A Singular Theta Lift in SU(1,1)

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Abstract

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Following the work of Bruinier and Funke in the orthogonal setting, we consider a regularised theta lift from weight 0 harmonic weak Maass forms on non-compact quotients of $\mathrm{SU}(1,1)$ to meromorphic modular forms of weight 2, and realise the result of the lift as a generating series of modular traces of those Maass forms on CM points. We also lift the non-holomorphic Eisenstein series of weight 0 and realise the derivative of a suitably normalised weight 2 Eisenstein series as the lift of the logarithm of the modular Δ function.

Declaration

The work in this thesis is based on research carried out in the Pure Mathematics Group at the Department of Mathematical Sciences, Durham University. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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Contents

Introduction

Hirzebruch and Zagier in [HZ76] show that the intersection numbers of certain algebraic cycles in Hilbert modular surfaces occur as Fourier coefficients of classical modular forms of weight 2. This result gave rise to widespread interest in geodesic cycles in other locally symmetric spaces and their relationship to modular forms, e.g. [Shi75], [Oda78].

Throughout the 1980's Kudla and Millson (see e.g., [KM86] and [KM87]) built a framework to vastly generalise these results using the theta correspondence. They consider the dual pairs $(\mathrm{O}(p,q), \mathrm{Sp}(n,\mathbb{R}))$ and $(\mathrm{U}(p,q), \mathrm{U}(n,n))$ to realize generating series of certain "special" cycles for the orthogonal group $\mathrm{O}(p,q)$ and the unitary group $\mathrm{U}(p,q)$ as holomorphic Siegel modular forms and as Hermitian modular forms, respectively. Their main tool is the explicit construction of certain theta series with values in the closed differential forms on the locally symmetric spaces attached to those orthogonal and unitary groups of signature (p,q). Furthermore, these forms give rise to Poincaré dual forms for the special cycles in question.

However, their result was subject to certain restrictions, e.g. it is assumed that the orthogonal or unitary locally symmetric space is compact. In the orthogonal case, several papers considered the non-compact situation. In [Fun02], the non-compact theta lift for $\mathrm{SO}(p,2)$ was considered. Together with the recent work [FM14] this properly recovers the orginal Hirzebruch-Zagier result (which is the $\mathbb{Q}$-rank 1 case for $\mathrm{SO}(2,2)$). Also, in a long running collaboration, Funke and Millson have been working on various aspects of the orthogonal case in this context, see [FM02], [FM06], [FM11], [FM13]. Further generalisations have been made in the orthogonal case in [BF04], [BF10] relating it to the Borcherds lifts, see [Bor95], [Bor98].

In [BF06], Bruinier and Funke considered in detail the non-compact case for

SO(1, 2), when the locally symmetric spaces in question are modular curves and the special cycles are the classical CM points on the upper half plane. In particular, they extended the Kudla-Millson lift beyond its original cohomological setting to allow for more general input, in particular for functions with exponential growth at the cusps such as the classical j-invariant.

The goal of this thesis is to study the analogous situation for the dual pair $(G, G') = (\mathrm{SU}(1,1), \mathrm{SL}_2(\mathbb{R}))$, where the relevant symmetric space for the unitary group is the upper half plane $\mathbb{H}$. (Note that we can identify the second factor with $\mathrm{SU}(1,1)$ to get to the setting described above). In this sense, this thesis can be viewed as the unitary equivalent of [BF06].

Some of the modular forms which arise as a result of the lift are also related to those appearing in [Hof11] and [Hof13], in which Hofmann constructs Borcherds products for unitary groups. For $\mathrm{SU}(1,1)$, the logarithmic derivative of these products are the same as the modular forms produced by the theta lift in this thesis. This is due to the inherent symmetry in the dual pair $(\mathrm{SU}(1,1), \mathrm{SL}_2(\mathbb{R})) \sim (\mathrm{SU}(1,1), \mathrm{SU}(1,1))$.

In Chapter 1, we define most of the basic objects we will be working with. Let F be an imaginary quadratic field of discriminant D and let V be a split 2-dimensional F Hermitian space of signature $(1,1)$, with inner product $\langle\ ,\ \rangle$ and an isotropic basis $\{\ell, \ell'\}$. We let $V_{\mathbb{R}} = V \otimes_F \mathbb{C}$ be the complexification of V. Then $\mathrm{SU}(V_{\mathbb{R}}) = \mathrm{SU}(1,1) \cong \mathrm{SL}_2(\mathbb{R})$. Using the map $\pi(w\ell + w'\ell') = [w : w'] \in \mathbb{P}^1(\mathbb{C})$, we identify the space of positive definite one-dimensional subspaces with the usual complex upper half plane $\mathbb{H}$, and the space of isotropic lines with the boundary of $\mathbb{H}$ in $\widehat{\mathbb{C}}$. If X has positive length then we denote its image in $\mathbb{H}$ under π by $\mathbb{D}_X$.

We let L be an integral $\mathcal{O}_F$-lattice in V. For simplicity, we assume in this introduction that $L = \mathcal{O}_F\ell \oplus \mathcal{O}_F\ell'$. It is unimodular (i.e., equal to its dual lattice) and its stabiliser is $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. In general, L will not be unimodular, and we treat the general case in the main body of the thesis by working in a vector-valued setting.

We can take the set of length m vectors as $L_m = \{X \in L : \frac{1}{2}\langle X, X\rangle = m\}$, and then we can form the special cycle

$$T(m) = 2 \sum_{X \in \Gamma \backslash L_m} \frac{1}{|\overline{\Gamma}_X|}[\mathbb{D}_X].$$

The 2 here comes from the fact that $-I$ acts non-trivially in L, but trivially on $\mathbb{H}$. For L as above we have explicitly

$$T(m) = 2 \sum_{\substack{ad=m \\ 0 \le b < d}} \left[\frac{a\zeta + b}{d}\right].$$

Here $\zeta = \frac{D+\sqrt{D}}{2}$, and the sum is taken as divisors in the modular curve $\Gamma\backslash\mathbb{H}$. Note that in this case we have that the degree of the cycle $T(m)$ is equal to the sum of divisors $2\sigma_1(m)$.

From this, we define the m-th modular trace (when $m > 0$) for a Γ-invariant function $\mathbb{H}$ by

$$\mathrm{tr}_f(m) = \sum_{z \in T(m)} f(z),$$

and the 0-th modular trace as

$$\mathrm{tr}_f(0) = -\frac{1}{4\pi} \int_{\Gamma\backslash\mathbb{H}}^{reg} f(z) \frac{dxdy}{y^2},$$

where the integral is defined by an appropriate regularization process. This is highly analogous to the $\mathrm{SO}(1,2)$ case outlined in [BF06]. However, note that while for $\mathrm{SO}(1,2)$ the cycles are given by CM points of discriminant $-m$ in the present situation they arise by Hecke correspondences. More precisely, we have that

$$\mathrm{tr}_f(m) = 2m(T_0(m)f)(\zeta),$$

where $T_0(m)f$ defines the weight 0 Hecke operator.

The main result of this thesis is to realize the generating series of these traces for various f as modular forms of weight 2 as the result of a theta lift.

In Chapter 2, we introduce the representation theoretic background needed for the rest of the thesis. We consider a dual reductive pair of type II, $(\mathrm{SU}(V_{\mathbb{R}}), \mathrm{SL}_2(\mathbb{R}))$. Then the Schrödinger model of the Weil representation with respect to a central character ψ is the space of Schwartz functions $\mathcal{S}(V_{\mathbb{R}})$ with the following actions of

$\mathrm{SU}(V_{\mathbb{R}}) \times \mathrm{SL}_2(\mathbb{R})$. The unitary group $\mathrm{SU}(V_{\mathbb{R}})$ acts by its natural linear action on $\mathcal{S}(V_{\mathbb{R}})$:

$$M_{\natural}[g]f(v) = f(g^{-1}v).$$

On the other hand, $\mathrm{SL}_2(\mathbb{R})$ acts by the following formulas:

$$\left(M_{Sch}\left[\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\right]f\right)(x) = |\alpha|^{1/2}f(\alpha x);$$
$$\left(M_{Sch}\left[\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\right]f\right)(x) = \psi(\tfrac{1}{2}\langle\beta x, x\rangle)f(x);$$
$$\left(M_{Sch}\left[\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]f\right)(x) = \hat{f}(x).$$

Here $\hat{f}$ is the Fourier transform with respect to ψ on $V_{\mathbb{R}}$. We define a theta function for a lattice L in V and $\varphi \in \mathcal{S}(V)$ by

$$\theta((g, g'), \varphi, L) = \theta((g, g') \cdot \varphi, L) = \sum_{x \in L} M_{\natural}[g]M_{Sch}[g']\varphi(x).$$

In Chapter 3, we construct a suitable Schwartz function φ for the theta kernel. This follows the work of Kudla and Millson, and we refer to it as φ_{KM}, the Kudla-Millson Schwartz function. We let e_1, e_2 be a standard Hermitian basis of $V_{\mathbb{R}}$ (so $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = 1$), and for $X = v_1e_1 + v_2e_2$ we write

$$\langle X, X\rangle_0 = |v_1|^2 + |v_2|^2$$

for the standard majorant of the Hermitian form $\langle X, X\rangle$. We let

$$\varphi_S(X) = \exp(-\pi\langle X, X\rangle_0)$$

be the standard Gaussian on $\mathcal{S}(V_{\mathbb{R}})$. The definition of φ_{KM} comes from [KM86], where it is the image of φ_S under the operation of a certain differential operator which they attribute to Howe. We have

$$\varphi_{KM}(X) = \frac{1}{8}\left(4|v_1|^2 - \frac{2}{\pi}\right)\exp(-\pi\langle X, X\rangle_0).$$

We let $K = \mathrm{SO}(2) \simeq \mathrm{U}(1)$ be the standard maximal compact subgroup of $\mathrm{SL}_2(\mathbb{R})$. Then it is fairly easy to see that φ_{KM} is invariant under the $M_{\natural}$-action of $K \subset \mathrm{SU}(V_{\mathbb{R}})$ (i.e. has weight 0), while for the M_{Sch}-action of $K \subset \mathrm{SL}_2(\mathbb{R})$ it has weight 2.

For ease of use, we want concrete formulas in terms of variables in $\mathbb{H}$ rather than elements of the groups $\mathrm{SU}(V_{\mathbb{R}})$, $\mathrm{SL}_2(\mathbb{R})$. We now construct $M_{\natural}[g]M_{Sch}[g']\varphi_{KM}(X)$ as a function in variables $z = x+iy$, $\tau = u+iv$ in $\mathbb{H}$. We let $g_z \in \mathrm{SU}(V_{\mathbb{R}}) \simeq \mathrm{SL}_2(\mathbb{R})$ and $g'_\tau \in \mathrm{SL}_2(\mathbb{R})$ be any element which maps the base point $i \in \mathbb{H}$ to z and τ respectively. We let $d\mu(z) = y^{-2}dxdy$ be the invariant measure on $\mathbb{H}$. Then we set

$$\varphi_{KM}(X, z, \tau) = v^{-1}M_{\natural}[g_z]M_{Sch}[g'_\tau]\varphi_{KM}(X)d\mu(z).$$

We can describe $\varphi_{KM}(X, z, \tau)$ a bit more explicitly. We first set

$$\langle X, X\rangle_z = \langle g_z^{-1}X, g_z^{-1}X\rangle_0.$$

We also set

$$R(X, z) = \frac{1}{2}\left(\langle X, X\rangle_z - \langle X, X\rangle\right).$$

This quantity is real and always greater than or equal to zero, with equality if and only if $z = \mathbb{D}_X$. We then set

$$\varphi_{KM}(X, z) = M_{\natural}[g_z]\varphi_{KM}(X)d\mu(z) = \varphi_{KM}(g_z^{-1}x)d\mu(z).$$

For convenience we also set

$$\varphi^0_{KM}(X, z) = \exp(\pi\langle X, X\rangle)\varphi_{KM}(X, z).$$

Finally, we then have

$$\varphi_{KM}(X, z, \tau) = \varphi^0_{KM}(\sqrt{v}X, z)\exp(\pi i\langle X, X\rangle\tau).$$

Writing θ directly as a function on $\mathbb{H} \times \mathbb{H}$ we get

$$\theta(\tau, z, L) = \sum_{X\in L}\varphi^0_{KM}(\sqrt{v}X, z)\exp(\pi i\langle X, X\rangle\tau).$$

Note that as a function of z, the theta series $\theta(\tau, z, L)$ defines a differential 2-form on $M = \Gamma\backslash\mathbb{H}$, while in τ it is a non-holomorphic modular form of weight 2 for (in general congruence subgroup of) Γ.

As mentioned earlier, we will realise the theta lift as the generating series of the modular traces. The following theorem will be the tool we use to accomplish this.

Theorem 3.2.9. Following Kudla for $\mathrm{O}(1,2)$ in [Kud97], we set for $X \neq 0$

$$\xi^0(X,z) = \mathrm{Ei}(-2\pi R(X,z)),$$

where $\mathrm{Ei}(t)$ is the exponential integral. Then $\xi^0(X,z)$ is a Green function for the cycle $\mathbb{D}_X$.

More precisely, we have that for any linear exponentially increasing function $f(z) = O(\exp(2\pi n y))$ with $n \in \mathbb{N}$, and for any non-isotropic vector X, the integral

$$\int_{\mathbb{D}} f(z)\varphi^0_{KM}(\sqrt{v}X, z)$$

converges for $v > \frac{2n\Im(\mathbb{D}_X)}{m}$ and in this range we have

$$\int_{\mathbb{D}} f(z)\varphi^0_{KM}(\sqrt{v}X,z) = \begin{cases} \pi^{-1}\int_{\mathbb{D}} \xi^0(\sqrt{v}X,z)\Delta_0(f)d\mu(z) & \text{if } \langle X,X\rangle < 0 \\ \pi^{-1}\int_{\mathbb{D}} \xi^0(\sqrt{v}X,z)\Delta_0(f)d\mu(z) + f(\mathbb{D}_X) & \text{if } \langle X,X\rangle \geq 0 \end{cases}$$

Here $m = \frac{1}{2}\langle X,X\rangle$ and $\Delta_0 = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is the hyperbolic Laplace operator.

The method of proof essentially is the same as Kudla's treatment of the analogous statement for $\mathrm{O}(1,2)$ in [Kud97], see also [BF06], where test functions of exponential growth are discussed. Note however, that in the present situation the growth considerations lead to a much more delicate statement.

In Chapter 4, we introduce the theta lift of a harmonic weak Maass form f of weight 0. The space of such forms was introduced by Bruinier and Funke in [BF04]. For simplicity we only describe in the introduction the lift for a weakly holomorphic modular form

$$f(z) = \sum_{n \geq -N} c_f^+(n)\exp(2\pi i n z) \in M_0^!(\Gamma).$$

We then define the theta lift $I(\tau, f)$ of f by

$$I(\tau,f) = \int_M f(z)\theta(\tau,z,L).$$

However, this integral does not converge for all τ, since the linear exponential growth $f(z)$ is not (completely) offset by θ which has only linear exponential decay. We have

Theorem 4.1.2. The theta integral $I(\tau, f)$ converges for $Im(\tau) = v > N_f\sqrt{|D|}$

This contrasts to the orthogonal case of signature $(1, 2)$ considered in [BF06] where the theta function has square exponential decay and hence the lift converges for all τ. We therefore regularise the lift by setting

$$I(\tau, f) = \lim_{T\to\infty} \int_{M_T} f(z)\theta(\tau, z, L),$$

which we also call the cut-off lift, as the domain of integration is cut off at height T of the standard fundamental domain for Γ.

Theorem 4.2.1. Let $f \in M_0^!(\Gamma)$ be a weakly holomorphic form of weight 0 for Γ. Then the regularised theta lift is defined for all $\tau \in \mathbb{H}$ except a discrete set of points. The singularities of $I(\tau, f)$ lie on the divisor

$$Z(f) = \sum_{\substack{X\in L\\ \langle X,X\rangle>0}} c_f^+(-\tfrac{1}{2}\langle X, X\rangle)[\mathbb{D}_X].$$

Hence $I(\tau, f)$ defines a modular form of weight 2 with singularities.

These singularities are of linear type, as defined in [Bor98], in the sense that for each point in the set described above there exists ρ such that $I(\tau, f) - \frac{\rho}{\tau-\mathbb{D}_X}$ is a smooth function in τ in a neighbourhood of $\mathbb{D}_X$, where ρ is given by $\rho = -\frac{c_f^+(-\frac{1}{2}\langle X,X\rangle)}{2\pi i\sqrt{|D|}}$. This theorem is proved by defining a more convenient domain of integration to calculate over, and showing that the singularities of both integrals are the same.

The next result characterises the behaviour of the lift under the Bruinier-Funke $\xi_k = 2iv^k\overline{\frac{\partial}{\partial\bar{\tau}}}$ operator.

Theorem 4.2.2. Let $f \in M_0^!(\Gamma)$ be a weakly holomorphic form of weight 0 for Γ. Then image of $I(\tau, f)$ under the map ξ_2 is

$$\xi_2(I(\tau, f)) = -\frac{\overline{c_f^+(0)}}{4\pi}.$$

In particular, if $c_f^+(0) = 0$, then $I(\tau, f)$ is a meromorphic modular form of weight 2.

In Chapter 5 we explicitly calculate a formula for the Fourier expansion of the lift, which is the generating function of the modular traces defined in Chapter 1. The main theorem is

Theorem 5.1.1. Let $f(z) \in M_0^!(\gamma)$. Then the Fourier expansion of $I(\tau, f)$, valid in the region $Im(\tau) = v > N_f\sqrt{|D|}$ of the theta integral, is given by

$$I(\tau, f) = \frac{c_f^+(0)}{4\pi v} + \sum_{m=0}^{\infty} \mathrm{tr}_f(m) q^m.$$

In order to prove this, we split up the theta function into several sums, and integrate over these separately. For the non-isotropic vectors, we can use the current equation to obtain the modular traces, and for $X = 0$, we just get the 0-th trace. However, for the sum over the non-zero isotropic vectors we cannot use the current equation. We instead use an idea from [Fun02] in order to subdivide the summation into something more manageable. After some careful manipulation of the sums, this allows us to use the Rankin-Selberg unfolding technique, after which everything becomes more manageable to compute. We still have to use a certain regularisation technique of Borcherds', after which we are left with the weighted sums of residues of the Epstein Zeta function, an idea which reappears in Chapter 6.

We also examine two interesting cases of an input function f. Firstly, we take $f \equiv 1$, the simplest of all possible examples; we then have

Theorem 5.1.3. Let $f = 1$. Then the theta integral $I(\tau, f)$ converges for all τ and we have

$$I(\tau, 1) = -\frac{1}{12} E_2(\tau).$$

Here

$$E_2(\tau) = -\frac{3}{\pi v} + 1 - 24 \sum_{m=1}^{\infty} \sigma_1(m) q^m$$

is the classical weight 2 Eisenstein series for Γ.

This result which we recover in a different way in Chapter 6 as well.

Let $J(z) = J_1(z) = j(z) - 744$, where $j(z)$ is the classical j-invariant, and set $J_m(z) = mT_0(m)J_1(z)$. By comparing principal parts we obtain

Theorem 5.2.1. Let $J_m'(\tau) = \frac{1}{2\pi i} \frac{d}{d\tau} J_m(\tau)$, then

$$I(\tau, J_m) = \frac{2}{m} \sum_{\gamma \in \Gamma \backslash M_2(m, \mathbb{Z})} \frac{J_m'(\tau)}{J_m(\gamma \cdot \zeta) - J_m(\tau)}.$$

When $m = 1$, we can also use the Fourier expansion formula to recover from this from the expression

$$\sum_{n\in\mathbb{Z}} J_n(\zeta)q^n = \frac{J'(\tau)}{J(\zeta) - J(\tau)},$$

which is a partial result of a theorem due to Faber.

In Chapter 6, we examine a couple of interesting examples of input function f for the lift which are not harmonic weak Maass forms. Let $\Gamma_{N,\infty} \subset \Gamma_0(N)$ be the stabiliser of cusp at infinity. The weight 0 Eisenstein series with respect to the group $\Gamma_0(N)$ at the cusp ∞ is defined by

$$E_{0,N}(z,s) = \sum_{\gamma\in\Gamma_{N,\infty}\backslash\Gamma_0(N)} (\Im(\gamma z))^s$$

and the modified Eisenstein series of weight 0 is

$$\mathcal{E}_{0,N}(z,s) = \zeta^*(2s)E_{0,N}(z,s)$$

where $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function. We also define a modified weight 2 Eisenstein series by

$$\mathcal{E}_2(\tau,s) = -\frac{1}{4\pi}\zeta^*(2s)sv^{s-1}\sum_{(c,d)=1}|c\tau+d|^{-2(s-1)}(c\tau+d)^{-2}.$$

Theorem 6.2.1. For $L = \mathcal{O}_F\ell \oplus N\mathcal{O}_F\ell'$ with $N \in \mathbb{N}$

$$I(\tau, \mathcal{E}_{0,N}(z,s)) = w_F\zeta_F^*(s,[\mathcal{O}_F])N^{1-s}\mathcal{E}_2(N\tau,s),$$

where

$$\zeta_F^*(s,[\mathcal{O}_F]) = D^{s/2}\pi^{-s}\Gamma(s)\sum_{I\in[\mathcal{O}_F]} N(I)^{-s}$$

is the extended partial Dedekind zeta function for the trivial ideal class $[\mathcal{O}_F]$.

By taking residues at $s = 1$ on both sides, we recover the lift of the constant function.

From the weight 0 Eisenstein series for $\Gamma_0(N)$, we can form a Kronecker limit formula, but we choose to do so in a slightly non-standard way. The formula we end up with is

$$\lim_{s\to1}\left(\mathcal{E}_{0,N}(z,s) - \tfrac{1}{2}d_N^{-1}w_F\zeta_F^*(s,[\mathcal{O}_F])\right) = -\frac{1}{d_N k_N}\log\left(\frac{|\Delta_N(z)|}{|\eta(\zeta)|^{2k_N}}(yD_N)^{k_N/2}\right).$$

The quantities in this equation need some explanation. The constants d_N, D_N and k_N are all dependent only on N. They are defined by

$$\begin{aligned} d_N &= [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] \\ k_N &= \mathrm{lcm}\left(4, \frac{1}{2}\phi(N)\frac{24}{(24, d_N)}\right) \\ D_N &= \exp(D_N(1)) \end{aligned}$$

where

$$D_N(s) = \frac{d}{ds}\log(J_{2s}(N))$$

and

$$J_k(n) = n^k \prod_{p|n}\left(1 - \frac{1}{p^k}\right)$$

is the Jordan totient function. By η, we mean the usual Dedekind eta function, and the variation of the usual modular Δ function, Δ_N used here is essentially defined by the above limit; it is a modular form for $\Gamma_0(N)$ of weight k_N. We then prove

Theorem 6.3.1.

$$-\frac{1}{k_N d_N} I\left(\tau, \log\left(\frac{|\Delta_N(z)|}{|\eta(\zeta)|^{2k_N}}(yD_N)^{k_N/2}\right)\right) = \mathcal{E}'_{2,N}(\tau, 1).$$

where the differentiation on the RHS is in the s variable.

This result should have an interpretation in Arakelov theory and the Kudla Programme in the same way as the analogous result for the CM points. Note that the unitary situation has been recently considered in [How15] and [BHY15]. It would be interesting to compare the approach suggested by this thesis systematically.

Chapter 1

Lattices and Modular Traces

In this chapter we establish some of the basic notation and the geometric framework we will be working in. We prove some simple results, showing that the specific objects we are working with are isomorphic to some very familiar ones, i.e. the groups $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{Z})$, and the complex upper half plane $\mathbb{H}$ as a symmetric space.

Following a brief discussion of the kinds of lattices we will be using, we then introduce the special cycles of Kudla and Millson. These are, in our case, divisors on the modular curve $\Gamma_L \backslash \mathbb{H}$ which have a close association to the Hecke operators. These are used to define the modular traces of weakly holomorphic modular forms which are, in this setting, similar to the traces of singular moduli found in [Zag02], for example.

1.1 Lattices and Symmetric Domains

This section establishes the fundamental notation used throughout this thesis. We also recall some number theoretic concepts which will be useful to us, and define most of the basic geometry of the setting we will be using. This includes realising the usual upper half plane $\mathbb{H}$ as the symmetric space associated to the group $\mathrm{SU}(V_{\mathbb{R}})$ of invertible linear maps of determinant 1 of a vector space with respect to a split signature $(1, 1)$ Hermitian form.

There then follows some discussion of the types of lattice we will be considering.

These are number theoretic in nature, with the prototypical example being defined by the ring of integers of the underlying imaginary quadratic space. Finally, we consider carefully the cusps of the modular domain, and define the cut-off domain, which essentially chops off the usual fundamental domain at a height T away from each cusp. This is necessary for a certain regularisation of integrals used in later chapters.

1.1.1 Basic Definitions

Let $F = \mathbb{Q}[\sqrt{-d}]$, where d is a positive, squarefree integer. Let D denote the discriminant of F, i.e.

$$D = \begin{cases} -d & d \equiv 1 \pmod 4 \\ -4d & d \equiv 2,3 \pmod 4 \end{cases},$$

We will be concerned with ideals of the ring of integers of the field F. It will be useful to write the ring of integers of F by $\mathcal{O}_F = \mathbb{Z} \oplus \mathbb{Z}[\zeta]$, where

$$\zeta = \begin{cases} \dfrac{1+\sqrt{-d}}{2} & -d \equiv 1 \pmod 4 \\ \sqrt{-d} & -d \equiv 2,3 \pmod 4 \end{cases}.$$

Definition 1.1.1. The $\mathbb{Z}$-dual of $\mathcal{O}_F$ with respect to the bilinear form $\mathrm{Tr}(xy)$ is $\mathfrak{d}^{-1}$, the inverse different ideal. Let $\delta = i\hat{\delta} = \sqrt{D}$. Then $\mathfrak{d}$ is generated by δ. We may also write the ring of integers of F as $\mathcal{O}_F = \mathbb{Z} \oplus \mathbb{Z}\left[\frac{D+\delta}{2}\right]$. We note that $\Im(\zeta) = \frac{\hat{\delta}}{2}$.

Let V be a 2-dimensional F-vector space with an Hermitian form of signature $(1,1)$. By this, we mean that V splits over F into $E^+ \perp E^-$, with the Hermitian form being positive definite on E^+ and negative definite on E^-, and $\dim E^+ = \dim E^- = 1$, as in [Lan02, Ch. XIV, §11]. We take the Hermitian form to be anti-linear in the first variable and linear in the second variable.

Furthermore, we assume V to be split, i.e. we can define an isotropic basis $\{\ell, \ell'\}$ over F for V such that $\langle \ell, \ell' \rangle = 2\delta^{-1}$. Let $V_\mathbb{R} = V \otimes_F \mathbb{C}$ be the equivalent $\mathbb{C}$-vector space via the usual extension of scalars. We may write this explicitly for $X, Y \in V$ as $\langle X, Y \rangle = {}^t\overline{X}HY$ where

$$H = \begin{pmatrix} 0 & 2\delta^{-1} \\ -2\delta^{-1} & 0 \end{pmatrix}.$$

There is another basis $\{e_1, e_2\}$ where $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = 1$ and $\langle e_1, e_2\rangle = 0$. Clearly then, $E^+ = \mathrm{span}(e_1)$ and $E^- = \mathrm{span}(e_2)$ represents a possible choice for E^+ and E^-. The coordinate transformations are given as follows:

$$\begin{array}{ll} e_1 = \frac{1}{2}\sqrt{\hat{\delta}}\,(\ell + i\ell') & e_2 = \frac{1}{2}\sqrt{\hat{\delta}}\,(\ell - i\ell') \\ \ell = \sqrt{\hat{\delta}}^{-1}\,(e_1 + e_2) & \ell' = i\sqrt{\hat{\delta}}^{-1}\,(e_2 - e_1)\,. \end{array}$$

Hence, for example,

$$\begin{aligned} w\ell + w'\ell' &= w\left(\sqrt{\hat{\delta}}^{-1}(e_1 + e_2)\right) + w'\left(i\sqrt{\hat{\delta}}^{-1}(e_2 - e_1)\right) \\ &= \sqrt{\hat{\delta}}^{-1}\left((w - iw')e_1 + (w + iw')e_2\right) \end{aligned}$$

We also have the following expression for the length of $X = w\ell + w'\ell'$:

$$\langle X, X\rangle = 4\hat{\delta}^{-1}\Im(w\overline{w}').$$

which is a scaling of the canonical symplectic form on $\mathbb{C}^2$.

We note that, in terms of co-ordinates, the isotropic vectors are those where either $w' = 0$ or $\Im(w/w') = 0$.

1.1.2 The group $\mathrm{SU}(V)$

We may now say what we mean by $\mathrm{SU}(V)$ (resp. $\mathrm{SU}(V_{\mathbb{R}})$).

Definition 1.1.2. Let $\mathrm{SU}(V)$ (resp. $\mathrm{SU}(V_{\mathbb{R}})$) be the group invertible linear maps of determinant 1 which preserve $\langle\cdot,\cdot\rangle$, i.e.

$$\mathrm{SU}(V) = \{A \in GL(V) : \langle Ax, Ay\rangle = \langle x, y\rangle\ \forall x, y \in V\ \ \det(A) = 1\}$$
$$\mathrm{SU}(V_{\mathbb{R}}) = \{A \in GL(V_{\mathbb{C}}) : \langle Ax, Ay\rangle = \langle x, y\rangle\ \forall x, y \in V_{\mathbb{R}}\ \ \det(A) = 1\}$$

Proposition 1.1.3. *The groups* $\mathrm{SU}(V_{\mathbb{R}}) = \mathrm{SU}(1,1)$ *and* $\mathrm{SL}_2(\mathbb{R})$ *are isomorphic.*

Proof. Is is well known, e.g. [Kud79], that $\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SU}(1,1)$ and, by the coordinate transformation from the basis $\{e_1, e_2\}$ to $\{\ell, \ell'\}$ it is easy to see that $\mathrm{SU}(V_{\mathbb{R}}) = \mathrm{SU}(1,1)$ by definition, as the change of co-ordinates from $X = v_1e_1 + v_2e_2$ to $X = w\ell + w'\ell'$ is represented by

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sqrt{\hat{\delta}}^{-1}\begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix}\begin{pmatrix} w \\ w' \end{pmatrix}$$

and hence we are done. However, we provide also a direct proof that $\mathrm{SU}(V_{\mathbb{R}}) \cong \mathrm{SL}_2(\mathbb{R})$ for completeness. This will partly illustrate the reason for choosing the basis $\{\ell, \ell'\}$, namely, that in this basis the isomorphism to $\mathrm{SL}_2(\mathbb{R})$ is given by the identity map.

