Renormalisation Group Flows in Lifshitz Holography

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Abstract

In this thesis we construct holographic duals of renormalisation group flows between field theories with conformal symmetries and the Lifshitz scaling symmetries. These take the form of spacetimes with a region asymptoting to AdS and another asymptoting to the Lifshitz metric of [1], with some domain wall smoothly interpolating between these regions. We first review the AdS/CFT correspondence in the context of Lorentz invariant boundary field theories, and then show how the holographic dictionary is modified by replacing the boundary field theory with one having the Lifshitz scaling symmetry.

We then consider a pair of actions capable of supporting both Lifshitz and AdS spacetimes. The first of these is a massive vector field coupled to gravity and the second is the 6 dimensional Romans $\mathcal{N} = 4$ massive gauged supergravity which supports 4D Lifshitz solutions. In each case we review the exact solutions that have been found previously, and then solve the linearised equations of motion around these solutions. These enable us to conjecture the existence of a variety of holographic RG flows. We then use numerical integration to confirm the existence of examples of each of these flows.

In both theories we find Lifshitz to Lifshitz, AdS to Lifshitz, and Lifshitz to AdS flows. In the supergravity we also find AdS to AdS flows, and a Lifshitz to AdS flow which has an intermediate AdS region with a different dilaton value. In addition the supergravity has flows from a non-compact 6D AdS space to each of the 4D compactifications.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

Chapter 1, 2 and 3 review known results and provide background for the rest of the thesis. The linearisations and holographic RG flows of chapters 4 and 5 are original work, done in collaboration with Simon Ross and Ruth Gregory.

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Chapter 1

Introduction

Since the introduction of the first explicit example of a holographic duality [2] in 1997, the AdS/CFT correspondence has been used to study a variety of conformal field theories in the limit of strong coupling, which is inaccessible to traditional techniques of quantum field theories such as perturbative expansions. This work has produced an example of confinement/deconfinement phase transitions [3] and predictions of viscosity [4] in strongly coupled Yang-Mills plasmas. More recent interest has focussed on reproducing phenomena from condensed matter, such as superconductivity [5] and the conductivity of strange metals [6]. These applications have provoked interest in symmetry groups other than relativistic conformal symmetry, such as the Schroedinger group [7] and the Lifshitz scaling symmetry [1].

The purpose of this thesis is to construct spacetimes that are holographically dual to renormalisation group flows in field theories between fixed points with relativistic conformal symmetry and the non-relativistic Lifshitz symmetry. Explicit constructions of such spacetimes could be used, for instance, as backgrounds on which to solve classical probe field equations, from which information about correlation functions in the dual field theory at strong coupling could then be extracted using the gauge/gravity correspondence. Correlation functions calculated in such a manner would be those appropriate to the field theory deformed from its UV limit by some relevant operator. We will find a wide variety of such flows, with all possible combinations of AdS/Lifshitz scalings at each end of the flow. The emergence conformal symmetry in the IR is common in condensed matter systems, for instance in graphene [8].

In section 2 we will briefly review some of the justification for the gauge gravity correspondence. We will describe the calculations that it allows us to perform, and demonstrate a simple example of these using a massive bulk scalar field in an asymptotically AdS spacetime. We will first find the near-boundary expansion of the solution, and then show how the dimension of the dual operator is encoded in this solution. We will show that the expectation value of the operator can be read off as one of the coefficients in this expansion, but that this coefficient is not determined solely by the near-boundary expansion.

The calculations of section 2 take place in an asymptotically AdS space and hence can only be holographically dual to theories with relativistic conformal symmetries in the UV. In section 3 we shall describe spacetimes dual to field theories with a different scaling symmetry in the UV, the Lifshitz symmetry. This is an anisotropic scaling symmetry, with no boost symmetry. The anisotropy is parametrised by the dynamical exponent, z, and reduces to the relativistic case at $z = 1$. We will describe how the treatment of the asymptotics of such spaces must differ from that of an asymptotically AdS space, due to the lack of a conformal boundary, and show how the asymptotics of such spaces can be treated using a conformal frame. We shall repeat the calculation of section 2 to find the near-boundary solution for a bulk scalar in these asymptotics. We will again use the holographic dictionary to extract the scaling dimension and expectation value of the dual operator from this solution. The solutions will differ from the AdS/CFT case, but they will match in the limit $z \to 1$.

In section 4 we will discuss renormalisation group flows in a phenomenological model for Lifshitz spacetimes. Our action will consist of Einstein gravity with a negative cosmological constant coupled to a vector field with a mass term in an arbitrary number, d, of spatial dimensions. We will review the equivalence of this model to the 2-form / $(d-1)$-form model, and discuss its relation to the more recent supergravity models supporting Lifshitz spacetimes. We note that for any negative value of the cosmological constant, Λ, this action supports an AdS spacetime. A single Lifshitz solution exists for $\Lambda/m_0^2 \leq -d/2$, and there exists a second Lifshitz solution with a different dynamical exponent when $-d/2 < \Lambda/m_0^2 \leq -(3d-4)/2(d-1)$, where m_0 is the mass of the vector field.

In section 4.4 we solve the linearised equations of motion around both the AdS and Lifshitz spacetimes for a simple ansatz preserving homogeneity and spatial isotropy on each radial slice. From the AdS/CFT results reviewed in section 2.2, these allow us to identify when the relativistic and Lifshitz duals possess relevant and irrelevant operators. The results of the linearisations suggest that there will exist holographic renormalisation group flows from AdS to Lifshitz for $\Lambda/m_0^2 < -d/2$, and both Lifshitz to Lifshitz and Lifshitz to AdS flows for $-d/2 < \Lambda/m_0^2 \leq -(3d-4)/2(d-1)$. In section 4.5 we use numerical integration to confirm the existence of examples of such flows in $d = 3$, 4 and 5.

In section 5 we consider a 6D supergravity model which supports solutions that are the product of a Lifshitz spacetime and a 2D hyperbolic metric. For the ansatz we consider, the equations of motion depends on a single parameter, $g^2\gamma^2$, the product of a gauge coupling and the flux along a compactified direction. In sections 5.2 and 5.3 we describe the AdS and Lifshitz solutions that this action supports. For $0 \leq g^2\gamma^2 \lesssim 0.227$ we find a single Lifshitz solution and two AdS solutions. For $0.227 \lesssim g^2\gamma^2 \lesssim 1.185$ there are two Lifshitz solutions and two AdS solutions, and for $1.185 \lesssim g^2\gamma^2$ there exist only the two Lifshitz solutions. For values of $g^2\gamma^2$ at which two AdS solutions exist, they are distinguished by having different curvature radii and dilaton values.

In section 5.4.1 we find analytically the linear perturbations about the AdS spacetimes. The linearisations about the Lifshitz solutions were found numerically, and the operator dimensions are given in section 5.4.2. These lead us to conjecture that there exist AdS to AdS, AdS to Lifshitz, Lifshitz to AdS and Lifshitz to Lifshitz flows. In section 5.5 we use numerical integration and a shooting technique to find examples of all these flows. We also find flows from one of the Lifshitz solutions in the UV to an AdS solution in the IR, which pass very close to the other AdS solution.

In addition to the above flows, in which the size of the compactified directions tends to a finite limit at both ends of the flows, we find a flow in which the size of the hyperbolic directions become large in the UV, and the metric in approaches an AdS_6 geometry.

Chapter 2

AdS/CFT background

The AdS/CFT correspondence is a large class of conjectured dualities between quantum field theories on a fixed background, and field theories coupled to dynamical gravity in a higher dimensional spacetime. The space on which the field theory lives (here denoted $\partial\mathcal{M}$) is identified as the boundary of the higher dimensional spacetime (denoted $\mathcal{M}$).

Originally the focus of research was on well-understood explicit examples of the correspondence and on tests of the duality. We will look below at the first example of the correspondence and briefly describe some of the tests of its validity. A comprehensive review of this early work is [9].

We will then show how the correspondence allows the calculation of some quantities in quantum field theories (at least in some limit) by solving classical equations of motion in the higher dimensional gravity theory. We will consider the simplest possible example illustrating this, a massive scalar in an asymptotically AdS spacetime. In section 2.1 we will derive the asymptotic expansion of such a field. In section 2.2 we will use the scaling behaviour of the field that we derive in 2.1 to find the dimension of the operator dual to this field. In section 2.3 we will show that the expectation value of this operator (and in fact all higher order correlation functions) follow from one of the coefficients in the expansion of section 2.1. We will find that a naive calculation of this expectation value gives a divergent result, but that renormalisation of the boundary field theory can also be implemented holographically.

While much of the early research in the gauge/gravity correspondence was aimed at applying the correspondence to high energy phenomena (eg. the quark-gluon plasma [4]) more recent work has investigated applications to strongly coupled condensed matter. This necessitates considering symmetry groups other than the conformal group. In chapter 3 we shall focus on a particular examples of this, the Lifshitz scaling symmetry. We will look at a spacetime with appropriate asymptotics to be a holographic dual to field theories with such symmetries, and repeat the calculations of this chapter to see how the results are changed. As there is a limit of the Lifshitz symmetry in which it reduces to the conformal scaling symmetry, the results of this chapter will also serve as a basic check of the calculations in chapter 3.

The first concrete example of a holographic duality was found in [2]. There the author considered N coincident D3 branes in type IIB string theory, which can be viewed as either end-points for open strings, or as sources in supergravity producing the background metric

$$ds^2 = \left(1 + \frac{4\pi g_s l_s^4 N}{r^4}\right)^{-\frac{1}{2}} \left(-dt^2 + \sum_{i=1}^{3} dx^{i2}\right) + \left(1 + \frac{4\pi g_s l_s^4 N}{r^4}\right)^{\frac{1}{2}} \left(dr^2 + r^2 d\Omega_5^2\right) \tag{2.1}$$

on which closed string propagate. The limit in which $l_s^2 \to 0$, $r \to 0$, taken such that $U = r/l_s^2$ is held fixed, reduces the metric to $AdS_5 \times S^5$

$$ds^2 = l_s^2 \left(\frac{U^2}{\sqrt{4\pi g_s N}} \left(-dt^2 + \sum_{i=1}^{3} dx^{i2}\right) + \sqrt{4\pi g_s N}\frac{dU^2}{U^2} + \sqrt{4\pi g_s N} d\Omega_5^2\right) \tag{2.2}$$

The same limit is known to decouple all the massive string modes, and the open string picture of the D3 branes reduces to $\mathcal{N} = 4$ $U(N)$ super Yang-Mills. Supergravity can only be trusted as a description of string theory at small curvatures, and here the curvature scalar scales like $(g_s N)^{-1}$. The Yang-Mills coupling constant is related to the string coupling by $g_s = g_{YM}^2$, so the duality can be trusted in the limit $g_{YM}^2 N \gg 1$, though [2] conjectures that it holds between the full type IIB string theory on $AdS_5 \times S^5$ and super Yang-Mills at finite $g_{YM}^2 N$. It is also observed in [2] that the radius in Planck units of both the S^5 and AdS_5 factors of the metric scale like $N^{1/4}$, so the $N \gg 1$ limit of the boundary field theory is the classical limit of the bulk supergravity theory. Similar arguments involving branes on different backgrounds have been used to argue for dualities between string theory and a large number of different field theories. A review of some of the other early examples can be found in [9].

The correspondence was made more precise in [10], [11] with the identification

$$Z_{\text{SUGRA}}\left(\phi_0\right) = \left\langle \exp\left(i \int_{\partial\mathcal{M}} \phi_0 \mathcal{O} \right) \right\rangle_{\text{boundary}} \tag{2.3}$$

In the strong coupling, large N limit of the quantum field theory, the partition function on the left hand side of (2.3) can computed using the saddle point approximation. That is, it should be evaluated from the action of the classical supergravity solution on $\mathcal{M}$ in which the fields (denoted schematically by ϕ - they need not be scalars) take the value[1] ϕ_0 on the boundary $\partial\mathcal{M}$. The expectation value on the right hand side would normally require computing the full path integral of the quantum field theory. In practice, for a strongly coupled field theory, computing even an approximation to such quantities is very difficult. However, in the classical limit the quantity on the left hand side of this *can* be computed in many cases, though often only numerically.

We may then take advantage of the fact that the right hand side of (2.3) is a generating function for the correlators of the operators $\{\mathcal{O}\}$, and that we may also calculate the variation of the left hand side with respect to the $\left\{\phi_{(0)}\right\}$. For a field theory with a holographic dual, we may therefore make use of

$$\left\langle \mathcal{O}_{(1)} \ldots \mathcal{O}_{(n)} \right\rangle = \frac{1}{Z_{\text{SUGRA}}\left(\phi_0\right)} \frac{\delta}{\delta\phi_{(1)0}} \cdots \frac{\delta}{\delta\phi_{(n)0}} Z_{\text{SUGRA}}\left(\phi_0\right) \tag{2.4}$$

at least in the field theory limit of strong coupling and large N, so we can evaluate the right hand side using the saddle-point approximation. We will see how to perform such a calculation for a scalar field in section 2.3.

In the original example of the correspondence of [2] the $U\left(1\right)$ factor of the boundary field theory can be shown not to be described by the bulk physics of the gravity theory [12], so we can use the bulk physics of IIB supergravity on $AdS_5 \times S^5$ to describe $\mathcal{N} = 4$ SU $\left(N\right)$ super-Yang-Mills. This case has been of particular interest due to its similarity to the strong force (SU (3) Yang-Mills without supersymmetry)

[1] The field ϕ will not typically tend to a finite limit at large r. What we mean by boundary value will be made more precise in 2.1.

and therefore its ability to model a system similar to the strongly-coupled quark-gluon plasma. It has been shown [3] to have a confinement-deconfinement phase transition when the field theory is placed on S^4 rather than $\mathbb{R}^4$, and [4] show that the ratio of shear viscosity to entropy density can be calculated for the fluid phase of the boundary theory.

$\mathcal{N} = 4$ SU (N) super Yang-Mills is also a theory about which it is possible to make some statements [9] even at strong coupling, which can be compared to results from type IIB supergravity. Operators in the boundary field theory of the form $\mathrm{Tr}\left(\phi^{(I_1} \ldots \phi^{I_n)}\right)$ for $n = 2, \ldots, N$, where ϕ is the scalar component of the vector super-multiplet, are known to be primary and chiral (they are constant under half of the covariant derivates). These should correspond to fields in type IIB supergravity compactified on $AdS_5 \times S^5$ if the correspondence is correct. It is shown in [9] that the spectrum of fields transforming under the same representation of the symmetry algebra matches that of the boundary field theory. A second test comes from a non-renormalisation theorem in $\mathcal{N} = 4$ SU (N) super Yang-Mills. The R-symmetry is anomalous when gauged, but the only contribution to the anomaly comes at the 1-loop level, and so the result will still hold away from weak coupling. Since the duality allows the calculation of 3-point functions using the supergravity theory, this can be computed in the strong coupling limit using the supergravity dual, and the results are found to match at leading order in N. Several other tests for this particular duality can be found in detail in [9], mostly taking advantage of operators in the field theory that are protected against renormalisation.

The most secure arguments for the duality have all come from considering D-branes and taking decoupling limits, restricting the gravity side of the field theory to a string theory compactification. The field theories have also possessed some degree of supersymmetry, and only in the large N limit of the field theories does the gravity side become classical. It may be the case that some of these are not necessary features, and it would be useful to have a clearer picture of which field theories possess a gravitational dual. Recently [13] conjectured that any conformal field theory with a large N expansion and a gap in the spectrum of anomalous dimensions of operators which grows with N has an AdS dual, and tested this to $O(N^{-2})$ by looking at 4-point correlators. In much recent work, particularly that focussing on condensed matter physics, the approach has been to consider a relatively simple set of fields in the bulk, and to try and produce a holographic dual to some interesting field theory behaviour (eg. superconductivity, see [5]) without worrying exactly what the dual field is, beyond that it has some scaling symmetry implemented as an isometry of the metric, and operators dual to the bulk fields that are considered. One review that takes this approach, with the aim of illustrating the behaviour found in superconductors, is [14].

A common feature of the earliest examples of the correspondences is that the field theory side of the duality possesses a relativistic conformal symmetry, at least at some energy scale. The spacetime may be only asymptotically AdS, corresponding to a conformal symmetry in the UV. For example, the interior of the spacetime could contain a black hole (corresponding to a thermal state of the dual theory [3]) or the interior could tend continuously to another AdS spacetime [15] (this case corresponds in the dual theory to a deformation by a relevant operator that drives a flow to another conformal fixed point in the IR.) Examples are also known where the field theory is conformal only in the IR. Such a duality was constructed in [16] using D2 branes.

However, not all field theories of physical interest have a conformal symmetry, even in some limit. We shall later look at examples from condensed matter physics that posses an anisotropic scaling symmetry, but not boost or special conformal symmetries.

Another common feature is that the above correspondences are between field theories in d dimensions and gravitational theories in $d+1$ dimensions. While this will be the case in all the examples we consider in this thesis, It should be emphasised that this property is not universal. In particular, holographic duals to theories with non-relativistic boost and scaling symmetries, which also have a conserved particle number, require $d+2$ dimensional gravity duals. The symmetry group of such theories is called the Galilean group, and more detail on duals to these can be found in [7], [17].

2.1 Asymptotic behaviour of fields in AdS space

We will first discuss the asymptotics of AdS spacetimes, and then review the simplest possible case that allows us to illustrate the holographic dictionary in such a background, namely a scalar field in the bulk coupled to Einstein gravity. We will follow the formalism of [18]. Very similar calculations with an emphasis on modelling condensed matter can be found in [14], which also covers the case of Einstein-Maxwell theory in the bulk. Since this system is not a supergravity, we cannot be confident that it has a field theory dual, let alone identify what theory the dual would actually be. The main results of this section will be that the radial fall-off of the field is determined by its mass, and that the near-boundary expansion is completely determined by recurrence relations once the coefficients of two terms, $r^{-\Delta_-}$ and $r^{-\Delta_+}$, (called respectively the slow and fast fall-off modes) are specified.

Our action is

$$S = \int d^{d+1}x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right) \tag{2.5}$$

and we will write our metric as

$$ds^2 = L^2\left(\frac{dr^2}{r^2}\right) + \gamma_{ab}dx^a dx^b \tag{2.6}$$

where $\gamma_{ab} = L^2r^2\eta_{ab} + O(1)$, and $L^2 = -d(d-1)/2\Lambda$. Asymptotically AdS metrics can always be put into such a form in some neighbourhood of the boundary by taking r^{-1} to be the affine parameter distance along geodesics eminating from the boundary [18]. We can see that, to leading order, the boundary directions are invariant under Poincaré transformations. The conformal scaling symmetry acts as

$$t \mapsto t' = \lambda t,\ x^i \mapsto x^{i\prime} = \lambda x^i,\ r \mapsto r' = \lambda^{-1}r \tag{2.7}$$

and again it can be checked that this is a symmetry of the leading order terms, so this metric has the correct symmetries to be dual to a CFT.

In what follows we will often wish to place the boundary, $\partial\mathcal{M}$ at some finite r. We will work with the induced metric on this boundary

$$\gamma_{ab} = \frac{\partial x^\mu}{\partial x^a}\frac{\partial x^\nu}{x^b}g_{\mu\nu} \tag{2.8}$$

With the metric in the coordinates of (2.6) this can be read off immediately. When we take the limit $r \to \infty$ the components of γ_{ab} diverge, so we will also define the conformal boundary metric

$$h_{ab} = \Omega^2 \left(x^a\right) \gamma_{ab} \tag{2.9}$$

such that h_{ab} tends to some finite, invertible metric as $r \to \infty$. For our current choice of surfaces $\Omega = r^{-1}$ will do. We will consistently work in coordinates such that the conformal boundary is at $r \to \infty$.

The metric (2.6) is a solution of the equations of motion derived from (2.5) for $\gamma_{ab} = L^2 r^2 \eta_{ab}$, $\phi = 0$. Suppose now that the scalar field is non-zero. The equation of motion for ϕ is

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi\right) - V' \left(\phi\right) = 0 \tag{2.10}$$

In some neighbourhood of the boundary, we rewrite $\phi\left(r, x^a\right)$ as

$$\phi\left(r, x^a\right) = r^{\Delta - d} \hat{\phi}\left(r, x^a\right) \quad \text{where} \quad \hat{\phi}\left(r, x^a\right) \to 1 \quad \text{as} \quad r \to \infty \tag{2.11}$$

A latin index denotes all coordinates except r. We will see in section 2.2 that labelling the leading power as $\Delta - d$ results in the scaling dimension of dual operator being Δ. Since ϕ satisfies a second order ODE, we expect to find two solutions for Δ. Only one of these will actually make $r^{\Delta - d}$ the leading term.

