GEOMETRY, DYNAMICS AND SPECTRAL ANALYSIS ON MANIFOLDS The Pestov Identity on Frame Bundles and Eigenvalue Asymptotics on Graph-like Manifolds

MICHELA EGIDI

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The Pestov Identity on Frame Bundles and Eigenvalue Asymptotics on Graph-like Manifolds

Michela Egidi

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Durham University

Pure Mathematics Research Group
Department of Mathematical Sciences
University of Durham
England

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Abstract

This dissertation is made up of two independent parts. In Part I we consider the Pestov Identity, an identity stated for smooth functions on the tangent bundle of a manifold and linking the Riemannian curvature tensor to the generators of the geodesic flow, and we lift it to the bundle T^kM of k-tuples of tangent vectors over a compact manifold M of dimension n. We also derive an integrated version over the bundle P^kM of orthonormal k-frames of M as well as a restriction to smooth functions on such a bundle. Finally, we present a dynamical application for the parallel transport of $\mathcal{G}^k_{or}(M)$, the Grassmannian of oriented k-planes of M. In Part II we consider a family of compact and connected n-dimensional manifolds X_ε, called graph-like manifold, shrinking to a metric graph as $\varepsilon \to 0$. We describe the asymptotic behaviour of the eigenvalues of the Hodge Laplacian acting on differential forms on X_ε in the appropriate limit. As an application, we produce manifolds and families of manifolds with arbitrarily large spectral gaps in the spectrum of the Hodge Laplacian.

Declaration

The work in this thesis is based on research carried out at the Pure Mathematics Research Group, the Department of Mathematical Sciences, England. No part of this thesis has been submitted elsewhere for any other degree or qualification. Part I and Appendix A are all my own work unless referenced to the contrary in the text. Part II is based on joint work with Olaf Post (Trier University). In particular, Chapter 7 is my own independent contribution unless referenced otherwise, and Chapter 8 contains our joint work.

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Contents

List of Figures

Part I

Chapter 1

Introduction

The Pestov Identity is an identity stated for smooth functions on the tangent bundle of a Riemannian manifold M. It links the generator of the geodesic flow with the Riemannian curvature tensor and other geometrically motivated differential operators and, therefore, it can be considered as a dynamical Weitzenböck identity on the tangent bundle.

It was first introduced by Pestov and Sharafudtinov in [PS88] to derive useful estimates on symmetric tensor fields and to give an answer to the question whether a smooth symmetric tensor field can be uniquely recovered from the knowledge of all its integrals along geodesics. Since then, it has been widely used to solve Geometric Inverse Problems such as tensor tomography, the boundary rigidity problem and spectral rigidity.

In this dissertation we lift this identity to the space of k-tuples of tangent vectors over a compact n-dimensional manifold M and we restrict it to the principal bundle of orthonormal k-frames. As an application, we use it to obtain an invariance property of smooth functions on Grassmannians under the parallel transport.

1.1 The Pestov Identity and its applications

Let (M, g) be a compact manifold of dimension n and let TM and SM be its tangent bundle and unit tangent bundle, respectively. Let $\pi : TM \longrightarrow M$ be the canonical projection of TM onto M. TM is a $2n$-dimensional manifold whose tangent space at $v \in TM$ splits as

$$T_vTM = \mathcal{H}_v \oplus \mathcal{V}_v \cong T_pM \times T_pM,$$

where $\mathcal{H}_v$ and $\mathcal{V}_v$ are called *horizontal* and *vertical distributions* at the point v, respectively. In fact, let $X : (-\varepsilon, \varepsilon) \longrightarrow TM$ be a curve in TM with $X(0) = v$, and let $\pi \circ X$ be its footpoint curve on M. Then,

$$T_vTM \ni X'(0) \cong \left(\frac{d}{dt}\Big|_{t=0} (\pi \circ X)(t), \frac{D}{dt}\Big|_{t=0} X(t) \right),$$

where $\frac{D}{dt}$ is the covariant derivative along $\pi \circ X$.

Hence, we define the two distributions as follows.

$$\mathcal{H}_v = \left\{ X'(0) \in T_vTM \;\Big|\; \frac{D}{dt}\Big|_{t=0} X(t) = 0 \right\} \cong \{(w, 0) \mid w \in T_{\pi(v)}M\} \cong T_{\pi(v)}M,$$

$$\mathcal{V}_v = \left\{ X'(0) \in T_vTM \;\Big|\; \frac{d}{dt}\Big|_{t=0} (\pi \circ X)(t) = 0 \right\} \cong \{(0, w) \mid w \in T_{\pi(v)}M\} \cong T_{\pi(v)}M.$$

Therefore, every vector $\xi \in T_vTM$ splits uniquely as $\xi = \xi^h + \xi^v$ with $\xi^h \in \mathcal{H}_v$ and $\xi^v \in \mathcal{V}_v$, called *horizontal* and *vertical component*, respectively.

We equip TM with the *Sasaki metric* [Dom62, GuKa02], defined as

$$\langle \xi, \eta \rangle_{TTM} = \langle \xi^h, \eta^h \rangle_{TM} + \langle \xi^v, \eta^v \rangle_{TM}. \tag{1.1.1}$$

The structure of TTM gives rise to horizontal and vertical differential operators, defined below.

Let $\psi \in C^\infty(TM)$ and denote by $u_w(t)$ the parallel transport of the vector u along the geodesic $c_w : (-\varepsilon, \varepsilon) \longrightarrow M$ with starting point $c_w(0) = \pi(w)$ and starting vector $c'_w(0) = w$. The gradient of ψ at $v \in TM$ is given by $\mathrm{grad}\psi(v) = (\overset{h}{\mathrm{grad}}\psi(v), \overset{v}{\mathrm{grad}}\psi(v))$ where horizontal and vertical component are define intrinsically as

$$\langle \overset{h}{\mathrm{grad}}\psi(v), w \rangle = \frac{d}{dt}\Big|_{t=0} \psi(v_w(t)) \qquad \text{and} \qquad \langle \overset{v}{\mathrm{grad}}\psi(v), w \rangle = \frac{d}{dt}\Big|_{t=0} \psi(v + tw).$$

In other words, they describe the derivative of ψ along the horizontal curve $t \mapsto v_w(t)$ and along the vertical curve $t \mapsto v + tw$.

Let $X : TM \longrightarrow TM$ be a *semi-basic vector field*, i.e., an element of $\mathfrak{X}(\pi^*(TM))$, $\pi^*(TM)$ being the pullback bundle of the vector bundle TM over M via the projection $\pi : TM \longrightarrow M$. The *horizontal* and *vertical covariant derivative* of X are given by

$$\overset{h}{\nabla}_w X(v) = \frac{D}{dt}\Big|_{t=0} X(v_w(t)) \qquad \text{and} \qquad \overset{v}{\nabla}_w X(v) = \frac{D}{dt}\Big|_{t=0} X(v + tw).$$

Consequently, the *horizontal* and *vertical divergence* of X are

$$\overset{h}{\mathrm{div}} X(v) = \sum_{i=1}^{n} \langle \overset{h}{\nabla}_{e_i} X(v), e_i \rangle \qquad \text{and} \qquad \overset{v}{\mathrm{div}} X(v) = \sum_{i=1}^{n} \langle \overset{v}{\nabla}_{e_i} X(v), e_i \rangle,$$

where $e_i, \dots, e_n$ is an orthonormal basis of T_pM for $p = \pi(v)$.

Finally, we define the *geodesic flow* on TM. Let $c_v : (-\varepsilon, \varepsilon) \longrightarrow M$ be the geodesic with initial vector $c_v'(0) = v$. The geodesic flow is the map

$$\phi^t : TM \longrightarrow TM, \qquad \text{such that} \qquad \phi^t(v) = c_v'(t), \tag{1.1.2}$$

and its generator is the vector

$$X_G(v) = \frac{d}{dt}\Big|_{t=0} \phi^t(v) \cong \left(\frac{d}{dt}\Big|_{t=0} (\pi \circ \phi^t)(v), 0 \right) = (v, 0),$$

i.e., $X_G(v)$ is a horizontal vector.

Theorem 1.1.1 (The Pestov Identity, [Kni02])**.** *Let M be a compact n-dimensional Riemannian manifold. For all $\psi \in C^\infty(TM)$, we have*

$$\begin{aligned} 2\langle \overset{h}{\mathrm{grad}} \psi(v), \overset{v}{\mathrm{grad}}(X_G \psi(v)) \rangle = \|\overset{h}{\mathrm{grad}} \psi(v)\|^2 + \overset{h}{\mathrm{div}} Y(v) + \overset{v}{\mathrm{div}} Z(v) \\ - \langle R(\overset{v}{\mathrm{grad}} \psi(v), v)v, \overset{v}{\mathrm{grad}} \psi(v) \rangle, \end{aligned} \tag{1.1.3}$$

where

$$Y(v) = \langle \overset{h}{\mathrm{grad}} \psi(v), \overset{v}{\mathrm{grad}} \psi(v) \rangle v - \langle v, \overset{h}{\mathrm{grad}} \psi(v) \rangle \overset{v}{\mathrm{grad}} \psi(v),$$

and

$$Z(v) = X_G \psi(v) \cdot \overset{h}{\mathrm{grad}} \psi(v) = \langle v, \overset{h}{\mathrm{grad}} \psi(v) \rangle \overset{h}{\mathrm{grad}} \psi(v).$$

The reader will find a coordinate-free proof in [Kni02, Appendix] and the original coordinate-based proof in [PS88] or [Sha94], both presented for manifolds of any dimension.

Most of the applications use the integrated version of this identity, where integration is performed over SM with respect to the Liouville measure $d\mu_L$.

Theorem 1.1.2 (Integrated Pestov's Identity, [Kni02])**.** *Let M be a compact n-dimensional Riemannian manifold and $\psi \in C^\infty(TM)$. Then,*

$$\begin{aligned} 2 \int_{SM} \langle \overset{h}{\mathrm{grad}} \psi(v), \overset{v}{\mathrm{grad}} X_G \psi(v) \rangle \, d\mu_L = \\ \int_{SM} \|\overset{h}{\mathrm{grad}} \psi(v)\|^2 \, d\mu_L + (n-1) \int_{SM} (X_G \psi(v))^2 \, d\mu_L \\ + \int_{SM} \langle \overset{v}{\nabla}_v Z(v), v \rangle \, d\mu_L - \int_{SM} \langle R(\overset{v}{\mathrm{grad}} \psi(v), v)v, \overset{v}{\mathrm{grad}} \psi(v) \rangle \, d\mu_L, \end{aligned} \tag{1.1.4}$$

where $Z(v)$ is as in Theorem 1.1.1.

The proof can be found in [Kni02, Theorem 1.1, Appendix]. It is based on the fact that the horizontal divergence vanishes under integration over SM while integration of the vertical divergence produces the second and third terms in the RHS (see [Kni02, Lemma 1.2, Appendix]).

The dynamical component of (1.1.3) and (1.1.4) lies in the presence of the generator of the geodesic flow, which is linked to the sectional curvature of the plane $span\{\overset{v}{\mathrm{grad}}\psi(v), v\}$. This indicates a close connection between properties of the geodesic flow and the curvature of the manifold. Such a connection is already well known in relation to ergodicity of the geodesic flow, see for example [Bal95].

Moreover, since the norm of the horizontal gradient (the first term in the RHS of (1.1.3)) is related to parallel transport, Theorems 1.1.1 and 1.1.2 provide a link between curvature conditions and invariant properties of functions under the action of the geodesic flow and under parallel transport.

Despite the strong dynamical flavour of these identities, their main applications are not in the field of pure Dynamics but in Integral Geometry and Inverse Problems. Below, we present a brief overview of three problems where the Pestov Identity plays a key role for the solution.

Tensor tomography. The material here presented has been extracted from [PSU13, PSU14a].

Tensor tomography is a subfield of integral geometry that studies how to recover a function or a tensor field by the knowledge of its integrals along curves. The simplest example is the X-ray (or Radon) transform in the plane, which aims at recovering a function f in $\mathbb{R}^2$ studying the integral of f along straight lines, i.e., geodesics in $\mathbb{R}^2$. This classical problem is nowadays well-known and well-studied. We refer the reader to [Hel11] for its properties and further information.

In the context of Riemannian manifolds, the problem of tensor tomography, or the geodesic ray transform problem, is posed as follows.

Let (M, g) be a compact, oriented Riemannian manifold of dimension $n \geqslant 2$ with boundary, and let ν be the unit outer normal to the boundary ∂M of M. Let SM

be its unit tangent bundle and let (p, v) be a point in SM, i.e., $v \in S_pM$, then

$$\partial(SM) = \partial_+(SM) \bigcup \partial_-(SM),$$

where $\partial_\pm(SM) = \{(p, v) \in \partial(SM) \mid \mp \langle \nu(p), v \rangle \geqslant 0\}$.

Without loss of generality, we think of M as embedded into a compact n-dimensional manifold N without boundary. The *exit time* $\tau : SM \longrightarrow [0, \infty]$ of a unit speed N-geodesic $\gamma_t(p, v)$ is $\tau(p, v) = \inf\{t > 0 \mid \gamma_t(p, v) \in N \backslash M\}$. If the geodesic $\gamma_t(p, v)$ never leaves the manifold M, we define $\tau(p, v) = \infty$. In the case $\gamma_t(p, v) < \infty$, the manifold M is called *non-trapping* and the *geodesic ray transform of a function* $f \in C^\infty(SM)$ is then defined as

$$If(p, v) = \int_0^{\tau(p,v)} f(\phi^t(p, v))\, dt, \qquad (p, v) \in \partial_+(SM),$$

where ϕ^t is the geodesic flow on M, and *the geodesic ray transform on a symmetric m-tensor* F is defined as $I_mF := If_m$, where f_m is the function on SM arising from the tensor F via $f_m(p, v) = F((p, v), \ldots, (p, v))$.

Given a smooth function f or a smooth m-tensor F, the geodesic ray transform problem explores what properties of f or F can be recovered from the knowledge of If or I_mF.

It is known [Sha94] that a sufficiently smooth tensor field F can be composed into a *solenoidal* and *potential part*, denoted by f^s and dp, respectively, i.e., $F = f^s + dp$, where f^s is a divergence free m-tensor field and p is a smooth $(m-1)$-tensor field vanishing at the boundary.

Using integration by parts and the fact that p vanishes on the boundary, it is easy to see that the geodesic ray transform of dp, the potential part of f, vanishes. Therefore, we can only aim at recovering the solenoidal part of f, which justifies the notion of *s-injectivity*, defined as follows. The X-ray transform on symmetric m-tensor fields, $m \geqslant 1$, is s-injective if $I_mF = 0$ implies $f^s = 0$ for any smooth m-tensor F. If $m = 0$, i.e., in the case of functions, I_0 is s-injective if $I_0 f = 0$ implies $f = 0$ for any $f \in C^\infty(SM)$.

A number of results are known about s-injectivity. I_0 and I_1 are s-injective [Muk77, AR97]. I_m is s-injective for all m on *simple surfaces* [PSU13], i.e., surfaces with strictly convex boundary and such that for any two points there exists a unique geodesic joining them and depending smoothly on the end-points. Moreover, I_m is known to be s-injective on manifolds of negative curvature [PS88], under other curvature restrictions [Sha94], or on higher dimensional simple or Anosov manifolds with certain conditions on modified Jacobi fields [PSU15]. I_2 is s-injective for manifolds of any dimension equipped with simple metrics including real-analytic ones [SU05b]. We remind the reader that a manifold M is said Anosov if the linearisation $D\phi^t$ of its geodesic flow splits the tangent bundle of SM into tree invariant subspaces: a one-dimensional subspace tangent to the direction of the flow, and two other subspaces on which $D\phi^t$ acts uniformly contracting and expanding, respectively.

In addition, tensor tomography has also been studied in other contexts. For example, it has been considered in the presence of an attenuation factor [SU11, PSU12], in the presence of a magnetic field (see [DP05, Ain13] and references therein), and for thermostats [DP07].

The main idea of the injectivity proof lies in observing that the transport equation $X_G u = -f$ in SM with $u_{|\partial(SM)} = 0$ is solved by the function $u(x,v) = \int_0^{\tau(x,v)} f(\phi^t(x,v))\,dt$. Therefore, it is enough to prove that u is constant, as this already implies $f = 0$ by the boundary condition. Then, the Pestov Identity enters the game giving an estimate of $\|X_G u\|^2$ that, together with other tools, allows us to conclude that $u = 0$.

The boundary rigidity problem. The material presented in this paragraph has been extracted from [SU05a].

The boundary rigidity problem addresses the question whether it is possible to recover uniquely the metric of a Riemannian manifold from the knowledge of the geodesic distance between any two points on the boundary. This problem arises in geophysics in an attempt to determine the inner structure of the Earth by measuring the travel times of seismic waves.

We observe that one can construct a metric g on a manifold M with boundary and find a point $x_0 \in M$ such that $d_g(x_0, \partial M) > \sup_{x,y\in\partial M} d_g(x,y)$. For such a metric, d_g is independent of a change of g in a small enough neighbourhood of x_0 [Uhl]. Therefore, it is natural to pose restrictions on the metric we want to recover. In 1981

Michel [Mic81] conjectured that every simple manifold is boundary rigid. Pestov and Uhlmann [PU05] proved the conjecture for simple two-dimensional manifolds with no restriction on the curvature. For higher dimensional manifolds, results are known for flat metrics, conformal metrics, and for locally symmetric spaces with negative curvature. We refer the reader to the survey [SU05a] and references therein.

There is a strong link between the boundary rigidity problem and the operator I_m introduced in the previous paragraph. In fact, the linearisation of the boundary rigidity problem near a simple metric g is given by showing that I_2 is s-injective on symmetric 2-tensor fields [Sha94]. Therefore, the Pestov Identity is a tool often used in proofs in a fashion similar to the one described in the previous paragraph (see for example [SU00] and [Dar06]).

As for tensor tomography, the boundary rigidity problem has been considered for other types of dynamics such as the magnetic flow. For results in this direction, we refer the reader to [DPSU07] and references therein.

Spectral rigidity. The material presented here has been extracted from [CS98].

Let (M, g) be a closed Riemannian manifold without boundary and let $\{g_t\}_{t\in[-\varepsilon,\varepsilon]}$ be a family of metrics with $g_0 = g$ and smoothly depending on t such that the spectra of the Laplacian on (M, g_t) coincide. Such a family is called *isospectral deformation*. Spectral rigidity is concerned with the question whether every isospectral deformation comes from a family of diffeomorphisms $\varphi^t : M \longrightarrow M$ such that $\varphi^0 = \mathrm{id}$ and $g_t = (\varphi^t)^* g_0$. The manifold M is *spectrally rigid* if for all isospectral deformations $\{g_t\}_t$ there exists a family of diffeomorphisms $\varphi^t : M \longrightarrow M$ smoothly depending on t and such that $\varphi^0 = \mathrm{id}$ and $g_t = (\varphi^t)^* g_0$.

This problem was initially posed by Guillemin and Kazhdan in [GK80a] where they proved that a two-dimensional, closed and negatively curved manifold is spectrally rigid. In a subsequent paper, they extended this result to manifolds of any dimension under a curvature pinching assumption [GK80b]. In both papers, the claim follows from the s-injectivity of I_0. Min-Oo [Min86] proved it for manifolds with negative definite curvature operator. These three results were proved without the use of the Pestov Identity. However, it appeared as a key feature in [CS98] where Croke and Sharafutdinov used it to prove that any closed negatively curved manifold of dimension n is spectrally rigid. Their result is again a consequence of the s-injectivity of the operator I_m on compact manifold of negative curvature as in [GK80a, GK80b], but they give an alternative proof of s-injectivity using the Pestov Identity.

Spectral rigidity has also been considered for a wider class of manifolds, namely, Anosov manifolds. Sharafutdinov and Uhlmann [SU00] proved that an Anosov surface with no focal points is spectrally rigid. This result relies on the s-injectivity of I_2 for Anosov surfaces with no focal points. More recently, Paternain, Salo and Uhlmann [PSU14b] extended this result to all closed oriented Anosov surfaces. Again, this is a consequence of the fact that I_2 is s-injective on a closed oriented Anosov surface.

1.2 Aim and main results

The aim of our investigation is to derive a Pestov-type identity for smooth functions on the bundle T^kM of k-tuples of tangent vectors over a compact n-dimensional manifold M and restrict it to the principal bundle P^kM of orthonormal k-frames, which we think of as a subspace of T^kM.

More precisely, the two bundles are defined as (see also Section 2.1)

$$T^kM := \bigcup_{p \in M} \underbrace{T_pM \times \ldots \times T_pM}_{k-times},$$

$$P^kM = \{(v_1, \ldots, v_k) \in T^kM \mid \langle v_i, v_j \rangle = \delta_{ij}\} \subset T^kM.$$

In particular, it is possible to describe $T_f T^kM$, $f \in T^kM$, mocking the splitting of TTM into horizontal and vertical component. This also apply to the tangent space of P^kM, where the splitting appears naturally (see Section 2.1 or [KN63] for a general overview on principal bundles).

This allows us to define horizontal and vertical differential operators following the description of Section 1.1 (see Section 2.3). Regarding the dynamics we use, it is given by the frame flows, the lifts of the geodesic flow, defined in the following way. Let $f = (v_1, \ldots, v_k) \in T^kM$ and choose the vector v_i for some i. The i-th frame flow,

$i = 1, \ldots, k$, is the map that parallel transports f along the geodesic c_{v_i} with starting vector v_i, i.e., every vector v_j component of f is parallel transported along c_{v_i}.

Such a framework allows us to state the Lifted Pestov Identity (Theorem 3.1.4, Section 3.1), and its integrated version (Theorem 3.2.3, Section 3.2) for smooth functions on T^kM. For sake of brevity we do not restate them here.

However, the real new results are the restriction of Theorem 3.2.3 to smooth functions on the principal bundle P^kM (Theorem 3.2.4, Subsection 3.2.1) and an equality for smooth functions on P^kM being invariant under one of the frame flows (Corollary 3.2.6, Subsection 3.2.1), where only the L^2-norm of the generators of the frame flows and the Riemannian curvature tensor are involved.

In particular, Corollary 3.2.6 is the key identity for our applications. In fact, it appears that if the manifold M is negatively curved, then any function invariant under one of the frame flows might also be invariant under the remaining ones. This is true when M is a two-dimensional negatively curved manifold or M is a n-dimensional manifold with constant sectional curvature (see Section 4.1).

However, the main application is for smooth functions on oriented k-th Grassmannians $\mathcal{G}^k_{or}(M)$, $k = 1, \ldots, n$, i.e., Grassmannians where the k-planes come with an intrinsic orientation (for a precise definition we refer the reader to Section 4.2). The bundle P^nM projects canonically onto $\mathcal{G}^k_{or}(M)$ and every function on $\mathcal{G}^k_{or}(M)$ can be lifted to a function on P^nM.

Moreover, on oriented Grassmannians we distinguish between intrinsic and non-intrinsic parallel transport of oriented k-planes. The parallel transport of an oriented k-plane A_{or} is called intrinsic if it is along a geodesic c_v with starting vector $v \in A_{or}$, it is called non-intrinsic, otherwise (see also Definition 4.2.2 in Section 4.2). Due to the projection of P^nM onto $\mathcal{G}^k_{or}(M)$, every smooth function on $\mathcal{G}^k_{or}(M)$ invariant under the parallel transports has a smooth lift on P^nM which is invariant under the first k frame flows.

This link allows us to prove the following theorem via Corollary 3.2.6.

Theorem 1.2.1. *Let M be a compact n-dimensional Riemannian manifold with non-positive curvature operator ($\mathcal{R} \leqslant 0$). Let $1 \leqslant k \leqslant n$ and $\varphi \in C^\infty(\mathcal{G}^k_{or}(M))$. If φ is invariant under all the intrinsic parallel transports then it is also invariant under all* *parallel transports.*

For the definition of the curvature operator, we refer the reader to (4.2.2) in Section 4.2.

We also point out that for $k = 1$, $\mathcal{G}^1_{or}(M) = SM$, the intrinsic parallel transport corresponds to the geodesic flow and the non-positivity of the curvature operator of M relaxes to the non-positivity of the curvature of M. Therefore, our result yields the following, which recovers an unpublished result of Knieper [Kni].

Corollary 1.2.2. *Let M be a compact Riemannian manifold with non-positive curvature. Let $\varphi \in C^\infty(SM)$ invariant under the geodesic flow, then φ is also invariant under parallel transport.*

Finally, combining the above theorem with Berger's holonomy classification, we obtain the following proposition.

Proposition 1.2.3. *Let M be a non-flat, compact Riemannian manifold with non-positive curvature operator $\mathcal{R}$. Then, the following statements hold:*

(i) If M is either a Kähler or a Quaternion-Kähler manifold of real dimension $2n \geqslant 4$ or $4n \geqslant 8$, respectively, or a locally symmetric space of non-constant curvature (i.e., not the real hyperbolic space), then there exist smooth, non-constant functions on $\mathcal{G}^2_{or}(M)$ or $\mathcal{G}^4_{or}(M)$ which are invariant under all intrinsic parallel transports.

(ii) If M is not one of the exceptions in (i), then, for all $k \leqslant \dim M$, any smooth function on $\mathcal{G}^k_{or}(M)$ which is invariant under intrinsic parallel transport is necessarily constant.

1.3 Overview of the text

The material is organized as follows. In Chapter 2 we describe the bundles T^kM and P^kM together with their differential operator and the frame flows. In Chapter 3 we present our main results. In Sections 3.1 and 3.2 we state and prove the Lifted Pestov Identity and the Integrated Lifted Pestov Identity, respectively. Moreover, in

Subsection 3.2.1, we restrict the Integrated Lifted Pestov Identity to smooth functions on P^kM. In Chapter 4 we present dynamical applications on smooth functions on P^nM over a two-dimensional manifold of negative sectional curvature and over a n-dimensional manifold of constant sectional curvature invariant under one of the frame flows, and on smooth functions on oriented Grassmannian invariant under intrinsic parallel transports.

Chapter 2

A new framework

We here introduce the two spaces we will work with in the next chapter, together with some of their features, the description of their differential operators and the dynamics we equip it with. The new spaces are the bundles T^kM and P^kM already introduced in Section 1.2.

This chapter is structured as follows. In Section 2.1 we describe the bundles T^kM and P^kM, their tangent spaces and the chosen metric on them. In Section 2.2 we explain how the geodesic flow and the frame flows are related and we describe the generators of the latter. Finally, in Section 2.3 we describe the geometrically motivated differential operators related to the structure of the tangent spaces of T^kM and P^kM, namely, horizontal and vertical gradient, covariant derivative and divergence.

2.1 The bundles T^kM and P^kM

2.1.1 The new bundle T^kM

Let (M, g) be a compact Riemannian manifold of dimension n. Let TM be its tangent bundle and $\pi : TM \longrightarrow M$, $v \mapsto p$ if $v \in T_pM$, be the canonical projection. Let $1 \leqslant k \leqslant n$, we define the *space of k-tuples of tangent vectors over M* as

$$T^kM := \bigcup_{p\in M} \underbrace{T_pM \times \ldots \times T_pM}_{k-times}.$$

This space projects canonically onto M via $\pi^k : T^kM \longrightarrow M$, $f = (u_1, \ldots, u_k) \mapsto p$ if $p = \pi(u_i)$ for all $i = 1, \ldots k$.

T^kM is a manifold of dimension $kn + n$ (for more details on its geometry, see Appendix A).

Let $f = (v_1, \ldots, v_k) \in T^kM$ and let $X = (V_1, \ldots V_k) : (-\varepsilon, \varepsilon) \longrightarrow T^kM$ be a curve on T^kM such that $X(0) = f$ and that the V_i's are all vector fields on M along the footpoint curve $\pi^k \circ X$ on M. Then,

$$T_f T^kM \ni X'(0) = \Big(\frac{d}{dt}\Big|_{t=0}(\pi^k \circ X)(t); \frac{D}{dt}\Big|_{t=0}V_1(t), \ldots, \frac{D}{dt}\Big|_{t=0}V_k(t)\Big).$$

Therefore, the tangent space of T^kM at f is given by

$$T_f T^kM = \underbrace{T_{\pi^k(f)}M \times \ldots \times T_{\pi^k(f)}M}_{(k+1)-times}. \tag{2.1.1}$$

We call the first $T_{\pi^k(f)}M$ copy in $T_f T^kM$ *horizontal distribution* and the product of the remaining k copies of $T_{\pi^k(f)}M$ *vertical distribution*. Consequently, any vector $x \in T_f T^kM$ is written as the sum of $\overset{h}{x} = (x_0; 0, \ldots, 0)$ and $\overset{v}{x} = (0; x_1, \ldots, x_k)$, called *horizontal* and *vertical component*, respectively. This construction allows us to define a *Sasaki-type metric* on T^kM.

Let $x = (x_0; x_1, \ldots, x_k), y = (y_0; y_1, \ldots, y_k) \in T_f T^kM$. Then,

$$\langle x, y \rangle_{T_f T^kM} := \langle x_0, y_0 \rangle_{T_{\pi^k(f)}M} + \sum_{i=1}^{k} \langle x_i, y_i \rangle_{T_{\pi^k(f)}M}. \tag{2.1.2}$$

Consequently, the horizontal and vertical distribution are pairwise orthogonal.

2.1.2 The frame bundle P^kM

Let (M, g) be as in Subsection 2.1.1. The *frame bundle of orthonormal k-frames over M* is denoted by

$$P^kM = \{(v_1, \ldots, v_k) \in T^kM \mid \langle v_i, v_j \rangle = \delta_{ij}\} \subset T^kM.$$

The orthogonal group $O(k)$ acts on the right on this space.

As for T^kM, P^kM projects canonically onto M and the projection map is again denoted by π^k. This map is a fibration where the fibre F_p is the Stiefel manifold of orthonormal k-frames over $\mathbb{R}^n$, i.e., $F_p \cong O(n)/O(n-k)$.

In particular, for $k = 1$ we have $P^1M = SM$, where SM denotes the unit tangent bundle of M. On the other hand, when $k = n$, P^nM is a principal bundle with fibre isomorphic to $O(n)$.

Let $f = (v_1, \ldots, v_k) \in P^kM$. Any curve in P^kM through the point f is given by $X = (V_1, \ldots, V_k) : (-\varepsilon, \varepsilon) \longrightarrow P^kM$ where the V_i's are orthonormal vector fields along the footpoint curve $\pi^k \circ X$ with $V_i(0) = v_i$. $X'(0) \in T_f P^kM$ is described as in (2.1.1) with an additional condition on the $\frac{D}{dt}\big|_{t=0} V_i(t)$'s. Since $\langle V_i(t), V_j(t)\rangle = \delta_{ij}$ for all t, differentiation at $t = 0$ yields

$$\langle \frac{D}{dt}\Big|_{t=0} V_i(t), v_j\rangle = -\langle \frac{D}{dt}\Big|_{t=0} V_j(t), v_i\rangle.$$

Therefore, the tangent space of P^kM at f is given by

$$T_f P^kM = \Big\{(u; w_1, \ldots, w_k) \in T_pM \times \ldots \times T_pM \;\Big|\; \big(\langle w_i, v_j\rangle\big)_{ij} \in \mathfrak{o}(k)\Big\}, \tag{2.1.3}$$

with $\mathfrak{o}(k)$ the Lie algebra of $O(k)$, i.e, the set of skew-symmetric real matrices of dimension $k \times k$. $T_f P^kM$ splits orthogonally into a *horizontal* and a *vertical distribution*, $\mathcal{H}^P_f$ and $\mathcal{V}^P_f$, described below.

$$\mathcal{H}^P_f = \Big\{(u; 0, \ldots, 0) \in T_pM \times \ldots \times T_pM\Big\} \cong T_pM,$$

$$\mathcal{V}^P_f = \Big\{(0; w_1, \ldots, w_k) \in T_pM \times \ldots \times T_pM \;\Big|\; \big(\langle w_i, v_j\rangle\big)_{ij} \in \mathfrak{o}(k)\Big\}.$$

We observe that for $k = n$, the vertical distribution is isomorphic to $\mathfrak{o}(n)$.

Consequently, any vector $u \in T_f P^kM$ splits as $u = \overset{h}{u} + \overset{v}{u}$ where $\overset{h}{u} \in \mathcal{H}^P_f$ and $\overset{v}{u} = (0; u_1, \ldots, u_k) \in \mathcal{V}^P_f$ called again *horizontal* and *vertical component*, respectively.

We point out that vectors on T^kM are not automatically vectors on P^kM, as they do not satisfy the constrain in (2.1.3). To obtain a vector on P^kM from a vector on T^kM, we need to perform an orthogonal projection of the vertical components of the latter onto $T_f P^kM$, or more precisely, onto the Lie algebra $\mathfrak{o}(k)$. We give an example below.

Let $f = (v_1, \ldots, v_k) \in P^kM$ and $(X(f); Y_1(f), \ldots, Y_k(f)) \in T_f T^kM$. Then, the vector $(X(f); Y_{1,\mathfrak{o}}(f), \ldots, Y_{k,\mathfrak{o}}(f)) \in T_f P^kM$ is defined component-wise as follows.

$$Y_{i,\mathfrak{o}}(f) := Y_i(f) - \frac{1}{2}\sum_{j=1}^{k} \Big(\langle Y_i(f), v_j\rangle + \langle Y_j(f), v_i\rangle\Big) v_j. \tag{2.1.4}$$

It is easy to check that the matrix $\left(\langle Y_{i,\mathfrak{o}}(f), v_j\rangle\right)_{ij}$ is skew-symmetric.

Finally since P^kM is a submanifold of T^kM, it inherits the metric described in (2.1.2) and horizontal and vertical component are again pairwise orthogonal.

2.2 Frame Flows

We now introduce the frame flows F_t^i, $i = 1, \dots, k$.

The first frame flow F_t^1 on T^kM is the lift of the geodesic flow and, more generally, the i-th frame flow F_t^i on T^kM is the parallel transport of the frame $f = (v_1, \dots, v_k)$ along the geodesic c_{v_i} with starting vector v_i.

The precise definition of the frame flows is given below.

Let $f = (v_1, \dots, v_k) \in T_f T^k M$, and let c_{v_i} be the geodesic on M such that $c_{v_i}(0) = \pi^k(f)$ and $c'_{v_i}(0) = v_i$. The *i-th frame flow*, $i = 1, \dots, k$, is the map

$$F_t^i : T^kM \longrightarrow T^kM$$
$$f = (v_1, \dots, v_k) \mapsto f_{v_i}(t) = ((v_1)_{v_i}(t), \dots, (v_k)_{v_i}(t))$$

where $f_{v_i}(t)$ denotes the parallel transport of the frame f along the geodesic c_{v_i}, i.e. every vector v_j of f is parallel transported along c_{v_i}. In particular, $(v_i)_{v_i}(t) = \phi^t(v_i)$, where ϕ^t is the geodesic flow on TM (see (1.1.2)).

Its *infinitesimal generator* is given by

$$G^i(f) = \frac{d}{dt}\Big|_{t=0} F_t^i(f) \cong \left(\frac{d}{dt}\Big|_{t=0} c_{v_i}(t); 0, \dots, 0\right) = (v_i; 0, \dots, 0),$$

i.e., $G^i(f)$ is a horizontal vector of $T_f T^k M$ for all $i = 1, \dots, k$.

The frame flows act on the frame bundle P^kM as well, and their generators are again horizontal vector on P^kM. The first frame flow on P^kM has been extensively studied in relation to ergodicity. Below, we list the conditions for which F_t^1 is ergodic.