We write out the conditions for a matrix A to be in $\mathrm{SU}(V_{\mathbb{R}})$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we must have

$$\begin{aligned} {}^t\overline{A} \begin{pmatrix} 0 & 2\delta^{-1} \\ -2\delta^{-1} & 0 \end{pmatrix} A &= \begin{pmatrix} 0 & 2\delta^{-1} \\ -2\delta^{-1} & 0 \end{pmatrix} \\ \Longleftrightarrow \qquad \begin{pmatrix} \overline{c}a - \overline{a}c & \overline{c}b - \overline{a}d \\ \overline{d}a - \overline{b}c & \overline{d}b - \overline{b}d \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

which is equivalent to the following system of equations:

$$\begin{aligned} \Re(a)\Re(d) - \Re(b)\Re(c) + \Im(a)\Im(d) - \Im(b)\Im(c) &= 1 \\ \Im(a)\Re(d) - \Re(a)\Im(d) - \Re(b)\Im(c) + \Im(b)\Re(c) &= 0 \\ \Im(a)\Re(c) - \Re(a)\Im(c) &= 0 \\ \Im(b)\Re(d) - \Re(b)\Im(d) &= 0. \end{aligned}$$

At least one of a, b, c, d must be non-zero. For simplicity we assume it is a. Then we may write $a = re^{i\theta}$, and let $A' = e^{-i\theta}A$. Since $(A'x, A'y) = (Ax, Ay)$, we may assume then that $\Im(a) = 0$. This clearly implies that $\Im(c) = 0$. Hence we are reduced to

$$\begin{aligned} \Re(a)\Re(d) - \Re(b)\Re(c) &= 1 \\ -\Re(a)\Im(d) + \Im(b)\Re(c) &= 0 \\ \Im(b)\Re(d) - \Re(b)\Im(d) &= 0 \end{aligned}$$

from which it is clear that we must have $\Im(b) = \Im(d) = 0$. Finally, since we must have $\det(A) = 1$, we must have $e^{-i\theta} = \pm 1$. Hence we have shown that $\mathrm{SU}(V_{\mathbb{R}}) \subseteq \mathrm{SL}_2(\mathbb{R})$. That $\mathrm{SL}_2(\mathbb{R}) \subseteq \mathrm{SU}(V_{\mathbb{R}})$ is relatively straightforward; it follows from noticing that if $A \in \mathrm{SL}_2(\mathbb{R})$ then $\langle Ax, Ay \rangle = \langle x, y \rangle$ by simple substitution into the system of equations defining $\mathrm{SU}(V_{\mathbb{R}})$. $\square$

The Siegel Domain Model

Let $\mathbb{P}^1(\mathbb{C})$ denote the complex projective line and let $\pi : (V_{\mathbb{R}} - \{0\}) \to \mathbb{P}^1(\mathbb{C})$ be the natural projection, i.e.

$$\pi(w\ell + w'\ell') = [w : w']$$

We let $V_{\mathbb{R}}^+$ be the positive cone of $V_{\mathbb{R}}$, i.e. the set of all $X \in V_{\mathbb{R}}$ such that $\langle X, X\rangle > 0$.

Proposition 1.1.4. *There is a bijection between the space of positive lines of* $V_{\mathbb{R}}$ *and the complex upper half plane* $\mathbb{H}$*. By the former, we mean the set of one (complex) dimensional subspaces generated by a vector in* $V_{\mathbb{R}}^+$*, which is the first Grassmannian of* $V_{\mathbb{R}}^+$*, which we denote* $\mathrm{Gr}_1(V_{\mathbb{R}}^+) = \mathbb{D}$*. This bijection is made explicit by the map* π*.*

Proof. If $X = w\ell + w'\ell'$ then $\langle X, X\rangle > 0$ is equivalent to $\Im(w\overline{w}') > 0$, which in particular implies that $w' \neq 0$. Hence $\pi(V_{\mathbb{R}}^+) = \{z \in \mathbb{C} : \Im(z) > 0\} = \mathbb{H}$, whence surjectivity. Injectivity comes from realising that the inverse image of any $z \in \mathbb{H}$ is the line in $V_{\mathbb{R}}^+$ generated by $z\ell + \ell'$. □

Proposition 1.1.5. *The linear action of* $\mathrm{SU}(V_{\mathbb{R}})$ *on* $V_{\mathbb{R}}$ *preserves* $V_{\mathbb{R}}^+$*, and so it acts on* $\mathbb{D}$*. It does so as fractional linear transformations, i.e. any* $A \in \mathrm{SL}_2(\mathbb{R})$ *just acts as* $A \cdot z = \frac{az+b}{cz+d}$ *for any* $z \in \mathbb{D}$*.*

Proof.

$$\pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w \\ w' \end{pmatrix}\right) = \pi\left(\begin{pmatrix} aw + bw' \\ cw + dw' \end{pmatrix}\right) = \frac{az+b}{cz+d} = A \cdot z$$ □

Definition 1.1.6. Let K be the stabiliser of the line z_0 generated by $i\ell + \ell' \in V_{\mathbb{R}}^+$. Clearly, $\pi(z_0) = i$ and we call $X(z_0) = i\ell + \ell'$ the basepoint. Hence $K = \mathrm{SO}(2)$.

We have realised $\mathbb{D} \cong \mathbb{H} \cong \mathrm{SU}(V_{\mathbb{R}})/K$ as the symmetric space for $\mathrm{SU}(V_{\mathbb{R}})$, where the group action on $\mathbb{D}$ is inherited via the bijection. This is well defined, as we have $\pi(AX) = A \cdot \pi(X)$ for any $X \in V_{\mathbb{R}}^+$ and any $A \in \mathrm{SU}(V_{\mathbb{R}})$. We note that $-I$ acts non-trivially on V, but acts trivially on $\mathbb{D}$.

Let g_z be any element in $\mathrm{SU}(V_{\mathbb{R}})$ such that $\pi(g_z z_0) = z$. We then define $X(z) = g_z z_0$. We note that g_z is not unique; it is only defined up to multiplication on the right by any element which stabilises z_0, i.e. it is a left K coset. We also see that this map behaves very nicely under the action of some $\gamma \in \mathrm{SL}_2(\mathbb{R})$,

$$X(\gamma z) = \gamma X(z)$$

where the action on the LHS is by fractional linear transformation and on the RHS by matrix multiplication from the left. Since we have that

$$\pi(X(z)) = z,$$

it is clear that $X(z)$ actually provides a section of the projection map π.

We may make the map $z \mapsto g_z$ explicit in the following way. Let $\mathrm{SL}_2(\mathbb{R}) = NAK$ be the Iwasawa decomposition, as in [Iwa02] defined by

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}, \quad A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^+ \right\},$$
$$K = \left\{ \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

Then, since $\mathbb{D} \cong \mathbb{H} \cong \mathrm{SL}_2(\mathbb{R})/K$, we have the identification of NA and $\mathbb{D}$ via the normal matrix multiplication on the basepoint $i\ell + \ell'$, regarded as a the vector $(i, 1)^t$. We can take

$$g_z = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix} = \begin{pmatrix} \sqrt{y} & \sqrt{y}^{-1} x \\ 0 & \sqrt{y}^{-1} \end{pmatrix}$$

as the matrix which takes the base point to $z = x + iy$. We note that $\langle X(z), X(z) \rangle = 4\hat{\delta}^{-1}$.

We now define the minimal majorant. Pick a positive definite (one complex dimensional) subspace of V and call it z. Then define

$$\langle X, X \rangle_z = \begin{cases} \langle X, X \rangle & X \in z \\ -\langle X, X \rangle & X \in z^{\perp} \end{cases}$$

which is positive definite Hermitian form. However, as we saw before, the space of positive lines is bijective to $\mathbb{H}$, so by abuse of notation we actually regard $z \in \mathbb{H}$ from now on.

We also note that

$$\langle X, X \rangle_z = \langle g_z^{-1} X, g_z^{-1} X \rangle_{z_0}.$$

The Poincaré Disk Model

Using the basis $\{e_1, e_2\}$, we see that the space of positive lines is

$$V_{\mathbb{R}}^{+} = \{X = v_1 e_1 + v_2 e_2 \in V_{\mathbb{R}} : |v_1|^2 - |v_2|^2 > 0\}.$$

Under the projection $v_1 e_1 + v_2 e_2 \mapsto [v_1 : v_2] \in \mathbb{P}^1(\mathbb{C})$ this is equivalent to the Poincaré disk - i.e. we must have $|\frac{v_2}{v_1}|^2 < 1$. We can also see this using the co-ordinate transformations between the two bases. We recall that

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sqrt{\hat{\delta}}^{-1} \begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix} \begin{pmatrix} w \\ w' \end{pmatrix}$$

and so, applying the projection on both sides, we obtain

$$z_1 = \begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix} \cdot z_2$$

which is exactly the Möbius transformation which takes the upper half to the Poincaré disk.

$\mathbf{SU}(1,1)$ and $\mathbf{SO}(2,2)$

Let $\iota : V_{\mathbb{R}} \to M_2(\mathbb{R})$ be the $\mathbb{R}$-linear map defined by

$$\iota\left((\zeta x_1 + x_2)\ell + (\zeta x_3 + x_4)\ell'\right) \mapsto \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \tag{1.1}$$

Viewing $M_2(\mathbb{R})$ as a 4-dimensional vector space with the det map as a real split $(2,2)$ quadratic form, we have that $\mathrm{SO}_0(M_2(\mathbb{R})) \cong \mathrm{SO}_0(2,2) \cong \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. In fact, this defines an isometry of $V_{\mathbb{R}}$ with $M_2(\mathbb{R})$. If $M \in M_2(\mathbb{R})$ and $(g,h) \in \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, then this action is explicitly matrix multiplication

$$(g,h)M = gMh^{-1}.$$

Then the action of $\mathrm{SU}(V_{\mathbb{R}})$ will transfer under ι to a subgroup of $\mathrm{SO}(2,2)$, producing a mapping $\iota_* : \mathrm{SU}(V_{\mathbb{R}}) \to \mathrm{SO}(2,2)$.

Lemma 1.1.7. *In this realisation, this is just projection onto the first factor, i.e. if $g \in \mathrm{SU}(V_{\mathbb{R}})$*

$$\iota_*(g) = (g, 1)$$

and, for $g \in \mathrm{SU}(V_{\mathbb{R}})$ *and* $X \in V_{\mathbb{R}}$*,*

$$\iota(gX) = \iota_*(g)\iota(X).$$

Proof. Explicitly, we have

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} ax_1 + bx_3 & ax_2 + bx_4 \\ cx_1 + dx_3 & cx_2 + dx_4 \end{pmatrix} \\ &= \iota\left(\begin{pmatrix} \zeta(ax_1 + bx_3) + (ax_2 + bx_4) \\ c\zeta(x_1 + dx_3) + (cx_2 + dx_4) \end{pmatrix}\right) \\ &= \iota_*\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \iota\left(\begin{pmatrix} \zeta x_1 + x_2 \\ \zeta x_3 + x_4 \end{pmatrix}\right)\end{aligned}$$

□

We can see this using the following formulation from [vdG88]. To each $z = (z_1, z_2) \in \mathbb{H} \times \mathbb{H}$, we define the set

$$V_z = \left\{ v \in M_2(\mathbb{R}) : (z_2\ 1)v \begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0 \right\}$$

which, under the determinant form, is a 2-dimensional positive definite linear subspace of $M_2(\mathbb{R})$. This gives a bijective map $\mathbb{H} \times \mathbb{H} \to Gr_2^+$, the Grassmannian of all 2-dimensional linear subspaces of $M_2(\mathbb{R})$ on which the determinant form is positive definite. This in turn can be identified with the symmetric space

$$\mathrm{SO}_0(2,2)/(\mathrm{SO}(2) \times \mathrm{SO}(2))$$

and hence we have an isomorphism of symmetric spaces $\mathbb{H} \times \mathbb{H} \cong \mathrm{SO}_0(2,2)/(\mathrm{SO}(2) \times \mathrm{SO}(2))$.

1.1.3 Hermitian Lattices

Let L be any lattice in V. As in [Hof11], we say that L is integral if $\langle X, X \rangle \in \mathfrak{d}^{-1}$ for all $X \in L$ and that L is even if $\langle X, X \rangle \in 2\mathbb{Z}$ for all $X \in L$. The dual of a lattice, denoted L', is defined by

$$L' = \{X \in V : \langle X, Y \rangle \in \mathfrak{d}^{-1}\ \forall\ Y \in L\}.$$

Clearly, if L is integral, then $L \subset L'$. If $L \cong L'$ then we say L is unimodular.

The stabiliser of a lattice is $\mathrm{Stab}(L) = \{\gamma \in \mathrm{SU}(V) : \gamma L = L\}$, which for an integral lattice is contained within $\mathrm{Stab}(L')$, however, we are interested in the subgroup Γ_L of $\mathrm{Stab}(L')$ which fixes all L-cosets of L', i.e.

$$\Gamma_L = \{\gamma \in \mathrm{SU}(V) : \gamma(L + \mathbf{h}) = L + \mathbf{h}, \text{ for all } \mathbf{h} \in L'/L\}$$

We also define Γ_X, the stabiliser of a point as

$$\Gamma_X = \{\gamma \in \Gamma_L : \gamma X = X\}.$$

Let $\mathfrak{p}$, $\mathfrak{q}$ be prime, integral $\mathcal{O}_F$-ideals which are not fixed under conjugation. Let $N(\mathfrak{p}) = \mathfrak{p}\overline{\mathfrak{p}} = p$ and $N(\mathfrak{q}) = \mathfrak{q}\overline{\mathfrak{q}} = q$, with $\gcd(p, q) = 1$. For $N \in \mathbb{N}$, let

$$\Gamma^0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : b \equiv 0 \mod N \right\}$$
$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \mod N \right\}$$

be the usual congruence subgroups.

Lemma 1.1.8. *Let $L = \mathfrak{p}\ell \oplus \mathfrak{q}\ell'$. Then L is even, $L' = \overline{\mathfrak{q}}^{-1}\ell \oplus \overline{\mathfrak{p}}^{-1}\ell'$, Stab$(L) = \Gamma_0(q) \cap \Gamma^0(p)$ and $\Gamma_L = \Gamma(pq)$.*

Proof. This follows directly from the relevant definitions. The lattice is even (and therefore integral) because for any $X \in L = \mathfrak{p}\ell \oplus \mathfrak{q}\ell'$, we have

$$\langle X, X \rangle = 4\hat{\delta}^{-1}\Im(w\overline{w}')$$

and, since $w \in \mathfrak{p} \subseteq \mathcal{O}_F$ and $w' \in \mathfrak{q} \subseteq \mathcal{O}_F$ are all in $\mathcal{O}_F$, it is clear that $\Im(w\overline{w}') \in \frac{1}{2}\hat{\delta}\mathbb{Z}$ and hence $\langle X, X \rangle \in 2\mathbb{Z}$. Since, if $X = w_1\ell + w_1'\ell'$ and $Y = w_2\ell + w_2'\ell'$

$$\langle X, Y \rangle = -2\delta^{-1}(w_2'\overline{w}_1 - w_2\overline{w}_1')$$

that $\overline{\mathfrak{q}}^{-1}\ell \oplus \overline{\mathfrak{p}}^{-1}\ell' \subset L'$ is fairly clear. To show that $L' \subset \overline{\mathfrak{q}}^{-1}\ell \oplus \overline{\mathfrak{p}}^{-1}\ell' \subset L'$, we note that for a vector X to be in L', the above must be true for *all* $Y \in L$, in particular, on each component individually, which shows that L' splits; then for

$Y = \mathfrak{p}\ell$, X is characterised by $2\delta^{-1}\mathfrak{p}\overline{w}_1' \subset \mathfrak{d}^{-1}$ and for $Y = \mathfrak{q}\ell'$, X is characterised by $2\delta^{-1}\mathfrak{p}\overline{w}_1 \subset \mathfrak{d}^{-1}$. Hence $X \subset \overline{\mathfrak{q}}^{-1}\ell \oplus \overline{\mathfrak{p}}^{-1}\ell'$.

To find $Stab(L)$, we need to find conditions on a, b, c, d such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w \\ w' \end{pmatrix} = \begin{pmatrix} aw + bw' \\ cw + dw' \end{pmatrix} \subset L$$

, which by the fact that $\mathfrak{p}$ and $\mathfrak{q}$ are prime and integral, means that we must have $q \mid c$ and $p \mid d$.

By a similar calculation, we see that the conditions on a, b, c, d to be in Γ_L are

$$aw + bw' \subset \mathfrak{p} + h$$

and

$$cw + dw' \subset \mathfrak{q} + h'$$

for $w \equiv h \pmod{\mathfrak{p}}$ and $w' \equiv h' \pmod{\mathfrak{q}}$ for all $\mathbf{h} \in L'/L$. Since $\gcd(p, q) = 1$ and $\mathfrak{p}$ and $\mathfrak{q}$ are both prime, the only possible solution which works for any $\mathbf{h} \in L'/L$ is $pq \mid c$, $pq \mid b$ and $a \equiv 1 \pmod{pq}$ and $d \equiv 1 \pmod{pq}$, where we have incorporated that we must still have $ad - bc = 1$. □

Remark 1.1.9. For any even $\mathcal{O}_F$-lattice L, it is possible to find some integer N such that

$$N^{-1}\mathcal{O}_F\ell \oplus N^{-1}\mathcal{O}_F\ell' \supset L' \supset L \supset N\mathcal{O}_F\ell \oplus N\mathcal{O}_F\ell'.$$

In the case $L = N\mathcal{O}_F\ell \oplus N\mathcal{O}_F\ell'$, then we have that $L' = N^{-1}\mathcal{O}_F\ell \oplus N^{-1}\mathcal{O}_F\ell'$ and $\Gamma_L = \Gamma(N^2)$.

Example 1.1.10. Let $L = \mathcal{O}_F\ell \oplus N\mathcal{O}_F\ell'$. Then $Stab(L) = \Gamma_0(N)$, and this group also stabilises the $\mathbf{h} = 0$ coset. This example will play an important role in Chapter 6.

Consider the extension of the projection map $\pi : \Gamma_L\backslash V_+ \to \Gamma_L\backslash\mathbb{H}$. This is well defined, and commutes with the natural projections $V \to \Gamma_L\backslash V$ and $\mathbb{H} \to \Gamma_L\backslash\mathbb{H}$. We denote the locally symmetric space $\Gamma_L\backslash\mathbb{H}$ by M.

1.1.4 Cusps

Since we have that the preimage of the real line and infinity under the projection π is just the isotropic vectors, it is clear that we can identify the cusps of Γ_L with the isotropic lines in $\Gamma_L \backslash V$. We let $\mathrm{Iso}(V)$ the space of isotropic lines in V.

We compactify $\Gamma_L \backslash \mathbb{H} = M$ by adding a point at each cusp. The compactified Riemann surface $\overline{M}$ contains a one-parameter family of submanifolds with boundary, which we call M_T. They are defined by taking an open neighbourhood around each cusp of height T and deleting it. We always assume that T is sufficiently large that no two of these neighbourhoods intersect. For example, if $M = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ there is only the cusp at infinity and this process amounts to literally cutting the standard fundamental domain off at height T. For surfaces with more than one cusp, we can imagine that we use the transformation which maps the cusp to the one at infinity, performing the same procedure, then doing the inverse of this transformation.

More formally, as in [BF06], let $\kappa \in \Gamma_L \backslash \mathrm{Iso}(V)$, and let $\kappa_0 \in \Gamma_L \backslash \mathrm{Iso}(V)$ be the line generated by ℓ, i.e. the line which corresponds to the cusp at infinity. Then there exists $\sigma_\kappa \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma_\kappa \kappa_0 = \kappa$. If Γ_κ is the stabiliser of the line κ, then

$$\sigma_\kappa^{-1} \Gamma_\kappa \sigma_\kappa = \left\{ \pm \begin{pmatrix} 1 & k\alpha_\kappa \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z} \right\}$$

where α_κ is the width of the cusp κ. Around each cusp in $\overline{M}$, we have an open neighbourhood U_κ, and we can write the chart $Q_\kappa \to \mathbb{C}$ as $Q_\kappa(z) = \exp(2\pi i \sigma_\kappa^{-1} z / \alpha_\kappa)$. For $T > 0$, let $D_{1/T} = \{z \in \mathbb{C} : |z| < \frac{1}{2\pi T}\}$. We now formally define

$$M_T = \overline{M} - \coprod_{\kappa \in \Gamma_L \backslash \mathrm{Iso}(V)} Q_\ell^{-1} D_{1/T}$$

as the cut-off domain.

1.2 Modular traces

1.2.1 Harmonic Weak Maass Forms

We recall that the slash operator $|_k[\gamma]$ for $k \in \mathbb{Z}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is defined, for a function $f : \mathbb{H} \to \mathbb{C}$ by

$$f|_k[\gamma](z) = j(\gamma, z)^{-k} f(\gamma z)$$

for the cocycle $j(\gamma, z)$, which if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $j(\gamma, z) = cz + d$.

Definition 1.2.1. A congruence subgroup is a discrete subgroup of $\mathrm{SL}_2(\mathbb{Z})$ containing a principal congruence subgroup of some level N, which is defined as

$$\Gamma(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv Id \pmod{N}\}$$

We define harmonic weak Maass forms as in [BF04]. Let Γ be a congruence subgroup, and $k \in \mathbb{Z}$. Let $f : \mathbb{H} \to \mathbb{C}$ such that

- $f|_k[\gamma](z) = f(z)$ for all $\gamma \in \Gamma$ and $z \in \mathbb{H}$
- for any cusp κ of $\Gamma \backslash \mathbb{H}$ there is a σ_κ such that $\sigma_\kappa \infty = \kappa$ and $f(\sigma_\kappa z) = O(\exp(Cy))$ as $y \to \infty$ uniformly in x
- $\Delta_k f(z) = 0$ where

  $$\Delta_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

  is the hyperbolic Laplacian in weight k.

We denote the space of harmonic weak Maass forms of weight k for Γ by $H_k(\Gamma)$. If f satisfies these conditions, then we have a Fourier expansion around each cusp κ given by

$$f(\sigma_\kappa z) = \sum_{n \in \mathbb{Z}} c^+_{f,\kappa}(n) \exp\left(\frac{2\pi i n z}{\alpha_\kappa} \right) + c^-_{f,\kappa}(0) y^{1-k} + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} c^-_{f,\kappa}(n) H_k\left(\frac{2\pi n y}{\alpha_\kappa} \right) \exp\left(\frac{2\pi i n x}{\alpha_\kappa} \right)$$

where here we define H_k as

$$H_k(w) = \exp(-w) \int_{-2w}^{\infty} \exp(-t) t^{-k} dt.$$

We sometimes use the notation

$$f^+(\sigma_\kappa z) = \sum_{n \in \mathbb{Z}} c^+_{f,\kappa}(n) \exp\left(\frac{2\pi i n z}{\alpha_\kappa}\right)$$
$$f^-(\sigma_\kappa z) = c^-_{f,\kappa}(0) y^{1-k} + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} c^-_{f,\kappa}(n) H_k\left(\frac{2\pi n y}{\alpha_\kappa}\right) \exp\left(\frac{2\pi i n x}{\alpha_\kappa}\right)$$

so that $f(\sigma_\kappa z) = f^+(\sigma_\kappa z) + f^-(\sigma_\kappa z)$. Given the growth condition on f in y, we must have that all but finitely many of the $c^+_f(n)$ for $n < 0$ (respectively, $c^-_f(n)$ for $n > 0$) must vanish. In [BF04], they also define the ξ_k operator by $(\xi_k f)(z) = y^{k-2}\overline{L_k f(z)} = R_{-k} y^k \overline{f(z)}$ for the raising and lowering operators given by

$$L_k = -2iv^2 \frac{\partial}{\partial \bar{z}}$$
$$R_k = 2i \frac{\partial}{\partial z} + k y^{-1}.$$

The image of $H_k(\Gamma)$ under ξ_k lies in $M^!_{2-k}(\Gamma)$, the space of weakly holomorphic modular forms of weight $2-k$. Under this mapping, we let $H^+_k(\Gamma)$ be the preimage of $S_{2-k}(\Gamma)$, the space of cusp forms. What this means for f is that all the $c^-_f(n)$ vanish for $n \geq 0$.

The space of harmonic weak Maass forms contains the space of weakly holomorphic modular forms, which are simply the harmonic weak Maass forms for which f^- is identically zero for all cusps.

Example 1.2.2. Let $q = e^{2\pi i z}$. The modular j-function

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$$

is a weakly holomorphic modular form of weight 0 for the full modular group. It clearly has a pole of order 1 at the cusp at infinity.

We examine the weakly holomorphic modular forms of weight 0 for the full modular group. It is well known (see [Zag02] for example) that the space of automorphic forms of weight 0 for $\mathrm{SL}_2(\mathbb{Z})$ is $\mathbb{C}[j]$, and it is clear that we can form a basis for this space indexed by the order of the principal part, i.e. each function in the basis is of the form $q^{-m} + O(q)$ for $m \in \mathbb{N}$. For technical reasons, we prefer to work with $J(z) = j(z) - 744$. We call this basis J_m. As suggested by the notation, $J_1 = J$. These functions will provide an important example later.

1.2.2 Hecke Operators

Though there are multiple ways to introduce Hecke operators, we take the point of view of double coset operators. Following [DS05, §5.1], let Γ be a congruence subgroup, and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then

$$\Gamma\alpha\Gamma = \{\gamma_1\alpha\gamma_2 : \gamma_1, \gamma_2 \in \Gamma\}$$

For any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, we can define as usual, an action on the upper half plane via fractional linear transformations, i.e. if $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then

$$\alpha\tau = \frac{a\tau + b}{c\tau + d}$$

and the cocycle $j(\alpha, \tau) = c\tau + d$.

We can extend the definition of the slash operator to matrices in $\mathrm{GL}_2^+(\mathbb{Q})$.

Definition 1.2.3. For any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ and $k \in \mathbb{Z}$, the weight k slash operator on functions $f : \mathbb{H} \to \mathbb{C}$ is defined by

$$(f|_k[\alpha])(\tau) = (\det\alpha)^{k/2} j(\alpha, \tau)^{-k} f(\alpha\tau)$$

Now let $\{\beta_j\}$ be a set of orbit representatives for the double coset, i.e. $\Gamma_1\alpha\Gamma_2 = \bigcup_j \Gamma_1\beta_j$ is a disjoint union.

Definition 1.2.4. The weight k double coset operator $\Gamma\alpha\Gamma$ is defined by

$$f \mid_k [\Gamma\alpha\Gamma] = \sum_j f|_k[\beta_j],$$

and is a linear map on the space of modular forms of weight k for the group Γ. This definition is independent of the choice of coset representatives.

Now let Γ be any congruence subgroup of level N, and define

$$\langle n\rangle = \begin{cases} \Gamma \begin{pmatrix} * & * \\ * & n \end{pmatrix} \Gamma, \text{ where } \begin{pmatrix} * & * \\ * & n \end{pmatrix} \in \Gamma_0(N) & \text{if } (n,N)=1 \\ 0 & \text{if } (n,N)>1 \end{cases}$$

and

$$T_m = \sum_{\substack{ad=m \\ a|d}} \langle a\rangle \Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma. \tag{1.2}$$

Then T_m is the usual Hecke operator, realised as a double coset operator. For more information, see [DS05, Ch. 5]. The following theorem is well known, see e.g. [Zag02], [BKO04].

Lemma 1.2.5. *Let T_m be the usual Hecke operator. Then we have that*

$$J_m(z) = m(T_m J)(z),$$

where the J_m form a $\mathbb{C}$-basis for the space of weight 0 weakly holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$.

Proof. We cite formula (5.13) from [DS05], which gives the formula

$$(T_m f)(z) = \sum_{n\in\mathbb{Z}} \left(\sum_{d|(n,m)} d^{k-1} a_{\frac{mn}{d^2}} \left(\langle d\rangle f\right) \right) q^n$$

for the Fourier coefficients under the action of T_m, then we see that, for $f = J$, we have that

$$(T_m)J(z) = \sum_{n\in\mathbb{Z}} \left(\sum_{d|(n,m)} d^{-1} a_{\frac{mn}{d^2}} (J) \right) q^n$$

and, since the principal part of J is just q^{-1}, the coefficients of the principal part of $T_m J$ are zero, unless $-mn = d^2$, which is only the case if $m = -n = d$ and so the principal part of $T_m J$ is $\frac{1}{m} q^{-m}$. □

1.2.3 Special Cycles

Let L be any even, integral lattice, and let $X \in L$ be such that $\langle X, X\rangle > 0$.

Definition 1.2.6. Let V_X be the complex line in the direction of X in V, which is given by

$$V_X = \{Y \in V_{\mathbb{R}} : Y = \lambda X \text{ for some } \lambda \in \mathbb{C}\}.$$

We denote by $(V_X)_+ = \{Y \in V_X : \langle Y, Y\rangle > 0\}$ the positive cone sitting inside V_X. In this case, we have that $(V_X)_+ = V_X - \{0\}$, however, this would not be so for the equivalent formulation in higher dimensions. We let $\mathbb{D}_X = \pi((V_X)_+)$. Thus, for any vector of positive length in the lattice L, we have associated to it a point $\mathbb{D}_X \in \mathbb{D} \cong \mathbb{H}$.