We make two further simplifications. We assume that ϕ has a sufficiently rapid fall-off as we approach the boundary that we can neglect its back reaction on the metric. We also take $V\left(\phi\right) = \frac{1}{2} m^2 \phi^2$. With these assumptions (2.10) can be rewritten as

$$\left(L^2 m^2 - \Delta\left(\Delta - d\right)\right) \hat{\phi} - r^{-2} \left(-\partial_t^2 + \partial_i^2\right) \hat{\phi} - r^2 \partial_r^2 \hat{\phi} - \left(2\Delta + 1 - d\right) r \partial_r \hat{\phi} = 0 \tag{2.12}$$

Taking the $r \to \infty$ limit all term except the first tend to zero, so Δ must satisfy

$$\Delta^2 - d\Delta - L^2 m^2 = 0, \text{ with solutions } \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + L^2 m^2} \tag{2.13}$$

Requiring that these be real imposes $m^2 L^2 \geq -d^2/4$, the Breitenlohner-Freedman stability bound [19]. Since we may neglect non-linear terms in ϕ at leading order, this expression for $\Delta_\pm$ will still hold in the case that $V\left(\phi\right)$ contains terms of higher order than ϕ^2.

As we expect for a second order differential equation, we've found two solutions. If $r^{\Delta-d}$ really is to be the leading term in ϕ, we need $\Delta = \Delta_+$. There are now two possibilities. If $\sqrt{d^2/4 + L^2m^2}$ is not an even integer, then the expansion should be written as

$$\phi\left(r, x^a\right) = r^{\Delta-d}\hat{\phi}\left(r, x^a\right) = r^{\Delta-d}\left(\sum_{n \text{ even}} \phi_{(n)}r^{-n} + \sum_{n \text{ even}} \phi_{(2\Delta-d+n)}r^{d-2\Delta-n}\right) \tag{2.14}$$

Now we can use (2.12) to derive a recurrence relation between these coefficients

$$\phi_{(n)}\left(x^a\right) = \frac{\left(-\partial_t^2 + \partial_i^2\right)\phi_{(n-2)}\left(x^a\right)}{n\left(2\Delta - d - n\right)} \quad \text{for } n \geq 2 \text{ and for } n \geq 2\Delta - d + 2 \tag{2.15}$$

Every term in the first series is determined in terms of $\phi_{(0)}$, and every term in the second by $\phi_{(2\Delta-d)}$. If $\sqrt{d^2/4 + L^2m^2}$ is an even integer, then (2.15) is not valid for $n = 2\Delta - d$. It turns out in this case to be necessary to modify the expansion to

$$\phi\left(r, x^a\right) = r^{\Delta-d}\hat{\phi}\left(r, x^a\right) = r^{\Delta-d}\left(\phi_{(0)} + \phi_{(2)}r^{-2} + \dots + \left(\phi_{(2\Delta-d)} + \psi_{(2\Delta-d)}\log r\right)r^{d-2\Delta} + \dots\right) \tag{2.16}$$

For $2 \leq n < 2\Delta - d$, plugging this expansion into (2.12) gives the same expression recurrence relation (2.15). [18] shows that

$$\psi_{(2\Delta-d)} = -\frac{1}{2^{2k-1}k!\left(k-1\right)!}\left(-\partial_t^2 + \partial_i^2\right)^k\phi_{(0)} \tag{2.17}$$

where $k = \Delta - d/2$. The $\phi_{(2\Delta-d)}$ term is still not fixed by any of the higher order terms, since the coefficient in front of $\phi_{(2\Delta-d)}$ in $-r^2\partial_r^2\hat{\phi} - \left(2\Delta + 1 - d\right)r\partial_r\hat{\phi}$ vanishes. It turns out that we do not need to know further terms in the expansion to do meaningful calculations.

In both cases the near-boundary analysis allows us to freely choose both $\phi_{(0)}$ and $\phi_{(2\Delta-d)}$. However, not all of these solutions will be acceptable. The condition that the field is regular as $r \to 0$ (or at the event horizon if there is a black hole in the bulk) will typically impose a relation between these two coefficients, and we will only need a single boundary condition as $r \to \infty$.

The above is still of use if ϕ is not small near the boundary. If we have some known exact solution Φ, including its backreaction on the metric, we may linearise (2.10) to find the asymptotic behaviour of linearised perturbations $\delta\phi$. (2.12) will be modified by $V''\left(\Phi\right)$ replacing m^2. More generally, we would find a similar relation to (2.13) for any component of a tensor field obeying a second order wave equation, though with L^2m^2 replaced by a term particular to that field. The property that $\Delta_+ + \Delta_- = d$ will continue to hold. Details of the asymptotic solution of the metric can be found in [18] and an example involving a linearised perturbation of a vector field, with some finite background field switched on, can be found in [14].

2.2 Operator dimensions and boundary conditions

In this section we will see that the scaling dimension of the operator, $\mathcal{O}$, dual to ϕ can be read off from the near-boundary expansion of ϕ as Δ. We see that this allows us to immediately identify whether $\mathcal{O}$ is relevant, irrelevant or marginal. We will give two arguments for when we may choose the faster fall-off term to be the boundary data, one based on the properties of a CFT, the other entirely based on the gravity side of the correspondence. Both of these show that we may take $\Delta = \Delta_-$ when $m^2L^2 < 1 - d^2/4$.

We solved the linearised wave equation for a scalar to find $\phi \sim \phi_{(0)} r^{\Delta - d} + \ldots$ as $r \to \infty$. Recall that from (2.3) we expect a coupling to exist between the 'boundary value' of this field, which we will take to mean $\phi_{(0)}$, and some operator in a conformal field theory,

$$\int_{\partial\mathcal{M}} d^dx\sqrt{-h}\phi_{(0)}\mathcal{O} \tag{2.18}$$

Recall that h_{ab} is the conformal metric on the boundary, which can be taken to be $h_{ab} = r^{-2}g_{ab}$ in our coordinates, and this is invariant under (2.7). Since ϕ is a scalar, under (2.7) it must transform as

$$\phi'\left(r', x'^a\right) = \phi\left(r, x^a\right) \tag{2.19}$$

and this preserves the form of the expansion only if

$$\phi'_{(0)}\left(x'^a\right) = \lambda^{\Delta - d}\phi_{(0)}\left(x^a\right) \tag{2.20}$$

For the operator to have dimension $[\mathcal{O}]$ we mean that

$$\mathcal{O}'\left(x'^a\right) = \lambda^{-[\mathcal{O}]}\mathcal{O}\left(x^a\right) \tag{2.21}$$

If the field theory is to be conformal then its action, and in particular the coupling (2.18), should be invariant under this transformation

$$\begin{aligned}\int_{\partial\mathcal{M}} d^d x' \sqrt{-h}\phi'_{(0)}(x')\,\mathcal{O}'(x') &= \int_{\partial\mathcal{M}} d^d x\sqrt{-h}\phi_{(0)}(x)\,\mathcal{O}(x)\\ &= \int_{\partial\mathcal{M}} \lambda^{-d} d^d x'\sqrt{-h}\lambda^{d-\Delta}\phi'_{(0)}(x')\,\lambda^{[\mathcal{O}]}\mathcal{O}'(x') \end{aligned} \tag{2.22}$$

allowing us to read off the operator dimension in terms of the scalar field asymptotics

$$[\mathcal{O}] = \Delta \tag{2.23}$$

In the case that $\sqrt{d^2/4 + L^2m^2}$ is an even integer, this get modified. In particular, under (2.7) $\mathcal{O}'$ now has a $\log\lambda$ contribution [18].

Under a conformal transformation (2.7), $\phi'(r, x^a) = \phi(\lambda r, \lambda^{-1}x^a)$, so such a transformation with $\lambda < 1$ will increase the length-scale of wave modes of this scalar field, and should be interpreted as a lowering of the energy scale. To determine whether the dual operator is relevant we need to know whether its coupling increases under such a transformation. From (2.20) we can read off that $\mathcal{O}$ is relevant, irrelevant or marginal if Δ is, respectively, less than, greater than or equal to d. This is what we would expect in any relativistic field theory. This will only tell us the leading order behaviour of the renormalisation group flow that this term drives - we will have to go beyond the linear level to find where the flow actually goes to.

We can also now find when $\Delta = \Delta_-$ is acceptable. Unitarity of a CFT requires [20] that no scalar operator has scaling dimension less than $\frac{1}{2}(d-2)$. This translates to $\Delta_- > \frac{1}{2}(d-2)$, which is satisfied for $m^2L^2 < 1 - d^2/4$.

We can also see this bound arising on the gravity side of the correspondence by considering the norm of the states in the Lorentzian theory [21], [22]. Setting $\Delta = \Delta_-$ is really saying that what we are going to take as boundary data, $\phi_{(0)}$, is the coefficient in front of $r^{-\Delta_+}$. We want to demand that the modes which are *not* fixed by the boundary conditions, and thus are varied when we vary the action, have finite norm.

The Klein-Gordon inner product between a pair of solutions, ϕ_1 and ϕ_2, to (2.10) can be defined as

$$-i\Omega(\phi_1, \phi_2) = -i\int_\Sigma d^d x\sqrt{g_\Sigma}\,(\pi_1\phi_2 - \pi_2\phi_1) \tag{2.24}$$

where Σ is some spacelike hypersurface with induced metric g_Σ, and the conjugate momenta are defined by

$$\pi = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \dot{\phi}} = N^\mu \partial_\mu \phi \tag{2.25}$$

where the dot denotes a derivative in a timelike direction, and N^μ is the unit lapse vector. Here it can be taken to be $N^t = r^{-1}$. Some motivation for this choice of inner product can be found in [23]. The Klein-Gordon norm of ϕ is then defined as

$$-i\Omega\left(\phi^*, \phi\right) \tag{2.26}$$

If we take Σ to be a $t =$ const surface, then the leading order contribution from the large r region is

$$-i \int^\infty dr\; r^{2\Delta-d-3} \int dx^1 \dots dx^{d-1} \left(\partial_t \phi^*_{(0)} \phi_{(0)} - \partial_t \phi_{(0)} \phi^*_{(0)}\right) \tag{2.27}$$

The $x^1, \dots, x^{d-1}$ integral should give a finite results if the field has some spread of wavenumbers. For the radial integral to give a finite result, we need

$$\Delta < 1 + \frac{d}{2} \tag{2.28}$$

This is always satisfied for Δ_-, and is true for Δ_+ provided that $m^2L^2 < 1 - d^2/4$. This then coincides with what we found above - for m^2L^2 above this bound, the norm of the $r^{\Delta_+ - d}$ mode is not finite, so we must take the coefficient of this to be the fixed boundary data.

There is a second condition we must satisfy. When the equations of motion are satisfied, $\delta\phi_{(0)} = 0$ must be a sufficient condition for $\delta S = 0$. Varying the scalar part of (2.5) gives

$$\delta S = \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} \delta\phi \left(\nabla_\mu \nabla^\mu \phi - V'\left(\phi\right)\right) - \int_{\partial\mathcal{M}} d^d x \sqrt{-\gamma} \delta\phi n^\mu \nabla_\mu \phi \tag{2.29}$$

where n^μ is the unit normal to $\partial\mathcal{M}$. The first term imposes the equation of motion (2.10), but $\delta\phi_{(0)} = 0$ only sets the second term to zero if this is the coefficient of the leading term in ϕ. When $\Delta = \Delta_-$, we can make $\delta\phi_{(0)} = 0$ a sufficient condition for $\delta S = 0$ by adding to the action the boundary term [14]

$$\int_{\partial\mathcal{M}} d^d x \sqrt{-\gamma} \phi n^\mu \nabla_\mu \phi \tag{2.30}$$

2.3 Expectation values and renormalisation

In this section we will see that the expectation value of the dual operator, $\mathcal{O}$, in the boundary field theory is encoded in the asymptotics of ϕ. It will turn out to be proportional to the coefficient of the r^{Δ} term. We will first attempt to calculate the expectation value from the action (2.5) using (2.4) and find that this is divergent. We will describe the minimal subtraction procedure of [18] for renormalising this action, and find the counter-terms in the case that $\sqrt{d^2/4 + L^2 m^2} \leq 1$. The general counter-terms for a scalar field can be found in [18] and those for a vector field in [24].

In the limit that we may use the saddle-point approximation in the bulk gravity theory, which we expect to correspond to a large N and strong coupling limit of the boundary theory, we can vary the identity (2.3) to find

$$\langle \mathcal{O} \rangle = -i \frac{\delta}{\delta \phi_{(0)}} \left\langle \exp \left(i \int_{\partial \mathcal{M}} \phi_{(0)} \mathcal{O} \right) \right\rangle_{\text{boundary}} = \frac{1}{\sqrt{-h}} \frac{\delta S_{\text{SUGRA}} \left(\phi_{(0)} \right)}{\delta \phi_{(0)}} \Bigg|_{\phi_{(0)}=0} \tag{2.31}$$

where the gravitational action should be evaluated using the classical solution with boundary data $\phi_{(0)}$.

We can do this calculation by first placing the boundary at finite r, and working in terms of fields rescaled by appropriate factors of r.

$$\langle \mathcal{O} \rangle = \lim_{r \to \infty} r^{\Delta} \frac{1}{\sqrt{-\gamma}} \frac{\delta S}{\delta \phi \left(r \right)} \tag{2.32}$$

The bulk piece of (2.29) vanishes on-shell, and the boundary piece gives

$$\langle \mathcal{O} \rangle = -r^{\Delta+1} \partial_r \phi \left(r, x^a \right) \sim \left(d - \Delta \right) \phi_{(0)} r^{2\Delta - d} \tag{2.33}$$

which is divergent for $\Delta = \Delta_+$, so we have not found a finite answer for $\langle \mathcal{O} \rangle$. This should not be hugely surprising. If we performed such a calculation in a field theory with a UV regulator, we would typically expect to have to renormalise the field theory before removing the cut-off. In our calculation the boundary acts as a UV regulator of the dual field theory, and we have failed to add any counter-terms.

Since we've found that the variation of the action diverges, then the action itself must be divergent. A minimal subtraction scheme is defined in [18] by determining the divergent contributions to the action, rewriting these in terms of local, bulk-covariant fields on the finite r boundary, and subtracting them from the original action. Integrating the scalar terms of (2.5) by parts, we get a piece that vanishes when the equations of motion hold, plus a boundary piece

$$-\frac{1}{2}\int_{\partial\mathcal{M}} d^d x\sqrt{-\gamma}\phi n^\mu \nabla_\mu \phi = -\frac{1}{2}\int_{\partial\mathcal{M}} d^d x\, r^{2\Delta-d}\left((\Delta-d)\,\hat{\phi}^2 + \hat{\phi} r\partial_r\hat{\phi}\right) \tag{2.34}$$

How many terms of this are actually divergent depends on the value of $\sqrt{d^2/4+L^2m^2}$. To illustrate the procedure we will look at the case $\sqrt{d^2/4+L^2m^2}\leq 1$. Here the boundary term can be rewritten as

$$-\frac{1}{2}\int_{\partial\mathcal{M}} d^d x\,(\Delta-d)\,r^{2\Delta-d}\phi_{(0)}^2 + O\left(1\right) \tag{2.35}$$

We need to rewrite $\phi_{(0)}$ in terms of bulk-covariant quantities. From (2.14) we see that $\phi_{(0)} = r^{d-\Delta}\left(\phi + O\left(r^{-1}\right)\right)$ so our counter-term is

$$S_{\text{ct}} = +\frac{1}{2}\int_{\partial\mathcal{M}} d^d x\sqrt{-\gamma}\,(\Delta-d)\left(\phi^2 + O\left(r^{-1}\right)\right) \tag{2.36}$$

We see that the subleading terms will not matter when we remove the regulator. If we had $\sqrt{d^2/4+L^2m^2} > 1$ we would have needed to subtract more terms, and to find $\phi_{(0)}$ to higher order.

Repeating the calculation with this counter term, we get

$$\langle\mathcal{O}\rangle = -r^{\Delta+1}\partial_r\phi + (\Delta-d)\,r^\Delta\phi = -r^{2\Delta-d}r\partial_r\hat{\phi} = (2\Delta-d)\,\phi_{(1)} + O\left(r^{-1}\right) \tag{2.37}$$

We can see that this coefficient is the only one which could be the expectation value by this by considering scaling dimensions, at least in the case that $\sqrt{d^2/4+L^2m^2}$ is not an even integer, and hence $\mathcal{O}$ transforms as (2.21). A general term in the expansion of ϕ transforms as

$$\phi'_{(n)}\left(x'^a\right) = \lambda^{\Delta-d+n}\phi_{(n)} \tag{2.38}$$

so only $\phi_{(2\Delta-d)}$ has the right scaling dimension to be the expectation value of $\mathcal{O}$. This behaviour, that the boundary date is a coupling and the coefficient not fixed by the near-boundary expansion gives an expectation value, is common to any field in the bulk, not just only scalars. In [18] and [25] it is shown that the exact relation is

$$\langle\mathcal{O}\rangle = (2\Delta-d)\,\phi_{(2\Delta-d)} \tag{2.39}$$

In the case that $\sqrt{d^2/4 + L^2m^2}$ is an even integer, there is an additional term in $\langle\mathcal{O}\rangle$ that depends directly through the near-boundary expansion on $\phi_{(0)}$. It can be removed by a change of counter-term action.

We stated earlier that a regularity condition in the interior would set $\phi_{(2\Delta-d)}$ as a function of $\phi_{(0)}$. To find the expectation value of $\mathcal{O}$ in this field theory in the absence of the coupling (2.18) we need only know $\phi_{(2\Delta-d)}\left(\phi_{(0)} = 0\right)$. To find the n-point function, we need to compute

$$\frac{\delta}{\delta\phi_{(0)}\left(x^a_n\right)}\cdots\frac{\delta}{\delta\phi_{(0)}\left(x^a_2\right)}\phi_{(2\Delta-d)}\left(x^a_1\right)\bigg|_{\phi_{(0)}=0} \tag{2.40}$$

That is, we need the n^{th} order dependence of $\phi_{(2\Delta-d)}$ on $\phi_{(0)}$. If we instead set $\phi_{(0)}$ to some non-zero value, we would be computing the correlator in the field theory deformed by the addition of the coupling (2.18).

2.4 The holographic stress tensor

The presence of a stress-energy tensor is a universal property of relativistic field theories, being sourced by the background metric. In particular, the coefficient of the fast-fall off term in the near-boundary expansion of g_{ab} will tell us the expectation value of the boundary stress-energy tensor, T_{ab}. The coefficient of the slow fall-off term will be a component of the background metric of the field theory. When we consider asymptotically Lifshitz spaces in section 3, these notions will have to be modified due to the lack of a conformal boundary.

This calculation is also of interest outside of AdS/CFT, due to there being no obvious way to define a local stress-energy tensor in general relativity (or indeed any coordinate-invariant theory of gravity). Varying the action with respect to the metric simply produces the equation of motion for the gravitational field. Calculations on surfaces at infinity for spacetimes with timelike Killing vectors can define an energy (the Komar mass), and if the spacetime possesses a spacelike Killing vector with a compact orbit the Komar angular momentum can be defined. However, these do not obviously generalise to spacetimes without such symmetries. The ADM formalism [26] provides a definition of energy and momentum for asymptotically flat spacetimes, but not angular momentum (see section 3.3 of [27] for a discussion).

In an asymptotically AdS spacetime, one option is to take the expectation value of the stress-energy tensor of the boundary field theory to define the stress-tensor of the classical bulk theory

$$T^{ab}_{\text{grav}} := \langle T^{ab}_{\text{bndry}} \rangle = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{grav}}}{\delta \gamma_{ab}} \tag{2.41}$$

where the first equality is our definition and the second comes from (2.3). This was first proposed in [28], where it was shown that for flat spacetimes this definition coincided with the ADM quantities. However, it was necessary to subtract off a contribution from a flat reference spacetime. [28] argue that this is indeed always possible for an asymptotically flat 4 dimensional manifold, but the spacetimes of interest in holography do not obey this condition.

In [29] the authors found boundary counter-terms which rendered (2.41) finite without the need to subtract off a contribution from a reference spacetime. For example, in AdS_4 it was found that the expectation value of the boundary stress-tensor could be rendered finite by adding to the action the counter-term

$$S_{\text{ct}} = -\frac{2}{L} \int_{\partial \mathcal{M}} \sqrt{-\gamma} \left(1 - \frac{L^2}{4} R \right) \tag{2.42}$$

where R denotes the Ricci scalar of the induced metric on the boundary, γ_{ab}. The form of this is consistent with this implementing renormalisation of the boundary quantum field theory holographically - if this really is dual to counter-terms in the boundary, we'd expect it to be built out of local quantities depending only on the intrinsic geometry of the boundary, which it is. With this new definition, (2.41) can be computed on any small closed surface around some small volume, without having to specify the entire interior of a reference spacetime. Hence this is sometimes referred to as the *quasi-local* stress-tensor.

When we consider non-relativistic field theories in section 3 we will still want to be able to compute quantities such as energy density holographically. However, in these cases the boundary no longer has a non-degenerate metric, and this procedure will have to be generalised.

