Conditional Limit Theorems for Renewal Random Walks

CLARE LOUISE WALLACE

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Clare Wallace

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Department of Mathematical Sciences
Durham University
United Kingdom

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Abstract: We consider the trajectories of a renewal random walk, that is, a random walk on the two-dimensional integer lattice whose jumps have positive horizontal component. In a contrast to the usual limit theorems for random walks, we do not require the jumps to have width 1, and we consider models in which the height of each jump may depend on its width. We prove a Functional Central Limit Theorem for these trajectories: the distribution of their fluctuations around a limiting profile converges weakly to that of Brownian motion. We also derive a conditional version of this theorem, under large-deviations conditions on the terminal height and the integral of the trajectories. We find the shape of the corresponding limiting profile, and prove the convergence of the distribution of the fluctuations around this profile to that of a conditioned Gaussian process.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification.

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Dedicated to

Jenny Wallace
Gran, we made it.

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List of Symbols

Symbol	Description	Page
η, ξ	a single increment and a sub-trajectory, formed of one or several increments	20,21
$\mathcal{F}_k, \mathcal{F}$	the sets containing all increments of width k, and all increments of any length, respectively	20
$\mathcal{H}_n$	the set of sub-trajectories ξ of width n	14
ν, $\mathcal{P}_{[a,b]}$	a partition of a segment into pieces of integer length , and the set of all partitions of the segment $[a, b]$	21
G, g	piecewise-constant and piecewise-linear versions of the height function	24
$Y_n(1), Y_n(\mathbf{t})$	the joint variable $(g(1), \int_0^1 g(t)dt)$, and the joint variable of the increments of G, together with $Y_n(1)$	25
λ, $\mathbb{P}_n$	a probability measure on $\mathcal{F}$ and the corresponding measure on $\mathcal{H}_n$	25,26
$\mathbb{P}_n^{\alpha_n,\beta_n}$	the measure obtained from $\mathbb{P}_n$, under the condition $Y_n(1) = (\alpha_n, \beta_n)$	26
$\mathbb{Q}_n^{\mathbf{u}}$	the Cramér transform of $\mathbb{P}_n$, with tilt vector $\mathbf{u}$	27
F_k	the partition function based on increments of width k	27
H_n	the partition function for the height, based on trajectories of width n	27
$\mathcal{B}_n$	the partition function for the height and area, based on trajectories of width n	28
m	the mass function obtained from $\lim \frac{1}{n} \log H_n$	31
$\mathcal{Z}, \mathcal{U}$	the neighbourhood on which the properties in Chapter 2 hold, and its intersection with the real line	35
$\mathcal{Z}^\Delta, \mathcal{U}^\Delta$	the set of pairs (z_1, z_0) such that $[z_1, z_1 + z_0] \subseteq \mathcal{Z}$, and its intersection with the real line	40

$z\,(x)$	the linear combination $z_1 + z_0(1-x)$	36
$\mathcal{B}_{[a,b]}$	the partition function for the contributions to the height and area arising from the segment $[a,b]$	38
Δ_T	the pair describing the distance to the nearest cutpoints to the left and right of T	43,47
Λ^T	the set of possible values of Δ_T	43,48
Λ_θ	the subset of Λ^T in which no pair has width greater than θ	45
$\mathcal{B}_{[a,b],(\mathbf{T},\theta)}$	the contribution to $\mathcal{B}_{[a,b]}$ from sub-trajectories in which each increment Δ_{T_j} has width at most θ	49
$\mathcal{B}^{\rho}_{[a,b]}$	the contribution to $\mathcal{B}_{[a,b]}$ from sub-trajectories in which no increment has gradient steeper than ρ	52
$\mathcal{Z}^{\Delta}_k$	the set of lists $\mathbf{z}_1, \ldots, \mathbf{z}_k$ in which each pair $\mathbf{z}_j$ has the same second argument, and $\mathbf{z}_j \in \mathcal{Z}^{\Delta}$ for each j	60
$\mathbf{z}_\mathbf{T}\,(x)$	a version of $z\,(x)$ in which z_1 varies with x, depending on its position relative to the T_js	60
$\mathcal{B}_{[a,b],\mathbf{T}}$	the partition function for the contributions to the height from each of the segments $[T_{j-1}, T_j]$ and the area	60
$\mathcal{B}_{n,\mathbf{T},(\mathbf{S},\theta)}$	the contribution to $\mathcal{B}_{[a,b],\mathbf{T}}$ from sub-trajectories in which the increments Δ_{S_j} have width at most θ	62
$F^{\star}_{[a,b]}$	the partition function F_{b-a}, rescaled by an exponential factor	64
$f_{\mathbf{t}}, \varphi_{[a,b]}$	the limit of the log-moment generating function of $Y_n(\mathbf{t})$, and an approximation to it	72
$\mathcal{B}^{\star}_{[a,b],T}$	the partition function $\mathcal{B}_{[b-a],\mathbf{T}}$, rescaled by an exponential factor	73
c_n, c_∞	the profiles obtained in the Law of Large Numbers, and their limit	87
$\tilde{G}, \tilde{g}$	the height functions which have been centred by nc_n and rescaled by $\sqrt{n}$	102,109

Chapter 1

Introduction

The aim of this thesis is to describe the asymptotic behaviour of the trajectories of a renewal random walk, when those trajectories are placed under large-deviations conditions on their area and terminal height. The question lies at the intersection of three areas of probability: renewal models, limit theorems for trajectories, and the study of Gaussian processes. Each is a significant field of study in its own right, so we can only touch on their significance before beginning a discussion of the particular model, the particular theorems, and the particular processes in which we are interested.

1.1 Historical Context

1.1.1 Renewal Models

Renewal models are a family of random models, which are used to count the number of randomly-occurring events that have happened up to a particular time. Examples of renewal models include a count of the number of customers arriving in a shop and the number of times we have had to replace a lightbulb, given the distribution of its length of life.

We suppose that the lifespans of the lightbulbs, $X_1, X_2, \ldots,$ are independent and identically distributed, and take strictly positive values, and let

$$N_t = \sup\left\{n : \sum_{j=1}^{n} X_j \leq t\right\}$$

be the number of lightbulbs we have replaced up to time t. When X has an exponential distribution with parameter λ, N_t is a Poisson process with intensity λ.

The early study of renewal models stemmed from the observation that the renewal function

$$m(t) = \mathbb{E}[N_t]$$

satisfies the renewal equation

$$m(t) = F(t) + \int_0^t m(t-x)dF(x), \tag{1.1.1}$$

where $F(t) = \mathbb{P}(X \leq t)$. In 1941, Feller composed a survey [23] of the literature about equations of the form

$$u(t) = g(t) + \int_0^t u(t-x)dF(x)$$

– a more general version of Equation (1.1.1) – in which he noted that an earlier paper by Lotka [45] had cited 74 references, and provided a further 16 which had appeared since. While these integral equations were widely studied throughout the 1930s and 1940s, renewal theory was brought into the realm of probability in 1948, when Doob [17] discussed the connection between the equations and our "lifespan" formulation of the models.

One of the principal results in renewal theory - described by Karlin and Taylor [39] as "one of the most basic theorems in applied probability" - is the renewal theorem. First proved for discrete random variables by Erdős, Feller, and Pollard in 1948 [20], it was generalised by Blackwell [5] to a form which holds for both continuous and discrete random variables. The strong law of large numbers applied to the renewal model tells us that as $t \to \infty$,

$$\frac{1}{t} N_t \to \frac{1}{\mathbb{E}[X_1]},$$

as long as $\mathbb{E}[X_1] < \infty$; the following local renewal theorem gives us more information about how $m(t+h) - m(t)$ behaves, when h is fixed and $t \to \infty$.

Theorem 1.1.1 (Blackwell's theorem). *If X is a discrete random variable and h is a multiple of its span, or if X is a continuous random variable and $h > 0$, then*

$$m(t+h) - m(t) \to \frac{h}{\mathbb{E}[X_1]}$$

as $t \to \infty$.

In other words, the expected number of renewals in an interval of width h converges to the ratio $\frac{h}{\mathbb{E}[X_1]}$.

An interesting, and not necessarily intuitive, property of renewal models is that, for any fixed time $t > 0$, the distribution of the lifetime of the lightbulb in use at time t is not in general the same as the distribution of the lifetime of the first (or fourth, or nth) lightbulb.

Within the family of renewal models, we are most interested in renewal-reward processes. One of the most famous illustrations of this type of process is the bank balance of an insurance company, as customers pay in premiums and take out claims. While the premiums arrive at a predictable, constant rate, both the timing and the size of each claim is random. We model the time delay between successive claims using a sequence $X_1, X_2, \ldots$ of independent, identically-distributed, strictly-positive-valued random variables; as in the classical renewal model, these time delays are often chosen to be exponentially distributed. Our running count of how many claims have been made is the renewal process

$$N_t = \sup \left\{ n : \sum_{j=1}^{n} X_j \leq t \right\}.$$

Next, we take a sequence $W_1, W_2, \ldots$ describing the cost of each of the claims. The costs are again independent and identically distributed and (unfortunately for the insurance company) positive valued, and are independent of $X_1, X_2, \ldots$. The running total of payments made by the company is

$$Y_t = \sum_{j=1}^{N_t} W_j.$$

At the same time, the company is receiving premiums at a constant and (unfortunately for the customers) positive rate c. If the initial bank balance is x, then at time t the company will hold a balance of

$$B_t = x + ct - Y_t \tag{1.1.2}$$

in its account, assuming there are no external factors like interest rates, overdraft fees, or credit limits. A question of some importance to the insurance company is: will this balance ever hit zero? And if so, how soon? These questions were posed in Lundberg's doctoral thesis [46] in 1903, and the theory was later clarified by Cramér in a review [10] which described Lundberg's proofs as "formulated with oracle-like abstruseness" [6]. Cramér's work on these models sparked the field of ruin theory, and the classical version, in which both X and W have exponential distributions, is called the *Cramér-Lundberg model*. The general setting, in which X can have any distribution, is known as the *Sparre Andersen model*, after his discussion of the model in a 1957 paper [51]. In the Cramér-Lundberg model, the probability of ruin, as a function of the initial capital, is given by

$$R(x) = \begin{cases} \left(1 - \frac{\mathbb{E}[W_1]}{c\mathbb{E}[X_1]}\right) \mathbb{1}\{x > 0\} & \mathbb{E}[W_1] < c\mathbb{E}[X_1] \\ 0 & \mathbb{E}[W_1] > c\mathbb{E}[X_1] \end{cases},$$

see for example Section 5.7.G of [39], or Example 2.6.11 of [18]. In other words, the company will be bankrupt almost surely if the average payout exceeds its income over an average claim interval, and the ratio of these two quantities determines the ruin probability otherwise.

The probability of ruin is not the only interesting feature of this model. In Section

10.5 of [30], Grimmett and Stirzaker describe the renewal-reward theorem, which uses the strong law of large numbers to tell us that

$$\frac{1}{t}Y_t \to \frac{\mathbb{E}[W_1]}{\mathbb{E}[X_1]},$$

as $t \to \infty$, both almost surely and in $\mathcal{L}_1$. They further discuss how the renewal-reward theorem can be applied even when the cost process Y_t is modelled as piecewise linear. In this context, the insurance company metaphor falls apart, but we can instead think of a freelancer (or research assistant!), taking contracts of random lengths and which pay fees at a random, constant rate. Then the situation of Equation (1.1.2) is reversed, and our freelancer's bank balance is given by

$$x - ct + Y_t,$$

where now $c > 0$ is the rate at which they spend money, and Y_t tracks their income.

The image of the freelancer is a helpful one to consider as we introduce the model studied in this thesis. In order to apply the results arising from the Sparre Andersen model, we would need to consider a situation in which the fee for each contract is independent of its duration. In the real world, this is an unrealistic stipulation: of course we expect that longer contracts should be associated to higher total fees, but we can also envisage a freelancer whose daily rate depends on the length of the contract, so that there might be a complicated dependence between X_j and W_j.

1.1.2 Limit Theorems for Trajectories

The next stop on our tour of probability is the study of limit theorems for trajectories. Where the Central Limit Theorem describes the convergence of real-valued random *variables*, the Functional Central Limit Theorem describes the convergence of random *functions*.

The Central Limit Theorem states that, for a sequence of independent and identically-distributed variables $X_1, X_2, \ldots$ with mean μ and variance σ^2,

$$Z_n = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{\mathcal{D}} Z.$$

Here $S_n = X_1 + \cdots + X_n$, $Z \sim \mathcal{N}(0,1)$, and $\xrightarrow{\mathcal{D}}$ means convergence in distribution, that is,

$$F_{Z_n}(x) \to F_Z(x)$$

for all continuity points x of F_Z.

To move from this theorem to a version which makes sense for the trajectories of random walks, we need to understand the process that will take the place of the Normal distribution, as well as what we mean by convergence in distribution.

In place of the Normal distribution, the limiting process is *Brownian motion.* Inspired by Robert Brown's observations of the movement of a particle of pollen suspended in water [8], Brownian motion was described as a financial model by Bachelier in 1900 [2], and separately as a physical process by Einstein in 1905 [19]. Strictly speaking, Brownian motion is a physical phenomenon, which is described using a stochastic process via the *Wiener measure* [55]. In practice, the two terms are often used interchangeably. See, for example, [48] for a nice discussion of the theory.

The standard Brownian motion can be defined by the following three criteria (cf [39]):

1. Every increment $B(t+s) - B(s)$ is normally distributed with mean 0 and variance s.

2. For every pair of time intervals with disjoint interiors $[s_1, s_2]$, $[t_1, t_2]$, $s_1 < s_2 \leq t_1 < t_2$, the increments $B(t_2) - B(t_1)$ and $B(s_2) - B(s_1)$ are independent random variables, and similarly for any n disjoint time intervals.

3. $B(0) = 0$, and the function $t \mapsto B(t)$ is continuous at time 0.

See Figure 1.1 for a sample path of Brownian motion, over the interval $[0, 1]$.

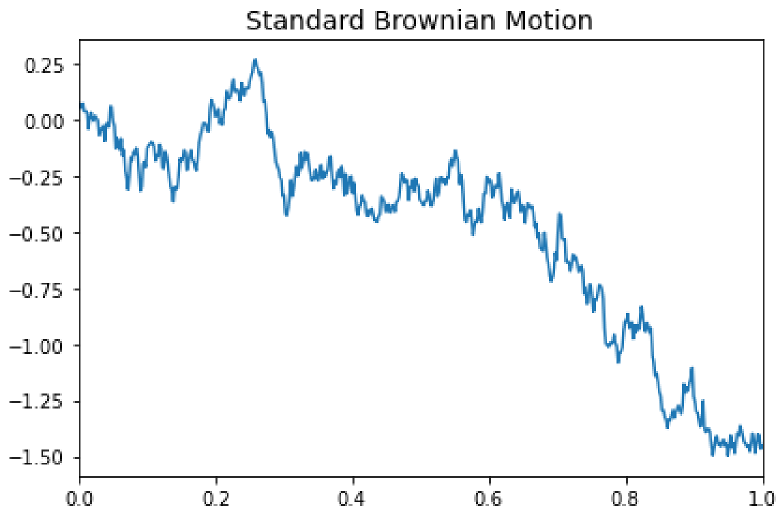

Figure 1.1: A sample path of Brownian motion on $[0, 1]$.

In Section XIV.6 of *An Introduction to Probability Theory and its Applications* (Vol. 1) [22], Feller describes how the trajectories of a simple random walk, appropriately scaled, converge to the trajectories of Brownian motion. More formally, Donsker [16] proved in 1951 that, given a sequence of independent and identically-distributed random variables $X_1, X_2, \ldots,$ with their partial sums $S_n = X_1 + \cdots + X_n$, the functions

$$t \mapsto \frac{1}{\sigma\sqrt{n}}\left(S_{\lfloor nt \rfloor} - \lfloor nt \rfloor \mu\right)$$

converge in distribution to the trajectories of Brownian motion.

We have still not defined what we mean by "convergence in distribution" in the context of random functions. While we were able to describe the convergence in distribution of random variables using their cumulative distribution functions, we cannot define such functions for our trajectories. Instead, we use an equivalent condition (see, for example, Chapter 3 of [18]): the variables X_n converge in distribution to X if and only if, for any bounded continuous function g, we have

$$\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)].$$

This version of convergence in distribution can be applied to random objects living in a general metric space – not only those living in $\mathbb{R}$. The Portmanteau Theorem (for example, Theorem 2.1 in [4]) gives three further equivalent conditions for this type of convergence, which we now call *weak convergence*. When this is the case, we write $X_n \Rightarrow X$ as $n \to \infty$, or we say that X_n converges weakly to X. Patrick Billingsley's book *Convergence of Probability Measures* [4] gives a thorough overview of the theory of weak convergence for trajectories of random walks in one dimension with independent or asymptotically independent increments. A survey of the application of this theory to random walks in higher dimensions is available in [44].

Theorems 8.1 and 15.1 of [4] give sufficient conditions for weak convergence of random functions with continuous and càdlàg trajectories, respectively. We will only quote Theorem 8.1 here. It deals with two conditions: convergence of the finite-dimensional distributions, and tightness of the distributions.

We say that the finite-dimensional distributions of a sequence of elements X_n of the set $C = C[0,1]$ of continuous functions on $[0,1]$ converge to those of $X \in C$ if, for every $k \in \mathbb{N}$ and $0 \leq t_1 < t_2 < \cdots < t_k \leq 1$, we have

$$(X_n(t_1), X_n(t_2), \ldots, X_n(t_k)) \to (X(t_1), X(t_2), \ldots, X(t_k)) \,. \tag{1.1.3}$$

If the convergence in Equation (1.1.3) is in probability or in distribution, we say that the finite-dimensional distributions converge in probability or in distribution. The name *finite-dimensional distributions* highlights the reason for our interest in these objects: they allow us to study the k-dimensional vectors $(X_n(t_1), \ldots, X_n(t_k))$, rather than the infinite-dimensional space C.

Next, we say that a family of distributions Π defined on a metric space (S, d) is *tight* if, for every $\varepsilon > 0$, there exists a set K which is compact under d such that

$$\inf_{P \in \Pi} P(K) > 1 - \varepsilon.$$

Several alternative conditions for tightness are established in [4]. The tightness criterion allows us to exclude the possibility that, although the finite-dimensional distributions converge, the process itself does not have a weak limit; see, for example, (3.5) in [4] or Example 4.16 in [44].

Now, we can state our theorem for weak convergence of random functions, a functional version of the Central Limit Theorem.

Theorem 1.1.2 (Theorem 8.1 in [4]). *Let $X_1, X_2, \dots$ be a sequence of random elements of C; denote the law of X_k by $\mathcal{L}(X_k)$. If, for some $X \in C$, the finite-dimensional distributions of X_n converge in distribution to those of X as $n \to \infty$, and if $\{\mathcal{L}(X_n)\}$ is tight, then $X_n \Rightarrow X$ as $n \to \infty$.*

Using Theorem 7.4.1 of [52], Whitt has shown in [53] that the distributions of the trajectories of the renewal random walk fulfil the conditions of Theorem 1.1.2, so that they converge weakly to the trajectories of Brownian motion. In this thesis, we will show that the distribution of sections of the trajectory, when placed under certain conditions, also fulfil both conditions of Theorem 1.1.2, so that they converge weakly to the distribution of the trajectories of an appropriate Gaussian process.

One additional limit theorem of interest to us is an extension of Slutsky's theorem to the space of random functions. For real-valued random variables, Slutsky's theorem (see for example Theorems 11.3 and 11.4 in [31]) states that

Theorem 1.1.3. *Let $X_1, X_2, \dots$ be a sequence of real-valued random variables converging in distribution to a random variable X, and $Y_1, Y_2, \dots$ be a sequence of real-valued random variables defined on the same probability space, which converge in probability to a constant c,*

$$\mathbb{P}\left(|Y_n - c| > \varepsilon\right) \to 0$$

for all $\varepsilon > 0$, as $n \to \infty$. Then

$$X_n + Y_n \xrightarrow{\mathcal{D}} X + c$$

$$X_n Y_n \xrightarrow{\mathcal{D}} cX$$

$$\frac{X_n}{Y_n} \xrightarrow{\mathcal{D}} \frac{X}{c},$$

where the last convergence holds as long as $c \neq 0$.

This theorem is sometimes attributed to Cramér [31].

Theorem 4.1 of [4] allows us to construct a version of Slutsky's theorem for random functions which are defined on the same separable metric space.

Theorem 1.1.4. *Let $X_1, X_2, \ldots$ be a sequence of random variables defined on a separable metric space (S, d), which converge weakly to a random variable X, and $Y_1, Y_2, \ldots$ be a sequence of random variables defined on the same space such that*

$$\mathbb{P}\left(d(X_n - Y_n) > \varepsilon\right) \to 0$$

for every $\varepsilon > 0$, as $n \to \infty$. Then Y_n converges weakly to X as $n \to \infty$.

1.1.3 Gaussian Processes

Finally, we take a closer look at some specific Gaussian processes. The most famous of these is Brownian motion, which we met in Subsection 1.1.2. Gaussian processes more generally are members of the family of stochastic processes $X(t)$ such that, for any sequence $t_1 < t_2 < \cdots < t_k$ and any real numbers $c_1, c_2, \ldots, c_k$, the random variable $c_1X(t_1) + c_2X(t_2) + \cdots + c_kX(t_k)$ is normally distributed.

A particularly useful property of Gaussian processes (see, for example, Lemma 13.1 in [38]) is their characterisation by their mean and covariance functions. In other words, the distribution of a Gaussian process $X(t)$ on $[0, T]$ is completely determined by the functions $\mathbb{E}[X(t)], 0 \leq t \leq T$ and $\mathrm{Cov}(X(s), X(t)), 0 \leq s \leq t \leq T$. Using this characterisation, for the standard Brownian motion we have $\mathbb{E}[B(t)] = 0$, and $\mathrm{Cov}(B(s), B(t)) = \min(s, t)$.

We are interested in a particular subset of Gaussian processes, known as *generalised Gaussian bridges*. The first example of these is the *Brownian bridge*. A heuristic definition of the Brownian bridge on $[0, 1]$ is that it is the Gaussian process which has the same distribution as Brownian motion on the same interval, under the condition that $B(1) = 0$. See Figure 1.2 for a realisation of a sample path of the Brownian bridge. We can equivalently define the Brownian bridge via its mean and covariance

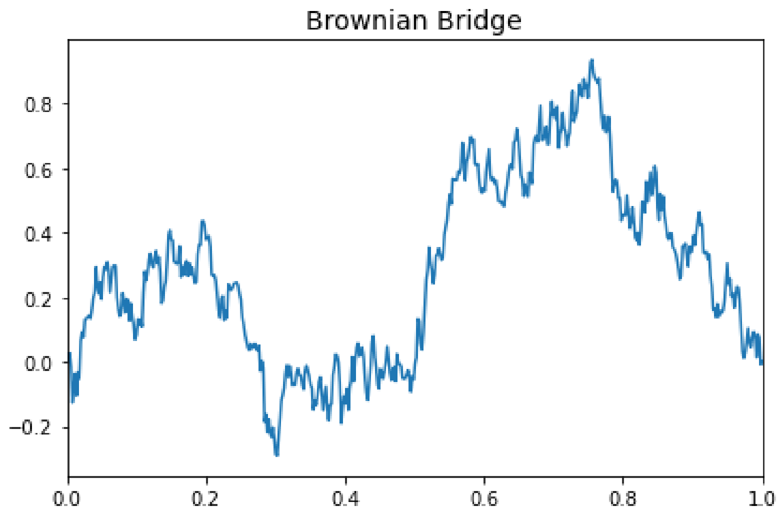

Figure 1.2: A sample path of the Brownian bridge on $[0, 1]$.

function: we have

$$\mathbb{E}[X(t)] = 0 \qquad 0 \le t \le 1$$

$$\mathrm{Cov}(X(s), X(t)) = s(1-t) \qquad 0 \le s \le t \le 1.$$

Another helpful interpretation of the Brownian bridge is written in terms of standard Brownian motion: we have

$$X(t) = B(t) - tB(1), \qquad 0 \le t \le 1$$

(see, for example, Section 13.6 in [30]).

Of course, since Brownian motion is not the only Gaussian process, the Brownian bridge is not the only Gaussian bridge. Given a general Gaussian process $G(t)$, we define the related Gaussian bridge on $[0, 1]$ by

$$X(t) = G(t) - tG(1) \qquad 0 \le t \le 1.$$

A nice summary of results about Gaussian bridges is in [26].

We can place further conditions on Gaussian bridges, to obtain *generalised Gaussian bridges*. Extensions of the Brownian bridge are studied in [3] and [1], and representations of a wider class of generalised Gaussian bridges are given in [50].

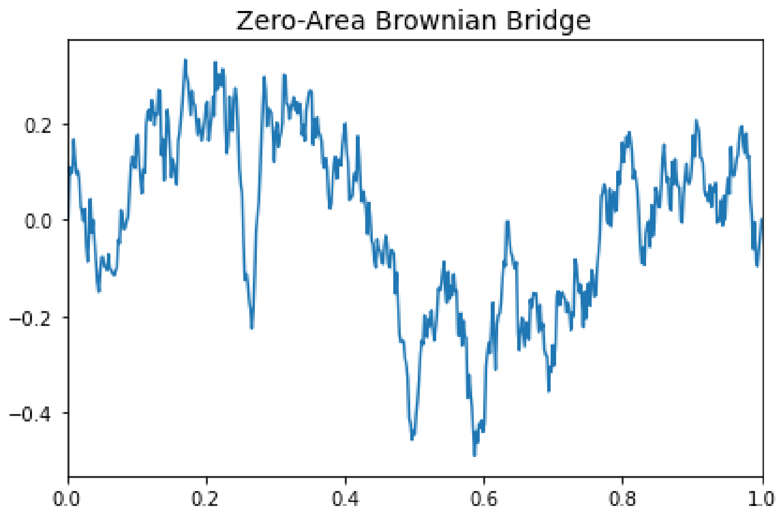

Figure 1.3: A sample path of the zero-area Brownian bridge on $[0, 1]$.

In the context of this thesis, we are interested in zero-area Gaussian bridges. As the name suggests, the zero-area Gaussian bridge on $[0, 1]$ has the same distribution as the Gaussian bridge on $[0, 1]$, under the additional condition that $\int_0^1 X(t)dt = 0$. The following characterisations of the zero-area Brownian bridge are from [28].

Using our classification of Gaussian processes by their mean and covariance functions, the zero-area Brownian bridge M is defined via

$$\mathbb{E}[M(t)] = 0 \qquad 0 \le t \le 1$$

$$\mathrm{Cov}(M(s), M(t)) = s - st - 3(s - s^2)(t - t^2) \qquad 0 \le s \le t \le 1.$$

We can also obtain the zero-area Brownian bridge from standard Brownian motion: we have

$$M(t) = B(t) - t(3t - 2)B(1) - 6t(1 - t) \int_0^1 B(s)ds.$$

See Figure 1.3 for a realisation of the zero-area Brownian bridge.

1.2 Motivation

In this thesis, we establish convergence results for sections of the trajectories of the renewal random walk introduced in the previous section, under large-deviations conditions on their area and terminal height. The initial motivation for such work comes from the study of skeletons arising from phase boundaries in statistical physics, particularly the Ising model. The approach in this thesis provides a general framework which can be applied not only in this context, but for contours arising from a wide range of two-dimensional models. Although such applications, including to the Ising model, are beyond the scope of this thesis, the next few paragraphs give an indication of the general structure.

We consider the Ising model on the two-dimensional lattice, at low temperature, and restrict our attention to configurations in a finite volume with fixed total magnetisation and periodic boundary conditions. In other words, we take a large rectangle $\Lambda \subset \mathbb{Z}^2$, and consider configurations $\sigma \in \{-1, +1\}^\Lambda$ such that

$$\sum_{x \in \Lambda} \sigma_x = \alpha_\Lambda,$$

for a suitably chosen constant α_Λ. Note in particular that α_Λ must be between $-|\Lambda|$ and $|\Lambda|$, and must be an integer with the same parity as $|\Lambda|$; we also suppose that, when we take $\Lambda \nearrow \mathbb{Z}^2$, the ratio $\frac{\alpha_\Lambda}{|\Lambda|}$ converges to some constant α.

The classical Dobrushin-Kotecký-Shlosman theory [15] establishes that the typical configurations in this setting exhibit phase separation, with a single macroscopic droplet whose rescaled shape can be described by the Wulff shape, as in Figure 1.4. By "zooming in" on a macroscopic section of this droplet and studying the behaviour of the phase boundary there, we can better understand the fluctuations around the Wulff profile and hence the behaviour of the droplet as a whole. In particular, a sharper description of typical configurations under the condition $\sum \sigma_x = \alpha_\Lambda$ will lead to sharp asymptotics for the large-deviations probabilities $\mathcal{P}\left(\sum \sigma_x = \alpha_\Lambda\right)$, and hence a large-deviations principle for the total magnetisation.

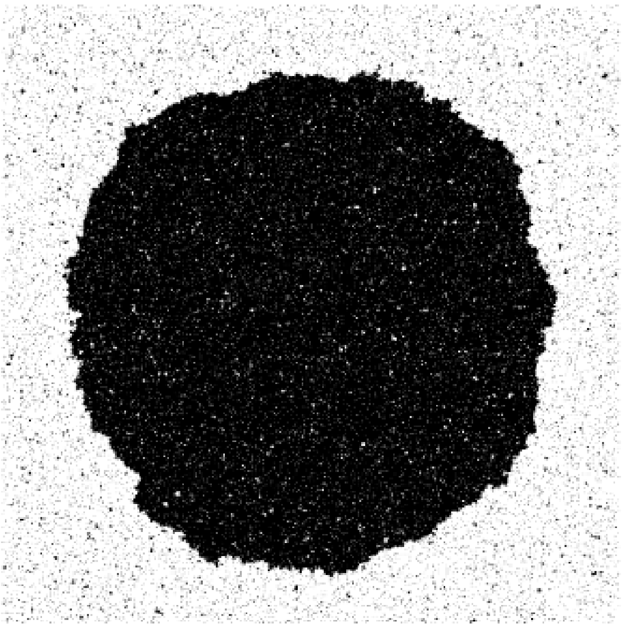

Figure 1.4: A sample configuration in the Ising model under fixed total magnetisation, in which the phase separation and Wulff profile can be seen. Taken from Figure 4.23 in Section 4.12 of [25], reproduced with permission of Cambridge University Press through PLSClear.

In order to represent a section of the phase boundary as a renewal random walk, we construct a skeleton. We take a configuration of the Ising model in a (smaller) finite volume, with Dobrushin boundary conditions, and identify the unique contour connecting the left-hand and right-hand boundaries of the volume. In order to construct a skeleton, we identify the points on the contour with unique horizontal projection, as shown in Figure 1.5. This separates the contour into distinct, indivisible "bridges", which form the increments of our renewal random walk. This construction produces increments whose widths are always positive integers, and which do not take values only on a sub-lattice of $\mathbb{Z}^2$; it is less straightforward to verify that the model fulfils the remaining assumptions 2.1.6, 2.2.7, and 2.2.15.

The construction of such skeleton models which represent contours, and which fulfil the assumptions set out in Chapter 2, should be applicable to a wider class of models arising from statistical physics; we hope to investigate such models in more detail in future work.

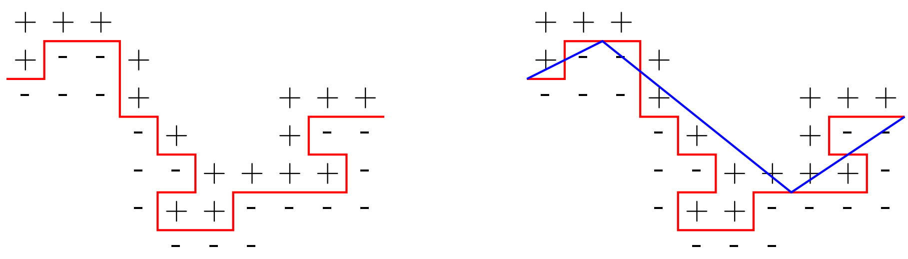

Figure 1.5: An illustration of the construction of a skeleton based on a section of phase boundary.

1.3 Main Results

In this section, we give an overview of the main results contained in this thesis.

We study the trajectories of a renewal random walk on the integer lattice. The renewal random walk resembles the renewal-reward processes described in Subsection 1.1.1, with the key distinction that we do not suppose that the horizontal and vertical parts of each increment (the "time" and "cost", in renewal-reward process terms) are independent of each other.

We take a set $\mathcal{F}$ containing all possible increments of our walk. To ensure that the model is well-defined, we suppose that every increment in $\mathcal{F}$ has strictly positive horizontal part. We can construct realisations of the renewal random walk by taking repeated independent samples from $\mathcal{F}$ according to some distribution λ, and attaching them in sequence. For $n \in \mathbb{N}$, we consider sub-trajectories whose total width is exactly n.

We assign height functions to each sub-trajectory ξ, in two different ways; both are defined on $[0,1]$, rather than $[0,n]$. The first, which we denote G, is a càdlàg function, which has constant value over intervals of random widths, with jumps of random heights between them. The second, g, is found by linearly interpolating between the jumps of G (see Figure 2.4 for an example of the two trajectories). By the end of this thesis we will have proved that, conditionally on large-deviations events of the type $\left\{g(1) \approx \alpha n, \int_0^1 g(t)dt \approx \beta n\right\}$, the distributions of the random functions $g(t)$ converge weakly to those of the sample paths of a Gaussian process, taken under a corresponding large-deviations condition.

In Theorem 3.2.1, we prove a Central Limit Theorem for the vector $\left(g(1), \int_0^1 g(t)dt\right)$ when it is appropriately centred and rescaled, and for vectors of the form

$$\left(G(t_1) - G(0), \ldots, G(t_k) - G(t_{k-1}), g(1) - G(t_k), \int_0^1 g(t)dt\right)$$

with $k \in \mathbb{N}$ and $0 < t_1 < t_2 < \cdots < t_k < 1$. In Theorems 4.1.2 and 4.2.2, we establish the corresponding Local Central Limit Theorems, and combine them to obtain the convergence of the finite-dimensional distributions, under large-deviations conditions on $\left(g(1), \int_0^1 g(t)dt\right)$.

In Chapter 5, we show that the functions

$$t \mapsto \frac{1}{\sqrt{n}} \left(g(t) - G(t)\right)$$

converge in probability to zero in the space $D([0,1])$ of càdlàg functions, with the uniform norm. Using Slutsky's Theorem 1.1.4, this allows us to obtain the convergence in distribution of the finite-dimensional distributions of G in Theorem 4.3.3. Finally, in Chapter 6, we establish in Theorem 6.0.3 that the distributions are tight.

This work extends the results found in [14] which apply in the $\mathcal{F} = \mathcal{F}_1$ case and only consider an area condition. As in that paper, it is partially motivated by the study of skeletons approximating contours arising in the low-temperature, two-dimensional Ising model. We anticipate that the more general framework we develop will allow for applications to a wider range of statistical mechanics models, as briefly discussed in Chapter 7.

Chapter 2

Definition and Properties of the Model

In the first part of this chapter, we introduce the model, most of the notation we will use to describe it, and state the assumptions necessary for the analysis. In the second, we establish properties of the model. We particularly focus on three partition functions associated with the model, and various ways in which they can be approximated. These partition functions, and their approximations, appear repeatedly in Chapters 3 to 5; to avoid repetition, this chapter collects the key properties of the partition functions in one place.

2.1 Definition of the model

We consider the trajectories of a renewal random walk. In this section, we will define what we mean by "renewal random walk", as well as "trajectories", and several other important features of the model. We will also introduce the key assumptions necessary for the analysis in the rest of this thesis.

It may be helpful to keep in mind the following example of a renewal random walk, as we define our notation.

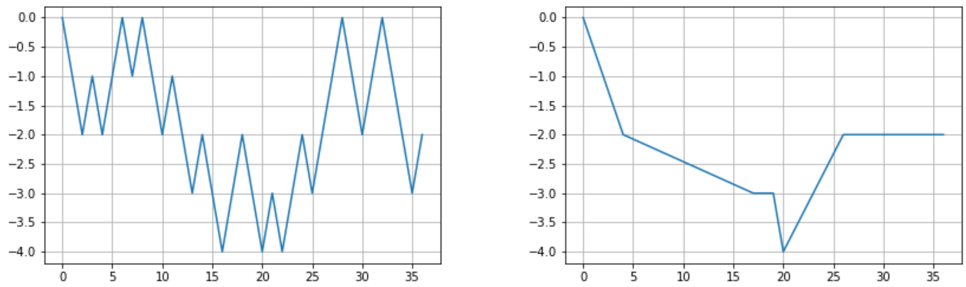

Figure 2.1: A realisation of SSRW, with $n = 36$ steps, and its decimation according to a sequence of independent Bernoulli random variables.

Example 2.1.1. Consider the simple symmetric random walk (SSRW) on $\mathbb{Z}$. At each time, SSRW moves up or down by one with equal probability, as in the first part of Figure 2.1. This is itself a (somewhat trivial) example of a renewal random walk.

To obtain a more interesting example, we consider the walk obtained by including or excluding the steps of the SSRW according to an independent sequence of Bernoulli random variables, shown in the second part of Figure 2.1. We may imagine that the SSRW is happening somewhere, but that due to some independent interference, we are only sometimes successful in observing it. Now, the width of each increment (or the time between consecutive successful observations) is strictly positive and geometrically distributed, while the heights follow a centred binomial distribution.

Let $\mathcal{F}$ be a non-empty subset of $\mathbb{N} \times \mathbb{Z}$. The elements of $\mathcal{F}$ are increments; we write

$$\eta = \big(w(\eta), h(\eta)\big)$$

and refer to $w(\eta)$ as the horizontal component, or width, of η, and $h(\eta)$ as its vertical component, or height.

For $k \in \mathbb{N}$, let

$$\mathcal{F}_k = \{\eta \in \mathcal{F} : w(\eta) = k\}$$

be the set of increments of width k.

Example 2.1.2. In the i.i.d. example, all increments have width 1 and so $\mathcal{F} = \mathcal{F}_1$.

We are already ready for our first assumption.

Assumption 2.1.3. *We assume that*

$$\gcd\{k \geq 1 : \mathcal{F}_k \neq \emptyset\} = 1.$$

In other words, we assume that the horizontal projections of the increments and sub-trajectories do not live on a sub-lattice $m\mathbb{Z}$ for some $m > 1$.

Example 2.1.4. If $h(\eta) = 0$ for all $\eta \in \mathcal{F}$, then the model is equivalent to a renewal model on the integers.

Given several increments $\eta_1, \eta_2, \ldots, \eta_m$, we can form a sub-trajectory ξ. We write

$$\xi = \eta_1 \sqcup \eta_2 \sqcup \cdots \sqcup \eta_m$$

for the sub-trajectory formed by appending $\eta_1, \eta_2, \ldots, \eta_m$ in order; note that this definition of ξ depends not only on the increments, but also on their order. We use $\#\xi = m$ to denote the number of increments that form ξ.

It is natural to define the width and height of ξ as the total width and height, respectively, of its increments, so that

$$w(\xi) = \sum_{j=1}^{m} w(\eta_j) \qquad\qquad h(\xi) = \sum_{j=1}^{m} h(\eta_j).$$

Just as we collected all increments of width k into $\mathcal{F}_k$, we define the set of all sub-trajectories of width n:

$$\mathcal{H}_n = \{\xi : w(\xi) = n\}.$$

Our assumption 2.1.3 is equivalent to the requirement that, for all n sufficiently large, $\mathcal{H}_n$ is non-empty.

Example 2.1.5. In the context of Example 2.1.2, every sub-trajectory of width n is formed of precisely n increments.

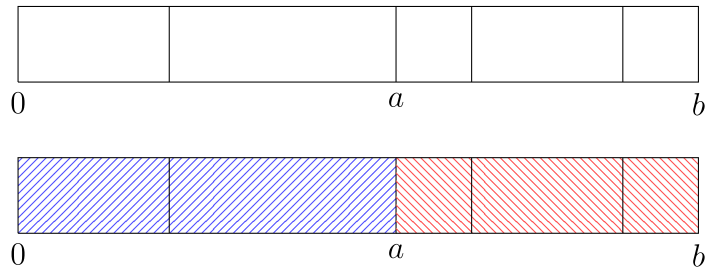

Figure 2.2: An illustration of a partition of $[0, b]$, represented using a partition of $[0, a]$ and another partition of $[a, b]$.

In order to describe the horizontal projections of the increments of ξ, we will often use partitions. We denote the set of partitions of the segment $[a, b]$ into integer pieces by $\mathcal{P}_{[a,b]}$. We represent partitions $\nu \in \mathcal{P}_{[a,b]}$ in the form

$$\nu = (\nu_1, \dots, \nu_{k-1}, b),$$

where $k \in \mathbb{N}$ and $\nu_j < \nu_{j+1}$ for each $0 \le j \le k-1$, with the convention $\nu_0 = a$. We write $|\nu| = k$, and use the shorthand $\mathcal{P}_{[0,n]} = \mathcal{P}_n$.

Now, every $\xi \in \mathcal{H}_n$ corresponds to exactly one $\nu \in \mathcal{P}_n$, with $|\nu| = \#\xi$, and, if $\xi = \eta_1 \sqcup \cdots \sqcup \eta_m$, we have

$$\nu_j = \sum_{i=1}^{j} w(\eta_i) \qquad\qquad 1 \le j \le m-1.$$

When this is the case, we write $\xi \sim \nu$; this decomposition of $\mathcal{H}_n$ according to the elements of $\mathcal{P}_n$ will be helpful as we define the partition functions. We describe the locations ν_j as the *cutpoints* of ξ (or ν). This is to emphasise that, at each of these cutpoints, ν can be separated into two smaller sub-partitions, as in Figure 2.2.

It may be useful to point out that partitions in the set $\mathcal{P}_{[a,a+n]}$ can be connected to elements of $\mathcal{P}_n$ via a shift of a: for each $\pi \in \mathcal{P}_{[a,a+n]}$ there is an element $\nu \in \mathcal{P}_n$ with $|\nu| = |\pi|$ and $\pi_j = a + \nu_j$ for each $j \le |\nu|$.

The description of sub-trajectories via partitions is immediately useful, as it allows us to describe the area associated to each trajectory. For a partition $\nu \in \mathcal{P}_n$, we

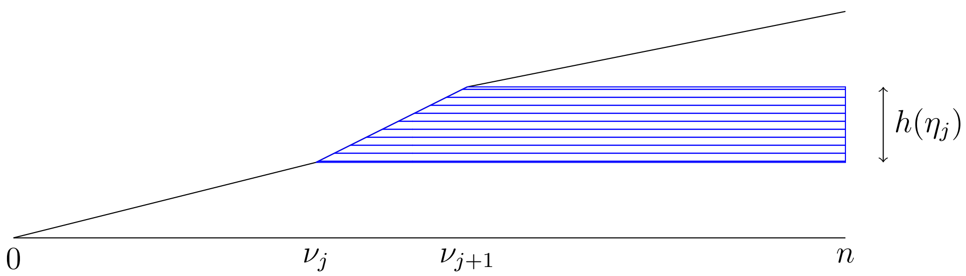

Figure 2.3: An illustration of our construction of the area; the contribution of the increment η_j is the area of the region shaded in blue.

write

$$c\left(\nu_j\right) = \frac{\nu_j + \nu_{j+1}}{2}$$

for the location of the central point of the block between ν_j and ν_{j+1}. Then, if $\xi \sim \nu$ and $\xi = \eta_1 \sqcup \eta_2 \sqcup \cdots \sqcup \eta_{|\nu|}$, we write

$$A(\xi) = \sum_{j=1}^{|\nu|} \left(n - c\left(\nu_j\right)\right) h(\eta_j). \tag{2.1.1}$$

As we see in Figure 2.3, this representation of the area separates the trajectory into horizontal strips corresponding to each of the increments. Note that this interpretation of the area is based on a linear interpolation between the heights, rather than strictly the heights themselves; we keep this in mind as we define the height functions.

We are not only interested in the total height of our sub-trajectories ξ, but in how the cumulative height progresses as we add the increments. In the insurance company model in Subsection 1.1.1, for example, we are interested in whether the company's balance has ever dipped below zero, not only in its precise value at some time t.

We represent the developing height of ξ using two functions, defined on the interval $[0,1]$. The first is piecewise-constant, and has the advantage of taking values in the integers; the second is piecewise-linear, and has the advantage of being continuous. For $t \in [0,1]$, let $N_{nt}(\xi)$ be the number of increments "to the left" of nt,

$$N_{nt} = N_{nt}(\xi) = \sup\left\{j \geq 0 : \nu_j \leq nt\right\}.$$

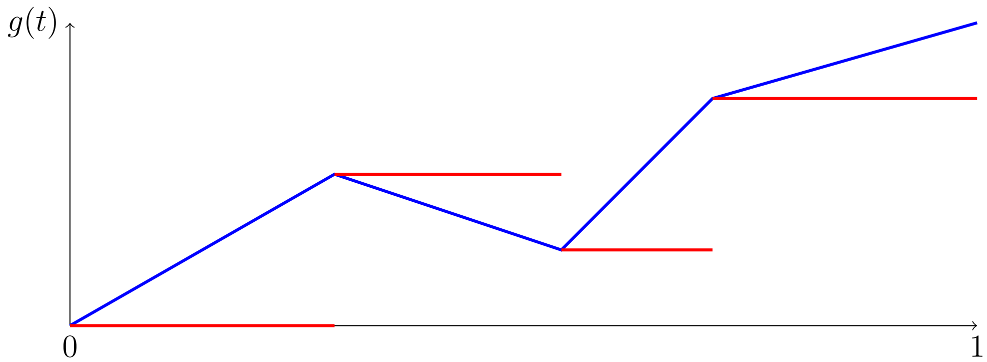

Figure 2.4: The height functions associated with a realisation of a sub-trajectory of width 24. The piecewise-constant height function G is shown in red, while the piecewise-linear version g is shown in blue.

Then we let

$$G(t) = \sum_{i=1}^{N_{nt}} h(\eta_i)$$

be the piecewise-constant cumulative height function and

$$g(t) = G(t) + \frac{nt - \nu_{N_{nt}}}{\nu_{N_{nt}+1} - \nu_{N_{nt}}} h(\eta_{N_{nt}+1})$$

be the piecewise linear interpolation between the cumulative heights, as shown in Figure 2.4. In the context of Subsection 1.1.1, $G(t)$ corresponds to the situation of the insurance company, where each claim arrives "all at once", whereas $g(t)$ represents the situation of the freelancer, where the payments arrive gradually over time.

Throughout this thesis, we use two different regimes to describe the behaviour of the sub-trajectories: when we talk about the height function G, which takes arguments in $[0, 1]$, we will use

$$\mathbf{t} = (t_1, \ldots, t_k)$$

with $0 \equiv t_0 < t_1 < \cdots < t_k < t_{k+1} \equiv 1$; when working with partitions, on the range $[0, n]$, we will instead write

$$\mathbf{T} = (T_1, \ldots, T_k),$$

with the convention that $0 \equiv T_0 < T_1 < \cdots < T_k < T_{k+1} \equiv n$. We say that such $\mathbf{t}$ (or $\mathbf{T}$) is a *vector of times.*

Note that while the rescaling $[0, n] \mapsto [0, 1]$ does not affect the height of the trajectory, the area is rescaled; in particular, we have

$$A(\xi) = n \int_0^1 g(t)dt.$$

Throughout this thesis, we will use the definition of area based on g rather than G, including in the sections dealing with the convergence of G. We prefer g because in the context of potential applications, we expect the continuous version to be of more interest.

In our study of the finite-dimensional distributions, we will deal with the increments of G and g. For $s < t$, let

$$G[s, t] = G(t) - G(s)$$
$$g[s, t] = g(t) - g(s),$$

so that, if $s < t < u$, $G[s, u] = G[s, t] + G[t, u]$.