Let $\mathbf{h} \in L'/L$. We can decompose the coset $L + \mathbf{h}$ of the lattice L' into subsets depending on the length m. It is clear that m takes values in $\mathbb{Z} + \frac{1}{2}\langle \mathbf{h}, \mathbf{h}\rangle$. We define

$$L_{m,\mathbf{h}} = \{X \in L + \mathbf{h} : \tfrac{1}{2}\langle X, X\rangle = m\}$$

to be set of vectors of a given length. For each such $m > 0$ we define the divisor

$$T(m, \mathbf{h}) = \epsilon \sum_{X \in \Gamma_L \backslash L_{m,\mathbf{h}}} \frac{1}{|\overline{\Gamma}_X|} [\mathbb{D}_X]$$

which naturally sits on $\Gamma_L\backslash\mathbb{H}$. Here, $\overline{\Gamma}_X$ is the projection of Γ_X into $\mathrm{PSL}_2(\mathbb{Z})$. We define ϵ by

$$\epsilon = \begin{cases} 2 & -I \in \Gamma_L \\ 1 & \text{otherwise} \end{cases}.$$

For example, if $\Gamma_L = \Gamma(N)$, a principal congruence subgroup, then

$$\epsilon = \begin{cases} 2 & N = 1, 2 \\ 1 & \text{otherwise} \end{cases}.$$

This divisor is closely related to certain Hecke operators. By using the embedding into $\mathrm{SO}(2, 2)$, i.e.

$$\iota((\zeta x_1 + x_2)\ell + (\zeta x_3 + x_4)\ell') = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

we may identify $L_{m,\mathbf{h}}$ with

$$\Delta_{m,\mathbf{h}} = \{\alpha \in \mathrm{GL}_2^+(\mathbb{Q}) : \pi_1(\alpha) \equiv \mathbf{h},\ \det(\alpha) = m\}.$$

For any orbit $\Gamma_L X_j$ in $L_{m,\mathbf{h}}$ we have

$$\iota(\Gamma_L X_j) = \iota_*(\Gamma_L)\iota(X_j)$$

and clearly, if we have a set of coset representatives $\Delta_{m,\mathbf{h}} = \coprod \iota_*(\Gamma_L)\beta_j$, then we have that

$$L_{m,\mathbf{h}} = \coprod \Gamma_L \beta_j(\zeta\ell + \ell')$$

We can relate these cosets to Hecke operators in certain cases. The divisor $T(m,0)$ is equivalent to the Hecke operator

$$\sum_{\alpha\in\Gamma_L\backslash\Delta_{m,0}/\Gamma_L} \Gamma_L\alpha\Gamma_L \tag{1.3}$$

acting on the point ζ, viewed on the modular curve $\Gamma_L\backslash\mathbb{H}$. For these Hecke operators, we have a finite set $\{\beta_j\}$ of orbit representatives for which (1.3) decomposes into disjoint Γ_L-cosets, i.e.

$$\sum_{\alpha\in\Gamma_L\backslash\Delta_{m,0}/\Gamma_L} \Gamma_L\alpha\Gamma_L = \sum_j \Gamma_L\beta_j.$$

We refer to [Shi71, Chap. 3] for more details. We can then write the divisor $T(m,0)$ in terms of the $\{\beta_j\}$ decomposition using the formula

$$T(m,0) = \frac{\epsilon}{\overline{\Gamma}_\zeta}\sum_j [\beta_j\cdot\zeta]$$

where β_j acts on ζ as a fractional linear transformation. Such a decomposition is valid for any $\mathbf{h}$, however, for nonzero $\mathbf{h}$ the set $\Delta_{m,\mathbf{h}}$ does not form a semi-group due to it not being closed under addition and so we cannot call it a Hecke operator in general. We give a simple example to make this clearer, using the case of a unimodular lattice (where the only coset is the $\mathbf{h}=0$ one):

Example 1.2.7. Let $L = \mathcal{O}_F\ell \oplus \mathcal{O}_F\ell'$, then $\Gamma_L = \mathrm{SL}_2(\mathbb{Z})$ and

$$\{\beta_j\} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = m,\ 0 \le b < d \right\}$$

and so

$$T(m,0) = 2\sum_{\substack{ad=m \\ 0\le b<d}} \left[\frac{a\zeta + b}{d}\right]$$

where, by a slight abuse of notation, we consider the RHS to sit on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. We note that the degree of this divisor is $2\sigma_1(m)$, and that these are the Fourier coefficients of the holomorphic part of $\frac{-1}{12}E_2(\tau)$, the Eisenstein series of weight 2. This is not a coincidence.

Example 1.2.8. Let $L = \mathcal{O}_F\ell \oplus N\mathcal{O}_F\ell'$. We recall that the $\mathbf{h} = 0$ coset is stablised by $\Gamma_0(N)$. Therefore the $T(m, 0)$ are characterised by the coset representatives of

$$\Gamma_0(N)\backslash\left\{\begin{pmatrix} x_1 & x_2 \\ Nx_3 & Nx_4 \end{pmatrix} : x_i \in \mathbb{Z},\ N(x_1x_4 - x_2x_3) = m\right\}.$$

In particular, we note that the length must always be divisible by N. We will see in Chapter 6 that in this case the degree of the divisor is $\sigma_1(m/N)$.

1.2.4 Modular traces

We will now define something which we call modular traces. They are analogous to the traces of singular moduli, however they are not equal to them - except in certain special cases.

We will later construct a process for producing a meromorphic modular form of weight 2 which has positive Fourier coefficients which are these modular traces.

Definition 1.2.9. For $m \neq 0$, the $(m, \mathbf{h})$-th trace of a modular function f is defined to be

$$\mathrm{tr}_f(m, \mathbf{h}) = \sum_{z \in T(m,\mathbf{h})} f(z)$$

This definition gives us two ways to think about the trace. On the one hand we can think of taking a function and evaluating over a set of points $T(m, \mathbf{h})$. On the other hand, using that $T(m, \mathbf{h}) = \frac{\epsilon}{\Gamma_\zeta} \sum_j [\beta_j \cdot \zeta]$ where $\Delta_{m,\mathbf{h}} = \coprod \iota_*(\Gamma_L)\beta_j$, we can (abusing the notation) let $T(m, \mathbf{h})$ act on f via

$$T(m, \mathbf{h})f = \sum_{\beta_j} f \mid_0 \beta_j$$

and simply evaluate $T(m, \mathbf{h})f$ at the point ζ. In other words, if we define a cohomological pairing by

$$\langle f, [z]\rangle = f(z),$$

then we have the following

Lemma 1.2.10.
$$\langle f, T(m,\mathbf{h})\rangle = \left\langle T(m,\mathbf{h})f, \frac{\epsilon}{\overline{\Gamma}_\zeta}[\zeta]\right\rangle.$$

Definition 1.2.11. The $(0,\mathbf{h})$-th trace is defined by the regularised integral

$$\mathrm{tr}_f(0,\mathbf{h}) = -\frac{\delta_{\mathbf{h},0}}{4\pi}\lim_{T\to\infty}\int_{M_T} f(z)\frac{dxdy}{y^2}$$

This definition seems arbitrary, but falls quite naturally out of the definition of the theta lift in Chapter 4.

Theorem 1.2.12. *As in [BF06] and [Bor98], we calculate the* $(0,\mathbf{h})$*-th trace to be, for a weakly holomorphic form with Fourier coefficients* $a_\kappa(n)$ *at the cusp* κ*,*

$$\mathrm{tr}_f(0,\mathbf{h}) = \frac{\delta_{\mathbf{h},0}c(0)}{4\pi}\sum_{\kappa\in\Gamma_L\backslash\mathrm{Iso}(V)}\alpha_\kappa\sum_{n\in\frac{1}{\alpha_\kappa}\mathbb{Z}_{\geq 0}} a_\kappa(-n)b_\kappa(n)$$

where

$$E_{2,\ell}(z) = j(\sigma_\ell,z)^{-2}E_2(\sigma_\ell z) = \left(b_\ell(0)+\frac{c(0)}{y}\right)+\sum_{n=1}^{\infty}b_\ell(n/\alpha_\ell)\exp(2\pi i n z/\alpha_\ell)$$

is the weight 2 *Eisenstein series at the cusp* ℓ*.*

Proof. See [BF06, Remark 4.9] and [Bor98, §9] □

We now discuss a few properties of these modular traces, and a simple example, to help get a feel for what they are like.

Proposition 1.2.13. *The values of* $\mathrm{tr}_J(m,\mathbf{h})$ *always lie in a real subfield of the Hilbert class field of the field* F*.*

Proof. This follows firstly from the fact that a general theorem of class field theory tells us that the J function, evaluated at quadratic irrationalities, always takes values in the Hilbert class field [Sil09, Appendix C, Thm. 11.2(c)]. Secondly, we observe that $\overline{J(z)} = J(-\overline{z})$, and it is easy to see therefore that in the set of modular divisors, all points occur in pairs of $(z,-\overline{z})$, hence, the modular trace will be real. □

Example 1.2.14. The case of $L = \mathcal{O}_F\ell\oplus\mathcal{O}_F\ell'$ and $f = J$ is the easiest to calculate. Since $L = L'$, there is only the $\mathbf{h} = 0$ coset to consider and so as discussed in Example 1.2.7, the cycles are just the usual T_m Hecke operators. Is therefore easy to calculate these traces because $T_m J = J_m$, and so the traces are, in this case, $J_m(\zeta)$. If the class number of F is 1, then $J(\zeta)$ obviously lies in $\mathbb{Z}$.

Using the tables below, it is easy to calculate $\mathrm{tr}_J(m, 0)$ for some low values of d and m.

d	$J_1(\zeta)$
-1	492
-2	7256
-3	-248
-5	$631256+282880\sqrt{5}$
-6	$2416728+1707264\sqrt{2}$
-7	-4119
-10	$212845656+95178240\sqrt{5}$
-11	-33512
-13	$3448439256+956448000\sqrt{13}$

m	Polynomial in J_1 generating J_m
1	x
2	$x^2 - 393768$
3	$x^3 - 590652x - 64481280$
4	$x^4 - 787536x^2 - 85975040x + 7406919032$

These traces also are related to some of the traces defined in [DIT11], namely the cases where the class number is 1. Then we only have one Hurwitz-Kronecker class, and so the (minimum polynomial of the) quadratic irrationality ζ must therefore be equivalent to (the bilinear form $Q(x, 1)$, whose solution gives rise to) τ_Q under the action of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. This explains why, in the cases $d = -1, -2, -3, -7, -11, -19,$ $-47, -163$ we have that

$$\mathrm{tr}_J(m, 0) = \mathrm{Tr}_{-D}(J_m) \tag{1.4}$$

where the left hand side comes from Definition 1.2.9 in this thesis, and the right hand side from [DIT11, pp. 6].

Example 1.2.15. Using the same lattice as in the previous example, we can consider the trace of the constant function, which is obviously the degree of the divisor $T(m, 0) = 2\sigma_1(m)$. As noted before, these are the Fourier coefficients of $-\frac{1}{12}E_2(\tau)$.

Chapter 2

The Weil Representation

We construct the Weil representation using the Schrödinger model, and discuss dual reductive pairs and the theta series map on the space V. Using a construction of Kudla we give formulas for the action of the Weil representation. The primary references are [LV80], [Pra93] and [Kud79].

2.1 The Weil Representation

Let W be a finite dimensional vector space over $\mathbb{R}$, equipped with a non-degenerating alternating form $\langle\ ,\ \rangle$. The pair $(W, \langle\ ,\ \rangle)$ form a symplectic space of even dimension $2n$. We define the Heisenberg group $H(W)$ as the set of all pairs

$$\{(w,t) : w \in W,\ t \in \mathbb{R}\}$$

with the group operation

$$(w_1,t_1)(w_2,t_2) = (w_1 + w_2, t_1 + t_2 + \tfrac{1}{2}\langle w_1, w_2\rangle).$$

We have the following well-known theorem:

Theorem 2.1.1 (Stone, von Neumann [LV80])**.** *The Heisenberg group $H(W)$ has an irreducible smooth representation on which $\mathbb{R}$ operates via the non-trivial central character ψ, which we call ρ_ψ. This representation is unique up to isomorphism.*

We now give a realisation of ρ_ψ. For any such space W, we can decompose it into maximal totally isotropic subspaces $W = W_1 \oplus W_2$. We call W_1 (resp. W_2)

a Lagrangian subspace. Let us take some additive character on $\mathbb{R}$, called ψ. Then there exists a smooth representation ρ_ψ of $H(W)$ on $\mathcal{S}(W_1)$, the Schwarz space of W_1 being the space of rapidly-decreasing functions on W_1. Formally, this is the set

$$\mathcal{S}(W_1) = \left\{ f \in \mathcal{C}^\infty : \sup_{x \in W_1} |x^\alpha \partial^\beta f(x)| < \infty \text{ for all multi-indices } \alpha, \beta \right\},$$

which we can understand as the smooth functions on W_1 all of whose derivatives decay faster than any inverse power. The representation ρ_ψ acts on $f \in \mathcal{S}(W_1)$ as follows:

$$\begin{aligned} \rho_\psi(w_1) f(x) &= f(x + w_1) \\ \rho_\psi(w_2) f(x) &= \psi(\langle x, w_2 \rangle) f(x) \\ \rho_\psi(t) f(x) &= \psi(t) f(x). \end{aligned}$$

This is known as the Schrödinger representation.

The Weil representation is a projective representation of the symplectic group constructed from the Schrödinger representation. We observe that the symplectic group $\mathrm{Sp}(W)$ acts on $H(W)$ via $g \cdot (w, t) = (gw, t)$. This defines another irreducible representation of $H(W)$ twisted by g, with the same central character; hence by Stone-von Neumann there exists an operator $\omega_\psi(g)$ on $\mathcal{S}$ such that

$$\rho_\psi(gw, t) \omega_\psi(g) = \omega_\psi(g) \rho_\psi(w, t) \tag{2.1}$$

for all $(w, t) \in H(W)$. By Schur's Lemma $\omega_\psi(g)$ is unique up to a non-zero complex scalar. This forms a group under pointwise multiplication,$\widetilde{Sp}_\psi(W)$, called the metaplectic group, defined as the set of all pairs $(g, \omega_\psi(g))$ such that (2.1) holds. We have the following short exact sequence:

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \widetilde{Sp}_\psi(W) \longrightarrow Sp(W) \longrightarrow 0.$$

The obvious projection onto the second factor $(g, \omega_\psi(g)) \mapsto \omega_\psi(g)$ gives a representation of the metaplectic group, called the Schrödinger model of the Weil representation.

We now give explicit formulas for the Schrödinger model M_{Sch} of the metaplectic representation. Let $\{e_1, \ldots, e_n,\ f_1, \ldots, f_n\}$ be a symplectic basis for $W = W_1 \oplus W_2$

where $e_i \in W_1$, $f_i \in W_2$ and $\langle e_i, f_j \rangle = \delta_{ij}$. The Metaplectic group is generated by

$$g(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{pmatrix} \text{ for any } \alpha \in GL(W_1)$$

$$t(\beta) = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \text{ for any } \beta =^t \beta$$

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then, using the map $\widetilde{g} \to g$ from the metaplectic group to the symplectic group, we have the following:

$$\left(M_{Sch}\left[\widetilde{g(\alpha)}\right] f\right)(x) = |\det \alpha|^{1/2} f({}^t\alpha x)$$

$$\left(M_{Sch}\left[\widetilde{t(\beta)}\right] f\right)(x) = \psi(\tfrac{1}{2}\langle \beta x, x \rangle) f(x)$$

$$\left(M_{Sch}\left[\widetilde{\sigma}\right] f\right)(x) = \gamma \hat{f}(x)$$

where $\hat{f}(x)$ is the Fourier transform defined by, using the convention $x = \sum x_i e_i$ and $y = \sum y_i e_i$,

$$\hat{f}(x) = \int_{W_1} f(y) \psi\left(\sum_{i=1}^{n} x_i y_i\right) dy$$

and γ is an eighth root of unity.

Remark 2.1.2. The specific situation we will be considering later will be a dual pair sitting inside $\mathrm{Sp}(8, \mathbb{R})$. In [KV78], they show that for $W = \mathrm{Sp}(8, \mathbb{R})$, the root of unity is $i^{n/2}$, and so for our case this root of unity is 1.

It is important to note that the Schrödinger representation depends on the choice of polarisation, i.e. on the choice of W_1. However, by the Stone-von Neumann theorem, there must exist some intertwining operator $\mathcal{F} : \mathcal{S}(W_1) \to \mathcal{S}(W_1')$ (uniquely defined up to a scalar) such that

$$\mathcal{F} \circ M_{Sch,W_1}[g] = M_{Sch,W_1'}[g] \circ \mathcal{F}$$

This operator is in fact the (partial) Fourier Transform [LV80], given by

$$(\mathcal{F} f)(y) = \int_{W_1/W_1 \cap W_1'} f(x) \psi(\tfrac{1}{2}\langle x, y \rangle) dx$$

for dx a positive W_1 invariant measure on the homogeneous space $W_1/W_1 \cap W_1'$.

Let $W = W_1 \oplus W_2$ be a polarisation. We now define a linear functional Θ on $\phi \in \mathcal{S}(W_1)$ by

$$\Theta(\phi, \Lambda) = \sum_{x \in \Lambda} \phi(x)$$

for some lattice Λ in W_1. We now define a function on $g \in \widetilde{Sp}(W)$ by

$$\Theta(g, \phi, \Lambda) = \Theta(g \cdot \phi, \Lambda).$$

For a dual reductive pair (G, G'), let $\widetilde{G}$ and $\widetilde{G}'$ be full inverse images of G and G' in $\widetilde{Sp}(W)$. Then the function above restricted to $\widetilde{G} \times \widetilde{G}'$ defines the theta kernel. If the groups G and G' lift to the metaplectic group (e.g. they are unitary groups, as will be the case in the next section) then the Weil representation realises this as a function on $G \times G'$.

2.2 Dual Pairs and Theta Functions

Following [Pra93], a dual reductive pair is a pair of subgroups (G, G') of a symplectic group $\mathrm{Sp}(W)$ such that

- they are mutual centralisers, i.e. G is the centraliser of G' in $\mathrm{Sp}(W)$ and G' is the centraliser of G
- the actions of G and G' are completely reducible on W, i.e. the complement of an invariant subspace of W under G or G' is itself invariant under that group.

If we have two dual reductive pairs (G_1, G_1') in $\mathrm{Sp}(W_1)$ and (G_2, G_2') in $\mathrm{Sp}(W_2)$ then $(G_1 \times G_1', G_2 \times G_2')$ is a dual reductive pair in $\mathrm{Sp}(W_1 \oplus W_2)$. Any dual reductive pair not constructible in this way is called irreducible.

Irreducible dual reductive pairs can be classified into one of two types, according to how $G \cdot G'$ acts on W. If the action is irreducible, we say that it is of type I, and if it is reducible then we say it is of type II.

In particular, for type II irreducible dual pairs, there exists a division algebra D and a right D-vector space W_1 and a left D-vector space W_2 such that

$$W = (W_1 \otimes_D W_2) \oplus (W_1 \otimes_D W_2)^*$$

and

$$(G, G') = (Aut_D(W_1), Aut_D(W_2))$$

We now make this more explicit by describing this for a specific dual pair which we will be concerned with for the rest of this thesis. Recall the vector space V and Hermitian form $\langle\ ,\ \rangle$ defined in Chapter 1. Kudla in [Kud79] gives a construction which realises $((U)(V_{\mathbb{R}}), U(V_2))$ (for V_2 described below) as a dual pair of type II, which we reproduce here.

Let V_2 be a 2 dimensional complex vector space with an Hermitian form given by $\langle\ ,\ \rangle_2$. Then we can form a symplectic vector space $W = V \otimes_{\mathbb{C}} V_2$ with the symplectic form given as follows. There is a natural Hermitian form on W given by

$$\langle v_1 \otimes v_2, v_1' \otimes v_2' \rangle_3 = \langle v_1, v_1' \rangle \langle v_2, v_2' \rangle_2$$

We then define a symplectic form by

$$\langle\langle\ ,\ \rangle\rangle = \Im\langle\ ,\ \rangle_3.$$

and we have homomorphisms

$$U(V) \times U(V_2) \longrightarrow U(W) \longrightarrow Sp(W, \langle\langle\ ,\ \rangle\rangle).$$

The space V_2 has an isotropic basis $\{u_2, u_2'\}$ such that $\langle u_2, u_2' \rangle_2 = i$. This allows the identification $W \cong V \times V$ via

$$v_1 \otimes u_2 + v_2 \otimes u_2' \longmapsto (v_1, v_2)$$

and identifies $\mathrm{SU}(V_2)$ with $\mathrm{SL}_2(\mathbb{R})$.

Kudla also calculates the action ω of the Weil representation on a Schwartz function on V. We are particularly interested in the action of $g \in \mathrm{SU}(V_{\mathbb{R}})$ and $g' \in \mathrm{SL}_2(\mathbb{R})$. This action is implicitly for the character $t \mapsto e^{2\pi i t}$ and the polarisation given by the isotropic basis on V_2, i.e. we choose for our Langrangian $V \otimes u_2' \cong V$. For $g' = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$

$$\omega(g')f(v) = |a|^2 e^{\pi i a d \langle v, v \rangle} f(av) = M_{Sch}[g']f(v).$$

and

$$\omega\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) f(v) = \int_V \exp(\pi i \langle v, u\rangle) f(u) du = \hat{f}(v)$$

Kudla does give a more general formula for the action (e.g. when $c \neq 0$), but we do not need it.

We have then for $g \in \mathrm{SU}(V_\mathbb{R})$ an alternative model which we call $M_\natural$, induced by the natural action of g on V, which is given by

$$\omega(g)f(v) = f(g^{-1}v) = M_\natural[g]f(v).$$

Of course, these formulas are dependent on the choice of Langrangian, which we took to be $V \otimes u_2'$. If we instead took the Langrangian to be $\ell' \otimes V_2$ then the operation of the Weil representation is given by the intertwining operator $\mathcal{F}$, which is the partial Fourier transform - as seen by the fact that

$$\begin{aligned}(W_1 \backslash W_1 \cap W_1')_\mathbb{C} &= (V_\mathbb{R} \otimes u_2)\backslash(V_\mathbb{R} \otimes u_2 \cap \ell' \otimes V_{2,\mathbb{C}}) \\ &= V_\mathbb{R} \otimes u_2' \backslash (\ell' \otimes u_2') \cong \mathbb{C}\ell.\end{aligned}$$

By symmetry, we can see that the action of $g \in \mathrm{SU}(V_\mathbb{R})$ is now given by the Schrödinger model, and hence we have the following formula

$$\mathcal{F}(M_\natural[g]M_{Sch}[g']f)(v) = M_{Sch}[g]M_\natural[g'](\mathcal{F}f)(v)$$

In the next chapter, we will be constructing a suitable test function φ. This function will behave particularly nicely under the maximal compact subgroup K of $\mathrm{SU}(V_\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. This will allow us to realise $\Theta((g, g'), \varphi, \Lambda)$ as a function (actually as a modular form) on $\mathbb{H} \times \mathbb{H}$.

Let $F : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ be a function on $\mathrm{SL}_2(\mathbb{R})$ which is semi-invariant with weight k under right multiplication by $r_\theta \in K$ the maximal compact subgroup, i.e.

$$F(gr_\theta) = e^{ik\theta}F(g)$$

and is left-invariant under some congruence subgroup Γ, i.e.

$$F(\gamma g) = F(g).$$

We can define a new function $f(z) : \mathbb{H} \to \mathbb{C}$ by

$$f(z) = F(g_z)j(g_z, i)^k$$

where the equation $g_z i = z$ defines the group element g_z, up to a multiple of $K = \mathrm{Stab}(i)$. First we establish that this is well defined, namely, if $g_z' i = z$ then clearly $g_z' = g_z r_\theta$ for

$$r_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix},$$

so we simply have

$$F(g_z')j(g_z', i)^k = F(g_z)e^{ik\theta}j(g_z, r_\theta \cdot i)^k j(r_\theta, i)^k = F(g_z)j(g_z, i)^k$$

by the semi-invariance of F, the co-cycle relation and the fact that $j(r_\theta, i)^k = e^{-ik\theta}$. This shows that f is a well defined function on $G/K = \mathbb{H}$. We can also show that f is a modular form for Γ of weight k, under the action of $\gamma \in \Gamma$ we have

$$f(\gamma z) = F(g_{\gamma z})j(g_{\gamma z}, i)^k.$$

Clearly, by the definition of g_z, we have that $g_{\gamma z} = \gamma g_z$, hence

$$\begin{aligned} f(\gamma z) &= F(\gamma g_z)j(\gamma g_z, i)^k \\ &= F(g_z)j(\gamma, g_z \cdot i)^k j(g_z, i)^k \\ &= j(\gamma, z)^k f(z) \end{aligned}$$

so we see that a function F on the group which is semi-invariant under K of weight k and left invariant under Γ defines a modular form of weight k for Γ. In a similar way, every modular form also defines a Γ invariant function on the group, by $F(g) = f(g \cdot i)j(g, i)^{-k}$.

We can apply this to the functions $\Theta((g, g'), \varphi, \Lambda)$. Assume that

$$M_\natural[g r_\theta]\varphi(X) = e^{ik\theta}\varphi_X$$

and

$$M_\natural[g' r_\theta'](\mathcal{F}\varphi)(X) = e^{ik'\theta}(\mathcal{F}\varphi)(X)$$

We can then realise $\Theta((g, g'), \varphi, \Lambda)$ as

$$\Theta(z, g', \varphi, \Lambda) = y^{-k} \sum_{X \in \Lambda} M_{\natural}[g_z] M_{Sch}[g'] \varphi(X)$$

and, using the mixed model

$$\Theta(g, \tau, \varphi, \Lambda) = v^{-k} \sum_{X \in \Lambda} \mathcal{F}^{-1}(M_{\natural}[g_\tau] M_{Sch}[g](\mathcal{F}\varphi)(X))$$

hence it makes sense to think of Θ as being a function on $\mathbb{H} \times \mathbb{H}$. If we assume that $g, g' \in Stab(\Lambda)$, then the fact that $\Theta(g, g')$ is invariant under the action $g \mapsto \gamma g$ is pretty obvious - it merely transfers to an inverse action on $X \in \Lambda$. If we also assume that Λ contains its own dual under Fourier inversion in the first co-ordinate, then we can, by using Poisson summation, conclude that Θ is invariant under $g' \mapsto \gamma g'$ by the same argument. Hence, under all these assumptions, Θ will be a modular form for $\Gamma = Stab(\Lambda)$ with weight k in z and weight k' in τ.

2.3 Vector Valued Modular Forms

The primary references used for this section are [Bru02] and [Kud79], both of which rely on calculations in [Shi75]. We first make some general definitions necessary for discussion of vector valued modular forms, before specialising to the case we need.

First, we recall that $\mathrm{SL}_2(\mathbb{Z})$ is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now let $L \subset V$ be an even $\mathcal{O}_F$-lattice of full rank and let $\{\mathfrak{e}_{\mathbf{h}}\}_{\mathbf{h} \in L'/L}$ be a basis of the group algebra $\mathbb{C}[L'/L]$. There is a unitary representation ϱ_L of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$, defined for the generators of $\mathrm{SL}_2(\mathbb{Z})$ given above by

$$\varrho_L(T)\mathfrak{e}_{\mathbf{h}} = \exp(\pi i \langle \mathbf{h}, \mathbf{h} \rangle)\mathfrak{e}_{\mathbf{h}}$$
$$\varrho_L(S)\mathfrak{e}_{\mathbf{h}} = \frac{1}{\sqrt{|L'/L|}} \sum_{\mathbf{k} \in L'/L} \exp(\pi i \langle \mathbf{k}, \mathbf{h} \rangle)\mathfrak{e}_{\mathbf{k}}.$$

These formulas follow from [Kud79, Prop. 2(i)], or from [BF04].

Since the signature of the lattice we are using is even, this representation in fact factors through $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ where N is the smallest positive integer such that $N\langle \mathbf{h}, \mathbf{h}\rangle \in \mathbb{Z}$ for all $\mathbf{h} \in L'$ (see [Bru02]). In general (i.e. in odd dimension) one needs to consider the Metaplectic group, the pre-image of $\mathrm{SL}_2(\mathbb{Z})$ in topological double cover of $\mathrm{SL}_2(\mathbb{R})$.

We now define the slash operator on $\mathbb{C}[L'/L]$-valued functions. Let $k \in \frac{1}{2}\mathbb{Z}$, $M \in \mathrm{SL}_2(\mathbb{Z})$ and $f : \mathbb{H} \to \mathbb{C}[L'/L]$. Then the weight k slash operator is defined as

$$(f \mid_k M)(\tau) = j(M, \tau)^{-k} \varrho_L(M)^{-1} f(M\tau)$$

We are now in a position to define a vector-valued modular form.

Definition 2.3.1. A vector valued modular form of weight k with respect to ϱ_L is a function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ which satisfies:

- $f \mid_k M = f$ for all $M \in \mathrm{SL}_2(\mathbb{Z})$
- f is holomorphic on $\mathbb{H}$
- f is holomorphic at the cusp ∞, i.e. has a Fourier expansion of the form

$$f(\tau) = \sum_{\mathbf{h} \in L'/L} \sum_{\substack{n \in \mathbb{Z} + \frac{1}{2}\langle \mathbf{h}, \mathbf{h}\rangle \\ n \geq 0}} c(\mathbf{h}, n) \exp(2\pi i n \tau) \mathfrak{e}_{\mathbf{h}}$$

We can extend this definition in several ways. For example, we can define vector valued forms for a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ by requiring that the function f need only be invariant under the $|_k$ operator for that subgroup, but must be holomorphic at all cusps associated to Γ. We can also define the notion of a weak vector valued form, by allowing poles of finite orders at cusps.

Chapter 3

Constructing the Theta function

We will begin this chapter by explicitly constructing some geometrical quantities which are key to the construction of the theta kernel. This theta kernel is constructed by summing over the lattice L a Schwartz function φ_{KM} which is from [KM86] and [KM87]. It is essentially the normal Gaussian function under the image of a certain differential operator.

