Contact interactions for point particles and strings

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A Thesis presented for the degree of
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Dedicated to

my Grandfather, *Harry Edwards*, and Grandmother, *Jean Fishwick*. Although you will not see this thesis you are at its heart.

Also to

Teresa Edwards, whose strength is inspiring, and *John Fishwick*, my own mad scientist.

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Abstract

We investigate δ-function contact interactions for theories of point particles and of strings. These interactions are introduced to reformulate the conventional theory of classical electrodynamics in terms of particles and strings which interact when they intersect. Upon quantisation we find that the tensionless limit of the spinning string theory generates well-known gauge invariant quantities in the worldline formulation of quantum field theory. Despite the off-shell nature of the interaction we find that the string theory does not encounter the expected break-down of conformal invariance. We further develop worldline techniques for non-Abelian theories and consider first quantised versions of some grand unified theories. This work can be seen as initiating the construction of a first-quantised version of the quantum field theory describing the standard model.

Declaration

The work in this thesis is based on research carried out in the Centre for Particle Theory, the Department of Mathematical Sciences, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text. Aspects of this thesis were carried out in collaboration with Prof. Paul Mansfield, the bulk of which work can be found in chapter 3. Any such shared progress is clearly signified through citation or other appropriate acknowledgement.

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Chapter 1

Introduction

String theory has provided much insight into modern physics, in particular to the dynamics of quantum fields. There are a great many examples of close relationships between the physics of field theory and some form of string theory [1–3]. Despite string theory being a first quantised theory its one-dimensional extended nature provides sufficiently large degrees of freedom to be applicable to problems in second quantised field theory [4–6]. One of the most prominent and popular examples of this has to be gauge / gravity duality and the AdS/CFT correspondence [7–9], where certain string theories on a curved space-time are related to conformal field theories on its spatial boundary. It is interesting, however, to return to the foundations of string theory and consider modifying it for application to the real world. One context where this is especially relevant is the consideration of interactions.

It is of course true that interactions of a sort are accommodated into string theory by summing over the topologies of worldsheets [10]. Furthermore the scattering of strings can be represented by introducing vertex operators to excite the required asymptotic states (via the state-operator correspondence) so that a typical picture of string theory interactions is a sum over an infinite set of physically distinct worldsheets with a puncture for each state taking part in the scattering. Physically this can be viewed as a smooth splitting and joining of strings which, for reasons we shall discuss, are on mass shell. The spectra of quantised string theories also contain states such as the photon, graviton, Kalb-Ramond tensor and dilaton and it is possible to consider propagation of a string coupled to background fields which take any of these forms [11, 12]. If this is done then by demanding the conformal invariance we shall discuss later it is possible to derive field theories which describe the long-wavelength limit of the interactions of these modes of the string.

It is not these types of interaction which we shall consider in this thesis. Instead we shall introduce interactions between fundamental strings by modifying the string action to include a contact term. We shall motivate the introduction of such an unconventional interaction by appealing to a recent demonstration that the electromagnetic field at a given space-time point of a pair of equal and opposite charges can be determined by carrying out a statistical average over string configurations in Polyakov's approach to string theory [13]. Only those string configurations whose end-points are fixed to the worldlines of the charges and which intersect that space-time point contribute to this average. We will determine the effect of this contact interaction by calculating the tree-level partition function for the new string theory to all orders of a perturbative expansion in the strength of the inter-string interaction. As a warm up we shall first consider an analogous contact interaction for point particles.

As we shall review below, requiring that the classical conformal invariance of the string is preserved upon quantisation places constraints on the spectrum of the theory and on the dimension of the target space in which the string propagates. Modification of the action faces the danger of altering these constraints, the introduction of further conditions or may even lead to an inconsistent theory. Remarkably the interacting string theory will be seen to evade the usual mass-shell constraints on the string spectra and will not encounter any change to the well-known Weyl anomaly, which in any case will decouple from the physical content the theory describes. For this reason the consistency of the inter-string interaction represents an important development in fundamental string theory.

This thesis aims to further advance this novel string theory by relating it to the physical observables which appear in quantum electrodynamics (QED). Second quantised field theory is an extremely powerful framework in which to consider processes which involve multi-particle states but having an equivalent quantum mechanical theory offers new conceptual understanding and calculational tools to address long standing problems. Experiment has verified the predictions of QED to extraordinary accuracy so it may seem unnecessary to consider a new approach. However, it is a low energy effective field theory which arises out of the standard model so eventually we shall seek to apply the proposed string theory to reformulate the non-Abelian field theories believed to describe fundamental interactions. The Abelian theory is somewhat simpler and supplies an ideal setting to develop new approaches to the full standard model.