(i) If M is a manifold of odd dimension different from 7 with negative curvature [BG80].

(ii) For the set of metrics with negative curvature that is open and dense in the C^3 topology [Br75].

(iii) If M is a manifold of even dimension different from 8 with pinched negative curvature, pinching constant bigger than 0.93 [BK84].

(iv) If M is a manifold of dimension 7 or 8 with pinched negative curvature, pinching constant bigger than 0.99023... [BP03].

However, in this dissertation, we are not interested in studying the ergodicity of the frame flows further. Our goal is to study a related property, namely, invariance properties of smooth functions on the frame bundle under the action of the frame flows.

2.3 Differential Operators

As in the classical case of Riemannian manifolds, we have differential operators on T^kM and P^kM such as the gradient of a smooth function, the covariant derivative and the divergence. However, here we need to distinguish between the horizontal and vertical distribution when defining these operators.

In what follows, all inner products are with respect to the metric on M, unless stated otherwise.

First, we introduce the notion of semi-basic vector field. We define the pullback bundle $\pi^*(T^kM) = \{(v,f) \in TM \times T^kM \mid \pi(v) = \pi^k(f)\}$ which is a vector bundle over T^kM. A *semi-basic vector field* is an element of $\mathfrak{X}(\pi^*(T^kM)) = \{X : T^kM \to TM \text{ smooth } \mid X(f) \in T_{\pi^k(f)}M \;\; \forall f \in T^kM\}$.

An example of semi-basic vector field is the vector field $V_i : T^kM \longrightarrow TM$ such that $f = (v_1, \ldots, v_k) \mapsto v_i$. It will appear extensively in the next chapter.

Let $\varphi : T^kM \longrightarrow \mathbb{R}$ be a smooth function and let $f = (v_1, \ldots, v_k) \in T^kM$ with $\pi^k(f) = p$. The gradient of φ with respect to the metric on T^kM is

$$\mathrm{grad}\varphi(f) = (\overset{h}{\mathrm{grad}}\varphi(f); \overset{v,1}{\mathrm{grad}}\varphi(f), \ldots, \overset{v,k}{\mathrm{grad}}\varphi(f)) \in T_f T^kM.$$

The horizontal and vertical component, called *horizontal* and *i-th vertical gradi-*

ent, are described intrinsically as follows. Let $u \in T_pM$, then

$$\langle \overset{h}{\text{grad}}\varphi(f), u\rangle = \frac{d}{dt}\Big|_{t=0}\varphi(f_u(t)),$$

$$\langle \overset{v,i}{\text{grad}}\varphi(f), u\rangle = \frac{d}{dt}\Big|_{t=0}\varphi(v_1, \ldots, v_{i-1}, v_i + tu, v_{i+1}, \ldots, v_k),$$

i.e., the horizontal and i-th vertical gradient of φ are the derivatives of φ along the horizontal curve $t \mapsto f_u(t)$ in T^kM and along the vertical curve $t \mapsto v_i + tu$ in the i-th T_pM copy of T^kM, respectively. We remind the reader that $f_u(t)$ is the parallel transport of the frame f along the geodesic c_u starting at $c_u(0) = \pi^k(f)$ with initial speed $c_u'(0) = u$.

We observe that $\overset{h}{\text{grad}}\varphi$ and $\overset{v,i}{\text{grad}}\varphi$ are semi-basic vector fields.

If $f \in P^kM$, $\text{grad}\varphi(f)$ defined as above is not an element of T_fP^kM, as we explained in Subsection 2.1.2. According to (2.1.4), the orthogonal projection of $\overset{v,i}{\text{grad}}\varphi(f)$ into T_fP^kM for $f = (v_1, \ldots, v_k)$ is

$$\overset{v,i}{\text{grad}}_{\mathbf{o}}\varphi(f) := \overset{v,i}{\text{grad}}\varphi(f) - \frac{1}{2}\sum_{j=1}^{k}\Big(\langle \overset{v,i}{\text{grad}}\varphi(f), v_j\rangle + \langle \overset{v,j}{\text{grad}}\varphi(f), v_i\rangle\Big)v_j. \tag{2.3.5}$$

Then, $(\overset{h}{\text{grad}}\varphi(f); \overset{v,1}{\text{grad}}_{\mathbf{o}}\varphi(f), \ldots, \overset{v,k}{\text{grad}}_{\mathbf{o}}\varphi(f)) \in T_fP^kM$.

Let $X : T^kM \longrightarrow TM$ be a semi-basic vector field. The *horizontal* and *i-th vertical covariant derivative* of X with respect to $u \in T_pM$ are given by

$$\overset{h}{\nabla}_u X(f) = \frac{D}{dt}\Big|_{t=0}X(f_u(t)),$$

$$\overset{v,i}{\nabla}_u X(f) = \frac{D}{dt}\Big|_{t=0}X(v_1, \ldots, v_{i-1}, v_i + tu, v_{i+1}, \ldots, v_k).$$

As the definitions of these two operators rely on the usual notion of covariant derivative, they satisfy additivity and the product rule, namely,

(i) $\dot{\nabla}_u\big[X(f) + Y(f)\big] = \dot{\nabla}_u X(f) + \dot{\nabla}_u Y(f)$,

(ii) $\dot{\nabla}_u(\varphi X)(f) = \varphi(f)\dot{\nabla}_u X(f) + \dot{\text{grad}}\varphi(f)X(f)$,

for all X, Y semi-basic vector fields and for all $\varphi \in C^\infty(T^kM)$.

Finally, we introduce the *horizontal* and *i-th vertical divergence* of a semi-basic vector field. They are defined as follows.

$$\overset{h}{\text{div}}X(f) = \sum_{i=1}^{n}\langle \overset{h}{\nabla}_{e_i}X(f), e_i\rangle \qquad \text{and} \qquad \overset{v,i}{\text{div}}X(f) = \sum_{i=1}^{n}\langle \overset{v,i}{\nabla}_{e_i}X(f), e_i\rangle,$$

where $e_1, \ldots, e_n$ is an orthonormal basis of T_pM for $p = \pi^k(f)$.

As a consequence of (i) and (ii) above, these operators are additive and satisfy the product rules

$$\overset{h}{\operatorname{div}}(\varphi X)(f) = \varphi(f)\overset{h}{\operatorname{div}}X(f) + \langle \overset{h}{\operatorname{grad}}\varphi(f), X(f)\rangle, \tag{2.3.6}$$

$$\overset{v,i}{\operatorname{div}}(\varphi X)(f) = \varphi(f)\overset{v,i}{\operatorname{div}}X(f) + \langle \overset{v,i}{\operatorname{grad}}\varphi(f), X(f)\rangle, \tag{2.3.7}$$

for all X, Y semi-basic vector fields and for all $\varphi \in C^\infty(T^kM)$.

We conclude the section giving an example of how to calculate the horizontal and i-th vertical covariant derivative and divergence of a very special semi-basic vector field, which we will use in the next chapter.

Example 2.3.1. Let $V_i : T^kM \longrightarrow TM$ be a semi-basic vector field such that $f = (v_1, \ldots, v_k) \mapsto v_i$. Then, we have the followings.

$$\overset{h}{\nabla}_u V_i(f) = \frac{D}{dt}\Big|_{t=0} V_i(f_u(t)) = \frac{D}{dt}\Big|_{t=0} (v_i)_u(t) = 0,$$
$$\overset{v,j}{\nabla}_u V_i(f) = \frac{d}{dt}\Big|_{t=0} v_i + \delta_{ij} t u = \delta_{ij} u.$$

Consequently,

$$\overset{h}{\operatorname{div}} V_i(f) = 0, \tag{2.3.8}$$

$$\overset{v,j}{\operatorname{div}} V_i(f) = n\delta_{ij}. \tag{2.3.9}$$

Chapter 3

The Pestov Identity on Frame Bundles

In this chapter we state the main results of Part I.

In Section 3.1 we present the Lifted Pestov Identity, a Pestov-type identity for smooth functions defined on T^kM, whose proof follows the steps of a coordinate free-proof given by Knieper in [Kni02]. In Section 3.2, we integrate the Lifted Pestov Identity over the frame bundle P^kM obtaining the Integrated Pestov Identity. Moreover, we restrict it to smooth functions defined on P^kM and we derive a new identity for smooth functions invariant under one of the frame flows.

We remind the reader that all the inner product are with respect to the metric on M, unless specified.

3.1 The Lifted Pestov Identity

We first prove some useful relations between the horizontal and vertical differential operators introduced in the previous chapter.

Lemma 3.1.1. *Let $\varphi \in C^\infty(T^kM)$, $f \in T^kM$ and $u, w \in T_pM$ with $p = \pi^k(f)$. Let $i = 1, \ldots, k$, then*

$$\langle \overset{v,i}{\nabla}_w \overset{h}{\mathrm{grad}}\varphi(f), u\rangle = \langle \overset{h}{\nabla}_u \overset{v,i}{\mathrm{grad}}\varphi(f), w\rangle. \tag{3.1.1a}$$

In particular, it follows

$$\overset{v,i}{\mathrm{div}}\overset{h}{\mathrm{grad}}\varphi(f) = \overset{h}{\mathrm{div}}\overset{v,i}{\mathrm{grad}}\varphi(f). \tag{3.1.1b}$$

Proof. Using the definitions of horizontal and i-th vertical covariant derivative and gradient we have

$$\begin{aligned}\langle \overset{v,i}{\nabla}_w \overset{h}{\text{grad}}\varphi(f), u\rangle &= \frac{d}{dt}\Big|_{t=0} \langle \overset{h}{\text{grad}}\varphi(v_1,\dots,v_{i-1}, v_i + tw, v_{i+1},\dots,v_k), u\rangle \\ &= \frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} \varphi((v_1)_u(s),\dots,(v_i)_u(s) + t(w)_u(s),\dots,(v_k)_u(s)) \\ &= \frac{d}{ds}\Big|_{s=0} \langle \overset{v,i}{\text{grad}}\varphi(f_u(s)), w\rangle \\ &= \langle \overset{h}{\nabla}_u \overset{v,i}{\text{grad}}\varphi(f), w\rangle,\end{aligned}$$

which proves (3.1.1a).

Equation (3.1.1b) follows by taking the trace. □

Lemma 3.1.2. *Let $\varphi \in C^\infty(T^kM)$, $f = (v_1,\dots,v_k) \in T^kM$ and $u, w \in T_pM$ with $p = \pi^k(f)$. Then*

$$\langle \overset{h}{\nabla}_w \overset{h}{\text{grad}}\varphi(f), u\rangle - \langle \overset{h}{\nabla}_u \overset{h}{\text{grad}}\varphi(f), w\rangle = \sum_{i=1}^{k} \langle R(\overset{v,i}{\text{grad}}\varphi(f), v_i)w, u\rangle, \tag{3.1.2a}$$

and

$$G^iG^j\varphi(f) - G^jG^i\varphi(f) = \sum_{l=1}^{k} \langle R(\overset{v,l}{\text{grad}}\varphi(f), v_l)v_i, v_j\rangle. \tag{3.1.2b}$$

Proof. We first prove (3.1.2a) since (3.1.2b) follows as a consequence.

Let $H(t,s) = \big(f_w(t)\big)_{u_{w(t)}}(s)$ be a variation in T^kM, i.e., $H(t,s) = (H_1(t,s),\dots,H_k(t,s))$ where $H_i(t,s) = \big((v_i)_w(t)\big)_{u_w(t)}(s)$. Then,

$$\begin{aligned}\langle \overset{h}{\nabla}_w \overset{h}{\text{grad}}\varphi(f), u\rangle &= \frac{d}{dt}\Big|_{t=0} \langle \overset{h}{\text{grad}}\varphi(f_u(t), u_w(t)\rangle \\ &= \frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} \varphi\Big((f_w(t))_{u_w(t)}(s)\Big) \\ &= \frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} \varphi(H(t,s)).\end{aligned}$$

Now, using (2.1.2), the definition of horizontal and i-th vertical component, and the definition of horizontal covariant derivative, we obtain

$$\begin{aligned}\frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} \varphi(H(t,s)) = &\frac{\partial}{\partial s}\Big|_{s=0} \Big\langle \overset{h}{\text{grad}}\varphi(H(0,s)), \underbrace{\Big[\frac{\partial}{\partial t}\Big|_{t=0} H(t,s)\Big]^h}_{\frac{\partial}{\partial t}\big|_{t=0}(\pi^k\circ H)(t,s)} \Big\rangle \\ &+ \frac{\partial}{\partial s}\Big|_{s=0} \sum_{i=1}^{k} \Big\langle \overset{v,i}{\text{grad}}\varphi(H(0,s)), \underbrace{\Big[\frac{\partial}{\partial t}\Big|_{t=0} H(t,s)\Big]^{v,i}}_{\frac{D}{dt}\big|_{t=0} H_i(t,s)} \Big\rangle\end{aligned}$$

$$= \langle \frac{D}{ds}\Big|_{s=0} \overset{h}{\mathrm{grad}}\varphi(H(0,s)), w\rangle + \langle \overset{h}{\mathrm{grad}}\varphi(f), \underbrace{\frac{D}{ds}\Big|_{s=0}\frac{\partial}{\partial t}\Big|_{t=0}(\pi^k \circ H)(t,s)}_{\frac{D}{dt}\big|_{t=0}\frac{\partial}{\partial s}\big|_{s=0}c_{u_w(t)}(s)=0}\rangle$$

$$+ \sum_{i=1}^{k}\langle \frac{D}{ds}\Big|_{s=0} \overset{v,i}{\mathrm{grad}}\varphi(H(0,s), \underbrace{\frac{D}{dt}\Big|_{t=0} H_i(t,0)}_{\frac{D}{dt}\big|_{t=0}(v_i)_w(t)=0}\rangle + \langle \overset{v,i}{\mathrm{grad}}\varphi(f), \frac{D}{ds}\Big|_{s=0}\frac{D}{dt}\Big|_{t=0} H_i(t,s)\rangle$$

$$= \langle \overset{h}{\nabla}_u \overset{h}{\mathrm{grad}}\varphi(f), w\rangle + \sum_{i=1}^{k}\langle \overset{v,i}{\mathrm{grad}}\varphi(f), \frac{D}{ds}\Big|_{s=0}\frac{D}{dt}\Big|_{t=0} H_i(t,s)\rangle,$$

where $c_{u_w(t)}(s)$ is the footpoint curve of $H(t,s)$.

Finally,

$$\frac{D}{ds}\Big|_{s=0}\frac{D}{dt}\Big|_{t=0} H_i(t,s) = \frac{D}{dt}\Big|_{t=0}\frac{D}{ds}\Big|_{s=0} H_i(t,s)$$
$$+ R\Big(\frac{\partial}{\partial s}\Big|_{s=0}(\pi^k \circ H_i)(0,s), \frac{\partial}{\partial t}\Big|_{t=0}(\pi^k \circ H_i)(t,0)\Big) H_i(0,0) = R(u,w)v_i.$$

Hence,

$$\langle \overset{h}{\nabla}_w \overset{h}{\mathrm{grad}}\varphi(f), u\rangle - \langle \overset{h}{\nabla}_u \overset{h}{\mathrm{grad}}\varphi(f), w\rangle = \sum_{i=1}^{k}\langle R(u,w)v_i, \overset{v,i}{\mathrm{grad}}\varphi(f)\rangle.$$

We now prove (3.1.2b). First we observe that

$$G^i\varphi(f) = \frac{d}{dt}\Big|_{t=0}\varphi(F_t^i(f)) = \big\langle \mathrm{grad}\varphi(f), \frac{d}{dt}\Big|_{t=0} F_t^i(f)\big\rangle_{T_f T^k M} = \langle \overset{h}{\mathrm{grad}}\varphi(f), v_i\rangle. \tag{3.1.3}$$

Therefore,

$$G^iG^j\varphi(f) = G^i\langle \overset{h}{\mathrm{grad}}\varphi(f), v_j\rangle = \frac{d}{dt}\Big|_{t=0}\langle \overset{h}{\mathrm{grad}}\varphi(f_{v_i}(t)), (v_j)_{v_i}(t)\rangle$$
$$= \langle \frac{D}{dt}\Big|_{t=0} \overset{h}{\mathrm{grad}}\varphi(f_{v_i}(t)), v_j\rangle = \langle \overset{h}{\nabla}_{v_i} \overset{h}{\mathrm{grad}}\varphi(f), v_j\rangle,$$

and $G^jG^i\varphi(f) = \langle \overset{h}{\nabla}_{v_j} \overset{h}{\mathrm{grad}}\varphi(f), v_i\rangle$.

Choosing $u = v_j$ and $w = v_i$ in (3.1.2a), we obtain (3.1.2b). □

Lemma 3.1.3. *Let $\varphi \in C^\infty(T^kM)$, $f = (v_1, \dots, v_k)$ and $w \in T_pM$ with $p = \pi^k(f)$. Then,*

$$\langle \overset{h}{\mathrm{grad}} G^j\varphi(f), w\rangle = G^j\langle \overset{h}{\mathrm{grad}}\varphi(f), w\rangle + \sum_{l=1}^{k}\langle R(\overset{v,l}{\mathrm{grad}}\varphi(f), v_l)w, v_j\rangle, \tag{3.1.4a}$$

$$\langle \overset{v,i}{\mathrm{grad}} G^j\varphi(f), w\rangle = G^j\langle \overset{v,i}{\mathrm{grad}}\varphi(f), w\rangle + \delta_{ij}\langle \overset{h}{\mathrm{grad}}\varphi(f), w\rangle. \tag{3.1.4b}$$

In particular,

$$\langle \overset{v,i}{\operatorname{grad}} G^j \varphi(f), v_l \rangle = G^j \langle \overset{v,i}{\operatorname{grad}} \varphi(f), v_l \rangle + \delta_{ij} G^l \varphi(f). \tag{3.1.4c}$$

Proof. We first prove (3.1.4a). Using (3.1.3) and (3.1.2a) we obtain,

$$\begin{aligned}
\langle \overset{h}{\operatorname{grad}} G^j \varphi(f), w \rangle &= \frac{d}{dt}\Big|_{t=0} G^j \varphi(f_w(t)) \\
&= \frac{d}{dt}\Big|_{t=0} \langle \overset{h}{\operatorname{grad}} \varphi(f_w(t)), (v_j)_w(t) \rangle \\
&= \langle \overset{h}{\nabla}_w \overset{h}{\operatorname{grad}} \varphi(f), v_j \rangle \\
&= \langle \overset{h}{\nabla}_{v_j} \overset{h}{\operatorname{grad}} \varphi(f), w \rangle + \sum_{l=1}^{k} \langle R(\overset{v,l}{\operatorname{grad}} \varphi(f), v_l) w, v_j \rangle \\
&= \frac{d}{dt}\Big|_{t=0} \langle \overset{h}{\operatorname{grad}} \varphi(f_{v_j}(t)), (w)_{v_j}(t) \rangle + \sum_{l=1}^{k} \langle R(\overset{v,l}{\operatorname{grad}} \varphi(f), v_l) w, v_j \rangle \\
&= G^j \langle \overset{h}{\operatorname{grad}} \varphi(f), w \rangle + \sum_{l=1}^{k} \langle R(\overset{v,l}{\operatorname{grad}} \varphi(f), v_l) w, v_j \rangle.
\end{aligned}$$

We now prove (3.1.4b). Using (3.1.3) and (3.1.1a), we have

$$\begin{aligned}
\langle \overset{v,i}{\operatorname{grad}} G^j \varphi(f), w \rangle &= \frac{d}{dt}\Big|_{t=0} G^j \varphi(v_1, \dots, v_i + tw, \dots, v_k) \\
&= \frac{d}{dt}\Big|_{t=0} \langle \overset{h}{\operatorname{grad}} \varphi(v_1, \dots, v_i + tw, \dots, v_k), v_j + \delta_{ij} tw \rangle \\
&= \langle \overset{v,i}{\nabla}_w \overset{h}{\operatorname{grad}} \varphi(f), v_j \rangle + \delta_{ij} \langle \overset{h}{\operatorname{grad}} \varphi(f), w \rangle \\
&= \langle \overset{h}{\nabla}_{v_j} \overset{v,i}{\operatorname{grad}} \varphi(f), w \rangle + \delta_{ij} \langle \overset{h}{\operatorname{grad}} \varphi(f), w \rangle \\
&= \frac{d}{dt}\Big|_{t=0} \langle \overset{v,i}{\operatorname{grad}} \varphi(f_{v_j}(t)), (w)_{v_j}(t) \rangle + \delta_{ij} \langle \overset{h}{\operatorname{grad}} \varphi(f), w \rangle \\
&= G^j \langle \overset{v,i}{\operatorname{grad}} \varphi(f), w \rangle + \delta_{ij} \langle \overset{h}{\operatorname{grad}} \varphi(f), w \rangle.
\end{aligned}$$

Setting $w = v_l$, in the last equality above, the second term of the RHS is $\langle \overset{h}{\operatorname{grad}} \varphi(f), v_l \rangle = G^l \varphi(f)$ by (3.1.3), and we obtain (3.1.4c). □

We are now ready to prove the Lifted Pestov Identity. We recall that the proof can be compared with [Kni02, Theorem 1.1 in Appendix].

Theorem 3.1.4 (Lifted Pestov Identity)**.** *Let $\varphi \in C^\infty(T^k M)$, Then,*

$$\begin{aligned}
&\overset{v,j}{\operatorname{div}} Z^i(f) + \overset{h}{\operatorname{div}} Y^{j,i}(f) + \delta_{ij} \|\overset{h}{\operatorname{grad}} \varphi(f)\|^2 = \\
&\qquad = \sum_{l=1}^{k} \langle R(\overset{v,l}{\operatorname{grad}} \varphi(f), v_l) v_i, \overset{v,j}{\operatorname{grad}} \varphi(f) \rangle + 2 \langle \overset{h}{\operatorname{grad}} \varphi(f), \overset{v,j}{\operatorname{grad}} G^i \varphi(f) \rangle, \quad (3.1.5)
\end{aligned}$$

where

$$Y^{j,i}(f) = \langle \overset{h}{\text{grad}}\varphi(f), \overset{v,j}{\text{grad}}\varphi(f)\rangle v_i - \big(G^i\varphi(f)\big)\overset{v,j}{\text{grad}}\varphi(f),$$

and

$$Z^i(f) = \big(G^i\varphi(f)\big)\overset{h}{\text{grad}}\varphi(f),$$

for all $i, j = 1, \ldots, k$.

Proof. Using equations (2.3.7) and (3.1.3) and the definitions of horizontal and j-th vertical gradient and covariant derivative, we have

$$\begin{aligned}
\overset{v,j}{\text{div}}Z^i(f) &= \overset{v,j}{\text{div}}\big(G^i\varphi(f)\big)\overset{h}{\text{grad}}\varphi(f) \\
&= G^i\varphi(f)\overset{v,j}{\text{div}}\overset{h}{\text{grad}}\varphi(f) + \underbrace{\langle \overset{v,j}{\text{grad}}G^i\varphi(f), \overset{h}{\text{grad}}\varphi(f)\rangle}_{\frac{d}{dt}\big|_{t=0}G^i\varphi(v_1,\ldots,v_j+t\overset{h}{\text{grad}}\varphi(f),\ldots,v_k)} \\
&= G^i\varphi(f)\overset{v,j}{\text{div}}\overset{h}{\text{grad}}\varphi(f) \\
&\qquad + \frac{d}{dt}\Big|_{t=0}\langle \overset{h}{\text{grad}}\varphi(v_1,\ldots,v_j + t\overset{h}{\text{grad}}\varphi(f),\ldots,v_k), v_i + \delta_{ij}t\overset{h}{\text{grad}}\varphi(f)\rangle \\
&= G^i\varphi(f)\overset{v,j}{\text{div}}\overset{h}{\text{grad}}\varphi(f) + \langle \overset{v,j}{\nabla}_{\overset{h}{\text{grad}}\varphi(f)}\overset{h}{\text{grad}}\varphi(f), v_i\rangle + \delta_{ij}\|\overset{h}{\text{grad}}\varphi(f)\|^2.
\end{aligned}$$

In the same way, using (2.3.6) in the second equality and (3.1.1b) in the third equality, we obtain

$$\begin{aligned}
\overset{h}{\text{div}}Y^{j,i}(f) &= \overset{h}{\text{div}}(\langle \overset{h}{\text{grad}}\varphi(f), \overset{v,j}{\text{grad}}\varphi(f)\rangle v_i) - \overset{h}{\text{div}}(G^i\varphi(f)\overset{v,j}{\text{grad}}\varphi(f)) \\
&= \langle \overset{h}{\text{grad}}\varphi(f), \overset{v,j}{\text{grad}}\varphi(f)\rangle \underbrace{\overset{h}{\text{div}}V_i(f)}_{=0\text{ by (2.3.8)}} + \langle \overset{h}{\text{grad}}(\langle \overset{h}{\text{grad}}\varphi(f), \overset{v,j}{\text{grad}}\varphi(f)\rangle), v_i\rangle \\
&\qquad - G^i\varphi(f)\overset{h}{\text{div}}\overset{v,j}{\text{grad}}\varphi(f) - \underbrace{\langle \overset{h}{\text{grad}}G^i\varphi(f), \overset{v,j}{\text{grad}}\varphi(f)\rangle}_{\frac{d}{dt}\big|_{t=0}G^i\varphi\big(f_{\overset{v,j}{\text{grad}}\varphi(f)}(t)\big)} \\
&= \frac{d}{dt}\Big|_{t=0}\langle \overset{h}{\text{grad}}\varphi(f_{v_i}(t)), \overset{v,j}{\text{grad}}\varphi(f_{v_i}(t))\rangle - G^i\varphi(f)\overset{v,j}{\text{div}}\overset{h}{\text{grad}}\varphi(f) \\
&\qquad - \frac{d}{dt}\Big|_{t=0}\langle \overset{h}{\text{grad}}\varphi(f_{\overset{v,j}{\text{grad}}\varphi(f)}(t), (v_i)_{\overset{v,j}{\text{grad}}\varphi(f)}(t)\rangle \\
&= \langle \overset{h}{\nabla}_{v_i}\overset{h}{\text{grad}}\varphi(f), \overset{v,j}{\text{grad}}\varphi(f)\rangle + \langle \overset{h}{\text{grad}}\varphi(f), \overset{h}{\nabla}_{v_i}\overset{v,j}{\text{grad}}\varphi(f)\rangle \\
&\qquad - G^i\varphi(f)\overset{v,j}{\text{div}}\overset{h}{\text{grad}}\varphi(f) - \langle \overset{h}{\nabla}_{\overset{v,j}{\text{grad}}\varphi(f)}\overset{h}{\text{grad}}\varphi(f), v_i\rangle.
\end{aligned}$$

Hence, summing these two divergences and using (3.1.1a) in the second equality and (3.1.2a) in the third equality, we have

$$\begin{aligned}
\overset{v,j}{\mathrm{div}} Z^i(f) + \overset{h}{\mathrm{div}} Y^{j,i}(f) &= \langle \overset{h}{\nabla}_{v_i} \overset{h}{\mathrm{grad}}\varphi(f), \overset{v,j}{\mathrm{grad}}\varphi(f)\rangle - \langle \overset{h}{\nabla}_{\overset{v,j}{\mathrm{grad}}\varphi(f)} \overset{h}{\mathrm{grad}}\varphi(f), v_i\rangle \\
&\quad + [\langle \overset{h}{\mathrm{grad}}\varphi(f), \overset{h}{\nabla}_{v_i} \overset{v,j}{\mathrm{grad}}\varphi(f)\rangle + \langle \overset{v,j}{\nabla}_{\overset{h}{\mathrm{grad}}\varphi(f)} \overset{h}{\mathrm{grad}}\varphi(f), v_i\rangle] + \delta_{ij}\|\overset{h}{\mathrm{grad}}\varphi(f)\|^2 \\
&= \langle \overset{h}{\nabla}_{v_i} \overset{h}{\mathrm{grad}}\varphi(f), \overset{v,j}{\mathrm{grad}}\varphi(f)\rangle - \langle \overset{h}{\nabla}_{\overset{v,j}{\mathrm{grad}}\varphi(f)} \overset{h}{\mathrm{grad}}\varphi(f), v_i\rangle \\
&\quad + 2\langle \overset{h}{\nabla}_{v_i} \overset{v,j}{\mathrm{grad}}\varphi(f), \overset{h}{\mathrm{grad}}\varphi(f)\rangle + \delta_{ij}\|\overset{h}{\mathrm{grad}}\varphi(f)\|^2 \\
&= \delta_{ij}\|\overset{h}{\mathrm{grad}}\varphi(f)\|^2 + 2\langle \overset{h}{\nabla}_{v_i} \overset{v,j}{\mathrm{grad}}\varphi(f), \overset{h}{\mathrm{grad}}\varphi(f)\rangle + \sum_{l=1}^{k} \langle R(\overset{v,l}{\mathrm{grad}}\varphi(f), v_l)v_i, \overset{v,j}{\mathrm{grad}}\varphi(f)\rangle.
\end{aligned}$$

Finally, using the definition of horizontal and vertical gradient, we have

$$\begin{aligned}
2\langle \overset{h}{\mathrm{grad}}\varphi(f), \overset{v,j}{\mathrm{grad}} G^i\varphi(f)\rangle &= 2\langle \overset{h}{\mathrm{grad}}\varphi(f), \overset{v,j}{\mathrm{grad}}(\langle \overset{h}{\mathrm{grad}}\varphi, V_i\rangle)(f)\rangle \\
&= 2\frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} \varphi\big((v_1)_{v_i}(s), \ldots, (v_j + t(\overset{h}{\mathrm{grad}}\varphi(f))_{v_i}(s)), \ldots, (v_k)_{v_i}(s)\big) \\
&\qquad\qquad + \delta_{ij}\|\overset{h}{\mathrm{grad}}\varphi(f)\|^2 \\
&= 2\frac{d}{dt}\Big|_{t=0} \langle \overset{h}{\mathrm{grad}}\varphi(v_1, \ldots, v_j + t\overset{h}{\mathrm{grad}}\varphi(f), \ldots, v_k), v_i + \delta_{ij} t \overset{h}{\mathrm{grad}}\varphi(f)\rangle \\
&= 2\langle \overset{h}{\nabla}_{v_i} \overset{v,j}{\mathrm{grad}}\varphi(f), \overset{h}{\mathrm{grad}}\varphi(f)\rangle + 2\delta_{ij}\|\overset{h}{\mathrm{grad}}\varphi(f)\|^2.
\end{aligned}$$

Substituting this expression in the previous identity we obtain the theorem. □

3.2 The Integrated Pestov Identity

We now prove the integrated version of (3.1.5) over the frame bundle P^kM with respect to the measure $d\mu_p(f) = \mathrm{dvol}(p)\,\mathrm{d}\nu_p(f)$ where $\mathrm{d}\nu$ is the measure on the fibre F_p of P^kM and dvol is the measure on M.

Before integrating, we show the behaviour of the horizontal divergence and of the generators of the frame flows under integration.

Lemma 3.2.1. *Let $X : T^kM \longrightarrow TM$ be a semi-basic vector field, then*

$$\int_{P^kM} \overset{h}{\mathrm{div}} X(f)\,\mathrm{d}\mu(f) = \int_M \int_{F_p} \overset{h}{\mathrm{div}} X(f)\,\mathrm{dvol}(p)\,\mathrm{d}\nu_p(f) = 0. \tag{3.2.6}$$

Proof. We first recall that $F_p \cong O(n)/O(n-k)$. To prove the Lemma, it suffices to show

$$\int_{F_p} \langle \overset{h}{\nabla}_v X, v\rangle \, \mathrm{d}\nu_p(f) = \langle \nabla_v \int_{F_p} X(f)\, \mathrm{d}\nu_p(f), v\rangle. \tag{3.2.7}$$

In fact, from the above it follows

$$\int_{O(n)/O(n-k)} \overset{h}{\mathrm{div}} X(f)\, \mathrm{d}\nu_p(f) = \mathrm{div} \int_{O(n)/O(n-k)} X(f)\, \mathrm{d}\nu_p(f).$$

Then, due to the compactness of M we obtain (3.2.6).

Let us prove (3.2.7).

We define $\varrho : P^n M \to P^k M$ with $\varrho((v_1, \dots, v_n)) = (v_1, \dots, v_k)$. In particular, $\varrho : \tilde{F}_p \to F_p$ where $\tilde{F}_p \cong O(n)$ is the fibre of the principal bundle $P^n M$ with $O(n)$ right action.

On the fibre $O(n)/O(n-k)$ there exists a unique (up to scalar) left $O(n)$-invariant Haar measure (see, e.g., [Fer98, Thereom 8.1.8]) obtained in the following way.

Fix $f_0 \in \tilde{F}_p$ and let $g \in O(n)$, Then, there exists $\psi_{f_0} : O(n)/O(n-k) \to F_p$ such that $g \cdot O(n-k) \mapsto \varrho(g \cdot f_0)$, canonical diffeomorphism. The pullback $\psi_{f_0}^*(\nu_p) = \theta_{O(n)/O(n-k)}$ gives the measure on $O(n)/O(n-k)$.

In what follows, we drop the arguments of the measure and the subscript of θ to ease the notation.

We have,

$$\begin{aligned}\int_{F_p} \langle \overset{h}{\nabla}_v X(f), v\rangle \, \mathrm{d}\nu_p(f) &= \int_{F_p} \langle \frac{D}{dt}\Big|_{t=0} X(f_v(t)), v\rangle \, \mathrm{d}\nu_p(f) \\ &= \int_{F_p} \frac{d}{dt}\Big|_{t=0} \langle X(f_v(t)), v\rangle \, \mathrm{d}\nu_p(f).\end{aligned}$$

Note that for $\tilde{f} = g \cdot f_0 \in \tilde{F}_p$,

$$X(f_v(t)) = X(\varrho(\tilde{f}_v(t))) = X(\varrho(g \cdot (f_0)_v(t))) = X(\psi_{(f_0)_v(t)}(g)).$$

Then,

$$\begin{aligned}\int_{F_p} \frac{d}{dt}\Big|_{t=0} \langle X(f_v(t)), v\rangle \, \mathrm{d}\nu_p(f) &= \int_{\psi_{f_0}\left(O(n)/O(n-k)\right)} \frac{d}{dt}\Big|_{t=0} \langle X(f_v(t)), \phi^t(v)\rangle \, \mathrm{d}\nu_p(f) \\ &= \int_{O(n)/O(n-k)} \frac{d}{dt}\Big|_{t=0} \langle X(\psi_{(f_0)_v(t)}(g)), \phi^t(v)\rangle \, \mathrm{d}(\psi_{f_0}^* \nu_p)(g) \\ &= \frac{d}{dt}\Big|_{t=0} \int_{O(n)/O(n-k)} \langle X(\psi_{(f_0)_v(t)}(g)), \phi^t(v)\rangle \, \mathrm{d}\theta(g \cdot O(n-k))\end{aligned}$$

$$= \frac{d}{dt}\Big|_{t=0} \langle \int_{O(n)/O(n-k)} X(\psi_{(f_0)_v(t)}(g))\,\mathrm{d}\theta \;, \phi^t(v)\;\rangle$$
$$= \langle \frac{D}{dt}\Big|_{t=0} \int_{O(n)/O(n-k)} X(\psi_{f_0}(g))\,\mathrm{d}\theta \;, v\;\rangle$$
$$= \langle \frac{D}{dt}\Big|_{t=0} \int_{F_p^k} X(f)\,\mathrm{d}\nu_p(f) \;, v\rangle,$$

which concludes the proof. □

Lemma 3.2.2. *Let* $\varphi, \psi \in C^\infty(T^kM)$, *then*

$$\int_{P^kM} \psi(f) G^i \varphi(f)\,\mathrm{d}\mu = -\int_{P^kM} \varphi(f) G^i \psi(f)\,\mathrm{d}\mu. \tag{3.2.8}$$

Proof. We observe that $G^i\varphi(f) = \langle \overset{h}{\mathrm{grad}}\varphi(f), v_i\rangle = \overset{h}{\mathrm{div}}(\varphi V_i)(f)$ by equations (3.1.3), (2.3.6) and (2.3.8). Therefore, $\int_{P^kM} G^i\varphi(f)\,\mathrm{d}\mu = 0$ by Lemma 3.2.1. This fact and the Leibniz rule prove the Lemma. □

We are now ready to state the integrated version of (3.1.5).