The function φ^0_{KM} depends on a vector X and point $z \in \mathbb{H}$. We will construct a function ξ^0 for which, as currents, we have the relation

$$dd^c[\xi^0] + \delta_{\mathbb{D}_X} = [\varphi^0_{KM}].$$

which is key in our calculation of the Fourier coefficients of the lift of harmonic weak Maass forms in Chapter 5.

3.1 Construction of φ_{KM}

Analagous to the real signature $(1,2)$ case in [Kud97], we make the following definitions. Let $X \in V$, so that for any $z \in \mathbb{H}$ we may decompose X as

$$X = X_z^{\perp} + \frac{\langle X, X(z)\rangle}{\langle X(z), X(z)\rangle} X(z)$$

where $X_z^{\perp}$ is the component of X orthogonal to $X(z)$. Hence we have

$$\langle X, X\rangle = \langle X_z^{\perp}, X_z^{\perp}\rangle + \frac{\hat{\delta}}{4}|\langle X, X(z)\rangle|^2.$$

We then define

$$R(X,z) = -\langle X_z^\perp, X_z^\perp \rangle = \frac{\hat{\delta}}{4}|\langle X, X(z)\rangle|^2 - \langle X, X\rangle. \tag{3.1}$$

Lemma 3.1.1. *$R(X,z)$ is real and always greater than or equal to zero, with equality when $X = 0$ or when z lies on $\mathbb{D}_X$, i.e. if we write $X = w\ell + w'\ell'$, then when $z = \frac{w}{w'}$.*

Proof. The case of $X = 0$ is obvious. For any $X \neq 0$, we must have that $X_z^\perp$ generates a negative definite subspace if it is not equal to 0, since $\langle X(z), X(z)\rangle > 0$ and hence $X(z)$ generates a positive definite one. If $X_z^\perp = 0$, then X lies in the direction of $X(z)$, which is equivalent to $z = \mathbb{D}_X$. □

We can relate $R(X,z)$ and $|\langle X, X(z)\rangle|^2$ to the minimal majorant by

$$\langle X, X\rangle_z = \frac{\hat{\delta}}{2}|\langle X, X(z)\rangle|^2 - \langle X, X\rangle = 2R(X,z) + \langle X, X\rangle. \tag{3.2}$$

We now give explicit formulas for each of these quantities.

Proposition 3.1.2. *Let $X = w\ell + w'\ell'$ and $z = x + iy$. The quantities mentioned above are given explicitly by*

$$\begin{aligned} |\langle X, X(z)\rangle|^2 &= 4\hat{\delta}^{-2}y^{-1}\left|\overline{z}w' - w\right|^2 \\ R(X,z) &= (y\hat{\delta})^{-1}\left|zw' - w\right|^2 \\ \langle X, X\rangle_z &= 2(y\hat{\delta})^{-1}\left(\left|xw' - w\right|^2 + y^2|w'|^2\right) \end{aligned}$$

Proof. We recall that

$$X(z) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{y}}\begin{pmatrix} z \\ 1 \end{pmatrix}$$

and hence

$$\begin{aligned} \langle X, X(z)\rangle &= (\overline{w}, \overline{w}')\begin{pmatrix} 0 & 2\delta^{-1} \\ -2\delta^{-1} & 0 \end{pmatrix}\begin{pmatrix} \sqrt{y}^{-1}z \\ \sqrt{y}^{-1} \end{pmatrix} \\ &= -\frac{2}{\delta\sqrt{y}}(z\overline{w}' - \overline{w}) \end{aligned}$$

and hence

$$|\langle X, X(z)\rangle|^2 = 4\hat{\delta}^{-2}y^{-1}\left|\overline{z}w' - w\right|^2.$$

We recall that

$$\langle X, X\rangle = 4\hat{\delta}^{-1}\Im(w\overline{w}')$$

and using Equation (3.1),

$$\begin{aligned} R(X,z) &= \frac{\hat{\delta}}{4}|\langle X, X(z)\rangle|^2 - \langle X, X\rangle \\ &= \hat{\delta}^{-1}y^{-1}\left|\overline{z}w' - w\right|^2 - 4\hat{\delta}^{-1}\Im(w\overline{w}') \\ &= (y\hat{\delta})^{-1}\left|zw' - w\right|^2. \end{aligned}$$

Finally, we can use these explicit formulas and Equation (3.2) to show that

$$\begin{aligned} \langle X, X\rangle_z &= 2R(X,z) + \langle X, X\rangle \\ &= 2(y\hat{\delta})^{-1}\left|zw' - w\right|^2 + 4\hat{\delta}^{-1}\Im(w\overline{w}') \\ &= 2(y\hat{\delta})^{-1}\left(\left|xw' - w\right|^2 + y^2|w'|^2\right) \end{aligned}$$

□

We now construct the Schwartz function we will use to construct the theta function. This Schwartz function was originally defined (in much greater generality) by Kudla and Millson in [KM86] and [KM87], where many properties of the lift in the compact case were considered.

Recall that V has a basis $\{e_1, e_2\}$ such that $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = 1$ and $\langle e_1, e_2\rangle = 0$. Let $X = v_1e_1 + v_2e_2$, and define the Gaussian

$$\varphi_S(X) = \exp(-\pi(|v_1|^2 + |v_2|^2))$$

noting that, in the notation established above,

$$\varphi_S(X) = \exp(-\pi\langle X, X\rangle_i).$$

This is weight 0, in the sense that it is invariant under K-action on the vector X. In [KM86], they define a certain differential operator

$$\nabla\overline{\nabla} = \frac{1}{8}\left(v_1 - \frac{1}{\pi}\frac{\partial}{\partial\overline{v_1}}\right)\left(\overline{v_1} - \frac{1}{\pi}\frac{\partial}{\partial v_1}\right)$$

and a new Schwartz function, the Kudla-Millson Schwartz function by

$$\widetilde{\varphi}_{KM}(X) = \nabla\overline{\nabla}\varphi_S(X).$$

Simply applying the operator in a straightforward manner gives

$$\widetilde{\varphi}_{KM}(X) = \frac{1}{8}\left(4|v_1|^2 - \frac{2}{\pi}\right)\exp(-\pi(|v_1|^2 + |v_2|^2)).$$

We can rewrite this formula in the following useful way:

$$\widetilde{\varphi}_{KM}(X) = \frac{1}{16\pi}\left(H_2\left(\sqrt{2\pi}\Re(v_1)\right) + H_2\left(\sqrt{2\pi}\Im(v_1)\right)\right)\varphi_S(X).$$

Here, $H_2(x) = 4x^2 - 2$ is the second Hermite polynomial.

We now define

$$\widetilde{\varphi}_{KM}(X, g) = M_\natural[g]\widetilde{\varphi}_{KM}(X) = \widetilde{\varphi}_{KM}(g^{-1}X)$$

and we claim that this function is right K-invariant. Since the action of g is simply the natural left action in the $\{\ell, \ell'\}$ basis, it follows by the co-ordinate transforms that

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \cdot (v_1e_1 + v_2e_2) = e^{i\theta}v_1e_1 + e^{-i\theta}v_2e_2$$

and therefore $|v_1|^2$ and $|v_2|^2$ are invariant under K and so is $\widetilde{\varphi}_{KM}(X, g)$. Hence we write $\widetilde{\varphi}_{KM}(X, z) = \widetilde{\varphi}_{KM}(X, g_z)$ with no ambiguity. Finally, we define

$$\varphi_{KM}(X, z) = \widetilde{\varphi}_{KM}(X, z)d\mu(z)$$

where $d\mu(z) = y^{-2}dxdy$, and we have,

Definition 3.1.3. For $X \in V$,

$$\varphi_{KM}(X, z) = \frac{1}{8}\left(\hat{\delta}|\langle X, X(z)\rangle|^2 - \frac{2}{\pi}\right)\exp(-\pi\langle X, X\rangle_z)d\mu(z)$$

where $z \in \mathbb{H}$.

For notational convenience later, we also define

$$\begin{aligned}\varphi^0_{KM}(X, z) &= \exp(\pi\langle X, X\rangle)\varphi_{KM}(X, z)\\ &= \frac{1}{8}\left(\hat{\delta}|\langle X, X(z)\rangle|^2 - \frac{2}{\pi}\right)\exp(-2\pi R(X, z))d\mu(z)\end{aligned}$$

Let $\alpha = \sqrt{\frac{\hat{\delta}}{2}}\Re(\langle X, X(z)\rangle)$ and $\beta = \sqrt{\frac{\hat{\delta}}{2}}\Im(\langle X, X(z)\rangle)$, then we have that

$$\varphi^0_{KM}(X, z) = \frac{1}{16\pi}(H_2(\sqrt{\pi}\alpha) + H_2(\sqrt{\pi}\beta))\exp(-\pi(\alpha^2 + \beta^2))\exp(2\pi\langle X, X\rangle)d\mu(z)$$

which will be useful later.

We now prove a few properties of $\varphi_{KM}(X, z)$.

Proposition 3.1.4. *The functions φ_{KM} and φ^0_{KM} are invariant under diagonal action of $\gamma \in \mathrm{SL}_2(\mathbb{R})$, i.e. $\varphi_{KM}(\gamma X, \gamma z) = \varphi_{KM}(X, z)$ and $\varphi^0_{KM}(\gamma X, \gamma z) = \varphi^0_{KM}(X, z)$*

Proof. Firstly, we note that the hyperbolic metric $y^{-2}dxdy$ is invariant under $\mathrm{SL}_2(\mathbb{R})$. Since, for any $\gamma \in \mathrm{SL}_2(\mathbb{R})$, by definition, $\langle \gamma X, \gamma Y\rangle = \langle X, Y\rangle$, and $X(\gamma z) = \gamma X(z)$, it is clear that

$$R(\gamma X, \gamma z) = \frac{\hat{\delta}}{4}|\langle \gamma X, X(\gamma z)\rangle|^2 - \langle \gamma X, \gamma X\rangle = \frac{\hat{\delta}}{4}|\langle X, X(z)\rangle|^2 - \langle X, X\rangle = R(X, z)$$

hence,

$$\begin{aligned}\varphi^0_{KM}(\gamma X, \gamma z) &= \frac{1}{8}\left(\hat{\delta}|\langle \gamma X, X(\gamma z)\rangle|^2 - \frac{2}{\pi}\right)\exp(-2\pi R(\gamma X, \gamma z))\frac{dx\wedge dy}{y^2} \\ &= \frac{1}{8}\left(\hat{\delta}|\langle X, X(z)\rangle|^2 - \frac{2}{\pi}\right)\exp(-2\pi R(X, z))\frac{dx\wedge dy}{y^2}\end{aligned}$$

and φ_{KM} follows similarly. □

Analogously to [BF06][Prop 3.2], we have

Theorem 3.1.5. *Assume that $\langle X, X\rangle = 2m > 0$. The form $\varphi^0_{KM}(X, z)$ has linear exponential decay in y and square linear exponential decay in x in all directions, i.e.*

$$O(\exp(-Cx^2)d\mu(z)) \text{ as } x \to \pm\infty$$

$$O(\exp(-Cy)d\mu(z)) \text{ as } y \to \infty$$

$$O(\exp(-C/y)d\mu(z)) \text{ as } y \to 0$$

for some constant $C > 0$ in each case, uniformly in y in the first case and uniformly in x in the other two. In particular, $C = \frac{\pi m}{\Im(\mathbb{D}_X)}$ in the case $y \to \infty$.

Proof. This is clear from the formula for $R(X, z)$. In particular, for $y \to \infty$ it is clear that the y term dominates, and its coefficient is $|w'|^2\hat{\delta}^{-1}$. By the formula for the length of a vector X, we see that $|w'|^2 = \frac{m\hat{\delta}}{2\Im(\mathbb{D}_X)}$, whence the result. □

Theorem 3.1.6. *The 2-form $\varphi^0_{KM}(X, z)$ is normalised to have a volume of 1 over the upper half plane, i.e.*

$$\int_{\mathbb{H}} \varphi^0_{KM}(X, z) = 1.$$

for any X such that $\langle X, X\rangle > 0$. Similarly, for X such that $\langle X, X\rangle < 0$ we have

$$\int_{\mathbb{H}} \varphi^0_{KM}(X, z) = 0.$$

Proof. The integral we are concerned with is

$$\int_{\mathbb{H}} \frac{1}{8}\left(\hat{\delta}|\langle X, X(z)\rangle|^2 - \frac{2}{\pi}\right)\exp(-2\pi R(X,z))\frac{dx\wedge dy}{y^2}.$$

The first step is to use the fact that $2R(X,z) + 2\langle X, X\rangle = \frac{\hat{\delta}}{2}|\langle X, X(z)\rangle|^2$,

$$\int_{\mathbb{H}} \frac{1}{8}\left(\hat{\delta}|\langle X, X(z)\rangle|^2 - \frac{2}{\pi}\right)\exp(-\pi\frac{\hat{\delta}}{2}|\langle X, X(z)\rangle|^2)\exp(2\pi\langle X, X\rangle)\frac{dxdy}{y^2}.$$

We now substitue in the formula for $|\langle X, X(z)\rangle|^2$

$$\int_{\mathbb{H}} \frac{1}{8}\left(4(\hat{\delta}y)^{-1}\left|\overline{z}w' - w\right|^2 - \frac{2}{\pi}\right)\exp(-\pi 2(\hat{\delta}y)^{-1}\left|\overline{z}w' - w\right|^2)\exp(2\pi\langle X, X\rangle)\frac{dxdy}{y^2}.$$

Since we are assuming that $\langle X, X\rangle \neq 0$, then we can assume that $w' \neq 0$, and hence the integral we are concerned with is

$$\int_{\mathbb{H}} \frac{1}{8}\left(4(\hat{\delta}y)^{-1}\left|w'\right|^2((x - \Re(w/w'))^2 + (y + \Im(w/w'))^2) - \frac{2}{\pi}\right)$$
$$\exp(-2\pi(\hat{\delta}y)^{-1}\left|w'\right|^2((x - \Re(w/w'))^2 + (y + \Im(w/w'))^2))\exp(2\pi\langle X, X\rangle)\frac{dxdy}{y^2},$$

but since we are integrating x between $-\infty$ and ∞, this is equal to

$$\frac{1}{4\pi}\int_{\mathbb{H}}\left(2\pi(\hat{\delta}y)^{-1}\left|w'\right|^2(x^2 + (y + \Im(w/w'))^2) - 1\right)$$
$$\exp(-2\pi(\hat{\delta}y)^{-1}\left|w'\right|^2(x^2 + (y + \Im(w/w'))^2))\exp(2\pi\langle X, X\rangle)\frac{dxdy}{y^2}.$$

From here it is relatively straightforward to calculate this integral using standard results, and we see that the value depends on the sign of $\Im(w/w')$, i.e. the sign $\langle X, X\rangle$.

$$\frac{1}{4\pi}\int_{\mathbb{H}}\left(2\pi(\hat{\delta}y)^{-1}\left|w'\right|^2(x^2 + (y + \Im(w/w'))^2) - 1\right)$$
$$\exp(-2\pi(\hat{\delta}y)^{-1}\left|w'\right|^2(x^2 + (y + \Im(w/w'))^2))\exp(2\pi\langle X, X\rangle)\frac{dxdy}{y^2}$$
$$= \begin{cases} \exp(-8\pi\hat{\delta}^{-1}\Im(w\overline{w}'))\exp(2\pi\langle X, X\rangle) = 1 & \langle X, X\rangle > 0 \\ 0 & \langle X, X\rangle < 0 \end{cases}$$

□

This is actually a prerequisite for Theorem 3.2.7, but in order to state it we need to introduce the notion of a current. In a loose sense, currents play a similar role to forms as distributions do to functions.

3.2 Currents and Greens Functions

Definition 3.2.1. [Lan88, Ch.1, §3] Let d be the usual exterior differential. For any function f on a complex manifold, we define operators ∂, $\overline{\partial}$ such that $d = \partial + \overline{\partial}$

$$\partial f = \frac{\partial f}{\partial z} dz, \qquad \overline{\partial} f = \frac{\partial f}{\partial \overline{z}} d\overline{z}.$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

We also define

$$d^c = \frac{1}{4\pi i}(\partial - \overline{\partial})$$

so that

$$dd^c = -\frac{1}{2\pi i}\partial\overline{\partial}.$$

Definition 3.2.2. [Sou92, §II.1] A current is a linear functional on $A^n(X)$, the space of complex valued differential forms with compact support of degree n of a smooth manifold X of complex dimension d, which is continuous in the sense that for any sequence $\{\omega_r\} \subset A^n(X)$ which are all supported in some fixed compact set K and for any T a current, we have $T(\omega_r) \to 0$ if $\omega_r \to 0$.

Example 3.2.3. Any differential form α of degree (p, q) defines a current on the differential forms of codegree $(d - p, d - q)$ via

$$[\alpha](\beta) = \int_X \alpha \wedge \beta$$

Proposition 3.2.4. *[Sou92, §II.1,1.2] For any α, a form of degree n, we have*

$$d[\alpha](\beta) = (-1)^{n+1}[\alpha](d\beta)$$

and similarly for ∂, $\overline{\partial}$ and d^c.

Example 3.2.5. The usual Dirac delta function defines a current on functions (i.e. 0-degree forms) in the obvious way, i.e.

$$[\delta_z](f) = f(z)$$

for some $z \in X$.

The standard definition of a current is too restrictive for our purposes. We wish to allow our currents to act on differential forms which do not have compact support. This necessitates a trade-off; consider Example 3.2.3 above, whatever we do, this integral must converge. Clearly the worse the behaviour of β we want to be able to deal with, the better behaved α must be. In distributions, this logic plays out with the notion of a *tempered* distribution, whereby in exchange for restricting ourselves to distributions with at most logarithmic growth, we are allowed test functions which are Schwartz functions (i.e. rapidly decaying), rather than compactly supported.

We wish to take as our input differential forms with linear-exponential growth. In order for this to be valid, we will take care that the (equivalent of the) integral given by Example 3.2.3 always converges.

We quote now a definition and theorem about currents on compactly supported forms.

Definition 3.2.6. A Greens current for a point P with respect to a Schwartz function φ is a function $g_P : M - P \to \mathbb{R}$ such that $dd^c g_P = \varphi$ away from P and, for a compactly-supported function f we have

$$\int_M g_P d^c df + f(P) = \int_M f\varphi \tag{3.3}$$

Equivalently, we say

$$dd^c[g_P] + \delta_P = [\varphi]$$

as currents.

Theorem 3.2.7. *[Lan88, Ch. II,§1, Thm 1.5]Let M be a Riemann surface over the complex numbers. Let φ be a normalised $(1,1)$-form, i.e.*

$$\int_M \varphi = 1$$

Let D be a divisor on M, represented by f. Then there exists a Greens current for $P \in M$ with respect to φ,

We do not prove this here, however we quote the following very useful Lemma which the proof essentially reduces to, and which we need later.

Lemma 3.2.8. *[Lan88, Ch. II,§1, p.p. 23] Let $C(P,a)$ be the anticlockwise oriented circle around P of radius a. If β is a smooth function in a neighbourhood of P and $\alpha = k\log r + O(1)$ for some constant k, then*

$$\lim_{\epsilon\to 0}\int_{C(P,\epsilon)} \alpha d^c\beta = 0.$$

If $\beta = \log r^2 + O(1)$ and α is continuous, then

$$\lim_{\epsilon\to 0}\int_{C(P,\epsilon)} \alpha d^c\beta = \alpha(P)$$

If we wish to extend Theorem 3.2.7 to f which do not have compact support, we must be able to make sense of Equation 3.3. This will depend highly upon the properties of g_P and φ of course - the "badness" of growth allowed in f must be balanced by well-behaved decay in g_P and φ, so that the relevant integrals converge.

3.2.1 A Greens function for φ^0_{KM}

We will construct a Green's function for the point $\mathbb{D}_X$ with respect to the function $\varphi^0_{KM}(\sqrt{v}X, z)$ for $v > 0$, following a method used by Kudla in [Kud97]. In order to do this, we now recall a few facts about the exponential integral function, $\mathrm{Ei}(z)$, defined for $z \in \mathbb{C}$ by

$$\mathrm{Ei}(z) = \int_{-\infty}^{z} \frac{e^t}{t} dt.$$

The path of integration stays away from the ray defined by $\{z \in \mathbb{C} : \Re(z) \leq 0,\ \Im(z) = 0\}$. We may rewrite this as

$$\mathrm{Ei}(z) = \gamma + \log(-z) + \int_0^z \frac{e^t - 1}{t} dt \tag{3.4}$$

where γ is Euler's constant. We note that the integral on the right hand side is an entire function, which implies that $\mathrm{Ei}(z)$ has a logarithmic singularity at 0 [GR07, Sec. 8.21]. Hence $Ei(z)$ has logarithmic growth as $z \to 0$ and linear exponential decay as $z \to -\infty$.

Theorem 3.2.9 (Extended Current Equation)**.** *Let*

$$\xi^0(\sqrt{v}X, z) = -\mathrm{Ei}(-2\pi R(\sqrt{v}X, z))$$

be defined for all $X \in V_{\mathbb{R}}$ *and for* $v > \frac{2n\Im(\mathbb{D}_X)}{m}$ *for some* $n \in \mathbb{N}$. *Then, for each* $X \neq 0$, $\xi^0(\sqrt{v}X, z)$ *is a Greens current for* $\mathbb{D}_X$ *with respect to* $\varphi^0_{KM}(\sqrt{v}X, z)$ *and for any function* $f(z) = O(exp(2\pi ny))$ *as* $y \to \infty$,

$$\int_{\mathbb{D}} f(z)\varphi^0(\sqrt{v}X, z) = \begin{cases} \int_{\mathbb{D}} \xi^0(\sqrt{v}X, z) d^c df & \text{if } \langle X, X\rangle < 0 \\ \int_{\mathbb{D}} \xi^0(\sqrt{v}X, z) d^c df + f(\mathbb{D}_X) & \text{if } \langle X, X\rangle \geq 0 \end{cases}$$

and hence, as currents,

$$dd^c[\xi^0(\sqrt{v}X, z)] + \delta_{\mathbb{D}_X} = [\varphi^0_{KM}(\sqrt{v}X, z)].$$

Proof. We first show that

$$dd^c\xi^0 = \varphi^0_{KM} \text{ for all } z \neq \mathbb{D}_X \tag{3.5}$$

and then we examine the behaviour in a neighbourhood of $\mathbb{D}_X$.

We firstly deal with behaviour on the complement $\mathbb{D} - \mathbb{D}_X$. This basically goes as in [Kud97, Prop 11.1]. We have that

$$-\partial\overline{\partial}\xi^0(X, z) = -2\pi\frac{e^{-2\pi R}}{R}\partial R \wedge \overline{\partial}R + \frac{e^{-2\pi R}}{R^2}(-\partial R \wedge \overline{\partial}R + R\partial\overline{\partial}R) \tag{3.6}$$

and we may calculate from Proposition 3.1.2 that

$$\partial R = \left(\frac{i}{2y}R + \frac{w'}{\hat{\delta}y}\left(\overline{w'z - w}\right)\right) dz \tag{3.7}$$

$$\overline{\partial}R = \left(-\frac{i}{2y}R + \frac{\overline{w}'}{\hat{\delta}y}(w'z - w)\right) d\overline{z} \tag{3.8}$$

and hence,

$$\partial R \wedge \overline{\partial}R = \frac{\hat{\delta}}{16}R|\langle X, X(z)\rangle|^2\frac{dz \wedge d\overline{z}}{y^2}$$
$$\partial\overline{\partial}R = \frac{1}{4}\left(R + \frac{\hat{\delta}}{4}|\langle X, X(z)\rangle|^2\right)\frac{dz \wedge d\overline{z}}{y^2}.$$

Substituting this into (3.6), and using that $-\partial\overline{\partial} = 2\pi i dd^c$ and $y^{-2}dz \wedge d\overline{z} = -2iy^{-2}dx \wedge dy$, we obtain that for z away from $\mathbb{D}_x$,

$$dd^c\xi^0(\sqrt{v}X, z) = \frac{1}{8}\left(\hat{\delta}|\langle\sqrt{v}X, X(z)\rangle|^2 - \frac{2}{\pi}\right)e^{-2\pi R}\frac{dx \wedge dy}{y^2} = \varphi^0_{KM}(\sqrt{v}X, z)$$

We now prove the full result. Firstly, we deal with convergence issues. We know that $f(z) = O(\exp(2\pi ny))$ and that $\xi^0(\sqrt{v}X, z)$ and $\varphi^0_{KM}(\sqrt{v}X, z)$ are $O(\exp(-\frac{\pi vmy}{\Im(\mathbb{D}_X)}))$.

Hence, since $v > \frac{2n\Im(\mathbb{D}_X)}{m}$ by assumption, $\xi^0(\sqrt{vX}, z)d^c df$ and $f(z)\varphi^0_{KM}(\sqrt{v}X, z)$ are $O(\exp(-2\pi\epsilon y))$ for some $\epsilon > 0$. Therefore these integrals converge. We note that the bound on v is essential, without it, the integrals diverge and none of what follows makes any sense.

Now, we slightly restate the claim. We put the integration all on one side, and break it up into two parts;

$$\lim_{\epsilon\to 0}\int_{X-C(\mathbb{D}_X,\epsilon)} \xi^0 d^c df - f\varphi^0_{KM}$$

and

$$\lim_{\epsilon\to 0}\int_{C(\mathbb{D}_X,\epsilon)} \xi^0 d^c df - f\varphi^0_{KM}.$$

Since

$$\xi^0 d^c df - f\varphi^0_{KM} = -\xi^0 dd^c f - f dd^c\xi^0 = -d(\xi^0 d^c f + f d^c\xi^0)$$

on the complement of $\mathbb{D}_X$, we can now apply Stokes' Theorem; clearly the second part vanishes and so we now need to prove

$$\lim_{\epsilon\to 0}\int_{C(\mathbb{D}_X,\epsilon)} \xi^0 d^c f + f d^c\xi^0 = f(\mathbb{D}_X)$$

which is readily apparent from Lemma 3.2.8 and the logarithmic growth of ξ^0. □

3.2.2 The theta function

We define our theta function on the group $\mathrm{SU}(V_{\mathbb{R}}) \times \mathrm{SL}_2(\mathbb{R})$ as

$$\Theta(g, g', L) = \sum_{X\in L'} M_\natural[g]M_{Sch}[g']\varphi_{KM}(X)\mathfrak{e}_X$$

As discussed above, by the right K-invariance of Θ in the $\mathrm{SL}_2(\mathbb{R})$ argument, it makes sense to think of this as a function on $G/K \cong \mathbb{H}$. Since, as we showed in Chapter 1, $\mathrm{SU}(V_{\mathbb{R}}) \cong \mathrm{SL}_2(\mathbb{R})$, it is natural to ask whether we can do the same on the g' component and thus realise Θ as a function on $\mathbb{H} \times \mathbb{H}$. The answer to this question is yes, and furthermore, we have that Θ is left-invariant under $\operatorname{Stab} L$ in the g' component as well. We refer to [KM86, Thm 3.1] for the K-invariance on g', and we will recover the transformation formula for τ later, which is equivalent to left Γ_L invariance on g'. Hence we make the following definitions:

Definition 3.2.10. Let $g_\tau \in \mathrm{SL}_2(\mathbb{R})$ be defined by $g_\tau i = \tau$. It is only defined up to a factor of K, one realisation of which is

$$g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{v} & 0 \\ 0 & \sqrt{v}^{-1} \end{pmatrix}.$$

The theta function associated to the lattice L is, for $z, \tau \in \mathbb{H}$

$$\Theta(\tau, z, L) = \sum_{X \in L'} v^{-1} M_{Sch}[g_\tau] \varphi^0_{KM}(X, z) \mathfrak{e}_X$$

We decompose the sum into L cosets to obtain theta functions for each $\mathbf{h} \in L'/L$ so that

$$\Theta(\tau, z, L) = \sum_{\mathbf{h} \in L'/L} \theta_{\mathbf{h}}(\tau, z, L) \mathfrak{e}_{\mathbf{h}}$$

where

$$\theta_{\mathbf{h}}(\tau, z, L) = \sum_{L+\mathbf{h}} v^{-1} M_{Sch}[g_\tau] \varphi^0_{KM}(X, z).$$

The factor of v^{-1} appears because of the weight in the τ variable, or more specifically, because of the way $M_{Sch}[g'] \varphi_{KM}(X, z)$ transforms under right K action, i.e.

$$M_{Sch}[g' r_\theta] \varphi_{KM}(X, z) = e^{2i\theta} M_{Sch}[g'] \varphi_{KM}(X, z).$$

Then, in order to make this a function on $\mathbb{H}$ rather than on the group, we need to multiply by $j(g_\tau, i)^2 = v^{-1}$. We note that when this was done previously to convert from a function in g to a function in z no additional factor was necessary as the weight is 0, i.e. the g variable is invariant under the right action of K.

By now we have established a number of different notations for expressing the theta function throughout this chapter. We collect them together in the proposition below for ease of reference.

Proposition 3.2.11. *We have the following equivalent expressions for* $\theta_{\mathbf{h}}(\tau, z, L)$

$$\begin{aligned} \theta_{\mathbf{h}}(\tau, z, L) &= \sum_{X \in L+h} \varphi_{KM}(X, \tau, z) \\ &= \sum_{X \in L+h} \varphi^0_{KM}(\sqrt{v} X, z) \exp(\pi i \langle X, X \rangle \tau) \\ &= \sum_{X \in L+h} \frac{1}{8} \left(\hat{\delta} v |\langle X, X(z) \rangle|^2 - \frac{2}{\pi} \right) \exp\left(-\frac{\pi v \hat{\delta}}{2} |\langle X, X(z) \rangle|^2 \right) \\ &\qquad \exp\left(\pi i \langle X, X \rangle \overline{\tau} \right) d\mu(z) \end{aligned}$$

Where we define φ_{KM} *by the equality*

$$\varphi_{KM}(X,\tau,z) = v^{-1}M_{Sch}[g_\tau]M_\natural[g_z]\varphi_{KM}(X)d\mu(z).$$

Additionally, if we make the following substitutions:

$$\alpha = \sqrt{\frac{v\hat{\delta}}{2}}\Re(\langle X, X(z)\rangle)\ , \qquad \beta = \sqrt{\frac{v\hat{\delta}}{2}}\Im(\langle X, X(z)\rangle)$$

this allows us to write

$$\varphi_{KM}(X,\tau,z) = \frac{1}{16\pi}(H_2(\sqrt{\pi}\alpha) + H_2(\sqrt{\pi}\beta))\exp(-\pi(\alpha^2+\beta^2))\exp(\pi i\langle X, X\rangle\overline{\tau})d\mu(z)$$

where $H_n(x)$ *is the* n*-th Hermite polynomial, which is defined by*

$$H_n = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2},$$

and, in particular,

$$H_2(x) = 4x^2 - 2.$$

For reference, we state the transformation laws for Θ in each argument.

