalgebra $\mathrm{End}(\mathcal{H})/[H, -]$ of endomorphisms, that commute with the Hamiltonian. This subalgebra forms a C*-algebras.

In this thesis, we are largely concerned with *finite* symmetries. In the setting of quantum mechanics, finite symmetries are described by finite-dimensional C*-algebras. In a quantum mechanics with finitely many physical states, and hence a finite-dimensional Hilbert space, this is automatic. In general we do not expect such a simplification however, instead we only ask that there are finitely many

[2] In the unoriented case, we should include also anti-linear maps.

isomorphism classes of physical states identified by the passage of time, this implies that the subalgebra of endomorphisms that commutes with the Hamiltonian is finite-dimensional.

By definition, any unitary operator $\mathcal{S}^*\mathcal{S} = 1$ satisfies

$$\langle \mathcal{S}x, \mathcal{S}y \rangle = \langle x, y \rangle \,, \tag{1.1.16}$$

for every $x, y \in \mathcal{H}$, and if that operator commutes with the Hamiltonian it will automatically describe a symmetry. Going the other way, there is a famous result of Wigner that states any invertible symmetry is equivalent via a rescaling to a unitary symmetry [3].

More generally we can apply this result to block unitary (or semi-unitary) operators that satisfy

$$\mathcal{S}^*\mathcal{S} = 1_{\mathrm{Im}(\mathcal{S})} \,, \tag{1.1.17}$$

where $1_{\mathrm{Im}(\mathcal{S})}$ is understood to be a projector onto the image of $\mathcal{S}$ in $\mathcal{H}$. We can build a 1-parameter family of unitary operators $\mathcal{S}_\tau = \mathcal{S} + \tau 1_{\mathrm{Ker}(\mathcal{S})}$ such that

$$\mathcal{S}_\tau^*\mathcal{S}_\tau = 1_{\mathrm{Im}(\mathcal{S})} + (\tau^*\tau)1_{\mathrm{Im}(\mathcal{S})} \tag{1.1.18}$$

defines a non-zero rescaling of the subspace $\mathrm{Ker}(\mathcal{S})$ for $\tau \in \mathbb{C}^\times$: these are equivalent to the identity up to rescaling and automatically preserve the cross-section. Then, if $\mathcal{S}$ commutes with the Hamiltonian, so too do all the $\mathcal{S}_\tau$, defining a symmetry for each $\tau \in \mathbb{C}^\times$. In reverse, Wigner's theorem holds for each symmetry $\mathcal{S}_\tau$, and so we have generalised Wigner's theorem to non-invertible symmetries:

every symmetry of a quantum mechanical system is equivalent to a block unitary endomorphism up to rescaling.

All this discussion of non-invertible symmetries is not to say that invertible, group-like, symmetries are uninteresting. To define a finite group G symmetry we first give

[3] For an unoriented quantum mechanics, we also have a similar result for anti-linear anti-unitary symmetries.

an assignment

$$\rho : G \to U(\mathcal{H}) \,, \tag{1.1.19}$$

where $U(\mathcal{H}) \subseteq \mathrm{Aut}(\mathcal{H})$ is the subgroup of unitary automorphisms, such that $[\rho(g), H] = 0$ for all $g \in G$. We further ask that this map defines a group homomorphism after passing to the projective Hilbert space $\mathcal{H}\mathbb{P} \simeq \mathcal{H}/\mathbb{C}^\times$. This is one way to define a unitary projective representation of G on $\mathcal{H}$ satisfying

$$\rho(e) = 1 \qquad \rho(g)^* = \alpha(g, g^{-1})^{-1}\rho(g^{-1}) \qquad \rho(g)\rho(h) = \alpha(g,h)\rho(gh) \tag{1.1.20}$$

for some (normalised) projective 2-cocycle $\alpha \in Z^2_{grp}(G, U(1))$ such that

$$\delta\alpha(g,h,k) = \frac{\alpha(h,k)\alpha(g,hk)}{\alpha(gh,k)\alpha(g,h)} = 1 \,. \tag{1.1.21}$$

Equivalence classes of homomorphisms identify different choices of α up to a 2-coboundary, suggesting that the different ways of implementing a finite G symmetry in quantum mechanics are labelled by classes in the group cohomology

$$[\alpha] \in H^2_{grp}(G, U(1)) \,. \tag{1.1.22}$$

This class is invariant under renormalisation group flow and represents a 't Hooft anomaly. Shifting the representative α corresponds to the addition of local counter-terms [35].

1.1.2 Another Perspective: Topological Operators

We now turn to an alternative perspective that we will adopt when we move to higher dimensions. Viewed as a local operator inserted on a $(0+1)$-dimensional QFT, a global symmetry $\mathcal{S}$ corresponds to a *topological* local operator. By topological we mean that due to $[H, \mathcal{S}] = 0$, in the absence of other local operators, we can freely translate $\mathcal{S}$ across the underlying 1-manifold. When we do meet another local operator, O say, the symmetry $\mathcal{S}$ will act upon it by time-ordered conjugation

$$^{\mathcal{S}}O = \mathcal{S}O\mathcal{S}^* \,, \tag{1.1.23}$$

by inserting the projection $id_{\mathrm{Im}(\mathcal{S})}$ onto a subspace and following equation (1.1.17), as illustrated in figure 1.1. The insertion of this projector should be viewed as restricting ourselves onto one of the twisted sectors of the theory [4], in the case where $\mathcal{S}$ is invertible it reduces to the identity.

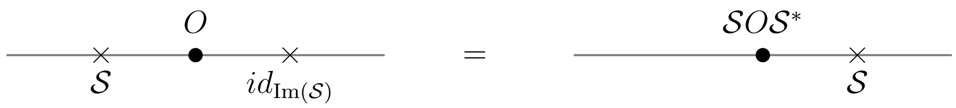

Figure 1.1

The association of symmetries with topological operators in QFT is not an uncommon idea, the most obvious example comes courtesy of Noether's theorem, which relates (continuous) global symmetries to conserved currents that generate extended topological operators. In the study of generalised symmetries we assert that this correspondence is fundamental in nature and that we should adopt as a new definition:

global symmetries in QFT are topological operators.

This is a very powerful organising principle it turns out; the consistency conditions we place on the mathematical structure that describes topological operators do not restrict us to groups. In one dimension we find C*-algebraic symmetries as we have already described, and in higher dimensions we will find higher *categorical* symmetries.

To further illustrate this correspondence for quantum mechanics, let us now return to the example of a group G symmetry with 't Hooft anomaly $\alpha \in Z^2_{grp}(G, U(1))$. For each group element $g \in G$, there is an associated topological local operator

[4]This interpretation becomes clearer when compared to the higher-dimensional analogue in subsection 1.2.2.

that we will label the same way. We can also take $\mathbb{C}$-linear combinations of these operators, so the full spectrum of local symmetries is the vector space $\mathbb{C}G$.

Since the local operators $g, h \in G$ are topological, their product admits a natural and well-defined (topological) regularisation at short distances. This in turn defines a well-defined bilinear product

$$\otimes : (g,h) \mapsto g \otimes h = \alpha(g,h)\; gh\,, \tag{1.1.24}$$

depicted in figure 1.2.

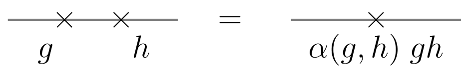

Figure 1.2

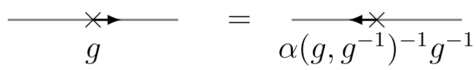

Figure 1.3

Including this product gives us the structure of a twisted group algebra ${}^{\alpha}\mathbb{C}G$. In addition to this structure, the topological operators also carry an orientation. The action of flipping this orientation, as illustrated in figure 1.3 [5], defines an anti-linear involution

$$* : g \mapsto g^* = \alpha(g, g^{-1})^{-1} g^{-1}\,. \tag{1.1.25}$$

This involution further endows ${}^{\alpha}\mathbb{C}G$ with the structure of a C*-algebra. The action of this symmetry on local operators in the quantum mechanics is defined as before in equation (1.1.23) and defines a unitary projective representation of G with projective 2-cocycle α.

One notion of equivalence for quantum mechanics stems from the question of whether or not two quantum mechanical theories share a gapped interface [6] in the form of a topological local operator that sits between them. The set of topological operators that sit at the interface do not generically define a symmetry in either

[5] In discussing orientation it becomes necessary to set a convention: in all figures in this work, unless otherwise indicated, time is assumed to proceed either from left to right, or from bottom to top, and the orientation of a defect is assumed to be in that direction.

[6] The adjectives "gapped" and "topological" are synonymous in this context, and will be used interchangeably throughout this work.

of the two theories, but instead forms a $(\mathcal{A}, \mathcal{A}')$-bimodule of the two associated C*-algebras $\mathcal{A}$ and $\mathcal{A}'$. Transpose complex conjugation further defines an adjoint $(\mathcal{A}', \mathcal{A})$-bimodule and these taken together define a *Morita* equivalence between C*-algebras. This means if we are interested in quantum mechanical systems only up to gapped interface, then the correct mathematical object to study in relation to symmetry is a Morita equivalence class of C*-algebras.

Before continuing, we should mention that in the same spirit, the situation for describing finite symmetries in $0 + 1$ dimensions is very degenerate. Every finite-dimensional C*-algebra $\mathcal{A}$ is semi-simple and admits a decomposition

$$A = M_{r_1}(\mathbb{C}) \oplus \cdots \oplus M_{r_n}(\mathbb{C})\,, \tag{1.1.26}$$

where each simple summand $M_{r_j}(\mathbb{C})$ is a *matrix algebra* describing the r_j^2-dimensional algebra of complex linear matrices action on $\mathbb{C}^{r_j}$, with canonical C*-structure given by the operator norm and transpose complex conjugation. This is demonstrated in any of our finite group $\mathcal{A} = {}^{\alpha}\mathbb{C}G$ examples, where the coefficients r_j are determined by the dimensions of the irreducible α^{-1}-projective representations of G. These matrix algebras are all Morita equivalent to one-another, and this equivalence class contains in particular the trivial algebra $M_1(\mathbb{C}) \simeq \mathbb{C}$.

From a physical perspective this demonstrates an important principle: the existence of non-trivial topological local operators in a quantum mechanics, and more generally in any quantum field theory, implies a decomposition of that theory into super-selection sectors. There is one super-selection sector for each matrix algebra component and in this way an "indecomposable" quantum mechanical system can only have a trivial symmetry (up to Morita equivalence).

1.2 Symmetry in Quantum Field Theory

The notion of describing symmetries in terms of topological defects naturally generalises to all dimensions. To exemplify this we now move to quantum field theory in $1+1$ dimensions, and rather than belabour an attempt at a rigorous definition thereof, we will take the point of view that the data of a theory $\mathcal{T}$ includes at least a space of operators $\mathcal{O}_{\mathcal{T}}$ which may be extended or local, an operator product expansion (OPE) that defines a notion of multiplication on $\mathcal{O}_{\mathcal{T}}$, and a correlation function

$$\langle \cdot \rangle : \mathcal{O}_{\mathcal{T}} \to \mathbb{C} \tag{1.2.1}$$

that respects the additive and multiplicative structures on $\mathcal{O}_{\mathcal{T}}$. The structure of symmetries is then captured by the topological sector $\mathcal{O}_{\mathcal{T}}^{top}$, where the most immediate difference from quantum mechanics is that the symmetry action on local operators is now implemented by topological lines wrapped around them.

Given a 1-submanifold γ we one way to define a topological line operator $\mathcal{S}$ supported on it by a (path-ordered) exponential

$$\mathcal{S}(\gamma) = \exp\left(i \int_{\gamma} \widetilde{\mathcal{S}}\right) . \tag{1.2.2}$$

By considering smooth deformations to γ it can be concluded from Stokes' theorem that this defect is topological if and only if the associated 1-form $\widetilde{\mathcal{S}}$ is closed

$$d\widetilde{\mathcal{S}} = 0 . \tag{1.2.3}$$

Not all defects arise in this way however, one example is the on-invertible Kramers-Wannier defect appearing in the two-dimensional critical Ising model, which we will return to momentarily in subsection 1.2.2.

One way to define the action of a line $\mathcal{S}(\gamma)$ on a local operator $O(x)$ is by fixing a foliation of space-time and considering time ordered conjugation, in analogy to subsection 1.1.2. It is equivalent however, and somewhat more natural, to define it as a wrapping

$$ {}^{\mathcal{S}}O(x) = \mathcal{S}(\gamma_x)O(x) = \exp\left(i \oint_{\gamma_x} \tilde{\mathcal{S}}\right) O(x)\,, \tag{1.2.4} $$

where γ_x is chosen to be a closed curve that winds once clock-wise around $O(x)$, as illustrated in figure 1.4.

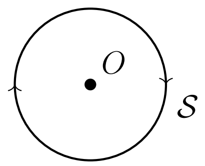

Figure 1.4

In addition to topological line operators, these theories can also have topological local operators. As we remarked at the end of the last section, these are trivial in an indecomposable theory with only one super-selection sector, but if we truly adopt this perspective that topological operators are symmetries, we cannot altogether exclude them. The result is that the mathematical structure needed to describe symmetries in $1+1$ dimensions is much richer than that of C*-algebras seen in the setting of quantum mechanics:

symmetries in a $(1+1)$*-dimensional QFT are described by category theory.*

1.2.1 Category Theory for Symmetries

In this subsection we will review some definitions from category theory, providing physical interpretations that cast them in the context of symmetries of a unitary oriented QFT $\mathcal{T}$ with operators $\mathcal{O}_{\mathcal{T}}$ and correlation function $\langle\cdot\rangle_{\mathcal{T}}$. We use unitary/oriented here to reflect that we are considering a unitary theory whose local excitations are all bosonic, and whose partition function can be defined on an oriented 2-manifold.

For more contemporary constructions concerning (higher) dagger categories and unitarity one can turn to [24, 25], and for the role of category theory in describing symmetries and TQFTs see [36–38].

A *category* C is a collection of *objects* $X \in \mathsf{C}$ and *morphisms* [7] $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ for every pair of objects $X, Y \in \mathsf{C}$, together with a *composition*

$$\circ : \mathrm{Hom}_{\mathsf{C}}(Y, Z) \times \mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{C}}(X, Z) \tag{1.2.5}$$

such that:

- The composition is associative:

$$(f \circ g) \circ h = f \circ (g \circ h)\,, \tag{1.2.6}$$

  for all morphisms $f \in \mathrm{Hom}_{\mathsf{C}}(Z, W)$, $g \in \mathrm{Hom}_{\mathsf{C}}(Y, Z)$, $h \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$, and objects $X, Y, Z, W \in \mathsf{C}$.

- The composition has a unique identity:

$$id_Y \circ f = f \circ id_X = f\,, \tag{1.2.7}$$

  for all morphisms $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ and objects $X, Y \in \mathsf{C}$.

Given a category C and objects $X, Y \in \mathsf{C}$, a morphism $f : X \to Y$ is:

- A *monomorphism* $f : X \hookrightarrow Y$, or *mono*, if

$$f \circ g = f \circ h \implies g = h \tag{1.2.8}$$

  for all morphisms $g, h \in \mathrm{Hom}_{\mathsf{C}}(Z, X)$ and objects $Z \in \mathsf{C}$. This can be thought of as a more general notion of injective.

- An *epimorphism* $f : X \twoheadrightarrow Y$, or *epic*, if

$$g \circ f = h \circ f \implies g = h \tag{1.2.9}$$

  for all morphisms $g, h \in \mathrm{Hom}_{\mathsf{C}}(Y, Z)$ and objects $Z \in \mathsf{C}$. This can be thought of as a more general notion of surjective.

[7] Often we will also write $f : X \to Y$.

- An *isomorphism* $f : X \simeq Y$, or *invertible*, if there exists $f^{-1} : Y \to X$ such that

$$f^{-1} \circ f = id_X \qquad f \circ f^{-1} = id_Y \, . \tag{1.2.10}$$

  This implies f, f^{-1} are both mono and epic, but not vice-versa.

- An *endomorphism* if $X \equiv Y$.

- An *automorphism* if it is both invertible and an endomorphism.

We define a *functor* $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ between categories C and D as an assignment $\mathcal{F} : X \mapsto \mathcal{F}X$, together with maps

$$\begin{aligned} \mathcal{F} &: \mathrm{Hom}_{\mathsf{C}}(X,Y) \to \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}X, \mathcal{F}Y) \\ &: f \mapsto \mathcal{F}f \end{aligned} \tag{1.2.11}$$

such that the composition agrees:

$$\mathcal{F}(f \circ_{\mathsf{C}} g) = (\mathcal{F}f) \circ_{\mathsf{D}} (\mathcal{F}g) \tag{1.2.12}$$

for all morphisms $f \in \mathrm{Hom}_{\mathsf{C}}(Y,Z)$, $g \in \mathrm{Hom}_{\mathsf{C}}(X,Y)$ and objects $X, Y, Z \in \mathsf{C}$.

We define a *natural transformation* $\eta : \mathcal{F} \to \mathcal{G}$ between functors $\mathcal{F}, \mathcal{G} : \mathsf{C} \to \mathsf{D}$ as a collection of morphisms

$$\eta_X : \mathcal{F}X \to \mathcal{G}X \, , \tag{1.2.13}$$

satisfying

$$\eta_Y \circ \mathcal{F}f = \mathcal{G}f \circ \eta_X \tag{1.2.14}$$

for all morphisms $f : X \to Y$ and objects $X, Y \in \mathsf{C}$.

In this way, we define a category $\mathrm{Fun}(\mathsf{C}, \mathsf{D})$ of functors from C to D, with morphisms given by natural transformations and composition inherited from the composition of morphisms in D.

For any category C, we may always define an identity functor $id_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}$ that assigns each object and morphism to itself. Further to this, functors also admit their own notion of composition

$$\circ_{\mathbf{Cat}} : \mathrm{Fun}(\mathsf{D}, \mathsf{F}) \times \mathrm{Fun}(\mathsf{C}, \mathsf{D}) \to \mathrm{Fun}(\mathsf{C}, \mathsf{F})\,, \tag{1.2.15}$$

where we have included the subscript **Cat** both to avoid confusion with the regular notion of composition on categories, and to emphasise that functors (up to natural isomorphism) form morphisms in the category of categories [8].

In $1+1$ dimensions, the subset $\mathcal{O}_{\mathcal{T}}^{top}$ of topological lines and local operators has the structure of a category that we denote $\mathcal{C}_{\mathcal{T}}$. The topological lines form the objects of this category and topological local operators sitting at the junction between two lines describe the morphisms. This picture of topological defects is illustrated further in figure 1.5.

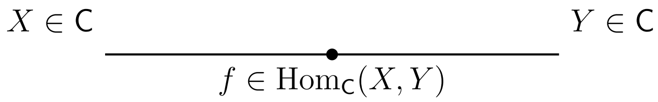

Figure 1.5

This alone is not enough structure to describe the symmetry of a quantum field theory however. Schematically, the correlation function is $\mathbb{C}$-linear

$$\langle a\mathcal{S}_1 + b\mathcal{S}_2 \rangle = a\langle \mathcal{S}_1 \rangle + b\langle \mathcal{S}_2 \rangle \tag{1.2.16}$$

for all operators $\mathcal{S}_1, \mathcal{S}_2$ and complex numbers $a, b \in \mathbb{C}$, and can support multiple insertions of defects that generate an operator product expansion

$$\langle \mathcal{S}_1(\gamma_1)\mathcal{S}_2(\gamma_2) \rangle = \sum_j a_{12}^j \langle \mathcal{S}_j(\gamma_2) \rangle\,. \tag{1.2.17}$$

The correlation function hence describes extra additive and multiplicative structure that the ordinary categorical structure does not account for, nor does it capture the orientation of topological defects or manipulations thereof. We now move on to

[8]This notion is improved by higher category theory, where **Cat** is properly understood as a 2-category with categories for objects, functors for 1-morphisms and natural transformations for 2-morphisms.

describing extra structure on the symmetry category $\mathcal{C}_\mathcal{T}$, starting with the additive structure.

A ($\mathbb{C}$-)*linear* category is defined as a category C such that:

- The set $\mathrm{Hom}_\mathsf{C}(X, Y)$ is a finite dimensional complex vector space for all objects $X, Y \in \mathsf{C}$.
- The composition defines a linear map between vector spaces

$$\circ : \mathrm{Hom}_\mathsf{C}(Y, Z) \otimes \mathrm{Hom}_\mathsf{C}(X, Y) \to \mathrm{Hom}_\mathsf{C}(X, Z)\,, \tag{1.2.18}$$

  for all objects $X, Y, Z \in \mathsf{C}$.

A prototypical example is Vec the category of finite-dimensional complex vector spaces, it is common to also see a linear category called a category *enriched* over Vec.

Given a linear category C, we define a *summand* of an object $X \in \mathsf{C}$ as an object $Y \in \mathsf{C}$ equipped with an *inclusion* monomorphism $\iota : Y \hookrightarrow X$, and a *projection* epimorphism $\pi : X \twoheadrightarrow Y$, satisfying

$$\pi \circ \iota = id_Y\,. \tag{1.2.19}$$

We define an *empty/zero object* as an object $\emptyset$ such that $(\emptyset, 0, 0)$ is a summand for every other object in C.

We further define the *direct sum* of two objects $X, Y \in C$ as an object $X \oplus Y \in C$ (unique up to isomorphism) such that (X, ι_X, π_X) and (Y, ι_Y, π_Y) are summands satisfying

$$\pi_Y \circ \iota_X = \pi_X \circ \iota_Y = 0 \tag{1.2.20}$$

and

$$\iota_Y \circ \pi_Y + \iota_X \circ \pi_X = id_{X\oplus Y}\,. \tag{1.2.21}$$

We define an object X to be *simple* if every non-zero monomorphism $\iota : Y \hookrightarrow X$ defines an isomorphism $\iota : Y \simeq X$. This implies summands of a simple object are isomorphic to that object. By definition then, simple objects cannot be written as a direct sum of non-zero objects. We define a linear category to be *semi-simple* if:

- The direct sum is closed:
$$X \oplus Y \in \mathsf{C} \tag{1.2.22}$$
for all objects $X, Y \in \mathsf{C}$.

- Every object is isomorphic to the direct sum of finitely many simple objects.

- There is a (unique) zero object $\emptyset$ that is an additive identity:
$$\emptyset \oplus X \simeq X \oplus \emptyset \simeq X \tag{1.2.23}$$
for all objects $X \in \mathsf{C}$.

We define a category to be *finite semi-simple* if it is semi-simple and contains finitely many simple objects (up to isomorphism).

Returning briefly to the context of (1+1)-dimensional QFT $\mathcal{T}$, the symmetry category $\mathcal{C}_{\mathcal{T}}$ should be considered linear, where for objects $X, Y \in \mathcal{C}_{\mathcal{T}}$ and $X \oplus Y \in \mathcal{C}_{\mathcal{T}}$ we identify
$$\langle X \oplus Y \rangle_{\mathcal{T}} = \langle X \rangle_{\mathcal{T}} + \langle Y \rangle_{\mathcal{T}} \,. \tag{1.2.24}$$

While it is perhaps natural to assume this direct sum is closed, and that we have a zero object $\langle \emptyset \rangle_{\mathcal{T}} = 0$, semi-simplicity is somewhat less natural. Indeed there are examples of symmetries in $1+1$ dimensions that do not exhibit semi-simplicity, a canonical example are those with twisted supersymmetry like the A/B-models appearing in topological string theories, and Landau-Ginzburg deformations thereof, where the non-trivial topological local operators form a chiral ring [39]. That said, those theories are not unitary, and the notion of unitarity we will adopt for categories in a moment actually enforces semi-simplicity [24].

There is a version of Schur's lemma for simple objects. Consider a morphism $f : X \to Y$ between simple objects $X, Y \in \mathsf{C}$, then let $(\mathrm{Ker}_f, \iota_f, \pi_f)$ be a summand of

X such that

$$f \circ \iota_f = 0 \,. \tag{1.2.25}$$

If this summand is non-zero, then by assumption $\iota_f : X \simeq \mathrm{Ker}_f$ is an isomorphism, and $f = 0$. If such a non-zero summand does not exist, then $f : X \simeq Y$ must be mono and hence an isomorphism. A similar argument for $f' = f - \lambda id_Y : X \to Y$ for $\lambda \in \mathbb{C}$ demonstrates that the space of isomorphisms is one-dimensional, ergo:

$$\mathrm{Hom}_{\mathsf{C}}(X,Y) = \begin{cases} \mathbb{C}\,, & X \simeq Y\,. \\ 0\,, & \text{otherwise.} \end{cases} \tag{1.2.26}$$

This demonstrates an important point about decomposition: if the trivial line operator in $\mathcal{C}_{\mathcal{T}}$ is simple, there are no non-trivial local operators and the QFT will not decompose into super-selection sectors. We now move onto the multiplicative structure.

We define a *monoidal* category as a category C equipped with a functor

$$\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C} \tag{1.2.27}$$

and *unit object* $1 \in \mathsf{C}$, together with left/right *unitor* and *associator* natural isomorphisms

$$\begin{gathered} l : (1 \otimes id_{\mathsf{C}}) \to id_{\mathsf{C}} \qquad\qquad r : (id_{\mathsf{C}} \otimes 1) \to id_{\mathsf{C}} \\ a : (id_{\mathsf{C}} \otimes id_{\mathsf{C}}) \otimes id_{\mathsf{C}} \to id_{\mathsf{C}} \otimes (id_{\mathsf{C}} \otimes id_{\mathsf{C}})\,, \end{gathered} \tag{1.2.28}$$

satisfying a triangle identity

$$(l_X \otimes id_Y) \circ a_{X,1,Y} = id_X \otimes r_Y \,, \tag{1.2.29}$$

and a pentagon identity

$$(id_X \otimes a_{Y,Z,W}) \circ a_{X,Y\otimes Z,W} \circ (a_{X,Y,Z} \otimes id_W) = a_{X,Y,Z\otimes W} \circ a_{X\otimes Y,Z,W} \,, \tag{1.2.30}$$

for all objects $X, Y, Z, W \in \mathsf{C}$. These conditions are illustrated schematically in figures 1.6 and 1.7.

By convention we will choose to suppress the left and right unitor data, by choosing

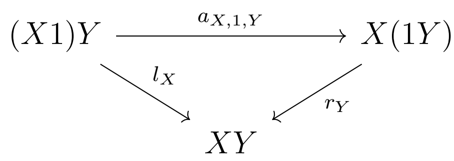

Figure 1.6

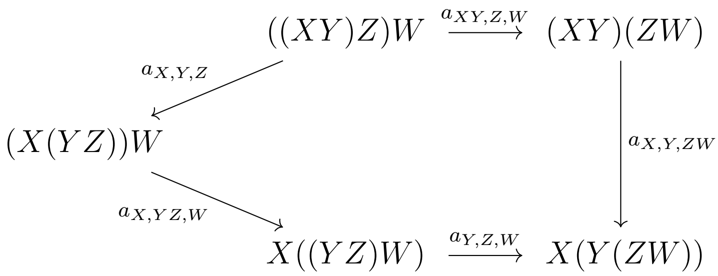

Figure 1.7

them to be $l_X = id_X = r_X$ up to a choice of natural isomorphism on $\otimes$. We will define a *monoidal* functor as a functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ between monoidal categories $(\mathsf{C}, \otimes_{\mathsf{C}}, a^{\mathsf{C}})$ and $(\mathsf{D}, \otimes_{\mathsf{D}}, a^{\mathsf{D}})$ together with a natural isomorphism

$$\mu_{X,Y} : \mathcal{F}(X \otimes Y) \simeq (\mathcal{F}X) \otimes (\mathcal{F}Y) \tag{1.2.31}$$

that is compatible with the associativity in such a way that:

$$\begin{aligned} &\mu_{X,Y\otimes Z} \circ_{\mathsf{D}} (id_{\mathcal{F}X} \otimes_{\mathsf{D}} \mu_{Y,Z}) \circ_{\mathsf{D}} a^{\mathsf{C}}_{X,Y,Z} \\ &\qquad\qquad = a^{\mathsf{D}}_{X,Y,Z} \circ \mu_{X\otimes Y,Z} \circ_{\mathsf{D}} (\mu_{X,Y} \otimes_{\mathsf{D}} id_{\mathcal{F}Z}) \, . \end{aligned} \tag{1.2.32}$$

In the context of our symmetry category $\mathcal{C}_{\mathcal{T}}$, this monoidal structure captures the operator product expansion of two topological lines as illustrated in figure 1.8. This product is well-defined in the sense that it is naturally (topologically) regularised. The trivial line defect automatically defines a unit object and the associator data is captured by the phases $a_{X,Y,Z}$ appearing in figure 1.9.

Thinking of the symmetry category $\mathcal{C}_{\mathcal{T}}$ as a finite semi-simple monoidal category begins to capture the structure of addition and multiplication for topological lines. Next we will discuss how to incorporate the orientation of topological defects.

Given a monoidal category $(\mathsf{C}, \otimes, a)$, we define the *left dual* of an object $X \in \mathsf{C}$ as

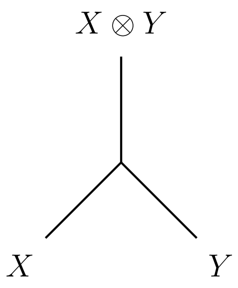

Figure 1.8

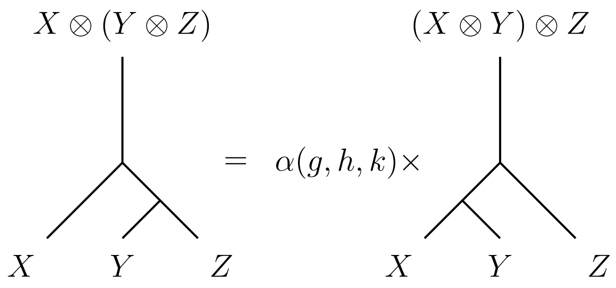

Figure 1.9

an object $X^* \in \mathsf{C}$ together with *evaluation* and *coevaluation* morphisms [9]

$$ev^l_X : X^* \otimes X \to 1 \qquad\qquad coev^l_X : 1 \to X \otimes X^* \,, \tag{1.2.33}$$

such that they satisfy snake relations

$$\begin{aligned}(id_X \otimes ev^l_X) \circ a_{X,X^*,X} \circ (coev^l_X \otimes id_X) = id_X \,, \\ (ev^l_X \otimes id_{X^*}) \circ a^{-1}_{X^*,X,X^*} \circ (id_{X^*} \otimes coev^l_X) = id_{X^*} \,.\end{aligned} \tag{1.2.34}$$

This is entirely equivalent to defining X as a *right dual* of X^*.

Notice this immediately sets up an isomorphism

$$(X \otimes Y)^* \simeq Y^* \otimes X^* \tag{1.2.35}$$

for all objects $X, Y \in \mathsf{C}$ admitting left duals.

An object $X \in \mathsf{C}$ is called *dualisable* if it admits both left and right duals. That monoidal category is further called *rigid/autonomous* if every object is dualisable. We define a *multi-fusion* category to be a rigid finite semi-simple monoidal category, and a *fusion* category to be a rigid finite semi-simple monoidal category whose unit object is simple.

Thinking of $\mathcal{T}$ as a (indecomposable) framed quantum field theory in 1+1 dimensions, that is, a theory defined on a framed 2-manifold (which are considered somewhat unphysical), the minimal data needed to describe the symmetry category $\mathcal{C}_{\mathcal{T}}$ is that

[9]These morphisms can be thought of as implementing a category-theoretic generalisation of semi-unitarity as seen in equation (1.1.17).

of a fusion category.

We define a *pivotal* category as a rigid monoidal category C where for each object $X^* \in \mathsf{C}$, the left and right duals $X, X^{**} \in \mathsf{C}$ are unique up to isomorphism and are identified by

$$S_X : X \simeq X^{**}\,, \tag{1.2.36}$$

such that the morphisms

$$\begin{aligned} ev^r_X &:= ev^l_{X^*} \circ (S_X \otimes id_{X^*}) : X \otimes X^* \to 1 \\ coev^r_X &:= (id_{X^*} \otimes S_X^{-1}) \circ coev^l_{X^*} : 1 \to X^* \otimes X \end{aligned} \tag{1.2.37}$$

identify X^* as a right dual of X.

Given an object $X \in \mathsf{C}$ in a pivotal category, the *left/right trace* of an endomorphism $f \in \mathrm{End}_{\mathsf{C}}(X)$ are defined as:

$$\begin{aligned} \mathrm{Tr}^l(f) &: ev^l_X \circ (f \otimes id_{X^*}) \circ coev^l_X : 1 \to 1\,, \\ \mathrm{Tr}^r(f) &: ev^r_X \circ (id_{X^*} \otimes f) \circ coev^r_X : 1 \to 1\,. \end{aligned} \tag{1.2.38}$$

We define a *spherical* category as a pivotal category where these traces coincide:

$$\mathrm{Tr}^l(f) = \mathrm{Tr}^r(f)\,, \tag{1.2.39}$$

for all endomorphisms $f \in \mathrm{End}_{\mathsf{C}}(X)$ and objects $X \in \mathsf{C}$.

In our QFT $\mathcal{T}$, the inclusion of left and right duals corresponds to "bending" the topological lines in $\mathcal{C}_{\mathcal{T}}$, as illustrated in figure 1.10.

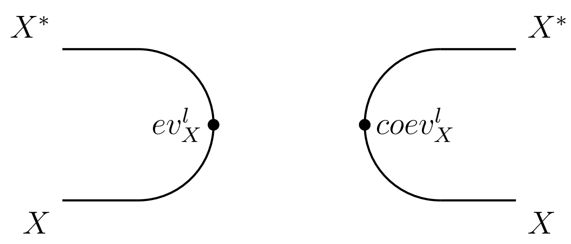

Figure 1.10: directed left to right

The consistency conditions in equation (1.2.34) correspond physically to our ability to straighten these bends as illustrated in figure 1.11.

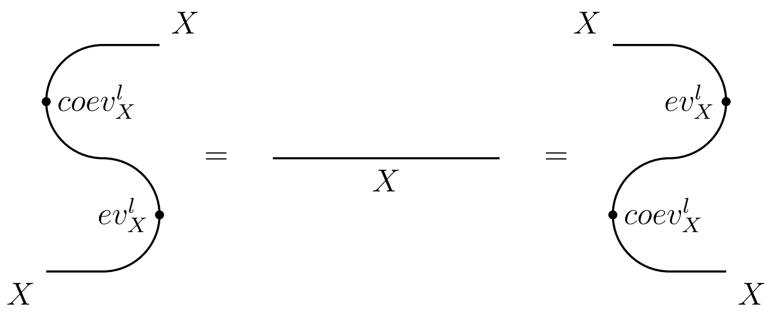

Figure 1.11: directed left to right

With the existence of left/right duals we are asking that the symmetry category $\mathcal{C}_\mathcal{T}$ is rigid, making it a multi-fusion category. Further asking that the theory does not decompose into super-selection sectors we have that the monoidal unit must be simple and hence the structure of $\mathcal{C}_\mathcal{T}$ is a fusion category.

We can take this further though: we expect the defects produced by bending a defect either to the left or to the right to be identified, further inducing a pivotal fusion structure on the symmetry category. The left and right traces correspond to oriented loops with the insertion of an endomorphism as illustrated in figure 1.12.

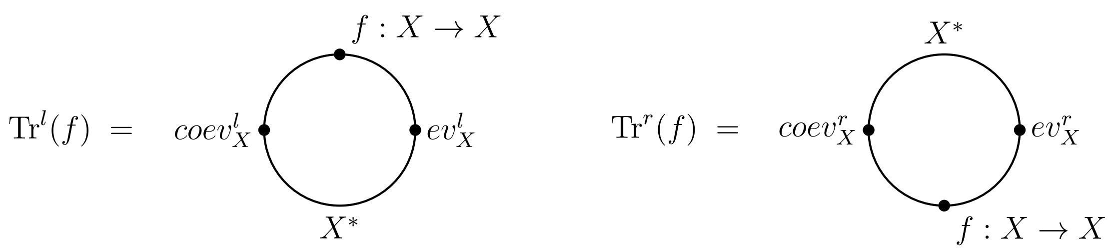

Figure 1.12: directed left to right

It is not immediately clear from topological manipulations that these traces should coincide until we consider placing the theory $\mathcal{T}$ on a 2-sphere. In that case stretching the loop over the equator of the sphere identifies the two traces and restricts us to spherical fusion categories.

Thinking of $\mathcal{T}$ as an (indecomposable) oriented quantum field theory $\mathcal{T}$ in $1+1$ dimensions, the minimal data needed to describe the symmetry category $\mathcal{C}_\mathcal{T}$ is that of a spherical fusion category.

If we want to describe the symmetry of a *unitary* oriented quantum field theory, we need further structure still.

We define a *dagger* category as a category C equipped with an anti-linear involution $\dagger$ such that for each morphism $f : X \to Y$, there exists a morphism

$$f^\dagger : Y \to X\,, \tag{1.2.40}$$

satisfying

$$id_X^\dagger = id_X \qquad (f \circ g)^\dagger = g^\dagger \circ f^\dagger \qquad f^{\dagger\dagger} = f \tag{1.2.41}$$

for all morphisms $f : X \to Y$, $g : Z \to X$ and objects $X, Y, Z \in \mathsf{C}$.

We define a *dagger* functor as a functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ between dagger categories C and D, that preserves the dagger structure:

$$\mathcal{F}(f^\dagger) = (\mathcal{F}f)^\dagger \tag{1.2.42}$$

for all morphisms $f : X \to Y$ and objects $X, Y \in \mathsf{C}$.

We define a *unitary* fusion category C is a dagger fusion category such that:

- The dagger is compatible with the fusion:

  $$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger \tag{1.2.43}$$

  for all morphisms $f : X \to Y$, $g : W \to Z$ and objects $X, Y, Z, W \in \mathsf{C}$.

- The daggered (co-)evaluation morphisms

  $$(ev_X^l)^\dagger =: coev_X^r \qquad\qquad (coev_X^l)^\dagger =: ev_X^r \tag{1.2.44}$$

  satisfy the identities (1.2.34), identifying each X^* as a right dual for all $X \in \mathsf{C}$, making C a pivotal fusion category.

- The pivotal structure is spherical:

  $$\mathrm{Tr}^l = \mathrm{Tr}^r =: \mathrm{Tr}\,, \tag{1.2.45}$$

and the trace

$$\langle f, g\rangle := \mathrm{Tr}(f^\dagger \circ g) \tag{1.2.46}$$

defines a consistent inner product for all morphisms $f, g \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$, inducing the structure of a finite-dimensional Hilbert space for each pair of objects $X, Y \in \mathsf{C}$ [10].

In the context of our quantum field theory $\mathcal{T}$, each line defect can be thought of equivalently as the insertion of a (topological) quantum mechanical theory on a choice of 1-submanifold. In the last section we explained that the topological local operators in a quantum mechanics are described by a finite-dimensional C*-algebra. Considering the symmetry category $\mathcal{C}_{\mathcal{T}}$ as a unitary fusion category makes this notion precise; the endomorphisms $\mathrm{End}_{\mathcal{C}_{\mathcal{T}}}(X)$ of a given topological line $X \in \mathcal{C}_{\mathcal{T}}$ describe a finite-dimensional C*-algebra with inner-product given by (1.2.46). The condition that the †-operation defines consistent right duals is illustrated for line defects in figure 1.13.

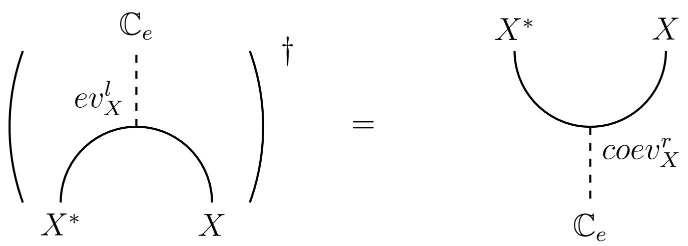

Figure 1.13: directed bottom to top

Another way to interpret the condition that the trace (1.2.46) defines a consistent inner product, is from the perspective of reflection-positivity [40]. Restricting our attention to a trivial defect loop $1 \in \mathcal{C}_{\mathcal{T}}$ centred on and perpendicular to some chosen line, then for some morphism $f : 1 \to 1$, the corresponding inner product with itself can then be identified as the correlation function for two (topological) local operators $f, f^\dagger$ whose locations are identified by reflection in the line. The

[10]Recent literature instead proposes a weaker constraint that the endomorphisms of a given object should form a C*-algebra. In the finite semi-simple setting of finite-dimensional Hilbert spaces we expect these definitions to coincide.

statement of reflection positivity is then just the positive-definiteness condition

$$\langle f, f \rangle = \mathrm{Tr}(f^\dagger \circ f) \geq 0\,. \tag{1.2.47}$$

For a (indecomposable) *unitary oriented* quantum field theory in $1+1$ dimensions, the minimal data needed to describe the symmetry category is that of a unitary fusion category. We will return to this construction more concretely for a group-like symmetry in subsection 3.1.1.

1.2.2 Non-Invertible Symmetries in QFT : Critical Ising Model

Just as we saw in quantum mechanics, identifying symmetries with topological operators lends the opportunity for more vibrant structure than that of group theory. In the rest of this thesis we will demonstrate the extent of this structure in a large class of examples obtained by starting with a finite group symmetry and gauging. In this section however, to illustrate the existence of non-invertible symmetries in QFT we will consider a much simpler example that specifically does not arise in this way: a conformal field theory (CFT) with central charge $c = \frac{1}{2}$, known as the critical Ising model.

Historically, the $(1+1)$-dimensional Ising model arose as a lattice QFT used to describe the physics of a $(1+1)$-dimensional ferromagnet [41]. These systems undergo a phase transition [42, 43] from ferromagnetic to paramagnetic as the temperature approaches the Curie temperature, and enjoy a Kramers-Wannier duality that identifies highly-ordered states at temperatures below the Curie temperature, with low-ordered states at temperatures above. The critical Ising model exists at the Curie temperature, where the physics become scale invariant, and describes a $(1+1)$-dimensional conformal field theory. In this critical limit, the Kramers-Wannier duality takes the critical Ising state to itself, describing a genuine symmetry of the underlying CFT [30, 44–47].

In our language of topological defects, we can imagine bifurcating a (1+1)-dimensional space and inserting a low-ordered state on one half, and a dual highly-ordered state on the other, identifying the observables on the interface with their Kramers-Wannier duals. Since these two states are dual to one-another and describe the same physics, the interface cannot be a measurable feature of the theory; it must be topological. In the critical case the two halves are automatically identified, making the interface a genuine topological defect of the theory that we call the Kramers-Wannier line defect.

We can equivalently construct the symmetry category by studying the Verlinde lines [48] corresponding to conformal primaries in the underlying CFT. This results in three simple line defects: two defects $1 \in \mathbb{Z}_2$ and $g \in \mathbb{Z}_2$ that describe a $\mathbb{Z}_2$ symmetry, and the non-invertible Kramers-Wannier line that we denote by $\mathcal{N}$. We then have fusion rules inherited from the operator product expansion (OPE) given by:

$$\begin{aligned} g \otimes g &\simeq 1\,, \\ g \otimes \mathcal{N} &\simeq \mathcal{N}\,, \\ \mathcal{N} \otimes \mathcal{N} &\simeq 1 \oplus g\,. \end{aligned} \tag{1.2.48}$$

Formally, this mathematical structure is that of a $\mathbb{Z}_2$ Tambara-Yamagami fusion category. The last of these fusion rules identifies the Kramers-Wannier line as being non-invertible. This non-invertibility carries interesting implications for the action on local operators; the resulting local operator is twisted in the sense it lives at the end of the line corresponding to $g \in \mathbb{Z}_2$ as depicted in figure 1.14 [46].

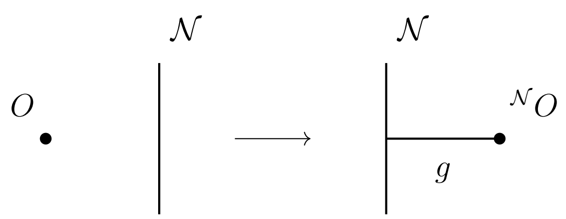

Figure 1.14

In the language of conformal primaries, this describes the mapping of the spin operator $O = \sigma$ to its dual the disorder operator ${}^{\mathcal{N}}O = \mu$ in the twisted sector for $g \in \mathbb{Z}_2$. In this sense the non-invertibility of $\mathcal{N}$ is necessary to capture the non-trivial mapping Kramers-Wannier duality describes between states occupying different twisted sectors. This behaviour is analogous to that we saw in quantum mechanics in subsection 1.1.2.

We note that the construction of topological defects we described is general; for any duality of a theory admitting a fixed point at which the theory becomes self-dual, the topological interface construction at the fixed point will result in a genuine symmetry. Furthermore if that duality non-trivially mixes twisted sectors of the theory, then the resulting symmetry *must* be non-invertible.

We also note that from a different point of view, the topological interfaces away from the critical point can be interpreted as domain walls, and that a small perturbation to the temperature away from the Curie temperature can be analogously interpreted as a (non-conformal) deformation to the action that spontaneously breaks the non-invertible Kramers-Wannier symmetry. This deformation can be interpreted more precisely as a Yukawa-type coupling to a scalar field, whose sign changes under the Kramers-Wannier symmetry, and whose vacuum generates a non-zero fermionic mass term that controls the temperature [49].

More generally we might expect the following generalisation:

dualities that have a self-dual point can be recast as the spontaneous breaking of a symmetry living in the self-dual theory, and if that duality non-trivially mixes the twisted sectors of a theory, then the corresponding symmetry must be non-invertible.

Recent work has focused on generalising these Tambara-Yamagami structures to higher dimensions [50–52]. A particularly interesting application of this mathematical structure appears in four dimensions, where Kramers-Wannier-like defects describe the nature of non-invertible symmetries known as duality defects appearing in self-dual gauge theories [5, 6].

In chapter 4 we will argue one of the central points of this thesis: that in three-dimensional (oriented) quantum field theories there are no symmetries of this sort. More precisely, we conjecture that the symmetries of all such theories are group-theoretic in the sense that they can be obtained by gauging a finite subgroup, and that their structure is determined entirely from finite group and representation theory [53].

1.2.3 Higher Dimensions and Higher Category Theory

A similar categorification story occurs in higher dimensions, with the key difference that from $1+1$ to $2+1$ dimensions we are categorifying categories to obtain *2-categories*, and in general from $d+1$ dimensions to $d+2$ dimensions we categorify from a d-category to a $(d+1)$-category. These subsequent categorifications are known as higher categories.

To start, we define an *2-category* (or *bicategory*) C analogously to a category:

- There are *objects* $X \in \mathsf{C}$ (sometimes fashioned as *0-morphisms*) that form the top level of the 2-category.
- There are *1-morphisms* $f : X \to Y$ between objects $X, Y \in \mathsf{C}$ that form the second level of the 2-category.
- There are *2-morphisms* $\theta : f \to g$ between 1-morphisms $f, g \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ that form the final level of the 2-category.
- There is a composition of 1-morphisms

$$\circ : \mathrm{Hom}_{\mathsf{C}}(Y, Z) \times \mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{C}}(x, Z) \tag{1.2.49}$$

  for each triple of objects $X, Y, Z \in \mathsf{C}$, together with:

  – A (unique up to 2-isomorphism) identity 1-morphism $id_X \in \mathrm{End}_{\mathsf{C}}(X)$ and *unitor* natural 2-isomorphisms

$$l_f : id_Y \circ f \Rightarrow f \qquad\qquad r_f : f \circ id_X \Rightarrow f\,, \tag{1.2.50}$$

for all 1-morphisms $f : X \to Y$ and objects $X, Y \in \mathsf{C}$.

– An *associator* natural 2-isomorphism

$$a_{f,g,h} : (f \circ g) \circ h \Rightarrow f \circ (g \circ g) \tag{1.2.51}$$

for all 1-morphisms $h : X \to Y$, $g : Y \to Z$, $f : Z \to W$ and objects $X, Y, Z, W \in \mathsf{C}$.

satisfying triangle and pentagon relations analogous to those illustrated in figures 1.6 and 1.7.

- There is a *vertical* composition of 2-morphisms

$$\circ : \mathrm{Hom}_{\mathsf{C}}(f, h) \times \mathrm{Hom}_{\mathsf{C}}(g, k) \to \mathrm{Hom}_{\mathsf{C}}(f \circ g, h \circ k) \tag{1.2.52}$$

for each quadruple of 1-morphisms $f, g : X \to Y$ and $h, k : Y \to Z$ and triple of objects $X, Y, Z \in \mathsf{C}$, satisfying associativity:

$$(\theta \circ \gamma) \circ \kappa = \theta \circ (\gamma \circ \kappa) \tag{1.2.53}$$

for all 2-morphisms $\theta : f \to h$, $\gamma : g \to k$, $\kappa : m \to n$, 1-morphisms $f, h : Z \to W$, $g, k : Y \to Z$, $m, n : X \to Y$, and objects $X, Y, Z, W \in \mathsf{C}$.

- There is a *horizontal* composition of 2-morphisms

$$\cdot : \mathrm{Hom}_{\mathsf{C}}(g, h) \times \mathrm{Hom}_{\mathsf{C}}(f, g) \to \mathrm{Hom}_{\mathsf{C}}(f, h) \tag{1.2.54}$$

for each triple of 1-morphisms $f, g, h : X \to Y$ and pair of objects $X, Y \in \mathsf{C}$, that is associative:

$$(\theta \cdot \gamma) \cdot \kappa = \theta \cdot (\gamma \cdot \kappa)\,, \tag{1.2.55}$$

and has a unique identity 2-automorphism $id_f \in \mathrm{End}_{\mathsf{C}}(f)$ such that

$$id_g \cdot \theta = \theta \cdot id_f = \theta \,, \tag{1.2.56}$$

for all 2-morphisms $\theta : h \to k$, $\gamma : g \to h$, $\kappa : f \to g$, 1-morphisms $f, g, h : X \to Y$, and objects $X, Y \in \mathsf{C}$.

One immediate result of this definition is that the category $\mathrm{End}_{\mathsf{C}}(X)$ of any object $X \in \mathsf{C}$ is monoidal.

In the context of a $(2 + 1)$-dimensional quantum field theory, the objects represent topological surface operators, the 1-morphisms are topological line operators between surfaces, and 2-morphisms topological local operators between lines.

Continuing the analogy, one can define linear, monoidal, rigid, pivotal and (multi-)fusion 2-categories. More recently there has been progress on generalising the notions of spherical [54], and unitary [25] to fusion 2-categories. In chapter 4 we will study 2-categories in more detail, focusing on concrete examples of fusion 2-categories in section 4.1. In fact, better understanding unitary fusion 2-categories forms part of the motivation for this thesis: our conjecture that all fusion 2-categories are group-theoretic provides a concrete platform to discuss unitarity, and we shall do so in subsection 4.3.3.

Similarly, one can also define higher $(n + 1)$-categories in an iterative fashion, where the 1-endomorphisms with their composition form monoidal n-categories. In this way, in a $(d + 1)$-dimensional QFT the objects of the symmetry n-category capture d-dimensional or codimension-1 topological defects, and in general the k-morphisms capture codimension-k topological defects, and we generalise the statement made at the beginning of this section:

symmetries in a QFT are described by (higher) category theory.

More introductory details on higher categories can be found in [55], and for more contemporary work on unitary and (∞, n)-constructions one can look in [24, 25, 56].

1.3 Topological Quantum Field Theories

In the previous section we remarked that topological line defects in a $(1+1)$-dimensional QFT can be recast as insertions of a quantum mechanics. Of course it is not the case that we can couple any choice of quantum mechanics and expect it to describe a topological defect; the quantum mechanics we couple to must itself be topological in that it should not depend on anything more than the orientation [11] of the 1-submanifold we insert it on. Viewed as a $(0+1)$-dimensional QFT, this restricts our focus to (oriented) topological quantum field theories, or TQFTs.

In a more mathematical setting there is the Atiyah-Segal axiomatisation of TQFTs [57], which casts an oriented $(d+1)$-dimensional TQFT in the language of category theory as a monoidal functor

$$Z : \mathsf{Cob}^{or}_{d+1} \to \mathsf{Vec}\,, \tag{1.3.1}$$

where Cob^{fr}_{d+1} is the (symmetric) monoidal category of oriented d-manifolds with the disjoint union and morphisms given by oriented $(d+1)$-dimensional (co)bordisms [12] between them. The idea being to assign to each d-manifold $\mathcal{M}_d$ a complex vector space $Z(\mathcal{M}_d)$ of states that characterises the QFT, and to each cobordism $c : \mathcal{M}_d \to \mathcal{M}'_d$ a linear map

$$Z(c) : Z(\mathcal{M}_d) \to Z(\mathcal{M}'_d)\,, \tag{1.3.2}$$

that interpolates between those spaces of states, formalising how excitations of the theory propagate over time. This is illustrated further in figure 1.15.

More generally we can talk about *n-extended* TQFTs which generalise the target

[11] In particular the theory should not depend on a choice of (induced) metric, as is often remarked in the physics literature.

[12] Cobordism and bordism are synonymous. This odd disparity in naming convention arises from the French word "bord" for "boundary", in the context of which "co-" actually refers to a "shared" boundary rather than a contravariant version thereof. That said, in the literature the notation **Cob** is often used to denote a category of manifolds and bordisms between them, while the notation **Bord** is usually reserved for the extended (∞, n)-categorical description of bordisms, which we will not study in detail here.

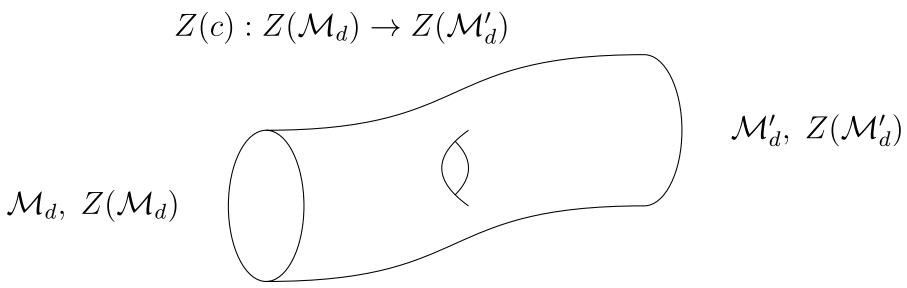

Figure 1.15: directed left to right

and replace this functor for one between (symmetric) monoidal $(n+1)$-categories [13]

$$Z : \mathsf{Bord}^{or}_{d+1,n} \to \mathsf{C}\,. \tag{1.3.3}$$

Here, C can be any symmetric monoidal $(n+1)$-category where the theory is said to be valued. The $(n+1)$-category Bord^{or}_{d+1} meanwhile contains objects corresponding to oriented $(d-n)$-manifolds, and k-morphisms corresponding to $(d-n+k)$-dimensional oriented cobordisms between $(k-1)$-morphisms. In the context of topological defects, we need the TQFTs living on those defects to be compatible with the inclusion of edges and corners and so on, and so we naturally expect some of them to be extended. When $n = d$, the theory is said to be *fully* extended. For more details on this construction one can turn to [36, 37].