2.5 The UV-IR correspondence and renormalisation group flows

Looking at the behaviour of $\phi_{(0)}$ in (2.20) as we move the cut-off boundary, and comparing this to the operator dimension, we see that the usual relation between an operator dimension and whether it is relevant, irrelevant or marginal is obeyed if moving the boundary to smaller r corresponds to flowing from the UV to the IR.

[30] make a more general argument that large distances in the bulk correspond to UV physics of the boundary field theory as follows. If some UV regulator mass μ is introduced in the boundary field theory, then the fact that the field theory is conformal implies that a correlator should in general have as its leading order term

$$\Delta\left(x_1^a, x_2^b\right) = \mu^{-p}\left|x_1^a - x_2^a\right|^{-p} \tag{2.43}$$

in the limit $|x_1^a - x_2^a| \ll \mu^{-1}$.

In the bulk, a typical propagator for a particle of mass m would have the form

$$e^{-m\left|x_1^a - x_2^a\right|} \tag{2.44}$$

and [30] calculate the length of a geodesic between x_1, x_2 in the bulk, regulated by placing the boundary at a radial distance of order δ^{-1}, and find the leading order piece $\log\left(\left|x_1^a - x_2^a\right|/\delta\right)$, giving a propagator the leading order form

$$\Delta\left(x_1, x_2\right) = \frac{\delta^m}{\left|x_1 - x_2\right|^m} \tag{2.45}$$

This only holds in the $|x_1^a - x_2^a| \gg \delta$ limit.

Since (2.3) implies that correlators should match between the supergravity theory and the boundary conformal field theory, we should also expect correlators to match in the regulated theories, at least well away from the regulator scales. The result in (2.43) is valid below a UV cut-off in the boundary, and (2.45) is valid above an IR cut-off in the bulk, and these results do indeed match. We must identify $\delta^m = \mu^{-p}$, so moving the boundary inward by increasing δ requires decreasing the mass cut-off, μ in the boundary theory. It should be noted that only the leading order pieces have been considered, and the relation between these two cut-offs is still not well-understood.

For us, the important feature of this UV-IR relation will be that different (approximate) isometries at large and small r will correspond, respectively, to different (approximate) scaling symmetries of the field theory at small and large wavelengths. We will assume that at least this feature of the correspondence continues to hold when we are far from the well-understood case of a conformal boundary and an asymptotically AdS bulk.

We are now able to ask what happens to the renormalisation group flow of the field theory when the coupling (2.18) to $\mathcal{O}$ is included in its action with finite $\phi_{(0)}$, rather than just thinking of this as an infinitesimal deformation. The sign of $\Delta - d$ only tells us the behaviour of the flow at the linear level, and we must solve the full equations of motion for ϕ, including its back-reaction on the metric and any other fields that may be present, to investigate whether the flow reaches a new fixed point in the IR. In the case of a new conformal fixed point, this would manifest itself in the bulk by the metric tending to an AdS metric at small r. The scaling behaviour of ϕ as $r \to 0$ will determine the dimension of the operator dual to ϕ in this new field theory.

The first explicit construction of such a flow was given in [15] numerically. By adding a perturbation of one of the scalar fields of $\mathcal{N} = 8$ gauged supergravity to the maximally supersymmetric point, a solution smoothly flowing (as $r \to 0$) to another critical point of the theory, with only $\mathcal{N} = 2$ supersymmetry, was found. In this case the authors were able to identify the field theories at the UV end of the flow as $\mathcal{N} = 4$ super Yang-Mills, and the deformation driving the renormalisation group flow as the addition of a mass to one of the adjoint chiral superfields.

The authors of [15] also prove a general theorem about a wide class of holographic flows. For even boundary dimension d, the stress-tensor of the conformal field theory possesses a trace anomaly. Writing the metric in the form

$$ds^2 = \frac{dr^2}{r^2} + e^{2A(r)}\eta_{ab}dx^a dx^b \tag{2.46}$$

the trace anomaly is

$$\langle T^a_a \rangle \propto \frac{1}{(r\partial_r A)^{d-1}} \tag{2.47}$$

Using Einstein's equation in the convention where the cosmological constant is included in the stress-energy tensor

$$-\left(d-1\right)r\partial_r\left(r\partial_r A\right)=2\left(T^t_t-T^r_r\right) \qquad (2.48)$$

and, assuming Poincaré invariance throughout the flow, the right hand side is non-negative if and only if the weak energy condition holds. Thus they show a renormalisation group flow of the dual field theory, preserving Poincaré invariance, cannot increase the trace anomaly $\langle T^a_a \rangle$. Since the trace anomaly is proportional to the central charge of a CFT [31], which parametrises the number of degrees of freedom, this is consistent with the intuition that an RG flow should integrate out degrees of freedom.

Chapter 3

Introduction to Lifshitz holography

Whilst many effective field theories in condensed matter possess relativistic conformal symmetries, by no means all do. However, there are other symmetries that a field theory may possess. The systems we shall consider in this chapter have the more general scaling symmetry $t \to \lambda^z t$, $x^i \to \lambda x^i$. This is commonly referred to as a *Lifshitz symmetry* [1] and an early investigation of field theories with such symmetries can be found in [32]. z is called the *dynamical exponent.* The lack of a boost symmetry is not unnatural for an effective field theory describing condensed matter - the preferred frame is set by the rest frame of the atomic lattice.

As with the conformal symmetry, many systems with this symmetry describe systems near phase transitions. $z = 2$ and 3 occur at the onset of antiferromagnetism [33] and ferromagnetism [32] respectively. Further details of these and several more example can be found in [14], including non-integer z.

The purpose of chapter 3 is to describe the geometry of holographic duals to such field theories, and to repeat the scalar field calculations of chapter 2 to illustrate how the results differ. We will consider the geometry of the boundary and the

[1]This should not be confused with the Galilean symmetry group, which shares the same generalised scale invariance, but also possesses a non-relativistic boost symmetry, and a conserved particle number. In particular, the holographic duals of such theories are very different to those of this chapter.

interior of such spacetimes in section 3.1. In section 3.2 we will solve the scalar wave equation in the near-boundary region of the bulk, and see that the expression for the expansion varies depending on the value of z. In section 3.3 we will use this to show that the operator dimension is again set by the exponent of the boundary data fall-off. We will repeat the calculation of the Klein-Gordon norm to find when we may choose the coefficient of the fast fall-off mode as boundary data. In section 3.4 we will calculate the expectation value of the dual operator in a simple case, and again find that we must add counter-terms to the action.

To find a holographic dual implementing the Lifshitz symmetry, we want a metric that is invariant under

$$t \to \lambda^z t,\ x^i \to \lambda x^i,\ r \to \lambda^{-1} r \tag{3.1}$$

A metric that possesses this isometry, along with isotropy and translation invariance of the $d-1$ boundary spatial directions, was first proposed in [1]

$$ds^2 = L^2 \left(-r^{2z} dt^2 + r^2 \sum_{i=1}^{d-1} dx^{i2} + \frac{dr^2}{r^2} \right) \tag{3.2}$$

From now on we will refer to this simply as a *Lifshitz spacetime*. This is not a solution to the vacuum Einstein equation - some matter content will be required to break the boost symmetry. We will consider the matter content of [1] in chapter 4, and an example of a matter which is a consistent truncation of supergravity in chapter 5. These are by no means the only examples. Further examples of phenomenological models can be found in [34], [35], [36], [37], [38] and constructions from supergravity in [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49]. If we wish to study the field theory dual at finite temperature, we would consider a spacetime with Lifshitz asymptotics and a black hole in the interior. Such spacetimes have been constructed and studied in [50], [51], [52], [53], [54], [36], [55], [56], [57], [37], [58], [59], [60], [38], [48], [61], [49]. We are not interested in $z < 1$, as this produces an unrealistic causal structure in the boundary field theory. It is also the case that if the gravitational part of the bulk theory is Einstein gravity, $z < 1$ bulks require matter violating the null energy condition [62]. One notable feature that (3.2) shares with AdS is that a radial null geodesic from some r_0 will reach the boundary in finite coordinate time, so the spacetime is not globally hyperbolic and boundary conditions are needed in addition to initial data to determine the evolution of a classical field.

If such dualities do in fact exist, we expect there to be the same limitations as in relativistic holography. Firstly, the boundary field theory will be strongly coupled when the gravity theory is weakly coupled. This is useful, as there exist condensed matter systems which are strongly coupled. Secondly, if the gravity side of the duality is to be well approximated classically, then the rank of some gauge group in the boundary theory will have to be large.

3.1 Geometry of Lifshitz and asymptotically Lifshitz spacetimes

3.1.1 Boundary

Recall that in section 2.1 we used the induced metric, γ_{ab}, to define a conformal boundary metric, h_{ab}, for asymptotically AdS spacetimes. This allowed us to treat the geometry of the boundary at infinity using a metric which did not become degenerate in this limit. For an asymptotically Lifshitz spacetime this does not work. Taking our cut-offs to be constant r surfaces, Σ_r, the induced metrics from (3.2) are

$$\gamma_{ab}dx^a dx^b = -r^{2z}dt^2 + r^2 \sum_{i=1}^{d-1} dx^{i2} \tag{3.3}$$

Choosing $\Omega = r^{-1}$ results in h_{tt} diverging, and choosing $\Omega = r^{-z}$ results in the spatial part of the metric vanishing. In order to understand what is happening here, it helps to look at the causal structure of the boundary. We can calculate the time it takes two points with spatial separation Δx in Σ_r to communicate with one another via a light ray

$$\Delta t = r^{1-z}\Delta x \tag{3.4}$$

For $z > 1$ this vanishes as $r \to \infty$, so if we're to find any sensible meaning for a 'boundary', any two points in the boundary at equal t must have the same causal futures (and causal pasts). Therefore we should not expect this theory to have many of the features associated with a relativistic field theory, such as a spacetime metric.

However, we still need a way to treat the geometry of the boundary if we're to find a stress-tensor. In the context of asymptotically AdS spacetimes, [63] treated the boundary using frame fields, a set of orthonormal 1-forms satisfying

$$g_{\mu\nu} = e^{(\alpha)}_{\mu} e^{(\beta)}_{\nu} \eta_{\alpha\beta} \tag{3.5}$$

This is applied to asymptotically Lifshitz spacetimes in [64], where instead of defining a conformal boundary metric the authors define a *conformal frame* as

$$\hat{e}^{(0)} = r^{-z} e^{(0)} \quad \hat{e}^{(i)} = r^{-1} e^{(i)} \text{ for } i = 1, \dots d-1 \tag{3.6}$$

where $e^{(d)}$ is chosen to normal to the boundary, and so has nothing to do with its intrinsic geometry. In our exactly Lifshitz space, we can take the frame fields to be $e^{(0)} = r^z dt$, $e^{(i)} = r dx^i$ and $e^{(d)} = r^{-1} dr$. The prescription above then gives $\hat{e}^{(0)} = dt$, $\hat{e}^{(i)} = dx^i$, which do indeed have finite components in the large r limit.

When we introduce frame fields, we would normally also introduce a new gauge symmetry, that of local Lorentz boosts. While the Lorentz group acting as $e^{(\alpha)} \mapsto \Lambda^{\alpha}{}_{\beta} e^{(\beta)}$ preserves (3.5), it does not preserve (3.6) as an appropriate choice of conformal frame. In the exactly Lifshitz space, a boost along the x^1 direction with rapidity ξ will map

$$\hat{e}^{(1)} \mapsto \cosh\xi dx^1 + r^{z-1} \sinh\xi dt \tag{3.7}$$

which clearly no longer satisfies the conditions we wanted. We really must ensure that our $e^{(0)}$ points in the t direction picked out by the metric (3.2), and orthogonality of $e^{(0)}$ to $e^{(i)}$ then ensures that our $\hat{e}^{(i)}$ are finite. We do still have $SO(d-1)$ symmetry acting on the $\hat{e}^{(i)}$.

We can now turn this around and use it to define what we mean by asymptotically Lifshitz. We will say that a spacetime is asymptotically Lifshitz if its induced metrics can be written as

$$\gamma_{ab} dx^a dx^b = -r^{2z} \left(\hat{e}^0\right)^2 + r^2 \sum_{i=1}^{d-1} \left(\hat{e}^i\right)^2 \tag{3.8}$$

and

$$\hat{e}^0 \to dt, \quad \hat{e}^i \to dx^i \text{ as } r \to \infty \tag{3.9}$$

The conformal frame also provides a natural way to define a volume form on the boundary, and hence perform integrals, even in the absence of a conformal metric. The volume form of the metric can be written as $\epsilon = e^{(0)} \wedge \cdots \wedge e^{(d-1)}$, so we take the rescaled volume form on our boundary as $\hat{\epsilon} = \hat{e}^{(0)} \wedge \cdots \wedge \hat{e}^{(d-1)}$. In the case of a flat boundary this reduces to $dt\ dx^1\ \ldots\ dx^{d-1}$. By varying the action with respect to the $\left\{\hat{e}^{(a)}\right\}$ [64] define energy density, momentum density, energy flux and stress. For the purposes of the rest of this chapter, we are interested in the fact that $\hat{\epsilon}$ allows us to define volume integrals on the boundary which remain finite as $r \to \infty$.

3.1.2 Behaviour as $r \to 0$

The metric (3.2) clearly has at least a coordinate singularity as $r \to 0$. A Poincaré patch of AdS also has a coordinate singularity, and we are able to continue through it and find that the spacetime is in fact geodesically complete, so we might hope that something similar occurs here. It is shown in [65] that this is not the case. For the purpose of searching for a curvature singularity, we want to ask if there is any possible way in which we can contract components of the Riemann tensor to obtain a divergent quantity, so we should look for divergences in the components of the Riemann tensor in an orthonormal basis. In a static basis,

$$e^{(0)} = -r^z dt, \quad e^{(d+1)} = Lr^{-1}dr, \quad e^{(i)} = Lrdx^i \tag{3.10}$$

all components of the Riemann tensor are constant, therefore we cannot build any curvature scalar that diverges as $r \to 0$.

However, an observer falling freely towards $r = 0$ with energy E making measurements of the curvature components would not do so in the above basis, but instead in a basis with one member parallel to his four-velocity,

$$e^{(0)} = -Edt - Er^{-1-z}\sqrt{1 - \frac{r^{2z}}{E^2}}dr, \quad e^{(d+1)} = -E\sqrt{1 - \frac{L^2 r^{2z}}{E^2}}dt - Er^{-1-z}dr, \quad e^{(i)} = Lrdx^i \tag{3.11}$$

In such a basis the R_{0i0i}, R_{1i1i} and R_{0i1i} components of the Riemann tensor diverge like r^{-2z}. While it seems surprising that static observers sat arbitrarily close to $r = 0$ see such a radically different curvature to an observer falling freely past them, note that the falling observer is boosted with rapidity $\xi = \cosh^{-1}(E/r^z)$ with respect to the static observer, and this diverges as $r \to 0$.

This is not specific to the metric being (3.2) to leading order at small r - [65] show that this singularity can also occur when the small r limit naively looks like a Poincaré horizon. In later chapters when we look for holographic flows, all those found will possess this singularity in the IR, including flows between different AdS spaces. It should also be pointed out that this singularity does not appear in the Euclidean version of this spacetime. Here (3.11) is no longer an orthonormal basis. The boost that allowed us to generate (3.11) from the static basis has been replaced by a rotation. In fact all orthonormal bases are now local $SO(d+1)$ rotations of the static basis, guaranteeing that all Riemann tensor components measured in such a basis are finite.

3.2 Asymptotic behaviour of fields in Lifshitz asymptotics

We wish to perform calculations analogous to those of section 2.1 and see how the results are modified by these new asymptotics. We will again find that ϕ has both a slow and a fast fall off mode, respectively $r^{-\Delta_-}$ and $r^{-\Delta_+}$, but the exponents of these will now depend on z, as well as d and m^2L^2. The relation between the various coefficients in the expansion of ϕ now depends on z, and we will not be able to give a general recurrence relation. We will no longer see the relativistic boundary wave operator, $(-\partial_t^2 + \partial_i^2)$, appearing in the expressions for the coefficients. This should not be surprising since the dual field theory no longer has the Lorentz group as a symmetry.

The example of scalar field will again be used. In this case we won't specify what the full action is, only that it contains a scalar minimally coupled to gravity through the term

$$S_\phi = \int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\left(-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right) \tag{3.12}$$

and that $V(\phi) = \frac{1}{2}m^2\phi^2$. We will take the metric to be (3.2) and assume that the back-reaction of ϕ on this can be neglected in the near boundary analysis. The equation of motion is again (2.10), and this time we will define $\hat{\phi}$ by

$$\phi\left(r,x^a\right)=r^{\Delta-d+1-z}\hat{\phi}\left(r,x^a\right)\quad\text{where}\quad\hat{\phi}\left(r,x^a\right)\to 1\quad\text{as}\quad r\to\infty \tag{3.13}$$

We again expect to find two solutions for Δ. Note that the definition of Δ depends on z. The equation of motion can be rewritten as

$$\left(L^2m^2-\Delta\left(\Delta-d+1-z\right)\right)\hat{\phi}-\left(-r^{-2z}\partial_t^2+r^{-2}\partial_i^2\right)\hat{\phi}-r^2\partial_r^2\hat{\phi}-\left(2\Delta+2-z-d\right)r\partial_r\hat{\phi} \tag{3.14}$$

As in the AdS case, the $r\to\infty$ limit requires that the coefficient of the first term vanishes, giving two solutions for Δ [1]

$$\Delta_{\pm}=\frac{d+z-1}{2}\pm\sqrt{\frac{\left(d+z-1\right)^2}{4}+L^2m^2} \tag{3.15}$$

Requiring that these are real imposes $L^2m^2\geq-\left(d+z-1\right)^2/4$. Again, this leading order result carries over to the case that $V\left(\phi\right)$ contains higher order terms.

It is straight forward to carry out the expansion to higher orders term by term, but the form of the series now depends on z. To illustrate this, suppose we name the next two terms in our series as

$$\hat{\phi}=\phi_{(0)}+\phi_{(\beta_1)}r^{-\beta_1}+\phi_{(\beta_2)}r^{-\beta_2}+\ldots\quad\text{where}\quad\beta_1<\beta_2 \tag{3.16}$$

Substituting this into (3.14) and keeping only the leading order contribution to each term, we find that we must have $\beta_1=2$ and

$$\phi_{(2)}=\frac{\partial_i^2\phi_{(0)}}{2\left(2\Delta-d-1-z\right)} \tag{3.17}$$

unless $2\Delta=d+1-z$, in which case we would have to introduce logarithmic terms, as in the asymptotically AdS expansions. Since we are always interested in $z>1$, $r^{-2z}\partial_t^2\phi$ is subleading at this level. At this point we might be concerned that this does not look like it will reduce to the AdS result (2.15) as $z\to 1$, however this will be remedied by the next term. The leading order contribution to each term is now

$$-r^{-2z}\partial_t^2\phi_{(0)}+r^{-4}\partial_i^2\phi_{(2)}+\beta_2\left(\beta_2-1+z+d-2\Delta\right)\phi_{(\beta_2)}r^{-\beta_2}=0 \tag{3.18}$$

Depending on the value of z, there are 3 possibilities. If $1 < z < 2$ then we must set $\beta_2 = 2z$ and

$$\phi_{(2z)} = \frac{-\partial_t^2 \phi_{(0)}}{2z\,(2\Delta - d + 1 - 3z)} \tag{3.19}$$

If $z = 2$ then we must set $\beta_2 = 4$ and

$$\phi_{(4)} = \frac{-\partial_t^2 \phi_{(0)} + \partial_i^2 \phi_{(2)}}{4\,(2\Delta - d - 3)} \tag{3.20}$$

If $z > 2$ then we must again set $\beta_2 = 4$ and

$$\phi_{(4)} = \frac{\partial_i^2 \phi_{(2)}}{4\,(2\Delta - d - 1 - z)} \tag{3.21}$$

In these cases we are assuming that, respectively, $2\Delta - d \neq 3z - 1$, 3 and $z + 1$. We can now see how we recover the correct value of $\phi_{(2)}$ in the relativistic limit - the $\phi_{(2)}$ and $\phi_{(2z)}$ terms merge to give a term with the same coefficient as that given by (2.15).

In general then we have the power series,

$$\hat{\phi} = \sum_a \phi_{(a)} r^{-a} \quad \text{where} \quad a \in \{2m + 2nz | m, n \in \mathbb{N}\} \tag{3.22}$$

The complication that there exist different values of (n, m) giving the same power, unless z is irrational, makes writing down a general recurrence relation difficult. We can at least show some similarity with the relativistic case. The coefficient in front of $r^{-\Delta}$ in the $r^2 \partial_r^2 \hat{\phi} + (2\Delta + 2 - z - d)\, r \partial_r \hat{\phi}$ term vanishes, so the coefficient $\phi_{(2\Delta - d + 1 - z)}$ is left undetermined by $\phi_{(0)}$ in the near boundary expansion. We would expect, as in the asymptotically AdS case, that it would be set by a regularity condition in the bulk. An example of an explicit solution for such a field can be found in [1] for $z = 2$ in $d = 3$, and regularity does restrict this to a one parameter set of solutions.