Theorem 3.2.12. *For any* $\gamma \in \Gamma_L$*, we have that*

$$\theta_{\mathbf{h}}(\tau, \gamma z, L) = \theta_{\mathbf{h}}(\tau, z, L)$$

Proof. This is fairly immediate by Proposition 3.1.4 and the definition of Γ_L. We have that

$$\begin{aligned}
\theta_{\mathbf{h}}(\tau, \gamma z, L) &= \sum_{X\in L+h}\varphi^0_{KM}(\sqrt{v}X, \gamma z)\exp(\pi i\langle X, X\rangle\tau)\\
&= \sum_{X\in L+h}\varphi^0_{KM}(\sqrt{v}\gamma^{-1}X, z)\exp(\pi i\langle \gamma^{-1}X, \gamma^{-1}X\rangle\tau)\\
&= \sum_{\gamma^{-1}X\in L+h}\varphi^0_{KM}(\sqrt{v}X, z)\exp(\pi i\langle X, X\rangle\tau)\\
&= \theta_{\mathbf{h}}(\tau, z, L)
\end{aligned}$$

□

Theorem 3.2.13. *For any* $\gamma \in \Gamma_L$*, we have that* Θ *is invariant under* $|_2[\gamma]$*, i.e.*

$$(\Theta|_2[\gamma])(\tau, z, L) = \Theta(\tau, z, L).$$

Proof. This is implicit in [KM86], [KM87]. However, we will recover this result in Section 3.4. □

3.3 Some commutativity relations

Definition 3.3.1. The lowering operator in τ is defined by

$$L_\tau = -2iv^2 \frac{\partial}{\partial \overline{\tau}}$$

A small calculation yields

$$\begin{aligned} L_\tau \xi^0 &= -2iv^2 \frac{\partial}{\partial \overline{\tau}} \Big(- \mathrm{Ei}(-2\pi v R(X,z)) \Big) \\ &= -v^2 \frac{\partial}{\partial v} \mathrm{Ei}(-2\pi v R(X,z)) \\ &= -v^2 \frac{\partial(-2\pi v R)}{\partial v} \frac{e^{-2\pi v R}}{-2\pi v R} \\ &= -v^2 \frac{-2\pi R}{-2\pi v R} e^{-2\pi v R} \\ &= -v^2 e^{-2\pi v R} = -\varphi_S. \end{aligned}$$

We now seek to establish the following commutative diagram:

$$\begin{array}{ccc} \xi^0 & \xrightarrow{L_\tau} & -\varphi_S \\ {\scriptstyle dd^c}\big\downarrow & & \big\downarrow{\scriptstyle dd^c} \\ \varphi^0_{KM} & \xrightarrow{L_\tau} & \phi \end{array}$$

Where ϕ is implicitly defined by the diagram.

It is clear that the operators must commute (hence ϕ is well defined), so really it really only remains to show that $L_\tau \xi^0 = \varphi_S$. However, we calculate ϕ directly in both ways for completenesses sake.

We now calculate $dd^c\varphi_0$ and $L_\tau \varphi_{KM}$ and see that they are equal. We first calculate

$$\begin{aligned} L_\tau \varphi^0_{KM} &= -2iv^2 \frac{1}{8} \frac{\partial}{\partial \overline{\tau}} \left(\hat{\delta} v |\langle X, X(z)\rangle|^2 - \frac{2}{\pi} \right) e^{-2\pi v R} d\mu(z) \\ &= \frac{v^2}{8} \left(\left(\hat{\delta} v |\langle X, X(z)\rangle|^2 - \frac{2}{\pi} \right) (-2\pi R) + \hat{\delta} |\langle X, X(z)\rangle|^2 \right) e^{-2\pi v R} d\mu(z) \\ &= \frac{v^2}{4} \left(2R + \tfrac{1}{2} \hat{\delta} |\langle X, X(z)\rangle|^2 - \pi \hat{\delta} v R |\langle X, X(z)\rangle|^2 \right) e^{-2\pi v R} d\mu(z) \\ &= \frac{v^2}{4} \left(2\langle X, X\rangle_z - \pi \hat{\delta} v R |\langle X, X(z)\rangle|^2 \right) e^{-2\pi v R} d\mu(z). \end{aligned}$$

This last equality follows from Equation 3.2. Alternatively, we can easily calculate that

$$\begin{aligned} dd^c\varphi_S &= \frac{-1}{2\pi i}\partial\overline{\partial}\varphi_S \\ &= -iv^2\left(\frac{\partial^2 R}{\partial z\partial\overline{z}} - 2\pi v\frac{\partial R}{\partial\overline{z}}\frac{\partial R}{\partial z}\right)e^{-2\pi vR}dz\wedge d\overline{z}. \end{aligned}$$

From Equation 3.7 we can see that

$$\frac{\partial R}{\partial\overline{z}}\frac{\partial R}{\partial z} = \frac{\hat{\delta}R}{16y^2}|\langle X, X(z)\rangle|^2$$

and also that

$$\frac{\partial^2 R}{\partial z\partial\overline{z}} = \frac{1}{4y^2}\langle X, X\rangle_z$$

hence,

$$dd^c\varphi_S = \frac{-iv^2}{8y^2}\left(2\langle X, X\rangle_z - \pi v\hat{\delta}R|\langle X, X(z)\rangle|^2\right)e^{-2\pi vR}dz\wedge d\overline{z}.$$

Finally, using that $dz\wedge d\overline{z} = -2idx\wedge dy$, we have

$$dd^c\varphi_S = \frac{v^2}{4}\left(2\langle X, X\rangle_z - \pi\hat{\delta}vR|\langle X, X(z)\rangle|^2\right)e^{-2\pi vR}d\mu(z),$$

as expected.

3.4 Rewriting theta

The definition of Θ is essentially as a Fourier series in the τ variable. For the purposes of our lift however, it would be much more useful to have Θ written as a Fourier series in z. It turns out this is possible via a relatively simple (but detailed) calculation, which is taking the Fourier transform on one component of the lattice. The explanation for this comes from changing our point of view, and again viewing Θ as a function on $\mathrm{SU}(V_{\mathbb{R}})\times\mathrm{SL}_2(\mathbb{R})$. Recall we have

$$\Theta(g, g', L) = \sum_{X\in L'} M_{\natural}[g]M_{Sch}[g']\varphi_{KM}(X)\mathfrak{e}_X$$

and also recall that the definition of the Schrödinger model of the Weil representation is implicitly dependent on a choice of polarisation of the underlying space. If we take a different polarisation, then the actions of the two different Schrödinger models are related - they have an intertwiner - which is exactly the partial Fourier transform! Initially, we chose a polarisation for $W = V \otimes_{\mathbb{C}} V_2$ based on $\ell_2 \in V_2$, however, there is another obvious polarisation based on $\ell \in V$. Now the inherent symmetry of our space comes into play, and what happens is that the two different models (natural and Schrödinger) of the Weil representation swap places. This is all made explicit in Theorem 3.4.2 below - if we rewite the formula below as a function on the group we have

$$\theta_{\mathbf{h}}(g, g', L) = \frac{1}{\mathrm{vol}_\ell(L)} \sum_{X \in L^*} \exp\left(4\pi i \hat{\delta}^{-1} \Im(w\overline{h})\right) M_{Sch}[g] M_\natural[g'] \varphi'_{KM}(X) dg$$

This expression contains the new terms $\varphi'_{KM}(X)$, vol_ℓ and L^* which we define below. As we will show later,

$$\varphi'_{KM}(X) = \overline{v_1} v_2 \varphi_S(X)$$

and by vol_ℓ we mean the volume of the lattice on the ℓ-component, which, if $\Lambda = \mathbb{Z}^2 S$ over the basis $\{1, \zeta\}$ for some integral matrix S, then $\mathrm{vol}_\ell(\Lambda) = 4\hat{\delta}^{-1} \det(S)$. Now let $(z_1, z_2) = 2\hat{\delta}^{-1} \Im(z_1 \overline{z_2})$ be a sesquilinear form on $\mathbb{C}$ and define

$$\mathcal{F}(f)(z_1) = \int_{\mathbb{C}} f(z_2) \exp(-2\pi i (z_1, z_2)) dx_2 dy_2$$

then what we really have is that

$$\theta_{\mathbf{h}}(g, g', L) = \sum_{X \in L + \mathbf{h}} \mathcal{F}^{-1}(M_{Sch}[g] M_\natural[g'] \varphi'_{KM})(X) dg$$

and

$$\varphi'_{KM}(X) = \mathcal{F}(\varphi_{KM})(X),$$

and the relationship between $L + \mathbf{h}$ and L^* is that L^* is $\mathbb{Z}$-dual to $L + \mathbf{h}$ under the form $(\ ,\ )$ on the ℓ component.

This also makes the transformation formula for τ more transparent; it follows from left Γ_L invariance and right K-invariance (up to a character) by the usual correspondence. As before,

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \cdot (v_1 e_1 + v_2 e_2) = e^{i\theta} v_1 e_1 + e^{-i\theta} v_2 e_2$$

so we have that

Lemma 3.4.1. *Let r_θ be defined as above and $\varphi'_{KM} = -\overline{v}_1 v_2 \exp(-\pi(|v_1|^2 + |v_2|^2))$, then*

$$M_\natural[r_\theta]\varphi'_{KM}(X) = e^{2i\theta}\varphi'_{KM}(X)$$

Proof.

$$M_\natural[r_\theta]\varphi'_{KM}(X) = \varphi'_{KM}(r_\theta^{-1}X) = -(e^{i\theta}\overline{v}_1)(e^{i\theta}v_2)\exp(-\pi(|v_1|^2+|v_2|^2)) = e^{2i\theta}\varphi'_{KM}(X)$$

□

and left Γ_L invariance simply follows from the fact that Γ_L stabilises the lattice.

Since the calculation for the taking the partial Fourier transform of just $\varphi_{KM}(X)$ is not that much simpler than just doing it over the whole sum in one go, we present below the partial Fourier transform calculation in full detail.

We wish to take the Fourier transform of $\theta_h(\tau, z, L)$ in the w variable on the u component. We recall that

$$\begin{aligned}\theta_{\mathbf{h}}(\tau, z, L) &= \sum_{X \in L+\mathbf{h}} \varphi^0(\sqrt{v}X, z)\exp(\pi i\langle X, X\rangle\tau)\\ &= \sum_{X \in L+\mathbf{h}} \frac{1}{8}\left(\hat{\delta}v|\langle X, X(z)\rangle|^2 - \frac{2}{\pi}\right)\exp\left(-\frac{\pi v\hat{\delta}}{2}|\langle X, X(z)\rangle|^2\right)\\ &\qquad \exp\left(\pi i\langle X, X\rangle\overline{\tau}\right)d\mu(z)\end{aligned}$$

We make the usual substitutions:

$$\alpha = \sqrt{\frac{v\hat{\delta}}{2}}\Re(\langle X, X(z)\rangle)\ , \qquad \beta = \sqrt{\frac{v\hat{\delta}}{2}}\Im(\langle X, X(z)\rangle)$$

which allows us to write

$$\theta_{\mathbf{h}}(\tau, z, L) = \frac{1}{16\pi}\sum_{X \in L+\mathbf{h}}(H_2(\sqrt{\pi}\alpha) + H_2(\sqrt{\pi}\beta))\exp(-\pi(\alpha^2+\beta^2))\exp(\pi i\langle X, X\rangle\overline{\tau})$$

where $H_n(x)$ is the n-th Hermite polynomial.

Theorem 3.4.2. *Let $L = \Lambda\ell + \Lambda'\ell'$ and $\mathbf{h} \in L'/L$. By taking the partial Fourier transform of the summand in $\theta_{\mathbf{h}}(\tau, z, L)$, we obtain the following formula for the theta function:*

$$\begin{aligned}\theta_{\mathbf{h}}(\tau, z, L) = \frac{-y^2}{v^2\hat{\delta}\,\mathrm{vol}_\ell(\Lambda)}\sum_{X \in L^*}&\exp\left(4\pi i\hat{\delta}^{-1}\Im(w\overline{h})\right)\exp(\pi ix\langle X, X\rangle)\\ &(\overline{\tau w'} - \overline{w})\,(\overline{\tau}w' - w)\exp(-\pi y\langle X, X\rangle_\tau)d\mu(z)\end{aligned}$$

where

$$L^* = \{w\ell + w'\ell' : w \in \Lambda^*,\ w' \in \Lambda' + h'\}\,.$$

and

$$\Lambda^* = \{w_1 \in \mathbb{C} : 2\hat{\delta}^{-1}\Im(w_1\overline{w_2}) \in \mathbb{Z},\ \text{for all } w_2 \in \Lambda\}$$

Proof. First, we prove a lemma concerning the partial Fourier transform of the sum. We then use this, combined with Poisson summation to prove the theorem. First, we recall that we may write the theta function in terms of the variables

$$\alpha = \sqrt{\frac{v\hat{\delta}}{2}}\Re(\langle X, X(z)\rangle)\ , \qquad \beta = \sqrt{\frac{v\hat{\delta}}{2}}\Im(\langle X, X(z)\rangle)$$

as

$$\theta_{\mathbf{h}}(\tau, z, L) = \frac{1}{16\pi}(H_2(\sqrt{\pi}\alpha) + H_2(\sqrt{\pi}\beta))\exp(-\pi(\alpha^2+\beta^2))\exp(\pi i\langle X, X\rangle\overline{\tau})d\mu(z).$$

We may write the length of a vector in terms of α and β so that we have

$$\langle X, X\rangle = 2\sqrt{\frac{2y}{\hat{\delta}v}}(\beta\Re(w') - \alpha\Im(w')) - 4\hat{\delta}^{-1}y|w'|^2$$

We now define

$$f(x_1, x_2) = (H_2(\sqrt{\pi}\alpha) + H_2(\sqrt{\pi}\beta))\exp(-\pi(\alpha^2+\beta^2))$$
$$\exp\left(\pi i\overline{\tau}\left(2\sqrt{\frac{2y}{\hat{\delta}v}}\,(\beta\Re(w') - \alpha\Im(w')) - 4\hat{\delta}^{-1}y|w'|^2\right)\right)$$

where

$$\alpha = \sqrt{\frac{2v}{\hat{\delta}y}}(x_1 - \Re(z\overline{w}'))\ , \qquad \beta = \sqrt{\frac{2v}{\hat{\delta}y}}(x_2 + \Im(z\overline{w}'))$$

so that

$$\theta_{\mathbf{h}}(\tau, z, L) = \frac{1}{16\pi}\sum_{X \in L+h} f(\Re(w), \Im(w))d\mu(z)$$

We now calculate the Fourier transform of f.

Lemma 3.4.3. *Let $Y = \frac{\hat{\delta}}{2}(-y_2 + iy_1)\ell + w'\ell'$. The Fourier transform of the function f as defined above is given by*

$$\hat{f}(y_1, y_2) = -\frac{4\pi y^2}{v^2}\left(\overline{\tau}\overline{w}' - \frac{\hat{\delta}}{2}(-y_2 - iy_1)\right)\left(\overline{\tau}w' - \frac{\hat{\delta}}{2}(-y_2 + iy_1)\right)$$
$$\exp(\pi i x\langle Y, Y\rangle)\exp(-\pi y\langle Y, Y\rangle_\tau)$$

Proof. By definition

$$\hat{f}(y_1, y_2) = \iint_{\mathbb{R}^2} f(x_1, x_2) \exp(-2\pi i(x_1 y_1 + x_2 y_2)) dx_1 dx_2.$$

Since we have that

$$x_1 = \sqrt{\frac{\hat{\delta} y}{2v}} \alpha + \Re(z\overline{w}'), \qquad x_1 = \sqrt{\frac{\hat{\delta} y}{2v}} \beta - \Im(z\overline{w}'),$$

we now write the integral in terms of the variables α and β:

$$\hat{f}(y_1, y_2) = \iint_{\mathbb{R}^2} f(x_1, x_2) \exp(-2\pi i(\alpha y_1 + \beta y_2)) \\ \exp(-2\pi i(y_1 \Re(z\overline{w}') - y_2 \Im(z\overline{w}')))) \frac{\hat{\delta} y}{2v} d\alpha d\beta.$$

Focussing on only the piece dependent on α and β, we will evaluate

$$\iint_{\mathbb{R}^2} (H_2(\sqrt{\pi}\alpha) + H_2(\sqrt{\pi}\beta)) \exp(-\pi(\alpha^2 + \beta^2)) \\ \exp\left(-2\pi i \sqrt{\frac{\hat{\delta} y}{2v}} \left(\alpha(y_1 - 2\hat{\delta}^{-1}\Im(w')) + \beta(y_2 + 2\hat{\delta}^{-1}\Re(w'))\right)\right) d\alpha d\beta \quad (3.9)$$

using the standard (see e.g. [GR07]) Fourier transforms

$$F_1(y) = \int_{\mathbb{R}} \exp(-\pi x^2) \exp(-2\pi i x y) dx = \exp(-\pi y^2)$$
$$F_2(y) = \int_{\mathbb{R}} H_2(\sqrt{\pi} x) \exp(-\pi x^2) \exp(-2\pi i x y) dx = -4\pi y^2 \exp(-\pi y^2).$$

Hence,

$$(3.9) = F_1\left(\sqrt{\frac{\hat{\delta} y}{2v}}(y_1 - 2\hat{\delta}^{-1}\Im(w'))\right) F_2\left(\sqrt{\frac{\hat{\delta} y}{2v}}(y_2 + 2\hat{\delta}^{-1}\Re(w'))\right) \\ + F_1\left(\sqrt{\frac{\hat{\delta} y}{2v}}(y_2 + 2\hat{\delta}^{-1}\Re(w'))\right) F_2\left(\sqrt{\frac{\hat{\delta} y}{2v}}(y_1 - 2\hat{\delta}^{-1}\Im(w'))\right) \\ = -\frac{2\pi\hat{\delta} y}{v}\left((y_1 - 2\hat{\delta}^{-1}\Im(w'))^2 + (y_2 + 2\hat{\delta}^{-1}\Re(w'))^2\right) \\ \exp\left(-\frac{\pi\hat{\delta} y}{2v}\left((y_1 - 2\hat{\delta}^{-1}\Im(w'))^2 + (y_2 + 2\hat{\delta}^{-1}\Re(w'))^2\right)\right)$$

It is now useful to make the substitution

$$(y_1 - 2\hat{\delta}^{-1}\Im(w'))^2 + (y_2 + 2\hat{\delta}^{-1}\Re(w'))^2$$
$$= 4\hat{\delta}^{-2}\left(\overline{\tau w}' - \frac{\hat{\delta}}{2}(y_2 + iy_1)\right)\left(\overline{\tau} w' - \frac{\hat{\delta}}{2}(y_2 - iy_1)\right).$$

Putting everything together, we have now arrived at

$$\hat{f}(y_1, y_2) = -\frac{4\pi y^2}{v^2}\left(\left(\overline{\tau w}' - \frac{\hat{\delta}}{2}(y_2 + iy_1)\right)^2 + \left(\overline{\tau} w' - \frac{\hat{\delta}}{2}(y_2 - iy_1)\right)^2\right)$$
$$\exp\left(-2\pi i(y_1\Re(z\overline{w}') - y_2\Im(z\overline{w}'))\right)\exp\left(-4\pi i\overline{\tau} y|w'|^2\right)$$
$$\exp\left(-\frac{2\pi y}{\hat{\delta} v}\left(\overline{\tau w}' - \frac{\hat{\delta}}{2}(y_2 + iy_1)\right)\left(\overline{\tau} w' - \frac{\hat{\delta}}{2}(y_2 - iy_1)\right)\right)$$

and by collecting x and y terms separately in the exponent, and using the explicit formulas for $\langle X, X\rangle$ and $\langle X, X\rangle_z$ given in Proposition 3.1.2, we arrive at the result.

□

Now we assume that $L = \Lambda\ell \oplus \Lambda'\ell'$ and let S be the matrix such that

$$\pi_*(\mathbb{Z}^2 S) = \Lambda$$

where, for $\mathcal{O}_F = \mathbb{Z}[\zeta]$,

$$\pi_*(x_1, x_2) = x_1 + \zeta x_2.$$

Then we have the following, by Poisson summation

$$\begin{aligned}
\theta_{\mathbf{h}}(\tau, z, L) &= \frac{1}{16\pi}\sum_{w'\in\Lambda'+h'}\sum_{w\in\Lambda+h} f(\Re(w), \Im(w))d\mu(z) \\
&= \frac{1}{16\pi}\sum_{w'\in\Lambda'+h'}\sum_{x_i\in\mathbb{Z}} f((x_1, x_2)S + \mathbf{h})d\mu(z) \\
&= \frac{1}{16\pi}\sum_{w'\in\Lambda'+h'}\sum_{x_i\in\mathbb{Z}} g_h(x_1, x_2)d\mu(z) \\
&= \frac{1}{16\pi}\sum_{w'\in\Lambda'+h'}\sum_{x_i\in\mathbb{Z}} \hat{g}_h(x_1, x_2)d\mu(z)
\end{aligned}$$

where we define

$$g_h(x_1, x_2) = f((x_1, x_2)S + (h_1, h_2)).$$

We have adopted the convention that $f((x_1, x_2)) = f(x_1, x_2)$ for notational convenience. Using the notation $\mathbf{x} = (x_1, x_2)$ and $H = (h_1, h_2)$, a simple calculation shows that

$$\hat{g}_h(x_1, x_2) = \frac{1}{\det S} \exp(2\pi i H^t(\mathbf{x}^t S^{-1}))\hat{f}(\mathbf{x}^t S^{-1})$$

We substitute this in, and, noting that ${}^tS^{-1} = \frac{1}{\det S} JSJ^{-1}$ where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we obtain

$$\begin{aligned}\sum_{x_i \in \mathbb{Z}} \hat{g}_h(x_1, x_2) &= \sum_{x_i \in \mathbb{Z}} \frac{1}{\det S} \exp(2\pi i H^t(\mathbf{x}^t S^{-1}))\hat{f}(\mathbf{x}^t S^{-1}) \\ &= \sum_{x_i \in \mathbb{Z}} \frac{1}{\det S} \exp(2\pi i H^t(\mathbf{x} JSJ^{-1}) \det S^{-1})\hat{f}(\mathbf{x} JSJ^{-1} \det S^{-1})\end{aligned}$$

Since the sum over $\mathbf{x}J$ (where $\mathbf{x}$ runs over all the integer pairs) is clearly just a reordering of the sum, we may write

$$\begin{aligned}\sum_{x_i \in \mathbb{Z}} \hat{g}_h(x_1, x_2) &= \sum_{\mathbf{x} \in \frac{1}{\det S}\mathbb{Z}^2 S} \frac{1}{\det S} \exp(2\pi i H^t(\mathbf{x} SJ^{-1}))\hat{f}(\mathbf{x} J^{-1}) \\ &= \sum_{w \in \frac{1}{\det S}\pi_*(\mathbb{Z}^2 S)} \frac{1}{\det S} \exp\left(2\pi i (\Im(w) h_1 - \Re(w) h_2)\right) \hat{f}\left(\Im(w), -\Re(w)\right) \\ &= \sum_{w \in \frac{\hat{\delta}}{2\det S}\pi_*(\mathbb{Z}^2 S)} \frac{1}{\det S} \exp\left(4\hat{\delta}^{-1}\pi i(\Im(w) h_1 - \Re(w) h_2)\right) \\ &\qquad \hat{f}\left(2\hat{\delta}^{-1}\Im(w), -2\hat{\delta}^{-1}\Re(w)\right) \\ &= \sum_{w \in \Lambda^*} \frac{4}{\hat{\delta}\operatorname{vol}_\ell(\Lambda)} \exp\left(4\pi i \hat{\delta}^{-1}\Im(w\overline{h})\right) \hat{f}\left(2\hat{\delta}^{-1}\Im(w), -2\hat{\delta}^{-1}\Re(w)\right)\end{aligned}$$

We note here that the sesquilinear form $\Im(z_1\overline{z_2})$ is, in some sense, equivalent to the form on $\mathbb{R}^2$ defined by $\mathbf{x}J^t\mathbf{y}$. This explains the appearance of Λ^*, which, up to scaling, is the $\mathbb{Z}$-dual of J pulled back under the π_* map.

Finally, substituting in with the formula for $\hat{f}$ from Lemma 3.4.3 and simplifying, we have that

$$\begin{aligned}\theta_{\mathbf{h}}(\tau, z, L) = \frac{-y^2}{v^2\hat{\delta}\operatorname{vol}_\ell(\Lambda)} \sum_{X \in L^*} \exp\left(4\pi i \hat{\delta}^{-1}\Im(w\overline{h})\right) \exp(\pi i x\langle X, X\rangle) \\ \left(\overline{\tau w}' - \overline{w}\right)\left(\overline{\tau} w' - w\right) \exp(-\pi y \langle X, X\rangle_\tau) d\mu(z)\end{aligned}$$

where

$$L^* = \{w\ell + w'\ell' : w \in \Lambda^*,\ w' \in \Lambda + h'\}. \qquad \square$$

In order to link this back to our earlier discussion (and prove our formula for $\mathcal{F}(\varphi_{KM}^0)(X)$), we observe that

$$M_\natural[g_\tau]M_{Sch}[g_z](-\overline{v}_1 v_2)\varphi_S(X)dg_z = -\frac{y^2}{v\hat{\delta}}(\overline{\tau}\overline{w}' - \overline{w})(\overline{\tau}w' - w)\exp(\pi i\langle X, X\rangle)$$
$$\exp(-\pi y\langle X, X\rangle_\tau)d\mu(z)$$

and hence

$$\theta_{\mathbf{h}}(\tau, z, L) = \frac{1}{v\,\mathrm{vol}_\ell(\Lambda)}\sum_{X\in L^*} M_\natural[g_\tau]M_{Sch}[g_z](-\overline{v}_1 v_2)\varphi_S(X)dg_z$$

which, by Poisson summation is

$$\theta_{\mathbf{h}}(\tau, z, L) = \sum_{X\in L+\mathbf{h}} \mathcal{F}^{-1}\left(v^{-1}M_\natural[g_\tau]M_{Sch}[g_z](-\overline{v}_1 v_2)\varphi_S(X)\right)dg_z.$$

Hence our formula for $\mathcal{F}(\varphi_{KM}^0)(X)$ was correct, and since Γ_L necessarily stabilises L^*, we have also recovered the transformation formula in τ.

Chapter 4

The Theta Lift

4.1 Defining the lift

We shall now let f be a harmonic weak Maass form of weight 0 in the space $H_0^+(\Gamma_L)$. This means in particular that there exists a polynomial $P_{f,\kappa} \in \mathbb{C}[\exp(-2\pi i z/\alpha_\kappa)]$, for each cusp ℓ such that

$$f(\sigma_\kappa z) - P_{f,\kappa}(z) = O(\exp(-Cy))$$

for some $C > 0$. This will turn out to mean that the convergence properties of the lift (and hence the location of the singularities of the lift) depend only on the principal parts $P_{f,\kappa}$. In particular, the degree of the polynomial will play a crucial role.

Definition 4.1.1. We define the lift of $f \in H_0^+(\Gamma_L)$ of weight 0 by

$$I(\tau, f) = \int_M f(z)\Theta(\tau, z, L) = \sum_{h \in L'/L} I_h(\tau, f)\mathfrak{e}_{\mathbf{h}},$$

where

$$I_h(\tau, f) = \int_M f(z)\theta_h(\tau, z, L).$$

There are serious questions to answer though, as to whether the definition as given even makes sense. The growth of such a $f(z)$ as $Im(\sigma_\kappa z) \to \kappa$ is $O(e^{2\pi N_\kappa y})$, whereas θ_h has decay of $O(e^{-2\pi\hat{\delta}^{-1} v N_{\Lambda'} y})$ for some constant $N_{\Lambda'}$ depending on the lattice $L = \Lambda\ell \oplus \Lambda'\ell'$ in this limit, as can be seen in Chapter 3. We then have

Theorem 4.1.2. *Let $v > \frac{N_\kappa \hat{\delta}}{N_{\Lambda'}}$ for all cusps κ, then $I(\tau, f)$ converges.*

Proof. It is clear from 3.4.2 that $\theta_{\mathbf{h}}(\tau, z, L) = O(e^{-2\pi\hat{\delta}^{-1}vN_{\Lambda'}y})$ as $y \to \infty$, as $\langle X, X\rangle_\tau > 2\hat{\delta}^{-1}v|w'|^2$, and since for non-zero w' we have that $|w'|^2 > N_{\Lambda'}$, hence the integrand is $O(-2\pi\epsilon y)$ for some $\epsilon > 0$. □

Of course, we would like to extend the lift beyond this boundary. We will show that, outside a discrete set of points and for a certain regularisation (the so called cut off integral) that the definition does indeed make sense for small v.

4.1.1 Regularisation

The regularisation used is the cut off integral, or capped lift. This is a more basic version of the regularisation used in [Bor98], [BF06], but is good enough for our purposes. We take

$$I(\tau, f) = \lim_{T\to\infty} \int_{M_T} f(z)\Theta(\tau, z, L)$$

where M_T is the canonical fundamental domain for $\Gamma_L\backslash\mathbb{D}$, but cut off at height T around each cusp. For example, if $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, then

$$M_T = \left\{ z = x + iy \in \mathbb{H} : |x| < \frac{1}{2},\ |z|^2 > 1,\ y < T \right\}.$$

For more complicated fundamental domains, with more than one cusp, we recall that, for each cusp $\kappa \in \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$, we have the existence of a matrix σ_κ such that $\sigma_\kappa\infty = \kappa$, and so when we consider $\sigma_\kappa^{-1}M_T$, this should be cut off at the height T in the natural way. See the discussion in 1.1.4 for more detail on this.