In this section however, we will restrict ourselves to discussing only some latent features of this construction relevant to classifying (oriented) TQFTs in low dimensions, due to [58, 59], and describing where and how they fit into the language of category theory.

1.3.1 TQFTs in $1+1$ Dimensions

Oriented (unextended) TQFTs in $1+1$ dimensions are described by commutative Frobenius algebras. The reasoning for this becomes clear from the functorial description above by considering the finite-dimensionsal vector space assigned to the circle $A := Z(S^1)$. This vector space is automatically equipped with the structure

[13] Or more formally, $(\infty, n+1)$-categories.

of a Frobenius algebra by the assignment of pair-of-pants and cap/cup bordisms as illustrated in figure 1.16.

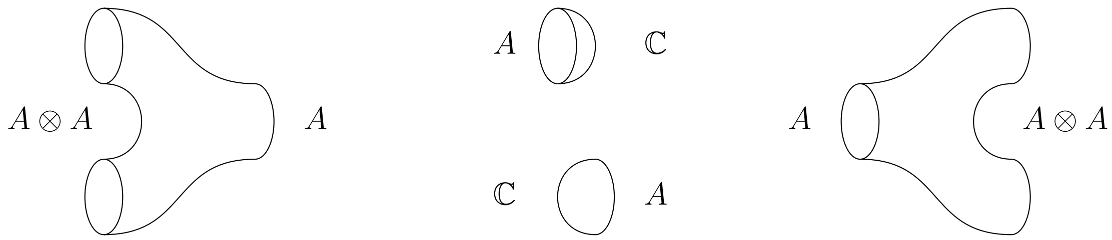

Figure 1.16: directed left to right

The pair-of-pants bordisms equip A with a multiplication and co-multiplication

$$m : A \otimes A \to A \qquad \Delta : A \to A \otimes A\,, \tag{1.3.4}$$

and the cup/cap bordisms further equip A with a unit and co-unit

$$i : \mathbb{C} \to A \qquad p : A \to \mathbb{C}\,. \tag{1.3.5}$$

The pair-of-pants bordisms are equivalent under the swapping of the legs, making these (co-)multiplications commutative, and the composition of these bordisms up to equivalence further demonstrate the associativity, unitality and Frobenius conditions (discussed in more detail in subsection 3.3.1).

Considering the space of states on S^1 is an important part of this construction as there is a state-operator correspondence: shrinking the circle to a point corresponds to replacing a boundary on which the states of the theory live, with the insertion of a topological local operator. In this way the algebra A living on S^1 admits a dual interpretation as the commutative Frobenius algebra of topological local operators living in the theory. From this perspective it is obvious why the product must be commutative: the topological local operators can wind around each other in $1+1$-dimensions.

The extra data that lifts these algebras to fully extended TQFTs is that which lifts A to a full (not necessarily commutative) Frobenius algebra $\mathcal{A}$. This is related to the previous construction by the centre

$$A = \mathcal{Z}(\mathcal{A}) \,. \tag{1.3.6}$$

In the setting of unitary TQFTs these are upgraded to Frobenius C*-algebras; A captures the local operators in the bulk of the $(1+1)$-dimensional TQFT, while the choice of $\mathcal{A}$ captures the local operators living on some corresponding choice $(0+1)$-dimensional topological boundary.

Just as we noted at the end of section 1.1.2, the choice of algebra $\mathcal{A}$ similarly admits a decomposition into matrix algebras. At the level of the centre this simplifies further to

$$A \simeq \mathbb{C}^{\oplus n} \,, \tag{1.3.7}$$

for some $n \in \mathbb{N}$. Hence, if we are to have an indecomposable $(1+1)$-dimensional TQFT, there is only one choice corresponding to $A \simeq \mathbb{C}$.

Something less trivial we can consider however, is central extensions of a finite group G by this trivial algebra $A \simeq \mathbb{C}$. These result in $(1+1)$-dimensional Dijkgraaf-Witten TQFTs with gauge group G and topological action determined by the extension class

$$[\psi] \in H^2_{grp}(G, \mathbb{C}^\times) \,. \tag{1.3.8}$$

These types of theories describe $(1+1)$-dimensional *symmetry protected* topological (SPT) phases, and we will study them in more detail in subsection 2.3.3.

1.3.2 TQFTs in $2+1$ Dimensions

In the previous subsection we saw that fully extended oriented TQFTs in $1+1$ dimensions correspond to Frobenius algebras of local operators living on the $(0+1)$-dimensional boundary, and whose centre captured local operators living in the bulk. There is an analogous formulation of a class of fully extended oriented TQFTs in $2+1$ dimensions known as *Turaev-Viro* TQFTs, which are described by (spherical) (multi-)fusion categories of topological line defects living on the $(1+1)$-dimensional boundary [60–62]. To describe this labelling further we must introduce some additional technology.

We define a *braided* monoidal/(multi-)fusion category to be a monoidal/(multi-)fusion category $(\mathsf{B}, \otimes, a)$ equipped with a set of *half-braiding* natural isomorphisms

$$b_{X,Y} : X \otimes Y \simeq Y \otimes X \tag{1.3.9}$$

satisfying hexagon identities:

$$\begin{aligned} a^{-1}_{Y,Z,X} \circ (id_Y \otimes b_{X,Z}) \circ a_{Y,X,Z} \circ (b_{X,Y} \otimes id_Z) &= b_{X,Y\otimes Z} \circ a_{X,Y,Z} \\ (b_{X,Z} \otimes id_Y) \circ a^{-1}_{X,Z,Y} \circ (id_X \otimes b_{Y,Z}) \circ a_{X,Y,Z} &= a^{-1}_{Z,X,Y} \circ b_{X\otimes Y,Z} \,, \end{aligned} \tag{1.3.10}$$

for all objects $X, Y, Z \in \mathsf{B}$.

A *braided* (monoidal) functor is a monoidal functor $\mathcal{F} : \mathsf{B} \to \mathsf{B}'$ compatible with the half-braidings b and b' such that

$$\mu_{X,Y} \circ_{\mathsf{B}'} \mathcal{F} b_{X,Y} = b'_{X,Y} \circ_{\mathsf{B}'} \mu_{X,Y} \,. \tag{1.3.11}$$

We define a *symmetric* (braided) monoidal/(multi-)fusion category as a braided monoidal/(multi-)fusion category B such that the half-braiding is symmetric

$$b_{X,Y} = b^{-1}_{Y,X} \,, \tag{1.3.12}$$

for all objects $X, Y \in \mathsf{B}$.

We define the *Müger centre* of a braided monoidal/(multi-)fusion category B to be the full (symmetric fusion) subcategory $\mathcal{Z}_2(\mathsf{B}) \subseteq \mathsf{B}$ of objects $X \in \mathsf{B}$ such that

$$b_{X,Y} = b^{-1}_{Y,X} \,, \tag{1.3.13}$$

for all objects $Y \in \mathsf{B}$. We define a braided fusion category B to be *non-degenerate* if it has trivial Müger centre

$$\mathcal{Z}_2(\mathsf{B}) \simeq \mathsf{Vec} \tag{1.3.14}$$

as symmetric fusion categories. Following [24, 63] we can define further structure on a non-degenerate braided fusion category turning it into a *unitary modular tensor* *category* (UMTC), however we have no need for the details of this construction and will not exposit it here.

We define the *Drinfeld* centre of a monoidal/(multi-)fusion category $(\mathsf{C}, \otimes, a)$ as the monoidal/(multi-)fusion category $\mathcal{Z}(\mathsf{C})$ whose categorical structure is given by:

- Objects $X \in \mathcal{Z}(\mathsf{C})$ are objects $X \in \mathsf{C}$ together with a half-braiding implemented by isomorphisms

$$\beta_Y : X \otimes Y \simeq Y \otimes X \tag{1.3.15}$$

  such that

$$\beta_{Y \otimes Z} = a^{-1}_{Y,Z,X} \circ (id_Y \otimes \beta_Z) \circ a_{Y,X,Z} \circ (\beta_Y \otimes id_Z) \circ a^{-1}_{X,Y,Z}\,, \tag{1.3.16}$$

  for all objects $Y, Z \in \mathsf{C}$. Their fusion $(X, \beta^X) \otimes (Y, \beta^Y) \simeq (X \otimes Y, \beta^{X \otimes Y})$ is inherited from C, where

$$\beta^{X \otimes Y}_Z = a_{Z,X,Y} \circ (\beta^X_Z \otimes id_Y) \circ a^{-1}_{X,Z,Y} \circ (id_X \otimes \beta^Y_Z) \circ a_{X,Y,Z}\,. \tag{1.3.17}$$

- Morphisms $f : (X, \beta_X) \to (Y, \beta_Y)$ are morphisms $f : X \to Y$ such that

$$(f \otimes g) \circ \beta_{X,Z} = \beta_{Y,W} \circ (f \otimes g) \tag{1.3.18}$$

  for all $g : Z \to W$ and objects $Z, W \in \mathsf{C}$, and composition inherited from C.

The Drinfeld centre is automatically a braided monoidal/(multi-)fusion category with the supplied half-braidings. When C is fusion, $\mathcal{Z}(\mathsf{C})$ is a non-degenerate braided fusion category. Further to this, when C is a unitary fusion category we expect $\mathcal{Z}(\mathsf{C})$ to have the structure of a UMTC [24, 63].

The existence of the Drinfeld centre naturally suggests a notion of equivalence between monoidal/(multi-)fusion categories

$$\mathsf{C} \sim \mathsf{C}' \iff \mathcal{Z}(\mathsf{C}) \simeq \mathcal{Z}(\mathsf{C})\,. \tag{1.3.19}$$

This notion coincides with the notion of *Morita* equivalence for monoidal/(multi-)fusion categories, which we will study in more detail in chapter 3.

We define the forgetful functor to be the monoidal functor

$$F : \mathcal{Z}(\mathsf{C}) \to \mathsf{C} \tag{1.3.20}$$

obtained by "forgetting" about the braiding information on each object in $\mathcal{Z}(\mathsf{C})$.

In analogy to the previous section, the (spherical) (multi-)fusion category C describing topological defects on the $(1+1)$-dimensional boundary is the natural lift of the Frobenius algebra $\mathcal{A}$, whereas its Drinfeld centre $\mathcal{Z}(\mathsf{C})$ describes the category of topological line defects living in the $(2+1)$-dimensional bulk, and is the natural lift of the algebraic centre $A = \mathcal{Z}(A)$. The braided structure of this category captures the phases accrued by winding one line defect around another. As we mentioned above, the Drinfeld centre is an invariant over a Morita class of fusion category, each of these representative fusion categories corresponds physically to a different choice of topological boundary condition, with bulk-boundary map given by the corresponding forgetful functor $F : \mathcal{Z}(\mathsf{C}) \to \mathsf{C}$. It has been shown that the TQFT resulting from this construction is fully extended, and is indecomposable if and only if the category C is fusion [60–62].

This said, not all $(2+1)$-dimensional TQFTs are expected to be Turaev-Viro. A more general class of TQFTs known as *Reshetikhin–Turaev* TQFTs can be constructed starting from any UMTC [64, 65]. For our purposes, this includes indecomposable $(2+1)$-dimensional TQFTs, the topological lines of which are described by non-degenerate braided fusion categories. While it is not clear if such TQFTs can be fully extended, we can at least identify that the class of Turaev-Viro TQFTs are contained within the class of Reshetikhin–Turaev TQFTs by picking those theories with non-degenerate braided fusion category

$$\mathsf{B} \simeq \mathcal{Z}(\mathsf{C})\,. \tag{1.3.21}$$

In section 4.3 we will see that different notions of gauging are captured by different classifications of TQFTs; in subsections 4.3.1 and 4.3.2 we will present formalisms that gauge a finite group coupled only to a choice of Turaev-Viro TQFT, whereas in subsection 4.3.3 we will see more general couplings to TQFTs corresponding to any Reshetikhin–Turaev TQFT.

It is a folklore result of condensed matter physics that fractional quantum Hall systems, that is, topological phases of matter that exhibit anyonic excitations [66–68], are described by (unitary) braided fusion categories (or UMTCs) [69, 70]. A simple example of this correspondence is illustrated by two-dimensional system with abelian anyons labelled by elements of an abelian group $a \in A$, each of whose space of states $\mathbb{C}_a$ satisfies statistics

$$e^{iB(a,b)} : \mathbb{C}_a \otimes \mathbb{C}_b \to \mathbb{C}_b \otimes \mathbb{C}_a \,, \tag{1.3.22}$$

for some characteristic symmetric bilinear form $B : A \times A \to U(1)$ admitting a quadratic refinement

$$B(a, b) = \theta(ab, ab) - \theta(a) - \theta(b) \tag{1.3.23}$$

by some quadratic form $\theta : A \to U(1)$. The spaces $\mathbb{C}_a$ are then precisely the simple objects of the braided fusion category $\mathsf{B} \simeq \mathsf{Vec}_A^B$ of A-graded vector spaces, with half-braiding determined by the bilinear form B, which is further non-degenerate if and only if B is non-degenerate [71–73].

The choices of quadratic form θ up to equivalence determine a class in the group cohomology

$$[\theta] \in H^2_{grp}(\Gamma A, U(1)) \,, \tag{1.3.24}$$

where ΓA denotes the universal quadratic group of A. This structure of abelian anyons has a different interpretation in three dimensions; the topological lines labelled by simple objects of $\mathsf{B} \simeq \mathsf{Vec}_A^B$ generate a 1-form A symmetry. Later in subsection 4.1.1 we will see that this classification admits a dual description as the ’t Hooft anomaly of A as a 1-form symmetry [74].

In analogy to the last subsection, we can also consider finite G-extensions of these TQFTs. For the trivial TQFT $\mathsf{B} \simeq \mathsf{Vec}$ this produces a class of invertible Turaev-Viro TQFTs known as $(2 + 1)$-dimensional Dijkgraaf-Witten theories, these correspond to $(2+1)$-dimensional SPT phases completing the analogy, but by repeating this for more general TQFTs in $2+1$ dimensions, we can then obtain a rich collection of so-called *symmetry-enriched* topological (SET) phases. We will study these in more detail in subsection 3.3.3.

Chapter 2

Gauging Finite Symmetries in $0 + 1$ Dimensions

We begin discussing finite symmetries and gauging in $(0 + 1)$-dimensional quantum field theory, or quantum mechanics. As we have already remarked at the end of section 1.1, this example is a slightly peculiar one in comparison to its higher-dimensional counterparts. This is because the only topological objects to speak of are local operators and we take the existence of such things to imply that there is a decomposition of our theory into super-selection sectors.

In spite of this subtlety, symmetry in quantum mechanics is still interesting to study as an introductory example, and studying the gauging of invertible symmetries in quantum mechanics will lay the foundations for work discussions in later chapters.

An essential idea for gauging a finite group in a quantum mechanics is that it is equivalent to inserting a space-filling Wilson line, which is in some sense badly or improperly quantised, because it lives at the boundary of some two-dimensional TQFT (namely, a Dijkgraaf-Witten theory). The classification of such gauged theories for a particular choice of group then reduces to studying local operators on which these Wilson lines can end, which transform in projective representations. Working out the details of this classification for group-like symmetries in the setting of quantum mechanics will illustrate simple working analogues to ideas we will encounter in higher-dimensional settings.

2.1 Finite Global Symmetries

As was discussed in the introduction, the most general mathematical structure for describing (oriented, unitary) finite symmetries in $(0+1)$ dimensions is that of a finite dimensional C*-algebra. Such a C*-algebra A can always be expressed as a finite direct sum of finite-dimensional matrix algebras

$$A = M_{r_1}(\mathbb{C}) \oplus \cdots \oplus M_{r_n}(\mathbb{C})\,, \tag{2.1.1}$$

with the operator norm on matrices and anti-linear $*$-involution given by transpose complex conjugation on those matrices. This decomposition is precisely equivalent to that of the associated quantum mechanics into super-selection sectors briefly mentioned in section 1.1.

On the surface it might then appear as though we still possess a great deal of freedom in the choices of matrix algebras $M_r(\mathbb{C})$, but in fact, our choice of matrix algebras doesn't matter all that much, since as we alluded to in section 1.1, they are all Morita equivalent to one-another, and in particular they are Morita equivalent to the trivial algebra $M_1(\mathbb{C}) \simeq \mathbb{C}$.

In this sense it it might be sensible to conclude that, up to gapped interface, there are no interesting symmetries in an indecomposable quantum mechanics. To keep things interesting then, and to provide a quantum mechanical analogy for later chapters, we will specifically consider those theories that do have non-trivial topological local operators and do admit a decomposition into super-selection sectors. In later chapters 3 and 4 however, we will make no such concession.

Dropping unitarity and working instead in a framed setting corresponds to forgetting C^*-structure and regarding A as a finite-dimensional semi-simple algebra. We might be concerned that in forgetting this structure we might lose some information, we are saved however by the fact that any finite-dimensional semi-simple algebra has the form (2.1.1) and thereby admits a unique canonical C^*-structure.

2.1.1 Invertible Symmetries

An finite invertible symmetry in $(0+1)$ dimensions contains the underlying structure of a finite group G. As we argued in subsection 1.1.1, the C*-algebras with such a structure are classified up to equivalence by classes in group cohomology

$$[\alpha] \in H^2_{grp}(G, U(1))\,. \tag{2.1.2}$$

This class describes the 't Hooft anomaly which is invariant along the renormalisation group flow, and shifting the representative α corresponds to adding a local counter-term. A theory $\mathcal{T}$ with symmetry (G, α) is described by the twisted group (C*-)algebra

$$A = {}^{\alpha}\mathbb{C}G\,. \tag{2.1.3}$$

A general element is now an arbitrary complex linear combination of basis vectors e_g labelled by $g \in G$ which satisfy

$$e_g e_h = \alpha(g,h) e_{gh}\,, \qquad e_g^{\dagger} = \alpha(g, g^{-1})^{-1} e_{g^{-1}}\,. \tag{2.1.4}$$

The summands in the matrix decomposition (2.1.1) for $A = {}^{\alpha}\mathbb{C}G$ are in 1-1 correspondence with irreducible unitary representations of ${}^{\alpha}\mathbb{C}G$ as a C*-algebra, or equivalently, α^{-1}-projective unitary representations of G.

Choosing a basis λ_j of irreducible projective representations of G, the decomposition (2.1.1) becomes

$${}^{\alpha}\mathbb{C}G \cong \bigoplus_j M_{d_j}(\mathbb{C}) \tag{2.1.5}$$

where $d_j = \dim(\lambda_j)$ is the dimension of the corresponding representation. This decomposition describes super-selection sectors containing states transforming in irreducible unitary projective representations of G. It is clear that dropping the C^*-structure is not a loss here: finite-dimensional projective representations of a finite group are (up to equivalence) always unitary.

2.2 Symmetries From Gauging

One way to construct more general symmetries is to start with a $(0+1)$-dimensional theory $\mathcal{T}$ with an invertible (G,α) symmetry and gauge a non-anomalous sub-symmetry. In $0+1$ dimensions this construction is somewhat trivial, once again due to the fact that there is only one Morita class of simple C*-algebra.

2.2.1 Gauging G With Trivial Anomaly

The most straightforward example is to take a quantum mechanical theory $\mathcal{T}$ with an invertible (G,α) symmetry such that the anomaly class $[\alpha]=0$ is trivial. In this case, we can choose to gauge the full G symmetry, and to do so we should pick a trivialisation

$$\delta\psi = \alpha^{-1}\,. \tag{2.2.1}$$

Equivalence classes of solutions up to co-boundaries naturally define an abelian group T. Any two solutions to this condition ψ,ψ' differ by a cocycle

$$\delta(\psi-\psi')=0\,, \tag{2.2.2}$$

and as result there is a natural action from the cohomology group $H^1_{grp}(G,U(1))$

$$\begin{aligned} a\ :\ H^1_{grp}(H,U(1))\times T\ &\to\ T \\ (\phi,\psi)\ &\mapsto\ \phi+\psi\,, \end{aligned} \tag{2.2.3}$$

such that the shear map

$$a\times id:(\phi,\psi)\mapsto(\phi+\psi,\psi) \tag{2.2.4}$$

is an isomorphism. This defines the space of trivialisations T to be a torsor over $H^1_{grp}(G,U(1))$, with representatives ψ corresponding to SPT phases or discrete torsions for G.

An equivalent interpretation is to identify each choice of trivialisation as a one-dimensional unitary projective representation of G. They can then in turn be thought of as Wilson lines for G that are badly/improperly quantised in the sense that they live at the boundary of a two-dimensional Dijkgraaf-Witten theory.

Usually when we speak of gauging a group G we mean coupling to and summing over principal G-bundles. For a finite group, these bundles are all necessarily flat, and a given set of transition functions can be replaced by topological defects labelled by elements of the group. This is a $(0+1)$-dimensional analogue of the picture presented in [21], and given a choice of trivialisation ψ, we gauge G by inserting a sufficiently fine [1] network of defects

$$e_\psi = \frac{1}{|G|} \sum_{g \in G} \psi(g)\, e_g\,, \tag{2.2.5}$$

that implements the sum over flat G-bundles twisted by ψ. Notice this defect is an idempotent $e_\psi^2 = e_\psi$, this feature is important as it ensures the gauging procedure is insensitive to topological manipulations of the network as illustrated in figure 2.1.

Figure 2.1

The correlation functions in the gauged theory $\mathcal{T}/_\psi\, G$ are then constructed from correlation functions in the original theory $\mathcal{T}$ together with the insertion of projection operators. Expanding such expressions using (2.2.5) generates weighted sums of correlation functions in $\mathcal{T}$ with the insertions of the generators $e_g \in {}^\alpha\mathbb{C}G$.

The resulting global symmetry after gauging is described by topological local operators $x \in {}^\alpha\mathbb{C}G$ that are compatible with the network of defects e_ψ in the sense

$$e_\psi x = x e_\psi = x\,. \tag{2.2.6}$$

If we decompose x as

$$x = \sum_{g \in G} x_g\, e_g\,, \tag{2.2.7}$$

[1] Working over one-dimensional compact manifolds, this just means inserting at least one copy of e_ψ on each connected component. In higher dimensions this will become more involved, and is described in general in subsection 3.2.1.

then this condition becomes

$$x_g = \frac{\psi(g)}{|G|} \sum_{h \in G} \frac{x_h}{\psi(h)} \,, \tag{2.2.8}$$

for each $g \in G$, which is clearly only solved by $x \in \mathbb{C}e_\psi$. Another way to see this is to note that since ψ describes an irreducible (projective) representation of G, the idempotent e_ψ is automatically primitive, and so by Schur's lemma we can only have $x \in \mathbb{C}e_\psi$. In any case, the symmetry after gauging G with trivial 't Hooft anomaly is described by topological local operators labelled by complex numbers $\mathbb{C}$.

2.2.2 Gauging a Subgroup of (G, α)

A more general thing to do when given an invertible (G, α) symmetry where the anomaly class might not vanish, is to identify a subgroup $H \subseteq G$ whereupon the restriction of the anomaly class vanishes. In this case, we can choose to gauge only the subgroup H, and to do so we should once again pick a trivialisation

$$\delta\psi = \alpha|_H^{-1} \,. \tag{2.2.9}$$

This space of trivialisations is again enumerated up to equivalence by a torsor over $H^1_{grp}(H, U(1))$, corresponding to inserting onto the theory a space-filling (improperly quantised) Wilson line labelled by a projective G-representation $\mathrm{Ind}_H^G \psi$, obtained by induction from the 1-dimensional projective H-representation specified by ψ.

Given a choice of ψ, we gauge H by inserting a sufficiently fine network of defects

$$e_\psi = \frac{1}{|H|} \sum_{h \in H} \psi(h)\, h \,, \tag{2.2.10}$$

that implements the sum over flat H-bundles twisted by ψ.

This defect is once again an idempotent $e_\psi^2 = e_\psi$, but is no longer primitive in general since for a proper subgroup $H \subset G$ the induced representation $\mathrm{Ind}_H^G \psi$ is not irreducible. A direct result of Mackey's restriction formula is that this induced representation has dimension

$$\dim(\mathrm{Ind}_H^G \psi) = |H\backslash G/H| \tag{2.2.11}$$

given by the number of double cosets over H.

Indeed the compatibility condition on topological local operators $x \in {}^\alpha\mathbb{C}G$

$$e_\psi x = x e_\psi = x \tag{2.2.12}$$

implies x must have support on at least a double coset $[g] \in H\backslash G/H$. Hence we see that the symmetry resulting from gauging a subgroup $H \subseteq G$ with SPT phase ψ consists of topological local operators labelled by $\mathbb{C}^{H\backslash G/H}$, with algebra structure inherited from the double coset ring, twisted by α. For $H = G$ this recovers $\mathbb{C}$ as from before, and for $H = 1$ this recovers the full algebra ${}^\alpha\mathbb{C}G$.

2.3 Generalised Gauging in $0+1$ Dimensions

We now turn our attention to the most general notion of gauging a finite group symmetry in $0+1$ dimensions, with the aim of presenting three equivalent methods for describing the possible gaugings of a theory $\mathcal{T}$ with symmetry algebra ${}^{\alpha}\mathbb{C}G$.

2.3.1 General Algebras Internal to ${}^{\alpha}\mathbb{C}G$

In the previous section, we saw that gauging is implemented by summing over a network of defects. We remarked there that the defect we insert needs to be an idempotent in order for the theory we produce to be immune to topological manipulations of the network. Something we did not remark however is that from a unitary perspective those topological operators were Hermitian, removing also the dependence on their orientation.

More generally, we may consider any Hermitian topological local operator e_λ that can duplicate freely on the line, implying that they are projectors

$$e_\lambda^2 = e_\lambda \,, \qquad e_\lambda^* = e_\lambda \,, \tag{2.3.1}$$

of the C*-algebra ${}^{\alpha}\mathbb{C}G$. As a helpful analogy for later chapters, we note that we could also call these operators (Hermitian) (0-)algebras internal to ${}^{\alpha}\mathbb{C}G$.

Continuing this analogy we define the space $\lambda = \mathrm{Mod}_{{}^{\alpha}\mathbb{C}G}(e_\lambda)$ of (right) (0-)modules over e_λ internal to ${}^{\alpha}\mathbb{C}G$ as the subspace of stable points $x \in {}^{\alpha}\mathbb{C}G$ such that

$$xe_\lambda = x \,. \tag{2.3.2}$$

The space λ naturally admits a natural (left) module action from ${}^{\alpha}\mathbb{C}G$, identifying λ as a unitary representation of ${}^{\alpha}\mathbb{C}G$ as a C*-algebra, or equivalently as a unitary projective representation

$$\lambda \in \mathrm{Rep}^{\dagger,\alpha^{-1}}(G) \,. \tag{2.3.3}$$

We define Morita equivalence of (0-)algebras $e_\lambda \sim e_{\lambda'}$ internal to ${}^{\alpha}\mathbb{C}G$ as an equivalence of unitary (projective) representations $\lambda \simeq \lambda'$.

A projector is said to be primitive if $\mathrm{Mod}_{^\alpha\mathbb{C}G}(e_\lambda)$ is irreducible as a (projective) representation. In this sense, the indecomposable gaugings of (G,α) in the theory $\mathcal{T}$ are classified up to Morita equivalence by irreducible projective representations of G with projective 2-cocycle α^{-1}.

To see this labelling more concretely, we note that given the matrix algebra decomposition

$$^\alpha\mathbb{C}G \cong \bigoplus_j M_{d_j}(\mathbb{C})\,, \tag{2.3.4}$$

for each irreducible projective representation λ_j, there are d_j Morita equivalent primitive projectors that act as $e^i_j : M_{d_j}(\mathbb{C}) \to \lambda_j$ for $i = 1...d_j$. Each e^i_j is given as a matrix in in $M_{d_j}(\mathbb{C})$ by setting just one of the diagonal elements to 1 and all other elements to 0.

2.3.2 Gapped Interfaces and Modules over $^\alpha\mathbb{C}G$

Another equivalent labelling of finite gauge theories in $0+1$ dimensions comes from looking at gapped interfaces between theories. This labelling is not far from what we might expect on physical grounds since the gapped interfaces between theories can be thought of as topological local operators on which badly quantised Wilson lines can end.

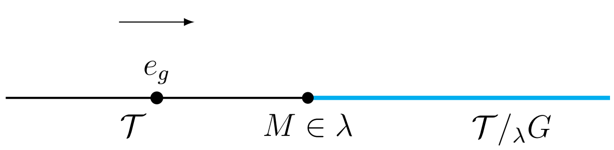

Figure 2.2

We start with a theory $\mathcal{T}$ with a (G,α) symmetry $^\alpha\mathbb{C}G$, and consider those theories that we will suggestively denote $\mathcal{T}/_\lambda G$, with which it shares a space λ of gapped interfaces.

Bringing local operators $e_g \in G$ to a junction $M \in \lambda$ between $\mathcal{T}$ and $\mathcal{T}/_\lambda G$ as in figure 2.2 describes a (left) module action of $^\alpha\mathbb{C}G$ on λ giving it the structure of a unitary projective representation $\lambda \in \mathrm{Rep}^{\alpha}(G)$. Identifying equivalence classes of theories $\mathcal{T}/_{\lambda}G$ with equivalence classes of unitary representations then induces a classification of gaugings of (G,α) by projective representations of G with projective 2-cocycle α.

The equivalence between this classification and the one derived from algebras can be made precise. Defining $\mathcal{T}/_{\lambda}G$ as $\mathcal{T}$ with the insertion of a network of projectors e_{λ}, having a gapped interface with $\mathcal{T}$ means we have local operators in $\mathcal{T}$ for which e_{λ} vanishes upon fusion as depicted in figure 2.3.

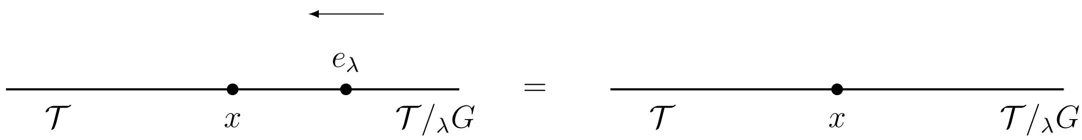

Figure 2.3

The local operators in ${}^{\alpha}\mathbb{C}G$ that have this behaviour are precisely those living in $\lambda = \mathrm{Mod}_{{}^{\alpha}\mathbb{C}G}(e_{\lambda})$, and so the unitary (projective) representation of gapped interfaces constructed in this way is identified with λ.

2.3.3 Symmetry TFTs and Lagrangian Algebras

These descriptions of gauging for ${}^{\alpha}\mathbb{C}G$ can be reformulated in terms of gapped boundary conditions in the associated sandwich construction in two dimensions [27, 28].

The starting point is the 2-dimensional unitary oriented Dijkgraaf-Witten theory labelled with gauge group G and a class

$$[\alpha] \in H^2_{grp}(G, U(1))\,. \tag{2.3.5}$$

These can be thought of as a generalisation of Chern-Simons theory to finite groups, but historically they were understood first from the perspective of orbifolding the Drinfeld/quantum double of G in $1+1$ dimensions [75, 76].

For our purposes, they are topological finite gauge theories supported on a 2-manifold M_2, described by a finite gauge field

$$\mathsf{a} : M_2 \to BG \,, \tag{2.3.6}$$

whose action is determined by a representative $\alpha \in Z^2_{grp}(G, U(1))$ satisfying

$$\delta\alpha(g,h,k) = \frac{\alpha(h,k)\alpha(g,hk)}{\alpha(gh,k)\alpha(g,h)} = 1 \,, \tag{2.3.7}$$

via the pull-back

$$exp\left(i \int_{M_2} \mathsf{a}^*\alpha\right) \,, \tag{2.3.8}$$

and is hence manifestly topological.

On a 1-submanifold $M_1 \subset M_2$ we have the restriction

$$\mathsf{a}|_{M_1} : M_1 \to BG \,. \tag{2.3.9}$$

Given a unitary representation λ over a vector space V, we can identify up to homotopy a map

$$\lambda : BG \to \mathrm{Aut}(V) \,, \tag{2.3.10}$$

whose (traced) pull-back

$$exp\left(i \int_{M_1} \mathrm{Tr}(\lambda^*\mathsf{a})\right) \tag{2.3.11}$$

defines a topological Wilson line corresponding to the representation λ in the Dijkgraaf-Witten theory. If we allow M_2 to have non-empty boundary ∂M_2, we must be more careful; picking $M_1 = \partial M_2$, we must instead pick the representation λ over V to be projective such that

$$\delta\lambda = \alpha^{-1} id_V \,. \tag{2.3.12}$$

In this way the contribution from (2.3.11) to action makes it fully topological. In this way we can see that the gapped boundaries $\mathcal{B}_\lambda$ of $(1+1)$-dimensional Dijkgraaf-Witten theory are labelled by unitary projective representations of G with projective 2-cocycle α^{-1}.

We will define the Dirichlet boundary condition $\mathcal{D}$ as the one that totally fixes a on the boundary. This corresponds to a $0+1$-dimensional topological boundary supporting a (G, α) symmetry described by

$$\mathcal{C}_{\mathcal{D}} = {}^{\alpha}\mathbb{C}G\,. \tag{2.3.13}$$

The existence of this canonical topological Dirichlet boundary reflects the fact that the Dijkgraaf-Witten theory is decomposable and that its topological local operators are described by the centre

$$\mathsf{DW}_{G,\alpha} \simeq \mathcal{Z}({}^{\alpha}\mathbb{C}G)\,, \tag{2.3.14}$$

as a commutative (Frobenius) C*-algebra. This is equipped with the Frobenius structure normalised such that $Z(S^2) = n$ is the number of irreducible unitary projective representations.

The symmetry of a QFT in d dimensions can be recast as the data of a (Turaev-Viro) topological field theory living in $d+1$ dimensions, called the symmetry TFT (or symTFT) [26, 27, 29]. From the discussion above we can see that for a quantum mechanical theory $\mathcal{T}$ with a (G, α) symmetry, the corresponding symmetry TFT is the $(1+1)$-dimensional (G, α) Dijkgraaf-Witten theory. The dynamics of the theory $\mathcal{T}$ are captured by a relative (non-topological) boundary condition $\mathcal{B}_{\mathcal{T}}$, and the theory itself can be recovered by interval compactification with the canonical gapped Dirichlet boundary condition $\mathcal{D}$. This is illustrated in figure 2.4.

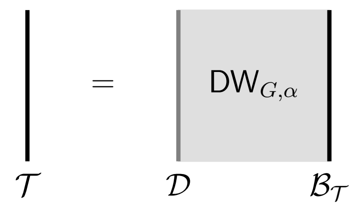

Figure 2.4

Interval compactification of $\mathcal{B}_{\mathcal{T}}$ with other choices of gapped boundary condition $\mathcal{B}_{\lambda}$ reproduces the gauged theories $\mathcal{T}/_{\lambda}\, G$. This is illustrated in figure 2.5.

Another perspective in this setting is that indecomposable gapped boundary condi-

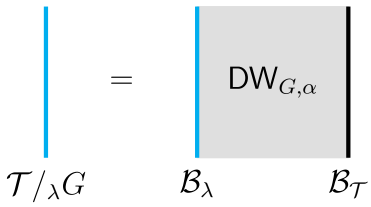

Figure 2.5

tions $\mathcal{B}_\lambda$ of the Dijkgraaf-Witten theory are labelled by primitive projectors internal to $\mathcal{Z}({}^\alpha\mathbb{C}G)$ that condense on the boundary:

$$\widehat{e}_\lambda^2 = \widehat{e}_\lambda \qquad \widehat{e}_\lambda^* = \widehat{e}_\lambda\,. \tag{2.3.15}$$

Such objects are precisely the primitive central projectors of ${}^\alpha\mathbb{C}G$, and correspond to irreducible projective representations. We can spell out the details of this correspondence concretely using the standard formula for primitive central projectors

$$\widehat{e}_\lambda = \frac{\dim(\lambda)}{|G|}\sum_{g\in G}\chi_\lambda(g^{-1})e_g\,, \tag{2.3.16}$$

where $\chi_\lambda : G \to U(1)$ is the projective character of the corresponding projective representation λ.

In checking (2.3.15), the following properties of projective characters are important:

- Complex conjugation: $\chi_\lambda(g^{-1}) = \overline{\chi_\lambda(g)}$.
- Twisted class function: $\chi_\lambda({}^hg) = \tau_g(\alpha)(h)\chi_\lambda(g)$.
- Orthonormality: $\sum_{g\in G}\chi_{\lambda_1}(g)\overline{\chi_{\lambda_2}(g)} = |G|\delta_{\lambda_1,\lambda_2}$.

The phases

$$\tau_g(\alpha)(h) := \frac{\alpha({}^hg,h)}{\alpha(h,g)} \tag{2.3.17}$$

are components of the transgression $\tau(\alpha) \in Z^1(G//G, U(1))$ of the 't Hooft anomaly.[2] The twisted class function property implies that $\chi_\lambda(g)$ vanishes unless $\alpha(g,h) = \alpha(h,g)$ for all $h \in C_G(g)$.

[2]Equivalently, $\tau_g(\alpha) \in Z^1(C_g(G), U(1))$ on restriction to $h \in C_g(G)$.

Once again we can see the correspondence more concretely using the matrix decomposition

$$^{\alpha}\mathbb{C}G \cong \bigoplus_j M_{d_j}(\mathbb{C})\,, \tag{2.3.18}$$

and primitive idempotents e^i_j for each irreducible representation λ_j. In this case the Morita class $\{e^i_j\}_i$ is lifted to a primitive central idempotent via the sum

$$\widehat{e}_j = \sum_{i=1}^{d_j} e^i_j\,, \tag{2.3.19}$$

or equivalently by the identity matrix in $M_{d_j}(\mathbb{C})$.

We see then, that the gapped boundary conditions of the symmetry TFT, and hence the gaugings of (G,α) in the theory $\mathcal{T}$, are labelled by projective representations of G with projective 2-cocycle α. We point out that the canonical Dirichlet boundary condition $\mathcal{D}$ itself is reproduced by choosing the regular unitary projective representation of G

$$\lambda^{reg} := \bigoplus_j \lambda_j^{\oplus d_j}\,. \tag{2.3.20}$$

Thus $\mathcal{D}$ is not irreducible and decomposes into super-selection sectors, this is a feature unique to one dimension related to the subtlety we observed at the outset of this chapter.

In the spirit of continued helpful analogies to later chapters, these primitive central idempotents can also be thought of as (0-)algebra objects internal to $\mathcal{Z}(^{\alpha}\mathbb{C}G)$. In particular they are *Lagrangian* algebras, in that the space of local (0-)modules described by commuting stable points

$$\widehat{e}_\lambda\, x = x\, \widehat{e}_\lambda = x\,, \tag{2.3.21}$$

all lie within a 1-dimensional subspace generated by $\widehat{e}_\lambda$. Here in one dimension this condition is trivially satisfied, however in higher dimensions we will see that this is not always the case.

2.4 Defects After Generalised Gauging

We now turn our attention to identifying the symmetry algebra in the gauged theories where a choice of (projective) representation has been made. Though we do not expect the results of these calculations to be very interesting, as the symmetry algebra of any indecomposable theory should just be (perhaps Morita equivalent to) $\mathbb{C}$, they are once again simpler illustrative analogues of calculations to come in higher dimensions.

2.4.1 0-Bimodules Over 0-Algebras

Given a representative idempotent e_λ for some projective representation $\lambda \in \mathrm{Rep}^\alpha(G)$, we expect topological local operators in the theory after gauging to be compatible with the network of idempotents. In practice an operator $x \in {}^\alpha\mathbb{C}G$ is compatible if

$$e_\lambda x = x e_\lambda = x \,. \tag{2.4.1}$$

Such elements of ${}^\alpha\mathbb{C}G$ describe linear maps on $\mathrm{Mod}_{{}^\alpha\mathbb{C}G}(e_\lambda)$ that commute with the ${}^\alpha\mathbb{C}G$-action. In the case that the corresponding projective representation λ is irreducible, Schur's lemma tells us that such operators can only be of the form $x \in \mathbb{C}e_\lambda$, meaning the topological local operators of the gauged theory $\mathcal{T}/_\lambda G$ are labelled by $\mathbb{C}$ for λ irreducible.

2.4.2 0-Modules Over Primitive Central Idempotents

We can also obtain the algebra of topological local operators from the symmetry TFT perspective. We recall that in addition to a gapped boundary condition $\mathcal{B}_\lambda$, a primitive central idempotent $\widehat{e}_\lambda$ also specifies a topological local operator in a $(1+1)$-dimensional symmetry bulk TFT. Further to this we will adopt the perspective that this local operator should condense on the $(0+1)$-dimensional boundary it corresponds to.

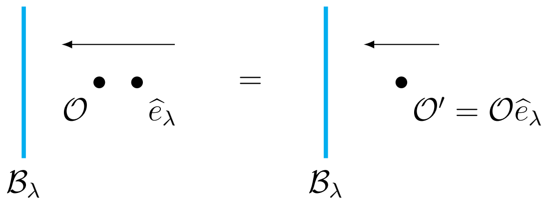

Figure 2.6

To make this more precise, an indecomposable boundary for the $(1+1)$-dimensional (G, α) Dijkgraaf-Witten theory describes a bulk-boundary map between algebras

$$\mathcal{F}_\lambda : \mathcal{Z}(^\alpha\mathbb{C}G) \to \mathbb{C}\,. \tag{2.4.2}$$

The statement that the bulk topological local operator corresponding to the primitive central idempotent $\widehat{e}_\lambda$ condenses on this boundary is equivalent to asking that

$$\mathcal{F}_\lambda(\widehat{e}_\lambda) = 1\,. \tag{2.4.3}$$

It follows then that bulk topological operators $\mathcal{O}' - \mathcal{O}\widehat{e}_\lambda$, related by a factor of $\widehat{e}_\lambda$, will map to the same boundary operator

$$\mathcal{F}_\lambda(\mathcal{O}') = \mathcal{F}_\lambda(\mathcal{O})\,, \tag{2.4.4}$$

as illustrated in figure 2.6.

Equivalence classes of bulk topological operators identified in this way are labelled by (0-)modules $x \in \mathcal{Z}(^\alpha\mathbb{C}G)$ such that

$$\widehat{e}_\lambda x = x\,. \tag{2.4.5}$$

Since the idempotent $\widehat{e}_\lambda$ is assumed primitive in $\mathcal{Z}(^\alpha\mathbb{C}G)$, its space of 0-modules describes a simple $\mathcal{Z}(^\alpha\mathbb{C}G)$-module, and Schur's lemma tells us once again that we can only have

$$x \in \mathbb{C}\widehat{e}_\lambda\,, \tag{2.4.6}$$

meaning topological local operators on the boundary for irreducible representations λ are just labelled by $\mathbb{C}$.

Chapter 3

Gauging Finite Symmetries in 1+1 Dimensions

We now move to quantum field theories in $1+1$ dimensions. Historically we can see that early examples of non-invertible symmetries were first observed in the context of $(1+1)$-dimensional conformal field theory; Verlinde lines catalogued by conformal primaries are topological, and inherit fusion rules from the associated operator product expansion [48, 77–79]. These observations foreshadowed a much more general point of view that the symmetry of any (unitary) $(1+1)$-dimensional quantum field theories is described by a (unitary) fusion category [3, 24].

A better known class of examples, and indeed the focus of this work, are group-theoretic symmetries obtained by gauging a finite group; Wilson lines labelled by representations of a finite group are topological, and fuse according to the tensor product of those representations [18–21]. The process of gauging a finite symmetry can be generalised by including additional degrees of freedom that admit a number of equivalent physical descriptions:

- Orbifolding a non-anomalous subgroup with a choice of discrete torsion [29, 80, 81].

- Stacking the theory with a $1+1$-dimensional TQFT that cancels the anomaly on a subgroup [2, 21].
- Choosing an opposing topological boundary for the $(2+1)$-dimensional symmetry TFT [27, 28].

In this chapter we demonstrate that these descriptions can all be recast in the language of higher representation theory; gauging a finite group symmetry is equivalent to inserting a badly quantised space-filling Wilson surface, the topological lines that these surfaces can end on then transform in unitary projective 2-representations.

Likewise there equivalent ways to describe the topological line defects that inhabit the resulting group-theoretic unitary fusion category after gauging:

- They are gauge-invariants defects of the original symmetry, together with Wilson lines of the finite gauge symmetry [18–21].
- They are equivalence classes of line defects in the $(2+1)$-dimensional symmetry TFT that are identified on the topological boundary.

These interpretations all represent direct categorifications to the ideas presented in chapter 2, and this additional categorical structure we have introduced leads to a much richer variety of symmetries in $1+1$ dimensions.

In this chapter we will first discuss the structure of finite unitary global symmetries in $1+1$ dimensions, paying special attention first to those that are invertible. Then we will construct more general non-invertible group-theoretic symmetries starting in the framed setting by gauging these finite groups, and explore some motivating examples. After that we will explore the various equivalent descriptions of finite invertible symmetries in the unitary setting, before finally returning to the construction of topological defects in this generalised gauging picture.

3.1 Finite Global Symmetries

We restrict our focus to oriented theories whose local excitations are purely bosonic, and whose partition function is defined on oriented 2-manifolds. Finite symmetries of unitary oriented $(1+1)$-dimensional quantum field theories are described by unitary fusion categories, as we defined them in subsection 1.2.1.

Unitarity in this context is meant more specifically that we are interested in $O^\dagger$-fusion categories that have a well defined notion of unitary duals on objects [24, 25]. The $O^\dagger$-fusion structure appears here because we are working in the oriented setting; more generally we expect a $\widehat{H}^\dagger$-fusion structure where $\widehat{H}$ is the extended tangential structure [24, 25, 40].

Unlike in $0+1$ dimensions, the symmetries here are assumed to have a simple unit object and as such do not admit a decomposition into super-selection sectors. This is equivalent to assuming there are no non-trivial topological local operators; unlike in chapter 2, we can make this assumption here without trivialising the entire symmetry.

In principle we could allow non-trivial topological local operators and widen our interpretation to allow multi-fusion categories. These categories, up to Morita equivalence, admit a decomposition into fusion categories enumerated over super-selection sectors, but such symmetries are not the objects of interest for this chapter.

Dropping unitarity and working in an oriented setting corresponds to forgetting the unitary structure and regarding the symmetry category as a spherical fusion category. We can further drop orientation and work in the framed setting which corresponds to forgetting the spherical structure and working with ordinary fusion categories.

3.1.1 Invertible Symmetries

A finite invertible symmetry in $1+1$ dimensions is described by a unitary fusion category with simple objects labelled by elements $g \in G$ of a finite group. The various unitary fusion structures compatible with the group multiplication are classified up to equivalence by group cohomology classes

$$[\alpha] \in H^3_{grp}(G, U(1))\,. \tag{3.1.1}$$

This class is an invariant of the renormalisation group flow and corresponds to a 't Hooft anomaly. We take the perspective that specifying a theory includes specifying a representative 3-cocycle α; shifting the representative by a 3-coboundary corresponds to adding local counter-terms [35].

A theory $\mathcal{T}$ with symmetry (G, α) is described by the unitary fusion category

$$\mathcal{C}_{\mathcal{T}} = \mathsf{Hilb}^{\alpha}_{G}\,, \tag{3.1.2}$$

consisting of finite-dimensional G-graded Hilbert spaces

$$X = \bigoplus_{g \in G} X_g\,, \tag{3.1.3}$$

and homogeneous linear maps between them. Simple objects in this category are 1-dimensional Hilbert spaces $\mathbb{C}_g$ labelled by $g \in G$, their fusion is given by the G-graded tensor product of Hilbert spaces depicted in figure 3.1.

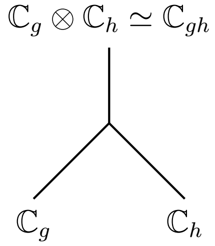

Figure 3.1

The associativity structure for this fusion is determined by α as depicted in figure 3.2.

The unitary structure incorporates both duals X^* of Hilbert spaces X acting on simple objects as

$$* : \mathbb{C}_g \mapsto (\mathbb{C}_g)^* \simeq \mathbb{C}^*_{g^{-1}} \tag{3.1.4}$$

for each $g \in G$, and adjoints $f^\dagger : Y \to X$ of homogenous linear maps $f : Y \to X$,

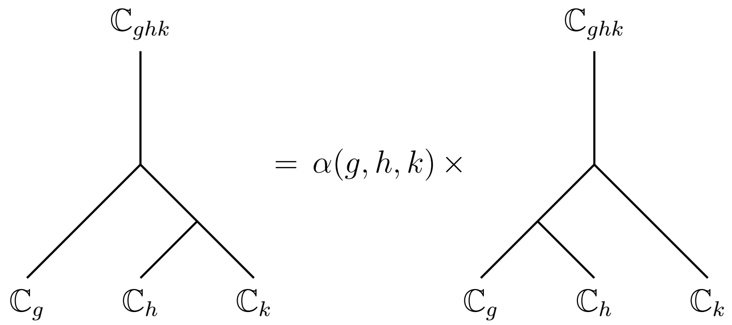

Figure 3.2

together with the unitary dual condition that identifies left duals as right duals via

$$coev^r_X := (ev^l_X)^\dagger : 1_e \to X^* \otimes X \qquad\qquad ev^r_X := (coev^l_X)^\dagger : X \otimes X^* \to 1_e\,, \tag{3.1.5}$$

as depicted schematically in figure 3.3.

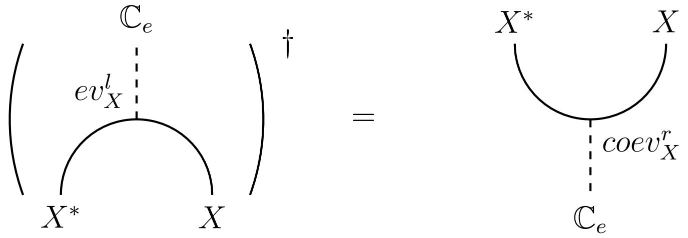

Figure 3.3

In addition to this structure, the unitary duals condition induces a canonical spherical structure [24, 25]

$$S_X = (ev^r_X \otimes id_{X^{**}}) \circ (id_X \otimes coev^l_{X^*}) : X \to X^{**}\,, \tag{3.1.6}$$

as described in subsection 1.2.1.

Working in the oriented setting corresponds to forgetting all but the unique spherical structure and working with the underlying spherical fusion category

$$\mathcal{C}_{\mathcal{T}} = \mathsf{Vec}^\alpha_G \tag{3.1.7}$$

of finite-dimensional G-graded vector spaces and linear transformations [1]. The

[1]It is then more appropriate to consider $\alpha \in Z^3_{grp}(G, \mathbb{C}^\times)$. However, $H^3_{grp}(G, \mathbb{C}^\times) \simeq H^3_{grp}(G, U(1))$ for finite G as it is always possible to choose a representative α to be a phase.

unique spherical structure reduces to precisely the canonical one for Vec. In same way, working in the framed setting corresponds to forgetting the spherical structure and considering Vec_G^α as an ordinary fusion category.

3.2 Symmetries from Gauging

One way to construct more general symmetries in $1+1$ dimensions is to start with a theory $\mathcal{T}$ with an invertible (G,α) symmetry, and gauge a non-anomalous sub-symmetry. This construction produces novel examples of non-invertible symmetries that are not group-like, but are still controlled by the properties of the underlying group and its representations; we refer to such symmetries as group-theoretic.

To see how this leads to non-invertible symmetries, we will opt to work in the framed setting of fusion categorical symmetries

$$\mathcal{C}_{\mathcal{T}} = \mathsf{Vec}_G^{\alpha}\,, \tag{3.2.1}$$

returning to the more general unitary construction later in section 3.3. The gauging of a non-anomalous sub-symmetry is then well understood in $1+1$ dimensions to be equivalent to populating the space with a network of topological defects describing algebra objects internal to Vec_G^{α} [21, 35].

Before continuing however, we should mention that it is an important result in $1+1$ dimensions that there exist non-trivial fusion-categorical symmetries that are not group-theoretic. A famous demonstrative class of these are the so-called Tambara-Yamagami fusion categories [82], an example of which arose in the form of the critical Ising model discussed in section 1.2.2. Nonetheless, group-theoretic symmetries still represent a large class of symmetries in $1+1$ dimensions and provide a direct construction of non-invertible symmetries that will have important generalisations in higher dimensions.

3.2.1 Gauging G With Trivial Anomaly

The most straightforward example is to take an invertible (G,α) symmetry such that the anomaly $\alpha = 0$ is trivial. In this case, we can choose to gauge the full G symmetry to produce a new theory $\mathcal{T}/G$ with group-theoretic symmetry category that we denote $\mathsf{C}(G|G)$. We do this by inserting a sufficiently fine network of line defects

$$A = \bigoplus_{g \in G} \mathbb{C}_g \,, \tag{3.2.2}$$

that implements the sum over flat G-bundles [8, 21].

We now take a moment to make more precise the notion of "sufficiently fine" in a way that naturally extends to all dimensions. Given a compact manifold and a corresponding triangulation by simplices, we construct a sufficiently fine network by taking the one dual to the simplicial complex. The vertices of this network sit at the centre of codimension-0 simplices, the edges at codimension-1 simplices, and so on. In practice one actually often takes a *pseudo-triangulation*; for example, the finest "sufficiently fine" network on the 2-torus is dual to a triangulation whose vertices are identified in a non-trivial way. For the purposes of intuition however, properly triangulated networks are always as fine or finer than these.

Here in two dimensions, given a network dual to some triangulation, we insert the defect A on the edges, and at the vertices we specify topological local operators $m : A \otimes A \to A$ that decompose as

$$id_{\mathbb{C}_{gh}} : \mathbb{C}_g \otimes \mathbb{C}_h \to \mathbb{C}_{gh} \,, \tag{3.2.3}$$

for each $g, h \in G$. The inclusion of these junctions categorifies the notion of idempotents studied in chapter 2 and ensures the gauging procedure is insensitive to the choice of network.

In addition to this data we also have a canonical unit map given by the inclusion $\mathbb{C}_e \hookrightarrow A$ which naturally endows A with the structure of a finite dimensional G-graded associative algebra. We can further equip it with a normalised Frobenius (or separable) algebra in the oriented setting [83], and a normalised special dagger-Frobenius algebra (or Q-system) in the unitary setting [84–86].