Theorem 3.2.3 (Integrated Pestov Identity)**.** *Let* $\varphi \in C^\infty(T^kM)$, *then*

$$\delta_{ij}\|\overset{h}{\mathrm{grad}}\varphi\|^2_{L^2(P^kM)} - \int_{P^kM} \sum_{l=1}^{k} \langle R(\overset{v,l}{\mathrm{grad}}\varphi(f), v_l)v_i, \overset{v,j}{\mathrm{grad}}\varphi(f)\rangle\,\mathrm{d}\mu =$$
$$= \int_{P^kM} \langle \overset{h}{\mathrm{grad}}\varphi(f), \overset{v,j}{\mathrm{grad}} G^i\varphi(f)\rangle + \langle \overset{h}{\mathrm{grad}} G^i\varphi(f), \overset{v,j}{\mathrm{grad}}\varphi(f)\rangle\,\mathrm{d}\mu. \tag{3.2.9}$$

Proof. Consider equation (3.1.5). Under integration over P^kM the horizontal divergence vanishes and the remaining non-zero terms are

$$\int_{P^kM} \overset{v,j}{\mathrm{div}} Z^i(f)\,\mathrm{d}\mu + \delta_{ij}\|\overset{h}{\mathrm{grad}}\varphi(f)\|^2_{L^2(P^kM)} = 2\int_{P^kM} \langle \overset{h}{\mathrm{grad}}\varphi(f), \overset{v,j}{\mathrm{grad}} G^i\varphi(f)\rangle\,\mathrm{d}\mu$$
$$+ \int_{P^kM} \sum_{l=1}^{k} \langle R(\overset{v,l}{\mathrm{grad}}\varphi(f), v_l v_i, \overset{v,j}{\mathrm{grad}}\varphi(f)\rangle\,\mathrm{d}\mu. \tag{3.2.10}$$

We have

$$\int_{P^kM} \overset{v,j}{\mathrm{div}} Z^i(f)\,\mathrm{d}\mu = \int_{P^kM} \overset{v,j}{\mathrm{div}}(G^i\varphi(f)\overset{h}{\mathrm{grad}}\varphi(f))\,\mathrm{d}\mu$$
$$= \int_{P^kM} \langle \overset{h}{\mathrm{grad}}\varphi(f), \overset{v,j}{\mathrm{grad}} G^i\varphi(f)\rangle + G^i\varphi(f)\overset{v,j}{\mathrm{div}}\overset{h}{\mathrm{grad}}\varphi(f)\,\mathrm{d}\mu$$
$$= \int_{P^kM} \langle \overset{h}{\mathrm{grad}}\varphi(f), \overset{v,j}{\mathrm{grad}} G^i\varphi(f)\rangle + G^i\varphi(f)\overset{h}{\mathrm{div}}\overset{v,j}{\mathrm{grad}}\varphi(f)\,\mathrm{d}\mu$$

$$= \int_{P^kM} \langle \overset{h}{\mathrm{grad}}\varphi(f), \overset{v,j}{\mathrm{grad}} G^i\varphi(f)\rangle - \langle \overset{h}{\mathrm{grad}} G^i\varphi(f), \overset{v,j}{\mathrm{grad}}\varphi(f)\rangle \,\mathrm{d}\mu +$$

$$+ \int_{P^kM} \overset{h}{\mathrm{div}}(G^i\varphi(f)\overset{v,j}{\mathrm{grad}}\varphi(f)) \,\mathrm{d}\mu$$

$$= \int_{P^kM} \langle \overset{h}{\mathrm{grad}}\varphi(f), \overset{v,j}{\mathrm{grad}} G^i\varphi(f)\rangle - \langle \overset{h}{\mathrm{grad}} G^i\varphi(f), \overset{v,j}{\mathrm{grad}}\varphi(f)\rangle \,\mathrm{d}\mu.$$

Substituting into (3.2.10) we have the theorem. □

3.2.1 Restriction to functions on the frame bundle P^kM

Theorem 3.2.3 can be restricted to smooth functions on P^kM. Below, we present its formulation.

Theorem 3.2.4. *Let $\varphi \in C^\infty(P^kM)$ and $f = (v_1, \ldots, v_k) \in P^kM$. Then*

$$k\|\overset{h}{\mathrm{grad}}\varphi\|^2_{L^2} - \frac{k+1}{2}\sum_{i=1}^{k}\|G^i\varphi\|^2_{L^2} - \int_{P^kM}\sum_{i,j=1}^{k}\langle R(\overset{v,j}{\mathrm{grad}_{\mathfrak{o}}}\varphi(f), v_j)v_i, \overset{v,i}{\mathrm{grad}_{\mathfrak{o}}}\varphi(f)\rangle \,\mathrm{d}\mu =$$

$$= \sum_{i=1}^{k}\int_{P^kM} \langle \overset{h}{\mathrm{grad}}\varphi(f), \overset{v,i}{\mathrm{grad}_{\mathfrak{o}}} G^i\varphi(f)\rangle + \langle \overset{h}{\mathrm{grad}} G^i\varphi(f), \overset{v,i}{\mathrm{grad}_{\mathfrak{o}}}\varphi(f)\rangle \,\mathrm{d}\mu, \quad (3.2.11)$$

where L^2 stands for the L^2-space on P^kM.

Proof. Let $\tilde{\varphi}$ be a smooth extension of φ on T^kM and consider equation (3.2.9). Setting $j = i$ and summing over $i = 1, \ldots, k$ we obtain

$$k\|\overset{h}{\mathrm{grad}}\tilde{\varphi}\|^2_{L^2} - \int_{P^kM}\sum_{i,l=1}^{k}\langle R(\overset{v,l}{\mathrm{grad}}\tilde{\varphi}(f), v_l)v_i, \overset{v,i}{\mathrm{grad}}\tilde{\varphi}(f)\rangle \,\mathrm{d}\mu =$$

$$= \int_{P^kM}\sum_{i=1}^{k}\langle \overset{h}{\mathrm{grad}}\tilde{\varphi}(f), \overset{v,i}{\mathrm{grad}} G^i\tilde{\varphi}(f)\rangle + \langle \overset{h}{\mathrm{grad}} G^i\tilde{\varphi}(f), \overset{v,i}{\mathrm{grad}}\tilde{\varphi}(f)\rangle \,\mathrm{d}\mu. \quad (3.2.12)$$

We consider the RHS and using equations (2.3.5) and (3.1.3) we obtain

$$\int_{P^kM} \langle \overset{h}{\mathrm{grad}}\tilde{\varphi}(f), \overset{v,i}{\mathrm{grad}} G^i\tilde{\varphi}(f)\rangle \,\mathrm{d}\mu = \int_{P^kM} \langle \overset{h}{\mathrm{grad}}\tilde{\varphi}(f), \overset{v,i}{\mathrm{grad}_{\mathfrak{o}}} G^i\tilde{\varphi}(f)\rangle \,\mathrm{d}\mu$$

$$+ \frac{1}{2}\sum_{j=1}^{k}\int_{P^kM} G^j\tilde{\varphi}(f)\langle \overset{v,i}{\mathrm{grad}} G^i\tilde{\varphi}(f), v_j\rangle + G^j\tilde{\varphi}(f)\langle \overset{v,j}{\mathrm{grad}} G^i\tilde{\varphi}(f), v_i\rangle \,\mathrm{d}\mu, \quad (3.2.13)$$

and

$$\begin{aligned}\int_{P^kM} \langle \overset{v,i}{\operatorname{grad}}\tilde{\varphi}(f), \overset{h}{\operatorname{grad}} G^i\tilde{\varphi}(f)\rangle \, d\mu &= \int_{P^kM} \langle \overset{v,i}{\operatorname{grad}_0}\tilde{\varphi}(f), \overset{h}{\operatorname{grad}} G^i\tilde{\varphi}(f)\rangle \, d\mu \\ &+ \frac{1}{2}\sum_{j=1}^{k}\int_{P^kM} G^jG^i\tilde{\varphi}(f)\langle \overset{v,j}{\operatorname{grad}}\tilde{\varphi}(f), v_i\rangle + G^jG^i\tilde{\varphi}(f)\langle \overset{v,i}{\operatorname{grad}}\tilde{\varphi}(f), v_j\rangle \, d\mu \\ = \int_{P^kM} \langle \overset{v,i}{\operatorname{grad}_0}\tilde{\varphi}(f), \overset{h}{\operatorname{grad}} G^i\tilde{\varphi}(f)\rangle \, d\mu &+ \frac{(k+1)}{2}\|G^i\tilde{\varphi}(f)\|^2_{L^2} \\ - \frac{1}{2}\sum_{j=1}^{k}\int_{P^kM} G^i\tilde{\varphi}(f)\langle \overset{v,i}{\operatorname{grad}} G^j\tilde{\varphi}(f), v_j\rangle &+ G^i\tilde{\varphi}(f)\langle \overset{v,j}{\operatorname{grad}} G^j\tilde{\varphi}(f), v_i\rangle \, d\mu,\end{aligned} \tag{3.2.14}$$

where we used Lemma 3.2.2 and 3.1.3 for the second equality.

Adding (3.2.13) and (3.2.14) and taking the sum over i, the RHS of (3.2.12) becomes

$$\int_{P^kM} \sum_{i=1}^{k} \langle \overset{h}{\operatorname{grad}}\tilde{\varphi}(f), \overset{v,i}{\operatorname{grad}_0} G^i\tilde{\varphi}(f)\rangle + \langle \overset{h}{\operatorname{grad}} G^i\tilde{\varphi}(f), \overset{v,i}{\operatorname{grad}_0}\tilde{\varphi}(f)\rangle \, d\mu + \frac{k+1}{2}\sum_{i=1}^{k}\|G^i\tilde{\varphi}\|^2_{L^2}. \tag{3.2.15}$$

Finally, using (2.3.5) and the antisymmetry of R, we have

$$\sum_{i,j=1}^{k} \langle R(\overset{v,j}{\operatorname{grad}}\varphi(f), v_j)v_i, \overset{v,i}{\operatorname{grad}}\varphi(f)\rangle = \sum_{i,j=1}^{k} \langle R(\overset{v,j}{\operatorname{grad}_0}\varphi(f), v_j)v_i, \overset{v,i}{\operatorname{grad}_0}\varphi(f)\rangle. \tag{3.2.16}$$

Substituting (3.2.15) and (3.2.16) into (3.2.12) proves the theorem. □

The form of the previous theorem allows us to derive a new identity assuming the function φ to be invariant under one the frame flows. The new formula contains only three terms, the Riemannian curvature tensor, the L^2 norms of the generators of the frame flows, and the L^2 norm of the horizontal gradient. Therefore, it shows a close connections between curvature properties of the manifold and properties of the frame flows.

Corollary 3.2.5. *Let $\varphi \in C^\infty(P^kM)$ and assume it is invariant under the i-th frame flow, i.e., $G^i\varphi(f) = 0$ for all $f \in P^kM$, and let $f = (v_1, \ldots, v_k) \in P^kM$. Then*

$$\frac{1}{2}\sum_{j=1, j\neq i}^{k} \|G^j\varphi\|^2_{L^2} + 2\sum_{l=1}^{n-k} \|\langle \overset{h}{\operatorname{grad}}\tilde{\varphi}(f), e_l\rangle\|_{L^2} = \sum_{j=1}^{n}\int_{P^nM} \langle R(w_j, v_j)v_i, w_i\rangle \, d\mu, \tag{3.2.17}$$

where $(v_1, \ldots, v_k, e_1, \ldots, e_{n-k})$ *is an orthonormal basis of* $T_{\pi^k(f)}M$, $w_j = \overset{v,j}{\mathrm{grad}}_{\mathfrak{o}}\varphi(f)$, *and* L^2 *stands for the* L^2*-space on* P^kM.

Proof. We prove the theorem for $i = 1$. The other cases follow in the same way.

Let $f = (v_1, \ldots, v_n) \in P^nM$ and $\tilde{\varphi}$ a smooth extension of φ on T^nM and let $(v_1, \ldots, v_k, e_1, \ldots, e_{n-k})$ be an orthonormal basis of T_pM, $p = \pi^k(f)$.

We first observe that due to equation (2.3.5), we have

$$\langle \overset{v,i}{\mathrm{grad}}_{\mathfrak{o}}\tilde{\varphi}(f), e_l \rangle = \langle \overset{v,i}{\mathrm{grad}}\tilde{\varphi}(f), e_l \rangle, \qquad \forall l = 1, \ldots, n-k. \tag{3.2.18}$$

We consider equation (3.2.11) and we aim at rewriting the horizontal gradient and its RHS in terms of L^2-norms of the generators of the frame flows and in terms of the Riemannian curvature tensor.

First of all, we observe that

$$\overset{h}{\mathrm{grad}}\tilde{\varphi}(f) = \sum_{i=1}^{n} G^i\tilde{\varphi}(f)v_i + \sum_{l=1}^{n-k} \langle \overset{h}{\mathrm{grad}}\tilde{\varphi}(f), e_l \rangle e_l, \tag{3.2.19a}$$

$$\|\overset{h}{\mathrm{grad}}\tilde{\varphi}\|_{L^2}^2 = \sum_{i=1}^{k} \|G^i\tilde{\varphi}\|_{L^2}^2 + \sum_{l=1}^{n-k} \|\langle \overset{h}{\mathrm{grad}}\tilde{\varphi}(f), e_l \rangle\|_{L^2}^2. \tag{3.2.19b}$$

We now look at the RHS of (3.2.11). From (2.3.5), we derive

$$\langle w_i, v_j \rangle = \frac{1}{2}\big(\langle \overset{v,i}{\mathrm{grad}}\tilde{\varphi}(f), v_j \rangle - \langle \overset{v,j}{\mathrm{grad}}\tilde{\varphi}(f), v_i \rangle\big), \qquad \forall\, j = 1, \ldots k. \tag{3.2.20}$$

$$\langle w_i, e_l \rangle = \langle \overset{v,i}{\mathrm{grad}}\tilde{\varphi}(f), e_l \rangle, \qquad \forall\, l = 1, \ldots, n-k. \tag{3.2.21}$$

Using these equations, Lemma 3.1.3 and equation (3.2.19a), we have

$$\begin{aligned}
\sum_{i=2}^{k}\int_{P^kM}\langle \overset{h}{\mathrm{grad}}\tilde{\varphi}(f), \overset{v,i}{\mathrm{grad}_{\mathfrak{o}}} G^i\tilde{\varphi}(f)\rangle \,\mathrm{d}\mu &= \sum_{i=2}^{k}\Big[\int_{P^kM}\sum_{j=2,j\neq i}^{k} G^j\tilde{\varphi}(f)\langle \overset{v,i}{\mathrm{grad}_{\mathfrak{o}}} G^i\tilde{\varphi}(f), v_j\rangle \,\mathrm{d}\mu \\
&\quad + \int_{P^kM}\sum_{l=1}^{n-k}\langle \overset{h}{\mathrm{grad}}\tilde{\varphi}(f), e_l\rangle\langle \overset{v,i}{\mathrm{grad}} G^i\tilde{\varphi}(f), e_l\rangle \,\mathrm{d}\mu\Big] \\
&= \sum_{i,j=2,j\neq i}^{k}\int_{P^kM}\frac{1}{2}G^j\tilde{\varphi}(f)\Big[\langle \overset{v,i}{\mathrm{grad}} G^i\tilde{\varphi}(f), v_j\rangle - \langle \overset{v,j}{\mathrm{grad}} G^i\tilde{\varphi}(f), v_i\rangle\Big] \,\mathrm{d}\mu \\
&\quad + \sum_{i=2}^{k}\sum_{l=1}^{n-k} + \int_{P^kM}\langle \overset{h}{\mathrm{grad}}\tilde{\varphi}(f), e_l\rangle\Big[\langle \overset{h}{\mathrm{grad}}\tilde{\varphi}(f), e_l\rangle + G^i(\langle \overset{v,i}{\mathrm{grad}}\tilde{\varphi}(f), e_l\rangle)\Big] \,\mathrm{d}\mu \\
&= \sum_{i,j=2,j\neq i}^{k}\int_{P^kM}\frac{1}{2}G^j\tilde{\varphi}(f)\Big[G^i\langle \overset{v,i}{\mathrm{grad}}\tilde{\varphi}(f), v_j\rangle - G^i\langle \overset{v,j}{\mathrm{grad}}\tilde{\varphi}(f), v_i\rangle + G^j\tilde{\varphi}(f)\Big] \,\mathrm{d}\mu \\
&\quad + \sum_{i=2}^{k}\sum_{l=1}^{n-k}\int_{P^kM}\langle \overset{h}{\mathrm{grad}}\tilde{\varphi}(f), e_l\rangle G^i(\langle \overset{v,i}{\mathrm{grad}}\tilde{\varphi}(f), e_l\rangle) \,\mathrm{d}\mu + (k-2)\sum_{l=1}^{n-k}\|\langle \overset{h}{\mathrm{grad}}\tilde{\varphi}(f), e_l\rangle\|_{L^2}^2 \\
&= \sum_{i,j=2,j\neq i}^{k}\int_{P^kM} -G^iG^j\tilde{\varphi}(f)\langle w_i, v_j\rangle \,\mathrm{d}\mu + \frac{k-2}{2}\sum_{i=2}^{k}\|G^i\tilde{\varphi}\|_{L^2}^2 \\
&\quad + \sum_{i=2}^{k}\sum_{l=1}^{n-k}\int_{P^kM}\langle \overset{h}{\mathrm{grad}}\tilde{\varphi}(f), e_l\rangle G^i(\langle \overset{v,i}{\mathrm{grad}}\tilde{\varphi}(f), e_l\rangle) \,\mathrm{d}\mu + (k-2)\sum_{l=1}^{n-k}\|\langle \overset{h}{\mathrm{grad}}\tilde{\varphi}(f), e_l\rangle\|_{L^2}^2,
\end{aligned} \tag{3.2.22}$$

and

$$\begin{aligned}
\sum_{i=2}^{k}\int_{P^kM}\langle \overset{h}{\mathrm{grad}} G^i\varphi(f), \overset{v,i}{\mathrm{grad}_{\mathfrak{o}}}\varphi(f)\rangle \,\mathrm{d}\mu &= \sum_{i=2}^{k}\sum_{j=1,j\neq i}^{k}\int_{P^kM}\langle \overset{h}{\mathrm{grad}} G^i\tilde{\varphi}(f), v_j\rangle\langle v_j, \overset{v,i}{\mathrm{grad}_{\mathfrak{o}}}\tilde{\varphi}(f)\rangle \,\mathrm{d}\mu \\
&\quad + \sum_{i=2}^{k}\sum_{l=1}^{n-k}\int_{P^kM}\langle \overset{h}{\mathrm{grad}} G^i\tilde{\varphi}(f), e_l\rangle\langle e_l, \overset{v,i}{\mathrm{grad}_{\mathfrak{o}}}\tilde{\varphi}(f)\rangle \,\mathrm{d}\mu \\
&= \sum_{i=2}^{k}\int_{P^kM} G^1G^i\tilde{\varphi}(f)\langle v_1, w_i\rangle \,\mathrm{d}\mu + \sum_{i,j=2,j\neq i}^{k}\int_{P^kM} G^jG^i\tilde{\varphi}(f)\langle w_i, v_j\rangle \,\mathrm{d}\mu \\
&\quad + \sum_{i=2}^{k}\sum_{l=1}^{n-k}\int_{P^kM} G^i(\langle \overset{h}{\mathrm{grad}}\tilde{\varphi}(f), e_l\rangle)\langle \overset{v,i}{\mathrm{grad}}\tilde{\varphi}(f), e_l\rangle \,\mathrm{d}\mu \\
&\quad - \sum_{i=2}^{k}\sum_{l=1}^{n-k}\int_{P^kM}\sum_{j=1}^{k}\langle R(\overset{v,j}{\mathrm{grad}}\tilde{\varphi}(f), v_j)v_i, e_l\rangle\langle \overset{v,i}{\mathrm{grad}}\tilde{\varphi}(f), e_l\rangle \,\mathrm{d}\mu.
\end{aligned} \tag{3.2.23}$$

Summing (3.2.22) and (3.2.23), using (3.1.2b), (3.2.8) and the skew-symmetry of the matrix $(\langle w_i, v_j\rangle)_{i,j}$, we have

$$
\begin{aligned}
&\sum_{i=1}^{k}\int_{P^kM}\langle\overset{h}{\text{grad}}\varphi(f),\overset{v,i}{\text{grad}_{\mathfrak{o}}}G^i\varphi(f)\rangle+\langle\overset{h}{\text{grad}}G^i\varphi(f),\overset{v,i}{\text{grad}_{\mathfrak{o}}}\varphi(f)\rangle\,\mathrm{d}\mu=\\
&=\frac{k-2}{2}\sum_{i=2}^{k}\|G^i\tilde{\varphi}\|_{L^2}^2+(k-2)\sum_{l=1}^{n-k}\|\langle\overset{h}{\text{grad}}\tilde{\varphi}(f),e_l\rangle\|_{L^2}^2\\
&+\sum_{i=2}^{k}\int_{P^kM}G^1G^i\tilde{\varphi}(f)\langle v_1,w_i\rangle\,\mathrm{d}\mu+\sum_{i,j=2,i\neq j}^{k}\int_{P^kM}(G^jG^i-G^iG^j)\tilde{\varphi}(f)\langle v_j,w_i\rangle\,\mathrm{d}\mu\\
&-\sum_{i=2}^{k}\sum_{l=1}^{n-k}\int_{P^kM}\sum_{j=1}^{k}\langle R(\overset{v,j}{\text{grad}}\tilde{\varphi}(f),v_j)v_i,e_l\rangle\langle\overset{v,i}{\text{grad}}\tilde{\varphi}(f),e_l\rangle\,\mathrm{d}\mu\\
&+\sum_{i=2}^{k}\sum_{l=1}^{n-k}\underbrace{\int_{P^kM}\langle\overset{h}{\text{grad}}\tilde{\varphi}(f),e_l\rangle G^i(\langle\overset{v,i}{\text{grad}}\tilde{\varphi}(f),e_l\rangle)+G^i(\langle\overset{h}{\text{grad}}\tilde{\varphi}(f),e_l\rangle)\langle\overset{v,i}{\text{grad}}\tilde{\varphi}(f),e_l\rangle\,\mathrm{d}\mu}_{\int_{P^kM}G^i(\langle\overset{h}{\text{grad}}\tilde{\varphi}(f),e_l\rangle\langle\overset{v,i}{\text{grad}}\tilde{\varphi}(f),e_l\rangle)\,\mathrm{d}\mu=0\text{ (see proof of Lemma 3.2.2)}}\\
&=\frac{k-2}{2}\sum_{i=2}^{k}\|G^i\tilde{\varphi}\|_{L^2}^2+(k-2)\sum_{l=1}^{n-k}\|\langle\overset{h}{\text{grad}}\tilde{\varphi}(f),e_l\rangle\|_{L^2}^2-\int_{P^kM}\sum_{i=2}^{k}\sum_{l=1}^{k}\langle R(w_i,v_l)v_i,w_i\rangle\,\mathrm{d}\mu.
\end{aligned}
\tag{3.2.24}
$$

Substituting (3.2.19b) and (3.2.24) into (3.2.11) concludes the proof. □

For $k = n$,the above corollary simplifies as follows.

Corollary 3.2.6. *Let $\varphi \in C^\infty(P^nM)$ and assume it is invariant under the i-th frame flow, i.e., $G^i\varphi(f) = 0$ for all $f \in P^nM$, and let $f = (v_1, \ldots, v_n) \in P^nM$. Then*

$$
\frac{1}{2}\sum_{j=1,j\neq i}^{n}\|G^j\varphi\|_{L^2}^2=\sum_{j=1}^{n}\int_{P^nM}\langle R(w_j,v_j)v_i,w_i\rangle\,\mathrm{d}\mu, \tag{3.2.25}
$$

where $w_j = \overset{v,j}{\text{grad}_{\mathfrak{o}}}\varphi(f)$ and L^2 stands for the L^2-space on P^nM.

Chapter 4

Dynamical Applications

In this chapter we present two dynamical consequences of Corollary 3.2.6 under curvature assumptions on the base manifold M. In Section 4.1 we consider the principal bundle over a n-dimensional compact manifold with non-positive constant curvature, and over a 2-dimensional manifold of non-positive curvature proving that any smooth functions invariant under one of the frame flows is also invariant under the remaining ones. In Section 4.2 we consider the Grassmannians of oriented k-planes over M with non-positive curvature operator and we show an invariance property of smooth functions on such Grassmannians under the action of the parallel transport.

4.1 An invariance property of smooth functions on frame bundles

Corollary 3.2.6 in Chapter 3 hints that if the curvature of the manifold could be non-positive, then function φ would also be invariant under the remaining frame flows. This is true in two cases, for 2-dimensional manifolds of non-positive curvature and for higher dimensional manifolds of non-positive constant curvature.

Corollary 4.1.1. *Let M be a 2-dimensional manifold of non-positive curvature. Let φ be a smooth function on the principal bundle P^2M. If φ is invariant under the i-th frame flow, $0 < i \leqslant 2$, i.e., $G^i\varphi(f) = 0$ for all $f \in C^\infty(P^2M)$, then it is invariant under the remaining frame flow.*

Proof. Consider equation (3.2.25) for $k = 2$ and assume $i = 1$, the case $i = 2$ being the same. Writing the vertical gradients in the RHS according to the orthonormal basis v_1, v_2, we obtain

$$\frac{1}{2}\|G^2\varphi\|_{L^2}^2 = \int_{P^2M} \langle R(v_2, v_1)v_1, v_2\rangle\langle \overset{v,1}{\mathrm{grad}_{\mathbf{o}}}\varphi(f), v_2\rangle^2 \, \mathrm{d}\mu \leqslant 0.$$

Hence, $\|G^2\varphi\|_{L^2}^2 = 0$ and therefore φ is also invariant under G^2. □

Corollary 4.1.2. *Let M be a n-dimensional manifold of non-positive constant sectional curvature K. Let φ be a smooth function on the principal bundle P^nM. If φ is invariant under one of the i-th frame flows, i.e., $G^i\varphi(f) = 0$ for all $f \in C^\infty(P^nM)$, then it is invariant under the remaining frame flows.*

Proof. We first notice that in case of constant sectional curvature K, the Riemannian curvature tensor decomposes as

$$\langle R(v_1, v_2)v_3, v_4\rangle = K\big(\langle v_1, v_4\rangle\langle v_2, v_3\rangle - \langle v_1, v_3\rangle\langle v_2, v_4\rangle\big).$$

Substituting this expression into the RHS of (3.2.25) we obtain

$$\frac{1}{2}\sum_{j=1, j\neq i}^{n} \|G^j\varphi\|_{L^2} = \int_{P^nM} \sum_{j=1}^{n} K\big(\langle w_j, w_i\rangle - \langle w_j, v_i\rangle\langle w_i, v_j\rangle\big) \, \mathrm{d}\mu = K\|w_i\|_{L^2} \leqslant 0.$$

Hence, $\sum_{j=1, j\neq i}^{n} \|G^j\varphi\|_{L^2} = 0$. This implies that each term of the sum is zero as they must all be non-negative. Therefore, φ is invariant under the remaining frame flows. □

4.2 An invariance property of smooth functions on oriented Grassmannians

Another consequence of Corollary 3.2.6 is for smooth functions on oriented Grassmannians. We begin explaining the relation between oriented Grassmannians and frame bundles.

4.2.1 Oriented Grassmannians and Parallel Transport

The *oriented k-th Grassmannian* of M, $\mathcal{G}^k_{or}(M)$ for $1 \leqslant k \leqslant n = \dim M$, is the union of all k-planes of T_pM for all $p \in M$ together with an intrinsic orientation, i.e.,

$$\mathcal{G}^k_{or}(M) = \bigcup_{p\in M} \big\{A_{or} \bigm| A \subset T_pM, \dim A = k\big\},$$

where A_{or} is a subspace of T_pM with an additional choice of intrinsic orientation.

This is a 2-fold of the non-oriented k-th Grassmannian. We observe that for $k = 1$ we have $\mathcal{G}^1_{or}(M) = SM$, the unit tangent bundle of M.

$\mathcal{G}^k_{or}(M)$ is linked to the principal bundle P^nM by the canonical projection

$$\tilde{\pi} : P^nM \longrightarrow \mathcal{G}^k_{or}(M)$$

$$f = (v_1, \dots, v_n) \mapsto \big(span\{v_1, \dots, v_k\}, (v_1, \dots, v_k)\big),$$

where $(v_1, \dots, v_k)$ stands for intrinsic orientation.

The existence of this projection implies that any function $\varphi \in C^\infty(\mathcal{G}^k_{or}(M))$ can be extended to a function $\phi \in C^\infty(P^nM)$ by setting $\phi = \varphi \circ \tilde{\pi}$. The function ϕ has two important properties. Firstly, since φ is invariant under the action of $SO(k)$, ϕ is invariant under the action of matrices of the form $\begin{pmatrix} SO(k) & 0 \\ 0 & SO(n-k) \end{pmatrix}$. Secondly, its vertical gradients satisfy the following conditions.

Lemma 4.2.1. *Let $\varphi \in C^\infty(\mathcal{G}^k_{or}(M))$ and $\phi = \varphi \circ \tilde{\pi} \in C^\infty(P^nM)$. Then,*

(i) $\overset{v,i}{\mathrm{grad}_{\mathfrak{o}}}\phi(f) \in span\{v_1, \dots, v_k\}$ *if* $i \geqslant k+1$;

(ii) $\overset{v,i}{\mathrm{grad}_{\mathfrak{o}}}\phi(f) \in span\{v_{k+1}, \dots, v_n\}$ *if* $i = 1, \dots k$.

Proof. To prove (i) and (ii), we need to show that $\langle \overset{v,i}{\mathrm{grad}_{\mathfrak{o}}}\phi(f), v_j\rangle = 0$ for $i, j \geqslant k+1$ and $\langle \overset{v,i}{\mathrm{grad}_{\mathfrak{o}}}\phi(f), v_j\rangle = 0$ for $i, j \leqslant k$, respectively.

Let $\tilde{\phi}$ be a smooth extension of ϕ on T^nM constructed as follows.

Let $h : T^nM \longrightarrow [0, \infty]$ be such that $h(w_1, \dots, w_n) = \det(\langle w_i, w_j\rangle)_{ij}$ and let $\psi : \{(w_1, \dots, w_n) \text{ lin. indep}\} \longrightarrow P^nM$ be the Gram-Schmidt process. We define the cut off function $H : [0, \infty] \longrightarrow [0, 1]$ be a cut off function such that $H(x) = 0$ for $x \leqslant \frac{1}{2}$, $H(x) = 1$ for $x \geqslant \frac{3}{4}$ and $0 \leqslant H(x) \leqslant 1$ for $\frac{1}{2} \leqslant x \leqslant \frac{3}{4}$. Then

$$\tilde{\phi}(w_1, \dots, w_n) = \begin{cases} H\big(h(w_1, \dots, w_n)\big)(\phi \circ \psi)(w_1, \dots, w_n) & \{w_1, \dots, w_n\} \text{ lin. indep.} \\ 0 & \text{otherwise} \end{cases}$$

Note that $\tilde{\phi}(w_1, \ldots, w_l + tw_j, \ldots, w_n) = \tilde{\phi}(w_1, \ldots, w_n)$ for all $l \geqslant k+1$.

This implies that, for $i, j \geqslant k+1$, we have

$$\langle \overset{v,i}{\mathrm{grad}}_{\mathbf{o}} \phi(f), v_j \rangle = \frac{1}{2} \langle \overset{v,i}{\mathrm{grad}} \tilde{\phi}(f), v_j \rangle - \frac{1}{2} \langle \overset{v,j}{\mathrm{grad}} \tilde{\phi}(f), v_i \rangle$$

$$= \frac{1}{2} \frac{d}{dt}\Big|_{t=0} \Big(\tilde{\phi}(v_1, \ldots, v_k, \ldots, v_i + tv_j, \ldots, v_n) - \tilde{\phi}(v_1, \ldots, v_k, \ldots, v_j + tv_i, \ldots, v_n) \Big) = 0,$$

which proves (i).

Part (ii) follows from the fact that $\tilde{\phi}$ depends only on the plane spanned by the first k vectors. □

Another consequence of the existence of $\tilde{\pi}$ is a correspondence between the generators of the frame flows on $P^n M$ and the parallel transport on $\mathcal{G}^k_{or}(M)$ along special directions.

In fact, let $\phi = \varphi \circ \tilde{\pi} \in C^\infty(P^n M)$ with $\varphi \in C^\infty(\mathcal{G}^k_{or}(M))$, $f = (v_1, \ldots, v_n) \in P^k M$ and $A_{or} = \big(span\{v_1, \ldots, v_k\}, (v_1, \ldots, v_k) \big) = \tilde{\pi}(f)$, then

$$G^i \phi(f) = G^i(\varphi \circ \tilde{\pi})(f) = \frac{d}{dt}\Big|_{t=0} (\varphi \circ \tilde{\pi})(f_{v_i}(t)) = \frac{d}{dt}\Big|_{t=0} \varphi((A_{or})_{v_i}(t)). \tag{4.2.1}$$

Definition 4.2.2. Let $(A_{or})_v(t)$ denote the parallel transport of the oriented k-plane $A_{or} \in \mathcal{G}^k_{or}(M)$ along a curve c_v on M with $c_v'(0) = v$. The parallel transport is called *intrinsic* if the vector v belongs to A_{or}, and *non-intrinsic* otherwise.

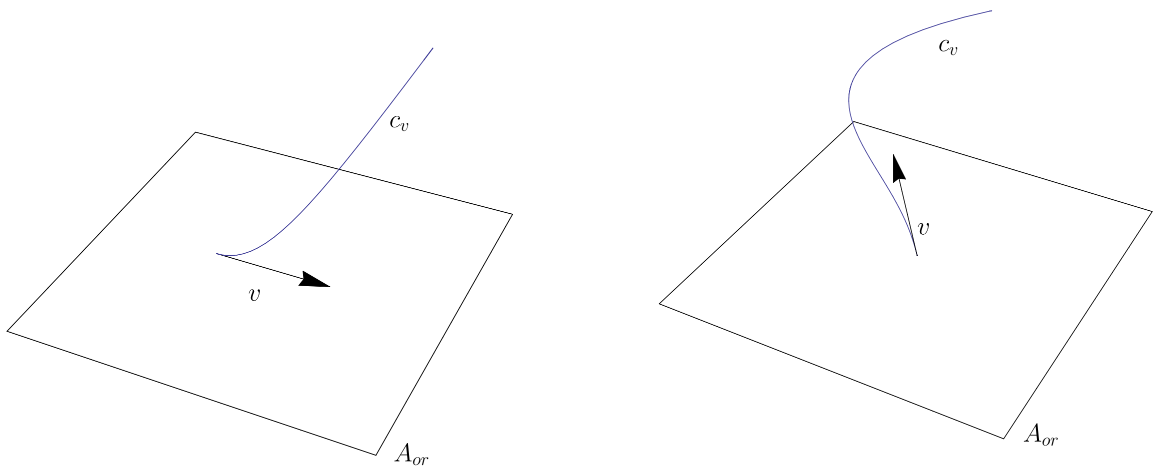

Figure 4.1: From left to right: example of intrinsic and non-intrinsic parallel transport of a plane A_{or} along the geodesic c_v.