4.2 Statement of Results

We will prove here that the regularised lift converges everywhere except a discrete set of points (and, if considered on the modular curve, then actually only a finite number of points).

Theorem 4.2.1. *The lift of the harmonic weak Maass form $f(z) \in H_0^+(\Gamma_L)$, converges everywhere except a discrete set of points. Furthermore, $I(\tau, f) = O(\exp(-Cv))$*

as $\Im(\sigma_\kappa\tau) \to \kappa$ *for* $\kappa \in \Gamma_L\backslash\mathbb{P}^1(\mathbb{Q})$, *for some* $C > 0$. *The singularities of* $I_{\mathbf{h}}(\tau, f)$ *lie on the divisor*

$$Z_{\mathbf{h}}(f) = \sum_{\substack{X \in L_{\mathbf{h}}^* \\ \langle X,X\rangle > 0}} \sum_{\kappa \in \Gamma_L\backslash\mathbb{P}^1(\mathbb{Q})} c_{f,\kappa}^+(-\tfrac{1}{2}\langle X, X\rangle)[\mathbb{D}_X]$$

These singularities are of linear type, as defined in [Bor98], in the sense that for each point in the set described above there exists ρ *such that*

$$I_{\mathbf{h}}(\tau, f) - \frac{\rho}{\sigma_\kappa\tau - \mathbb{D}_X}$$

is a smooth function in u, v *in a neighbourhood of* $\mathbb{D}_X$, *where* ρ *is given by*

$$\rho = -\frac{1}{2\pi i}\frac{\alpha_\kappa c_{f,\kappa}^+(-\frac{1}{2}\langle X, X\rangle)}{\hat{\delta}\operatorname{vol}_\ell(\Lambda)}\exp(4\pi i\hat{\delta}^{-1}\Im(w\overline{h})).$$

In fact we can make a stronger statement than this. Using the ξ_k operator defined in [BF04], which for weight 2 is given by $-2iv^2\overline{\frac{\partial}{\partial\overline{\tau}}}$, we have

Theorem 4.2.2. *Let* $f(z)$ *be a harmonic weak Maass form of weight* 0 *for the group* Γ_L, *with Fourier expansion*

$$f(\sigma_\kappa z) = \sum_{n \geq -N_\kappa} c_{f,\kappa}^+(n)\exp(2\pi i n z/\alpha_\kappa) + \sum_{n<0} c_{f,\kappa}^-(n) H_0(2\pi n y/\alpha_\kappa)\exp(2\pi i n x/\alpha_\kappa)$$

around each cusp $\kappa \in \Gamma_L\backslash\mathbb{P}^1(\mathbb{Q})$. *The image of* $I(\tau, f)$ *under the map*

$$\xi_2 = -2iv^2\overline{\frac{\partial}{\partial\overline{\tau}}}$$

is

$$\xi_2(I_{\mathbf{h}}(\tau, f)) = -\sum_{\kappa \in \Gamma_L\backslash\mathbb{P}^1(\mathbb{Q})} \frac{\delta_{\mathbf{h},0}\overline{c_{f,\kappa}^+(0)}}{\pi\hat{\delta}\operatorname{vol}_\ell(\Lambda)}$$

and hence $I(\tau, f)$ *is a harmonic Maass form of weight 2 with singularities, described in Theorem 4.2.1.*

This theorem clearly implies that the lift is harmonic; it also implies a simple condition on the input function to ensure meromorphicity of the lift, namely the vanishing of the constant terms at all the cusps.

Corollary 4.2.3. *Assuming* $c_{f,\kappa}^+(n) \neq 0$ *and* $c^+(0, \kappa) = 0$, $I(\tau, f)$ *is a meromorphic form of weight 2 for* Γ_L *with poles where* $\sigma_\kappa\tau = \mathbb{D}_X$, *for* $X \in \Gamma_L\backslash L_{n,\mathbf{h}}^*$ *for all* $n < 0$ *and* $\kappa \in \Gamma_L\backslash\mathbb{P}^1(\mathbb{Q})$.

4.3 Proof of Theorem 4.2.1

Proof. We define the cut-off box region $\mathcal{B}_T$ by

$$\mathcal{B}_T = \{z \in \mathbb{H} : 0 \leq x \leq \alpha_\infty,\ 1 \leq y \leq T\}.$$

where α_∞ is the width of the cusp at infinity. We prove the theorem by comparing the lift of f to the boxed cut-off lift

$$\tilde{I}_{\mathbf{h}}(\tau, f) = \lim_{T\to\infty} \sum_{\kappa \in \Gamma_L \backslash \mathbb{P}^1(\mathbb{Q})} \int_{\mathcal{B}_T} f(\sigma_\kappa z)\theta_{\mathbf{h}}(\tau, \sigma_\kappa z, L)$$

The convergence properties of $\tilde{I}$ are the same as I, as the domains of integration differ only by a compact region.

Any harmonic weak Maass form has a Fourier expansion around each cusp κ given by

$$\begin{aligned} f(\sigma_\kappa z) &= \sum_{n \geq -N_\kappa} c^+_{f,\kappa}(n) \exp(2\pi i n z/\alpha_\kappa) + \sum_{n<0} c^-_{f,\kappa}(n) H_0(2\pi n y/\alpha_\kappa) \exp(2\pi i n x/\alpha_\kappa) \\ &= f^+(\sigma_\kappa z) + f^-(\sigma_\kappa z) \end{aligned}$$

where α_κ is the width of the cusp, and N_κ is the order of the pole at κ.

Expanding the formula on the RHS, we get the equation

$$\tilde{I}_{\mathbf{h}}(\tau, f) = \lim_{T\to\infty} \sum_{\kappa \in \Gamma_L \backslash \mathbb{P}^1(\mathbb{Q})} \int_1^T \int_0^{\alpha_\kappa} f(\sigma_\kappa z)\theta_{\mathbf{h}}(\tau, \sigma_\kappa z, L).$$

We now recall that $\theta_{\mathbf{h}}(\tau, z, L)$ is of order $O(e^{-Cy})$, and at the cusp κ, the order of f is $O(e^{2\pi N_\kappa y/\alpha_\kappa})$, which is potentially problematic. However, for the non-holomorphic part, we have rapid decay everywhere, since we assume that $c^-_{f,\kappa}(n) = 0$ for all non-negative n. Hence we only have to that part of the integral which is over f^+.

We recall that we have the following expression for the theta function associated to the lattice $L = \Lambda\ell \oplus \Lambda'\ell'$:

$$\theta_{\mathbf{h}}(\tau, z, L) = \frac{-y^2}{2v^2\hat{\delta}\,\mathrm{vol}_\ell(\Lambda)} \sum_{X \in L^*_{\mathbf{h}}} \exp\left(4\pi i \hat{\delta}^{-1}\Im(w\overline{h})\right) \exp(\pi i x\langle X, X\rangle)$$
$$(\overline{\tau}\overline{w}' - \overline{w})\,(\overline{\tau}w' - w)\exp(-\pi y\langle X, X\rangle_\tau)d\mu(z)$$

where $X = w\ell + w'\ell'$ and

$$L^*_{\mathbf{h}} = \left\{ w\ell + w'\ell' : 2\hat{\delta}^{-1}\Im(w\overline{w_1}) \in \mathbb{Z} \text{ for all } w_1 \in \Lambda,\ w' \in \mathfrak{q} + h' \right\}.$$

We examine this expression for $\theta_{\mathbf{h}}$, and we note that the integral in the x variable is,

$$\int_0^{\alpha_\kappa} \exp(\pi i x \langle X, X\rangle + 2\pi i x n/\alpha_\kappa) dx = \begin{cases} \alpha_\kappa & \text{if } \frac{1}{2}\langle X, X\rangle = -\frac{n}{\alpha_\kappa} \\ 0 & \text{else} \end{cases}$$

hence,

$$\tilde{I}_{\mathbf{h}}(\tau, f) = -\lim_{T\to\infty} \sum_{\kappa \in \Gamma_L \backslash \mathbb{P}^1(\mathbb{Q})} \frac{\alpha_\kappa}{2v^2\hat{\delta}\,\mathrm{vol}_\ell(\Lambda)} \left(\sum_{n \geq -N_\kappa} \sum_{X \in (\sigma_\kappa^{-1} L^*)_{-n}} c^+_{f,\kappa}(n) A(X, \tau, h) B^+(T, X, \tau, z) \right)$$

where

$$A(X, \tau, h) = \exp(4\pi i \hat{\delta}^{-1}\Im(w\overline{h}))(\overline{\tau w}' - \overline{w})(\overline{\tau} w' - w)$$
$$B^+(T, X, \tau, z) = \int_1^T \exp(-\pi y \langle X, X\rangle_\tau) \exp(-2\pi y n/\alpha_\kappa) dy.$$

We can now examine the integral B^+. For the positive coefficients $c^+_{f,\kappa}(n)$, we have, since $\frac{1}{2}\langle X, X\rangle = -n/\alpha_\kappa$, that

$$B^+(T, X, \tau, z) = \int_1^T \exp(-2\pi y R(X, \tau)) dy$$
$$= \frac{\exp(-2\pi R(X, \tau)) - \exp(-2\pi T R(X, \tau))}{2\pi R(X, \tau)}.$$

If we have that $R(X, \tau) > 0$, the second term goes to 0 in the limit, and the sum is convergent in this case. In which case it is also clear that $I(\tau, f) = O(\exp(-Cv))$ as $v \to \kappa$ for some $C > 0$.

However, for $R(X, \tau) = 0$ the sum does does *not* converge. This only happens when $\sigma_\kappa \tau = \mathbb{D}_X$ for $X \in L^*_{\mathbf{h},-n}$, which necessarily means that $n < 0$. Hence there are only a finite number of places on each copy on the modular curve embedded into $\mathbb{H}$ where the series does not converge, and they are where

$$\sigma_\kappa \tau = \mathbb{D}_X$$

for $X \in \Gamma_L \backslash L^*_{\mathbf{h},n}$ where $\kappa \in \Gamma_L \backslash \mathbb{P}^1(\mathbb{Q})$, and $n < 0$ and $c^+_{f,\kappa}(n) \neq 0$.

We now calculate the residues at each pole. Again, since the domain of integration of the integrals $\tilde{I}_{\mathbf{h}}(\tau, f)$ and $I_{\mathbf{h}}(\tau, f)$ differs only by a compact region, this means that, as functions in τ, they differ only by a bounded function in τ. Since they share poles, the residue at a pole of $I_{\mathbf{h}}(\tau, f)$ is equal to the residue of the same pole of $\tilde{I}_{\mathbf{h}}(\tau, f)$.

Pick a pair κ and X from the set of singularities. Then, in the notation established above, we have

$$\lim_{\sigma_\kappa \tau \to \mathbb{D}_X} (\sigma_\kappa \tau - \mathbb{D}_X) \tilde{I}_{\mathbf{h}}(\tau, f) = \lim_{\sigma_\kappa \tau \to \mathbb{D}_X} (\sigma_\kappa \tau - \mathbb{D}_X) \lim_{T \to \infty} \sum_{\kappa' \in \Gamma_L \backslash \mathbb{P}^1(\mathbb{Q})} \frac{-\alpha_{\kappa'}}{2v^2 \hat{\delta} \operatorname{vol}_\ell(\Lambda)} \left(\sum_{n \geq -N_{\kappa'}} \sum_{Y \in (\sigma_{\kappa'}^{-1} L^*)_{-n}} c^+_{f,\kappa'}(n) A(Y, \tau, h) B^+(T, Y, \tau, z) \right)$$

by definition; at this point all we have done is very fancily rewrite the inner limit. We know that this limit converges except when $\sigma_\kappa \tau = \mathbb{D}_X$ for a certain set of X described above. In the outer limit, all the terms will vanish except for when we have $X = Y$ and $\kappa = \kappa'$, and so, writing explicitly, we are left only with the term

$$\lim_{\sigma_\kappa \tau \to \mathbb{D}_X} (\sigma_\kappa \tau - \mathbb{D}_X) \tilde{I}_{\mathbf{h}}(\tau, f) = \lim_{\sigma_\kappa \tau \to \mathbb{D}_X} (\sigma_\kappa \tau - \mathbb{D}_X) \frac{-\alpha_\kappa c^+_{f,\kappa}(-n)}{2v^2 \hat{\delta} \operatorname{vol}_\ell(\Lambda)} \exp(4\pi i \hat{\delta}^{-1} \Im(w \overline{h})) (\overline{\tau w}' - \overline{w})(\overline{\tau} w' - w) \frac{\exp(-2\pi R(X, \tau))}{2\pi R(X, \tau)}$$

and hence by the explicit formula for R, we have

$$\lim_{\sigma_\kappa \tau \to \mathbb{D}_X} (\sigma_\kappa \tau - \mathbb{D}_X) \tilde{I}_{\mathbf{h}}(\tau, f) = \lim_{\sigma_\kappa \tau \to \mathbb{D}_X} (\sigma_\kappa \tau - \mathbb{D}_X) \frac{-\alpha_\kappa c^+_{f,\kappa}(-n)}{4\pi v \hat{\delta} \operatorname{vol}_\ell(\Lambda)} \exp(4\pi i \hat{\delta}^{-1} \Im(w \overline{h})) \frac{(\overline{\sigma_\kappa \tau} - \overline{\mathbb{D}_X})(\overline{\sigma_\kappa \tau} - \mathbb{D}_X)}{|\sigma_\kappa \tau - \mathbb{D}_X|^2}$$

which simplifies to

$$\lim_{\sigma_\kappa \tau \to \mathbb{D}_X} (\sigma_\kappa \tau - \mathbb{D}_X) \tilde{I}_{\mathbf{h}}(\tau, f) = \lim_{\sigma_\kappa \tau \to \mathbb{D}_X} \frac{-\alpha_\kappa c^+_{f,\kappa}(-n)}{4\pi v \hat{\delta} \operatorname{vol}_\ell(\Lambda)} (\overline{\sigma_\kappa \tau} - \mathbb{D}_X) \exp(4\pi i \hat{\delta}^{-1} \Im(w \overline{h}))$$

and hence

$$\lim_{\sigma_\kappa \tau \to \mathbb{D}_X} (\sigma_\kappa \tau - \mathbb{D}_X) \tilde{I}_{\mathbf{h}}(\tau, f) = \frac{-\alpha_\kappa c^+_{f,\kappa}(-n)}{2\pi i \hat{\delta} \operatorname{vol}_\ell(\Lambda)} \exp(4\pi i \hat{\delta}^{-1} \Im(w \overline{h})). \qquad \square$$

4.4 Proof of Theorem 4.2.2

Proof. First, we recall that

$$\frac{\partial}{\partial\overline{\tau}}\xi^0(\sqrt{v}X,z) = \frac{1}{2iv}\exp(-2\pi vR(X,z)) = \frac{-1}{2iv^2}\varphi_S^0(\sqrt{v}X,z)$$

and we define a differential 1-form

$$\begin{aligned}\psi(X,\tau,z) &= \frac{\partial}{\partial\overline{\tau}}\partial\xi(X,\tau,z)\\ &= \frac{-\pi v}{2\hat{\delta}y}\left(\hat{\delta}R(X,z) - 2iw'(\overline{w'z-w})\right)\exp(-2\pi vR(X,z))\exp(-\pi\langle X,X\rangle\tau)dz.\end{aligned}$$

Lemma 4.4.1. *Let $f(z)$ be a harmonic function. Then*

$$d\left(f(z)\frac{\partial}{\partial\overline{\tau}}\partial\xi^0 - \overline{\partial}f(z)\frac{\partial}{\partial\overline{\tau}}\xi^0\right) = f(z)\overline{\partial}\partial\frac{\partial}{\partial\overline{\tau}}\xi^0$$

Proof. By definition, $d = \partial + \overline{\partial}$, so

$$\begin{aligned}d\left(f(z)\frac{\partial}{\partial\overline{\tau}}\partial\xi^0 - \overline{\partial}f(z)\frac{\partial}{\partial\overline{\tau}}\xi^0\right) &= \partial f(z)\partial\frac{\partial}{\partial\overline{\tau}}\xi^0 + f(z)\partial^2\frac{\partial}{\partial\overline{\tau}}\xi^0 + \overline{\partial}f(z)\partial\frac{\partial}{\partial\overline{\tau}}\xi^0\\ &\quad + f(z)\overline{\partial}\partial\frac{\partial}{\partial\overline{\tau}}\xi^0 - \partial\overline{\partial}f(z)\frac{\partial}{\partial\overline{\tau}}\xi^0 - \overline{\partial}f(z)\frac{\partial}{\partial\overline{\tau}}\partial\xi^0\\ &\quad - \overline{\partial}^2 f(z)\frac{\partial}{\partial\overline{\tau}}\xi^0 - \overline{\partial}f(z)\overline{\partial}\frac{\partial}{\partial\overline{\tau}}\xi^0,\end{aligned}$$

and by the fact that $\partial^2 = \overline{\partial}^2 = dz\wedge dz = d\overline{z}\wedge d\overline{z} = 0$, we have that virtually all the terms above vanish. Using the fact that $f(z)$ is harmonic, i.e. $\partial\overline{\partial}f(z) = 0$ we are simply left with the result. □

We have that

$$\begin{aligned}\frac{\partial}{\partial\overline{\tau}}I_{\mathbf{h}}(\tau,f(z)) &= \frac{\partial}{\partial\overline{\tau}}\lim_{T\to\infty}\int_{M_T} f(z)\theta(z,\tau,\varphi_{KM}^0)\\ &= \frac{-1}{2\pi i}\lim_{T\to\infty}\int_{M_T} f(z)\frac{\partial}{\partial\overline{\tau}}\theta(z,\tau,\overline{\partial}\partial\xi^0)\\ &= \frac{-1}{2\pi i}\lim_{T\to\infty}\int_{M_T} f(z)\theta(z,\tau,\overline{\partial}\psi)\\ &= \frac{-1}{2\pi i}\lim_{T\to\infty}\int_{M_T} d\left(f(z)\theta(z,\tau,\psi) - \overline{\partial}f(z)\theta\left(z,\tau,\frac{-1}{2iv^2}\varphi_S\right)\right)\\ &= \frac{-1}{2\pi i}\lim_{T\to\infty}\int_{\partial M_T}\left(f(z)\theta(z,\tau,\psi) - \overline{\partial}f(z)\theta\left(z,\tau,\frac{-1}{2iv^2}\varphi_S\right)\right)\end{aligned}$$

by Stokes' Theorem, and the Lemma above. Since $\overline{\partial} f(z)$ is a cusp form, as f^+ vanishes under $\overline{\partial}$ and f^- has no constant term by assumption, the second integral vanishes. Hence we are reduced to calculating the first integral,

$$\frac{-1}{2\pi i} \lim_{T\to\infty} \int_{\partial M_T} f(z) \sum_{X \in L+\mathbf{h}} \psi(X,\tau,z).$$

This proves to be a longer and more challenging task. What follows is essentially the calculation from Section 3.4 repeated for the kernel function ψ. Doing this calculation has similar results, in that it changes the sum from being written essentially as a Fourier series in τ to being a Fourier series in z which allows us to perform the integration explicitly. We do this by first explicitly calculating the differential one form $\frac{\partial}{\partial \overline{\tau}}\partial\xi^0(\sqrt{v}X,z)$, and then we will use partial Poisson summation.

We again recall that

$$\xi^0(X,z) = -\mathrm{Ei}(-2\pi R(X,z))$$

and hence

$$\frac{\partial}{\partial\overline{\tau}}\partial\xi^0(\sqrt{v}X,z) = \frac{\partial}{\partial\overline{\tau}}\left(-\exp(-2\pi R(\sqrt{v}X,z))\frac{\partial R(\sqrt{v}X,z)}{R(\sqrt{v}X,z)}\right).$$

Since the RHS has no dependence on u, and $R(\sqrt{v}X,z) = vR(X,z)$, we have

$$\frac{\partial}{\partial\overline{\tau}}\partial\xi^0(\sqrt{v}X,z) = \frac{-i}{2}\left(2\pi\exp(-2\pi vR(X,z))\partial R(\sqrt{v}X,z)\right).$$

Using the substitutions

$$\alpha = \sqrt{\frac{2v}{\hat{\delta}y}}\left(\Re(w) - x\Re(w') - y\Im(w')\right)$$

$$\beta = \sqrt{\frac{2v}{\hat{\delta}y}}\left(\Im(w) - x\Im(w') + y\Re(w')\right)$$

we have that

$$2vR(\sqrt{v}X,z) = \alpha^2+\beta^2$$

$$\partial R(X,z) = \frac{i}{2vy}\left(\left(\alpha + i\overline{w}'\sqrt{\frac{2vy}{\hat{\delta}}}\right)^2 + \left(\beta - \overline{w}'\sqrt{\frac{2vy}{\hat{\delta}}}\right)^2\right)dz$$

and hence

$$\frac{\partial}{\partial\overline{\tau}} I_{\mathbf{h}}(\tau, f(z)) = \lim_{T\to\infty}\int_{\partial M_T} f(z)\sum_{x\in L} F(w,w',z,\tau)dz,$$

where

$$F(w,w',z,\tau)=\frac{i}{4vy}\left(\left(\alpha+i\overline{w}'\sqrt{\frac{2vy}{\hat{\delta}}}\right)^2+\left(\beta-\overline{w}'\sqrt{\frac{2vy}{\hat{\delta}}}\right)^2\right)$$
$$\cdot\exp(-\pi(\alpha^2+\beta^2))\exp(-\pi i\langle X,X\rangle\tau).$$

We now want to do partial Poisson summation in the w variable. For this, we now regard F as being a function only of w, holding all other variables constant, and by abuse of notation write it as $F(x_1,x_2)$, so that

$$\frac{\partial}{\partial\overline{\tau}}I(\tau,f(z))=\lim_{T\to\infty}\int_{\partial M_T}f(z)\sum_{X\in L+\mathbf{h}}F(\Re(w),\Im(w))dz$$

where the dependence on the variables w', z and τ has been suppressed in the notation.

Lemma 4.4.2. *Let $Y=\frac{\hat{\delta}}{2}(y_2-iy_1)\ell+w'\ell'$. The Fourier transform of the function $F(x_1,x_2)$ defined above is*

$$\widehat{F}(y_1,y_2)=-\frac{i\hat{\delta}}{8v^2}\left(2yR(Y,\tau)-\frac{1}{\pi}\right)\exp(-2\pi yR(Y,\tau))\exp(\pi i\langle Y,Y\rangle)$$

Proof. We can rewrite the length of a vector in terms of α and β:

$$\begin{aligned}\langle X,X\rangle&=4\hat{\delta}^{-1}\Im(w\overline{w}')\\&=4\hat{\delta}^{-1}\left(\left(\sqrt{\frac{\hat{\delta}y}{2v}}\beta+\Im(zw')\right)\Re(w')-\left(\sqrt{\frac{\hat{\delta}y}{2v}}\alpha+\Re(zw')\right)\Im(w')\right)\\&=4\hat{\delta}^{-1}\sqrt{\frac{\hat{\delta}y}{2v}}(\beta\Re(w')-\alpha\Im(w'))+4\hat{\delta}^{-1}y|w'|^2.\end{aligned}$$

Hence

$$F(\Re(w),\Im(w))=\frac{i}{4vy}((\alpha+i\overline{w}'\sqrt{\frac{2vy}{\hat{\delta}}})^2+(\beta-\overline{w}'\sqrt{\frac{2vy}{\hat{\delta}}})^2)\exp(-\pi(\alpha^2+\beta^2))$$
$$\exp(4\pi i\hat{\delta}^{-1}\sqrt{\frac{\hat{\delta}y}{2v}}(\beta\Re(w')-\alpha\Im(w'))\tau+4\pi i\hat{\delta}^{-1}y|w'|^2\tau)$$

The Fourier transform of F is

$$\begin{aligned}\widehat{F}(y_1, y_2) &= \int_{\mathbb{R}^2} F(x_1, x_2) \exp(-2\pi i(x_1 y_1 + x_2 y_2)) dx_1 dx_2 \\ &= \frac{i\hat{\delta}}{8v} \exp(-2\pi i(y_1 \Re(zw') + y_2 \Im(zw'))) \exp(4\pi i y |w'|^2 \tau) \\ &\int_{\mathbb{R}^2} ((\alpha + i\overline{w}'\sqrt{\frac{2vy}{\hat{\delta}}})^2 + (\beta - \overline{w}'\sqrt{\frac{2vy}{\hat{\delta}}})^2) \exp(-\pi(\alpha^2 + \beta^2)) \\ &\exp(4\pi i \hat{\delta}^{-1}\sqrt{\frac{\hat{\delta}y}{2v}}(\beta\Re(w') - \alpha\Im(w'))\tau)) \exp(-2\pi i\sqrt{\frac{\hat{\delta}y}{2v}}(\alpha y_1 + \beta y_2)) d\alpha d\beta\end{aligned}$$

We now calculate the integral above. We do this by using the standard result

$$\int_{\mathbb{R}} \exp(-2\pi xy) \exp(-\pi x^2) H_n(\sqrt{2\pi}x) dx = \exp(-\pi y^2) H_n(\sqrt{2\pi}y) i^n$$

from [GR07, Sec. 7.376]. For brevity, we now write

$$\begin{aligned}a &= \sqrt{\frac{\hat{\delta}y}{2v}}(y_1 + 2\hat{\delta}^{-1}\Im(w')\tau) \\ b &= \sqrt{\frac{\hat{\delta}y}{2v}}(y_2 - 2\hat{\delta}^{-1}\Re(w')\tau) \\ k &= \sqrt{\frac{2vy}{\hat{\delta}}}\overline{w}'\end{aligned}$$

so that the integral we are interested in calculating can be written more compactly as

$$\int_{\mathbb{R}^2} ((\alpha + ik)^2 + (\beta - k)^2) \exp(-\pi(\alpha^2 + \beta^2)) \exp(-2\pi i(\alpha a + \beta b)) d\alpha d\beta.$$

This is completely separable, so we write it as

$$\begin{aligned}&\left(\int_{\mathbb{R}} (\alpha + ik)^2 \exp(-\pi\alpha^2) \exp(-2\pi\alpha a) d\alpha\right) \left(\int_{\mathbb{R}} \exp(-\pi\beta^2)) \exp(-2\pi i\beta b) d\beta\right) \\ &+ \left(\int_{\mathbb{R}} (\beta - k)^2 \exp(-\pi\beta^2) \exp(-2\pi\beta b) d\beta\right) \left(\int_{\mathbb{R}} \exp(-\pi\alpha^2)) \exp(-2\pi i\alpha a) d\alpha\right)\end{aligned}$$

and solve each of these. Two of them are are very simple:

$$\begin{aligned}\int_{\mathbb{R}} \exp(-\pi\alpha^2)) \exp(-2\pi i\alpha a) d\alpha &= \exp(-\pi a^2) \\ \int_{\mathbb{R}} \exp(-\pi\beta^2)) \exp(-2\pi i\beta b) d\beta &= \exp(-\pi b^2)\end{aligned}$$

are standard formulas, see e.g.[GR07]. We calculate the other two integrals by rewriting the polynomials in front as sums of Hermite polynomials and using (4.4). It is easy to see that

$$(\alpha+ik)^2 = \frac{1}{2\pi}\left(\frac{1}{4}H_2(\sqrt{2\pi}\alpha) + ik\sqrt{2\pi}H_1(\sqrt{2\pi}\alpha) + H_0(\sqrt{2\pi}\alpha)\left(k^2-\frac{1}{2}\right)\right)$$
$$(\beta-k)^2 = \frac{1}{2\pi}\left(\frac{1}{4}H_2(\sqrt{2\pi}\beta) - k\sqrt{2\pi}H_1(\sqrt{2\pi}\beta) + H_0(\sqrt{2\pi}\beta)\left(k^2-\frac{1}{2}\right)\right)$$

so, by applying (4.4) and linearity, we obtain

$$\int_{\mathbb{R}^2}((\alpha+ik)^2+(\beta-k)^2)\exp(-\pi(\alpha^2+\beta^2))\exp(-2\pi i(\alpha a+\beta b))d\alpha d\beta$$
$$= -\exp(-\pi(a^2+b^2))\left((a+k)^2+(b+ik)^2-\frac{1}{\pi}\right)$$

We will now substitute a, b and k back out of the equation. First however, we note that we have following tidy expression:

$$(a+k)^2+(b+ik)^2 = \frac{2y}{\hat{\delta}v}\left|\frac{\hat{\delta}}{2}(y_2-iy_1)-w'\tau\right|^2.$$

Similarly, in the exponent, we have

$$-\pi(a^2+b^2) - 2\pi i\sqrt{\frac{\hat{\delta}}{2v}}\,(y_1\Re(zw')+y_2\Im(zw')) - 4\pi i\hat{\delta}^{-1}y|w'|^2\tau$$
$$= -\frac{2\pi y}{\hat{\delta}v}\left|\frac{\hat{\delta}}{2}(y_2-iy_1)-w'\tau\right|^2 + \pi iz\langle Y,Y\rangle$$

where $Y = \frac{\hat{\delta}}{2}(y_2-iy_1)\ell + w'\ell'$. Using this, and the explicit formula for R, we see that

$$\widehat{F}(y_1,y_2) = -\frac{i\hat{\delta}}{8v^2}\left(2yR(Y,\tau)-\frac{1}{\pi}\right)\exp(-2\pi yR(Y,\tau))\exp(\pi i\langle Y,Y\rangle)$$