We will also note at this stage that the relation $\Delta_+ + \Delta_- = d + z - 1$ will continue to hold for any field obeying a second order wave equation at the linear level.

3.3 Operator dimensions and boundary conditions in Lifshitz asymptotics

In this section we will first clarify how several 1-forms and vectors in the bulk scale under (3.1) and then go on to find the scaling dimension of the dual operator using the results of section 3.2. It turns out that, with our choice of labelling for the exponents, $\mathcal{O}$ has scaling dimension Δ. We then calculate the Klein-Gordon norm of both the fast and slow fall-off modes of ϕ, and find that we may only choose $\Delta = \Delta_-$ when $L^2 m^2 < \frac{1}{4}\left(3z - (d+z-1)^2\right)$.

To fix what we mean by dimension here, we will say that an object ϕ (either a classical bulk field or a boundary operator or expectation value) has *scaling dimension* α if under the Lifshitz scaling transformation (3.1) it scales as $\phi \to \lambda^{-\alpha}\phi$, and denote this by $[\phi] = \alpha$.

From (3.1) we get the scaling dimensions of the 1-forms and hence the vectors

$$[dt] = -z,\ \left[dx^i\right] = -1,\ [dr] = +1,\ [\partial_t] = +z,\ [\partial_i] = +1,\ [\partial_r] = -1 \tag{3.23}$$

We assume that a duality of the form (2.3) exists, but we no longer have a conformal metric, so our coupling should take the form

$$\int_{\partial\mathcal{M}} \hat{\epsilon}\phi_{(0)}\mathcal{O} \tag{3.24}$$

If the space is asymptotically Lifshitz, then

$$\hat{\epsilon} \mapsto \hat{\epsilon}' = \lambda^{z+d-1}\hat{\epsilon} \tag{3.25}$$

under (3.1) to leading order at large r. By demanding that ϕ is a scalar, ie. $\phi'(r', x'^a) = \phi(r, x^a)$, we get the transformation of its coefficients

$$\phi'_{(\beta)}(x'^a) = \lambda^{\Delta-d+1-z-\beta}\phi_{(\beta)}(x^a) \tag{3.26}$$

We now demand that the coupling of this operator in our dual field theory is invariant under a Lifshitz scaling, to get

$$\int_{\partial\mathcal{M}} \hat{\epsilon}'\phi'_{(0)}\mathcal{O}' = \int_{\partial\mathcal{M}} \hat{\epsilon}\phi_{(0)}\mathcal{O} = \int_{\partial\mathcal{M}} \lambda^{-d+1-z}\hat{\epsilon}'\lambda^{d-\Delta-1+z}\phi'_{(0)}\lambda^{[\mathcal{O}]}\mathcal{O}' \tag{3.27}$$

so we see that the dimension of $\mathcal{O}$ is still determined entirely by the asymptotic behaviour of the scalar field

$$[\mathcal{O}] = \Delta \tag{3.28}$$

We can again look at what happens to the coupling under an RG flow from the UV to the IR (ie. a scale transformation (3.1) with $\lambda < 1$) The condition on the operator dimension is now that $\mathcal{O}$ is irrelevant, marginal or relevant if $[\mathcal{O}]$ is, respectively, greater than, equal to, or less than $d + z - 1$.

Recall that in section 2.2 we used the dimension of the operators in the dual field theory to restrict the range of L^2m^2 for which $\Delta = \Delta_-$ was acceptable. Here we do not have a conformal field theory on the boundary, so we cannot employ the same result. However, we can still look at the Klein-Gordon norm of the bulk field. The lapse vector must now be $N^t = r^{-z}$ so our conjugate momenta become $\pi = r^{-z}\partial_t\phi$ and the contribution from the large r region is

$$-i\int^{\infty} dr\, r^{2\Delta-d-3z}\int dx^1 \dots dx^{d-1}\left(\partial_t\phi^*_{(0)}\phi_{(0)} - \partial_t\phi_{(0)}\phi^*_{(0)}\right) \tag{3.29}$$

which receives a divergent contribution at large r unless

$$\Delta < \frac{3z-1}{2} + \frac{d}{2} \tag{3.30}$$

This is always satisfied for Δ_-, and is satisfied for Δ_+ if

$$L^2m^2 < \frac{1}{4}\left(3z - (d+z-1)^2\right) \tag{3.31}$$

This is the range of L^2m^2 for which we can set $\Delta = \Delta_-$, and have the coefficient in front of $r^{\Delta_- - d+1-z}$ as boundary data. We would again need a boundary term such that $\delta\phi_{(0)} = 0$ is a sufficient condition for $\delta S = 0$.

3.4 Expectation values

In this section we will calculate $\langle\mathcal{O}\rangle$, using the same minimal subtraction procedure as in section 2.3. As in the asymptotically AdS spacetime, we will consider only the simplest possible case, namely that $\sqrt{(d+z-1)^2/4 + L^2m^2} \leq 1$, and find a non-relativistic boundary counter-term that gives a finite result for $\langle\mathcal{O}\rangle$. We expect to find $\langle \mathcal{O} \rangle \propto \phi_{(2\Delta-d+1-z)}$, since (3.26) shows that this is the only component of the field with the correct scaling dimension, and this is indeed what we find below.

Since we no longer have a conformal metric we should rewrite (2.31) as

$$\langle \mathcal{O} \rangle = \frac{1}{\sqrt{\det \hat{\epsilon}}} \frac{\delta S_{\text{SUGRA}}\left(\phi_{(0)}\right)}{\delta \phi_{(0)}} \bigg|_{\phi_{(0)}=0} \tag{3.32}$$

Again we want to work with a boundary at a finite distance, and in terms of bulk covariant quantities, before taking a large r limit. The limit we need to take turns out to be the same as in AdS

$$\langle \mathcal{O} \rangle = \lim_{r\to\infty} r^{\Delta} \frac{1}{\sqrt{-\gamma}} \frac{\delta S}{\delta \phi\left(r\right)} \tag{3.33}$$

Since the scalar part of the action is the same as in our AdS example, our contribution can again be read off from the boundary term in 2.29

$$\langle \mathcal{O} \rangle = -r^{\Delta+1} \partial_r \phi\left(r, x^a\right) \sim \left(d - \Delta - 1 + z\right) \phi_{(0)} r^{2\Delta-d+1-z} \tag{3.34}$$

This is always divergent for $\Delta = \Delta_+$. We should again be able to find counter-terms by looking for a quantity which we can subtract from the action to leave the scalar part of it finite.

We now restrict to $\sqrt{\left(d+z-1\right)^2/4 + L^2 m^2} \leq 1$, equivalently $2\Delta - d + 1 - z \leq 1$. The only divergent term in the on-shell action is

$$-\frac{1}{2} \int_{\partial\mathcal{M}} d^d x r^{2\Delta-d+1-z} \left(\Delta - d + 1 - z\right) \phi_{(0)}^2 \tag{3.35}$$

Again, we need only invert the expansion of ϕ to first order, and get the following large r form for the counter-term

$$S_{\text{ct}} = \frac{1}{2} \int_{\partial\mathcal{M}} d^d x \sqrt{-\gamma} \left(\Delta - d + 1 - z\right) \left(\phi^2 + O\left(r^{-1}\right)\right) \tag{3.36}$$

Our regulated expectation value is now

$$\begin{aligned} \langle \mathcal{O} \rangle &= -r^{\Delta+1} \partial_r \phi + \left(\Delta - d + 1 - z\right) r^{\Delta} \phi = -r^{2\Delta-d-z+1} r \partial_r \hat{\phi} \\ &= \left(2\Delta - d + 1 - z\right) \phi_{(2\Delta-d-z+1)} + O\left(r^{-1}\right) \end{aligned} \tag{3.37}$$

This is reassuring - we might have worried that renormalising the dual field theory would require us to include in the gravity theory a counter-term that was not a bulk scalar (for instance, something containing t derivatives.) These results also reduce to those of section 2.3 if we set $z = 1$, as we should expect.

Chapter 4

The Massive Vector Model

Having discussed the geometry of asymptotically Lifshitz spacetimes and how the holographic dictionary is modified by these asymptotics, we will now consider a concrete example of such a spacetime as a stationary point of a particular action.

The theory that we will consider is a massive vector field of mass m_0 coupled to Einstein gravity with a cosmological constant, Λ, in an arbitrary number of bulk dimensions, $d+1$. In section 4.1 we will review the equivalence of this theory to another phenomenological theory with a 2-form/$(d-1)$-form as its matter content, which was the subject of several early papers on Lifshitz holography(eg. [1], [50], [51]). We will also discuss the relation between this model and some of the recent string theory constructions of Lifshitz spacetimes. In section 4.3 we will show that this model supports Lifshitz spacetimes with arbitrary radius of curvature and dynamical exponent, for appropriate choices of m_0, Λ. We show that the number of Lifshitz spacetimes that the action supports is determined by Λ/m_0^2, and that the parameter space as split into three regions variously possessing one, two or zero Lifshitz solutions.

We will perform linearisations around the AdS and Lifshitz solutions that this models supports in section 4.4, and identify which of these possess a relevant operator capable of driving an RG flow to a new fixed point in the IR. Having conjectured the existence of Lifshitz to AdS, Lifshitz to Lifshitz and AdS to Lifshitz flows, we perform numerical integration in section 4.5 to confirm the existence of examples of each of these in $d = 3$, 4 and 5.

4.1 Equivalence to the 2-form/$(d-1)$-form model

We will briefly review the on-shell equivalence between two early phenomenological models for Lifshitz spacetimes. Since the bulk calculations we perform are entirely classical, all of our results will be valid in both of these models.

An action that admits the metric (3.2) in $d = 3$ as a solution to its equations of motion was first written down in [1], and is directly generalised to arbitrary d

$$S_1 = \int d^{d+1}x \left(\sqrt{-g}\,(R - 2\Lambda) - \frac{1}{2} F_{(2)} \wedge *F_{(2)} - \frac{1}{2} F_{(d)} \wedge *F_{(d)} - \gamma B_{(d-1)} \wedge F_{(2)} \right) \tag{4.1}$$

where $F_{(2)} = dA_{(1)}$, and $F_{(d)} = dB_{(d-1)}$ are respectively the field strengths of abelian 1-form and $(d-1)$-form fields. It has been shown that this is equivalent on-shell to a much simpler model [34], a massive vector field coupled to gravity

$$S_2 = \int d^{d+1}x \sqrt{-g} \left(R - 2\Lambda - \frac{1}{4} F_{ab}F^{ab} - \frac{m_0^2}{2} A_a A^a \right) \tag{4.2}$$

where F is now the field strength of the 1-form A. Note that the mass term means that this field does not have a gauge symmetry, and hence there is no conserved charge associated to it.

The equation of motion of $B_{(d-1)}$ from (4.1) is $d * F_{(d)} = \gamma dA_{(1)}$, so we can write

$$*F_{(d)} = \gamma A_{(1)} - C \tag{4.3}$$

where C is some closed 1-form (the requirement that C be exact, and hence that the space be simply connected, in [65] does not seem to be necessary.) If we now define a vector field A by

$$A = A_{(1)} - \frac{1}{\gamma} C \tag{4.4}$$

then this has field strength $F = F_{(2)}$. We also have $F_{(d)} = (-1)^d A$. Therefore the $F_{(2)} \wedge *F_{(2)}$ term of (4.1) becomes the kinetic term of (4.2) and the $F_{(d)} \wedge *F_{(d)}$ becomes a mass term for A. Replacing $F_{(2)}$ by F and integrating by parts, it can be seen that, up to a surface term, the Chern-Simons term of (4.1) also becomes a mass term for A. The total mass is $m_0^2 = \gamma^2$, and the action coincides with (4.2).

4.2 Relation to string theory

As the justification for gauge/gravity dualities is best supported within string theory, we should really only expect a gravity theory to be dual to a Lifshitz field theory if it is a consistent truncation of a supergravity. As such theories have been found recently (e.g. [39], [40], [41]) including some that preserve some supersymmetry [45], it might seem that there is no longer any reason to study the massive vector model. Indeed, in chapter 5 we will see an even richer structure of holographic flows within the theory of [41].

However, many such truncations contain a massive vector (or equivalent form fields) within their matter content. A recently published example illustrating this very clearly is [48]. There the authors use the results of [45] to obtain a 5D consistent truncation of type IIB supergravity retaining the Ramond-Ramond scalar, C_0, and the dilaton, ϕ. They then compactify on an S_1 to a 4D spacetime, gaining a scalar, T, parameterizing the size of the S_1, and a vector $\mathcal{A}$ gauging the reparametrization invariance of the circle coordinate. $\mathcal{A}$ has a 2-form field strength, and dC_0 can be dualised to a 3-form field strength, giving the fields of [1], and generating the required Chern-Simons coupling in the action.

Unfortunately this also illustrates the limitations of looking at only the massive vector part of the model. The final action of [48] still contains the scalars ϕ, T, the presence of which significantly alters the details of the theory. For instance, previous numerics in [51], [50] found that the massive vector model possess extremal black holes in the limit of vanishing horizon size, whereas the numerical work of [48] showed that this truncation of IIB supergravity does not possess extremal black holes, and that the size of the S_1 sets a minimum horizon radius. It is not simply the case that the scalars are 'cutting off' some of the family of black hole solutions. A more significant qualitative difference is that the supergravity black holes of [48] have a range of horizon sizes for which they have a negative specific heat (similar to small black holes in AdS spaces), whereas [51] showed that those of the massive vector model do not.

As there seems to be no way to predict exactly which features of the massive vector model will be left intact under interactions with scalars (or indeed any other fields left over from truncations and compactifications) its usefulness for making predications about genuine field theory duals seems to be severely limited. However, its simplicity makes it an attractive toy model, and the existence of such a wide range of holographic flows within the theory suggests that they are a common feature, and that we might expect to find some within a supergravity theory. We will see in chapter 5 that this is indeed the case.

4.3 Lifshitz solutions

In this section we will introduce the equations of motion for the massive vector model, and our ansatz for the metric and the gauge field. We will show that, for an appropriate choice of m_0 and Λ, this matter content supports Lifshitz spacetimes with arbitrary dynamical exponent, and that the same conditions on m_0, Λ are necessary for the spacetime to be asymptotically Lifshitz at large or small r. We show that the number of Lifshitz spacetimes supported by the model is either zero, one or two, depending on the value of Λ/m_0^2.

Working with the massive vector model makes a little clearer what our ansatz for the vector field should be if it is to support a Lifshitz spacetime. We do not wish to break spatial isotropy or homogeneity of the boundary, so the A_i components should vanish, and neither A_r nor A_t should depend on the x_i coordinates. We do not wish to break time translation invariance either, so neither of these should depend on t. This leaves us with $A_r(r)$, $A_t(r)$.

The equations of motion are

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = \frac{1}{2}\left(F_{ac}F_b{}^c - \frac{1}{4}F_{cd}F^{cd}g_{ab}\right) + \frac{m_0^2}{4}\left(2A_aA_b - A_cA^cg_{ab}\right) \quad (4.5)$$

$$\nabla_b F^{ba} = m_0^2 A^a \quad (4.6)$$

$\nabla_b F^{br} = m_0^2 A^r$ imposes $A_r(r) = 0$, so we may restrict our attention to $A_t(r)$.

We will write down the most general ansatz that we will consider in this section

$$ds^2 = L^2\left(-e^{2F(r)}dt^2 + e^{2D(r)}\frac{dr^2}{r^2} + r^2\sum_{i=1}^{d-1}dx_i^2\right) \quad (4.7)$$

$$A = \alpha(r)\,e^{F(r)}dt \quad (4.8)$$

The rr and tt components of (4.5) and the t components of (4.6) then respectively become

$$rF' + \frac{d-2}{2} + \frac{L^2\Lambda}{d-1}e^{2D} + \frac{1}{4(d-1)L^2}\left(\left(e^{-F}r\left(\alpha e^F\right)'\right)^2 - m_0^2L^2e^{2D}\alpha^2\right) = 0 \quad (4.9)$$

$$rD' - \frac{d}{2} - \frac{L^2\Lambda}{d-1}e^{2D} - \frac{1}{4(d-1)L^2}\left(\left(e^{-F}r\left(\alpha e^F\right)'\right)^2 + m_0^2L^2e^{2D}\alpha^2\right) = 0 \quad (4.10)$$

$$r\left(r\left(\alpha e^f\right)'\right)' + (d-2)\,r\left(\alpha e^F\right)' = \frac{m_0^2}{2(d-1)}e^{2D}\alpha^2 r\left(\alpha e^F\right)' + m_0^2L^2e^{2D}\alpha e^F \quad (4.11)$$

These are satisfied [1], [34] for a Lifshitz solution with dynamical exponent z,

$$F = z\log r, \qquad D = 0, \qquad \alpha = \frac{\sqrt{2(d-1)(z-1)}}{m_0} \quad (4.12)$$

provided that

$$\Lambda = -\frac{z^2 + (d-2)z + (d-1)^2}{2L^2}, \qquad m_0^2 = \frac{z(d-1)}{L^2} \quad (4.13)$$

We might worry that to support a spacetime that is merely asymptotically Lifshitz, these conditions on z and Λ might not be necessary. We can show that they are as follows [65]. Summing (4.9) and (4.10) gives

$$\frac{m_0^2}{2(d-1)}e^{2D}\alpha^2 = rF' + rD' - 1 \quad (4.14)$$

For (4.7) to be asymptotically Lifshitz with radius L, we need $F = z\log r + O(1)$ and $D = 0 + O(r^{-1})$. Then (4.14) implies

$$\alpha^2 = \frac{2(d-1)(z-1)}{m_0^2} + O\left(r^{-1}\right) \quad (4.15)$$

Substituting these asymptotics into (4.11) gives

$$m_0^2 = \frac{z(d-1)}{L^2} \quad (4.16)$$

and substituting this and the asymptotics of the fields into either of (4.9), (4.10) gives

$$\Lambda = -\frac{z^2 + (d-2)z + (d-1)^2}{2l^2} \quad (4.17)$$

This argument works at small r if the subleading term in D is replaced by $O(r)$. Note that the Lifshitz solutions with $z < 1$ are not acceptable as the vector field no longer takes a real value.

The pair (m_0^2, Λ) does not always specify a unique Lifshitz solution [53]. We can eliminate L between (4.16) and (4.17) to get

$$z^2 + \left(d - 2 + 2(d-1)\frac{\Lambda}{m_0^2}\right) z + (d-1)^2 = 0 \tag{4.18}$$

To have at least one Lifshitz solution with $z \geq 1$, we need to have $\Lambda/m_0^2 \leq -(3d-4)/2(d-1)$. There is a second Lifshtiz solution for $-d/2 \leq \Lambda/m_0^2 \leq -(3d-4)/2(d-1)$.

In addition to the Lifshitz solutions found above, for every $\Lambda < 0$ we also have an AdS solution with

$$F = \log r, \quad D = 0, \quad \alpha = 0, \quad L^2 = -\frac{d(d-1)}{2\Lambda} \tag{4.19}$$

4.4 Linearisations

In a similar fashion to the examples in section 2.1, we will look for solutions of the equations of motion, linearised about either a Lifshitz or AdS background. We wish to identify when one of the fields of our ansatz is dual to a relevant operator, as this allows a perturbation to the UV theory that may drive a renormalisation group flow to another fixed point in the IR. The results of this section will not be sufficient to justify the existence of these RG flows, and we will need to resort to numerics in section 4.5 to do this. However, the results of this section will rule out the existence of some flows.

4.4.1 Linearisation around AdS

Setting $F = z\log r + \delta F$, $D = 0 + \delta D$ and $\alpha = 0 + \delta\alpha$, the linearised equations of motion (4.9)-(4.11) are quite simple

$$r\delta F' = d\delta D, \quad r\delta D' = -d\delta D, \quad r\left(r\delta\alpha'\right)' = -dr\delta\alpha' + \left(m_0^2 L^2 + 1 - d\right)\delta\alpha \tag{4.20}$$

These have 4 independent solutions

$$\delta F = F_0 + F_1 r^{-d}, \quad \delta D = F_1 r^{-d}, \quad \alpha = \alpha_1 r^{-\Delta_1} + \alpha_2 r^{-\Delta_2} \tag{4.21}$$

where

$$\Delta_{1,2} = \frac{d}{2} \mp \frac{\sqrt{4m_0^2 L^2 + (d-2)^2}}{2} \tag{4.22}$$

The F_0 solution corresponds to rescaling the t coordinate, and the F_1 mode to the expectation value of the energy of the boundary field theory. This operator has dimension d, and is marginal at the linear level. From the fact that it corresponds to a global rescaling of t, we in fact know that this is exactly marginal, and will not drive an interesting RG flow.