The resulting global symmetry $\mathsf{C}(G|G)$ after gauging is known to be described by topological Wilson lines labelled by representations of G. To replicate this we consider defects $V \in \mathsf{Vec}_G^\alpha$ compatible with the insertion of a network of defects. The compatibility data of a topological defect

$$V = \bigoplus_{g \in G} V_g \,, \tag{3.2.4}$$

amounts to us specifying left and right morphisms

$$\ell : A \otimes V \to V \qquad\qquad r : V \otimes A \to V \,, \tag{3.2.5}$$

giving V the structure of a bimodule over A internal to $\mathcal{C}_\mathcal{T} = \mathsf{Vec}_G$; this identifies the symmetry with a fusion category of bimodule objects over A

$$\mathsf{C}(G|G) \simeq \mathrm{Bimod}_{\mathcal{C}_\mathcal{T}}(A) \,, \tag{3.2.6}$$

we will return to this perspective in more detail later in section 3.4. These left and right morphisms decompose to

$$\ell_{h|g} : \mathbb{C}_h \otimes V_g \to V_{hg} \qquad\qquad r_{g|h} : V_g \otimes \mathbb{C}_h \to V_{gh} \,, \tag{3.2.7}$$

for each pair $g, h \in G$. This compatibility is illustrated further in figure 3.4. These 1-morphisms are subject to their own compatibility conditions.

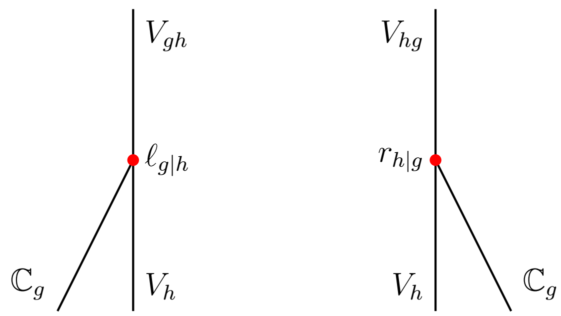

Figure 3.4

First we ask that the left and right 1-morphisms satisfy normalisation conditions that ensure compatibility with the bulk unit

$$\ell_{e|g} = r_{g|e} = V_g \,. \tag{3.2.8}$$

where we have used a short-hand notation that V_g denotes the identity 1-automorphism on the same object. Next we ask that they satisfy compatibility conditions with the bulk fusion

$$\ell_{h_1 h_2|g} \;=\; \ell_{h_1|h_2 g} \circ (1_{h_1} \otimes \ell_{h_2|g})\,, \tag{3.2.9}$$

$$r_{g|h_1 h_2} \;=\; r_{g h_1|h_2} \circ (r_{g|h_1} \otimes 1_{h_1})\,. \tag{3.2.10}$$

Finally we ask that the two module actions commute as

$$r_{h_1 g|h_2} \circ (\ell_{h_1|g} \otimes 1_{h_2}) \;=\; \ell_{h_1|g h_2} \circ (1_{h_1} \otimes r_{g|h_2})\,, \tag{3.2.11}$$

turning the left and right morphisms into compatible bimodule actions.

One way to present solutions to these criteria is to note that the left and right morphisms are each invertible, giving isomorphisms $V_g \simeq V_h$ for all $g, h \in G$. Restricting our focus then to the trivially-graded component V_e, we see it carries a representation Φ of G that is constructed from the left and right morphisms as

$$\Phi(g) \;:=\; r_{g|g^{-1}} \circ (\ell_{g|e} \otimes 1_{g^{-1}})\,. \tag{3.2.12}$$

The interpretation of this combination of morphisms is a symmetry defect intersecting V as illustrated in 3.5.

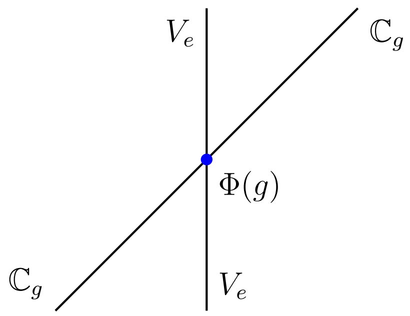

Figure 3.5

A straightforward consequence of the consistency conditions (3.2.9), (3.2.10) and (3.2.11) is that this combination of morphisms indeed defines a representation in the sense that

$$\Phi(gh) \;=\; \Phi(g) \circ \Phi(h) \tag{3.2.13}$$

for all elements $g, h \in G$, illustrated with intersections in figure 3.6.

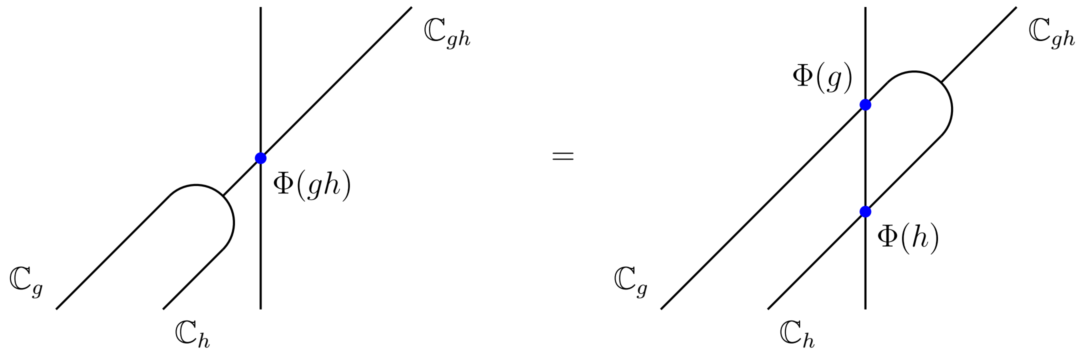

Figure 3.6

This construction concretely demonstrates the equivalence

$$\mathsf{C}(G|G) \simeq \text{Rep}(G)\,. \tag{3.2.14}$$

More generally one could gauge G with a non-trivial discrete torsion $[\psi] \in H^2_{grp}(G, U(1))$. For a given representative ψ this just corresponds to a different choice of junctions $m : A \otimes A \to A$ given by

$$\psi(g,h) \circ id_{gh} : \mathbb{C}_g \otimes \mathbb{C}_h \to \mathbb{C}_{gh}\,, \tag{3.2.15}$$

producing the gauged theory $\mathcal{T}/_{\psi}G$. Including such a phase acts on the resulting symmetry category by an auto-equivalence, one way to see this is to consider interfaces between a pair of theories $\mathcal{T}/_{\psi_1}G$ and $\mathcal{T}/_{\psi_2}G$ with mismatched phases ψ_1 and ψ_2. In that case the picture we had in figure 3.6 changes to that of figure 3.7, and equation (3.2.13) becomes

$$\Phi(gh) \quad = \quad \frac{\psi_2(g,h)}{\psi_1(g,h)}\Phi(g) \,\circ\, \Phi(h)\,. \tag{3.2.16}$$

Hence interfaces between $\mathcal{T}/_{\psi_1}G$ and $\mathcal{T}/_{\psi_2}G$ are labelled by projective representations of G with projective 2-cocycle ψ_1/ψ_2. This projective 2-cocycle vanishes when $\psi_1 = \psi_2$, returning us once again to the ordinary representations of G.

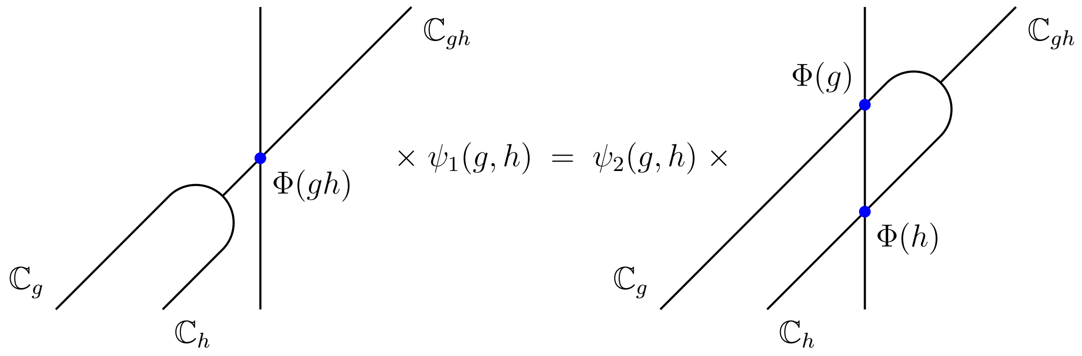

Figure 3.7

3.2.2 Gauging a Subgroup of (G, α)

A more general thing to do when given an invertible (G, α) symmetry where the anomaly class might not vanish, is to identify a subgroup $H \subseteq G$ whereupon the restriction of the anomaly class vanishes. In this case, we can choose to gauge only the subgroup H, and to do so we should pick a trivialisation

$$\delta\psi = \alpha|_H^{-1} \,. \tag{3.2.17}$$

Equivalence classes of trivialisations are, in analogy to subsection 2.2.2, classified by a torsor over $H^2_{grp}(H, U(1))$, with representatives ψ corresponding to SPT phases or discrete torsions for H. This theory produced by performing this gauging then has the group-theoretic symmetry category $\mathsf{C}(G, \alpha|H, \psi)$.

Given a choice of ψ, we gauge H by inserting a sufficiently fine network of defects

$$A = \bigoplus_{h \in H} \mathbb{C}_h \,, \tag{3.2.18}$$

that implements the sum over flat H-bundles. At the junctions, we specify a topological local operators $m : A \otimes A \to A$ that decompose as

$$\psi(g, h) \cdot id_{\mathbb{C}_{gh}} : \mathbb{C}_g \otimes \mathbb{C}_h \to \mathbb{C}_{gh} \,, \tag{3.2.19}$$

for each $g, h \in H$. The inclusion of these junctions ensures the theory is insensitive to changes to the network of defects, contingent on the condition

$$\delta\psi(g,h,k) = \frac{\psi(h,k)\psi(g,hk)}{\psi(gh,k)\psi(g,h)} = \alpha|_H^{-1}\,. \tag{3.2.20}$$

This is just the trivialisation condition we observed in (3.2.17).

Once again in addition to this data we also have a canonical unit map given by the inclusion $\mathbb{C}_e \hookrightarrow A$ which naturally endows A with the structure of a G-graded associative algebra. We can further equip it with a normalised Frobenius algebra in the oriented setting, and a normalised special dagger-Frobenius algebra (or Q-system) in the unitary setting.

Defects corresponding to objects in $\mathsf{C}(G,\alpha|H,\psi)$ are then described by topological defects $V \in \mathsf{Vec}_G^\alpha$ compatible with the insertion of a network of defects. Given a choice of gauging $\lambda = (H,\psi)$ with associated algebra object A, the compatibility data of a topological defect

$$V = \bigoplus_{g\in G} V_g\,, \tag{3.2.21}$$

amounts to us specifying left and right morphisms

$$\ell : A\otimes V \to V \qquad\qquad r : V\otimes A \to V\,, \tag{3.2.22}$$

giving V the structure of a bimodule over A internal to $\mathcal{C}_\mathcal{T}$; like before this identifies

$$\mathsf{C}(G,\alpha|H,\psi) \simeq \mathrm{Bimod}_{\mathcal{C}_\mathcal{T}}(A) \tag{3.2.23}$$

as fusion categories. These left and right morphisms decompose to

$$\ell_{h|g} : \mathbb{C}_h \otimes V_g \to V_{hg} \qquad\qquad r_{g|h} : V_g\otimes \mathbb{C}_h \to V_{gh}\,, \tag{3.2.24}$$

for each pair of $h\in H$ and $g \in G$ as was illustrated in figure 3.4.

We again ask that the left and right module actions are normalised in the sense

$$\ell_{e|g} = r_{g|e} = 1\,, \tag{3.2.25}$$

however their compatibility conditions with the bulk fusion

$$\ell_{h_1h_2|g} \circ \psi(h_1,h_2) \;=\; \ell_{h_1|h_2g} \circ (1_{h_1} \otimes \ell_{h_2|g}) \circ \alpha(h_1,h_2,g)\,, \tag{3.2.26}$$

$$r_{g|h_1h_2} \circ \psi(h_1,h_2) \;=\; r_{gh_1|h_2} \circ (r_{g|h_1} \otimes 1_{h_2}) \circ \alpha(g,h_1,h_2)^{-1}\,, \tag{3.2.27}$$

and the commutation compatibility condition

$$r_{h_1g|h_2} \circ (\ell_{h_1|g} \otimes 1_{h_2}) \;=\; \ell_{h_1|gh_2} \circ (1_{h_1} \otimes r_{g|h_2}) \circ \alpha(h_1,g,h_2)\,, \tag{3.2.28}$$

are now twisted by α.

From the form of the left and right morphisms in (3.2.24), it is clear that any solution to these constraints will decompose as a direct sum of solutions supported on double H-cosets in G. Let us therefore restrict our attention to a solution supported on a single double coset $[g] \in H\backslash G/H$ with representative $g \in G$.

The left and right morphisms are each invertible, giving isomorphisms $V_{g_1} \simeq V_{g_2}$ for all $g_1, g_2 \in H\backslash G/H$, hence we may restrict our focus to the component V_g, which carries a projective representation Φ_g of the subgroup $H_g := H \cap gHg^{-1}$ that is constructed from the left and right morphisms as

$$\Phi_g(h) \;:=\; r_{hg|(h^g)^{-1}} \circ (\ell_{h|g} \otimes 1_{(h^g)^{-1}})\,, \tag{3.2.29}$$

where $h \in H_g$ and $h^g := g^{-1}hg$. The interpretation of this combination of morphisms is a symmetry defect intersecting V as illustrated in figure 3.8.

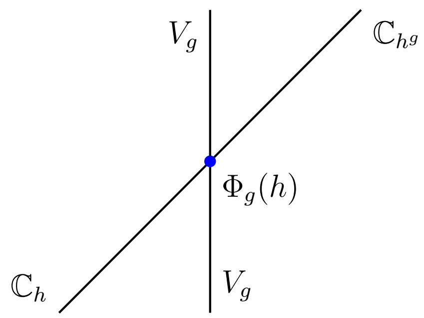

Figure 3.8

A straightforward consequence of the consistency conditions (3.2.26), (3.2.27) and (3.2.28) is that this combination of morphisms indeed defines a projective representation in the sense that

$$\Phi_g(h_1 h_2) \;=\; c_g(h_1,h_2)\cdot\Phi_g(h_1)\circ\Phi_g(h_2) \tag{3.2.30}$$

for all elements $h_1, h_2 \in H_g$, where the 2-cocycle $c_g \in Z^2_{grp}(H_g, U(1))$ is given up to co-boundary by

$$c_g(h_1,h_2) \;:=\; \frac{\psi(h_1^g,h_2^g)}{\psi(h_1,h_2)}\cdot\frac{\alpha(h_1,h_2,g)\,\alpha(g,h_1^g,h_2^g)}{\alpha(h_1,g,h_2^g)}\,. \tag{3.2.31}$$

It is known that conversely such a projective representation determines a solution to the compatibility constraints for left and right morphisms [87, 88]. The above construction then sets up a bijection between isomorphism classes of simple objects in $\mathsf{C}(G,\alpha|H,\psi)$ and isomorphism classes of pairs (g,Φ_g) consisting of

1. A double coset $[g] \in H\backslash G/H$ with representative $g \in G$.
2. An irreducible projective representation Φ_g of H_g with 2-cocycle

$$c_g(h_1,h_2) \;=\; \frac{\psi(h_1^g,h_2^g)}{\psi(h_1,h_2)}\cdot\frac{\alpha(h_1,h_2,g)\,\alpha(g,h_1^g,h_2^g)}{\alpha(h_1,g,h_2^g)}\,. \tag{3.2.32}$$

The isomorphism class of a simple object depends on the double coset representative g and the 2-cocycle c_g only up to isomorphism.

The above description of simple topological lines allows for the following alternative physical interpretation: Let us consider the line $g \in G$ in $\mathcal{T}$. This is left invariant under the action of $H_g \subset H$ and therefore supports a H_g symmetry group. However, due to the bulk 't Hooft anomaly and its trivialisation, the topological line has an anomaly captured by the representative 2-cocycle $\bar{c}_g \in Z^2_{grp}(H_g, U(1))$. In order to define a consistent topological line when gauging $H \subset G$, this anomaly must be cancelled by dressing with a 1-dimensional TQFT with H_g symmetry and 't Hooft anomaly c_g. This is precisely specified by a vector space supporting a projective representation of H_g with 2-cocycle c_g. It may simultaneously be regarded as a badly quantized Wilson line for H_g whose anomalous transformation cancels that of the symmetry defect.

3.2.3 Examples

To conclude this section, let us study some examples of symmetries constructed in the ways we have just described.

Example 1 : $G = \mathbb{Z}_4$

First we consider a theory $\mathcal{T}$ with symmetry group $G = \mathbb{Z}_4$ and trivial 't Hooft anomaly, viewed as an extension of $A = \mathbb{Z}_2$ by $K = \mathbb{Z}_2$

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1\,, \tag{3.2.33}$$

with non-trivial class $[e] \in H^2_{grp}(\mathbb{Z}_2, \mathbb{Z}_2) \simeq \mathbb{Z}_2$. If we denote the generators of $A = \mathbb{Z}_2$ and $K = \mathbb{Z}_2$ by x and y respectively, the normalised 2-cocycle e is completely determined by the condition $e(y, y) = x$.

There is no possibility for discrete torsion since $H^2_{grp}(\mathbb{Z}_4, U(1)) = 0$. Gauging the whole symmetry G leads to a theory $\mathcal{T}/G$ with symmetry category

$$\mathcal{C}(\mathbb{Z}_4 \,|\, \mathbb{Z}_4) \;=\; \mathrm{Rep}(\mathbb{Z}_4) \;\simeq\; \mathsf{Vec}_{\mathbb{Z}_4}\,. \tag{3.2.34}$$

Alternatively, we may gauge the symmetry in steps by first gauging the subgroup $A = \mathbb{Z}_2$ and subsequently gauging $K = \mathbb{Z}_2$ in $\mathcal{T}/A$.

- First gauging $A = \mathbb{Z}_2$ results in a theory $\mathcal{T}/A$ with symmetry group $\widetilde{G} = \widehat{A} \times K = \mathbb{Z}_2 \times \mathbb{Z}_2$ and mixed anomaly $\alpha \in Z^3_{grp}(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$ determined by the extension e [35]. This anomaly may be represented by the SPT phase

$$\frac{1}{2} \int_X \hat{\mathsf{a}} \cup \mathsf{k} \cup \mathsf{k} \tag{3.2.35}$$

in terms of the background fields $\hat{\mathsf{a}}, \mathsf{k} \in H^1_{grp}(X, \mathbb{Z}_2)$ for $\widetilde{G}$. There is no possibility for discrete torsion since $H^2_{grp}(\mathbb{Z}_2, U(1)) = 1$. The symmetry category of $\mathcal{T}/A$ is thus

$$\mathsf{C}(\mathbb{Z}_4 \,|\, \mathbb{Z}_2) \;\simeq\; \mathsf{Vec}^{\alpha}(\mathbb{Z}_2 \times \mathbb{Z}_2)\,. \tag{3.2.36}$$

- Now consider subsequently gauging $K = \mathbb{Z}_2$, which again does not allow for discrete torsion. The simple objects are labelled by pairs (χ, Φ), where $\chi \in \widehat{A}$ and Φ is an irreducible projective representation of K with 2-cocycle $\langle \chi, e \rangle$. Let us denote the generators of $\widehat{A} = \mathbb{Z}_2$ and $\widehat{K} = \mathbb{Z}_2$ by $\widehat{x}$ and $\widehat{y}$, respectively. For $\chi = 1$, we obtain two simple objects

$$U_0 \;:=\; (1,1) \qquad \text{and} \qquad U_2 \;:=\; (1,\widehat{y})\,. \tag{3.2.37}$$

For $\chi = \widehat{x}$, we obtain two additional simple objects

$$U_3 \;:=\; (\widehat{x}, f) \qquad \text{and} \qquad U_1 \;:=\; (\widehat{x}, f \cdot \widehat{y})\,, \tag{3.2.38}$$

where the normalised 1-cochain $f : K \to U(1)$ is defined by $f(y) = i$. Using $f^2 = \widehat{y}$, the fusion of the simple objects can then be determined to be

$$(U_1)^n \;=\; U_{n \bmod 4}\,. \tag{3.2.39}$$

This reproduces the symmetry category $\mathsf{C}(\mathbb{Z}_2 \times \mathbb{Z}_2, \alpha \,|\, \mathbb{Z}_2) = \mathsf{Vec}_{\mathbb{Z}_4}$, which agrees with that of $\mathcal{T}/G$.

Example 2 : $G = D_{2n}$

Next we consider a theory $\mathcal{T}$ with a non-anomalous finite dihedral symmetry group

$$G \;=\; D_{2n} \;\simeq\; A \rtimes H \;\simeq\; \mathbb{Z}_n \rtimes \mathbb{Z}_2\,, \tag{3.2.40}$$

with n even. The normal subgroup $H \simeq \mathbb{Z}_2$ with group elements $\{1, h\}$ acts on $A \simeq \mathbb{Z}_n$ with group elements $\{1, a, \dots, a^{n-1}\}$ through $h : a \mapsto a^{-1}$. Gauging the subgroup A generates another theory $\widehat{\mathcal{T}}$ with isomorphic symmetry group

$$\widehat{G} = D_{2n} \simeq \widehat{A} \rtimes H\,, \tag{3.2.41}$$

where A has been replaced by its Pontryagin dual $\widehat{A} \simeq \mathbb{Z}_n$ with elements $\{1, \chi, \dots, \chi^{n-1}\}$ and H-action $h : \chi \mapsto \chi^{-1}$.

Gauging $H \cong \mathbb{Z}_2$ produces a pair of theories with symmetry category $\mathsf{Rep}(D_{2n})$, as

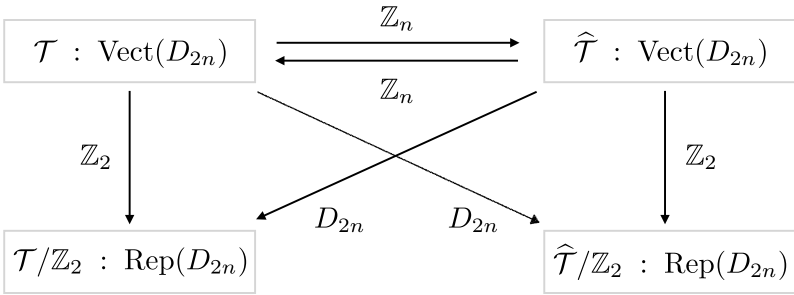

Figure 3.9

shown in figure 3.9. We can reproduce the symmetry category of $\mathcal{T}/D_{2n}$ by starting from $\widehat{\mathcal{T}}$ and gauging the subgroup H. We can view this by-steps gauging procedure a physical version of Mackey's construction for representations of semi-direct product groups.

There are the following simple objects:

- The 1-dimensional orbit $1 = \{1\}$ may be supplemented by irreducible representations 1, w of its stabiliser $\mathbb{Z}_2$. We denote the corresponding simple objects by 1, w. [2]

- The 1-dimensional orbit $o = \{\chi^{\frac{n}{2}}\}$ may be supplemented by irreducible representations 1, w of its stabiliser $\mathbb{Z}_2$. We denote the corresponding simple objects by o, ow.

- The 2-dimensional orbits $\{\chi^i, \chi^{n-i}\}$ with $j = 1, \ldots, \frac{n}{2}-1$ have trivial stabilisers. We denote the corresponding simple objects by $\mathcal{O}_j$, $j = 1, \ldots, \frac{n}{2} - 1$.

The fusion rules for irreducible representations may be computed following the recipe above and are given by

$$w \otimes w = 1 \qquad o \otimes o = 1 \qquad o \otimes w = ow \tag{3.2.42}$$

$$w \otimes \mathcal{O}_j = \mathcal{O}_j \qquad o \otimes \mathcal{O}_j = \mathcal{O}_j \tag{3.2.43}$$

$$\mathcal{O}_i \otimes \mathcal{O}_j = \mathcal{O}_{i+j} \oplus \mathcal{O}_{i-j}\,, \tag{3.2.44}$$

[2]They are pure topological Wilson lines for $H \cong \mathbb{Z}_2$.

where in the final line it is understood that $\mathcal{O}_0 = 1 \oplus w$ and $\mathcal{O}_{\frac{n}{2}} = o \oplus ow$ and $\mathcal{O}_j = \mathcal{O}_{\frac{n}{2}+j}$ for $j \neq 0, \frac{n}{2}$ mod n.

Example 3 : $G = D_8$

Finally lets restrict our focus from $G = D_{2n}$ to $G = D_8$, so that we might fully exposit the gauging of subgroups. In $1+1$ dimensions, an example is the $c = 1$ CFT or $\mathbb{Z}_2$-orbifold theory. In addition to the symmetry group $G = D_8$ considered here, this theory has a rich spectrum of non-invertible topological defects due to the fact that it is invariant under gauging of various subgroups [89]. We therefore emphasise that the symmetry categories discussed below form only part of the full fusion category symmetry in this example.

It is convenient to introduce generators r, s of D_8 corresponding to rotation by $\pi/2$ and reflection such that

$$D_8 \;=\; \langle r, s \,|\, r^4 = s^2 = 1,\; srs^{-1} = r^{-1} \rangle \,, \tag{3.2.45}$$

which manifests its presentation as a semi-direct product $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$. Alternatively, one may introduce generators $a := rs$ and $b := sr$ such that

$$D_8 \;=\; \langle a, b, s \,|\, a^2 = b^2 = s^2 = 1,\; ab = ba,\; sas^{-1} = b \rangle \,, \tag{3.2.46}$$

which manifests its presentation as a semi-direct product $D_4 \rtimes \mathbb{Z}_2$, where we denoted by $D_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$ the dihedral group of order four.

The automorphism group of D_8 is again D_8: There is a D_4 subgroup of inner automorphisms generated by the conjugations $x \mapsto {}^{rs}x$ and $x \mapsto {}^{s}x$ as well as a $\mathbb{Z}_2$ subgroup of outer automorphisms generated by the automorphism that sends $r \mapsto r^3$ and $s \mapsto rs$. The latter acts on D_4 by sending ${}^{rs}(.) \mapsto {}^{s}(.)$, so that the total automorphism group is indeed given by $D_4 \rtimes \mathbb{Z}_2 \cong D_8$.

There are 10 subgroups $H \subset D_8$ forming 8 conjugacy classes, whose structure is summarised in figure 3.10. The subgroups are organized in rows according to their orders 1, 2, 4 and 8 from bottom to top. Normal subgroups are coloured in red whereas non-normal subgroups are coloured in black with red arrows indicating their transformation behaviour under conjugation. The encircled subgroup is the centre of D_8 and grey arrows denote inclusion as a normal subgroup. The blue arrow indicates the transformation behaviour of subgroups under the generator of outer automorphisms, which acts by reflection of the diagram.

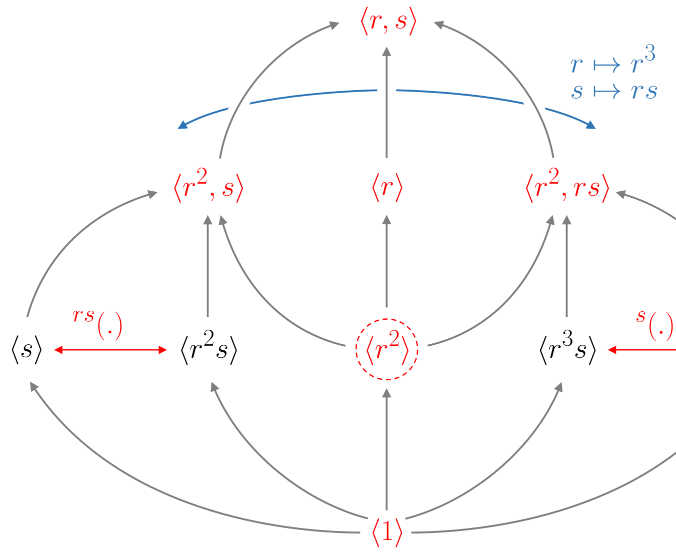

Figure 3.10

The starting point is the symmetry category $\mathsf{C}(D_8 \,|\, 1) = \mathsf{Vec}_{D_8}$. We consider the symmetry categories that result from gauging subgroups with discrete torsion, beginning with subgroups of the smallest order and working upwards in figure 3.10.

Order two subgroups

We begin by gauging order 2 subgroups $H \cong \mathbb{Z}_2$. There is no possibility of discrete torsion since $H^2_{grp}(\mathbb{Z}_2, U(1)) = 1$. There are 5 order 2 subgroups forming 3 conjugacy classes, two of which are related by an outer automorphism. Thus there are only two substantive cases to consider.

- The center $H = \langle r^2 \rangle \cong \mathbb{Z}_2$ of D_8 forms a non-split extension

$$1 \to \mathbb{Z}_2 \to D_8 \to D_4 \to 1 \tag{3.2.47}$$

with non-trivial extension class $[e] \in H^2_{grp}(D_4, \mathbb{Z}_2)$. Gauging the center therefore leads to a symmetry group $\mathbb{Z}_2 \times D_4$ with 't Hooft anomaly determined by $[e]$, which can be represented by the cubic SPT phase

$$\frac{1}{2} \int_X \widehat{\mathsf{a}} \cup \mathsf{a}_1 \cup \mathsf{a}_2 \tag{3.2.48}$$

in terms of the background fields for $\mathbb{Z}_2 \times D_4$. More concretely, we can describe the simple objects as follows: there are four double H-cosets $[1]$, $[r]$, $[s]$ and $[rs]$, all of whose stabilisers are given by H. The double coset ring is given by

$$[r]^2 = [s]^2 = [1] \qquad [r] * [s] = [rs]\,. \tag{3.2.49}$$

There are therefore 8 simple objects corresponding to the following pairs of double cosets and irreducible representations

$$([1], \chi^n)\,, \quad ([r], \chi^n)\,, \quad ([s], \chi^n)\,, \quad ([rs], \chi^n)\,, \tag{3.2.50}$$

where $n = 0, 1$ and χ denotes the generator of $\widehat{H} \cong \mathbb{Z}_2$. The fusion ring contains a $\mathbb{Z}_2$ subgroup generated by $C = ([1], \chi)$ as well as a D_4 subgroup generated by $Y = ([r], 1)$ and $Z = ([s], 1)$, which commute with each other

$$C \otimes Y = Y \otimes C \qquad C \otimes Z = Z \otimes C. \tag{3.2.51}$$

The symmetry can thus be identified with the product group $\mathbb{Z}_2 \times D_4$ as stated above. The corresponding symmetry category is given by $\mathsf{C}(D_8 \,|\, \langle r^2 \rangle) = \mathsf{Vec}^{\alpha}_{\mathbb{Z}_2 \times D_4}$.

- Now consider the two non-normal subgroups $H = \langle s \rangle, \langle r^2 s \rangle \cong \mathbb{Z}_2$, which are related to each other by conjugation. For concreteness, consider gauging $H = \langle s \rangle$. There are three double cosets $[1]$, $[r]$, $[r^2]$ with stabilisers H, 1, H respectively. The double coset ring is given by

$$[r] * [r] \;=\; [1] + [r^2] \qquad [r] * [r^2] \;=\; [r] \qquad [r^2] * [r^2] \;=\; [1]\,. \tag{3.2.52}$$

There are therefore 5 simple objects corresponding to the following pairs of double cosets and irreducible representations

$$1 = ([1], 1)\,, \quad U = ([r^2], 1)\,, \quad V = ([1], \chi)\,, \quad W = ([r^2], \chi)\,, \quad X = ([r], 1)\,, \tag{3.2.53}$$

where χ denotes the generator of $\widehat{H} \cong \mathbb{Z}_2$. The fusion ring contains a D_4 subgroup generated by U and V with $U \otimes V = W$ and additional relations

$$U \otimes X = X \qquad V \otimes X = X \qquad X \otimes X = 1 \oplus U \oplus V \oplus W\,. \tag{3.2.54}$$

The symmetry category is therefore a Tambara-Yamagami category of type D_4. A computation of the associator shows that $\mathsf{C}(D_8 \,|\, \langle s \rangle) = \mathsf{Rep}(D_8)$.

- Now consider the non-normal subgroups $H = \langle rs \rangle, \langle r^3 s \rangle \cong \mathbb{Z}_2$. They are related to each other by conjugation and to the subgroups in the previous bullet point by an outer-automorphism. The computation of the symmetry category is therefore the same up to relabelling, which implies $\mathsf{C}(D_8 \,|\, \langle rs \rangle) = \mathsf{C}(D_8 \,|\, \langle r^3 s \rangle) = \mathsf{Rep}(D_8)$.

Order four subgroups

There are three order 4 subgroups: one is isomorphic to $\mathbb{Z}_4$ and invariant under the outer automorphism, and the remaining two are isomorphic to D_4 and exchanged by the outer automorphism. In the latter case, there is the potential for discrete torsion because $H^2_{grp}(D_4, U(1)) = \mathbb{Z}_2$. There are therefore only two substantive cases to consider.

- Consider gauging the normal subgroup $H = \langle r \rangle \cong \mathbb{Z}_4$. There are two double cosets, $[1]$ and $[s]$, both of which have H as their stabiliser. The double coset ring is

$$[s] * [s] = [1]\,. \tag{3.2.55}$$

There are therefore 8 simple objects corresponding to the following pairs of double cosets and irreducible representations

$$([1], \chi^n)\,, \qquad ([s], \chi^n)\,, \tag{3.2.56}$$

where $n = 0, ..., 3$ and χ denotes the generator of $\widehat{H} \cong \mathbb{Z}_4$. The fusion ring is generated by $R := ([1], \omega)$ and $S := ([s], 1)$ subject to the relations

$$R^4 = S^2 = 1 \qquad S \otimes R \otimes S^{-1} = R^{-1}\,. \tag{3.2.57}$$

The symmetry can therefore be identified with the semi-direct product $\mathbb{Z}_4 \rtimes \mathbb{Z}_2 \cong D_8$, so that the corresponding symmetry category is given by $\mathsf{C}(D_8 \,|\, \langle r \rangle) = \mathsf{Vec}(D_8)$.

- Now consider the normal subgroup $H = \langle r^2, s \rangle \cong D_4$. There are again two double cosets [1] and $[r]$, both of which have H as their stabiliser. The double coset ring is

$$[r] * [r] = [1]\,. \tag{3.2.58}$$

There are therefore 8 simple objects corresponding to the following pairs of double cosets and irreducible representations

$$([1], \chi^n \omega^m) \qquad \text{and} \qquad ([r], \chi^n \omega^m)\,, \tag{3.2.59}$$

where $n, m = 0, 1$ and χ, ω denote the generators of $\widehat{H} \cong D_4$. The fusion ring is generated by $A := ([1], \chi)$, $B := ([1], \omega)$ and $D := ([r], 1)$ subject to the relations

$$A^2 = B^2 = D^2 = 1 \qquad D \otimes A \otimes D^{-1} = B\,. \tag{3.2.60}$$

The symmetry can therefore be identified with $D_4 \rtimes \mathbb{Z}_2 \cong D_8$ and the symmetry category is again given by $\mathsf{C}(D_8 \,|\, \langle r^2, rs \rangle) = \mathsf{Vec}_{D_8}$.

Adding a discrete torsion element $\psi \in H^2_{grp}(D_4, U(1)) = \mathbb{Z}_2$ leads to the same result, i.e. acts as an auto-equivalence of symmetry categories. This can be understood from the point of view of spectral sequences, interpreting $H^2_{grp}(D_4, U(1))$ as $H^0_{grp}(\mathbb{Z}_2, H^2_{grp}(D_4, U(1)))$. There are then no non-trivial differentials in the spectral sequence, which collapses at the second page. In particular, there is no obstruction in lifting ψ to a class in $H^2_{grp}(D_8, U(1))$.

- The normal subgroup $H = \langle r^2, rs\rangle \cong D_4$ is obtained from the bullet point above by an outer automorphism and therefore the computation of the symmetry category is the same up to relabelling. Adding discrete torsion again acts by an auto-equivalence of the symmetry category. We conclude that $\mathsf{C}(D_8 \,|\, \langle r^2, s\rangle) = \mathsf{Vec}_{D_8}$.

Note that gauging both order four subgroups, including with discrete torsion, results in an identical symmetry category Vec_{D_8}, up to equivalence. It is therefore possible that a theory $\mathcal{T}$ is invariant under gauging these subgroups, resulting in a rich spectrum of additional non-invertible duality defects that we have not not considered here. It was shown that this scenario is indeed realised when $\mathcal{T}$ is the $\mathbb{Z}_2$-orbifold CFT in [89].

Whole group

Finally, we gauge the entire symmetry group leading to the symmetry category $\mathsf{Rep}(D_8)$. Plugging in $n = 4$ to the previous example simplifies $\mathsf{C}(G|G)$ to

$$w \otimes w = 1 \qquad o \otimes o = 1 \qquad o \otimes w = ow \tag{3.2.61}$$

$$w \otimes \mathcal{O} = \mathcal{O} \qquad o \otimes \mathcal{O} = \mathcal{O} \tag{3.2.62}$$

$$\mathcal{O} \otimes \mathcal{O} = 1 \oplus w \oplus o \oplus ow \tag{3.2.63}$$

and so we see the symmetry category $\mathsf{C}(G|G) \simeq \mathsf{Rep}(D_8)$ is a Tambara-Yamagami fusion category based on the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$ [82].

Adding a discrete torsion element $\psi \in H^2_{grp}(D_8, U(1)) \cong \mathbb{Z}_2$ results in the same symmetry category up to equivalence. The results are summarised in figure 3.11.

There are various consistency checks on these results that correspond to taking different routes from bottom to top in figure 3.11. Due to the reflection symmetry of the diagram, it is sufficient to perform these checks for left hand side:

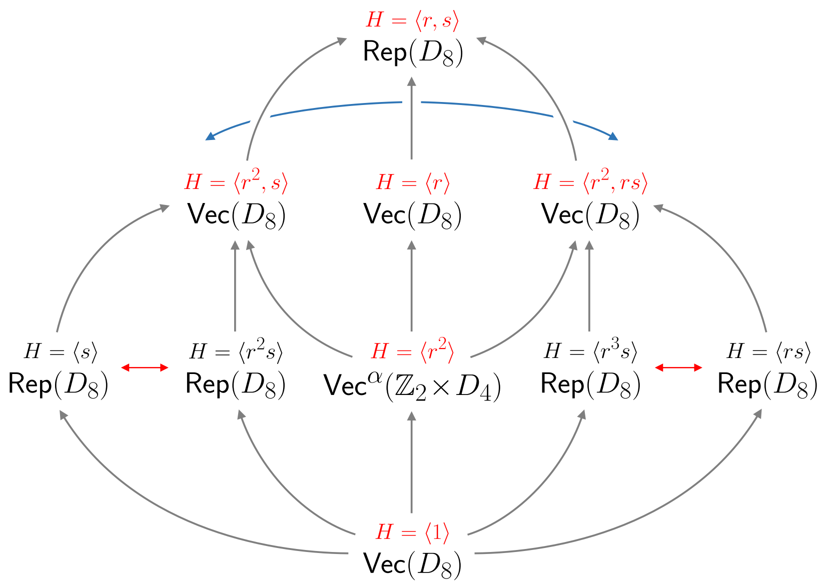

Figure 3.11

- Starting from the theory $\mathcal{T}$ with symmetry category $\mathsf{Vec}(D_8)$ we can gauge the central subgroup $\langle r^2 \rangle \cong \mathbb{Z}_2$ to obtain the theory $\mathcal{T}/\langle r^2 \rangle$ whose symmetry category is given by $\mathsf{Vec}^{\alpha}_{\mathbb{Z}_2 \times D_4}$ as described in the first bullet point in 3.2.3. This contains a $D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup generated by defects Y, Z, whose factors may be gauged independently:

  - Gauging $\langle Y \rangle \cong \mathbb{Z}_2$ reproduces the theory $\mathcal{T}/\langle r \rangle$ with symmetry category given by Vec_{D_8}. The latter contains a $\mathbb{Z}_2$ subgroup generated by the defect S, whose gauging reproduces the theory $\mathcal{T}/\langle r, s \rangle$ with symmetry category $\mathsf{Rep}(D_8)$.

  - Gauging $\langle Z \rangle \cong \mathbb{Z}_2$ reproduces the theory $\mathcal{T}/\langle r^2, s \rangle$ whose symmetry category is also Vec_{D_8}. The latter contains a $\mathbb{Z}_2$ subgroup generated by the defect D, whose gauging reproduces the theory $\mathcal{T}/\langle r, s \rangle$ with symmetry category $\mathsf{Rep}(D_8)$.

- Starting from $\mathcal{T}$ we can gauge the non-normal subgroup $\langle s \rangle \cong \mathbb{Z}_2$ to obtain the theory $\mathcal{T}/\langle s \rangle$ with symmetry category $\mathsf{Rep}(D_8)$ as described in the second bullet point in 3.2.3. The latter contains a $\mathbb{Z}_2$ subgroup generated by the defect U, whose gauging reproduces the theory $\mathcal{T}/\langle r^2, s \rangle$ with symmetry category

Vec_{D_8}.

3.3 Generalised Gauging in $1+1$ Dimensions

In the previous section we studied the defects after gauging a finite subgroup with discrete torsion in the framed setting, now we turn our attention to general notions of gauging a finite group symmetry of a $(1+1)$-dimensional unitary oriented quantum field theory.

The three equivalent methods presented in section 2.3 all admit natural lifts in one dimension higher, producing three equivalent methods for gauging a theory $\mathcal{T}$ with symmetry category Hilb_G^α, or Vec_G^α in lieu of unitarity.

The key takeaway from this section will be that these lifts are in fact categorifications of those in 2.3, as we shall see in each case the gaugings are labelled by unitary projective 2-representations of G that categorify unitary projective representations.

3.3.1 Gauging and Algebras internal to Hilb_G^α

First we consider a more general version of the algebra picture presented in the last section, as a generalisation to section 2.3.1, where gauging corresponds to summing over networks of symmetry defects. In the unitary setting this is implemented by choosing a (normalised) special dagger-Frobenius (1-)algebra object in

$$\mathsf{C}_{\mathcal{T}} = \mathsf{Hilb}_G^\alpha \,. \tag{3.3.1}$$

In the framed setting, this reduces to the study of (1-)algebra objects in

$$\mathsf{C}_{\mathcal{T}} = \mathsf{Vec}_G^\alpha \,, \tag{3.3.2}$$

or normalised Frobenius algebra objects in the oriented setting.

An algebra object $A \in \mathcal{C}_{\mathcal{T}}$ is equipped with multiplication and unit morphisms

$$m : A \otimes A \to A \qquad i : \mathbb{C} \to A \,, \tag{3.3.3}$$

subject to associativity and unitality conditions summarised by the commuting diagrams in figures 3.12 and 3.13.

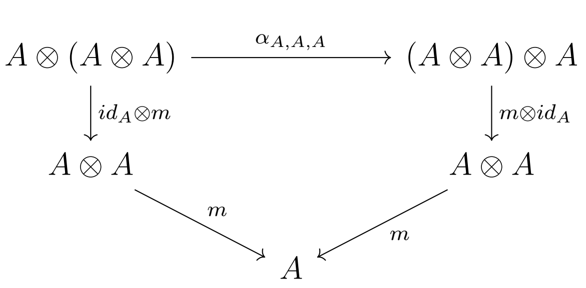

Figure 3.12

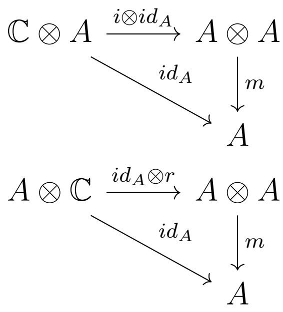

Figure 3.13

This preliminary data equips A with the structure of a finite-dimensional α^{-1}-twisted associative G-graded algebra, much like the structure we used in the previous section 2.2.

The dagger structure on morphisms then determine a co-multiplication and co-unit

$$m^\dagger : A \to A \otimes A \qquad\qquad i^\dagger : A \to \mathbb{C}_e \,, \tag{3.3.4}$$

satisfying analogous co-associativity and co-unity conditions. We also ask that they satisfy a Frobenius condition that all maps

$$A^{\otimes m+1} \to A^{\otimes n+1} \tag{3.3.5}$$

that can be built from m copies of the mulitplication and n copies of the co-multiplication are equivalent, equipping A with the structure of a finite-dimensional Frobenius algebra.

The combination $i^\dagger \circ m : A \otimes A \to \mathbb{C}_e$ forms a bilinear product which further defines two 1-morphisms

$$\begin{aligned} \sigma_l &:= ((i^\dagger \circ m) \otimes id_{A^*}) \circ \alpha^{-1}_{A,A,A^*} \circ (id_A \otimes coev_A) : A \to A^* \\ \sigma_r &:= (id_{A^*} \otimes (i^\dagger \circ m)) \circ \alpha_{A^*,A,A} \circ (ev^\dagger_A \otimes id_A) : A \to A^* \,. \end{aligned} \tag{3.3.6}$$

We ask that these two morphisms are equivalent $\sigma_l = \sigma_r$, then unitary duals imply that $\sigma_l^\dagger \circ \sigma_l = \sigma_r^\dagger \circ \sigma_r = id_A$. In particular this makes $\sigma_l : A \to A^*$ an isomorphism, defining an anti-linear involution on A and giving it the structure of a (normalised [3])

[3] We expect the normalisation to be uniquely determined by the constraint that the algebra is special/symmetric, which by definition strongly constrains the choice of bilinear form.

symmetric or *special* dagger-Frobenius algebra [84], or equivalently a (normalised) Q-system [85, 86].

In analogy to section 2.3.1, we define the Morita equivalence of special dagger-Frobenius algebras A, A' internal to $\mathcal{C}_{\mathcal{T}} = \mathsf{Hilb}_G^\alpha$ as an equivalence of (left) unitary module categories $\mathrm{Mod}_{\mathcal{C}_{\mathcal{T}}}(A) \simeq \mathrm{Mod}_{\mathcal{C}_{\mathcal{T}}}(A')$ of dagger-Frobenius module objects over A, A', internal to $\mathcal{C}_{\mathcal{T}}$. These comprise of objects $M \in \mathcal{C}_{\mathcal{T}}$ together with a (right) module action from A

$$\mu : M \otimes A \to M\,, \tag{3.3.7}$$

satisfying compatibility conditions with the multiplication and unit of A summarised in figures 3.14 and 3.15.

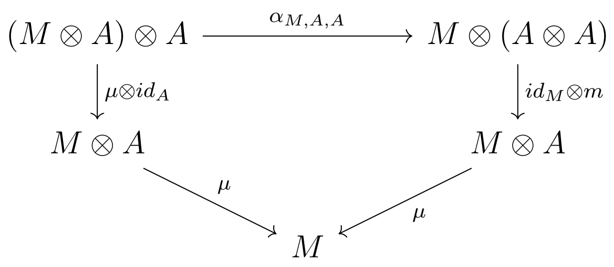

Figure 3.14

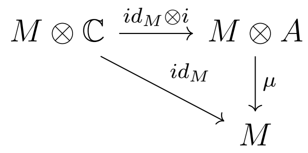

Figure 3.15

This preliminary data equips M with the structure of a finite-dimensional α^{-1}-twisted G-graded module over A.

The dagger structure on morphisms determines a co-module action which must satisfy analogous conditions. We also ask that they satisfy a Frobenius condition that all the maps

$$M \otimes A^{\otimes n} \to M \otimes A^{\otimes m}\,, \tag{3.3.8}$$

that can be built from n copies of the action and m copies of the co-action are equivalent, equipping M with the structure of a Frobenius module over A.

The combination of 1-morphisms

$$(id_M \otimes (i^\dagger \circ m)) \;\circ\; \alpha_{M,A,A} \;\circ\; (\mu^\dagger \otimes id_A) : M \otimes A \to M \tag{3.3.9}$$

also defines a right action of A on M; to make M a dagger-Frobenius module over

A, we demand that this is equivalent to the given module action μ. This condition can be equivalently viewed as constraining

$$\mu^{*\dagger} \circ (\sigma_l \otimes id_{M^*}) : A \otimes M^* \to M^* \tag{3.3.10}$$

to be a consistent left module action of A on M^*.

In the framed setting, the space of (right) modules $\lambda = \mathrm{Mod}_{\mathsf{Vec}_G^\alpha}(A)$ over an associative algebra object A forms a (left) module category over $\mathcal{C}_\mathcal{T} = \mathsf{Vec}_G^\alpha$: this is precisely how we define an α^{-1}-projective 2-representation of G [90, 91]. This induces a labelling of Morita equivalence classes of associative algebras internal to Vec_G^α by simple/irreducible projective 2-representations with projective 3-cocycle α^{-1}, which in turn are are classified by [92, 93]:

- A conjugacy class of subgroup $[H \subseteq G]$.
- A class ψ in a torsor over $H^2_{grp}(H, \mathbb{C}^\times)$ such that $\delta\psi = \alpha^{-1}$.

This classification overlaps with the gaugings studied in section 3.2; the minimal choices of gauging of an anomalous G symmetry in $1+1$ dimensions are given by a choice of non-anomalous subgroup $H \subseteq G$ to gauge, together with a choice of trivialisation $\delta\psi = \alpha^{-1}$.

Similarly, in the unitary setting the category $\lambda = \mathrm{Mod}_{\mathcal{C}_\mathcal{T}}(A)$ of (right) dagger-Frobenius A-modules then admits a natural (left) unitary module action from $\mathcal{C}_\mathcal{T} = \mathsf{Hilb}_G^\alpha$: this is precisely how we define unitary projective 2-representation G with projective 3-cocycle α^{-1}. This induces a labelling of Morita equivalence classes of special dagger-Frobenius algebras by unitary projective 2-representations

$$\lambda \in 2\mathrm{Rep}^{\dagger,\alpha^{-1}}(G)\,. \tag{3.3.11}$$

We now construct these special dagger-Frobenius algebras concretely. Given a representative subgroup $H \subseteq G$, the corresponding algebra objects are faithfully graded and decompose as

$$A = \bigoplus_{h \in H} \mathbb{C} \cdot e_h \,, \tag{3.3.12}$$

where we have elected to define a basis $\{e_h\}_{h \in H}$. The choice of multiplication $m : A \otimes A \to A$ is specified by a 2-cochain $\psi \in C^2_{grp}(H, \mathbb{C}^\times)$

$$m : e_{h_1} e_{h_2} \mapsto \psi(h_1, h_2) e_{h_1 h_2} \,, \tag{3.3.13}$$

and the associativity constraint 3.12 enforces the trivialisation condition

$$\delta\psi = \alpha|_H^{-1} \,. \tag{3.3.14}$$

We equip A with a dagger-Frobenius structure by adding a non-degenerate bilinear form $\beta : A \otimes A \to \mathbb{C}$ with non-vanishing components

$$\beta(e_h, e_{h^{-1}}) = \psi(h, h^{-1}) \,, \tag{3.3.15}$$

and an anti-linear involution

$$e_h^* = \psi(h, h^{-1})^{-1} e_{h^{-1}} \,. \tag{3.3.16}$$

This structure will be special/symmetric if and only if ψ is such that

$$\psi(1, h) = \psi(h, 1) = 1 \iff \psi(h, h^{-1}) = \psi(h^{-1}, h) \,. \tag{3.3.17}$$

Notice that from the framed perspective this is just a choice of normalisation for ψ that we can always make when G is finite. This categorifies the statement that all representations of finite groups are equivalent to unitary representations: all (projective) 2-representations of finite groups can be lifted to unitary (projective) 2-representations by picking the appropriate normalisation for α.

3.3.2 Gapped Interfaces and Module Categories over Hilb_G^α

As we just saw, classifying gaugings as Morita classes of algebra objects is equivalent to classifying projective 2-representations. We can arrive at this result in an equivalent way by considering gapped interfaces between $\mathcal{T}$ and the gauged theory $\mathcal{T}/_\lambda G$.

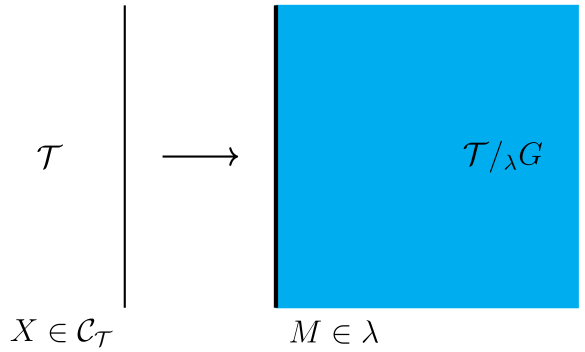

Figure 3.16

The gapped interfaces between $\mathcal{T}$ and $\mathcal{T}/_\lambda G$ form a finite semi-simple category λ, whose objects $M \in \lambda$ are gapped/topological interfaces, and morphisms are topological local operators supported on junctions between interfaces.

In the framed setting, the action from fusing topological lines in $\mathcal{T}$ onto the interface as depicted in figure 3.16 endows λ with the structure of a (left) module category over

$$\mathcal{C}_\mathcal{T} = \mathsf{Vec}_G^\alpha \,. \tag{3.3.18}$$

This is precisely how we define α^{-1}-projective 2-representations of G, directly categorifying the notion of projective representations in section 2.3.2 as modules/representations over ${}^\alpha\mathbb{C}G$.

An alternative formulation of these module categories is in terms of functors of the form

$$\mathcal{R} : \mathbf{B}G \to 2\mathsf{Vec} \,, \tag{3.3.19}$$

where $\mathbf{B}G$ of G denotes the delooping of G thought of as a fusion 2-category with a single simple object $\star$, and endomorphisms $End(\star) = \mathsf{Vec}_G$ [90–93]. For $\alpha = 0$ these functors are monoidal, otherwise they are monoidal up to α^{-1} which twists its compatibility with the associator, we call this an α^{-1}-projective monoidal functor.

We define the 2-vector spaces that make up the fusion 2-category $2\mathsf{Vec}$ as module categories over Vec equivalent to $\mathsf{Vec}^{\oplus n}$ for some $n \in \mathbb{N}$. This definition can be interpreted in two different ways:

1. A 2-vector space equivalent to $\mathsf{Vec}^{\oplus n}$ can be thought of as a finite semi-simple category with n simple objects.

2. The objects of a module 2-category over Vec up to equivalence can be thought of as modules over a (Morita class of) algebra object internal to Vec corresponding to an associative algebra.

Thought of as a finite semi-simple category, the 2-vector space $\mathcal{R}(\star)$ is equipped with a module action from Vec_G^{α} via the assignment of elements $g \in G$ to automorphisms of $\mathcal{R}(\star)$. This reproduces a module category from a choice of projective monoidal functors.

To relate this picture to the previous one in terms of algebras, we note that to construct an interface between $\mathcal{T}$ and $\mathcal{T}/_{\lambda}G$ obtained by gauging an algebra object A, we must first choose an object $M \in \mathcal{C}_{\mathcal{T}}$ and then specify how the algebra object ends on it. The data that implements this is precisely that of a module over A, identifying the category of gapped interfaces with the earlier module category

$$\lambda = \mathrm{Mod}_{\mathcal{C}_{\mathcal{T}}}(A)\,. \tag{3.3.20}$$

In the oriented setting we can formulate a direct argument for the reverse statement by considering $\mathcal{T}/_{\lambda}G$ with insertions of the identity operator and resolve them into oriented loops containing $\mathcal{T}$. Expanding the loops, eventually the interfaces will collide and produce a network of topological defects A_{λ} in $\mathcal{T}$. In order for the resulting theory to be independent of the way the expansion is performed, the topological defects A must describe normalised Frobenius algebra objects internal to $\mathcal{C}_{\mathcal{T}}$ [94].