□

The proof of the theorem now proceeds similarly to Theorem 3.4.2. As before, we assume that $L=\Lambda\ell\oplus\Lambda'\ell'$ and we let S be the matrix such that

$$\pi_*(\mathbb{Z}^2S)=\Lambda$$

where

$$\pi_*(x_1,x_2)=x_1+\zeta x_2.$$

We recall that, in such an arrangement, we have that $4\hat{\delta}^{-1}\det S = \mathrm{vol}_\ell(\Lambda)$. Now let

$$G(x_1, x_2) = F\left(\mathbf{x}S + H\right)$$

so that

$$\begin{aligned}\frac{\partial}{\partial\overline{\tau}} I(\tau, f(z)) &= \lim_{T\to\infty}\int_{\partial M_T} f(z)\sum_{w'\in\Lambda'+h'}\sum_{x_i\in\mathbb{Z}} G(x_1, x_2)dz\\ &= \lim_{T\to\infty}\int_{\partial M_T} f(z)\sum_{w'\in\Lambda'+h'}\sum_{x_i\in\mathbb{Z}} \widehat{G}(x_1, x_2)dz\end{aligned}$$

by Poisson summation. We may easily calculate the Fourier transform of G in terms of the Fourier transform of F, in a calculation almost identical to that in Theorem 3.4.2:

$$\widehat{G}_h(x_1, x_2) = \frac{1}{\det S}\exp(2\pi i H^t(\mathbf{x}^t S^{-1}))\widehat{F}(\mathbf{x}^t S^{-1})$$

We have adopted the convention that $f((x_1, x_2)) = f(x_1, x_2)$ for notational convenience. Using the notation $\mathbf{x} = (x_1, x_2)$ and $H = (h_1, h_2)$, a simple calculation shows that

$$\widehat{G}_h(x_1, x_2) = \frac{1}{\det S}\exp(2\pi i H^t(\mathbf{x}^t S^{-1}))\widehat{F}(\mathbf{x}^t S^{-1})$$

We substitute this in, and, noting that ${}^tS^{-1} = \frac{1}{\det S} JSJ^{-1}$ where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we obtain

$$\begin{aligned}\sum_{x_i\in\mathbb{Z}} \widehat{G}_h(x_1, x_2) &= \sum_{x_i\in\mathbb{Z}} \frac{1}{\det S}\exp(2\pi i H^t(\mathbf{x}^t S^{-1}))\widehat{F}(\mathbf{x}^t S^{-1})\\ &= \sum_{x_i\in\mathbb{Z}} \frac{1}{\det S}\exp(2\pi i H^t(\mathbf{x}JSJ^{-1})\det S^{-1})\widehat{F}(\mathbf{x}JSJ^{-1}\det S^{-1})\end{aligned}$$

Since the sum over $\mathbf{x}J$ (where $\mathbf{x}$ runs over all the integer pairs) is clearly just a reordering of the sum, we may write

$$\begin{aligned}\sum_{x_i\in\mathbb{Z}}\widehat{G}_h(x_1,x_2) &= \sum_{\mathbf{x}\in\frac{1}{\det S}\mathbb{Z}^2S}\frac{1}{\det S}\exp(2\pi i H^t(\mathbf{x}SJ^{-1}))\widehat{F}(\mathbf{x}J^{-1})\\ &= \sum_{w\in\frac{1}{\det S}\pi_*(\mathbb{Z}^2S)}\frac{1}{\det S}\exp\left(2\pi i(\Im(w)h_1-\Re(w)h_2)\right)\widehat{F}\left(\Im(w),-\Re(w)\right)\\ &= \sum_{w\in\frac{\hat\delta}{2\det S}\pi_*(\mathbb{Z}^2S)}\frac{1}{\det S}\exp\left(4\hat\delta^{-1}\pi i(\Im(w)h_1-\Re(w)h_2)\right)\\ &\qquad\widehat{F}\left(2\hat\delta^{-1}\Im(w),-2\hat\delta^{-1}\Re(w)\right)\\ &= \sum_{w\in\Lambda^*}\frac{4}{\hat\delta\operatorname{vol}_\ell(\Lambda)}\exp\left(4\pi i\hat\delta^{-1}\Im(w\overline{h})\right)\\ &\qquad\widehat{F}\left(2\hat\delta^{-1}\Im(w),-2\hat\delta^{-1}\Re(w)\right)\end{aligned}$$

We now use the formula for $\widehat{F}$ from the lemma to obtain

$$\begin{aligned}\frac{\partial}{\partial\overline{\tau}}I_{\mathbf{h}}(\tau,f(z)) = \frac{1}{2iv^2\hat\delta\operatorname{vol}_\ell(\Lambda)}\lim_{T\to\infty}\int_{\partial M_T}f(z)\sum_{X\in L^*_{\mathbf{h}}}(2yR(X,\tau)-\frac{1}{\pi})\exp(-2\pi yR(X,\tau))\\ \exp\left(4\pi i\hat\delta^{-1}\Im(w\overline{h})\right)\exp(\pi iz\langle X,X\rangle)dz\end{aligned}$$

We recall that

$$f(\sigma_\kappa z) = \sum_{n\geq -N_\kappa}c^+_{f,\kappa}(n)\exp(2\pi inz/\alpha_\kappa)+\sum_{n<0}c^-_{f,\kappa}(n)H_0(2\pi ny)\exp(2\pi inx/\alpha_\kappa)$$

so that

$$\begin{aligned}\frac{\partial}{\partial\overline{\tau}}I_{\mathbf{h}}(\tau,f(z)) = \frac{1}{2iv^2\hat\delta\operatorname{vol}_\ell(\Lambda)}\lim_{T\to\infty}\sum_{\kappa\in\Gamma_L\backslash\mathbb{P}^1(\mathbb{Q})}\int_{iT}^{iT+\alpha_\kappa}F(\sigma_\kappa z)\\ \sum_{X\in\sigma_\kappa^{-1}L^*_{\mathbf{h}}}(2yR(X,\tau)-\frac{1}{\pi})\exp(-2\pi yR(X,\tau))\exp(\pi iz\langle X,X\rangle)\\ \exp\left(4\pi i\hat\delta^{-1}\Im(w\overline{\sigma_\kappa^{-1}h})\right)dz\end{aligned}$$

We can think of this as an integral in x, i.e. we just need to solve

$$\int_0^{\alpha_\kappa}\exp(2\pi inx/\alpha_\kappa)\exp(\pi ix\langle X,X\rangle)dx = \begin{cases}\alpha_\kappa & \frac{1}{2}\langle X,X\rangle = -n/\alpha_\kappa\\ 0 & \text{else}\end{cases}$$

Hence, we have

$$\frac{\partial}{\partial\overline{\tau}} I_{\mathbf{h}}(\tau, f(z)) = \frac{1}{2iv^2\hat{\delta}\operatorname{vol}_\ell(\Lambda)} \lim_{T\to\infty} \sum_{\kappa\in\Gamma_L\backslash\mathbb{P}^1(\mathbb{Q})} \sum_{n\in\mathbb{Z}} c^+_{f,\kappa}(n)$$
$$\sum_{X\in\sigma_\kappa^{-1}L^*_{\mathbf{h},-n}} \left(2TR(X,\tau) - \frac{1}{\pi}\right) \exp(-2\pi TR(X,\tau))$$
$$\exp\left(4\pi i\hat{\delta}^{-1}\Im(w\overline{\sigma_\kappa^{-1}h})\right)$$

It is clear that the limit converges to 0 if and only if $R(X,\tau) > 0$. However, by definition $R(X,\tau) \geq 0$, and we have $R(X,\tau) = 0$ if and only if $\sigma_\kappa\tau = \mathbb{D}_X$ for $X \in L^*_{\mathbf{h},-n}$ or in the case of $X = 0$, which we deal with below.

Since $\sigma_\kappa\tau = \mathbb{D}_X$ is only possible for X of positive length, and $c^+_{f,\kappa}(-n)$ is always equal to 0 for n big enough, we only have a finite number of terms where we do not have have convergence, coming from from the negative powers in the Fourier expansion of F. More explicitly then, the finite set of points on the curve M where the series diverges are the set of $\tau \in \mathbb{H}$ such that

$$\sigma_\kappa\tau = \mathbb{D}_X$$

for all $X \in \Gamma_L\backslash L^*_{\mathbf{h},n}$ and all $\kappa \in \Gamma_L\backslash\mathbb{P}^1(\mathbb{Q})$ provided $c^+_{f,\kappa}(n) \neq 0$ and $n < 0$. This is completely expected, as these are exactly the singularities from Theorem 4.2.1

It remains only to deal with the case of $X = 0$, in which we note that we must have $\mathbf{h} = 0$, so we get

$$\frac{\partial}{\partial\overline{\tau}} I_{\mathbf{h}}(\tau, f(z)) = \sum_{\kappa\in\Gamma_L\backslash\mathbb{P}^1(\mathbb{Q})} \frac{-\delta_{\mathbf{h},0}c^+_{f,\kappa}(0)}{2\pi iv^2\hat{\delta}\operatorname{vol}_\ell(\Lambda)}$$

hence

$$\xi_2(I_{\mathbf{h}}(\tau, f)) = -2iv^2\overline{\frac{\partial}{\partial\overline{\tau}} I_{\mathbf{h}}(\tau, f(z))} = \sum_{\kappa\in\Gamma_L\backslash\mathbb{P}^1(\mathbb{Q})} \frac{-\delta_{\mathbf{h},0}\overline{c^+_{f,\kappa}(0)}}{\pi\hat{\delta}\operatorname{vol}_\ell(\Lambda)}$$

□

Chapter 5

The Fourier Expansion

We can now calculate the Fourier expansion of the lift of a harmonic weak Maass form. We do this by utilising the current equation which was proved in Chapter 3. Since the current equation is only valid for $v >> 0$, the Fourier expansion we obtain for the lift will also only be valid in this domain. The Fourier coefficients will turn out to be the harmonic weak Maass form f evaluated on the divisor described in Chapter 1.

5.1 The unfolding method

In this method we use an unfolding technique, similar to that of Rankin-Selberg unfolding. This unfolding is only valid for the non-isotropic vectors of the lattice however, and so we are forced to deal with the isotropic vectors separately.

Theorem 5.1.1. *Let $f(z)$ be a harmonic weak Maass form for Γ_L, lying in $H_0^+(\Gamma_L)$, with Fourier expansion*

$$f(\sigma_\kappa z) = \sum_{n \geq -N_f} c_{f,\kappa}^+(n) \exp(2\pi i n z/\alpha_\kappa) + \sum_{n<0} c_{f,\kappa}^-(n) H_0(4\pi n y/\alpha_\kappa) \exp(2\pi i n x/\alpha_\kappa)$$

around each cusp $\kappa \in \Gamma_L \backslash \mathbb{P}^1(\mathbb{Q})$ with width α_κ. Then, for $v > N_{\Lambda'} N_f \hat{\delta}$, (with $N_{\Lambda'} \in \mathbb{Z}_+$ defined below)

$$I(\tau, f) = \sum_{\mathbf{h} \in L'/L} \left(\sum_{\kappa \in \Gamma_L \backslash \mathbb{P}^1(\mathbb{Q})} \delta_\kappa \frac{\alpha_\kappa c_{f,\kappa}^+(0)}{4\pi v} + \sum_{m \in \mathbb{Z} + \frac{1}{2}\langle \mathbf{h}, \mathbf{h} \rangle} \left(\sum_{z \in T(m, \mathbf{h})} f(z) \right) q^m \right) \mathfrak{e}_{\mathbf{h}}$$

where $\delta_\kappa = 1$ *if the line corresponding to* κ *in* V *has non-empty intersection with* $L + \mathbf{h}$*, and is* 0 *otherwise.*

Proof. Recalling that

$$\theta_{\mathbf{h}}(z, \tau, L) = \sum_{X \in L+\mathbf{h}} \varphi^0_{KM}(\sqrt{v}X, z) q^{\frac{1}{2}\langle X, X\rangle}$$

we define

$$\theta^0_{\mathbf{m}}(v) = \sum_{X \in L_{\mathbf{h},m}} \varphi^0_{KM}(\sqrt{v}X, z)$$

so that

$$I_{\mathbf{h}}(\tau, f) = \sum_{m \in \mathbb{Z}+\frac{1}{2}\langle \mathbf{h}, \mathbf{h}\rangle} \left(\int_M f(z)\theta^0_m(v) \right) q^m$$

For $m \neq 0$, we can unfold the integral:

$$\begin{aligned} \int_M \theta^0_m(v) f(z) &= \int_{\Gamma_L \backslash \mathbb{D}} \sum_{X \in \Gamma_L \backslash L_{\mathbf{h},m}} \sum_{\gamma \in \Gamma_L} f(z) \varphi^0_{KM}(\sqrt{v}\gamma X, z) \\ &= \sum_{X \in \Gamma_L \backslash L_{\mathbf{h},m}} \int_{\Gamma_L \backslash \mathbb{D}} \sum_{\gamma \in \Gamma_L} f(\gamma^{-1} z) \varphi^0_{KM}(\sqrt{v}X, \gamma^{-1} z) \\ &= \sum_{X \in \Gamma_L \backslash L_{\mathbf{h},m}} \frac{1}{|\overline{\Gamma}_X|} \int_{\mathbb{D}} f(z) \varphi^0_{KM}(\sqrt{v}X, z) \end{aligned}$$

which follows by the invariance of the two form $\frac{dxdy}{y^2}$ under any $\gamma \in \mathrm{SL}_2(\mathbb{R})$ and Proposition 3.1.4. It is important to note here that such unfolding will not work for $m = 0$, as then interchanging summation and integration is not valid as it an infinite sum in that case. We also need to be careful that the inner integral converges - this is the reason for the condition on v. By our previous analysis, we know that this integral will converge for all $v > \frac{2N_f \Im(\mathbb{D}_X)}{m}$, but we would like a more uniform bound (not depending on X) on this. We note that $2\Im(\mathbb{D}_X)/m = \hat{\delta}|w'|^2$, and so if $N_{\Lambda'}$ is the smallest positive integer such that $N_{\Lambda'}|w'|^2 \in \mathbb{Z}$ for all non-zero $w' \in \Lambda' + h'$, then we can take $v > N_{\Lambda'} N_f \hat{\delta}$. Since this bound will be attained by the definition of $N_{\Lambda'}$, it is sharp.

We now recall the current equation

$$dd^c[\xi^0(X, z)] + \delta_X = [\varphi^0(X, z)]$$

and apply it above (which is valid for our restriction on v), by Theorem 3.2.9 this shows that

$$\int_M f(z)\theta_m^0(v) = \epsilon \sum_{X \in \Gamma_L \backslash L_{\mathbf{h},m}} \frac{1}{|\overline{\Gamma}_X|} f(\mathbb{D}_X)$$

since $dd^c f(z) = 0$.

To deal with the case $m = 0$, first we look at the case of $X = 0$: this gives the term

$$-\frac{\delta_{\mathbf{h},0}}{4\pi} \int_M f(z) \frac{dxdy}{y^2}$$

which is exactly the constant term that we expect. For $X \neq 0$ things are more tricky. We will show that the integral over the non-zero isotropic vectors must be a multiple of the constant term in the Fourier expansion of $f(\sigma_\kappa z)$. This calculation is very similar to the one in Section 4.3.2(C) of [Fun02].

Let $\kappa_1, \ldots, \kappa_t$ be a set of Γ_L-representatives of the isotropic lines in V. For each κ_i we let δ_{κ_i} be equal to 1 if the intersection $\kappa_i \cap (L + h)$ is non-empty, and be 0 otherwise. Moreover, for each κ_i there exists $\sigma_i \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma_i \ell = X_i$, which is clearly seen by the fact that under the projection map π, we can identify the κ_i with the cusps of $\Gamma_L \backslash \mathbb{H}^*$. We then have the usual formulation for the stabiliser of X_i as for the stabiliser of the cusp, namely that

$$\sigma_i^{-1} \Gamma_{X_i} \sigma_i = \left\{ \begin{pmatrix} 1 & n\alpha_i \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

where the width of the cusp κ_i is denoted α_i. We now have

$$\int_M \sum_{\substack{X \in (L+h)_0 \\ X \neq 0}} \varphi_{KM}^0(\sqrt{v}X, z) f(z) = \sum_{i=1}^t \delta(\kappa_i) \int_M \sum_{\substack{X \in \Gamma_L(\kappa_i \cap (L+h)) \\ X \neq 0}} f(z) \varphi_{KM}^0(\sqrt{v}X, z).$$

Examining the integral we have

$$\int_M \sum_{\substack{X \in \Gamma_L(\kappa_i \cap (L+h)) \\ X \neq 0}} f(z) \varphi_{KM}^0(\sqrt{v}X, z) = \int_M \sum_{\substack{X \in \kappa_i \cap (L+h) \\ X \neq 0}} \sum_{\gamma \in \Gamma_i \backslash \Gamma_L} f(\gamma z) \varphi_{KM}^0(\sqrt{v}X, \gamma z).$$

We identify M with $\Gamma_L\backslash\mathbb{H}$, and for notational convenience we now write $\Gamma' = \sigma_i^{-1}\Gamma_L\sigma_i$, so that,

$$\int_M \sum_{\substack{X\in(L+h)_0\\X\neq 0}} f(z)\varphi^0_{KM}(\sqrt{v}X,z) =$$

$$\int_{\Gamma'\backslash\mathbb{H}} \sum_{\gamma\in\Gamma_i\backslash\Gamma_L} \sideset{}{'}\sum_{X\in\kappa_i\cap(L+h)} f(\gamma\sigma_i z)\varphi^0_{KM}(\sigma_i^{-1}\sqrt{v}X, \sigma_i^{-1}\gamma\sigma_i z).$$

where the $'$ on the sum means we omit $X=0$ if $h=0$.

Now, we observe that since $X\in\kappa_i$, $\sigma_i^{-1}X$ must therefore be some multiple of $X_0=\ell$, as this is the preimage of the cusp at ∞. Therefore, for some set of values, which we call K_i we have that

$$\int_M \sum_{\substack{X\in(L+h)_0\\X\neq 0}} f(z)\varphi^0_{KM}(\sqrt{v}X,z) =$$

$$\int_{\Gamma'\backslash\mathbb{H}} \sum_{\gamma\in\Gamma'_{X_0}\backslash\Gamma'} \sideset{}{'}\sum_{k\in K_i} f(\sigma_i z)\varphi^0_{KM}(\sqrt{v}kX_0, z).$$

Now, we observe that we may unfold the integral, and that we have a nice expression for φ^0_{KM} at these values, namely,

$$\varphi^0_{KM}(\sqrt{v}kX_0,z) = \frac{1}{4\pi}\left(\frac{2\pi v}{\hat{\delta}y}|k|^2-1\right)\exp\left(-\frac{2\pi v}{\hat{\delta}y}|k|^2\right)d\mu(z)$$

which we note has no dependence on x. Indeed, after unfolding, we have

$$\int_{\Gamma'_{X_0}\backslash\mathbb{H}} \sideset{}{'}\sum_{k\in K_i} f(\sigma_i z)\varphi^0_{KM}(\sqrt{v}kX_0,z)$$

and the x part of the integral is simply

$$\int_0^{\alpha_i} f(\sigma_i z)dx = \alpha_i c^+_{f,\kappa_i}(0)$$

which picks out the constant term in the Fourier expansion. In order to then complete the calculation and evaluate the integral in y, we need to employ a trick (also used by Borcherds in [Bor98]) of multiplying inside the integral by y^{-s} and then taking the limit as $s\to\infty$ after calculating the integral. Hence, we wish to calculate

$$\sum_{i=1}^t \delta_{\kappa_i}\int_0^\infty \alpha_i c^+_{f,\kappa_i}(0)\frac{1}{4\pi}\sum_{k\in K_i}\left(\frac{2\pi v}{\hat{\delta}y}|k|^2-1\right)\exp\left(-\frac{2\pi v}{\hat{\delta}y}|k|^2\right)\frac{dy}{y^{2+s}}. \tag{5.1}$$

To do this, we need

Lemma 5.1.2. *For $A \neq 0$,*

$$\int_0^\infty (Ay^{-1} - 1)\exp(-Ay^{-1})y^{-2-s}dy = s\Gamma(s+1)A^{-1-s},$$

Proof. We make the substitution $Ay^{-1} = t$, to obtain

$$\int_0^\infty (Ay^{-1} - 1)\exp(-Ay^{-1})y^{-2-s}dy = A^{-1-s}\int_0^\infty (t-1)\exp(-t)t^s dt$$

which, using the integral representation

$$\Gamma(s) = \int_0^\infty \exp(-t)t^{s-1}dt$$

and the fact that $\Gamma(s+1) = s\Gamma(s)$, we get

$$\begin{aligned} A^{-1-s}\int_0^\infty (t-1)\exp(-t)t^s dt &= A^{-1-s}\left(\int_0^\infty \exp(-t)t^{s+1}dt - \int_0^\infty \exp(-t)t^s dt\right) \\ &= A^{-1-s}\left(\Gamma(s+2) - \Gamma(s+1)\right) \\ &= A^{-1-s}\left((s+1)\Gamma(s+1) - \Gamma(s+1)\right) \\ &= A^{-1-s}s\Gamma(s+1) \end{aligned}$$

□

So using the lemma above,5.1 is equal to

$$\sum_{i=1}^t \delta_{\kappa_i}\alpha_i c_{f,\kappa_i}^+(0)\frac{1}{4\pi}s\Gamma(1+s)\sum_{k\in K_i}\left(\frac{2\pi v}{\hat{\delta}}|k|^2\right)^{-1-s}.$$

So we need to characterise the set K_i. But it is clear that all we need to recognise is that K_i is an $\mathcal{O}_F$ ideal. Then we have that

$$\sum_{k\in K_i}|k|^{-2-2s} = w_F\zeta_F(1+s,[K_i])$$

where w_F is the number of roots of unity in F and $\zeta_F(s,[K_i])$ is the partial Dedekind zeta function for the class containing K_i in the ideal class group.

Now, in the limit $s \to 0$, this is equal to

$$\sum_{i=1}^t \delta_{\kappa_i}\alpha_i c_{f,\kappa}^+(0)\frac{\hat{\delta}w_F}{8\pi^2 v}\operatorname*{Res}_{s=1}\zeta_F(s,[K_i]).$$

By the analytic class number formula, the value of this residue is $2\pi w_F \hat{\delta}^{-1}$, hence the contribution for the non-zero isotropic vectors is

$$\sum_{i=1}^{t} \delta_{\kappa_i} \frac{\alpha_i c^{+}_{f,\kappa_i}(0)}{4\pi v}.$$

which is equivalent to the expression in the statement of the theorem by the identification of the Γ_L equivalence classes of isotropic lines with the cusps of $\Gamma_L \backslash \mathbb{H}$. □

We examine a few different cases of input f, beginning with the most simple.

Theorem 5.1.3. *Let $L = \mathcal{O}_F \ell \oplus \mathcal{O}_F \ell'$, and so $\Gamma_L = \mathrm{SL}_2(\mathbb{Z})$. Then $L = L' = L^*$, so there is only the trivial coset $\mathbf{h} = 0$ to consider (which we drop from the notation), and only the one cusp at ∞. If we lift the constant function 1, we obtain the following formula, using Theorem 5.1.1*

$$I(\tau, 1) = -\frac{1}{12}\left(-\frac{3}{\pi v} + 1 - 24 \sum_{n \geq 0} \sigma_1(m) q^m\right) = -\frac{1}{12} E_2(\tau)$$

using the formula for the traces of 1 we calculated in Chapter 1. We noted then that the fact that the traces of 1 produced the Fourier coefficients of the weight 2 Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ was not a coincidence and we see now that it is not; we obtain the constant term from the seemingly arbitrary (at the time) definition of the 0-th trace, and we obtain the non-holomorphic part from the non-zero isotropic vectors.

This gives us a general principle: non-holomorphic parts of the Fourier expansion are due only to Eisenstein series. We can see why by examining the formula in the statment of Theorem 5.1.1 - the non-holomorphic parts come from the constant term (at each cusp) of the input function f. However, the lift of the constant term is an Eisenstein series, and so it is possible (if messy) to subtract on the one hand, all the constant terms at all the cusps from the input function, whilst on the other, subtracting the Eisenstein series which are the images of these from the Fourier expansion, to obtain the holomorphic (in a neighbourhood on infinity) function which is the lift of the Maass cusp form we generated by this procedure.

5.2 The case of the j function

In the following section, we use the notation

$$J'(\tau) = \frac{1}{2\pi i}\frac{d}{d\tau}J(\tau).$$

We do this to follow convention; this is equivalent to taking the derivative in q.

Theorem 5.2.1. *Let $M_2(m,\mathbb{Z})$ be the 2×2 matrices with integer entries and determinant m. For $\tau > \zeta$*

$$I(\tau, J_m) = \frac{2}{m}\sum_{\gamma\in\Gamma\backslash M_2(m,\mathbb{Z})}\frac{J'_m(\tau)}{J_m(\gamma\cdot\zeta) - J_m(\tau)} \tag{5.2}$$

Proof. From Theorem 4.2.1, we can see that the poles of both functions are the same, namely, they lie on the divisor $T(1,0)$. Hence, their difference,

$$f(\tau) = I(\tau, J_m) - \frac{2}{m}\sum_{\gamma\in\Gamma\backslash M_2(m,\mathbb{Z})}\frac{J'_m(\tau)}{J_m(\gamma\cdot\zeta) - J_m(\tau)}$$

is, *a fortiori*, a holomorphic modular function of weight 2 for the full modular group, and hence much be equal to a constant. By examining the limit $\tau \to \infty$, we can calculate what this constant is. This amounts to calculating the constant terms in both $I(\tau, J_m)$ and $\frac{2}{m}\sum_{\gamma\in\Gamma\backslash M_2(m,\mathbb{Z})}\frac{J'_m(\tau)}{J_m(\gamma\cdot\zeta)-J_m(\tau)}$. The constant term of $I(\tau, J_m)$ is given by Theorem 1.2.12, which says, if $J_m(\tau) = \sum_{n\in\mathbb{Z}} a(n)q^n$,

$$\mathrm{tr}_{J_m}(0,0) = 2\sum_{n\in\mathbb{Z}_{\geq 0}} a(-n)\sigma_1(n),$$

and we can calculate that $\mathrm{tr}_{J_m}(0,0) = 2\sigma_1(m)$.

The constant term of $\frac{2}{m}\sum_{\gamma\in\Gamma\backslash M_2(m,\mathbb{Z})}\frac{J'_m(\tau)}{J_m(\gamma\cdot\zeta)-J_m(\tau)}$ can be calculated by simply adding up the constant term over each summand, so for each γ in the sum we have

$$\begin{aligned}\lim_{q\to 0}\frac{J'_m(\tau)}{J_m(\gamma\cdot\zeta) - J_m(\tau)} &= \lim_{q\to 0}\frac{(-mq^{-m} + O(q))}{J_m(\gamma\cdot\zeta) - q^{-m} + O(q)}\\ &= \lim_{q\to 0}\frac{(-m + O(q))}{q^m J_m(\gamma\cdot\zeta) - 1 + O(q)}\\ &= m\end{aligned}$$

and so,

$$\begin{aligned}\lim_{q\to 0}\frac{2}{m}\sum_{\gamma\in\Gamma\backslash M_2(m,\mathbb{Z})}\frac{J'_m(\tau)}{J_m(\gamma\cdot\zeta) - J_m(\tau)} &= \frac{2}{m}\sum_{\gamma\in\Gamma\backslash M_2(m,\mathbb{Z})} m\\ &= 2\sigma_1(m)\end{aligned}$$

hence $f(\tau)$ is identically 0, and the proof is complete. □

As we recall from Example 1.2.14, for $m = 1$, we have

$$I(\tau, J) = 2 \sum_{m \in \mathbb{Z}} J_m(\zeta) q^m$$

hence, in combination with the previous theorem, we have (partially) recovered an old result due to Faber, namely

$$\sum_{m \in \mathbb{Z}} J_m(\zeta) q^m = \frac{J'(\tau)}{J(\zeta) - J(\tau)}.$$

We have only shown that this is true when $\tau > \zeta$ for ζ a quadratic irrationality.

Chapter 6

Lifting the Eisenstein series

We now calculate the lift of some functions which are not harmonic weak Maass forms. First, we use the weight 0 Eisenstein series as input, and the result of this is a modified version of the Eisenstein series of weight 2 with respect to the group $\Gamma_0(N)$, which we call $\mathcal{E}_2(\tau, s)$. Using this result, we take residues in the s variable to show that the lift of the constant function is the usual weight 2 Eisenstein series with respect to Γ_L, up to a certain multiplicative factor which we calculate.

We then use these results, combined with a Kronecker limit formula, to show that the lift of the logarithm of a level N analogue of the modular Δ function is the derivative of the extended weight 2 Eisenstein series, which was carried out in the SO(2, 1) case in [BF06].

For this chapter, we take a non-vector valued approach. Let $L = \mathcal{O}_F\ell \oplus N\mathcal{O}_F\ell'$. Then, by Theorem 3.2.13,

$$\theta(z, \tau, L) = \sum_{X \in L} \varphi_{KM}(X, z, \tau)$$

is a modular form in z for weight 0 and in τ for weight 2 for the congruence subgroup $\Gamma_0(N)$.

6.1 Eisenstein Series

We define an Eisenstein series for $\Gamma_0(N)$ based at the cusp ∞.

Definition 6.1.1. Let $\Gamma_{N,\infty} \subset \Gamma_0(N)$ be the stabiliser of cusp at infinity. The weight 0 Eisenstein series with respect to the group $\Gamma_0(N)$ at the cusp ∞ is defined by

$$E_{0,N}(z,s) = \sum_{\gamma \in \Gamma_{N,\infty} \backslash \Gamma_0(N)} (\Im(\gamma z))^s$$

and the modified Eisenstein series of weight 0 is

$$\mathcal{E}_{0,N}(z,s) = \zeta^*(2s) E_{0,N}(z,s)$$

where $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function. This has an analytic continuation to the whole of the complex plane via the functional equation [Miy89, Section 7.2]

$$\mathcal{E}_{0,N}(z,s) = \mathcal{E}_{0,N}(z,1-s).$$

We note for later use that $\mathcal{E}_{0,N}(z,s)$ has a simple pole at $s=1$ with residue equal to $\frac{1}{2}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]^{-1}$. In order to save space in later formulae, we set $d_N = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$.

We also define a modified weight 2 Eisenstein series.