We will regard α_1 as boundary data for the vector field, so that α_2 is the expectation value of some operator in the dual field theory. The operator has dimension Δ_2, and is relevant for $\Lambda/m_0^2 < -d/2$. In this range we might find the AdS spacetime at the UV end of a renormalisation group flow. For $\Lambda/m_0^2 > -d/2$ we might find it at the IR end. When we look for flows numerically in section 4.5 we will see that this is the case. Since both the equations of motion and the AdS background are unchanged under $\alpha \mapsto -\alpha$, changing the sign of this perturbation will only change the sign of α along the flow. There is really only a single direction to perturb along.

4.4.2 Linearisation around Lifshitz

The linearisation around a Lifshitz solution is more complicated, as the gauge field no longer decouples from the metric components. Our fields are now

$$F = z \log r + \delta F, \quad D = \delta D, \quad \alpha = \frac{\sqrt{2(d-1)(z-1)}}{m_0} + \delta\alpha \tag{4.23}$$

The general solutions are

$$\delta F = F_0 + \frac{d-1-z}{d-1+z}\frac{F_1}{r^{z+d-1}} + (z+d-2)(z+d-1+\beta)\frac{\alpha_1}{r^{\frac{1}{2}(z+d-1-\beta)}} + (z+d-2)(z+d-1-\beta)\frac{\alpha_2}{r^{\frac{1}{2}(z+d-1+\beta)}} \tag{4.24}$$

$$\delta D = \frac{F_1}{r^{z+d-1}} + (z-1)(z-3d+3+\beta)\frac{\alpha_1}{r^{\frac{1}{2}(z+d-1-\beta)}} + (z-1)(z-3d+3-\beta)\frac{\alpha_2}{r^{\frac{1}{2}(z+d-1+\beta)}} \tag{4.25}$$

$$\delta\alpha = -L(d-2+z)\sqrt{\frac{2}{z(z-1)}}\frac{F_1}{r^{z+d-1}} + L(z+d-2)(3z-d+1-\beta)\sqrt{\frac{2(z-1)}{z}}\frac{\alpha_1}{r^{\frac{1}{2}(z+d-1-\beta)}} + L(z+d-2)(3z-d+1+\beta)\sqrt{\frac{2(z-1)}{z}}\frac{\alpha_2}{r^{\frac{1}{2}(z+d-1+\beta)}} \tag{4.26}$$

where $\beta(z,d) = \sqrt{9z^2-(2+6d)z+(d+7)(d-1)}$. For $d=3$, these modes were found in [64] - F_0, F_1, α_1 and α_2 correspond respectively to c_1, c_4, c_3 and c_2.

The F_0 mode again corresponds to globally rescaling the t coordinate, and so we can assume this is exactly marginal. F_1 corresponds to the energy density of the field theory. The stress-energy tensor in such asymptotics for $d=3$, including more general modes, is discussed in [64].

We assume that we are working with boundary conditions such that α_1, the coefficient of the slow fall-off mode, is fixed (though it should be noted that this interpretation may not be valid above some value of z [64]). Then the vector field is dual to an operator in the boundary theory with dimension $\frac{1}{2}(z+d-1+\beta)$. This is relevant for $1<z<(d-1)$. Since the Lifshitz background does not have the $\alpha \mapsto -\alpha$ symmetry that the AdS case did, we expect the two signs of the perturbation to drive different flows.

Based on the linearisations of sections 4.4.2 and 4.4.1 we can now restrict our search for flows, since the spacetime at the UV end of the flow must have an irrelevant operator, and the spacetime in the IR must have a relevant operator. To confirm that such flows actually exist we must go beyond the linear level, and the only way to proceed is through numerics. This will be the subject of section 4.5.

4.5 Numerical Flows

In the preceding section we found the perturbations around the Lifshitz and AdS solutions to linear order. We can conjecture the existence of flows from the spacetimes with relevant operators to those with irrelevant operators from this, and rule out any other such flows. We now turn to numerics to confirm the existence of these flows, and to find their exact profiles.

We will work in terms of the radial variable $\rho = \log r$. We will do all our integration using RK4 with a fixed step-length of $\Delta\rho = 0.01$. All the spacetimes we are interested in have at least 3 negative eigenvalues in their linearisation, so trying to integrate from large to small r will be expected to be very sensitive to initial conditions. Generically we would expect numerical error to introduce contributions to the expectation values F_1 and α_2, so we would find the IR appropriate to some non-trivial state. Therefore in all examples we 'shoot' from small to large r.[1] The vacua we shoot from have a single unstable direction, so we need only choose the sign of the perturbation.

We already know that this model is capable of supporting one example of such a geometry. Working in the equivalent 1-form/$(d-1)$-form model in $d = 3$, [1] numerically found a spacetime that is asymptotically AdS at small r and asymptotically Lifshitz with $z = 2$ at large r. The UV-IR correspondence interpretation of this is the gravitational dual of a field theory with a renormalisation group flow from a Lifshitz fixed point at high energy to a conformal fixed point at low energy.[2] However, this is a rather special case - we see from section 4.3 that this is at the highest value of Λ/m_0^2 for a Lifshitz spacetime exists.

We will show in $d = 3$, 4 and 5 that there exist flows from Lifshitz spactimes with $1 < z \leq (d-1)$ to AdS spacetimes. There also exist flows from these Lifshitz spaces with $1 < z \leq (d-1)$ to Lifshitz spacetimes with dynamical exponent $(d-1)^2/z$. These two classes of flow exist within the parameter range

[1] Note that the 'shot' goes in the opposite direction to the 'flow'.

[2] We will use the language 'Lifshitz to AdS flow' from now on to describe such a spacetime without further comment.

$-d/2 < \Lambda/m_0^2 \leq -(3d-4)/2(d-1)$. We will also find flows from AdS spacetimes to Lifshitz spacetimes with dynamical exponents $z \geq (d-1)^2$. These exist for $\Lambda/m_0^2 \leq -d/2$.

4.5.1 Lifshitz → AdS flows

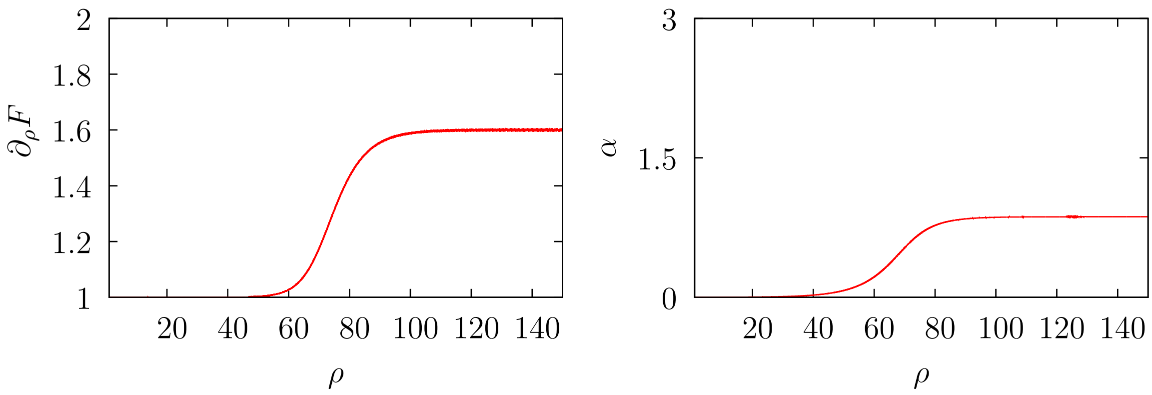

Figure 4.1: Holographic RG flow in $d = 3$ from a Lifshitz spacetime with $z = 1.6$ in the UV to an AdS_4 spacetime in the IR.

Based on the linearisations of section 4.4, we expect there to exist flows from any Lifshitz spacetimes with $1 < z \leq (d-1)$ in the UV to an AdS space with

$$L_{\text{AdS}}^2 = -\frac{d(d-1)}{2\Lambda} = \frac{d(d-1)}{z^2 + (d-2)z + (d-1)^2} L_{\text{Li}}^2 \tag{4.27}$$

in the IR. This is possible within the parameter range $-d/2 < \Lambda/m_0^2 \leq -(3d-4)/2(d-1)$. Such a flow was found in [1] in the case $z = 2$, $d = 3$, however this is a slightly special case since it is the value of Λ at which the two Lifshitz spaces 'merge'.

Since the AdS vacuum has $\alpha = 0$ and the irrelevant perturbation in section 4.4.1 involves only the vector field, the $\alpha \mapsto -\alpha$ symmetry of (4.9)-(4.11) means that the sign of the perturbation does not matter here - there is only a single direction we can shoot in. The value of Λ for these numerics is set such that the spacetime in the IR has curvature length 1, and then m_0^2 is chosen to support a Lifshitz spacetime with the desired value of z.

In $d = 3$, numerical shots were made from AdS spacetimes with values of m_0^2 chosen such that they would support Lifshitz spacetimes with $z = 1.2$, 1.4, 1.6, 1.8 and 2, and we did hit such spacetimes in the UV. Plots of $\partial_\rho F$ and α for the $z = 1.6$ case are included here as figure 4.1. In all cases $\partial_\rho F$ comes within 10^{-5} of the expected value, and the curvature length comes within 10^{-3} of the value we would expect from (4.27). In the $z = 2$ case ∂_ρ and α decay very slowly to their expected values. This is to be expected, since in this case the direction we are approaching the Lifshitz point along is marginal at the linear level, so the decay should be logarithmic rather than a power-law.

This behaviour seems persist in higher dimensions. Flows were found in $d = 4$ from Lifshitz spacetimes with $z = 1.5$, 2, 2.5 and 3 to AdS_5, and in $d = 5$ from Lifshitz spacetimes with $z = 1.5$, 2, 3 and 4 to AdS_5. The behaviour of α and each of the metric components is qualitatively similar to the $d = 3$ case, so I have not included figures for these flows.

4.5.2 Lifshitz $\rightarrow$ Lifshitz flows

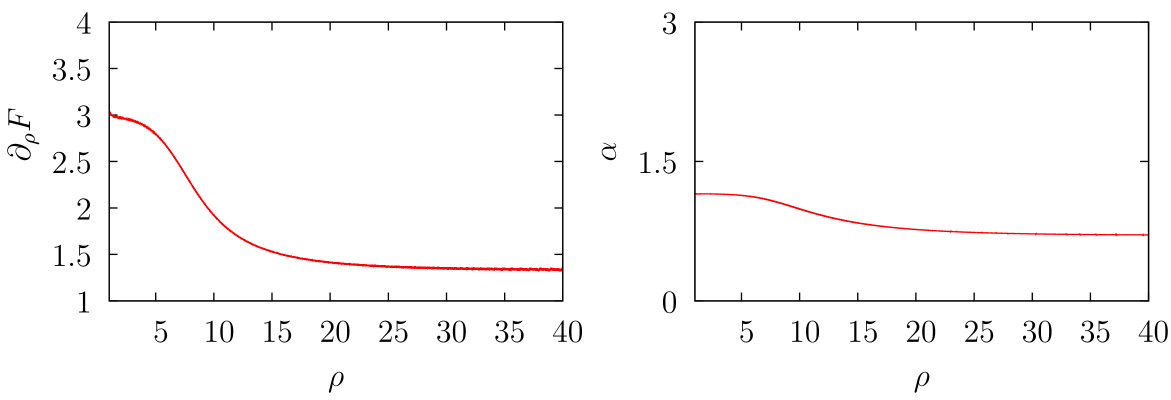

Figure 4.2: Holographic RG flow in $d = 3$ from a Lifshitz spacetime with $z = 1.333$ in the UV to one with $z = 3$ in the IR.

We can conjecture, based on the linearisations of section 4.4, that these flows will exist from any Lifshitz spacetime with dynamical exponent $1 < z < (1 - d)$ to one with dynamical exponent $(d-1)^2/z \in \left((1-d), (1-d)^2\right)$. As in the previous section, these flows require $-d/2 < \Lambda/m_0^2 \leq -(3d-4)/2(d-1)$.

In $d = 3$ we searched for flows to the $z = 2.5$, 3.0 and 3.5 spacetimes. Shooting from the IR with a perturbation such that $\delta\alpha < 0$ we hit another Lifshitz spacetime in the UV. In each of these cases the $\partial_\rho F$ plot showed that we hit the expected values

$$z_{UV} = \frac{(d-1)^2}{z_{IR}}, \quad L_{UV}^2 = \frac{(d-1)^2}{z_{IR}^2} L_{IR}^2 \tag{4.28}$$

to within 10^{-4}. The flow from $z = 1.333$ in the UV to $z = 3$ in the IR is reproduced here as figure 4.2. Shooting in the $\delta\alpha > 0$ direction resulted in F and D and α becoming numerically infinite within finite ρ.

In $d = 4$ flows from $z = 1.125$, 1.5, 1.8 and 2.25 in the UV to, respectively, $z = 8$, 6, 5, and 4 in the IR were found. In $d = 5$ such flows were found from $z = 1.067$, 1.6 and 3.2 to, respectively, $z = 15$, 10 and 5. These flows were all qualitively similar to those found in the $d = 3$ case, so plots have not been reproduced here.

4.5.3 AdS $\to$ Lifshitz flows

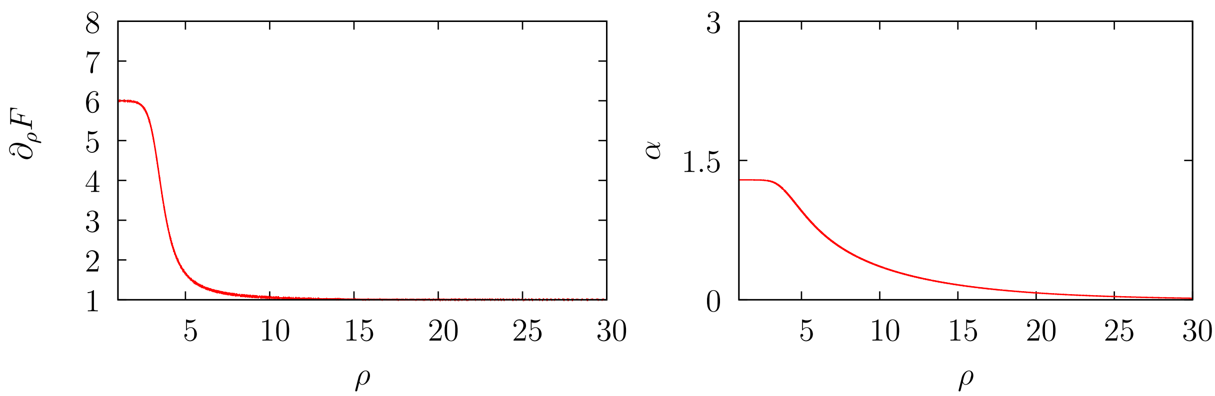

Figure 4.3: Holographic RG flow in $d = 3$ from an AdS_4 spacetime in the UV to a Lifshitz spacetime with $z = 6$ in the IR.

We expect these to exist from the AdS spacetime within the range $\Lambda/m_0^2 \leq -d/2$, to a Lifshitz spacetime with $z \geq (d-1)^2$.

Again, shooting in the $\delta\alpha > 0$ direction from the Lifshitz IR resulted in divergences within finite ρ. Shooting in the $\delta\alpha < 0$ direction, we found flows in $d = 3$ from AdS spacetimes to Lifshitz spacetime with $z = 4$, 6 and 10 in the IR. The length-scale of the AdS space we expect in the UV from (4.27) matched those found to within 10^{-4}. The flow to the $z = 6$ Lifshitz spacetime is included as figure 4.3.

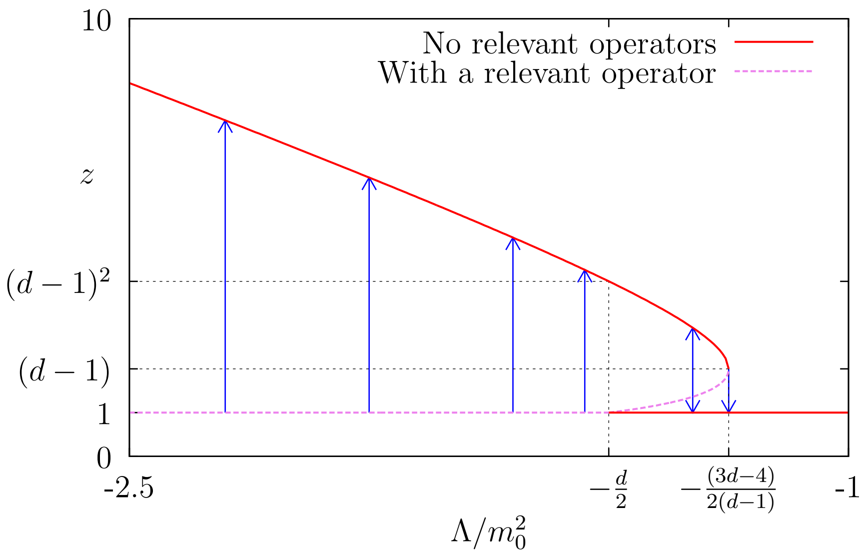

Figure 4.4: The vacua of the massive vector model in $d+1$ dimensions, labelled according to whether or not they possess an irrelevant perturbation within our ansatz. This plot was made using $d = 3$, but is qualitatively the same at higher d. The arrows denote the holographic RG flows. The existence of these can be guessed from the linearisations, but is not fully justified without the numerics. We have checked that examples of each of these flows exist in $d = 3$, 4 and 5.

In $d = 4$ such flows were checked to exist to $z = 9$, 15 and 20, and in $d = 5$ for $z = 16$, 25 and 30. The plots of these were qualitatively similar to figure 4.3, so they are not reproduced here.

We summarise the exact Lifshitz and AdS solutions, whether they possess a relevant operator and the holographic RG flows that we have found in figure 4.4. We should also note that we can rule out the existence of any other flows to these points in the IR - we have tried shooting in each irrelevant direction.

Chapter 5

6D $\mathcal{N} = 4$ gauged massive supergravity

5.1 Field content, ansatz and equations of motion

We now wish to find holographic flows involving Lifshitz spacetimes as solutions of a supergravity theory. It has already been shown in [41] that $\mathcal{N} = 4$ 6D gauged massive supergravity is capable of supporting such solutions over a range of z, with some region of the parameter space having two Lifshitz solutions, and this theory also possess an AdS solution. Therefore we might hope to reproduce each species of flow found in section 4.5.

The bosonic matter content of the theory consists of a dilaton, ϕ, a 2-form, $B_{\mu\nu}$, an $SU(2)$ vector field, $A_\mu^{(i)}$ and a $U(1)$ vector field $\mathcal{A}_\mu$. We ignore the fermion content. In our conventions (which differ from those of [41] in both the signature of the metric and the sign of the curvature tensor) the bosonic part of the action is

$$\begin{aligned} S = \int \sqrt{-g} \Bigg(& \frac{1}{4}R - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{e^{-\sqrt{2}\phi}}{4}\left(\mathcal{H}^{\mu\nu}\mathcal{H}_{\mu\nu} + F^{(i)\mu\nu}F^{(i)}_{\mu\nu}\right) \\ & - \frac{e^{2\sqrt{2}\phi}}{12}G_{\mu\nu\rho}G^{\mu\nu\rho} - \frac{1}{8}\epsilon^{\mu\nu\rho\lambda\sigma\tau}B_{\mu\nu}\left(\mathcal{F}_{\rho\lambda}\mathcal{F}_{\sigma\tau} + mB_{\rho\lambda}\mathcal{F}_{\sigma\tau} + \frac{m^2}{3}B_{\rho\lambda}B_{\sigma\tau} + F^{(i)}_{\rho\lambda}F^{(i)}_{\sigma\tau}\right) \\ & + \frac{1}{8}\left(g^2e^{\sqrt{2}\phi} + 4gme^{-\sqrt{2}\phi} - m^2e^{-3\sqrt{2}\phi}\right)\Bigg) \end{aligned} \tag{5.1}$$

where $\mathcal{H} = \mathcal{F} + mB$, and $\mathcal{F}$, F and G are respectively the field strengths for the $U(1)$ and $SU(2)$ gauge fields and the 2-form.