The above definition and equation (4.2.1) implies the following.

Corollary 4.2.3. *Let $\tilde{\pi} : P^nM \longrightarrow \mathcal{G}^k_{or}(M)$ be the canonical projection, $\varphi \in C^\infty(\mathcal{G}^k_{or}(M))$, and $\phi = \varphi\circ\tilde{\pi} \in C^\infty(P^nM)$. Then, φ is invariant under all intrinsic parallel transports if and only if ϕ is invariant under the i-th frame flows, for $i = 1, \dots, k$.*

4.2.2 Invariance Property

We now assume that M has non-positive curvature operator $\mathcal{R}$. The *curvature operator* is a linear operator $\mathcal{R} : \Lambda^2(TM) \longrightarrow \Lambda^2(TM)$ defined as

$$\langle \mathcal{R}(X \wedge Y), Z \wedge W\rangle_{\Lambda^2(TM)} = \langle R(X, Y)W, Z\rangle_{TM} \tag{4.2.2}$$

for all vector fields X, Y, Z, W on M.

The curvature operator $\mathcal{R}$ is symmetric and we say that $\mathcal{R}$ is non-positive ($\mathcal{R} \leqslant 0$) if all of its real eigenvalues are non-positive.

A manifold M with non-positive curvature operator has non-positive curvature. However, the inverse implication is not true, for details on this topic we refer the reader to [AF04] and [Ara10].

Theorem 4.2.4. *Let M be a compact n-dimensional Riemannian manifold with non-positive curvature operator ($\mathcal{R} \leqslant 0$). Let $1 \leqslant k \leqslant n$ and $\varphi \in C^\infty(\mathcal{G}^k_{or}(M))$. If φ is invariant under all the intrinsic parallel transports then it is also invariant under all parallel transports.*

Proof. Let $\phi = \varphi \circ \tilde{\pi} \in C^\infty(P^nM)$.

Since φ is invariant under intrinsic parallel transports, ϕ is invariant under $G^1, \dots, G^k$ due to (4.2.1).

Considering equation (3.2.25) and summing over $i = 1, \dots, k$ we obtain

$$\begin{aligned} \frac{k}{2}\sum_{j=1}^{n} \|G^j\phi\|^2_{L^2(P^nM)} &= \int_{P^nM} \sum_{i=1}^{k}\sum_{j=1}^{n} \langle R(w_j, v_j)v_i, w_i\rangle \, \mathrm{d}\mu \\ &= \int_{P^nM} \langle \mathcal{R}\Big(\sum_{j=1}^{n} w_j \wedge v_j\Big), \sum_{i=1}^{k} w_i \wedge v_i\rangle_{\Lambda^2(TM)} \, \mathrm{d}\mu, \end{aligned} \tag{4.2.3}$$

where $w_i = \overset{v,i}{\operatorname{grad}_{\mathfrak{o}}}\phi(f)$.

Since the matrix $\big(\langle w_i, v_j\rangle\big)_{i,j}$ is skew-symmetric and making use of Lemma 4.2.1, we obtain

$$\sum_{j=1}^{n} w_j \wedge v_j = \sum_{j=1}^{n}\sum_{l=1,l\neq j}^{n} \langle w_j, v_l\rangle v_l \wedge v_i = \sum_{j=1}^{k}\sum_{l=k+1}^{n} \langle w_j, v_l\rangle v_l \wedge v_j + \sum_{j=k+1}^{n}\sum_{l=1}^{k}\langle w_j, v_l\rangle v_l \wedge v_j$$
$$= 2\sum_{j=1}^{k} w_j \wedge v_j.$$

Since $\mathcal{R} \leqslant 0$, the RHS of (4.2.3) is non-positive forcing the LHS to be zero. We conclude that ϕ is invariant under all frame flows, and so φ is invariant under all parallel transports. □

For $k = 1$ the theorem above allows us to recover an unpublished result of Knieper [Kni] on the geodesic flow. In fact, $\mathcal{G}^1_{or}(M) = SM$, the intrinsic parallel transport corresponds to the geodesic flow and the assumption on the non-positivity of the curvature operator can be weakened to the non-positivity of the sectional curvature, as it is clear from the proof of Theorem 4.2.4. Therefore, we can state the following.

Corollary 4.2.5. *Let M be a compact Riemannian manifold with non-positive curvature. Let $\varphi \in C^\infty(SM)$ invariant under the geodesic flow, then φ is also invariant under parallel transport.*

In view of Theorem 4.2.4, it is natural to investigate density properties of orbits of k-planes A_{or} under all intrinsic parallel transports. In fact, let $I(A_{or})$ be the set of all k-planes obtained by finitely many moves along intrinsic parallel transport and $G(A_{or})$ be the set of all k-planes obtained by finitely many moves along general parallel transport. Theorem 4.2.4 suggests that even though there might be many k-planes A_{or} such that $I(A_{or})$ is much smaller and not dense in $G(A_{or})$, there might always be a k-plane A'_{or} arbitrarily close to A_{or} such that $I(A'_{or})$ is dense in $G(A'_{or})$. This is at least true in the case of the flat torus and in the case of constant negative curvature. In the general case, this seems to be a difficult question to answer.

However, we can give an answer to the related, easier question, whether a smooth function φ invariant under all intrinsic parallel transports is necessarily constant.

Proposition 4.2.6. *Let M be a non-flat, compact Riemannian manifold with non-positive curvature operator $\mathcal{R}$. Then the following statements hold:*

(i) *If M is either a Kähler or a Quaternion-Kähler manifold of real dimension $2n \geqslant 4$ or $4n \geqslant 8$, respectively, or a locally symmetric space of non-constant curvature (i.e., not the real hyperbolic space), then there exist smooth, non-constant functions on $\mathcal{G}^2_{or}(M)$ or $\mathcal{G}^4_{or}(M)$ which are invariant under all intrinsic parallel transports.*

(ii) *If M is not one of the exceptions in (i), then, for all $k \leqslant \dim M$, any smooth function on $\mathcal{G}^k_{or}(M)$ which is invariant under intrinsic parallel transport is necessarily constant.*

Proof. (i) Let M be a Kähler manifold of dimension $2n \geqslant 4$. The almost complex structure J is parallel and it gives rise to a smooth function φ on oriented 2-planes, which is invariant under parallel transports but it is not constant. This function is defined via $\varphi(A_{or}) = \langle v_1, Jv_2 \rangle$ where v_1, v_2 is an oriented orthonormal basis of $A_{or} \in \mathcal{G}^2_{or}(M)$.

Now, let M be a Quaternion-Kähler manifold of real dimension $4n \geqslant 8$ with non-positive curvature operator. The canonical 4-forms Ω (see for example [Ish74] or [Gray69]) globally defined on M is parallel. This gives rise to the smooth function $\varphi : \mathcal{G}^4_{or}(M) \to \mathbb{R}$, via $\varphi(A_{or}) = \Omega_p(v_1, \dots, v_4)$ where $v_1, \dots, v_4$ is an oriented orthonormal basis of $A_{or} \in \mathcal{G}^4_{or}(M)$ and $A_{or} \in T_pM$. This function is invariant under parallel transports and non constant.

Finally, let M be a locally symmetric space of non-constant non-positive curvature, then M is a compact quotient of a symmetric space with non-constant curvature and its Riemannian curvature tensor is parallel. We define $\varphi \in C^\infty(\mathcal{G}^2_{or}(M))$ as $\varphi(A_{or}) = \langle R(v_1, v_2)v_2, v_1 \rangle$ where v_1, v_2 is an oriented orthonormal basis of $A_{or} \in \mathcal{G}^2_{or}(M)$. Now, φ is invariant under all parallel transports but it is not constant.

(ii) If M is a n-dimensional manifold which is not one of the exceptions above, the holonomy group of M is $SO(n)$ (see [Ber93] or [Bes87]). Therefore, any smooth function on $\mathcal{G}^k_{or}(M)$ invariant under all intrinsic parallel transports is also invariant under the non-intrinsic parallel transports by Theorem 4.2.4 and, hence, is constant due to the transitive action of $SO(n)$ on oriented k-planes in the tangent space. □

Remark 4.2.7. It seems to be an open question whether there exist compact non-locally symmetric Quaterion-Kähler manifolds with non-positive curvature operator. We refer the reader to [CDL14, CNS13, LeB91, LeB88] for an overview concerning this question.

Part II

Chapter 5

Introduction

A graph-like manifold is a family of compact, oriented and connected Riemannian manifolds $\{X_\varepsilon\}_{\varepsilon>0}$ made of building blocks according to the structure of metric graph X_0. Roughly speaking, a metric graph is a graph where every edge e is associated to a length ℓ_e. The manifold X_ε has the property that it shrinks to the metric graph in the limit $\varepsilon \to 0$. More precisely, a graph-like manifold is made up of edge neighbourhoods $X_{\varepsilon,e} = [0, \ell_e] \times Y_{\varepsilon,e}$ and vertex neighbourhoods $X_{\varepsilon,v}$ according to the structure of the underlying metric graph. The parameter ε can be considered as the radius of the edge neighbourhoods. $X_{\varepsilon,e}$ and $X_{\varepsilon,v}$ are n-dimensional manifolds with boundary whose intersection is a boundary-less $(n-1)$-dimensional manifold $Y_{\varepsilon,e}$ if the edge e emanates from the vertex v. $Y_{\varepsilon,e}$ is called the transversal manifold (see Section 6.4 for a complete description).

Graph-like manifolds are widely used in Mathematics as well as in Physics. In Mathematics, they are used to prove spectral properties of manifolds. The main example is given by Colin de Verdièr [CdV86] who proved that the first non-zero eigenvalue of a compact manifold of dimension $n \geqslant 3$ can have arbitrarily large multiplicity. Other authors used them to prove the existence of metric with arbitrarily large eigenvalues. Gentile and Pagliara [GP95] proved that any manifolds of dimension $n \geqslant 4$ admits a metric such that the first non-zero eigenvalues of the Hodge Laplacian acting on differential forms is arbitrarily large. Similar results were also proved in [CM10] and [CG14] where the authors analysed the first non-zero eigenvalue of the Hodge Laplacian on surfaces with boundary and of the rough Laplacian on differential forms on manifolds of any dimension, respectively. The key feature of these ‘spectral engineering’ articles is to modify the manifolds with shrinking ‘pieces’ without affecting its topology and carry out the analysis on the new manifold, that can then be considered as a graph-like manifold. In Physics, graph-like manifolds, or more concrete, small neighbourhoods of metric graphs embedded in $\mathbb{R}^3$, are used to model electronic and optic nano-structures. The natural question arising is whether the underlying graph is a good approximation for the graph-like manifold. This leads to the study of the asymptotic behaviour of the eigenvalues of the manifold. In this direction, the convergence of the Laplacian on functions on graph-like manifolds, the scalar Laplacian, has been analysed in details, and the convergence of various objects such as resolvents, spectrum, etc. is established in many contexts (see [EP05], [Pos12], [EP13] and references therein).

Related work. Graph-like manifolds can also be considered as collapsing manifold with no curvature control. In this context, the focus is on understanding how well the manifold approaches its limit space. An example is given in [AC95] where the authors considered manifolds with shrinking handles. Similar partial collapsing have also been considered in [AT12] and [AT14], where the authors studied the limit spectrum of the Hodge-Laplacian of the collapsing of one part of a connected sum.

Another research line for collapsing manifolds is to study the collapse under curvature bounds. In general, this assumption gives rise to extra structures, we refer the reader to [Jam05], [Lot02], [Lot14] for a detailed overview.

5.1 Aim and main results

The aim of our investigation is to describe the behaviour of the eigenvalues of the Hodge Laplacian $\Delta^p_{X_\varepsilon}$ acting on differential p-forms on a compact n-dimensional graph-like manifold X_ε for $p = 1, \dots, n-1$ as $\varepsilon \to 0$.

The main idea beyond our description is that any p-forms can be orthogonally decomposed into three components: a harmonic one; an exact one; and a co-exact one (see [deR55] or [McG93] and references therein or Section 6.5). Consequently, the spectrum of $\Delta^p_{X_\varepsilon}$ is the union of the spectra of the $\Delta^{p,ex}_{X_\varepsilon}$ and of $\Delta^{p,co-ex}_{X_\varepsilon}$, the Hodge

Laplacian acting on exact and co-exact forms, respectively, hence it is sufficient to describe the eigenvalues of these two operators in the limit $\varepsilon \to 0$ to obtain a full description of the asymptotic behaviour of the eigenvalues of the Hodge Laplacian acting on p-forms on X_ε. We will denote the j-th eigenvalue, counted with multiplicity, of $\Delta_{X_\varepsilon}^{p,ex}$ and $\Delta_{X_\varepsilon}^{p,co-ex}$ by $\bar{\lambda}_j^p(X_\varepsilon)$ and $\bar{\bar{\lambda}}_j^p(X_\varepsilon)$, respectively. For short, we will call them exact and co-exact p-form eigenvalues. By Hodge duality and the fact that the exterior derivative d and its adjoint d^* are isomorphisms between the space of exact and co-exact eigenforms (see also end of Section 6.5), the exact and co-exact eigenvalues are related as follows.

$$\bar{\lambda}_j^p(X_\varepsilon) = \bar{\bar{\lambda}}_j^{p-1}(X_\varepsilon) \qquad \text{and} \qquad \bar{\lambda}_j^p(X_\varepsilon) = \bar{\bar{\lambda}}_j^{n-p}(X_\varepsilon) \qquad \forall\, j \geqslant 1. \tag{5.1.1}$$

Hence, the asymptotic behaviour of the exact p-form eigenvalues up to the middle degree gives the description of all the eigenvalues of $\Delta_{X_\varepsilon}^p$. We observe that for a graph-like manifold of dimension two, the spectrum of the Laplacian in all degree forms is entirely determined by its spectrum on functions. Since the convergence of the function eigenvalues on X_ε has already been established [Pos12, EP13], we easily conclude the converge of the eigenvalues of the Laplacian on X_ε for all degree forms by (5.1.1) (see Section 7.2, in particular Corollary 7.2.3). For higher dimensional graph-like manifolds, we still have convergence for the 1-form eigenvalues, but it is not sufficient to describe the form eigenvalues in higher degrees: by Hodge duality we only obtain the convergence of the co-exact $(n-1)$-form eigenvalues. The remaining p-form eigenvalues are all divergent under the hypothesis that X_ε is transversally trivial (Theorem 7.3.3), i.e., the transversal manifolds Y_e have trivial p-th cohomology groups for $p = 1, \dots, n-2$. This is proved using the natural structure of X_ε and the McGowan Lemma stated in Proposition 6.6.4. Removing the topological assumption, we have divergence for some of the higher form eigenvalues, namely, for eigenvalues indexed by $j \geqslant N$, where N depends on the dimension of the cohomology of Y_e for all $e \in E$ (Theorem 7.3.5).

We summarise our results as follows.

Theorem 5.1.1. *Let X_ε be a compact n-dimensional Riemannian graph-like manifold and let X_0 be its associated metric graph. The following statements are true.*

(i) The 0-form eigenvalues, or equivalently the exact 1-form eigenvalues, of X_ε converge to the eigenvalues of X_0, i.e.,

$$\lambda_j(X_\varepsilon) = \bar{\lambda}_j^1(X_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \lambda_j(X_0) \qquad \forall\, j \geqslant 1.$$

(ii) Let $n \geqslant 3$ and $2 \leqslant p \leqslant n-1$ and that X_ε is a transversally trivial manifold. Then, the first exact p-form (co-exact $(p-1)$-form) eigenvalues diverge, i.e,

$$\bar{\lambda}_1^p(X_\varepsilon) = \bar{\bar{\lambda}}_1^{p-1}(X_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \infty,$$

and so do all the exact p-form (co-exact $(p-1)$-form) eigenvalues.
(iii) Let $n \geqslant 3$ and $2 \leqslant p \leqslant n-1$. Then, j-th exact p-form (co-exact $(p-1)$-form) eigenvalues diverge, i.e.,

$$\bar{\lambda}_N^p(X_\varepsilon) = \bar{\bar{\lambda}}_N^{p-1}(X_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \infty, \qquad \textit{for} \qquad N = 1 + 2 \sum_{e \in E} \dim H^{p-1}(Y_e),$$

and so does all the j-th exact and co-exact form eigenvalues (in the right dimension) for $j \geqslant N$.

We also asked ourselves about the relation between spectral gaps in the spectrum of the Laplacian acting on 1-forms on X_ε and X_0. The interval (a, b) is a spectral gap for the Laplacian if it does not belong to its spectrum. Our asymptotic description, in particular Theorem 7.2.2 and Corollary 8.1.1, yields the following.

Corollary 5.1.2. *Assume that the graph-like manifold X_ε is transversally trivial and suppose that (a_0, b_0) is a spectral gap for the metric graph X_0, then there exist a_ε, b_ε with $a_\varepsilon \to a_0$ and $b_\varepsilon \to b_0$ such that $(a_\varepsilon, b_\varepsilon)$ is a spectral gap for the Hodge Laplacian on X_ε in all degrees, i.e., $\sigma(\Delta^\bullet_{X_\varepsilon}) \cap (a_\varepsilon, b_\varepsilon) = \varnothing$.*

In all our applications (presented in Chapter 8) we assume $a_\varepsilon = a = 0$, hence the interval $(0, b_\varepsilon)$ is a spectral gap (0 is always an eigenvalue). Considering a single graph-like manifold with constant volume, we are able to recover a result of Gentile and Pagliara [GP95] on the divergence of the first non-zero p-form eigenvalue (see Proposition 8.2.1, Section 8.2). Moreover, we consider families of graph-like manifolds arising from families of either Ramanujan graphs or general graphs. In this setting, we manage to find spectral gap properties in relation to volume properties of the manifolds (Section 8.3).

5.2 Overview of the text

The material is organised as follows. In Chapter 6 we present all the necessary preliminaries to our analysis. We describe metric graphs and graph-like manifolds together with their associate Laplacians. We also review some basic facts about the Hodge-Laplacian acting on differential forms on a manifold and we state the McGowan Lemma (Proposition 6.6.4), the key ingredient in our proofs. In Chapter 7 we state the main results, namely we prove convergence for the exact 1- form eigenvalues and divergence for higher dimensional degree form eigenvalues. For completeness, we also discuss the dimension of the space of harmonic forms. Finally, in Chapter 8 we present some applications. We establish the existence of spectral gaps for graph-like manifolds with underlying metric graph having a spectral gap in the spectrum of its Laplacian. Moreover, we construct examples of (family of) manifolds with (upper or lower) bounds on the first eigenvalues of the Laplacian acting on functions or forms.

Chapter 6

Preliminaries

We here introduce all the basic notions needed in the next chapters and we set the notation.

In Section 6.1 we define discrete graphs and introduce their discrete Laplacians. These definitions are the basis to treat metric graphs and the corresponding Laplacians, defined in Section 6.2. In Section 6.3 we describe discrete and metric Ramanujan graphs, which will be used to construct families of manifolds with "special" spectral gap (see Section 8.3). Section 6.4 is dedicated to the definition of graph-like manifolds and to a brief description of their Hodge Laplacian on differential 1-forms. In Sections 6.5 and 6.6 we recall some facts about the Hodge Laplacian acting on forms and its eigenvalues. In particular, we describe how eigenvalues behave under scaling and we present the McGowan Lemma, an eigenvalue estimate from below. This lemma will be crucial in the proof of the divergence behaviour of the 1-form eigenvalues (see Section 7.3).

6.1 Graphs and their Laplacians

A *finite discrete graph* is a triple $G = (V, E, \partial)$ where $V = V(G)$ and $E = E(G)$ are finite sets, called vertices and edges sets respectively, and $\partial\colon E \to V \times V$ is such that $e \mapsto (\partial_- e, \partial_+ e)$ associates to an edge its initial and terminal vertex. This map fixes an orientation for the graph, crucial when working with 1-forms. In what follows, we will assume, without stating it each time, that all discrete graphs are connected.

For a vertex $v \in V$ we denote by

$$E_v^{\pm} = \{e \in E \,|\, \partial_\pm e = v\}$$

the set of incoming and outgoing edges at a vertex v, and with

$$E_v = E_v^- \mathbin{\dot\cup} E_v^+$$

(disjoint union) the set of edges emanating from v. The *degree of a vertex* is the number of emanating edges, i.e.,

$$\deg v := |E_v|.$$

In our graphs, we allow *loops*, i.e., edges e such that $\partial_- e = \partial_+ e = v$, and each loop is counted twice in $\deg v$ and it appears twice in E_v, due to the disjoint union. We also allow *multiple edges*, i.e., edges $e_1 \neq e_2$ with the same starting and ending point.

To consider various types of discrete Laplacian, it is convenient to introduce *edge and vertex weights*, defined as follows.

$$\begin{aligned} \mu = \mu_E : E &\longrightarrow (0, +\infty), & e &\mapsto \mu_e > 0, \\ \mu = \mu_V : V &\longrightarrow (0, +\infty), & v &\mapsto \mu_v > 0. \end{aligned}$$

The graph (G, μ) is called *weighted discrete graph.* There are two natural choices for the weights, the combinatorial weight and the standard weight. The *combinatorial weight* is defined such that $\mu_e = 1$ and $\mu_v = 1$, while the *standard weight* is such that $\mu_e = 1$ and $\mu_v = \deg v$.

Throughout this dissertation, we will consider the following weights.

$$\begin{aligned} \mu_E = \ell^{-1} : E &\longrightarrow (0, +\infty), & e &\mapsto \ell_e^{-1} > 0, \\ \mu_V = \deg : V &\longrightarrow (0, +\infty), & v &\mapsto \deg v. \end{aligned}$$

In particular, we define the function $\ell : E \longrightarrow (0, +\infty)$ such that $e \mapsto \ell_e$ as one over weight and we call it the *length function associated to G.*

Given a function $F : V \longrightarrow \mathbb{C}$ on the vertex space of G , the *discrete Laplacian* on functions is defined as

$$(\Delta_G F)(v) = -\frac{1}{\deg v} \sum_{e \in E_v} \frac{1}{\ell_e} \big(F(v_e) - F(v)\big),$$

where v_e is the vertex on the opposite side of v on $e \in E_v$.

We observe that Δ_G can also be defined as $\Delta_G = d_G^* d_G$ where

$$d_G : \ell_2(V, \deg) \longrightarrow \ell_2(E, \ell^{-1}), \qquad \text{such that} \qquad (d_G F)_e = F(\partial_+ e) - F(\partial_- e),$$

and where $\ell_2(V, \deg) = \mathbb{C}^V$ and $\ell_2(E, \ell^{-1}) = \mathbb{C}^E$ carry the norms given by

$$\|F\|^2_{\ell_2(V,\deg)} = \sum_{v \in V} |F(v)|^2 \deg v \qquad \text{and} \qquad \|\eta\|^2_{\ell_2(E,\ell^{-1})} = \sum_{e \in E} |\eta|^2 \frac{1}{\ell_e},$$

respectively, and where d_G^* is its adjoint operator with respect to the corresponding inner products.

We can equally define a Laplacian on 1-forms by $\Delta_G^1 := d_G d_G^*$, acting on $\ell_2(E, \ell^{-1})$.

Remark 6.1.1. For a general weighted graph (G, μ), the *weighted discrete Laplacian* is defined as

$$\Delta_{(G,\mu)} F(v) = \frac{1}{\mu_v} \sum_{e \in E_v} \mu_e (F(v_e) - F(v)),$$

acting on the space $\ell_2(V, \mu) = \{F = (F(v))_{v \in V} \mid \|F\|^2_{\ell_2(V,\mu)} := \sum_{v \in V} |F(v)|^2 \mu_v < \infty\}$.

For further readings on discrete graphs and discrete Laplacians we refer the reader to [Pos12, Section 2.1]

6.2 Metric Graphs and their Laplacians

Let $G = (V, E, \partial)$ be a discrete graph and let $\ell : E \to (0, \infty)$ be the associated length function introduced in the previous section.

A *metric graph* X_0 associated to the discrete graph G is the quotient

$$X_0 := \dot{\bigcup_{e \in E}} I_e / \sim,$$

where $I_e := [0, \ell_e]$ and $\sim$ is the relation identifying the end points of the intervals I_e according to the graph, i.e., $x \sim y$ if and only if $\psi(x) = \psi(y)$ where

$$\psi : \dot{\bigcup_{e \in E}} I_e \to V, \qquad \begin{cases} 0 \in I_e \mapsto \partial_- e, \\ \ell_e \in I_e \mapsto \partial_+ e, \\ x \in \dot{\bigcup}_{e \in E} (0, \ell_e) \mapsto x. \end{cases}$$

An alternative way to describe a metric graph is to think of G as a topological graph where each edge $e \in E$ (topologically an interval) is associated to a length $\ell_e > 0$.

Metric graphs carry a natural measure on each interval, the Lebesgue measure $\mathrm{d}s_e$. This allows us to define a natural Hilbert space of functions and 1-forms by

$$L^2(X_0) = \bigoplus_{e \in E} L^2(I_e) \qquad \text{and} \qquad L^2(\Lambda^1(X_0)) = \bigoplus_{e \in E} L^2(\Lambda^1(I_e))$$

with norms given by

$$\|f\|^2_{L^2(X_0)} := \sum_{e \in E} \|f_e\|^2_{L^2(I_e)} \qquad \text{and} \qquad \|\alpha\|^2_{L^2(\Lambda^1(X_0))} := \sum_{e \in E} \|\alpha_e\|^2_{L^2(\Lambda^1(I_e))}$$

for functions $f \colon X_0 \longrightarrow \mathbb{C}$, $f = (f_e)_{e \in E}$ and 1-forms $\alpha = (\alpha_e)_{e \in E} = (g_e \, \mathrm{d}s_e)_{e \in E}$ on X_0, respectively. We remark that functions on I_e can be identified with 1-forms via $f_e \mapsto f_e \, \mathrm{d}s_e$, the difference of forms and functions appears only in the domain of the corresponding differential operators below.

We define the *exterior derivative* $d = d_{X_0}$ on X_0 as the operator

$$d \colon \operatorname{dom} d \longrightarrow L^2(\Lambda^1(X_0)) \qquad \text{such that} \qquad d(f_e)_{e \in E} = (f'_e \, \mathrm{d}s_e)_{e \in E}$$

whose domain is

$$\operatorname{dom} d = H^1(X_0) \cap C(X_0),$$

where $H^1(X_0) = \{f \in L^2(X_0) \,|\, f' = (f'_e)_{e \in E} \in L^2(X_0)\} = \bigoplus_{e \in E} H^1(I_e)$ and $C(X_0)$ denotes the space of continuous functions on X_0.

It is not difficult to see that $d = d_{X_0}$ is a closed operator with adjoint given by

$$d^*(\alpha_e)_{e \in E} = -(\alpha'_e)_{e \in E},$$

with domain

$$\operatorname{dom} d^* = \Big\{ \alpha \in H^1(\Lambda^1(X_0)) \,\Big|\, \sum_{e \in E} \vec{\alpha}_e(v) = 0 \Big\},$$

where $H^1(\Lambda^1(X_0)) = \{\alpha = (g_e \, \mathrm{d}s_e)_{e \in E} \in L^2(\Lambda^1(X_0)) \,|\, g'_e \in L^2(I_e)\}$ and $\vec{\alpha}$ is the *oriented evaluation* of α, i.e.,

$$\vec{\alpha}_e(v) = \begin{cases} -g_e(0), & v = \partial_- e \\ g_e(\ell_e), & v = \partial_+ e. \end{cases}$$

The *(metric) Laplacians acting on functions and* 1*-forms* defined on X_0 are the operators

$$\Delta^0_{X_0} = d^* d \qquad \text{and} \qquad \Delta^1_{X_0} = dd^*,$$

respectively. The domains of $\Delta^0_{X_0}$ and $\Delta^1_{X_0}$ are

$$\operatorname{dom} \Delta^0_{X_0} = \{f \in \operatorname{dom} d \,|\, df \in \operatorname{dom} d^*\},$$
$$\operatorname{dom} \Delta^1_{X_0} = \{\alpha \in \operatorname{dom} d^* \,|\, d^*\alpha \in \operatorname{dom} d\},$$

respectively.

Writing the vertex conditions for the Laplacian on functions explicitly, we obtain the conditions

$$f_e(v) := \begin{cases} f_e(0), & v = \partial_- e \\ f_e(\ell_e), & v = \partial_+ e \end{cases} \quad \text{is independent of } e \in E_v \text{ and} \quad \sum_{e \in E_v} \overrightarrow{f'_e}(v) = 0, \quad (6.2.1)$$

called *standard* or *Kirchhoff vertex conditions*. The first condition can be rephrased as continuity of f on the metric graph, while the second is interpreted as a flux conservation where the 1-form $df = (f'_e)_e$ is considered as a vector field (we remind the reader that there is one-to-one correspondence between vector fields and 1-forms through the musical isomorphism [GHL90, p.75]).

As for the discrete Laplacian, it is possible to define a weighted metric Laplacian. We do not give details of this operator and of the various types of metric Laplacians here as we do not treat them. For further details we refer the reader to [Ku04, Ku05] and references therein.

The Laplacians $\Delta^0_{X_0}$ and $\Delta^1_{X_0}$ are both self-adjoint and non-negative operators and, since X_0 is compact, they have purely discrete spectrum [Pos12, Proposition 2.2.10, and 2.2.14]. Moreover, they fulfil a supersymmetry condition in the sense of [Pos09, Sec. 1.2], as explained below.

Set $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, a Hilbert space, then d has *supersymmetry* if $d : \operatorname{dom} d \longrightarrow \mathcal{H}_1$ and $\operatorname{dom} d \subset \mathcal{H}_0$. In our case, $\mathcal{H}_0 = L^2(X_0)$ and $\mathcal{H}_1 = L^2(\Lambda^1(X_0))$, and so $d = d_{X_0}$ has supersymmetry.

As a consequence, the spectra of $\Delta^0_{X_0}$ and $\Delta^1_{X_0}$ away from zero coincide including multiplicity, i.e., let $\lambda^0_j(X_0)$ and $\lambda^1_j(X_0)$ denote the eigenvalues of $\Delta^0_{X_0}$ and $\Delta^1_{X_0}$ in increasing order and repeated according to their multiplicity, then

$$\lambda_j^1(X_0) = \lambda_j^0(X_0) \qquad \forall\, j \geqslant 1. \tag{6.2.2}$$

A general proof of this fact can be found in [Pos09, Proposition 1.2].

We conclude this section presenting a relation between discrete and metric Laplacian for equilateral graphs [Nic85, Cat97].

A graph is said to be *equilateral* if $\ell_e = \ell_0$ for all $e \in E$ and the spectra of its discrete and metric Laplacian on functions (or 0-forms) satisfy the following.

Let $\Sigma := \{(j\pi/\ell_0)^2 \mid j = 1, 2, \dots\}$ be the Dirichlet spectrum of the interval $[0, \ell_0]$, then

$$\lambda \in \sigma(\Delta^0_{X_0}) \qquad \text{if and only if} \qquad \phi(\lambda) := 1 - \cos(\ell_0\sqrt{\lambda}) \in \sigma(\Delta_G) \tag{6.2.3}$$

for all $\lambda \notin \Sigma$.

As a consequence of the supersymmetry condition mentioned above, the same relation holds for the spectra of discrete and metric Laplacian on 1-forms.

There is also a relation at the bottom of the spectrum of Δ_G and Δ_{X_0} for general (not necessarily equilateral) metric graphs for which we refer to [Pos09, Sec. 6.1] or [Pos12, Sec. 2.4.2] for more details.

6.3 Ramanujan Graphs

A discrete graph G is *k-regular*, if all its vertices have degree k. For ease of notation we assume here that the graph $G = (V, E, \partial)$ is simple, and we write $v \sim w$ for adjacent vertices.

Definition 6.3.1. Let G be a k-regular discrete graph with n vertices and let Δ_G be its (normalised) discrete Laplacian. The graph G is said to be *Ramanujan* if

$$\max\big\{|1-\mu| \,\big|\, \mu \in \sigma(\Delta_G)\big\} \leqslant \frac{2\sqrt{k-1}}{k}.$$

We remark that many authors use the eigenvalues of the adjacency matrix A_G as the spectrum of a graph. The *adjacency matrix* is given by $(A_G)_{v,w} = 1$ if $v \sim w$ and $(A_G)_{v,w} = 0$ otherwise. As $v \sim w$ is equivalent with $w \sim v$, the adjacency matrix is symmetric. For more details about the adjacency matrix and its spectra we refer the reader to [BH12]. For a k-regular graph, we have the relation

$$A_G = k(\mathrm{id} - \Delta_G), \qquad \text{or}, \qquad \Delta_G = \mathrm{id} - \frac{1}{k} A_G \tag{6.3.4}$$

with the discrete graph Laplacian (with length function $\ell_e = 1$). We observe that $\sigma(A_G) \subset [-k, k]$ and $\sigma(\Delta_G) \subset [0, 2]$. Moreover, k and 0 are always eigenvalues of A_G and Δ_G, respectively, as well as $-k$ and 2 are always eigenvalues of A_G and Δ_G, respectively, if and only if the graph is bipartite (recall that we assume that G is a finite graph).

We define the (maximal) *spectral gap length* of a discrete graph by

$$\gamma(G) := \min\bigl\{\mu, 2 - \mu \,\big|\, \mu \in \sigma(\Delta_G) \backslash \{0, 2\}\bigr\} = 1 - \frac{1}{k} \max\bigl\{|\alpha| \,\big|\, \alpha \in \sigma(A_G), |\alpha| < k\bigr\}, \tag{6.3.5}$$

i.e., $\gamma(G)$ is the distance of the non-trivial spectrum of the Laplacian Δ_G from the extremal points 0 and 2. Hence, a graph is Ramanujan if its spectral gap length has size at least

$$\gamma(G) \geqslant 1 - \frac{2\sqrt{k-1}}{k}.$$

It has been shown that the lower bound is optimal, i.e., for any k-regular graph (or even for any graph with maximal degree k) with diameter large enough, the spectral gap length is smaller than $1 - 2\sqrt{k-1}/k + \eta$, where $1/\eta$ is of the same order as the diameter (see [Nil91, Thm. 1] and references therein). In this sense, Ramanujan graphs are optimal expanders, i.e., optimal highly connected sparse graphs. Expander graphs have been characterised in several ways in a number of different contexts and are used in a number of applications in pure mathematics as well as in computer science. For a survey on expander graphs we refer the reader to [HLW06, Lub10, Lub12] and references therein.