We denote the finite-dimensional distributions of G corresponding to $\mathbf{t}$ by

$$G_n(\mathbf{t}) = \Big(G[0, t_1], G[t_1, t_2], \ldots, G[t_{k-1}, t_k]\Big), \tag{2.1.2}$$
$$Y_n(\mathbf{t}) = \Big(G[0, t_1], G[t_1, t_2], \ldots, G[t_{k-1}, t_k], G[t_k, 1], \int_0^1 g(t)dt\Big). \tag{2.1.3}$$

In the particular case $k = 0$, we write

$$Y_n(1) = \left(G(1), \int_0^1 g(t)dt\right).$$

With our height functions defined, we now introduce a probability distribution on the sub-trajectories. Place a measure λ on $\mathcal{F}$, such that $\lambda(\eta) > 0$ for all $\eta \in \mathcal{F}$, and such that

$$\sum_{\eta \in \mathcal{F}} \lambda(\eta) = 1.$$

Our choice of λ allows us to define a probability measure on $\mathcal{H}_n$. For $\xi \in \mathcal{H}_n$ with $\xi = \eta_1 \sqcup \eta_2 \sqcup \cdots \sqcup \eta_m$, let

$$\lambda(\xi) = \prod_{j=1}^{m} \lambda(\eta_j).$$

This defines a measure $\mathbb{P}$ on $(\mathcal{F}, 2^{\mathcal{F}})$, via $\mathbb{P}(\{\xi\}) = \lambda(\xi)$.

Now, for any $\mathcal{A} \subset \mathcal{H}_n$, let

$$\mathbb{P}_n(\mathcal{A}) = \frac{\sum_{\xi \in \mathcal{H}_n} \mathbb{1}\{\xi \in \mathcal{A}\} \lambda(\xi)}{\sum_{\xi' \in \mathcal{H}_n} \lambda(\xi')}.$$

We denote the related expectation by $\mathbb{E}_n$.

In this thesis, we are interested in the behaviour of $t \mapsto g(t)$ under conditions on the value of $Y_n(1)$; for $\mathcal{A} \subset \mathcal{H}_n$, let

$$\begin{aligned}\mathbb{P}_n^{\alpha,\beta}(\mathcal{A}) &= \mathbb{P}_n\left(\mathcal{A} \mid Y_n(1) = (\alpha, \beta)\right) \\ &= \frac{\mathbb{P}_n\left(\mathcal{A} \,\cap\, \{Y_n(1) = (\alpha, \beta)\}\right)}{\mathbb{P}_n\left(Y_n(1) = (\alpha, \beta)\right)}\end{aligned}$$

whenever (α, β) is such that $\mathbb{P}_n\left(Y_n(1) = (\alpha, \beta)\right) > 0$. (Note in particular that we must have $(\alpha, \beta) \in \mathbb{Z} \times \frac{1}{2n}\mathbb{Z}$.) We write $\mathbb{E}_n^{\alpha,\beta}$ for expectations under this conditional distribution.

Here, we introduce some assumptions about λ. Without loss of generality, we suppose that, for all c,

$$\mathbb{P}\left(\frac{h(\eta)}{w(\eta)} = c\right) < 1;$$

in other words, our random walk is not composed solely of increments with gradient c. (In this case, we always have $g(t) = nct$ and our description of $g(t) - nct$ would be very short!) For notational simplicity, we assume that $\mathbb{E}[h(\eta)] = 0$, which is indeed the case in many models which arise in applications. Otherwise, all of our results still hold in the general case $\mathbb{E}[h(\eta)] \neq 0$, with the condition $Y_n(1) = (\alpha, \beta)$ replaced with $Y_n(1) - \mathbb{E}[Y_n(1)] = (\alpha, \beta)$, and corresponding adjustments to the tilts, measures and constants introduced later.

We suppose that increments selected according to λ have finite exponential moments:

Assumption 2.1.6. *There exist $u_1 > 0$ and $u_2 > 0$ such that*

$$\sum_{\eta \in \mathcal{F}} e^{u_1 |h(\eta)| + u_2 w(\eta)} \lambda(\eta) < \infty.$$

This is equivalent to the requirement that, under λ, $\|\eta\|$ has exponential moments for any norm $\|\cdot\|$ on $\mathbb{R}^2$; in particular, the height and width of increments of our random walk have jointly finite exponential moments.

We also introduce the family of tilted probability distributions associated with λ. First, for $u \in \mathbb{R}$ and $\mathcal{A} \subset \mathcal{F}_k$, let

$$\mathbb{Q}_k^u(\mathcal{A}) = \frac{\sum_{\eta \in \mathcal{F}_k} \mathbb{1}\{\eta \in \mathcal{A}\} \lambda(\eta) \exp\{uh(\eta)\}}{\sum_{\eta' \in \mathcal{F}_k} \lambda(\eta') \exp\{uh(\eta')\}}.$$

Next, for $\mathbf{u} = (u_1, u_0) \in \mathbb{R}^2$ and $\mathcal{A} \subset \mathcal{H}_n$, let

$$\mathbb{Q}_n^{\mathbf{u}}(\mathcal{A}) = \frac{\sum_{\xi \in \mathcal{H}_n} \mathbb{1}\{\xi \in \mathcal{A}\} \lambda(\xi) \exp\left\{u_1 h(\xi) + \frac{u_0}{n} A(\xi)\right\}}{\sum_{\xi' \in \mathcal{H}_n} \lambda(\xi') \exp\left\{u_1 h(\xi') + \frac{u_0}{n} A(\xi')\right\}}. \tag{2.1.4}$$

Note that we use the area corresponding to the functional version of the trajectory, which is equal to $\frac{1}{n} A(\xi)$. The expectations associated with $\mathbb{Q}^u$ and $\mathbb{Q}_n^{\mathbf{u}}$ will be denoted $\bar{\mathbb{E}}^u$ and $\bar{\mathbb{E}}_n^{\mathbf{u}}$, respectively.

These distributions are referred to as the *Cramér transforms* of $\mathbb{P}_n$ by den Hollander [33]; they also appear in Fredrik Esscher's article *On the probability function in the collective theory of risk* [21], and so are referred to as Esscher transforms in the field of actuarial science.

For $k \geq 1$, $n \geq 1$, and $z \in \mathbb{C}$, we write

$$F_k(z) = \sum_{\eta \in \mathcal{F}_k} \lambda(\eta) \exp\{zh(\eta)\},$$

$$H_n(z) = \sum_{\xi \in \mathcal{H}_n} \lambda(\xi) \exp\{zh(\xi)\},$$

and, for $\mathbf{z} = (z_1, z_0) \in \mathbb{C}^2$,

$$\mathcal{B}_n(\mathbf{z}) = \sum_{\xi \in \mathcal{H}_n} \lambda(\xi) \exp\left\{ z_1 h(\xi) + \frac{z_0}{n} A(\xi) \right\}.$$

We use the convention $H_0(z) \equiv \mathcal{B}_0(z) \equiv 1$.

We will see in the following sections how the partition functions $F_k(z)$, $H_n(z)$, and $\mathcal{B}_n(\mathbf{z})$ are connected to each other. For now, we note that, for $v \in \mathbb{C}$,

$$\bar{\mathbb{E}}_k^u \left[\exp\{vh(\eta)\}\right] = \frac{F_k(u+v)}{F_k(u)},$$

while for $\mathbf{u} \in \mathbb{R}^2$ and $\mathbf{v} \in \mathbb{C}^2$,

$$\bar{\mathbb{E}}_n^{\mathbf{u}} \left[\exp\{\mathbf{v} \cdot Y_n(1)\}\right] = \frac{\mathcal{B}_n(\mathbf{u}+\mathbf{v})}{\mathcal{B}_n(\mathbf{u})}.$$

A particularly useful property of the Cramér transforms arises from the observation that, for any $\mathbf{u} \in \mathbb{R}^2$, we have

$$\mathbb{Q}_n^{\mathbf{u}}\left(Y_n(1) = (\alpha, \beta)\right) = \frac{e^{u_1\alpha + u_0\beta}}{\mathcal{B}_n(\mathbf{u})} \mathbb{P}_n\left(Y_n(1) = (\alpha, \beta)\right) \tag{2.1.5}$$

for any $(\alpha, \beta) \in \mathbb{N} \times \frac{1}{2}\mathbb{N}$, and

$$\mathbb{Q}_n^{\mathbf{u}}\left(Y_n(\mathbf{t}) = \mathbf{x}\right) = \frac{e^{u_1\alpha + u_0\beta}}{\mathcal{B}_n(\mathbf{u})} \mathbb{P}_n\left(Y_n(\mathbf{t}) = \mathbf{x}\right) \tag{2.1.6}$$

for any $\mathbf{x} \in \mathbb{N}^{k+1} \times \frac{1}{2n}\mathbb{N}$ such that $x_1 + \cdots + x_{k+1} = \alpha$, $x_{k+2} = \beta$.

In particular, for any such $\mathbf{x}$,

$$\mathbb{P}_n\left(Y_n(\mathbf{t}) = \mathbf{x} | Y_n(1) = (\alpha, \beta)\right) = \frac{\mathbb{P}_n\left(Y_n(\mathbf{t}) = \mathbf{x}\right)}{\mathbb{P}_n\left(Y_n(1) = (\alpha, \beta)\right)} = \frac{\mathbb{Q}_n^{\mathbf{u}}\left(Y_n(\mathbf{t}) = \mathbf{x}\right)}{\mathbb{Q}_n^{\mathbf{u}}\left(Y_n(1) = (\alpha, \beta)\right)}.$$

We can therefore select a convenient tilt $\mathbf{u}$ - or sequence of tilts $\mathbf{u}_n$ - under which to study the asymptotics of such ratios of probabilities.

Throughout this thesis, we will use $\|\cdot\|$ to represent the Euclidean norm on $\mathbb{R}^k$ for $k \geq 1$, and we write

$$B(a, r) = \left\{ x \in \mathbb{R}^k : \|x - a\| < r \right\}.$$

We denote complex numbers by z, and their real and imaginary parts by u and v respectively, so that $z = u + iv$.

2.2 Properties of the Partition Functions: Part One

The rest of this chapter is dedicated to establishing the relationships between the partition functions F_k, H_n, and $\mathcal{B}_n$, as well as approximations found by considering only sub-trajectories ξ in which the width, height, or gradient of increments is bounded. We begin with an analysis of F_k and H_n, based on the renewal structure of the horizontal projections of our walk.

We begin with a property which will hold for all three partition functions.

Proposition 2.2.1. *For all $k \geq 1$ and $z = u + iv$,*

$$|F_k(z)| \leq F_k(u).$$

Proof. We have

$$\begin{aligned} |F_k(z)| &= \left| \sum_{\eta \in \mathcal{F}_k} \lambda(\eta) e^{zh(\eta)} \right| \\ &\leq \sum_{\eta \in \mathcal{F}_k} \left| \lambda(\eta) e^{zh(\eta)} \right| \\ &\leq \sum_{\eta \in \mathcal{F}_k} \lambda(\eta) e^{uh(\eta)} \\ &\leq F_k(u). \end{aligned}$$

□

To see the corresponding result for H_n, we use a polymer decomposition (see, for instance, [41]).

Proposition 2.2.2. *For all $z \in \mathbb{C}$ and $n \in \mathbb{N}$,*

$$H_n(z) = \sum_{\nu \in \mathcal{P}_n} \prod_{j=1}^{|\nu|} F_{\nu_j - \nu_{j-1}}(z).$$

Proof. We choose $\nu \in \mathcal{P}_n$ and consider all $\xi \in \mathcal{H}_n$ such that $\xi \sim \nu$. For such trajectories, we have

$$\begin{aligned}
&\sum_{\xi\sim\nu}\prod_{j=1}^{|\nu|}\lambda(\eta_j)\exp\{zh(\eta_j)\}\\
&=\prod_{j=1}^{|\nu|}\sum_{\eta_j\in\mathcal{F}_{\nu_j-\nu_{j-1}}}\lambda(\eta_j)\exp\{zh(\eta_j)\}\\
&=\prod_{j=1}^{|\nu|}F_{\nu_j-\nu_{j-1}}(z)\,.
\end{aligned}$$

Taking the sum over all possible profiles $\nu \in \mathcal{P}_n$ we get

$$H_n(z)=\sum_{\nu\in\mathcal{P}_n}\prod_{j=1}^{|\nu|}F_{\nu_j-\nu_{j-1}}(z)\,,$$

as claimed. □

Combining Propositions 2.2.1 and 2.2.2, we have the following two corollaries.

Corollary 2.2.3. *For all $n \geq 1$ and $z = u + iv$,*

$$|H_n(z)| \leq H_n(u)\,.$$

Corollary 2.2.4. *For all $z \in \mathbb{C}$ and $n \in \mathbb{N}$,*

$$H_n(z)=\sum_{k\leq n}F_k(z)\,H_{n-k}(z)\,. \tag{2.2.1}$$

Proof. For $k \leq n$, consider the set of partitions $\nu \in \mathcal{P}_n$ such that $\nu_1 = k$. This set is in one-to-one correspondence with $\mathcal{P}_{[k,n]}$, and hence with $\mathcal{P}_{n-k}$, so that

$$\begin{aligned}
H_n(z)&=\sum_{k\leq n}\sum_{\nu\in\mathcal{P}_n:\nu_1=k}\prod_{j=1}^{|\nu|}F_{\nu_j-\nu_{j-1}}(z)\\
&=\sum_{k\leq n}F_k(z)\sum_{\nu\in\mathcal{P}_n:\nu_1=k}\prod_{j=2}^{|\nu|}F_{\nu_j-\nu_{j-1}}(z)\\
&=\sum_{k\leq n}F_k(z)\sum_{\pi\in\mathcal{P}_{[k,n]}}\prod_{j=1}^{|\pi|}F_{\pi_j-\pi_{j-1}}(z)\\
&=\sum_{k\leq n}F_k(z)\,H_{n-k}(z)\,,
\end{aligned}$$

where we use Proposition 2.2.2 in the final line. □

Corollary 2.2.5. *For any partition* $\nu \in \mathcal{P}_n$ *and* $u \in \mathbb{R}$,

$$H_n(u) \geq \prod_{j=1}^{|\nu|} F_{\nu_j - \nu_{j-1}}(u).$$

Equation (2.2.1) represents the renewal structure of the horizontal projections, which we can separate according to the location of the first renewal. It also allows us to place ourselves in the context of [37], and to inherit many of Ioffe's results, both about the convergence of $\frac{1}{n}\log H_n(u)$, and about the behaviour of the renormalised increment width distribution.

Wherever it exists and is finite, let

$$m(u) = \lim_{n\to\infty} \frac{1}{n} \log H_n(u).$$

Remark 2.2.6. The existence of m at $u = u_1$ is a consequence of Assumption 2.1.6, as can be seen, for example, in Section XIII.10 of [22]. As a result, m exists for $u \in [0, u_1]$.

Note that, in the i.i.d. model in Example 2.1.2, the partition functions are multiplicative and we have

$$H_n(u) = (F_1(u))^n$$

for all n and u, so that

$$m(u) = \log F_1(u)$$

and

$$\frac{1}{n} \log H_n(u) = m(u).$$

In the more general setting, we might heuristically consider m as representing the i.i.d. random walk model which "best approximates" the renewal random walk, in the sense that the rescaled log-characteristic function

$$\frac{1}{n} \log \frac{H_n(u + iv)}{H_n(u)}$$

is well-approximated by $m(u+iv) - m(u)$.

Assumption 2.2.7. *We assume that there exists $u^\star \in \mathbb{R}$ such that*

1. *$u^\star$ is in the interior of the set*

$$\{u \in \mathbb{R} : m(u) < \infty\}.$$

2. *We have*

$$\lim_{n\to\infty} \frac{1}{n} \log H_n(u^\star) > \limsup_{k\to\infty} \frac{1}{k} \log F_k(u^\star).$$

Remark 2.2.8. A weaker version of part 2, namely

$$\limsup_{n\to\infty} \frac{1}{n} \log H_n(u) \geq \limsup_{n\to\infty} \frac{1}{n} \log F_n(u),$$

holds for all $u \in \mathbb{R}$, as a consequence of the renewal structure.

To see this, fix $u \in \mathbb{R}$ and choose $k \in \mathbb{N}$ such that $F_k(u) > 0$, and consider the subsequence $\{H_{nk}(u)\}_{n\in\mathbb{N}}$. From Corollary 2.2.5, we know that

$$\begin{aligned} H_{nk}(u) &\geq \prod_{j=1}^{n} F_k(u) \\ &\geq (F_k(u))^n \end{aligned}$$

and hence

$$\log H_{nk}(u) \geq n \log F_k(u).$$

Dividing by nk and letting $k \to \infty$ gives the desired inequality.

Under Assumption 2.2.7, we have the following result.

Proposition 2.2.9. *[Theorem 2.1, [37]] There exist a constant c and a complex neighbourhood $\mathcal{Z}$ of $u^\star$, on which m can be extended to an analytic function, the function*

$$\mu(z) = \left(\sum_k k e^{-km(z)} F_k(z)\right)^{-1}$$

is non-zero and analytic and, uniformly in $z \in \mathcal{Z}$, as $n \to \infty$,

$$\frac{1}{n}\log H_n(z) - m(z) = \frac{1}{n}\log \mu(z) + o(e^{-cn}).$$

The following two properties arise as part of the proof of Theorem 2.1 in [37], and are particularly useful.

Corollary 2.2.10. *For all $u \in \mathcal{Z} \cap \mathbb{R}$, the sequence $\{e^{-km(u)}F_k(u)\}_{k\in\mathbb{N}}$ is a probability distribution, and there exists $\rho = \rho(u) > 0$ such that*

$$\sum_{k\geq 1} e^{k(-m(u)+\rho)} F_k(u) < \infty.$$

In other words, the probability distribution has exponential tails.

Note that the mean of the distribution $\{e^{-km(u)}F_k(u)\}$ is $\frac{1}{\mu(u)} \geq 1$.

Remark 2.2.11. Proposition 2.2.9 allows us to obtain a sharper asymptotic for the convergence of $\frac{1}{n}\log H_n(u)$ to $m(u)$, in terms of the mean step length under $\{e^{km(u)}F_k(u)\}$. This, in turn, will help us to understand the behaviour of $\mathcal{B}_n(u)$ in greater detail, while Corollary 2.2.10 allows us to estimate the impact of trajectories with particularly wide increments.

Proposition 2.2.12. *There exists a compact region $\mathcal{Z}' \subset \mathcal{Z}$, on which, for all $j \geq 1$, the function*

$$f_j(z) = F_j(z)\, e^{-jm(z)}$$

is analytic and bounded, along with its derivatives. In particular, there exist positive constants $\tau > 0$ and $c_0, c_1, \ldots$ for which the following uniform bounds hold:

$$\sup_{z\in\mathcal{Z}'} |f_j(z)| \leq c_0 e^{-j\tau}$$

$$\sup_{z\in\mathcal{Z}'} \left| \left(\frac{d}{dz}\right)^k f_j(z) \right| \leq c_k e^{-j\tau}$$

Proof. Firstly, the functions f_j are analytic since both F_j and $\exp m$ are analytic on $\mathcal{Z}$.

Next, for $u \in \mathcal{Z} \cap \mathbb{R}$, $f_j(u) \geq 0$ and Corollary 2.2.10 implies that for each u there exist finite positive constants C and τ such that

$$f_j(u) \leq C_0 e^{-j\tau} \tag{2.2.2}$$

holds for each j. We can find a compact interval $\mathcal{U}$ of the real line such that $\mathcal{U} \subset \mathcal{Z} \cap \mathbb{R}$, on which the upper bounds in Equation (2.2.2) can be made uniform.

Now, for $z \in \mathcal{Z}$, we have

$$|f_j(z)| \leq \left| e^{j(m(u)-m(z))} \right| f_j(u)$$

and, since m is analytic, there exist a bounded region $\mathcal{Z}' \subset \mathcal{Z}$ with $\mathcal{Z}' \cap \mathbb{R} = \mathcal{U}$, and a constant $\tau' < \tau$, such that

$$\sup_{z \in \mathcal{Z}'} \left| e^{j(m(u)-m(z))} \right| \leq e^{j(\tau - \tau')}$$

and hence

$$\sup_{z \in \mathcal{Z}'} |f_j(z)| \leq C_0 e^{-j\tau'}.$$

Finally, we take $\varepsilon > 0$ such that $\mathcal{Z}^\varepsilon = \{z \in \mathcal{Z} : B(z, \varepsilon) \subset \mathcal{Z}'\}$ is non-empty. Using the estimates in Proposition B.1.1, we have

$$\left| \left(\frac{d}{dz} \right)^k f_j(z) \right| \leq \frac{k!}{\varepsilon^k} C_0 e^{-j\tau'}$$

for all $k \in \mathbb{N}$ and $z \in \mathcal{Z}^\varepsilon$.

□

The following Proposition is a result of the proof of Theorem 2.1, together with Proposition 2.3, in [37].

Proposition 2.2.13. *Let $\varepsilon > 0$ be such that $\mathcal{Z}^\varepsilon = \{z \in \mathcal{Z} : B(z, \varepsilon) \subset \mathcal{Z}\}$ is non-empty. There exists a constant $a > 0$ such that, for all n large enough,*

$$\frac{|H_n(u + iv)|}{H_n(u)} \leq e^{-an|v|^2} \tag{2.2.3}$$

holds uniformly in $z = u + iv \in \mathcal{Z}^{\varepsilon}$ *such that* $|v| < \pi$*.*

Remark 2.2.14. Starting from the neighbourhood $\mathcal{Z}$ from Proposition 2.2.9, we obtain a new, smaller neighbourhood from which we will draw our arguments. First, as in Proposition 2.2.12, we take a compact subset $\mathcal{Z}' \subseteq \mathcal{Z}$; we then find $\varepsilon > 0$ such that $\mathcal{Z}^{2\varepsilon} = \{z \in \mathcal{Z}' : B(z, 2\varepsilon) \subset \mathcal{Z}'\}$ is non-empty.

Then Proposition 2.2.13 holds for all $z \in \mathcal{Z}^{\varepsilon}$, and upper bounds on derivatives such as those in Proposition B.1.1 hold for all $z \in \mathcal{Z}^{2\varepsilon}$. Moreover, the constant ρ in Corollary 2.2.10 can be made uniform in u.

From now on we will refer to this smaller neighbourhood as $\mathcal{Z}$, and we will write $\mathcal{U}$ for its intersection with the real line.

Assumption 2.2.15. *We suppose that there exist constants* C *and* c *such that, for all* $u \in \mathcal{U}$*,* $k \in \mathbb{N}$*, and* $\rho > 0$*, we have*

$$\mathbb{Q}^u(|h(\eta)| > \rho k | w(\eta) = k) \leq Ce^{-c\rho}.$$

Remark 2.2.16. The existence of such constants for fixed k and ρ, at $u = 0$, is a consequence of Assumption 2.1.6; however, the existence of *uniform* constants for all k and ρ is not in general guaranteed.

Example 2.2.17. Any model in which there are only finitely-many possible values of $w(\eta)$ will fulfil Assumption 2.2.15. In particular, under the i.i.d. model from Example 2.1.2, Assumption 2.1.6 implies Assumption 2.2.15.

2.3 Properties of the Partition Functions: Part Two

In this section, we focus on the partition function $\mathcal{B}_n$, and its relationship to F_k and H_n. Several of the results in this section are analogous to those in the previous section; when comparing the two, it is helpful to remember that, if we exclude the area condition by setting $z_0 = 0$, we have

$$\begin{aligned}\mathcal{B}_n\left(z_1, 0\right) &= \sum_{\xi \in \mathcal{H}_n} \lambda(\xi) \exp\left\{z_1 h(\xi)\right\} \\ &= H_n\left(z_1\right).\end{aligned}$$

Throughout the rest of this chapter (and the rest of this thesis), $[a, b]$ will always refer to a segment with integer endpoints.

Proposition 2.3.1. *Let* $\mathbf{z} = (z_1, z_0)$*.*

Then

$$\mathcal{B}_n\left(\mathbf{z}\right) = \sum_{\nu \in \mathcal{P}_n} \prod_{j=1}^{|\nu|} F_{\nu_j - \nu_{j-1}}\left(z_1 + \frac{z_0}{n}\left(n - c\left(\nu_j\right)\right)\right). \tag{2.3.1}$$

The polymer decomposition, analogous to Proposition 2.2.2, forms the basis of our analysis of $\mathcal{B}_n$. To simplify the notation throughout this section, we introduce the following shorthand.

Definition 2.3.2. For $\mathbf{z} \in \mathbb{C}^2$ and $x \in [0, 1]$, let

$$z\left(x\right) = z_1 + z_0(1 - x).$$

Definition 2.3.3. For intervals $[a, b] \subset [0, n]$, let

$$\begin{aligned}F_{[a,b]}\left(\mathbf{z}\right) &= F_{b-a}\left(z_1 + z_0\left(1 - \frac{b+a}{2n}\right)\right) = F_{b-a}\left(z\left(\frac{a+b}{2n}\right)\right), \\ H_{[a,b]}\left(\mathbf{z}\right) &= H_{b-a}\left(z_1 + z_0\left(1 - \frac{b+a}{2n}\right)\right) = H_{b-a}\left(z\left(\frac{a+b}{2n}\right)\right).\end{aligned}$$

Note that, for any partition ν, we can write

$$F_{[\nu_j, \nu_{j+1}]}\left(\mathbf{z}\right) = F_{\nu_{j+1} - \nu_j}\left(z_1 + z_0\left(1 - c(\nu_j)\right)\right).$$

Remark 2.3.4. In our new notation, Equation (2.3.1) becomes

$$\mathcal{B}_n\left(\mathbf{z}\right) = \sum_{\nu \in \mathcal{P}_n} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1}, \nu_j]}\left(\mathbf{z}\right). \tag{2.3.2}$$

Corollary 2.3.5. *For all $n \geq 1$ and $\mathbf{z} = \mathbf{u} + i\mathbf{v}$,*

$$|\mathcal{B}_n(\mathbf{z})| \leq \mathcal{B}_n(\mathbf{u}).$$

Corollary 2.3.6. *For any partition $\nu \in \mathcal{P}_n$ and $\mathbf{u} \in \mathbb{R}^2$,*

$$\mathcal{B}_n(\mathbf{u}) \geq \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}(\mathbf{u}).$$

Proof of Proposition 2.3.1. As in the previous section, we select $\nu \in \mathcal{P}_n$ and consider the trajectories $\xi \in \mathcal{H}_n$ such that $\xi \sim \nu$.

Using our construction of the area (recall Equation (2.1.1), if $\xi \sim \nu$ then

$$\begin{aligned} z_1 h(\xi) + \frac{z_0}{n} A(\xi) &= \sum_{j=1}^{|\nu|} \left(z_1 + \frac{z_0}{n}(n - c(\nu_j)) \right) h(\eta_j) \\ &= \sum_{j=1}^{|\nu|} z\left(\frac{c(\nu_j)}{n}\right) h(\eta_j). \end{aligned}$$

Now,

$$\begin{aligned} \sum_{\xi\sim\nu} \exp\left\{ z_1 h(\xi) + \frac{z_0}{n} A(\xi) \right\} \lambda(\xi) &= \sum_{\xi\sim\nu} \exp\left\{ \sum_{j=1}^{|\nu|} z\left(\frac{c(\nu_j)}{n}\right) h(\eta_j) \right\} \lambda(\xi) \\ &= \sum_{\xi\sim\nu} \prod_{j=1}^{|\nu|} \lambda(\eta_j) \exp\left\{ z\left(\frac{c(\nu_j)}{n}\right) h(\eta_j) \right\} \\ &= \prod_{j=1}^{|\nu|} \sum_{\eta_j \in \mathcal{F}_{\nu_j - \nu_{j-1}}} \lambda(\eta_j) \exp\left\{ z\left(\frac{c(\nu_j)}{n}\right) h(\eta_j) \right\} \\ &= \prod_{j=1}^{|\nu|} F_{\nu_j - \nu_{j-1}}\left(z\left(\frac{c(\nu_j)}{n}\right)\right). \end{aligned}$$

Taking the sum over all partitions $\nu \in \mathcal{P}_n$ gives Equation (2.3.2), as claimed. □

Using this polymer decomposition, we can add a third partition function to those introduced in Definition 2.3.3.

Definition 2.3.7. For any interval $[a, b] \subset [0, n]$, and $\mathbf{z} = (z_1, z_0) \in \mathbb{C}^2$, let

$$\mathcal{B}_{[a,b]}(\mathbf{z}) = \sum_{\pi \in \mathcal{P}_{[a,b]}} \prod_{j=1}^{|\pi|} F_{[\pi_{j-1},\pi_j]}(\mathbf{z}).$$

Proposition 2.3.8. *For any* $0 \leq a \leq b \leq n$ *and* $\mathbf{z} = (z_1, z_0) \in \mathbb{C}^2$*, we have*

$$\mathcal{B}_{[a,b]}(\mathbf{z}) = \mathcal{B}_{b-a}\left(z_1 + z_0\frac{n-b}{n}, z_0\frac{b-a}{n}\right).$$

Proof. Let $\pi \in \mathcal{P}_{[a,b]}$, and let $\nu \in \mathcal{P}_{b-a}$ be such that

$$\nu_j = \pi_j - a$$

for $1 \leq j \leq |\pi|$. Now,

$$\begin{aligned} z_1 + \frac{z_0}{n}(n - c(\pi_j)) &= z_1 + \frac{z_0}{n}(n - (a + c(\nu_j))) \\ &= z_1 + z_0\frac{n-b}{n} + \frac{1}{b-a}z_0\frac{b-a}{n}(b - a - c(\nu_j)). \end{aligned}$$

As a result,

$$\begin{aligned} \mathcal{B}_{[a,b]}(\mathbf{z}) &= \sum_{\nu \in \mathcal{P}_{b-a}} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}\left(z_1 + z_0\frac{n-b}{n}, z_0\frac{b-a}{n}\right) \\ &= \mathcal{B}_{b-a}\left(z_1 + z_0\frac{n-b}{n}, z_0\frac{b-a}{n}\right). \end{aligned}$$

□

Remark 2.3.9. We can view $\mathcal{B}_{[a,b]}(\mathbf{z})$ in two different ways. First,

$$\frac{\mathcal{B}_{[a,b]}(\mathbf{u} + i\mathbf{v})}{\mathcal{B}_{[0,n]}(\mathbf{u})} = \bar{\mathbb{E}}_n^{\mathbf{u}}\left[\exp\left\{iv_1 g([a,b]) + iv_0\int_{\frac{a}{n}}^{\frac{b}{n}} g(t)dt\right\}\right].$$

(Note that when $[a,b] = [0,n]$ we recover $\mathcal{B}_{[0,n]}(\mathbf{z}) = \mathcal{B}_n(\mathbf{z})$.)

In light of Proposition 2.3.8, we also have

$$\frac{\mathcal{B}_{[a,b]}(\mathbf{u} + i\mathbf{v})}{\mathcal{B}_{[a,b]}(\mathbf{u})} = \bar{\mathbb{E}}_{b-a}^{\mathbf{u}[a,b]}\left[\exp\left\{i\mathbf{v}[a,b] \cdot Y_{b-a}(1)\right\}\right],$$

where $\mathbf{z}[a,b] = \left(z_1 + z_0\frac{n-b}{n}, z_0\frac{b-a}{n}\right)$.

Proposition 2.3.10. *For any* $n \in \mathbb{N}$ *and* $\mathbf{z} \in \mathbb{C}^2$*,*

$$\mathcal{B}_n(\mathbf{z}) = \sum_{k \leq n} F_{[0,k]}(\mathbf{z})\,\mathcal{B}_{[k,n]}(\mathbf{z}).$$

Proof. As in the proof of Corollary 2.2.5, fix $k \leq n$ and consider the set of partitions

$\nu \in \mathcal{P}_n$ such that $\nu_1 = k$. This set corresponds exactly with $\mathcal{P}_{[k,n]}$, so that we have

$$\begin{aligned}
\mathcal{B}_n(\mathbf{z}) &= \sum_{k \le n} \sum_{\nu \in \mathcal{P}_n : \nu_1 = k} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1}, \nu_j]}(\mathbf{z}) \\
&= \sum_{k \le n} F_{[0,k]}(\mathbf{z}) \sum_{\pi \in \mathcal{P}_{[k,n]}} \prod_{j=1}^{|\pi|} F_{[\pi_{j-1}, \pi_j]}(\mathbf{z}) \\
&= \sum_{k \le n} F_{[0,k]}(\mathbf{z}) \mathcal{B}_{[k,n]}(\mathbf{z}). \qquad \square
\end{aligned}$$

Proposition 2.3.11. *For any $n \in \mathbb{N}$, $0 < a < n$, and $\mathbf{u} \in \mathbb{R}^2$,*

$$\mathcal{B}_n(\mathbf{u}) \ge \mathcal{B}_{[0,a]}(\mathbf{u}) \mathcal{B}_{[a,n]}(\mathbf{u}).$$

Proof. Consider

$$\mathcal{P}_n^a := \{\nu \in \mathcal{P}_n : \exists j \le |\nu| : \nu_j = a\}.$$

Each partition $\nu \in \mathcal{P}_n^a$ corresponds with exactly one pair $(\pi^1, \pi^2) \in \mathcal{P}_{[0,a]} \times \mathcal{P}_{[a,n]}$, so that

$$\mathcal{P}_n^a = \left\{\nu = (\pi^1, \pi^2) : \pi^1 \in \mathcal{P}_{[0,a]}, \pi^2 \in \mathcal{P}_{[a,n]}\right\}.$$

Now, since $F_{[a,b]}(\mathbf{u}) \ge 0$ for all $[a,b] \subseteq [0,n]$, we have

$$\begin{aligned}
\mathcal{B}_n(\mathbf{u}) &= \sum_{\nu \in \mathcal{P}_n} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1}, \nu_j]}(\mathbf{u}) \\
&\ge \sum_{\nu \in \mathcal{P}_n^a} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1}, \nu_j]}(\mathbf{u});
\end{aligned}$$

writing $\nu = (\pi^1, \pi^2)$, we can separate

$$\prod_{j=1}^{|\nu|} F_{[\nu_{j-1}, \nu_j]}(\mathbf{u})$$

as

$$\prod_{j=1}^{|\pi^1|} F_{\left[\pi^1_{j-1}, \pi^1_j\right]}(\mathbf{u}) \times \prod_{k=1}^{|\pi^2|} F_{\left[\pi^2_{j-1}, \pi^2_j\right]}(\mathbf{u}).$$

Writing $\mathcal{P}_n^a = \mathcal{P}_{[0,a]} \times \mathcal{P}_{[a,n]}$, we have

$$\sum_{\nu \in \mathcal{P}_n^a} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}(\mathbf{u}) = \sum_{\pi^1 \in \mathcal{P}_{[0,a]}} \prod_{j=1}^{|\pi^1|} F_{\left[\pi^1_{j-1},\pi^1_j\right]}(\mathbf{u}) \sum_{\pi^2 \in \mathcal{P}_{[a,n]}} \prod_{k=1}^{|\pi^2|} F_{\left[\pi^2_{j-1},\pi^2_j\right]}(\mathbf{u})$$
$$= \mathcal{B}_{[0,a]}(\mathbf{u}) \mathcal{B}_{[a,n]}(\mathbf{u}). \qquad \square$$

Corollary 2.3.12. *For any integers $a_0 = 0 < a_1 < a_2 < \cdots < a_k < n = a_{k+1}$, and any $\mathbf{u} \in \mathbb{R}^2$,*

$$\mathcal{B}_n(\mathbf{u}) \geq \prod_{j=1}^{k+1} \mathcal{B}_{[a_{j-1},a_j]}(\mathbf{u}).$$

For any integers $b_0 = 0 < b_1 < b_2 < \cdots < b_{2k} < n = b_{2k+1}$ and any $\mathbf{u} \in \mathbb{R}^2$,

$$\mathcal{B}_n(\mathbf{u}) \geq \prod_{j=0}^{k} \mathcal{B}_{[b_{2j},b_{2j+1}]}(\mathbf{u}) \prod_{j=1}^{k} F_{[b_{2j-1},b_{2j}]}(\mathbf{u}).$$

Remark 2.3.13. We have

$$\mathbb{Q}_n^{\mathbf{u}}(\xi \text{ has cutpoints at } a_j, 1 \leq j \leq k) = \frac{1}{\mathcal{B}_n(\mathbf{u})} \prod_{j=1}^{k+1} \mathcal{B}_{[a_{j-1},a_j]}(\mathbf{u}),$$

and

$$\mathbb{Q}_n^{\mathbf{u}}(\xi \text{ has } \textit{consecutive} \text{ cutpoints at } b_{2j-1} \text{ and } b_{2j}, 1 \leq j \leq k)$$
$$= \frac{1}{\mathcal{B}_n(\mathbf{u})} \prod_{j=0}^{k} \mathcal{B}_{[b_{2j},b_{2j+1}]}(\mathbf{u}) \prod_{j=1}^{k} F_{[b_{2j-1},b_{2j}]}(\mathbf{u}).$$

Definition 2.3.14. Let

$$\mathcal{Z}^\Delta = \left\{(z_1, z_0) \in \mathbb{C}^2 : \forall t \in [0,1], z_1 + tz_0 \in \mathcal{Z}\right\}.$$

Similarly, let

$$\mathcal{U}^\Delta = \left\{(u_1, u_0) \in \mathbb{R}^2 : \forall t \in [0,1], u_1 + tu_0 \in \mathcal{U}\right\}.$$

We can also consider $\mathcal{Z}^\Delta$ and $\mathcal{U}^\Delta$ in terms of the complex line segments $[z_1, z_1 + z_0]$ and $[u_1, u_1 + u_0]$.

Proposition 2.3.15. *For any* $[a,b] \subseteq [0,n]$*, and* $\mathbf{u} \in \mathcal{U}^{\Delta}$*, we have*

$$\mathcal{B}_{[a,b]}(\mathbf{u}) \geq H_{[a,b]}(\mathbf{u}).$$

Proof. Take $\nu \in \mathcal{P}_{[a,b]}$. We write

$$\prod_{j=1}^{|\nu|} F_{\nu_j-\nu_{j-1}}\left(u\left(\frac{c(\nu_j)}{n}\right)\right) = \prod_{j=1}^{|\nu|} F_{\nu_j}\left(u\left(\frac{a+b}{2n}\right)\right) \frac{F_{\nu_j}\left(u\left(\frac{c(\nu_j)}{n}\right)\right)}{F_{\nu_j}\left(u\left(\frac{a+b}{2n}\right)\right)};$$

as we noted in Section 2.1, the ratio of the partition functions can be viewed as an expectation, so that

$$\frac{F_{\nu_j}\left(u\left(\frac{c(\nu_j)}{n}\right)\right)}{F_{\nu_j}\left(u\left(\frac{a+b}{2n}\right)\right)} = \bar{\mathbb{E}}^{u\left(\frac{a+b}{2n}\right)}_{\nu_j-\nu_{j-1}}\left[\exp\left\{\frac{u_0}{n}\left(c(\nu_j) - \frac{a+b}{2}\right)h(\eta_j)\right\}\right]$$

and

$$\begin{aligned}\prod_{j=1}^{|\nu|} \frac{F_{\nu_j}\left(u\left(\frac{c(\nu_j)}{n}\right)\right)}{F_{\nu_j}\left(u\left(\frac{a+b}{2n}\right)\right)} &= \prod_{j=1}^{|\nu|} \bar{\mathbb{E}}^{u\left(\frac{a+b}{2n}\right)}_{\nu_j-\nu_{j-1}}\left[\exp\left\{\frac{u_0}{n}\left(c(\nu_j) - \frac{a+b}{2}\right)h(\eta_j)\right\}\right] \\ &= \bar{\mathbb{E}}^{u\left(\frac{a+b}{2n}\right)}_{\nu}\left[\exp\left\{\frac{u_0}{n}\sum_{j=1}^{|\nu|}\left(c(\nu_j) - \frac{a+b}{2}\right)h(\eta_j)\right\}\right],\end{aligned}$$

where $\bar{\mathbb{E}}^u_\nu$ is the expectation based on the product measure $\mathbb{Q}^u_{\nu_1} \times \mathbb{Q}^u_{\nu_2-\nu_1} \times \cdots \times \mathbb{Q}^u_{n-\nu_{|\nu|}}$.

Now, consider the partition $\pi \in \mathcal{P}_{[a,b]}$ obtained by inverting the order of the blocks of ν, so that

$$\pi_j = b + a - \nu_j \qquad\qquad 1 \leq j \leq |\nu|.$$

We have

$$\prod_{j=1}^{|\nu|} F_{\nu_j}\left(u\left(\frac{a+b}{2n}\right)\right) = \prod_{j=1}^{|\pi|} F_{\pi_j}\left(u\left(\frac{a+b}{2n}\right)\right),$$

while

$$c(\nu_j) - \frac{a+b}{2} = \frac{a+b}{2} - c(\pi_j).$$

We have

$$\prod_{j=1}^{|\nu|} \frac{F_{\nu_j}\left(u\left(\frac{c(\nu_j)}{n}\right)\right)}{F_{\nu_j}\left(u\left(\frac{a+b}{2n}\right)\right)} = \frac{1}{2}\left(\prod_{j=1}^{|\nu|} \frac{F_{\nu_j}\left(u\left(\frac{c(\nu_j)}{n}\right)\right)}{F_{\nu_j}\left(u\left(\frac{a+b}{2n}\right)\right)} + \prod_{j=1}^{|\pi|} \frac{F_{\pi_j}\left(u\left(\frac{c(\pi_j)}{n}\right)\right)}{F_{\pi_j}\left(u\left(\frac{a+b}{2n}\right)\right)}\right),$$

so that

$$\prod_{j=1}^{|\nu|} \frac{F_{\nu_j}\left(u\left(\frac{c(\nu_j)}{n}\right)\right)}{F_{\nu_j}\left(u\left(\frac{a+b}{2n}\right)\right)} = \bar{\mathbb{E}}_{\nu}^{u\left(\frac{a+b}{2n}\right)}\left[\frac{1}{2}\left(e^{\frac{u_0}{n}\sum_{j=1}^{|\nu|}\left(c(\nu_j)-\frac{a+b}{2}\right)h(\eta_j)} + e^{-\frac{u_0}{n}\sum_{j=1}^{|\nu|}\left(c(\nu_j)-\frac{a+b}{2}\right)h(\eta_j)}\right)\right]$$
$$= \bar{\mathbb{E}}_{\nu}^{u\left(\frac{a+b}{2n}\right)}\left[\cosh\left(\frac{u_0}{n}\sum_{j=1}^{|\nu|}\left(c\left(\nu_j\right)-\frac{a+b}{2n}\right)h(\eta_j)\right)\right] \geq 1.$$

Now

$$\begin{aligned}\mathcal{B}_{[a,b]}\left(\mathbf{u}\right) &= \sum_{\nu\in\mathcal{P}_{a,b}}\prod_{j=1}^{|\nu|} F_{\nu_j}\left(u\left(\frac{c\left(\nu_j\right)}{n}\right)\right) \\ &\geq \sum_{\nu\in\mathcal{P}_n}\prod_{j=1}^{|\nu|} F_{\nu_j-\nu_{j-1}}\left(u\left(\frac{a+b}{2n}\right)\right) \\ &\geq H_{[a,b]}\left(\mathbf{u}\right).\end{aligned}$$

□

2.4 Approximation Based on the Width of Increments

In the following sections, we see how restricting our attention to sub-trajectories $\xi \in \mathcal{H}_n$ whose increments do not exceed a given height, width, or gradient affects the value of $\mathcal{B}_n\left(z\right)$. As long as our upper limit on these quantities grows suitably with n, the resulting approximations are close to the true partition function $\mathcal{B}_n\left(z\right)$.

In Proposition 2.3.10, we found a decomposition of $\mathcal{B}_n$ according to the width of the first increment of the sub-trajectory. In this section, we find similar decompositions of $\mathcal{B}_n$ based on the position of an increment elsewhere in the sub-trajectory - for example, the increment exactly in the middle.

2.4.1 Approximations Based on a Single Increment

Definition 2.4.1. For $T \in [0, n]$, let Δ_T be the pair describing the increment at location T in the sub-trajectory ξ; in other words, we write

$$\Delta_T = (\ell, r)$$

if the sub-trajectory contains an increment with endpoints at $T - \ell$ and $T + r$. If T is a cutpoint of the sub-trajectory, we write $\ell = r = 0$. See Figure 2.5 for an example. We write $w_T = \ell_T + r_T$.

If ν is the partition associated with ξ, we write

$$\nu \overset{T}{\sim} (\ell, r)$$

to indicate that, for some j, $\nu_j = T - \ell$, and $\nu_{j+1} = T + r$.

Let

$$\Lambda^T = \{(\ell, r) : 0 < \ell \leq T, 0 < r \leq n - T,\ T - \ell, T - r \in \mathbb{N}\}$$

be the set of all possible values of Δ_T.

When $T \in \mathbb{N}$, we include $(0, 0)$ as an element of Λ^T to represent the possibility that T is a cutpoint.

If $T = 0$ or $T = n$, we define $\Delta_T = (0, 0)$ for every sub-trajectory ξ. In this case, $\Lambda^T = \{(0, 0)\}$.

Let

$$w_T^+ = \begin{cases} w_{T+\frac{1}{2}} & T \in \mathbb{N} \\ w_T & T \notin \mathbb{N} \end{cases}.$$

Then w_T^+ is positive, even when $w_T = 0$; this construction will be useful when we discuss the gradient of the height function g.

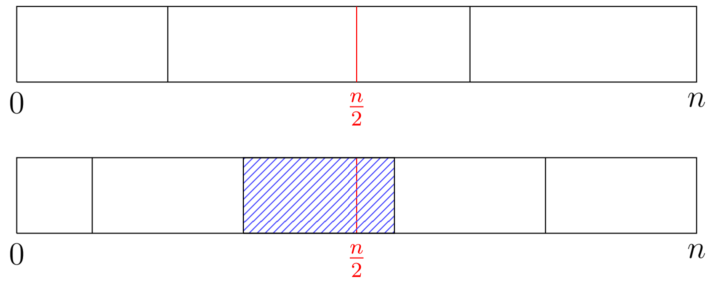

Figure 2.5: Two partitions of $[0, n]$ into integer pieces. In the first, there is a cutpoint at $\frac{n}{2}$; in the second, the increment $\Delta_{\frac{n}{2}}$ is shaded in blue.

Proposition 2.4.2. *For any integer $0 \leq T \leq n$ and $\mathbf{z} \in \mathcal{Z}^{\Delta}$, we have*

$$\mathcal{B}_n(\mathbf{z}) = \sum_{(\ell,r)\in\Lambda^T} \mathcal{B}_{[0,T-\ell]}(\mathbf{z})\, F_{[T-\ell,T+r]}(\mathbf{z})\, \mathcal{B}_{[T+r,n]}(\mathbf{z}). \tag{2.4.1}$$

Proof. We have

$$\mathcal{B}_n(\mathbf{z}) = \sum_{\nu\in\mathcal{P}_n} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}(\mathbf{z}).$$

As in the proof of Proposition 2.3.11, we categorise the partitions $\nu \in \mathcal{P}_n$ using the elements of Λ^T. For $(\ell, r) \in \Lambda^T$, consider the set

$$\mathcal{P}_n^{(\ell,r)} = \left\{\nu \in \mathcal{P}_n : \nu \overset{T}{\sim} (\ell, r)\right\}.$$

The cutpoints of ν "to the left" of Δ_T form a partition of $[0, T-\ell]$, while those "to the right" form a partition of $[T+r, n]$. We can therefore describe the elements of $\mathcal{P}_n^{(\ell,r)}$ using pairs $(\pi, \mu) \in \mathcal{P}_{[0,T-\ell]} \times \mathcal{P}_{[T+r,n]}$, and this characterisation forms a bijection between the set $\mathcal{P}_n^{(\ell,r)}$ and $\mathcal{P}_{[0,T-\ell]} \times \mathcal{P}_{[T+r,n]}$.