The crucial stepping stone between field theory and first quantised theory will be the so-called worldline formalism of quantum field theory. This approach has its roots in the work of Feynman [14] but was revived and developed by Strassler [15,16] and is steadily growing in popularity. Briefly, a typical field theory is described by an action which consists of matter coupled to gauge fields. Integrating over the matter content leads to an effective action for the gauge field which involves the functional determinant and propagators of the kinetic term describing the dynamics of the free theory. The worldline formalism re-expresses the determinant and the propagators in terms of an ensemble of first quantised theories defined on closed and open worldlines respectively. These worldline theories take the form of point particles coupled to the gauge field via the Wilson line – the path ordered exponential of the gauge connection transported along the worldline. The connection to string theory will then be found by identifying these worldlines as the boundaries of fundamental interacting strings which serve to produce the interaction of the worldline theory with the gauge field.

Within this framework the quantum field theory describing the low energy interaction of spinor matter with the electromagnetic field will be reformulated in terms of a quantum theory of strings which interact on contact; their boundaries will be fixed to the Wilson lines which emerge in the worldline formalism of the field theory. The standard model, however, contains far more information than this. In order to go on to describe the known matter content of the universe it is therefore imperative to develop worldline techniques for non-Abelian interactions and to describe the intrinsic chirality of spin 1/2 particles. Furthermore, whilst the standard model is hugely successful, there have been many attempts to unify the electro-weak and strong forces into a single theory to address problems such as the unification of their coupling strengths. It is consequently of significant interest that such unified theories are considered in the first quantised framework that we propose.

With these aims in sight this thesis is laid out as follows. We form the context for the proposed reformulation of QED in the remainder of this chapter by briefly reviewing the worldline formalism of quantum field theory and the quantisation of conventional string theory. Throughout the thesis we shall make use of functional methods so we will present this material using Polyakov's approach to quantisation. We will also demonstrate the passage from quantum field theory to quantum mechanics on the worldline and describe the application of this formalism to simple one-loop scattering amplitudes. We also revise the functional integration over string worldsheet geometries and how the recovery of conformal invariance leads to the critical dimension of space-time and the mass-shell conditions.

In Chapter 2 the contact interaction will be considered in the context of point particle theories. We shall first re-derive a previous result showing how the functional averaging over worldlines passing through a spatial point provides the electric field at that point. That calculation was for bosonic particles but this will be generalised to a supersymmetric model applicable to fermionic particles. We shall then turn to developing a quantum theory of a set of point particles which interact when their worldlines intersect. The limitations of this model will provide sufficient motivation to turn instead to a theory of interacting strings.

Chapter 3 forms the backbone of the thesis and contains the reformulation of spinor QED in terms of spinning strings which interact upon contact. Specifically we shall propose that the partition function of the string theory coincides with the expectation value of the product of Wilson lines defined by the boundaries of the strings. First the case of scalar QED is considered which is postulated to be equivalent to an interacting bosonic string theory. Calculation will show that potential divergences spoil this claim which will motivate us to consider spinor QED. Here we will see that the extra structure of supersymmetry that the spinning string enjoys is sufficient to ensure that these problems are no longer encountered. By regulating the ultra violet divergences on the string worldsheet we will demonstrate that super-conformal invariance can be restored and that the familiar expression for exponentiated line integrals in QED arises naturally out of the model.

The application of the worldline approach to non-Abelian theories and chiral fermions is considered in Chapter 4. We build upon a worldline approach to the standard model to consider unified theories in the same context. There we will show that the familiar SU (5) unified theory arises quite naturally in this formalism; the chiral projection operators, path ordering and group representations of non-Abelian field theories can be represented by introducing further worldline degrees of freedom and we show that the well-known Georgi-Glashow model can be extracted by considering a functional determinant associated to these new fields. We also discuss some of the other well-known unified theories such as that with gauge group $SO(10)$. We shall then discuss the consequences of the work in this thesis and state our conclusions. Further detailed calculations and subsidiary arguments will be provided in the appendices.

1.0.1 The worldline approach to quantum field theory

The world-line formulation of Strassler (which has been elaborated by a number of authors [17–19]) will play a crucial role in this thesis. Strassler was motivated by the well-known fact that the infinite tension limit of various string theories reduces to familiar field theories describing the massless states of the string. Bern and Kosower analysed the scattering of strings in this limit [20, 21] and derived a series of rules which enabled the systematic calculation of one-loop scattering amplitudes in corresponding field theories [22]. Although derived from string theory, calculations do not depend in any way on the underlying string model. They also represent a non-trivial reorganisation of the physical content of the theory and do not resemble the usual perturbative expansion which Feynman diagrams provide. Strassler's achievement was to uncover the same rules directly from field theory, without any reference to strings. More recently this approach has been studied in the context of pair production [23], on curved backgrounds [24] and in a non-commutative setting [25].