The existence of infinite families $\{G^i\}_{i \in \mathbb{N}}$ of k-regular Ramanujan graphs has been shown whenever k is a prime or a power of a prime (see e.g. [LPS88, Mar88, Mor94]). The existence of infinite families of bipartite k-regular Ramanujan graphs for every $k > 2$ has been proved in [MSS15a] by showing that any bipartite Ramanujan graph has a 2-lift which is again Ramanujan, bipartite and has twice as many vertices. Recently, the same authors proved the existence of bipartite Ramanujan graphs of every degree and every number of vertices [MSS15b] by showing that a random m-regular bipartite graph, obtained as a union of m random perfect matchings across a bipartition of an even number of vertices, is Ramanujan with nonzero probability.

Let $\{G^i\}_{i\in\mathbb{N}}$ be a family of Ramanujan graphs such that

$$\nu_i := |V(G^i)| \to \infty \tag{6.3.6}$$

and consider the associated family of equilateral metric graphs $\{X_0^i\}_{i\in\mathbb{N}}$ of length ℓ_0. By (6.2.3), the metric graph Laplacians $\Delta_{X_0^i}$ all have a spectral gap

$$(a_0, b_0) = \Big(0, \frac{h}{\ell_0^2}\Big) \qquad \text{with} \qquad h = h_k := \arccos^2\Big(1 - \frac{2\sqrt{k-1}}{k}\Big) > 0 \tag{6.3.7}$$

at the bottom of the spectrum.

6.4 Graph-like Manifolds and their Hodge Laplacian

A *graph-like manifold* associated with a metric graph X_0 is a family of oriented and connected n-dimensional Riemannian manifolds $(X_\varepsilon)_{0<\varepsilon\leqslant\varepsilon_0}$ (ε_0 small enough) shrinking to X_0 as $\varepsilon \to 0$ in the following sense. We assume that X_ε decomposes as

$$X_\varepsilon = \bigcup_{e\in E} X_{\varepsilon,e} \cup \bigcup_{v\in V} X_{\varepsilon,v}, \tag{6.4.8}$$

where $X_{\varepsilon,v}$ and $X_{\varepsilon,e}$ are called *edge* and *vertex neighbourhood*, respectively. More precisely, we assume that $X_{\varepsilon,v}$ and $X_{\varepsilon,e}$ are closed subsets of X_ε such that

$$X_{\varepsilon,v} \cap X_{\varepsilon,e} = \begin{cases} Y_{\varepsilon,e} & e \in E_v \\ \varnothing & e \notin E_v, \end{cases}$$

with $Y_{\varepsilon,e}$ a boundaryless smooth connected Riemannian manifold of dimension $n-1$ (see Figure 6.1). We will often refer to $Y_{\varepsilon,e}$ as the *transversal manifold*.

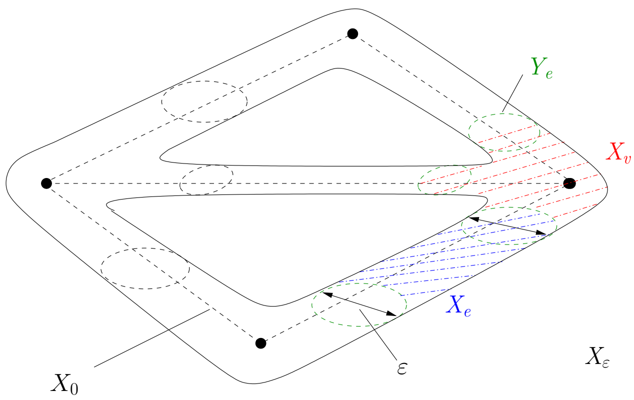

Figure 6.1: An example of a two dimensional graph-like manifold X_ε with associated graph X_0.

Furthermore, we assume that the manifolds $(X_{\varepsilon,v}, g_{e,v})$ and $(Y_{\varepsilon,e}, h_{\varepsilon,e})$ are conformally equivalent to the Riemannian manifolds (X_v, g_v) and (Y_e, h_e), respectively, with (constant) conformal factor ε^2, i.e.,

$$g_{\varepsilon,v} = \varepsilon^2 g_v \qquad \text{and} \qquad h_{\varepsilon,e} = \varepsilon^2 h_e. \tag{6.4.9}$$

For short, we will write $X_{\varepsilon,v} = \varepsilon\, X_v$ and $Y_{\varepsilon,e} = \varepsilon\, Y_e$.

Moreover, we assume that $X_{\varepsilon,e}$ is isometric to the product $I_e \times \varepsilon\, Y_e$. Let $g_{\varepsilon,e}$ denotes the metric on $X_{\varepsilon,e}$, it satisfies

$$g_{\varepsilon,e} = \mathrm{d}s^2 + \varepsilon^2 h_e. \tag{6.4.10}$$

We often refer to a single manifold X_ε as graph-like manifold instead of the family $(X_\varepsilon)_\varepsilon$ as in the definition above.

Throughout this dissertation, we will assume that $\mathrm{vol}_{n-1} Y_e = 1$ for all $e \in E$, for simplicity. The general case would lead to the weighted vertex condition $\sum_{e \in E_v} (\mathrm{vol}_{n-1} Y_e) \vec{f}_e'(v) = 0$ instead of (6.2.1) for the metric graph Laplacian (see [Pos12, EP09] for details).

We call a graph-like manifold X_ε *transversally trivial* if all transversal manifolds Y_e are Moore spaces, i.e., if $H^p(Y_e) = 0$ for all $1 \leqslant p \leqslant n-2$ and all $e \in E$, where $H^p(\cdot)$ denotes the p-th cohomology group. We observe that a member of a transversally trivial graph-like manifold X_ε is not necessarily homotopy equivalent to the metric graph X_0, as the vertex neighbourhoods do not need to be contractible.

Below, we give an example of how to construct a transversally trivial graph-like manifold.

Example 6.4.1. Let $n \geqslant 2$. For each vertex v fix a manifold $\hat{X}_v$. Remove $\deg v$ open balls from $\hat{X}_v$ hence the resulting manifold X_v has a boundary consisting of $\deg v$ many components each diffeomorphic to a $(n-1)$-sphere $\mathbb{S}^{n-1}$. For $e \in E_v$ let $Y_e = \mathbb{S}^{n-1}$ with a metric such that its volume is 1. As (unscaled) edge neighbourhood, we choose $X_{1,e} := [0, \ell_e] \times Y_e$ with the product metric. Then we can construct a graph-like (topological) manifold X_1 with a canonical decomposition as in (6.4.8) (for $\varepsilon = 1$) by identifying the e-th boundary component of X_v with the corresponding end of the edge neighbourhood $X_{1,e}$. By a small local change we can assume that the resulting manifold X_1 is smooth. The corresponding family of graph-like manifolds $(X_\varepsilon)_{\varepsilon>0}$ is now given as above by choosing the metric accordingly.

Remark 6.4.2. Given a closed (i.e., compact and boundaryless) manifold X and a metric graph X_0, it is possible to turn X into a graph-like manifold with underlying metric graph being X_0. As an example, we assume X_0 to be a metric finite tree graph, then X turns into a graph-like manifold letting the tree "grow" out of the original manifold. More formally, we construct a graph-like manifold according to a tree graph and leave one cylinder of a leaf (a vertex of degree 1) "uncapped". Then, we glue the original manifold X with one disc removed together with the free cylinder. Obviously, the resulting manifold is homeomorphic to the original manifold X and can be turned into the a graph-like manifold by the above choice of metric.

We can now define on X_ε the corresponding Hodge Laplacian $\Delta^p_{X_\varepsilon} = d\delta + \delta d$ acting on differential p-forms. The operators d and δ are the classical exterior derivative and its formal adjoint on manifolds, as unbounded operators in the corresponding L^2 spaces.

We give further details on the Hodge Laplacian on manifolds in the next section.

6.5 Hodge Theory

Let (M, g) be a compact, oriented and connected n-dimensional Riemannian manifold. The Riemannian metric g induces the L^2-space of p-forms

$$L^2(\Lambda^p(M,g)) = \left\{\omega\colon M \longrightarrow \mathbb{C} \,\middle|\, \|\omega\|^2_{L^2(\Lambda^p(M,g)} = \int_M |\omega|_g^2 \,\mathrm{dvol}_g\, M < \infty\right\}$$

where

$$\|\omega\|^2_{L^2(\Lambda^p(M,g))} = \langle\omega,\omega\rangle_{L^2(\Lambda^p(M,g))} := \int_M |\omega|_g^2 \,\mathrm{dvol}_g\, M = \int_M \omega \wedge *\omega,$$

and $*$ denotes the Hodge star operator (depending on the metric g).

The *Laplacian on p-forms* on M is formally defined as

$$\Delta^p_{(M,g)} = \Delta^p = d\delta + \delta d,$$

where d is the classical exterior derivative and $\delta = (-1)^{np+n+1} {*d*}$ is its formal adjoint with respect to the inner product induced by g.

If M has no boundary, then δ is the L^2-adjoint of d and Δ^p is a non-negative self-adjoint operator with discrete spectrum denoted by $\lambda_j^p(M, g)$ (repeated according to multiplicity).

We allow the manifold M to have a boundary ∂M, itself a smooth manifold of dimension $n-1$. As in the function case, it is possible to impose boundary conditions for functions in the domain of the Hodge Laplacian, called absolute and relative boundary conditions. To do so, we first decompose a p-form ω in its tangential and normal components on ∂M, i.e., $\omega = \omega_{\mathrm{tan}} + \omega_{\mathrm{norm}}$ where ω_{tan} can be considered as a form on ∂M while $\omega_{\mathrm{norm}} = \mathrm{d}r \wedge \omega^\perp$ with $\omega^\perp$ being a form on ∂M and r being the distance from ∂M.

Absolute boundary conditions require that ω satisfies

$$\omega_{\mathrm{norm}} = 0 \quad \text{and} \quad (d\omega)_{\mathrm{norm}} = 0$$

while *relative boundary conditions* require

$$\omega_{\mathrm{tan}} = 0 \quad \text{and} \quad (\delta\omega)_{\mathrm{tan}} = 0.$$

These boundary conditions give rise to two unbounded and self-adjoint operators Δ^{abs} and Δ^{rel} with discrete spectrum, the Hodge Laplacians with absolute and relative boundary conditions, respectively (see e.g. [Cha84] or [McG93]). We remark that for functions, the absolute correspond to Neumann while the relative correspond to Dirichlet boundary conditions.

Furthermore, since the Hodge star operator exchanges absolute and relative boundary conditions, there is a correspondence between the spectrum of Δ^{abs} and the spectrum of Δ^{rel}, which allows us to study just one of them to cover both cases. In the sequel, we will only consider absolute boundary conditions if the manifold has a boundary, and hence we will mostly suppress the label $(\cdot)^{\mathrm{abs}}$ for ease of notation.

In an L^2-framework, we consider d and δ_0 as unbounded operators, defined as the closures $\overline{d}$ and $\overline{\delta}_0$ of d and δ_0 on

$$\begin{aligned} \operatorname{dom} d &= \bigl\{\omega \in C^\infty(\Lambda^p(M,g)) \bigm| d\omega \in L^2(\Lambda^{p+1}(M,g))\bigr\}, \\ \operatorname{dom} \delta_0 &= \bigl\{\omega \in C^\infty(\Lambda^p(M,g)) \bigm| \delta\omega \in L^2(\Lambda^{p-1}(M,g)),\ \omega_{\mathrm{norm}} = 0\bigr\}, \end{aligned}$$

respectively. The Hodge Laplacian with absolute boundary condition is then given by

$$\Delta = \Delta^{\mathrm{abs}} = \overline{d}\,\overline{\delta}_0 + \overline{\delta}_0\overline{d}.$$

For this operator, Hodge Theory is still valid. In particular, the de Rham theorem holds (see [deR55] or [McG93, Sec. 2.1] and references therein), i.e.,

$$\mathcal{H}^p(M,g) \cong H^p(M),$$

where $\mathcal{H}^p(M,g)$ is the space of harmonic p-forms (with absolute boundary conditions if the boundary is non-empty) and $H^p(M)$ is the p-th de Rham cohomology, and any p-form $\omega \in L^2(\Lambda^p(M,g))$ can be orthogonally decomposed into an exact ($d\bar{\omega}$), co-exact ($\delta\bar{\bar{\omega}}$) and harmonic (ω_0) component, i.e.,

$$\omega = d\bar{\omega} + \delta\bar{\bar{\omega}} + \omega_0, \tag{6.5.11}$$

where $\bar{\omega} \in \operatorname{dom}\overline{d}$ is a $(p-1)$-form, $\bar{\bar{\omega}} \in \operatorname{dom}\overline{\delta}_0$ is a $(p+1)$-form and ω_0 is a harmonic p-form. Moreover, the Hodge Laplacian leaves these spaces invariant and maps p-forms into p-forms. In particular, we can consider the Hodge Laplacian acting on exact and co-exact p-forms as the operators $\overline{d}\,\overline{\delta}_0$ and $\overline{\delta}_0\overline{d}$, respectively. We call their respective eigenvalues exact and co-exact (absolute) p-form eigenvalues, and we denote them by $\bar{\lambda}_j^p(M,g)$ and $\bar{\bar{\lambda}}_j^p(M,g)$, respectively.

Let $\bar{E}^p(\lambda) = \ker(d\,\delta_0 - \lambda)$ and $\bar{\bar{E}}^p(\lambda) = \ker(\delta_0 d - \lambda)$ denote the eigenspaces of exact and co-exact p-forms with eigenvalue λ (as the eigenforms are smooth by elliptic regularity, we can omit the closures). Since d is an isomorphism between $\bar{\bar{E}}^{p-1}(\lambda)$ and $\bar{E}^p(\lambda)$, we have

$$\bar{\lambda}_j^p(X_\varepsilon) = \bar{\bar{\lambda}}_j^{p-1}(X_\varepsilon), \qquad \forall\, j \geqslant 1. \tag{6.5.12}$$

In addition, due to Hodge duality, i.e., the star operator interchanges absolute and relative boundary conditions, the following relation holds

$$\bar{\lambda}_j^p(X_\varepsilon) = \bar{\bar{\lambda}}_j^{n-p}(X_\varepsilon) \qquad \forall\, j \geqslant 1. \tag{6.5.13}$$

6.6 Some useful facts about eigenvalues

We finally collect some useful facts about the eigenvalues of the Hodge Laplacian on a Riemannian manifold. In particular, we describe their behaviour under scaling of the metric, their characterization, and we present a crucial estimate from below.

Scaling behaviour

We consider a manifold $(M_\varepsilon, g_\varepsilon)$ conformally equivalent to (M, g) with conformal factor ε^2 (meaning that $g_\varepsilon = \varepsilon^2 g$). Again for short, we write $M_\varepsilon = \varepsilon M$ (see also Section 6.4).

We have the following result for L^2 norms of p-forms on M and εM and for the eigenvalues of the Hodge Laplacian on M and εM.

Lemma 6.6.1. *Let ω be a p-form on a n-dimensional Riemannian manifold M with metric g, and let εM be the Riemannian manifold $(M, \varepsilon^2 g)$, then we have*

$$\|\omega\|^2_{L^2(\Lambda^p(\varepsilon M))} = \varepsilon^{n-2p} \|\omega\|^2_{L^2(\Lambda^p(M))} \qquad \text{and} \tag{6.6.14a}$$

$$\bar{\lambda}_j^p(\varepsilon M) = \varepsilon^{-2} \bar{\lambda}_1^p(M). \tag{6.6.14b}$$

Proof. The first assertion follows from the fact that we have $|w|^2_{\varepsilon^2 g} = \varepsilon^{-2p}|w|^2_g$ and $\mathrm{dvol}_{\varepsilon^2 g} M = \varepsilon^n \,\mathrm{dvol}_g M$ pointwise. The second follows from the variational characterisation of the j-th eigenvalue of Proposition 6.6.2, as we have the scaling behaviour

$$\frac{\|\eta\|^2_{L^2(\Lambda^p(\varepsilon M))}}{\|\theta\|^2_{L^2(\Lambda^{p-1}(\varepsilon M))}} = \frac{\varepsilon^{n-2p}\|\eta\|^2_{L^2(\Lambda^p(M))}}{\varepsilon^{n-2(p-1)}\|\theta\|^2_{L^2(\Lambda^{p-1}(M))}} = \varepsilon^{-2}\frac{\|\eta\|^2_{L^2(\Lambda^p(M))}}{\|\theta\|^2_{L^2(\Lambda^{p-1}(M))}}.$$

Note that the condition $\eta = d\theta$ is independent of the metric, see Proposition 6.6.2.

□

Eigenvalue characterisation

Here we present a useful characterisation of eigenvalues of the Hodge Laplacian acting on p-forms due to Dodziuk [Dod82, Prop. 3.1], whose proof can be found in [McG93, Prop. 2.1]. Its advantage is that it does not make use of the adjoint δ of the exterior derivative, and hence no derivation of the metric g or of its coefficients are needed. The metric g enters only via the L^2 norms.

We remind the reader that a form ω satisfies absolute boundary conditions when $w_{norm} = 0$ and $(d\omega)_{norm} = 0$ on the boundary (see also p.57, Section 6.5).

Proposition 6.6.2. *Let M be a compact Riemannian manifold, then the spectrum of the Laplacian $0 < \bar{\lambda}_1^p \leqslant \bar{\lambda}_2^p \leqslant \dots$ on exact p-forms on M satisfying absolute boundary conditions can be computed by*

$$\bar{\lambda}_j^p(M) = \inf_{V_j} \sup \Big\{ \frac{\langle \eta, \eta \rangle_{L^2(\Lambda^p(M))}}{\langle \theta, \theta \rangle_{L^2(\Lambda^{p-1}(M))}} \,\Big|\, \eta \in V_j \backslash \{0\} \text{ such that } \eta = d\theta \Big\},$$

where V_j ranges over all j-dimensional subspaces of smooth exact p-forms and θ is a smooth $(p-1)$-form.

As a consequence we have (see [Dod82, Prop. 3.3] or [McG93, Lem. 2.2]),

Proposition 6.6.3. *Assume that g and $\widetilde{g}$ are two Riemannian metrics on M such that $c_-^2 g \leqslant \widetilde{g} \leqslant c_+^2 g$ for some constants $0 < c_- \leqslant c_+ < \infty$, i.e.,*

$$c_-^2 g_x(\xi, \xi) \leqslant \widetilde{g}_x(\xi, \xi) \leqslant c_+^2 g_x(\xi, \xi) \quad \text{for all } \xi \in T_x^* M \text{ and } x \in M,$$

then the eigenvalues of exact p-forms ω with absolute boundary conditions fulfil

$$\frac{1}{c_-^2} \Big(\frac{c_-}{c_+}\Big)^{n+2p} \bar{\lambda}_j^p(M, g) \leqslant \bar{\lambda}_j^p(M, \widetilde{g}) \leqslant \frac{1}{c_+^2} \Big(\frac{c_+}{c_-}\Big)^{n+2p} \bar{\lambda}_j^p(M, g)$$

for all $j \geqslant 1$.

As a result, the eigenvalues $\bar{\lambda}_j^p(M, g)$ depend continuously on g in the sup-norm defined, e.g., in [Pos12, Sec. 5.2]. In particular, this proposition allows us to consider also perturbation of graph-like manifolds. For a discussion of possible cases we refer to [Pos12, Sec. 5.2–5.6]).

An estimate from below for exact eigenvalues

Finally, we introduce a simplified but useful version of an estimate from below on the first eigenvalue of the exact p-form Laplacian on a manifold by McGowan ([McG93, Lemma 2.3]) also used by Gentile and Pagliara in [GP95, Lemma 1].

Let (M, g) be a n-dimensional compact Riemannian manifold without boundary and let $\{U_i\}_{i=1}^m$ be an open cover of M such that $U_{ij} = U_i \cap U_j$ have a smooth boundary. Moreover, we denote by

$$I_i := \{j \in \{1, \dots, i-1, i+1, \dots, m\} \,|\, U_i \cap U_j \neq \varnothing\}$$

the index set of neighbours of U_i. We say that the cover $\{U_i\}_i$ has *no intersection of degree* r if and only if $U_{i_1} \cap \dots \cap U_{i_r} = \varnothing$ for any r-tuple $(i_1, \dots, i_r)$ with $1 \leqslant i_1 < i_2 < \dots < i_r \leqslant m$. We choose a fixed partition of unity $\{\rho_j\}_{j=1}^m$ subordinate to the open cover and we set $\|d\rho\|_\infty := \max_i \sup_{x \in U_i} |d\rho_i(x)|_g$.

Furthermore, we denote by $\bar{\lambda}_1^{p,\mathrm{abs}}(U)$ the first positive eigenvalue on exact p-forms on U satisfying absolute boundary conditions on ∂U. Finally, denote by $H^p(U_{ij})$ the p-th cohomology group of U_{ij}.

Proposition 6.6.4. *Let M and $\{U_i\}_{i=1}^m$ be as above and let $p \geqslant 2$. Assume that the open cover has no intersection of degree higher than 2 and $H^{p-1}(U_{ij}) = 0$ for all i, j. Then, the first positive eigenvalue of the Laplacian acting on exact p-forms (without boundary conditions) on M satisfies*

$$\bar{\lambda}_1^p(M) \geqslant \frac{2^{-3}}{\sum_{i=1}^m \left(\frac{1}{\bar{\lambda}_1^{p,\mathrm{abs}}(U_i)} + \sum_{j \in I_i} \left(\frac{c_{n,p}\|d\rho\|_\infty^2}{\bar{\lambda}_1^{p-1,\mathrm{abs}}(U_{ij})} + 1 \right) \left(\frac{1}{\bar{\lambda}_1^{p,\mathrm{abs}}(U_i)} + \frac{1}{\bar{\lambda}_1^{p,\mathrm{abs}}(U_j)} \right) \right)} \tag{6.6.15}$$

where $c_{n,p}$ is a combinatorial constant depending only on p and n.

We remark that these assumptions impose a topological restriction on the manifold as such an open cover does not necessarily exist. Actually, the following general version holds for higher exact eigenvalues.

Proposition 6.6.5. *Let M and $\{U_i\}_i$ be as above and let $p \geqslant 2$. Assume that the open cover has no intersection of degree higher than 2. We set $N_1 = \sum_{i,j} \dim H^{p-1}(U_{ij})$*

and $N = N_1 + 1$. Then, the N-th eigenvalue of the Laplacian on exact p-forms on M satisfies

$$\bar{\lambda}^p_N(M) \geqslant \frac{1}{\sum_{i=1}^{m}\left(\frac{1}{\bar{\lambda}_1^{p,\mathrm{abs}}(U_i)} + \sum_{j\in I_i}\left(\frac{\|d\rho\|_\infty^p}{\bar{\lambda}_1^{p-1,\mathrm{abs}}(U_{ij})}+1\right)\left(\frac{1}{\bar{\lambda}_1^{p,\mathrm{abs}}(U_i)}+\frac{1}{\bar{\lambda}_1^{p,\mathrm{abs}}(U_j)}\right)\right)}$$

The proof of this proposition uses the same argument of the proof of McGowan's lemma (Lemma 2.3 in [McG93]). The first step is to consider $\bar{\lambda}^p_N(M)$ as characterised in Proposition 6.6.2 and observe that $\bar{\lambda}^p_N(M) \geqslant \frac{(\eta,\eta)}{(\theta,\theta)}$ for η one of its eigenform and θ a $(p-1)$-form such that $\theta = d\eta$. The second and main step is to construct θ such that its L^2-norm can be bounded from above. The construction is local and uses the knowledge of eigenvalues on the pieces U_i and on the double intersections U_{ij} extracted from the Cech-de Rham sequence [BT82, Chapter 2], a generalised Meyer-Vietoris sequence. The argument is then completed using a partition of unity.

The generalisation to p-forms is trivial since we have particular assumptions on the cover, i.e., no intersections of degree higher than 2 (see the remark after Lemma 2.3 in [McG93]).

Chapter 7

Asymptotic behaviour

We now present the main result of Part II, namely, the asymptotic behaviour of the (non-trivial) spectrum of the Hodge Laplacian of a graph-like manifold. To obtain a full description, it is sufficient to analyse the spectrum of the Laplacian acting on exact (resp. co-exact forms) away from zero, due to the orthogonal splitting in (6.5.11). We remark that the dimension of the class of harmonic forms depends on the topological properties of the manifold, and this is the reason why we always consider the non-trivial spectrum.

This chapter is organised as follows. In Section 7.1 we describe the space of harmonic forms on a graph-like manifold. In Section 7.2 we review the convergence result for the spectrum of the scalar Laplacian, i.e., the Laplacian acting on functions, from which we will recover a convergence result for the Hodge Laplace spectrum for exact 1-forms. We will then focus on the spectrum of the Hodge Laplacian acting on co-exact 1-forms (see Section 7.3), which is divergent in the limit. To show this behaviour, we will make use of Proposition 6.6.4, assuming the cohomology of the transversal manifolds Y_e to be non-trivial, together with asymptotic estimates on the building blocks. The same proposition allows us to study the asymptotic behaviour of the spectrum of the Hodge Laplacian on p-forms for $2 \leqslant p \leqslant n-2$ under the same assumption. Finally, we will briefly explain how the same argument works, when no assumptions on Y_e are made, using a result of McGowan (see Proposition 6.6.5).

7.1 Harmonic forms

We first analyse the dimension of the class of harmonic forms, eigenforms for the zero (trivial) eigenvalue, for graph-like manifolds. We have already explained in the introduction that this dimension depends on topological properties of the manifold. In particular, the de Rham Theorem (Section 6.5) establishes an isomorphism between the class of harmonic p-forms on X_ε and the p-th cohomology group of X_ε. Therefore, it is sufficient to calculate the cohomology groups of the graph-like manifold, to know the dimension of the class of its harmonic forms in all degrees.

Since the graph-like manifold X_ε arises from X_0, it is intuitive that X_ε will inherit some topological properties of the graph. We will see that for a general graph-like manifold, the dimension of its first cohomology group is the sum of the first Betti number of the graph (also equal to the dimension of its first cohomology gruop) and of the dimension of a subset of the first cohomology group of $\bigcup_{v\in V} X_{\varepsilon,v}$.

For transversally trivial graph-like manifolds, i.e., $H^p(Y_e) = 0$ for $1 \leqslant p \leqslant n-2$ and for all $e \in E$, the following lemma holds.

Lemma 7.1.1. *Let X_ε be a transversally trivial graph-like manifold of dimension n with underlying metric graph X_0. Then, the cohomology groups of X_ε are given by*

$$H^k(X_\varepsilon) = \begin{cases} \mathbb{R} & k \in \{0, n\} \\ \bigoplus_{v\in V} H^1(X_v) \bigoplus H^1(X_0) & k \in \{1, n-1\} \\ \bigoplus_{v\in V} H^k(X_v) & k \in \{2, \dots, n-2\}. \end{cases}$$

Proof. We use the natural decomposition of X_ε in (6.4.8) and the Mayer-Vietoris sequence. Set $A = \bigcup_{e\in E} X_e \cong \bigcup_{e\in E} I_e \times Y_e$ and $B = \bigcup_{v\in V} X_v$ and fix $\varepsilon = 1$. Then, our graph-like (topological) manifold is X_1. Note that we have avoided any reference to the metric carried by each space as they do not enter into the topological argument.

The Mayer-Vietoris sequence in dimension k is

$$\dots \longrightarrow H^k(X_1) \longrightarrow H^k(A) \oplus H^k(B) \longrightarrow H^k(A \cap B) \longrightarrow \dots$$

We have $H^k(A) \cong \bigoplus_{e\in E} H^k(Y_e)$ and, by the assumption on the transversal manifolds, $H^k(A) = 0$ for $k = 1, \dots, n-2$, and $H^0(A) = H^{n-1}(A) \cong \mathbb{R}^{|E|}$.

We also have $H^k(B) \cong \bigoplus_{v\in V} H^k(X_v)$. In particular, $H^0(B) \cong \mathbb{R}^{|V|}$.

Finally, $A \cap B \cong \bigcup_{v\in V} \bigcup_{e\in E_v} Y_e$. Hence, $H^k(A \cap B) \cong \mathbb{R}^{2|E|}$ for $k \in \{0, n-1\}$, and $H^k(A \cap B) = 0$ otherwise.

By compactness, we derive $H^0(X_1) \cong \mathbb{R}$. Then, since the long exact sequence splits into short exact sequences, we obtain $H^k(X_1) \cong \bigoplus_{v\in V} H^k(X_v)$ for $2 \leqslant k \leqslant n-1$. A dimensional argument yields $H^1(X_1) \cong \bigoplus_{v\in V} H^1(X_v) \bigoplus H^1(X_0)$.

The use of Poincaré duality concludes the claim. □

We observe that this computation agrees with the results in [AC95] where the authors considered a manifold with shrinking handles, i.e., a graph-like manifold where the shrinking parameter involves only the edge neighbourhood.

We also remark that, although the dimension of the class of harmonic 0-forms on X_ε coincides with the dimension of the ones on the graphs ($H^0(X_0) = \mathbb{Z}$ since the graph is connected), for harmonic forms of higher degree this is not valid. The class of the harmonic p-forms on X_ε is larger than the graph's one, both in the transversal trivially case and in the general case.

When some or all of the Y_e have non-trivial p-th cohomology groups for $1 \leqslant p \leqslant n-2$, we do not have a general formula. In this case, the Mayer-Vietoris sequence and a dimensional argument only allows us to deduce the dimension of the first cohomology group of X_ε, equal to $b_1(X_0) + q$ where $b_1(X_0) = |E| - |V| + 1$ is the first Betti number of X_0 and q is the dimension of a subset of $\bigoplus_{v\in V} H^1(X_v)$. However, it is possible to compute the cohomology groups explicitly for concrete examples of edge and vertex neighbourhoods.

7.2 Convergence for functions and exact 1-forms

The Laplacian on functions on graph-like manifolds has been analysed in details in a series of papers [EP05, EP09, EP13, Pos06, Pos12] where the convergence of several objects has been established. For the proof of the eigenvalues convergence we particularly refer to [EP05] (see also [Pos12]) where the authors proved the following (based on the results [KuZ01, RS01]).

Proposition 7.2.1 ([EP05, Pos12]). *Let X_ε be a compact graph-like manifold associated with a metric graph X_0 and let $\lambda_j(X_\varepsilon)$ and $\lambda_j(X_0)$ denote the eigenvalues (in increasing order, repeated according to their multiplicity) of the Laplacian acting on functions on X_ε and on X_0. Then we have*

$$|\lambda_j(X_\varepsilon) - \lambda_j(X_0)| = \mathcal{O}(\varepsilon^{1/2}/\ell_0) \qquad \textit{for all } j \geqslant 1,$$

where $\ell_0 = \min_{e \in E}\{\ell_e, 1\} > 0$ denotes the truncated minimal edge length. Moreover, the error depends only on j, and the building blocks X_v, Y_e of the graph-like manifold.

The proof in [EP05] is based on two sides eigenvalue estimates obtained by an average process on the vertex neighbourhoods $X_{\varepsilon,v}$ which corresponds to projection onto the lowest (constant) eigenvalue using the variational principle or min-max principle as in Proposition 6.6.2.

In following works, the convergence of the resolvents (in a suitable sense), spectral projections, eigenfunctions, and discrete and essential spectrum has been proved using an abstract setting that deals with operators acting in different Hilbert spaces (first used in [Pos06] without requiring compactness of the manifold). The basic idea is that we need to define a "distance" between the operators Δ_{X_ε} and Δ_{X_0} with suitable identification operators. For a detailed overview and proofs of these techniques we refer to the reader to [Pos06] and [Pos12, Ch. 4].

We have already noticed in (6.2.2) and (6.5.12), that the exact 1-form eigenvalues equal the 0-form (function) eigenvalues both on the graph X_0 and on the graph-like manifold X_ε. Therefore, the previous result immediately gives the convergence for exact 1-forms, using a simple supersymmetry argument as in [Pos09, Sec. 1.2].

Theorem 7.2.2. *Let X_ε be a graph-like manifold with underlying metric graph X_0. Denote by $\bar\lambda^1_j(X_\varepsilon)$ and $\bar\lambda^1_j(X_0)$ the j-th exact 1-form eigenvalue on X_ε and X_0, respectively. Then,*

$$\bar\lambda^1_j(X_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \bar\lambda^1_j(X_0) \qquad \textit{for all } j \geqslant 1.$$

Proof. We will just show that the eigenspaces for non-zero eigenvalues of $\Delta^1_{X_\varepsilon} = \Delta^1 = dd^*$ and $\Delta^0_{X_\varepsilon} = \Delta^0 = d^*d$ are isomorphic (the argument works for $\varepsilon > 0$ and $\varepsilon = 0$ as well). Then, the convergence result follows immediately from Proposition 7.2.1.

As isomorphism, we choose

$$d\colon \ker(\Delta^0 - \lambda) \longrightarrow \ker(\Delta^1 - \lambda)$$

for $\lambda \neq 0$. First, note that if $f \in \ker(\Delta^0 - \lambda)$, then

$$\Delta^1 df = dd^* df = d\Delta^0 f = \lambda df,$$

i.e., $df \in \ker(\Delta^1 - \lambda)$, hence the above map is properly defined. The map d as above is injective. If $df = 0$ for $f \in \ker(\Delta^0 - \lambda)$ then $\lambda f = \Delta^0 f = d^* df = 0$. As $\lambda \neq 0$ we have $f = 0$. For the surjectivity, let $\alpha \in \ker(\Delta^1 - \lambda)$. Set $f := \lambda^{-1} d^* \alpha$ (we use again that $\lambda \neq 0$). Then,

$$df = d(\lambda^{-1} d^* \alpha) = \lambda^{-1} \Delta^1 \alpha = \alpha,$$

i.e., d as above is surjective. In particular, we have shown that the spectrum of Δ^0 and Δ^1 away from 0 is the same, including multiplicity. □

We remark that if $n = \dim X_\varepsilon = 2$, the above theorem is sufficient to determine the spectra of Laplacian in all degree forms. In fact, by (6.5.12) and (6.5.13), the exact 1-form eigenvalues coincide with the 0-form eigenvalues, the co-exact 1-form eigenvalues coincide with the (exact) 2-form eigenvalues, and these eigenvalues coincide with the 0-form eigenvalues. Therefore, we can state the following.

Corollary 7.2.3. *Let X_ε be a graph-like Riemannian compact manifold of dimension 2 associated to a metric graph X_0. Then,*

$$\begin{aligned} \bar{\lambda}^1_j(X_\varepsilon) &\underset{\varepsilon\to 0}{\longrightarrow} \lambda_j(X_0), \\ \bar{\bar{\lambda}}^1_j(X_\varepsilon) = \lambda^2_j(X_\varepsilon) = \lambda_j(X_\varepsilon) &\underset{\varepsilon\to 0}{\longrightarrow} \lambda_j(X_0), \end{aligned} \tag{7.2.1}$$

for all $j \geqslant 1$.

In addition, by Hodge duality (see (6.5.13)), Theorem 7.2.2 gives convergence for n-forms on graph-like manifolds of any dimension.

7.3 Divergence for co-exact p-forms

If $n \geqslant 3$, the behaviour of the co-exact p-forms for $1 \leqslant p \leqslant n-2$ cannot be known using duality. In order to study their limit behaviour, we analyse the limit behaviour of the exact $(p+1)$-forms eigenvalues, due to (6.5.12). In particular, we first give some eigenvalue asymptotics for eigenvalues of exact p-forms with absolute boundary conditions on the building blocks of the graph-like manifold, which are needed to make use of Proposition 6.6.4 and Proposition 6.6.5.

7.3.1 Eigenvalue asymptotics on the building blocks

A vertex neighbourhood $X_{\varepsilon,v}$ is conformally equivalent to X_v by definition. As a result of Lemma 6.6.1, we have the following corollary.