As a result,

$$\begin{aligned}\sum_{\nu\overset{T}{\sim}(\ell,r)} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}(\mathbf{z}) &= \sum_{\pi\in\mathcal{P}_{[0,T-\ell]}} \prod_{j=1}^{|\pi|} F_{[\pi_{j-1},\pi_j]}(\mathbf{z})\, F_{[T-\ell,T+r]}(\mathbf{z}) \sum_{\mu\in\mathcal{P}_{[T+r,n]}} \prod_{j=1}^{|\mu|} F_{[\mu_{j-1},\mu_j]}(\mathbf{z}) \\ &= \mathcal{B}_{[0,T-\ell]}(\mathbf{z})\, F_{[T-\ell,T+r]}(\mathbf{z})\, \mathcal{B}_{[T+r,n]}(\mathbf{z}).\end{aligned}$$

Taking the sum over all pairs $(\ell, r) \in \Lambda^T$ gives

$$\mathcal{B}_n(\mathbf{z}) = \sum_{(\ell,r)\in\Lambda^T} \mathcal{B}_{[0,T-\ell]}(\mathbf{z})\, F_{[T-\ell,T+r]}(\mathbf{z})\, \mathcal{B}_{[T+r,n]}(\mathbf{z}),$$

as claimed. Note that the term corresponding to $(\ell, r) = (0,0)$ is

$$\mathcal{B}_{[0,T]}(\mathbf{z})\,\mathcal{B}_{[T,n]}(\mathbf{z})\,.$$

□

We limit our attention to pairs (ℓ, r) such that $\ell + r$ is not "too big".

Definition 2.4.3. For $0 \le \theta \le n$, let

$$\Lambda_\theta = \left\{(\ell, r) \in \Lambda^T : \ell + r < \theta\right\}.$$

Let

$$\mathcal{B}_{n,(T,\theta)}(\mathbf{z}) = \sum_{(\ell,r)\in\Lambda_\theta} \sum_{\nu \overset{T}{\sim} (\ell,r)} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}(\mathbf{z})\,.$$

In practice, we will use $\theta = n^\gamma$, for some $\gamma < 1$; in the next proposition we will see that the penalty for excluding such trajectories from the characteristic function is exponentially small in θ. Later, we will find asymptotic bounds on the behaviour of $F_\theta(z)$ which allow us to control the contribution from such increments in the other direction.

Proposition 2.4.4. *For all $0 < \varepsilon < 1$, there exists a constant $c > 0$ such that for all $\mathbf{z} = \mathbf{u} + i\mathbf{v} \in \mathcal{Z}^\Delta$ and large enough n,*

$$\left|\mathcal{B}_n(\mathbf{z}) - \mathcal{B}_{n,(T,W)}(\mathbf{z})\right| \le Ce^{-cW}\mathcal{B}_n(\mathbf{u}) \tag{2.4.2}$$

holds uniformly in $0 \le T \le n$.

Remark 2.4.5. The left-hand side of Equation 2.4.2 is exactly the part of Equation 2.4.1 coming from partitions in which the increment around T is wider than θ.

Corollary 2.4.6. *In particular there exists a finite constant C such that, for any $0 < t < 1$ and $\mathbf{u} \in \mathcal{U}^\Delta$ and for any $W > 0$ large enough, we have*

$$\mathbb{Q}_n^{\mathbf{u}}(w_{nt} > W) \le Ce^{-cW}.$$

Proof of Proposition 2.4.4. We have

$$\left|\mathcal{B}_n\left(\mathbf{z}\right)-\mathcal{B}_{n,(T,W)}\left(\mathbf{z}\right)\right|=\left|\sum_{\nu\notin\Lambda_W}\prod_{j=1}^{|\nu|}F_{[\nu_{j-1},\nu_j]}\left(\mathbf{z}\right)\right|.$$

For $(\ell,r)\notin\Lambda_W$, we have

$$\frac{1}{\mathcal{B}_n\left(\mathbf{u}\right)}\sum_{\nu\overset{T}{\sim}(\ell,r)}\prod_{j=1}^{|\nu|}F_{[\nu_{j-1},\nu_j]}\left(\mathbf{z}\right)=\frac{\mathcal{B}_{[0,T-\ell]}\left(\mathbf{z}\right)F_{[T-\ell,T+r]}\left(\mathbf{z}\right)\mathcal{B}_{[T+r,n]}\left(\mathbf{z}\right)}{\mathcal{B}_n\left(\mathbf{u}\right)},$$

and by Corollaries 2.3.5 and 2.3.12,

$$\begin{aligned}\left|\frac{\mathcal{B}_{[0,T-\ell]}\left(\mathbf{z}\right)F_{[T-\ell,T+r]}\left(\mathbf{z}\right)\mathcal{B}_{[T+r,n]}\left(\mathbf{z}\right)}{\mathcal{B}_n\left(\mathbf{u}\right)}\right| &\le \frac{\mathcal{B}_{[0,T-\ell]}\left(\mathbf{u}\right)F_{[T-\ell,T+r]}\left(\mathbf{u}\right)\mathcal{B}_{[T+r,n]}\left(\mathbf{u}\right)}{\mathcal{B}_n\left(\mathbf{u}\right)}\\ &\le \frac{\mathcal{B}_{[0,T-\ell]}\left(\mathbf{u}\right)F_{[T-\ell,T+r]}\left(\mathbf{u}\right)\mathcal{B}_{[T+r,n]}\left(\mathbf{u}\right)}{\mathcal{B}_{[0,T-\ell]}\left(\mathbf{u}\right)\mathcal{B}_{[T-\ell,T+r]}\left(\mathbf{u}\right)\mathcal{B}_{[T+r,n]}\left(\mathbf{u}\right)}\\ &\le \frac{F_{[T-\ell,T+r]}\left(\mathbf{u}\right)}{\mathcal{B}_{[T-\ell,T+r]}\left(\mathbf{u}\right)},\end{aligned}$$

Now, we know from Proposition 2.3.15 that

$$\frac{F_{[T-\ell,T+r]}\left(\mathbf{u}\right)}{\mathcal{B}_{[T-\ell,T+r]}\left(\mathbf{u}\right)}\le\frac{F_{[T-\ell,T+r]}\left(\mathbf{u}\right)}{H_{[T-\ell,T+r]}\left(\mathbf{u}\right)}.$$

For $k\in\mathbb{N}$ and $u\in\mathcal{U}$, we have

$$\frac{F_k\left(u\right)}{H_k\left(u\right)}=\frac{F_k\left(u\right)e^{-km(u)}}{H_k\left(u\right)e^{-km(u)}}.$$

For k large enough, Proposition 2.2.9 implies that

$$H_k\left(u\right)e^{-km(u)}\ge\frac{1}{2}\mu(u)$$

for all $u\in\mathcal{U}$, while Corollary 2.2.10 allows us to find constants c and C such that

$$F_k\left(u\right)e^{-km(u)}\le Ce^{-ck}.$$

Since μ is bounded on $\mathcal{U}$, there exists C' such that

$$\frac{F_k\left(u\right)}{H_k\left(u\right)}\le C'e^{-ck}.$$

Now, for each $k \geq W$ there are $k-1$ pairs (ℓ, r) with $\ell + r = k$, so that

$$\sum_{(\ell,r)\notin\Lambda_W} \frac{F_{[T-\ell,T+r]}(\mathbf{u})}{H_{[T-\ell,T+r]}(\mathbf{u})} \leq \sum_{(\ell,r)\notin\Lambda_W} C' e^{-c(\ell+r)}$$
$$\leq \sum_{k\geq W} (k-1)C' e^{-ck}$$
$$\leq C' e^{-c'W},$$

for some constant $c' > 0$ which does not depend on W. □

Proof of Corollary 2.4.6. We have

$$\mathbb{Q}_n^{\mathbf{u}}(w_{nt} > W) = \frac{\sum_{\xi\in\mathcal{H}_n} \mathbb{1}\{w_{nt} > W\}\lambda(\xi)\exp\left\{u_1 h(\xi) + \frac{u_0}{n}A(\xi)\right\}}{\sum_{\xi'\in\mathcal{H}_n}\lambda(\xi')\exp\left\{u_1 h(\xi') + \frac{u_0}{n}A(\xi')\right\}}$$
$$= \frac{\mathcal{B}_n(\mathbf{u}) - \mathcal{B}_{n,(T,W)}(\mathbf{u})}{\mathcal{B}_n(\mathbf{u})};$$

by Proposition 2.4.4, we therefore have

$$\mathbb{Q}_n^{\mathbf{u}}(w_{nt} > W) \leq Ce^{-cW}.$$

□

2.4.2 Approximations Based on Several Increments

Proposition 2.4.4 tells us that the set of sub-trajectories with a particularly wide increment around T have an exponentially small contribution to the behaviour of $\mathcal{B}_n$. We can extend this analysis to require that multiple increments are, simultaneously, not wider than n^ε. Recalling Definition 2.4.1, we introduce the following notation.

Definition 2.4.7. For $\mathbf{T} = (T_1, \ldots, T_k)$, let $\Delta_{\mathbf{T}}$ be the list of pairs describing the increments at locations $T_1, \ldots, T_k$ in the sub-trajectory ξ, so that

$$\Delta_{\mathbf{T}} = (\boldsymbol{\ell}, \mathbf{r}) = \Big((\ell_1, r_1), \ldots, (\ell_k, r_k)\Big)$$

if $\Delta_{T_j} = (\ell_j, r_j)$ for each $1 \leq j \leq k$. Note that the pairs (ℓ_j, r_j) may not all describe distinct intervals, for example in the event that $T_j + r_j > T_{j+1}$ holds for some j.

If ν is the partition associated with ξ, we write

$$\nu \overset{\mathbf{T}}{\sim} (\boldsymbol{\ell}, \mathbf{r})$$

to indicate that $\nu \overset{T_j}{\sim} (\ell_j, r_j)$ for $1 \leq j \leq k$.

For $0 \leq \theta \leq n$ and $k \in \mathbb{N}$, let

$$\Lambda_{k,\theta} = \{(\boldsymbol{\ell}, \mathbf{r}) : (\ell_j, r_j) \in \Lambda_\theta, 1 \leq j \leq k\}.$$

Where necessary, we use the convention $\ell_{k+1} = r_0 = 0$.

In the rest of this section, we focus on an interval $[a, b] \subseteq [0, n]$. We suppose that

$$a = T_0 < T_1 < T_2 < \cdots < T_k < b = T_{k+1},$$

and that

$$\min_{1 \leq j \leq k+1} T_j - T_{j-1} > 2\theta.$$

In particular we must have $b - a > 2\theta$.

Proposition 2.4.8. *For all* $(\boldsymbol{\ell}, \mathbf{r}) \in \Lambda_{k,\theta}$ *and all* $\mathbf{z} \in \mathcal{Z}^\Delta$,

$$\sum_{\nu \overset{\mathbf{T}}{\sim} (\boldsymbol{\ell}, \mathbf{r})} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1}, \nu_j]}(\mathbf{z}) = \prod_{j=1}^{k+1} \mathcal{B}_{[T_{j-1}+r_{j-1}, T_j - \ell_j]}(\mathbf{z}) \prod_{j=1}^{k} F_{[T_j - \ell_j, T_j + r_j]}(\mathbf{z}).$$

Proof. The requirement

$$\max_{1 \leq j \leq k} (\ell_j + r_j) < \theta < \frac{1}{2} \min_{1 \leq j \leq k} T_j - T_{j-1}$$

ensures that the increments Δ_{T_j} are distinct. As in Proposition 2.4.2, each element of the set

$$\left\{\nu \in \mathcal{P}_n : \nu \overset{\mathbf{T}}{\sim} (\boldsymbol{\ell}, \mathbf{r})\right\}$$

corresponds to precisely one element of the set

$$\left\{(\pi^1, \pi^2 \ldots, \pi^k) : \pi^j \in \mathcal{P}_{[T_{j-1}+r_{j-1}, T_j - \ell_j]}, 1 \leq j \leq k+1\right\}.$$

As a result,

$$\sum_{\nu \overset{\mathbf{t}}{\sim} (\boldsymbol{\ell},\mathbf{r})} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}(\mathbf{z}) = \left(\sum_{\pi^1} \prod_{j=1}^{|\pi^1|} F_{\left[\pi^1_{j-1},\pi^1_j\right]}(\mathbf{z})\right) \times F_{[t_1-\ell_1,t_1+r_1]}(\mathbf{z}) \times$$
$$\times \left(\sum_{\pi^2} \prod_{j=1}^{|\pi^2|} F_{\left[\pi^2_{j-1},\pi^2_j\right]}(\mathbf{z})\right) \times \cdots \times \left(\sum_{\pi^{k+1}} \prod_{j=1}^{|\pi^{k+1}|} F_{\left[\pi^{k+1}_{j-1},\pi^{k+1}_j\right]}(\mathbf{z})\right)$$
$$= \prod_{j=1}^{k+1} \mathcal{B}_{[t_{j-1}+r_{j-1},T_j-\ell_j]}(\mathbf{z}) \prod_{j=1}^{k} F_{[T_j-\ell_j,T_j+r_j]}(\mathbf{z}).$$

□

Remark 2.4.9. In Proposition 2.4.8, we separate each sub-trajectory ξ into $k+1$ smaller sub-trajectories, joined by k increments. We refer to the horizontal projections of each of these increments as

$$\Delta_{\mathbf{T},j} = [T_j - \ell_j, T_j + r_j],$$

and those of the sub-trajectories between them as

$$\xi_{\mathbf{T},j} = [T_{j-1} + r_{j-1}, T_j - \ell_j].$$

Definition 2.4.10. We write

$$\mathcal{B}_{[a,b],(\mathbf{T},\theta)}(\mathbf{z}) = \sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{k,\theta}} \sum_{\nu \overset{\mathbf{T}}{\sim} (\boldsymbol{\ell},\mathbf{r})} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}(\mathbf{z}).$$

When $[a,b]=[0,n]$ we write

$$\mathcal{B}_{[0,n],(\mathbf{T},\theta)}(\mathbf{z}) = \mathcal{B}_{n,(\mathbf{T},\theta)}(\mathbf{z})$$

Remark 2.4.11. For all segments of non-zero width, we have

$$\mathcal{B}_{[a,b],(\mathbf{T},\theta)}(z_1,z_0) = \mathcal{B}_{b-a,(\mathbf{T}',\theta)}\left(z_1 + z_0\frac{n-b}{n}, z_0\frac{b-a}{n}\right),$$

where $T'_j = \frac{1}{b-a}(T_j - a)$.

Remark 2.4.12. Proposition 2.4.8 allows us to write

$$\mathcal{B}_{[a,b],(\mathbf{T},\theta)}(\mathbf{z}) = \sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{k,\theta}} \prod_{j=1}^{k+1} \mathcal{B}_{[T_{j-1}+r_{j-1},T_j-\ell_j]}(\mathbf{z}) \prod_{j=1}^{k} F_{[T_j-\ell_j,T_j+r_j]}(\mathbf{z})$$

$$= \sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{k,\theta}} \prod_{j=1}^{k+1} \mathcal{B}_{\xi_{\mathbf{T},j}}(z) \prod_{j=1}^{k} F_{\Delta_{\mathbf{T},j}}(z).$$

Corollary 2.4.13. *For any* $\mathbf{z} = \mathbf{u} + i\mathbf{v} \in \mathcal{Z}^{\Delta}$,

$$\left|\mathcal{B}_{[a,b],(\mathbf{t},\theta)}(\mathbf{z})\right| \leq \mathcal{B}_{[a,b],(\mathbf{t},\theta)}(\mathbf{u}).$$

Remark 2.4.14. Since

$$\left\{\nu \overset{\mathbf{T}}{\sim} (\boldsymbol{\ell},\mathbf{r}) : (\boldsymbol{\ell},\mathbf{r}) \in \Lambda_{k,\theta}\right\} \subset \mathcal{P}_{[a,b]},$$

and $F_k(x)$ is non-negative on $\mathcal{U}$, we have

$$\mathcal{B}_{[a,b],(\mathbf{T},\theta)}(\mathbf{u}) \leq \mathcal{B}_{[a,b]}(\mathbf{u}).$$

Remark 2.4.15. Just as the ratio

$$\frac{\mathcal{B}_n(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_n(\mathbf{u})}$$

represents the characteristic function of $Y_n(1)$ under the tilt $\mathbf{u}$, we can view $\mathcal{B}_{n,(\mathbf{t},\theta)}(\mathbf{u})$ as the partition function associated with a restricted distribution. Let $\mathbb{P}^{\mathbf{u}}_{n,(\mathbf{t},\theta)}$ be the measure obtained by restricting $\mathbb{Q}^{\mathbf{u}}_n$ to the set of sub-trajectories ξ for which $\Delta_{\mathbf{T}} \in \Lambda_{k,\theta}$. Writing $\mathbb{E}^{\mathbf{u}}_{n,(\mathbf{T},\theta)}$ for the related expectation, we have

$$\mathbb{E}^{\mathbf{u}}_{n,(\mathbf{T},\theta)}\left[e^{i\mathbf{v}\cdot Y_n(1)}\right] = \frac{\mathcal{B}_{n,(\mathbf{T},\theta)}(\mathbf{u}+\mathbf{i}\mathbf{v})}{\mathcal{B}_{n,(\mathbf{T},\theta)}(\mathbf{u})}.$$

Proposition 2.4.16. *There exists c such that for all n large enough, and all* $\mathbf{z} = \mathbf{u} + i\mathbf{v} \in \mathcal{Z}^{\Delta}$,

$$\frac{\left|\mathcal{B}_{[a,b]}(\mathbf{z}) - \mathcal{B}_{[a,b],(\mathbf{T},\theta)}(\mathbf{z})\right|}{\mathcal{B}_{[a,b]}(\mathbf{u})} = \mathcal{O}(\exp\{-c\theta\}).$$

Remark 2.4.17. The asymptotic in Proposition 2.4.16 holds even when $|\mathbf{T}|$ grows polynomially with n.

Corollary 2.4.18. *In particular there exists a finite constant C such that, for any finite collection $0 < t_1 < \cdots < t_j < 1$, any $\mathbf{u} \in \mathcal{U}^{\Delta}$, and any $W > 0$ large enough,* *we have*

$$\mathbb{Q}_n^{\mathbf{u}}\left(\max_{1\le i\le j} w_{nt_i} > W\right) \le Ce^{-cjW}.$$

Corollary 2.4.19. *For any $\delta \in (0,1)$, there exists n_0 such that, for all $n > n_0$ and any segment $[a,b] \subseteq [0,n]$,*

$$\mathcal{B}_{[a,b]}\left(\mathbf{u}\right) \le (1+\delta)\mathcal{B}_{[a,b],(\mathbf{T},n^{\varepsilon})}\left(\mathbf{u}\right).$$

Proof of Proposition 2.4.16. As in Proposition 2.4.4, $\mathcal{B}_{[a,b]}\left(\mathbf{z}\right) - \mathcal{B}_{[a,b],(\mathbf{t},\theta)}\left(\mathbf{z}\right)$ represents the contribution to $\mathcal{B}_{[a,b]}\left(\mathbf{z}\right)$ from partitions $\nu \overset{\mathbf{T}}{\sim} (\boldsymbol{\ell},\mathbf{r})$ in which at least one of the (ℓ_j, r_j) pairs has $\ell_j + r_j > \theta$. Using Proposition 2.2.1, we have

$$\begin{aligned}\left|\mathcal{B}_{[a,b]}\left(\mathbf{z}\right) - \mathcal{B}_{[a,b],(\mathbf{T},\theta)}\left(\mathbf{z}\right)\right| &= \left|\sum_{(\boldsymbol{\ell},\mathbf{r})\notin\Lambda_{k,\theta}} \sum_{\nu\overset{\mathbf{T}}{\sim}(\boldsymbol{\ell},\mathbf{r})} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}\left(\mathbf{z}\right)\right| \\ &\le \sum_{(\boldsymbol{\ell},\mathbf{r})\notin\Lambda_{k,\theta}} \sum_{\nu\overset{\mathbf{T}}{\sim}(\boldsymbol{\ell},\mathbf{r})} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}\left(\mathbf{u}\right).\end{aligned}$$

We separate $\Lambda_{k,\theta}$ according to the index associated with the widest block. For $1 \le j \le k$, let

$$\mathcal{A}_j = \left\{(\boldsymbol{\ell},\mathbf{r}) : \ell_j + r_j = \max_{1\le i\le k} \ell_i + r_i\right\}.$$

Now, if $\nu \overset{T}{\sim} (\boldsymbol{\ell},\mathbf{r})$ for some $(\boldsymbol{\ell},\mathbf{r}) \in \mathcal{A}_j$, then in particular $\nu \overset{T_j}{\sim} (\ell_j, r_j)$, so that

$$\sum_{(\boldsymbol{\ell},\mathbf{r})\in\mathcal{A}_j} \sum_{\nu\overset{\mathbf{T}}{\sim}(\boldsymbol{\ell},\mathbf{r})} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}\left(\mathbf{u}\right) \le \sum_{(\ell,r)\notin\Lambda_\theta} \sum_{\nu\overset{T_j}{\sim}(\ell,r)} \prod_{i=1}^{|\nu|} F_{[\nu_{i-1},\nu_i]}\left(\mathbf{u}\right).$$

By Proposition 2.4.4,

$$\sum_{(\boldsymbol{\ell},\mathbf{r})\in\mathcal{A}_j} \sum_{\nu\overset{\mathbf{T}}{\sim}(\boldsymbol{\ell},\mathbf{r})} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}\left(\mathbf{u}\right) \le e^{-c\theta}\mathcal{B}_n\left(\mathbf{u}\right),$$

and

$$\begin{aligned}\left|\mathcal{B}_{[a,b]}\left(\mathbf{z}\right) - \mathcal{B}_{[a,b],(\mathbf{T},\theta)}\left(\mathbf{z}\right)\right| &= \sum_{j=1}^{k} \sum_{(\boldsymbol{\ell},\mathbf{r})\in\mathcal{A}_j} \sum_{\nu\overset{\mathbf{T}}{\sim}(\boldsymbol{\ell},\mathbf{r})} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}\left(\mathbf{u}\right) \\ &\le ke^{-c\theta}\mathcal{B}_n\left(\mathbf{u}\right).\end{aligned}$$

We can therefore find constants c' and C' such that

$$\left|\mathcal{B}_{[a,b]}(\mathbf{z}) - \mathcal{B}_{[a,b],(\mathbf{T},\theta)}(\mathbf{z})\right| \leq C' e^{-c'\theta} \mathcal{B}_n(\mathbf{u}).$$

□

2.5 Approximation Based on the Gradient of Increments

In this section, we restrict our attention to sub-trajectories $\xi \in \mathcal{H}_n$ in which no increment has a gradient steeper than some quantity ρ. We will see how this restriction affects the values of $H_{[a,b]}$ and $\mathcal{B}_{[a,b]}$, and use these results to find an upper bound on the difference $\mathcal{B}_{[a,b]}(\mathbf{z}) - H_{[a,b]}(\mathbf{z})$ for certain segments $[a,b] \subset [0,n]$.

Definition 2.5.1. For $k \in \mathbb{N}$ and $\rho > 0$, let

$$\mathcal{F}_k^\rho = \{\eta \in \mathcal{F}_k : |h(\eta)| \leq \rho k\}.$$

Let

$$F_k^\rho(z) = \sum_{\eta \in \mathcal{F}_k^\rho} e^{zh(\eta)} \lambda(\eta),$$

and for $[a,b] \subseteq [0,n]$ let

$$F_{[a,b]}^\rho(\mathbf{z}) = F_{b-a}^\rho\left(z_1 + z_0\left(1 - \frac{a+b}{2n}\right)\right).$$

Let

$$H_n^\rho(z) = \sum_{\nu \in \mathcal{P}_n} \prod_{j=1}^{|\nu|} F_{\nu_j - \nu_{j-1}}^\rho(z),$$

$$H_{[a,b]}^\rho(\mathbf{z}) = H_{b-a}^\rho\left(z\left(\frac{a+b}{2n}\right)\right),$$

and

$$\mathcal{B}_{[a,b]}^\rho(\mathbf{z}) = \sum_{\nu \in \mathcal{P}_{[a,b]}} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}^\rho(\mathbf{z}).$$

Proposition 2.5.2. *There exist constants C and c such that, for all $z = u+iv \in \mathcal{Z}$, $k \in \mathbb{N}$, and $\rho > 0$,*

$$|F_k(z) - F_k^{\rho}(z)| \leq Ce^{-c\rho} F_k(u). \tag{2.5.1}$$

Proof. It is sufficient to prove Proposition 2.5.2 for arguments $u \in \mathcal{U}$, since

$$\begin{aligned} |F_k(z) - F_k^{\rho}(z)| &= \left| \sum_{\eta \in \mathcal{F}_k \setminus \mathcal{F}_k^{\rho}} \lambda(\eta) e^{zh(\eta)} \right| \\ &\leq \sum_{\eta \in \mathcal{F}_k \setminus \mathcal{F}_k^{\rho}} \lambda(\eta) e^{uh(\eta)} \\ &\leq F_k(u) - F_k^{\rho}(u). \end{aligned}$$

Next, if $F_k(u) = 0$ then $F_k^{\rho}(u) = 0$, while if $F_k(u) > 0$ we have

$$\frac{F_k(u) - F_k^{\rho}(u)}{F_k(u)} = \mathbb{Q}_k^u(|h(\eta)| \geq \rho k).$$

By Assumption 2.2.15,

$$\frac{F_k(u) - F_k^{\rho}(u)}{F_k(u)} \leq Ce^{-c\rho}.$$

□

Proposition 2.5.3. *There exist constants C' and c' such that, for all $\mathbf{z} = \mathbf{u} + i\mathbf{v} \in \mathcal{Z}^{\Delta}$, $n \in \mathbb{N}$, and $\rho > 0$,*

$$|\mathcal{B}_n(\mathbf{z}) - \mathcal{B}_n^{\rho}(\mathbf{z})| \leq C' n e^{-c'\rho} \exp\{C' n e^{-c'\rho}\} \mathcal{B}_n(\mathbf{u}).$$

By setting $z_0 = 0$, we obtain the following corollary.

Corollary 2.5.4. *For all $z \in \mathcal{Z}$, $n \in \mathbb{N}$, and $\rho > 0$,*

$$|H_n(z) - H_n^{\rho}(z)| \leq C' n e^{-c'\rho} \exp\{C' n e^{-c'\rho}\} H_n(\mathbf{u})$$

Proof of Proposition 2.5.3. As in the proof of Proposition 2.5.2, the proof for complex arguments follows directly from the proof for real arguments, since for $\mathbf{z} = \mathbf{u} + i\mathbf{v}$,

$$\left| \sum_{\xi \in \mathcal{H}_n \setminus \mathcal{H}_n^\rho} \lambda(\xi) e^{z_1 h(\xi) + z_0 A_\xi(1)} \right| \leq \sum_{\xi \in \mathcal{H}_n \setminus \mathcal{H}_n^\rho} \lambda(\xi) e^{u_1 h(\xi) + u_0 A_\xi(1)}.$$

For $\mathbf{u} \in \mathcal{U}^\Delta$,

$$\begin{aligned} \left| \mathcal{B}_n(\mathbf{u}) - \mathcal{B}_n^\rho(\mathbf{u}) \right| &= \left| \sum_{\nu \in \mathcal{P}_n} \left(\prod_{j=1}^{|\nu|} F_{[\nu_{j-1}, \nu_j]}(\mathbf{u}) - \prod_{j=1}^{|\nu|} F^\rho_{[\nu_{j-1}, \nu_j]}(\mathbf{u}) \right) \right| \\ &\leq \sum_{\nu \in \mathcal{P}_n} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1}, \nu_j]}(\mathbf{u}) \left| \prod_{j=1}^{|\nu|} \frac{F^\rho_{[\nu_{j-1}, \nu_j]}(\mathbf{u})}{F_{[\nu_{j-1}, \nu_j]}(\mathbf{u})} - 1 \right|. \end{aligned}$$

By Proposition 2.5.2, we have

$$\left| \frac{F^\rho_{[\nu_{j-1}, \nu_j]}(\mathbf{u})}{F_{[\nu_{j-1}, \nu_j]}(\mathbf{u})} - 1 \right| \leq C e^{-c\rho}$$

for each $1 \leq j \leq |\nu|$; by Proposition B.1.7,

$$\begin{aligned} \left| \prod_{j=1}^{|\nu|} \frac{F^\rho_{[\nu_{j-1}, \nu_j]}(\mathbf{u})}{F_{[\nu_{j-1}, \nu_j]}(\mathbf{u})} - 1 \right| &\leq \left(\left(1 + C e^{-c\rho}\right)^n - 1 \right) \\ &\leq n C e^{-c\rho} \exp n C e^{-c\rho}, \end{aligned}$$

so that

$$\left| \mathcal{B}_n(\mathbf{z}) - \mathcal{B}_n^\rho(\mathbf{z}) \right| \leq C n e^{-c\rho} \exp\{C n e^{-c\rho}\} \mathcal{B}_n(\mathbf{u}).$$

□

Proposition 2.5.3 implies that, for any sequence ρ_n such that $\frac{\rho_n}{\log n} \to \infty$, the restriction "consider only sub-trajectories in which the gradient never exceeds ρ_n" has an increasingly small effect relative to the value of $\mathcal{B}_n$. Next, we will see that if ρ_n does not grow too quickly, then the value of $\mathcal{B}^{\rho_n}_{[a,b]}(\mathbf{z})$ is essentially $H^{\rho_n}_{[a,b]}(z)$ on certain intervals $[a, b] \subset [0, n]$.

Proposition 2.5.5. *Let $0 < \gamma < \frac{1}{2}$, and let $\{\rho_n\}$ be a sequence of positive numbers such that $\rho_n n^{2\gamma - 1} \to 0$. Then for all n large enough, uniformly in intervals $[a, b] \subset$*

$[0, n]$ *such that* $b - a \leq n^\gamma$, *and for all* $\mathbf{z} \in \mathcal{Z}^\Delta$,

$$\left| \mathcal{B}^{\rho_n}_{[a,b]} (\mathbf{z}) - H^{\rho_n}_{[a,b]} (\mathbf{z}) \right| \leq |z_0|^2 \rho_n^2 n^{4\gamma - 2} H^{\rho_n}_{[a,b]} (\mathbf{u}) .$$

Remark 2.5.6. For any $k \in \mathbb{N}$, $u \in \mathcal{U}$, and $\rho > 0$, the ratio

$$\frac{F^\rho_k (u + v)}{F^\rho_k (u)}$$

can be viewed as the moment-generating function of the height of an increment η drawn from $\mathcal{F}^\rho_k$, according to the distribution

$$P^{u,\rho}_k(\mathcal{A}) = \frac{\sum_{\eta \in \mathcal{F}^\rho_k} \lambda(\eta) e^{uh(\eta)} \mathbb{1}\{\eta \in \mathcal{A}\}}{\sum_{\eta' \in \mathcal{F}^\rho_k} \lambda(\eta') e^{uh(\eta')}}.$$

Then

$$\frac{F^\rho_k (u + v)}{F^\rho_k (u)} = \mathbb{E}^{u,\rho}_k \left[e^{vh(\eta)} \right].$$

Proof. We begin with the case $\mathbf{z} = \mathbf{u} \in \mathcal{U}^\Delta$, in which the proof follows similar lines to that of Proposition 2.3.15. We write

$$\begin{aligned}
\mathcal{B}^{\rho_n}_{[a,b]} (\mathbf{u}) &= \sum_{\nu \in \mathcal{P}_{[a,b]}} \prod_{j=1}^{|\nu|} F^{\rho_n}_{\nu_j - \nu_{j-1}} \left(u \left(\frac{c(\nu_j)}{n} \right) \right) \\
&= \sum_{\nu \in \mathcal{P}_{[a,b]}} \prod_{j=1}^{|\nu|} F^{\rho_n}_{\nu_j - \nu_{j-1}} \left(u \left(\frac{a+b}{2n} \right) \right) \frac{F^{\rho_n}_{\nu_j - \nu_{j-1}} \left(u \left(\frac{c(\nu_j)}{n} \right) \right)}{F^{\rho_n}_{\nu_j - \nu_{j-1}} \left(u \left(\frac{a+b}{2n} \right) \right)};
\end{aligned}$$

Using Remark 2.5.6, we have

$$\prod_{j=1}^{|\nu|} \frac{F^{\rho_n}_{\nu_j - \nu_{j-1}} \left(u \left(\frac{c(\nu_j)}{n} \right) \right)}{F^{\rho_n}_{\nu_j - \nu_{j-1}} \left(u \left(\frac{a+b}{2n} \right) \right)} = \bar{\mathbb{E}}^{u\left(\frac{a+b}{2n}\right),\rho_n}_\nu \left[\exp \left\{ \frac{u_0}{n} \sum_{j=1}^{|\nu|} \left(c(\nu_j) - \frac{a+b}{2} \right) h(\eta_j) \right\} \right],$$

where $\bar{\mathbb{E}}^{u,\rho}_\nu$ is the expectation based on the product measure $\mathbb{Q}^{u,\rho}_{\nu_1} \times \mathbb{Q}^{u,\rho}_{\nu_2 - \nu_1} \times \cdots \times \mathbb{Q}^{u,\rho}_{n - \nu_{|\nu|}}$.

Let $\pi \in \mathcal{P}_{[a,b]}$ be the partition obtained by inverting the blocks of ν, i.e.

$$\pi_j = b + a - \nu_j \qquad\qquad 1 \leq j \leq |\nu|.$$

Then

$$\prod_{j=1}^{|\nu|} F^{\rho_n}_{\nu_j-\nu_{j-1}}\left(u\left(\frac{a+b}{2n}\right)\right) = \prod_{j=1}^{|\pi|} F^{\rho_n}_{\pi_j-\pi_{j-1}}\left(u\left(\frac{a+b}{2n}\right)\right),$$

and

$$c\left(\nu_j\right) - \frac{a+b}{2} = -\left(c\left(\pi_j\right) - \frac{a+b}{2}\right),$$

so that

$$\begin{aligned}\frac{1}{2}\left(\prod_{j=1}^{|\nu|} \frac{F^{\rho_n}_{\nu_j-\nu_{j-1}}\left(u\left(\frac{c(\nu_j)}{n}\right)\right)}{F^{\rho_n}_{\nu_j-\nu_{j-1}}\left(u\left(\frac{a+b}{2n}\right)\right)} + \prod_{j=1}^{|\nu|} \frac{F^{\rho_n}_{\pi_j-\pi_{j-1}}\left(u\left(\frac{c(\pi_j)}{n}\right)\right)}{F^{\rho_n}_{\pi_j-\pi_{j-1}}\left(u\left(\frac{a+b}{2n}\right)\right)}\right)&\\ = \bar{\mathbb{E}}_\nu^{u\left(\frac{a+b}{2n}\right),\rho_n}\left[\cosh\left(\frac{u_0}{n}\sum_{j=1}^{|\nu|}\left(c\left(\nu_j\right) - \frac{a+b}{2}\right)h(\eta_j)\right)\right].&\end{aligned}$$

For each $\nu \in \mathcal{P}_{[a,b]}$, the contribution from ν to the difference $\mathcal{B}^{\rho_n}_{[a,b]}\left(\mathbf{u}\right) - H^{\rho_n}_{[a,b]}\left(\mathbf{u}\right)$ is therefore

$$\prod_{j=1}^{|\nu|} F^{\rho_n}_{\nu_j-\nu_{j-1}}\left(u\left(\frac{a+b}{2n}\right)\right) \bar{\mathbb{E}}_\nu^{u\left(\frac{a+b}{2n}\right),\rho_n}\left[\cosh\left(\frac{u_0}{n}\sum_{j=1}^{|\nu|}\left(c\left(\nu_j\right) - \frac{a+b}{2}\right)h(\eta_j)\right) - 1\right]. \tag{2.5.2}$$

Under the restrictions $\eta_j \in \mathcal{F}^{\rho_n}_{\nu_j}$ and $|b-a| \le n^\gamma$, we have

$$\begin{aligned}\left|\frac{u_0}{n}\sum_{j=1}^{|\nu|}\left(c\left(\nu_j\right) - \frac{a+b}{2}\right)h(\eta_j)\right| &\le \frac{|u_0|}{n}\sum_{j=1}^{|\nu|}(b-a)\rho_n(\nu_j - \nu_{j-1})\\ &\le \frac{|u_0|}{n}n^{2\gamma}\rho_n,\end{aligned}$$

so that for n large enough,

$$\begin{aligned}&\left|\bar{\mathbb{E}}_\nu^{u\left(\frac{a+b}{2n}\right),\rho_n}\left[\cosh\left(\frac{u_0}{n}\sum_{j=1}^{|\nu|}\left(c\left(\nu_j\right) - \frac{a+b}{2}\right)h(\eta_j)\right) - 1\right]\right|\\ &\quad\le \left(\frac{u_0}{n}\sum_{j=1}^{|\nu|}\left(c\left(\nu_j\right) - \frac{a+b}{2}\right)h(\eta_j)\right)^2\\ &\quad\le |u_0|^2\frac{\rho_n^2}{n^{2-4\gamma}},\end{aligned}$$

as long as $n^{2\gamma-1}\rho_n \to 0$ as $n \to \infty$.

We therefore have

$$\left|\mathcal{B}^{\rho_n}_{[a,b]}(\mathbf{u}) - H^{\rho_n}_{[a,b]}(\mathbf{u})\right| \le |u_0|^2 \rho_n^2 n^{4\gamma-2} \sum_{\nu\in\mathcal{P}_n} \prod_{j=1}^{|\nu|} F^{\rho_n}_{\nu_j-\nu_{j-1}}\left(u\left(\frac{a+b}{2n}\right)\right)$$
$$\le |u_0|^2 \rho_n^2 n^{4\gamma-2} H^{\rho_n}_{[a,b]}(\mathbf{u}).$$

Now in the complex case,

$$\mathcal{B}^{\rho_n}_{[a,b]}(\mathbf{z}) = \sum_{\nu\in\mathcal{P}_{[a,b]}} \prod_{j=1}^{|\nu|} F^{\rho_n}_{\nu_j-\nu_{j-1}}\left(u\left(\frac{c(\nu_j)}{n}\right)\right)$$
$$= \sum_{\nu\in\mathcal{P}_{[a,b]}} \prod_{j=1}^{|\nu|} F^{\rho_n}_{\nu_j-\nu_{j-1}}\left(u\left(\frac{a+b}{2n}\right)\right) \frac{F^{\rho_n}_{\nu_j-\nu_{j-1}}\left(z\left(\frac{c(\nu_j)}{n}\right)\right)}{F^{\rho_n}_{\nu_j-\nu_{j-1}}\left(u\left(\frac{a+b}{2n}\right)\right)}.$$

We have

$$z\left(\frac{c(\nu_j)}{n}\right) - u\left(\frac{a+b}{2n}\right) = \frac{z_0}{n}\left(c(\nu_j) - \frac{a+b}{2n}\right) + iv\left(\frac{a+b}{2n}\right),$$

so that

$$\prod_{j=1}^{|\nu|} \frac{F^{\rho_n}_{\nu_j-\nu_{j-1}}\left(z\left(\frac{c(\nu_j)}{n}\right)\right)}{F^{\rho_n}_{\nu_j-\nu_{j-1}}\left(u\left(\frac{a+b}{2n}\right)\right)}$$
$$= \mathbb{E}_\nu^{u\left(\frac{a+b}{2n}\right),\rho_n}\left[\exp\left\{\sum_{j=1}^{|\nu|}\left(\frac{z_0}{n}\left(c(\nu_j) - \frac{a+b}{2}\right) + iv\left(\frac{a+b}{2n}\right)\right)h(\eta_j)\right\}\right].$$

Using our method of pairing up partitions, this expectation becomes

$$\mathbb{E}_\nu^{u\left(\frac{a+b}{2}\right),\rho_n}\left[\cosh\left(\frac{z_0}{n}\sum_{j=1}^{|\nu|}\left(c(\nu_j) - \frac{a+b}{2}\right)h(\eta_j)\right)\exp\left\{iv\left(\frac{a+b}{2}\right)\sum_j h(\eta_j)\right\}\right].$$

Meanwhile,

$$H^{\rho_n}_{[a,b]}(z) = \sum_{\nu\in\mathcal{P}_{[a,b]}} \prod_{j=1}^{|\nu|} F^{\rho_n}_{\nu_j-\nu_{j-1}}\left(u\left(\frac{a+b}{2n}\right)\right) \frac{F^{\rho_n}_{\nu_j-\nu_{j-1}}\left(z\left(\frac{a+b}{2n}\right)\right)}{F^{\rho_n}_{\nu_j-\nu_{j-1}}\left(u\left(\frac{a+b}{2n}\right)\right)},$$

with

$$z\left(\frac{a+b}{2n}\right) - u\left(\frac{a+b}{2n}\right) = iv\left(\frac{a+b}{2n}\right)$$

so that

$$H^{\rho_n}_{[a,b]}(z) = \sum_{\nu\in\mathcal{P}_n} \prod_{j=1}^{|\nu|} F^{\rho_n}_{\nu_j-\nu_{j-1}}\left(u\left(\frac{a+b}{2n}\right)\right) \mathbb{E}_\nu^{u\left(\frac{a+b}{2n}\right),\rho_n}\left[\exp\left\{iv\left(\frac{a+b}{2n}\right)\sum_j h(\eta_j)\right\}\right].$$

Subtracting terms, we need an upper bound on

$$\mathbb{E}_{\nu}^{u\left(\frac{a+b}{2}\right),\rho_n}\left[e^{iv\left(\frac{a+b}{2}\right)\sum_j h(\eta_j)}\left(\cosh\left(\frac{z_0}{n}\sum_{j=1}^{|\nu|}\left(c\left(\nu_j\right)-\frac{a+b}{2}\right)h(\eta_j)\right)-1\right)\right].$$

Now

$$\begin{aligned}
&\left|e^{iv\left(\frac{a+b}{2}\right)\sum_j h(\eta_j)}\left(\cosh\left(\sum_{j=1}^{|\nu|}\frac{z_0}{n}\left(c\left(\nu_j\right)-\frac{a+b}{2}\right)h(\eta_j)\right)-1\right)\right| \\
&\quad\leq\left|\cosh\left(\frac{z_0}{n}\sum_{j=1}^{|\nu|}\left(c\left(\nu_j\right)-\frac{a+b}{2}\right)h(\eta_j)\right)-1\right| \\
&\quad\leq\left|\frac{z_0}{n}\sum_{j=1}^{|\nu|}\left(c\left(\nu_j\right)-\frac{a+b}{2}\right)h(\eta_j)\right|^2 \\
&\quad\leq|z_0|^2\rho_n^2 n^{4\gamma-2},
\end{aligned}$$

so that

$$\begin{aligned}
\left|\mathcal{B}_{[a,b]}^{\rho_n}\left(\mathbf{z}\right)-H_{[a,b]}^{\rho_n}\left(\mathbf{z}\right)\right| &\leq\sum_{\nu\in\mathcal{P}_n}\prod_{j=1}^{|\nu|}F_{\nu_j-\nu_{j-1}}^{\rho_n}\left(u\left(\frac{a+b}{2n}\right)\right)|z_0|^2\rho_n^2 n^{4\gamma-2} \\
&\leq|z_0|^2\rho_n^2 n^{4\gamma-2}H_{[a,b]}^{\rho_n}\left(\mathbf{z}\right).
\end{aligned}$$

□

Corollary 2.5.7. *Let $0<\gamma<\frac{1}{2}$, and let ρ_n be a sequence of positive numbers such that*

$$\lim_{n\to\infty}\frac{\rho_n}{\log n}=\infty \qquad \text{and} \qquad \lim_{n\to\infty}\frac{\rho_n}{n^{1-2\gamma}}=0.$$

Then there exist constants c_1, c_2 such that uniformly in segments $[a,b]\subset[0,n]$ with $b-a\leq n^{\gamma}$, and in $\mathbf{u}\in\mathcal{U}^{\Delta}$,

$$\left|\mathcal{B}_{[a,b]}\left(\mathbf{z}\right)-H_{[a,b]}\left(\mathbf{z}\right)\right|\leq\left(3c_1 n e^{-c_2\rho_n}\exp\{c_1 n e^{-c_2\rho_n}\}+|z_0|^2\rho_n^2 n^{4\gamma-2}\right)H_{[a,b]}\left(\mathbf{u}\right),$$

for all $n\in\mathbb{N}$.

Remark 2.5.8. In particular, when $\rho_n=n^{1-2\gamma-\delta}$ for some $\delta\in(0,1-2\gamma)$, and given a threshold n_0 we can find constants C_1,C_2,C_3,C_4 such that

$$\left|\mathcal{B}_{[a,b]}\left(\mathbf{z}\right)-H_{[a,b]}\left(\mathbf{z}\right)\right|\leq\left(C_1 e^{-C_2 n^{1-2\gamma-\delta}}+C_3|z_0|^2 n^{-2\delta}\right)H_{[a,b]}\left(\mathbf{u}\right)$$

$$\leq C_4 n^{-2\delta} H_{[a,b]}(\mathbf{u})$$

holds uniformly in $\mathbf{z} \in \mathcal{Z}^{\Delta}$ for all $n > n_0$.

Note that, when $|z_0| = 0$, $\mathcal{B}_{[a,b]}(\mathbf{z}) = H_{[a,b]}(\mathbf{z})$ so that these asymptotics hold whatever the value of $|z_0|$.

Proof. By Proposition 2.5.3,

$$\mathcal{B}_{[a,b]}(\mathbf{u}) \leq \frac{1}{1 - nCe^{-c\rho_n} \exp\{nCe^{-c\rho_n}\}} \mathcal{B}^{\rho_n}_{[a,b]}(\mathbf{u}),$$

while by Proposition 2.5.5,

$$\begin{aligned} \mathcal{B}^{\rho_n}_{[a,b]}(\mathbf{u}) &\leq \left(\exp\left\{ \frac{1}{8} \left(u_0 \frac{\rho_n}{n^{1-2\gamma}} \right)^2 \right\} - 1 \right) H^{\rho_n}_{[a,b]}\left(u\left(\frac{a+b}{2} \right) \right) \\ &\leq \left(\exp\left\{ \frac{1}{8} \left(u_0 \frac{\rho_n}{n^{1-2\gamma}} \right)^2 \right\} - 1 \right) H_{[a,b]}(\mathbf{u}). \end{aligned}$$

As a result,

$$\mathcal{B}_{[a,b]}(\mathbf{u}) \leq 2 H_{[a,b]}(\mathbf{u})$$

as long as n is large enough.

Now, writing

$$\begin{aligned} \left| \mathcal{B}_{[a,b]}(\mathbf{z}) - H_{[a,b]}(\mathbf{z}) \right| \leq \left| \mathcal{B}_{[a,b]}(\mathbf{z}) - \mathcal{B}^{\rho_n}_{[a,b]}(\mathbf{z}) \right| + \left| \mathcal{B}^{\rho_n}_{[a,b]}(\mathbf{z}) - H^{\rho_n}_{[a,b]}\left(z\left(\frac{a+b}{2} \right) \right) \right| \\ + \left| H^{\rho_n}_{[a,b]}\left(z\left(\frac{a+b}{2} \right) \right) - H_{[a,b]}(\mathbf{z}) \right| \end{aligned}$$

and applying Proposition 2.5.3 to the first term, Proposition 2.5.5 to the second, and Corollary 2.5.4 to the third, we have

$$\left| \mathcal{B}_{[a,b]}(\mathbf{z}) - H_{[a,b]}(\mathbf{z}) \right| \leq \left(3c_1 n e^{-c_2\rho_n} \exp\{c_1 n e^{-c_2\rho_n}\} + |z_0|^2 \rho_n^2 n^{4\gamma-2} \right) H_{[a,b]}(\mathbf{u}).$$

□

2.6 A New Version of the Partition Function

In the final part of this chapter, we introduce a version of $\mathcal{B}_n$ in which z_1 takes different values depending on the location of each increment. We will see that this version of $\mathcal{B}_n$ allows us to create a good approximation for the characteristic function of $Y_n(\mathbf{t})$, just as the polymer decompositions in the previous sections allowed us to find approximations for the characteristic function of $Y_n(1)$.