In the unitary picture, the category λ of interfaces is now a finite semi-simple dagger category. The action from fusing topological lines in $\mathcal{T}$ onto the interface depicted in figure 3.16 endows λ with the structure of a (left) module category over

$$\mathcal{C}_{\mathcal{T}} = \mathsf{Hilb}_G^{\alpha}\,. \tag{3.3.21}$$

This action is compatible with the unitary structure of Hilb_G^{α} in the sense that:

1. For each pair of morphisms $\gamma : a \to b$ in λ and $f : X \to Y$ in $\mathcal{C}_{\mathcal{T}}$, the module action $f \triangleright \gamma$ satisfies
$$(f \triangleright \gamma)^{\dagger} = f^{\dagger} \triangleright \gamma^{\dagger}\,. \tag{3.3.22}$$
making λ a dagger-module category over Hilb_G^{α}.

2. Pre-composing the (left) module action with the unitary dual $*$ defines a consistent (right) module category, making λ a *unitary* module category over Hilb_G^{α}. This property may be thought of physically as capturing the identification between two gapped interfaces related by a reflection.

This is how we expect to define unitary α^{-1}-projective 2-representations of G, directly categorifying the notion of unitary projective representations from section 2.3.2 as unitary representations over ${}^{\alpha}\mathbb{C}G$ as a C*-algebra. Alternatively we expect that we can also formulate these unitary module categories as unitary α^{-1}-projective monoidal functors

$$\mathcal{R} : \mathbf{B}G \to 2\mathsf{Hilb}\,, \tag{3.3.23}$$

where we would define $2\mathsf{Hilb}$ analogously to $2\mathsf{Vec}$ as the unitary fusion 2-category [4] of module categories over Hilb equivalent to $\mathsf{Hilb}^{\oplus n}$.for some $n \in \mathbb{N}$.

3.3.3 Symmetry TFTs and Lagrangian Algebras

Now we consider the same gauging procedures from the perspective of gapped boundary conditions in the sandwich construction [27, 28]. The starting point is

[4]We will return to these objects in chapter 4, for now it suffices to say that a generally accepted definition of these objects is still lacking.

2+1-dimensional unitary oriented Dijkgraaf-Witten theory labelled by a gauge group G and a class

$$[\alpha] \in H^3_{grp}(G, U(1)) \,. \tag{3.3.24}$$

These are gauge theories supported on a 3-manifold M_3, described by a finite gauge field

$$\mathsf{a} : M_3 \to BG \,, \tag{3.3.25}$$

whose action is determined by a representative $\alpha \in Z^3_{grp}(G, U(1))$ satisfying

$$\delta\alpha(g,h,k,l) = \frac{\alpha(h,k,l)\alpha(g,hk,l)\alpha(g,h,k)}{\alpha(gh,k,l)\alpha(g,h,kl)} = 1 \,, \tag{3.3.26}$$

via the pull-back

$$\int_{M_3} \mathsf{a}^* \alpha \,, \tag{3.3.27}$$

and is hence manifestly topological. When the boundary ∂M_3 is non-empty, we can specify topological boundary conditions by fixing the restriction

$$\mathsf{a}|_{\partial M_3} : \partial M_3 \to BH \,, \tag{3.3.28}$$

for some subgroup $H \subseteq G$ such that $\alpha|_H^{-1} = \delta\psi$ trivialises. The pull-back

$$\int_{\partial M_3} \psi^* \mathsf{a} \tag{3.3.29}$$

then defines a consistent contribution to the topological action on the boundary that makes the total theory topological. This construction accounts for all possible topological boundary conditions for $(2+1)$-dimensional Dijkgraaf-Witten theories.

We will define the Dirichlet boundary condition $\mathcal{D}$ as the one that totally fixes a on the boundary by setting $H = 1$ to be the trivial subgroup. In the framed setting this corresponds to a $1+1$-dimensional topological boundary supporting a (G, α) symmetry described by

$$\mathcal{C}_{\mathcal{D}} = \mathsf{Vec}^{\alpha}_{G} \,. \tag{3.3.30}$$

The existence of this canonical topological Dirichlet boundary reflects the fact that the Dijkgraaf-Witten theory is a Turaev-Viro type TQFT whose symmetry in the

$(2+1)$-dimensional bulk contains topological lines described by the Drinfeld centre

$$\mathsf{DW}_{G,\alpha} \simeq \mathcal{Z}(\mathsf{Vec}_G^\alpha)\,. \tag{3.3.31}$$

This is a braided fusion category whose objects are objects $X \in \mathsf{Vec}_G^\alpha$ together with a half-braiding that comes in the forms of 1-isomorphisms

$$b_{X,Y} : X \otimes Y \to Y \otimes X\,, \tag{3.3.32}$$

for each other object $Y \in \mathsf{Vec}_G^\alpha$, together with a fusion compatibility condition, as we defined in subsection 1.3.2. We can think of these isomorphisms physically as capturing how the topological lines braid with one-another in $2+1$ dimensions. Concretely these objects are characterised by the following data:

1. A finite-dimensional G-graded vector space

$$X = \bigoplus_{g \in G} X_g\,. \tag{3.3.33}$$

2. A G-action by automorphisms $\rho_{g,h} : X_h \to X_{ghg^{-1}}$ satisfying twisted composition

$$\rho_{g,{}^h f} \circ \rho_{h,f} = \tau_f(\alpha)(g,h)\rho_{gh,f} \tag{3.3.34}$$

and twisted distributivity

$$\rho_{f,g} \otimes \rho_{f,h} = \widetilde{\tau}_f(\alpha)(g,h)\rho_{f,gh}\,, \tag{3.3.35}$$

for all $g, h, f \in G$.

The collections of phases

$$\tau_f(\alpha)(g,h) := \frac{\alpha(g, {}^h f, h)}{\alpha({}^{gh} f, g, h)\alpha(g,h,f)} \qquad \widetilde{\tau}_f(\alpha)(g,h) := \frac{\alpha({}^f g, f, h)}{\alpha(f,g,h)\alpha({}^f g, {}^f h, f)} \tag{3.3.36}$$

define groupoid 2-cocycles

$$\tau(\alpha),\, \widetilde{\tau}(\alpha) \in Z^2_{grp}(G/\!/G, U(1))\,, \tag{3.3.37}$$

or equivalently a collection of group 2-cocycles $\tau_f(\alpha)$, $\widetilde{\tau}_f(\alpha) \in Z^2_{grp}(C_f(G), U(1))$ upon restriction to arguments in the centralizer $g, h \in C_f(G)$. These satisfy the same properties as the transgression of α, and in particular they identifies the Drinfeld centre with projective representations of the Drinfeld double

$$\mathcal{Z}(\mathsf{Vec}^{\alpha}_{G}) \simeq \mathrm{Rep}^{\alpha^{-1}}(G//G)\,. \tag{3.3.38}$$

The symmetry of a (1+1)-dimensional quantum field theory can be recast as a (2+1)-dimensional symmetry TFT. For a (G, α) symmetry the corresponding symmetry TFT is the (G, α) Dijkgraaf-Witten theory. The dynamics of $\mathcal{T}$ are captured by a relative (non-topological) boundary condition, and the theory itself can be recovered by interval compactification with the canonical gapped Dirichlet boundary condition $\mathcal{D}$.

For other choices of gapped boundary $\lambda = (H, \psi)$, interval compactification produces the theory $\mathcal{T}/_{\lambda}G$. This has an alternative description of starting from the canonical Dirichlet boundary condition $\mathcal{D}$ and gauging an anomaly free subgroup $H \subset G$ with trivialisation ψ.

The statement that the Drinfeld centers of $\mathcal{C}_{\mathcal{T}}$, $\mathcal{C}_{\mathcal{T}/_{\lambda}G}$ coincide is equivalent to the statement that they are Morita equivalent. This is general: two unitary fusion categories are unitarily Morita equivalent if and only if their Drinfeld centers are unitarily equivalent. In the sandwich construction, this is the statement that all gapped boundary conditions admit invertible junctions between them. We will see that the analogous statement fails in higher dimensions in chapter 4.

We now connect to another description that sets gapped boundary conditions in the language of Lagrangian algebras; it is known that the gapped boundary conditions of a $(2+1)$-dimensional TQFT of Turaev-Viro type built from a fusion category C can be identified with Lagrangian algebras in $Z(\mathsf{C})$ [95, 96]. These represent topological line defects in the symmetry bulk that condense on their corresponding topological boundary.

An algebra object A in $\mathcal{Z}(\mathsf{Vec}_G^\alpha)$ as a braided fusion category is said to be *braided* (or *commutative*) if its multiplication is compatible with that braiding in the sense of figure 3.17.

$$\begin{array}{ccc} A\otimes A & \xrightarrow{b_{A,A}} & A\otimes A \\ & \searrow^{m} & \downarrow m \\ & & A \end{array}$$

Figure 3.17

$$\begin{array}{ccc} M\otimes A & \xrightarrow{b_{M,A}} & A\otimes M \\ b_{A,M}\uparrow & & \downarrow \mu \\ A\otimes M & \xrightarrow{\mu} & M \end{array}$$

Figure 3.18

The Lagrangian condition can be formulated in terms of local modules. A module object M over A in $\mathcal{C}_\mathcal{T} \simeq \mathsf{Vec}_G^\alpha$ is *local* if its module action is compatible with the braiding in the sense of figure 3.18. We say then that a connected braided algebra object A is *Lagrangian* if the category of local modules over A trivialises to

$$\mathrm{Mod}^{loc}_{\mathcal{Z}(\mathcal{C}_\mathcal{T})}(A) \simeq \mathsf{Vec} \tag{3.3.39}$$

as a braided fusion category.

A braided algebra object in $\mathcal{Z}(\mathsf{Vec}_G^\alpha)$ is an α^{-1}-twisted G-crossed commutative extension of an associative algebra in Vec. The data of such an object is as follows:

1. A finite-dimensional G-graded vector space $A = \bigoplus_{g\in G} A_g$.

2. A G-action by automorphisms $\rho_{g,h} : A_h \to A_{ghg^{-1}}$ satisfying twisted composition

$$\rho_{g,{}^h f}(\rho_{h,f}(x)) = \tau_f(\alpha)(g,h)\,\rho_{gh,f}(x) \tag{3.3.40}$$

   for all $x \in A_f$ and $g,h,f \in G$.

3. A G-graded multiplication $A_g A_h \subseteq A_{gh}$ satisfying:

   - Twisted associativity

$$(xy)z = \alpha(g,h,k)x(yz) \tag{3.3.41}$$

     for all $x \in A_g$, $y \in A_h$, $z \in A_k$, and $g,h,k \in G$.

- G-crossed commutativity

$$xy = \rho_{g,h}(y)x \tag{3.3.42}$$

for all $x \in A_g$, $y \in A_h$, and $g, h \in G$.

- Twisted distributivity

$$\rho_{f,gh}(xy) = \widetilde{\tau}_f(\alpha)(g,h)\rho_{f,g}(x)\rho_{f,h}(y) \tag{3.3.43}$$

for all $x \in A_g$, $y \in A_h$ and $g, h, f \in G$.

It is important for us to stop here for a moment and appreciate that if we forget the multiplicative structure, this data is that of a projective 2-character which categorifies that of the projective characters seen in section 2.3.3:

- We have an assignment $X : G \to \mathsf{Vec}$, this lifts the assignment of $\chi : G \to \mathbb{C}$ for projective characters.
- We have isomorphisms $\rho_{g,h} : X_h \to X_{{}^gh}$ that lift the projective class function property for projective characters.

We will now show that for a Lagrangian algebra internal to $2\mathsf{Vec}_G^\alpha$, this is the data of an irreducible/simple α^{-1}-projective 2-character.

It is known that Lagrangian algebra objects in $\mathcal{Z}(\mathsf{C})$ correspond to full centres of indecomposable algebra objects in C [83]. These are braided algebra objects $\mathcal{Z}(A)$ in $\mathcal{Z}(\mathsf{C})$ together with a 1-morphism $\mathcal{Z}(A) \to A$ that restricts to an indecomnposable algebra object A in C. We will not construct these objects in full generality but instead restrict our attention back to the group-theoretic setting by constructing explicit examples.

The Lagrangian algebras in $\mathcal{Z}(\mathsf{Vec}_G^\alpha)$ then are described by indecomposable algebra in Vec_G^α labelled by simple α^{-1}-projective 2-representations of G, coinciding with our previous analyses. To do this concretely, recall that indecomposable algebra objects in Vec_G^α corresponding to a projective 2-representation $\lambda = (H, \psi)$ are written

$$A_\lambda = \bigoplus_{h \in H} \mathbb{C}_h, \tag{3.3.44}$$

with multiplication twisted by the 2-cochain $\psi \in Z^2_{grp}(G, U(1))$ satisfying $\delta\psi = (\alpha|_H)^{-1}$. Computation of the full centre requires promoting A_λ to a Lagrangian algebra $\mathcal{Z}(A_\lambda)$ in $\mathcal{Z}(\mathsf{Vec}_G^\alpha)$. To this end, we start choosing coset representatives $a_i H$ that determine a permutation representation σ on the quotient G/H by

$$g \cdot a_j H = a_{\sigma_g(j)} H \tag{3.3.45}$$

with compensating transformations

$$\ell_{g,j} = a^{-1}_{\sigma_g(j)} \cdot g \cdot a_j \, . \tag{3.3.46}$$

The 2-cochain ψ then induces a 2-cochain $\{c_1, ..., c_n\} \in C^2_{grp}(G, U(1)^{G/H})$ that trivialises α^{-1} via the Shapiro isomorphism,

$$c_j(g_1, g_2) = \psi(\ell_{g_1, \sigma^{-1}_{g_1}(j)}, \ell_{g_2, \sigma^{-1}_{g_1 g_2}(j)}) \, . \tag{3.3.47}$$

We then construct the full centre by summing

$$\mathcal{Z}(A_\lambda) = \bigoplus_{i \in G/H} A_{({}^{a_i}H, \psi^{a_i})} \, , \tag{3.3.48}$$

where ${}^{a_i}H = a_i H a_i^{-1}$ and

$$\psi^{a_i}(h_1, h_2) = \psi({}^{a_i^{-1}}h_1, {}^{a_i^{-1}}h_2) \, , \tag{3.3.49}$$

overlaps with $c_i(h_1, h_2)$ on $h_1, h_2 \in {}^{a_i}H$. In doing this we capture all equivalent choices of subgroup H for a given irreducible projective 2-representation λ, making $\mathcal{Z}(A_\lambda)$ central by construction. To equip $\mathcal{Z}(A_\lambda)$ with a half-braiding we introduce generators

$$\mathcal{Z}(A_\lambda) = \bigoplus_{i \in G/H} \bigoplus_{g \in {}^{a_i}H} \mathbb{C} \cdot e^i_g \tag{3.3.50}$$

indexed by $i \in G/H$ and $g \in {}^{a_i}H$, then the half-braiding takes the form of a G-action

$$\rho_{g,h}(e^i_h) = \frac{c_i(g,h)}{c_i({}^g h, g)} e^{\sigma_g(i)}_{ghg^{-1}} \,. \tag{3.3.51}$$

We further equip $\mathcal{Z}(A_\lambda)$ with the structure of a braided algebra in $\mathcal{Z}(\mathsf{Vec}^\alpha_G)$ by equipping it with an α^{-1}-twisted associative G-crossed commutative multiplication

$$e^i_{h_1} e^i_{h_2} = \psi^{a_i}(h_1, h_2)\, e^i_{h_1 h_2} \tag{3.3.52}$$

for each $i \in G/H$ and $h_1, h_2 \in {}^{a_i}H$.

In the oriented setting we can further equip this braided algebra with a normalised Frobenius structure by defining a bilinear map

$$\beta(e^i_h, e^i_{h^{-1}}) = \psi^{a_i}(h, h^{-1})\,. \tag{3.3.53}$$

Unitarity

This entire story admits a natural lift to the unitary setting; the (G,α) Dijkgraaf-Witten theory has a natural unitary structure. the Dirichlet boundary condition $\mathcal{D}$ supports a global (G,α) symmetry described by

$$\mathcal{C}_{\mathcal{D}} = \mathsf{Hilb}^\alpha_G \,. \tag{3.3.54}$$

The $(2+1)$-dimensional bulk symmetry now contains topological lines described by the Drinfeld centre

$$\mathsf{DW}_{G,\alpha} \simeq \mathcal{Z}(\mathsf{Hilb}^\alpha_G)\,, \tag{3.3.55}$$

now thought of as a unitary braided fusion category. The objects of this category are characterised concretely by the following data:

1. A finite-dimensional G-graded Hilbert space

$$X = \bigoplus_{g \in G} X_g \,. \tag{3.3.56}$$

2. A G-action by $*$-automorphisms $\rho_{g,h} : X_h \to X_{ghg^{-1}}$ satisfying twisted composition and twisted distributivity.

In this setting the Drinfeld centre is identified with unitary projective representations of the Drinfeld double

$$\mathcal{Z}(\mathsf{Hilb}_G^{\alpha}) \simeq \mathrm{Rep}^{\dagger,\alpha^{-1}}(G//G)\,. \tag{3.3.57}$$

We can further extend the full-centre construction to the unitary setting by normalising $\psi(1,g) = \psi(g,1) = 1$ and equipping $\mathcal{Z}(A_\lambda)$ with an anti-linear involution

$$(e_h^i)^* = \psi^{a_i}(h,h^{-1})^{-1} e^i_{h^{-1}}\,. \tag{3.3.58}$$

This $*$ operation is automatically compatible with the G-action and multiplication, making $\mathcal{Z}(A_\lambda)$ a (normalised) braided special dagger-Frobenius algebra object in $\mathcal{Z}(\mathsf{Hilb}_G^{\alpha})$.

3.4 Defects After Generalised Gauging

We now revisit the study of symmetry defects that result from this more general picture of gauging a (G, α) symmetry in $1+1$ dimensions.

3.4.1 Bimodules Over Algebras

Earlier in section 3.2 we showed that the defects obtained after gauging an algebra object A internal to $\mathsf{Vec}^{a}\alpha_G$ were described by bimodule objects over A

$$\mathsf{C}(G, \alpha | H, \psi) \simeq \mathrm{Bimod}_{\mathcal{C}_{\mathcal{T}}}(A) \,. \tag{3.4.1}$$

We will now return to this perspective in greater generality.

Starting in the framed setting, given an algebra object A internal to Vec_G^α, we can construct the corresponding fusion category of bimodule objects over A. A Bimodule object over A is an object $M \in \mathsf{Vec}_G^\alpha$ together with left and right module actions implemented by 1-morphisms

$$\mu_l : A \otimes M \to A \qquad\qquad \mu_r : M \otimes A \to M \,, \tag{3.4.2}$$

satisfying compatibility conditions like those appearing in figures 3.14 and 3.15, and now additionally a commutation relation depicted by the commuting diagram in figure 3.19.

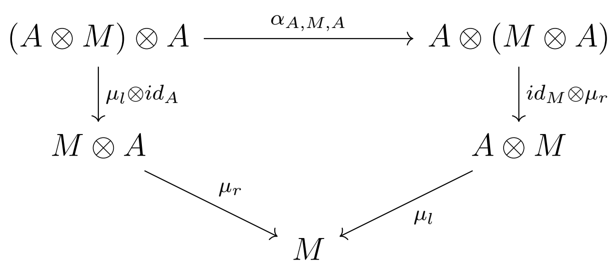

Figure 3.19

For an indecomposable algebra object A corresponding to a simple projective 2-representation λ, these conditions coincide with the earlier ones from section 3.2.2 stated in equations (3.2.26), (3.2.27) and (3.2.28).

In the unitary setting with a special dagger-Frobenius structure on A, we further make these objects special dagger-Frobenius bimodules by imposing similar conditions to those described in section 3.3.1. Importantly since we are working in the finite setting, this does not change the earlier construction of defects; all equivalence classes of representations over a finite group admit a basis that makes them unitary.

3.4.2 Modules Over Lagrangian Algebras

We can now also recover the symmetry defects after gauging by considering the symmetry category on the topological boundary corresponding to $\mathcal{T}/_\lambda G$ in the sandwich construction.

Starting in the framed setting, with the Dijkgraaf-Witten TQFT with gauge group G and topological action α, we take the perspective that a Lagrangian algebra L_λ labelled by projective 2-representations $\lambda = (H, \psi)$ specifies a topological line in the symmetry bulk that condenses on its corresponding topological boundary condition $\mathcal{B}_\lambda$. To state this precisely, we note that the boundary condition $\mathcal{B}_\lambda$ determines a monoidal functor

$$\mathcal{F}_\lambda : \mathcal{Z}(\mathsf{Vec}_G^\alpha) \to \mathsf{C}(G, \alpha | H, \psi) \tag{3.4.3}$$

corresponding to bringing bulk topological lines to the boundary. By identifying $\mathcal{Z}(\mathsf{Vec}_G^\alpha) \simeq \mathcal{Z}(\mathcal{C}_{\mathcal{T}/\lambda G})$ and $\mathcal{C}_{\mathcal{T}/_\lambda G} \simeq \mathsf{C}(G, \alpha | H, \psi)$, we can recast this functor as the forgetful functor defined in subsection 1.3.2

$$\mathcal{F}_\lambda \sim F_\lambda : \mathcal{Z}(\mathcal{C}_{\mathcal{T}/_\lambda G}) \to \mathcal{C}_{\mathcal{T}/_\lambda G} \,. \tag{3.4.4}$$

The statement that L_λ condenses on the boundary means that its image under $\mathcal{F}_\lambda$ is the trivial line/monoidal unit

$$\mathcal{F}_\lambda \, L_\lambda \simeq 1 \tag{3.4.5}$$

in $\mathcal{C}_{\mathcal{T}/_\lambda G}$.

We next consider pairs of bulk topological lines $l, l' \in \mathcal{Z}(\mathsf{Hilb}_G^\alpha)$ that map to the same boundary line

$$\mathcal{F}_\lambda\, l \;\simeq\; \mathcal{F}_\lambda\, l' \;\simeq\; l_0 \in \mathsf{C}_{\mathcal{T}/_\lambda G}\,. \tag{3.4.6}$$

The boundary condition hence induces an equivalence relation $l \sim l'$ which in particular implies the existence of a morphism

$$\mu_{l,l'} : l \otimes \mathcal{Z}(A_\lambda) \to l'\,, \tag{3.4.7}$$

that specifies how to consistently end L_λ on the junction, as depicted in figure 3.20, together with compatibility conditions with the algebra structure.

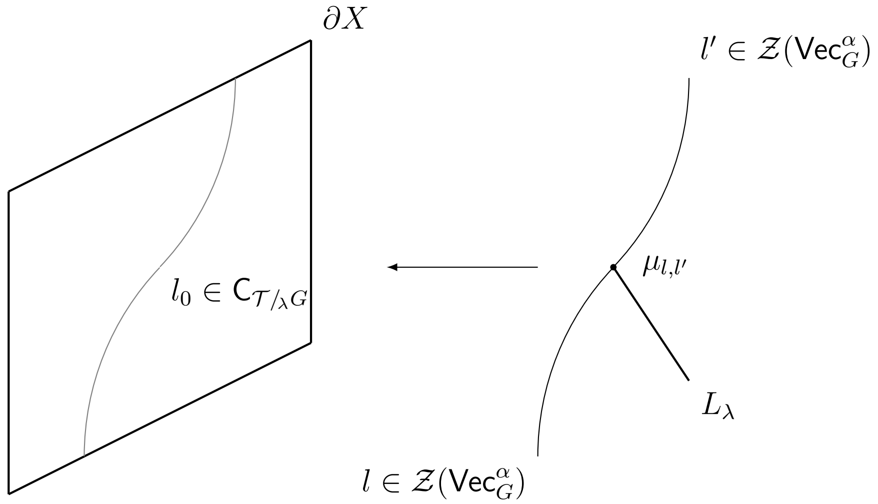

Figure 3.20

Topological lines on the boundary $\mathcal{B}_\lambda$ are hence identified with lines in the bulk modulo this equivalence relation. These equivalence classes correspond to modules over L_λ in $\mathcal{Z}(\mathsf{Vec}_G^\alpha)$ and so we identify the symmetry category on the boundary as

$$\mathsf{C}_{\mathcal{T}/_\lambda G} = \mathrm{Mod}_{\mathcal{Z}(\mathsf{Vec}_G^\alpha)}(L_\lambda)\,. \tag{3.4.8}$$

This identification replicates our earlier construction from bimodules. In particular, we conjecture that for any associative algebra object internal to a fusion category C, should it admit a full-centre lift to a braided algebra object $\mathcal{Z}(A)$ internal to $\mathcal{Z}(\mathsf{C})$, then there should be a general equivalence of fusion categories

$$\mathrm{Bimod}_{\mathsf{C}}(A) \simeq \mathrm{Mod}_{\mathcal{Z}(\mathsf{C})}(\mathcal{Z}(A))\,. \tag{3.4.9}$$

We do not aim to prove this statement in generality here, rather we will demonstrate it in the group-theoretic setting with an explicit example.

First recall that from the full-centre construction, a Lagrangian algebra L_λ can be expressed in terms of generators e^i_g indexed by cosets with representatives $a_i \in G$ and group elements $g \in {}^{a_i}H$. The G-graded multiplication, twisted G-action and G-crossed braiding are all determined as before.

A (right) module over L_λ internal to $\mathcal{Z}(\mathsf{Vec}_G^\alpha)$ is specified by the following data:

1. A finite-dimensional G-graded vector space $V = \bigoplus_{g\in G} V_g$.
2. A G-action by unitary linear maps $\omega_g(h) : V_h \to V_{{}^g h}$ satisfying twisted composition
$$\omega_{g,{}^h f} \circ \omega_{h,f} = \tau_f(\alpha)(g,h)\omega_{gh,f}\,. \tag{3.4.10}$$
3. A right module action r from the generators e^i_g satisfying
$$r(e^i_{h_2}) \circ r(e^i_{h_1})|_{V_f} = \psi^{a_i}(h_1,h_2)\alpha(a,h_1,h_2)r(e^i_{h_1h_2})|_{V_f}\,, \tag{3.4.11}$$
and
$$\omega_{g,hf} \circ r(e^i_h) = r(\rho_{g,h}(e^i_h)) \circ \omega_{g,f}\,, \tag{3.4.12}$$
for each graded component V_f and $f \in G$, and
$$\sum_{i\in G/H} r(e^i_1) = id_V. \tag{3.4.13}$$

Using the braided structure of $\mathcal{Z}(\mathsf{Vec}_G^\alpha)$ we can absorb the ω action and induce from the right action r, a left action

$$l(e^i_h)|_{V_f} = r(\rho_{f^{-1},h}(e^{\sigma_f(i)}_h)) \tag{3.4.14}$$

satisfying
$$l(e^i_{h_1}) \circ l(e^i_{h_2})|_{V_f} = \frac{\psi(h_1,h_2)}{\alpha(h_1^f,h_2^f,f)}l(e^i_{h_1h_2})|_{V_f} \tag{3.4.15}$$

and
$$r(e^i_{h_2}) \circ l(e^i_{h_1})|_{V_f} = \alpha(h_1,f,h_2^f)l(e^i_{h_1}) \circ r(e^i_{h_2})|_{V_f}\,, \tag{3.4.16}$$

for each $f \in G$. These formulae should be compared with the earlier ones appearing in section 3.2.2 in the form of equations (3.2.26), (3.2.27) and (3.2.28), with which they coincide; this demonstrates the equivalence (3.4.9) for indecomposable algebra objects internal to Vec_G^α.

In particular, given a choice of double coset with representative $g \in G$, the combination

$$\Phi(h) = r(e^H_{(h^g)^{-1}}) \circ l(e^H_h)\,, \tag{3.4.17}$$

for $h \in H_g = H \cap {}^g H$ reconstructs the projective representation of H_g. The defects after gauging are hence labelled by

1. A double coset $[a] \in H\backslash G/H$.
2. A projective representation Φ of $H_a = H \cap {}^a H$ with projective 2-coycle $c_g \in Z^2_{grp}(H_a, U(1))$ defined by the formula

$$c_g(h_1, h_2) := \frac{\psi^a(h_1,h_2)}{\psi(h_1,h_1)} \frac{\alpha(h_1,h_2,a)\alpha(a,h_1^a,h_2^a)}{\alpha(h_1,a,h_2^a)}\,. \tag{3.4.18}$$

reproducing precisely the expected structure of a group-theoretic symmetry category $\mathsf{C}_{\mathcal{T}/_\lambda G} = \mathsf{C}(G,\alpha|H,\psi)$ obtained earlier.

In the unitary setting with a special dagger-Frobenius structure on L_λ, we further make these objects special dagger-Frobenius modules by imposing similar conditions to those described in section 3.3.1. Once again since we are working in the finite setting, this does not change the earlier construction of defects; all equivalence classes of representations over a finite group admit a basis that makes them unitary.

Chapter 4

Gauging Finite Symmetries in 2+1 Dimensions

We now move to symmetries of quantum field theories occupying $2+1$ dimensions. The construction of non-invertible symmetries in $2+1$ dimensions has seen many advancements in recent years, and is still an area of active development [1, 2, 8, 22, 53].

In analogy to chapter 3, the most general process of gauging a finite group symmetry includes additional degrees of freedom that we can describe physically in a few ways:

1. Orbifolding a non-anomalous subgroup with a choice of discrete torsion and symmetry fractionalisation [97, 98].

2. Stacking the theory with a $2+1$-dimensional TQFT that cancels the anomaly on a subgroup [2, 8].

3. Choosing an opposing topological boundary for the $(3+1)$-dimensional symmetry TFT [27, 28, 53].

In this chapter we will demonstrate that the findings of chapter 3 generalise to $2+1$ dimensions, and that these descriptions can all be recast in the language of higher representation theory; gauging a finite group symmetry corresponds to inserting a badly quantised space-filling Wilson volume/hyper-surface, and the topological surfaces these volumes can end on transform in a projective 3-representation.

Topological line defects in $2+1$ dimensions continue to be labelled by representations, however in addition to these we also observe topological surface defects that have a couple of equivalent descriptions:

- They are insertions of extended $(1 + 1)$-dimensional TQFTs that transform under the finite gauge symmetry [8].
- They are surface defects of the original symmetry, together with additional data that makes them compatible with the insertion of a network of defects [1, 2].
- They are equivalence classes of surface defects in the $(3 + 1)$-dimensional symmetry TFT that are identified on the topological boundary [1].

These interpretations represent further categorifications to the ideas presented in chapters 2 and 3. The appearance of non-trivial topological lines together with topological surfaces that can act on them in a non-trivial way leads to a far richer variety of symmetries in $2 + 1$ dimensions. We can see one aspect of this richness immediately from the class of invertible *2-group* symmetries, which categorify the notion of a group and have been the topic of research for many recent lines of research [74, 99–102].

In this chapter we will first discuss the structure of finite global symmetries in $2 + 1$ dimensions [2], focusing our attention on invertible symmetries described generally by 2-groups. Then we will construct more general non-invertible group-theoretic symmetries in the framed setting by gauging groups and 2-groups, and explore some motivating examples. Finally, we will explore the most general way to gauge

[1] We will not discuss this perspective in full detail in this chapter, we will return to in in chapter 5.

[2] In truth we would like to perform this exposition for unitary symmetries, but as we will note again in a moment, this is presently still an area of ongoing research.

a symmetry in $2 + 1$ dimensions, in an attempt to construct all oriented fusion 2-categorical symmetries via a generalised notion of gauging finite subgroups.

4.1 Finite Global Symmetries

We restrict our focus to oriented theories whose local excitations are purely bosonic, and whose partition function is defined on oriented 3-manifolds. Finite symmetries of unitary oriented $(2+1)$-dimensional quantum field theories are expected to be described by unitary fusion 2-categories.

However, while we certainly expect that unitary fusion 2-categories can be defined consistently, and also that we can do so in a manner conducive to categorifying the arguments we had in chapter 3, a precise description of their structure is still very much a topic of active research in the field. Recent developments point towards constructions of $O^\dagger$ unitary structures for oriented fusion 2-categories, and more generally for fusion n-categories [25], which generalises the notion of unitary dual functors set out in [24], and the notion of unitary fusion category we exposited in subsection 1.2.1.

Dropping unitarity and working in an oriented setting corresponds to regarding the symmetry as a spherical fusion 2-category, which by comparison are very well understood [54]. Further dropping orientation and working in a framed setting corresponds to regarding the symmetry category as an ordinary fusion 2-category.

4.1.1 Invertible Symmetries

Group Symmetries

We start with the simpler case of a 0-form invertible symmetry in $2+1$ dimensions with no non-trivial topological line defects. We expect the various unitary fusion 2-categorical structures compatible with the group multiplication to be classified up to equivalence by group cohomology classes

$$[\alpha] \in H^4_{grp}(G, U(1)) \,. \tag{4.1.1}$$

This class is an invariant of the renormalisation group flow and corresponds to a 't Hooft anomaly. As usual, we take the perspective that specifying a theory includes specifying a representative 4-cocycle α, and that shifting the representative by a 4-coboundary corresponds to adding local counter-terms [35].

Working in the framed/oriented setting, the symmetry of a theory $\mathcal{T}$ with symmetry (G, α) is described by the (spherical) fusion 2-category

$$\mathcal{C}_{\mathcal{T}} = 2\mathsf{Vec}_G^\alpha \,, \tag{4.1.2}$$

of G-graded 2-vector spaces [3], whose structure is summarised as follows:

- Simple objects are denoted by 1_g for $g \in G$, and fuse according to $1_g \otimes 1_h = 1_{gh}$ as illustrated in figure 4.1. They correspond to the two-dimensional topological surfaces generating the finite group symmetry G.
- The 1-morphisms form categories $\mathrm{Hom}(1_g, 1_h) = \delta_{g,h}\mathsf{Vec}$; there are no non-trivial topological lines in the theory, nor any topological interfaces between inequivalent surface defects.
- The pentagonator for the fusion of four simple objects 1_g, 1_h, 1_k, 1_l is twisted by the phase $\alpha(g, h, k, l)$ as illustrated in figure 4.2. This phases satisfies a consistency condition described abstractly by an associahedron diagram and here concretely by the cocycle condition

$$\delta\alpha(g, h, k, l, m) = \frac{\alpha(h, k, l, m)\alpha(g, hk, l, m)\alpha(g, h, k, lm)}{\alpha(gh, k, l, m)\alpha(g, h, kl, m)\alpha(g, h, k, l)} = 1 \,. \tag{4.1.3}$$

[3] It is then more appropriate to consider $\alpha \in Z^4_{grp}(G, \mathbb{C}^\times)$. However, $H^4_{grp}(G, \mathbb{C}^\times) \simeq H^4_{grp}(G, U(1))$ for finite G as it is always possible to choose a representative α to be a phase.

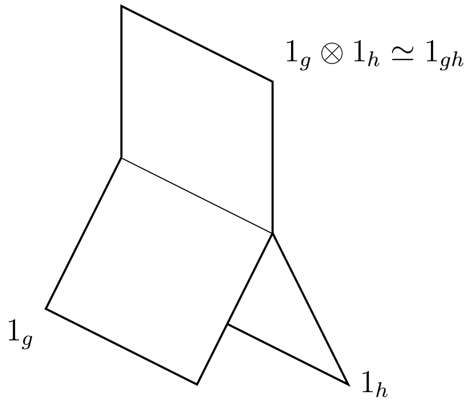

Figure 4.1

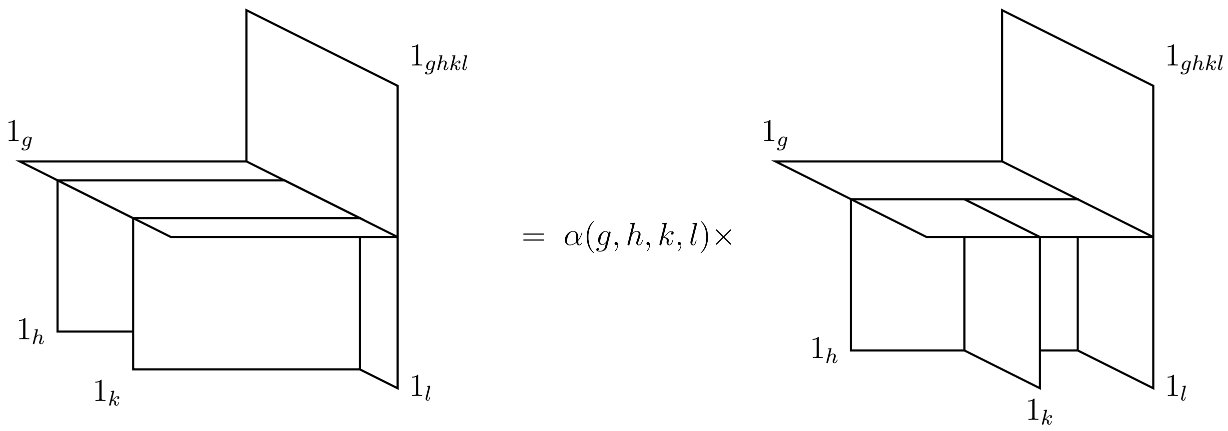

Figure 4.2

We define 2-vector spaces in the same way as we did in subsection 3.3.2, and in the same way, we can consider G-graded 2-vector spaces as finite semi-simple G-graded categories with endomorphisms given by homogeneous functors.

A new interpretation we will now consider is that a 2-vector space equivalent to $\mathsf{Vec}^{\oplus n}$ can be thought of as a set of n elements with automorphisms described by permutations. In the setting of G-graded 2-vector spaces, this identifies objects with G-graded sets with automorphisms described by homogeneous permutations.

2-Group Symmetries

More generally, an invertible symmetry in $2 + 1$ dimensions may be described by a 2-group $\mathcal{G}$. These symmetries represent a small generalisation of group symmetries in that they have non-trivial categorical structure while still being invertible.

The data of a finite 2-group $\mathcal{G} = (G, A, \rho, e)$ includes:

- A finite (0-form) group symmetry G.
- A finite abelian (1-form) group symmetry A.
- An action $\rho : G \to \mathrm{Aut}(A)$.
- A Postnikov extension class classified by group cohomology $[e] \in H^3_{grp,\rho}(G, A)$, where A is thought of as a G-module such that

  $$\delta e(g,h,k,l) := \rho_{g^{-1}} e(h,k,l) - e(gh,k,l) + e(g,hk,l) - e(g,h,kl) + e(g,h,k) = 0\,, \tag{4.1.4}$$

  for all $g, h, k, l \in G$.

We construct the classifying space $B\mathcal{G}$ of $\mathcal{G}$ via a Serre fibration, or Postnikov system

$$1 \to B^2A \to B\mathcal{G} \to BG \to 1\,, \tag{4.1.5}$$

with characteristic map $BG \to B^3A$ determined up to homotopy by $[e]$. The possible 't Hooft anomalies are classified by

$$[\alpha] \in H^4_{grp}(\mathcal{G}, U(1)) \simeq H^4(B\mathcal{G}, U(1))\,, \tag{4.1.6}$$

where we are extending the definition of group cohomology to 2-group cohomology via the singular cohomology over the fibration $B\mathcal{G}$ [74].

We can construct these classifying cohomology groups explicitly using a Serre spectral sequence

$$E_2^{p,q} = H^p_{grp,\rho}(G, H^q(B^2A, U(1))) \Rightarrow H^{p+q}_{grp}(\mathcal{G}, U(1))\,, \tag{4.1.7}$$

for $p+q = 4$. If the 2-group extension $e = 0$ vanishes, the sequence (4.1.5) splits and we expect the spectral sequence above to collapse at the second page. In that case $H^4_{grp}(\mathcal{G}, U(1))$ decomposes into components. The spectral components $E_2^{3,1}$ and $E_2^{1,3}$ vanish automatically, but those remaining admit various physical interpretations:

1. The component
$$E_2^{4,0}(\alpha) \in Z^4_{grp}(G, U(1)) \tag{4.1.8}$$
captures the 't Hooft anomaly of the 0-form symmetry G as we had in the previous subsection.

2. The component
$$E_2^{2,2}(\alpha) \in Z^2_{grp,\rho}(G, H^2(B^2A, U(1))) \simeq Z^2_{grp,\rho}(G, \widehat{A}) \tag{4.1.9}$$
captures the mixed 't Hooft anomaly between G and $A[1]$. This corresponds to the four-dimensional SPT phase
$$\int_{M_4} \mathsf{a} \cup \mathsf{k}^* E_2^{2,2}(\alpha) \tag{4.1.10}$$
on a 4-manifold M_4 supporting background fields $\mathsf{a} \in H^2(M_4, A)$ and $\mathsf{k} : M_4 \to BG$.

3. The component
$$E_2^{0,4}(\alpha) \in Z^4(B^2A, U(1))^G \,, \tag{4.1.11}$$
captures the 1-form 't Hooft anomaly of A. The cohomology group in this case admits a dual description
$$H^4(B^2A, U(1)) \simeq \mathrm{Hom}(\Gamma A, U(1)) \,, \tag{4.1.12}$$
where ΓA is the universal quadratic group of A. As we remarked in subsection 1.3.2, this classifies the space of quadratic forms on A, and hence identifies the 1-form 't Hooft anomaly with a choice braiding for the abelian anyons labelled by A, equipping the endomorphism category Vec_A with the structure of a braided fusion category [74, 103].

In principle the more general case with non-vanishing Postnikov extension class $[e] \in H^3_{grp,\rho}(G, A)$ will induce obstructions to this decomposition that come in the form of higher derivatives in the spectral sequence. This merely restricts for us the space of possible 't Hooft anomalies for a particular choice of 2-group, but the essential physical interpretations remain the same.

The symmetry of a theory $\mathcal{T}$ with symmetry $(\mathcal{G}, \alpha) = (G, A, \rho, e, \alpha)$ is described by the fusion 2-category

$$\mathcal{C}_{\mathcal{T}} = 2\mathsf{Vec}_{\mathcal{G}}^{\alpha}\,, \tag{4.1.13}$$

whose structure is summarised as follows:

- Simple objects are denoted by 1_g for $g \in G$, and fuse according to $1_g \otimes 1_h = 1_{gh}$. They correspond to the two-dimensional topological surfaces generating the finite 0-form group symmetry G.
- The 1-morphisms form braided fusion categories $\mathrm{Hom}(1_g, 1_h) = \delta_{g,h}\mathsf{Vec}_A$, with simple objects $\mathbb{C}_a$ labelled by elements $a \in A$ and half-braidings determined by $E_2^{0,4}(\alpha)$, as we set out in subsection 1.3.2.
- The associator for the fusion of three simple objects 1_g, 1_h, 1_k is determined by the Postnikov extension $\mathbb{C}_{e(g,h,k)}$.
- The linking of a line $\mathbb{C}_a$ with the junction $1_g \otimes 1_h \to 1_{gh}$ is determined by $E_2^{2,2}(g,h)(\alpha)$.
- The pentagonator for the fusion of four simple objects 1_g, 1_h, 1_k, 1_l is twisted by the phase $E_2^{4,0}(\alpha)(g, h, k, l)$.

4.2 Symmetries From Gauging

We can begin to construct more general symmetries in $2+1$ dimensions by starting with a theory $\mathcal{T}$ with an invertible 0-form (G, α) symmetry, and gauging a non-anomalous sub-symmetry. To see how this produces novel examples of non-invertible symmetries, we will opt to work in the framed setting of fusion 2-categorical symmetries

$$\mathcal{C}_{\mathcal{T}} = 2\mathsf{Vec}_G^{\alpha}\,, \tag{4.2.1}$$

leaving any details concerning spherical or unitary fusion 2-categories for later sections.

Unlike in chapter 3 however, there is much greater freedom in how we choose to gauge a finite symmetry. The reasoning for this can be traced back to a single important difference: in subsection 3.3.3 we noted that in $1+1$ dimensions, stating the Morita equivalence of two fusion categories $\mathsf{C}_1 \sim \mathsf{C}_2$ is equivalent to instead stating their Drinfeld centres coincide

$$\mathcal{Z}(\mathsf{C}_1) \simeq \mathcal{Z}(\mathsf{C}_2) \tag{4.2.2}$$

as braided fusion categories. The analogous statement fails in $2+1$ dimensions for fusion 2-categories, where instead the former implies the latter

$$\mathsf{C}_1 \sim \mathsf{C}_2 \implies \mathcal{Z}(\mathsf{C}_1) \simeq \mathcal{Z}(\mathsf{C}_2)\,, \tag{4.2.3}$$

but the reverse statement is false. The space of counter-examples is spanned by fusion 2-categories built from non-degenerate braided fusion categories. Physically these are oriented $(2+1)$-dimensional TQFTs containing topological lines that braid non-trivially, and which in particular do not admit a gapped boundary.

The degrees of freedom in gauging a finite symmetry in $2+1$ dimensions then correspond precisely to TQFTs of this type that we may couple the theory to prior to gauging. Group-theoretic symmetries for example are obtained by taking a QFT with an invertible (G, α) symmetry, coupling to a (2+1)-dimensional TQFT with a G

symmetry that cancels α on a subgroup, and gauging the subgroup. More concretely, we define a fusion 2-category C to be group-theoretic if we have an equivalence

$$\mathcal{Z}(\mathsf{C}) \simeq \mathcal{Z}(2\mathsf{Vec}_G^{\alpha}) \tag{4.2.4}$$

as braided fusion 2-categories, for some (G, α). At face value this extra variety in gauging procedures might appear to imply that the classification problem is more complicated. However, unlike in $1+1$ dimensions where it was not the case that all symmetries were group-theoretic, in $2+1$ dimensions we are afforded a remarkable simplification in that we expect the symmetry of an oriented quantum field theory to *always* be group-theoretic.

To be more precise, while there is a rich variety of oriented TQFTs in 2+1 dimensions, in $3+1$ dimensions there are comparatively fewer oriented TQFTs, which under reasonable assumptions can all be described up to equivalence by a (3+1)-dimensional Dijkgraaf-Witten model [104, 105]. From the perspective of an oriented QFT in $2+1$ dimensions, this means the associated (oriented) symmetry TFT in $3+1$ dimensions is equivalent to a Dijkgraaf-Witten theory for some choice of finite group and topological action, and that we must have an equivalence like that seen in (4.2.4), we hence conjecture:

Conjecture 4.2.1. *Every fusion 2-category is group-theoretic.*

That said, since we are working in $2+1$ dimensions we can in principle also consider gauging a theory with an invertible 2-group symmetry $(\mathcal{G}, \alpha) = (G, A, \rho, e, \alpha)$ and work with the fusion 2-category

$$\mathcal{C}_{\mathcal{T}} = 2\mathsf{Vec}_{\mathcal{G}}^{\alpha}\,. \tag{4.2.5}$$

We might imagine such a choice would result in "2-group-theoretic" fusion 2-categories; but we have just argued that group-theoretic should be sufficient to describe all fusion 2-categories.

From the perspective of 2-group symmetries, we might take this naively to imply that for an oriented QFT we only have access to those anomalies for which the 1-form anomaly $E_2^{2,2}(\alpha) = 0$ vanishes, such that we can "gauge away" the 1-form symmetry, but in fact even when this class does not vanish we can show that the corresponding fusion 2-category is still group-theoretic.

Indeed, starting then from a theory $\mathcal{T}$ with $(\mathcal{G}, \alpha)$ symmetry for which the 1-form anomaly vanishes, we may gauge the 1-form A symmetry to produce a theory $\mathcal{T}/A$ with a pure 0-form group symmetry Γ given by the extension

$$1 \to \widehat{A} \to \Gamma \to G \to 1\,, \tag{4.2.6}$$

of G by the Pontryagin dual $\widehat{A}$. The action ρ induces an action $\widehat{\rho} : G \to \mathrm{Aut}(\widehat{A})$, and the spectral component

$$E_2^{2,2}(\alpha) \in Z^2_{grp,\rho}(G, H^2_s(B^2A, U(1))) \simeq Z^2_{grp,\widehat{\rho}}(G, \widehat{A})\,, \tag{4.2.7}$$

induces the extension class of Γ. The spectral component

$$E_2^{4,0}(\alpha) \in Z^4_{grp}(G, U(1)) \tag{4.2.8}$$

continues to describe a pure anomaly on G, and the original Postnikov class

$$e \in Z^3_{grp,\rho}(G, \widehat{A}) \tag{4.2.9}$$

induces a new mixed anomaly between G and $\widehat{A}$ [35]. From an anomaly-inflow perspective this corresponds to the four-dimensional SPT phase

$$\int_{M_4} \widehat{\mathsf{a}} \cup \mathsf{k}^* e \tag{4.2.10}$$

on a 4-manifold M_4 supporting background fields $\widehat{\mathsf{a}} \in H^1(M_4, \widehat{A})$ and $\mathsf{k} : M_4 \to BG$. In this way, we have returned to the setting of an ordinary 0-form group symmetry and so we see that those 2-group symmetries with vanishing 1-form anomaly are all group-theoretic examples of fusion 2-categories.

More generally however, we can imagine a situation where the 1-form anomaly does not completely vanish, and is instead only non-trivial on a subgroup. We then write the 1-form symmetry A as an extension

$$1 \to C \to A \to B \simeq A/C \to 1\,, \tag{4.2.11}$$

where C is the maximal subgroup for which the 1-form anomaly $E^{0,4}(\alpha)$ trivialises. In that case, interpreted as a quadratic form on $B \simeq A/C$, $E^{0,4}(\alpha)$ defines a non-degenerate braiding on the corresponding fusion category Vec_B. The rest then follows as before, we can gauge C to produce a 0-form group symmetry Γ', now extended by a non-degenerate braided fusion category

$$\mathrm{Mod}(\mathsf{Vec}_B^\alpha) \to \mathcal{C}_{\mathcal{T}/C} \to 2\mathsf{Vec}_{\Gamma'}^\alpha\,, \tag{4.2.12}$$

where we have used a shorthand notation that Vec_B^α is the braided fusion category with braiding coming from $E_2^{0,4}(\alpha)|_B$. This suggests every finite 2-group symmetry can be obtained via the following steps: Start from a theory containing an ordinary 0-form group symmetry, couple to a certain $(2+1)$-dimensional TQFT characterised by a non-degenerate braided fusion category as above, and gauge a subgroup.

This result demonstrates our earlier conjecture 4.2.1 for *all* 2-groups. We will see later on in subsection 4.3.3 in more detail how these types of situations are accounted for within this new framework of generalised gauging.

In this section however, to simplify matters for the time being, we will restrict our attention to those symmetries produced by coupling to invertible TQFTs and gauging a finite subgroup. Starting from a 0-form group symmetry this means coupling only to $(2+1)$-dimensional SPT phases, starting from a 2-group this means assuming the 1-form anomaly vanishes. While all the fusion 2-categories produced this way will be strictly Morita equivalent to $2\mathsf{Vec}_G^\alpha$ for some G, this is in principle a much stronger restriction that rules out also non-trivial Turaev-Viro TQFTs which admit gapped boundaries but are not invertible.

4.2.1 Gauging G With Trivial Anomaly

Now consider gauging the finite symmetry G of a theory $\mathcal{T}$ in which the 't Hooft anomaly α vanishes. In addition to topological Wilson lines labelled by representations of G, the symmetry that results from gauging also contains topological surfaces labelled by simple objects in the fusion 2-category of 2-representations

$$\mathcal{C}_{\mathcal{T}/G} = 2\mathrm{Rep}(G)\,. \tag{4.2.13}$$

One perspective is that these surfaces are condensation defects, which appear in condensed matter physics in the context of transitions between topologically ordered phases [106–108]. More recently these condensation have been described in terms of gauging higher-form symmetries [7, 109], a perspective we can see here in steps:

1. First, we take a trivial surface defect and regard the $\mathrm{Rep}(G)$ symmetry generated by Wilson lines supported on the surface as a localised 0-form symmetry. From the perspective of (1+1)-dimensional QFTs this is the symmetry obtained by gauging a 0-form G symmetry.

2. Second, continuing to regard the surface as its own QFT in $1+1$ dimensions, we can "un-gauge" to produce topological lines labelled by Vec_G supported on the surface.

3. Finally, we may gauge the resulting G symmetry localised on the surface as we did in chapter 3.3; the various possible ways to do this produce distinct topological defects labelled by 2-representations $2\mathrm{Rep}(G)$ of G.

This process makes apparent that the Wilson surfaces fill the role of condensation defects built from Wilson lines [4]. More mathematically we can see this fact from the equivalence of fusion categories

$$2\mathrm{Rep}(G) \simeq \mathrm{Mod}(\mathrm{Rep}(G))\,, \tag{4.2.14}$$

[4] In physics it has been common to only identify the trivial SPT phases as condensations, in contrast mathematicians consider all 2-representations to be condensations.

which captures how 2-representations arise from the condensation completion [5] of Rep(G) [94, 109]. Following this process through to its conclusion suggests isomorphism classes of topological surfaces should be labelled by:

- A conjugacy class of subgroup $[H \subseteq G]$.
- A cohomology class $[\psi] \in H^2_{grp}(H, U(1))$.

These objects can be thought of equivalently as surfaces on which the finite gauge group is broken to a subgroup $H \subseteq G$, dressed with a gauge-invariant invertible TQFT corresponding to a discrete torsion, or SPT phase. The appearance of group cohomology in this classification lends another analogy to Wilson lines; Wilson lines valued in 1-dimensional representations of G are labelled by $H^1_{grp}(G, U(1))$.

More generally we can also consider theories $\mathcal{T}/_{\psi}G$ obtained by gauging G with the inclusion of a discrete torsion

$$\psi \in Z^3_{grp}(G, U(1)) \,. \tag{4.2.15}$$

Similarly to subsection 3.2.1, the addition of such a phase acts on the resulting symmetry category by an auto-equivalence. Topological surfaces that sit at the interface between two theories $\mathcal{T}/_{\psi_1}G$ and $\mathcal{T}/_{\psi_2}G$ with different SPT phases can be constructed in a similar manner to above, except now the 0-form G symmetry localised on the surface picks up an anomaly $\psi_2 - \psi_1$. The defects that result from gauging non-anomalous sub-symmetries on the surface are then classified in the same way as section 3.3 by projective 2-representations

$$2\text{Rep}^{\psi_1 - \psi_2}(G) \,. \tag{4.2.16}$$

We will now reproduce this classification by considering topological defects in $\mathcal{C}_{\mathcal{T}} = 2\mathsf{Vec}_G$ that are compatible with the insertion of a network of defects. To compute the symmetry category of $\mathcal{T}/G$ more concretely, we begin by inserting a sufficiently

[5] Also called Karoubi/idempotent/orbifold completion in the literature

fine network of surface defects

$$A = \bigoplus_{g \in G} 1_g \,, \tag{4.2.17}$$

that implements the sum over flat G-bundles. At the junction of surfaces, we specify topological line operators $m : A \otimes A \to A$ that decompose as

$$id_{1_{gh}} : 1_g \otimes 1_h \to 1_{gh} \,, \tag{4.2.18}$$

for each $g, h \in G$, and at the junction of lines we specify topological local operators $a : m(m \otimes id_A) \to m(id_A \otimes m)$ that decompose as

$$id_{id_{1_{ghk}}} : id_{1_{ghk}} \circ (id_{1_g} \otimes id_{1_{hk}}) \to id_{1_{ghk}} \circ (id_{1_{gh}} \otimes id_{1_k}) \,, \tag{4.2.19}$$

for each $g, h, k \in G$. The inclusion of these junctions categorifies the notion of algebra objects studied in subsection 2.3.1, which itself was a categorification of the idempotents studied in chapter 2, and ensures the gauging procedure is insensitive to the choice of network.