The ansatz we take for the metric is

$$ds^2 = -e^{2F(r)}dt^2 + r^2 \left(dx_1^2 + dx_2^2\right) + e^{2d(r)}\frac{dr^2}{r^2} + e^{2h(r)}\frac{1}{y_2^2}\left(dy_1^2 + dy_2^2\right) \tag{5.2}$$

where the hyperbolic directions y_1 and y_2 are compactified by modding out some discrete subgroup, and our ansatz for the matter fields is

$$\mathcal{F} = 0, \quad F^{(3)} = \frac{\alpha(r)\, e^{F(r)+d(r)}}{r} dt \wedge dr + \frac{\gamma}{y_2^2} dy_1 \wedge dy_2, \quad B = \frac{\bar{\beta}(r)}{2} r^2 dx_2 \wedge dx_2, \quad \phi = \phi(r) \tag{5.3}$$

γ is required to be constant by $dF^{(3)} = 0$ and from now on we will treat it as a parameter of the theory. The t component of the equation of motion for $F^{(3)}$ can be integrated to give

$$\alpha = \gamma \bar{\beta} e^{\sqrt{2}\phi} e^{-2h} \tag{5.4}$$

and we will replace all occurrences of α with this in the equations of motion below.

Before proceeding further we will make some field redefinitions to absorb some factors of g and m into the matter fields, and work with

$$\varphi = \sqrt{\frac{m}{g}} e^{-\sqrt{2}\phi}, \quad e^{-2H} = \frac{\gamma}{\sqrt{gm}} e^{-2h}, \quad \beta = \sqrt{\frac{m}{g}} \bar{\beta}, \quad e^{-2D} = \frac{1}{\sqrt{g^3 m}} e^{-2d} \tag{5.5}$$

and f. With these redefinitions the equations of motion will turn out to depend only on the combination $g^2\gamma^2$.

We will work with the radial coordinate $\rho = \log r$, as this further simplifies the equations of motion, and will be a more practical variable to work in for the numerical integration. We get 4 independent equations of motion

$$\partial_\rho\partial_\rho\beta = -2\partial_\rho\beta - (2\beta + \partial_\rho\beta)\left(\partial_\rho F + 2\partial_\rho H - 2\varphi^{-1}\partial_\rho\varphi - \partial_\rho D\right) + \varphi\left(\varphi^2 + 4e^{-4H}\right)e^{2D}\beta \tag{5.6}$$

$$\partial_\rho\partial_\rho\varphi = -\partial_\rho\varphi\left(\partial_\rho F + 2\partial_\rho H - \varphi^{-1}\partial_\rho\varphi - \partial_\rho D + 2\right) + \frac{1}{4}\left(\left(\varphi^2 - 4e^{-4H}\right)\beta^2 + 1 - 4\varphi^2 + 3\varphi^4 + 4\varphi^2 e^{-4H}\right)e^{2D} - \frac{1}{2}\varphi^{-1}\left(\partial_\rho\beta + 2\beta\right)^2 \tag{5.7}$$

$$\partial_\rho\partial_\rho F = -\partial_\rho F\left(2 + \partial_\rho F + 2\partial_\rho H - \partial_\rho D\right) + \frac{1}{8}e^{2D}\varphi^{-1}\left(\left(\varphi^2 + 12e^{-4H}\right)\beta^2 + 1 + 4\varphi^2 - \varphi^4 + 4\varphi^2 e^{-4H}\right) + \frac{1}{4}\varphi^{-1}\left(\partial_\rho\beta + 2\beta\right)^2 \tag{5.8}$$

$$\partial_\rho\partial_\rho H = \frac{1}{8}\varphi^{-1}e^{2D}\left(\left(\varphi^2 - 4e^{-4H}\right)\beta^2 + 1 + 4\varphi^2 - \varphi^4 - 12\varphi^2 e^{-4H}\right) + \frac{1}{4}\varphi^{-2}\left(\partial_\rho\beta + 2\beta\right)^2 - \left(\partial_\rho F + 4\partial_\rho H - \partial_\rho D + 2\right)\partial_\rho H - \frac{1}{g\gamma}e^{-2H}e^{2D} + \left(\partial_\rho H\right)^2 \tag{5.9}$$

We also have the following

$$e^{-2D} = \left(-\frac{1}{g\gamma}e^{-2H} + \frac{1}{4}\varphi^{-1}\left(-\left(\varphi^2 + 4e^{-4H}\right)\beta^2 + 1 + 4\varphi^2 - \varphi^4 - 4\varphi^2 e^{-4H}\right)\right) \Big/ \left(1 + \left(\partial_\rho H\right)^2 + 4\left(1 + \partial_\rho H\right)\partial_\rho F - \frac{1}{2}\left(\varphi^{-1}\partial_\rho\varphi\right)^2 + 4\partial_\rho H - \frac{1}{4}\varphi^{-2}\left(\partial_\rho\beta + 2\beta\right)^2\right) \tag{5.10}$$

$$\partial_\rho D = \left(\partial_\rho F + 2\partial_\rho H + 2 + \frac{1}{4}\varphi^{-2}\left(\partial_\rho\beta + 2\beta\right)^2\right) - \frac{1}{8}\varphi^{-1}\left(-\left(3\varphi^2 + 4e^{-4H}\right)\beta^2 + 1 + 4\varphi^2 - \varphi^4 + 4\varphi^2 e^{-4H}\right) \tag{5.11}$$

which should be regarded as algebraic equations for e^{-2D} and $\partial_\rho e^{-2D}$. (5.11) is redundant, however it simplifies matter to have to available in this form. After substituting for (5.10) and (5.11), we have 4 coupled second order ODEs for the 4 fields (β, φ, F, H), or equivalently an 8 dimensional first order vector ODE. We do not explicitly do this substitution as the equations that result would be unwieldy.

5.2 Lifshitz solutions

Translating into our conventions[1], [41] find that the equations of motion generated by (5.1) are satisfied by

$$F = z\rho, \quad e^{-2H} = \frac{\left((z+2)(z-3) \pm 2\sqrt{2z+8}\right)^{1/2}}{2\sqrt{z}\,(z+4)^{1/4}\left(6+z\mp 2\sqrt{2z+8}\right)^{1/4}}$$

$$\varphi = \left(\frac{6+z\mp 2\sqrt{2z+8}}{z^2(z+4)}\right)^{\frac{1}{4}}, \quad \beta = \left(\frac{6+z\mp 2\sqrt{2z+8}}{z^2(z+4)}\right)^{\frac{1}{4}}\sqrt{z-1} \qquad (5.12)$$

provided that

$$g^2\gamma^2 = \frac{(z+4)\left((z+2)(z-3)\pm 2\sqrt{2z+8}\right)}{\left(3z+6\mp 2\sqrt{2z+8}\right)^2} \qquad (5.13)$$

I will refer to these as the *upper sign* and *lower sign* Lifshitz solutions. We might worry that $\beta \to -\beta$ is a symmetry of the equations of motion, and produces another pair of Lifshitz solutions, but these are identical in every other field. It is easy to show that $\{\partial_\rho \beta = \beta = 0\}$ is an invariant manifold, and therefore that we cannot find a holographic RG flow between $\beta > 0$ and $\beta < 0$, so we ignore $\beta < 0$ from now on.

For the lower sign solution, the requirement that $g^2\gamma^2 > 0$ restricts us to $z > \sqrt{2} + \frac{1+\sqrt{\sqrt{32}+17}}{2} \approx 4.294$. This branch of solutions exists for all $g^2\gamma^2 > 0$.

For the upper sign solution, we are restricted to $z > 1$ by reality of β. This solution exists for $g^2\gamma^2 > \frac{30-10\sqrt{10}}{36\sqrt{10}-121} \approx 0.227$. As $z \to 1$ this branch connects to one of the branches of AdS solutions found in the next section.

5.3 AdS solutions

This theory also possess AdS solutions, but only in a restricted range of $g^2\gamma^2$. It suffices to label the AdS spacetimes by φ^2, and then

$$g^2\gamma^2 = -\frac{(1-\varphi^2)(1-3\varphi^2)}{(1-2\varphi^2+2\varphi^4)^2} \qquad (5.14)$$

For $0 < g^2\gamma^2 < \frac{9-\sqrt{216}}{\sqrt{1536}-44} \approx 1.185$ this has two solutions for φ^2, one either side of $1 - 1/\sqrt{6} \approx 0.592$. I refer to a solution with $\varphi^2 \in \left(\frac{1}{3}, 1 - \frac{1}{\sqrt{6}}\right)$ as a *small* φ *AdS*

[1] In particular, note that γ here is not the same as in [41]

solution and one with $\varphi^2 \in \left(1 - \frac{1}{\sqrt{6}}, 1\right)$ as a *large* φ *AdS solution*. These have

$$F = \rho, \quad \beta = 0, \quad e^{-2H} = \frac{\sqrt{(1-\varphi^2)(3\varphi^2 - 1)}}{2\varphi} \tag{5.15}$$

Note that as $z \to 1$ the upper sign branch of Lifshitz solutions joins on to the small φ branch of AdS solutions, at $\varphi^2 \approx 0.3675$.

It can be shown that $\{\partial_\rho \beta = \beta = 0, \partial_\rho F = 1,\ F = \rho\}$ is also an invariant manifold of this dynamical system.

5.4 Linearisations

We will now linearise the equations of motion around the AdS and Lifshitz solutions. This will tell us the dimensions of the operators in the dual field theory, or equivalently the behaviour of a small perturbation from one of these spacetimes as we integrate the equations of motions radially.

In the case of the AdS spacetimes, many of the fields decouple from one another and we are able to solve the linearised equations analytically. As we vary φ, we find that there are AdS spaces with one, two and three irrelevant operators within this ansatz. There is also a range of φ in which the dimension of one of the operators is complex, indicating a Breitenlohner-Freedman type instability.

Due to the complexity of the Lifshitz solutions, the equations of motion were linearised around these using a computer algebra system, and the eigenvalue problem had to be solved numerically. We will find that the lower sign Lifshitz solution always has two irrelevant operators. For sufficiently large z this space only has real operator dimensions, and so does not suffer from an instability. The upper-sign Lifshitz solution always possesses a single irrelevant operator and has only a small range of z for which the operator dimensions are all real.

Our ansatz has only considered modes which vary with r, whereas Breitenlohner-Freedman instabilities are really dynamical instabilities in which the magnitude of small amplitude Fourier modes grow exponentially. However, looking at (3.14) suggests that when Δ is complex we might expect eigenvalues of the ∂_t^2 operator to be complex, which would lead to such growing modes. When such an instability is present we do not necessarily expect the spacetime to have a field theory dual, and so our use of the term 'operator dimension' above to refer to Δ may be inappropriate in these cases.

5.4.1 Linearisations around AdS

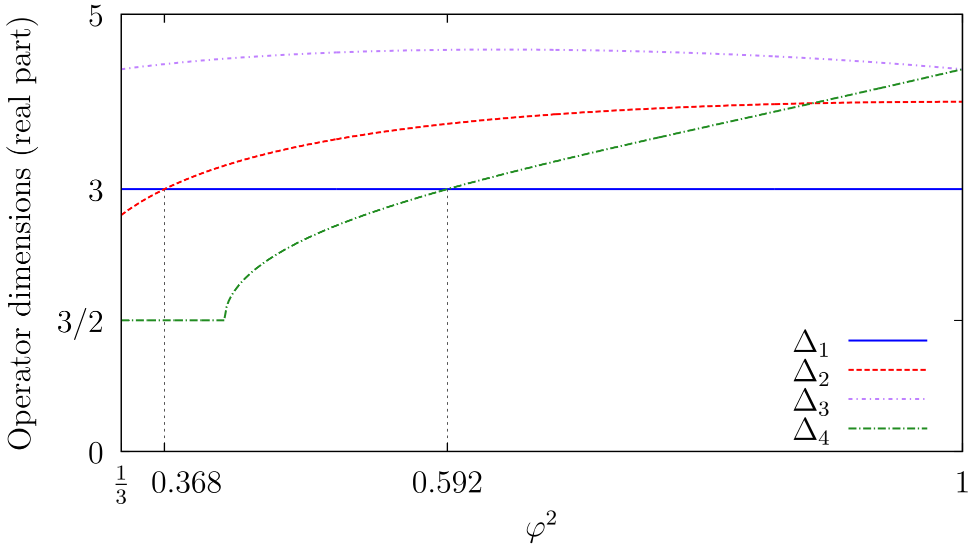

Figure 5.1: Real part of operator dimensions of the field theory dual to AdS solutions. Note that an operator is irrelevant if $\Delta > 3$.

This can be done analytically. The fact that $\{\partial_\rho \beta = \beta = 0\}$ and $\{\partial_\rho \beta = \beta = 0, \partial_\rho F = 1,\ F = \rho\}$ are invariant suggest that the β, F directions should decouple, and they do.

At the linear level we have solutions $\delta F \sim F_0 + F_1 r^{-3}$, so we have an operator (the energy density) of dimension $\Delta_1 = 0$.

We also have $\delta\beta \sim \beta_0 r^{\Delta_2 - 3} + \beta_1 r^{-\Delta_2}$ where

$$\Delta_2 = \frac{3}{2} + \frac{1}{2\varphi}\sqrt{\frac{(12 - 7\varphi^2)(7\varphi^2 - 2)}{(2 - \varphi^2)}} \tag{5.16}$$

This operator is irrelevant for $\varphi^2 > 1 - \sqrt{\frac{2}{5}} \approx 0.368$.

The φ and H directions mix, and have solutions

$$\begin{aligned}
\delta\varphi &= \varphi_5 r^{\Delta_3 - 3} + \varphi_6 r^{-\Delta_3} + \varphi_7 r^{\Delta_4 - 3} + \varphi_8 r^{-\Delta_4} \\
\delta H &= H_5 r^{\Delta_3 - 3} + H_6 r^{-\Delta_3} + H_7 r^{\Delta_4 - 3} + H_8 r^{-\Delta_4}
\end{aligned} \tag{5.17}$$

with operator dimensions

$$\Delta_3 = \frac{3}{2} + \frac{\sqrt{3}}{2\varphi}\sqrt{\frac{-7\varphi^4 + 22\varphi^2 - 4 + 4\left(1-\varphi^2\right)\sqrt{25\varphi^4 - 6\varphi^2 + 1}}{\left(2-\varphi^2\right)}} \tag{5.18}$$

$$\Delta_4 = \frac{3}{2} + \frac{\sqrt{3}}{2\varphi}\sqrt{\frac{-7\varphi^4 + 22\varphi^2 - 4 - 4\left(1-\varphi^2\right)\sqrt{25\varphi^4 - 6\varphi^2 + 1}}{\left(2-\varphi^2\right)}} \tag{5.19}$$

with the coefficients satisfying

$$\begin{aligned} \frac{\varphi_5}{H_5} = -\frac{\varphi_6}{H_6} &= \frac{2\varphi^2\left(4\varphi^2 + \sqrt{25\varphi^4 - 6\varphi^2 + 1}\right)}{\left(3\varphi^2 - 1\right)^{3/2}\left(1-\varphi^2\right)^{1/2}}, \\ \frac{\varphi_7}{H_7} = -\frac{\varphi_8}{H_8} &= \frac{2\varphi^2\left(4\varphi^2 - \sqrt{25\varphi^4 - 6\varphi^2 + 1}\right)}{\left(3\varphi^2 - 1\right)^{3/2}\left(1-\varphi^2\right)^{1/2}} \end{aligned} \tag{5.20}$$

Δ_4 is complex for $\varphi^2 < \frac{37-\sqrt{433}}{39} \approx 0.4152$.

The operator dimensions are summarised in figure 5.1. Note that $-\Delta < 0$ for all operators, making shooting from large r to small r is impractical - numerical error will generate some perturbation along these directions, and we would miss the fixed point we were aiming for at small r. In field theory language, the numerical error would move us out of the vacuum state, and this non-trivial state would dominate the IR physics (typically by being at finite temperature and hence introducing an event horizon in the bulk.) Note than from now on we will consistently use the verb *shoot* to mean to numerically integrate *from small* r *to large* r. This is opposite to the direction of a *flow*.

Since $\Delta_4 - 3$ changes sign at the boundary between the small φ and large φ branches, we can guess that shooting along this direction will give us a flow from small φ AdS in the UV to large φ AdS in the IR. The fact that the operator associated to Δ_2 changes from being relevant to irrelevant suggests that there will be both Lifshitz to AdS and AdS to Lifshitz flows. We could also shoot along the Δ_3 direction, however I postpone discussion of this to section 5.5.1 after we have seen the other linearisations.

5.4.2 Linearisation around Lifshitz

Nothing obviously decouples around the Lifshitz fixed points, so a computer algebra system was used to do these linearisations and to compute the eigenvalues of the

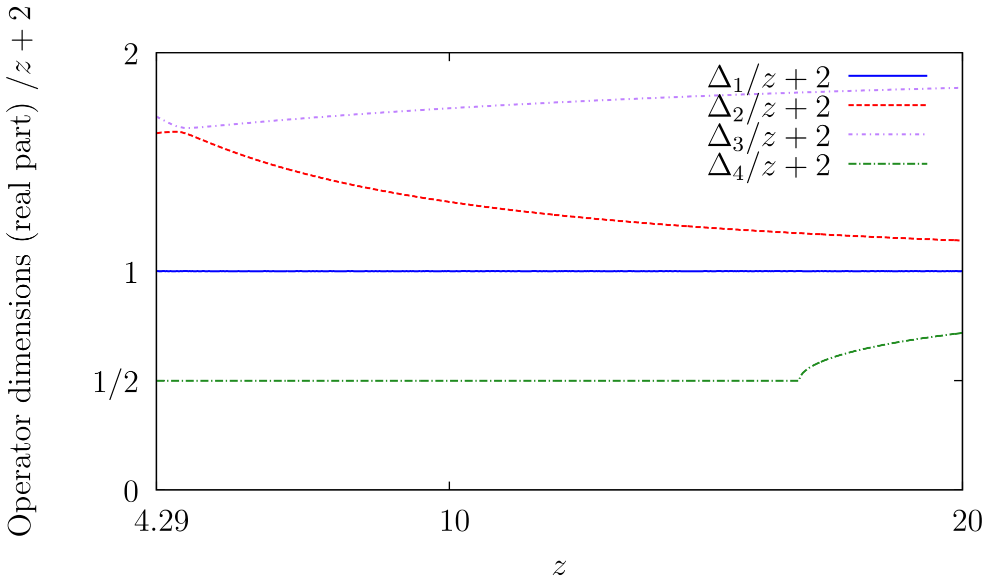

Figure 5.2: Real part of operator dimensions of the field theory dual to lower sign Lifshitz solution. Note that an operator is irrelevant if $\Delta > z+2$.

flow matrix numerically. The operator dimensions are plotted in figures 5.2 and 5.3.

None of the operators change from being relevant to irrelevant within the range plotted. I have looked at up to $z = 100$ and found no such changes. It can be seen that $\Delta_1 = z+2$, and it appears that Δ_2, $\Delta_4 \sim z+2$ as $z \to \infty$. The perturbation associated to the Δ_1 source is entirely in the field F, so this still corresponds to globally rescaling the t coordinate.

Around the lower sign Lifshitz solution, Δ_4 is complex for $z \lesssim 16.8221$. Around the upper sign Lifshitz solution, Δ_4 is complex for $z \lesssim 5.6927$, and both Δ_4 and Δ_2 are complex for $z \gtrsim 5.8329$.

5.5 Numerical flows

The numerical integration was done using RK4 with a fixed step-length of $\Delta\rho = 0.001$. The dynamical system was initialised to either an AdS or Lifshitz point at $\rho = 1$, plus some small perturbation (of size 0.001 in our field variables). Since such a perturbation does not necessarily put us on exactly the trajectory we want, or indeed a trajectory that decays to it in the case that there is more than one positive eigenvalue, we searched (by interval bisection) in the space of possible directions on

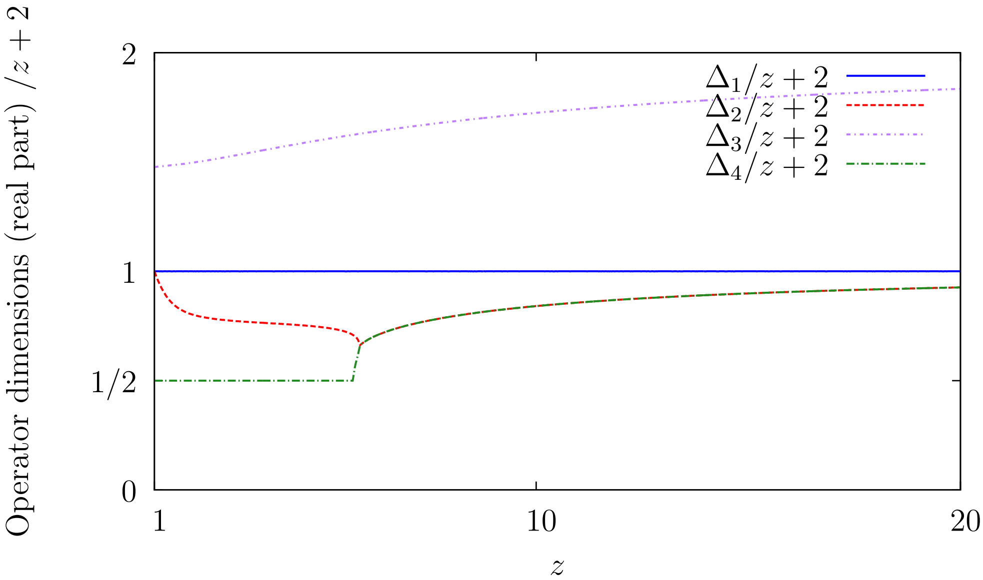

Figure 5.3: Real part of operator dimensions of the field theory dual to upper sign Lifshitz solution. Note that an operator is irrelevant if $\Delta > z + 2$.

a 2D plane spanned by two unstable eigenvectors. This should be sufficient to do any necessary fine tuning, unless we are shooting from the AdS fixed point with $\varphi \gtrsim 0.592$. In practise it turns out that we can still make some progress within this range anyway.