Definition 2.6.1. Let $\mathcal{Z}^{\Delta}_{k+1}$ be the set

$$\mathcal{Z}^{\Delta}_{k+1} = \left\{ \mathbf{z} = (z_1, \ldots, z_{k+1}, z_0) \in \mathbb{C}^{k+2} : u_j = u_1,\ (z_j, z_0) \in \mathcal{Z}^{\Delta},\ 1 \leq j \leq k+1 \right\}$$

Each element $\mathbf{z}$ of $\mathcal{Z}^{\Delta}_{k+1}$ can be viewed as a collection of $k+1$ elements $\mathbf{z}_j = (z_j, z_0) \in \mathcal{Z}$ whose real parts, and second components, all coincide. For $\mathbf{z} \in \mathcal{Z}^{\Delta}_{k+1}$, we will not distinguish between $\mathbf{u} = (u_1, \ldots, u_1, u_0) \in \mathbb{R}^{k+2}$ and $\mathbf{u} = (u_1, u_0) \in \mathbb{R}^2$; particularly when discussing the characteristic function of $Y_n(\mathbf{t})$, we will use $\mathbf{u}$ to refer to the tilt vector $(u_1, u_0) \in \mathcal{U}^{\Delta}$.

For $S \in [0, n]$ and $\mathbf{z} \in \mathcal{Z}^{\Delta}_{k+1}$, let

$$\mathbf{z}^{\star}_{\mathbf{T}}(S) = \sum_{j \leq k+1} \mathbf{z}_j \mathbb{1}\left\{T_{j-1} < S < T_j\right\}.$$

Let

$$\begin{aligned} \mathbf{z}_{\mathbf{T}}(x) &= \sum_{j \leq k+1} z_j \mathbb{1}\left\{T_{j-1} \leq nx < T_j\right\} + z_0(1-x) \\ &= \sum_{j \leq k+1} z_j(x) \mathbb{1}\left\{T_{j-1} \leq nx < T_j\right\}. \end{aligned} \tag{2.6.1}$$

Let

$$\mathcal{B}_{[a,b],\mathbf{T}}(\mathbf{z}) = \sum_{\nu \in \mathcal{P}_{[a,b]}} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1}, \nu_j]}\left(\mathbf{z}_{\mathbf{T}}(\nu_{j-1})\right). \tag{2.6.2}$$

This is an analogue of $\mathcal{B}_{[a,b]}(\mathbf{z})$, in which we replace the fixed pair $\mathbf{z} = (z_1, z_0)$ with the varying pairs $\mathbf{z}_j = (z_j, z_0)$, according to the position of the left endpoint of each increment. As usual, when $[a, b] = [0, n]$ we write $\mathcal{B}_{[0,n],\mathbf{T}}(\mathbf{z}) = \mathcal{B}_{n,\mathbf{T}}(\mathbf{z})$.

We denote

$$z_j^\star(a,b) = z_j + z_0\left(1 - \frac{2T_j + a - b}{2n}\right).$$

Remark 2.6.2. If $k = 0$ and $\mathbf{T} = (T)$, we have when $x < T$

$$\mathbf{z}_T^\star(x) = z_1 + \frac{z_0}{n}(n - x)$$

while when $x > T$,

$$\mathbf{z}_T^\star(x) = z_2 + \frac{z_0}{n}(n - x).$$

In particular,

$$\mathbf{z}_T^\star(a) = z_1 + \frac{z_0}{n}(n - a)$$
$$\mathbf{z}_T^\star(b) = z_2 + \frac{z_0}{n}(n - b).$$

Remark 2.6.3. When $\mathbf{u} \in \mathcal{Z}_{k+1}^\Delta \cap \mathbb{R}^{k+1}$,

$$\mathcal{B}_{n,\mathbf{T}}(\mathbf{u}) = \mathcal{B}_n(\mathbf{u}).$$

Remark 2.6.4. We have

$$\bar{\mathbb{E}}_n^{\mathbf{u}}\left[\exp\left\{i\mathbf{v}\cdot Y_n(\mathbf{t})\right\}\right] = \frac{\mathcal{B}_{n,n\mathbf{t}}(\mathbf{z})}{\mathcal{B}_n(\mathbf{u})}.$$

Proposition 2.6.5. *Let $\rho \le \min_j T_j - T_{j-1}$. For each $(\boldsymbol{\ell},\mathbf{r}) \in \Lambda_{k,\rho}$, we have*

$$\sum_{\nu \overset{\mathbf{T}}{\sim} (\boldsymbol{\ell},\mathbf{r})} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}\left(\mathbf{z}_{\mathbf{T}}(\nu_{j-1})\right) = \prod_{j=1}^{k+1} \mathcal{B}_{[T_{j-1}+r_{j-1},T_j-\ell_j]}(\mathbf{z}_j) \prod_{j=1}^{k} F_{\ell_j+r_j}\left(z_j^\star(\ell_j,r_j)\right).$$

Proof. For this proposition, the proof will be given only in the case $k = 1$; the arguments remain the same by induction for larger values of k, with increasing complexity in the notation.

Let $(\boldsymbol{\ell},\mathbf{r}) \in \Lambda_{1,\rho}$. As in the proof of Proposition 2.4.2, every $\nu \sim (\boldsymbol{\ell},\mathbf{r})$ may be written in terms of two sub-partitions $\pi \in \mathcal{P}_{[0,T-\ell]}$ and $\mu \in \mathcal{P}_{[T+r,n]}$, describing the cutpoints to the left and right of Δ_T respectively. Now, the definition

$$\mathbf{z_T}\left(\nu_{j-1}\right) = \begin{cases} \mathbf{z}_1 & \text{if } \nu_{j-1} < T \\ \\ \mathbf{z}_2 & \text{if } \nu_{j-1} > T \end{cases}$$

assigns the argument $\mathbf{z}_1$ to precisely the blocks of ν corresponding to π, and $\mathbf{z}_2$ to the blocks corresponding to μ, so that

$$\begin{aligned} &\sum_{\nu \overset{T}{\sim} (\boldsymbol{\ell},\mathbf{r})} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}\left(\mathbf{z_T}\left(\nu_{j-1}\right)\right) \\ &\qquad = \sum_{\pi \in \mathcal{P}_{T-\ell}} \prod_{m=1}^{|\pi|} F_{[\pi_{m-1},\pi_m]}\left(\mathbf{z}_1\right) F_{\ell+r}\left(z_1^{\star}(\ell,r)\right) \sum_{\mu \in \mathcal{P}_{n-T-r}} \prod_{m=1}^{|\mu|} F_{[\mu_{m-1},\mu_m]}\left(\mathbf{z}_2\right) \\ &\qquad = \mathcal{B}_{[0,T-\ell]}\left(\mathbf{z}_1\right) F_{\ell+r}\left(z_1^{\star}(\ell,r)\right) \mathcal{B}_{[T+r,n]}\left(\mathbf{z}_2\right), \end{aligned}$$

as claimed. For the case $k > 1$, the similar statement follows by induction. □

We can use this polymer decomposition to construct a good approximation for $\bar{\mathbb{E}}_n^{\mathbf{u}}\left[\exp\left\{i\mathbf{v}\cdot Y_n(\mathbf{t})\right\}\right]$. We consider a restriction of the partition function $\mathcal{B}_{n,\mathbf{T}}\left(\mathbf{z}\right)$ to the set of trajectories in which the increments at locations T_j, as well as certain other increments, are of width at most n^{ε}.

Definition 2.6.6. Let $\mathbf{S} = (S_1, \ldots, S_\ell)$ be a vector of times, such that each of the elements of $\mathbf{T}$ appears in $\mathbf{S}$. Note that we can separate $\mathbf{S}$ into $k+1$ sub-vectors with endpoints T_{j-1} and T_j. We denote these sub-vectors $\mathbf{S}_j$. Let

$$\mathcal{B}_{n,\mathbf{T},(\mathbf{S},\theta)}\left(\mathbf{z}\right) = \sum_{(\boldsymbol{\ell},\mathbf{r}) \in \Lambda_{|\mathbf{S}|,\theta}} \sum_{\nu \overset{\mathbf{S}}{\sim} (\boldsymbol{\ell},r)} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}\left(\mathbf{z}^{\star}(\nu_{j-1})\right).$$

Notice that if $z_j = z_1$ for $1 \le j \le k+1$, then we recover the version seen in Definition 2.4.3, and $\mathcal{B}_{n,\mathbf{T},(\mathbf{S},\theta)}\left(\mathbf{z}\right) = \mathcal{B}_{n,(\mathbf{S},\theta)}\left(\mathbf{z}\right)$.

We have the following polymer decomposition, an extension of that in Proposition 2.6.5.

Corollary 2.6.7. *We have*

$$\mathcal{B}_{n,\mathbf{T},(\mathbf{S},\theta)}\left(\mathbf{z}\right) = \sum_{(\boldsymbol{\ell},\mathbf{r}) \in \Lambda_{k,\theta}} \prod_{j=1}^{k+1} \mathcal{B}_{\xi_{\mathbf{T},j},(\mathbf{S}_j,\theta)}\left(\mathbf{z_j}\right) \prod_{j=1}^{k} F_{\ell_j+r_j}\left(z_j^{\star}(\ell_j,r_j)\right)$$

and

$$\mathcal{B}_{n,\mathbf{T},(\mathbf{S},\theta)}(\mathbf{u}) = \mathcal{B}_{n,(\mathbf{S},\theta)}(u_1, u_0).$$

Remark 2.6.8. As in Remark 2.4.15, we view $\mathcal{B}_{n,\mathbf{T},(\mathbf{S},\theta)}(\mathbf{z})$ as the partition function associated with a restricted distribution. Let $\mathbb{P}^{(u_1,u_0)}_{n,(\mathbf{S},\theta)}$ be the measure associated with the restriction of our distribution $\mathbb{Q}^{(u_1,u_0)}_n$ to trajectories in which the increments at the locations indexed in $\mathbf{S}$ have width at most θ; and write $\mathbb{E}^{(u_1,u_0)}_{n,(\mathbf{S},\theta)}$ for the related expectation, so that

$$\mathbb{E}^{(u_1,u_0)}_{n,(\mathbf{S},\theta)}\left[e^{i\mathbf{v}\cdot Y_n(\mathbf{t})}\right] = \frac{\mathcal{B}_{n,n\mathbf{t},(\mathbf{S},\theta)}(\mathbf{z})}{\mathcal{B}_{n,n\mathbf{t},(\mathbf{S},\theta)}(\mathbf{u})}.$$

Note that our requirement that $u_1 = u_2 = \cdots = u_{k+1}$ allows us to use the tilt vector $\mathbf{u} = (u_1, u_0) \in \mathbb{R}^2$ as in the previous sections, without further complication.

The following proposition is an analogue of Proposition 2.4.16.

Proposition 2.6.9. *Let $\varepsilon > 0$. Then there exists c such that for all n large enough, and all $\mathbf{z}$ with $\mathbf{z}_j \in \mathcal{Z}^{\Delta}$ for each $1 \leq j \leq k+1$,*

$$\frac{1}{\mathcal{B}_n(\mathbf{u})}\left|\mathcal{B}_{n,\mathbf{T}}(\mathbf{z}) - \mathcal{B}_{n,\mathbf{T},(\mathbf{S},n^{\varepsilon})}(\mathbf{z})\right| = \mathcal{O}(\exp\{-cn^{\varepsilon}\}),$$

as $n \to \infty$.

Proof. First, note that

$$\frac{1}{\mathcal{B}_n(\mathbf{u})}\left|\mathcal{B}_{n,\mathbf{T}}(\mathbf{z}) - \mathcal{B}_{n,\mathbf{T},(\mathbf{S},n^{\varepsilon})}(\mathbf{z})\right| = \frac{1}{\mathcal{B}_n(\mathbf{u})}\left|\sum_{\nu}\prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}(\mathbf{z}^{\star}(\nu_{j-1}))\right|, \quad (2.6.3)$$

where the sum is over partitions ν that do not correspond to any $(\boldsymbol{\ell}, \mathbf{r}) \in \Lambda_{|\mathbf{s}|,n^{\varepsilon}}$.

Now, for all $\nu \in \mathcal{P}_n$ and $j \leq |\nu|$, the real part of $\mathbf{z}^{\star}(\nu_{j-1})$ is $\mathbf{u}_1$, so that, by Proposition 2.2.1,

$$\left|F_{[\nu_{j-1},\nu_j]}(\mathbf{z}^{\star}(\nu_{j-1}))\right| \leq F_{[\nu_{j-1},\nu_j]}(\mathbf{u}_1).$$

As a result, the right hand side of Equation (2.6.3) can be bounded above by

$$\frac{1}{\mathcal{B}_n(\mathbf{u})}\sum_{\nu}\prod_{j=1}^{|\nu|}F_{[\nu_{j-1},\nu_j]}(\mathbf{u}_1)$$

and, as we saw in the proof of Proposition 2.4.16, this is at most $\mathcal{O}(\exp\{-cn^{\varepsilon}\})$. □

Remark 2.6.10. As in the case of Proposition 2.4.16, this Proposition holds also when $|\mathbf{S}|$ grows with n, in particular when $|\mathbf{S}| \approx n^{1-\gamma}$ for some $\varepsilon < \gamma < 1$.

2.7 A Rescaling of a Partition Function

As we saw in Corollary 2.2.10, the collection $\{e^{km(u)}F_k(u)\}$ forms a probability distribution, with mean $\frac{1}{\mu(u)}$; in other words, for any $u \in \mathcal{U}$,

$$\sum_{k\geq 1} kF_k(u)\,e^{-km(u)}\mu(u) - 1 = 0.$$

In this section, we will establish versions of Corollary 2.2.10 and Proposition 2.2.12 which hold on $\mathcal{Z}$, including when the arguments of $\exp\{-km(\cdot)\}$ and $\mu(\cdot)$ differ from that of $F_k\cdot$ by a small amount.

Definition 2.7.1. For $[a,b] \subseteq [0,n]$ and $\mathbf{z} \in \mathcal{Z}^{\Delta}$, let

$$F^{\star}_{[a,b]}(\mathbf{z}) = F_{[a,b]}(\mathbf{z})\exp\left\{-\int_a^b m\left(z\left(\frac{x}{n}\right)\right)dx\right\}\sqrt{\mu\left(z\left(\frac{a}{n}\right)\right)\mu\left(z\left(\frac{b}{n}\right)\right)}.$$

For a point $T \in [0,n]$, and a segment $[a,b] \subseteq [0,n]$ such that $a < T \leq b$, and a pair of arguments $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{Z}^{\Delta}$, let

$$F^{\star}_{[a,b]}(\mathbf{z}_1,\mathbf{z}_2) = F_{[a,b]}(\mathbf{z}_1)\exp\left\{-\int_a^b m\left(\mathbf{z}^{\star}_T(x)\right)dx\right\}\sqrt{\mu\left(\mathbf{z}^{\star}_T(a)\right)\mu\left(\mathbf{z}^{\star}_T(b)\right)}.$$

Remark 2.7.2. When $z_0 = 0$, we take the left endpoint of the interval to obtain

$$F^{\star}_{[a,b]}(\mathbf{z}) = F_{b-a}(z_1)\,e^{-(b-a)m(z_1)}\mu(z_1).$$

Proposition 2.7.3. *There exist positive constants $C, c,$ and τ such that, for all*

$[a,b] \subseteq [0,n]$ and $\mathbf{z}_1 \in \mathcal{Z}^\Delta$,

$$\left|F^\star_{[a,b]}(\mathbf{z}_1)\right| \leq C_0 \exp\left\{-(b-a)\tau + c_0|z_0|^2\frac{(b-a)^2}{2n}\right\}$$

$$\left|\frac{\partial}{\partial z_j}F^\star_{[a,b]}(\mathbf{z}_1)\right| \leq C_1 \exp\left\{-(b-a)\tau + c_1|z_0|^2\frac{(b-a)^2}{2n}\right\} \qquad j=0,1$$

$$\left|\frac{\partial^2}{\partial z_j \partial z_k}F^\star_{[a,b]}(\mathbf{z}_1)\right| \leq C_2 \exp\left\{-(b-a)\tau + c_2|z_0|^2\frac{(b-a)^2}{2n}\right\} \qquad j,k=0,1.$$

Moreover, there exist constants $\tau, C, c,$ and c' such that, for all $[a,b] \subseteq [0,n]$ with $a < T \leq b$, and whenever $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{Z}^\Delta$ coincide in their second argument - that is, $\mathbf{z}_1 = (z_1, z_0)$ and $\mathbf{z}_2 = (z_2, z_0)$ - we have

$$\left|F^\star_{[a,b]}(\mathbf{z}_1,\mathbf{z}_2)\right| \leq C \exp\left\{-(b-a)\tau + c|z_0|^2\frac{(b-a)^2}{2n} + (c'(b-a)+c'')|z_1-z_2|(b-T)\right\}.$$

Proof. These estimates follow from those in Proposition 2.2.12, and in Section B.2.

□

Lemma 2.7.4. *Let $0 < \varepsilon < \frac{1}{3}$. Then there exist constants a_1, a_2, a_3 such that for all integers $0 < T < n$,*

$$\left|\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} F^\star_{[T-\ell,T+r]}(\mathbf{z}) - 1\right| \leq a_1\frac{|z_0|^2}{n^2} + a_2 e^{-a_3 n^\varepsilon}$$

holds uniformly in $\mathbf{z} \in \mathcal{Z}^\Delta$.

Proof of Lemma 2.7.4. We use a series of approximations to establish the convergence of $\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} F^\star_{[t-\ell,t+r]}(\mathbf{z})$ to 1.

Let

$$\hat{F}_{[a,b]}(z,x) = F_{b-a}(z(x))\, e^{-(b-a)m(z(x))}\mu(z(x)). \tag{2.7.1}$$

We write

$$\left|\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} F^\star_{[T-\ell,T+r]}(\mathbf{z}) - 1\right| \leq \sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} \left|F^\star_{[T-\ell,T+r]}(\mathbf{z}) - \hat{F}_{[T-\ell,T+r]}\left(z, \frac{2T-\ell+r}{2n}\right)\right|$$
$$+ \left|\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} \hat{F}_{[T-\ell,T+r]}\left(z, \frac{2T-\ell+r}{2n}\right) - \hat{F}_{[T-\ell,T+r]}\left(z, \frac{T}{n}\right)\right| \tag{2.7.2}$$

$$+\left|\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} \hat{F}_{[T-\ell,T+r]}\left(z,\frac{T}{n}\right)-1\right|.$$

We will show that each of the parts of Equation (2.7.2) converges to zero.

First,

$$F^\star_{[T-\ell,T+r]}(\mathbf{z}) = \hat{F}_{[T-\ell,T+r]}\left(z,\frac{2T-\ell+r}{2n}\right)\frac{e^{-\int_{T-\ell}^{T+r} m\left(z\left(\frac{x}{n}\right)\right)dx}}{e^{-(\ell+r)m\left(z\left(\frac{T}{n}+\frac{r-\ell}{2n}\right)\right)}}\frac{\sqrt{\mu\left(z\left(\frac{T-\ell}{n}\right)\right)\mu\left(z\left(\frac{T+r}{n}\right)\right)}}{\mu\left(z\left(\frac{T}{n}+\frac{r-\ell}{2n}\right)\right)}$$

so that we can use Equation (B.2.1) and Proposition 2.2.12 to write

$$\begin{aligned}\left|F^\star_{[T-\ell,T+r]}(\mathbf{z})-\hat{F}_{[T-\ell,T+r]}\left(z,\frac{2T-\ell+r}{2n}\right)\right| &\le C|z_0|^2\frac{(\ell+r)^3}{n^2}\left|\hat{F}_{[T-\ell,T+r]}\left(z,\frac{2T-\ell+r}{2n}\right)\right|\\ &\le C|z_0|^2\frac{(\ell+r)^3}{n^2}\sup_{z\in\mathcal{Z}}|\mu(z)|\ c_0e^{-(\ell+r)\tau}.\end{aligned}$$

As a result,

$$\begin{aligned}\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}}\left|F^\star_{[T-\ell,T+r]}(\mathbf{z})-\hat{F}_{[T-\ell,T+r]}(z,0)\right| &\le C_1\frac{|z_0|^2}{n^2}\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}}(\ell+r)^3e^{-(\ell+r)\tau}\\ &\le C_2\frac{|z_0|^2}{n^2}.\end{aligned}$$

For the second term, the contribution to the sum from pairs (ℓ,r) with $\ell=r$ is clearly zero. We note that our definition of $\hat{F}_{[T-\ell,T+r]}$ only depends on the value of $\ell+r$, and not on ℓ and r themselves; by matching up the pairs (ℓ,r) and (r,ℓ), we see that

$$\begin{aligned}&\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}}\hat{F}_{[T-\ell,T+r]}\left(z,\frac{2T-\ell+r}{2n}\right)-\hat{F}_{[T-\ell,T+r]}\left(z,\frac{T}{n}\right)\\ &=\frac{1}{2}\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}}\hat{F}_{[T-\ell,T+r]}\left(z,\frac{2T-\ell+r}{2n}\right)+\hat{F}_{[T-\ell,T+r]}\left(z,\frac{2T-r+\ell}{2n}\right)-2\hat{F}_{[T-\ell,T+r]}\left(z,\frac{T}{n}\right).\end{aligned}$$

Proposition 2.7.3 allows us to use the upper bound in Equation (B.1.3) to write

$$\begin{aligned}&\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}}\hat{F}_{[T-\ell,T+r]}\left(z,\frac{2T-\ell+r}{2n}\right)-\hat{F}_{[T-\ell,T+r]}\left(z,\frac{T}{n}\right)\\ &\le\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}}|z_0|^2C\frac{(r-\ell)^2}{n^2}e^{-(\ell+r)\tau}\\ &\le C_3\frac{|z_0|^2}{n^2},\end{aligned}$$

Finally, we note that for each $k \in \mathbb{N}$ there are $k-1$ pairs (ℓ, r) with $\ell + r = k$, so that by Proposition 2.2.9 we have

$$\sum_{k\in\mathbb{N}} \sum_{\ell+r=k} \hat{F}_{[T-\ell,T+r]}\left(z, \frac{T}{n}\right) = 1$$

and

$$\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} \hat{F}_{[T-\ell,T+r]}\left(z, \frac{T}{n}\right) - 1 = -\sum_{k>n^\varepsilon} \sum_{\ell+r=k} \hat{F}_{[T-\ell,T+r]}\left(z, \frac{T}{n}\right).$$

By Proposition 2.7.3, there exist constants C_0, C_4, C_5 such that

$$\begin{aligned}\left|\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} \hat{F}_{[T-\ell,T+r]}\left(z, \frac{T}{n}\right) - 1\right| &\leq \sum_{k>n^\varepsilon} \sum_{\ell+r=k} C_0 e^{-(\ell+r)\tau} \\ &\leq C_4 e^{-C_5 n^\varepsilon}.\end{aligned}$$

Now we have

$$\left|\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} F^\star_{[T-\ell,t+r]}(\mathbf{z}) - 1\right| \leq (C_2 + C_3)\frac{|z_0|^2}{n^2} + C_4 e^{-C_5 n^\varepsilon},$$

as claimed. □

Lemma 2.7.5. *Let $0 < \varepsilon < \frac{1}{3}$. Then there exists a constant $C < \infty$ such that, with the constants a_1, a_2, a_3 as in Lemma 2.7.4, and for all $1 \leq j \leq k$,*

$$\begin{aligned}\left|\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} F^\star_{[T_j-\ell,T_j+r]}(\mathbf{z}_{j-1}, \mathbf{z}_j) - 1\right| &\leq Cn^\varepsilon |v_j - v_{j-1}| \left(1 + a_1\frac{|z_0|^2}{n^2} + a_2 e^{-a_3 n^\varepsilon}\right) \\ &\quad + a_1\frac{|z_0|^2}{n^2} + a_2 e^{-a_3 n^\varepsilon}\end{aligned}$$

holds uniformly in $\mathbf{z} \in \mathcal{Z}^\Delta_{k+1}$.

Remark 2.7.6. In the case $\mathbf{z}_{j-1} = \mathbf{z}_j$, this lemma becomes Lemma 2.7.4. We will use this similarity to simplify the proof.

Corollary 2.7.7. *In the context of Lemma 2.7.5, there exists $C < \infty$ such that, if additionally $|v_{j-1}|, |v_j| \leq n^{-\frac{1}{2}+\zeta}$ for some $\zeta < \frac{1}{2} - \varepsilon$, then*

$$\left|\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} F^\star_{[T_j-\ell,T_j+r]}(\mathbf{z}_{j-1}, \mathbf{z}_j) - 1\right| \leq Cn^{\varepsilon+\zeta-\frac{1}{2}}.$$

Proof of Lemma 2.7.5. We first write

$$\begin{aligned}&\left|\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} F^\star_{[T_j-\ell,T_j+r]}(\mathbf{z}_{j-1},\mathbf{z}_j)-1\right|\\&\quad\le\left|\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} F^\star_{[T_j-\ell,T_j+r]}(\mathbf{z}_{j-1},\mathbf{z}_j)-F^\star_{[T_j-\ell,T_j+r]}(\mathbf{z}_j)\right|+\left|\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} F^\star_{[T_j-\ell,T_j+r]}(\mathbf{z}_j)-1.\right|\end{aligned}$$

As noted in Remark 2.7.6, an upper bound on the second term is given in Lemma 2.7.4.

For the first term, we write

$$F^\star_{[T_j-\ell,T_j+r]}(\mathbf{z}_{j-1},\mathbf{z}_j)=F^\star_{[T_j-\ell,T_j+r]}(\mathbf{z}_j)\cdot\frac{\exp\left\{-\int_{T_j}^{T_j+r} m\left(z_j\left(\frac{x}{n}\right)\right)dx\right\}\sqrt{\mu\left(z_j\left(\frac{T_j+r}{n}\right)\right)}}{\exp\left\{-\int_{T_j}^{T_j+r} m\left(z_{j-1}\left(\frac{x}{n}\right)\right)dx\right\}\sqrt{\mu\left(z_{j-1}\left(\frac{T_j+r}{n}\right)\right)}}.$$

We use the upper bound in Corollary B.2.6 to write

$$\begin{aligned}&\left|\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} F^\star_{[T_j-\ell,T_j+r]}(\mathbf{z}_{j-1},\mathbf{z}_j)-F^\star_{[T_j-\ell,T_j+r]}(\mathbf{z}_j)\right|\\&\quad\le\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}}\left|F^\star_{[T_j-\ell,T_j+r]}(\mathbf{z}_j)\right|(c(\ell+r)+C)|z_j-z_{j-1}|e^{(c(\ell+r)+C)|z_j-z_{j-1}|};\end{aligned}$$

since $\ell+r\le n^\varepsilon$ and $|z_j-z_{j-1}|\le n^{-\frac{1}{2}+\zeta}$, we have that

$$\begin{aligned}&\left|\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} F^\star_{[T_j-\ell,T_j+r]}(\mathbf{z}_{j-1},\mathbf{z}_j)-F^\star_{[T_j-\ell,T_j+r]}(\mathbf{z}_j)\right|\\&\quad\le C|v_j-v_{j-1}|n^\varepsilon\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}}\left|F^\star_{[T_j-\ell,T_j+r]}(\mathbf{z}_j)\right|\\&\quad\le cn^\varepsilon|v_j-v_{j-1}|\left(1+a_1\frac{|z_0|^2}{n^2}+a_2e^{-a_3n^\varepsilon}\right),\end{aligned}$$

so that

$$\begin{aligned}&\left|\sum_{(\ell,r)\in\Lambda_{1,n^\varepsilon}} F^\star_{[T_j-\ell,T_j+r]}(\mathbf{z}_{j-1},\mathbf{z}_j)-1\right|\\&\quad\le cn^\varepsilon|v_j-v_{j-1}|\left(1+a_1\frac{|z_0|^2}{n^2}+a_2e^{-a_3n^\varepsilon}\right)+a_1\frac{|z_0|^2}{n^2}+a_2e^{-a_3n^\varepsilon}.\end{aligned}$$

□

Chapter 3

Convergence of Finite-Dimensional Distributions of a Piecewise-Constant Trajectory

We return to the "functional" perspective of our trajectories, through the lens of the height functions g and G. Recall that our aim is to prove convergence of the distributions of the fluctuations of the trajectories $g(t)$ around some limiting profile $c_\infty(t)$, under conditions on $g(1)$ and $\int_0^1 g(t)dt$. In this chapter, we will use properties of the partition functions derived in Chapter 2 to establish Central Limit Theorem results for $Y_n(1)$ and $Y_n(\mathbf{t})$ under the tilted distributions $\mathbb{Q}_n^{\mathbf{u}}$, and discuss how our choice of condition determines $c_\infty(t)$, as well as the best choice of $\mathbf{u}$. These will help us to prove Local Central Limit Theorems for $Y_n(1)$ and $Y_n(\mathbf{t})$ under unconditional, tilted distributions in Chapter 4, as well as a Local Central Limit Theorem for $G_n(\mathbf{t})$ under our conditional distribution.

3.1 The Distribution

We begin this chapter with a reminder of some important notation, as well as definitions of some new objects.

Recall that

$$G_n[s,t] = G(t) - G(s),$$

and that for $\mathbf{t} = (t_1, \ldots, t_k)$, we defined the finite-dimensional distributions of G_n and Y_n in Equation (2.1.2) by

$$\begin{aligned} G_n(\mathbf{t}) &= \Big(G[0,t_1], G[t_1,t_2], \ldots, G[t_{k-1},t_k], G[t_k,1] \Big), \\ Y_n(\mathbf{t}) &= \Big(G[0,t_1], G[t_1,t_2], \ldots, G[t_{k-1},t_k], G[t_k,1], \int_0^1 g(t)dt \Big). \end{aligned}$$

In Equation (2.6.1), we defined

$$\mathbf{z}_{\mathbf{T}}(x) = \sum_{j \le k+1} z_j(x) \, \mathbb{1}\left\{ T_{j-1} \le nx < T_j \right\};$$

for $\mathbf{z} \in \mathcal{Z}_{k+1}^{\Delta}$, let

$$\begin{aligned} f_{\mathbf{t}}(\mathbf{z}) &= \int_0^1 m\left(\mathbf{z}_{n\mathbf{t}}(x)\right) dx \\ &= \sum_{j=1}^{k+1} \int_{t_{j-1}}^{t_j} m\left(z_j + z_0(1-x)\right) dx. \end{aligned}$$

We denote the gradient of f at $\mathbf{z}$ by $\nabla f(\mathbf{z})$. Note that for $k = 0$, we have

$$f_1(\mathbf{z}) = \int_0^1 m\left(z_1 + z_0(1-x)\right) dx.$$

For $[a,b] \subseteq [0,n]$, let

$$\varphi_{[a,b]}(\mathbf{z}) = \int_a^b m\left(\mathbf{z}_{n\mathbf{t}}\left(\frac{x}{n}\right)\right) dx + \frac{1}{2}\log \mu\left(\mathbf{z}_{n\mathbf{t}}\left(\frac{a}{n}\right)\right) \mu\left(\mathbf{z}_{n\mathbf{t}}\left(\frac{b}{n}\right)\right).$$

We draw attention to the value of $\varphi_{[a,b]}$ in two specific situations. When $[a,b] = [0,n]$, we write $\varphi_{[0,n]}(\mathbf{z}) = \varphi_n(\mathbf{z})$ and note that

$$\varphi_n(\mathbf{z}) = n f_{\mathbf{t}}(\mathbf{z}) + \frac{1}{2}\log \mu\left(z_1 + z_0\right) \mu\left(z_{k+1}\right).$$

Meanwhile, if $nt_{j-1} \le a < b \le nt_j$, and $z_0 = 0$, then

$$\varphi_{[a,b]}(\mathbf{z}) = (b-a) m\left(z_j\right) + \log \mu\left(z_j\right).$$

Recall that, as in Equation 2.6.1,

$$\mathcal{B}_{[a,b],\mathbf{T}}(\mathbf{z}) = \sum_{\nu \in \mathcal{P}_{[a,b]}} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}\left(\mathbf{z}_{\mathbf{T}}\left(\nu_{j-1}\right)\right);$$

let

$$\mathcal{B}^{\star}_{[a,b],\mathbf{T}}(\mathbf{z}) = \mathcal{B}_{[a,b],\mathbf{T}}(\mathbf{z}) \exp\left\{-\varphi_{[a,b]}(\mathbf{z})\right\}. \tag{3.1.1}$$

Similarly, let

$$\mathcal{B}^{\star}_{[a,b],\mathbf{T},(\mathbf{S},\theta)}(\mathbf{z}) = \mathcal{B}_{[a,b],\mathbf{T},(\mathbf{S},\theta)}(\mathbf{z}) \exp\left\{-\varphi_{[a,b]}(\mathbf{z})\right\}$$

be the version of $\mathcal{B}^{\star}_{[a,b],\mathbf{T}}(\mathbf{z})$ in which we consider only the trajectories in which the increments around locations S_j have width at most θ. When $k = 0$ we refer to $\mathcal{B}^{\star}_{[a,b],1,(\mathbf{S},n^{\varepsilon})}(\mathbf{z})$ or $\mathcal{B}^{\star}_{[a,b],(\mathbf{S},n^{\varepsilon})}(z)$.

For $0 < \varepsilon < \gamma < \frac{1}{3}$, let $\rho_n = n^{\varepsilon}$, and $k_n = \lfloor n^{\gamma} \rfloor$. Let

$$\mathbf{S}_n[0,n] = \left(k_n, 2k_n, \ldots, \left\lfloor \frac{n}{k_n} \right\rfloor k_n\right).$$

For $[a,b] \subseteq [0,n]$ with $b - a \geq k_n$, let

$$\mathbf{S}_n[a,b] = \left(a, a + k_n, a + 2k_n, \ldots, a + \left\lfloor \frac{b-a}{k_n} \right\rfloor k_n\right). \tag{3.1.2}$$

The vector $\mathbf{S}_n[a,b]$ divides the interval $[a,b]$ into segments of width approximately n^{γ}; note that the number of such points is approximately $(b-a)n^{-\gamma}$.

In this chapter and the next, our usual choice of $\mathbf{S} = \mathbf{S}_n$ will be found by constructing $\mathbf{S}_n[nt_{j-1}, nt_j]$ for each j, so that

$$\mathbf{S}_n = \Big(\mathbf{S}_n[0, nt_1], \mathbf{S}_n[nt_1, nt_2], \ldots, \mathbf{S}_n[nt_k, n]\Big). \tag{3.1.3}$$

In Section 2.7, we found several properties of the rescaled partition functions $F^{\star}_{[a,b]}$, which took different forms in the two cases $a < T_j < b$ and $T_{j-1} < a < b < T_j$. Note that, in the notation of this chapter, both definitions of $F^{\star}_{[a,b]}$ can be expressed in the form

$$F^{\star}_{[a,b]}(\mathbf{z}) = F_{[a,b]}(\mathbf{z}) \exp\left\{-\varphi_{[a,b]}(\mathbf{z})\right\} \mu\left(\mathbf{z}_T\left(\frac{a}{n}\right)\right) \mu\left(\mathbf{z}_T\left(\frac{b}{n}\right)\right). \tag{3.1.4}$$

3.2 Central Limit Theorems

In this section, we prove the following Central Limit Theorem.

Theorem 3.2.1. *Let $\mathbf{u} = (u_1, u_0) \in \mathcal{U}^{\Delta}$. Under the measures $\mathbb{Q}_n^{\mathbf{u}}$, we have for all $k \in \mathbb{N}$ and $\mathbf{t} = (t_1, \dots, t_k)$,*

$$\frac{1}{\sqrt{n}}\left(Y_n(\mathbf{t}) - n\nabla f_{\mathbf{t}}(\mathbf{u})\right) \xrightarrow{\mathcal{D}} Z$$

as $n \to \infty$, where Z has a $(k+2)$-dimensional Normal distribution with mean $\mathbf{0}$ and covariance matrix Σ_{k+2} given by

$$(\Sigma_{k+2})_{j,\ell} = \frac{\partial}{\partial v_j \partial v_\ell} f_{\mathbf{t}}(\mathbf{u} + i\mathbf{v})\Big|_{\mathbf{v}=\mathbf{0}}.$$

Moreover, for any sequence $\mathbf{u}_n \in \mathcal{U}^{\Delta}$ such that $\|\mathbf{u}_n - \mathbf{u}\| \to 0$ as $n \to \infty$, under the measures $\mathbb{Q}_n^{\mathbf{u}_n}$,

$$\frac{1}{\sqrt{n}}\left(Y_n(\mathbf{t}) - n\nabla f_{\mathbf{t}}(\mathbf{u}_n)\right) \xrightarrow{\mathcal{D}} Z \tag{3.2.1}$$

as $n \to \infty$, where, as above, Z has a $(k+2)$-dimensional Normal distribution with mean $\mathbf{0}$ and covariance matrix Σ_{k+2}.

Remark 3.2.2. If additionally $\sqrt{n}\|\mathbf{u}_n - \mathbf{u}\| \to 0$ as $n \to \infty$, we can replace the centering terms $n\nabla f_{\mathbf{t}}(\mathbf{u}_n)$ in Equation (3.2.1) with $n\nabla f_{\mathbf{t}}(\mathbf{u})$.

In order to prove Theorem 3.2.1, we study the behaviour of the characteristic functions. As outlined in Theorem A.0.1, it is sufficient to prove that, as $n \to \infty$,

$$\left|\log \bar{\mathbb{E}}_n^{u}\left[e^{i\mathbf{v}\cdot Y_n(\mathbf{t})}\right] - n\left(f_{\mathbf{t}}(\mathbf{u} + i\mathbf{v}) - f_{\mathbf{t}}(\mathbf{u})\right)\right|$$

converges to zero uniformly in $\mathbf{z} = \mathbf{u} + i\mathbf{v} \in \mathcal{Z}_{k+1}^{\Delta}$ such that $\|\mathbf{v}\| \le n^{-\frac{1}{2}+\zeta}$, for some $\zeta > 0$.

In fact, it is sufficient to prove that we have uniform convergence of

$$\left|\log \bar{\mathbb{E}}_n^u \left[e^{i\mathbf{v}\cdot Y_n(\mathbf{t})}\right] - \Big(\varphi_n(\mathbf{z}) - \varphi_n(\mathbf{u})\Big)\right| \to 0,$$

as we can find an upper bound on the difference

$$\left|\Big(\varphi_n(\mathbf{z}) - \varphi_n(\mathbf{u})\Big) - n\Big(f_{\mathbf{t}}(\mathbf{z}) - f_{\mathbf{t}}(\mathbf{u})\Big)\right| = \frac{1}{2}\left|\log \frac{\mu(z_1+z_0)\,\mu(z_{k+1})}{\mu(u_1+u_0)\,\mu(u_{k+1})}\right|$$

using the fact that $\log \mu$ is analytic: there exists a constant $c < \infty$ such that

$$\left|\Big(\varphi_n(\mathbf{z}) - \varphi_n(\mathbf{u})\Big) - n\Big(f_{\mathbf{t}}(\mathbf{z}) - f_{\mathbf{t}}(\mathbf{u})\Big)\right| \leq c|v_1+v_0| + c|v_k|$$

and, if $\|\mathbf{v}\| \leq n^{-\frac{1}{2}+\zeta}$, this difference converges to zero as $n \to \infty$.

3.2.1 Central Limit Theorem in 2 Dimensions

We begin with the case $k = 0$, and proceed with an auxiliary result, using the distributions $\mathbb{P}^{\mathbf{u}}_{n,(\mathbf{S},\theta)}$ discussed in Remark 2.4.15.

Proposition 3.2.3. *Let* $0 < \varepsilon < \gamma < \frac{1}{3}$*,* $\frac{1-\gamma}{2} < \delta < 1-2\gamma$*, and* $0 < \zeta < \frac{\gamma}{2}$*. Let* $\mathbf{S}_n$ *be as in Equation* (3.1.2)*. Then there exist constants* C_1 *and* C_2 *such that, for all* n *large enough and any segment* $[a,b] \subseteq [0,n]$ *with* $b-a \geq n^\gamma$*,*

$$\left|\mathcal{B}^\star_{[a,b],1,(\mathbf{S},n^\varepsilon)}(\mathbf{z}) - 1\right| \leq C_1|z_0|^2 n^{2\gamma-1} + C_2 n^{1-\gamma-2\delta}$$

holds for all $\mathbf{z} \in \mathcal{Z}^\Delta$ *such that* $\|\mathbf{v}\| \leq n^{-\frac{1}{2}+\zeta}$*.*

Corollary 3.2.4. *Under the same conditions as in Proposition 3.2.3, there exist finite constants* C_1' *and* C_2' *such that*

$$\left|\log \mathcal{B}_{[a,b],1,(\mathbf{S},n^\varepsilon)}(\mathbf{z}) - \varphi_{[a,b]}(\mathbf{z})\right| \leq C_1'|z_0|^2 n^{2\gamma-1} + C_2' n^{1-\gamma-2\delta}$$

holds for all $\mathbf{z} \in \mathcal{Z}^\Delta$ *such that* $\|\mathbf{v}\| \leq n^{-\frac{1}{2}+\zeta}$ *for some* $\zeta \in (0, \frac{\gamma}{2})$*.*

Remark 3.2.5. In light of Remark 2.4.11, there exists an integer n_0 such that, for all $n \geq n_0$ and all segments $[a,b] \subseteq [0,n]$ with $b-a \geq n_0$, we have

$$\left|\log \mathcal{B}_{[a,b],1,(\mathbf{S},n^\varepsilon)}(\mathbf{u}) - \varphi_{[a,b]}(\mathbf{u})\right| \leq C_1'|u_0|^2 n^{2\gamma-1} + C_2' n^{1-\gamma-2\delta}$$

for all $\mathbf{u} \in \mathcal{U}^{\Delta}$.

Lemma 3.2.6. *Let $0 < \varepsilon < \gamma < \frac{1}{3}$. Then there exist finite positive constants C and C', and $0 < \delta < 1 - 2\gamma$ such that for all n large enough, we have for every segment $[a,b]$ with $n^{\varepsilon} < b - a < n^{\gamma}$,*

$$\left|\mathcal{B}^{\star}_{[a,b],1}(\mathbf{z}) - 1\right| \leq Cn^{-2\delta} + C'n^{3\gamma-2}|z_0|^2$$

holds uniformly in $\mathbf{z} \in \mathcal{Z}^{\Delta}$ such that $\|\mathbf{v}\| \leq n^{-\frac{1}{2}+\zeta}$ for $0 < \zeta < \frac{\gamma}{2}$.

Proof of Proposition 3.2.3. Recalling the polymer decomposition of $\mathcal{B}_{[a,b],1,(\mathbf{S},n^{\varepsilon})}(\mathbf{z})$ in Remark 2.4.12, along with Equations (3.1.1) and (3.1.4), we write

$$\mathcal{B}^{\star}_{[a,b],1,(\mathbf{S},n^{\varepsilon})}(\mathbf{z}) = \sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{|\mathbf{S}|,n^{\varepsilon}}} \prod_{j=1}^{|\mathbf{S}|+1} \mathcal{B}^{\star}_{\xi_{\mathbf{S},j}}(\mathbf{z}) \prod_{j=1}^{|\mathbf{S}|} F^{\star}_{\Delta_{\mathbf{S},j}}(\mathbf{z}),$$

so that

$$\begin{aligned}\left|\mathcal{B}^{\star}_{[a,b],1,(\mathbf{S},n^{\varepsilon})}(\mathbf{z}) - 1\right| \leq \sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{|\mathbf{S}|,n^{\varepsilon}}} \left|\prod_{j=1}^{|\mathbf{S}|+1} \mathcal{B}^{\star}_{\xi_{\mathbf{S},j}}(\mathbf{z}) - 1\right| \left|\prod_{j=1}^{|\mathbf{S}|} F^{\star}_{\Delta_{\mathbf{S},j}}(\mathbf{z})\right| \\ + \left|\sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{|\mathbf{S}_n|,n^{\varepsilon}}} \prod_{j=1}^{|\mathbf{S}|} F^{\star}_{\Delta_{\mathbf{S},j}}(\mathbf{z}) - 1\right|. \end{aligned} \tag{3.2.2}$$

First, note that

$$\left|\sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{|\mathbf{S}_n|,n^{\varepsilon}}} \prod_{j=1}^{|\mathbf{S}_n|} F^{\star}_{\Delta_{\mathbf{S}_n,j}}(\mathbf{z}) - 1\right| = \left|\prod_{j=1}^{|\mathbf{S}_n|} \sum_{(\ell_j,r_j)\in\Lambda_{1,n^{\varepsilon}}} F^{\star}_{\Delta_{\mathbf{S}_n,j}}(\mathbf{z}) - 1\right|;$$

applying Proposition B.1.7 to the upper bound in Lemma 2.7.4, we have for n large enough and positive constants C_1, C_2, c_1,

$$\begin{aligned}\left|\sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{|\mathbf{S}_n|,n^{\varepsilon}}} \prod_{j=1}^{|\mathbf{S}_n|} F^{\star}_{\Delta_{\mathbf{S}_n,j}}(\mathbf{z}) - 1\right| &\leq \exp\left\{|\mathbf{S}_n|\left(C_1\frac{|z_0|^2}{n^2} + C_2 e^{-c_1 n^{\varepsilon}}\right)\right\} - 1 \\ &\leq \exp\left\{C_1|z_0|^2 n^{\gamma-1} + C_2 e^{-2c_1 n^{\varepsilon}}\right\} - 1. \end{aligned} \tag{3.2.3}$$

We once again use Proposition B.1.7, this time with the upper bound in Lemma 3.2.6, to see that for some constants C_3, C_4,

$$\left|\prod_{j=1}^{|\mathbf{S}_n|+1} \mathcal{B}^{\star}_{\xi_{\mathbf{S}_n,j}}(\mathbf{z}) - 1\right| \leq \exp\left\{(|\mathbf{S}_n|+1)\left(C_3 n^{-2\delta} + C_4 n^{3\gamma-2}|z_0|^2\right)\right\} - 1$$

$$\leq \exp\left\{C_3 n^{1-\gamma-2\delta} + C_4|z_0|^2 n^{2\gamma-1}\right\} - 1$$
$$\leq 2 \tag{3.2.4}$$

for n large enough.