In addition to this data we also have a canonical unit map given by the inclusion $1_e \hookrightarrow A$, which together with canoncial unitor data from $2\mathsf{Vec}_G$ naturally endows A with the structure of a finite semi-simple G-graded monoidal category. We can further equip it with a G-graded fusion structure in the framed setting [110], a spherical G-graded fusion structure in the oriented setting [110, 111], and we expect to be able to equip it with a unitary G-graded fusion structure in the unitary setting.

Topological surfaces in $\mathcal{T}/G$ then correspond to surface defects in $\mathcal{T}$ labelled by objects of

$$\mathcal{C}_{\mathcal{T}} = 2\mathsf{Vec}_G \,, \tag{4.2.20}$$

together with compatibility data for how networks of symmetry defects end on them. A surface defect corresponding to a G-graded 2-vector space

$$\mathcal{S}_G = \bigoplus_{g \in G} \mathcal{S}_g \simeq \bigoplus_{g \in G} 1_g^{\oplus n_g} \,, \tag{4.2.21}$$

can be thought of equivalently as a G-graded set with $n_g \in \mathbb{N}$ elements for each

$g \in G$. The compatibility data is then implemented by 1-morphisms in $\mathcal{C}_{\mathcal{T}}$

$$\ell : A \otimes \mathcal{S}_G \to \mathcal{S}_G \qquad r : \mathcal{S}_G \otimes A \to \mathcal{S}_G \tag{4.2.22}$$

corresponding to topological lines supported on $\mathcal{S}_G$ that the surfaces A can end on from the left and right, together with further compatibility data that, in analogy to subsection 3.2.1 gives $\mathcal{S}_G$ the structure of a (2-)bimodule over A internal to $\mathcal{C}_{\mathcal{T}} = 2\mathsf{Vec}_G$. In this way we have identified the symmetry category with a category of bimodule objects

$$\mathcal{C}_{\mathcal{T}/G} \simeq \mathsf{C}(G|G) \simeq \mathrm{Bimod}_{\mathcal{C}_{\mathcal{T}}}(A)\,, \tag{4.2.23}$$

categorifying the construction we saw in subsection 3.4.1. We will now compute the properties of this fusion 2-category.

Objects

The objects we are computing are G-graded bimodule categories over the G-graded fusion category corresponding to A. As we saw earlier the topological surfaces they correspond to are also labelled by 2-representations of G, we will now show that this description is correctly reproduced by the bimodule construction.

Over the G-grading, the bimodule 1-morphisms decompose to

$$\ell_{h|g} : 1_h \otimes \mathcal{S}_g \to \mathcal{S}_{hg} \qquad r_{g|h} : \mathcal{S}_g \otimes 1_h \to \mathcal{S}_{gh}\,, \tag{4.2.24}$$

for each pair $g, h \in G$. These actions are illustrated further in figures 4.3.

In section 3.2, we observed that the analogous junctions had to satisfy some compatibility conditions to ensure consistency with topological manipulations of the gauging network. In $2+1$ dimensions we instead have that these conditions are not equalities, but extra compatibility data in the form of 2-isomorphisms in $\mathcal{C}_{\mathcal{T}}$, implemented by invertible topological local operators:

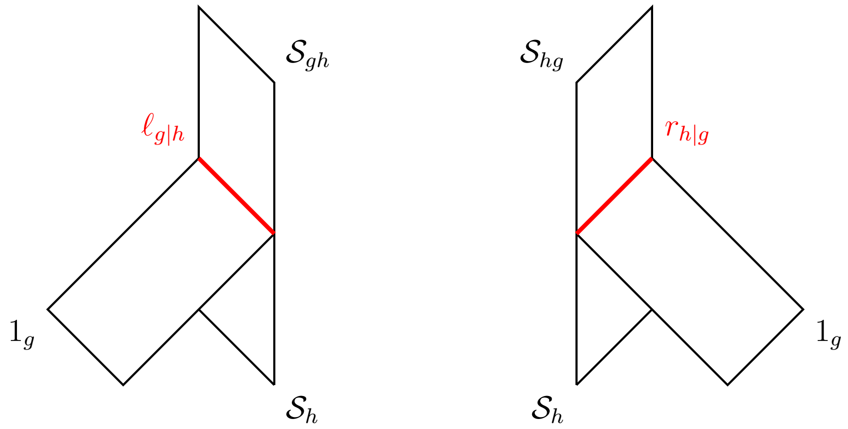

Figure 4.3

- There are normalisation 2-isomorphisms

$$\Psi^\ell_g : \mathcal{S}_g \Rightarrow \ell_{e|g} \qquad \Psi^r_g : \mathcal{S}_g \Rightarrow r_{g|e}\,, \tag{4.2.25}$$

  that capture topological local operators on which the lines $l_{e|g}, r_{g|e}$ that the trivial surface 1_e ends on may end.

- There are left and right multiplication 2-isomorphisms

$$\Psi^\ell_{h,h'|g} : \ell_{h|h'g} \circ (1_h \otimes l_{h'|g}) \Rightarrow \ell_{hh'|g} \qquad \Psi^r_{g|h,h'} : r_{gh|h'} \circ (r_{g|h} \otimes 1_{h'}) \Rightarrow r_{g|hh'}\,, \tag{4.2.26}$$

  that capture the fusion of symmetry defects.

- There are commutator 2-isomorphisms

$$\Psi^{\ell r}_{h|g|h'} : \ell_{h|gh'} \circ (1_h \otimes r_{g|h'}) \Rightarrow r_{hg|h'} \circ (\ell_{h|g} \otimes 1_{h'})\,, \tag{4.2.27}$$

  that capture the commutativity of the left and right actions.

We are using a shorthand notation here that $\mathcal{S}_g$ denotes the identity 1-isomorphism on the same surface. The interpretation of the multiplication 2-isomorphisms is illustrated in figure 4.4. For clarity, we have flattened the surfaces and the attached symmetry defects are omitted: one must imagine symmetry defects attached to $l_{h,g}/r_{g,h}$ pointing out of/into the page.

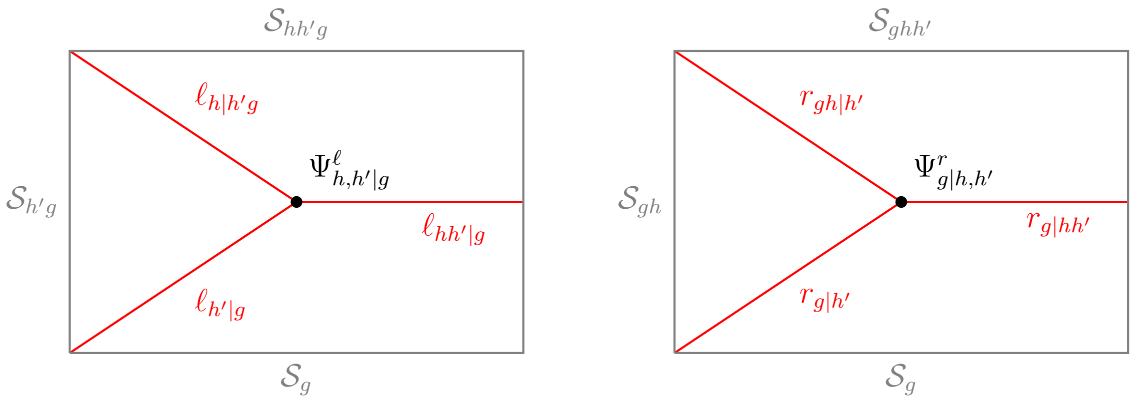

Figure 4.4

The 2-morphisms must themselves satisfy further compatibility conditions. The first set of conditions may be viewed as a normalisation condition for the 2-isomorphisms in equation (4.2.25) and take the form

$$\begin{aligned} \Psi^{\ell}_{h,1|g} &= \ell_{h|g} \circ \Psi^{\ell}_{g} \qquad & \Psi^{\ell}_{1,h|g} &= \Psi^{\ell}_{hg} \circ \ell_{h|g} \, , \\ \Psi^{r}_{g|1,h} &= r_{g|h} \circ \Psi^{r}_{g} \qquad & \Psi^{r}_{g|h,1} &= \Psi^{r}_{gh} \circ r_{g|h} \, . \end{aligned} \tag{4.2.28}$$

The second set of conditions ensure compatibility of the 2-isomorphisms with associativity of the fusion of symmetry defects in equation (4.2.26),

$$\begin{aligned} \Psi^{\ell}_{h_1h_2,h_3|g} \cdot (\Psi^{\ell}_{h_1,h_2|h_3g} \circ (1_{h_1h_2} \otimes \ell_{h_3|g})) &= \Psi^{\ell}_{h_1,h_2h_3|g} \cdot (\ell_{h_1|h_2h_3g} \circ (1_{h_1} \otimes \Psi^{\ell}_{h_2,h_3|g})) \, , \\ \Psi^{r}_{g|h_1,h_2h_3} \cdot (\Psi^{r}_{gh_1|h_2,h_3} \circ (r_{g|h_1} \otimes 1_{h_2h_3})) &= \Psi^{r}_{g|h_1h_2,h_3} \cdot (r_{gh_1h_2|h_3} \circ (\Psi^{r}_{g|h_1,h_2} \otimes 1_{h_3})) \, . \end{aligned} \tag{4.2.29}$$

The final set of conditions ensure compatibility between the fusion and commutativity 2-isomorphisms in equation (4.2.27),

$$\begin{aligned} \Psi^{\ell r}_{h_1h_2|g|h'} \cdot \left[\Psi^{\ell}_{h_1,h_2|gh'} \circ (1_{h_1h_2} \otimes r_{g|h'})\right] &= \left[r_{h_1h_2g|h'} \circ (\Psi^{\ell}_{h_1,h_2|g} \otimes 1_{h'})\right] \\ &\quad \cdot \left[\Psi^{\ell r}_{h_1|h_2g|h'} \circ (1_{h_1} \otimes \ell_{h_2|g} \otimes 1_{h'})\right] \cdot \left[\ell_{h_1|h_2gh'} \circ (1_{h_1} \otimes \Psi^{\ell r}_{h_2|g|h'})\right] \, , \\ \Psi^{\ell r}_{h'|g|h_1h_2} \cdot \left[(\ell_{h'|g} \otimes 1_{h_1h_2}) \circ \Psi^{r}_{h'g|h_1,h_2}\right] &= \left[\Psi^{r}_{h'g|h_1,h_2} \circ (\ell_{h'|g} \otimes 1_{h_1h_2})\right] \\ &\quad \cdot \left[r_{h'gh_1|h_2} \circ (\Psi^{\ell r}_{h'|g|h_1} \otimes 1_{h_2})\right] \cdot \left[\Psi^{\ell r}_{h'|g|h_2} \circ (1_{h'} \otimes r_{g|h_1} \otimes 1_{h_2})\right] \, . \end{aligned} \tag{4.2.30}$$

We are using a similar shorthand notation to earlier here where $l_{h|g}$, $r_{g|h}$ and 1_g all denote identity 2-isomorphisms on their corresponding topological lines. The interpretation of these conditions for the left and right multiplication 1-morphisms is illustrated in figure 4.5.

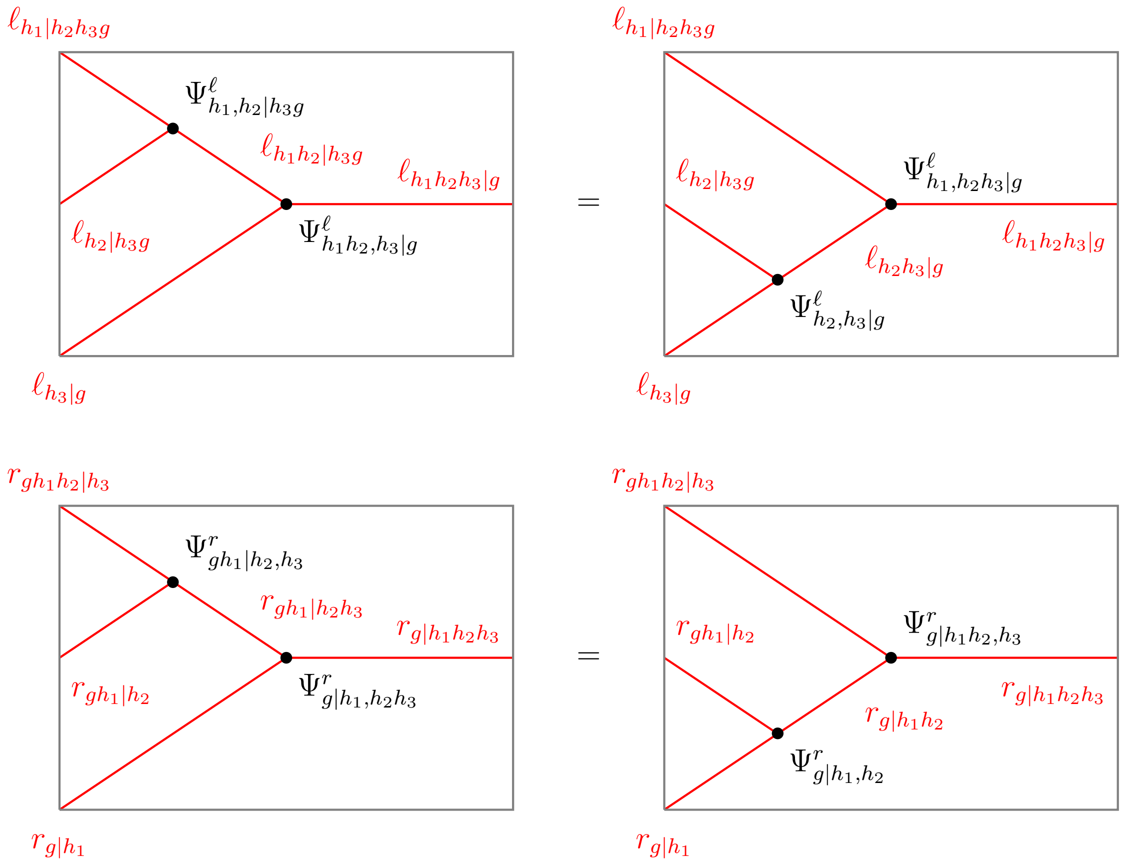

Figure 4.5

The existence of 2-isomorphisms in (4.2.25) and (4.2.26) imply the 1-morphisms $l_{h|g}$, $r_{g|h}$ are weakly invertible with inverting 2-isomorphisms

$$\begin{aligned} \Psi^\ell_{h^{-1},h|g} \circ \Psi^\ell_g \quad &: \quad l_{h^{-1}|hg} \otimes l_{h|g} \Rightarrow \mathcal{S}_g\,, \\ \Psi^r_{h,h^{-1}|g} \circ \Psi^r_g \quad &: \quad r_{gh|h^{-1}} \otimes r_{g|h} \Rightarrow \mathcal{S}_g\,. \end{aligned} \tag{4.2.31}$$

These hence identify $\mathcal{S}_g \simeq \mathcal{S}_e =: \mathcal{S}$ as sets of size $|\mathcal{S}| = n_e =: n$ for all $g \in G$. We can hence restrict our focus to $\mathcal{S}$ with compatibility 1-morphisms $l_{g|e}$, $r_{e|g}$ and the remaining component 1-morphisms may be constructed using combinations of the 2-isomorphisms in equations (4.2.25) (4.2.26) and (4.2.27).

We formulate the remaining data on $l_{g,e}$, $r_{e,g}$ using the combination

$$\rho_g := r_{g|g^{-1}} \circ (\ell_{g|e} \otimes 1_{g^{-1}}) \in \mathrm{Hom}(\mathcal{S}, \mathcal{S})\,. \tag{4.2.32}$$

This captures the topological line sitting at the intersection of the trivially-graded component $\mathcal{S}$ and a symmetry defect 1_g, as is illustrated in figure 4.6.

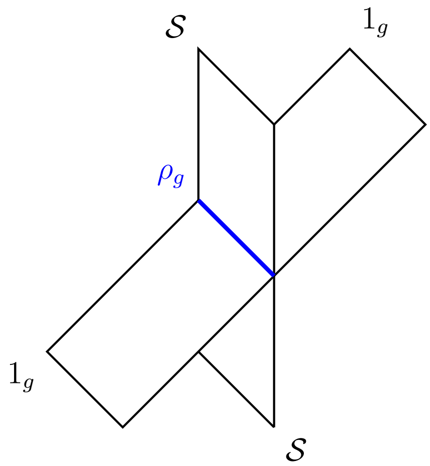

Figure 4.6

The remaining 2-isomorphisms may be organised into combinations of the form

$$\Psi_e : \mathcal{S} \Rightarrow \rho_e \qquad\qquad \Psi_{g,h} : \rho_{gh} \Rightarrow \rho_g \circ \rho_h\,, \tag{4.2.33}$$

subject to the conditions

$$\begin{gathered}\Psi_{e,g} = \Psi_e \circ \rho_g \qquad \Psi_{g,e} = \rho_g \circ \Psi_e \\ \Psi_{h_1h_2,h_3} \cdot (\Psi_{h_1,h_2} \circ \rho_{h_3}) \;=\; \Psi_{h_1,h_2h_3} \cdot (\rho_{h_1} \circ \Psi_{h_2,h_3})\,.\end{gathered} \tag{4.2.34}$$

We illustrate the multiplication condition in figure 4.7. These exhaust the remaining 1-morphisms, 2-isomorphisms and the conditions they satisfy.

In summary, a topological surface in $\mathcal{T}/G$ is specified by the following data:

1. A set $\mathcal{S} \simeq \{1, \ldots, n\} \in 2\mathsf{Vec}$ corresponding to a 2-vector space.
2. A collection of 1-morphisms $\rho_g \in \mathrm{Hom}(\mathcal{S}, \mathcal{S})$ that act as permutations.
3. A 2-isomorphism $\Psi_e : 1_{\mathcal{S}} \Rightarrow \rho_e$ subject to the conditions (4.2.34).

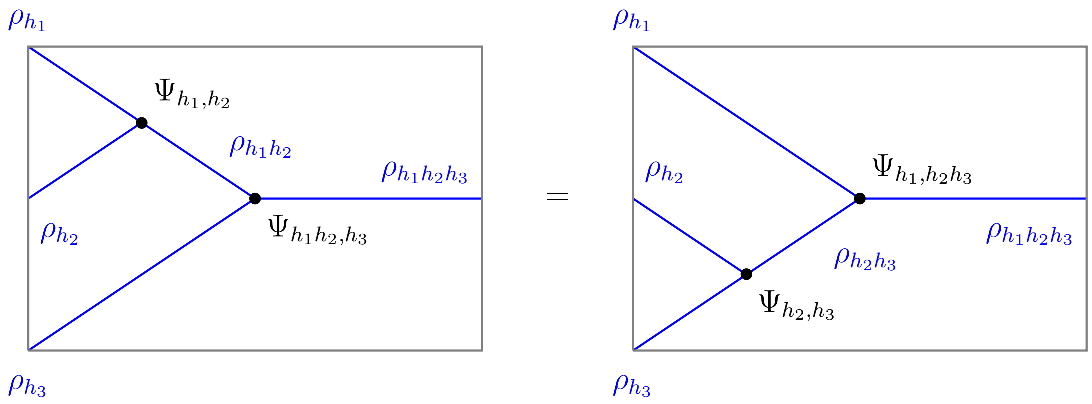

Figure 4.7

4. 2-isomorphisms $\Psi_{g,h} : \rho_{gh} \Rightarrow \rho_g \circ \rho_h$ subject to the conditions (4.2.34).

As expected from our earlier analysis, this labelling of surface defects is precisely the data of a 2-representation of the finite group G in $2\mathsf{Vec}$ [92, 93, 112, 113].

To see this in more detail, let us now summarise the classification of 2-representations following [93]. First, the 2-isomorphisms imply that the 1-morphisms $\rho_g \in \mathrm{Hom}(\mathcal{S}, \mathcal{S})$ are weakly invertible, with inverting 2-isomorphisms

$$\Psi_{g,g^{-1}} \cdot \Psi_e : 1_{\mathcal{S}} \Rightarrow \rho_g \otimes \rho_{g^{-1}} \,. \tag{4.2.35}$$

As a consequence, they endow $\mathcal{S}$ with the structure of a G-set with action

$$\sigma : G \to \mathrm{Aut}(\mathcal{S}) \,. \tag{4.2.36}$$

More concretely, writing $\mathcal{S} = \{1, \ldots, n\}$, the 1-isomorphisms ρ_g should be understood as $n \times n$ permutation 2-matrices whose non-zero entries are 1-dimensional vector spaces up to isomorphism. It is therefore entirely determined by the associated permutation representation $\sigma : G \to S_n$. This is an analogue of topological Wilson lines being labelled by linear representations.

In the same way, the 2-isomorphisms

$$\Psi_{g,h} : \rho_{gh} \Rightarrow \rho_g \circ \rho_h \tag{4.2.37}$$

should be understood as a collection of linear isomorphisms for each entry of in the 2-matrix. Since ρ_{gh} and $\rho_g \circ \rho_h$ are both $n \times n$ permutation 2-matrices, they have only one non-zero entry per row and column, which is a 1-dimensional vector space. The 2-isomorphism $\Psi_{g,h}$ is therefore completely determined by a sequence of n non-zero complex numbers $\{c_j(g,h) \in \mathbb{C}^\times\}$ specifying the isomorphism between the 1-dimensional vector spaces in the j-th row. By varying the group elements g and h, we can think of this sequence as a 2-cochain

$$c : G \times G \to (\mathbb{C}^\times)^n. \tag{4.2.38}$$

Condition (4.2.34) then translates into the 2-cocycle condition

$$c_{\sigma_g^{-1}(j)}(h,k) - c_j(gh,k) + c_j(g,hk) - c_j(g,h) = 0 \tag{4.2.39}$$

for all group elements $g, h, k \in G$ and $i = 1, \dots, n$. Thus, c defines a class

$$[c] \in H^2_{grp,\sigma}(G, (\mathbb{C}^\times)^{\mathcal{S}}) \,, \tag{4.2.40}$$

where $(\mathbb{C}^\times)^{\mathcal{S}}$ is the abelian group $(\mathbb{C}^\times)^{|\mathcal{S}|}$ supplemented with the structure of a G-module via the permutation representation σ.

Simple topological surfaces correspond to simple/irreducible 2-representations. Simple 2-representations are those $(\mathcal{S}, c)$ for which the action $\sigma : G \to \mathrm{Aut}(\mathcal{S})$ is transitive, making $\mathcal{S}$ a G-orbit. In that case we may ivoke the orbit-stabiliser theorem to identify $\mathcal{S}$ with conjugacy class of subgroup $[H \subseteq G]$ via

$$\mathcal{S} \simeq G/H \qquad\qquad \sigma_g : kH \mapsto gkH \,, \tag{4.2.41}$$

for all $g \in G$, $kH \in G/H$. These simple 2-representations can be thought of equivalently as inductions from 1-dimensional 2-representations of a subgroup. The 1-dimensional 2-representations of the subgroup $H \subseteq G$ are labelled up to equivalence by cohomology classes that we can see concretely by the Shapiro isomorphism

$$H^2_{grp,\sigma}(G, (\mathbb{C}^\times)^{G/H}) \simeq H^2_{grp}(H, \mathbb{C}^\times) \,. \tag{4.2.42}$$

In contrast to ordinary irreducible representations, we see here that all simple 2-

representations arise from induction in this way.

In summary, topological surfaces in $\mathcal{T}/G$ are 2-representations of the finite group G, which can be labelled by pairs $(\mathcal{S}, c)$ consisting of

- a G-set $\mathcal{S}$.
- a cohomology class $c \in H^2_{grp,\sigma}(G, (\mathbb{C}^\times)^{\mathcal{S}})$.

The dimension of a 2-representation $(\mathcal{S}, c)$ is $|\mathcal{S}| = n$.

Simple topological surfaces in $\mathcal{T}/G$ are simple 2-representations, labelled by

- a conjugacy class of subgroup $[H \subseteq G]$.
- a cohomology class $\psi \in H^2_{grp}(H, U(1))$.

This construction thus reproduces the classification of 2-representations described in chapter 3.

1-Morphisms

Topological lines sitting at the junctions between topological surfaces are described by 1-morphisms between 2-representations. The 1-morphisms between two topological surfaces in $\mathcal{T}/G$ may be constructed from 1-morphisms between parent topological surfaces together with instructions on how they interact with networks of symmetry defects in $\mathcal{T}$.

Let us first consider a 1-morphism category to a trivial 2-representation

$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}/G}}(1, (\mathcal{S}, c))\,, \tag{4.2.43}$$

which describes topological lines bounding or screening a topological surface $(\mathcal{S}, c)$. The starting point is then the 1-morphisms of the parent topological surface in $\mathcal{T}$

$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(1_e, \mathcal{S})\,, \tag{4.2.44}$$

These are $\mathcal{S}$-graded vector spaces or equivalently collections of vector spaces $\{W_j\}$ indexed by $j \in \{1, \ldots, n\}$.

The component 1-morphisms must satisfy compatibility conditions involving the topological lines $\rho_g \in \mathrm{Hom}(\mathcal{S}, \mathcal{S})$ arising from the intersection with symmetry defects. In particular, these topological lines may end at the boundary on topological local operators corresponding to 2-isomorphisms in $\mathcal{T}$,

$$\Phi_g : \quad \mathrm{Hom}_{\mathcal{C}_\mathcal{T}}(1_e, \mathcal{S}) \;\Rightarrow\; \mathrm{Hom}_{\mathcal{C}_\mathcal{T}}(1_e, \mathcal{S})\,. \tag{4.2.45}$$

Concretely, such a 2-isomorphism is a collection of linear maps $\Phi_{g,j} : W_j \to W_{\sigma_g(j)}$ for all $j = 1, \ldots, n$. This is illustrated in figure 4.8.

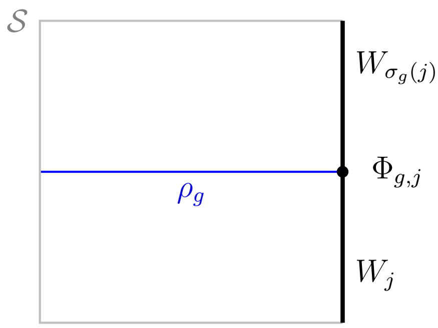

Figure 4.8

The compatibility with the fusion of symmetry defects intersecting the parent topological surface in $\mathcal{T}$ requires that the 2-morphisms compose as

$$\Phi_{gh,j} \;=\; c_j(g,h) \cdot \Phi_{g,\sigma_h(j)} \cdot \Phi_{h,j}\,. \tag{4.2.46}$$

The additional phase arises due to the same anomaly inflow mechanism described toward the end of subsection 3.2.1. This condition is illustrated further in figure 4.9. In adddition to this we also require a normalisation

$$\Phi_{e,j} \;=\; id_{W_j}\,, \tag{4.2.47}$$

making each $\Phi_{g,j}$ invertible.

To summarise, an object in the 1-morphism category $\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}/G}}(1, (\mathcal{S}, c))$ is determined by the following data:

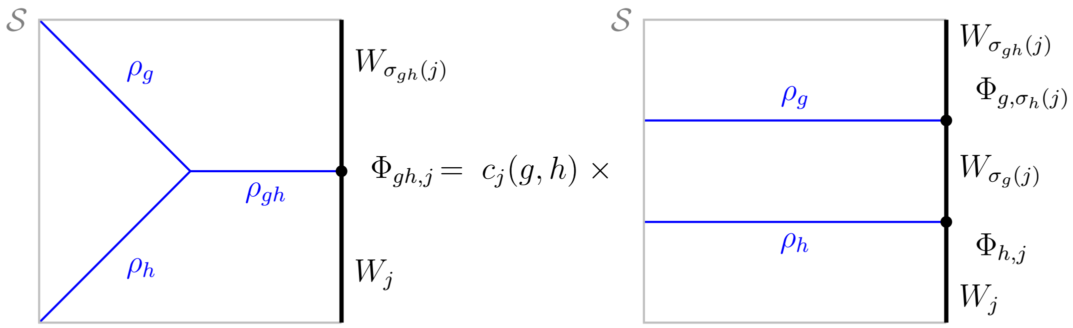

Figure 4.9

- A collection of vector spaces $\{W_1, \ldots, W_n\}$ indexed by $\mathcal{S} \cong \{1, \ldots, n\}$,
- a collection of linear maps $\Phi_{g,j} : W_j \to W_{\sigma_g(j)}$ satisfying

$$\Phi_{gh,j} \;=\; c_j(g,h) \cdot (\Phi_{g,j} \cdot \Phi_{h,j})$$

and

$$\Phi_{e,j} = id_{W_j} \,.$$

We call this an $\mathcal{S}$-graded projective representation of G.

For a simple representation with representatives (H, ψ), the vector spaces $W_j \simeq \mathbb{C}_j$ are 1-dimensional for all $j \in G/H$. Further to this, since the action σ is now transitive on $\mathcal{S} \simeq G/H$, they are all identified. The remaining information is then exhausted by

$$\Psi_{h,i} : \mathbb{C}_i \to \mathbb{C}_i \,, \tag{4.2.48}$$

for $h \in H$ and i corresponding to the trivial coset H, and the fusion condition reduces to

$$\Psi_{gh,i} = \psi(g,h) \cdot (\Psi_{g,i} \cdot \Psi_{h,i}) \,, \tag{4.2.49}$$

for all $g, h \in H$. This data specifies a projective representation of H with projective 2-cocycle c. Physically we interpret this as a badly quantised Wilson line on which the Wilson surface (H, ψ) supporting the H gauge symmetry can end.

In summary, we have found that

$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}/G}}(1, (\mathcal{S}, c)) \simeq \mathrm{Rep}^{(\mathcal{S},c)}(G) \tag{4.2.50}$$

is the category of $\mathcal{S}$-graded projective representations of G with cocycle c, and that for simple 2-representations

$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}/G}}(1,(H,\psi)) \simeq \mathrm{Rep}^{\psi}(H)\,. \tag{4.2.51}$$

We can now generalise these results to 1-morphisms between arbitrary pairs of topological surfaces, with the result

$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}/G}}((\mathcal{S},c),(\mathcal{S}',c')) \simeq \mathrm{Rep}^{(\mathcal{S}\otimes\mathcal{S}',\,c'-c)}(G)\,, \tag{4.2.52}$$

or for simple surfaces

$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}/G}}((H,\psi),(K,\phi')) \simeq \bigoplus_{[g]\in H\backslash G/K} \mathrm{Rep}^{\phi^g-\psi}(H\cap {}^gK)\,, \tag{4.2.53}$$

where $\phi^g(k_1,k_2) = \phi(k_1^g,k_2^g)$ and ${}^gK = gKg^{-1}$. These may be computed directly by generalising the line of reasoning above, or alternatively using a folding trick to equate the result with 1-morphisms from the trivial topological surface to the tensor product of 2-representations $(\mathcal{S},c)^*\otimes(\mathcal{S}',c')$.

Fusion

Thus far we have demonstrated the structure of 2Rep(G) as a 2-category, now we will show that it inherits the structure of a fusion 2-category from $\mathcal{C}_{\mathcal{T}}$. The fusion and sum of topological surfaces in $\mathcal{T}/G$ are inherited from those of parent topological surfaces in $\mathcal{T}$ and correspond to the direct sum and tensor product in the symmetry category $2\mathsf{Vec}_G$. They correspond to natural ways in which to combine the data labelling 2-representations fo G and are described in generality below.

First, given two G-sets $\mathcal{S}$ and $\mathcal{S}'$, we define their direct sum and tensor product via disjoint union and Cartesian product respectively, i.e.

$$\begin{aligned} \mathcal{S}\oplus\mathcal{S}' &= \mathcal{S}\sqcup\mathcal{S}' \\ \mathcal{S}\otimes\mathcal{S}' &= \mathcal{S}\times\mathcal{S}' \end{aligned} \tag{4.2.54}$$

with the appropriate induced G-actions. More concretely, let us write $\mathcal{S} = \{1, \ldots, n\}$ and $\mathcal{S}' = \{1, \ldots, n'\}$ with permutations $\sigma, \sigma' : G \to S_n, S_{n'}$. Then

$$\begin{aligned}(\sigma \oplus \sigma')_g(j) &= \begin{cases} \sigma_g(j) & j \in \mathcal{S} \\ \sigma'_g(n-j)+n & j-n \in \mathcal{S}' \end{cases} \\ (\sigma \otimes \sigma')_g(j) &= (\sigma_g(i), \sigma'_g(i')) \qquad j = (i,i') \in \mathcal{S} \times \mathcal{S}'\,. \end{aligned} \tag{4.2.55}$$

provide explicit permutation actions on $\mathcal{S} \oplus \mathcal{S}'$ and $\mathcal{S} \otimes \mathcal{S}'$.

These definitions provide motivation for our earlier statement that a defect $(\mathcal{S}, c)$ is simple if and only if its G-action is transitive; were this not the case, we would be able to further decompose $(\mathcal{S}, c) = (\mathcal{S}_1, c_1) \oplus (\mathcal{S}_2, c_2)$ into simple objects.

Similarly, given two classes $c \in H^2_{grp}(G, U(1)^{\mathcal{S}})$ and $c' \in H^2_{grp}(G, U(1)^{\mathcal{S}'})$, we define their direct sum and tensor product

$$\begin{aligned} c \oplus c' &\in H^2_{grp}(G, U(1)^{\mathcal{S}\oplus\mathcal{S}'}) \\ c \otimes c' &\in H^2_{grp}(G, U(1)^{\mathcal{S}\otimes\mathcal{S}'}) \end{aligned} \tag{4.2.56}$$

by setting for each $g, h \in G$

$$\begin{aligned}(c \oplus c')_j(g,h) &= \begin{cases} c_j(g,h) & j \in \mathcal{S} \\ c'_{j-n}(g,h) & j-n \in \mathcal{S}' \end{cases} \\ (c \otimes c')_j(g,h) &= c_i(g,h) + c'_{i'}(g,h) \quad j = (i,i') \in \mathcal{S} \times \mathcal{S}'\,. \end{aligned} \tag{4.2.57}$$

It is straightforward to check that these satisfy the appropriate 2-cocycle conditions. Combining these formulae provides a combinatorial definition of the direct sum and fusion of topological surfaces $(\mathcal{S}, c)$ and $(\mathcal{S}', c')$ in $\mathcal{T}/G$.

The direct sum and tensor product give $\mathcal{C}_{\mathcal{T}/G} = 2\text{Rep}(G)$ the structure of a fusion [6] category, with duals given for each object by the conjugate 2-representation $(\mathcal{S}, c)^* := (\mathcal{S}, -c)$ [114].

[6] Specifically fusion and not multi-fusion since the unit $1 = (\{1\}, 1) \simeq (G, 1)$ is simple.

4.2.2 Gauging a Subgroup of (G, α)

A more general thing to do when given an invertible 0-form symmetry (G, α) where the anomaly class might not vanish is to identify a subgroup $H \subseteq G$ whereupon the restriction of the anomaly class vanishes. In this case, we can choose only to gauge the subgroup H, and to do so we must specify a trivialisation

$$\delta\psi = \alpha|_H^{-1} . \tag{4.2.58}$$

Any two solutions ψ, ψ' to this condition differ by a 3-cocycle

$$\delta(\psi - \psi_0) = 0 , \tag{4.2.59}$$

and so we see the space of trivialisations is enumerated up to equivalence by a torsor over $H^3_{grp}(H, U(1))$. This corresponds physically to a choice of discrete torsion for the H gauge symmetry.

Given a choice of ψ, we gauge H by inserting a sufficiently fine network of defects

$$A = \bigoplus_{h \in H} 1_h , \tag{4.2.60}$$

that implements the sum over flat H-bundles. At the junction of surfaces, we specify topological line operators $m : A \otimes A \to A$ that decompose as

$$id_{1_{gh}} : 1_g \otimes 1_h \to 1_{gh} , \tag{4.2.61}$$

for each $g, h \in H$, and at the junction of lines we specify topological local operators $a : m(m \otimes id_A) \to m(id_A \otimes m)$ that decompose as

$$\psi(g,h,k) \cdot id_{id_{1_{ghk}}} : id_{1_{ghk}} \circ (id_{1_g} \otimes id_{1_{hk}}) \to id_{1_{ghk}} \circ (id_{1_{gh}} \otimes id_{1_k}) , \tag{4.2.62}$$

for each $g, h, k \in H$. The inclusion of these junctions categorifies the notion of algebra objects studied in chapter 3, which itself was a categorification of idempotents from chapter 2, and ensures the gauging procedure is insensitive to the choice of network provided

$$\delta\psi(g,h,k,l) = \frac{\psi(h,k,l)\psi(g,hk,l)\psi(g,h,k)}{\psi(gh,k,l)\psi(g,h,kl)} = \alpha(g,h,k,l)^{-1} \tag{4.2.63}$$

for all $g,h,k,l \in H.$, this is just the trivialisation (4.2.58).

As before, in addition to this data we also have a canonical unit map given by the inclusion $1_e \hookrightarrow A$, which together with canoncial unitor data from $2\mathsf{Vec}_G^\alpha$ naturally endows A with the structure of a finite semi-simple α^{-1}-twisted G-graded monoidal category. In analogy to the previous subsection, we expect that we can further equip A with an α^{-1}-twisted G-graded fusion structure in the framed setting, an α^{-1}-twisted spherical G-graded fusion structure in the oriented setting, and an α^{-1}-twisted unitary G-graded fusion structure in the unitary setting.

The result of this gauging is a new theory $\mathcal{T}/_\psi H$ whose symmetry is captured by a group-theoretic fusion 2-category we denote by

$$\mathsf{C}(G,\alpha \,|\, H,\psi)\,. \tag{4.2.64}$$

We will now describe the structure of these fusion 2-categories in greater detail.

Objects

Topological surfaces in $\mathcal{T}/G$ correspond to surface defects in $\mathcal{T}$ labelled by objects of

$$\mathcal{C}_\mathcal{T} = 2\mathsf{Vec}_G^\alpha\,, \tag{4.2.65}$$

together with compatibility data for how networks of symmetry defects end on them. A surface defect

$$\mathcal{S}_G = \bigoplus_{g\in G} \mathcal{S}_g \simeq \bigoplus_{g\in G} 1_g^{\oplus n_g}\,, \tag{4.2.66}$$

can be thought of equivalently as a G-graded set with $n_g \in \mathbb{N}$ elements for each $g \in G$. The compatibility data is then implemented by 1-morphisms in $\mathcal{C}_\mathcal{T}$

$$\ell : A \otimes \mathcal{S}_G \to \mathcal{S}_G \qquad\qquad r : \mathcal{S}_G \otimes A \to \mathcal{S}_G \tag{4.2.67}$$

corresponding to topological lines supported on $\mathcal{S}_G$ that the surfaces A can end on from the left and right, together with further compatibility data that gives $\mathcal{S}_G$ the structure of a (2-)bimodule over A internal to $\mathcal{C}_{\mathcal{T}} = 2\mathsf{Vec}_G^\alpha$. These morphisms decompose to

$$\ell_{h|g} : 1_h \otimes \mathcal{S}_g \to \mathcal{S}_{hg} \qquad\qquad r_{g|h} : \mathcal{S}_g \otimes 1_h \to \mathcal{S}_{gh} \,, \tag{4.2.68}$$

for each pair $h \in H$ and $g \in G$.

The compatibility data for these defects is implemented by 2-morphisms:

- There are normalisation 2-isomorphisms

$$\Psi^\ell_g : \mathcal{S}_g \Rightarrow \ell_{e|g} \qquad\qquad \Psi^r_g : \mathcal{S}_g \Rightarrow r_{g|e} \,, \tag{4.2.69}$$

  that capture topological local operators on which the lines $l_{e|g}, r_{g|e}$ that the trivial surface 1_e ends on may end.

- There are left and right 2-isomorphisms

$$\Psi^\ell_{h,h'|g} : \ell_{hh'|g} \Rightarrow \ell_{h|h'g} \circ (1_h \otimes l_{h'|g}) \qquad\qquad \Psi^r_{g|h,h'} : r_{g|hh'} \Rightarrow r_{gh|h'} \circ (r_{g|h} \otimes 1_{h'}) \,, \tag{4.2.70}$$

  implementing compatibility with fusion of symmetry defects.

- There are 2-isomorphisms

$$\Psi^{\ell r}_{h|g|h'} : \ell_{h|gh'} \circ (1_h \otimes r_{g|h'}) \Rightarrow r_{hg|h'} \circ (\ell_{h|g} \otimes 1_{h'}) \,, \tag{4.2.71}$$

  implementing the commutativity of left and right 1-morphisms.

These 2-morphisms must themselves satisfy further compatibility conditions. The first set of conditions may be viewed as a normalisation condition for the 2-isomorphisms in equation (4.2.69) and take the form

$$\begin{aligned} \Psi^\ell_{h,1|g} &= \ell_{h|g} \circ \Psi^\ell_g \qquad & \Psi^\ell_{1,h|g} &= \Psi^\ell_{hg} \circ \ell_{h|g} \,, \\ \Psi^r_{g|1,h} &= r_{g|h} \circ \Psi^r_g \qquad & \Psi^r_{g|h,1} &= \Psi^r_{gh} \circ r_{g|h} \,. \end{aligned} \tag{4.2.72}$$

The second set of conditions ensure compatibility of the 2-isomorphisms with associativity of the fusion of symmetry defects in equation (4.2.70),

$$\begin{aligned}\psi(h_1,h_2,h_3)\cdot\Psi^{\ell}_{h_1h_2,h_3|g}\cdot(\Psi^{\ell}_{h_1,h_2|h_3g}\circ(1_{h_1h_2}\otimes\ell_{h_3|g})) \;&=\; \alpha(h_1,h_2,h_3,g)\cdot\Psi^{\ell}_{h_1,h_2h_3|g}\\ &\quad\cdot(\ell_{h_1|h_2h_3g}\circ(1_{h_1}\otimes\Psi^{\ell}_{h_2,h_3|g}))\,,\\ \alpha(g,h_1,h_2,h_3)\cdot\Psi^{r}_{g|h_1,h_2h_3}\cdot(\Psi^{r}_{gh_1|h_2,h_3}\circ(r_{g|h_1}\otimes 1_{h_2h_3})) \;&=\; \psi(h_1,h_2,h_3)\cdot\Psi^{r}_{g|h_1h_2,h_3}\\ &\quad\cdot(r_{gh_1h_2|h_3}\circ(\Psi^{r}_{g|h_1,h_2}\otimes 1_{h_3}))\,.\end{aligned} \tag{4.2.73}$$

The final set of conditions ensure compatibility between the fusion and commutativity 2-isomorphisms in equation (4.2.71),

$$\begin{aligned}\Psi^{\ell r}_{h_1h_2|g|h'}\cdot\Big[\Psi^{\ell}_{h_1,h_2|gh'}\circ(1_{h_1h_2}\otimes r_{g|h'})\Big] \;&=\; \alpha(h_1,h_2,g,h')\cdot\Big[r_{h_1h_2g|h'}\circ(\Psi^{\ell}_{h_1,h_2|g}\otimes 1_{h'})\Big]\\ &\quad\cdot\Big[\Psi^{\ell r}_{h_1|h_2g|h'}\circ(1_{h_1}\otimes\ell_{h_2|g}\otimes 1_{h'})\Big]\cdot\Big[\ell_{h_1|h_2gh'}\circ(1_{h_1}\otimes\Psi^{\ell r}_{h_2|g|h'})\Big]\,,\\ \alpha(h',g,h_1,h_2)\cdot\Psi^{\ell r}_{h'|g|h_1h_2}\cdot\Big[(\ell_{h'|g}\otimes 1_{h_1h_2})\circ\Psi^{r}_{h'g|h_1,h_2}\Big] \;&=\; \Big[\Psi^{r}_{h'g|h_1,h_2}\circ(\ell_{h'|g}\otimes 1_{h_1h_2})\Big]\\ &\quad\cdot\Big[r_{h'gh_1|h_2}\circ(\Psi^{\ell r}_{h'|g|h_1}\otimes 1_{h_2})\Big]\cdot\Big[\Psi^{\ell r}_{h'|g|h_2}\circ(1_{h'}\otimes r_{g|h_1}\otimes 1_{h_2})\Big]\,.\end{aligned} \tag{4.2.74}$$

From the form of the left and right morphisms (4.2.68), it is clear that any solution $\mathcal{S}_G$ will decompose as a direct sum of solutions supported on double H-cosets in G. As before the left and right 1-morphisms are invertible, so let us restrict our focus to a solution supported on a single double coset $[g] \in H\backslash G/H$ with representative $g \in G$ and its corresponding g-graded component $\mathcal{S}_g$.

The associated set $\mathcal{S}_g$ carries a projective 2-representation Ψ_g of the subgroup $H_g := H \cap {}^gH \subset H$ that is constructed from the left and right 1- and 2-morphisms as follows. First, we define 1-morphisms

$$\rho^g_h \;:=\; r_{hg|(h^g)^{-1}} \,\circ\, \ell_{h|g} \in \mathrm{Hom}(\mathcal{S}_g,\mathcal{S}_g) \tag{4.2.75}$$

with $h \in H_g$ and $h^g := g^{-1}hg$, which describe how symmetry defects pierce through $\mathcal{S}_g$ as illustrated in figure 4.10.

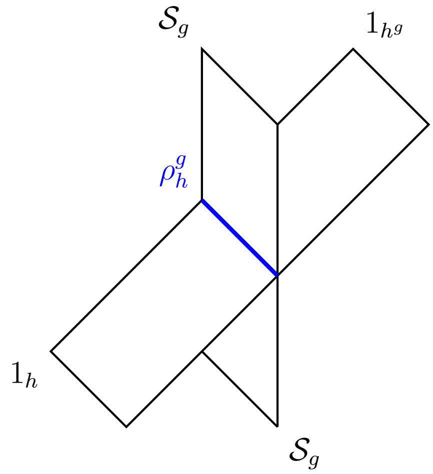

Figure 4.10

The remaining 2-isomorphisms may then be organised into combinations of the form

$$\Psi^g_e : \mathcal{S}_g \Rightarrow \rho^g_e \qquad \Psi^g_{h,h'} \; : \; \rho^g_{hh'} \;\Rightarrow\; \rho^g_h \otimes \rho^g_{h'} \,, \tag{4.2.76}$$

subject to conditions that define a projective 2-representation on H_g on $\mathcal{S}_g$ in the sense that

$$\Psi^g_{e,h} = \Psi^g_e \circ \rho^g_h \qquad \Psi^g_{h,e} = \rho^g_h \circ \Psi^g_e$$

$$\begin{aligned} &(\Psi^g_{h_1,h_2} \otimes \rho^g_{h_3}) \;\circ\; \Psi^g_{h_1h_2,h_3} \\ &\quad = \; c_g(h_1,h_2,h_3) \,\cdot\, \Big[(\rho^g_{h_1} \otimes \Psi^g h_2, h_3) \,\circ\, \Psi^g h_1, h_2h_3 \Big] \,, \end{aligned} \tag{4.2.77}$$

where the 3-cocycle $c_g \in Z^3_{grp}(H_g, U(1))$ depends on the anomaly α and its trivialisation ψ. Upon renormalising $\Psi^g \to \gamma_g \cdot \Psi^g$ by an appropriate 2-cochain $\gamma_g \in C^2_{grp}(H_g, U(1))$, the 3-cocycle c_g can be brought into the canonical form [2]

$$c_g(h_1,h_2,h_3) \;=\; \frac{\psi(h^g_1,h^g_2,h^g_3)}{\psi(h_1,h_2,h_3)} \cdot \frac{\alpha(h_1,h_2,h_3,g)\,\alpha(h_1,g,h^g_2,h^g_3)}{\alpha(h_1,h_2,g,h^g_3)\,\alpha(g,h^g_1,h^g_2,h^g_3)} \,. \tag{4.2.78}$$

The interpretation of the projective 2-representation is illustrated in figure 4.11, where it is shown to represent the compatibility with topological moves of the network of H-defects.

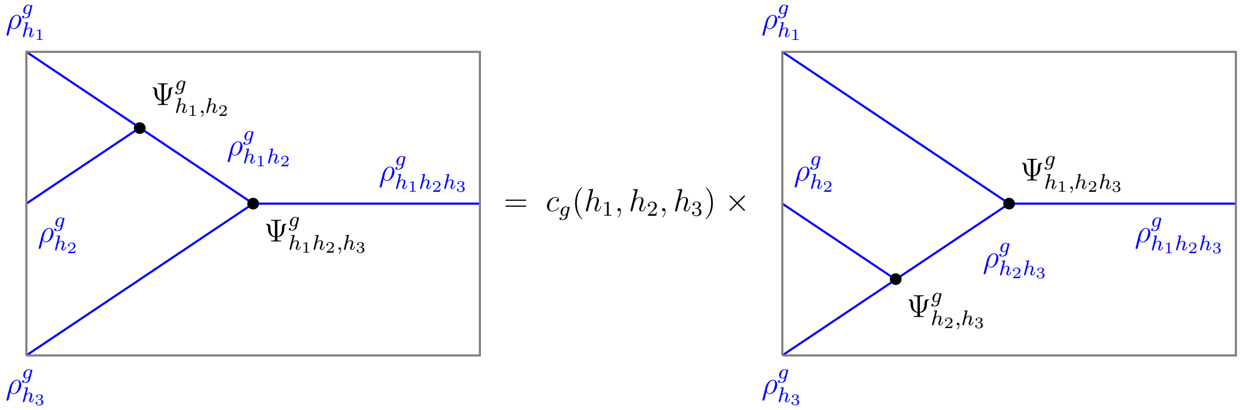

Figure 4.11

We claim that conversely any such projective 2-representation determines a solution to the compatibility constraints for left and right morphisms. The above construction then sets up a bijection between isomorphism classes of simple objects and isomorphism classes of pairs (g, Ψ_g) consisting of

1. A representative $g \in G$ of a double coset $[g] \in H\backslash G/H$.

2. An irreducible projective 2-representation Ψ_g of H_g with 3-cocycle

$$c_g(h_1,h_2,h_3) \;:=\; \frac{\psi(h_1^g,h_2^g,h_3^g)}{\psi(h_1,h_2,h_3)} \cdot \frac{\alpha(h_1,h_2,h_3,g)\,\alpha(h_1,g,h_2^g,h_3^g)}{\alpha(h_1,h_2,g,h_3^g)\,\alpha(g,h_1^g,h_2^g,h_3^g)}\,. \tag{4.2.79}$$

The isomorphism class of a simple object depends on the double coset representative g and the 3-cocycle representative c_g only up to isomorphism.

We can give an alternative description of simple objects using induction of projective 2-representations: In this context, every irreducible projective 2-representation of H_g may be seen as being induced by a 1-dimensional 2-representation of a subgroup of $K \subset H_g$. The latter is completely determined by a choice of 2-cochain $\phi \in C^2_{grp}(K, U(1))$ satisfying $\delta\phi = c_g|_K$, which slightly generalises the considerations in [93, 113].

In summary, simple objects are classified by

1. A representative $g \in G$ of a double coset $[g] \in H\backslash G/H$.

2. A subgroup $K \subset H_g$.

3. A 2-cochain $\phi \in C^2_{grp}(K, U(1))$ satisfying $\delta\phi = c_g|_K$.

The above description of simple topological lines again allows for an alternative physical interpretation: The topological surface labelled by $g \in G$ in $\mathcal{T}$ is invariant under the action of $H_g \subset H$ and therefore supports a H_g symmetry group. However, due to the bulk 't Hooft anomaly and choice of trivialisation, it has an anomaly $c_g^{-1} \in Z^3_{grp}(H_g, U(1))$. To define a consistent topological surface when gauging, the anomaly must be cancelled by dressing with an irreducible 2-dimensional TQFT with H_g symmetry and opposite 't Hooft anomaly. This is a projective 2-representation of the above type.

1-morphisms

The 1-morphisms in the gauged theory $\mathcal{T}/_\psi H$ are obtained from morphisms in the ungauged theory $\mathcal{T}$ together with compatibility conditions for how they intersect with networks of H-defects. Unlike in the previous subsection however, we are no longer afforded the simplification that every defect admits a morphism to the identity object; we need to be more general.

Concretely, given two simple objects (g, Ψ^g) and $(g', \Psi^{g'})$, a 1-morphism between them is obtained from a 1-morphism $\mathcal{V} : \mathcal{S}_g \to \mathcal{S}'_{g'}$ in $\mathsf{2Vec}^\alpha(G)$. Since this must preserve the grading of the 2-vector spaces $\mathcal{S}_g$ and $\mathcal{S}'_{g'}$, such a morphism can only exist when $g = g'$.

In addition, the 1-morphism $\mathcal{V}$ needs to be equipped with 2-isomorphisms

$$\Phi^g_h : \rho'^g_h \circ \mathcal{V} \to \mathcal{V} \circ \rho^g_h \tag{4.2.80}$$

in $\mathsf{2Vec}^\alpha(G)$ that describe the intersection of $\mathcal{V}$ with networks of H-defects as illustrated in figure 4.12.

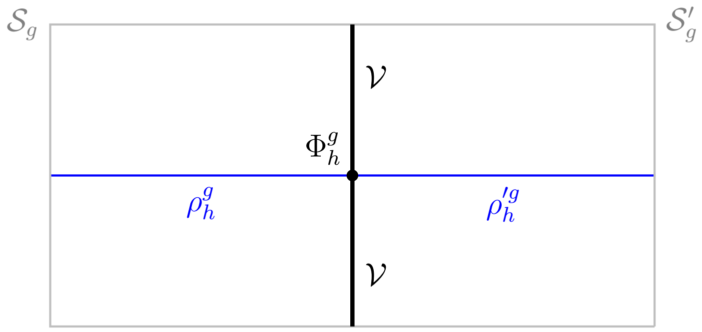

Figure 4.12

These 2-morphisms must be compatible with topological manipulations of H-defects intersecting $\mathcal{S}_g$ and $\mathcal{S}'_g$ in the sense that

$$\Phi^g_{hh'} = \frac{\Psi'^g_{h,h'}}{\Psi^g_{h,h'}} \cdot \left[(\rho'^g_h \otimes \Phi^g_h) \circ (\Phi^g_{h'} \otimes \rho^g_h)\right] \tag{4.2.81}$$

for all $h, h' \in H_g$, which is illustrated in figure 4.13. This allows us to identify 1-morphims in $\mathcal{T}/_\psi H$ with graded projective representations, or equivalently 1-intertwiners between 2-representations [115]. For our purposes, any simple graded projective representation of H_g can be seen as being induced by an ordinary projective representation of a subgroup $K \subset H_g$.