Corollary 7.3.1. *Let $X_{\varepsilon,v}$ be a vertex neighbourhood of a graph-like manifold X_ε. Then, the smallest positive eigenvalue of the Laplacian acting on exact p-forms on $X_{\varepsilon,v}$ with absolute boundary conditions satisfies*

$$\bar{\lambda}_1^p(X_{\varepsilon,v}) = \varepsilon^{-2}\bar{\lambda}_1^p(X_v). \tag{7.3.2}$$

To describe the asymptotic behaviour of the edge neighbourhood, there is a bit more work to do. We recall that the edge neighbourhood $X_{\varepsilon,e}$ is isomorphic to $I_e \times Y_{\varepsilon,e}$ with the product metric. However, we cannot make use of the product structure as it does not respect exact and co-exact forms.

Proposition 7.3.2. *Let $X_{\varepsilon,e}$ be an edge neighbourhood of a n-dimensional graph-like manifold X_ε. Then, the smallest eigenvalue of the Laplacian acting on exact p-forms ($2 \leqslant p \leqslant n-1$) with absolute boundary conditions satisfies*

$$\bar{\lambda}_1^p(X_{\varepsilon,e}) = \varepsilon^{-2}c_p(\varepsilon), \tag{7.3.3}$$

where $c_p(\varepsilon) \to \bar{\lambda}_1^p(Y_e) > 0$ as $\varepsilon \to 0$, and where $\bar{\lambda}_1^p(Y_e)$ denotes the first eigenvalue of the Laplacian acting on exact p-forms on Y_e.

Proof. By Proposition 6.6.2 we have to analyse the quotient $\|\eta\|^2/\|\theta\|^2$ for an exact p-form η and a $(p-1)$-form θ such that $\eta = d\theta$. Recall that $X_{\varepsilon,e} = I_e \times \varepsilon\, Y_e$ (i.e., $I_e \times Y_e$ with metric $g_{\varepsilon,e} = \mathrm{d}s^2 + \varepsilon^2 h_e$). Then, the $(p-1)$-form θ on $X_{\varepsilon,e}$ can be written uniquely as

$$\theta = \theta_1 \wedge \mathrm{d}s + \theta_2 \tag{7.3.4}$$

where θ_1 and θ_2 are a $(p-2)$-form and $(p-1)$-form on Y_e, respectively. Using the scaling behaviour of the metric in a similar way as in Lemma 6.6.1, we have

$$\begin{aligned}\|\theta\|^2_{L^2(\Lambda^{p-1}(X_{\varepsilon,e}))} &= \int_{X_{\varepsilon,e}} |\theta|^2_{g_{\varepsilon,e}} \,\mathrm{dvol}\, X_{\varepsilon,e} \\ &= \int_{I_e}\int_{Y_e} \left(\varepsilon^{-2(p-2)}|\theta_1|^2_{h_e} + \varepsilon^{-2(p-1)}|\theta_2|^2_{h_e}\right)\varepsilon^{n-1}\,\mathrm{d}s\,\mathrm{dvol}\, Y_e \\ &= \varepsilon^{n-2p+1}\int_{I_e}\int_{Y_e}\left(\varepsilon^2|\theta_1|^2_{h_e} + |\theta_2|^2_{h_e}\right)\mathrm{d}s\,\mathrm{dvol}\, Y_e,\end{aligned} \tag{7.3.5}$$

where the ε factor appears due to the scaled metric $\varepsilon^2 h_e$. The decomposition of $d\theta$ according to (7.3.4) is given by

$$d\theta = (d_{Y_e}\theta_1 + \partial_s\theta_2)\wedge \mathrm{d}s + d_{Y_e}\theta_2. \tag{7.3.6}$$

Hence,

$$\begin{aligned}\|d\theta\|^2_{L^2(\Lambda^{p}(X_{\varepsilon,e}))} &= \int_{X_{\varepsilon,e}} |d\theta|^2_{g_{\varepsilon,e}} \,\mathrm{dvol}\, X_{\varepsilon,e} \\ &= \int_{I_e}\int_{Y_e} \left(\varepsilon^{-2(p+1)}|d_{Y_e}\theta_1 + \partial_s\theta_2|^2_{h_e} + \varepsilon^{-2p}|d_{Y_e}\theta_2|^2_{h_e}\right)\varepsilon^{n-1}\,\mathrm{d}s\,\mathrm{dvol}\, Y_e \\ &= \varepsilon^{n-2p-1}\int_{I_e}\int_{Y_e}\left(\varepsilon^2|d_{Y_e}\theta_1 + \partial_s\theta_2|^2_{h_e} + |d_{Y_e}\theta_2|^2_{h_e}\right)\mathrm{d}s\,\mathrm{dvol}\, Y_e\,.\end{aligned} \tag{7.3.7}$$

In particular, if we substitute (7.3.5) and (7.3.7) into the quotient $\|\eta\|^2/\|\theta\|^2$ we conclude

$$\frac{\|d\theta\|^2_{L^2(\Lambda^{p}(X_{\varepsilon,e}))}}{\|\theta\|^2_{L^2(\Lambda^{p-1}(X_{\varepsilon,e}))}} = \varepsilon^{-2}\frac{\int_{I_e}\int_{Y_e}\left(\varepsilon^2|d_{Y_e}\theta_1 + \partial_s\theta_2|^2_{h_e} + |d_{Y_e}\theta_2|^2_{h_e}\right)\mathrm{d}s\,\mathrm{dvol}\, Y_e}{\int_{I_e}\int_{Y_e}\left(\varepsilon^2|\theta_1|^2_{h_e} + |\theta_2|^2_{h_e}\right)\mathrm{d}s\,\mathrm{dvol}\, Y_e}.$$

In particular, together with Proposition 6.6.2 this yields

$$\bar\lambda_1^p(X_{\varepsilon,e}) = \varepsilon^{-2}c_p(\varepsilon)$$

with

$$c_p(\varepsilon) = \sup\left\{\frac{\int_{I_e}\int_{Y_e}\varepsilon^2\left(|d_{Y_e}\theta_1 + \partial_s\theta_2|^2_{h_e} + |d_{Y_e}\theta_2|^2_{h_e}\right)\mathrm{d}s\,\mathrm{dvol}\, Y_e}{\int_{I_e}\int_{Y_e}\left(\varepsilon^2|\theta_1|^2_{h_e} + |\theta_2|^2_{h_e}\right)\mathrm{d}s\,\mathrm{dvol}\, Y_e}\;\middle|\; \begin{array}{l}\theta = \theta_1\wedge \mathrm{d}s + \theta_2 \neq 0,\\ \theta_1\ (p-2)\text{-form},\\ \theta_2\ (p-1)\text{-form}\end{array}\right\}.$$

In the limit $\varepsilon\to 0$, this constant tends to a number $c_p(0)$ given by

$$c_p(0) = \sup\left\{\frac{\int_{I_e}\int_{Y_e}|d_{Y_e}\theta_2|^2_{h_e}\,\mathrm{d}s\,\mathrm{dvol}\, Y_e}{\int_{I_e}\int_{Y_e}|\theta_2|^2_{h_e}\,\mathrm{d}s\,\mathrm{dvol}\, Y_e}\;\middle|\;\theta_2\neq 0\ (p-1)\text{-form}\right\}.$$

This constant is the min-max characterisation of the first eigenvalue of the operator $\mathrm{id} \otimes \bar{\Delta}^p_{Y_e}$ acting on $L^2(I_e) \otimes L^2(\Lambda^p(Y_e))$, whose spectrum agrees with the spectrum of $\bar{\Delta}^p_{Y_e}$ (see e.g. [RS78, Thm. XIII.34]). Hence, we have $c_p(0) = \bar{\lambda}^p_1(Y_e)$. □

7.3.2 Main theorems

We are now ready to prove the divergence behaviour of the spectrum of the co-exact p-forms, for $1 \leqslant p \leqslant n-2$, on a n-dimensional graph-like manifold X_ε. For the rest of this section we assume that $n \geqslant 3$, as the 2-dimensional case has been already explained in Corollary 7.2.3. We remind the reader that by (6.5.12), we analyse the exact p-forms for $2 \leqslant p \leqslant n-1$.

We first analyse the case when X_ε is transversally trivial, making use of Proposition 6.6.4. Let

$$\mathcal{U}_\varepsilon = \{U_{\varepsilon,v}\}_{v \in V} \cup \{U_{\varepsilon,e}\}_{e \in E}$$

be an open cover of X_ε, where $U_{\varepsilon,v}$ and $U_{\varepsilon,e}$ are open ε-neighbourhoods of $X_{\varepsilon,v}$ and $X_{\varepsilon,e}$ in X_ε, respectively, or in other words, a slightly enlarged vertex and edge neighbourhoods to ensure that $\mathcal{U}_\varepsilon$ is an open cover.

It is easy to see that $\mathcal{U}_\varepsilon$ has intersections up to degree 2 only (three or more different sets of $\mathcal{U}_\varepsilon$ have always trivial intersection). The intersections of degree 2 are given by $X_{\varepsilon,v,e} = U_{\varepsilon,v} \cap U_{\varepsilon,e}$ which is empty if $e \notin E_v$ or otherwise isometric to the product $(0,\varepsilon) \times Y_{\varepsilon,e}$, hence conformally equivalent to the product $(0,1) \times Y_e$ with conformal factor ε^2, as we enlarged $X_{\varepsilon,v}$ by an ε-neighbourhood. Moreover, $X_{\varepsilon,v,e}$ is homeomorphic to $(0,1) \times Y_e$, and hence homotopy equivalent to Y_e. In particular, $H^{p-1}(X_{\varepsilon,v,e}) = H^{p-1}(Y_e)$.

Theorem 7.3.3. *Let X_ε be a graph-like manifold of dimension $n \geqslant 3$ with underlying metric graph X_0. Assume that $2 \leqslant p \leqslant n-1$ and that the $(p-1)$-th cohomology group of the transversal manifold Y_e vanishes for all $e \in E$, i.e., $H^{p-1}(Y_e) = 0$. Then, the first eigenvalue of the Hodge Laplacian acting on exact p-forms on X_ε satisfies*

$$\bar{\lambda}^p_1(X_\varepsilon) \geqslant \tau_p \varepsilon^{-2},$$

where $\tau_p > 0$ is a constant depending only on the building blocks X_v and Y_e of the graph-like manifold, the truncated minimal length $\ell_0 = \min_{e \in E}\{\ell_e, 1\}$, and p. In particu-

lar, all eigenvalues $\bar{\lambda}^p_j(X_\varepsilon)$ of exact p-forms and all eigenvalues $\bar{\bar{\lambda}}^{p-1}_j(X_\varepsilon)$ of co-exact $(p-1)$-forms tend to ∞ as $\varepsilon \to 0$.

Proof. We apply Proposition 6.6.4 as the cover $\mathcal{U}_\varepsilon$ has no intersections of degree higher than 2 and $H^{p-1}(X_{\varepsilon,v,e}) = H^{p-1}(Y_e) = 0$.

We first look at the denominator of the right hand side of the estimate in Proposition 6.6.4. We note that our open cover $\mathcal{U}_\varepsilon$ is labelled by $v \in V$ and $e \in E$ and therefore, the sum over $i = 1, \ldots, m$ of the estimate in Proposition 6.6.4 becomes a sum over $v \in V$ and $e \in E$. Moreover, the sum over the edges can easily be rewritten as a sum over the vertices taking care of some appearing extra factors. Using these observations and equations (7.3.2) and (7.3.3), we obtain

$$\begin{aligned}
&\sum_{v\in V}\left(\frac{1}{\bar{\lambda}^p_1(X_{\varepsilon,v})} + \sum_{e\in E_v}\left(\frac{c_{n,p}\|d\rho_\varepsilon\|^2_\infty}{\bar{\bar{\lambda}}^{p-1}_1(X_{\varepsilon,v,e})} + 1\right)\left(\frac{1}{\bar{\lambda}^p_1(X_{\varepsilon,v})} + \frac{1}{\bar{\lambda}^p_1(X_{\varepsilon,e})}\right)\right)\\
&\qquad + \sum_{e\in E}\left(\frac{1}{\bar{\lambda}^p_1(X_{\varepsilon,e})} + \sum_{v=\partial_\pm e}\left(\frac{c_{n,p}\|d\rho_\varepsilon\|^2_\infty}{\bar{\bar{\lambda}}^{p-1}_1(X_{\varepsilon,v,e})} + 1\right)\left(\frac{1}{\bar{\lambda}^p_1(X_{\varepsilon,v})} + \frac{1}{\bar{\lambda}^p_1(X_{\varepsilon,e})}\right)\right)\\
&= \sum_{v\in V}\left(\frac{1}{\bar{\lambda}^p_1(X_{\varepsilon,v})} + \frac{\deg v}{\bar{\lambda}^p_1(X_{\varepsilon,e})} + 2\sum_{e\in E_v}\left(\frac{c_{n,p}\|d\rho_\varepsilon\|^2_\infty}{\bar{\bar{\lambda}}^{p-1}_1(X_{\varepsilon,v,e})} + 1\right)\left(\frac{1}{\bar{\lambda}^p_1(X_{\varepsilon,v})} + \frac{1}{\bar{\lambda}^p_1(X_{\varepsilon,e})}\right)\right)\\
&= \varepsilon^2\sum_{v\in V}\left(\frac{1}{\bar{\lambda}^p_1(X_v)} + \frac{\deg v}{c_p(\varepsilon)} + 2\sum_{e\in E_v}\left(\frac{c_{n,p}\varepsilon^2\|d\rho_\varepsilon\|^2_\infty}{\bar{\bar{\lambda}}^{p-1}_1(X_{v,e})} + 1\right)\left(\frac{1}{\bar{\lambda}^p_1(X_v)} + \frac{1}{c_p(\varepsilon)}\right)\right) =: \varepsilon^2 C_p(\varepsilon),
\end{aligned}$$

where the extra term with $\deg v$ and the factor 2 are due to the transformation of the sum over the edges into a sum over the vertices.

We now analyse the constant $C_p(\varepsilon)$ as $\varepsilon \to 0$.

First, we have seen in Proposition 7.3.2 that $c_p(\varepsilon) \to \bar{\lambda}^p_1(Y_e) > 0$. Moreover, the norm of the derivative of the partition of unit norm depends on ε as these functions have to change from 0 to 1 on a length scale of order ε on the vertex neighourhoods and on a length scale of order ℓ_0 on the edge neighbourhood, hence the derivative is of order $\varepsilon^{-1} + \ell_0^{-1}$ and $\varepsilon^2\|d\rho_\varepsilon\|^2_\infty = \mathcal{O}(1) + \mathcal{O}((\varepsilon/\ell_0)^2)$. In particular, $C_p(\varepsilon) \to C_p(0)$ as $\varepsilon \to 0$ provided ε/ℓ_0 remains bounded, where $C_p(0)$ depends only on some data of the building blocks.

Therefore, by Proposition 6.6.4 we can conclude

$$\bar{\lambda}^p_1(X_\varepsilon) \geqslant \frac{2^{-3}}{\varepsilon^2 C_p(\varepsilon)},$$

which proves the theorem. □

We observe that, by duality and supersymmetry (see (6.5.12) and (6.5.13)), Theorem 7.3.3 gives divergence for the spectrum of p-forms for $2 \leqslant p \leqslant n-2$.

We also point out that Theorem 7.2.2 and Theorem 7.3.3 for $p = 2$ gives a complete description of the behaviour of the spectrum on 1-forms, and by duality we also have a description of the spectrum on $(n-1)$-forms. We remark that this spectrum is partially convergent (exact eigenvalues) and partially divergent (co-exact eigenvalues). Hence, we can state the following.

Corollary 7.3.4. *Let X_ε be a graph-like Riemannian compact manifold of dimension $n \geqslant 3$ associate to a metric graph X_0. Assume that all transversal manifolds Y_e have trivial cohomology for $p = 1, \dots, n-2$. Then,*

$$\begin{aligned} \bar{\bar{\lambda}}_j^{n-1}(X_\varepsilon) = \bar{\lambda}_j^1(X_\varepsilon) &\underset{\varepsilon\to 0}{\longrightarrow} \lambda_j^0(X_0), \\ \bar{\lambda}_j^{n-1}(X_\varepsilon) = \bar{\bar{\lambda}}_j^1(X_\varepsilon) &\underset{\varepsilon\to 0}{\longrightarrow} \infty, \qquad (7.3.8) \\ \lambda_j^p(X_\varepsilon) &\underset{\varepsilon\to 0}{\longrightarrow} \infty, \end{aligned}$$

for all $j \geqslant 1$ and $2 \leqslant p \leqslant n-2$,

We remark that the case $n = 2$ has been treated in Corollary 7.2.3.

Removing the assumption of vanishing cohomology groups of the transversal manifolds, the following theorem holds.

Theorem 7.3.5. *Let X_ε be a graph-like manifold of dimension $n \geqslant 3$ with underlying metric graph X_0. Then, the N-th eigenvalue of the Laplacian acting on exact p-forms on X_ε satisfies*

$$\bar{\lambda}_N^p(X_\varepsilon) \geqslant \tilde{\tau}_p \varepsilon^{-2},$$

where $\tilde{\tau}_p > 0$ is as before and where

$$N = 1 + \sum_{v\in V} \sum_{e\in E_v} \dim H^{p-1}(Y_e) = 1 + 2\sum_{e\in E} \dim H^{p-1}(Y_e).$$

Its proof follows the line of the previous one with the difference that we use Proposition 6.6.5 to estimate a higher eigenvalue for exact p-forms on X_ε.

Remark 7.3.6. We point out that the first $N-1$ eigenvalues of the theorem above are strictly positive since we consider the spectrum away from zero. The theorem states that $\bar{\lambda}_j^p(X_\varepsilon) = \bar{\bar{\lambda}}_{(j-1)}^p(X_\varepsilon)$ are divergent for $j \geqslant N$ in the limit $\varepsilon \to 0$. However, it remains an open question how the first $(N-1)$ eigenvalues behave asymptotically as $\varepsilon \to 0$.

Chapter 8

Manifolds with spectral gap

In this chapter we discuss some applications of the asymptotic behaviours described in Chapter 7. We will state some general facts about the existence of spectral gaps in the spectrum of the Hodge-Laplacian on a graph-like manifold X_ε in relation to existing spectral gaps in the spectrum of the Laplacian on its associated metric graph X_0. Moreover, we will construct manifolds and families of manifolds with spectral gaps.

In Section 8.1 we define the Hausdorff convergence and we state a weaker version of Corollary 7.3.4 in relation to this definition (see Corollary 8.1.1). Moreover, we give the definition of spectral gap and a general result on how to produce graph-like manifolds with spectral gaps in their spectrum. In Section 8.2 we construct manifolds with constant volume and arbitrarily large form eigenvalues, i.e., manifolds with an arbitrarily large spectral gap in their Hodge-Laplacian on p-forms for $2 \leqslant p \leqslant n-1$. In Section 8.3 we construct families of manifolds with spectral gaps arising from families of Ramanujan graphs and of arbitrary graphs.

8.1 Hausdorff convergence of the spectrum and spectral gaps

Let $A, B \subset \mathbb{R}$ be two compact sets. The *Hausdorff distance of A and B* is defined as

$$d(A,B) := \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}, \quad \text{where} \quad d(a,B) := \inf_{b\in B} |a-b|. \tag{8.1.1}$$

A sequence $(A_n)_n$ of compact sets $A_n \subset \mathbb{R}$ *converges in Hausdorff distance* to A_0 if and only if $d(A_n, A) \to 0$ as $n \to \infty$. In particular, $d(A_n, A_0) \to 0$ if and only if for all $\lambda_0 \in A_0$ there exists $\lambda_n \in A_n$ such that $|\lambda_0 - \lambda_n| \to 0$ and for all $x \in \mathbb{R} \backslash A_0$ there exists $\eta > 0$ such that $[x - \eta, x + \eta] \cap A_n = \varnothing$ for n sufficiently large (see e.g. [Pos12, Proposition A.1.6]).

In view of this definition, a weaker result than Corollary 7.3.4 is as follows.

Corollary 8.1.1. *Let X_ε be a transversally trivial graph-like manifold with associated metric graph X_0. Then, for all $\lambda_0 > 0$ we have that $\sigma(\Delta^\bullet_{X_\varepsilon}) \cap [0, \lambda_0]$ converges in Hausdorff distance to $\sigma(\Delta_{X_0}) \cap [0, \lambda_0]$.*

In fact, in a compact interval $[0, \lambda_0]$, eventually all divergent eigenvalues from higher forms leave this interval, and the remaining ones converge.

Furthermore, we asked ourselves about the relation between spectral gaps in the spectrum of the Laplacian acting on 1-forms on X_ε and X_0, i.e., about intervals (a, b) not belonging to the spectrum. More precisely, a *spectral gap* of an operator $\Delta \geqslant 0$ is a non-empty interval (a, b) such that

$$\sigma(\Delta) \cap (a, b) = \varnothing.$$

As a consequence of the asymptotic description of the spectrum in Theorems 7.2.2, 7.3.3 and in Corollary 8.1.1, we have the following result on spectral gaps (i.e., intervals disjoint with the spectrum).

Corollary 8.1.2. *Assume that the graph-like manifold X_ε is transversally trivial and suppose that (a_0, b_0) is a spectral gap for the metric graph X_0, then there exist a_ε, b_ε with $a_\varepsilon \to a_0$ and $b_\varepsilon \to b_0$ such that $(a_\varepsilon, b_\varepsilon)$ is a spectral gap for the Hodge Laplacian on X_ε in all degree forms, i.e., $\sigma(\Delta^\bullet_{X_\varepsilon}) \cap (a_\varepsilon, b_\varepsilon) = \varnothing$.*

Examples of manifolds with spectral gaps can be generated in different ways. In [Pos03, LP08] the authors constructed (non-compact) abelian covering manifolds having an arbitrary large number of gaps in their essential spectrum of the scalar Laplacian, and in [ACP09], the analysis was extended to the Hodge Laplacian on certain cyclic covering manifolds.

One can construct metric graphs with spectral gaps, and hence graph-like manifolds with spectral gaps, with a technique called graph decoration that works as follows. We consider a finite metric graph X_0 and a second finite metric graph $\widetilde{X}_0$. For each $v \in V(X_0)$, let $\widetilde{X}_0 \times \{v\}$ be a copy of a finite metric graph $\widetilde{X}_0$. Fix a vertex $\widetilde{v}$ of $\widetilde{X}_0$. Then the graph decoration of X_0 with the graph $\widetilde{X}_0$ is the graph obtained from X_0 by identifying the vertex $\widetilde{v}$ of $\widetilde{X}_0 \times \{v\}$ with v. This decoration opens up a gap in the spectrum of the Laplacian on function on X_0 as described in [Ku05] and therefore in its 1-form Laplacian. Consequently, the associated graph-like manifold has a spectral gap in its 1-form Laplacian, and no spectrum away from 0 for higher forms, as all the form eigenvalues diverge.

More examples of manifolds with spectral gap and family of manifolds with a spectral gap are given in the next sections.

8.2 Manifolds with arbitrarily large spectral gap

Let $(X_\varepsilon)_{\varepsilon>0}$ be a graph-like manifold constructed from a metric graph X_0 with underlying (discrete) graph (V, E, ∂). We assume the graph-like manifold to be transversally trivial (i.e., $H^p(Y_e) = 0$ for all $1 \leqslant p \leqslant n-2$ and for all $e \in E$).

For simplicity, we assume that X_0 is equilateral, i.e., all edge lengths are given by a number $\ell > 0$. The result can be easily extended to the case when $c_-\ell \leqslant \ell_e \leqslant c_+\ell$ for all $e \in E$ and some constants $c_\pm > 0$.

We write

$$a_\varepsilon \lesssim b_\varepsilon, \qquad a_\varepsilon \gtrsim b_\varepsilon, \qquad a_\varepsilon \eqsim b_\varepsilon \tag{8.2.2}$$

if

$$a_\varepsilon \leqslant \text{const}_+ \, b_\varepsilon, \qquad a_\varepsilon \geqslant \text{const}_- \, b_\varepsilon, \qquad \text{const}_- \, a_\varepsilon \leqslant b_\varepsilon \leqslant \text{const}_+ \, a_\varepsilon \tag{8.2.2'}$$

for all $\varepsilon > 0$ small enough and constants $\text{const}_\pm$ independent of ε.

We first summarise the asymptotic spectral behaviour of a graph-like manifold X_ε and its dependence on the parameters ε, ℓ, $|V|$, and $|E|$. In particular, for the volume, the 0-forms (functions), and the exact p-forms and co-exact $(p-1)$-forms, we have

$$\operatorname{vol} X_\varepsilon \approx \varepsilon^n |V| + \varepsilon^{n-1}\ell|E| \tag{8.2.3a}$$

$$|\lambda_j^0(X_\varepsilon) - \lambda_j^0(X_0)| \lesssim \frac{\varepsilon^{1/2}}{\ell_0} \qquad (\ell_0 = \min\{\ell, 1\}) \tag{8.2.3b}$$

$$\bar{\lambda}_1^p(X_\varepsilon) = \bar{\bar{\lambda}}_1^{p-1}(X_\varepsilon) \gtrsim \frac{1}{\varepsilon^2|E|(1+\varepsilon^2/\ell^2)} \qquad 2 \leqslant p \leqslant n-1, \tag{8.2.3c}$$

where the constants in $\lesssim$ etc. depend only on the building blocks X_v and Y_e of the (unscaled, i.e, $\varepsilon = 1$) graph-like manifold. Equation (8.2.3a) is a direct consequence of the structure of the graph-like manifold described in (6.4.8). Equation (8.2.3b) is a direct consequence of Proposition 7.2.1. Equation (8.2.3c) follows from analysing the lower bound constant τ_p in Theorem 7.3.3 (or Theorem 7.3.5). We see that the constant $C_p(\varepsilon)$ in its proof is bounded from above by

$$C_p(\varepsilon) \lesssim \big(|V| + |E|(1+\varepsilon^2/\ell^2)\big) \lesssim |E|(1+\varepsilon^2/\ell^2),$$

where again the constants in $\lesssim$ depend only on the building blocks and where we used $|V| \leqslant \sum_{v\in V} \deg v = 2|E|$ for any graph G, assuming that there are no isolated vertices, i.e., vertices of degree 0.

We now assume that $\ell = \ell_\varepsilon = \varepsilon^\gamma$ depends on ε for some $\gamma \in \mathbb{R}$ (negative γ's are not excluded). In particular, X_0 now also depends on ε, and we write $\varepsilon^\gamma X_0$ for a metric graph with all edge lengths multiplied by ε^γ. If we plug $\ell = \varepsilon^\gamma$ into equations (8.2.3a)–(8.2.3c), we observe the following.

(i) In (8.2.3a), the dominant term is ε^n for $\gamma \geqslant 1$ and it is $\varepsilon^{n-1+\gamma}$ otherwise.

(ii) For the metric graph eigenvalues, we have $\lambda_j^0(\varepsilon^\gamma X_0) = \varepsilon^{-2\gamma}\lambda_j^0(X_0)$.

(iii) In (8.2.3b) we need $\gamma < 1/2$ for convergence to hold, as the error term is of order $\varepsilon^{1/2}/\min\{\varepsilon^\gamma, 1\} = \varepsilon^{1/2-\max\{\gamma,0\}}$. We also need $\gamma > -1/4$ for the metric graph eigenvalue (of order $\varepsilon^{-2\gamma}$) to be dominant with respect to the error (of order $\varepsilon^{1/2-\max\{\gamma,0\}}$).

(iv) In (8.2.3c) we need $\gamma < 2$ for divergence to hold. Moreover, ε^2 is the dominant term in the denominator of the RHS for $\gamma \leqslant 1$, and it is $\varepsilon^{4-2\gamma}$ otherwise.

Therefore, equations (8.2.3a)–(8.2.3c) become

$$\operatorname{vol} X_\varepsilon \eqsim \varepsilon^n |V| + \varepsilon^{n-1+\gamma}|E| \eqsim \begin{cases} \varepsilon^{n-1+\gamma}|E|, & \gamma \leqslant 1 \\ \varepsilon^n |V|, & \gamma \geqslant 1, \end{cases} \tag{8.2.3a'}$$

$$\lambda_j^0(X_\varepsilon) \begin{cases} \eqsim \varepsilon^{-2\gamma}, & -1/4 < \gamma\ (< 1/2) \\ \lesssim \varepsilon^{1/2}, & \gamma \leqslant -1/4, \end{cases} \tag{8.2.3b'}$$

$$\bar\lambda_1^p(X_\varepsilon) \gtrsim \begin{cases} \varepsilon^{-2}, & \gamma \leqslant 1 \\ \varepsilon^{-4+2\gamma}, & 1 \leqslant \gamma\ (< 2). \end{cases} \tag{8.2.3c'}$$

These equations and statements (i)–(iv) give the existence of manifolds with constant volume and arbitrarily large form eigenvalues, i.e, manifolds with an arbitrarily large spectral gap in their form spectrum. Our proposition below states an analogous result than the one in [GP95, Theorem 1], where the authors state that for any closed manifold M of dimension $n \geqslant 4$ there exits a metric of volume 1 such that the first non-zero p-form eigenvalue $\lambda_1^p(M)$ is unbounded. In particular, they give an answer to a question of Tanno [Tan83], whether there exists a constant $k(M)$ such that the first non-zero p-form eigenvalue satisfies $\lambda_1^p(M) \leqslant k(M)(\operatorname{vol}(M,g))^{-n/2}$ for all Riemannian metrics g on M. The same question was previously posed by Berger [Ber73] on the first non-zero function eigenvalue and answered positively (see [GP95] and references therein for further contributions). We observe that the construction of Gentile and Pagliara in [GP95] corresponds to a simple graph with one edge and two vertices. Therefore, we conclude the following.

Proposition 8.2.1. *On any transversally trivial graph-like manifold of dimension $n \geqslant 3$ there exists a family of metrics $\widetilde g_\varepsilon$ of volume 1 such that for the first eigenvalue on exact p-forms we have*

$$\bar\lambda_1^p(X, \widetilde g_\varepsilon) \to \infty \qquad \textit{as } \varepsilon \to 0$$

for $2 \leqslant p \leqslant n-1$. Moreover, the function ($p = 0$) and exact 1-form spectrum converges to 0, i.e.,

$$\lambda_1^0(X, \widetilde g_\varepsilon) = \bar\lambda_1^1(X, \widetilde g_\varepsilon) \to 0 \qquad \textit{as } \varepsilon \to 0.$$

Proof. Let g_ε be the metric of the graph-like manifold as constructed in Section 6.4. For any $\gamma < 1$, we have

$$\bar{\lambda}_1^p(X, g_\varepsilon)(\mathrm{vol}(X, g_\varepsilon))^{2/n} \gtrsim \varepsilon^{-2}\varepsilon^{2(n-1+\gamma)/n} = \varepsilon^{-2(1-\gamma)/n} \to \infty \qquad \text{as } \varepsilon \to 0$$

by (8.2.3c') and (8.2.3a'). Set now $\widetilde{g}_\varepsilon := \mathrm{vol}(X, g_\varepsilon)^{-2/n} g_\varepsilon$, then $\mathrm{vol}(X, \widetilde{g}_\varepsilon) = 1$ and

$$\bar{\lambda}_1^p(X, \widetilde{g}_\varepsilon) = \mathrm{vol}(X, g_\varepsilon)^{2/n}\bar{\lambda}_1^p(X, g_\varepsilon) \gtrsim \varepsilon^{-2(1-\gamma)/n} \to \infty \qquad \text{as } \varepsilon \to 0.$$

If $-1/4 < \gamma < 1/2$, then the 0-form (and exact 1-form) eigenvalues of the metric graph and the manifold are close and $\lambda_j^0(X, g_\varepsilon) \approx \varepsilon^{-2\gamma}$, hence

$$\lambda_j^0(X, \widetilde{g}_\varepsilon) = \mathrm{vol}(X, g_\varepsilon)^{2/n}\lambda_j^0(X, g_\varepsilon) \approx \varepsilon^{2(n-1+\gamma)/n}\varepsilon^{-2\gamma} = \varepsilon^{2(n-1)(1-\gamma)/n} \to 0. \qquad \square$$

We observe that for manifolds as constructed in the proof, the transversal length scale (the one of the transversal manifolds Y_e) is $\varepsilon^{(1-\gamma)/n} \to 0$, while the longitudinal length scale (the one of the metric graph edges I_e) is $\varepsilon^{-(1-1/n)(1-\gamma)} \to \infty$ as $\varepsilon \to 0$. This implies that the edge neighbourhoods become thinner but longer in the limit.

Unfortunately, we cannot extend the result of [GP95] to the case $n = 3$ and 1-forms, as the exact 1-form spectrum converges.

8.3 Families of manifolds with special spectral properties arising from families of graphs

We now consider families of graph-like manifolds constructed according to a sequence of graphs $\{G^i\}_{i\in\mathbb{N}}$. We assume for simplicity that the vertex degree is uniformly bounded, say by $k_0 \in \mathbb{N}$. Then, if there are no isolated vertices, we have

$$|V(G^i)| \leqslant \sum_{v \in V(G^i)} \deg_{G^i} v = 2|E(G^i)| \leqslant 2k_0|V(G^i)|,$$

i.e., $\nu_i := |V(G^i)| \simeq |E(G^i)|$ as $i \to \infty$.

We begin with a general statement about the spectral convergence. We assume that $\{G^i\}_{i\in\mathbb{N}}$ is a family of discrete graphs and that $\{X_0^i\}_{i\in\mathbb{N}}$ is the family of associated equilateral metric graphs, each graph X_0^i having edge lengths equal to ℓ_i (for the definition of equilateral graph, see Section 6.2). Accordingly, we construct a family of graph-like manifolds $\{X_\varepsilon^i\}_{i\in\mathbb{N}}$ where the building blocks X_v and Y_e are isometric to a given number of prototypes (independent of i), such that Y_e all have trivial cohomology for $1 \leqslant p \leqslant n-2$ (see Example 6.4.1), so that all graph-like manifolds X_ε^i are transversally trivial and hence our estimates (8.2.3a)–(8.2.3c) are uniform in the building blocks and (8.2.3c) holds for the first exact eigenvalue. We call such a family of graph-like manifolds *uniform*.

We now assume that ε_i and ℓ_i are dependent on the number of vertices ν_i of G^i. Specifically, we set

$$\varepsilon_i = \nu_i^{-\alpha} \qquad \text{and} \qquad \ell_i = \nu_i^{-\beta} \tag{8.3.5}$$

for some $\alpha > 0$ and $\beta \in \mathbb{R}$ (negative values of β are not excluded). In particular, X_0^i now also depends on ε, and we write $\nu_i^{-\beta}X_0^i$ for the metric graph X_0^i with all edge lengths being $\nu_i^{-\beta}$. Substituting conditions (8.3.5) into equations (8.2.3a)–(8.2.3b), we observe the following.

(i') The volume is now given by $\operatorname{vol} X_\varepsilon^i \eqsim \nu_i^{-n\alpha+1} + \nu_i^{-(n-1)\alpha-\beta+1}$.

(ii') For the metric graph eigenvalue, we have $\lambda_j^0(\nu_i^{-\beta}X_0^i) = \nu_i^{2\beta}\lambda_j^0(X_0^i)$.