5.5.1 Flows from 6D AdS

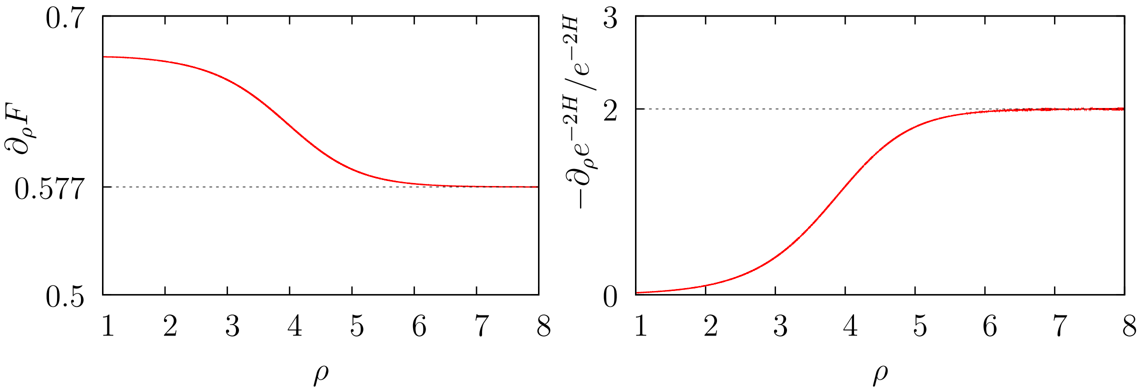

Figure 5.4: Holographic RG flow from the 6D spacetime (5.21) to the $\varphi^2 = 0.45$ AdS solution. $F = \rho$, $\beta = 0$ throughout this flow.

Before attempting to find flows between the fixed points described in sections

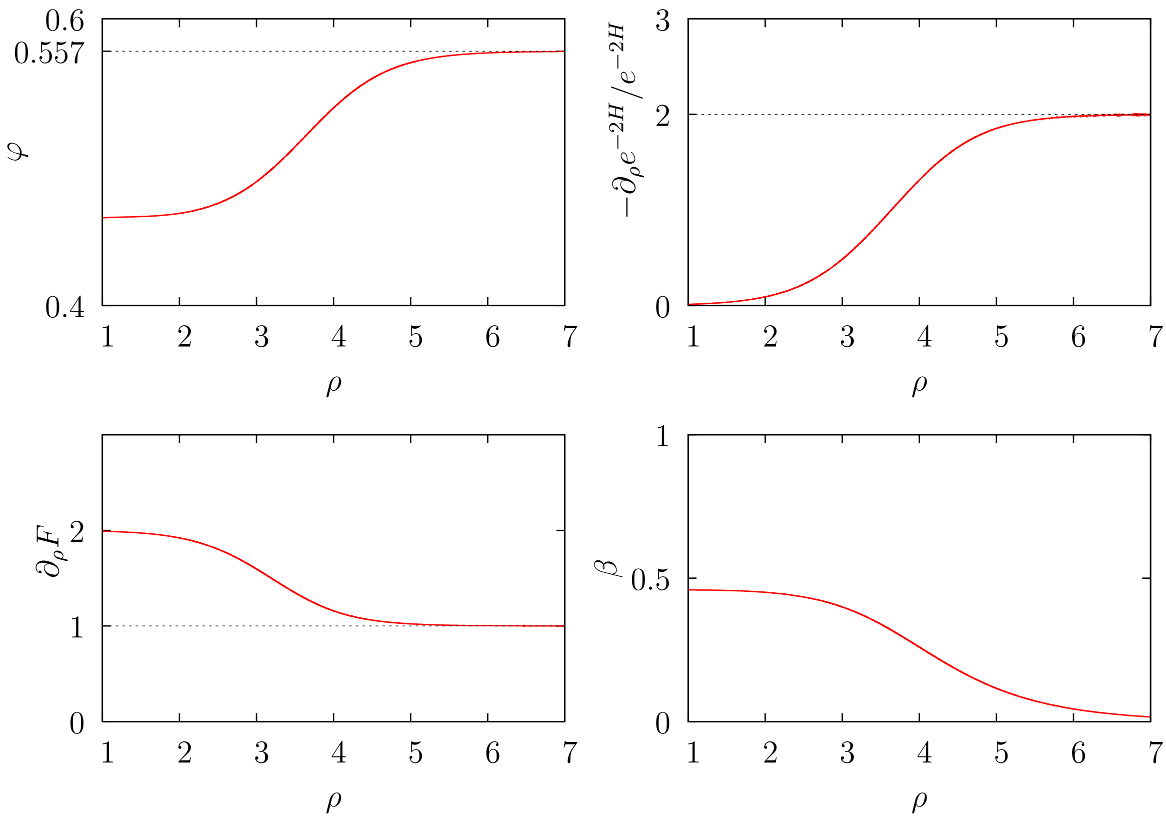

Figure 5.5: Holographic RG flow from the 6D spacetime (5.21) to the $z = 2$ upper sign Lifshitz solution.

5.2 and 5.3 we shall first shoot along the direction associated to the Δ_3 operator about the AdS and Lifshitz fixed points. Note that as this is the "most irrelevant" operator, and in the later sections most of our numerical efforts will be spent tuning out perturbations along this direction, and the results of this section will help us to do so.

Shots along the Δ_3 direction from the AdS solutions with $\varphi^2 = 0.35, 0.45$ and 0.8, with $\delta\varphi < 0$ were all qualitatively the same. The shot from $\varphi^2 = 0.45$ is reproduced as figure 5.4. $\varphi^2 \to \frac{1}{3}$ was a common feature of these shots. We can see that e^{2H} scales like r^2 in the UV, and that e^{2D} tends to some finite, non-zero value. This is interpreted as a holographic flow from a 6 dimensional spacetime

$$ds^2 = -r^2 dt^2 + r^2\left(dx_1^2 + dx_2^2\right) + \sqrt{g^3 m} e^{2D}\frac{dr^2}{r^2} + C\frac{r^2}{y_2^2}\left(dy_1^2 + dy_2^2\right) \qquad (5.21)$$

where C is some constant. Shooting in the opposite direction to this, $\delta\varphi > 0$, resulted in $e^H \sim r^{-6}$, but also $\varphi^2 \sim r^4$.

Shooting from either the upper or lower sign Lifshitz solutions with $\delta\varphi > 0$ also gave a flow to (5.21). This was tried with $z = 2$, 4, and 10 from upper sign branch and $z = 5$, 10 and 25 from the lower sign branch. The shot from $z = 2$ is reproduced as figure 5.5. Shooting in the $\delta\varphi < 0$ direction resulted in $\varphi \to \infty$, $e^{2H} \to 0$, but in these cases there is no power law scaling of either of these variables.

5.5.2 AdS $\to$ AdS flows

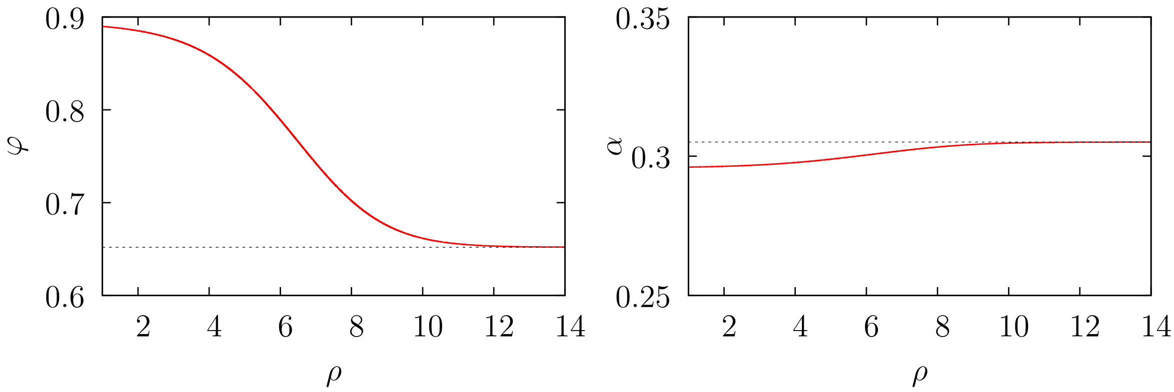

Figure 5.6: Holographic RG flow from an AdS space with $\varphi^2 = 0.425$ to an AdS space with $\varphi^2 = 0.8$. The dashed lines show the exact values of φ and e^{-2H} of the small φ AdS space we expected to hit. $F = \rho$ and $\beta = 0$ throughout the flow, as expected.

The change of the operator associated to Δ_4 from relevant to irrelevant as φ is increased past $\varphi^2 = 1 - \frac{1}{\sqrt{6}}$ strongly suggests that every small φ AdS solution possesses a flow to the corresponding large φ AdS solution. It initially looks like shooting from $\varphi^2 \in \left(1 - \frac{1}{\sqrt{6}}\right)$ will involve searching among 3 unstable directions. However, one of these (Δ_2) only involves the $\delta\beta$, $\delta\partial_r\beta$ directions. This does not 'mix' with the other directions, even at the non-linear level, due to the invariance of $\{\partial_\rho\beta = \beta = 0\}$, nor will numerical error produce a perturbation along this direction. Therefore we can ignore this direction and just search in the plane spanned by the unstable eigenvectors of the linearisation associated to the Δ_3, Δ_4 directions.

Looking at the unstable manifold of the small φ AdS point, and ignoring the β directions, we find that we can label which half of it a flow has been attracted to entirely by whether φ is larger or smaller than the value corresponding to the fixed point. We now have some sense of which direction we've missed by, and can proceed to tune a shot to hit the fixed point. We expect the flow that hits the other AdS point to be very close to the Δ_4 direction. Indeed we found that the starting direction asymptotes to this eigenvector (with $\varphi_7 < 0$ in the notation of (5.20)) as we reduce the size of the perturbation.

Shots were made from each of $\varphi^2 = 0.62$,0.65,0.7,0.75,0.8,0.85,0.9,0.95. For $\varphi^2 > 0.7$ we could tune the shot to come very close to the small φ AdS point in the UV. For the $\varphi^2 = 0.62$ and $\varphi^2 = 0.65$ shots, the interval we are bisecting becomes smaller than the precision of the variable we used before we came very close to the fixed point. However, the fact that shots from either end of this small interval miss by different directions suggest that such a flow exists, can could be found to arbitrary accuracy by using higher precision variables. A typical example of such a flow, from $\varphi^2 = 0.425$ to $\varphi^2 = 0.8$, is reproduced as figure 5.6.

5.5.3 AdS $\to$ Lifshitz flows

Any flow from AdS to Lifshitz (or vice versa) must involve the source corresponding to Δ_2 at the AdS end (since only this can take us off the $\{\partial_\rho\beta = \beta = 0\}$ subspace.) So to have the AdS solution at the UV end of the flow requires $\Delta_2 < 3$, hence $g^2\gamma^2 \lesssim 0.227$. We understand what happens when we shoot from either of the Lifshitz solutions along their Δ_4 direction, so we need the Lifshitz end of the flow to have a second irrelevant operator, which only the lower sign Lifshitz solution possesses (Δ_2). The range $0 < g^2\gamma^2 \lesssim 0.227$ corresponds to $4.294 \lesssim z \lesssim 7.066$. We already understand the unstable manifold around the space we are aiming at, so we can proceed to find flows by shooting.

Trying shots from the lower sign Lifshitz point with $z = 4.5$,5,6 and 6.5 we were able to tune the shot to come very close to the field values of the AdS fixed point in the UV. The shot from $z = 5$ is reproduced as figure 5.7. The 'dip' in the φ plot in figure 5.7 is also present in the shot from $z = 4.5$, but not in the shots for $z = 6$ or 6.5. This seems to be the only qualitative difference between different flows of this type. As predicted these fine tuned flows do indeed depart the IR end along the Δ_2

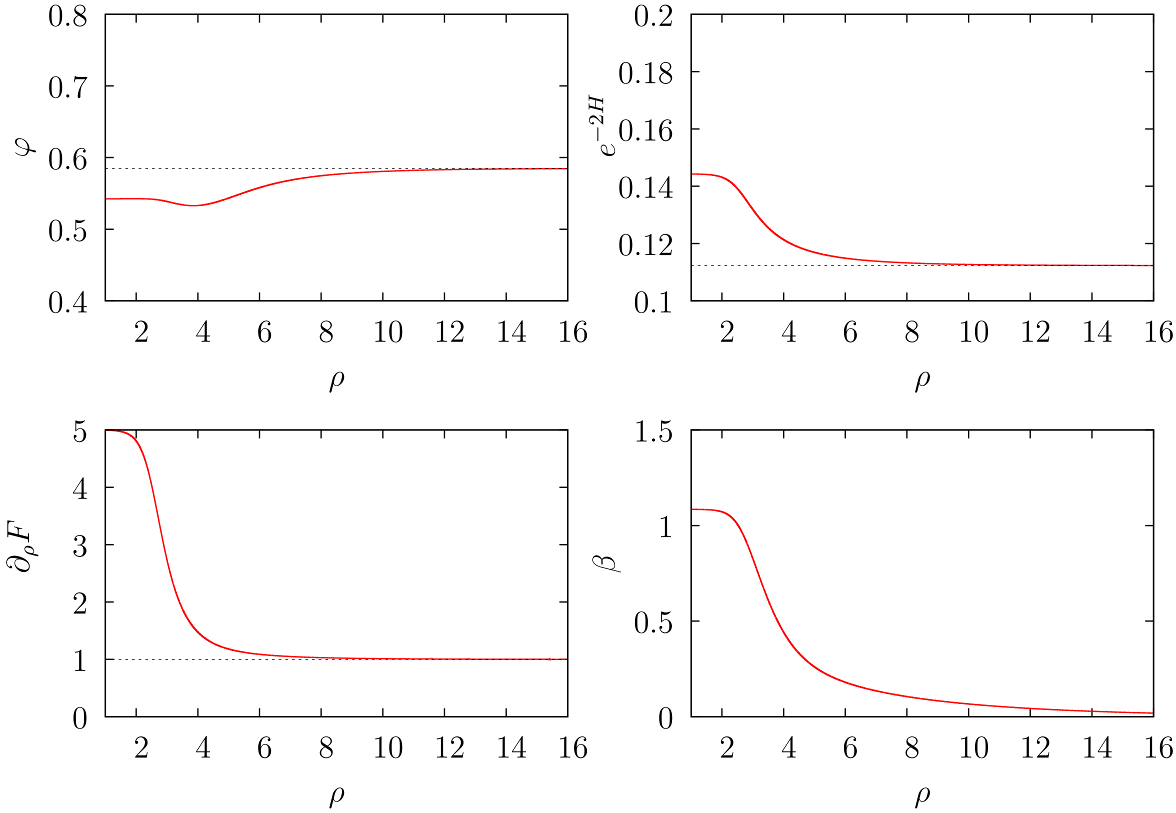

Figure 5.7: Holographic RG flow from an AdS space with $\varphi^2 = 0.342$ to a Lifshitz space on the lower sign branch with $z = 5$. The dashed lines show the exact field values of the AdS space I expected to find in the UV. Note that $\partial_\rho F$ provides an estimate of z.

direction.

5.5.4 Lifshitz $\rightarrow$ Lifshitz flows

One might raise the question of what happens to the above flows at $z = 7.066$, since nothing special seems to occur in the IR, based on figure 5.2. We will see that such shots now hit the upper sign branch of Lifshitz solutions, which branches off from the small φ AdS solutions at $g^2\gamma^2 = 0.227$. Looking at figure 5.3 we can see that such points have a one dimensional unstable manifold, and it turns out that we can again identify which half of this our flow has been attracted to purely from the magnitude of φ.

Shooting from the lower sign Lifshitz solutions with $z = 7.5$ and $z = 9$ it was

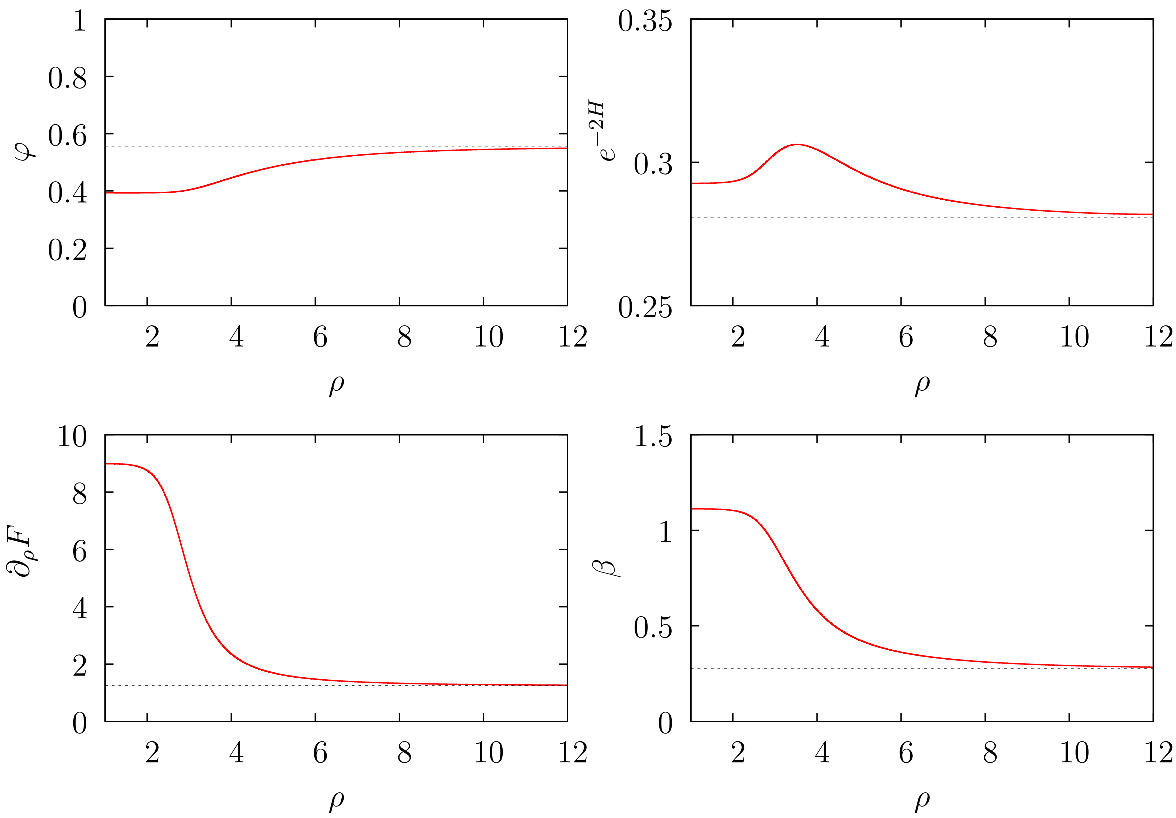

Figure 5.8: Holographic RG flow from an upper sign Lifshitz space with $z = 1.248$ to a Lifshitz space on the lower sign branch with $z = 9$. The dashed lines show the exact field values of the Lifshitz space I expected to find in the UV. Note that $\partial_\rho F$ provides an estimate of z.

possible to get very close to the field values for the appropriate upper sign Lifshitz solution in the UV. Shooting from $z = 15$ was more difficult, and required increasing the size of the initial perturbation to 0.005 to get reasonably close. As in previous cases, the existence of flows coming close to the fixed point, and then being attracted to different directions along the unstable manifold, suggests that such flows exist, but it would require the use of higher precision variables to find them accurately.

The shot from $z = 9$ is included as figure 5.8. At some point between this and the $z = 15$ shot, the value of e^{-2H} of the IR solution becomes less than that of the UV solution, and the 'bump' in this plot disappears.