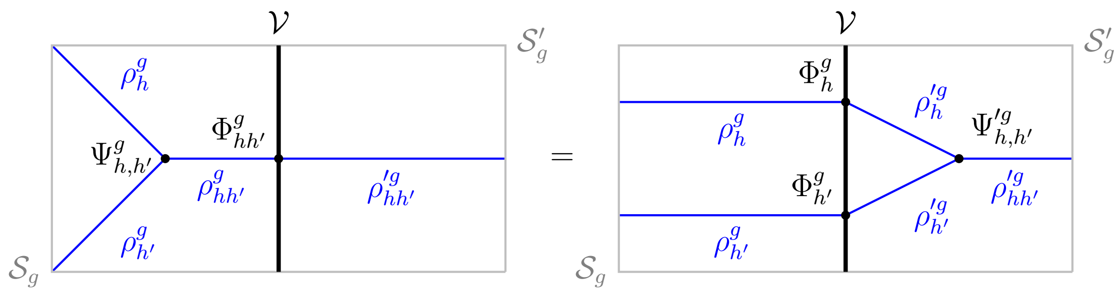

Figure 4.13

In summary, we obtain a decomposition

$$\mathsf{C}(G, \alpha \,|\, H, \psi) \;\cong \bigoplus_{[g] \in H\backslash G/H} \mathsf{2Rep}^{c_g}(H_g)\,. \tag{4.2.82}$$

at the level of 2-categories. A generic object will thus be given by a collection of projective 2-representations of subgroups $H_g \subset H$ with 3-cocycles c_g indexed by representatives of double cosets $[g] \in H\backslash G/H$.

Similarly to the two-dimensional case, taking both H and ψ to be trivial reproduces the expected result

$$\mathsf{C}(G,\alpha\,|\,1) \;=\; \mathsf{2Vec}^{\alpha}(G) \tag{4.2.83}$$

at the level of categories. On the other hand, taking $H = G$ with trivial anomaly gives

$$\mathsf{C}(G\,|\,G,\psi) \;=\; \mathsf{2Rep}(G) \tag{4.2.84}$$

at the level of categories as anticipated from the discussion in subsection 4.2.1.

Fusion

The fusion of objects is determined by the tensor product of 2-bimodules for the 2-algebra object A in $\mathsf{2Vec}_G^{\alpha}$ associated to H and ψ. We will not present the general formula, but restrict ourselves to some salient features.

Consider two simple objects $\mathcal{S}_{G,1}$ and $\mathcal{S}_{G,2}$ supported on double cosets $[g_1]$ and $[g_2]$ respectively. As defects in $\mathcal{C}_{\mathcal{T}} \simeq \mathsf{2Vec}_G^{\alpha}$ they have the form

$$\mathcal{S}_{G,i} \simeq \bigoplus_{g\in H\backslash g_i/H} 1_g^{\oplus \dim(\Psi_i^{g_i})}\,, \tag{4.2.85}$$

and so their fusion should hence be supported on the decomposition of $[g_1]\cdot[g_2]$ into double cosets.

We define the support of a generic object $\mathcal{S}_G$ inside the double coset ring $\mathbb{Z}[H\backslash G/H]$ by

$$\text{Supp}(S) \;:=\; \sum_{[g]\,\in\, H\backslash G/H} \dim(\Psi^g)\cdot[g]\,, \tag{4.2.86}$$

where we regarded S as a collection $\{\Psi^g\}$ of projective 2-representations indexed by double cosets $[g]\in H\backslash G/H$ as above. The fusion of two objects S and S' must then preserves their support in the sense that

$$\text{Supp}(S\otimes S') \;=\; \text{Supp}(S)\,*\,\text{Supp}(S')\,, \tag{4.2.87}$$

where $*$ denotes the ring product on $\mathbb{Z}[H\backslash G/H]$.

In this way, the double coset ring forms the backbone of fusion with respect to the sum decomposition (4.2.82). The remaining fusion structure corresponds to decomposing and combining projective 2-representations.

4.2.3 Gauging a 2-Subgroup of $(\mathcal{G},\alpha)$ With Trivial 1-Form Anomaly

Let us now finally consider a $(2+1)$-dimensional theory $\mathcal{T}$ with a finite 2-group symmetry $\mathcal{G}=(G,A,\rho,e)$, with anomaly $[\alpha]\in H^4_{grp}(\mathcal{G},U(1))$. This is specified by a 0-form symmetry group G, an abelian 1-form symmetry group $A[1]$, a group action $\rho: G\to \mathrm{Aut}(A)$ and a Postnikov class $[e]\in H^3_{grp,\rho}(G,A)$. In our conventions, specifying local counter terms in the background fields amounts to choosing a representative $e\in Z^3_{grp}(G,A)$ of the Postnikov class. If the Postnikov class vanishes, one must choose a trivialisation. In this case, shifts of the trivialisation correspond to a choice of symmetry fractionalisation and form a torsor over $H^2_{grp}(G,A)$.

Our ambition is to gauge an anomaly-free 2-subgroup $\mathcal{H}\subset\mathcal{G}$. This consists of subgroups $L\subset K$ and $B\subset A$ such that the group action $\rho: G\to\mathrm{Aut}(A)$ restricts to a group action $\rho: L\to\mathrm{Aut}(B)$ and $e|_L\in Z^3_{grp}(L,A)$ is valued in B. The condition that $\mathcal{H}$ be anomaly-free requires $\alpha|_{\mathcal{H}}^{-1}=(\delta\psi)$ for some trivialisation $\psi\in C^3_{grp}(\mathcal{H},U(1))$. This gauging will result in a 2-group-theoretical fusion 2-category

$$\mathsf{C}(\mathcal{G},\alpha\,|\,\mathcal{H},\psi)\,. \tag{4.2.88}$$

As we explained earlier, we will restrict our attention here to cases where the 't Hooft anomaly does not obstruct gauging the whole 1-form symmetry $A[1]$. In this case, $A[1]$ may be gauged first to obtain an ordinary 0-form group symmetry Γ given by an extension of G by the Pontryagin dual of A

$$1\;\to\;\widehat{A}\;\to\;\Gamma\;\to\;G\to\;1\,, \tag{4.2.89}$$

with action $\widehat{\rho}: G\to\mathrm{Aut}(\widehat{A})$ and Postnikov class $\widehat{e}=E_2^{2,2}(\alpha)\in Z^2_{grp}(G,\widehat{A})$. In the case that $\widehat{e}$ vanishes, we see that the sequence splits and describes a semi-direct product group

$$\Gamma = \widehat{A} \rtimes_{\widehat{\rho}} G\,. \tag{4.2.90}$$

As we explained earlier, we determine the 't Hooft anomaly of Γ in components;

1. There is a pure G anomaly

$$E_2^{4,0}(\alpha) \in Z^4_{grp}(G, U(1))\,. \tag{4.2.91}$$

2. There is a mixed anomaly

$$e \in Z^3_{grp}(G, \widehat{A})\,. \tag{4.2.92}$$

We can then apply the machinery from previous section to the 0-form Γ symmetry. To illustrate this procedure we now turn to the example of gauging a 2-subgroup $\mathcal{H} \subset \mathcal{G}$ of an anomaly-free 2-group without discrete torsion.

- First, we gauge A without discrete torsion to obtain a theory $\mathcal{T}/A$ with symmetry group $\Gamma = \widehat{A} \rtimes_{\widehat{\rho}} G$. In the presence of a non-trivial Postnikov class, this symmetry has a 't Hooft anomaly $[\gamma] \in H^4_{grp}(\Gamma, U(1))$ with 4-cocycle representative

$$\gamma\Big((\chi_1, k_1), (\chi_2, k_2), (\chi_3, k_3), (\chi_4, k_4)\Big) \;=\; \langle\, \widehat{\rho}_{k_1k_2k_3}(\chi_4)\,,\, e(k_1, k_2, k_3)\,\rangle\;, \tag{4.2.93}$$

  where we have used $\langle \cdot, \cdot \rangle$ to denote the evaluation of $\widehat{A}$ on A. The symmetry category of $\mathcal{T}/A$ is given by

$$\mathsf{C}(\mathcal{G}|A) \;=\; 2\mathsf{Vec}^{\gamma}(\Gamma)\,. \tag{4.2.94}$$

- Next, we note that we can relate the 2-subgroup $\mathcal{H} \subset \mathcal{G}$ in $\mathcal{T}$ to a corresponding ordinary subgroup $H \subset \Gamma$ in $\mathcal{T}/A$ as follows:

  - Given a 2-subgroup $\mathcal{H} = (L, B)$ of $\mathcal{G}$, there is an associated short exact sequence for the 1-form parts

$$1 \;\to\; B \;\xrightarrow{\imath}\; A \;\xrightarrow{\pi}\; C := A/B \;\to\; 1\,, \tag{4.2.95}$$

which can be dualised to obtain a short exact sequence

$$1 \to \widehat{C} \overset{\widehat{\pi}}{\to} \widehat{A} \overset{\widehat{\imath}}{\to} \widehat{B} \to 1 \tag{4.2.96}$$

for the corresponding Pontryagin dual groups. Let now $l \in L$ and $\chi \in \widehat{C}$. Using that by assumption the group action ρ restricts to a group action of L on B, it is then straighforward to check that

$$\langle \widehat{\imath} \circ \widehat{\rho}_l \circ \widehat{\pi}(\chi), b \rangle = \langle \chi, (\pi \circ \imath \circ \rho_{l^{-1}}(b) \rangle \equiv 1 \tag{4.2.97}$$

for all $b \in B$, which implies that $\widehat{\rho}_l \circ \widehat{\pi}(\chi) \in \ker(\widehat{\imath}) = \mathrm{im}(\widehat{\pi})$. Thus, the Pontryagin dual action $\widehat{\rho}$ restricts to an action of L on $\widehat{\pi}(\widehat{C}) \subset \widehat{A}$, which allows us to define a subgroup $H := \widehat{\pi}(\widehat{C}) \rtimes_{\widehat{\rho}} L$ of Γ. Furthermore, since $e|_L$ is valued in B by assumption, we have that

$$\langle \widehat{\pi}(\chi), e(l_1, l_2) \rangle = \langle \chi, (\pi \circ \imath)(e(l_1, l_2)) \rangle \equiv 1 \tag{4.2.98}$$

for all $\chi \in \widehat{C}$ and $l_1, l_2 \in L$, which is equivalent to saying that the anomaly γ from (4.2.93) becomes trivial upon restriction to $H \subseteq \Gamma$.

- Conversely, running through the above arguments backwards shows that any subgroup $H \subseteq \Gamma$ with $\gamma|_H = 1$ uniquely determines a 2-subgroup $\mathcal{H}$ of $\mathcal{G}$.

In summary, there is a 1-1 correspondence between 2-subgroups $\mathcal{H} \subset \mathcal{G}$ and subgroups $H \subset \Gamma$ with $\gamma|_H = 1$ given by

$$\mathcal{H} = (L, B) \quad \leftrightarrow \quad H = \widehat{A/B} \rtimes L\,. \tag{4.2.99}$$

Gauging the 2-subgroup $\mathcal{H} \subseteq \mathcal{G}$ in $\mathcal{T}$ is achieved by gauging the subgroup $H \subseteq \Gamma$ in $\mathcal{T}/A$ using the machinery from the previous section. The symmetry category of $\mathcal{T}/H$ is therefore given by

$$\mathsf{C}(\mathcal{G}\,|\,\mathcal{H}) = \mathsf{C}(\Gamma, \gamma\,|\,H)\,. \tag{4.2.100}$$

We can point out some immediate results of this for some basic choices of 2-subgroup:

- Gauging the trivial 2-subgroup $\mathcal{H} = (1,1) \subset \mathcal{G}$ in $\mathcal{T}$ means not gauging at all, and corresponds to gauging the subgroup $\widehat{A} \subset \Gamma$ in $\mathcal{T}/A$, reproducing the 1-form A symmetry.

- Gauging the 2-subgroup $\mathcal{H} = (1,A)$ in $\mathcal{T}$ means gauging only the 1-form A symmetry, and corresponds to gauging the trivial subgroup $1 \subset \Gamma$ in $\mathcal{T}/A$, which means not gauging at all.

- Gauging the 2-subgroup $\mathcal{H} = (G,1) \subset \mathcal{G}$ in $\mathcal{T}$ means gauging only the 0-form symmetry G, this is possible only when e vanishes and corresponds to gauging the full group Γ in $\mathcal{T}/A$.

- Gauging the full 2-group $\mathcal{H} = \mathcal{G}$ in $\mathcal{T}$ corresponds to gauging the subgroup $G \subset \Gamma$ in $\mathcal{T}/A$, which completes the gauging sequence for gauging all $\mathcal{G}$.

4.2.4 Examples

To conclude this section, let us now take a moment to study some examples of symmetries constructed in the way we have described.

Example 1 : $G = \mathbb{Z}_2$

Let us consider the simplest example $G = \mathbb{Z}_2$. The resulting "pure" $\mathbb{Z}_2$-gauge theory should not be confused with similarly named theories appearing in the literature [116, 117] which can be thought of as "$\mathbb{Z}_2$-gauge theories coupled to 't Hooft defects", and differ by the appearance of an addition $\mathbb{Z}_2$ 1-form symmetry that comes from coupling a singular $\mathbb{Z}_2$ gauge field [35].

The theory $\mathcal{T}$ has symmetry category $2\mathsf{Vec}_{\mathbb{Z}_2}$ with two simple objects $1, s$ with fusion $s \otimes s = 1$ and non-trivial 1-morphism categories $\mathrm{Hom}_{\mathcal{T}}(1,1) = \mathrm{Hom}_{\mathcal{T}}(s,s) = \mathsf{Vec}$.

Upon gauging the symmetry G, the resulting theory $\mathcal{T}/G$ has topological Wilson lines generating the Pontryagin dual $\mathbb{Z}_2$ 1-form symmetry. However, there is also condensation surface defect for the topological Wilson lines and the full symmetry category is the fusion 2-category $2\mathrm{Rep}(\mathbb{Z}_2)$.

The simple objects are irreducible 2-representations. There are only two $\mathbb{Z}_2$-orbits: the trivial orbit with stabiliser $\mathbb{Z}_2$, and the maximal orbit with trivial stabiliser. There are no SPT phases because $H^2_{grp}(\mathbb{Z}_2, U(1)) = 0$. Let us denote the corresponding simple objects by 1, X, respectively. The physical interpretation of these objects is clear: 1 is the identity surface, while X is the condensation defect for the $\mathbb{Z}_2$ 1-form symmetry.

Their fusion is determined by

$$X \otimes X = 2X \,, \tag{4.2.101}$$

which follows from the fact that the cartesian product of two maximal orbits decomposes as a sum of two orbits. The 1-morphism categories are

$$\mathrm{End}_{\mathcal{T}/G}(1) \simeq \mathrm{Rep}(\mathbb{Z}_2) \simeq \mathsf{Vec}_{\mathbb{Z}_2} \tag{4.2.102}$$

$$\mathrm{Hom}_{\mathcal{T}/G}(1, X) \simeq \mathrm{Hom}_{\mathcal{T}/G}(X, 1) = \mathsf{Vec} \tag{4.2.103}$$

$$\mathrm{End}_{\mathcal{T}/G}(X) \simeq \mathsf{Vec}_{\mathbb{Z}_2} \,. \tag{4.2.104}$$

This symmetry category can be interpreted alternatively as the condensation completion of the 1-form symmetry

$$\mathcal{C}_{\mathcal{T}/G} \simeq 2\mathsf{Vec}_{\mathcal{G}} \,, \tag{4.2.105}$$

with the 2-group $\mathcal{G} = (1, \mathbb{Z}_2, 1, 1)$ equivalent to a pure 1-form $\mathbb{Z}_2$ symmetry.

Example 2 : $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

As a slightly more involved example, let us consider $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. The theory $\mathcal{T}$ has symmetry category $2\mathsf{Vec}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ with four simple objects 1, s_+, s_0 and s_-, fusion

$$\begin{gathered} s_+^2 = s_0^2 = s_-^2 = 1 \\ s_\pm \cdot s_0 = s_\mp \\ s_+ \cdot s_- = s_0 \end{gathered} \qquad (4.2.106)$$

and non-trivial 1-morphisms $\mathrm{Hom}(1,1) = \mathrm{Hom}(s_\pm, s_\pm) = \mathrm{Hom}(s_0, s_0) = \mathsf{Vec}$.

Upon gauging the symmetry G, the symmetry category of the resulting theory $\mathcal{T}/G$ is the fusion 2-category $2\mathrm{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2)$. There are now five orbits, corresponding to the five subgroups of $\mathbb{Z}_2\times\mathbb{Z}_2$ acting as stabilizers of the orbits: the group $G = \mathbb{Z}_2\times\mathbb{Z}_2$ itself, three subgroups of order 2, and the trivial subgroup. In particular, the trivial orbit with stabilizer $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ can be supplemented by an SPT phase

$$\alpha \ \in \ H^2_{grp}(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \ \cong \ \mathbb{Z}_2 \,. \qquad (4.2.107)$$

Let us denote the corresponding simple objects by 1^α, X_i and Y respectively (where $i = 1, 2, 3$). Their fusion is determined by

$$1^\alpha \otimes 1^\beta \ = \ 1^{\alpha+\beta}\,, \qquad (4.2.108)$$

$$X_i \otimes X_j \ = \ \begin{cases} 2\,X_i & \text{if } i = j, \\ Y & \text{if } i \neq j, \end{cases} \qquad (4.2.109)$$

$$X_i \otimes Y \ = \ 2Y\,, \qquad (4.2.110)$$

$$Y \otimes Y \ = \ 4Y\,. \qquad (4.2.111)$$

As before, we can interpret this symmetry category alternatively as the condensation completion of the 1-form symmetry

$$\mathcal{C}_{\mathcal{T}/G} \simeq 2\mathsf{Vec}_{\mathcal{G}}\,, \qquad (4.2.112)$$

with the 2-group $\mathcal{G} = (1, \mathbb{Z}_2 \times \mathbb{Z}_2, 1, 1)$ equivalent to a pure 1-form $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry.

Example 3 : $G = D_8$

As a final example, let us consider a theory $\mathcal{T}$ with anomaly free symmetry $G = D_8$ and systematically gauge all possible subgroups $H \subset G$ with discrete torsion.

Our primary example will be $(2+1)$-dimensional Yang-Mills theory with gauge group $PSO(N)$ with N even, whose magnetic and charge conjugation symmetries combine to form D_8. Gauging subgroups of this symmetry will provide a systematic analysis of the fusion 2-category symmetries of various global forms of gauge theories based on the Lie algebra $\mathfrak{so}(N)$, including those with disconnected gauge groups and discrete theta angles [118].

If $N = 4k + 2$, we introduce standard generators r and s and present D_8 as

$$D_8 \;=\; \langle r, s \,|\, r^4 = s^2 = 1,\; srs^{-1} = r^{-1} \rangle \,, \tag{4.2.113}$$

which identifies the symmetry group with the semi-direct product $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$. In this formulation, $\mathbb{Z}_4$ corresponds to the magnetic symmetry $\pi_1(PSO(N))^\vee$ and $\mathbb{Z}_2$ to the charge conjugation symmetry $\mathrm{Out}(PSO(N))$.

If $N = 4k$, we introduce generators $a = rs$ and $b = sr$ and present D_8 as

$$D_8 \;=\; \langle a, b, s \,|\, a^2 = b^2 = s^2 = 1,\; ab = ba,\; sas^{-1} = b \rangle \,, \tag{4.2.114}$$

which identifies the symmetry group with the semi-direct product

$$D_8 \simeq (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \,. \tag{4.2.115}$$

In this formulation, the normal subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ corresponds to the magnetic symmetry given by the fundamental group $\pi_1(PSO(N))^\vee$, while the subgroup $\mathbb{Z}_2$ corresponds to the charge conjugation symmetry given by outer-automorphisms $\mathrm{Out}(PSO(N))$.

Recall that we summarised the subgroup and automorphism structure of D_8 in figure 3.10. We now consider the symmetry categories that result from gauging subgroups with discrete torsion, beginning with subgroups of the smallest order and working upwards.

Order two subgroups

We begin by gauging the order 2 subgroups isomorphic to $H \simeq \mathbb{Z}_2$. In this case, it is possible to gauge with discrete torsion corresponding to the non-trivial class in $H^3_{grp}(\mathbb{Z}_2, U(1)) \simeq \mathbb{Z}_2$, which on a 3-manifold M_3 corresponds physically to adding a local counter-term of the form

$$\int_{M_3} \mathsf{a} \cup \mathsf{a} \cup \mathsf{a}\,, \tag{4.2.116}$$

in terms of the background field $\mathsf{a} \in H^1(M_4, \mathbb{Z}_2)$ for the $\mathbb{Z}_2$ symmetry. There are 5 order two subgroups forming 3 conjugacy classes, two of which are related by an outer automorphism. There are therefore only two substantive cases to consider:

- The center $H = \langle r^2 \rangle \simeq \mathbb{Z}_2$ of D_8 forms a non-split extension

  $$1 \to \mathbb{Z}_2 \to D_8 \to D_4 \to 1 \tag{4.2.117}$$

  with non-trivial extension class $[e] \in H^2_{grp}(D_4, \mathbb{Z}_2)$. Gauging the centre will result in an $SO(N)$ gauge theory. However, the global structure and symmetry category will depend on the choice of discrete torsion. We denote the choice of discrete torsion by $\psi \in \mathbb{Z}_2$ and the resulting global form may be expressed as

  $$SO(N)_\psi \;=\; \frac{SO(N) \times \mathsf{DW}_{\mathbb{Z}_2,\psi}}{\mathbb{Z}_2[1]}\,, \tag{4.2.118}$$

  where the quotient means gauging the diagonal $\mathbb{Z}_2$ 1-form symmetry [118, 119]. Here we have denoted by $\mathsf{DW}_{H,\psi}$ the $(2+1)$-dimensional Dijkgraaf-Witten theory associated to $\psi \in H^3_{grp}(H, U(1))$ as discussed in subsection 1.3.2.

  - In the absence of discrete torsion ($\psi = 0 \bmod 2$), gauging $H \simeq \mathbb{Z}_2$ results in a split 2-group symmetry $\mathbb{Z}_2[1] \times D_4$ with 't Hooft anomaly determined by the extension class $[e]$, which can be represented by the cubic $(3+1)$-

dimensional SPT phase

$$\int_{M_4} \hat{\mathsf{a}} \cup \mathsf{a}^*_{1,2} e \;=\; \frac{1}{2}\int_{M_4} \hat{\mathsf{a}} \cup \mathsf{a}_1 \cup \mathsf{a}_2 \,, \tag{4.2.119}$$

where $\hat{\mathsf{a}} \in H^2(M_4, \mathbb{Z}_2)$ is the background for the $\mathbb{Z}_2[1]$ symmetry, and $\mathsf{a}_1, \mathsf{a}_2 \in H^1(M_4, \mathbb{Z}_2)$ are the backgrounds for the 0-form $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. The corresponding global form is the plain $SO(N)_0$ gauge theory.

- Now consider gauging with non-trivial discrete torsion ($\psi = 1 \bmod 2$). This can be understood via the Lyndon-Hochschild-Serre spectral sequence associated to the short exact sequence of groups (4.2.117). In this instance, the first obstruction vanishes and the second obstruction corresponding to the differential

$$d_3^{0,3} :\; H^3_{grp}(\mathbb{Z}_2, U(1)) \;\to\; H^3_{grp}(D_4, U(1)) \tag{4.2.120}$$

sends the discrete torsion to an additional contribution to the 't Hooft anomaly represented by the SPT phase

$$\frac{1}{2}\int_{M_4} \mathcal{P}(\mathsf{a}_1 \cup \mathsf{a}_2)\,, \tag{4.2.121}$$

where $\mathcal{P} : H^2_{grp}(-, \mathbb{Z}_2) \to H^4_{grp}(-, \mathbb{Z}_4)$ is the Pontryagin square operation. The spectral sequence computation is performed explicitly in [120]. The same computation is performed in [118] using an explicit Chern-Simons theory representation. This corresponds to a distinct global form $SO(N)_1$.

In summary, gauging the centre $H = \langle r^2 \rangle$ with discrete torsion $\psi \in \mathbb{Z}_2$ leads to the global form $SO(N)_\psi$ with symmetry category

$$\mathsf{C}(D_8 \,|\, \langle r^2 \rangle, \psi) \;=\; 2\mathsf{Vec}^{\alpha_\psi}_{\mathbb{Z}_1[1] \times D_4}\,, \tag{4.2.122}$$

where the anomaly α_ψ is represented by the SPT phase

$$\frac{1}{2}\int_X \hat{\mathsf{a}} \cup \mathsf{a}_1 \cup \mathsf{a}_2 \;+\; \frac{\psi}{2}\int_X \mathcal{P}(\mathsf{a}_1 \cup \mathsf{a}_2)\,. \tag{4.2.123}$$

The result of adding discrete torsion is thus to shift 't Hooft anomaly in the resulting symmetry category.

- Now consider the two non-normal subgroups $H = \langle s\rangle, \langle r^2 s\rangle \cong \mathbb{Z}_2$, which are related to each other by conjugation. For concreteness, consider gauging charge conjugation $H = \langle s\rangle$. Gauging this subgroup results in a $PO(N)$ gauge theory. However, the specific global form and symmetry category will depend on the choice of discrete torsion when gauging.

  - First consider the case without discrete torsion. The simple objects can be determined as follows. There are three double cosets [1], $[r]$, $[r^2]$ with stabilisers H, 1, H respectively and double coset ring

$$[r]*[r] \;=\; [1]+[r^2] \qquad [r]*[r^2] \;=\; [r] \qquad [r^2]*[r^2] \;=\; [1]\,. \tag{4.2.124}$$

There are therefore 5 simple objects corresponding to the following pairs of double cosets and irreducible representations

$$\begin{aligned} 1 &= ([1],1)\,, \qquad & X &= ([1],\omega)\,, \\ V &= ([r^2],1)\,, \qquad & X' &= ([r^2],\omega)\,, \end{aligned} \qquad D = ([r],1)\,, \tag{4.2.125}$$

where ω denotes the non-trivial irreducible 2-representation (or condensation defect) of $\mathbb{Z}_2$. The fusion ring takes the following form:

$$\begin{aligned} V\otimes V &= 1 \qquad & D\otimes D &= X\oplus X' \\ V\otimes D &= D \qquad & X\otimes D &= D\oplus D \\ V\otimes X &= X' \qquad & X\otimes X &= 2X\,. \end{aligned} \tag{4.2.126}$$

The symmetry category is identified with

$$\mathsf{C}(D_8\,|\,\langle s\rangle) \;=\; 2\mathrm{Rep}(\mathbb{Z}_4[1]\rtimes\mathbb{Z}_2)\,. \tag{4.2.127}$$

To understand this result, note that one may first gauge the subgroup $\langle r\rangle \simeq \mathbb{Z}_4$ to obtain a dual 2-group symmetry $\mathbb{Z}_4[1]\rtimes\mathbb{Z}_2$. Then, gauging the entire 2-group symmetry reproduces the $PO(N)$ theory and symmetry category $\mathsf{2Rep}(\mathbb{Z}_4[1]\rtimes\mathbb{Z}_2)$. An analogous statement holds if we replace

$\langle r\rangle \simeq \mathbb{Z}_4$ by $\langle rs, r^3s\rangle \simeq D_4$, making use of the fact that

$$\mathsf{2Rep}(\mathbb{Z}_4[1] \rtimes \mathbb{Z}_2) \;\simeq\; \text{2Rep}(D_4[1] \rtimes \mathbb{Z}_2)\,. \tag{4.2.128}$$

The above results are compatible with the fusion rules derived in [5]. The non-invertible defect $\mathcal{N}_1$ there is identified with the 2-dimensional 2-representation D, while the symmetry defect W is identified with the 1-dimensional 2-representation V, and X is the condensation.

∘ Adding a non-trivial discrete torsion when gauging results in a $PO(N)$ gauge theory with a discrete theta angle

$$\frac{1}{2}\int w_1 \cup w_1 \cup w_1\,, \tag{4.2.129}$$

where w_1 denotes the first Stiefel-Whitney class obstructing the restriction of a $PO(N)$ bundle to a $PSO(N)$ bundle [119]. Since $H = \langle s\rangle$ is not a normal subgroup of D_8, we cannot utilise a spectral sequence construction to determine the symmetry category.

- Now consider the two non-normal subgroups $H = \langle rs\rangle, \langle r^3s\rangle \simeq \mathbb{Z}_2$, which are related to reach other by conjugation. Gauging these subgroups results in $Ss(N)$ and $Sc(N)$ gauge theories respectively. The two subgroups are related to those considered in the previous bullet point by an outer automorphism and therefore the construction of the symmetry category is identical to above.

Order four subgroups

Recall that there are three order four subgroups, all of which are normal: one is isomorphic to $\mathbb{Z}_4$ and invariant under the outer automorphism and the remaining two are isomorphic to D_4 and exchanged by the outer automorphism. In both cases there is the opportunity to add discrete torsion since

$$\begin{aligned} H^3_{grp}(\mathbb{Z}_4, U(1)) \;&=\; \mathbb{Z}_4\,, \\ H^3_{grp}(D_4, U(1)) \;&=\; \mathbb{Z}_2^3\,. \end{aligned} \tag{4.2.130}$$

We consider the resulting symmetry 2-categories in turn:

- Let us first consider the normal subgroup $H = \langle r^2, s\rangle \cong D_4$. Gauging this subgroup results in a 2-group symmetry $D_4[1] \rtimes \mathbb{Z}_2$. Since H forms a split short exact sequence with D_8, there are no obstructions and discrete torsion acts on the resulting symmetry 2-category by an auto-equivalence. In summary,

$$\mathsf{C}(D_8 \,|\, D_4, \psi) \;=\; 2\mathsf{Vec}_{D_4[1]\rtimes\mathbb{Z}_2} \,. \tag{4.2.131}$$

  In our running example, this results in an $O(N)^0$ gauge theory and the effect of adding discrete torsion is to alternate between different global forms. On the one hand, introducing discrete torsion for the $\mathbb{Z}_2$ subgroup $\langle s\rangle \subset H$ corresponds to adding a discrete theta angle

$$\frac{1}{2}\int w_1 \cup w_1 \cup w_1 \,, \tag{4.2.132}$$

  where w_1 now denotes the first Stiefel-Whitney class obstructing the lift of an $O(N)$-bundle to an $SO(N)$-bundle. On the other hand, introducing discrete torsion for the $\mathbb{Z}_2$ subgroup $\langle r^2\rangle \subset H$ corresponds to the global form

$$O(N)_\psi \;=\; \frac{O(N)\times \mathsf{DW}_{\mathbb{Z}_2,\psi}}{\mathbb{Z}_2[1]} \,. \tag{4.2.133}$$

  There is one further generator of discrete torsion and 8 possible global forms given the $\mathbb{Z}_2^3$ classification in (4.2.130). Our analysis shows that all of these global forms share the same symmetry category up to equivalence.

- The remaining normal D_4 subgroup $H = \langle r^2, rs\rangle$ is related to the one above by an outer automorphism and therefore leads to an identical analysis for the symmetry categories. They correspond to $Spin(N)$ gauge theories with discrete torsion resulting in different global forms

$$Spin(N)_\psi \;=\; \frac{Spin(N)\times \mathsf{DW}_{D_4,\psi}}{D_4[1]} \,, \tag{4.2.134}$$

  where $\psi \in H^3_{grp}(D_4, U(1)) \cong \mathbb{Z}_2^3$.

- Finally, consider the normal subgroup $H = \langle r \rangle \cong \mathbb{Z}_4$. Gauging this subgroup leads to a split 2-group symmetry $\mathbb{Z}_4[1] \rtimes \mathbb{Z}_2$. Since H forms a split short exact sequence with D_8, there are no obstructions and discrete torsion $[\psi] \in H^3_{grp}(\mathbb{Z}_4, U(1))$ acts on the resulting symmetry 2-category by an auto-equivalence. In summary,

$$\mathsf{C}(D_8 \,|\, \mathbb{Z}_4, \psi) \;=\; 2\mathsf{Vec}_{\mathbb{Z}_4[1] \rtimes \mathbb{Z}_2} \,. \tag{4.2.135}$$

In our running example, gauging $H = \mathbb{Z}_4$ leads to a $O(N)^1$ gauge theory, where the superscript 1 denotes the presence of the discrete theta angle

$$\frac{1}{2} \int w_1 \cup w_2 \,. \tag{4.2.136}$$

Here, w_1 and w_2 are the first and second Stiefel-Whitney class of $O(N)$-bundles. One way to understand this interpretation is to gauge in steps. Recall that first gauging the central subgroup $\langle r^2 \rangle$ reproduces an $SO(N)$ gauge theory. The remaining 0-form symmetries correspond to the magnetic symmetry $\langle rs \rangle \cong \mathbb{Z}_2$ and charge conjugation $\langle s \rangle \cong \mathbb{Z}_2$. Subsequently gauging the diagonal combination of these symmetries, which in our notation corresponds to gauging $\langle r \rangle$, reproduces the $O(N)^1$ theory [119].

The effect of adding discrete torsion $\psi \in H^3_{grp}(\mathbb{Z}_4, U(1)) = \mathbb{Z}_4$ corresponds to different global forms of an $O(N)^1$ gauge theory

$$O(N)^1_\psi \;=\; \frac{O(N)^1 \times \mathsf{DW}_{\mathbb{Z}_4,\psi}}{\mathbb{Z}_2[1]} \,. \tag{4.2.137}$$

Our analysis shows that these global forms share the same symmetry 2-category up to equivalence.

Gauging the whole group

Finally, we may gauge the entire symmetry group $H = D_8$ together with discrete torsion

$$[\psi] \in H^3_{grp}(D_8, U(1)) \;\simeq\; \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \,. \tag{4.2.138}$$

The resulting symmetry 2-category is given by $\mathsf{C}(D_8 \,|\, D_8, \psi) \simeq 2\mathrm{Rep}(D_8)$.

In our running example, this corresponds to a $Pin^{\pm}(N)$ gauge theory, where the choice of $\pm$ and specific global form depends on the choice of discrete torsion. In order to enumerate the possibilities and understand their physical interpretation, it is convenient to use as an organisational tool the Lyndon-Hochschild-Serre spectral sequence to enumerate possible discrete torsion. This does not necessarily reproduce the group structure on (4.2.138), but it is a convenient way to identify specific discrete torsion elements and their physical interpretation. There are many ways to do this and we provide two illustrative examples below.

Let us first consider the split short exact sequence

$$1 \to D_4 \to D_8 \to \mathbb{Z}_2 \to 1 \qquad (4.2.139)$$

that is associated to the semi-direct product structure $D_8 \cong D_4 \rtimes \mathbb{Z}_2$. One discrete torsion element of interest arises from the term

$$E_2^{3,0} = H^3_{grp}(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2 \,. \qquad (4.2.140)$$

This corresponds to gauging the $\mathbb{Z}_2$ charge conjugation symmetry of $Spin(N)$ gauge theory with discrete torsion and reproduces the $Pin^+(N)$ gauge theory with discrete theta angle

$$\frac{1}{2} \int w_1 \cup w_1 \cup w_1 \,, \qquad (4.2.141)$$

where w_1 denotes the first Stiefel-Whitney class that obstructs lifting a $Pin^+(N)$-bundle to a $Spin(N)$-bundle.

Now consider instead the short exact sequence

$$1 \to \mathbb{Z}_4 \to D_8 \to \mathbb{Z}_2 \to 1 \qquad (4.2.142)$$

associated to the semi-direct product structure $D_8 \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_2$. We now consider the discrete torsion element arising from the term

$$E_2^{2,1} = H^2_{grp}(\mathbb{Z}_2, \mathbb{Z}_4) \cong \mathbb{Z}_2 \,, \qquad (4.2.143)$$

where $\mathbb{Z}_4$ is understood as a non-trivial $\mathbb{Z}_2$-module. This corresponds to first gauging the $\mathbb{Z}_4$ symmetry of the $PSO(N)$ theory with a local counter term

$$\frac{1}{4}\int \mathsf{k}^*(\psi) \cup \mathsf{a}\,, \tag{4.2.144}$$

where a is the dynamical $\mathbb{Z}_4$ background and k denotes the background for the remaining $\mathbb{Z}_2$ symmetry. The result is a $O(N)^1$ gauge theory where the background $\widehat{\mathsf{a}}$ for the emergent $\mathbb{Z}_4[1]$ symmetry is shifted by

$$\widehat{\mathsf{a}} \;\to\; \widehat{\mathsf{a}} + \mathsf{k}^*(\psi)\,. \tag{4.2.145}$$

If ψ is non-trivial, this corresponds to adding a non-trivial symmetry fractionalisation. Subsequently gauging the remaining $\mathbb{Z}_2$ symmetry then results in a $Pin^-(N)$ gauge theory [119].

4.3 Generalised Gauging in $2+1$ Dimensions

In the previous section we restricted ourselves to those theories obtained by gauging a finite subgroup after coupling to an invertible TQFT, or discrete torsion. Now we turn our attention to general notions of gauging a finite group symmetry of a $(2+1)$-dimensional quantum field theory.

In general we wish to consider theories obtained by gauging a finite subgroup coupled to *any* $(2+1)$-dimensional TQFT. Starting from a theory $\mathcal{T}$ with a (G, α) symmetry, the data of such a gauging is captured by:

- A (conjugacy class of) subgroup $H \subseteq G$.
- A $(2+1)$-dimensional TQFT corresponding to a non-degenerate braided fusion category $\mathcal{B}$.
- Some additional higher data that captures how we couple $\mathcal{B}$ to the H symmetry.

The data of this coupling can be interpreted physically as an action of the surface defects labelled by $h \in H$ on the line defects $b \in \mathcal{B}$, as depicted in figure 4.14, together with a symmetry fractionalisation of H by invertible lines $\mathcal{B}^{\times}$, as depicted in figure 4.15, and a discrete torsion corresponding to a choice of trivialisation for the combined anomaly of the original theory $\mathcal{T}$ and the TQFT on a subgroup.

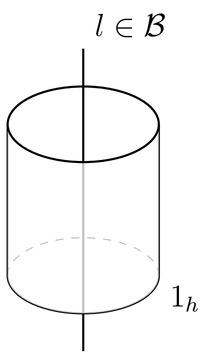

Figure 4.14

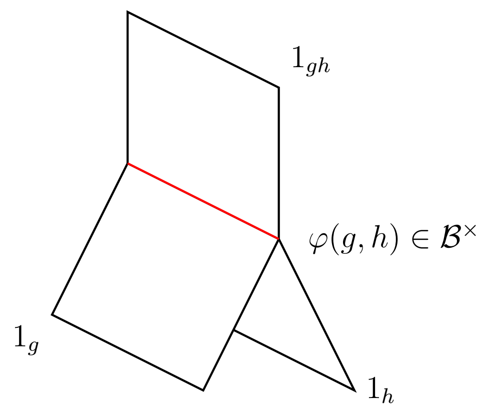

Figure 4.15

The data we are describing is precisely that of a H-symmetric topological order, with topological lines labelled by objects of $\mathcal{B}$; these topological orders have been studied extensively in the maths and physics literature and are known to be described by H-crossed braided extensions of $\mathcal{B}$ classified up to equivalence by [98, 121]:

- A H-action $\rho : H \to Aut(\mathcal{B})$.
- A class $[\varphi]$ in a torsor over $H^2_{grp,\rho}(H, \mathcal{B}^\times)$, where $\mathcal{B}^\times$ is the group of invertible objects thought of as a H-module.
- A class $[\psi]$ in a torsor over $H^3_{grp}(H, \mathbb{C}^\times)$.

The algebra and module pictures that generalise those we studied in subsections 3.3.1 and 3.3.2 reproduce this labelling only for those $(2+1)$-dimensional TQFTs that admit a gapped boundary, called Turaev-Viro TQFTs. Mathematically, this means restricting the data above to those non-degenerate braided fusion categories that admit an equivalence

$$\mathcal{B} \simeq \mathcal{Z}(\mathsf{C})\,, \tag{4.3.1}$$

for some fusion category C, and for a choice of subgroup $H \subseteq G$ corresponds to classifying H-extensions of C [121]. We will argue in subsections 4.3.1 and 4.3.2 that this classification is equivalent to a labelling by α^{-1}-projective 3-representations of G valued in $3\mathsf{Vec}$. In the unitary setting we expect this labelling to be replaced by unitary projective 3-representations valued in $3\mathsf{Hilb}$.

The symmetry TFT perspective meanwhile, which generalises the picture from subsubsection 3.3.3, faithfully reproduces this labelling of gaugings in its entirety; Lagrangian algebra objects in the Drinfeld centre $\mathcal{Z}(2\mathsf{Vec}^\alpha_G)$ correspond precisely to G-crossed braided extensions of non-degenerate braided fusion categories. We will argue in subsection 4.3.3 that these are equivalently described by α^{-1}-projective 3-representations of G valued in a certain fusion 3-category whose simple objects are non-degenerate braided fusion categories. In the unitary setting we expect this labelling to be replaced by unitary 3-representations values in a certain unitary fusion 3-category whose simple objects are given by unitary braided fusion categories [7].

[7]With, as we will address in further detail in subsection 4.3.3, vanishing central chiral charge (mod 8).

4.3.1 Gauging and 2-Algebras internal to $2\mathsf{Vec}_G^\alpha$

We return to the algebra construction presented in the previous section, which categorifies those constructions seen in subsections 2.3.1 and 3.3.1 where gauging corresponds to summing over a network of topological defects.

The goal then is to classify theories $\mathcal{T}/_\lambda G$ obtained by starting with a theory $\mathcal{T}$ with a finite (G,α) symmetry, coupling to a $(2+1)$-dimensional Turaev-Viro TQFT, and gauging a non-anomalous subgroup. In the unitary setting we expect this gauging to be implemented by picking a (2-)algebra object in

$$\mathcal{C}_\mathcal{T} \simeq 2\mathsf{Hilb}_G^\alpha\,, \tag{4.3.2}$$

and for this to be equivalent to classifying those unitary fusion 2-categories which lie in the same Morita class.

In the framed setting, this reduces to studying (2-)algebra objects in

$$\mathcal{C}_\mathcal{T} \simeq 2\mathsf{Vec}_G^\alpha\,, \tag{4.3.3}$$

and corresponds to classifying fusion 2-categories in the same Morita class.

An algebra object consists of an object in $\mathcal{C}_\mathcal{T}$

$$A = \bigoplus_{g\in G} A_g \tag{4.3.4}$$

together with multiplication and unit 1-morphisms

$$m : A\otimes A \to A \qquad\qquad i : 1_e \to A\,, \tag{4.3.5}$$

and before in subsection 3.3.1 these were subject to further compatibility constraints that captured how to make sense of A as a network of defects. Here in the 2-categorical setting these constraints are replaced by higher compatibility data in the form of unitor and associator 2-isomorphisms

$$\begin{gathered} u_l : m\circ(i\otimes id_A) \Rightarrow id_A \qquad\qquad u_r : m\circ(id_A\otimes i) \Rightarrow id_A \\ a : m\circ(id_A\otimes m) \Rightarrow m\circ(m\otimes id_A) \end{gathered} \tag{4.3.6}$$

that control the commutativity of diagrams analogous to those in figures 3.12 and 3.13. This extra compatibility data is then subject to further constraints

$$\begin{aligned}&[id_m \circ (a \otimes id_{id_A})] \cdot [a \circ (id_{id_A} \otimes id_m \otimes id_{id_A})] \cdot [id_m \circ (id_{id_A} \otimes a)] \\ &\qquad = [a \circ (id_{id_A} \otimes id_m)] \cdot \alpha^{-1}_{A,A,A,A} \cdot [a \circ (id_m \otimes id_{id_A})]\,,\end{aligned} \tag{4.3.7}$$

and

$$[id_m \circ (u_r \otimes id_{id_A})] \cdot [a \circ (id_{id_A} \otimes id_i \otimes id_{id_A})] = id_m \circ (id_{id_A} \otimes u_l)\,. \tag{4.3.8}$$

From the point of view of A being an object in $2\mathsf{Vec}_G^\alpha$, these conditions can be expressed abstractly as two tetrahedral and one associahedral commuting diagram, twisted by α^{-1}. Alternatively, thinking of A as a finite semi-simple G-graded category, these conditions can be expressed more concretely as a pentagon and triangle identities, twisted by α^{-1}. This latter point of view gives A the structure of what we will call a finite semi-simple G-graded α^{-1}-twisted monoidal category.

It was at this point in subsection 3.3.1 we remarked that in the unitary setting, the dagger structure of Hilb_G^α led naturally to a co-algebra structure, and further more a special dagger-Frobenius structure. We expect that in $2+1$ dimensions we should have a similar result, and that $2\mathsf{Hilb}_G^\alpha$ as a dagger fusion 2-category should likewise automatically equip a natural co-algebra structure satisfying analogous dagger-Frobenius compatibility conditions. We further expect the full unitary fusion 2-categorical structure to endow A with the structure of a unitary multi-fusion category.

In analogy to subsections 2.3.1 and 3.3.1, we define the Morita equivalence of rigid algebra objects A, A' internal to $\mathcal{C}_\mathcal{T} \simeq 2\mathsf{Vec}_G^\alpha$ as an equivalence of (left) module 2-categories $\mathrm{Mod}_{\mathcal{C}_\mathcal{T}}(A) \simeq \mathrm{Mod}_{\mathcal{C}_\mathcal{T}}(A)$ of (2-)module objects over A, A', internal to $\mathcal{C}_\mathcal{T}$. These comprise of objects $M \in \mathcal{C}_\mathcal{T}$ together with a (right) module action from A

$$\mu : M \otimes A \to M\,, \tag{4.3.9}$$

and higher compatibility data comprising of 2-isomorphisms

$$a^\mu : \mu \circ (id_M \otimes m) \Rightarrow \mu \circ (\mu \otimes id_A) \qquad\qquad u^\mu : \mu \circ (id_M \otimes i) \Rightarrow id_M \,, \tag{4.3.10}$$

that lift those conditions we depicted back in figures 3.14 and 3.15. These must satisfy further compatibility conditions

$$\begin{aligned}[id_\mu \circ (a^\mu \otimes id_{id_A})] \cdot [a^\mu \circ (id_{id_M} \otimes id_{id_M} \otimes id_{id_A})] \cdot [id_\mu \circ (id_{id_M} \otimes a)] \\ = [a^\mu \circ (id_\mu \otimes id_{id_A})] \cdot \alpha^{-1}_{M,A,A,A} \cdot [a^\mu \circ (id_{id_{M\otimes A}} \otimes id_{id_M})]\end{aligned} \tag{4.3.11}$$

and

$$[id_\mu \circ (u^\mu \otimes id_{id_A})] \cdot [a^\mu \circ (id_{id_M} \otimes id_i \otimes id_{id_A})] = id_\mu \circ (id_{id_M} \otimes u_l)\,, \tag{4.3.12}$$

that give M the structure of what we will call a finite semi-simple G-graded α^{-1}-twisted module category over A.

The full 2-category of (right) module objects $\lambda = \mathrm{Mod}_{\mathcal{C}_\mathcal{T}}(A)$ naturally forms a (left) module 2-category over $\mathcal{C}_\mathcal{T} \simeq 2\mathsf{Vec}_G^\alpha$ which, in analogy to subsections 3.3.1 and 3.3.2, is how we define finite-dimensional α^{-1}-projective 3-representations of G. In the framed setting this identifies Morita equivalence classes of algebra objects internal to $2\mathsf{Vec}_G^\alpha$ with projective 3-representations

$$\lambda \in 3\mathrm{Rep}^{\alpha^{-1}}(G)\,. \tag{4.3.13}$$

In the unitary setting we expect this to lift to a labelling by unitary projective 3-representations.

To classify these 3-representations, we need a slight enhancement on the structure available to us in the framed setting; we will opt to include the co-algebra structure that we expect to come for free in the unitary setting, by hand. Concretely we ask that A as an algebra object in $2\mathsf{Vec}_G^\alpha$ admits a compatible co-algebra structure

$$\Delta : A \to A \otimes A \qquad\qquad p : A \to 1_e\,, \tag{4.3.14}$$

with higher compatibility data analogous to above that makes A a co-algebra object.

We ask also for additional higher compatibility data in the form 2-morphisms

$$\epsilon : \Delta \circ m \Rightarrow id_A \qquad\qquad \eta : id_{A\otimes A} \Rightarrow m \circ \Delta \,, \tag{4.3.15}$$

and two 2-isomorphisms that lift the Frobenius conditions from subsection 3.3.1. This extra data must then satisfy further compatibility conditions, the total sum of which makes A a *rigid* [8] algebra object in $2\mathsf{Vec}_G^\alpha$ [122].

Let's summarise: to better understand this extra data, let us simplify to $G = 1$ for a moment. An algebra object A in $\mathcal{C}_\mathcal{T} = 2\mathsf{Vec}$ describes a finite semi-simple monoidal category, and the usual notion of Morita equivalence for these categories is precisely that which we have described for algebras internal to $2\mathsf{Vec}$. The compatibility data and conditions for rigid algebra objects can be summarised as a promotion of A to a multi-fusion category by consistently defining left and right duals. The Morita equivalence of (rigid) algebras internal to $2\mathsf{Vec}$ coincides with the usual notion of Morita equivalence for monoidal/multi-fusion categories.

Returning to (G, α) then, an algebra object A in $\mathcal{C}_\mathcal{T} \simeq 2\mathsf{Vec}_G^\alpha$ describes a finite semi-simple G-graded α^{-1}-twisted monoidal category. The compatibility data and conditions for rigid algebra object promote A to a G-graded α^{-1}-twisted multi-fusion category [122]. The Morita equivalence of (rigid) algebras internal to $2\mathsf{Vec}_G^\alpha$ implies the usual notion of Morita equivalence on the trivially-graded component as a monoidal/multi-fusion category.

Indecomposable gauge theories $\mathcal{T}/_\lambda G$ correspond to *connected* rigid algebra objects, which are characterised by having a simple unit morphism:

$$End(i) \simeq \mathbb{C} \,. \tag{4.3.16}$$

A connected rigid algebra object in $2\mathsf{Vec}$ corresponds to a fusion category, and all multi-fusion categories decompose into a direct sum of fusion categories up to Morita equivalence. A connected rigid algebra object in $2\mathsf{Vec}_G^\alpha$ describes a G-graded α^{-1}-

[8] For the fusion 2-category $2\mathsf{Vec}_G^\alpha$ this is actually equivalent to separable algebra objects, which are the natural ones to talk about in the oriented setting of spherical fusion 2-categories [111, 122].

twisted fusion category [122]. The reason for needing this extra data now becomes clear: the property of A having left and right duals implies the grading of A includes inverses, and further forms a subgroup $H \subseteq G$.

For the case $\alpha = 0$, the connected rigid algebra objects are faithfully H-graded fusion categories for some $H \subseteq G$, also known as H-extensions of fusion categories in the literature. The classification for a given (Morita class of) fusion category $A_e \simeq \mathsf{C}$ that describes the trivially graded component follows from [121]:

- An action $\rho : H \to Aut(\mathcal{Z}(\mathsf{C}))$, where $\mathcal{Z}(\mathsf{C})$ is the Drinfeld centre.
- A class $[\varphi]$ in a torsor over $H^2_{grp,\rho}(H, \mathcal{Z}(\mathsf{C})^\times)$, where the abelian group of invertible objects $\mathcal{Z}(\mathsf{C})^\times$ is thought of as a H-module.
- A class $[\psi]$ in a torsor over $H^3_{grp}(H, \mathbb{C}^\times)$.

The choice of fusion category C up to Morita equivalence corresponds physically to a choice of $(2+1)$-dimensional Turaev-Viro TQFT with topological lines given by $\mathcal{Z}(\mathsf{C})$. The appearance of torsors over group cohomology arise as a consequence of obstructions to constructing a H-graded fusion category with trivially graded component $A_e \simeq \mathsf{C}$, which admit their own physical interpretations:

1. A choice of fusion category C and action ρ determines an obstruction class $[\mathcal{O}_3(\mathsf{C};\rho)] \in H^3_{grp,\rho}(H, \mathcal{Z}(\mathsf{C})^\times)$. This is an obstruction to constructing a H-extension and needs to be trivialised via

$$\delta\varphi = \mathcal{O}_3(\mathsf{C}\,;\,\rho)\,. \tag{4.3.17}$$

Different choices of trivialisation are then distinguished by 2-cocycles and are classified up to equivalence by a torsor over $H^2_{grp,\rho}(H, \mathcal{Z}(\mathsf{C})^\times)$. Physically we should interpret these classes as determining a symmetry fractionalisation of H by the lines in $\mathcal{Z}(\mathsf{C})$ as depicted in 4.15.

2. The choice of trivialisation $[\varphi]$ further determines a second obstruction class $[\mathcal{O}_4(\mathsf{C};\rho,\varphi)] \in H^4_{grp}(H,\mathbb{C}^\times)$. This is yet another obstruction to constructing a H-extension that needs to be trivialised via

$$\delta\psi = \mathcal{O}_4(\mathsf{C}\,;\,\rho,\,\psi)\,. \tag{4.3.18}$$

Different choices of trivialisation are then distinguished by 3-cocycles and are classified up to equivalence by a torsor over $H^3_{grp}(H,\mathcal{Z}(\mathsf{C})^\times)$. Physically we should interpret these classes as determining a discrete torsion for H.

As a G-graded fusion category, the 3-cocycle ψ plays the role of the associator [121]. This means for the case $\alpha \neq 0$, it must satisfy equation (4.3.7). This has the effect of shifting the $\mathcal{O}_4$ obstruction to

$$\delta\psi = \alpha|_H^{-1}\,\mathcal{O}_4(\mathsf{C}\,;\,\rho,\,\varphi)\,. \tag{4.3.19}$$

The appearance of the $\mathcal{O}_4$ obstruction in this more general setting suggests that in contrast to chapter 3 where we could only gauge a truly non-anomalous subgroup, here we can in fact gauge an anomalous subgroup, provided we can find a TQFT to couple to with a H-symmetry that cancels that of the subgroup.