(iii') In (8.2.3b) we need $\max\{\beta, 0\} < \alpha/2$, for the convergence to hold, as the error term is of order $\varepsilon_i^{1/2}/\min\{\ell_i, 1\} = \nu_i^{-\alpha/2+\max\{\beta,0\}}$ (Figure 8.1 (a) below). We also need $\beta \geqslant -\alpha/2$ and $\beta \geqslant 0$, or $\beta \geqslant -\alpha/4$ and $\beta \leqslant 0$, for the metric graph eigenvalue (of order $\nu_i^{2\beta}$) to be dominant with respect to the error (of order $\nu_i^{-\alpha/2+\max\{\beta,0\}}$) (Figure 8.1 (b) below).

(iv') In (8.2.3c), we need $\alpha > 1/2$ (resp. $2\alpha > 1+\beta$), for the divergence to hold. Moreover, if $\alpha \geqslant \beta$ the dominant term in the denominator of the RHS is $\nu^{-2\alpha+1}$, it is $\nu^{-4\alpha+2\beta+1}$ otherwise (Figure 8.1 (c) below).

Therefore, in view of the above statements, we can rewrite (8.2.3a)–(8.2.3c) as

$$\operatorname{vol} X_\varepsilon^i \eqsim \begin{cases} \nu_i^{-(n-1)\alpha-\beta+1}, & \alpha \geqslant \beta, \\ \nu_i^{-n\alpha+1}, & \alpha \leqslant \beta. \end{cases} \tag{8.2.3a''}$$

$$\lambda_j^0(X_\varepsilon^i) \begin{cases} \eqsim \nu_i^{2\beta}\lambda_j(X_0^i), & (\beta \geqslant -\alpha/2, \beta \geqslant 0) \text{ or } (\beta \geqslant -\alpha/4, \beta \leqslant 0), \\ \lesssim \nu_i^{-\alpha/2}, & \text{otherwise.} \end{cases} \tag{8.2.3b''}$$

$$\bar{\lambda}_1^p(X^i_{\varepsilon_i}) \gtrsim \begin{cases} \nu_i^{2\alpha-1}, & \alpha \geqslant \beta, \\ \nu_i^{4\alpha-2\beta-1}, & \alpha \leqslant \beta, \end{cases} \qquad \text{for } 2 \leqslant p \leqslant n-1. \tag{8.2.3c”}$$

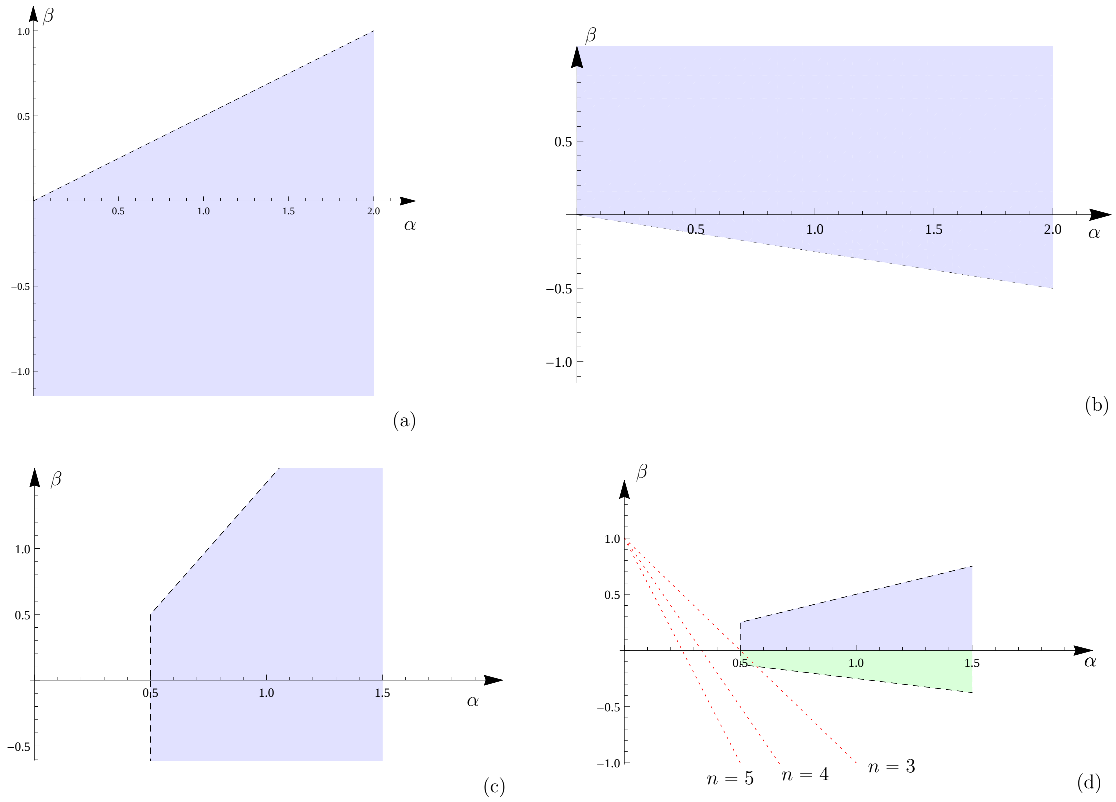

Figure 8.1: (a) Region where the 0-form eigenvalue convergence in (8.2.3b) holds ($\max\{\beta, 0\} < \alpha/2$); (b) Region where $\lambda_j^0(X^i_{\varepsilon_i}) \approx \nu_i^{2\beta}\lambda_j(X^i_0)$ ($\beta > -\alpha/2, \beta \geqslant 0$ or $\beta > -\alpha/4, \beta \leqslant 0$); (c) Region where $\bar{\lambda}_1^p(X^i_{\varepsilon_i})$ diverges. ($\alpha > 1/2, \alpha \geqslant \beta$ or $4\alpha - 2\beta - 1 > 0$, $\alpha \leqslant \beta$); (d) Blue region: all eigenvalues diverge. Green region: the form eigenvalues diverge and the function eigenvalues converge to 0. Above the red dotted line the volume tends to 0, below it tends to ∞.

We now discuss some examples using statements (i’)–(iv’) and equations (8.2.3a”)–(8.2.3c”).

Families of manifolds arising from a sequence of Ramanujan graphs

We consider a sequence of discrete Ramanujan graph $(G^i)_i$ with $\nu_i = |V(G^i)|$ many vertices and the associate sequence of equilateral metric graphs $(X^i_0)_i$ with all edge lengths equal to 1 (for a formal definition see Section 6.3). Then, the (metric) graph Laplacians have a common spectral gap $(0,h)$ (see (6.3.7)). Accordingly, we construct a uniform family of graph-like manifolds $(X^i_{\varepsilon_i})_i$, as described at the beginning of this section, and assuming conditions (8.3.5) for the parameters ε_i and ℓ_i. Consequently, the edge length of the sequence of metric graphs becomes $\nu^{-\beta}$ and the common spectral gap is now dependent on ℓ_i, i.e., it is given by $(0,h_i)$ where $h_i = h/\ell_i^2$ (see again (6.3.7)).

If we choose (α,β) from the blue region of picture (d) we have the following.

Proposition 8.3.1. *There is a uniform family of graph-like manifolds $(X^i_{\varepsilon_i})_i$ constructed as above such that the Hodge Laplacian in all degree forms has an arbitrarily large spectral gap, i.e., there exists $h_i \approx \nu_i^{\min\{2\beta,2\alpha-1\}} \to \infty$ such that*

$$\sigma(\Delta^\bullet_{X^i_{\varepsilon_i}}) \cap (0,h_i) = \varnothing,$$

and such that the volume shrinks to 0, more precisely, $\operatorname{vol} X^i_{\varepsilon_i} \approx \nu_i^{-(n-1)\alpha-\beta+1}$.

In particular, if $\beta = 0$, i.e, if $\ell_i = 1$ for all i, then there exists a common spectral gap $(0,h)$ of the Hodge Laplacian. If, additionally, $n = 3$, then the volume decay can be made arbitrarily small as $\alpha \searrow 1/2$, i.e., of order $\nu_i^{-2\alpha+1}$.

Proof. The proof follows from considerations (i')–(iv') above and choosing (α,β) such that $\alpha > 1/2$, $\beta \geqslant 0$ and $\beta \leqslant \alpha/2$ (see Figure 8.1 (d)). We observe that for a sequence of Ramanujan graphs, there exists $h > 0$ such that the first non-zero eigenvalue of the metric graph Laplacian with unit edge length fulfils $\lambda_1(X^i_0) \geqslant h$ for all i, hence we can conclude divergence from the first line of (8.2.3b"). □

We observe that the length scale of the underlying metric graphs is of order $\nu_i^{-\beta}$, but the radius is of order $\varepsilon_i = \nu_i^{-\alpha}$, which is smaller; hence the injectivity radius of $X^i_{\varepsilon_i}$ is of order $\varepsilon_i = \nu_i^{-\alpha}$, and the curvature is of order $\varepsilon_i^{-2} = \nu_i^{2\alpha}$.

It is also possible to construct families of manifolds with fixed volume arising from families of Ramanujan graphs. In order to do so, we need to rescale the metric. We set $\widetilde{g}_i := (\operatorname{vol}(X^i_{\varepsilon_i}, g_{\varepsilon_i}))^{-2/n} g_{\varepsilon_i})$ and we consider $\widetilde{X}^i := (X^i_{\varepsilon_i}, \widetilde{g}_i)$. Then, the latter manifold has volume 1. Unfortunately, we cannot have divergence at all degrees at the same time. In fact, for $n = 3$ the conditions are $\beta > \alpha - 1/2$ for divergence of the eigenvalues of degree 0, while $\beta < \alpha - 1/2$ is needed for divergence of exact 2-forms. But we can have divergence of 0-forms and higher degree forms separately.

Proposition 8.3.2. *For all $n \geqslant 2$ there exists a family of graph-like manifolds $(\widetilde{X}^i)_i$ of volume 1 with underlying Ramanujan graphs such that the first non-zero eigenvalue on functions (0-forms) diverges.*

Proof. The rescaling factor $\tau_i = (\mathrm{vol}(X^i_{\varepsilon_i}, g_{\varepsilon_i}))^{-1/n}$ is of order $\nu_i^{(1-1/n)\alpha+\beta/n-1/n}$. The rescaled eigenvalue on functions fulfils

$$\lambda_1^0(\widetilde{X}^i) = \tau_i^{-2}\lambda_1^0(X^i_{\varepsilon_i}) \approx \tau_i^{-2}\nu_i^{2\beta}\lambda_1^0(X_0^i) \approx \nu_i^{2/n-2(1-1/n)(\alpha-\beta)}\lambda_1^0(X_0^i), \tag{8.3.6}$$

and the latter exponent is positive if and only if $\beta > \alpha - 1/(n-1)$. The allowed parameters (α, β) lie inside the triangle $(0,0)$, $(4,-1)/(5(n-1))$, $(2,1)/(n-1)$ such that $\lambda_1(\widetilde{X}^i) \approx \nu_i^{2/n-\delta}$ (see Figure 8.2). The difference $\beta - \alpha$ approaches its maximum on this triangle at the vertex $(0,0)$. Hence for any $\delta > 0$ there exists (α, β) inside the triangle such that $\lambda_1(\widetilde{X}^i) \approx \nu_i^{2/n-\delta}$. □

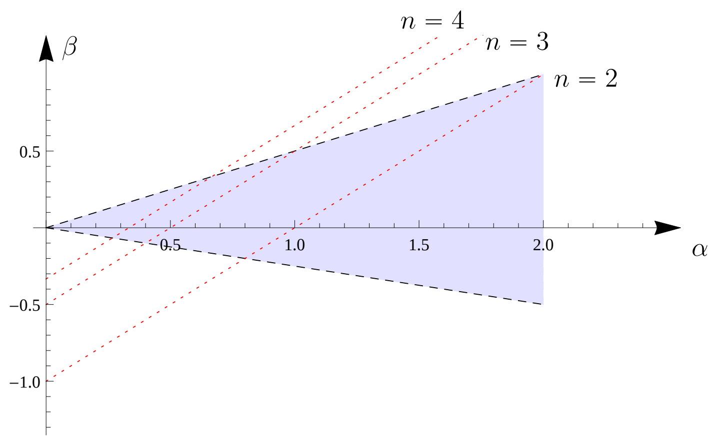

Figure 8.2: Parameter region where the rescaled 0-form eigenvalue $\lambda_1^0(\widetilde{X}^i_{\varepsilon_i})$ diverges, i.e., region where $\max\{\beta, 0\} < \alpha/2$ and $\lambda_1^0(\widetilde{X}^i_{\varepsilon_i}) \approx \nu^{2\beta}\lambda_1(X_0^i)$ are both satisfied. Above the dotted lines, the exponent in (8.3.6) is positive and the eigenvalue diverges.

In particular, for $n = 2$ Proposition 8.3.2 yields the following corollary.

Corollary 8.3.3. *There exists a sequence of graph-like surfaces $(\widetilde{X}^i)_i$ of area 1 and genus $\gamma(\widetilde{X}^i)$ with underlying Ramanujan graphs such that the first non-zero eigenvalue* *on functions diverges. Moreover, for any* $\delta > 0$ *there exists a sequence* $(\widetilde{X}^i)_i$ *such that*

$$\lambda_1^0(\widetilde{X}^i) \approx \gamma(\widetilde{X}^i)^{1-\delta}.$$

Proof. We have to choose $Y_e = \mathbb{S}^1$ here, moreover we let the vertex neighbourhood be a sphere with k discs removed (as in Example 6.4.1). In this case, the genus of the surface $\widetilde{X}^i$ is given by $1 - \chi(G^i)$ where $\chi(G^i)$ is the Euler characteristic of the graph G^i, and hence

$$\gamma(\widetilde{X}^i) = 1 - |V(G^i)| + |E(G^i)| = 1 - \nu_i + \frac{k}{2}\nu_i = 1 + \Big(\frac{k}{2} - 1\Big)\nu_i \to \infty$$

as $i \to \infty$ as $k \geqslant 3$ for a Ramanujan graph. In particular, $\gamma(\widetilde{X}^i) \approx \nu_i$. □

Family of manifolds arising from a sequence of arbitrary graphs

We now consider a sequence of arbitrary discrete graphs (G^i), with $\nu_i = |V(G^i)| \to \infty$ as $i \to \infty$ and with degrees bounded by k, and the associated sequence of metric graphs $(X_0^i)_i$. Then, we construct a sequence of graph-like manifolds with underlying metric graph X_0^i as explained at the beginning of the section assuming (8.3.5) for ε_i and ℓ_i. We show the existence of families of manifolds with constant volume, arbitrarily large form spectrum and convergent function spectrum, hence we do not need that the underlying graphs are Ramanujan. To obtain manifolds with constant volume we again set $\widetilde{g}_i = (\mathrm{vol}(X^i_{\varepsilon_i}))^{-2/n} g_{\varepsilon_i})$ so that the manifolds $\widetilde{X}^i$ equipped with the metric $\widetilde{g}^i$ will have constant volume 1.

We immediately have the following result.

Proposition 8.3.4. *For all* $n \geqslant 3$ *there exists a family of graph-like manifolds* $(\widetilde{X}^i)_i$ *of volume* 1*, constructed as described above, such that the first eigenvalue on exact* p*-forms diverges* ($2 \leqslant p \leqslant n-1$)*. Moreover, the first non-zero eigenvalue on functions converges.*

Proof. The rescaled eigenvalue on p-forms fulfils

$$\bar{\lambda}_1^p(\widetilde{X}^i) = \tau_i^{-2}\bar{\lambda}_1^p(X^i_{\varepsilon_i}) \gtrsim \tau_i^{-2}\nu_i^{2\alpha-1} \approx \nu_i^{2(\alpha-\beta+1)/n-1},$$

(as $\alpha \geqslant \beta$, see (8.2.3c")) and the latter exponent is positive if and only if $\beta < \alpha - (n/2 - 1)$. The allowed parameters (α, β) lie below this line (see Figure 8.3 below).

For the first non-zero eigenvalue on functions, note first that $\lambda_1(X_0^i)$ (the first non-zero eigenvalue of the unilateral metric graph X_0^i) can be bounded from above by π^2, this follows immediately from the spectral relation (6.2.3). Therefore, we conclude from (8.3.6) that $\lambda_1^0(\widetilde{X}^i) \to 0$ as $i \to \infty$ as $\beta < \alpha - (n/2 - 1)$ implies that $2/n - 2(1-n)(\alpha - \beta) < 0$. □

Actually, comparing the speed of divergence and convergence, we obtain

$$\bar{\lambda}_1^p(\widetilde{X}^i) \gtrsim \nu_i^{\frac{n^2}{2(n-1)}} \lambda_1(\widetilde{X}^i)^{-\frac{n^2}{4(n-1)}},$$

confirming again that we cannot have divergence for both function and form eigenvalues with our construction.

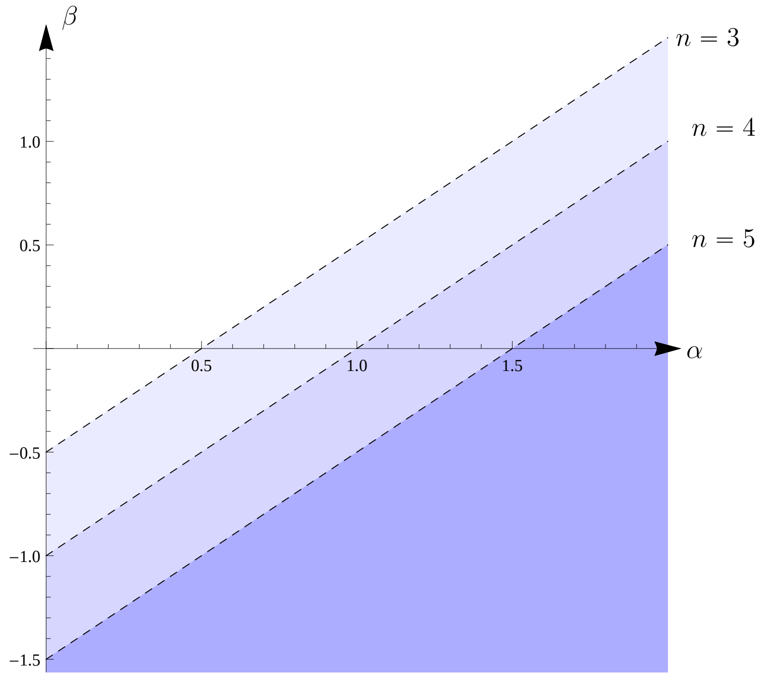

Figure 8.3: Above the dotted lines the p-forms eigenvalues diverge ($2 \leqslant p \leqslant n-1$), depending on the dimension.

In the special case that our family of graphs consists only of trees, we can modify any given manifold X to become a graph-like manifold (see Remark 6.4.2). In particular, we can show the following corollary.

Corollary 8.3.5. *On any compact manifold X of dimension $n \geqslant 3$, there exists a sequence of metrics g_i of volume 1 such that the infimum of the (non-zero) function spectrum converges to 0, while the exact p-form eigenvalues ($2 \leqslant p \leqslant n-1$) diverge.*

Appendix A

Geometry of $T^k M$

We here describe the space T^kM introduced in Chapter 2. We have already noticed that vectors on T^kM can be decomposed into horizontal and vertical components, resembling the structure of the tangent spaces of TM. In this Appendix we will analyse the Lie brackets and the covariant derivative of horizontal and vertical vectors, and we will look at curvature properties of T^kM in relation to the curvature of the base manifold M.

The reader will find it useful to compare these results with the ones in [Dom62, GuKa02, KS05] and references therein. In [Dom62, GuKa02], the authors describe the geometry of TM equipped with the Sasaki metric, while in [KS05], the authors describe the geometry of the linear frame bundle LM over a manifold equipped with a Sasaki-type metric. In both articles, the authors use local coordinates in their geometric descriptions of TM and LM. Moreover, they also discuss other types of metric on TM and LM.

Let (M, g) be a compact n-dimensional manifold with tangent bundle TM, and let $\pi : TM \longrightarrow M$ be the canonical projection. We remind the reader that for $k = 1, \ldots, n$, T^kM is defined as

$$T^kM = \bigcup_{p\in M} \{f = (v_1, \ldots, v_k) \in T_pM \times \ldots \times T_pM \mid \pi(v_i) = p \quad \forall\, i = 1, \ldots, k\},$$

and that there is a canonical projection $\pi^k : T^kM \longrightarrow M$ such that $\pi^k(f) = p$ if $v_i \in T_pM$ for all $i = 1, \ldots, k$.

T^kM is a manifold of dimension $n+nk$ and its tangent space at a point f is given by

$$T_f T^k M = \underbrace{T_pM \times \ldots \times T_pM}_{(k+1)-\text{times}}.$$

In fact, any tangent vector on T^kM is described as

$$X'(0) = \Big(\frac{d}{dt}\Big|_{t=0}(\pi^k \circ X)(t); \frac{D}{dt}\Big|_{t=0}V_1(t), \ldots, \frac{D}{dt}\Big|_{t=0}V_k(t)\Big),$$

where $X = (V_1, \ldots, V_k) : (-\varepsilon, \varepsilon) \longrightarrow T^kM$ is a curve on T^kM. In view of this, and having in mind the decomposition of vectors on TM into horizontal and vertical component, we consider any vector u on T^kM as the sum of $\overset{h}{u} = (u_0; 0, \ldots, 0)$ and $\overset{v}{u} = (0; u_1, \ldots, u_k)$. Moreover, we define

$$\mathcal{H}_f = \bigcup_{f \in T^kM} \{\overset{h}{u} \mid u \in T_f T^kM\} \cong T_{\pi^k(f)}M \quad \text{and}$$

$$\mathcal{V}_f = \bigcup_{f \in T^kM} \{\overset{v}{u} \mid u \in T_f T^kM\} \cong \underbrace{T_{\pi^k(f)}M \times \ldots \times T_{\pi^k(f)}M}_{k-\text{times}}$$

to be the horizontal and vertical distributions at the point f. Consequently,

$$T_f T^kM = \mathcal{H}_f + \mathcal{V}_f.$$

We also remind the reader that we equip T^kM with the Sasaki-type metric

$$\overline{g}_f(u, w) = g_p(u_0, w_0) + \sum_{i=1}^{k} g_p(u_i, w_i),$$

for every $u = (u_0; u_1, \ldots, u_k)$ and $w = (w_0; w_1, \ldots, w_k)$ vectors in $T_f T^kM$ with $\pi^k(f) = p$. Hence, horizontal and vertical components are pairwise orthogonal.

A.1 Horizontal and vertical lifts

Let $f \in T^kM$ with $\pi^k(f) = p$, and let $u = (u_0; u_1, \ldots, u_k) \in T_f T^kM$. For all $i = 0, \ldots, k$, we define the map

$$\pi_i : T_f T^kM \longrightarrow T_pM \quad \text{such that} \quad \pi_i(u) = u_i.$$

Definition A.1.1. Let $w \in T_pM$. The *horizontal lift* of w to a point $f \in T^kM$ is the unique vector $w^h \in T_f T^kM$ such that $\pi_0(w^h) = w$ and $\pi_i(w^h) = 0$ for $i \neq 0$, i.e., $w^h = (w; 0, \ldots, 0)$.

The horizontal lift of a vector field X on M is the unique vector field X^h on T^kM such that $\pi_0(X^h) = X(p) \in T_pM$ and $\pi_i(X^h) = 0$ for $i \neq 0$ for all $f \in T^kM$ with $\pi^k(f) = p$.

Definition A.1.2. Let $w \in T_pM$. The *j-th vertical lift* of w to a point $f \in T^kM$ is the unique vector w_j^v such that $\pi_j(w_j^v) = w$ and $\pi_i(w_j^v) = 0$ for $i \neq j$, i.e., $w_j^v = (0; 0, \ldots, 0, \underbrace{w}_{j\text{-th place}}, 0, \ldots, 0)$.

The j-th vertical lift of a vector field X on M is the unique vector field X_j^v on T^kM such that $\pi_j(X_j^v) = X(p) \in T_pM$ and $\pi_i(X_j^v) = 0$ for $i \neq j$ for all $f \in T^kM$ with $\pi^k(f) = p$.

We observe that the maps $w \mapsto w^h$ and $w \mapsto w_j^v$ for all j are vector isomorphisms between T_pM and $\mathcal{H}_f$ and between T_pM and the j-th copy of T_pM in $\mathcal{V}_f$, respectively. Therefore, the horizontal and vertical component of any vector $z \in T_f T^kM$ can be interpreted as horizontal and vertical lift, i.e., we have

$$z = \overset{h}{z} + \overset{v}{z} = z_0^h + \sum_{i=1}^{k} z_i^v = (z_0; z_1, \ldots, z_k).$$

Therefore, it is sufficient to look at the horizontal and vertical lifts to recover the behaviour of the horizontal and vertical component of a vector on T^kM.

We also note that for every $h \in C^\infty(M)$ and every $w \in T_fT^kM$, we have

$$w^h(\underbrace{h \circ \pi^k}_{\in C^\infty(T^kM)}) = w_0(h) \qquad \text{and} \qquad w_j^v(h \circ \pi^k) = 0, \tag{A.1.1}$$

while for every $H \in C^\infty(T^kM)$, we have

$$w^h(H)(f) = \frac{d}{dt}\Big|_{t=0} H(f_w(t)), \tag{A.1.2}$$

$$w_j^v(H)(f) = \frac{d}{dt}\Big|_{t=0} H(f + tJ(w_j^v)), \tag{A.1.3}$$

where $J : TT^kM \longrightarrow T^kM$ is such that $J(u) = J((u_0; u_1, \ldots, u_k)) = (u_1, \ldots, u_k)$. In fact, we can think of w^h and w_j^v to be the generators of the local 1-parameter groups $\varphi_t(f) = f_w(t)$ and $\widetilde{\varphi}_t(f) = f + tJ(w_j^v) = (v_1, \ldots, v_{j-1}, v_j + tw, v_{j+1}, \ldots, v_k)$ for $f = (v_1, \ldots, v_k)$.

We also point out that (A.1.2) and (A.1.3) correspond to the definition of horizontal and j-th vertical gradient of a smooth function on T^kM as introduced in Section 2.3.

A.2 Lie Brackets

Proposition A.2.1. *Let X, Y be vector fields on M and consider X^h, X_j^v, Y^h, Y_j^v their horizontal and j-th vertical lifts for $j = 1, \ldots, k$. Then, for all $f = (v_1, \ldots, v_k) \in T^kM$ and $\pi^k(f) = p$, we have*

$$\begin{aligned}[X^h, Y^h](f) &= ([X, Y](p))^h - \sum_{i=1}^{k} (R_p(X, Y)v_i)_i^v \\ &= \Big([X, Y](f); -R_p(X, Y)v_1, \ldots, -R_p(X, Y)v_k\Big),\end{aligned} \tag{A.2.4}$$

$$[X^h, Y_j^v](f) = (\nabla_{X(p)Y})_j^v = (0; 0, \ldots, \underbrace{\nabla_{X(p)}Y}_{j\text{-th place}}, 0 \ldots, 0), \tag{A.2.5}$$

$$[X_j^v, Y_i^v](f) = 0 \qquad \forall\, i, j = 1 \ldots, k, \tag{A.2.6}$$

where R_p is the Riemannian curvature tensor on M evaluated at the point p.

Proof. Let $J : T_fT^kM \longrightarrow T_pM \times \ldots \times T_pM$ be such that $J(u_0; u_1, \ldots, u_k) = (u_1, \ldots, u_k)$. Let $\varphi_s, \widetilde{\varphi}_s$ and $\varphi_t, \widetilde{\varphi}_t$ be local 1-parameter groups associated to $\widetilde{X}^h, \widetilde{X}_j^v$ and $\widetilde{Y}^h, \widetilde{Y}_j^v$, respectively, i.e.,

$$\begin{aligned}\varphi_s : \mathbb{R} \times T^kM &\longrightarrow T^kM \\ (s, f) &\mapsto f_{X(p)}(s)\end{aligned} \qquad\qquad \begin{aligned}\widetilde{\varphi}_s : \mathbb{R} \times T^kM &\longrightarrow T^kM \\ (s, f) &\mapsto f + sJ(\widetilde{X}_j^v(f)),\end{aligned}$$

the same for $\varphi_t, \widetilde{\varphi}_t$ associated to $\widetilde{Y}^h, \widetilde{Y}_j^v$, respectively.

Note that $\varphi_\bullet^{-1} = \varphi_{-\bullet}$.

Using [KN63, Proposition 1.9], we have

$$[X^h, Y^h](f) = \lim_{s \to 0} \frac{1}{s}\big(Y^h(f) - (d\varphi_s(Y^h))(f)\big).$$

Then,

$$\begin{aligned}(d\varphi_s(\widetilde{Y}^h))(f) &= \frac{d}{dt}\Big|_{t=0}(\varphi_s\circ\varphi_t\circ\varphi_s^{-1})(f)\\ &= \frac{d}{dt}\Big|_{t=0}\varphi_s((\varphi_{-s}(f))_{Y(p)})(t)\\ &= \frac{d}{dt}\Big|_{t=0}(f_{Y(p)}(t))_{X(q)}(s),\end{aligned}$$

where $q = c(t)$, and c is the geodesic with starting vector $Y(p)$.

We now define the variation $H(s,t) = \big(f_{Y(p)}(t)\big)_{X(q)}(s) = (H_1(s,t),\ldots,H_k(s,t))$ where $H_j(s,t) = ((v_j)_{Y(p)}(t))_{X(q)}(s)$. Then,

$$(d\varphi_s(Y^h))(f) = \frac{\partial}{\partial t}\Big|_{t=0}H(s,t) = \Big(\frac{\partial}{\partial t}\Big|_{t=0}(\pi^k\circ H)(s,t);\frac{D}{dt}\Big|_{t=0}H_1(s,t),\ldots,\frac{D}{dt}\Big|_{t=0}H_k(s,t)\Big).$$

Since $Y^h(f) = (Y(p);0,\ldots,0) = \big(\frac{\partial}{\partial t}\big|_{t=0}(\pi^k\circ H)(0,t);0,\ldots,0\big)$, we have

$$\begin{aligned}[X^h,Y^h](f) &= \Big(\lim_{s\to 0}\frac{1}{s}\Big(\frac{\partial}{\partial t}\Big|_{t=0}(\pi^k\circ H)(0,t) - \frac{\partial}{\partial t}\Big|_{t=0}(\pi^k\circ H)(s,t)\Big);\\ &\qquad -\lim_{s\to 0}\frac{1}{s}\frac{D}{dt}\Big|_{t=0}H_1(s,t),\ldots,-\lim_{s\to 0}\frac{1}{s}\frac{D}{dt}\Big|_{t=0}H_k(s,t)\Big)\\ &= \Big([X,Y](p);-\frac{D}{ds}\Big|_{s=0}\frac{D}{dt}\Big|_{t=0}H_1(s,t),\ldots,-\frac{D}{ds}\Big|_{s=0}\frac{D}{dt}\Big|_{t=0}H_k(s,t)\Big).\end{aligned}$$

Now,

$$\begin{aligned}-\frac{D}{ds}\Big|_{s=0}\frac{D}{dt}\Big|_{t=0}H_i(s,t) &= R\Big(\frac{\partial}{\partial t}\Big|_{t=0}(\pi^k\circ H)(0,t),\frac{\partial}{\partial s}\Big|_{s=0}(\pi^k\circ H)(s,0)\Big)H_i(0,0)\\ &\qquad \underbrace{-\frac{D}{dt}\Big|_{t=0}\frac{D}{ds}\Big|_{s=0}H_i(s,t)}_{-\frac{D}{dt}\big|_{t=0}(v_i)_{Y(p)}(t)=0}\\ &= R_p(Y,X)v_i.\end{aligned}$$

Hence,

$$[X^h,Y^h](f) = ([X,Y](f);-R_p(X,Y)v_1,\ldots,-R_p(X,Y)v_k),$$

which proves (A.2.4).

To prove (A.2.5), we proceed as before using the local 1-parameter groups φ_s and $\widetilde{\varphi}_t$ generating $\widetilde{X}^h$ and $\widetilde{Y}_j^v$, respectively. Then, we have

$$[X^h,Y_j^v](f) = \lim_{s\to 0}\frac{1}{s}\big(Y_j^v(f) - (d\varphi_s)(Y_j^v)(f)\big)$$

and

$$\begin{aligned}(d\varphi_s)(Y_j^v)(f) &= \frac{d}{dt}\Big|_{t=0}(\varphi_s \circ \widetilde{\varphi}_t \circ \varphi_s^{-1})(f)\\ &= \frac{d}{dt}\Big|_{t=0}\varphi_s(\varphi_{-s}(f) + tJ(Y_j^v(\varphi_{-s}(f))))\\ &= Y_j^v(\varphi_{-s}(f))_{X(\pi^k(\varphi_{-s}(f)))}(s).\end{aligned}$$

Therefore,

$$\begin{aligned}[X^h, Y_j^v](f) &= \lim_{s\to 0}\frac{1}{s}\big(Y_j^v(\varphi_0(f)) - Y_j^v(\varphi_{-s}(f))_{X(\pi^k(\varphi_{-s}(f)))}(s)\big)\\ &= (Z_0(f); Z_1(f), \ldots, Z_k(f))\end{aligned}$$

where $Z_i(f) = 0$ for all $i \neq j$ and

$$Z_j(f) = \lim_{s\to 0}\frac{1}{s}(Y(\pi^k(\varphi_0(f))) - Y(\pi^k(\varphi_{-s}(f))_{X(\pi^k(\varphi_{-s}(f)))}(s)) = \nabla_{X(p)}Y.$$

Finally, we consider the local 1-parameter groups $\widetilde{\varphi}_s, \widetilde{\varphi}_t$ to prove (A.2.6). It is easy to see that $\widetilde{\varphi}_s$ and $\widetilde{\varphi}_t$ commute. Therefore, by [KN63, Corollary 1.11] we conclude $[X_i^v, Y_j^v] = 0$ for all $i, j = 1, \ldots, k$. □

A.3 Covariant Derivative

Let $\nabla, \overline{\nabla}$ be the Levi-Civita connection on (M, g) and $(T^kM, \overline{g})$ respectively. We recall that for any V, U, W vector fields on T^kM, Kozul formula holds [Sak96, Equation 1.13, p. 28].

$$\begin{aligned}\overline{g}_f(\overline{\nabla}_V U, W) = \frac{1}{2}\big(&V(\overline{g}_f(U, W)) + U(\overline{g}_f(V, W)) - W(\overline{g}_f(V, U))\\ &- \overline{g}_f(V, [U, W]) - \overline{g}_f(U, [V, W]) - \overline{g}_f(W, [U, V])\big) \quad \text{(A.3.7)}\end{aligned}$$

Using the formula above, we are able to compute the covariant derivatives of horizontal and vertical lifts.