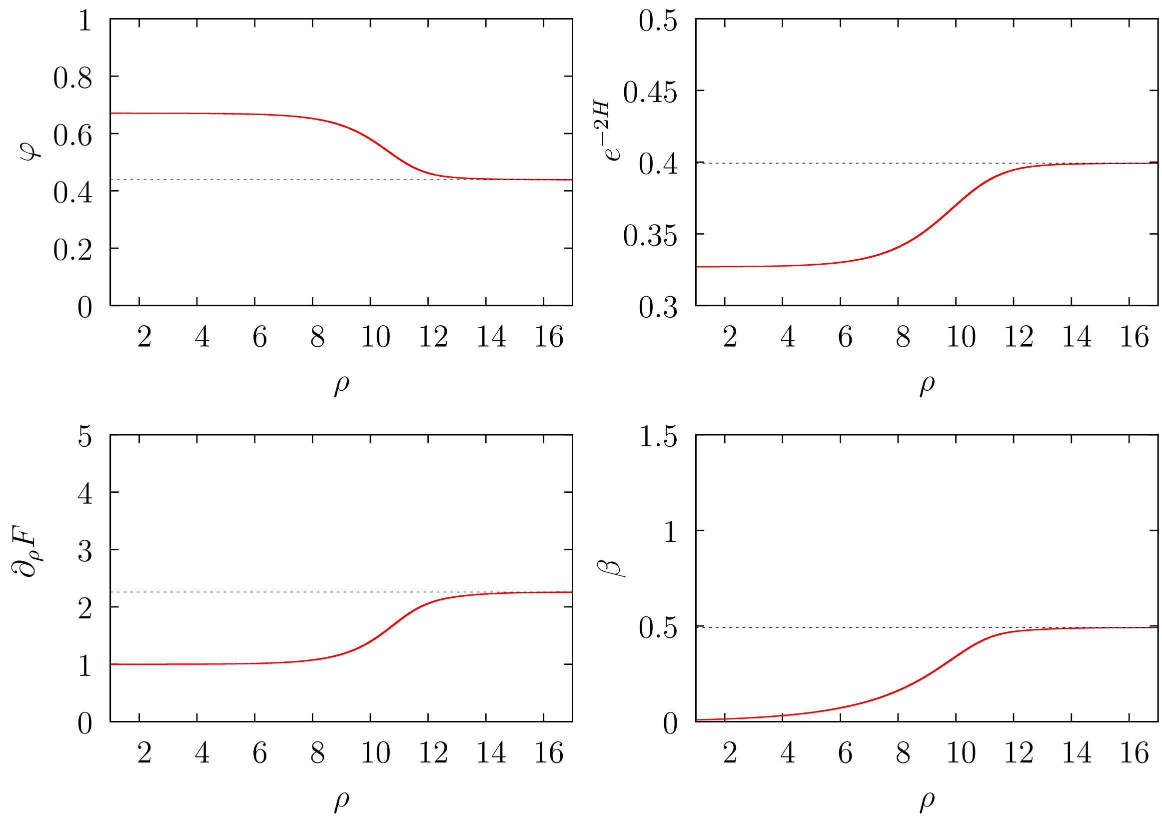

Figure 5.9: Holographic RG flow from an upper sign Lifshitz space with $z = 2.258$ to an AdS space on the small φ branch with $\varphi^2 = 0.45$. The dashed lines show the exact field values of the Lifshitz space I expected to find in the UV. Note that $\partial_\rho F$ provides an estimate of z.

5.5.5 Lifshitz $\rightarrow$ AdS flows

The linearisation suggests that an AdS spacetime with $\varphi^2 > 1 - \sqrt{\frac{2}{5}}$ could sit at the IR end of a flow with a Lifshitz spacetime in the UV. We should be able to make such shots easily from the small φ AdS branch. In fact, it will not even matter if we set off in the 'opposite direction' due to the $\beta \leftrightarrow -\beta$ symmetry. However, on the large φ AdS branch there are three unstable directions - in general I would expect small perturbations along the Δ_4 direction introduced through numerical error to make these shots impractical.

A shot from AdS with $\varphi^2 = 0.4$ came fairly close to the upper sign Lifshitz solution, and the shots from $\varphi^2 = 0.45$,0.5,0.55 and 0.58 come very close. The shots from $\varphi^2 = 0.65$,0.7,0.75 and 0.8 also hit the upper sign Lifshitz solution, despite our

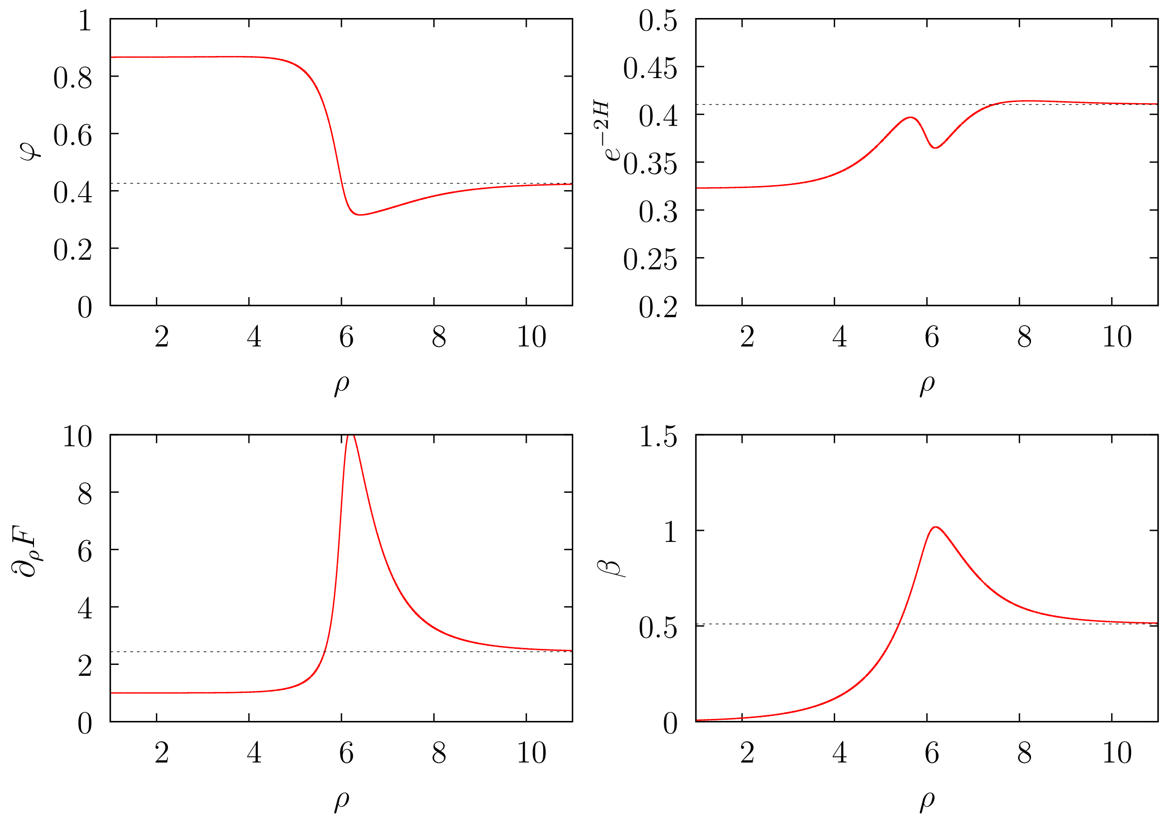

Figure 5.10: Holographic RG flow from an upper sign Lifshitz space with $z = 2.437$ to an AdS space on the large φ branch with $\varphi^2 = 0.75$. The dashed lines show the exact field values of the Lifshitz space I expected to find in the UV. Note that $\partial_\rho F$ provides an estimate of z.

initial pessimism. The shots from $\varphi^2 = 0.45$ and $\varphi^2 = 0.75$ are included as figures 5.9 and 5.10 respectively. The differences between these two flows is typical of the difference between flows from the same Lifshitz space to AdS spaces on different branches.

Since AdS solutions only exist for $0 < g^2\gamma^2 < \frac{9-\sqrt{216}}{\sqrt{1536}-44} \approx 1.185$, they can only be at the IR end of flows from upper sign Lifshitz spaces with $1 < z \lesssim 4.367$. Nothing obviously changes about the upper sign Lifshitz branch above this value of z, leading us to wonder what would happen if we could shoot inwards from such spaces.

Another question is what would happen if we shot from an AdS space with $0.9086 \lesssim \varphi^2 < 1$, since in this range Δ_2 is irrelevant, but the upper sign Lifshitz solutions do not exist. Shooting from $\varphi^2 = 0.95$ we have not been able to hit anything other than 6D spaces of section 5.5.1. However, our code doesn't allow us to search systematically in this region, due to the presence of 3 positive eigenvalues.

5.5.6 Li → AdS → AdS flows

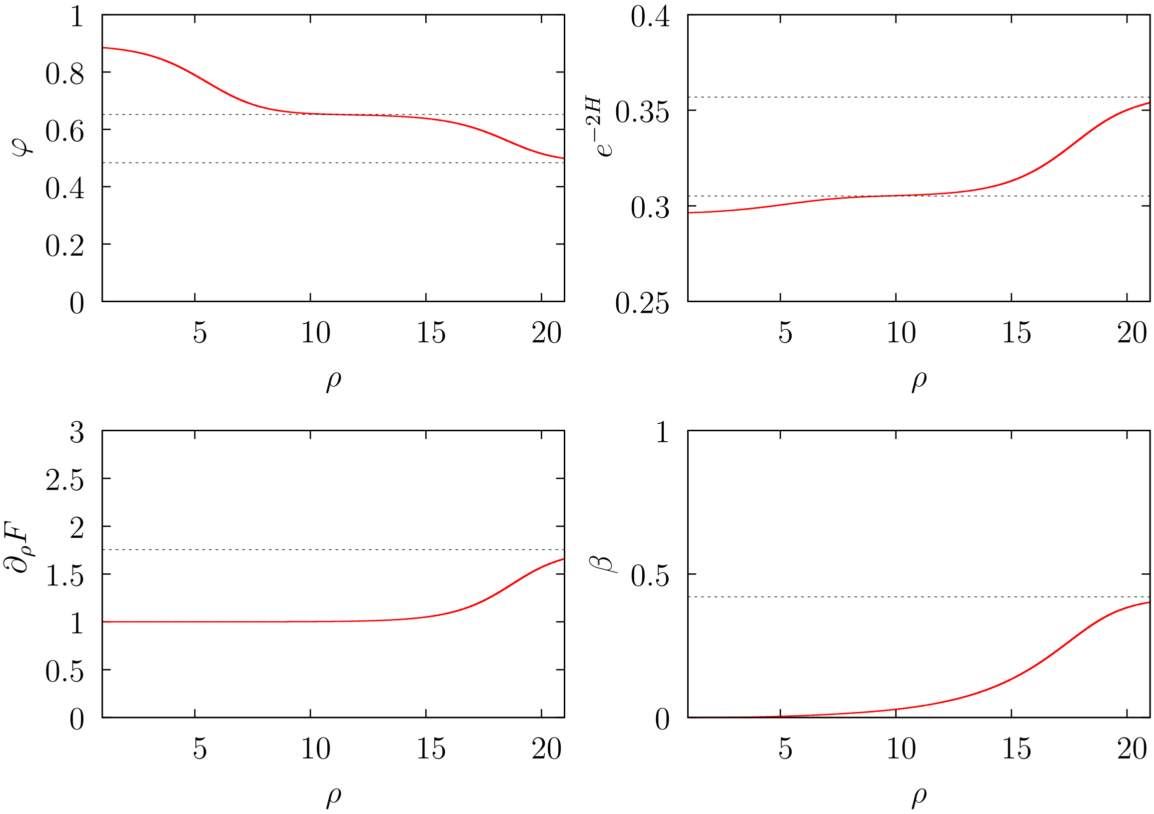

Figure 5.11: Holographic RG flow from an upper sign Lifshitz space with $z = 1.756$ to *a point close to* the AdS space on the small φ branch with $\varphi^2 = 0.234$, and finally to the AdS space on the large φ branch with $\varphi^2 = 0.8$. The dashed lines show the exact field values of the Lifshitz space I expected to find in the UV. Note that $\partial_\rho F$ provides an estimate of z.

The results of subsections 5.5.2 and 5.5.5 suggest the existence of another species of flow that we should be able to find numerically. If we were to make a shot from a large φ AdS solution along the direction that we expect to take us to the small φ AdS solution, plus a smaller perturbation in the direction associated to Δ_2, we would expect to pass very close to the small φ AdS solution, before being attracted to a shot leaving the small φ AdS point in its Δ_2 direction, finally hitting the upper sign Lifshitz solution in the UV.

Note that such a perturbation is within the unstable manifold of the large φ AdS point, and therefore that point genuinely is the IR limit of the flow. Note also that this will only work for the region of parameter space in which Δ_2 is positive for both AdS solutions and the upper sign Lifshitz solution exists, namely $0.227 \lesssim g^2\gamma^2 \lesssim 1.185$. Equivalently, we must shoot from an AdS spacetime in the IR with $0.592 \lesssim \varphi^2 \lesssim 0.909$.

Such a shot was made, from the $\varphi^2 = 0.8$ AdS point. The results are plotted as figure 5.11, and show that we do indeed have such a flow.

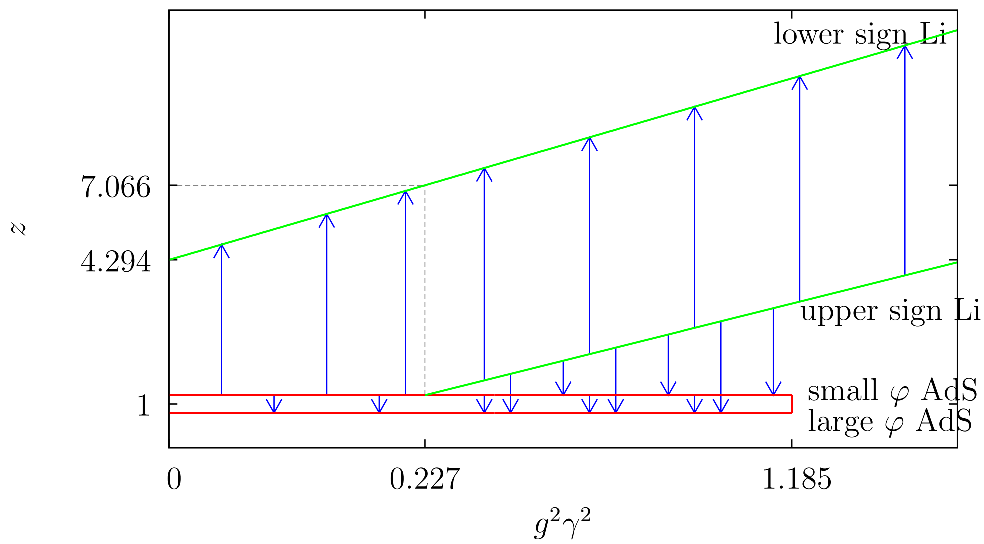

Figure 5.12: Summary of the solutions described in sections 5.2 and 5.3, and the flows between them found in sections 5.5.2 to 5.5.6. Note that this figure is purely schematic. It is not to scale, and the allowed values of z are not linear functions of $g^2\gamma^2$.

The flows found in sections 5.5.2 to 5.5.6 between the 4 dimensional solutions are summarised schematically in figure 5.12. We have found all the flows we expected to based on the linearisations in section 5.4. For each of these, based on the results of section 5.5.1, there will exist a very similar flow from (5.21) in the UV, connecting onto one of the flows in figure 5.12.

Chapter 6

Conclusions

We have reviewed a particular example of non-relativistic holography, where the boundary field theory possesses the Lifshitz scaling symmetry. In chapter 3 we have considered spacetimes dual to such theories, and described how the basic holographic dictionary get modified by the asymptotics of such spaces.

In chapter 4 we described a phenomenological model, a massive vector field coupled to Einstein gravity, capable of producing such spacetimes in an arbitrary number of dimensions. We noted that whether this model supported zero, one or two different Lifshitz spacetimes was dependent on the ratio of the cosmological constant to the square of the vector mass, Λ/m_0^2, and also that there exists an AdS solution for all $\Lambda < 0$. We then solved the linearised field equations analytically for perturbations around these solutions. Using the results of chapters 2 and 3 we were able to identify when the boundary field theory possesses a relevant operator dual to one of these fields, and hence when we might be able to perturb this theory to generate a renormalisation group flow to another of the fixed points in the IR.

We used numerical integration in section 4.5 to explicitly find examples of holographic renormalisation group flows and hence verify that our intuition based on the linearisations was correct. In the region $\Lambda/m_0^2 \leq -d/2$ there exists an AdS solution and a single Lifshitz solution which has $z \geq (d-1)^2$. There is an RG flow from the AdS solution to the Lifshitz solution. In the region $-d/2 < \Lambda/m_0^2 < -(3d-4)/2(d-1)$ there exists an AdS solution, and a pair of Lishitz solutions with dynamical exponents in the ranges $(1,(1-d))$ and $\left((1-d),(1-d)^2\right)$. Within this range, there exist two RG flows from the Lifshitz solution with smaller z - one to the AdS solution, and the other to the Lifshitz solution with larger z. At $\Lambda/m_0^2 = -(3d-4)/2(d-1)$ there is a single Lifshitz solution, which has a holographic RG flow to the AdS solution, as previously found in [1] for $d = 3$. We summarised these as figure 4.4.

In chapter 5 we essentially repeated the procedures of chapter 4, but in $\mathcal{N} = 4$ 6D massive gauged supergravity. This theory was shown in [41] to support 4D Lifshitz solutions with a range of dynamical exponents, and for some values of the parameters of the theory this also has AdS solutions. We showed that the equations of motion can be reduced to depend on the combination $g^2\gamma^2$, and therefore we use this to label regions of the parameter space. We linearised and solved analytically the equations of motion for perturbations around the AdS spacetimes. In the Lifshitz case we were unable to solve the linearised equations of motion analytically, and used numerics to extract the eigenvalues of the flow matrix. In both cases we were able to identify how many relevant and irrelevant operators the field theory dual possessed, and make educated guesses as to which RG flows should exist.

We first used numerics to show that each of the 4D spacetimes is at the IR end of a holographic RG flow from a 6D AdS spacetime, with boundary geometry $\mathbb{R}^{1,2} \times \mathbb{H}^2$.

We then used numerical integration to verify the existence of our conjectured flows between the 4D spacetimes. In the region $0 < g^2\gamma^2 \lesssim 0.227$ there exists a pair of AdS solutions and a single Lifshitz solution with $z \gtrsim 4.294$. Two RG flows exist from the AdS space with the larger dilaton (smaller φ) value - one to the AdS space with smaller dilaton value, and one to the Lifshitz space. In the region $0.227 \lesssim g^2\gamma^2 \lesssim 1.185$ there exist a pair of Lifshitz solutions with different dynamical exponents, z, and a pair of AdS solutions. The AdS to AdS flows still exists. There is an RG flow from the Lifshitz solution with smaller z to the one with larger z. The Lifshitz solution with smaller z also has RG flows to both AdS spaces. In the case of flows from this Lifshitz space to the small dilaton (large φ) AdS solution, it is possible to tune the flow to pass very close the large dilaton AdS solution. For $1.185 \lesssim g^2\gamma^2$, only the two Lifshitz solutions exist. There still exists an RG flow from the space with smaller z to the one with larger z. These 4D flows are summarised as figure 5.12.

These results extend previous work on holographic RG flows, such as the AdS to AdS flows of [15] and the Lifshitz to AdS flow found in [1]. However, the AdS to Lifshiz flows and the Lifshitz to Lifshitz flows are new, although asymptotically AdS spacetimes with Lifshitz scaling in the IR were found in [66] when a perfect fluid was included in the matter content. The Lifshitz to AdS flows are applicable to holographic condensed matter physics due to the existence of systems with emergent relativistic conformal symmetries in the IR. The fact that many of the Lifshitz solutions in the supergravity model are dynamically unstable is disappointing, particularly as we do not have a stable solution for $z = 2$ or 3, which are of particular physical interest. It would be interesting to see whether any of the other known supergravity constructions of Lifshitz spacetimes for $z = 2$ are dynamically stable. We are not aware of an analogue of the c-theorem of [15] for Lifshitz field theories. If a similar measure of the number of degrees of freedom could be found for these field theories, it would be interesting to see whether this quantity does decrease along all the holographic flows found here.

One use of the spacetimes constructed here would be as backgrounds on which to solve the equations of motion of probe fields to obtain correlation functions in a relevant deformation of the boundary field theory. Another potentially interesting piece of further work would be to investigate black holes in these spacetimes. In a different supergravity model [48] black hole solutions in Lifshitz asymptotics were found which had a minimum horizon radius and a horizon radius at which the specific heat of the black hole changed sign. In the supergravity model of chapter 5 it might be possible to numerically construct black holes that asymptote to a Lifshitz spacetime, with a non-zero source for one of the relevant operators. It would then be interesting to see whether such a black hole could be constructed with a sufficiently small horizon radius that we would find an intermediate region with AdS scaling. It might be possible to produce changes in the behaviour of probe field correlators by increasing the horizon radius to 'hide' the IR conformal scaling. Similar investigations could be performed with AdS asymptotics and intermediate Lifshitz regions.

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