To summarise, the ways to gauge a (G,α) symmetry corresponding to connected rigid algebras up to Morita equivalence are labelled by:

- A (conjugacy class of) subgroup $H \subseteq G$, corresponding to a choice of subgroup to gauge.
- A (Morita class of) fusion category C, corresponding to a Turaev-Viro TQFT with line defects in $\mathcal{Z}(\mathsf{C})$.
- An action $\rho : H \to (Aut)(\mathcal{Z}(\mathsf{C}))$, corresponding to H-defects wrapping lines, as depicted in figure 4.14.
- A class of 2-cochain $\varphi \in C^2_{grp,\rho}(H,\mathcal{Z}(\mathsf{C})^\times)$ satisfying $\delta\varphi = \mathcal{O}_3(\mathsf{C};\rho)$, corresponding to a symmetry fractionalisation of H by $\mathcal{Z}(\mathsf{C})^\times$, as depicted in figure 4.15.

- A class of 3-cochain $\psi \in C^3_{grp}(H, \mathbb{C}^\times)$ satisfying $\delta\psi = \alpha|_H^{-1}\, \mathcal{O}_4(\mathsf{C}; \rho, \varphi)$, corresponding to a choice of discrete torsion for H.

We can restrict to invertible TQFTs by choosing $\mathsf{C} \simeq \mathsf{Vec}$, in this case the Drinfeld centre $\mathcal{Z}(\mathsf{Vec}) = \mathsf{Vec}$, automorphism group $\mathrm{Aut}(\mathsf{Vec}) = 1$, and invertible lines $\mathsf{Vec}^\times = 1$, all trivialise. This means the action ρ and obstructions $\mathcal{O}_3$ and $\mathcal{O}_4$ automatically vanish, and further trivialises the choice of symmetry fractionalisation φ. The only remainind data then is the discrete torsion ψ, for which the trivialisation condition (4.3.19) reduces to that of (4.2.58) observed in subsection 4.2.2.

4.3.2 Gapped Interfaces and Module Categories over $2\mathsf{Vec}_G^\alpha$

As we just saw, classifying gaugings as Morita classes of (2-)algebra objects is equivalent to classifying module 2-categories over

$$\mathcal{C}_\mathcal{T} \simeq 2\mathsf{Vec}_G^\alpha\,, \tag{4.3.20}$$

which is how we define projective 3-representations. We can arrive at this result in an equivalent way by considering gapped interfaces between $\mathcal{T}$ and the gauged theory $\mathcal{T}/_\lambda G$.

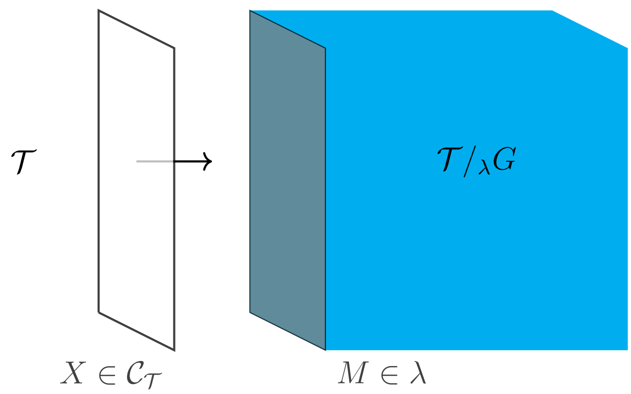

Figure 4.16

The gapped interfaces between $\mathcal{T}$ and $\mathcal{T}/_\lambda G$ form a finite semi-simple 2-category λ, whose objects $M \in \lambda$ are gapped interfaces, 1-morphisms are topological lines supported on junctions between interfaces, and 2-morphisms are topological local operators supported on junctions between the lines.

In the framed setting, the action from fusing topological surfaces in $\mathcal{T}$ onto the interface as depicted in figure 4.16 endows λ with the structure of a (left) module 2-category over

$$\mathcal{C}_{\mathcal{T}} = 2\mathsf{Vec}_G^{\alpha} \,. \tag{4.3.21}$$

This is precisely how we define α^{-1}-projective 3-representations of G, categorifying the notion of projective 2-representations in subsection 3.3.2, which themselves were categorifications of modules/representations in subsection 2.3.2.

An alternative formulation of these module 2-categories is in terms of functors of the form

$$\mathcal{R} : \mathbf{B}G \to 3\mathsf{Vec} \,, \tag{4.3.22}$$

where $\mathbf{B}G$ is the delooping of G thought of as a fusion 3-category. For $\alpha = 0$ these functors are monoidal, otherwise they are monoidal up to α^{-1} which twists its compatibility with the pentagonator, we call this an α^{-1}-projective monoidal functor.

We define the 3-vector spaces that make up the fusion 3-category $3\mathsf{Vec}$ analogously to Kapronov-Voevodsky 2-vector spaces [123] as module 2-categories over $2\mathsf{Vec}$ equivalent to $2\mathsf{Vec}^{\oplus n}$ for some $n \in \mathbb{N}$. This definition can be interpreted in two different ways:

1. A 3-vector space equivalent to $2\mathsf{Vec}^{\oplus n}$ can be thought of as a finite semi-simple 2-category with n simple objects.
2. The objects of a module 2-category over $2\mathsf{Vec}$ up to equivalence can be thought of as module categories over a Morita class of (rigid) algebra object internal to $2\mathsf{Vec}$ corresponding to a monoidal/(multi-)fusion category.

Thought of as a finite semi-simple category, the 3-vector space $\mathcal{R}(\star)$ is equipped with a module action from $2\mathsf{Vec}_G^{\alpha}$ via the assignment of elements $g \in G$ to automorphisms of $\mathcal{R}(\star)$. This reproduces a module 2-category from a choice of projective monoidal functor.

Thought of instead as a Morita class of multi-fusion category C, we can reconstruct the classification from [121] detailed in the previous subsection. We formulate the remaining data of $\mathcal{R} : \star \mapsto [\mathsf{C}]$ as a homomorphism between 3-groups

$$\lambda : \widetilde{\mathsf{G}} \longrightarrow \mathsf{Aut}(\mathsf{C})\,. \tag{4.3.23}$$

The domain and codomain have the following descriptions:

- The domain $\widetilde{\mathsf{G}}$ is a 3-group extension with homotopy groups

$$\begin{aligned} \pi_0 &= G \\ \pi_1 &= 1 \\ \pi_2 &= \mathbb{C}^\times \end{aligned} \tag{4.3.24}$$

  and Postnikov invariant $\alpha^{-1} \in H^4_{grp}(G, U(1))$.

- The codomain $\mathsf{Aut}(\mathsf{C})$ is the automorphism 3-group of C with

$$\begin{aligned} \pi_0 &= \mathrm{Aut}(\mathcal{Z}(\mathsf{C})) \\ \pi_1 &= \mathcal{Z}(\mathsf{C})^\times \\ \pi_2 &= (\mathbb{C}^\times)^{|\mathsf{C}|}\,. \end{aligned} \tag{4.3.25}$$

  There is potential Postnikov data given by

$$\begin{aligned} [\mathcal{O}_3] &\in H^3_{grp}(\mathrm{Aut}(\mathcal{Z}(\mathsf{C})), \mathcal{Z}(\mathsf{C})^\times)\,, \\ [\mathcal{O}_4] &\in H^4_{grp}(\pi_{\leq 1}, (\mathbb{C}^\times)^{|\mathsf{C}|})\,, \end{aligned} \tag{4.3.26}$$

  where $\pi_{\leq 1}$ is shorthand for the 2-group truncation of $\mathsf{Aut}(\mathsf{C})$ determined by $\mathrm{Aut}(\mathcal{Z}(\mathsf{C})), \mathcal{Z}(\mathsf{C})^\times$, and $[\mathcal{O}_3]$. The possibility of non-trivial Postnikov data is novel compared to the corresponding construction for 2-representations in $1+1$ dimensions.

An α^{-1}-projective 3-representation on $[\mathsf{C}]$ is then determined by:

1. A homomorphism $\widetilde{\rho} : G \to \mathrm{Aut}(\mathcal{Z}(\mathsf{C}))$.

2. A 2-cochain $\widetilde{\varphi} \in C^2_{grp,\widetilde{\rho}}(G, \mathcal{Z}(\mathsf{C})^\times)$ satisfying $\delta\widetilde{\varphi} = \widetilde{\rho}^*\mathcal{O}_3$.

3. A 3-cochain $\widetilde{\psi} \in C^3_{grp}(G, (\mathbb{C}^\times)^{|\mathsf{C}|})$ satisfying $\delta\widetilde{\psi} = \alpha^{-1}\,(\widetilde{\rho}, \widetilde{\varphi})^*\mathcal{O}_4$.

where we view the combination $(\widetilde{\rho}, \widetilde{\varphi}) : BG \to \pi_{\leq 1}$.

To restrict to simple projective 3-representations we ask that $\widetilde{\rho} : G \to \mathrm{Aut}(\mathcal{Z}(\mathsf{C}))$ determines a transitive action on $\mathcal{Z}(\mathsf{C})$, which via the orbit stabiliser theorem corresponds to picking a conjugacy class of subgroup $[H \subseteq G]$ and setting

$$\mathsf{C} \simeq \mathsf{c}^{\oplus G/H} \qquad\qquad \mathcal{Z}(\mathsf{C}) \simeq \mathcal{Z}(\mathsf{c})^{\oplus G/H}\,, \tag{4.3.27}$$

for some fusion category c, together with a choice of $\rho : H \to \mathrm{Aut}(\mathcal{Z}(\mathsf{c}))$. Using the Shapiro isomorphism we then surmise that a simple α^{-1}-projective 3-representation of G is determined by:

1. A (conjugacy class of) subgroup $H \subseteq G$.

2. A (Morita class of) fusion category c.

3. A homomorphism $\rho : H \to \mathrm{Aut}(\mathcal{Z}(\mathsf{c}))$.

4. A 2-cochain $\varphi \in C^2_{grp,\rho}(H, \mathcal{Z}(\mathsf{c})^\times)$ satisfying $\delta\varphi = [\widetilde{\rho}^*\mathcal{O}_3]_H$.

5. A 3-cochain $\psi \in C^3_{grp}(H, (\mathbb{C}^\times))$ satisfying $\delta\psi = \alpha|_H^{-1}\,[(\widetilde{\rho}, \widetilde{\varphi})^*\mathcal{O}_4]_H$.

which reconstructs the classification from the previous subsection.

To further relate this picture to the previous one in terms of algebras, we note that to construct an interface between $\mathcal{T}$ and $\mathcal{T}/_\lambda G$ obtained by gauging an algebra object A, we must first choose an object $M \in \mathcal{C}_\mathcal{T}$ and then specify how the algebra object ends on it. The data that implements this is precisely that of a module object over A, identifying the category of gapped interfaces with the earlier module category

$$\lambda = \mathrm{Mod}_{\mathcal{C}_\mathcal{T}}(A)\,. \tag{4.3.28}$$

In the oriented setting we can formulate a direct argument for the reverse statement by considering $\mathcal{T}/_\lambda G$ with insertions of the identity operator and resolve them into oriented spheres containing $\mathcal{T}$. Expanding the spheres, eventually the interfaces will collide and produce a network of topological defects A_λ in $\mathcal{T}$. In analogy to subsection 3.3.2, we expect that in order for the resulting theory to be independent of the way the expansion is performed, the topological defects A must describe rigid/separable algebra objects internal to $\mathcal{C}_\mathcal{T}$ [94].

In the unitary setting, we expect the 2-category λ of interfaces to now be a finite semi-simple dagger 2-category [25]. We further expect that the action from fusing topological lines in $\mathcal{T}$ onto the interface depicted in figure 3.16 endows λ with the structure of a (left) unitary module 2-category over

$$\mathcal{C}_\mathcal{T} = 2\mathsf{Hilb}_G^\alpha \,. \tag{4.3.29}$$

This is how we expect to define unitary α^{-1}-projective 3-representations of G. Alternatively we expect that we can also formulate these unitary module 2-categories as unitary α^{-1}-projective monoidal functors

$$\mathcal{R} : \mathbf{B}G \to 3\mathsf{Hilb} \,, \tag{4.3.30}$$

where we define $3\mathsf{Hilb}$ analogously to $3\mathsf{Vec}$ as the 3-category of (unitary) module 2-categories over $2\mathsf{Hilb}$ equivalent to $2\mathsf{Hilb}^{\oplus n}$ for some $n \in \mathbb{N}$.

4.3.3 Symmetry TFTs and Lagrangian Algebras

Now we consider the same gauging procedures from the perspective of gapped boundary conditions in the sandwich construction. The starting point is $3+1$-dimensional unitary oriented Dijkgraaf-Witten theory labelled by a gauge group G and a class

$$[\alpha] \in H^4_{grp}(G, U(1)) \,. \tag{4.3.31}$$

Similar to those we discussed in subsection 3.3.3, they are gauge theories supported on a 4-manifold M_4, described by a finite gauge field

$$\mathsf{a} : M_4 \to BG \,, \tag{4.3.32}$$

whose action is determined by a representative $\alpha \in Z^4_{grp}(G, U(1))$ satisfying

$$\delta\alpha(g,h,k,l,m) = \frac{\alpha(h,k,l,m)\alpha(g,hk,l,m)\alpha(g,h,k,lm)}{\alpha(gh,k,l,m)\alpha(g,h,kl,m)\alpha(g,h,k,l)} = 1 \,, \tag{4.3.33}$$

via the pull-back

$$\int_{M_4} \mathsf{a}^*\alpha \,, \tag{4.3.34}$$

and is hence manifestly topological. When the boundary ∂M_4 is non-empty, we can specify topological boundary conditions by fixing the restriction

$$\mathsf{a}|_{\partial M_4} : \partial M_4 \to BH \,, \tag{4.3.35}$$

for some subgroup $H \subseteq G$ such that $\alpha|_H^{-1} = \delta\psi$ trivialises. The pull-back

$$\int_{\partial M_4} \psi^*\mathsf{a} \tag{4.3.36}$$

then defines a consistent contribution to the topological action on the boundary that makes the total theory topological [76].

Unlike in subsection 3.3.3 however, this labelling by (H, ψ) is not sufficient to capture all gapped/topological boundary conditions for Dijkgraaf-Witten theory in $3+1$ dimensions. The reasoning is the same as we have seen elsewhere in this section: there is a proliferation of non-trivial TQFTs in $2+1$ dimensions to which we can couple the boundary, and these naive choices only make up for the subclass of invertible TQFTs.

Continuing the analogy to subsection 3.3.3, we can define the Dirichlet boundary condition as the boundary condition $\mathcal{D} = (H = 1, \psi = 1)$. In the framed setting this corresponds to a $2+1$-dimensional topological boundary supporting a (G, α) symmetry described by

$$\mathcal{C}_{\mathcal{D}} = 2\mathsf{Vec}^{\alpha}_{G} \,. \tag{4.3.37}$$

The existence of this canonical topological Dirichlet boundary reflects the fact that the Dijkgraaf-Witten theory is a Turaev-Viro type TQFT whose symmetry in the $(3+1)$-dimensional bulk contains topological surfaces and lines described by the Drinfeld centre

$$\mathsf{DW}_{G,\alpha} \simeq \mathcal{Z}(2\mathsf{Vec}_G^\alpha)\,. \tag{4.3.38}$$

This is a braided fusion 2-category whose objects are objects $X \in 2\mathsf{Vec}_G^\alpha$ together with a half-braiding that comes in the forms of 1-isomorphisms

$$b_{X,Y} : X \otimes Y \to Y \otimes X\,, \tag{4.3.39}$$

for each other object $Y \in 2\mathsf{Vec}_G^\alpha$. The compatibility conditions are now exchanged for extra compatibility data implemented by 2-isomorphisms

$$\begin{aligned} b_{X;Y,Z} &: (id_Y \otimes b_{X,Z}) \circ (b_{X,Y} \otimes id_Z) \Rightarrow b_{X,Y\otimes Z} \\ b_{X,Y;Z} &: (b_{X,Z} \otimes id_Y) \circ (id_X \otimes b_{Y,Z}) \Rightarrow b_{X\otimes Y,Z}\,, \end{aligned} \tag{4.3.40}$$

satisfying further compatibility conditions with the pentagonator α [111, 124]. Concretely these objects are characterised by the following data:

1. A finite-dimensional G-graded 2-vector space

$$X = \bigoplus_{g\in G} X_g\,. \tag{4.3.41}$$

2. A G-action by 1-automorphisms

$$\rho_{g,h} : X_h \to X_{ghg^{-1}} \tag{4.3.42}$$

3. Compositor 2-isomorphisms

$$\rho^\circ_{g,h;f} : \rho_{g,{}^h f} \circ \rho_{h,f} \Rightarrow \rho_{gh,f} \tag{4.3.43}$$

satisfying a twisted composition tetrahedron condition:

$$\rho^\circ_{g,hk;f} \cdot [id_{\rho_{g,{}^{hk}f}} \circ \rho^\circ_{h,k;f}] = \tau_f\alpha(g,h,k) \cdot \rho^\circ_{gh,k;f} \cdot [\rho^\circ_{g,h;{}^k f} \circ id_{\rho_{k,f}}] \tag{4.3.44}$$

for all $g, h, k, f \in G$.

4. Distributor 2-isomorphisms

$$\rho^{\otimes}_{f;g,h} : \rho_{f,g} \otimes \rho_{f,h} \Rightarrow \rho_{f,gh} \,, \tag{4.3.45}$$

satisfying a twisted fusion tetrahedron condition:

$$\rho^{\otimes}_{f}(gh,k) \cdot [\rho^{\otimes}_{f;g,h} \otimes id_{\rho_{f,k}}] = \widetilde{\tau}_f \alpha(g,h,k) \cdot \rho^{\otimes}_{f;g,hk} \cdot [id_{\rho_{f,g}} \otimes \rho^{\otimes}_{f;h,k}] \tag{4.3.46}$$

for all $g, h, f \in G$.

The collections of phases

$$\tau_f(\alpha)(g,h,k) = \frac{\alpha(g,h,{}^k f,k)\alpha({}^{ghk}f,g,h,k)}{\alpha(g,h,k,f)\alpha(g,{}^{hk}f,h,k)} \tag{4.3.47}$$

and

$$\widetilde{\tau}_f(\alpha)(g,h,k) = \frac{\alpha({}^f g,f,h,k)\alpha({}^f g,{}^f h,{}^f k,f)}{\alpha(f,g,h,k)\alpha({}^f g,{}^f h,f,k)} \tag{4.3.48}$$

indexed by $f \in G$, define a groupoid 3-cocycle

$$\tau(\alpha) \in Z^3(G//G, U(1)) \tag{4.3.49}$$

or equivalently, collections of 3-cocycles $\tau_f(\alpha) \in Z^3(C_f(G), U(1))$ upon restriction to the centraliser $g, h, k \in C_f(G)$.

The symmetry of a (2+1)-dimensional quantum field theory can be recast as a (3+1)-dimensional symmetry TFT. For *any* oriented QFT, the corresponding symmetry TFT is a Dijkgraaf-Witten theory for some choice of (G, α) [105]. The dynamics of $\mathcal{T}$ are captured by a relative (non-topological) boundary condition, and the theory itself can be recovered by interval compactification with the canonical gapped Dirichlet boundary condition $\mathcal{D}$.

For other choices of gapped boundary λ, interval compactification produces the theory $\mathcal{T}/_\lambda G$. This has an alternative description of starting from the canonical Dirichlet boundary condition $\mathcal{D}$ coupling a $(2+1)$-dimensional TQFT and gauging a non-anomalous subgroup. Unlike in the previous two subsections, where we were restricted to Turaev-Viro TQFTs, we really mean all (unitary oriented) TQFTs here, and so we expect a much richer classification. For an invertible choice of TQFT $\lambda = (H, \psi)$ this reproduces the theories studied in subsection 4.2.2.

In analogy to subsection 3.3.3 we now reconstruct the gapped boundary conditions of a $(3+1)$-dimensional TQFT of Turaev-Viro type built from a fusion 2-category C, via the 3-category of Lagrangian algebra objects internal to $Z(\mathsf{C})$. Continuing the analogy, we expect these to represent topological line defects in the symmetry bulk that condense on their corresponding topological boundary.

A braided algebra object B in $\mathcal{Z}(2\mathsf{Vec}_G^\alpha)$ is equipped with a braiding 2-isomorphism

$$\beta : m \circ b_{\mathsf{B},\mathsf{B}} \Rightarrow m \,, \tag{4.3.50}$$

which lifts the commuting diagram from figure 3.17, this can be thought of as defining a braiding on B as a monoidal category. This satisfies further compatibility constraints

$$\begin{aligned} a^{-1} \cdot [id_m \circ (\beta \otimes id_{id_\mathsf{B}})] \cdot [a \circ (id_{b_{\mathsf{B},\mathsf{B}}} \otimes id_{id_\mathsf{B}})] \cdot [id_m \circ (id_{id_\mathsf{B}} \otimes \beta) \circ (id_{b_{\mathsf{B},\mathsf{B}}} \otimes id_{id_\mathsf{B}})] \\ = a \cdot [\beta \circ (id_m \otimes id_{id_\mathsf{B}})] \cdot [id_m \circ (id_{id_\mathsf{B}} \otimes id_m) \circ b_{\mathsf{B};\mathsf{B},\mathsf{B}}] \,, \end{aligned} \tag{4.3.51}$$

and

$$\begin{aligned} a \cdot [id_m \circ (id_{id_\mathsf{B}} \otimes \beta)] \cdot [a^{-1} \circ (id_{id_\mathsf{B}} \otimes id_{b_{\mathsf{B},\mathsf{B}}})] \cdot [id_m \circ (\beta \otimes id_{id_\mathsf{B}}) \circ (id_{id_\mathsf{B}} \otimes id_{b_{\mathsf{B},\mathsf{B}}})] \\ = a^{-1} \cdot [beta \circ (id_{id_\mathsf{B}} \otimes id_m)] \cdot [id_m \circ (id_{id_\mathsf{B}} \otimes id_m) \circ b_{\mathsf{B},\mathsf{B};\mathsf{B}}] \,. \end{aligned} \tag{4.3.52}$$

These can be equivalently thought of as twisted hexagon relations for the braiding on B.

Before continuing, let us simplify to $G = 1$ and $\mathcal{C}_\mathcal{T} \simeq 2\mathsf{Vec}$. In this case the Drinfeld centre is again $\mathcal{Z}(2\mathsf{Vec}) \simeq 2\mathsf{Vec}$, now thought of as a braided fusion 2-category with trivial braiding. A a braided algebra object is a finite semi-simple braided monoidal category, a rigid braided algebra object is a braided multi-fusion category, and a connected rigid braided algebra object is a braided fusion category.

Returning to $\mathcal{Z}(2\mathsf{Vec}_G^\alpha)$, a braided algebra object has the structure of what we will call a finite semi-simple α^{-1}-twisted G-crossed braided monoidal category. Concretely, this consists of the following data [98, 121]:

- A finite semi-simple G-graded category $\mathsf{B} = \bigoplus_{g\in G} \mathsf{B}_g$.
- A G-action $\rho_{g,h} : \mathsf{B}_h \to \mathsf{B}_{{}^g h}$, with composition described by natural isomorphisms $\rho^\circ_{g,h;f}$ defined as above for all $g, h, f \in G$.
- A G-graded monoidal structure $\mathsf{B}_g \otimes \mathcal{B}_h \subseteq \mathsf{B}_{gh}$, with distributivity with respect to ρ described by natural isomorphisms $\rho^\otimes_{f;g,h}$ defined as above for all $g, h, f \in G$.
- A G-crossed braiding implemented by natural isomorphisms $\beta_{X,Y} : X \otimes Y \to \rho_{g,h}(Y) \otimes X$ for every pair $X \in \mathsf{B}_g$ and $Y \in \mathsf{B}_h$, satisfying twisted hexagon relations determined from equations (4.3.51) and (4.3.52).

Similarly, a rigid braided algebra object is an α^{-1}-twisted G-crossed braided multi-fusion category, and a connected rigid braided algebra object is an α^{-1}-twisted G-crossed braided multi-fusion category with ρ defining a transitive G-action upon restriction to the trivially graded component B_e as a braided multi-fusion category [122].

It is worth belabouring that final statement: restricting to rigid braided algebra objects such that $\mathrm{End}(i) \simeq \mathbb{C}$ does not impose that the identity object in B is simple. Instead we define in tandem a notion of *strongly* connected to be a (connected) rigid braided algebra object such that the unit $i : 1_e \to \mathsf{B}$ defines a simple identity object in B. In this way, strongly connected rigid braided algebra objects in $\mathcal{Z}(2\mathsf{Vec}_G^\alpha)$ correspond to α^{-1}-twisted G-crossed braided *fusion* categories.

The Lagrangian condition can be formulated in terms of local modules. Recall in subsection 3.3.3 the locality of a module action was determined by its commutation with the braiding. Here this condition is replaced by a 2-isomorphism that implements the commuting diagram we had in figure 3.18, subject to some extra compatibility data. We say then that a connected rigid braided algebra object B is *Lagrangian* if the 2-category of local modules over B trivialises to

$$\mathrm{Mod}_{\mathsf{B}}^{loc}(A) \simeq 2\mathsf{Vec} \tag{4.3.53}$$

as a braided fusion 2-category.

For a strongly connected rigid braided algebra object, this reduces further to asking that the Müger centre of B vanishes, defined in the sense of [111]. There, the Müger centre of B is defined in terms of 1-morphisms $1_e \to \mathsf{B}$ that identify objects in the trivially graded component $X \in \mathsf{B}_e$, together with isomorphisms $\eta_{X,g} : X \to \rho_{g,e}(X)$, for every $g \in G$; these pairs (X, η_X) define objects in the G-equivariantisation B^G of B. We then ask that these morphisms are *transparent*; that is, the braiding is symmetric in the sense

$$\beta_{X,Y} = \beta_{Y,X}^{-1} \circ (\eta_{X,g} \otimes id_Y) \tag{4.3.54}$$

for all other $Y \in \mathsf{B}_g$ and $g \in \mathrm{Supp}(\mathsf{B})$, where $\mathrm{Supp}(\mathsf{B}) \subseteq G$ is the subgroup of G where B is supported. For a given object $X \in \mathsf{B}_e$, this restriction fixes an isomorphism $\eta_{X,g}$ for each $g \in \mathrm{Supp}(\mathsf{B})$. This in turn identifies the Müger centre of B with the Müger centre $\mathcal{Z}_2(\mathsf{B}_e^G)$ defined in the usual sense [9], modulo representations of $\mathrm{Supp}(G)$. Clearly, the Müger centre defined this way vanishes if

1. The grading $\mathrm{Supp}(G) = G$ is faithful.

2. The Müger centre of B_e^G is non-degenerate over $\mathrm{Rep}(G)$:

$$\mathcal{Z}_2(\mathsf{B}_e^G) \simeq \mathrm{Rep}(G)\,. \tag{4.3.55}$$

This is true if and only if B_e is a non-degenerate braided fusion category [125].

We expect that this exhausts all strongly connected Lagrangian algebra objects internal to $\mathcal{Z}(2\mathsf{Vec}_G^\alpha)$. Armed with this technology, we are now ready to classify Lagrangian algebra objects more generally.

[9]That is, the full symmetric braided subcategory of B_e^G.

For a general rigid braided algebra object B, the support $\mathrm{Supp}(\mathsf{B})$ need not be a subgroup; thought of as a G-graded multi-fusion category, the support only needs to be at least a disjoint union of subgroups. Taking into account also the action from G, we see that this union needs to include all subgroups in a conjugacy class, and in the minimal case when the object is connected and this action is transitive, we can use the orbit-stabiliser theorem to express the support as

$$\mathrm{Supp}(G) = \bigsqcup_{gH \in G/H} {}^{g}H\,, \tag{4.3.56}$$

for a given conjugacy class of subgroup $[H \subseteq G]$. Up to equivalence, the algebra object B then decomposes into $|G/H|$ fusion components described by $\alpha|^{-1}_{{}^{g}H}$-twisted ${}^{g}H$-crossed braided fusion categories for each $gH \in G/H$.

The remaining components of ρ identify each of these fusion components with one-another. The data of a connected rigid braided algebra object internal to $\mathcal{Z}(2\mathsf{Vec}_G^{\alpha})$ can hence be reduced to that of the fusion component described by a faithfully graded $\alpha|^{-1}_{H}$-twisted H-crossed braided fusion category. We note that this data is equivalent to that of a strongly connected rigid braided algebra object internal to $\mathcal{Z}(2\mathsf{Vec}_H^{\alpha|_H})$.

This simplification sets up an inclusion of strongly connected rigid braided algebra objects in $\mathcal{Z}(2\mathsf{Vec}_H^{\alpha|_H})$ as connected rigid braided algebra objects in $\mathcal{Z}(2\mathsf{Vec}_G^{\alpha})$. This in turn sets up a pull-back from module objects in $\mathcal{Z}(2\mathsf{Vec}_G^{\alpha})$ to module objects in $\mathcal{Z}(2\mathsf{Vec}_H^{\alpha|_H})$. For local module objects we expect that this functor is essentially surjective and hence that a connected algebra in $\mathcal{Z}(2\mathsf{Vec}_G^{\alpha})$ is Lagrangian only if the corresponding strongly connected algebra in $\mathcal{Z}(2\mathsf{Vec}_H^{\alpha|_H})$ is Lagrangian.

For $\alpha = 0$ and a choice of (conjugacy class of) subgroup H, Lagrangian algebra objects in $\mathcal{Z}(2\mathsf{Vec}_H^{\alpha|_H})$ are faithfully graded H-crossed braided fusion categories whose trivially graded components are non-degenerate. For a given non-degenerate braided fusion category that describes the trivially graded component $\mathsf{B}_e := \mathcal{B}$, these are also known as H-crossed braided extensions of $\mathcal{B}$, and are classified by [98, 121]:

- A H-action $\rho : H \to \mathrm{Aut}(\mathcal{B})$.
- A class $[\varphi]$ in a torsor over $H^2_{grp,\rho}(H, \mathcal{B}^\times)$, where the abelian group $\mathcal{B}^\times$ of invertible objects in $\mathcal{B}$ is thought of as a H-module.
- A class $[\psi]$ in a torsor over $H^3_{grp}(H, \mathbb{C}^\times)$.

The choice of non-degenerate braided fusion category $\mathcal{B}$ corresponds physically to the category of topological lines in a corresponding choice of $(2+1)$-dimensional TQFT with trivial $(3+1)$-dimensional bulk, this restriction ensures the coupling does not affect the symmetry TFT of the original theory. The appearance of torsors over group cohomology here is once again a consequence of obstructions to constructing a H-crossed braided fusion category with trivially graded component $\mathsf{B}_e = \mathcal{B}$:

1. A choice of non-degenerate braided fusion category $\mathcal{B}$ and action ρ determines an obstruction class $[\mathcal{O}_3(\mathsf{C};\rho)] \in H^3_{grp,\rho}(H, \mathcal{B}^\times)$. This is an obstruction to constructing a H-crossed extension and needs to be trivialised via
$$\delta\varphi = \mathcal{O}_3(\mathsf{C}\,;\,\rho)\,. \tag{4.3.57}$$
Different choices of trivialisation are then distinguished by 2-cocycles and are classified up to equivalence by a torsor over $H^2_{grp,\rho}(H, \mathcal{B}^\times)$. Physically we should interpret these classes as determining a symmetry fractionalisation of H by the lines in $\mathcal{B}$ as depicted in 4.15.

2. The choice of trivialisation $[\varphi]$ further determines a second obstruction class $[\mathcal{O}_4(\mathsf{C};\rho,\varphi)] \in H^4_{grp}(H, \mathbb{C}^\times)$. This is yet another obstruction to constructing a H-crossed extension that needs to be trivialised via
$$\delta\psi = \mathcal{O}_4(\mathsf{C}\,;\,\rho,\,\psi)\,. \tag{4.3.58}$$
Different choices of trivialisation are then distinguished by 3-cocycles and are classified up to equivalence by a torsor over $H^3_{grp}(H, \mathcal{Z}(\mathsf{C})^\times)$. Physically we should interpret these classes as determining a discrete torsion for H.

In analogy to subsection 4.3.1, for $\alpha \neq 0$, we expect to shift the $\mathcal{O}_4$ obstruction to

$$\delta\psi = \alpha|_H^{-1}\, \mathcal{O}_4(\mathsf{C}\,;\, \rho,\, \varphi)\,. \tag{4.3.59}$$

Once again the appearance of the $\mathcal{O}_4$ obstruction suggests that rather than gauging only strictly non-anomalous subgroups, we can in fact gauge any subgroup, provided its anomaly can be cancelled by that of the TQFT we couple to.

Unitarity

In the unitary setting we expect the symmetry on the Dirichlet boundary $\mathcal{D}$ to be described by the unitary fusion 2-category

$$\mathcal{C}_{\mathcal{T}} \simeq 2\mathsf{Hilb}_G^{\alpha}\,. \tag{4.3.60}$$

We further expect the bulk symmetry of the Dijkgraaf-Witten theory to be described by a unitary braided fusion 2-category in the Drinfeld centre

$$\mathcal{Z}(2\mathsf{Hilb}_G^{\alpha})\,, \tag{4.3.61}$$

and for gapped boundary conditions to correspond to Lagrangian algebras objects internal to the Drinfeld centre. We might have then expected that lifting the construction we have presented would only require exchanging braided fusion categories for unitary braided fusion categories, but there is a subtle complication. There is a non-trivial class of anomalies for unitary oriented TQFTs in $2+1$ dimensions, called the chiral central charge, that can be phrased as a (3+1)-dimensional symmetry TFT constructed from the signature. The signature of a given 4-manifold is a topological invariant that can be understood as a characteristic homomorphism

$$\sigma_{M_4} : \Omega_4^X \to \mathbb{Z}\,, \tag{4.3.62}$$

from the bordism group with tangential structure X to the integers. In the framed setting we have $\Omega_4^{fr} = 1$, and hence the signature on the corresponding framed 4-manifold can only ever be trivial. In the oriented setting we instead have $\Omega_4^{SO} = \mathbb{Z}$

and the possibility for non-trivial signatures of oriented 4-manifolds. We can hence define a non-trivial $(3+1)$-dimensional TFT with action

$$e^{\frac{2\pi i c}{8}\sigma_{M_4}}\,, \tag{4.3.63}$$

where c is the chiral central charge, which controls the anomaly of the $(2+1)$-dimensional topological boundary. The value of this chiral central charge should be a property of the corresponding unitary braided fusion category, and so if we wish to couple a theory $\mathcal{T}$ to a $(2+1)$-dimensional unitary TQFT without changing the symmetry TFT, we must ask that in addition to non-degeneracy, the unitary braided fusion category corresponding to that TQFT also has chiral central charge $c = 0 \pmod 8$ [10]. For more details on these anomalies, also for more general choices of tangential structure, one can turn to [126–129].

To summarise, the gaugings of G corresponding to gapped boundary conditions of $(3+1)$-dimensional (G,α) Dijkgraaf-Witten theory are classified by:

- A (conjugacy class of) subgroup $H \subseteq G$, corresponding to a choice of subgroup to gauge.
- A (equivalence class of) non-degenerate braided-fusion category $\mathcal{B}$, corresponding to a TQFT with line defects described by objects of $\mathcal{B}$.
- An action $\rho : H \to (Aut)(\mathcal{B})$, corresponding to H-defects wrapping lines, as depicted in figure 4.14.
- A class of 2-cochain $\varphi \in C^2_{grp,\rho}(H,\mathcal{B}^\times)$ satisfying $\delta\varphi = \mathcal{O}_3(\mathcal{B};\rho)$, corresponding to a symmetry fractionalisation of H by $\mathcal{B}^\times$, as depicted in figure 4.15.
- A class of 3-cochain $\psi \in C^3_{grp}(H,\mathbb{C}^\times)$ satisfying $\delta\psi = \alpha|_H^{-1}\,\mathcal{O}_4(\mathcal{B};\rho,\varphi)$, corresponding to a choice of discrete torsion for H.

[10] Upon restriction to unitary Turaev-Viro TQFTs, this constraint is automatic, the reverse is not true however.

Restricting to Turaev-Viro TQFTs $\mathcal{B} \simeq \mathcal{Z}(\mathsf{C})$ reproduces the classification in terms of projective 3-representations valued in $3\mathsf{Vec}$ seen in the previous two subsections. Restricting further to invertible TQFTs by choosing $\mathsf{C} \simeq \mathsf{Vec}$, the trivialisation (4.3.59) reduces to (4.2.58) and the classification reduces to that studied in subsection 4.2.2.

4.3.4 Projective 3-Representations

In the last subsection we gave a comprehensive classification of gapped boundaries for Dijkgraaf-Witten theories in $3+1$ dimensions, but at the outset of this section we claimed that these can alternatively be viewed as projective 3-representations valued in a particular fusion 3-category of non-degenerate braided fusion categories.

We did outline how the Lagrangian algebras correspond to projective 3-characters, but to understand this statement concretely, let us now return to the functorial description presented in subsection 4.3.2, with a slight modification.

The starting point is α^{-1}-projective monoidal functors

$$\mathcal{R} : \mathbf{B}G \to \mathrm{Pic}\,, \tag{4.3.64}$$

where we have suggestively denoted the target 3-category Pic in analogy to the Picard 2-group [121]. Rather than define the full structure, we restrict our focus to the invertible structure we expect it to have:

- The simple objects $\mathcal{B} \in \mathrm{Pic}$ are (equivalence classes of) non-degenerate braided fusion categories, that is, braided multi-fusion categories admitting a decomposition into non-degenerate braided fusion categories.

- The 1-automorphisms of an object $\mathcal{B}$ are captured by objects of the Picard 2-group $\mathrm{Pic}(\mathcal{B})$, which are labelled up to equivalence by $\mathrm{Aut}(\mathcal{B})$, the group of braided automorphisms of $\mathcal{B}$ [11].

[11] This equivalence only holds if $\mathcal{B}$ is non-degenerate

- The 2-automorphisms are given by 1-morphisms in the Picard 2-group $\mathrm{Pic}(\mathcal{B})$, which are labelled up to equivalence by $\mathcal{B}^\times$, the group of invertible objects in $\mathcal{B}$.

- The 3-automorphisms of a given 2-automorphism are natural isomorphisms of bimodule equivalences, which are labelled up to equivalence by $(\mathbb{C}^\times)^{|\mathbb{C}|}$, where $|\mathcal{B}|$ is the number of fusion summands in $\mathcal{B}$.

For a choice of non-degenerate braided multi-fusion category $\mathcal{B}$, we then formulate the remaining data of $\mathcal{R} : \star \mapsto \mathcal{B}$ as a homomorphism between 3-groups

$$\lambda : \widetilde{\mathsf{G}} \longrightarrow \mathsf{Aut}(\mathcal{B})\,. \tag{4.3.65}$$

The domain and codomain have the following descriptions:

- The domain $\widetilde{\mathsf{G}}$ is a 3-group extension with homotopy groups

$$\begin{aligned} \pi_0 &= G \\ \pi_1 &= 1 \\ \pi_2 &= \mathbb{C}^\times \end{aligned} \tag{4.3.66}$$

  and Postnikov invariant $\alpha^{-1} \in H^4_{grp}(G, U(1))$.

- The codomain $\mathsf{Aut}(\mathcal{B})$ is the automorphism 3-group of $\mathcal{B}$ with

$$\begin{aligned} \pi_0 &= \mathrm{Aut}(\mathcal{B}) \\ \pi_1 &= \mathcal{B}^\times \\ \pi_2 &= (\mathbb{C}^\times)^{|\mathcal{B}|}\,. \end{aligned} \tag{4.3.67}$$

  There is potential Postnikov data given by

$$\begin{aligned} [\mathcal{O}_3] &\in H^3_{grp}(\mathrm{Aut}(\mathcal{B})), \mathcal{B}^\times)\,, \\ [\mathcal{O}_4] &\in H^4_{grp}(\pi_{\leq 1}, (\mathbb{C}^\times)^{|\mathcal{B}|})\,, \end{aligned} \tag{4.3.68}$$

  where $\pi_{\leq 1}$ is shorthand for the 2-group truncation of $\mathsf{Aut}(\mathcal{B})$, determined by $\mathrm{Aut}(\mathcal{B}), \mathcal{B}^\times$, and $[\mathcal{O}_3]$.

An α^{-1}-projective 3-representation on $\mathcal{B}$ is then determined by:

1. A homomorphism $\widetilde{\rho} : G \to \mathrm{Aut}(\mathcal{B})$.

2. A 2-cochain $\widetilde{\varphi} \in C^2_{grp,\widetilde{\rho}}(G, \mathcal{B}^\times)$ satisfying $\delta\widetilde{\varphi} = \widetilde{\rho}^*\mathcal{O}_3$.

3. A 3-cochain $\widetilde{\psi} \in C^3_{grp}(G, (\mathbb{C}^\times)^{|\mathcal{B}|})$ satisfying $\delta\widetilde{\psi} = \alpha^{-1}\ (\widetilde{\rho}, \widetilde{\varphi})^*\mathcal{O}_4$.

where we view the combination $(\widetilde{\rho}, \widetilde{\varphi}) : BG \to \pi_{\leq 1}$.

To restrict to simple projective 3-representations we ask that $(\widetilde{\rho}, \widetilde{\varphi}) : G \to \mathrm{Aut}(\mathcal{B})$ determines a transitive action on $\mathcal{B}$, which via the orbit stabiliser theorem corresponds to picking a conjugacy class of subgroup $[H \subseteq G]$ and setting

$$\mathcal{B} \simeq \mathsf{b}^{\oplus G/H}\,, \tag{4.3.69}$$

for some non-degenerate braided fusion category b, together with a choice of $\rho : H \to \mathrm{Aut}(\mathsf{b})$. Using the Shapiro isomorphism we then surmise that a simple α^{-1}-projective 3-representation of G is then determined by:

1. A (conjugacy class of) subgroup $H \subseteq G$.

2. A non-degenerate braided fusion category b.

3. A homomorphism $\rho : H \to \mathrm{Aut}(\mathsf{b})$.

4. A 2-cochain $\varphi \in C^2_{grp,\rho}(H, \mathsf{b}^\times)$ satisfying $\delta\varphi = [\widetilde{\rho}^*\mathcal{O}_3]_H$.

5. A 3-cochain $\psi \in C^3_{grp}(H, (\mathbb{C}^\times))$ satisfying $\delta\psi = \alpha|_H^{-1}\ [(\widetilde{\rho}, \widetilde{\varphi})^*\mathcal{O}_4]_H$.

This reconstructs the classification from the previous subsection and demonstrates concretely that gapped boundaries of a (G, α) Dijkgraaf-Witten theory are in 1-1 correspondence with α^{-1}-projective representations of G valued in Pic.

Chapter 5

Future Work and Research Directions

In this chapter we will briefly comment on some future directions this research could take. These comments are largely focused on extending this formalism to higher dimensions and as a result will be mostly speculative, but before that we will attempt to be a bit more concrete and turn to ideas that directly generalise those seen in sections 2.4 and 3.4 to $2+1$ dimensions.

5.1 Defects after Generalised Gauging in $2+1$ Dimensions

One obvious direction for future research is the fusion 2-category of defects produced after the generalised gauging of a finite invertible symmetry in $2+1$ dimensions.

Recall that in section 4.3 we had two classifications of gauging that did not totally agree; the algebra and module pictures discussed in subsections 4.3.1 and 4.3.2 produced gaugings of a finite group coupled to a Turaev-Viro TQFT, whereas the symmetry TFT perspective presented in subsection 4.3.3 was more general in that it extended this classification to all (Reshetikhin-Turaev) TQFTs.

This restriction in the algebra construction of gauge theories means the corresponding bimodule construction of defects after gauging we presented in section 4.2 only extends to those gauge theories produced by coupling to a Turaev-Viro TQFT. Concretely, starting from a $(2+1)$-dimensional theory $\mathcal{T}$ with a (G, α) symmetry we should consider (2-)bimodule objects over (rigid) algebra objects internal to

$$\mathcal{C}_{\mathcal{T}} \simeq 2\mathsf{Vec}_G^{\alpha} \,. \tag{5.1.1}$$

Choosing a gauging labelled by an α^{-1}-projective 2-representation $\lambda = (H, \mathsf{C}; \rho, \varphi, \psi)$ with corresponding rigid algebra object A_λ as described in subsection 4.3.2, we denote the resulting fusion 2-category of defects

$$\mathsf{C}(G, \alpha|\lambda) \simeq \mathrm{Bimod}_{\mathcal{C}_{\mathcal{T}}}(A_\lambda) \,. \tag{5.1.2}$$

If we wish to discuss the defects of gauge theories produced by coupling to more general TQFTs, we need a generalisation of the symmetry TFT construction presented in subsections 2.4.2 and 3.4.2 to $2+1$ dimensions. Concretely, starting from a $(3+1)$-dimensional (G, α) Dijkgraaf-Witten theory, we should consider (2-)module objects over Lagrangian algebra objects internal to the Drinfeld centre

$$\mathsf{DW}_{G,\alpha} \simeq \mathcal{Z}(2\mathsf{Vec}_G^{\alpha}) \,. \tag{5.1.3}$$

Choosing a gauging labelled by an α^{-1}-projective 2-representation $\lambda = (H, \mathcal{B}; \rho, \varphi, \psi)$ with corresponding Lagrangian algebra object L_λ as described in subsection 4.3.3, we denote the resulting fusion 2-category of defects in a similar fashion as

$$\mathsf{C}(G, \alpha|\lambda) \simeq \mathrm{Mod}_{\mathsf{DW}_{G,\alpha}}(L_\lambda) \,. \tag{5.1.4}$$

In analogy to section 3.4 we expect that when these two classifications coincide on Turaev-Viro TQFTs $\mathcal{Z}(\mathsf{C}) \simeq \mathcal{B}$, the corresponding Lagrangian algebra object should generalise the full centre construction seen in subsection 3.3.3 to $2+1$ dimensions

$$L_\lambda \simeq \mathcal{Z}(A_\lambda) \,, \tag{5.1.5}$$

and that these fusion 2-categories coincide, generalising equation (3.4.9):

$$\mathrm{Mod}_{\mathsf{DW}_{G,\alpha}}(\mathcal{Z}(A_\lambda)) \simeq \mathrm{Bimod}_{\mathcal{C}_\mathcal{T}}(A_\lambda)\,. \tag{5.1.6}$$

5.1.1 Defects From Extensions

To be a little more concrete, we can try to build off of the results of section 4.2. There we studied the defects of gauge theories produced by coupling to strictly invertible TQFTs corresponding in our new notation to $\mathcal{B} \simeq \mathsf{Vec}$. In this setting we can write the fusion 2-category $\mathsf{C}(G,\alpha|H,\psi)$ for a normal subgroup $H \trianglelefteq G$ as an extension

$$2\mathrm{Rep}(H) \to \mathsf{C}(G,\alpha|H,\psi) \to 2\mathsf{Vec}_K^{\alpha|_K}\,, \tag{5.1.7}$$

where $K \simeq G/H \simeq H\backslash G/H$. Here it is useful to once again note the equivalence that identified 2-representations as condensations:

$$2\mathrm{Rep}(H) \simeq \mathrm{Mod}(\mathrm{Rep}(H)) \simeq \mathrm{Mod}(\mathsf{Vec}^H)\,, \tag{5.1.8}$$

where in the last equivalence we have utilised that $\mathrm{Rep}(H)$ is the H-equivariantisation (defined in the same way as subsection 4.3.3) of Vec, thought of in particular as a H-crossed braided fusion category with trivial grading and H-action.

Maintaining our assumption that $H \trianglelefteq G$, but now allowing more general non-degenerate braided fusion categories $\mathcal{B}$, we still expect the fusion 2-category of defects to describe an extension

$$X_\lambda \to \mathsf{C}(G,\alpha|\lambda) \to 2\mathsf{Vec}_K^{\alpha|_K}\,, \tag{5.1.9}$$

for some appropriate fusion 2-category X_λ. Given a H-crossed braided extension B_λ of $\mathcal{B}$, we might then expect that this mystery fusion 2-category is built in a similar fashion to equation (5.1.8), by module categories over the H-equivariantisation

$$X_\lambda \simeq \mathrm{Mod}(\mathsf{B}_\lambda^H)\,. \tag{5.1.10}$$

In this way, the restriction to invertible TQFTs reproduces the extension (5.1.6).

5.2 Symmetries from Gauging in $3+1$ Dimensions

Another clear direction for future research is to try and use the notion of 3-representations detailed in subsections 4.3.3 and 4.3.4 to describe defects in $3+1$ dimensions.

Gauging G With Trivial Anomaly

Starting from a $(3+1)$ dimensional theory $\mathcal{T}$ with a non-anomalous finite G symmetry, one perspective we can take is analogous to those described in sections 3.2 and 4.2, that the theory resulting from gauging all G will contain, at the very least, topological Wilson lines labelled by representations of G. We can then construct higher dimensional defects as condensations:

1. A trivial surface defect supports a localised $\text{Rep}(G)$ symmetry generated by the Wilson lines. Un-gauging this produces a surface supporting a localised non-anomalous 0-form G symmetry. We can then produce more general surface defect by the various $(1+1)$-dimensional gaugings of G, labelled by 2-representations of G.

2. We see then that a trivial volume defect supports a localised $2\text{Rep}(G)$ symmetry. Un-gauging on this produces a volume defect supporting a localised non-anomalous 0-form G symmetry. Similarly to before, we can produce more general surface defects by the various $(2+1)$-dimensional gaugings of G, labelled by 3-representations of G.

We see then, that the pure finite G gauge theory in $3+1$ dimensions has a symmetry described by $3\text{Rep}(G)$. As we have already seen though, we should be careful to specify which version of 3-representations we really mean.

To understand which notion is appropriate, we can consider folding the gauged theory $\mathcal{T}/G$ around a volume defect, producing a finite $G\times G$ gauge theory $(\mathcal{T}\times\mathcal{T}^\vee)/(G\times G)$

with a topological boundary, where we have used $\mathcal{T}^\vee$ to denote the orientation-reversal of $\mathcal{T}$. The topological sector of this particular gauge theory shares a canonical topological interface with the $(3+1)$-dimensional $(G,1)$ Dijkgraaf-Witten theory

$$\mathsf{DW}_{(G,1)} \simeq \mathcal{Z}(2\mathsf{Vec}_G)\,, \tag{5.2.1}$$

obtained by gauging the diagonal subgroup of a global $G \times G$ symmetry. As a result we can formulate a sandwich-like construction for the gapped boundaries of $(\mathcal{T} \times \mathcal{T}^\vee)/(G \times G)$ as illustrated in figure 5.1.

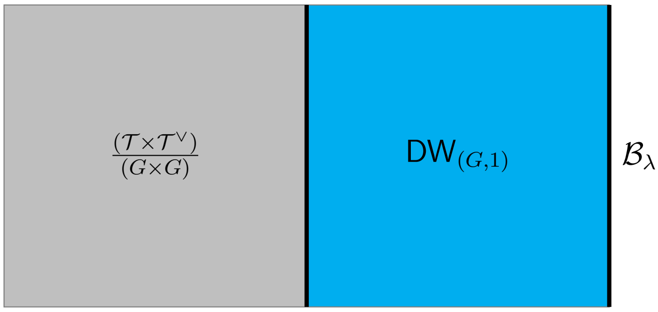

Figure 5.1

In this way, the topological boundaries of $(\mathcal{T} \times \mathcal{T}^\vee)/(G \times G)$, and hence the topological volume defects of $\mathcal{T}/G$ are reconstructed from the boundary conditions of the Dijkgraaf-Witten theory, which are labelled by 3-representations of G in the sense of subsection 4.3.3.

We might want to be careful with language however, for example, we might only want to say that those $(2+1)$-dimensional defects produced by coupling to an invertible TQFT labelled by a subgroup $H \subseteq G$ and discrete torsion $\psi \in Z^3_{grp}(H, U(1))$ can really be thought of as condensations.

In contrast, those $(2+1)$-dimensional defects produced by coupling to a more general Turaev-Viro TQFT, although they can end due to the fact that the underlying TQFT admits a gapped boundary, the TQFT itself carries extra data, making these defects slightly less-trivial and slightly more exotic than an ordinary condensation.

Even more generally, those $(2+1)$-dimensional defects produced by coupling to a general (Reshetikhin-Turaev) TQFT might not admit any gapped boundary at all, in this sense they are even less trivial and it certainly makes less sense to talk about them as condensation defects.

Gauging a Subgroup of (G, α)

More generally, a $(3+1)$-dimensional theory $\mathcal{T}$ with a finite G symmetry can have a 't Hooft anomaly determined up to equivalence by a group cohomology class

$$[\alpha] \in H^5_{grp}(G, U(1)) \,. \tag{5.2.2}$$

Then we might want to know the fusion 2-category of defects produced after gauging a non-anomalous subgroup $H \subseteq G$ with a discrete torsion corresponding to a choice of trivialisation

$$\delta\psi = \alpha|_H^{-1} \,. \tag{5.2.3}$$

This gauging is implemented by picking a (3-)algebra object A_ψ internal to

$$\mathcal{C}_{\mathcal{T}} \simeq 3\mathsf{Vec}_G^\alpha \,, \tag{5.2.4}$$

the fusion 3-category of G-graded 3-vector spaces twisted by α, where 3-vector spaces are defined in the same way as they were in subsection 4.3.2. Without discussing too much the details of these algebra objects, we expect that in analogy to sections 3.2 and 4.2, that the fusion 3-category of defects produced by this gauging has the structure of (3-)bimodule objects over A_ψ.

A conservative guess at the outcome of this calculation following what we have seen in lower dimensions, is that the defects populating $\mathcal{T}/_\psi H$ should be labelled by:

- A double coset $[g] \in H\backslash G/H$ with representative $g \in G$.

- A c_g-projective 3-representation of $H_g \simeq H \cup gHg^{-1}$, where the projective 4-cocycle c_g should be determined entirely from (g, α, ψ), in analogy to equation (4.2.79).

We expect that we can also see this labelling from the perspective of the sandwich construction in the previous section. Gauging the subgroup H produces, in the first instance, defects labelled by double cosets $[g] \in H\backslash G/H$. Folding the theory around one such defect produces the theory $(\mathcal{T} \times \mathcal{T}^\vee)/_{(\psi,-\psi)}(H \times H)$ with a gapped boundary corresponding to $g \in G$.

That boundary breaks the gauge symmetry to $H_g \times H_g$, and carries a localised 't Hooft anomaly that we shall denote $c_g \in Z^4_{grp}(H_g, U(1))$, the appearance of which is analogous to the inflow mechanisms mentioned in subsections 3.2.2 and 4.2.2. We hence expect to be able to attach it to the boundary of a $(3+1)$-dimensional (H_g, c_g) Dijkgraaf-Witten theory as illustrated in figure 5.2.