Proposition A.3.1. *Let X, Y be two vector fields on M, X^h, Y^h be their horizontal lifts on T^kM and X_j^v, Y_i^v be their j-th and i-th vertical lifts for $i, j = 1, \ldots, k$. Then, for all $f = (v_1, \ldots, v_k) \in T^kM$ with $\pi^k(f) = p$, we have*

$$\overline{\nabla}_{X_j^v(f)} Y_i^v = 0, \qquad \text{(A.3.8)}$$

$$\overline{\nabla}_{X^h(f)} Y^h = (\nabla_{X(p)} Y)^h - \frac{1}{2} \sum_{i=1}^{k} \big(R_p(X,Y)v_i\big)_i^v$$
$$= \Big(\nabla_{X(p)} Y; -\frac{1}{2} R_p(X,Y)v_1, \dots, -\frac{1}{2} R_p(X,Y)v_k\Big), \tag{A.3.9}$$

$$\overline{\nabla}_{X^h(f)} Y_i^v = \big(\frac{1}{2} R_p(v_i, Y)X\big)^h + (\nabla_{X(p)} Y)_i^v$$
$$= (\frac{1}{2} R_p(v_i, Y)X; 0, \dots, 0, \nabla_{X(p)} Y, 0, \dots, 0), \tag{A.3.10}$$

$$\overline{\nabla}_{Y_i^v(f)} X^h = \frac{1}{2} (R_p(v_i, Y)X)^h = \Big(\frac{1}{2} R_p(v_i, Y)X; 0, \dots, 0\Big). \tag{A.3.11}$$

Proof. Let $W = \overset{h}{W} + \overset{v}{W} = (W_0; W_1, \dots, W_k)$ be a vector field on T^kM. In order to understand the horizontal and vertical components of the vectors in (i)–(iv), we take their inner products against $\overset{h}{W}$ and the l-th component of $\overset{v}{W}$, denoted by $\overset{v}{W}_l$. In what follows, we will make use of the definition of $\overline{g}$, Proposition A.2.1 and equations (A.1.1) and (A.3.7).

We have

$$\overline{g}_f(\overline{\nabla}_{X_j^v} Y_i^v, \overset{v}{W}_l) = \frac{1}{2}\big(\widetilde{X}_j^v(\overline{g}_f(Y_i^v, \overset{v}{W}_l)) + Y_i^v(\overline{g}_f(X_j^v, \overset{v}{W}_l)) - \overset{v}{W}_l(\overline{g}_f(X_j^v, Y_i^v))$$
$$- \overline{g}_f(X_i^v, [Y_j^v, \overset{v}{W}_l]) - \overline{g}(Y_j^v, [X_j^v, \overset{v}{W}_l]) - \overline{g}(\overset{v}{W}_l, [Y_i^v, X_j^v])\big) = 0,$$

and

$$\overline{g}_f(\overline{\nabla}_{X_j^v} Y_i^v, \overset{h}{W}) = \frac{1}{2}\big(X_j^v(\overline{g}_f(Y_i^v, \overset{h}{W})) + Y_i^v(\overline{g}_f(X_j^v, \overset{h}{W})) - \overset{h}{W}(\overline{g}_f(X_j^v, Y_i^v))$$
$$- \overline{g}_f(X_i^v, [Y_j^v, \overset{h}{W}]) - \overline{g}(Y_j^v, [X_j^v, \overset{h}{W}]) - \overline{g}(\overset{h}{W}, [Y_i^v, X_j^v])\big)$$
$$= \frac{1}{2}\big(- W_0(g_p(X_j, Y_i)) + g_p(X_j, \nabla_W Y_i) + g_p(Y_i, \nabla_W X_j)\big).$$

If $i \neq j$, then each term of the above formula is zero due to the definition of $\overline{g}$. If $i = j$, then the above sum is zero due to the Riemannian property of ∇. This proves (A.3.8).

Now, we prove (A.3.9). As before,

$$\overline{g}_f(\overline{\nabla}_{X^h} Y^h, \overset{h}{W}) = \frac{1}{2}\big(X^h(\overline{g}_f(Y^h, \overset{h}{W})) + Y^h(\overline{g}_f(X^h, \overset{h}{W})) - oshW(\overline{g}_f(X^h, Y^h))$$
$$- \overline{g}_f(X^h, [Y^h, \overset{h}{W}]) - \overline{g}_f(Y^h, [X^h, \overset{h}{W}]) - \overline{g}_f(\overset{h}{W}, [Y^h, X^h])\big)$$

$$=\frac{1}{2}\big(X(g_p(Y,W_0)) + Y(g_p(X,W_0)) - W_0(g_p(X,Y))$$
$$- \overline{g}_p(X,[Y,W_0]) - g_p(Y,[X,W_0]) - g_p(W_0,[Y,X])\big)$$
$$=g_p(\nabla_X Y, W_0),$$

and

$$\overline{g}_f(\overline{\nabla}_{X^h}Y^h,\overset{v}{W}_l) = \frac{1}{2}\big(X^h(\overline{g}_f(Y^h,\overset{v}{W}_l)) + Y^h(\overline{g}_f(X^h,\overset{v}{W}_l)) - \overset{v}{W}_l(\overline{g}_f(X^h,Y^h))$$
$$- \overline{g}_f(X^h,[Y^h,\overset{v}{W}_l]) - \overline{g}_f(Y^h,[X^h,\overset{v}{W}_l]) - \overline{g}_f(\overset{v}{W}_l,[Y^h,X^h])\big)$$
$$= -\frac{1}{2}\overline{g}_f(\overset{v}{W}_l,[Y^h,X^h])$$
$$= \frac{1}{2}g_p(W_l, R(Y,X)v_l).$$

Hence, the l-th vertical component of $\overline{\nabla}_{X^h(f)}Y^h$ is $\frac{1}{2}R(Y(p),X(p))v_l$ for all $l = 1,\dots k$, and so (A.3.9) is proved.

Now, we look at (A.3.10). Again,

$$\overline{g}_f(\overline{\nabla}_{X^h}Y_i^v,\overset{h}{W}) = \frac{1}{2}\big(X^h(\overline{g}_f(Y_i^v,\overset{h}{W})) + Y_i^v(\overline{g}_f(X^h,\overset{h}{W})) - \overset{h}{W}(\overline{g}_f(X^h,Y_i^v))$$
$$- \overline{g}_f(X^h,[Y_i^v,\overset{h}{W}]) - \overline{g}_f(Y_i^v,[X^h,\overset{h}{W}]) - \overline{g}_f(\overset{h}{W},[Y_i^v,X^h])\big)$$
$$= -\frac{1}{2}g_f(Y_i^v,[X^h,\overset{h}{W}])$$
$$= \frac{1}{2}g_p(Y_i, R(X,W_0)v_i)$$
$$= \frac{1}{2}g_p(R(v_i,Y_i)X, W_0),$$

and

$$\overline{g}_f(\overline{\nabla}_{X^h}Y_i^v,\overset{v}{W}_l) = \frac{1}{2}\big(X^h(\overline{g}_f(Y_i^v,\overset{v}{W}_l)) + Y_i^v(\overline{g}_f(X^h,\overset{v}{W}_l)) - \overset{v}{W}_l(\overline{g}_f(X^h,Y_i^v))$$
$$- \overline{g}_f(X^h,[Y_i^v,\overset{v}{W}_l]) - \overline{g}_f(Y_i^v,[X^h,\overline{v}W_l]) - \overline{g}_f(\overset{v}{W}_l,[Y_i^v,X^h])\big)$$
$$= \frac{1}{2}\big(X^h(\overline{g}_f(Y_i^v,\overset{v}{W}_l)) - \overline{g}_f(Y_i^v,[X^h,\overline{v}W_l]) - \overline{g}_f(\overset{v}{W}_l,[Y_i^v,X^h])\big)$$
$$= \frac{1}{2}\delta_{il}\big(X(g_p(Y,W_l) - g_p(Y,\nabla_X W_l) + g_p(W_l,\nabla_X Y)\big)$$
$$= \delta_{il}g_p(\nabla_X Y_i, W_l),$$

where the last equality is due to the Riemannian property of ∇. Therefore, (A.3.10) is proved since the only non-zero component is the i-th ($l = i$).

Finally, we look at (A.3.11). Using (A.2.5) and (A.3.10), we obtain.

$$\begin{aligned}\overline{\nabla}_{Y_i^v(f)}X^h &= \overline{\nabla}_{X^h(f)}Y_i^v - [X^h, Y_i^v](f)\\ &= \Big(\frac{1}{2}R_p(v_i, Y)X\Big)^h + (\nabla_{X(p)}Y)_i^v - (\nabla_{X(p)}Y)_i^v\\ &= \Big(\frac{1}{2}R_p(v_i, Y)X\Big)^h,\end{aligned}$$

which concludes the proof. □

We analyse the Levi-Civita connection of the horizontal and i-th vertical lift of a semi-basic vector field. We remind the reader that a semi-basic vector field is a map $F : T^kM \longrightarrow TM$ such that $F(f) \in T_pM$ if $\pi^k(f) = p$ (see also Section 2.3).

Definition A.3.2. Let $F : T^kM \longrightarrow TM$ be a semi-basic vector field. The *horizontal* and *i-th vertical lift of F* are the maps

$$F^h : T^kM \longrightarrow TT^kM, \qquad F^h(f) = \big(F(f)\big)^h$$

and

$$F_i^v : T^kM \longrightarrow TT^kM, \qquad F_i^v(f) = \big(F(f)\big)_i^v$$

We now consider a very special semi-basic vector field. We define $P_i : T^kM \longrightarrow TM$ such that $P_i(f) = v_i$ for every $f = (v_1, \ldots, v_k)$, i.e., P_i is the projection of f on the i-th component of T^kM, and we consider $G : TM \longrightarrow TM$, an endomorphism on TM. Then, $H = (G \circ P_i)$ is a semi-basic vector field on T^kM. We have the following result.

Proposition A.3.3. *Let H be the semi-basic vector field defined above. Let X^h, X_j^v be the horizontal and j-th vertical lift of $X \in \mathfrak{X}(M)$ and let $\overline{\nabla}$ be the Levi-Civita connection on T^kM. Then, for all $f = (v_1, \ldots, v_k)$, we have*

$$\overline{\nabla}_{X^h(f)}H^h = \overline{\nabla}_{X^h(f)}(H \circ V)^h, \tag{A.3.12}$$

$$\overline{\nabla}_{X^h(f)}H_i^v = \overline{\nabla}_{X^h(f)}(H \circ V)_i^v, \tag{A.3.13}$$

$$\overline{\nabla}_{X_j^v(f)}H_i^v = \big(H(J(X_j^v(f)))\big)_i^v, \tag{A.3.14}$$

$$\overline{\nabla}_{X_j^v(f)}H^h = \big(H(J(X_j^v(f))) + \frac{1}{2}R_p(v_j, X)H(f)\big)^h, \tag{A.3.15}$$

where $V = (V_1, \ldots, V_k)$ is a realization of f, i.e., $f = V(f) = (V_1(p), \ldots, V_k(p))$, and $J : T_fT^kM \longrightarrow T_pM \times \ldots \times T_pM$ is such that $J(u_0; u_1, \ldots, u_k) = (u_1, \ldots, u_k)$.

Proof. We consider the curve φ_t generating X^h, i.e., $\varphi_t(f) = f_{X(p)}(t)$. Then,

$$\begin{aligned}
\overline{\nabla}_{X^h(f)} H^h &= \frac{d}{ds}\Big|_{s=0} H(f_{X(p)}(s))^h \\
&= \frac{d}{ds}\Big|_{s=0} H(V(f_{X(p)}(s)))^h \\
&= \frac{d}{ds}\Big|_{s=0} ((H \circ V) \circ \varphi_s(f)))^h \\
&= \overline{\nabla}_{X^h(f)} (H \circ V)^h,
\end{aligned}$$

which proves (A.3.12).

Equation (A.3.13) follows from the same considerations.

To prove (A.3.14) and (A.3.15), we first observe that for $v_i = \sum_{\alpha=1}^n dx_\alpha(v_i) \frac{\partial}{\partial x_\alpha}\big|_p$, we have

$$H(f) = (G \circ P_i)(f) = G(v_i) = \sum_{\alpha=1}^n a_\alpha(p) G\Big(\frac{\partial}{\partial x_\alpha}\Big|_p\Big) = \sum_{\alpha=1}^n dx_\alpha(v_i) G\Big(\frac{\partial}{\partial x_\alpha}\Big|_p\Big).$$

Therefore,

$$H_i^v(f) = \sum_{\alpha=1}^n dx_\alpha(v_i) \Big((G \circ \frac{\partial}{\partial x_\alpha} \circ \pi^k(f)\Big)_i^v$$

and

$$H^h(f) = \sum_{\alpha=1}^n dx_\alpha(v_i) \Big((G \circ \frac{\partial}{\partial x_\alpha} \circ \pi^k)(f)\Big)^h.$$

Hence,

$$\begin{aligned}
\overline{\nabla}_{X_j^v(f)} H_i^v &= \sum_{\alpha=1}^n \overline{\nabla}_{X_j^v(f)} \big(dx_\alpha (G \circ \frac{\partial}{\partial x_\alpha} \circ \pi^k)_i^v\big) \\
&= \sum_{\alpha=1}^n \big(X_j^v(dx_\alpha)\big)(f) \Big((G \circ \frac{\partial}{\partial x_\alpha} \circ \pi^k)(f)\Big)_i^v + dx_\alpha(v_i) \underbrace{\overline{\nabla}_{X_j^v(f)} (G \circ \frac{\partial}{\partial x_\alpha} \circ \pi^k)_i^v}_{=0 \text{ by (A.2.6)}}
\end{aligned}$$

Since $X_j^v(f)$ is generated by $\varphi_t(f) = f + tJ(X_j^v(f))$, we have

$$\begin{aligned}
X_j^v(dx_\alpha)(f) &= \big(X_j^v(f)\big)(dx_\alpha(P_i(f))) \\
&= \frac{d}{dt}\Big|_{t=0} dx_\alpha\big(P_i(f + tJ(X_j^v(f)))\big) \\
&= \frac{d}{dt}\Big|_{t=0} dx_\alpha(P_i(f)) + t dx_\alpha(P_i(J(X_j^v(f)))) \\
&= dx_\alpha\big(P_i(J(X_j^v(f)))\big),
\end{aligned} \tag{A.3.16}$$

where the third and fourth equality are due to the fact that dx_α and P_i are linear.

Hence, we conclude

$$\overline{\nabla}_{X_j^v(f)} H_i^v = \sum_{\alpha=1}^{n} dx_\alpha\big(P_i(J(X_j^v(f)))\big)\Big((G \circ \frac{\partial}{\partial x_\alpha} \circ \pi^k)(f)\Big)_i^v = \big(H(J(X_j^v(f)))\big)_i^v,$$

which proves (A.3.8).

We proceed in the same way to prove (A.3.11).

$$\begin{aligned}\overline{\nabla}_{X_j^v(f)} H^h &= \sum_{\alpha=1}^{n} \overline{\nabla}_{X_j^v(f)}\big(dx_\alpha(G \circ \frac{\partial}{\partial x_\alpha} \circ \pi^k)^h\big)\\ &= \sum_{\alpha=1}^{n} \big(X_j^v(dx_\alpha)\big)(f)\Big((G \circ \frac{\partial}{\partial x_\alpha} \circ \pi^k)(f)\Big)^h + dx_\alpha(v_i)\overline{\nabla}_{X_j^v(f)}(G \circ \frac{\partial}{\partial x_\alpha} \circ \pi^k)^h\\ &= H(J(X_j^v(f)))^h + \Big(\frac{1}{2}R_p(v_j, X)H(f)\Big)^h,\end{aligned}$$

where the last equality is due to (A.3.16) and (A.3.11). □

A.4 The Riemannian curvature tensor

Let $\overline{R}$ and R be the Riemannian curvature tensor on T^kM and M, respectively, and let $\overline{R}_f, R_p$ be their evaluations at the point f and p, respectively. For any $V, W, U \in \mathfrak{X}(T^kM)$, we have

$$\overline{R}_f(V, W)U = \overline{\nabla}_{V(f)}\overline{\nabla}_{W(f)}U - \overline{\nabla}_{W(f)}\overline{\nabla}_{V(f)}U - \overline{\nabla}_{[V,W](f)}U. \tag{A.4.17}$$

Proposition A.4.1. *Let $X, Y \in \mathfrak{X}(M)$ and consider their horizontal and vertical lifts, denoted as usual. Let $f = (v_1, \ldots, v_k) \in T^kM$ with $\pi^k(f) = p$. Then,*

$$\overline{R}_f(X_i^v, Y_j^v)Z_l^v = 0 \qquad \forall\, i, j, l = 1, \ldots, k, \tag{A.4.18}$$

$$\overline{R}_f(X^h, Y_j^v)Z_l^v = \Big(-\frac{1}{4}R_p(v_j, Z)(R_p(v_l, Z)X) - \frac{1}{2}\delta_{jl}R_p(Y, Z)X\Big)^h, \tag{A.4.19}$$

$$\begin{aligned}\overline{R}_f(X_i^v, Y_j^v)Z^h = \Big(\delta_{ij}R_p(X, Y)Z &+ \frac{1}{4}R_p(v_i, X)(R_p(v_j, Y)Z)\\ &- \frac{1}{4}R_p(v_j, Y)(R_p(v_i, X)Z)\Big)^h,\end{aligned} \tag{A.4.20}$$

$$\begin{aligned}\overline{R}_f(X^h, Y_j^v)Z^h = \frac{1}{2}\Big(\nabla_{X(p)}R(v_j, Y)Z\Big)^h &+ \frac{1}{2}\Big(R_p(X, Z)Y\Big)_j^v\\ &- \frac{1}{4}\sum_{i=1}^{k}\Big(R_p(X, R_p(v_j, Y)Z)v_i\Big)_i^v,\end{aligned} \tag{A.4.21}$$

$$\overline{R}_f(X^h,Y^h)Z_l^v = \frac{1}{2}\Big(\nabla_{X(p)}R(v_l,Z)Y) - \nabla_{Y(p)}R(v_l,Z)X\Big)^h + \Big(R_p(X,Y)Z\Big)_l^v$$
$$\sum_{i=1}^k \frac{1}{4}\Big(R_p(Y,R(v_l,Z)X)v_i - R_p(X,R(v_l,Z)Y)v_i\Big)_i^v, \quad \text{(A.4.22)}$$

$$\overline{R}(X^h,Y^h)Z^h = \Big(R_p(X,Y)Z + \frac{1}{4}\sum_{i=1}^k R_p(v_i,R(X,Z)v_i)Y$$
$$- R_p(v_i,R(Y,Z)v_i)X + 2R_p(v_i,R(X,Y)v_i)Z\Big)^h$$
$$-\frac{1}{2}\sum_{i=1}^k\Big(\nabla_{Z(p)}R(X,Y)v_i\Big)_i^v. \quad \text{(A.4.23)}$$

Proof. We will make use of (A.4.17) to prove the proposition.

Equation (A.4.18) is an easy consequence of (A.4.17), (A.2.6), and (A.3.8).

Before proving the remaining equations, we observe that the map $F(f) = F(v_1,\dots,v_k) = R(v_i,X(p))Y(p)$ is a semi-basic vector field for any $X,Y \in \mathfrak{X}(M)$ and any index i. In particular, Lemma A.3.3 applies to such a map.

We now prove (A.4.19). Using (A.2.5), Proposition A.3.1 and Lemma A.3.3, we have

$$\begin{aligned}
\overline{R}_f(X^h,Y_j^v)Z_l^v &= \overline{\nabla}_{X^h(f)}\overline{\nabla}_{Y_j^v(f)}Z_l^v - \overline{\nabla}_{Y_j^v(f)}\overline{\nabla}_{X^h(f)}Z_l^v - \overline{\nabla}_{[X^h,Y_j^v](f)}Z_l^v\\
&= -\overline{\nabla}_{Y_j^v(f)}\overline{\nabla}_{X^h(f)}Z_l^v\\
&= -\frac{1}{2}\overline{\nabla}_{Y_j^v(f)}(R(v_l,Z)X)^h - \underbrace{\overline{\nabla}_{Y_j^v(f)}(\nabla_{X(p)}Z)_l^v}_{=0}\\
&= -\frac{1}{2}\Big(\delta_{jl}R_p(Y,Z)X - \frac{1}{2}R_p(v_j,Y)(R_p(v_l,Z)X\Big)^h.
\end{aligned}$$

To prove (A.4.20), we use the Bianchi Identity and (A.4.19). Then,

$$\begin{aligned}
\overline{R}_f(X_i^v,Y_j^v)Z^h)(f) &= -\overline{R}_f(Z^h,X_i^v)Y_j^v - \overline{R}_f(Y_j^v,Z^h)X_i^v\\
&= -\overline{R}_f(Z^h,X_i^v)Y_j^v + \overline{R}_f(Z^h,Y_j^v)X_i^v\\
&= \Big(\delta_{ij}R_p(X,Y)Z + \frac{1}{4}R_p(v_i,X)(R_p(v_j,Y)Z)\\
&- \frac{1}{4}R_p(v_j,Y)(R_p(v_i,X)Z)\Big)^h.
\end{aligned}$$

Now,

$$\overline{R}_f(X^h,Y_j^v)Z^h = \overline{\nabla}_{X^h(f)}\overline{\nabla}_{Y_j^v(f)}Z^h - \overline{\nabla}_{Y_j^v(f)}\overline{\nabla}_{X^h(f)}Z^h - \overline{\nabla}_{[X^h,Y_j^v](f)}Z^h$$

$$
\begin{aligned}
=&\frac{1}{2}\overline{\nabla}_{X^h(f)}(R(v_j,Y)X)^h - \overline{\nabla}_{Y_j^v(f)}(\nabla_{X(p)}Y)^h + \sum_{i=1}^k \frac{1}{2}\overline{\nabla}_{Y_j^v(f)}(R(X,Z)v_i)_i^v \\
&\qquad - \overline{\nabla}_{(\nabla_{X(p)}V)_j^v} Z^h \\
=&\frac{1}{2}(\nabla_{X(p)}R(v_j,Y)Z)^h, -\frac{1}{4}\sum_{i=1}^k \Big(R_p(X,R_p(v_j,Y)Z)v_i\Big)_i^v - \frac{1}{2}(R_p(v_j,Y)(\nabla_{X(p)}Z))^h \\
&\qquad + \frac{1}{2}(R_p(v_j,\nabla_{X(p)}Y)Z)^h + \frac{1}{2}(R_p(X,Z)Y)_i^v \\
=&\frac{1}{2}\Big(\nabla_{X(p)}R(v_j,Y)Z\Big)^h + \frac{1}{2}\Big(R_p(X,Z)Y\Big)_j^v - \frac{1}{4}\sum_{i=1}^k \Big(R_p(X,R_p(v_j,Y)Z)v_i\Big)_i^v,
\end{aligned}
$$

which proves (A.4.21).

We again use the Bianchi Identity to prove (A.4.22).

$$
\begin{aligned}
\overline{R}_f(X^h,Y^h)Z_l^v &= -\overline{R}_f(Z_l^v,X^h)Y^h - \overline{R}_f(Y^h,Z_l^v)X^h \\
&= \overline{R}_f(X^h,X_l^v)Y^h - \overline{R}_f(Y^h,Z_l^v)X^h \\
&= \frac{1}{2}\Big(\nabla_{X(p)}R(v_l,Z)Y - \nabla_{Y(p)}R(v_l,Z)X\Big)^h + (R_p(X,Y)Z)_l^v \\
&\qquad + \frac{1}{4}\sum_{i=1}^k \Big(R_p(Y,R_p(v_l,Z)Y)v_i - R_p(X,R_p(v_l,Z)Y)v_i\Big)_i^v.
\end{aligned}
$$

Finally, we prove (A.4.23).

$$
\begin{aligned}
&\overline{R}_f(X^h,Y^h)Z^h = \overline{\nabla}_{X^h(f)}\overline{\nabla}_{Y^h(f)}Z^h - \overline{\nabla}_{Y^h(f)}\overline{\nabla}_{X^h(f)}Z^h - \overline{\nabla}_{[X^h,Y^h](f)}Z^h \\
&= \overline{\nabla}_{X^h(f)}\Big((\nabla_{Y(p)}Z)^h - \frac{1}{2}\sum_{i=1}^k (R_p(Y,Z)v_i)_i^v\Big) - \overline{\nabla}_{Y^h(f)}\Big((\nabla_{X(p)}Z)^h - \frac{1}{2}\sum_{i=1}^k (R_p(X,Z)v_i)_i^v\Big) \\
&\qquad - \overline{\nabla}_{([X,Y](p))^h}\widetilde{Z}^h + \sum_{i=1}^k \overline{\nabla}_{(R_p(X,Y)v_i)_i^v} Z^h \\
&= (\nabla_{X(p)}\nabla_{Y(p)}Z)^h - \frac{1}{2}\sum_{i=1}^k (R_p(X,\nabla_{Y(p)}Z)v_i)_i^v - (\nabla_{Y(p)}\nabla_{X(p)}Z)^h \\
&+ \frac{1}{2}\sum_{i=1}^k (R_p(Y,\nabla_{X(p)}Z)v_i)_i^v - \frac{1}{4}\sum_{i=1}^k \Big((R_p(v_i,R_p(Y,Z)v_i)X)^h + 2(\nabla_{X(p)}R(Y,Z)v_i)_i^v\Big) \\
&+ \frac{1}{4}\sum_{i=1}^k \Big((R_p(v_i,R_p(X,Z)v_i)Y)^h + 2(\nabla_{Y(p)}R(X,Z)v_i)_i^v\Big) - (\nabla_{[X,Y](p)}Z)^h \\
&+ \frac{1}{2}\sum_{i=1}^k (R_p([X,Y],Z)v_i)_i^v + \frac{1}{2}\sum_{i=1}^k (R_p(v_i,R_p(X,Y)v_i)Z)^h
\end{aligned}
$$

$$= \Big(R_p(X,Y)Z + \frac{1}{4}\sum_{i=1}^{k}\big[R_p(v_i, R_p(X,Z)v_i)Y - R_p(v_i, R_p(Y,Z)v_i)X$$
$$+ 2R_p(v_i, R_p(X,Y)v_i)Z\big]\Big)^h - \frac{1}{2}\sum_{i=1}^{k}\Big(\nabla_{Z(p)}R(X,Y)v_i\Big)_i^v.$$

□

A.5 Curvature

Let $\overline{K}, \overline{\mathrm{Ric}}, \overline{S}$ be the sectional curvature, the Ricci curvature and the scalar curvature on T^kM, respectively. For any $v, w \in T_f T^kM$ we have

$$\overline{K}(\{v,w\}) = \frac{\overline{g}_f(\overline{R}(v,w)w, v)}{\|v\|^2\|w\|^2 - \overline{g}_f(v,w)}, \tag{A.5.24}$$

$$\overline{\mathrm{Ric}}(v,v) = \sum_{i=1}^{n(k+1)} \overline{K}(\{v, e_i\}), \tag{A.5.25}$$

$$\overline{S} = \sum_{i=1}^{n(k+1)} \overline{\mathrm{Ric}}(e_i, e_i), \tag{A.5.26}$$

where $\{v,w\}$ is the plane spanned by v and w, and where $e_1, \ldots, e_{n(k+1)}$ is an orthonormal basis of $T_f T^kM$.

Proposition A.5.1. *Let $X, Y \in \mathfrak{X}(M)$ be two unitary vector fields and consider their horizontal and vertical lifts as usually denoted. The sectional curvature $\overline{K}$ of T^kM with respect to the Sasaki-type metric satisfies the followings.*

$$\overline{K}(\{X_i^v(f), Y_j^v(f)\}) = 0 \qquad \forall\, i,j = 1,\ldots,k, \tag{A.5.27}$$

$$\overline{K}(\{X^h(f), Y_j^v(f)\}) = \frac{1}{4}\|R(v_i, Y(p))X(p)\|^2, \tag{A.5.28}$$

$$\overline{K}(\{X^h(f), Y^h(f)\}) = K(\{X(p), Y(p)\}) - \frac{3}{4}\sum_{i=1}^{k}\|R_p(X,Y)v_i\|^2. \tag{A.5.29}$$

Proof. To prove the proposition we plug in (A.5.24) X^h, Y^h, X_i^v, Y_j^v.

Equation (A.5.27) holds due to (A.4.18).

Since, X, Y are unitary, then $\|X^h\|^2 = \|Y_j^v\|^2 = \|Y^h\|^2 = 1$. Using (A.4.19), we have

$$
\begin{aligned}
\overline{K}(\{X^h(f), Y_j^v(f)\}) &= \overline{g}_f(\overline{R}(X^h, Y_j^v)Y_j^v, X^h) \\
&= -\frac{1}{4}g_p(R(v_j, Y)(R(v_j, Y)X), X) - \frac{1}{2}g_p(R(Y, Y)X, X) \\
&= \frac{1}{4}\|R_p(v_i, Y)X\|^2,
\end{aligned}
$$

which proves (A.5.28).

Finally, using (A.4.22), we obtain

$$
\begin{aligned}
\overline{K}(\{X^h(f), Y^h(f)\}) &= \overline{g}_f(\overline{R}(X^h, Y^h)Y^h, X^h) \\
&= g_p(R(X, Y)Y, X) \\
&+ \frac{1}{4}\sum_{i=1}^{k}\Big(g_p(R(v_i, R(X, Y)v_i)Y, X) - g_p(v_i, R(Y, Y)v_i)X, X) \\
&+ 2g(R_p(v_i, R(X, Y)v_i)Y, X)\Big) \\
&= K(\{X(p), Y(p)\} - \frac{3}{4}\sum_{i=1}^{k}\|R_p(X, Y)v_i\|^2,
\end{aligned}
$$

which proves (A.5.29). □

From the above proposition, we derive some relations between the curvature of T^kM and M.

Proposition A.5.2. *Let (M, g) be a Riemannian manifold and let $(T^kM, \overline{g})$ be the bundle of k-frames on M equipped with a Sasaki-type metric. Then, T^kM is flat if and only if M is.*

Proof. Statement (i) is a direct consequence of Proposition A.5.1 or of Proposition (A.4.1). □

Proposition A.5.3. *Let (M, g) and $(T^kM, \overline{g})$ be as in the above proposition. Then, the following statements are true.*

(i) If T^kM has bounded sectional curvature, then it is flat,

(ii) If T^kM has bounded sectional curvature, then M is flat.

Proof. By contradiction, we assume that T^kM is not flat. Then, M is not flat by Proposition A.5.2. Hence, there exist a point $p \in M$ and a pair of orthonormal vectors $u_1, u_2 \in T_pM$ such that $R_p(u_1, u_2)V \neq 0$ for some V vector field on M. Then, $\overline{K}(\{u_1^h, u_2^h\}) = K(\{u_1, u_2\}) - \frac{3}{4}\sum_{i=1}^{k} \|R_p(u_1, u_2)v_i\|^2$ for u_1^h, u_2^h the horizontal lifts of u_1, u_2 at the point $f = (v_1, \ldots, v_k)$ such that $\pi^k(f) = p$. Since the set of v_i satisfying this condition is unbounded, so is the set of f which has v_i has one of the components. In a similar way, we show that $\overline{K}(\{u_1^h, u_2^h\})$ is unbounded from above using the (A.5.28). This proves statement (i).

Statement (ii) is a consequence of statement (i) and of Proposition A.5.2. □

We now look at the Ricci and scalar curvature of T^kM. We fist observe that given $e_1, \ldots, e_n$ an orthonormal basis for T_pM, then $e_1^h, \ldots, e_n^h$ is an orthonormal basis for $\mathcal{H}_f$ and $(e_1)_i^v, \ldots (e_n)_i^v$ for $i = 1, \ldots, k$ is an orthonormal basis for $\mathcal{V}_f$.

Corollary A.5.4. *Let $e_1^h, \ldots, e_k^h$ be an orthonormal basis for $\mathcal{H}_f$, $(e_j)_i^v$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n$ be an orthonormal basis for $\mathcal{V}_f$ as described above. Let u^h, u_l^v be the horizontal and l-th vertical lifts of $u \in T_pM$ at the point $f = (v_1, \ldots, v_k)$ with $\pi^k(f) = p$. Then,*

$$\overline{\mathrm{Ric}}(u^h, u^h) = \mathrm{Ric}(u, u) - \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{n} \|R_p(e_j, v_i)u\|^2, \tag{A.5.30}$$

$$\overline{\mathrm{Ric}}(u_l^v, u_l^v) = \frac{1}{4}\sum_{i=1}^{n} \|R(v_l, u)e_i\|^2. \tag{A.5.31}$$

Proof. We apply Proposition A.5.1. Then,

$$\begin{aligned}
\overline{\mathrm{Ric}}(u^h, u^h) &= \sum_{j=1}^{n} \overline{K}(\{u^h, e_j^h\}) + \sum_{j=1}^{n}\sum_{i=1}^{k} \overline{K}(\{u^h, (e_j)_i^v\}) \\
&= \sum_{j=1}^{k} \Big(K(\{u, e_j\}) - \frac{3}{4}\sum_{l=1}^{k} \|R_p(u, e_j)v_l\|^2\Big) + \frac{1}{4}\sum_{i=1}^{k}\sum_{j=1}^{n} \|R_p(v_i, e_j)u\|^2 \\
&= \mathrm{Ric}(u, u) + \sum_{j=1}^{n}\sum_{i=1}^{k} \frac{1}{4}\|R_p(v_i, e_j)u\|^2 - \frac{3}{4}\|R_p(u, e_j)v_i\|^2.
\end{aligned}$$

We now observe that for $v_i = \sum_{\alpha=1}^{n} v_i^\alpha e_\alpha$ and $u = \sum_{l=1}^{n} u_l e_l$, we have

$$
\begin{aligned}
\sum_{j+1}^{n} \|R_p(u,e_j)v_i\|^2 &= \sum_j \|\sum_{l,\alpha} u_l v_i^\alpha R_p(e_l,e_j)e_\alpha\|^2 \\
&= \sum_{j,q,l,\alpha,\beta,s} u_l u_q v_i^\alpha v_i^\beta g_p(R(e_l,e_j)e_\alpha,e_s) g_p(R(e_q,e_j)e_\beta,e_s) \\
&= \sum_{j,q,l,\alpha,\beta,s} u_l u_q v_i^\alpha v_i^\beta g_p(R(e_s,e_\alpha)e_l,e_j) g_p(R(e_\beta,e_s)e_q,e_j) \\
&= \sum_{j,q,l,\alpha,\beta} u_l u_q v_i^\alpha v_i^\beta g_p(R(e_j,e_\alpha)e_l, R(e_j,e_\beta)e_q) \\
&= \sum_j \|R_p(e_j,v_i)u\|^2.
\end{aligned}
$$

Therefore,

$$
\overline{\mathrm{Ric}}(u^h,u^h) = \mathrm{Ric}(u,u) - \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{n} \|R_p(e_j,v_i)u\|^2,
$$

which proves (A.5.30).

Equation (A.5.31) is an easy consequence of (A.5.28). □

Corollary A.5.5. *Let $e_1^h,\dots,e_k^h$ and let $(e_j)_i^v$ for $i=1,\dots,k$ and $j=1,\dots,n$ be as above. Let $f=(v_1,\dots,v_k)$, then*

$$
\overline{S} = S - \frac{1}{4}\sum_{i,j=1}^{n}\sum_{l=1}^{k} \|R_p(e_j,v_l)e_i\|^2 \tag{A.5.32}
$$

Proof. This a consequence of the definition of scalar curvature and of Corollary A.5.4. □

Proposition A.5.6. *Let (M,g) be a Riemannian manifold and let T^kM equipped with the Sasaki-type metric $\overline{g}$. Then, $(T^kM,\overline{g})$ has constant scalar curvature if and only if M is flat.*

Proof. This is a direct consequence of Corollary A.5.5. □

Corollary A.5.7. *Let M, T^kM as above. Then, T^kM has constant scalar curvature with respect to the metric $\overline{g}$ if and only if the scalar curvature is zero.*

Proof. This corollary is a consequence of Propositions A.5.6 and A.5.2. □

Corollary A.5.8. *Let M, T^kM as above. Then, T^kM is Einstein with respect to the metric $\overline{g}$ if and only if it is flat.*

Proof. Again, this is a direct consequence of Proposition A.5.6. □

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