Finally, we note that

$$\left|\sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{|\mathbf{S}_n|+1,n^\varepsilon}} \prod_{j=1}^{|\mathbf{S}_n|+1} \left|F^\star_{\Delta\mathbf{S}_{n,j}}(\mathbf{z})\right| - 1\right| = \left|\prod_{j=1}^{|\mathbf{S}_n|+1} \sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{1,n^\varepsilon}} \left|F^\star_{\Delta\mathbf{S}_{n,j}}(\mathbf{z})\right| - 1\right|.$$

We have

$$\left|\sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{1,n^\varepsilon}} \left|F^\star_{\Delta\mathbf{S}_{n,j}}(\mathbf{z})\right| - 1\right| \leq \sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{1,n^\varepsilon}} \left|\left|F^\star_{\Delta\mathbf{S}_{n,j}}(\mathbf{z})\right| - F^\star_{\Delta\mathbf{S}_{n,j}}(\mathbf{u})\right| + \left|\sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{1,n^\varepsilon}} F^\star_{\Delta\mathbf{S}_{n,j}}(\mathbf{u}) - 1\right|.$$

The upper bounds in Proposition 2.7.3 allow us to use Proposition B.1.3 to write

$$\left|\left|F^\star_{\Delta\mathbf{S}_{n,j}}(\mathbf{z})\right| - F^\star_{\Delta\mathbf{S}_{n,j}}(\mathbf{u})\right| \leq C_5\|\mathbf{v}\|^2 e^{-c_2(\ell_j+r_j)}$$
$$\leq C_5 n^{-1+2\zeta} e^{-c_2(\ell_j+r_j)},$$

so that

$$\sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{1,n^\varepsilon}} \left|\left|F^\star_{\Delta\mathbf{S}_{n,j}}(\mathbf{z})\right| - F^\star_{\Delta\mathbf{S}_{n,j}}(\mathbf{u})\right| \leq C_6 n^{-1+2\zeta}$$

and

$$\left|\sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{1,n^\varepsilon}} \left|F^\star_{\Delta\mathbf{S}_{n,j}}(\mathbf{z})\right| - 1\right| \leq C_6 n^{-1+2\zeta} + C_1\frac{|u_0|^2}{n^2} + C_2 e^{-c_1 n^\varepsilon}$$
$$\leq C_1\frac{|u_0|^2}{n^2} + C_7 e^{-c_1 n^\varepsilon}.$$

By Proposition B.1.7, we have

$$\left|\sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{|\mathbf{S}_n|+1,n^\varepsilon}} \prod_{j=1}^{|\mathbf{S}_n|+1} \left|F^\star_{\Delta\mathbf{S}_{n,j}}(\mathbf{z})\right|\right| \leq \left|\sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{|\mathbf{S}_n|+1,n^\varepsilon}} \prod_{j=1}^{|\mathbf{S}_n|+1} \left|F^\star_{\Delta\mathbf{S}_{n,j}}(\mathbf{z})\right| - 1\right| + 1$$
$$\leq \exp\left\{(|\mathbf{S}_n|+1)\left(C_1\frac{|u_0|^2}{n^2} + C_7 e^{-c_1 n^\varepsilon}\right)\right\}$$
$$\leq \exp\left\{C_8|u_0|^2 n^{\gamma-1} + C_9 e^{-c_1 n^\varepsilon}\right\}. \tag{3.2.5}$$

Combining the upper bounds in Equations (3.2.3), (3.2.4), and (3.2.5), we have

$$\begin{aligned}\left|\mathcal{B}^{\star}_{[a,b],1,(\mathbf{S},n^{\varepsilon})}(\mathbf{z})-1\right| &\leq \exp\left\{C_8|z_0|^2 n^{\gamma-1}+C_9 e^{-c_1 n^{\varepsilon}}\right\}-1\\ &\leq C_8|z_0|^2 n^{\gamma-1}+C_9 e^{-c_1 n^{\varepsilon}}.\end{aligned}$$

□

Proof of Lemma 3.2.6. As in the proof of Lemma 2.7.4, we use a series of approximations. Let $\mathbf{z}\in\mathcal{Z}^{\Delta}_1=\mathcal{Z}^{\Delta}$.

Let

$$H^{\star}_{[a,b]}(\mathbf{z})=H_{[a,b]}(\mathbf{z})\exp\left\{-\varphi_{[a,b]}(\mathbf{z})\right\},$$

and

$$\hat{H}_{[a,b]}(\mathbf{z})=H_{[a,b]}(\mathbf{z})\exp\left\{-\varphi_{[a,b]}\left(z\left(\frac{a+b}{2n}\right),0\right)\right\}.$$

Then

$$\left|\mathcal{B}^{\star}_{[a,b],1}(\mathbf{z})-1\right|\leq\left|\mathcal{B}^{\star}_{[a,b],1}(\mathbf{z})-H^{\star}_{[a,b]}\right|+\left|H^{\star}_{[a,b]}-\hat{H}_{[a,b]}(\mathbf{z})\right|+\left|\hat{H}_{[a,b]}(\mathbf{z})-1\right|.$$

First,

$$\left|\mathcal{B}^{\star}_{[a,b],1}(\mathbf{z})-H^{\star}_{[a,b]}\right|=\left|\mathcal{B}_{[a,b]}(\mathbf{z})-H_{[a,b]}(\mathbf{z})\right|\left|\exp\left\{-\varphi_{[a,b]}(\mathbf{z})\right\}\right|.$$

By Remark 2.5.8, we have

$$\left|\mathcal{B}_{[a,b]}(\mathbf{z})-H_{[a,b]}(\mathbf{z})\right|\leq Cn^{-2\delta}H_{[a,b]}(\mathbf{u})$$

and by Corollary B.2.4,

$$\left|\exp\left\{-\varphi_{[a,b]}(\mathbf{z})\right\}\right|\leq C\left(1+|z_0|^2\frac{(b-a)^3}{n^2}\right)\exp\left\{-\varphi_{[a,b]}\left(u\left(\frac{a+b}{2n}\right),0\right)\right\},$$

so that

$$\left|\mathcal{B}^{\star}_{[a,b],1}(\mathbf{z})-H^{\star}_{[a,b]}\right|\leq\left(Cn^{-2\delta}+C|z_0|^2\frac{(b-a)^3}{n^2}\right)\hat{H}_{[a,b]}(\mathbf{u}).\tag{3.2.6}$$

Next,

$$\left|H^{\star}_{[a,b]} - \hat{H}_{[a,b]}(\mathbf{z})\right| = \left|\frac{\exp\left\{-\varphi_{[a,b]}(\mathbf{z})\right\}}{\exp\left\{-\varphi_{[a,b]}\left(z\left(\frac{a+b}{2n}\right),0\right)\right\}} - 1\right| \left|\hat{H}_{[a,b]}(\mathbf{z})\right|$$

so that using Corollary B.2.3 we have

$$\left|H^{\star}_{[a,b]} - \hat{H}_{[a,b]}(\mathbf{z})\right| \leq C|z_0|^2 \frac{(b-a)^3}{n^2} \left|\hat{H}_{[a,b]}(\mathbf{z})\right|. \tag{3.2.7}$$

Finally, we establish upper bounds on $\left|\hat{H}_{[a,b]}(\mathbf{z})\right|$ and $\left|\hat{H}_{[a,b]}(\mathbf{z}) - 1\right|$ using Proposition 2.2.9: we have

$$\left|\hat{H}_{[a,b]}(\mathbf{z}) - 1\right| \leq Ce^{-c(b-a)} \leq Ce^{-cn^{\varepsilon}} \leq Cn^{-2\delta}, \tag{3.2.8}$$

and

$$\left|\hat{H}_{[a,b]}(\mathbf{z})\right| \leq 1 + Ce^{-cn^{\varepsilon}} \leq 2, \tag{3.2.9}$$

as long as n is large enough. Combining Equations (3.2.6) - (3.2.9), we have

$$\begin{aligned}\left|\mathcal{B}^{\star}_{[a,b],1}(\mathbf{z}) - 1\right| &\leq Cn^{-2\delta} + C'|z_0|^2 \frac{(b-a)^3}{n^2} \\ &\leq Cn^{-2\delta} + C'n^{3\gamma-2}|z_0|^2.\end{aligned}$$

□

Proposition 3.2.3 implies the convergence of

$$\frac{1}{\sqrt{n}}\left(Y_n(1) - n\nabla f_1(\mathbf{u})\right)$$

under the measures $\mathbb{P}^{\mathbf{u}_n}_{n,(\mathbf{s}_n,\rho_n)}$. It will also be helpful in the proof of our Central Limit Theorem under the unrestricted measures $\mathbb{Q}^{\mathbf{u}}_n$.

Proof of Theorem 3.2.1, in the case k=0. As we noted earlier in this section, it is sufficient to prove that

$$\left|\log \bar{\mathbb{E}}^{\mathbf{u}}_n\left[e^{i\mathbf{v}\cdot Y_n(1)}\right] - \left(\varphi_n(\mathbf{z}) - \varphi_n(\mathbf{u})\right)\right| \to 0$$

as $n \to \infty$, uniformly in $\mathbf{z} = \mathbf{u} + i\mathbf{v} \in \mathcal{Z}^{\Delta}$ such that $\|v\| \leq n^{-\frac{1}{2}+\zeta}$. We have

$$\begin{aligned}
\left|\log \bar{\mathbb{E}}_n^{\mathbf{u}}\left[e^{i\mathbf{v}\cdot Y_n(1)}\right] - \Big(\varphi_n(\mathbf{z}) - \varphi_n(\mathbf{u})\Big)\right| \\
&= \left|\log \frac{\mathcal{B}_n(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_n(\mathbf{u})} - \Big(\varphi_n(\mathbf{z}) - \varphi_n(\mathbf{u})\Big)\right| \\
&\leq \left|\log \frac{\mathcal{B}_n(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_n(\mathbf{u})} - \log \frac{\mathcal{B}_{n,(\mathbf{S},n^{\varepsilon})}(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_{n,(\mathbf{S},n^{\varepsilon})}(\mathbf{u})}\right| \\
&\quad + \left|\log \frac{\mathcal{B}_{n,(\mathbf{S},n^{\varepsilon})}(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_{n,(\mathbf{S},n^{\varepsilon})}(\mathbf{u})} - \Big(\varphi_n(\mathbf{z}) - \varphi_n(\mathbf{u})\Big)\right|.
\end{aligned}$$

We know that the second term converges to zero, as a consequence of Corollary 3.2.4. To see that the first also converges to zero, we note that by Proposition 2.4.16, we have

$$\begin{aligned}
&\left|\frac{\mathcal{B}_n(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_n(\mathbf{u})} - \frac{\mathcal{B}_{n,(\mathbf{S},n^{\varepsilon})}(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_{n,(\mathbf{S},n^{\varepsilon})}(\mathbf{u})}\right| \\
&\quad \leq \left|\frac{\mathcal{B}_n(\mathbf{u}+i\mathbf{v}) - \mathcal{B}_{n,(\mathbf{S},n^{\varepsilon})}(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_n(\mathbf{u})}\right| + \left|\frac{\mathcal{B}_{n,(\mathbf{S},n^{\varepsilon})}(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_{n,(\mathbf{S},n^{\varepsilon})}(\mathbf{u})}\right| \left|\frac{\mathcal{B}_n(\mathbf{u}) - \mathcal{B}_{n,(\mathbf{s}_n,\rho_n)}(\mathbf{u})}{\mathcal{B}_n(\mathbf{u})}\right| \\
&\quad = \mathcal{O}\left(\exp\{-cn^{\varepsilon}\}\right).
\end{aligned}$$

As a result, there exists a radius $r < 1$ such that, for n large enough,

$$\frac{\mathcal{B}_n(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_n(\mathbf{u})}, \quad \frac{\mathcal{B}_{n,(\mathbf{S},n^{\varepsilon})}(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_{n,(\mathbf{S},n^{\varepsilon})}(\mathbf{u})}, \text{ and } \exp\left\{-\frac{1}{2}\sum_{1\leq j,k\leq 2} v_j v_k (\Sigma_2)_{j,k}\right\}$$

all lie within $B(1, r)$ in the complex plane, so that they lie on the same branch of the complex logarithm and in particular

$$\left|\log \frac{\mathcal{B}_n(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_n(\mathbf{u})} - \log \frac{\mathcal{B}_{n,(\mathbf{S},n^{\varepsilon})}(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_{n,(\mathbf{S},n^{\varepsilon})}(\mathbf{u})}\right|$$

converges to zero as $n \to \infty$.

As a result, by Theorem A.0.1, we have that

$$\frac{1}{\sqrt{n}}\left(Y_n(1) - n\nabla f_1(\mathbf{u})\right) \xrightarrow{\mathcal{D}} Z$$

as $n \to \infty$, under the measures $\mathbb{Q}_n^{\mathbf{u}}$, and

$$\frac{1}{\sqrt{n}}\left(Y_n(1) - n\nabla f_1(\mathbf{u}_n)\right) \xrightarrow{\mathcal{D}} Z$$

as $n \to \infty$, under the measures $\mathbb{Q}_n^{\mathbf{u}_n}$. □

3.2.2 Central Limit Theorem in $k+2$ Dimensions

In this section, we extend the results of the previous section for $k > 0$. The proofs will largely follow a similar structure, with the principal difference being more complicated notation. As before, we begin with an auxiliary result concerning the distributions $\mathbb{P}^{\mathbf{u}}_{n,(\mathbf{S}_n,\theta)}$. Throughout this section, we will use the definition of $\mathbf{S}_n$ given in Equation (3.1.3).

Proposition 3.2.7. *Let $0 < \varepsilon < \gamma < \frac{1}{3}$, $\frac{1-\gamma}{2} < \delta < 1 - 2\gamma$, and $0 < \zeta < \frac{\gamma}{2}$. Let $k \in \mathbb{N}$, and let $\mathbf{t} = (t_1, t_2, \ldots, t_k)$ be a vector of times. Then there exist finite constants C_1 and C_2 such that for all n large enough and any segment $[a,b] \subseteq [0,n]$ with $b - a \geq n^\gamma$,*

$$\left|\mathcal{B}^\star_{[a,b],n\mathbf{t},(\mathbf{S},n^\varepsilon)}(\mathbf{z}) - 1\right| \leq C_1|z_0|^2 n^{2\gamma-1} + C_2 n^{1-\gamma-2\delta} + C_3 n^{\varepsilon+\zeta-\frac{1}{2}}$$

holds for all $\mathbf{z} \in \mathcal{Z}^\Delta_{k+1}$ such that $\|\mathbf{v}\| \leq n^{-\frac{1}{2}+\zeta}$.

Corollary 3.2.8. *Under the same conditions as in Proposition 3.2.7, there exist finite constants C_1', C_2', and C_3' such that*

$$\left|\log \mathcal{B}_{[a,b],n\mathbf{t},(\mathbf{S},n^\varepsilon)}(\mathbf{z}) - \varphi_{[a,b]}(\mathbf{z})\right| \leq C_1'|z_0|^2 n^{2\gamma-1} + C_2' n^{1-\gamma-2\delta} + C_3' n^{\varepsilon+\zeta-\frac{1}{2}}$$

holds for all $\mathbf{z} \in \mathcal{Z}^\Delta_{k+1}$ such that $\|\mathbf{v}\| \leq n^{-\frac{1}{2}+\zeta}$, for some $\zeta \in (0, \frac{\gamma}{2})$.

Proof of Proposition 3.2.7. The polymer decomposition in Corollary 2.6.7 allows us to write

$$\mathcal{B}^\star_{[a,b],n\mathbf{t},(\mathbf{S},n^\varepsilon)}(\mathbf{z}) = \sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{k,n^\varepsilon}} \prod_{j=1}^{k+1} \mathcal{B}^\star_{\xi_{n\mathbf{t},j},1,\left(\mathbf{S_j},n^\varepsilon\right)}(\mathbf{z}) \prod_{j=1}^{k} F^\star_{\Delta_{n\mathbf{t},j}}(\mathbf{z}).$$

Note that, in the notation of Section 2.7,

$$F^\star_{\Delta_{\mathbf{T},j}}(\mathbf{z}) = F^\star_{[T_j-\ell_j,T_j+r_j]}(\mathbf{z}_{j-1},\mathbf{z}_j).$$

Now,

$$\left|\mathcal{B}^{\star}_{[a,b],n\mathbf{t},(\mathbf{S},n^{\varepsilon})}(\mathbf{z})-1\right| \leq \sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{k,n^{\varepsilon}}} \left|\prod_{j=1}^{k+1}\mathcal{B}^{\star}_{\xi_{n\mathbf{t},j},1,\left(\mathbf{S_j},n^{\varepsilon}\right)}(\mathbf{z})-1\right|\left|\prod_{j=1}^{k}F^{\star}_{\Delta_{n\mathbf{t},j}}(\mathbf{z})\right| + \left|\sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{k,n^{\varepsilon}}}\prod_{j=1}^{k}F^{\star}_{\Delta_{n\mathbf{t},j}}(\mathbf{z})-1\right|. \tag{3.2.10}$$

Applying Proposition B.1.7 to the upper bound in Corollary 2.7.7, we have

$$\begin{aligned}\left|\sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{k,n^{\varepsilon}}}\prod_{j=1}^{k}F^{\star}_{\Delta_{n\mathbf{t},j}}(\mathbf{z})-1\right| &= \left|\prod_{j=1}^{k}\sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{1,n^{\varepsilon}}}F^{\star}_{\Delta_{n\mathbf{t},j}}(\mathbf{z})-1\right| \\ &\leq \left(1+Cn^{\varepsilon+\zeta-\frac{1}{2}}\right)^{k}-1 \\ &\leq C'n^{\varepsilon+\zeta-\frac{1}{2}}. \end{aligned} \tag{3.2.11}$$

Using the upper bound in Proposition 2.7.3, we have

$$\left|F^{\star}_{\Delta_{n\mathbf{t},j}}(\mathbf{z})\right| \leq C\exp\left\{-(\ell_j+r_j)\tau + c|z_0|^2\frac{(\ell_j+r_j)^2}{2n} + c'(\ell_j+r_j)|v_{j-1}-v_j|r_j\right\}.$$

The restrictions $\ell_j+r_j\leq n^{\varepsilon}$ and $|v_{j-1}-v_j|\leq n^{-\frac{1}{2}+\zeta}$ allow us to find a constant C' such that, if n is large enough,

$$\left|F^{\star}_{\Delta_{n\mathbf{t},j}}(\mathbf{z})\right| \leq C'e^{-(\ell_j+r_j)\tau},$$

so that

$$\begin{aligned}\sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{k,n^{\varepsilon}}}\prod_{j=1}^{k}\left|F^{\star}_{\Delta_{n\mathbf{t},j}}(\mathbf{z})\right| &= \prod_{j=1}^{k}\sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{1,n^{\varepsilon}}}\left|F^{\star}_{\Delta_{n\mathbf{t},j}}(\mathbf{z})\right| \\ &\leq \prod_{j=1}^{k}\sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{1,n^{\varepsilon}}}C'e^{-(\ell_j+r_j)\tau}\leq C'' \end{aligned} \tag{3.2.12}$$

for some constant C''.

Finally, for every $(\boldsymbol{\ell},\mathbf{r})\in\Lambda_{k,n^{\varepsilon}}$, we can use Proposition B.1.7 with the upper bound in Proposition 3.2.3 to see that

$$\begin{aligned}\left|\prod_{j=1}^{k+1}\mathcal{B}^{\star}_{\xi_{n\mathbf{t},j},1,\left(\mathbf{S_j},n^{\varepsilon}\right)}(\mathbf{z})-1\right| &\leq \left(1+C_1|z_0|^2n^{2\gamma-1}+C_2n^{1-\gamma-2\delta}\right)^{k+1}-1 \\ &\leq C_1'|z_0|^2n^{2\gamma-1}+C_2'n^{1-\gamma-2\delta}. \end{aligned} \tag{3.2.13}$$

Combining the upper bounds in Equations (3.2.11), (3.2.12), and (3.2.12), we have

$$\left|\mathcal{B}^{\star}_{[a,b],n\mathbf{t},(\mathbf{S},n^{\varepsilon})}(\mathbf{z}) - 1\right| \leq C_1''|z_0|^2 n^{2\gamma-1} + C_2'' n^{1-\gamma-2\delta} + C_3'' n^{\varepsilon+\zeta-\frac{1}{2}}.$$

□

Proof of Theorem 3.2.1. As in the $k = 0$ case, it is sufficient to prove that, for $\mathbf{u} + i\mathbf{v} \in \mathcal{Z}^{\Delta}_{k+1}$ such that $\|\mathbf{v}\| \leq n^{-\frac{1}{2}+\zeta}$,

$$\left|\log \bar{\mathbb{E}}^{\mathbf{u}}_n \left[e^{i\mathbf{v}\cdot Y_n(\mathbf{t})}\right] - \Big(\varphi_n(\mathbf{z}) - \varphi_n(\mathbf{u})\Big)\right| \to 0$$

as $n \to \infty$, uniformly in $\mathbf{z} = \mathbf{u} + i\mathbf{v} \in \mathcal{Z}^{\Delta}$ such that $\|v\| \leq n^{-\frac{1}{2}+\zeta}$. Now

$$\begin{aligned}\left|\log \bar{\mathbb{E}}^{\mathbf{u}}_n \left[e^{i\mathbf{v}\cdot Y_n(\mathbf{t})}\right]\right. & \left.-\Big(\varphi_n(\mathbf{z}) - \varphi_n(\mathbf{u})\Big)\right| \\ & \leq \left|\log \frac{\mathcal{B}_{[0,n],n\mathbf{t}}(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_{[0,n],n\mathbf{t}}(\mathbf{u})} - \log \frac{\mathcal{B}_{[0,n],n\mathbf{t},(\mathbf{S},n^{\varepsilon})}(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_{[0,n],n\mathbf{t},(\mathbf{S},n^{\varepsilon})}(\mathbf{u})}\right| \\ & \quad + \left|\log \frac{\mathcal{B}_{[0,n],n\mathbf{t},(\mathbf{S},n^{\varepsilon})}(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_{[0,n],n\mathbf{t},(\mathbf{S},n^{\varepsilon})}(\mathbf{u})} - \Big(\varphi_n(\mathbf{u}+i\mathbf{v}) - \varphi_n(\mathbf{u})\Big)\right|.\end{aligned}$$

The second term converges to zero as a result of Corollary 3.2.8, while we can use Proposition 2.6.9 to write

$$\left|\log \frac{\mathcal{B}_{[0,n],n\mathbf{t}}(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_{[0,n],n\mathbf{t}}(\mathbf{u})} - \log \frac{\mathcal{B}_{[0,n],n\mathbf{t},(\mathbf{S},n^{\varepsilon})}(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_{[0,n],n\mathbf{t},(\mathbf{S},n^{\varepsilon})}(\mathbf{u})}\right| = \mathcal{O}(\exp\{-cn^{\varepsilon}\}).$$

As in the previous section, there exists a radius $r < 1$ such that, for large enough n,

$$\frac{\mathcal{B}_{[0,n],n\mathbf{t}}(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_{[0,n],n\mathbf{t}}(\mathbf{u})}, \quad , \frac{\mathcal{B}_{[0,n],n\mathbf{t},(\mathbf{S},n^{\varepsilon})}(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_{[0,n],n\mathbf{t},(\mathbf{S},n^{\varepsilon})}(\mathbf{u})}, \text{ and } \exp\left\{-\frac{1}{2}\sum_{1\leq j,\ell\leq k+2} v_j v_k (\Sigma_{k+2})_{j,k}\right\}$$

all lie in $B(1, r)$ in the complex plane. As a result, the difference

$$\left|\log \bar{\mathbb{E}}^{u}_n \left[e^{i\mathbf{v}\cdot Y_n(\mathbf{t})}\right] - \Big(\varphi_n(\mathbf{u}+i\mathbf{v}) - \varphi_n(\mathbf{u})\Big)\right|$$

converges to zero uniformly in $\mathbf{v}$ such that $\|\mathbf{v}\| \leq n^{-\frac{1}{2}+\zeta}$, and by Theorem A.0.1, we have that

$$\frac{1}{\sqrt{n}}\left(Y_n(\mathbf{t}) - n\nabla f_1(\mathbf{u})\right) \xrightarrow{\mathcal{D}} Z$$

as $n \to \infty$, under the measures $\mathbb{Q}_n^{\mathbf{u}}$, and

$$\frac{1}{\sqrt{n}}\left(Y_n(\mathbf{t}) - n\nabla f_1(\mathbf{u}_n)\right) \xrightarrow{\mathcal{D}} Z$$

as $n \to \infty$, under $\mathbb{Q}_n^{\mathbf{u}_n}$. □

3.3 Tilts

We are now in a position to discuss the optimal choice of tilt $\mathbf{u}$ for each n. While our Central Limit Theorem 3.2.1 holds for all $\mathbf{u} \in \mathcal{U}^{\Delta}$, we will obtain sharper asymptotics in the Local Limit Theorem by carefully choosing the tilt applied with reference to α_n and β_n.

As noted in Theorem A.0.1, under a fixed constant tilt $\mathbf{u}$, we have

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \bar{\mathbb{E}}_n^{\mathbf{u}}[Y_n(1)] = \nabla f_1(\mathbf{u}),$$

where

$$f_1(\mathbf{z}) = \int_0^1 m(z_1 + (1-x)z_0)dx.$$

In other words,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \bar{\mathbb{E}}_n^{\mathbf{u}}[G(1)] &= \frac{\partial}{\partial v_1} f_1(\mathbf{u} + i\mathbf{v})\Big|_{\mathbf{v}=\mathbf{0}} \\ &= \int_0^1 m'(u_1 + (1-x)u_0)dx \\ &= m(u_1) - m(u_1 + u_0),\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \bar{\mathbb{E}}_n^{\mathbf{u}}\left[\int_0^1 g(t)dt\right] &= \frac{\partial}{\partial z_0} f_1(\mathbf{u} + i\mathbf{v})\Big|_{\mathbf{v}=\mathbf{0}} \\ &= \int_0^1 (1-x)m'(u_1 + (1-x)u_0)dx.\end{aligned}$$

These two equations define a set of possible limiting values for $\bar{\mathbb{E}}_n^{\mathbf{u}}[Y_n(1)]$. Let

$$\mathcal{A} = \left\{ \left(m(u_1 + u_0) - m(u_1), \int_0^1 (1-x)m'(u_1 + (1-x)u_0)dx \right) : (u_1, u_0) \in \mathcal{U}^\Delta \right\};$$

then for every $(\alpha, \beta) \in \mathcal{A}$, there exists $\mathbf{u} \in \mathcal{U}^\Delta$ such that

$$\lim_{n \to \infty} \frac{1}{n} \bar{\mathbb{E}}_n^{\mathbf{u}}[Y_n(1)] = (\alpha, \beta).$$

We are interested in the behaviour of the trajectory under the large-deviations condition

$$Y_n(1) \approx (\alpha n, \beta n),$$

for pairs $(\alpha, \beta) \in \mathcal{A}$; however, as $Y_n(1)$ takes values in $\mathbb{Z} \times \frac{1}{2n}\mathbb{Z}$, we can only meaningfully discuss the behaviour of the trajectories under large deviations conditions of the form

$$Y_n(1) = (\alpha_n, \beta_n),$$

where $\alpha_n \in \mathbb{Z}$, $\beta_n \in \frac{1}{2n}\mathbb{Z}$, and $(\alpha_n, \beta_n) \to (\alpha, \beta)$.

There is some flexibility in the choice of sequence (α_n, β_n), but we require that

$$\begin{aligned} \frac{1}{\sqrt{n}} \left| \alpha n - \alpha_n \right| &\to 0 \\ \frac{1}{\sqrt{n}} \left| \beta n - \beta_n \right| &\to 0, \end{aligned} \tag{3.3.1}$$

while

$$\begin{aligned} \alpha_n &\in \mathbb{Z} \\ 2n\beta_n &\in \mathbb{Z} \\ \frac{1}{n}(\alpha_n, \beta_n) &\in \mathcal{A}. \end{aligned} \tag{3.3.2}$$

For $n \in \mathbb{N}$, we let $\mathbf{u}_n$ be the tilt such that

$$\bar{\mathbb{E}}_n^{\mathbf{u}_n}[Y_n(1)] = (\alpha_n, \beta_n),$$

and let $\mathbf{u}$ be the tilt such that

$$\lim_{n\to\infty} \frac{1}{n} \bar{\mathbb{E}}_n^{\mathbf{u}}[Y_n(1)] = (\alpha, \beta). \tag{3.3.3}$$

Then under the conditions (3.3.1),

$$\|\mathbf{u}_n - \mathbf{u}\| \to 0,$$

as $n \to \infty$ by the implicit function theorem.

We can also view our choice of $\mathbf{u}$ and $\mathbf{u}_n$ in terms of the convex conjugate of the log-moment generating functions.

For $\mathbf{u} \in \mathcal{U}^{\Delta}$, let

$$\begin{aligned} \mathcal{L}_{Y_n}(\mathbf{u}) &= \log \bar{\mathbb{E}}_n^{\mathbf{0}} \left[e^{\mathbf{u}\cdot Y_n(1)} \right] \\ &= \log \frac{\mathcal{B}_n(\mathbf{u})}{\mathcal{B}_n(\mathbf{0})}, \end{aligned} \tag{3.3.4}$$

and let

$$\mathcal{L}^{\star}_{Y_n}(\mathbf{a}) = \sup_{\mathbf{u}\in\mathcal{U}^{\Delta}} \mathbf{a}\cdot\mathbf{u} - \mathcal{L}_{Y_n}(\mathbf{u}).$$

Then our requirement

$$\bar{\mathbb{E}}_n^{\mathbf{u}_n}[Y_n(1)] = (\alpha_n, \beta_n)$$

is equivalent to

$$\nabla \mathcal{L}_{Y_n}(\mathbf{u}_n) = (\alpha_n, \beta_n),$$

and using the properties of the convex conjugate as seen in [29], we see that

$$\mathcal{L}^{\star}_{Z_n}(\mathbf{a}_n) = \mathbf{a}_n \cdot \mathbf{u}_n - \mathcal{L}_{Y_n}(\mathbf{u}_n) \tag{3.3.5}$$

when $\mathbf{a}_n = (\alpha_n, \beta_n)$.

Now, for any sequence (α_n, β_n) satisfying conditions (3.3.1) and (3.3.2), the corresponding sequence of tilts $\mathbf{u}_n$ is suitable for use in our Central Limit Theorem 3.2.1, for any value of k.

We can also use the Central Limit Theorem result in this chapter to find an explicit form for our limiting profile, $c_\infty(t)$. For $t \in [0,1]$, consider

$$Y_n(t) = \left(G(t), G(1) - G(t), \int_0^1 g(s)ds\right).$$

From Theorem 3.2.1, we know that

$$\frac{1}{\sqrt{n}}\left(Y_n(t) - n\nabla f_t(\mathbf{u}_n)\right) \tag{3.3.6}$$

converges in distribution to a Normally-distributed random variable. We write

$$c_n(t) = \frac{\partial}{\partial u_1} f_t(\mathbf{u}_n), \tag{3.3.7}$$

and

$$c_\infty(t) = \frac{\partial}{\partial u_1} f_t(\mathbf{u}).$$

To get an explicit formula for c_∞, we only need to find the partial derivative of f_t with respect to v_1, evaluated at $\mathbf{v} = 0$. We have

$$\begin{aligned} c_\infty(t) &= \frac{\partial}{\partial v_1} \int_0^t m(z_1 + z_0(1-x))dx + \int_t^1 m(z_2 + z_0(1-x))dx \bigg|_{\mathbf{v}=0} \\ &= \int_0^t m'(u_1 + u_0(1-x))dx \\ &= m(u_1 + u_0(1-t)) - m(u_1 + u_0). \end{aligned}$$

We note that c_∞, given in this form, corresponds with the Wulff construction as set out in [9]. We also have the following Law of Large Numbers, a consequence of Theorem 3.2.1.

Theorem 3.3.1. *The trajectories g obey a weak law of large numbers, in the sense that for every t and ε,*

$$\mathbb{Q}_n^{\mathbf{u}_n}\left(\left|\frac{1}{n}g(t) - c_\infty(t)\right| > \varepsilon\right) \to 0$$

as $n \to \infty$.

If α_n and β_n have been chosen such that $\sqrt{n}\|\mathbf{u}_n - \mathbf{u}\| \to 0$, then we can use c_∞ in place of c_n as the centering term in Equation (3.3.6). It may not in general be possible to choose such α_n and β_n; in the rest of this thesis, we use c_n as the centering term.

Chapter 4

Local Central Limit Theorems

In the previous chapters, our analysis of the trajectories has been in terms of the *unconditioned* distributions $\mathbb{P}_n$ and $\mathbb{Q}_n^{\mathbf{u}_n}$. In order to move from these results to a discussion of the trajectories under $\mathbb{P}_n^{\alpha_n,\beta_n}$, we need to take one more step. By finding Local Central Limit Theorems for $Y_n(1)$ and $Y_n(\mathbf{t})$ under the unconditional distributions, we can at last establish convergence results for $G_n(\mathbf{t})$ under the *conditional* distributions $\mathbb{P}_n^{\alpha_n,\beta_n}$. Recalling Equations (2.1.5) and (2.1.6), we have

$$\mathbb{P}_n\left(G_n(\mathbf{t}) = \mathbf{x} | Y_n(1) = (\alpha_n, \beta_n)\right) = \frac{\mathbb{Q}_n^{\mathbf{u}_n}\left(\{G_n(\mathbf{t}) = \mathbf{x}\} \cap \{Y_n(1) = (\alpha_n, \beta_n)\}\right)}{\mathbb{Q}_n^{\mathbf{u}_n}\left(Y_n(1) = (\alpha_n, \beta_n)\right)}. \tag{4.0.1}$$

In order to understand the asymptotics of the probability on the left, we will first study each of the numerator and denominator in the fraction on the right.

4.1 Local Central Limit Theorem in 2 Dimensions

In this section, we begin with the Local Central Limit Theorem for $Y_n(1)$. The proof of this result contains most of the analysis necessary for the corresponding theorem for $Y_n(\mathbf{t})$ in the next section.

Definition 4.1.1. Let

$$\Lambda_n = \left[-\pi\sqrt{n}, \pi\sqrt{n}\right] \times \left[-\pi\sqrt{n}^3, \pi\sqrt{n}^3\right].$$

For $\mathbf{v} \in \Lambda_n$, we write $\mathbf{v} = (v_1, v_0)$. Let

$$\tilde{Y}_n(1) = \frac{1}{\sqrt{n}}\left(g(1) - n\alpha, \int_0^1 g(t)dt - n\beta\right),$$

and let

$$\begin{aligned}\chi &= \left\{(x_1, x_0) : \mathbb{P}_n(\tilde{Y}_n(1) = (x_1, x_0)) > 0\right\} \\ &\subseteq \left\{(x_1, x_0) : \sqrt{n}x_1 + n\alpha \in \mathbb{Z}, \sqrt{n}x_0 + n\beta \in \frac{1}{2n}\mathbb{Z}\right\}.\end{aligned}$$

Note that, for any $\mathbf{u} \in \mathcal{U}^\Delta$,

$$\mathbb{Q}_n^{\mathbf{u}}(\tilde{Y}_n(1) = (x_1, x_0)) > 0 \iff \mathbb{P}_n(\tilde{Y}_n(1) = (x_1, x_0)) > 0.$$

Theorem 4.1.2. *Let* $\mathbf{u}$ *and* $\mathbf{u}_n$ *be a sequence of tilts satisfying* $\sqrt{n}\|\mathbf{u}_n - \mathbf{u}\| \to 0$ *as* $n \to \infty$. *Let* $\phi_2(x, y)$ *be the density of the two-dimensional Normal distribution with mean* $\mathbf{0}$, *and covariance matrix* $\Sigma_2 = \Sigma_2(\mathbf{u})$ *as in Theorem 3.2.1. Then*

$$\sup_{\mathbf{x}\in\chi} \left|n^2 \mathbb{Q}_n^{\mathbf{u}_n}\left(\tilde{Y}_n(1) = \mathbf{x}\right) - \phi_2(\mathbf{x})\right| \to 0,$$

as $n \to \infty$.

In order to compare the behaviour of $\mathbb{Q}_n^{\mathbf{u}_n}\left(\tilde{Y}_n(1) = \mathbf{x}\right)$ and $\phi(\mathbf{x})$, we rewrite them using the Inversion Formula. See, for example, Section 36 of [27] for the derivation, or Proposition 2.2.2 of [42] for the lattice case.

Proposition 4.1.3. *For any* $\mathbf{u} \in \mathcal{U}^\Delta$, *and any* $\mathbf{x} \in \chi$, *we have*

$$\mathbb{Q}_n^{\mathbf{u}}\left(\tilde{Y}_n(1) = \mathbf{x}\right) = \frac{1}{4\pi^2 n^2}\int_{\Lambda_n} \bar{\mathbb{E}}_n^{\mathbf{u}}\left[\exp\left\{i\mathbf{v}\cdot(\tilde{Y}_n(1) - \mathbf{x})\right\}\right] d\mathbf{v}.$$

If N *has a 2-dimensional Normal distribution with mean* $\mathbf{0}$ *and covariance matrix* Σ, *then its density* ϕ_2 *satisfies*

$$\phi_2(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{R}^2} \exp\left\{-i\mathbf{v}\cdot\mathbf{x} - \frac{1}{2}\mathbf{v}^\intercal\Sigma\mathbf{v}\right\} d\mathbf{v}.$$

Lemma 4.1.4. *Let $k \in \mathbb{N}$ be such that $F_k(0) > 0$. For $d \in \mathbb{N}$, let*

$$\mathcal{P}_{d,s} = \{\nu \in \mathcal{P}_n : \nu \text{ has two consecutive cutpoints at } sd \text{ and } sd+k\},$$

and

$$\mathcal{P}'_s = \begin{cases} \mathcal{P}_{d,s} & 1 \le s \le n_d \\ \mathcal{P}_{d+1,s} & n_d + 1 \le s \le 2n_d \end{cases},$$

where $n_d = \lfloor \frac{n}{2(d+1)} \rfloor - 2$. Let

$$\mathcal{A}'_s = \mathbb{1}\{\exists \nu \in \mathcal{P}'_s : \xi \sim \nu\}.$$

Then there exist $p \in (0,1)$, and integers d_0 and n_0 such that, if $d > d_0$ and $n > n_0$, for any finite sequences $1 \le s_1 < \cdots < s_j < t \le 2n_d$ and $\{e_1, \ldots, e_j\} \in \{0,1\}^j$ we have

$$\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1 \;\middle|\; \mathcal{A}'_{s_1} = e_1, \ldots, \mathcal{A}'_{s_j} = e_j\right) \ge p,$$

for all $\mathbf{u} \in \mathcal{U}^\Delta$.

Proof of Theorem 4.1.2. Let $\varepsilon > 0$. By the Inversion Formula, we have

$$\begin{aligned}(2\pi)^2 &\left(n^2 \mathbb{Q}_n^{\mathbf{u}_n}\left(\tilde{Y}_n(1) = \mathbf{x}\right) - \phi_2(\mathbf{x})\right) \\ &= \int_{\Lambda_n} \exp\{-i\mathbf{v}\cdot\mathbf{x}\}\, \bar{\mathbb{E}}_n^{\mathbf{u}_n}\left[e^{i\mathbf{v}\cdot\tilde{Y}_n(1)}\right] d\mathbf{v} - \int_{\mathbb{R}^2} \exp\left\{-i\mathbf{v}\cdot\mathbf{x} - \frac{1}{2}\mathbf{v}^\intercal \Sigma \mathbf{v}\right\} d\mathbf{v}.\end{aligned}$$

For $n \in \mathbb{N}$ and $A > 0$, let

$$\begin{aligned} R_1 &= [-A, A]^2 \\ R_2 &= \mathbb{R}^2 \setminus R_1 \\ R_3(n) &= \Lambda_n \setminus R_1, \end{aligned}$$

and let

$$I_1 = \int_{R_1} \exp\{-i\mathbf{v}\cdot\mathbf{x}\}\left(\bar{\mathbb{E}}_n^{\mathbf{u}_n}\left[e^{i\mathbf{v}\cdot\tilde{Y}_n(1)}\right] - \exp\left\{-\frac{1}{2}\mathbf{v}^\intercal\Sigma\mathbf{v}\right\}\right) d\mathbf{v}$$

$$I_2 = \int_{R_2} \exp\left\{-i\mathbf{v}\cdot\mathbf{x} - \frac{1}{2}\mathbf{v}^\intercal\Sigma\mathbf{v}\right\} d\mathbf{v}$$
$$I_3 = \int_{R_3(n)} \exp\left\{-i\mathbf{v}\cdot\mathbf{x}\right\} \bar{\mathbb{E}}_n^{\mathbf{u}_n}\left[e^{i\mathbf{v}\cdot\tilde{Y}_n(1)}\right] d\mathbf{v}$$

Then

$$(2\pi)^2\left(n^2\mathbb{Q}_n^{\mathbf{u}_n}\left(\tilde{Y}_n(1) = \mathbf{x}\right) - \phi_2(\mathbf{x})\right) = I_1 + I_2 + I_3;$$

we will show that we can choose A such that for all n large enough, we have $|I_j| \leq \frac{\varepsilon}{3}$ for each of $j = 1, 2, 3$.

First,

$$|I_1| \leq \int_{R_1}\left|\bar{\mathbb{E}}_n^{\mathbf{u}_n}\left[e^{i\mathbf{v}\cdot\tilde{Y}_n(1)}\right] - \exp\left\{-\frac{1}{2}\mathbf{v}^\intercal\Sigma\mathbf{v}\right\}\right| d\mathbf{v}.$$

By Theorem 3.2.1, for any $A > 0$ we can find n_1 such that for $n > n_1$, the difference of the characteristic functions is sufficiently small that

$$|I_1| < \frac{\varepsilon}{3}.$$

Next, there exists $A > 0$ such that we have

$$\begin{aligned}
|I_2| &= \left|\int_{R_2} \exp\left\{-i\mathbf{v}\cdot\mathbf{x} - \frac{1}{2}\mathbf{v}^\intercal\Sigma\mathbf{v}\right\} d\mathbf{v}\right| \\
&\leq \int_{R_2} \exp\left\{-\frac{1}{2}\mathbf{v}^\intercal\Sigma\mathbf{v}\right\} d\mathbf{v} \\
&\leq \int_{R_2} \exp\left\{-c\|\mathbf{v}\|^2\right\} d\mathbf{v} \\
&\leq \frac{\varepsilon}{3}.
\end{aligned}$$

For the remainder, we use a stochastic dominance argument.

Recalling Lemma 4.1.4, take $k \in \mathbb{N}$ such that $F_k(0) > 0$, and $d \in \mathbb{N}$ such that $d > 2d_0 + 2k$. We also suppose that $n > n_0$ as established in Lemma 4.1.4.

Since $F_k(z)$ is analytic, and 2π-periodic in the imaginary direction, there exists a constant $a > 0$ such that

$$|F_k(u + iv)| \leq \exp\left\{a\Big(\cos(v) - 1\Big)\right\} F_k(u) \tag{4.1.1}$$

holds for all $u \in \mathcal{U}$, and $|v| < \pi$.

As in Lemma 4.1.4, we write

$$\mathcal{P}_{d,s} = \{\nu \in \mathcal{P}_n : \nu \text{ has two consecutive cutpoints at } sd \text{ and } sd + k\},$$

and

$$\mathcal{P}'_s = \begin{cases} \mathcal{P}_{d,s} & 1 \le s \le n_d \\ \mathcal{P}_{d+1,s} & n_d + 1 \le s \le 2n_d \end{cases},$$

where $n_d = \lfloor \frac{n}{2(d+1)} \rfloor - 2$. Let

$$\mathcal{A}'_s = \mathbb{1}\{\exists \nu \in \mathcal{P}'_s : \xi \sim \nu\},$$

and for $\mathbf{z} = (z_1, z_0) \in \mathcal{Z}^\Delta$, let

$$z^\star_s = \begin{cases} z_1 + \frac{z_0}{n}\left(n - sd - \frac{k}{2}\right) & 1 \le s \le n_d \\ z_1 + \frac{z_0}{n}\left(n - s(d+1) - \frac{k}{2}\right) & n_d + 1 \le s \le 2n_d \end{cases},$$

and denote the real and imaginary parts of $z^\star_s$ by $u^\star_s$ and $v^\star_s$, respectively. Recall that for a partition $\nu \in \mathcal{P}_n$ and a sub-trajectory $\xi \in \mathcal{H}_n$, we write $\xi \sim \nu$ if the cutpoints of ξ correspond with the cutpoints of ν.

Then, combining Equation (4.1.1) with Proposition 2.2.1, for any $\nu \in \mathcal{P}_n$ and $1 \le j \le |\nu|$ we have

$$\left|F_{[\nu_{j-1},\nu_j]}(\mathbf{z})\right| \le F_{[\nu_{j-1},\nu_j]}(\mathbf{u}) \exp\left\{\sum_{s=1}^{2n_d} \mathbb{1}\{\nu_{j-1} = sd, \nu_j = sd + k\}\, a\Big(\cos(v^\star_s) - 1\Big)\right\},$$

so that for any $\mathbf{u} + i\mathbf{v} \in \mathcal{Z}^\Delta$,

$$\begin{aligned} |\mathcal{B}_n(\mathbf{u} + i\mathbf{v})| &= \left|\sum_{\nu \in \mathcal{P}_n} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}(\mathbf{z})\right| \\ &\le \sum_{\nu \in \mathcal{P}_n} \prod_{j=1}^{|\nu|} F_{[\nu_{j-1},\nu_j]}(\mathbf{u}) \exp\left\{\sum_{s=1}^{2n_d} \mathbb{1}\{\nu \in \mathcal{P}'_s\}\, a\Big(\cos(v^\star_s) - 1\Big)\right\}. \end{aligned}$$

As a result, we have

$$\begin{aligned}\left|\bar{\mathbb{E}}_n^{\mathbf{u}}[e^{i\mathbf{v}\cdot Y_n(1)}]\right| &= \left|\frac{\mathcal{B}_n(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_n(\mathbf{u})}\right| \\ &\leq \frac{1}{\mathcal{B}_n(\mathbf{u})}\sum_{\nu\in\mathcal{P}_n}\prod_{j=1}^{|\nu|}F_{[\nu_{j-1},\nu_j]}(\mathbf{u})\exp\left\{\sum_{s=1}^{2n_d}\mathbb{1}\{\nu\in\mathcal{P}'_s\}\,a\Big(\cos(v_s^\star)-1\Big)\right\} \\ &= \bar{\mathbb{E}}_n^{\mathbf{u}}\left[\exp\left\{\sum_{s=1}^{2n_d}\mathcal{A}'_s a\Big(\cos(v_s^\star)-1\Big)\right\}\right].\end{aligned}$$

Now, Lemma 4.1.4 allows us to use Lemma 1.1 of [43] to show that the law of the variables $\mathcal{A}'_s$ stochastically dominates the law of a sequence of independent Bernoulli random events X_s with the same index set $1 \leq s \leq 2n_d$ and some parameter $p > 0$, so that

$$\left|\frac{\mathcal{B}_n(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_n(\mathbf{u})}\right| \leq \mathbb{E}_X\left[\exp\left\{\sum_{s=1}^{2n_d}X_s a\Big(\cos(v_s^\star)-1\Big)\right\}\right],$$

where $\mathbb{E}_X$ is the expectation based on the joint law of the X_ss.

Let

$$Z_n(\mathbf{v}) = \frac{1}{2}\sum_{s=1}^{n_d}X_s\Big(1-\cos(v_s^\star)\Big),$$

and

$$Z'_n(\mathbf{v}) = \frac{1}{2}\sum_{s=n_d+1}^{2n_d}X_s\Big(1-\cos(v_s^\star)\Big);$$

then

$$\left|\frac{\mathcal{B}_n(\mathbf{u}+i\mathbf{v})}{\mathcal{B}_n(\mathbf{u})}\right| \leq \mathbb{E}_X\Big[\exp\{-2aZ_n(\mathbf{v})\}\Big]\mathbb{E}_X\Big[\exp\{-2aZ'_n(\mathbf{v})\}\Big].$$

We will show that, for all $\mathbf{v} \in R_3(n)$, this product of expectations is suitably bounded so that

$$\int_{R_3(n)}\left|\frac{\mathcal{B}_n\left(\mathbf{u}+i\frac{1}{\sqrt{n}}\mathbf{v}\right)}{\mathcal{B}_n(\mathbf{u})}\right|d\mathbf{v} \leq \frac{\varepsilon}{3}$$

holds for all n large enough.

We note that $Z_n(\mathbf{v})$ fulfils the conditions of Corollary 2.4.14 in [13], with

$$g_{n_d}(x_1, \ldots, x_{n_d}) = \frac{1}{2}\sum_{s=1}^{n_d} x_s\Big(\cos\Big(\frac{1}{\sqrt{n}}v_s^\star\Big) - 1\Big),$$

so that for $y > 0$ we have

$$\mathbb{P}_X(Z_n(\mathbf{v}) < \mathbb{E}_X[Z_n(\mathbf{v})] - y) \leq \exp\left\{-2\frac{y^2}{n_d}\right\}.$$

As a result, we can write

$$\mathbb{E}_X[\exp\{-2aZ_n(\mathbf{v})\}] \leq \exp\Big\{ - 2a(\mathbb{E}_X[Z_n(\mathbf{v})] - y)\Big\} + \exp\left\{-2\frac{y^2}{n_d}\right\}. \tag{4.1.2}$$

It is therefore sufficient to establish the asymptotic behaviour of $\mathbb{E}_X\left[Z_n\left(\frac{1}{\sqrt{n}}\mathbf{v}\right)\right]$ and $\mathbb{E}_X\left[Z_n'\left(\frac{1}{\sqrt{n}}\mathbf{v}\right)\right]$, for $\mathbf{v} \in R_3(n)$.