Definition 6.1.2. Let $\zeta^*(2s)$ be as above, and $\tau = u + iv$ as usual. We define a modified weight 2 Eisenstein series

$$\mathcal{E}_2(\tau,s) = -\frac{1}{4\pi}\zeta^*(2s) s v^{s-1} \sum_{(c,d)=1} |c\tau+d|^{-2(s-1)}(c\tau+d)^{-2}.$$

Initially, it is not clear that this makes sense if $\Re(s) \leq 1$, however, from [Miy89, Ch. 7] we have the following

Proposition 6.1.3. *The Eisenstein series*

$$E_2(z,s) = \sum_{(c,d)=1} |c\tau+d|^{-2s}(c\tau+d)^{-2}$$

is analytically continued to a meromorphic function on the upper half plane. It is a modular form of weight 2 for the full modular group. Miyake also calculates the Fourier expansion of $E_2(z,s)$, but we only need that

$$\lim_{s\to 0} E_2(z,s) = -\frac{3}{\pi y} + 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)\exp(2\pi i n z).$$

Later in this chapter, we will need the partial Dedekind zeta function for the trivial ideal class, given by

$$\zeta_F(s, [\mathcal{O}_F]) = \sum_{I \in [\mathcal{O}_F]} N(I)^{-s}.$$

This is related to the Epstein zeta function $\zeta_Q(s)$ for a positive definite quadratic form $Q(m, n) = am^2 + 2bmn + cn^2$ by

$$\zeta_F(s, [\mathcal{O}_F]) = \frac{1}{w_F}\zeta_Q(s),$$

where w_F is the number of units in the imaginary quadratic field F. Presently, we prefer to work with the Epstein zeta function using [Sie65] as our source, however later in the chapter we will prefer to write formulae in terms of the partial Dedekind zeta function.

We now state some properties of the Epstein zeta function. Let the Epstein Zeta function for a positive definite quadratic form $Q(m, n) = am^2 + 2bmn + cn^2$ be defined as

$$\zeta_Q(s) = {\sum_{m,n\in\mathbb{Z}}}' Q(m, n)^{-s},$$

and let $b^2 - ac = -d$ be the discriminant of the quadratic form in [Sie65]. Let the completed Epstein Zeta function be defined as

$$\zeta_Q^*(s) = (b^2 - ac)^{s/2}\pi^{-s}\Gamma(s)\zeta_Q(s).$$

We also need a Kronecker Limit Formula for $\zeta_Q(s)$.

Proposition 6.1.4. *We recall the definition of $\zeta_Q(s)$ as*

$$\begin{aligned}\zeta_Q(s) &= w_F {\sum_{z\in\mathcal{O}_F}}' N(z)^{-s} = {\sum_{m,n\in\mathbb{Z}}}' (Q(m, n))^{-s} \\ &= \frac{\pi}{d^{1/2}(s-1)} + C + O(s-1), \qquad (6.1)\end{aligned}$$

where the $'$ indicates we omit summands with vanishing denominator. Let $\omega = \frac{b+i\sqrt{d}}{a}$. The value C in (6.1) is

$$C = 2\pi d^{-1/2}\left(\gamma - \log(2) - \log(\sqrt{\Im(\omega)}|\eta(\omega)|^2)\right)$$

where $\eta(z)$ is the Dedekind eta function and γ is the Euler-Mascheroni constant, which can be defined by $\gamma = -\Gamma'(1)$ and whose value is approximately 0.577.

A proof for this result appears in [Sie65, Thm. 1, p.p. 13]. In particular, we will want to use this result when Q represents the norm form on the field F. Then we have

$$\zeta_Q^*(s) = \frac{1}{s-1} + \gamma - \log(4\pi|\eta(\zeta)|^4) + O(s-1)$$

where $Q(m,n) = m^2 + 2Dmn + \frac{D(D-1)}{2}n^2$, and $\mathcal{O}_F = \mathbb{Z}[\zeta]$ as usual. We also write

$$w_F \zeta_F^*(s, [\mathcal{O}_F]) = \zeta_Q^*(s)$$

in order to make the connection with the underlying field F clearer.

Finally, we define a level N version of the modular Δ function. This is motivated quite naturally from a Kronecker limit type formula for $\mathcal{E}_{0,N}(z,s)$. From [Vas96], we have the following expansion for $E_{0,N}(z,s)$ at $s=1$

$$E_{0,N}(z,s) = \frac{\alpha_N \pi}{s-1} + \beta_N \pi - 2\pi\alpha_N \log(2) - \alpha_N 2\pi \log\left(y^{1/2}|\eta_N(z)|^2\right) + O(s-1)$$

where

$$\alpha_N = \frac{3}{\pi^2}\phi(N)\left(N^2 \prod_{p|N}(1-p^{-2})\right)^{-1}$$

and

$$\beta_N = \alpha_N\left(2\gamma - D_N(1) - 2\frac{\zeta'(2)}{\zeta(2)}\right).$$

where

$$D_N(s) = \frac{d}{ds}\log\left(J_{2s}(N)\right)$$

where $J_{2s}(N)$ is the Jordan totient function, defined by

$$J_k(n) = n^k \prod_{p|n}\left(1 - \frac{1}{p^k}\right).$$

Also from [Vas96] we have

$$\eta_N(z)^{\phi(N)} = \prod_{\nu|N} \eta(\nu z)^{\mu(N/\nu)\nu}$$

which clearly implies that, for $\gamma \in \Gamma_0(N)$

$$\eta_N(\gamma z)^{\phi(N)} = j(\gamma,z)^{\phi(N)/2} \prod_{\nu|N} \epsilon_N(\gamma)^{\mu(N/\nu)\nu} \eta(\nu z)^{\mu(N/\nu)\nu}$$

In particular, there exists an integer l_N such that $\epsilon_N(\gamma)^{l_N} = 1$ for all $\gamma \in \Gamma_0(N)$. We then define

$$\Delta_N(z) = (\eta_N(z))^{\phi(N) l_N}$$

which is a modular form of weight $k_N = \frac{1}{2}\phi(N) l_N$ for $\Gamma_0(N)$.

In order to find a formula for l_N we use a theorem from [Raj06]

Theorem 6.1.5. *Let*

$$f_1(z) = \prod_{\nu | N} \eta(\nu z)^{r_\nu}$$

where r_ν are integers such that

$$\sum_{\nu | N} r_\nu \nu \equiv 0 \pmod{24}$$
$$\sum_{\nu | N} \frac{N}{\nu} r_\nu \equiv 0 \pmod{24}$$

then, f_1 is a modular form for $\Gamma_0(N)$ of weight

$$k_N = \frac{1}{2} \sum_{\nu | N} r_\nu$$

and character

$$\chi(d) = \left(\frac{(-1)^k \prod_{\nu | N} \nu^{r_\nu}}{d} \right).$$

Lemma 6.1.6. *If we set*

$$l_N = 2\phi(N)^{-1} \operatorname{lcm}\left(4, \tfrac{1}{2}\phi(N) \frac{24}{(24, d_N)}\right)$$

then $\Delta_N(z)$ satisfies the conditions for Theorem 6.1.5, and $\epsilon_N(\gamma)^{\ell_N} = \chi(d)^{l_N} = 1$ for all $\gamma \in \Gamma_0(N)$ and is thus a modular form for $\Gamma_0(N)$ of weight k_N. The given value for l_N is the smallest for which this is the case.

Proof. Since we have that $r_\nu = \mu(N/\nu)\nu l_N$, the conditions in Theorem 6.1.5, and the condition that forces $\epsilon_N(\gamma)^{l_N} = \chi(d)^{l_N} = 1$ for all $\gamma \in \Gamma_0(N)$ translate to

$$d_N \phi(N) l_N \equiv 0 \pmod{24}$$
$$l_1 \equiv 0 \pmod{24}$$
$$l_N \equiv 0 \pmod{4},$$

which are all definitely satisfied by l_N. That this is the smallest possible l_N that does the job is not too hard to see, indeed, for $N = 1$ this all reduces down to the usual formulas for the definition of the Δ function, and of course we have that $k_1 = 12$ and $l_1 = 24$. $\square$

There is a Kronecker limit formula for $\mathcal{E}_{0,N}(z,s)$, which is similar in spirit to the one for the usual real analytic Eisenstein series for the full modular group.

Theorem 6.1.7. *Let ζ be the generator over $\mathbb{Z}$ of the ring of integers of F, so that $\mathcal{O}_F = \mathbb{Z}[\zeta]$. Let N be a positive integer and $\mathcal{E}_{0,N}(z,s)$, $\Delta_N(z)$, d_N, and k_N be defined as above, and let $D_N = e^{D_N(1)}$. Then we have that*

$$\lim_{s\to 1}\left(\mathcal{E}_{0,N}(z,s) - \tfrac{1}{2}d_N^{-1}w_F\zeta_F^*(s,[\mathcal{O}_F])\right) = -\frac{1}{d_N k_N}\log\left(\frac{|\Delta_N(z)|}{|\eta(\zeta)|^{2k_N}}(yD_N)^{k_N/2}\right).$$

Proof. This is mainly a matter of using the expansions already given and juggling the terms. We note that

$$\zeta^*(2s) = \frac{\pi}{6} + \frac{\pi}{6}\left(-\gamma - \log\pi + 2\frac{\zeta'(2)}{\zeta(2)}\right)(s-1) + O((s-1)^2),$$

and hence

$$\mathcal{E}_{0,N}(z,s) = \frac{\alpha_N\pi^2}{6(s-1)} + \frac{\beta_N\pi^2}{6} - \frac{2\pi^2\alpha_N}{6}\log(2) + \frac{\alpha_N\pi^2}{6}\left(-\gamma - \log(\pi) + \frac{2\zeta'(2)}{\zeta(2)}\right) - \frac{\alpha_N 2\pi^2}{6}\log\left(y^{1/2}|\eta_N(z)|^2\right) + O(s-1)$$

We use that

$$\frac{\alpha_N\pi^2}{6} = \frac{1}{2}d_N^{-1}$$

and

$$\beta_N = \frac{6}{\pi^2}d_N^{-1}\left(\gamma - \tfrac{1}{2}D_N(1) - \frac{\zeta'(2)}{\zeta(2)}\right),$$

to obtain the more compact

$$\mathcal{E}_{0,N}(z,s) = \frac{1}{2}d_N^{-1}\left(\frac{1}{s-1} + \gamma + D_N(1) - \log(4\pi) - 2\log\left(y^{1/2}|\eta_N(z)|^2\right)\right) + O(s-1)$$

and we compare this to the expansion

$$\tfrac{1}{2}d_N^{-1}w_F\zeta_F^*(s,[\mathcal{O}_F]) = \frac{1}{2}d_N^{-1}\left(\frac{1}{s-1} + \gamma - \log(4\pi|\eta(\zeta)|^4)\right) + O(s-1)$$

where now it should be obvious that the $(s-1)^{-1}$ terms cancel in the limit, so we do indeed only get a constant. Moreover, we note that all the γ terms cancel as well, and the 4π in the logarithm also cancels. Thus we have

$$\lim_{s\to 1}\left(\mathcal{E}_{0,N}(z,s)-\tfrac{1}{2}d_N^{-1}w_F\zeta_F^*(s,[\mathcal{O}_F])\right)=\frac{1}{2}d_N^{-1}\left(-D_N(1)-2\log\left(y^{1/2}\frac{|\eta_N(z)|^2}{|\eta(\zeta)|^2}\right)\right)$$

All that is left to do now is to collect all of the terms into the logarithm. When we pull down the correct factor so that $\Delta_N(z)$ appears, we obtain the result. □

Corollary 6.1.8. *For $N=1$, we simply have*

$$\lim_{s\to 1}\left(\mathcal{E}_{0,1}(z,s)-\frac{1}{2}w_F\zeta_F^*(s,[\mathcal{O}_F])\right)=\frac{-1}{12}\log\left(\frac{|\Delta(z)|}{|\Delta(\zeta)|}y^6\right).$$

It is worth pointing out some differences here between Theorem 6.1.7 and the usual presentation of the Kronecker limit formula. In the normal Kronecker limit formula one subtracts the (completed) Riemann zeta function, rather than the partial Dedekind zeta function. Although it presently seems arbitrary to change the zeta function in this way, it will turn out that this is the most natural choice in the situation of Theorem 6.3.1. The effect of doing so is to make sense of the extra terms in the logarithm, which now depend entirely on either F, the underlying field, or on N.

6.2 Lift of $\mathcal{E}_{0,N}(z,s)$

Theorem 6.2.1. *Let $L=\mathcal{O}_F\ell\oplus N\mathcal{O}_F\ell'$ for $N\in\mathbb{N}$. Then the lift of $\mathcal{E}_{0,N}(z,s)$, as defined above, is*

$$I(\tau,\mathcal{E}_{0,N}(z,s))=w_F\zeta_F^*(s,[\mathcal{O}_F])N^{1-s}\mathcal{E}_2(N\tau,s).$$

Proof.

$$I(\tau,\mathcal{E}_{0,N}(z,s))=\int_{\Gamma_0(N)\backslash\mathbb{H}}\theta(\tau,z,L)\mathcal{E}_{0,N}(z,s)\frac{dxdy}{y^2},$$

which, by the Rankin-Selberg unfolding trick, is

$$I(\tau,\mathcal{E}_{0,N}(z,s))=\zeta^*(2s)\int_0^\infty\int_0^1\theta(\tau,z,L)y^s\frac{dxdy}{y^2}.$$

We now substitute in our expression for $\theta(z,\tau,L)$ to obtain

$$I(\tau,\mathcal{E}_{0,N}(z,s)) = \frac{-\zeta^*(2s)}{2\hat{\delta}v^2}\int_0^\infty\int_0^1\sum_{X\in L}\exp(\pi ix\langle X,X\rangle)\,(\overline{\tau w}'-\overline{w})\,(\overline{\tau}w'-w)$$
$$\exp(-\pi y\langle X,X\rangle_\tau)y^s dxdy.$$

When we do the integration in the x variable, we obtain

$$I(\tau,\mathcal{E}_{0,N}(z,s)) = \frac{-\zeta^*(2s)}{2\hat{\delta}v^2}\int_0^\infty\sum_{X\in L_0}(\overline{\tau w}'-\overline{w})\,(\overline{\tau}w'-w)\exp(-\pi y\langle X,X\rangle_\tau)y^s dy,$$

where $L_0=\{X\in L:\langle X,X\rangle=0\}$. We now use the standard integral

$$\int_0^\infty \exp(-Ay)y^s dy = A^{-1-s}\Gamma(s+1)$$

to complete the integration in the y variable, and we are left with

$$I(\tau,\mathcal{E}_{0,N}(z,s)) = \frac{-\zeta^*(2s)}{2\hat{\delta}v^2}\sum_{X\in L_0}(\overline{\tau w}'-\overline{w})\,(\overline{\tau}w'-w)\,(\pi\langle X,X\rangle_\tau)^{-1-s}\Gamma(1+s).$$

We recall the following formulas for the minimal majorant

$$\langle X,X\rangle_\tau = 2R(X,\tau)+\langle X,X\rangle,$$
$$R(X,\tau) = (v\hat{\delta})^{-1}|\tau w'-w|^2,$$

and substitute these into the sum, to now obtain

$$I(\tau,\mathcal{E}_{0,N}(z,s)) = \frac{-\zeta^*(2s)}{2\hat{\delta}v^2}\left(\frac{\hat{\delta}v}{2\pi}\right)^{s+1}\Gamma(1+s)\sum_{X\in L_0}|\tau w'-w|^{-2(s-1)}(\tau w'-w)^{-2}\frac{\overline{\tau}w'-w}{\overline{\tau w}'-\overline{w}}.$$

We now examine the lattice L_0. For any $w\ell+Nw'\ell'\in L_0$ we must have that $\Im(w\overline{Nw}')=0$, indeed, this condition is equivalent to the statement that $X\in L_0$. However, since $N\in\mathbb{N}$, this is again equivalent to $\Im(w\overline{w}')$. Hence,

$$I(\tau,\mathcal{E}_{0,N}(z,s)) = \frac{-\zeta^*(2s)}{2\hat{\delta}v^2}\left(\frac{\hat{\delta}v}{2\pi}\right)^{s+1}\Gamma(1+s)$$
$$\sum_{\substack{w,w'\in\mathcal{O}_F\\ \Im(w\overline{w}')=0}}|\tau Nw'-w|^{-2(s-1)}(\tau Nw'-w)^{-2}\frac{\overline{\tau}Nw'-w}{\overline{\tau}N\overline{w}'-\overline{w}}$$

We can see that for any $w,w'\in\mathcal{O}_F$ such that $\Im(w\overline{w}')$ we must have that

$$\frac{w}{w'}=\frac{d}{c}$$

for some $d/c \in \mathbb{Q}^*$. Assume now that this fraction is written in lowest terms, i.e. $(c,d)=1$. Then

$$\frac{w}{w'} = \frac{d}{c},$$

for which the only solutions are $w = ck$ and $w' = dk$ for some (non-zero) $k \in \mathcal{O}_F$. Hence the sum over L_0 may be written,

$$\sum_{k \in \mathcal{O}_F} \sum_{(c,d)=1} |\tau Nck - dk|^{-2(s-1)} (\tau Nck - dk)^{-2} \frac{\overline{\tau} Nck - Ndk}{\overline{\tau} Nc\overline{k} - d\overline{k}}.$$

We can pull out all of the k dependence from the inner sum to obtain

$$\sum_{k \in \mathcal{O}_F} |k|^{-2s} \sum_{(c,d)=1} |\tau Nc - d|^{-2(s-1)} (\tau Nc - d)^{-2}.$$

This is just $\zeta_Q(s)$ times the inner sum. Hence, using the definitions given above, we clearly have

$$I(\tau, \mathcal{E}_{0,N}(z,s)) = w_F \zeta_F^*(s, [\mathcal{O}_F]) N^{1-s} \mathcal{E}_2(N\tau, s). \qquad \square$$

Corollary 6.2.2. *Taking residues at $s = 1$ on both sides in (6.2.1) gives the lift of the constant function, i.e.*

$$I(\tau, 1) = 2 d_N \mathcal{E}_2(N\tau, 1)$$

We can use this formulation to say something about the Fourier coefficients of the holomorphic part of $\mathcal{E}_2(N\tau, 1)$. Since, by the same unfolding argument as in Therorem 5.1.1, we have that the m-th Fourier coefficient for the lift of 1 must be

$$c(m) = 2 \sum_{X \in \Gamma_0(N) \backslash L_m} \frac{1}{|\Gamma_X|}$$

which is essentially the degree of the divisor defined by the cosets of $\Gamma_0(N)\backslash L_m$. By the isometry ι defined in (1.1) we know that a vector X in L_m is in bijection with a matrix

$$\begin{pmatrix} x_1 & x_2 \\ Nx_3 & Nx_4 \end{pmatrix}$$

with determinant m. This of course implies that $N \mid m$ and so were we to form the generating series of the number of such matrices modulo $\Gamma_0(N)$, with the action being matrix multiplication from the left, we only get terms where the index is divisible by N. Examining $\mathcal{E}_2(N\tau, 1)$ we see that this is so, in fact, the m-th Fourier coefficient is

$$2\sigma_1(m/N) = \begin{cases} 2\sigma_1(m/N) & \text{if } N \mid m \\ 0 & \text{else} \end{cases},$$

which gives an extraordinarily roundabout proof of

Corollary 6.2.3. *The size of*

$$\Gamma_0(N)\backslash \left\{ \begin{pmatrix} x_1 & x_2 \\ Nx_3 & Nx_4 \end{pmatrix} : N(x_1x_4 - x_2x_3) = m \right\}$$

is equal to $\sigma_1(m/N)$.

6.3 Lift of $\log|\Delta_N(z)|$

Theorem 6.3.1. *Let* N *be a positive integer,* $L = \mathcal{O}_F u \oplus N\mathcal{O}_F u'$, *where* $F = \mathbb{Z} + \mathbb{Z}[\zeta]$, *and let* k_N, D_N *and* $\Delta_N(z)$ *be defined as above. Then*

$$-\frac{1}{k_N d_N} I\left(\tau, \log\left(\frac{|\Delta_N(z)|}{|\eta(\zeta)|^{2k_N}}(yD_N)^{k_N/2}\right)\right) = \mathcal{E}'_{2,N}(\tau, 1).$$

where the $'$ *on the RHS indicates differentiation in the* s *variable.*

Proof. We use Theorem 6.1, and lift both sides. Since we have shown that $\Delta_N(z)$ is a modular form of weight k_N for $\Gamma_0(N)$, then we have that $|\Delta_N(z)|y^{k_n/2}$ is a modular function (i.e. is of weight 0) for $\Gamma_0(N)$. We can then calculate

$$\begin{aligned} I\big(\tau, \lim_{s\to 1}\big(\mathcal{E}_{0,N}(z,s) - \tfrac{1}{2}d_N^{-1}w_F\zeta_F^*(s,[\mathcal{O}_F])\big)\big) \\ &= \lim_{s\to 1}\big(I(\tau, \mathcal{E}_{0,N}(z,s)) - \tfrac{1}{2}d_N^{-1}w_F\zeta_F^*(s,[\mathcal{O}_F])I(\tau,1)\big) \\ &= \lim_{s\to 1}\left(w_F\zeta_F^*(s,[\mathcal{O}_F])\mathcal{E}_{2,N}(z,s) - w_F\zeta_F^*(s,[\mathcal{O}_F])\mathcal{E}_{2,N}(\tau,1)\right) \\ &= \lim_{s\to 1}\frac{\mathcal{E}_{2,N}(z,s) - \mathcal{E}_{2,N}(\tau,1)}{s-1} \end{aligned}$$

using Corollary 6.2.2 and the expansion of $\zeta_Q^*(s)$ at $s = 1$. The right hand side is then obviously the definition of $\mathcal{E}'_{2,N}(\tau, 1)$. □

We compare the above Theorem to Theorem 7.3 in [BF06]. The choice made in Theorem 6.1.7 to use the partial Dedekind zeta function now reveals its utility, namely, to exchange the need for the constant $(4\pi)^3 e^{-3\gamma}$ in favour of using functions which depend on N and F and have some interpretation. In [BF06] they go on the explain how to interpret the lift of $\log|\Delta|$ in terms of arithmetic geometry. This is done by realising the Fourier expansion as the generating function of an arithmetic intersection pairing, see [Yan04]. We hope that a similar interpretation exists here in the unitary setting.

Bibliography

[BF04] J. H. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. **125** (2004), no. 1, 45–90. MR 2097357 (2005m:11089)

[BF06] ______, *Traces of CM values of modular functions*, J. Reine Angew. Math. **594** (2006), 1–33. MR 2248151 (2007e:11050)

[BF10] ______, *On the injectivity of the Kudla-Millson lift and surjectivity of the Borcherds lift*, Moonshine: the first quarter century and beyond, London Math. Soc. Lecture Note Ser., vol. 372, Cambridge Univ. Press, Cambridge, 2010, pp. 12–39. MR 2681774 (2012a:11056)

[BHY15] J. Bruinier, B. Howard, and T. Yang, *Heights of Kudla-Rapoport divisors and derivatives of L-functions*, To appear in Invent. Math. (2015).

[BKO04] Jan H. Bruinier, Winfried Kohnen, and Ken Ono, *The arithmetic of the values of modular functions and the divisors of modular forms*, Compos. Math. **140** (2004), no. 3, 552–566. MR 2041768 (2005h:11083)

[Bor95] R. E. Borcherds, *Automorphic forms on* $\mathrm{O}_{s+2,2}(\mathbf{R})$ *and infinite products*, Invent. Math. **120** (1995), no. 1, 161–213. MR 1323986 (96j:11067)

[Bor98] ______, *Automorphic forms with singularities on grassmannians*, Inventiones Mathematicae **132** (1998), 491–562, 10.1007/s002220050232.

[Bru02] Jan H. Bruinier, *Borcherds products on O(2, l) and Chern classes of Heegner divisors*, Lecture Notes in Mathematics, vol. 1780, Springer-Verlag, Berlin, 2002. MR 1903920 (2003h:11052)

[DIT11] W. Duke, Ö. Imamoḡlu, and Á. Tóth, *Cycle integrals of the j-function and mock modular forms*, Ann. of Math. (2) **173** (2011), no. 2, 947–981. MR 2776366 (2012g:11084)

[DS05] F. Diamond and J. Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005. MR 2112196 (2006f:11045)

[FM02] J. Funke and J. J. Millson, *Cycles in hyperbolic manifolds of non-compact type and Fourier coeffients of Siegel modular forms*, Manuscripta Mathematica **107** (2002), no. 4, 409–449.

[FM06] ———, *Cycles with local coefficients for othogonal groups and vector-valued Siegel modular forms*, American Journal of Mathematics **128** (2006), 899–948.

[FM11] Jens Funke and John Millson, *Spectacle cycles with coefficients and modular forms of half-integral weight*, Arithmetic geometry and automorphic forms, Adv. Lect. Math. (ALM), vol. 19, Int. Press, Somerville, MA, 2011, pp. 91–154. MR 2906907

[FM13] ———, *Boundary behaviour of special cohomology classes arising from the Weil representation*, J. Inst. Math. Jussieu **12** (2013), no. 3, 571–634. MR 3062872

[FM14] J. Funke and J. J. Millson, *The geometric theta correspondence for Hilbert modular surfaces*, Duke Math. J. **163** (2014), no. 1, 65–116. MR 3161312

[Fun02] J. Funke, *Heegner divisors and nonholomorphic modular forms*, Compositio Mathematica **133** (2002), 289–321.

[GR07] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, seventh ed., Elsevier/Academic Press, Amsterdam, 2007, Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX). MR 2360010 (2008g:00005)

[Hof11] E. Hofmann, *Automorphic products on unitary groups*, Ph.D. thesis, Technischen Universität Darmstadt, 2011.

[Hof13] Eric Hofmann, *Borcherds products for* $\mathrm{U}(1,1)$, Int. J. Number Theory **9** (2013), no. 7, 1801–1820. MR 3130151

[How15] B. Howard, *Complex multiplication cycles and Kudla-Rapoport divisors II*, To appear in Amer. J. Math. (2015).

[HZ76] F. E. P. Hirzebruch and D. B. Zagier, *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus*, Inventiones mathematicae **36** (1976), 57–113.

[Iwa02] H. Iwaniec, *Spectral methods of automorphic forms*, Graduate Studies in Mathematics, Vol. 53, The American Mathematical Society, 2002.

[KM86] S. S. Kudla and J. J. Millson, *The theta correspondence and harmonic forms. I*, Mathematiche Annalen **274** (1986), 353–378.

[KM87] ———, *The theta correspondence and harmonic forms. II*, Mathematiche Annalen **277** (1987), 267–314.

[Kud79] S. S. Kudla, *On certain arithmetic automorphic forms for* $\mathrm{SU}(1,\, q)$, Invent. Math. **52** (1979), no. 1, 1–25. MR 532744 (80f:10033)

[Kud97] S.S Kudla, *Central derivatives of Eisenstein series and height pairings*, The Annals of Mathematics, 2nd Series **146** (1997), 545–646.

[KV78] M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil representations and harmonic polynomials*, Invent. Math. **44** (1978), no. 1, 1–47. MR 0463359 (57 #3311)

[Lan88] S. Lang, *Introduction to Arakelov theory*, Springer-Verlag, New York, 1988. MR 969124 (89m:11059)

[Lan02] ———, *Algebra*, third ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002. MR 1878556 (2003e:00003)

[LV80] Gérard Lion and Michèle Vergne, *The Weil representation, Maslov index and theta series*, Progress in Mathematics, vol. 6, Birkhäuser, Boston, Mass., 1980. MR 573448 (81j:58075)

[Miy89] T. Miyake, *Modular forms*, Springer Monographs in Mathematics, Springer, 1989.

[Oda78] T. Oda, *On modular forms associated with indefinite quadratic forms of signature* $(2, n-2)$, Math. Ann. **231** (1977/78), no. 2, 97–144. MR 0466026 (57 #5909)

[Pra93] Dipendra Prasad, *Weil representation, Howe duality, and the theta correspondence*, Theta functions: from the classical to the modern, CRM Proc. Lecture Notes, vol. 1, Amer. Math. Soc., Providence, RI, 1993, pp. 105–127. MR 1224052 (94e:11043)

[Raj06] W. Raji, *Modular forms representable as eta products*, Ph.D. thesis, Temple University, 2006.

[Shi71] Goro Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo, 1971, Kanô Memorial Lectures, No. 1. MR 0314766 (47 #3318)

[Shi75] T. Shintani, *On construction of holomorphic cusp forms of half integral weight*, Nagoya Math. J. **58** (1975), 83–126. MR 0389772 (52 #10603)

[Sie65] C. L. Siegel, *Lectures on advanced analytic number theory*, Notes by S. Raghavan. Tata Institute of Fundamental Research Lectures on Mathematics, No. 23, Tata Institute of Fundamental Research, Bombay, 1965.

[Sil09] Joseph H. Silverman, *The arithmetic of elliptic curves*, second ed., Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009. MR 2514094 (2010i:11005)

[Sou92] C. Soulé, *Lectures on Arakelov geometry*, Cambridge Studies in Advanced Mathematics, vol. 33, Cambridge University Press, Cambridge, 1992, With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer. MR 1208731 (94e:14031)

[Vas96] I. N. Vassileva, *Dedekind eta function, kronecker limit formula and dedekind sum for the hecke group*, Ph.D. thesis, University of Massachusetts Amherst, 1996, Doctoral Dissertations Available from Proquest, Paper AAI9639046. http://scholarworks.umass.edu/dissertations/AAI9639046.

[vdG88] Gerard van der Geer, *Hilbert modular surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 16, Springer-Verlag, Berlin, 1988. MR 930101 (89c:11073)

[Yan04] Tonghai Yang, *Faltings heights and the derivative of Zagier's Eisenstein series*, Heegner points and Rankin L-series, Math. Sci. Res. Inst. Publ., vol. 49, Cambridge Univ. Press, Cambridge, 2004, pp. 271–284. MR 2083215 (2005e:11086)

[Zag02] D. B. Zagier, *Traces of singular moduli*, Motives, polylogarithms and Hodge theory, Part I(Irvine, CA, 1998), Int. Press Lect. Ser., vol. 3, Int. Press, Somerville, MA, 2002, pp. 211–244. MR 1977587 (2004h:11037)