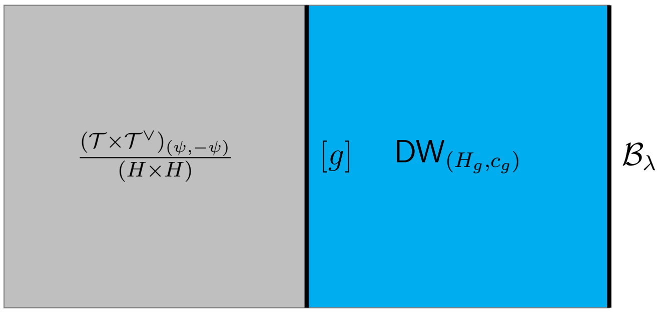

Figure 5.2

Picking the canonical Dirichlet boundary reproduces the $[g]$ defect, whereas picking different gapped boundaries thence reproduces the labelling of volume defects in $\mathcal{T}/_\psi H$ by double cosets and projective 3-representations.

5.2.1 Generalised Gauging in $3 + 1$ Dimensions

In the last section we mentioned some constructions of defects in $(3+1)$-dimensional theories where we gauged after coupling to an invertible TQFT. Another route for future research however would be to consider more general gaugings stemming from more general choices of $(3 + 1)$-dimensional TQFT.

As we have noted multiple times in this thesis however, in the setting of oriented TQFTs, we expect all of these to be equivalent to a Dijkgraaf-Witten theory for some finite group Γ and topological action determined by a 4-cocycle $\gamma \in Z^4_{grp}(\Gamma, U(1))$ [104, 105].

For our purposes this somewhat simplifies the construction of generalised gauge theories. Starting from a $(3+1)$-dimensional theory $\mathcal{T}$ with a finite (G, α) symmetry, we expect the generalised gaugings to be labelled in analogy to section 4.3, by:

- A (conjugacy class) of subgroup $H \subseteq G$.
- A $(3+1)$-dimensional (Γ, γ) Dijkgraaf-Witten Theory $\mathsf{DW}_{(\Gamma,\gamma)}$.
- Some additional data that captures how we couple $\mathsf{DW}_{(\Gamma,\gamma)}$ to the H-symmetry.

We then expect that in analogy to the $\mathcal{O}_4$ obstruction appearing in section 4.3, that this choice of coupling should in general produce a 't Hooft anomaly for the H-symmetry corresponding to some obstruction class $\mathcal{O}_5 \in Z^5_{grp}(H, U(1))$ that we require to cancel the restriction $\alpha|_H$.

We expect that a reasonable suggestion for this data should include:

- An action $\rho : H \to \mathrm{Aut}(\mathcal{Z}(2\mathsf{Vec}^{\gamma}_{\Gamma}))$.
- Symmetry fractionalisation classes

  $$\varphi^2 \in C^2_{grp,\rho}(H, \mathcal{Z}(\Gamma)) \qquad \varphi^3 \in C^3_{grp,\rho}(H, \widehat{\mathcal{Z}(\Gamma)})\,, \tag{5.2.5}$$

  where the centre $\mathcal{Z}(\Gamma)$ plays the role of invertible surface defects in $\mathcal{Z}(2\mathsf{Vec}^{\gamma}_{\Gamma})$, and its Pontryagin dual $\widehat{\mathcal{Z}(\Gamma)}$ plays the role of invertible line defects. We expect these might come with their own subsequent obstructions $\mathcal{O}_3(\Gamma, \gamma; \rho)$ and $\mathcal{O}_4(\Gamma, \gamma; \rho, \varphi^2)$, for which they solve the conditions

  $$\delta\varphi^2 = \mathcal{O}_3 \qquad \delta\varphi^3 = \mathcal{O}_4\,. \tag{5.2.6}$$

- A discrete torsion

  $$\psi \in Z^4_{grp}(H, U(1))\,, \tag{5.2.7}$$

which we expect to trivialise a twisted obstruction $\mathcal{O}_5(\Gamma, \gamma; \rho, \varphi^2, \varphi^3)$ via

$$\delta\psi = \alpha|_H^{-1}\, \mathcal{O}_5\,. \tag{5.2.8}$$

Since all of the TQFTs we are considering admit a gapped boundary, we expect the algebra, module and symmetry TFT pictures we have presented to coincide in $(3+1)$ dimensions. Further to this we expect that in analogy to subsections 2.3.1, 3.3.1 and 4.3.1, one can reconstruct the data from above for the simpler case of $\alpha = 0$ and $H \simeq G$ by considering G-extensions of fusion 2-categories, or equivalently G-crossed braided extensions of braided fusion 2-categories, and their classifications as a categorification of that seen in [98, 130].

For more general choices of subgroup and anomaly, we expect the generalisation to look similar to that we have studied in this work, in analogy to section 4.3.

Bibliography

[1] Thomas Bartsch et al. ‘Non-invertible Symmetries and Higher Representation Theory I’. In: (Aug. 2022). arXiv: 2208.05993 [hep-th].

[2] Thomas Bartsch et al. ‘Non-invertible Symmetries and Higher Representation Theory II’. In: (Dec. 2022). arXiv: 2212.07393 [hep-th].

[3] Davide Gaiotto et al. ‘Generalized Global Symmetries’. In: *JHEP* 02 (2015), p. 172. DOI: 10.1007/JHEP02(2015)172. arXiv: 1412.5148 [hep-th].

[4] Ben Heidenreich et al. ‘Non-invertible global symmetries and completeness of the spectrum’. In: *JHEP* 09 (2021), p. 203. DOI: 10.1007/JHEP09(2021)203. arXiv: 2104.07036 [hep-th].

[5] Justin Kaidi, Kantaro Ohmori and Yunqin Zheng. ‘Kramers-Wannier-like Duality Defects in (3+1)D Gauge Theories’. In: *Phys. Rev. Lett.* 128.11 (2022), p. 111601. DOI: 10.1103/PhysRevLett.128.111601. arXiv: 2111.01141 [hep-th].

[6] Yichul Choi et al. ‘Non-Invertible Duality Defects in 3+1 Dimensions’. In: (Nov. 2021). arXiv: 2111.01139 [hep-th].

[7] Konstantinos Roumpedakis, Sahand Seifnashri and Shu-Heng Shao. ‘Higher Gauging and Non-invertible Condensation Defects’. In: (Apr. 2022). arXiv: 2204.02407 [hep-th].

[8] Lakshya Bhardwaj et al. ‘Non-invertible higher-categorical symmetries’. In: *SciPost Phys.* 14.1 (2023), p. 007. DOI: 10.21468/SciPostPhys.14.1.007. arXiv: 2204.06564 [hep-th].

[9] Guillermo Arias-Tamargo and Diego Rodriguez-Gomez. 'Non-Invertible Symmetries from Discrete Gauging and Completeness of the Spectrum'. In: (Apr. 2022). arXiv: 2204.07523 [hep-th].

[10] Yichul Choi et al. 'Non-invertible Condensation, Duality, and Triality Defects in 3+1 Dimensions'. In: (Apr. 2022). arXiv: 2204.09025 [hep-th].

[11] Yichul Choi, Ho Tat Lam and Shu-Heng Shao. 'Non-invertible Global Symmetries in the Standard Model'. In: (May 2022). arXiv: 2205.05086 [hep-th].

[12] Clay Cordova and Kantaro Ohmori. 'Non-Invertible Chiral Symmetry and Exponential Hierarchies'. In: (May 2022). arXiv: 2205.06243 [hep-th].

[13] Justin Kaidi, Gabi Zafrir and Yunqin Zheng. 'Non-Invertible Symmetries of $\mathcal{N} = 4$ SYM and Twisted Compactification'. In: (May 2022). arXiv: 2205.01104 [hep-th].

[14] Andrea Antinucci, Giovanni Galati and Giovanni Rizi. 'On Continuous 2-Category Symmetries and Yang-Mills Theory'. In: (June 2022). arXiv: 2206.05646 [hep-th].

[15] Vladimir Bashmakov, Michele Del Zotto and Azeem Hasan. 'On the 6d Origin of Non-invertible Symmetries in 4d'. In: (June 2022). arXiv: 2206.07073 [hep-th].

[16] Jeremias Aguilera Damia, Riccardo Argurio and Eduardo Garcia-Valdecasas. 'Non-Invertible Defects in 5d, Boundaries and Holography'. In: (July 2022). arXiv: 2207.02831 [hep-th].

[17] Iñaki García Etxebarria and Nabil Iqbal. 'A Goldstone theorem for continuous non-invertible symmetries'. In: (Nov. 2022). arXiv: 2211.09570 [hep-th].

[18] Jurg Frohlich et al. 'Defect lines, dualities, and generalised orbifolds'. In: *16th International Congress on Mathematical Physics*. Sept. 2009. DOI: 10.1142/9789814304634_0056. arXiv: 0909.5013 [math-ph].

[19] Nils Carqueville and Ingo Runkel. ‘Orbifold completion of defect bicategories’. In: *Quantum Topol.* 7 (2016), p. 203. DOI: 10.4171/QT/76. arXiv: 1210.6363 [math.QA].

[20] Ilka Brunner, Nils Carqueville and Daniel Plencner. ‘A quick guide to defect orbifolds’. In: *Proc. Symp. Pure Math.* 88 (2014). Ed. by Ron Donagi et al., pp. 231–242. DOI: 10.1090/pspum/088/01456. arXiv: 1310.0062 [hep-th].

[21] Lakshya Bhardwaj and Yuji Tachikawa. ‘On finite symmetries and their gauging in two dimensions’. In: *JHEP* 03 (2018), p. 189. DOI: 10.1007/JHEP03(2018)189. arXiv: 1704.02330 [hep-th].

[22] Lakshya Bhardwaj, Sakura Schafer-Nameki and Jingxiang Wu. ‘Universal Non-Invertible Symmetries’. In: *Fortsch. Phys.* 70.11 (2022), p. 2200143. DOI: 10.1002/prop.202200143. arXiv: 2208.05973 [hep-th].

[23] Lakshya Bhardwaj, Sakura Schafer-Nameki and Apoorv Tiwari. ‘Unifying Constructions of Non-Invertible Symmetries’. In: (Dec. 2022). arXiv: 2212.06159 [hep-th].

[24] David Penneys. *Unitary dual functors for unitary multitensor categories.* 2018. arXiv: 1808.00323 [math.QA]. URL: https://arxiv.org/abs/1808.00323.

[25] Giovanni Ferrer et al. ‘Dagger n-categories’. In: (Mar. 2024). arXiv: 2403.01651 [math.CT].

[26] Daniel S. Freed and Constantin Teleman. ‘Relative quantum field theory’. In: *Commun. Math. Phys.* 326 (2014), pp. 459–476. DOI: 10.1007/s00220-013-1880-1. arXiv: 1212.1692 [hep-th].

[27] Daniel S. Freed, Gregory W. Moore and Constantin Teleman. ‘Topological symmetry in quantum field theory’. In: (Sept. 2022). arXiv: 2209.07471 [hep-th].

[28] Daniel S. Freed. ‘Introduction to topological symmetry in QFT’. In: (Nov. 2022). arXiv: 2212.00195 [hep-th].

[29] Davide Gaiotto and Justin Kulp. 'Orbifold groupoids'. In: *JHEP* 02 (2021), p. 132. DOI: 10.1007/JHEP02(2021)132. arXiv: 2008.05960 [hep-th].

[30] Daniel S. Freed and Constantin Teleman. 'Topological dualities in the Ising model'. In: *Geom. Topol.* 26 (2022), pp. 1907–1984. DOI: 10.2140/gt.2022.26.1907. arXiv: 1806.00008 [math.AT].

[31] Fabio Apruzzi et al. 'Symmetry TFTs from String Theory'. In: *Commun. Math. Phys.* 402.1 (2023), pp. 895–949. DOI: 10.1007/s00220-023-04737-2. arXiv: 2112.02092 [hep-th].

[32] Fabio Apruzzi. 'Higher form symmetries TFT in 6d'. In: *JHEP* 11 (2022), p. 050. DOI: 10.1007/JHEP11(2022)050. arXiv: 2203.10063 [hep-th].

[33] T. Daniel Brennan and Zhengdi Sun. 'A SymTFT for Continuous Symmetries'. In: (Jan. 2024). arXiv: 2401.06128 [hep-th].

[34] Justin Kaidi, Kantaro Ohmori and Yunqin Zheng. 'Symmetry TFTs for Non-Invertible Defects'. In: (Sept. 2022). arXiv: 2209.11062 [hep-th].

[35] Yuji Tachikawa. 'On gauging finite subgroups'. In: *SciPost Phys.* 8.1 (2020), p. 015. DOI: 10.21468/SciPostPhys.8.1.015. arXiv: 1712.09542 [hep-th].

[36] Bruce H. Bartlett. *Categorical Aspects of Topological Quantum Field Theories.* 2005. arXiv: math/0512103 [math.QA]. URL: https://arxiv.org/abs/math/0512103.

[37] John C. Baez. 'Quantum quandaries: A Category theoretic perspective'. In: (Apr. 2004). Ed. by Dean Rickles, Steven French and Juha Saatsi, pp. 240–265. arXiv: quant-ph/0404040.

[38] Lakshya Bhardwaj et al. 'Lectures on generalized symmetries'. In: *Phys. Rept.* 1051 (2024), pp. 1–87. DOI: 10.1016/j.physrep.2023.11.002. arXiv: 2307.07547 [hep-th].

[39] K. Hori et al. *Mirror symmetry.* Vol. 1. Clay mathematics monographs. Providence, USA: AMS, 2003.

[40] Daniel S. Freed and Michael J. Hopkins. 'Reflection positivity and invertible topological phases'. In: *Geom. Topol.* 25 (2021), pp. 1165–1330. DOI: 10.2140/gt.2021.25.1165. arXiv: 1604.06527 [hep-th].

[41] Ernst Ising. 'Contribution to the Theory of Ferromagnetism'. In: *Z. Phys.* 31 (1925), pp. 253–258. DOI: 10.1007/BF02980577.

[42] Lars Onsager. 'Crystal statistics. 1. A Two-dimensional model with an order disorder transition'. In: *Phys. Rev.* 65 (1944), pp. 117–149. DOI: 10.1103/PhysRev.65.117.

[43] C. H. Chang. 'The Spontaneous Magnetization of a Two-Dimensional Rectangular Ising Model'. In: *Phys. Rev.* 88 (1952), pp. 1422–1422. DOI: 10.1103/PhysRev.88.1422.

[44] David Aasen, Roger S. K. Mong and Paul Fendley. 'Topological Defects on the Lattice I: The Ising model'. In: *J. Phys. A* 49.35 (2016), p. 354001. DOI: 10.1088/1751-8113/49/35/354001. arXiv: 1601.07185 [cond-mat.stat-mech].

[45] Ryan Thorngren and Yifan Wang. 'Fusion Category Symmetry I: Anomaly In-Flow and Gapped Phases'. In: (Dec. 2019). arXiv: 1912.02817 [hep-th].

[46] Chi-Ming Chang et al. 'Topological Defect Lines and Renormalization Group Flows in Two Dimensions'. In: *JHEP* 01 (2019), p. 026. DOI: 10.1007/JHEP01(2019)026. arXiv: 1802.04445 [hep-th].

[47] Jürgen Fuchs et al. 'Algebraic structures in two-dimensional conformal field theory'. In: May 2023. arXiv: 2305.02773 [math.QA].

[48] Erik P. Verlinde. 'Fusion Rules and Modular Transformations in 2D Conformal Field Theory'. In: *Nucl. Phys. B* 300 (1988), pp. 360–376. DOI: 10.1016/0550-3213(88)90603-7.

[49] S. C. Park. *Massive Scaling Limit of the Ising Model: Subcritical Analysis and Isomonodromy*. 2019. arXiv: 1811.06636 [math.PR]. URL: https://arxiv.org/abs/1811.06636.

[50] Theo Johnson-Freyd and Matthew Yu. 'Fusion 2-categories With no Line Operators are Grouplike'. In: *Bull. Austral. Math. Soc.* 104.3 (2021), pp. 434–442. DOI: 10.1017/S0004972721000095. arXiv: 2010.07950 [math.QA].

[51] Thibault D. Décoppet and Matthew Yu. 'Fiber 2-Functors and Tambara-Yamagami Fusion 2-Categories'. In: (June 2023). arXiv: 2306.08117 [math.CT].

[52] Zhengdi Sun and Yunqin Zheng. 'When are Duality Defects Group-Theoretical?' In: (July 2023). arXiv: 2307.14428 [hep-th].

[53] Mathew Bullimore and Jamie Pearson. *Toward All Categorical Symmetries in 2+1 Dimensions*. 2024.

[54] Christopher L. Douglas and David J. Reutter. 'Fusion 2-categories and a state-sum invariant for 4-manifolds'. In: (Dec. 2018). arXiv: 1812.11933 [math.QA].

[55] John C. Baez. *An Introduction to n-Categories*. 1997. arXiv: q-alg/9705009 [q-alg]. URL: https://arxiv.org/abs/q-alg/9705009.

[56] Julia E Bergner et al. '2–Segal objects and the Waldhausen construction'. In: *Algebraic and Geometric Topology* 21.3 (Aug. 2021), 1267–1326. DOI: 10.2140/agt.2021.21.1267. URL: http://dx.doi.org/10.2140/agt.2021.21.1267.

[57] M. Atiyah. 'Topological quantum field theories'. In: *Inst. Hautes Etudes Sci. Publ. Math.* 68 (1989), pp. 175–186. DOI: 10.1007/BF02698547.

[58] J. C. Baez and J. Dolan. 'Higher dimensional algebra and topological quantum field theory'. In: *J. Math. Phys.* 36 (1995), pp. 6073–6105. DOI: 10.1063/1.531236. arXiv: q-alg/9503002.

[59] Jacob Lurie. *On the Classification of Topological Field Theories*. 2009. arXiv: 0905.0465 [math.CT]. URL: https://arxiv.org/abs/0905.0465.

[60] John W. Barrett and Bruce W. Westbury. 'Spherical categories'. In: *Adv. Math.* 143 (1999), pp. 357–375. DOI: 10.1006/aima.1998.1800. arXiv: hep-th/9310164.

[61] John W. Barrett and Bruce W. Westbury. 'Invariants of piecewise linear three manifolds'. In: *Trans. Am. Math. Soc.* 348 (1996), pp. 3997–4022. DOI: 10.1090/S0002-9947-96-01660-1. arXiv: hep-th/9311155.

[62] Alexander Kirillov Jr. au2 and Benjamin Balsam. *Turaev-Viro invariants as an extended TQFT*. 2010. arXiv: 1004.1533 [math.GT]. URL: https://arxiv.org/abs/1004.1533.

[63] César Galindo. *On braided and ribbon unitary fusion categories*. 2013. arXiv: 1209.2022 [math.QA]. URL: https://arxiv.org/abs/1209.2022.

[64] N. Reshetikhin and V. G. Turaev. 'Invariants of three manifolds via link polynomials and quantum groups'. In: *Invent. Math.* 103 (1991), pp. 547–597. DOI: 10.1007/BF01239527.

[65] VLADIMIR G. TURAEV. 'MODULAR CATEGORIES AND 3-MANIFOLD INVARIANTS'. In: *International Journal of Modern Physics B* 06.11n12 (1992), pp. 1807–1824. DOI: 10.1142/S0217979292000876. eprint: https://doi.org/10.1142/S0217979292000876. URL: https://doi.org/10.1142/S0217979292000876.

[66] Frank Wilczek. 'Quantum Mechanics of Fractional Spin Particles'. In: *Phys. Rev. Lett.* 49 (1982), pp. 957–959. DOI: 10.1103/PhysRevLett.49.957.

[67] D. Arovas, J. R. Schrieffer and Frank Wilczek. 'Fractional Statistics and the Quantum Hall Effect'. In: *Phys. Rev. Lett.* 53 (1984), pp. 722–723. DOI: 10.1103/PhysRevLett.53.722.

[68] Gregory W. Moore and N. Read. 'Nonabelions in the fractional quantum Hall effect'. In: *Nucl. Phys. B* 360 (1991), pp. 362–396. DOI: 10.1016/0550-3213(91)90407-O.

[69] Alexei Kitaev. 'Anyons in an exactly solved model and beyond'. In: *Annals Phys.* 321.1 (2006), pp. 2–111. DOI: 10.1016/j.aop.2005.10.005. arXiv: cond-mat/0506438.

[70] Chetan Nayak et al. 'Non-Abelian anyons and topological quantum computation'. In: *Reviews of Modern Physics* 80.3 (Sept. 2008), 1083–1159. DOI: 10.1103/revmodphys.80.1083. URL: http://dx.doi.org/10.1103/RevModPhys.80.1083.

[71] Dmitriy Belov and Gregory W. Moore. 'Classification of Abelian spin Chern-Simons theories'. In: (May 2005). arXiv: hep-th/0505235.

[72] Spencer D. Stirling. 'Abelian Chern-Simons theory with toral gauge group, modular tensor categories, and group categories'. PhD thesis. Texas U., Math Dept., 2008. arXiv: 0807.2857 [hep-th].

[73] Anton Kapustin and Natalia Saulina. 'Topological boundary conditions in abelian Chern-Simons theory'. In: *Nucl. Phys. B* 845 (2011), pp. 393–435. DOI: 10.1016/j.nuclphysb.2010.12.017. arXiv: 1008.0654 [hep-th].

[74] Francesco Benini, Clay Córdova and Po-Shen Hsin. 'On 2-Group Global Symmetries and their Anomalies'. In: *JHEP* 03 (2019), p. 118. DOI: 10.1007/JHEP03(2019)118. arXiv: 1803.09336 [hep-th].

[75] Robbert Dijkgraaf et al. 'The Operator Algebra of Orbifold Models'. In: *Commun. Math. Phys.* 123 (1989), p. 485. DOI: 10.1007/BF01238812.

[76] Robbert Dijkgraaf and Edward Witten. 'Topological Gauge Theories and Group Cohomology'. In: *Commun. Math. Phys.* 129 (1990), p. 393. DOI: 10.1007/BF02096988.

[77] Gregory W. Moore and Nathan Seiberg. 'Classical and Quantum Conformal Field Theory'. In: *Commun. Math. Phys.* 123 (1989), p. 177. DOI: 10.1007/BF01238857.

[78] Jurg Frohlich et al. 'Duality and defects in rational conformal field theory'. In: *Nucl. Phys. B* 763 (2007), pp. 354–430. DOI: 10.1016/j.nuclphysb.2006.11.017. arXiv: hep-th/0607247.

[79] Jurg Frohlich et al. 'Kramers-Wannier duality from conformal defects'. In: *Phys. Rev. Lett.* 93 (2004), p. 070601. DOI: 10.1103/PhysRevLett.93.070601. arXiv: cond-mat/0404051.

[80] Lance J. Dixon et al. 'Strings on Orbifolds'. In: *Nucl. Phys. B* 261 (1985). Ed. by B. Schellekens, pp. 678–686. DOI: 10.1016/0550-3213(85)90593-0.

[81] Lance J. Dixon et al. 'Strings on Orbifolds. 2.' In: *Nucl. Phys. B* 274 (1986), pp. 285–314. DOI: 10.1016/0550-3213(86)90287-7.

[82] Daisuke Tambara and Shigeru Yamagami. 'Tensor categories with fusion rules of self-duality for finite abelian groups'. In: *Journal of Algebra* 209.2 (1998), pp. 692–707.

[83] Alexei Davydov and Darren Simmons. 'On Lagrangian algebras in group-theoretical braided fusion categories'. In: *Journal of Algebra* 471 (2017), pp. 149–175. DOI: https://doi.org/10.1016/j.jalgebra.2016.09.016. URL: https://www.sciencedirect.com/science/article/pii/S0021869316303246.

[84] Jurgen Fuchs, Ingo Runkel and Christoph Schweigert. 'TFT construction of RCFT correlators 1. Partition functions'. In: *Nucl. Phys. B* 646 (2002), pp. 353–497. DOI: 10.1016/S0550-3213(02)00744-7. arXiv: hep-th/0204148.

[85] Quanlin Chen and David Penneys. 'Q-system completion is a 3-functor'. In: 2021. URL: https://api.semanticscholar.org/CorpusID:235606022.

[86] Quan Chen et al. ‘Q-system completion for C∗ 2-categories’. In: *J. Funct. Anal.* 283 (2022), p. 109524. DOI: 10.1016/j.jfa.2022.109524. arXiv: 2105.12010 [math.OA].

[87] Viktor Ostrik. ‘Module categories over the Drinfeld double of a finite group’. In: *International Mathematics Research Notices* 2003.27 (Jan. 2003), pp. 1507–1520. DOI: 10.1155/S1073792803205079. eprint: https://academic.oup.com/imrn/article-pdf/2003/27/1507/1931507/2003-27-1507.pdf. URL: https://doi.org/10.1155/S1073792803205079.

[88] Shlomo Gelaki and Deepak Naidu. ‘Some properties of group-theoretical categories’. In: *Journal of Algebra* 322.8 (2009), pp. 2631–2641. DOI: https://doi.org/10.1016/j.jalgebra.2009.05.047. URL: https://www.sciencedirect.com/science/article/pii/S0021869309004165.

[89] Ryan Thorngren and Yifan Wang. ‘Fusion Category Symmetry II: Categoriosities at $c = 1$ and Beyond’. In: (June 2021). arXiv: 2106.12577 [hep-th].

[90] Nora Ganter and Mikhail Kapranov. *Representation and character theory in 2-categories*. 2007. arXiv: math/0602510 [math.KT].

[91] Nora Ganter and Robert Usher. *Representation and character theory of finite categorical groups*. 2017. arXiv: 1407.6849 [math.CT].

[92] Josep Elgueta. ‘Representation theory of 2-groups on Kapranov and Voevodsky’s 2-vector spaces’. In: *Advances in Mathematics* 213.1 (2007), pp. 53–92. DOI: https://doi.org/10.1016/j.aim.2006.11.010. URL: https://www.sciencedirect.com/science/article/pii/S0001870806003835.

[93] Angélica M. Osorno. ‘Explicit formulas for 2-characters’. In: *Topology and its Applications* 157.2 (2010), pp. 369–377. DOI:

https://doi.org/10.1016/j.topol.2009.09.005. URL: https://www.sciencedirect.com/science/article/pii/S0166864109004416.

[94] Nils Carqueville and Ingo Runkel. 'Orbifold completion of defect bicategories'. In: *Quantum Topology* 7.2 (2016), pp. 203–279. DOI: 10.4171/qt/76. URL: https://doi.org/10.4171%2Fqt%2F76.

[95] Deepak Naidu and Dmitri Nikshych. 'Lagrangian Subcategories and Braided Tensor Equivalences of Twisted Quantum Doubles of Finite Groups'. In: *Communications in Mathematical Physics* 279.3 (Feb. 2008), 845–872. DOI: 10.1007/s00220-008-0441-5. URL: http://dx.doi.org/10.1007/s00220-008-0441-5.

[96] Salman Beigi, Peter W. Shor and Daniel Whalen. 'The Quantum Double Model with Boundary: Condensations and Symmetries'. In: *Commun. Math. Phys.* 306.3 (2011), pp. 663–694. DOI: 10.1007/s00220-011-1294-x. arXiv: 1006.5479 [quant-ph].

[97] Nils Carqueville, Ingo Runkel and Gregor Schaumann. 'Orbifolds of n-dimensional defect TQFTs'. In: *Geom. Topol.* 23 (2019), pp. 781–864. DOI: 10.2140/gt.2019.23.781. arXiv: 1705.06085 [math.QA].

[98] Maissam Barkeshli et al. 'Symmetry Fractionalization, Defects, and Gauging of Topological Phases'. In: *Phys. Rev. B* 100.11 (2019), p. 115147. DOI: 10.1103/PhysRevB.100.115147. arXiv: 1410.4540 [cond-mat.str-el].

[99] Clay Córdova, Thomas T. Dumitrescu and Kenneth Intriligator. 'Exploring 2-Group Global Symmetries'. In: *JHEP* 02 (2019), p. 184. DOI: 10.1007/JHEP02(2019)184. arXiv: 1802.04790 [hep-th].

[100] Clay Cordova, Thomas T. Dumitrescu and Kenneth Intriligator. '2-Group Global Symmetries and Anomalies in Six-Dimensional Quantum Field Theories'. In: *JHEP* 04 (2021), p. 252. DOI: 10.1007/JHEP04(2021)252. arXiv: 2009.00138 [hep-th].

[101] Lakshya Bhardwaj. '2-Group symmetries in class S'. In: *SciPost Phys.* 12.5 (2022), p. 152. DOI: 10.21468/SciPostPhys.12.5.152. arXiv: 2107.06816 [hep-th].

[102] Fabio Apruzzi et al. 'The Global Form of Flavor Symmetries and 2-Group Symmetries in 5d SCFTs'. In: (May 2021). arXiv: 2105.08724 [hep-th].

[103] Anton Kapustin and Ryan Thorngren. 'Topological Field Theory on a Lattice, Discrete Theta-Angles and Confinement'. In: *Adv. Theor. Math. Phys.* 18.5 (2014), pp. 1233–1247. DOI: 10.4310/ATMP.2014.v18.n5.a4. arXiv: 1308.2926 [hep-th].

[104] Tian Lan, Liang Kong and Xiao-Gang Wen. 'Classification of (3+1)D Bosonic Topological Orders: The Case When Pointlike Excitations Are All Bosons'. In: *Phys. Rev. X* 8.2 (2018), p. 021074. DOI: 10.1103/PhysRevX.8.021074.

[105] Theo Johnson-Freyd. 'On the Classification of Topological Orders'. In: *Commun. Math. Phys.* 393.2 (2022), pp. 989–1033. DOI: 10.1007/s00220-022-04380-3. arXiv: 2003.06663 [math.CT].

[106] F. A. Bais and J. K. Slingerland. 'Condensate induced transitions between topologically ordered phases'. In: *Phys. Rev. B* 79 (2009), p. 045316. DOI: 10.1103/PhysRevB.79.045316. arXiv: 0808.0627 [cond-mat.mes-hall].

[107] F. J. Burnell. 'Anyon condensation and its applications'. In: *Ann. Rev. Condensed Matter Phys.* 9 (2018), pp. 307–327. DOI: 10.1146/annurev-conmatphys-033117-054154. arXiv: 1706.04940 [cond-mat.str-el].

[108] Liang Kong. 'Anyon condensation and tensor categories'. In: *Nucl. Phys. B* 886 (2014), pp. 436–482. DOI: 10.1016/j.nuclphysb.2014.07.003. arXiv: 1307.8244 [cond-mat.str-el].

[109] Davide Gaiotto and Theo Johnson-Freyd. 'Condensations in higher categories'. In: (May 2019). arXiv: 1905.09566 [math.CT].

[110] Thibault D. Décoppet. 'Rigid and separable algebras in fusion 2-categories'. In: *Advances in Mathematics* 419 (Apr. 2023), p. 108967. DOI: 10.1016/j.aim.2023.108967. URL: http://dx.doi.org/10.1016/j.aim.2023.108967.

[111] Theo Johnson-Freyd and David Reutter. 'Minimal nondegenerate extensions'. In: *Journal of the American Mathematical Society* 37.1 (July 2023), pp. 81–150. DOI: 10.1090/jams/1023. URL: http://dx.doi.org/10.1090/jams/1023.

[112] John W. Barrett and Marco MacKaay. 'Categorical Representations of Categorical Groups'. In: *THEORY APPL. CATEG* 16.20 (2006), pp. 529–557.

[113] Nora Ganter and Mikhail Kapranov. 'Representation and character theory in 2-categories'. In: *Advances in Mathematics* 217.5 (2008), pp. 2268–2300. DOI: https://doi.org/10.1016/j.aim.2007.10.004. URL: https://www.sciencedirect.com/science/article/pii/S0001870807003027.

[114] Bruce Bartlett. *On unitary 2-representations of finite groups and topological quantum field theory*. 2009. arXiv: 0901.3975 [math.QA].

[115] Josep Elgueta. *Representation theory of 2-groups on finite dimensional 2-vector spaces*. 2005. arXiv: math/0408120 [math.CT].

[116] Tom Banks and Nathan Seiberg. 'Symmetries and Strings in Field Theory and Gravity'. In: *Phys. Rev. D* 83 (2011), p. 084019. DOI: 10.1103/PhysRevD.83.084019. arXiv: 1011.5120 [hep-th].

[117] Tom Rudelius and Shu-Heng Shao. 'Topological Operators and Completeness of Spectrum in Discrete Gauge Theories'. In: *JHEP* 12 (2020), p. 172. DOI: 10.1007/JHEP12(2020)172. arXiv: 2006.10052 [hep-th].

[118] Po-Shen Hsin and Ho Tat Lam. 'Discrete theta angles, symmetries and anomalies'. In: *SciPost Phys.* 10.2 (2021), p. 032. DOI: 10.21468/SciPostPhys.10.2.032. arXiv: 2007.05915 [hep-th].

[119] Clay Cordova, Po-Shen Hsin and Nathan Seiberg. 'Global Symmetries, Counterterms, and Duality in Chern-Simons Matter Theories with Orthogonal Gauge Groups'. In: *SciPost Phys.* 4.4 (2018), p. 021. DOI: 10.21468/SciPostPhys.4.4.021. arXiv: 1711.10008 [hep-th].

[120] Anton Kapustin and Ryan Thorngren. 'Anomalies of discrete symmetries in three dimensions and group cohomology'. In: *Phys. Rev. Lett.* 112.23 (2014), p. 231602. DOI: 10.1103/PhysRevLett.112.231602. arXiv: 1403.0617 [hep-th].

[121] Pavel Etingof et al. *Fusion categories and homotopy theory*. 2009. arXiv: 0909.3140 [math.QA].

[122] Thibault D. Décoppet. 'Rigid and separable algebras in fusion 2-categories'. In: *Advances in Mathematics* 419 (2023), p. 108967. DOI: https://doi.org/10.1016/j.aim.2023.108967. URL: https://www.sciencedirect.com/science/article/pii/S000187082300110X.

[123] M. M. Kapranov and V. A. Voevodsky. '2-categories and Zamolodchikov tetrahedra equations'. In: *Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991).* Vol. 56. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1994, 177–259.

[124] John C. Baez and Martin Neuchl. *Higher-Dimensional Algebra I: Braided Monoidal 2-Categories*. 2020. arXiv: q-alg/9511013 [q-alg].

[125] Vladimir Drinfeld et al. *On braided fusion categories I*. 2010. arXiv: 0906.0620 [math.QA].

[126] Maissam Barkeshli et al. 'Reflection and time reversal symmetry enriched topological phases of matter: path integrals, non-orientable manifolds, and anomalies'. In: *Commun. Math. Phys.* 374.2 (2019), pp. 1021–1124. DOI: 10.1007/s00220-019-03475-8. arXiv: 1612.07792 [cond-mat.str-el].

[127] Liang Kong and Xiao-Gang Wen. 'Braided fusion categories, gravitational anomalies, and the mathematical framework for topological orders in any dimensions'. In: (May 2014). arXiv: 1405.5858 [cond-mat.str-el].

[128] Andrey Gromov et al. 'Framing Anomaly in the Effective Theory of the Fractional Quantum Hall Effect'. In: *Phys. Rev. Lett.* 114.1 (2015). [Erratum: Phys.Rev.Lett. 114, 149902 (2015)], p. 016805. DOI: 10.1103/PhysRevLett.114.149902. arXiv: 1410.6812 [cond-mat.str-el].

[129] Ryohei Kobayashi. 'Anomaly constraint on chiral central charge of (2+1)d topological order'. In: *Phys. Rev. Res.* 3.2 (2021), p. 023107. DOI: 10.1103/PhysRevResearch.3.023107. arXiv: 2101.01018 [cond-mat.str-el].

[130] Pavel Etingof et al. 'Fusion categories and homotopy theory'. In: (Sept. 2009). arXiv: 0909.3140 [math.QA].

[131] Giovanni Ferrer et al. *Dagger n-categories*. 2024. arXiv: 2403.01651 [math.CT].

[132] Lakshya Bhardwaj et al. 'Non-invertible symmetry webs'. In: *SciPost Phys.* 15.4 (2023), p. 160. DOI: 10.21468/SciPostPhys.15.4.160. arXiv: 2212.06842 [hep-th].

[133] Theo Johnson-Freyd. 'On the Classification of Topological Orders'. In: *Communications in Mathematical Physics* 393.2 (2022), pp. 989–1033. DOI: 10.1007/s00220-022-04380-3. URL: https://doi.org/10.1007/s00220-022-04380-3.

[134] THEO JOHNSON-FREYD and MATTHEW YU. 'FUSION 2-CATEGORIES WITH NO LINE OPERATORS ARE GROUPLIKE'. In: *Bulletin of the Australian Mathematical Society* 104.3 (Feb. 2021), pp. 434–442. DOI: 10.1017/s0004972721000095. URL: http://dx.doi.org/10.1017/S0004972721000095.

[135] Thibault D. D'ecoppet. 'Drinfeld Centers and Morita Equivalence Classes of Fusion 2-Categories'. In: 2022. URL: https://api.semanticscholar.org/CorpusID:256358901.

[136] Thibault D. Décoppet. 'The Morita Theory of Fusion 2-Categories'. In: *Higher Structures* 7.1 (May 2023), pp. 234–292. DOI: 10.21136/hs.2023.07. URL: http://dx.doi.org/10.21136/HS.2023.07.

[137] Quan Chen et al. 'Q-system completion for C∗ 2-categories'. In: *Journal of Functional Analysis* 283.3 (2022), p. 109524. DOI: https://doi.org/10.1016/j.jfa.2022.109524. URL: https://www.sciencedirect.com/science/article/pii/S0022123622001446.

[138] Sebastian Burciu and Sonia Natale. 'Fusion rules of equivariantizations of fusion categories'. In: *Journal of Mathematical Physics* 54.1 (Jan. 2013), p. 013511. DOI: 10.1063/1.4774293. eprint: https://pubs.aip.org/aip/jmp/article-pdf/doi/10.1063/1.4774293/15648704/013511_1_online.pdf. URL: https://doi.org/10.1063/1.4774293.

[139] Nora Ganter. 'Inner products of 2-representations'. In: *Advances in Mathematics* 285 (2015), pp. 301–351. DOI: https://doi.org/10.1016/j.aim.2015.05.015. URL: https://www.sciencedirect.com/science/article/pii/S0001870815002224.

[140] Thomas Bartsch, Mathew Bullimore and Andrea Grigoletto. 'Representation theory for categorical symmetries'. In: (May 2023). arXiv: 2305.17165 [hep-th].

[141] Gregory W Moore and Graeme Segal. 'D-branes and K-theory in 2D topological field theory'. In: *arXiv preprint hep-th/0609042* (2006).

[142] Pavel Etingof, Dmitri Nikshych and Viktor Ostrik. 'On fusion categories'. In: *Annals of mathematics* (2005), pp. 581–642.

[143] David Reutter. ‘Uniqueness of Unitary Structure for Unitarizable Fusion Categories’. In: *Communications in Mathematical Physics* 397.1 (2023), pp. 37–52. DOI: 10.1007/s00220-022-04425-7. URL: https://doi.org/10.1007/s00220-022-04425-7.

[144] Lukas Müller and Richard J. Szabo. ‘’t Hooft Anomalies of Discrete Gauge Theories and Non-abelian Group Cohomology’. In: *Commun. Math. Phys.* 375.3 (2019), pp. 1581–1627. DOI: 10.1007/s00220-019-03546-w. arXiv: 1811.05446 [hep-th].

[145] Victor Ostrik. *Module categories over the Drinfeld double of a finite group.* 2006. arXiv: math/0202130 [math.QA].

[146] Chi-Ming Chang, Jin Chen and Fengjun Xu. ‘Topological Defect Lines in Two Dimensional Fermionic CFTs’. In: (Aug. 2022). arXiv: 2208.02757 [hep-th].

[147] E. Sharpe. ‘Topological operators, noninvertible symmetries and decomposition’. In: (Aug. 2021). arXiv: 2108.13423 [hep-th].

[148] Mendel Nguyen, Yuya Tanizaki and Mithat Ünsal. ‘Noninvertible 1-form symmetry and Casimir scaling in 2D Yang-Mills theory’. In: *Phys. Rev. D* 104.6 (2021), p. 065003. DOI: 10.1103/PhysRevD.104.065003. arXiv: 2104.01824 [hep-th].

[149] Chi-Ming Chang and Ying-Hsuan Lin. ‘Lorentzian dynamics and factorization beyond rationality’. In: *JHEP* 10 (2021), p. 125. DOI: 10.1007/JHEP10(2021)125. arXiv: 2012.01429 [hep-th].

[150] Ying-Hsuan Lin and Shu-Heng Shao. ‘Duality Defect of the Monster CFT’. In: *J. Phys. A* 54.6 (2021), p. 065201. DOI: 10.1088/1751-8121/abd69e. arXiv: 1911.00042 [hep-th].

[151] Wenjie Ji, Shu-Heng Shao and Xiao-Gang Wen. ‘Topological Transition on the Conformal Manifold’. In: *Phys. Rev. Res.* 2.3 (2020), p. 033317. DOI:

10.1103/PhysRevResearch.2.033317. arXiv: 1909.01425 [cond-mat.str-el].

[152] Po-Shen Hsin, Ho Tat Lam and Nathan Seiberg. 'Comments on One-Form Global Symmetries and Their Gauging in 3d and 4d'. In: *SciPost Phys.* 6.3 (2019), p. 039. DOI: 10.21468/SciPostPhys.6.3.039. arXiv: 1812.04716 [hep-th].

[153] WEI WANG. 'ON THE 3-REPRESENTATIONS OF GROUPS AND THE 2-CATEGORICAL TRACES.' In: *Theory & Applications of Categories* 30.56 (2015).

[154] Alex Bullivant and Clement Delcamp. 'Gapped boundaries and string-like excitations in (3+1)d gauge models of topological phases'. In: *JHEP* 07 (2021), p. 025. DOI: 10.1007/JHEP07(2021)025. arXiv: 2006.06536 [cond-mat.str-el].

[155] Juven Wang, Xiao-Gang Wen and Edward Witten. 'Symmetric Gapped Interfaces of SPT and SET States: Systematic Constructions'. In: *Phys. Rev. X* 8 (3 2018), p. 031048. DOI: 10.1103/PhysRevX.8.031048. URL: https://link.aps.org/doi/10.1103/PhysRevX.8.031048.

[156] Hongyu Wang et al. 'Gapped Boundary Theory of the Twisted Gauge Theory Model of Three-Dimensional Topological Orders'. In: *JHEP* 10 (2018), p. 114. DOI: 10.1007/JHEP10(2018)114. arXiv: 1807.11083 [cond-mat.str-el].

[157] Salman Beigi, Peter W. Shor and Daniel Whalen. 'The Quantum Double Model with Boundary: Condensations and Symmetries'. In: *Communications in Mathematical Physics* 306.3 (2011), pp. 663–694. DOI: 10.1007/s00220-011-1294-x. URL: https://doi.org/10.1007/s00220-011-1294-x.

[158] Alex Bullivant, Yuting Hu and Yidun Wan. 'Twisted quantum double model of topological order with boundaries'. In: *Phys. Rev. B* 96.16 (2017), p. 165138. DOI: 10.1103/PhysRevB.96.165138. arXiv: 1706.03611 [cond-mat.str-el].

[159] Tzu-Chen Huang, Ying-Hsuan Lin and Sahand Seifnashri. 'Construction of two-dimensional topological field theories with non-invertible symmetries'. In: *JHEP* 12 (2021), p. 028. DOI: 10.1007/JHEP12(2021)028. arXiv: 2110.02958 [hep-th].

[160] Maissam Barkeshli and Meng Cheng. 'Relative Anomalies in (2+1)D Symmetry Enriched Topological States'. In: *SciPost Phys.* 8 (2020), p. 028. DOI: 10.21468/SciPostPhys.8.2.028. arXiv: 1906.10691 [cond-mat.str-el].

[161] Daniel Bulmash and Maissam Barkeshli. 'Absolute anomalies in (2+1)D symmetry-enriched topological states and exact (3+1)D constructions'. In: *Phys. Rev. Res.* 2.4 (2020), p. 043033. DOI: 10.1103/PhysRevResearch.2.043033. arXiv: 2003.11553 [cond-mat.str-el].

[162] Yichul Choi, Ho Tat Lam and Shu-Heng Shao. 'Non-invertible Gauss Law and Axions'. In: (Dec. 2022). arXiv: 2212.04499 [hep-th].

[163] Ivan M. Burbano, Justin Kulp and Jonas Neuser. 'Duality defects in E_8'. In: *JHEP* 10 (2022), p. 186. DOI: 10.1007/JHEP10(2022)187. arXiv: 2112.14323 [hep-th].

[164] Daniel Robbins, Eric Sharpe and Thomas Vandermeulen. 'Decomposition, Trivially-Acting Symmetries, and Topological Operators'. In: (Nov. 2022). arXiv: 2211.14332 [hep-th].

[165] Thibault D. Décoppet and Matthew Yu. 'Gauging Noninvertible Defects: A 2-Categorical Perspective'. In: (Nov. 2022). arXiv: 2211.08436 [math.CT].

[166] Clay Cordova et al. 'Neutrino Masses from Generalized Symmetry Breaking'. In: (Nov. 2022). arXiv: 2211.07639 [hep-ph].

[167] Avner Karasik. 'On anomalies and gauging of U(1) non-invertible symmetries in 4d QED'. In: (Nov. 2022). arXiv: 2211.05802 [hep-th].

[168] Vladimir Bashmakov et al. 'Non-invertible Symmetries of Class $\mathcal{S}$ Theories'. In: (Nov. 2022). arXiv: 2211.05138 [hep-th].

[169] Shi Chen and Yuya Tanizaki. 'Solitonic symmetry beyond homotopy: invertibility from bordism and non-invertibility from TQFT'. In: (Oct. 2022). arXiv: 2210.13780 [hep-th].

[170] Juven Wang and Yi-Zhuang You. 'Gauge Enhanced Quantum Criticality Between Grand Unifications: Categorical Higher Symmetry Retraction'. In: (Nov. 2021). arXiv: 2111.10369 [hep-th].

[171] V. B. Petkova and J. B. Zuber. 'Generalized twisted partition functions'. In: *Phys. Lett. B* 504 (2001), pp. 157–164. DOI: 10.1016/S0370-2693(01)00276-3. arXiv: hep-th/0011021.

[172] Mendel Nguyen, Yuya Tanizaki and Mithat Ünsal. 'Semi-Abelian gauge theories, non-invertible symmetries, and string tensions beyond N-ality'. In: *JHEP* 03 (2021), p. 238. DOI: 10.1007/JHEP03(2021)238. arXiv: 2101.02227 [hep-th].

[173] Pierluigi Niro, Konstantinos Roumpedakis and Orr Sela. 'Exploring Non-Invertible Symmetries in Free Theories'. In: (Sept. 2022). arXiv: 2209.11166 [hep-th].

[174] Heidar Moradi, Seyed Faroogh Moosavian and Apoorv Tiwari. 'Topological Holography: Towards a Unification of Landau and Beyond-Landau Physics'. In: (July 2022). arXiv: 2207.10712 [cond-mat.str-el].

[175] Da-Chuan Lu and Zhengdi Sun. 'On Triality Defects in 2d CFT'. In: (Aug. 2022). arXiv: 2208.06077 [hep-th].

[176] Laurens Lootens et al. ‘Dualities in one-dimensional quantum lattice models: symmetric Hamiltonians and matrix product operator intertwiners’. In: (Dec. 2021). arXiv: 2112.09091 [quant-ph].

[177] Ying-Hsuan Lin et al. ‘Asymptotic density of states in 2d CFTs with non-invertible symmetries’. In: (Aug. 2022). arXiv: 2208.05495 [hep-th].

[178] Ling Lin, Daniel G. Robbins and Eric Sharpe. ‘Decomposition, condensation defects, and fusion’. In: *Fortsch. Phys.* 70 (2022), p. 2200130. DOI: 10.1002/prop.202200130. arXiv: 2208.05982 [hep-th].

[179] Masataka Koide, Yuta Nagoya and Satoshi Yamaguchi. ‘Non-invertible topological defects in 4-dimensional $\mathbb{Z}_2$ pure lattice gauge theory’. In: *PTEP* 2022.1 (2022), 013B03. DOI: 10.1093/ptep/ptab145. arXiv: 2109.05992 [hep-th].

[180] Kansei Inamura. ‘Fermionization of fusion category symmetries in 1+1 dimensions’. In: (June 2022). arXiv: 2206.13159 [cond-mat.str-el].

[181] Tzu-Chen Huang et al. ‘Numerical Evidence for a Haagerup Conformal Field Theory’. In: *Phys. Rev. Lett.* 128.23 (2022), p. 231603. DOI: 10.1103/PhysRevLett.128.231603. arXiv: 2110.03008 [cond-mat.stat-mech].

[182] Jonathan J. Heckman et al. ‘The Branes Behind Generalized Symmetry Operators’. In: (Sept. 2022). arXiv: 2209.03343 [hep-th].

[183] Noppadol Mekareeya and Matteo Sacchi. ‘Mixed Anomalies, Two-groups, Non-Invertible Symmetries, and 3d Superconformal Indices’. In: (Oct. 2022). arXiv: 2210.02466 [hep-th].

[184] Yui Hayashi and Yuya Tanizaki. ‘Non-invertible self-duality defects of Cardy-Rabinovici model and mixed gravitational anomaly’. In: *JHEP* 08 (2022), p. 036. DOI: 10.1007/JHEP08(2022)036. arXiv: 2204.07440 [hep-th].

[185] Iñaki García Etxebarria. 'Branes and Non-Invertible Symmetries'. In: (Aug. 2022). DOI: 10.1002/prop.202200154. arXiv: 2208.07508 [hep-th].

[186] Michele Del Zotto and Iñaki García Etxebarria. 'Global Structures from the Infrared'. In: (Apr. 2022). arXiv: 2204.06495 [hep-th].

[187] Jeremias Aguilera Damia, Riccardo Argurio and Luigi Tizzano. 'Continuous Generalized Symmetries in Three Dimensions'. In: (June 2022). arXiv: 2206.14093 [hep-th].

[188] Clay Cordova, Kantaro Ohmori and Tom Rudelius. 'Generalized symmetry breaking scales and weak gravity conjectures'. In: *JHEP* 11 (2022), p. 154. DOI: 10.1007/JHEP11(2022)154. arXiv: 2202.05866 [hep-th].

[189] Yichul Choi, Ho Tat Lam and Shu-Heng Shao. 'Non-invertible Time-reversal Symmetry'. In: (Aug. 2022). arXiv: 2208.04331 [hep-th].

[190] Xie Chen et al. 'Loops in 4+1d Topological Phases'. In: (Dec. 2021). arXiv: 2112.02137 [cond-mat.str-el].

[191] Ilka Brunner, Nils Carqueville and Daniel Plencner. 'Orbifolds and topological defects'. In: *Commun. Math. Phys.* 332 (2014), pp. 669–712. DOI: 10.1007/s00220-014-2056-3. arXiv: 1307.3141 [hep-th].

[192] Francesco Benini, Christian Copetti and Lorenzo Di Pietro. 'Factorization and global symmetries in holography'. In: (Mar. 2022). arXiv: 2203.09537 [hep-th].

[193] Fabio Apruzzi et al. 'Non-Invertible Symmetries from Holography and Branes'. In: (Aug. 2022). arXiv: 2208.07373 [hep-th].

[194] Andrea Antinucci et al. 'The holography of non-invertible self-duality symmetries'. In: (Oct. 2022). arXiv: 2210.09146 [hep-th].

[195] Ryan Thorngren and Curt von Keyserlingk. 'Higher SPT's and a generalization of anomaly in-flow'. In: (Nov. 2015). arXiv: 1511.02929 [cond-mat.str-el].

[196] Anton Kapustin and Ryan Thorngren. 'Anomalies of discrete symmetries in various dimensions and group cohomology'. In: (Apr. 2014). arXiv: 1404.3230 [hep-th].

[197] Sebastian Burciu and Sonia Natale. 'Fusion rules of equivariantizations of fusion categories'. In: *Journal of Mathematical Physics* 54.1 (2013), p. 013511. DOI: 10.1063/1.4774293. eprint: https://doi.org/10.1063/1.4774293. URL: https://doi.org/10.1063/1.4774293.

[198] Pavel Etingof, Dmitri Nikshych and Viktor Ostrik. 'On fusion categories'. In: *Ann. of Math. (2)* 162.2 (2005), pp. 581–642. DOI: 10.4007/annals.2005.162.581. URL: https://doi.org/10.4007/annals.2005.162.581.

[199] John C Baez et al. *Infinite-dimensional representations of 2-groups.* American Mathematical Soc., 2012.

[200] Yasunori Lee, Kantaro Ohmori and Yuji Tachikawa. 'Matching higher symmetries across Intriligator-Seiberg duality'. In: *JHEP* 10 (2021), p. 114. DOI: 10.1007/JHEP10(2021)114. arXiv: 2108.05369 [hep-th].

[201] Michele Del Zotto, Iñaki García Etxebarria and Sakura Schafer-Nameki. '2-Group Symmetries and M-Theory'. In: (Mar. 2022). arXiv: 2203.10097 [hep-th].

[202] Mirjam Cvetič et al. '0-Form, 1-Form and 2-Group Symmetries via Cutting and Gluing of Orbifolds'. In: (Mar. 2022). arXiv: 2203.10102 [hep-th].

[203] Michele Del Zotto and Kantaro Ohmori. '2-Group Symmetries of 6D Little String Theories and T-Duality'. In: *Annales Henri Poincare* 22.7 (2021), pp. 2451–2474. DOI: 10.1007/s00023-021-01018-3. arXiv: 2009.03489 [hep-th].

[204] Alberto G. Raggi-Cárdenas and Luis Valero-Elizondo. 'Global representation rings'. In: *Journal of Algebra* 441 (2015), pp. 426–440. DOI: https://doi.org/10.1016/j.jalgebra.2015.07.001. URL: https://www.sciencedirect.com/science/article/pii/S0021869315003208.

[205] S.J. Witherspoon. 'The Ring of Equivariant Vector Bundles on Finite Sets'. In: *Journal of Algebra* 175.1 (1995), pp. 274–286. DOI: https://doi.org/10.1006/jabr.1995.1186. URL: https://www.sciencedirect.com/science/article/pii/S0021869385711866.

[206] Christopher L. Douglas and David J. Reutter. 'Fusion 2-categories and a state-sum invariant for 4-manifolds'. In: *arXiv e-prints*, arXiv:1812.11933 (Dec. 2018), arXiv:1812.11933. arXiv: 1812.11933 [math.QA].

[207] Tzu-Chen Huang and Ying-Hsuan Lin. 'Topological field theory with Haagerup symmetry'. In: *J. Math. Phys.* 63.4 (2022), p. 042306. DOI: 10.1063/5.0079062. arXiv: 2102.05664 [hep-th].

[208] Zohar Komargodski et al. 'Symmetries and strings of adjoint QCD_2'. In: *JHEP* 03 (2021), p. 103. DOI: 10.1007/JHEP03(2021)103. arXiv: 2008.07567 [hep-th].

[209] Lakshya Bhardwaj et al. 'Anomalies of Generalized Symmetries from Solitonic Defects'. In: (May 2022). arXiv: 2205.15330 [hep-th].