First, we will show that there exist $\delta > 0$ and a constant $c_1 \in (0,1)$ such that, for all $\mathbf{v} \in R_3(n)$ with $|v_0| > \delta\sqrt{n}$,

$$\max\left(\mathbb{E}_X\left[Z_n\left(\frac{1}{\sqrt{n}}\mathbf{v}\right)\right], \mathbb{E}_X\left[Z_n'\left(\frac{1}{\sqrt{n}}\mathbf{v}\right)\right]\right) \geq c_1 n_d. \tag{4.1.3}$$

Then, using $y = \frac{1}{2}c_1 n_d$ in Equation (4.1.2), we have

$$\begin{aligned}\left|\frac{\mathcal{B}_n\left(\mathbf{u} + i\frac{1}{\sqrt{n}}\mathbf{v}\right)}{\mathcal{B}_n\left(\mathbf{u}\right)}\right| &\leq \exp\{-ac_1 n_d\} + \exp\left\{-\frac{c_1^2}{2}n_d\right\} \\ &\leq \exp\left\{-c_1' n_d\right\}\end{aligned}$$

for some constant $c_1' > 0$, so that

$$\begin{aligned}\int_{\mathbf{v}\in R_3(n):|v_0|>\delta\sqrt{n}} \left|\frac{\mathcal{B}_n\left(\mathbf{u} + i\frac{1}{\sqrt{n}}\mathbf{v}\right)}{\mathcal{B}_n\left(\mathbf{u}\right)}\right| d\mathbf{v} &\leq |R_3(n)|e^{-c' n_d} \\ &\leq \frac{\varepsilon}{6}\end{aligned}$$

holds for all n large enough.

Next, we will show that if $\mathbf{v} \in R_3(n)$ and $|v_0| < \delta\sqrt{n}$, we have

$$\max\left(\mathbb{E}_X\left[Z_n\left(\frac{1}{\sqrt{n}}\mathbf{v}\right)\right], \mathbb{E}_X\left[Z_n'\left(\frac{1}{\sqrt{n}}\mathbf{v}\right)\right]\right) \geq c_2\|\mathbf{v}\|^2, \tag{4.1.4}$$

for some constant $c_2 > 0$; as a result, there exists a constant $c_2' > 0$ such that Equation (4.1.2) becomes

$$\begin{aligned}\left|\frac{\mathcal{B}_n\left(\mathbf{u}+i\frac{1}{\sqrt{n}}\mathbf{v}\right)}{\mathcal{B}_n\left(\mathbf{u}\right)}\right| &\leq \exp\left\{-ac_2\|\mathbf{v}\|^2\right\} + \exp\left\{-\frac{c_2\|\mathbf{v}\|^4}{2n_d}\right\} \\ &\leq \exp\left\{-c_2'\|\mathbf{v}\|^2\right\}\end{aligned}$$

for all $\mathbf{v} \in R_3(n)$ with $|v_0| \leq \delta\sqrt{n}$.

As a result

$$\begin{aligned}\int_{\mathbf{v}\in R_3(n):|v_0|<\delta\sqrt{n}} \left|\frac{\mathcal{B}_n\left(\mathbf{u}+i\frac{1}{\sqrt{n}}\mathbf{v}\right)}{\mathcal{B}_n\left(\mathbf{u}\right)}\right| d\mathbf{v} &\leq \int_{R_2} e^{-c_2'\|\mathbf{v}\|^2} d\mathbf{v} \\ &\leq \frac{\varepsilon}{6}\end{aligned}$$

holds for all n large enough.

We only have to verify the lower bounds in Equations (4.1.3) and (4.1.4). We first note that, for all $\mathbf{v} \in R_3(n)$, we have

$$\mathbb{E}_X\left[Z_n\left(\frac{1}{\sqrt{n}}\mathbf{v}\right)\right] = \frac{p}{2}\sum_{s=1}^{n_d}\left(1-\cos\left(\frac{1}{\sqrt{n}}v_s^\star\right)\right),$$

and we can use the identities in [40] to write

$$\begin{aligned}\left|\sum_{s=1}^{n_d}\cos\left(\frac{1}{\sqrt{n}}v_s^\star\right)\right| &= \left|\frac{\sin(\frac{1}{2}dn_d x)}{\sin(\frac{1}{2}dx)}\cos\left(a+d\frac{n_d-1}{2}x\right)\right| \\ &\leq \left|\frac{\sin(\frac{1}{2}dn_d x)}{\sin(\frac{1}{2}dx)}\right| \qquad (4.1.5)\end{aligned}$$

where $a = \frac{1}{\sqrt{n}}\left(v_1 + v_0(1-\frac{2d+k}{2n})\right)$, and $x = -\frac{v_0}{\sqrt{n}^3}$. Similarly,

$$\mathbb{E}_X\left[Z_n'\left(\frac{1}{\sqrt{n}}\mathbf{v}\right)\right] = \frac{p}{2}\sum_{s=n_d+1}^{2n_d}\left(1-\cos\left(\frac{1}{\sqrt{n}}v_s^\star\right)\right),$$

and we have

$$\begin{aligned}\left|\sum_{s=n_d+1}^{2n_d}\cos\left(\frac{1}{\sqrt{n}}v_s^\star\right)\right| &= \left|\frac{\sin(\frac{1}{2}(d+1)n_d x)}{\sin(\frac{1}{2}(d+1)x)}\cos\left(a'+(d+1)\frac{n_d-1}{2}x\right)\right| \\ &\leq \left|\frac{\sin(\frac{1}{2}(d+1)n_d x)}{\sin(\frac{1}{2}(d+1)x)}\right|, \qquad (4.1.6)\end{aligned}$$

where $a' = \frac{1}{\sqrt{n}}\left(v_1 + v_0(1 - \frac{2(d+1)+k}{2n})\right)$.

As we see in Proposition B.2.7, the upper bound in Equation (4.1.5) is $\frac{2\pi}{d}$-periodic in x. Moreover, whenever x is not within $\frac{2\pi}{dn_d}$ of any integer multiple of $\frac{2\pi}{d}$, we have

$$\left|\frac{\sin(\frac{1}{2}dn_d x)}{\sin(\frac{1}{2}dx)}\right| \leq \frac{1}{3}n_d.$$

Similarly, the upper bound in Equation (4.1.6) is $\frac{2\pi}{d+1}$-periodic in x, and is bounded above by the same constant $\frac{1}{3}n_d$ whenever x is not within $\frac{2\pi}{(d+1)n_d}$ of any multiple of $\frac{2\pi}{d+1}$. Since d and $d+1$ are coprime, and the width of these neighbourhoods is proportional to n_d^{-1}, for all n large enough they do not overlap except at integer multiples of 2π.

Since $x = -\frac{v_0}{\sqrt{n}^3}$ takes values between $-\pi$ and π, the only such integer multiple available is at zero. As a result, there exists a constant δ which only depends on d such that $|v_0| > \delta\sqrt{n}$ implies $|x| > \frac{2\pi}{dn_d}$, and hence

$$\min\left(\left|\sum_{s=1}^{n_d}\cos\left(\frac{1}{\sqrt{n}}v_s^\star\right)\right|, \left|\sum_{s=n_d+1}^{2n_d}\cos\left(\frac{1}{\sqrt{n}}v_s^\star\right)\right|\right) \leq \frac{1}{3}n_d,$$

and

$$\mathbb{E}_X\left[Z_n\left(\frac{1}{\sqrt{n}}\mathbf{v}\right)\right] \geq \frac{p}{3}n_d$$

Finally, if $|v_0| < \delta\sqrt{n}$, we can use the quadratic part of the upper bound in Proposition B.2.7 to write

$$\left|\frac{\sin(\frac{1}{2}dn_d x)}{\sin(\frac{1}{2}dx)}\right| \leq n_d\left(1 - \left(\frac{xdn_d}{2\pi}\right)^2\right),$$

while

$$\cos\left(a + d\frac{n_d - 1}{2}x\right) \leq 1 - \frac{2}{\pi^2}\left(a + d\frac{n_d - 1}{2}x\right)^2$$

holds for all a and x under consideration; as a result,

$$\sum_{s=1}^{n_d}\cos\left(\frac{1}{\sqrt{n}}v_s^\star\right) \leq n_d\left(1 - \left(\frac{xdn_d}{2\pi}\right)^2\right)\left(1 - \frac{2}{\pi^2}\left(a + d\frac{n_d - 1}{2}x\right)^2\right).$$

.

Evaluating this upper bound when $a = \frac{1}{\sqrt{n}}\left(v_1 + v_0(1 - \frac{2d+k}{2n})\right)$ and $x = -\frac{v_0}{\sqrt{n}^3}$ allows us to find a constant c such that

$$\sum_{s=1}^{n_d} \cos\left(\frac{1}{\sqrt{n}} v_s^\star\right) \leq n_d \left(1 - c \left\| \frac{\mathbf{v}}{\sqrt{n}} \right\|^2\right),$$

so that

$$\mathbb{E}_X \left[Z_n \left(\frac{1}{\sqrt{n}} \mathbf{v} \right) \right] \geq c_2 \|\mathbf{v}\|^2$$

holds for $\mathbf{v} \in R_3(n)$ with $|v_0| < \delta\sqrt{n}$.

□

Proof of Lemma 4.1.4. Recall

$$\varphi_{[a,b]}\left(\mathbf{z}\right) = \int_a^b m\left(\mathbf{z}_{n\mathbf{t}}\left(\frac{x}{n}\right)\right) dx + \frac{1}{2}\log\mu\left(\mathbf{z}_{n\mathbf{t}}\left(\frac{a}{n}\right)\right)\mu\left(\mathbf{z}_{n\mathbf{t}}\left(\frac{b}{n}\right)\right);$$

by Remarks 2.4.11 and 3.2.5, for any $\delta > 0$ there exist d_0 and n_0 such that, for any $n > n_0$ and any integer segment $[a, b] \subset [0, n]$ with $b - a \geq d_0$, we have

$$(1-\delta)\exp\{\varphi_{[a,b]}\left(\mathbf{u}\right)\} \leq \mathcal{B}_{[a,b]}\left(\mathbf{u}\right) \leq (1+\delta)\exp\{\varphi_{[a,b]}\left(\mathbf{u}\right)\}. \tag{4.1.7}$$

We fix $\delta > 0$, and choose $d > 2(d_0 + k)$.

We also note that, for any $0 \leq a < b < c \leq n$, we have

$$\begin{aligned}
&\exp\left\{\varphi_{[a,b]}\left(\mathbf{u}\right) + \varphi_{[b+k,c]}\left(\mathbf{u}\right) - \varphi_{[a,c]}\left(\mathbf{u}\right)\right\} F_{[b,b+k]}\left(\mathbf{u}\right) \\
&= \sqrt{\mu\left(u\left(\frac{b}{n}\right)\right)\mu\left(u\left(\frac{b+k}{n}\right)\right)} \exp\left\{\int_b^{b+k} m\left(u\left(\frac{x}{n}\right)\right) dx\right\} F_{[b,b+k]}\left(\mathbf{u}\right),
\end{aligned}$$

so that we can find a positive constant ρ such that

$$\frac{\exp\left\{\varphi_{[a,b]}\left(\mathbf{u}\right) + \varphi_{[b+k,c]}\left(\mathbf{u}\right)\right\}}{\exp\left\{\varphi_{[a,c]}\left(\mathbf{u}\right)\right\}} F_{[b,b+k]}\left(\mathbf{u}\right) \geq \rho \tag{4.1.8}$$

holds uniformly in $\mathbf{u} \in \mathcal{U}^\Delta$, and in a, b, and c.

We first consider the case $e_j = 1$. If $j = 1$, then

$$\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}_t' = 1 \middle| \mathcal{A}_s' = 1\right) = \frac{\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}_t' = 1 \cap \mathcal{A}_s' = 1\right)}{\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}_s' = 1\right)}$$

$$= \frac{\mathcal{B}_{[0,sd]}(\mathbf{u})\, F_{[sd,sd+k]}(\mathbf{u})\, \mathcal{B}_{[sd+k,td]}(\mathbf{u})\, F_{[td,td+k]}(\mathbf{u})\, \mathcal{B}_{[td+k,n]}(\mathbf{u})}{\mathcal{B}_{[0,sd]}(\mathbf{u})\, F_{[sd,sd+k]}(\mathbf{u})\, \mathcal{B}_{[sd+k,n]}(\mathbf{u})}$$

$$= \frac{\mathcal{B}_{[sd+k,td]}(\mathbf{u})\, F_{[td,td+k]}(\mathbf{u})\, \mathcal{B}_{[td+k,n]}(\mathbf{u})}{\mathcal{B}_{[sd+k,n]}(\mathbf{u})}. \tag{4.1.9}$$

Note that, since $t \le n_d = \lfloor \frac{n}{d} \rfloor - 2$, we have $n - (td + k) > d > d_0$, so that we can use the asymptotics in Equations (4.1.7) and (4.1.8) to write

$$\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1 \middle| \mathcal{A}'_s = 1\right) \ge \frac{(1-\delta)^2 \exp\left\{\varphi_{[sd+k,td]}(\mathbf{u}) + \varphi_{[td+k,n]}(\mathbf{u})\right\}}{(1+\delta)\exp\left\{\varphi_{[sd+k,n]}(\mathbf{u})\right\}} F_{[td,td+k]}(\mathbf{u})$$

$$\ge \frac{(1-\delta)^2}{1+\delta}\rho.$$

Now, if $j > 1$, we can write $\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1 \;\middle|\; \mathcal{A}'_{s_1} = e_1, \ldots, \mathcal{A}'_{s_j} = 1\right)$ in the form

$$\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1 \;\middle|\; \mathcal{A}'_{s_1} = e_1, \ldots, \mathcal{A}'_{s_j} = 1\right) = \frac{\hat{\mathcal{B}}_{[0,sd+k]}(\mathbf{u})\mathcal{B}_{[sd+k,td]}(\mathbf{u})\, F_{[td,td+k]}(\mathbf{u})\, \mathcal{B}_{[td+k,n]}(\mathbf{u})}{\hat{\mathcal{B}}_{[0,sd+k]}(\mathbf{u})\mathcal{B}_{[sd+k,n]}(\mathbf{u})},$$

where $\hat{\mathcal{B}}_{[0,sd+k]}(\mathbf{u})$ is the contribution to $\mathcal{B}_{[0,sd+k]}(\mathbf{u})$ from partitions $\nu \in \mathcal{P}_n$ corresponding to the conditions $\mathcal{A}'_{s_1} = e_1, \ldots, \mathcal{A}'_{s_{j-1}} = e_{j-1}$; that is,

$$\hat{\mathcal{B}}_{[0,sd+k]}(\mathbf{u}) = \sum_{\nu \in \mathcal{P}(e_1,\ldots,e_j)} \prod_{\ell=1}^{|\nu|} F_{[\nu_{\ell-1},\nu_\ell]}(\mathbf{u}),$$

with

$$\mathcal{P}(e_1, \ldots, e_j) = \left\{\nu \in \mathcal{P}_{[0,s_j d+k]} : \nu \in \mathcal{P}'_{s_\ell} \text{ iff } e_\ell = 1\right\}.$$

We see that

$$\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1 \;\middle|\; \mathcal{A}'_{s_1} = e_1, \ldots, \mathcal{A}'_{s_j} = 1\right) = \mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1 \;\middle|\; \mathcal{A}'_{s_j} = 1\right)$$

$$= \frac{\mathcal{B}_{[sd+k,td]}(\mathbf{u})\, F_{[td,td+k]}(\mathbf{u})\, \mathcal{B}_{[td+k,n]}(\mathbf{u})}{\mathcal{B}_{[sd+k,n]}(\mathbf{u})}$$

$$\ge \frac{(1-\delta)^2}{1+\delta}\rho.$$

Now when $e_j = 0$, we again begin with the case $j = 1$, and note that

$$\mathbb{Q}_n^{\mathbf{u}}(\mathcal{A}'_s = 0) = 1 - \mathbb{Q}_n^{\mathbf{u}}(\mathcal{A}'_s = 1)$$

$$= 1 - \frac{\mathcal{B}_{[0,sd]}(\mathbf{u})\, F_{[sd,sd+k]}(\mathbf{u})\, \mathcal{B}_{[sd+k,n]}(\mathbf{u})}{\mathcal{B}_{[0,n]}(\mathbf{u})}$$

$$\geq 1 - \frac{(1+\delta)^2}{1-\delta}\rho. \tag{4.1.10}$$

Now, let

$$\Lambda'_{sd+k} = \left\{(\ell, r) : 1 \leq \ell \leq sd + k, 1 \leq r \leq \frac{d}{2} \text{ or } \ell = r = 0\right\}.$$

Recalling Definition 2.4.1, we write $\Delta_S = (\ell, r)$ if the nearest cutpoints of the trajectory to the left and right of S are at $S - \ell$ and $S + r$ respectively. We have

$$\begin{aligned}
&\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1 \middle| \mathcal{A}'_s = 0\right) \\
&\quad \geq \sum_{(\ell,r)\in\Lambda'_{sd+k}} \mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1, \Delta_{sd+k} = (\ell, r) \middle| \mathcal{A}'_s = 0\right) \\
&\quad \geq \sum_{(\ell,r)\in\Lambda'_{sd+k}} \mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1 \middle| \Delta_{sd+k} = (\ell, r), \mathcal{A}'_s = 0\right) \mathbb{Q}_n^{\mathbf{u}}\left(\Delta_{sd+k} = (\ell, r) \middle| \mathcal{A}'_s = 0\right).
\end{aligned}$$

For each pair $(\ell, r) \in \Lambda'_{sd+k}$, we have $td - sd - k - r > d_0$, so that if $\ell, r > 0$ we have

$$\begin{aligned}
&\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1 \middle| \Delta_{sd+k} = (\ell, r), \mathcal{A}'_s = 0\right) \\
&\quad = \frac{\mathcal{B}_{[0,sd+k-\ell]}(\mathbf{u})\, F_{[sd+k-\ell,sd+k+r]}(\mathbf{u})\, \mathcal{B}_{[sd+k+r,td]}(\mathbf{u})\, F_{[td,td+k]}(\mathbf{u})\, \mathcal{B}_{[td+k,n]}(\mathbf{u})}{\mathcal{B}_{[0,sd+k-\ell]}(\mathbf{u})\, F_{[sd+k-\ell,sd+k+r]}(\mathbf{u})\, \mathcal{B}_{[sd+k+r,n]}(\mathbf{u})} \\
&\quad = \frac{\mathcal{B}_{[sd+k+r,td]}(\mathbf{u})\, F_{[td,td+k]}(\mathbf{u})\, \mathcal{B}_{[td+k,n]}(\mathbf{u})}{\mathcal{B}_{[sd+k+r,n]}(\mathbf{u})} \\
&\quad \geq \frac{(1-\delta)^2}{1+\delta}\rho,
\end{aligned}$$

while if $\ell = r = 0$,

$$\begin{aligned}
&\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1 \middle| \Delta_{sd+k} = (\ell, r), \mathcal{A}'_s = 0\right) \\
&\quad = \frac{\mathcal{B}_{[0,sd+k]}(\mathbf{u})\, \mathcal{B}_{[sd+k,td]}(\mathbf{u})\, F_{[td,td+k]}(\mathbf{u})\, \mathcal{B}_{[td+k,n]}(\mathbf{u})}{\mathcal{B}_{[0,sd+k]}(\mathbf{u})\, \mathcal{B}_{[sd+k,n]}(\mathbf{u})}
\end{aligned}$$

and the same upper bound holds.

Meanwhile,

$$\sum_{(\ell,r)\in\Lambda'_{sd+k}} \mathbb{Q}_n^{\mathbf{u}}\left(\Delta_{sd+k} = (\ell, r) \middle| \mathcal{A}'_s = 0\right) = 1 - \sum_{(\ell,r)\notin\Lambda'_{sd+k}} \frac{\mathbb{Q}_n^{\mathbf{u}}\left(\Delta_{sd+k} = (\ell, r), \mathcal{A}'_s = 0\right)}{\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_s = 0\right)}$$

$$\geq 1 - \sum_{(\ell,r)\notin\Lambda'_{sd+k}} \frac{\mathbb{Q}_n^{\mathbf{u}}\left(\Delta_{sd+k} = (\ell,r)\right)}{\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_s = 0\right)}$$
$$\geq 1 - \frac{\mathbb{Q}_n^{\mathbf{u}}\left(w_{sd+k} > \frac{d}{2}\right)}{\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_s = 0\right)}.$$

Using Corollary 2.4.6 and the lower bound in Equation (4.1.10), there exist constants C and c such that

$$\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1 \middle| \mathcal{A}'_s = 0\right) \geq \frac{(1-\delta)^2}{1+\delta}\rho\left(1 - \frac{Ce^{-c\frac{d}{2}}}{1 - \frac{(1+\delta)^2}{1-\delta}\rho}\right);$$

we can choose d large enough that, for example,

$$\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1 \middle| \mathcal{A}'_s = 0\right) \geq \frac{(1-\delta)^3}{1+\delta}\rho.$$

Finally, for $j > 1$ and $e_j = 0$, we use a similar approach as in the $e_j = 1$ case, to write

$$\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1 \,\middle|\Delta_{s_jd+k} = (\ell,r),\ \mathcal{A}'_{s_1} = e_1, \ldots, \mathcal{A}'_{s_j} = 0\right)$$
$$= \frac{\hat{\mathcal{B}}_{[0,s_jd+k-\ell]}(\mathbf{u}) F_{[s_jd+k-\ell,s_jd+k+r]}(\mathbf{u})\, \mathcal{B}_{[s_jd+k+r,td]}(\mathbf{u})\, F_{[td,td+k]}(\mathbf{u})\, \mathcal{B}_{[td+k,n]}(\mathbf{u})}{\hat{\mathcal{B}}_{[0,s_jd+k-\ell]}(\mathbf{u}) F_{[s_jd+k-\ell,s_jd+k+r]}(\mathbf{u})\, \mathcal{B}_{[s_jd+k+r,n]}(\mathbf{u})},$$

where as before $\hat{\mathcal{B}}_{[0,s_jd+k-\ell]}(\mathbf{u})$ represents the contribution to $\mathcal{B}_{[0,s_jd+k-\ell]}(\mathbf{u})$ from partitions corresponding to the conditions $\mathcal{A}'_{s_1} = e_1, \ldots, \mathcal{A}'_{s_{j-1}} = e_{j-1}$

Now, for $j > 1$, if the last j_0 values among $e_1, \ldots, e_j$ are all zero - in other words, $e_{j-j_0} = 1$, $e_{j-j_0+1} = \cdots = e_j = 0$, then

$$\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1 \,\middle|\, \mathcal{A}'_{s_1} = e_1, \ldots, \mathcal{A}'_{s_j} = 0\right)$$
$$\geq \sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda'_s} \mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1, \Delta_s = (\boldsymbol{\ell},\mathbf{r}) \,\middle|\, \mathcal{A}'_{s_1} = e_1, \ldots, \mathcal{A}'_{s_j} = 0\right)$$
$$= \sum_{(\boldsymbol{\ell},\mathbf{r})\in\Lambda' s} \mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1 \,\middle|\, \Delta_s = (\boldsymbol{\ell},\mathbf{r}), \mathcal{A}'_{s_1} = e_1, \ldots, \mathcal{A}'_{s_j} = 0\right)$$
$$\times \mathbb{Q}_n^{\mathbf{u}}\left(\Delta_s = (\boldsymbol{\ell},\mathbf{r}) \middle| \mathcal{A}'_{s_1} = e_1, \ldots, \mathcal{A}'_{s_j} = 0\right),$$

where

$$\Lambda'_s = \left\{(\boldsymbol{\ell},\mathbf{r}) : 1 \leq \ell_i \leq s_i d + k, 1 \leq r_i \leq \frac{d}{2}, 1 \leq i \leq j_0\right\}.$$

As in the $j = 1$ case, we can use the asymptotics in Equation (4.1.7) to write

$$\mathbb{Q}_n^{\mathbf{u}}\left(\mathcal{A}'_t = 1 \;\middle|\; \Delta_s = (\boldsymbol{\ell}, \mathbf{r}), \mathcal{A}'_{s_1} = e_1, \ldots, \mathcal{A}'_{s_j} = 0\right) \geq \frac{(1-\varepsilon)^2}{1+\varepsilon} c$$

whenever $(\boldsymbol{\ell}, \mathbf{r}) \in \Lambda'_s$, while

$$\begin{aligned}\sum_{(\boldsymbol{\ell}, \mathbf{r}) \in \Lambda' s} \mathbb{Q}_n^{\mathbf{u}}\left(\Delta_s = (\boldsymbol{\ell}, \mathbf{r}) \middle| \mathcal{A}'_{s_1} = e_1, \ldots, \mathcal{A}'_{s_j} = 0\right) &\geq \mathbb{Q}_n^{\mathbf{u}}\left(\max_{i \leq j_0} w_{s_{j-i}} \leq \frac{d}{2}\right) \\ &\geq (1 - Ce^{-cj_0 W}),\end{aligned}$$

by Corollary 2.4.18.

□

4.2 Local Central Limit Theorem in $k+2$ Dimensions

In this section, we move on to the Local Central Limit Theorem for $Y_n(\mathbf{t})$. We suppose that $k \in \mathbb{N}$, and $\mathbf{t} = (t_1, \ldots, t_k)$, are fixed throughout the section.

Definition 4.2.1. Let

$$\Lambda_n^k = \left[-\pi\sqrt{n}, \pi\sqrt{n}\right]^{k+1} \times \left[-\pi\sqrt{n}^3, \pi\sqrt{n}^3\right].$$

For $t \in [0,1]$, let

$$\tilde{G}(t) = \frac{G(t) - nc_n(t)}{\sqrt{n}},$$

and for $s < t \in [0,1]$ let

$$\tilde{G}[s,t] = \tilde{G}(t) - \tilde{G}(s)$$

$$c_n[s,t] = c_n(t) - c_n(s).$$

Let

$$\tilde{G}_n(\mathbf{t}) = \left(\tilde{G}[0,t_1], \tilde{G}[t_1,t_2], \ldots, \tilde{G}[t_{k-1},t_k]\right),$$

$$\tilde{Y}_n(\mathbf{t}) = \left(\tilde{G}[0,t_1], \tilde{G}[t_1,t_2], \ldots, \tilde{G}[t_{k-1},t_k], \tilde{G}[t_k,1], \frac{1}{\sqrt{n}}\left(\int_0^1 g(t)dt - nb\right)\right),$$

and let

$$\begin{aligned}\chi_k &= \left\{\mathbf{x} \in \mathbb{R}^{k+2} : \mathbb{P}_n(\tilde{Y}_n(\mathbf{t}) = \mathbf{x}) > 0\right\} \\ &\subseteq \left\{(x_1, \ldots, x_{k+1}, x_0) : \sqrt{n}x_j + nc_n[t_{j-1}, t_j] \in \mathbb{Z}, 1 \le j \le k+1, \sqrt{n}x_0 + nb \in \frac{1}{2n}\mathbb{Z}\right\}.\end{aligned}$$

Theorem 4.2.2. *Let* $\mathbf{u}$ *and* $\mathbf{u}_n$ *be a sequence of tilts such that* $\sqrt{n}\|\mathbf{u}_n - \mathbf{u}\| \to 0$ *as* $n \to \infty$. *Let* $\phi_{k+2}(\mathbf{x})$ *be the density of the* $(k+2)$*-dimensional Normal distribution with mean* $\mathbf{0}$, *and covariance matrix* $\Sigma_{k+2} = \Sigma_{k+2}(\mathbf{u})$ *as in Theorem 3.2.1. Then*

$$\sup_{\mathbf{x}\in\chi_k} \left| n^{\frac{k}{2}+2}\mathbb{Q}_n^{\mathbf{u}_n}\left(\tilde{Y}_n(\mathbf{t}) = \mathbf{x}\right) - \phi_{k+2}(\mathbf{x})\right| \to 0,$$

as $n \to \infty$.

We once again use the Inversion Formula.

Proposition 4.2.3. *For any* $\mathbf{u} \in \mathcal{U}^{\Delta}$, *and any* $\mathbf{x} \in \chi_k$, *we have*

$$\mathbb{Q}_n^{\mathbf{u}}\left(\tilde{Y}_n(\mathbf{t}) = \mathbf{x}\right) = \left(\frac{1}{2\pi\sqrt{n}}\right)^{k+2} \frac{1}{n}\int_{\Lambda_n^k} \bar{\mathbb{E}}_n^{\mathbf{u}}\left[\exp\left\{i\mathbf{v}\cdot(\tilde{Y}_n(\mathbf{t}) - \mathbf{x})\right\}\right] d\mathbf{v}.$$

If N *has a* $(k+2)$*-dimensional Normal distribution with mean* $\mathbf{0}$ *and covariance matrix* Σ, *then its density* $\phi_{k+2}(\mathbf{x})$ *satisfies*

$$\phi_{k+2}(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^{k+2} \int_{\mathbb{R}^{k+2}} \exp\left\{-i\mathbf{v}\cdot\mathbf{x} - \frac{1}{2}\mathbf{v}^{\intercal}\Sigma\mathbf{v}\right\} d\mathbf{v}.$$

Proof of Theorem 4.2.2. As in the 2-dimensional case, we note that

$$\begin{aligned}&(2\pi)^{k+2}\left(n^{\frac{k}{2}+2}\mathbb{Q}_n^{\mathbf{u}_n}\left(\tilde{Y}_n(\mathbf{t}) = \mathbf{x}\right) - \phi_{k+2}(\mathbf{x})\right) \\ &\qquad = \int_{\Lambda_n^k} \exp\{-i\mathbf{v}\cdot\mathbf{x}\}\, \bar{\mathbb{E}}_n^{\mathbf{u}_n}\left[e^{i\mathbf{v}\cdot\tilde{Y}_n(\mathbf{t})}\right] d\mathbf{v} - \int_{\mathbb{R}^{k+2}} \exp\left\{-i\mathbf{v}\cdot\mathbf{x} - \frac{1}{2}\mathbf{v}^{\intercal}\Sigma\mathbf{v}\right\} d\mathbf{v}.\end{aligned}$$

Let

$$R_1 = [-A, A]^{k+2}$$

$$R_2 = \mathbb{R}^{k+2} \setminus R_1$$

$$R_3(n) = \Lambda_n \setminus R_1,$$

and let

$$I_1 = \int_{R_1} \exp\{-i\mathbf{v}\cdot\mathbf{x}\}\left(\bar{\mathbb{E}}_n^{\mathbf{u}_n}\left[e^{i\mathbf{v}\cdot\tilde{Y}_n(\mathbf{t})}\right] - \exp\left\{-\frac{1}{2}\mathbf{v}^\intercal\Sigma\mathbf{v}\right\}\right)d\mathbf{v}$$
$$I_2 = \int_{R_2} \exp\left\{-i\mathbf{v}\cdot\mathbf{x} - \frac{1}{2}\mathbf{v}^\intercal\Sigma\mathbf{v}\right\}d\mathbf{v}$$
$$I_3 = \int_{R_3(n)} \exp\{-i\mathbf{v}\cdot\mathbf{x}\}\,\bar{\mathbb{E}}_n^{\mathbf{u}_n}\left[e^{i\mathbf{v}\cdot\tilde{Y}_n(\mathbf{t})}\right]d\mathbf{v}.$$

As before, we have

$$(2\pi)^{k+2}\left(n^{k+2}\mathbb{Q}_n^{\mathbf{u}_n}\left(\tilde{Y}_n(\mathbf{t}) = \mathbf{x}\right) - \phi_{k+2}(\mathbf{x})\right) = I_1 - I_2 + I_3;$$

we will show that we can choose A such that, for all n large enough, we have $|I_j| \le \frac{\varepsilon}{3}$ for each of $j = 1, 2, 3$.

The upper bounds for I_1 and I_2 follow the same arguments as in the 2-dimensional case, while for I_3 we have

$$\left|\bar{\mathbb{E}}_n^{\mathbf{u}_n}\left[e^{i\mathbf{v}\cdot\tilde{Y}_n(\mathbf{t})}\right]\right| \le \left|\bar{\mathbb{E}}_{nt_1}^{\mathbf{u}_n}\left[e^{i\mathbf{v}\cdot\tilde{Y}_{nt_1}(1)}\right]\right|.$$

The exponential decay of $\left|\bar{\mathbb{E}}_n^{\mathbf{u}_n}\left[e^{i\mathbf{v}\cdot\tilde{Y}_n(1)}\right]\right|$ established in the previous section for $\mathbf{v} \in R_3(n)$ means that there exist a finite box size A and a constant $C > 0$ such that

$$\begin{aligned}\int_{\mathbf{v}\in R_3(n)}\left|\bar{\mathbb{E}}_n^{\mathbf{u}_n}\left[\exp\left\{i\mathbf{v}\cdot\tilde{Y}_n(\mathbf{t})\right\}\right]\right|d\mathbf{v} &\le \int_{\mathbf{v}\in R_3(n)}\left|\bar{\mathbb{E}}_{nt_1}^{\mathbf{u}_n}\left[\exp\left\{i\mathbf{v}\cdot\tilde{Y}_{nt_1}(1)\right\}\right]\right|d\mathbf{v}\\ &\le \int_{R_3(n)} e^{-Cn_d}d\mathbf{v}\\ &\le |R_3(n)|e^{-Cn_d} \le \frac{\varepsilon}{3}\end{aligned}$$

holds for all n large enough. □

4.3 Local Central Limit Theorem Under the Conditional Distribution

We are now able to establish a Local Central Limit Theorem for the finite-dimensional distributions $G(\mathbf{t})$, under the conditional distribution $\mathbb{P}_n^{\alpha_n,\beta_n}$.

We begin with two intermediate results, which are specific cases of Theorems 4.1.2 and 4.2.2.

Corollary 4.3.1. *As $n \to \infty$, we have*

$$\left| n^2 \mathbb{Q}_n^{\mathbf{u}_n} \left(Y_n(1) = (\alpha_n, \beta_n) \right) - \phi_2(0,0) \right| \to 0,$$

where $\phi_2(x,y)$ is the density of the two-dimensional Normal distribution with mean $\mathbf{0}$, and covariance matrix $\Sigma_2 = \Sigma_2(\mathbf{u})$ as in Theorem 3.2.1.

Proof. We have

$$\mathbb{Q}_n^{\mathbf{u}_n} \left(Y_n(1) = (\alpha_n, \beta_n) \right) = \mathbb{Q}_n^{\mathbf{u}_n} \left(\tilde{Y}_n(1) = \left(\frac{\alpha_n - n\alpha}{\sqrt{n}}, \frac{\beta_n - n\beta}{\sqrt{n}} \right) \right),$$

so that Theorem 4.1.2 implies that

$$\left| n^2 \mathbb{Q}_n^{\mathbf{u}_n} \left(Y_n(1) = (\alpha_n, \beta_n) \right) - \phi_2 \left(\frac{\alpha_n - n\alpha}{\sqrt{n}}, \frac{\beta_n - n\beta}{\sqrt{n}} \right) \right| \to 0$$

as $n \to \infty$. Meanwhile, our choice of (α_n, β_n) as in Equations (3.3.1), means that, as $n \to \infty$,

$$\left| \phi_2 \left(\frac{\alpha_n - n\alpha}{\sqrt{n}}, \frac{\beta_n - n\beta}{\sqrt{n}} \right) - \phi_2(0,0) \right| \to 0.$$

□

Recalling Equation (2.1.2), for a vector of times $\mathbf{t}$ we have

$$\tilde{G}_n(\mathbf{t}) = \left(\tilde{G}[0, t_1], \tilde{G}[t_1, t_2], \ldots, \tilde{G}[t_{k-1}, t_k] \right).$$

Let

$$\chi_{k,\alpha,\beta} = \left\{\mathbf{x} \in \mathbb{R}^k : \mathbb{P}_n^{\alpha,\beta}\left(\tilde{G}_n(\mathbf{t}) = \mathbf{x}\right) > 0\right\}$$
$$\subseteq \left\{(x_1, \ldots, x_k) : \sqrt{n}x_j + nc_n[t_{j-1}, t_j] \in \mathbb{Z}, 1 \leq j \leq k\right\}.$$

For $\mathbf{x} \in \chi_{k,\alpha,\beta}$, we write

$$(\mathbf{x}, a, b) = \left(x_1, \ldots, x_k, a - \sum_{j=1}^{k} x_j, b\right).$$

Then, we have

$$\left\{\tilde{G}_n(\mathbf{t}) = \mathbf{x}, Y_n(1) = (\alpha_n, \beta_n)\right\} \iff \left\{\tilde{Y}_n(\mathbf{t}) = \left(\mathbf{x}, \frac{\alpha_n - n\alpha}{\sqrt{n}}, \frac{\beta_n - n\beta}{\sqrt{n}}\right)\right\}.$$

Corollary 4.3.2. *As $n \to \infty$, we have*

$$\sup_{\mathbf{x} \in \chi_{k,\alpha_n,\beta_n}} \left| n^{\frac{k}{2}+2} \mathbb{Q}_n^{\mathbf{u}_n}\left(\tilde{Y}_n(\mathbf{t}) = \left(\mathbf{x}, \frac{\alpha_n - n\alpha}{\sqrt{n}}, \frac{\beta_n - n\beta}{\sqrt{n}}\right)\right) - \phi_{k+2}(\mathbf{x}, 0, 0)\right| \to 0,$$

where $\phi_{k+2}(x_1, \ldots, x_{k+1}, x_0)$ is the density of the $(k+2)$-dimensional Normal distribution with mean $\mathbf{0}$, and covariance matrix $\Sigma_{k+2} = \Sigma_{k+2}(\mathbf{u})$ as in Theorem 3.2.1.

Proof. As in Corollary 4.3.2, we can use a Local Central Limit Theorem from earlier in the chapter. We have that

$$\sup_{\mathbf{x} \in \chi_{k,\alpha_n,\beta_n}} \left| n^{\frac{k}{2}+2} \mathbb{Q}_n^{\mathbf{u}_n}\left(\tilde{Y}_n(\mathbf{t}) = \left(\mathbf{x}, \frac{\alpha_n - n\alpha}{\sqrt{n}}, \frac{\beta_n - n\beta}{\sqrt{n}}\right)\right) - \phi_{k+2}\left(\mathbf{x}, \frac{\alpha_n - n\alpha}{\sqrt{n}}, \frac{\beta_n - n\beta}{\sqrt{n}}\right)\right| \to 0$$

as $n \to \infty$, by Theorem 4.2.2, while

$$\sup_{\mathbf{x} \in \chi_{k,\alpha_n,\beta_n}} \left| \phi_{k+2}\left(\mathbf{x}, \frac{\alpha_n - n\alpha}{\sqrt{n}}, \frac{\beta_n - n\beta}{\sqrt{n}}\right) - \phi_{k+2}\left(\mathbf{x}, 0, 0\right)\right| \to 0$$

as $n \to \infty$ through our choice of α_n and β_n. □

With Corollaries 4.3.1 and 4.3.2 in place, we have our first result about the behaviour of the conditional trajectories.

Theorem 4.3.3. *The finite-dimensional distributions of $\tilde{G}(t)$, under $\mathbb{P}_n^{\alpha_n,\beta_n}$, converge in distribution to the finite-dimensional distributions of a Generalised Gaussian bridge on $[0,1]$, $M(t)$.*

Proof. Combining Corollaries 4.3.1 and 4.3.2, and recalling Equation 4.0.1, we have

$$\sup_{\mathbf{x}\in\chi_{k,\alpha_n,\beta_n}} \left| n^{\frac{k}{2}} \mathbb{P}_n^{\alpha_n,\beta_n}\left(\tilde{G}_n(\mathbf{t}) = \mathbf{x}\right) - \frac{\phi_{k+2}(\mathbf{x},0,0)}{\phi_2(0,0)} \right|$$
$$= \sup_{\mathbf{x}\in\chi_{k,\alpha_n,\beta_n}} \left| \frac{n^{\frac{k}{2}+2}\, \mathbb{Q}_n^{\mathbf{u}_n}\left(\tilde{Y}_n(\mathbf{t}) = \left(\mathbf{x}, \frac{\alpha_n - n\alpha}{\sqrt{n}}, \frac{\beta_n - n\beta}{\sqrt{n}}\right)\right)}{n^2\, \mathbb{Q}_n^{\mathbf{u}_n}\left(Y_n(1) = (\alpha_n,\beta_n)\right)} - \frac{\phi_{k+2}(\mathbf{x},0,0)}{\phi_2(0,0)} \right| \to 0$$

as $n \to \infty$.

Finally, we note that the ratio $\frac{\phi_{k+2}(\mathbf{x},0,0)}{\phi_2(0,0)}$ is the density of the appropriate finite-dimensional distribution of a generalised Gaussian bridge. The initial Gaussian process X_t has zero drift, and the covariance between X_s and X_t, $s \le t$, is given by

$$\mathrm{Cov}(X_s, X_t) = \int_s^t m(u_1 + u_0(1-x))dx.$$

The density $\phi_{k+2}(\mathbf{x}, y, z)$ describes the joint distribution of the increments of X_t with its integral; when we set $y = z = 0$ and divide by $\phi_2(0,0)$, we obtain its distribution conditional on the event $X_1 = 0$, $\int_0^1 X_t dt = 0$.

□

Chapter 5

Convergence in Finite-Dimensional Distributions of a Piecewise-Linear Trajectory

So far, we have established the convergence of the finite-dimensional distributions of the piecewise-constant trajectories $G(t)$, under conditions on the values of $g(1)$ and $\int_0^1 g(s)ds$. In this chapter, we obtain similar asymptotics for the piecewise-linear trajectories $g(t)$.

We will prove that, under the conditional distributions $\mathbb{P}_n^{\alpha_n,\beta_n}$, the finite-dimensional distributions of $g(t) - G(t)$, when properly rescaled, converge in probability to zero.

Definition 5.0.1. Recalling that

$$c_n(t) = \frac{\partial}{\partial u_1} f_{\mathbf{t}}(\mathbf{u}_n)$$

where u_n is chosen according to the conditions α_n and β_n as in Equation (3.3.7), let

$$\tilde{g}(t) = \frac{g(t) - nc_n(t)}{\sqrt{n}}.$$

Theorem 5.0.2. *The finite-dimensional distributions of the trajectories $\tilde{G}(t) - \tilde{g}(t)$ converge in probability to zero; in particular, there exist constants c and C such that,* *for any $k \in \mathbb{N}$ and $\mathbf{t} = (t_1, \ldots, t_k)$, and any $\varepsilon > 0$,*

$$\mathbb{P}_n^{\alpha_n,\beta_n}\left(\max_{1\le j\le k}\left|\tilde{g}(t_j) - \tilde{G}(t_j)\right| > \varepsilon\right) \le Cke^{-c\varepsilon\sqrt{n}}.$$

Corollary 5.0.3. *The finite-dimensional distributions of the trajectories $\tilde{g}(t)$, under the measures $\mathbb{P}_n^{\alpha_n,\beta_n}$, converge in distribution to the finite-dimensional distributions of the zero-area Gaussian bridge $M(t)$.*

Definition 5.0.4. Recall Definition 2.4.1, in which we describe the horizontal projection of the increment at location T in the sub-trajectory using $\Delta_T = (\ell_T, r_T)$ if the endpoints of the increment are $T - \ell_T$ and $T + r_T$.

Let

$$h_T = h(\Delta_T)$$

be the height of that increment; then we write $\Delta_T^h = (\ell_T, r_T, h_T)$.

Proof of Theorem 5.0.2. We begin with the case $k = 1$; let $t \in [0,1]$. We suppose that $w_{nt} > 0$; otherwise, $g(t) = G(t)$. As we see in Figure 5.1,

$$g(t) - G(t) = \frac{\ell_{nt}}{\ell_{nt} + r_{nt}} h_{nt},$$

so that

$$\begin{aligned}\tilde{g}(t) - \tilde{G}(t) &= \frac{1}{\sqrt{n}}\Big(G(t) - g(t)\Big)\\ &= \frac{1}{\sqrt{n}}\frac{\ell_{nt}}{\ell_{nt} + r_{nt}} h_{nt}.\end{aligned}$$

We therefore have

$$\begin{aligned}\mathbb{P}_n^{\alpha_n,\beta_n}\left(\left|\tilde{g}(t) - \tilde{G}(t)\right| > \varepsilon\right) &= \mathbb{P}_n^{\alpha_n,\beta_n}\left(\left|\frac{\ell_{nt}}{\ell_{nt} + r_{nt}} h_{nt}\right| > \varepsilon\sqrt{n}\right)\\ &\le \mathbb{P}_n^{\alpha_n,\beta_n}\left(|h_{nt}| > \varepsilon\sqrt{n}\right).\end{aligned}$$

Now, for any $\mathbf{u} \in \mathcal{U}^\Delta$, we have

$$\mathbb{P}_n^{\alpha_n,\beta_n}\left(|h_{nt}| > \varepsilon\sqrt{n}\right) = \frac{\mathbb{P}_n\left(|h_{nt}| > \varepsilon\sqrt{n}, Y_n(1) = (\alpha_n, \beta_n)\right)}{\mathbb{P}_n\left(Y_n(1) = (\alpha_n, \beta_n)\right)}$$

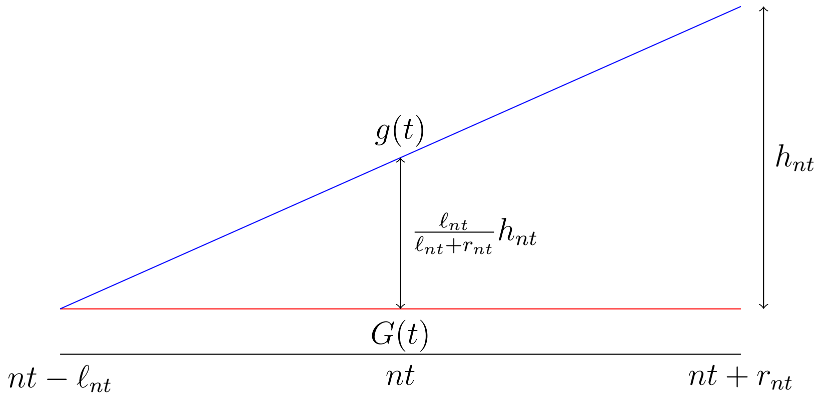

Figure 5.1: The difference between $g(t)$ and $G(t)$, in terms of the increment Δ_{nt}.

$$\begin{aligned} &= \frac{\mathbb{Q}_n^{\mathbf{u}}\left(|h_{nt}| > \varepsilon\sqrt{n}, Y_n(1) = (\alpha_n, \beta_n)\right)}{\mathbb{Q}_n^{\mathbf{u}}\left(Y_n(1) = (\alpha_n, \beta_n)\right)} \\ &\leq \frac{\mathbb{Q}_n^{\mathbf{u}}\left(|h_{nt}| > \varepsilon\sqrt{n}\right)}{\mathbb{Q}_n^{\mathbf{u}}\left(Y_n(1) = (\alpha_n, \beta_n)\right)}. \end{aligned}$$

We have

$$\mathbb{Q}_n^{\mathbf{u}}\left(|h_{nt}| > \varepsilon\sqrt{n}\right) = \sum_{\Delta \in \Lambda^{nt}} \mathbb{Q}_n^{\mathbf{u}}\left(|h_{nt}| > \varepsilon\sqrt{n} | \Delta_{nt} = \Delta\right) \mathbb{Q}_n^{\mathbf{u}}\Big(\Delta_{nt} = \Delta\Big).$$

We separate this sum into two parts, according to the value of $\ell + r$. Recalling Definition 2.4.3, we have

$$\Lambda_{1,\rho} = \Big\{(\ell, r) \in \Lambda^T : \ell + r < \theta\Big\}.$$

Now

$$\mathbb{Q}_n^{\mathbf{u}}\left(|h_{nt}| > \varepsilon\sqrt{n}\right) \leq \sum_{\Delta \in \Lambda_{1,\rho}} \mathbb{Q}_n^{\mathbf{u}}\left(|h_{nt}| > \varepsilon\sqrt{n} | \Delta_{nt} = \Delta\right) + \mathbb{Q}_n^{\mathbf{u}}\left(w_{nt} > \rho\right).$$

We use $\rho = n^{\frac{1}{3}}$; by Corollary 2.4.6, we have

$$\mathbb{Q}_n^{\mathbf{u}}\left(w_{nt} > n^{\frac{1}{3}}\right) \leq e^{-cn^{\frac{1}{3}}}.$$

We now turn to the range of Δ such that $\ell + r < n^{\frac{1}{3}}$. Note that, under the condition $\Delta_{nt} = \Delta = (\ell, r)$, the height h_{nt} is independent of the behaviour of the rest of the sub-trajectory, so that

$$\mathbb{Q}_n^{\mathbf{u}}\left(h_{nt}=h|\Delta_{nt}=\Delta\right)=\mathbb{Q}_{\ell+r}^{u(c(\Delta))}\left(h(\eta)=h\right).$$

When we consider the whole sub-trajectory of width n, the tilt applied to the increment Δ is $u\left(c(\Delta)\right)$; as a result, it is the correct tilt to use when we consider only the increment Δ.

Now, by Assumption 2.2.15, there exists a constant $c'>0$ such that

$$\begin{aligned}\mathbb{Q}_{\ell+r}^{u(c(\Delta))}\left(|h(\eta)|>\varepsilon\sqrt{n}\right)&\leq\exp\left\{-c'\frac{\varepsilon\sqrt{n}}{\ell+r}\right\}\\&\leq\exp\left\{-c'\varepsilon n^{\frac{1}{6}}\right\}.\end{aligned}$$

As a result, for all $\mathbf{u}\in\mathcal{U}^{\Delta}$ and n large enough, we have

$$\begin{aligned}\mathbb{Q}_n^{\mathbf{u}}\left(|h_{nt}|>\varepsilon\sqrt{n}\right)&\leq e^{-c'\varepsilon n^{\frac{1}{6}}}+e^{-cn^{\frac{1}{3}}}\\&\leq e^{-c''\varepsilon n^{\frac{1}{6}}},\end{aligned}$$

and we can find $a>0$ such that

$$\begin{aligned}\sum_{\Delta\in\Lambda_{1,\rho}}\mathbb{Q}_n^{\mathbf{u}}\left(|h_{nt}|>\varepsilon\sqrt{n}|\Delta_{nt}=\Delta\right)&\leq|\Lambda_{1,\rho}|e^{-c''\varepsilon n^{\frac{1}{6}}}\\&\leq n^{\frac{2}{3}}e^{-c''\varepsilon n^{\frac{1}{6}}}\\&\leq e^{-a\varepsilon n^{\frac{1}{6}}}\end{aligned}$$

Now, as we saw in Theorem 4.1.2, there exists $C>0$ such that

$$\mathbb{Q}_n^{\mathbf{u}_n}\left(Y_n(1)=(\alpha_n,\beta_n)\right)\geq\frac{1}{n^2}C\phi(0,0)$$

holds for all $n\in\mathbb{N}$. We therefore have

$$\begin{aligned}\mathbb{P}_n^{\alpha_n,\beta_n}\left(\left|\tilde{g}(t)-\tilde{G}(t)\right|>\varepsilon\right)&=\mathbb{P}_n^{\alpha_n,\beta_n}\left(\left|\frac{\ell_{nt}}{\ell_{nt}+r_{nt}}h_{nt}\right|>\varepsilon\sqrt{n}\right)\\&\leq\mathbb{P}_n^{\alpha_n,\beta_n}\left(|h_{nt}|>\varepsilon\sqrt{n}\right)\\&\leq\frac{1}{C\phi(0,0)}n^2e^{-a''\varepsilon n^{\frac{1}{6}}}\to 0\end{aligned}\tag{5.0.1}$$

as $n \to \infty$.

Now, for $k > 1$, we write

$$\left\{ \max_{1 \le j \le k} \left| \tilde{g}(t_j) - \tilde{G}(t_j) \right| > \varepsilon \right\} = \bigcup_{j=1}^{k} \left\{ \left| \tilde{g}(t_j) - \tilde{G}(t_j) \right| > \varepsilon \right\},$$

so that

$$\begin{aligned} \mathbb{P}_n^{\alpha_n,\beta_n} \left(\max_{1 \le j \le k} \left| \tilde{g}(t_j) - \tilde{G}(t_j) \right| > \varepsilon \right) &\le \sum_{j=1}^{k} \mathbb{P}_n^{\alpha_n,\beta_n} \left(\left| \tilde{g}(t_j) - \tilde{G}(t_j) \right| > \varepsilon \right) \\ &\le \frac{k}{C\phi(0,0)} n^2 e^{-a''\varepsilon n^{\frac{1}{6}}}. \end{aligned}$$

As a result, the finite-dimensional distributions of $\tilde{g}$ converge in probability to those of $\tilde{G}$.

□

Proof of Corollary 5.0.3. By Theorem 4.3.3, the finite-dimensional distributions of $\tilde{G}$, under the conditional measures $\mathbb{P}_n^{\alpha_n,\beta_n}$ converge in distribution to those of the Gaussian process X. As we have just seen, under the same measures the finite-dimensional distributions of $\tilde{g}$ converge in probability to those of $\tilde{G}$. By Slutsky's theorem (Theorem 1.1.4), the finite-dimensional distributions of $\tilde{g}$, under $\mathbb{P}_n^{\alpha_n,\beta_n}$, converge in distribution to those of X. □

Chapter 6

Tightness of the Conditional Distributions

We have now finished our investigation into the convergence of the finite-dimensional distributions, both of G and of g. As we noted in Theorem 1.1.2, the convergence of the finite-dimensional distributions is not sufficient for weak convergence of the trajectories: we also need to prove that the family of distributions induced by $\mathbb{P}_n^{\alpha_n,\beta_n}$ is tight, in order to ensure that the trajectories themselves also converge as functions.

Once we have done so, we will be able to prove the following result.

Theorem 6.0.1. *The distributions of the trajectories $\tilde{g}(t)$, under the measures $\mathbb{P}_n^{\alpha_n,\beta_n}$, converge weakly to those of the zero-area Gaussian bridge $M(t)$.*

We use the following tightness criterion, a particular case of Theorem 12.3 in [4] or Corollary 16.9 in [38].

Proposition 6.0.2. *The family of distributions associated with a sequence of random functions $X_1, X_2, \ldots$ in $C([0,1])$ is tight if it satisfies these two conditions:*

1. *The family of distributions associated with $X_1(0), X_2(0), \ldots$ is tight.*

2. *There exist constants $\gamma > 0$, $\alpha > 1$, and a finite constant c such that*

$$\mathbb{E}\left[(X_n(t) - X_n(s))^\gamma\right] \leq c(t-s)^\alpha.$$

holds for all $n \in \mathbb{N}$, and all $0 \leq s < t \leq 1$.

Since we have $g(0) = 0$ for all n, condition 1 holds for our trajectories and it is sufficient to prove the following theorem.

Theorem 6.0.3. *There exists a finite constant $c > 0$ such that*

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq s < t \leq 1} \frac{1}{(t-s)^{\frac{3}{2}}} \mathbb{E}_n^{\alpha_n, \beta_n} \left[(\tilde{g}(t) - \tilde{g}(s))^4 \right] \leq c;$$

in particular, the family of distributions associated with $\{\tilde{g}_n\}$ is tight.

As part of the proof of Theorem 6.0.3, we establish that the gradient of g at a given point t has finite exponential moments.

Proposition 6.0.4. *For $t \in [0,1]$, let h_{nt} and w_{nt}^+ be as in Definitions 5.0.4 and 2.4.1. There exist a neighbourhood $\mathcal{V}$ of the origin, and a constant c, such that for all $t \in [0,1]$, all $v \in \mathcal{V}$, and all $n \in \mathbb{N}$,*

$$\mathbb{E}_n^{\alpha_n, \beta_n} \left[\exp \left\{ v \frac{h_{nt}}{w_{nt}^+} \right\} \right] \leq c.$$

Proof of Theorem 6.0.3. Let $n \in \mathbb{N}$; we consider $s, t \in [0,1]$ according to two different regimes. Throughout, we use the notation $f[s,t] = f(t) - f(s)$, for the sake of notational simplicity.

First, we consider $0 \leq s < t \leq n$ such that $(t-s) < n^{-\frac{4}{5}}$.

Recalling Definition 5.0.1, we have

$$\tilde{g}(t) = \frac{g(t) - nc_n(t)}{\sqrt{n}},$$

where

$$c_n(t) = \frac{\partial}{\partial u_1} f_{\mathbf{t}}(\mathbf{u}_n)$$

with $\mathbf{u}_n$ chosen according to α and β.

We note that, as we see in Appendix A,

$$\frac{1}{\sqrt{n}} \left| nc_n(t) - \mathbb{E}_n^{\alpha_n, \beta_n}[g(t)] \right| \to 0$$

so that

$$\mathbb{E}_n^{\alpha_n,\beta_n}\left[(\tilde{g}[s,t])^4\right] \le \mathbb{E}_n^{\alpha_n,\beta_n}\left[\left(\frac{g[s,t]-\mathbb{E}_n^{\alpha_n,\beta_n}[g[s,t]]}{\sqrt{n}}\right)^4\right] + C(t-s)^{\frac{3}{2}}$$

and it is sufficient to prove that

$$\mathbb{E}_n^{\alpha_n,\beta_n}\left[\left(\frac{g[s,t]-\mathbb{E}_n^{\alpha_n,\beta_n}[g[s,t]]}{\sqrt{n(t-s)}}\right)^4\right]$$

is bounded by a constant for all n and all s,t under consideration.

First, if there exists $k \in \mathbb{N}$ such that $\frac{k-1}{n} \le s < t \le \frac{k}{n}$, we have $\Delta_{ns} = \Delta_{nt}$ and as we noted in Figure 5.1, we have

$$g(t) - g(s) = n(t-s)\frac{h_{ns}}{w_{ns}^+}.$$

As a result,

$$\begin{aligned}\mathbb{E}_n^{\alpha_n,\beta_n}&\left[\left(\frac{g[s,t]-\mathbb{E}_n^{\alpha_n,\beta_n}[g[s,t]]}{\sqrt{n}}\right)^4\right] \\ &= \left(\sqrt{n}(t-s)\right)^4 \mathbb{E}_n^{\alpha_n,\beta_n}\left[\left(\frac{h_{ns}}{w_{ns}^+} - \mathbb{E}_n^{\alpha_n,\beta_n}\left[\frac{h_{ns}}{w_{ns}^+}\right]\right)^4\right].\end{aligned}$$

In light of the exponential moments of $\frac{h_{ns}}{w_{ns}^+}$ established in Proposition 6.0.4, we can find a finite constant C such that, for all n and s,

$$\mathbb{E}_n^{\alpha_n,\beta_n}\left[\left(\frac{h_{ns}}{w_{ns}^+} - \mathbb{E}_n^{\alpha_n,\beta_n}\left[\frac{h_{ns}}{w_{ns}^+}\right]\right)^4\right] \le C,$$

so that

$$\mathbb{E}_n^{\alpha_n,\beta_n}\left[\left(\frac{g[s,t]-\mathbb{E}_n^{\alpha_n,\beta_n}[g[s,t]]}{\sqrt{n}}\right)^4\right] \le (t-s)^2\, C \le \frac{1}{n^2}C \tag{6.0.1}$$

holds whenever $\frac{k-1}{n} \le s < t \le \frac{k}{n}$.

Next, if s and t are such that ns and nt are integers, and $(t-s) < n^{-\frac{4}{5}}$, then by Jensen's inequality in Proposition B.1.5,

$$\mathbb{E}_n^{\alpha_n,\beta_n}\left[\left(\frac{g[s,t]-\mathbb{E}_n^{\alpha_n,\beta_n}[g[s,t]]}{\sqrt{n}}\right)^4\right]$$

$$= \mathbb{E}_n^{\alpha_n,\beta_n}\left[\left(\sum_{j=ns+1}^{nt} \frac{g[\frac{j-1}{n},\frac{j}{n}] - \mathbb{E}_n^{\alpha_n,\beta_n}[g[\frac{j-1}{n},\frac{j}{n}]]}{\sqrt{n}}\right)^4\right]$$
$$\leq (n(t-s))^3 \sum_{j=ns+1}^{nt} \mathbb{E}_n^{\alpha_n,\beta_n}\left[\left(\frac{g[\frac{j-1}{n},\frac{j}{n}] - \mathbb{E}_n^{\alpha_n,\beta_n}[g[\frac{j-1}{n},\frac{j}{n}]]}{\sqrt{n}}\right)^4\right].$$

Using the upper bound in Equation (6.0.1), we therefore have

$$\mathbb{E}_n^{\alpha_n,\beta_n}\left[\left(\frac{g[s,t] - \mathbb{E}_n^{\alpha_n,\beta_n}[g[s,t]]}{\sqrt{n}}\right)^4\right] \leq (n(t-s))^3 n(t-s)\frac{C}{n^2} \qquad \leq (t-s)^{\frac{3}{2}}C$$

whenever ns and nt are integers and $n^2(t-s)^{\frac{5}{2}} < 1$ – that is, $(t-s) < n^{-\frac{4}{5}}$.

Now if $\frac{j-1}{n} < s < \frac{j}{n}$ and $\frac{k-1}{n} < t < \frac{k}{n}$ with $k - j < n^{\frac{1}{5}}$, we can combine the previous two estimates using Jensen's inequality, so that

$$\mathbb{E}_n^{\alpha_n,\beta_n}\left[\left(\frac{g[s,t] - \mathbb{E}_n^{\alpha_n,\beta_n}[g[s,t]]}{\sqrt{n}}\right)^4\right] \leq 27C(t-s)^{\frac{3}{2}}. \tag{6.0.2}$$

Second, we consider s and t such that $n^{-\frac{4}{5}} < t - s < 1$. Let

$$Z_n(s,t) = \frac{g[s,t] - \mathbb{E}_n^{\alpha_n,\beta_n}[g[s,t]]}{\sqrt{t-s}},$$

and note that for any random variable X we have

$$\mathbb{E}[X^4] \leq \sum_{k\geq 0}(k+1)^4\mathbb{P}(|X| \in (k,k+1))$$
$$\leq \sum_{k\geq 0}(k+1)^4\mathbb{P}(|X| > k).$$

Now

$$\mathbb{E}_n^{\alpha_n,\beta_n}\left[\left(\frac{g[s,t] - \mathbb{E}_n^{\alpha_n,\beta_n}[g[s,t]]}{\sqrt{n(t-s)}}\right)^4\right] = \mathbb{E}_n^{\alpha_n,\beta_n}\left[\left(\frac{1}{\sqrt{n}}Z_n(s,t)\right)^4\right]$$
$$\leq \sum_{k\geq 0}(k+1)^4\mathbb{P}_n^{\alpha_n,\beta_n}\left(\left|\frac{1}{\sqrt{n}}Z_n(s,t)\right| > k\right),$$

and in particular we can write

$$\mathbb{E}_n^{\alpha_n,\beta_n}\left[\left(\frac{1}{\sqrt{n}}Z_n(s,t)\right)^4\right] \leq \sum_{k\geq 0}(k+1)^4\frac{\mathbb{P}_n\left(|Z_n(s,t)| > k\sqrt{n}, Y_n(1) = (\alpha_n,\beta_n)\right)}{\mathbb{P}_n\left(Y_n(1) = (\alpha_n,\beta_n)\right)}. \tag{6.0.3}$$

In order to evaluate the fraction, we introduce a new tilted probability measure. In addition to the tilts u_1 and u_0 corresponding to $Y_n(1)$, we apply a further tilt $w^\star$ corresponding to $Z_n(s,t)$.

For segments $[ns, nt] \subseteq [0, n]$, $\xi \in \mathcal{H}_n$, and $\mathbf{w} = (w^\star, w_1, w_0) \in \mathbb{R}^3$, let

$$\mathbf{w} \cdot \mathbf{Z}(\xi) = w^\star Z_n(s,t)(\xi) + w_1 h(\xi) + \frac{w_0}{n} A(\xi),$$

and define

$$\hat{\mathbb{Q}}^{\mathbf{w}}_{[s,t]}(\mathcal{A}) = \frac{\sum_{\xi \in \mathcal{P}_n} \mathbb{1}\{\xi \in \mathcal{A}\} \lambda(\xi) \exp\{\mathbf{w} \cdot \mathbf{Z}(\xi)\}}{\sum_{\xi' \in \mathcal{P}_n} \lambda(\xi') \exp\{\mathbf{w} \cdot \mathbf{Z}(\xi')\}}.$$

We have already met the denominator in this fraction: writing $\mathbf{t} = (s,t)$ and recalling Definition 2.6.1, we have

$$\sum_{\xi' \in \mathcal{P}_n} \lambda(\xi') \exp\{\mathbf{w} \cdot \mathbf{Z}(\xi')\} = \mathcal{B}_{n,n\mathbf{t}}\left(w_1, w_1 + \frac{w^\star}{\sqrt{t-s}}, w_1, w_0\right).$$

Note that when the event $\mathcal{A}$ is of the form

$$\mathcal{A}_k = \left\{Z_n(s,t) = k\sqrt{n}, Y_n(1) = (a,b)\right\},$$

we have the following analogue of Equations (2.1.5) and (2.1.6):

$$\begin{aligned}\hat{\mathbb{Q}}^{\mathbf{w}}_{[s,t]}\left(\mathcal{A}_k\right) &= \frac{e^{w^\star k\sqrt{n}} \mathcal{B}_n\left(w_1, w_0\right)}{\mathcal{B}_{n,n\mathbf{t}}\left(w_1, w_1 + \frac{w^\star}{\sqrt{t-s}}, w_1, w_0\right)} \mathbb{Q}^{w_1,w_0}_n\left(\mathcal{A}_k\right) \\ &= \frac{e^{w^\star k\sqrt{n} + w_1 a + w_0 b} \mathcal{B}_n\left(0,0\right)}{\mathcal{B}_{n,n\mathbf{t}}\left(w_1, w_1 + \frac{w^\star}{\sqrt{t-s}}, w_1, w_0\right)} \mathbb{P}_n\left(\mathcal{A}_k\right).\end{aligned}$$

By choosing $\mathbf{w}$ as a function of k, we can understand the behaviour of $\hat{\mathbb{Q}}^{\mathbf{w}}_{[s,t]}\left(\mathcal{A}_k\right)$ and hence that of $\mathbb{P}_n\left(\mathcal{A}_k\right)$. To do so, we recall Equation (3.3.4), in which we introduced the log-moment generating function of $Y_n(1)$,

$$\mathcal{L}_{Y_n}(\mathbf{u}) = \log \frac{\mathcal{B}_n\left(\mathbf{u}\right)}{\mathcal{B}_n\left(\mathbf{0}\right)}.$$

Similarly, let $\mathcal{L}_{Z_n}$ be the joint log-moment generating function of $Z_n(s,t)$ and $Y_n(1)$, so that for $\mathbf{w} = (w^\star, w_1, w_0)$ we have

$$\mathcal{L}_{Z_n}(\mathbf{w}) = \log \frac{\mathcal{B}_{n,n\mathbf{t}}\left(w_1, w_1 + \frac{w^\star}{\sqrt{t-s}}, w_1, w_0\right)}{\mathcal{B}_n(\mathbf{0})}.$$

Now for $k \in \mathbb{Z}$, we define $\mathbf{w}_k$ as the argument $(w^\star, w_1, w_0) \in \mathbb{R}^3$ such that

$$\nabla \mathcal{L}_{Z_n}(\mathbf{w}_k) = \mathbf{a}_{k,n} := \left(k\sqrt{n}, \alpha_n, \beta_n\right). \tag{6.0.4}$$

Note that we can also connect $\mathbf{w}_k$ and $\mathbf{a}_{k,n}$ using the convex conjugate of the log-moment generating functions, as in Equation (3.3.5): let

$$\mathcal{L}^\star_{Z_n}(\mathbf{a}) = \sup_{\mathbf{w}} \left\{\mathbf{a} \cdot \mathbf{w} - \mathcal{L}_{Z_n}(\mathbf{w})\right\}.$$

Then $\mathbf{w}_k$ is the maximal argument in $\mathcal{L}^\star_{Z_n}(\mathbf{a}_{k,n})$, that is,

$$\mathcal{L}^\star_{Z_n}(\mathbf{a}_{k,n}) = \mathbf{a}_{k,n} \cdot \mathbf{w}_k - \mathcal{L}_{Z_n}(\mathbf{w}_k).$$

In particular, we have

$$\mathbb{P}_n(\mathcal{A}_k) = \exp\left\{-\mathcal{L}^\star_{Z_n}(\mathbf{a}_{k,n})\right\} \hat{\mathbb{Q}}^{\mathbf{w}_k}_{[s,t]}(\mathcal{A}_k). \tag{6.0.5}$$

Note that these are not the probabilities in Equation (6.0.3), where we instead require $|Z_n(s,t)| > k\sqrt{n}$. In order to relate the two, we look closer at the case $k = 0$. First, when $w^\star = 0$, the expectation under $\hat{\mathbb{Q}}^{0,w_1,w_0}_{s,t}$ coincides with that under $\mathbb{Q}^{w_1,w_0}_n$. As a result, we have

$$\mathcal{L}_{Z_n}(0, w_1, w_0) = \mathcal{L}_{Y_n}(w_1, w_0), \tag{6.0.6}$$

and because of our choice of centering, if $\mathbf{u}_n$ as defined in Equation (3.3.5) is given by (u_1, u_0), we have

$$\nabla \mathcal{L}_{Z_n}(0, u_1, u_0) = (0, \alpha_n, \beta_n), \tag{6.0.7}$$

so that (using the properties in [29]) we also have

$$\nabla \mathcal{L}^\star_{Z_n}(0, \alpha_n, \beta_n) = (0, u_1, u_0).$$

Since $\mathcal{L}_{Z_n}$ is strictly convex, so is $\mathcal{L}^\star_{Z_n}$, and $w^\star > 0$ if and only if $k > 0$. As a result, for $k > 0$ we have

$$\begin{aligned}\mathbb{P}_n(\mathcal{A}_{k+1}) &= \exp\left\{-w^\star - \mathcal{L}^\star_{Z_n}(\mathbf{a}_{k,n})\right\}\hat{\mathbb{Q}}^{\mathbf{w}_k}_{[s,t]}\left(\mathcal{A}_{k+1}\right)\\ &\leq \exp\left\{-\mathcal{L}^\star_{Z_n}(\mathbf{a}_{k,n})\right\}\hat{\mathbb{Q}}^{\mathbf{w}_k}_{[s,t]}\left(\mathcal{A}_{k+1}\right),\end{aligned}$$

so that for $k > 0$,

$$\begin{aligned}&\mathbb{P}_n\left(Z_n(s,t) > k\sqrt{n}, Y_n(1) = (\alpha_n, \beta_n)\right)\\ &\quad\leq \exp\left\{-\mathcal{L}^\star_{Z_n}(\mathbf{a}_{k,n})\right\}\hat{\mathbb{Q}}^{\mathbf{w}_k}_{[s,t]}\left(Z_n(s,t) > k\sqrt{n}, Y_n(1) = (\alpha_n, \beta_n)\right),\end{aligned}$$

while we can use a similar argument for $k < 0$, to obtain

$$\begin{aligned}&\mathbb{P}_n\left(Z_n(s,t) < k\sqrt{n}, Y_n(1) = (\alpha_n, \beta_n)\right)\\ &\quad\leq \exp\left\{-\mathcal{L}^\star_{Z_n}(\mathbf{a}_{k,n})\right\}\hat{\mathbb{Q}}^{\mathbf{w}_k}_{[s,t]}\left(Z_n(s,t) < k\sqrt{n}, Y_n(1) = (\alpha_n, \beta_n)\right).\end{aligned}$$

Using the asymptotics in Equation (2.1.5), along with Equation (6.0.6), we have

$$\mathbb{P}_n\left(Y_n(1) = (\alpha_n, \beta_n)\right) = \exp\left\{-\mathcal{L}^\star_{Z_n}(\mathbf{a}_{0,n})\right\}\mathbb{Q}^{\mathbf{u}_n}_n\left(Y_n(1) = (\alpha_n, \beta_n)\right),$$

so that

$$\begin{aligned}&\frac{\mathbb{P}_n\left(|Z_n(s,t)| > k\sqrt{n}, Y_n(1) = (\alpha_n, \beta_n)\right)}{\mathbb{P}_n\left(Y_n(1) = (\alpha_n, \beta_n)\right)}\\ &\quad\leq \frac{\exp\left\{-\mathcal{L}^\star_{Z_n}(\mathbf{a}_{k,n})\right\}}{\exp\left\{-\mathcal{L}^\star_{Z_n}(\mathbf{a}_{0,n})\right\}}\frac{\hat{\mathbb{Q}}^{\mathbf{w}_k}_{[s,t]}\left(|Z_n(s,t)| > k\sqrt{n}, Y_n(1) = (\alpha_n, \beta_n)\right)}{\mathbb{Q}^{\mathbf{u}_n}_n\left(Y_n(1) = (\alpha_n, \beta_n)\right)}.\end{aligned} \tag{6.0.8}$$

We study the asymptotic behaviour of each of the fractions in Equation (6.0.8) according to the magnitude of k.

Let

$$\mathbf{w}^\star = \left(w_1, w_1 + \frac{w^\star}{\sqrt{t-s}}, w_1, w_0\right).$$

First, we note that if (w_1, w_0) and $(w_1 + \frac{w^\star}{\sqrt{t-s}}, w_0)$ are in $\mathcal{U}^\Delta$, by Proposition 2.6.9, we have, for $0 < \varepsilon < \frac{1}{3}$,

$$\left|\mathcal{B}_{n,n\mathbf{t}}\left(\mathbf{w}^{\star}\right) - \mathcal{B}_{n,n\mathbf{t},(n\mathbf{t},n^{\varepsilon})}\left(\mathbf{w}^{\star}\right)\right| = \mathcal{O}(\exp\{-cn^{\varepsilon}\}),$$

where as in Definition 2.6.6, $\mathcal{B}_{n,n\mathbf{t},(n\mathbf{t},n^{\varepsilon})}\left(\mathbf{w}\right)$ represents the contribution to $\mathcal{B}_{n,n\mathbf{t}}(\mathbf{w})$ from sub-trajectories $\xi \in \mathcal{H}_n$ in which the increments at s and t are not wider than n^{ε}.

Next, by Corollary 3.2.8, there exist C and γ, with $0 < \varepsilon < \gamma < \frac{1}{3}$, such that

$$\left|\log \mathcal{B}_{n,n\mathbf{t},(n\mathbf{t},n^{\varepsilon})}\left(\mathbf{w}^{\star}\right) - \varphi_{[0,n]}\left(\mathbf{w}^{\star}\right)\right| \leq Cn^{\frac{1-\gamma}{2}},$$

where

$$\varphi_{[a,b]}\left(\mathbf{z}\right) = \int_a^b m\left(\mathbf{z}_{n\mathbf{t}}\left(\frac{x}{n}\right)\right)dx + \frac{1}{2}\log\mu\left(\mathbf{z}_{n\mathbf{t}}\left(\frac{a}{n}\right)\right)\mu\left(\mathbf{z}_{n\mathbf{t}}\left(\frac{b}{n}\right)\right).$$

As a result, if n is large and (w_1, w_0) and $(w_1 + \frac{w^{\star}}{\sqrt{t-s}}, w_0)$ are in $\mathcal{U}^{\Delta}$,

$$\left|\mathcal{L}_{Z_n}(\mathbf{w}^{\star}) - \varphi_{[0,n]}\left(\mathbf{w}^{\star}\right)\right| \leq Cn^{\frac{1-\gamma}{2}}, \tag{6.0.9}$$

and by the Cauchy integral formula in Proposition B.1.1, the derivatives of $\mathcal{L}_{Z_n}$ also converge to those of $\varphi_{[0,n]}$. In particular, since

$$\varphi_{[0,n]}\left(\mathbf{w}^{\star}\right) = n\int_0^1 m\left(\mathbf{w}_{n\mathbf{t}}\left(t\right)\right)dt + \frac{1}{2}\log\mu\left(\mathbf{w}_{n\mathbf{t}}\left(0\right)\right)\mu\left(\mathbf{w}_{n\mathbf{t}}\left(0\right)\right),$$

$\mathcal{L}_{Z_n}$ and all of its partial derivatives are of order n.

Now, in order to identify for which k these estimates are appropriate, we recall that $\mathbf{w}_0 = (0, u_1, u_0)$, and (u_1, u_0) is in the interior of $\mathcal{U}$. In particular, it is sufficient to find the range of k for which $\mathbf{w}_k^{\star} \in B(\mathbf{w}_0^{\star}, \rho(t-s))$, with ρ taking the role of ε in Remark 2.2.14. In light of this constraint, Equation (6.0.4), and our estimates on the partial derivatives of $\mathcal{L}_{Z_n}$, there exists $\delta > 0$ such that, for all $k \leq \delta\sqrt{n(t-s)}$, we have $\mathbf{w}_k^{\star} \in B(\mathbf{w}_0^{\star}, \rho(t-s))$.

As a result, for such k, we can use the fact that the Hessian matrix of $\mathcal{L}_{Z_n}$ is related to that of $\mathcal{L}^\star_{Z_n}$ via

$$\mathbf{H}_{\mathcal{L}}(\mathbf{w}_k) = \left(\mathbf{H}_{\mathcal{L}^\star}(\mathbf{a}_{k,n})\right)^{-1},$$

(see, for example, [29]). Now the convergence result in Equation (6.0.9) implies that for some $c > 0$ we also have

$$\frac{\partial^2}{\partial k^2}\mathcal{L}^\star_{Z_n}(\mathbf{a}_{k,n}) \geq \frac{1}{cn} \tag{6.0.10}$$

for all $|k| \leq \delta\sqrt{n(t-s)}$.

As a result, we can use the convexity of $\mathcal{L}^\star_{Z_n}$, together with Equations (6.0.7) and (6.0.10) to write

$$\mathcal{L}^\star_{Z_n}(\mathbf{a}_{k,n}) - \mathcal{L}^\star_{Z_n}(\mathbf{a}_{0,n}) \geq \frac{c}{n}(k\sqrt{n})^2 \tag{6.0.11}$$

for all $|k| \leq \delta\sqrt{n(t-s)}$.

Moreover, this lower bound and the strict convexity of $\mathcal{L}^\star_{Z_n}$ imply that, for $k > \delta\sqrt{n(t-s)}$, we have

$$\begin{aligned}\mathcal{L}^\star_{Z_n}(\mathbf{a}_{k,n}) - \mathcal{L}^\star_{Z_n}(\mathbf{a}_{0,n}) &\geq 2c\delta\sqrt{n(t-s)}k \\ &\geq 2c\delta n^{\frac{1}{10}}k,\end{aligned}$$

where in the final line we recall that $n(t-s) > n^{\frac{1}{5}}$.

As a result, we have

$$\frac{\exp\left\{-\mathcal{L}^\star_{Z_n}(\mathbf{a}_{k,n})\right\}}{\exp\left\{-\mathcal{L}^\star_{Z_n}(\mathbf{a}_{0,n})\right\}} \leq \begin{cases} e^{-ck^2} & |k| \leq \delta\sqrt{n(t-s)} \\ e^{-2c\delta n^{\frac{1}{4}}k} & |k| > \delta\sqrt{n(t-s)}. \end{cases}$$

Turning to the second ratio in Equation (6.0.8), we recall that for $|k| \leq \delta\sqrt{n(t-s)}$, $\mathbf{w}_k$ satisfies $(w_1, v_0), (w_1 + w^\star, v_0) \in \mathcal{U}^\Delta$, so that the Local Limit Theorem 4.1.2 applies, and we can find a constant $C > 0$ such that

$$\begin{aligned}\hat{\mathbb{Q}}^{\mathbf{w}_k}_{[s,t]}\left(|Z_n(s,t)| > k\sqrt{n}, Y_n(1) = (\alpha_n, \beta_n)\right) &\leq \hat{\mathbb{Q}}^{\mathbf{w}_k}_{[s,t]}\left(Y_n(1) = (\alpha_n, \beta_n)\right) \\ &\leq Cn^{-2}\phi(0,0)\end{aligned}$$

holds for all $n \in \mathbb{N}$ and $k \leq \delta\sqrt{n(t-s)}$.

We can also use Theorem 4.1.2 to find a constant $C' > 0$ such that, for all n,

$$\mathbb{Q}_n^{\mathbf{u}_n}\left(Y_n(1) = (\alpha_n, \beta_n)\right) \geq C' n^{-2}\phi(0,0),$$

so that there exist C_1 and C_1' such that

$$\frac{\hat{\mathbb{Q}}_{[s,t]}^{\mathbf{w}_k}\left(|Z_n(s,t)| > k\sqrt{n}, Y_n(1) = \alpha_n, \beta_n)\right)}{\mathbb{Q}_n^{\mathbf{u}_n}\left(Y_n(1) = (\alpha_n, \beta_n)\right)} \leq \begin{cases} C_1 & |k| \leq \delta\sqrt{n(t-s)} \\ C_1' n^2 & |k| > \delta\sqrt{n(t-s)}. \end{cases}$$

Now, we have

$$\frac{\mathbb{P}_n\left(|Z_n(s,t)| > k\sqrt{n}, Y_n(1) = (\alpha_n, \beta_n)\right)}{\mathbb{P}_n\left(Y_n(1) = (\alpha_n, \beta_n)\right)} \leq \begin{cases} C_1 e^{-ck^2} & |k| \leq \delta\sqrt{n(t-s)} \\ C_1 n^2 e^{-2c\sqrt{n}k} & |k| > \delta\sqrt{n(t-s)}. \end{cases}$$

As a result,

$$\sum_{k\geq 0}(k+1)^4 \frac{\mathbb{P}_n\left(|Z_n(s,t)| > k\sqrt{n}, Y_n(1) = (\alpha_n, \beta_n)\right)}{\mathbb{P}_n\left(Y_n(1) = (\alpha_n, \beta_n)\right)}$$

is finite and bounded for all $n \in \mathbb{N}$, and in particular there exists a constant C such that

$$\mathbb{E}_n^{\alpha_n,\beta_n}\left[\left(\frac{g[s,t] - nc_n[s,t]}{\sqrt{n}}\right)^4\right] \leq C(t-s)^2 \leq C(t-s)^{\frac{3}{2}}$$

holds for all n, and all $0 \leq s < t \leq 1$ with $(t-s) > n^{-\frac{4}{5}}$.

In conjunction with the upper bound for $0 \leq s < t \leq 1$ with $(t-s) < n^{-\frac{4}{5}}$ in Equation (6.0.2), this proves the result. □

Proof of Proposition 6.0.4. Recall Definition 5.0.4, in which we write $\Delta_{nt}^h = (\ell, r, k)$ if the increment at location nt has endpoints at $nt - \ell$ and $nt + r$, and height k.

Using this notation, we can write

$$\mathbb{E}_n^{\alpha_n,\beta_n}\left[\exp\left\{v\frac{h_{nt}}{w_{nt}^+}\right\}\right] = \sum_{\Delta=(\ell,r,k)} \exp\left\{v\frac{k}{\ell+r}\right\} \mathbb{P}_n^{\alpha_n,\beta_n}\left(\Delta_{nt}^h = \Delta\right),$$

where the sum ranges over all possible values of Δ_{nt}^h - that is, $(\ell, r) \in \Lambda^{nt}$, and $k \in \mathbb{Z}$.

We have

$$\mathbb{P}_n^{\alpha_n,\beta_n}\left(\Delta_{nt}^h=\Delta\right)=\mathbb{P}_n\left(\Delta_{nt}^h=\Delta\middle|Y_n(1)=(\alpha_n,\beta_n)\right)$$
$$=\mathbb{P}_n\left(\Delta_{nt}^h=\Delta\right)\frac{\mathbb{P}_n\left(Y_n(1)=(\alpha_n,\beta_n)\middle|\Delta_{nt}^h=\Delta\right)}{\mathbb{P}_n\left(Y_n(1)=(\alpha_n,\beta_n)\right)}$$

and, using Equation (2.1.5), we can rewrite this equality using tilted probabilities: for any $\mathbf{u}\in\mathcal{U}^\Delta$, we have

$$\mathbb{P}_n^{\alpha_n,\beta_n}\left(\Delta_{nt}^h=\Delta\right)=\mathbb{Q}_n^{\mathbf{u}}\left(\Delta_{nt}^h=\Delta\right)\frac{\mathbb{Q}_n^{\mathbf{u}}\left(Y_n(1)=(\alpha_n,\beta_n)\middle|\Delta_{nt}^h=\Delta\right)}{\mathbb{Q}_n^{\mathbf{u}}\left(Y_n(1)=(\alpha_n,\beta_n)\right)},$$

so that

$$\mathbb{E}_n^{\alpha_n,\beta_n}\left[e^{v\frac{h_{nt}}{w_{nt}^+}}\right]=\sum_{\Delta=(\ell,r,k)}e^{v\frac{k}{\ell+r}}\,\mathbb{Q}_n^{\mathbf{u}}\left(\Delta_{nt}^h=\Delta\right)\frac{\mathbb{Q}_n^{\mathbf{u}}\left(Y_n(1)=(\alpha_n,\beta_n)\middle|\Delta_{nt}^h=\Delta\right)}{\mathbb{Q}_n^{\mathbf{u}}\left(Y_n(1)=(\alpha_n,\beta_n)\right)}. \tag{6.0.12}$$

In order to estimate the sum in Equation (6.0.12), we separate it according to the magnitudes of $\ell+r$ and k. We first recall some preliminary estimates: note that

$$\mathbb{Q}_n^{\mathbf{u}}\left(\Delta_{nt}^h=\Delta\right)=\mathbb{Q}_n^{\mathbf{u}}\left(h_{nt}=k\middle|\Delta_{nt}=(\ell,r)\right)\mathbb{Q}_n^{\mathbf{u}}\left(\Delta_{nt}=(\ell,r)\right);$$

by Assumption 2.2.15, there exist constants C and c such that

$$\mathbb{Q}_n^{\mathbf{u}}\left(|h_{nt}|>k\middle|\Delta_{nt}=(\ell,r)\right)\le C\exp\left\{-c\frac{k}{\ell+r}\right\}. \tag{6.0.13}$$

In the rest of this proof, we will suppose that $|v|<\frac{c}{2}$.

Meanwhile, by Corollary 2.4.6, there exist constants C' and c' such that

$$\mathbb{Q}_n^{\mathbf{u}}\left(\ell_{nt}+r_{nt}>W\right)\le C'e^{-c'W} \tag{6.0.14}$$

holds uniformly in W.

Finally, by Theorem 4.1.2, there exist constants γ and γ' such that for all n, and taking $\mathbf{u}$ as identified in Equation (3.3.3), we have

$$\frac{1}{n^2}\gamma\le\mathbb{Q}_n^{\mathbf{u}}\left(Y_n(1)=(\alpha_n,\beta_n)\right)\le\frac{1}{n^2}\gamma'. \tag{6.0.15}$$

Now for each $(\ell, r) \in \Lambda^{nt}$ we can use the upper bounds in Equations (6.0.13) and (6.0.15) to write

$$\sum_{|k|>n^{\frac{1}{6}}(\ell+r)} e^{v\frac{k}{\ell+r}}\, \mathbb{Q}_n^{\mathbf{u}}\left(\Delta_{nt}^h = (\ell, r, k)\right) \frac{\mathbb{Q}_n^{\mathbf{u}}\left(Y_n(1) = (\alpha_n, \beta_n)\middle|\Delta_{nt}^h = (\ell, r, k)\right)}{\mathbb{Q}_n^{\mathbf{u}}\left(Y_n(1) = (\alpha_n, \beta_n)\right)}$$
$$\leq \frac{C}{\gamma} n^2 \exp\left\{-c\frac{1}{\ell + r} n^{\frac{1}{6}}\right\},$$

while by Equations (6.0.14) and (6.0.15) we have

$$\sum_{\ell+r>n^{\frac{1}{6}}} \sum_{|k|<n^{\frac{1}{6}}(\ell+r)} e^{v\frac{k}{\ell+r}}\, \mathbb{Q}_n^{\mathbf{u}}\left(\Delta_{nt}^h = (\ell, r, k)\right) \frac{\mathbb{Q}_n^{\mathbf{u}}\left(Y_n(1) = (\alpha_n, \beta_n)\middle|\Delta_{nt}^h = (\ell, r, k)\right)}{\mathbb{Q}_n^{\mathbf{u}}\left(Y_n(1) = (\alpha_n, \beta_n)\right)}$$
$$\leq \frac{C'}{\gamma} n^2 \exp\left\{-c' n^{\frac{1}{6}}\right\}.$$

We can therefore restrict our attention to $\Delta = (\ell, r, k)$ such that $\ell + r < n^{\frac{1}{6}}$, and $|k| < n^{\frac{1}{3}}$. In this case, we further separate our sum according to the height of the trajectory at the left endpoint of Δ. We write $H_{nt-\ell}$ for this quantity; note that although it is equal to, for example, $G(t)$, we prefer to work in terms of the unrescaled sub-trajectories $\xi \in \mathcal{H}_n$ to avoid confusion later in the proof. Now, we have

$$\mathbb{Q}_n^{\mathbf{u}}\Big(Y_n(1) = (\alpha_n, \beta_n)\Big|\Delta_{nt}^h = \Delta\Big)$$
$$= \sum_{j\in\mathbb{Z}} \mathbb{Q}_n^{\mathbf{u}}\left(Y_n(1) = (\alpha_n, \beta_n), H_{nt-\ell} = j\middle|\Delta_{nt}^h = \Delta\right); \tag{6.0.16}$$

in order to understand the probabilities in the sum, we construct a new sub-trajectory ξ', obtained by removing the increment described by Δ from ξ. Note that, since the width of ξ is n, ξ' will have width $n - \ell - r$.

We can use Figure 6.1 to see how the height and area of ξ' differ from those of ξ. In particular, we note that the difference in area does not only come from the horizontal strip, shown in red in the figure and corresponding to our interpretation of the area in Figure 2.3; we also have to consider the vertical part, shown in blue. Now, if

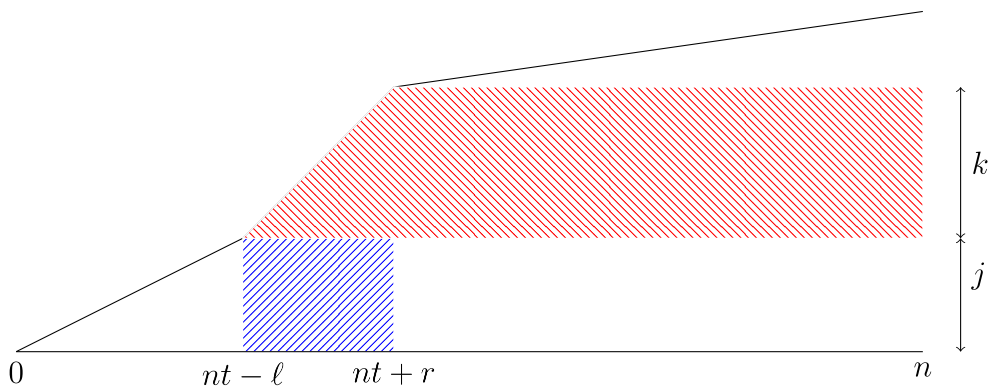

Figure 6.1: A trajectory of width n, in which the single increment between $nt-\ell$ and $nt+r$ is highlighted along with its contributions to the height and area of the trajectory.

$\xi \in \mathcal{H}_n$ has $h(\xi) = h$, $A(\xi) = A$, and $H_{nt-\ell}(\xi) = j$, then we must have

$$\begin{aligned} h(\xi') &= h - k \\ A(\xi') &= A - \left(n - nt + \frac{\ell - r}{2}\right) k - j(\ell + r), \end{aligned} \tag{6.0.17}$$

and ξ' must have a cutpoint at $nt - \ell$, with $H_{nt-\ell}(\xi') = j$. We let

$$\begin{aligned} h_{\Delta,j} &= \alpha_n - k \\ A_{\Delta,j} &= n\beta_n - \left(n - nt + \frac{\ell - r}{2}\right) k - j(\ell + r), \end{aligned}$$

and we write

$$\begin{aligned} \mathcal{R}_{nt-\ell} &= \{\text{there is a renewal at } nt - \ell\} \\ \mathcal{R}_{nt-\ell,j} &= \{\text{there is a renewal at } nt - \ell,\ H_{nt-\ell} = j\} \end{aligned}$$

For the sake of space, we will write $n' = n - \ell - r$,

$$\lambda^{\mathbf{u}}(\xi) = \exp\left\{u_1 h(\xi) + \frac{u_0}{n} A(\xi)\right\} \lambda(\xi),$$

and

$$\lambda^{\mathbf{u}}(\xi') = \exp\left\{u_1 h(\xi') + \frac{u_0}{n'} A(\xi')\right\} \lambda(\xi');$$

let $\mathbf{u}' = \left(u_1, u_0 \frac{n'}{n}\right)$.

Now, using Equations (6.0.17), we have

$$\lambda^{\mathbf{u}}(\xi) = \exp\left\{u^c k + \frac{u_0}{n} j(\ell + r)\right\} \lambda(\Delta)\ \lambda^{\mathbf{u}'}(\xi'),$$

where $u^c = u_1 + u_0\left(1 - t - \frac{\ell - r}{2n}\right)$ is the value of $u(x)$ when x is taken at the centre of the increment Δ.

In order to rewrite the probabilities in Equation (6.0.16) in the context of these new sub-trajectories, we first write

$$\begin{aligned}
&\sum_{\xi\in\mathcal{H}_n} \mathbb{1}\left\{\Delta^h_{nt} = \Delta\right\} \mathbb{1}\left\{Y_n(1) = (\alpha_n, \beta_n)\right\} \lambda^{\mathbf{u}}(\xi) \\
&= e^{u^c k}\lambda(\Delta) e^{\frac{u_0}{n} j(\ell + r)} \sum_{\xi'\in\mathcal{H}_{n'}} \mathbb{1}\left\{\mathcal{R}_{nt-\ell,j}\right\} \mathbb{1}\left\{Y_{n'}(1) = (h_{\Delta,j}, A_{\Delta,j})\right\} \lambda^{\mathbf{u}'}(\xi').
\end{aligned}$$

Meanwhile, as in Remark 2.3.13 we have

$$\begin{aligned}
\sum_{\xi\in\mathcal{H}_n} \mathbb{1}\left\{\Delta^h_{nt} = \Delta\right\} \lambda^{\mathbf{u}}(\xi) &= \mathcal{B}_n\left(u_1, u_0\right) \mathbb{Q}^{\mathbf{u}}_n(\Delta^h_{nt} = \Delta) \\
&= \mathcal{B}_{[0,nt-\ell]}\left(\mathbf{u}\right) e^{u^c k}\lambda(\Delta)\mathcal{B}_{[nt+r,n]}\left(\mathbf{u}\right),
\end{aligned}$$

while similarly

$$\sum_{\xi'\in\mathcal{H}_{n'}} \mathbb{1}\left\{\mathcal{R}_{nt-\ell}\right\} \lambda^{\mathbf{u}'}(\xi') = \mathcal{B}_{[0,nt-\ell]}\left(\mathbf{u}'\right) \mathcal{B}_{[nt-\ell,n']}\left(\mathbf{u}'\right).$$

Note that here, the segments $[0, nt - \ell]$ and $[nt - \ell, n']$ are considered relative to the interval $[0, n']$, instead of our usual context of $[0, n]$. Writing

$$\mathcal{B}_{[nt-\ell,n']}\left(\mathbf{u}'\right) = \mathcal{B}_{n(1-t)-r}\left(u_1, u_0\frac{n(1-t) - r}{n}\right),$$

we see that

$$\mathcal{B}_{[nt-\ell,n']}\left(\mathbf{u}'\right) = \mathcal{B}_{[nt+r,n]}\left(\mathbf{u}\right),$$

where $[nt + r, n]$ is taken relative to $[0, n]$ in the right-hand side.

Meanwhile, in the context of $[0, n']$ we have

$$\mathcal{B}_{[0,nt-\ell]}\left(\mathbf{u}'\right) = \mathcal{B}_{nt-\ell}\left(u_1 + \frac{n' - nt + \ell}{n}u_0, u_0\right)$$

$$= \mathcal{B}_{nt-\ell}\left(u_1 + \frac{n-nt-r}{n} u_0, u_0\right);$$

since $\log \mathcal{B}_{nt-\ell}$ is analytic, there exists a constant c such that

$$\mathcal{B}_{nt-\ell}\left(u_1 + u_0 \frac{n-nt-r}{n} u_0, u_0\right) \leq e^{c|\frac{u_0}{n}|(\ell+r)} \mathcal{B}_{nt-\ell}\left(u_1 + u_0 \frac{n-nt+\ell}{n} u_0, u_0\right).$$

Now, the identity

$$\mathcal{B}_{nt-\ell}\left(u_1 + \frac{n-nt+\ell}{n} u_0, u_0\right) = \mathcal{B}_{[0,nt-\ell]}\left(\mathbf{u}\right)$$

allows us to place this term in the context of $[0, n]$, so that

$$\begin{aligned} \frac{\sum_{\xi' \in \mathcal{H}_{n'}} \mathbb{1}\left\{\mathcal{R}_{nt-\ell}\right\} \lambda^{\mathbf{u}'}(\xi')}{\sum_{\xi \in \mathcal{H}_n} \mathbb{1}\left\{\Delta^h_{nt} = \Delta\right\} \lambda^{\mathbf{u}}(\xi)} &= \frac{\mathcal{B}_{[0,nt-\ell]}\left(\mathbf{u}'\right) \mathcal{B}_{[nt-\ell,n']}\left(\mathbf{u}'\right)}{\mathcal{B}_{[0,nt-\ell]}\left(\mathbf{u}\right) e^{u^c k} \lambda(\Delta) \mathcal{B}_{[nt+r,n]}\left(\mathbf{u}\right)} \\ &\leq e^{c|\frac{u_0}{n}|(\ell+r)} \frac{1}{e^{u^c k}\lambda(\Delta)}. \end{aligned}$$

Since we are only interested in pairs (ℓ, r) with $\ell + r < n^{\frac{1}{6}}$, there exists a constant C such that

$$\frac{\sum_{\xi' \in \mathcal{H}_{n'}} \mathbb{1}\left\{\mathcal{R}_{nt-\ell}\right\} \lambda^{\mathbf{u}'}(\xi')}{\sum_{\xi \in \mathcal{H}_n} \mathbb{1}\left\{\Delta^h_{nt} = \Delta\right\} \lambda^{\mathbf{u}}(\xi)} \leq C \frac{1}{e^{u^c k}\lambda(\Delta)}.$$

Now, we have

$$\begin{aligned} &\mathbb{Q}^{\mathbf{u}}_n\left(Y_n(1) = (\alpha_n, \beta_n), H_{nt-\ell} = j \middle| \Delta^h_{nt} = \Delta\right) \\ &\quad = \frac{\sum_{\xi \in \mathcal{H}_n} \mathbb{1}\left\{\Delta^h_{nt} = \Delta, H_{nt-\ell} = j\right\} \mathbb{1}\left\{Y_n(1) = (\alpha_n, \beta_n)\right\} \lambda^{\mathbf{u}}(\xi)}{\sum_{\xi \in \mathcal{H}_n} \mathbb{1}\left\{\Delta^h_{nt} = \Delta\right\} \lambda^{\mathbf{u}}(\xi)} \\ &\quad \leq C e^{u_0 \frac{j(\ell+r)}{n}} \frac{\sum_{\xi' \in \mathcal{H}_{n'}} \mathbb{1}\left\{\mathcal{R}_{nt-\ell,j}\right\} \mathbb{1}\left\{Y_{n'}(1) = (\alpha', \beta')\right\} \lambda^{\mathbf{u}'}(\xi')}{\sum_{\xi' \in \mathcal{H}_{n'}} \mathbb{1}\left\{\mathcal{R}_{nt-\ell}\right\} \lambda^{\mathbf{u}'}(\xi')} \\ &\quad \leq C e^{u_0 \frac{j(\ell+r)}{n}} \mathbb{Q}^{\mathbf{u}'}_{n'}\left(Y_{n'}(1) = (h_{\Delta,j}, A_{\Delta,j}), H_{nt-\ell} = j \middle| \mathcal{R}_{nt-\ell}\right). \end{aligned}$$

By Lemma 4.1.4, there exist k and p such that

$$\mathbb{Q}^{\mathbf{u}'}_{n'}(\mathcal{R}_T) > p$$

holds for all $\mathbf{u} \in \mathcal{U}^{\Delta}$, whenever $\min(nt - \ell, n - nt - r) > k$.

Since there are only finitely-many other values of T to consider, we can find a constant $p_1 > 0$ such that

$$\mathbb{Q}_{n'}^{\mathbf{u}'}(\mathcal{R}_T) > p_1$$

holds for all T such that $\mathbb{Q}_n^{\mathbf{0}}(\mathcal{R}_T) > 0$, all n large enough, and all $\mathbf{u} \in \mathcal{U}^\Delta$.

As a result, we have

$$\begin{aligned}
&\mathbb{Q}_{n'}^{\mathbf{u}'}\left(Y_{n'}(1) = (h_{\Delta,j}, A_{\Delta,j}), H_{nt-\ell} = j \middle| \mathcal{R}_{nt-\ell}\right) \\
&\quad \le \frac{1}{p_1}\mathbb{Q}_{n'}^{\mathbf{u}'}\left(Y_{n'}(1) = (h_{\Delta,j}, A_{\Delta,j}), H_{nt-\ell} = j,\ \mathcal{R}_{nt-\ell}\right) \\
&\quad \le \frac{1}{p_1}\mathbb{Q}_{n'}^{\mathbf{u}'}\left(Y_{n'}(1) = (h_{\Delta,j}, A_{\Delta,j}), H_{nt-\ell} = j\right)
\end{aligned}$$

for all n large enough, all $t \in [0,1]$ and all $\ell \le n^{\frac{1}{6}}$.

Moreover, by writing

$$\begin{aligned}
&e^{u_0 \frac{j(\ell+r)}{n}}\mathbb{Q}_{n'}^{\mathbf{u}'}\left(Y_{n'}(1) = (h_{\Delta,j}, A_{\Delta,j}), H_{nt-\ell} = j\right) \\
&\quad = e^{u_0 \frac{j(\ell+r)}{n}}\mathbb{Q}_{n'}^{\mathbf{u}'}\left(Z_{n'}([0, nt-\ell]) = j, Y_{n'}(1) = (h_{\Delta,j}, A_{\Delta,j})\right)
\end{aligned}$$

in the notation of our previous proof, we can use the analysis of $\mathcal{L}_{Z_n}$ there to note that, if v_1, v_0 are fixed, there exist $\delta > 0$ and a finite constant c such that for all $z \in \mathbb{C}$ with $|z| < \delta$,

$$\mathcal{L}_{Z_n}(z, v_1, v_0) \le \mathcal{L}_{Z_n}(0, v_1, v_0) + C.$$

Since here we have $z = u_0 \frac{\ell+r}{n}\sqrt{nt-\ell}$, and $\ell + r \le n^{\frac{1}{6}}$, we do have $|z| < \delta$ for all $t \in [0,1]$ and all n large enough. As a result,

$$\begin{aligned}
&e^{u_0 \frac{j(\ell+r)}{n}}\mathbb{Q}_{n'}^{\mathbf{u}'}\left(Y_{n'}(1) = (h_{\Delta,j}, A_{\Delta,j}), H_{nt-\ell} = j\right) \\
&\quad \le e^{C}\mathbb{Q}_{n'}^{\mathbf{u}'}\left(Y_{n'}(1) = (h_{\Delta,j}, A_{\Delta,j}), H_{nt-\ell} = j\right)
\end{aligned}$$

holds for all $j \in \mathbb{Z}$, as long as n is large enough.

Using the upper bounds in Equations (6.0.13) to (6.0.15), we are therefore left to find a constant upper bound for

$$\sum_{\ell+r>n^{\frac{1}{6}}} \sum_{|k|<n^{\frac{1}{6}}(\ell+r)} \exp\left\{-\frac{c}{2}\frac{k}{\ell+r} - c(\ell+r)\right\} \\ \times n^2 \sum_{j\in\mathbb{Z}} \mathbb{Q}_{n'}^{\mathbf{u}'}\left(Y_{n'}(1) = (h_{\Delta,j}, A_{\Delta,j}), H_{nt-\ell} = j\right).$$

First, there exist constants C_2 and C_2' such that, if

$$\left|j - \bar{\mathbb{E}}_{n'}^{\mathbf{u}'}[H_{nt}]\right| < C_2\sqrt{n},$$

then we also have

$$|h_{\Delta,j} - n'\alpha| \leq C_2'\sqrt{n},$$

and

$$\frac{1}{n'}|j(\ell+r)| \leq \frac{C_2'}{2}n^{\frac{1}{3}},$$

so that

$$\left|\frac{1}{n'}A_{\Delta,j} - n'\beta\right| \leq C_2'\sqrt{n}.$$

Now, using Equation (6.0.15),

$$\begin{aligned} \mathbb{Q}_{n'}^{\mathbf{u}'}\left(Y_{n'}(1) = (h_{\Delta,j}, A_{\Delta,j}), H_{nt-\ell} = j\right) &\leq \mathbb{Q}_{n'}^{\mathbf{u}'}\left(Y_{n'}(1) = (h_{\Delta,j}, A_{\Delta,j})\right) \\ &\leq C'\frac{1}{(n-\ell-r)^2}\phi(0,0) \\ &\leq \frac{C'}{2}\frac{1}{n^2}\phi(0,0). \end{aligned} \tag{6.0.18}$$

Meanwhile, the Local Central Limit Theorem 4.2.2 implies that, for n large enough, the distribution of $H_{nt-\ell}$ is approximately normal and in particular has exponential moments. We can find a constant c such that for all ρ and all n large enough,

$$\mathbb{Q}_{n'}^{\mathbf{u}'}\left(\frac{1}{\sqrt{n}}|H_{nt-\ell} - \bar{\mathbb{E}}_{n'}^{\mathbf{u}'}[H_{nt}]| > \rho\right) \leq e^{-c\rho}. \tag{6.0.19}$$

As a result, the sum

$$n^2 \sum_{j\in\mathbb{Z}} \mathbb{Q}_{n'}^{\mathbf{u}'} \left(Y_{n'}(1) = (h_{\Delta,j}, A_{\Delta,j}), H_{nt-\ell} = j\right)$$

is bounded above by a constant which is uniform in n. Now there exists a constant C such that for any $t \in [0,1]$, any $n \in \mathbb{N}$, and any v with $|v| < \frac{c}{2}$ as in Equation (6.0.13),

$$\mathbb{E}_n^{\alpha_n,\beta_n}\left[\exp\left\{v\frac{h_{nt}}{w_{nt}^+}\right\}\right] \le C.$$

□

Proof of Theorem 6.0.1. As noted in Theorem 1.1.2, we need both convergence in finite-dimensional distributions and tightness. The first is proved in Corollary 5.0.3, and the second follows from Proposition 6.0.2 and Theorem 6.0.3. □

Chapter 7

Conclusion

The motivation of this work is to develop a framework under which to study a range of statistical mechanical models, and in particular to obtain sharp asymptotics for the shape of the phase boundary in the low-temperature, two-dimensional Ising model. The models in question possess contours which can be locally well approximated by discrete skeletons supported on the integer lattice, and hence related to the renewal random walk studied here. The connection between these frameworks is a topic for future research, in addition to that in [14] and [36] which we hope to further explore in future work.

The study of Functional Central Limit Theorems for random walks stretches back to Donsker's work in the 1950s. In this thesis, we have proved a Functional Central Limit Theorem for a very specific random walk, which extends the classical i.i.d. case in two ways. First, each "step" of the walk is not constrained to have width 1, but takes a random width from among the positive integers. A section of the walk whose width is n therefore consists of at most – but not necessarily exactly – n steps. This introduces a dependence structure between the widths of the increments: within a section which has been conditioned to have width n, the knowledge that one increment is unusually wide gives us some information about the widths (and hence the heights) of all the other increments.

In addition to this dependence between increments, we further place large-deviations conditions on the trajectories of the walk. We describe a section of the walk whose total width is n using the height function $g(t)$ with time rescaled so that $t \in [0,1]$. We ask the question: what does $g(t)$ look like, when $g(1)$ is close to $n\alpha$ and $\int_0^1 g(s)ds$ is close to $n\beta$?

Using the results in Chapter 3, we can pose this question more precisely. Based on the original distribution of the increments, we find a neighbourhood $\mathcal{A}$ of viable conditions (α,β). We construct a sequence (α_n,β_n) of conditions which are asymptotic to $(n\alpha, n\beta)$, such that $\mathbb{P}_n\left(g(1)=\alpha_n, \int_0^1 g(s)ds=\beta_n\right)>0$, and study the behaviour of the trajectories under the conditional distributions

$$\mathbb{P}_n^{\alpha_n,\beta_n}(\,\cdot\,) = \mathbb{P}_n\left(\,\cdot\,\Big| g(1)=\alpha_n, \int_0^1 g(s)ds=\beta_n\right).$$

Using an exponential tilting argument applied to the unconditioned distributions, we prove the following theorem.

Theorem 7.0.1. *For every $(\alpha,\beta)\in\mathcal{A}$, there exist functions $c_n(t)$ converging to a limit $c_\infty(t)\in C([0,1])$ such that, under the conditional distributions $\mathbb{P}_n^{\alpha_n,\beta_n}$, the trajectories*

$$\frac{g(t)-nc_n(t)}{\sqrt{n}}$$

converge weakly to the trajectories of a zero-area Gaussian bridge $M(t)$.

In Chapter 3, we give an explicit form for the limiting profile $c_\infty(t)$; we prove convergence of the finite-dimensional distributions of a different version the trajectories under the conditional distributions in Chapter 4, and relate this to the "true" trajectories in 5. In Chapter 6, we prove that the family $\{\mathbb{P}_n^{\alpha_n,\beta_n}\}$ is tight.

Along the way, we also prove convergence in finite-dimensional distributions of the height function under the unconditional, tilted distributions $\mathbb{Q}_n^{\mathbf{u}_n}$, as well as of the piecewise-constant height function G under the conditional distributions. The arguments of Chapter 6 apply here too, so that we can obtain full weak convergence in both cases.

This work offers (at least) two possible avenues for extension. Firstly, our conditions $g(1) \approx n\alpha, \int_0^1 g(s)ds \approx n\beta$ were chosen with certain applications in mind, but could be generalised to other additive functionals of the trajectories. Using the techniques in this thesis with an appropriate version of the partition functions, it should be possible to prove Functional Central Limit Theorems for renewal random walks whose trajectories have been placed under a more general finite set of conditions.

Secondly, we consider the importance of Assumption 2.2.15 in the model. We require the existence of constants c and C such that, uniformly in $\rho > 0$, $k \in \mathbb{N}$, and $u \in \mathcal{U}$, we have

$$\mathbb{Q}^u \left(|h(\eta)| > \rho k | w(\eta) = k \right) \leq C e^{-c\rho}.$$

In the particular case of $u = 0$, these constants exist for fixed k and ρ, as a result of Assumption 2.1.6; the additional constraint is the requirement that they hold uniformly, not only for different choices of k and ρ, but also for each of the measures $\mathbb{Q}^u$. It should be possible to relax this assumption by using different techniques in the proofs in Chapters 2 and 5.

Finally, we want to draw attention to an application of this work. Renewal random walks of the type discussed in this thesis appear in representations of certain models in statistical mechanics. Similar results in the i.i.d ($\mathcal{F} = \mathcal{F}_1$) case [14] have been used to describe interfaces in the one-dimensional Solid-On-Solid model [35]. Meanwhile, the shape of macroscopic sections of the phase boundary in the low-temperature expansion of the two-dimensional Ising model with fixed total magnetisation is well-approximated by skeletons which obey the renewal random walk framework. In this case, the limiting profile c_∞ describes the shape of the corresponding section of the Wulff shape obtained from the surface tension using the classical Dobrushin-Kotecký-Shlosman theory [15] and whose formula is given explicitly in [47], while our results about the fluctuations of the trajectories around c_∞ allow us to derive a sharp large deviations principle for the total magnetisation. A shortened proof of these ideas is presented in [36]; in future work we plan to further expand them. Moreover, the renewal random walk model provides a framework, with which we can study a wider class of two-dimensional models which can be represented by contours. We expect our exploration of this class to produce sharp asymptotic results which will have implications in a wide range of settings, including those discussed in [56].

Appendix A

Characteristic Functions and Convergence

In Chapter 3, we prove that the log-characteristic functions of the variables $Y_n(1)$ and $Y_n(\mathbf{t})$ take a particular form as $n \to \infty$, as part of the proof of Central Limit Theorems for those variables. In this appendix, we justify this approach and make the connection between the asymptotic behaviour of the log-characteristic function of a variable, and the convergence in distribution of the variable after centering and rescaling.

Throughout this appendix, we will refer to random variables $\mathbf{X}_1, \mathbf{X}_2, \dots$ taking values in $\mathbb{R}^d$, with characteristic functions defined for $\mathbf{t} \in \mathbb{R}^d$ by

$$\varphi_{\mathbf{X}_n}(\mathbf{t}) = \mathbb{E}\left[e^{i\mathbf{t}\cdot\mathbf{X}_n}\right].$$

We will suppose that each $\mathbf{X}_n$ has finite first and second moments, and write $\mu_n = \mathbb{E}[\mathbf{X}_n]$, and $\Sigma_n = \mathbb{E}[(\mathbf{X}_n - \mu_n)^\intercal(\mathbf{X}_n - \mu_n)]$.

We denote the d-dimensional Normal distribution with mean μ and covariance matrix Σ by $\mathcal{N}_d(\mu, \Sigma)$.

We write $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^d$.

We take a connected, open neighbourhood $\mathcal{Z} \subset \mathbb{C}^d$, and write $\mathcal{Z} \cap \mathbb{R}^d = \mathcal{U}$. We take a function f which is analytic on $\mathcal{Z}$, and write ∇f for the gradient vector $(\nabla f(\mathbf{z}))_j = \frac{\partial}{\partial z_j} f(\mathbf{z})$, and $\mathbf{H}_f$ for the Hessian matrix, $(\mathbf{H}_f(\mathbf{z}))_{j,k} = \frac{\partial^2}{\partial z_j \partial z_k} f(\mathbf{z})$.

Our main focus will be to prove the following result.

Theorem A.0.1. *Let* $(\mathbf{u}_n)$ *be a sequence in* $\mathcal{U}$*, such that for some* $\mathbf{u} \in \mathrm{int}\mathcal{U}$*, we have* $\|\mathbf{u}_n - \mathbf{u}\| \to 0$ *as* $n \to \infty$*. Let*

$$\begin{aligned} \mathbf{Y}_n &= \frac{1}{\sqrt{n}} (\mathbf{X}_n - \mu_n) \\ \mathbf{Y}'_n &= \frac{1}{\sqrt{n}} (\mathbf{X}_n - n\nabla f(\mathbf{u}_n)) \\ \mathbf{Y}''_n &= \frac{1}{\sqrt{n}} (\mathbf{X}_n - n\nabla f(\mathbf{u})) , \end{aligned}$$

and let $\mathbf{N}, \mathbf{N}', \mathbf{N}''$ *be random vectors with distribution* $\mathcal{N}_d(\mathbf{0}, \mathbf{H}_f(\mathbf{u}))$*. Then a sufficient condition for*

$$\mathbf{Y}_n \xrightarrow{\mathcal{D}} \mathbf{N} \tag{A.0.1}$$

$$\mathbf{Y}'_n \xrightarrow{\mathcal{D}} \mathbf{N}' \tag{A.0.2}$$

as $n \to \infty$ *is*

$$\sup \Big|\log \varphi_{\mathbf{X}_n}(\mathbf{v}) - n\Big(f(\mathbf{u_n} + i\mathbf{v}) - f(\mathbf{u}_n)\Big)\Big| \to 0,$$

where the supremum is over $\mathbf{v}$ *such that* $\mathbf{u}_n + i\mathbf{v} \in \mathcal{Z}$*, and* $\|\mathbf{v}\| \le n^{-\frac{1}{2}}$*. If additionally* $\sqrt{n}\|\mathbf{u}_n - \mathbf{u}\| \to 0$ *as* $n \to \infty$*, then under the same condition*

$$\mathbf{Y}''_n \xrightarrow{\mathcal{D}} \mathbf{N}''. \tag{A.0.3}$$

Example A.0.2. If $\mathbf{X}_n = \sum_{j=1}^n \xi_j$, where the ξ_js are independent and identically distributed with finite mean and covariance matrix, and with common characteristic function ψ, then

$$\varphi_{\mathbf{X}_n}(\mathbf{v}) = \psi(\mathbf{v})^n,$$

and Theorem A.0.1 becomes the Central Limit Theorem in d dimensions.

In the proof of Theorem A.0.1, we will use some elementary properties of characteristic functions. First, the Cramér-Wold device [11] allows us to place ourselves in a 1-dimensional context.

Proposition A.0.3. *Let* $\mathbf{X}$ *be a random variable taking values in* $\mathbb{R}^d$. *Then*

$$\mathbf{X}_n \xrightarrow{\mathcal{D}} \mathbf{X} \text{ as } n \to \infty$$

if and only if

$$\mathbf{v} \cdot \mathbf{X}_n \xrightarrow{\mathcal{D}} \mathbf{v} \cdot \mathbf{X}$$

as $n \to \infty$ *for all* $\mathbf{v} \in \mathbb{R}^d$.

Next, we can establish the convergence of $\mathbf{v} \cdot \mathbf{X}_n$ using the following proposition - see, for example,Theorem 5.9.1 in [31].

Proposition A.0.4. *Let* $X, X_1, X_2, \ldots$ *be random variables taking values in* $\mathbb{R}$. *Then* X_n *converges in distribution to* X *as* $n \to \infty$ *if and only if for each* $t \in \mathbb{R}$,

$$\varphi_{X_n}(t) \to \varphi_X(t)$$

as $n \to \infty$.

Using the results about convergence of logarithms in Appendix B, we can replace the condition in Proposition A.0.4 with

$$\log \varphi_{X_n}(t) \to \log \varphi_X(t).$$

Finally, we note some properties of the log-characteristic functions $\varphi_{\mathbf{X}_n}$.

Remark A.0.5. For any random variable $\mathbf{X} = (X_1, \ldots, X_d)$ with characteristic function φ, we have

$$\begin{aligned}
\log \varphi(\mathbf{0}) &= 0 \\
\frac{\partial}{\partial v_j} \log \varphi(\mathbf{v}) \Big|_{\mathbf{v}=\mathbf{0}} &= i\mathbb{E}[X_j] && \forall 1 \le j \le d \\
\frac{\partial^2}{\partial v_j \partial v_k} \log \varphi(\mathbf{v}) \Big|_{\mathbf{v}=\mathbf{0}} &= -\mathrm{Cov}(X_j, X_k) && \forall 1 \le j, k \le d.
\end{aligned}$$

If $\mathbf{N}$ has distribution $\mathcal{N}_d(\mu, \Sigma)$, then

$$\varphi_{\mathbf{N}}(\mathbf{v}) = \exp\left\{ i\mathbf{v}\mu - \frac{1}{2}\mathbf{v}^\intercal \Sigma \mathbf{v} \right\}.$$

Proof of Theorem A.0.1. We begin with the convergence of $\mathbf{Y}'_n$. By Proposition A.0.3 it is sufficient to show that

$$\mathbf{v} \cdot \mathbf{Y}'_n \xrightarrow{\mathcal{D}} \mathbf{v} \cdot \mathbf{N}'$$

for any $\mathbf{v} \in \mathbb{R}^d$. Using Proposition A.0.4, we can reframe this in terms of the convergence of the functions

$$t \mapsto \log \varphi_{\mathbf{Y}'_n}(t\mathbf{v})$$

to $\log \varphi_{\mathbf{N}}(t\mathbf{v})$.

We have

$$\log \varphi_{\mathbf{Y}'_n}(t\mathbf{v}) = \log \varphi_{\mathbf{X}_n}\left(\frac{t}{\sqrt{n}}\mathbf{v}\right) - i\frac{t}{\sqrt{n}}\mathbf{v} \cdot n\nabla f(\mathbf{u}_n),$$

so that

$$\begin{aligned}
&\left|\log \varphi_{\mathbf{Y}'_n}(t\mathbf{v}) + \frac{1}{2}t^2\mathbf{v}^\intercal \mathbf{H}_f(\mathbf{u})\mathbf{v}\right| \\
&\leq \left|\log \varphi_{\mathbf{X}_n}\left(\frac{t}{\sqrt{n}}\mathbf{v}\right) - n\left(f\left(\mathbf{u}_n + i\frac{t}{\sqrt{n}}\mathbf{v}\right) - f(\mathbf{u}_n)\right)\right| \\
&\quad + n\left|f\left(\mathbf{u}_n + i\frac{t}{\sqrt{n}}\mathbf{v}\right) - f(\mathbf{u}_n) - i\frac{t}{\sqrt{n}}\mathbf{v}\cdot\nabla f(\mathbf{u_n}) - \frac{1}{2}\left(i\frac{t}{\sqrt{n}}\right)^2\mathbf{v}^\intercal \mathbf{H}_f(\mathbf{u}_n)\mathbf{v}\right| \qquad \text{(A.0.4)} \\
&\quad + \frac{1}{2}t^2\left|\mathbf{v}^\intercal \mathbf{H}_f(\mathbf{u}_n)\mathbf{v} - \mathbf{v}^\intercal \mathbf{H}_f(\mathbf{u})\mathbf{v}\right|.
\end{aligned}$$

By assumption, the first term converges to zero for every t and $\mathbf{v}$, as $n \to \infty$. Meanwhile, the second term is the remainder in the second-order Taylor expansion of the function

$$g : s \mapsto nf\left(\mathbf{u}_n + is\mathbf{v}\right),$$

evaluated at $s = \frac{t}{\sqrt{n}}$. By Proposition B.1.2, we have

$$\left| g\left(\frac{t}{\sqrt{n}}\right) - g(0) - \frac{t}{\sqrt{n}} g'(0) - \frac{t^2}{2n} g''(0) \right| \leq \left| \left(\frac{t}{\sqrt{n}}\right)^3 \int_0^1 \frac{(1-s)^2}{2} g^{(3)}(st) dt \right|.$$

For any $\delta > 0$, there exists n large enough such that $\mathbf{u}_n + i\frac{t}{\sqrt{n}}\mathbf{v} \in B(\mathbf{u}, \delta)$; we can choose δ such that this ball is contained in $\mathcal{Z}$, where f is analytic and so has bounded derivatives. As a result, there exists c such that, when n is large enough,

$$\left| \int_0^1 \frac{(1-s)^2}{2} g^{(3)}(st) dt \right| \leq cn,$$

so that the second term in Equation (A.0.4) is bounded above by

$$\left| \frac{t}{\sqrt{n}} \right|^3 \times cn$$

and so converges to zero as $n \to \infty$.

Finally, since f is analytic on $\mathcal{Z}$, whenever n is large enough that $\mathbf{u}_n + i\frac{t}{\sqrt{n}}\mathbf{v} \in \mathcal{Z}$ we can bound the third term in Equation (A.0.4) from above by

$$c\|t\mathbf{v}\|^2 \|\mathbf{u}_n - \mathbf{u}\|,$$

where c is a constant depending on the second partial derivatives of f. For any t and $\mathbf{v}$ fixed, we can make this term arbitrarily small through our choice of n, so that

$$\left| \log \varphi_{\mathbf{Y}'_n}(t\mathbf{v}) + \frac{1}{2} t^2 \mathbf{v}^\intercal \mathbf{H}_f(\mathbf{u}) \mathbf{v} \right| \to 0$$

as $n \to \infty$, and hence $\mathbf{Y}'_n \xrightarrow{\mathcal{D}} \mathbf{N}'$.

Now, to show that $\mathbf{Y}_n$ and $\mathbf{Y}''_n$ converge in distribution, we use Slutsky's theorem (Theorem 1.1.4); it is sufficient to show that, for every $\varepsilon > 0$,

$$\mathbb{P}\left(\|\mathbf{Y}_n - \mathbf{Y}'_n\| > \varepsilon\right) \to 0$$

$$\mathbb{P}\left(\|\mathbf{Y}''_n - \mathbf{Y}'_n\| > \varepsilon\right) \to 0$$

as $n \to \infty$, or equivalently that

$$\frac{1}{\sqrt{n}} \|\mu_n - n\nabla f(\mathbf{u}_n)\| \to 0$$

$$\frac{1}{\sqrt{n}}\left\|n\nabla f(\mathbf{u}) - n\nabla f(\mathbf{u}_n)\right\| \to 0.$$

The second assertion follows from the analyticity of f on $\mathcal{Z}$ and our assumption that $\sqrt{n}\|\mathbf{u}_n - \mathbf{u}\| \to 0$ as $n \to \infty$.

Finally, we note that the components of the vector

$$\mu_n - n\nabla f(\mathbf{u}_n)$$

are the partial derivatives of

$$\varphi_{\mathbf{X}_n}(\mathbf{v}) - nf\left(\mathbf{u_n} + i\mathbf{v}\right);$$

using Cauchy's estimate in Proposition B.1.1, and the same choice of δ as in the earlier part of the proof, there exist constants c_j, $1 \leq j \leq d$, such that

$$|(\mu_n - n\nabla f(\mathbf{u}_n))_j| \leq \frac{c_j}{\delta} \sup_{\mathbf{v}:\mathbf{u}_n + i\mathbf{v} \in B(\mathbf{u},\delta)} \left|\log \varphi_{\mathbf{X}_n}(\mathbf{v}) - n\Big(f(\mathbf{u_n} + i\mathbf{v}) - f(\mathbf{u}_n)\Big)\right|.$$

As a result, both $\mathbf{Y}_n - \mathbf{Y}'_n$ and $\mathbf{Y}''_n - \mathbf{Y}'_n$ converge in probability to 0, and we have convergence in distribution of all three variables. □

Appendix B

Appendix

In this appendix, we collect several results from probability theory and functional analysis. The first part contains general results, which we apply in the second part to find estimates relating to the functions m and $\log \mu$.

B.1 General Results

In this section we present some useful results from probability and functional analysis.

We begin with the Taylor expansion, and Cauchy's integral formula, for holomorphic functions on $\mathbb{C}$. They are stated here as in Theorems 2.13, 2.15, and 2.33 in [32].

Proposition B.1.1. *Let $\Omega \subset \mathbb{C}$ be an open set, and let $a \in \Omega$ and $R > 0$ be such that $B(a, R) \subseteq \Omega$. Suppose that f is a holomorphic function on Ω. Then*

- *The sum*

$$f(z) = \sum_{n=1}^{\infty} a_n (z-a)^n$$

converges uniformly on all open subsets of $B(a, R)$, where the coefficients a_n are given by

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

- *For* $\gamma(t) = a + re^{it}$*, where* $r < R$ *and* $t \in [0, 2\pi]$*, we have*

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz,$$

for all $n \in \mathbb{N}$*. In particular, if*

$$\sup_{z \in \Omega} |f(z)| \leq c,$$

we have

$$\left| f^{(n)}(a) \right| \leq n! \frac{c}{R^n}.$$

We will use the Taylor expansion with the following specific form of the remainder, first proved by Darboux in 1876 [12]. It corresponds to the case $\phi(t) = \frac{(1-t)^2}{2}$ in the original paper.

Proposition B.1.2. *In the setting of Proposition B.1.1, we have*

$$f(a+h) - f(a) = hf'(a) + \frac{h^2}{2} f''(a) + h^3 \int_0^1 \frac{(1-t)^2}{2} f^{(3)}(a+ht) dt,$$

whenever $a + h$ *is in the interior of* $B(a, R)$*.*

Proposition B.1.3. *Let* $f : \mathbb{C}^k \to \mathbb{C}$ *be an analytic function defined over* $\mathbb{C}^k$*, for some* $k \in \mathbb{N}$*. Let* $\mathcal{Z}^k \subset \mathbb{C}^k$ *be a neighbourhood, such that* $\mathcal{U}^k = \mathcal{Z}^k \cap \mathbb{R}^k$ *is non-empty.*

Suppose that there exist constants $c_0, c_j, c_{i,j}$ *such that*

$$\sup_{\mathbf{z} \in \mathcal{Z}^k} |f(\mathbf{z})| < c_0$$

$$\sup_{\mathbf{z} \in \mathcal{Z}^k} \left| \frac{\partial}{\partial z_j} f(\mathbf{z}) \right| < c_j \qquad \text{for each } 1 \leq j \leq k$$

$$\sup_{\mathbf{z} \in \mathcal{Z}^k} \left| \frac{\partial^2}{\partial z_i \partial z_j} f(\mathbf{z}) \right| < c_{i,j} \qquad \text{for each } 1 \leq i, j \leq k,$$

and that

$$\inf_{\mathbf{u} \in \mathcal{U}^k} f(\mathbf{u}) > 0.$$

Then there exist finite constants $C, C' > 0$ *such that, for all* $\mathbf{z} = \mathbf{x} + i\mathbf{y} \in \mathcal{Z}^k$,

$$|f(\mathbf{z}) - f(\mathbf{x})| \leq C\|\mathbf{y}\| \tag{B.1.1}$$

$$||f(\mathbf{z})| - f(\mathbf{x})| \leq C'\|\mathbf{y}\|^2 \tag{B.1.2}$$

Proposition B.1.4. *Let* f *be an analytic function defined on a connected neighbourhood* $\mathcal{Z} \subset \mathbb{C}$*, such that* $\mathcal{U} = \mathcal{Z} \cap \mathbb{R}$ *is non-empty. Then there exists a constant* C'' *such that*

$$\left| f(x_1) + f(x_2) - 2f\left(\frac{x_1 + x_2}{2}\right) \right| \leq C''(x_1 - x_2)^2 \tag{B.1.3}$$

and

$$\left| \int_{x_1}^{x_2} f(t)dt - (x_1 - x_2) f\left(\frac{x_1 + x_2}{2}\right) \right| \leq C''(x_1 - x_2)^3 \tag{B.1.4}$$

hold for all $x_1, x_2 \in \mathcal{U}$.

Proposition B.1.5 (Jensen's Inequality)**.** *Let* f *be a convex function on* $\mathbb{R}$*. Then for any* $k \geq 1$*, and any* $x_1, \ldots, x_k \in \mathbb{R}$ *and* $a_1, \ldots, a_k > 0$ *such that* $\sum_{j=1}^k a_j = 1$,

$$f\left(\sum_{j=1}^k a_j x_j\right) \leq \sum_{j=1}^k a_j f(x_j).$$

Proposition B.1.6. *For any compact subset* $\mathcal{Z}$ *of* $\mathbb{C} \setminus \{0\}$*, there exist constants* c_1 *and* c_2 *such that*

$$|\log(1 - z)| \leq c_1|z|$$

$$|e^z - 1| \leq c_2|z|.$$

Proposition B.1.7. *Let* $\{z_j\}$ *be a sequence of* n *complex numbers, and* $c > 0$ *such that, for each* $j \leq n$*, we have*

$$|z_j - 1| \leq c.$$

Then for any $K \leq n$,

$$\left| \prod_{j=1}^{K} z_j - 1 \right| \leq (1+c)^K - 1 \tag{B.1.5}$$
$$\leq e^{cK} - 1.$$

Proof. This can be neatly seen using a proof by induction. For $K = 1$ it is clear that

$$|z_1 - 1| \leq c = (1+c)^1 - 1.$$

Now, supposing that Equation (B.1.5) holds for $K = k-1$, we note that $|z_k| \leq 1+c$, so that we have

$$\begin{aligned} \left| \prod_{j=1}^{k} z_j - 1 \right| &\leq |z_k| \left| \prod_{j=1}^{k-1} z_j - 1 \right| + |z_k - 1| \\ &\leq (1+c)\left((1+c)^{k-1} - 1\right) + c \\ &\leq (1+c)^k - 1. \end{aligned}$$

Finally, since $c \geq 0$, we have $(1+c)^K - 1 \leq e^{cK} - 1$ for all $k \leq n$. □

B.2 Specific Estimates

By applying the upper bounds in Equations (B.1.3) and (B.1.4) to the functions

$$\begin{aligned} f &: t \mapsto m(z_1 + z_0(1-t)) = m\left(z\left(t\right)\right) \\ g &: t \mapsto \log \mu(z_1 + z_0(1-t)), \end{aligned}$$

we find the following upper bounds.

Proposition B.2.1. *There exist constants c, c', and c'' such that, for any segment $[a,b] \subset [0,n]$, and $(z_1, z_0) \in \mathcal{Z}^\Delta$,*

$$\begin{aligned} \left| \int_a^b m\left(z\left(\frac{x}{n}\right)\right) - m\left(z\left(\frac{a+b}{2n}\right)\right) dx \right| &\leq c|z_0|^2 \frac{(b-a)^3}{n^2} \\ &\leq c'|z_0|^2 \frac{(b-a)^2}{n}. \end{aligned}$$

Moreover, if $|z_0|^2 \frac{(b-a)^2}{n}$ *is bounded, then*

$$\left| \exp\left\{ -\int_a^b m\left(z\left(\frac{x}{n}\right)\right) - m\left(z\left(\frac{a+b}{2n}\right)\right) dx \right\} \right| \leq e^{c'|z_0|^2 \frac{(b-a)^2}{2n}}$$

and

$$\left| \exp\left\{ -\int_a^b m\left(z\left(\frac{x}{n}\right)\right) - m\left(z\left(\frac{a+b}{2n}\right)\right) dx \right\} - 1 \right| \leq c''|z_0|^2 \frac{(b-a)^3}{n^2}.$$

Proof. The first upper bound is a direct application of Proposition B.1.3, with an additional factor of $\frac{|z_0|^2}{n^2}$ arising from (for example) the substitution $u = \frac{z_0}{n}t$ inside the integral. The remaining upper bounds come from applications of Proposition B.1.6. □

Proposition B.2.2. *There exist positive constants* C, C'*, and* C'' *such that, for any segment* $[a, b] \subseteq [0, n]$*, and* $(z_1, z_0) \in \mathcal{Z}^\Delta$*,*

$$\left| \frac{1}{2} \log \mu\left(z\left(\frac{a}{n}\right)\right) \mu\left(z\left(\frac{b}{n}\right)\right) - \log \mu\left(z\left(\frac{a+b}{2n}\right)\right) \right| \leq C'|z_0|^2 \frac{(b-a)^2}{n}.$$

Moreover,

$$\left| \frac{\sqrt{\mu\left(z\left(\frac{a}{n}\right)\right)\mu\left(z\left(\frac{b}{n}\right)\right)}}{\mu\left(z\left(\frac{a+b}{2n}\right)\right)} \right| \leq e^{C'|z_0|^2 \frac{(b-a)^2}{n}},$$

and

$$\left| \frac{\sqrt{\mu\left(z\left(\frac{a}{n}\right)\right)\mu\left(z\left(\frac{b}{n}\right)\right)}}{\mu\left(z\left(\frac{a+b}{2n}\right)\right)} - 1 \right| \leq C''|z_0|^2 \frac{(b-a)^3}{n^2}.$$

Using Proposition B.1.7, we can combine these results to obtain the following Corollary.

Corollary B.2.3. *There exists a constant* C *such that, for any segment* $[a, b] \subset [0, n]$ *and* $(z_1, z_0) \in \mathcal{Z}^\Delta$*,*

$$\left| \exp\left\{ -\int_a^b m\left(z\left(\frac{x}{n}\right)\right) - m\left(z\left(\frac{a+b}{2n}\right)\right) dx \right\} \frac{\sqrt{\mu\left(z\left(\frac{a}{n}\right)\right)\mu\left(z\left(\frac{b}{n}\right)\right)}}{\mu\left(z\left(\frac{a+b}{2n}\right)\right)} - 1 \right| \leq C|z_0|^2 \frac{(b-a)^3}{n^2} \tag{B.2.1}$$

and

$$\left|\exp\left\{-\int_a^b m\left(z\left(\frac{x}{n}\right)\right)-m\left(z\left(\frac{a+b}{2n}\right)\right)dx\right\}\frac{\mu\left(z\left(\frac{a+b}{2n}\right)\right)}{\sqrt{\mu\left(z\left(\frac{a}{n}\right)\right)\mu\left(z\left(\frac{b}{n}\right)\right)}}-1\right|\leq C|z_0|^2\frac{(b-a)^3}{n^2}. \tag{B.2.2}$$

Corollary B.2.4. *There exist constants C, c, c' such that, for any segment $[a,b]\subset[0,n]$ and $(z_1,z_0)\in\mathcal{Z}^\Delta$,*

$$\left|\frac{\exp\left\{-\int_a^b m\left(z\left(\frac{x}{n}\right)\right)dx\right\}}{\sqrt{\mu\left(z\left(\frac{a}{n}\right)\right)\mu\left(z\left(\frac{b}{n}\right)\right)}}\right|\leq\left(1+C|z_0|^2\frac{(b-a)^3}{n^2}\right)\left|\frac{\exp\left\{-(b-a)m\left(z\left(\frac{b+a}{2n}\right)\right)\right\}}{\mu\left(z\left(\frac{a+b}{2n}\right)\right)}\right|$$

and

$$\left|\frac{\exp\left\{-(b-a)m\left(z\left(\frac{b+a}{2n}\right)\right)\right\}}{\mu\left(z\left(\frac{a+b}{2n}\right)\right)}\right|\leq e^{(c(b-a)+c')\left|v\left(\frac{a+b}{2n}\right)\right|^2}\frac{\exp\left\{-(b-a)m\left(u\left(\frac{b+a}{2n}\right)\right)\right\}}{\mu\left(u\left(\frac{a+b}{2n}\right)\right)}.$$

In particular, if $(b-a)\left|v\left(\frac{a+b}{2n}\right)\right|^2$ is bounded,

$$\left|\frac{\exp\left\{-\int_a^b m\left(z\left(\frac{x}{n}\right)\right)dx\right\}}{\sqrt{\mu\left(z\left(\frac{a}{n}\right)\right)\mu\left(z\left(\frac{b}{n}\right)\right)}}\right|\leq C\left(1+C|z_0|^2\frac{(b-a)^3}{n^2}\right)\frac{\exp\left\{-(b-a)m\left(u\left(\frac{b+a}{2n}\right)\right)\right\}}{\mu\left(u\left(\frac{a+b}{2n}\right)\right)}.$$

We also use the following upper bounds on the differences

$$m\left(z_1+z_0(1-t)\right)-m\left(z_2+z_0(1-t)\right)=m\left(z_1\left(t\right)\right)-m\left(z_2\left(t\right)\right)$$

$$\log\mu\left(z_1+z_0(1-t)\right)-\log\mu\left(z_2+z_0(1-t)\right)=\log\mu\left(z_1\left(t\right)\right)-\log\mu\left(z_2\left(t\right)\right).$$

Proposition B.2.5. *There exist constants c, C and c', C' such that, for any segment $[a,b]\subset[0,n]$ and $(z_1,z_0),(z_2,z_0)\in\mathcal{Z}^\Delta$,*

$$\left|\int_a^b m\left(z_1\left(\frac{x}{n}\right)\right)-m\left(z_2\left(\frac{x}{n}\right)\right)dx\right|\leq c|z_1-z_2|(b-a)$$

$$\frac{1}{2}\left|\log\mu\left(z_1\left(\frac{a}{n}\right)\right)-\log\mu\left(z_2\left(\frac{a}{n}\right)\right)\right|\leq C|z_1-z_2|.$$

Moreover, if $|z_1-z_2|(b-a)$ is bounded, then

$$\left|\exp\left\{-\int_a^b m\left(z_1\left(\frac{x}{n}\right)\right)-m\left(z_2\left(\frac{x}{n}\right)\right)dx\right\}\right|\leq e^{c|z_1-z_2|(b-a)} \tag{B.2.3}$$

$$\left|\sqrt{\frac{\mu\left(z_2\left(\frac{a}{n}\right)\right)}{\mu\left(z_1\left(\frac{a}{n}\right)\right)}}\right| \leq e^{C|z_1-z_2|}. \tag{B.2.4}$$

Proof. For the first pair of upper bounds, the analyticity of

$$y \mapsto m\left(y + z_0(1-t)\right)$$

(respectively $\log \mu\left(y + z_0(1-t)\right)$) allows us to find an upper bound on its first derivative; the remaining bounds are, as in the previous Proposition, consequences of Proposition B.1.6. □

Corollary B.2.6. *There exist constants c and C such that, for any segment $[a,b] \subset [0,n]$ and $(z_1,z_0),(z_2,z_0) \in \mathcal{Z}^{\Delta}$,*

$$\left|\exp\left\{-\int_a^b m\left(z_1\left(\frac{x}{n}\right)\right) - m\left(z_2\left(\frac{x}{n}\right)\right)dx\right\}\sqrt{\frac{\mu\left(z_1\left(\frac{b}{n}\right)\right)}{\mu\left(z_2\left(\frac{b}{n}\right)\right)}} - 1\right|$$
$$\leq (c(b-a)+C)|z_1-z_2|e^{(c(b-a)+C)|z_1-z_2|}.$$

Proposition B.2.7. *For any $k, d \in \mathbb{N}$, the function*

$$\frac{\sin\left(x\frac{kd}{2}\right)}{\sin\left(x\frac{d}{2}\right)}$$

is $\frac{2\pi}{d}$-periodic in x and we have

$$\left|\frac{\sin\left(x\frac{kd}{2}\right)}{\sin\left(x\frac{d}{2}\right)}\right| \leq \begin{cases} k\left(1-\left(\frac{xdk}{2\pi}\right)^2\right) & 0 \leq |x| \leq \frac{2\pi}{dk} \\ \frac{2}{3}k & \frac{2\pi}{dk} \leq |x| \leq \frac{\pi}{d}. \end{cases}$$

Meanwhile, if $|x| < \pi$, we have

$$\cos(x) \leq 1 - \frac{2}{\pi^2}x^2$$

Proof. First, for $0 \leq y \leq \pi$, we have

$$y - \frac{y^3}{6} \leq \sin(y) \leq y,$$

so that

$$\left|\frac{\sin(y)}{\sin\left(\frac{y}{k}\right)}\right| \leq k\left(1 - \frac{y^2}{\pi^2}\right);$$

setting $y = x\frac{kd}{2}$ gives the first part of the inequality.

Next, if $\frac{2\pi}{dk} < |x| < \frac{\pi}{d}$, then

$$\left|\frac{\sin\left(x\frac{kd}{2}\right)}{\sin\left(x\frac{d}{2}\right)}\right| \leq \left|\frac{1}{\sin\left(\frac{\pi}{k}\right)}\right| \leq \frac{k}{\pi} < \frac{1}{3}k. \qquad \square$$

Finally, we note that

$$\cos(x) - 1 + \frac{2}{\pi^2}x^2$$

has roots at 0 and at $\pm\pi$, and is negative between $-\pi$ and π, from which we obtain the final upper bound.

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