Scalar matter

We begin with scalar electrodynamics with a single complex scalar field ϕ coupled to the electromagnetic field which has Euclidean action

$$S\left[\bar{\phi},\phi\right] = \int d^4x\,\bar{\phi}\left(-\mathcal{D}^2 + m^2\right)\phi, \tag{1.0.1}$$

where $\mathcal{D} = \partial + iA$ is the usual covariant derivative (we absorb the coupling constant into A). The generating functional for Green functions is written by introducing sources $\bar{J}$ and J as follows

$$Z[\bar{J},J] = \int \mathscr{D}(A,\bar{\phi},\phi)\,e^{-S[\bar{\phi},\phi]-S[A]+\int d^4x\,\bar{J}\phi+\bar{\phi}J}. \tag{1.0.2}$$

where $S\left[A\right]$ describes the dynamics of the gauge field. We integrate over the matter content for

$$Z[\bar{J},J] = \int \mathscr{D}A\,\frac{1}{\det\left(-\mathcal{D}^2+m^2\right)}e^{-S[A]-\int d^4x\,\bar{J}\left(-\mathcal{D}^2+m^2\right)^{-1}J} \tag{1.0.3}$$

Strassler noted that the logarithm of the functional determinant can be represented by using the Schwinger proper time trick [26]:

$$-\log\left[\det\left(-\mathcal{D}^2+m^2\right)\right] = \int_0^\infty \frac{dT}{T}\mathrm{Tr}\,e^{-\frac{1}{2}eT\left(-\mathcal{D}^2+m^2\right)}. \tag{1.0.4}$$

In this equation e is an arbitrary constant. Strassler interpreted this exponent as specifying a Hamiltonian for quantum mechanical evolution and therefore gave a path integral representation for the trace by introducing a fictitious particle ω^μ. The trace requires the one dimensional worldline of this particle to be closed (so identifying the initial and final positions) and we must functionally integrate over all such closed loops. Strassler originally gave the worldline action

$$S = \int_0^T \frac{\dot{\omega}^2}{2e} + \frac{e}{2}m^2 + i\dot{\omega}\cdot A\left[\omega\left(\tau\right)\right]\,d\tau \tag{1.0.5}$$

but we prefer to give reparameterisation invariant expressions by recognising that the constant e plays the role of the square root of the one dimensional metric (or einbein): (1.0.4) can be expressed as

$$-\oint \frac{\mathscr{D}(h,\omega)}{Z}\, e^{-S[\omega,\, h]} \tag{1.0.6}$$

where we integrate over closed curves and ascribe the following action to each world-line [27]

$$S[\omega,h] = S_0[\omega,h] + i\int d\omega\cdot A; \quad S_0[\omega,h] = \frac{1}{2}\int_0^1 \left(h^{-1}(\xi)\left(\frac{d\omega(\xi)}{d\xi}\right)^2 + m^2\right)\sqrt{h(\xi)}\, d\xi \tag{1.0.7}$$

This form of the action was first given by Brink, di Vecchia and Howe and contains $\omega(\xi)$, $0 \leq \xi \leq 1$, a parametrisation of a world-line depending on the arbitrary parameter ξ, and $h(\xi) > 0$, an intrinsic metric on the worldline. The dimensions of h will be taken as $[\text{length}]^4$ so that the action is dimensionless. This action is invariant under diffeomorphisms $\xi \to \tilde{\xi}$ that preserve the parameter interval if under such reparameterisations $h(\xi)$ transforms as a metric and $\omega\,(\xi)$ as a scalar:

$$\tilde{h}(\tilde{\xi})\, d\tilde{\xi}^2 = h(\xi)\, d\xi^2; \qquad \tilde{\omega}(\tilde{\xi}) = \omega(\xi) \tag{1.0.8}$$

In our discussion of functional quantisation below we shall show that upon gauge fixing the reparameterisation invariance by choosing $h = T^2$ the integral over metrics becomes $\mathscr{D}h \propto \frac{dT}{T}$. In this gauge (1.0.6) becomes equivalent to Strassler's expression.

In a similar fashion the Green function can also be represented as

$$\left\langle b\right| \left(-\mathcal{D}^2 + m^2\right)^{-1} \left|a\right\rangle = -\int_{\omega(0)=a}^{\omega(1)=b} \frac{\mathscr{D}(h,\omega)}{Z}\, e^{-S[\omega,h]} \tag{1.0.9}$$

where now we integrate over open worldlines whose endpoints are fixed to the space-time points a and b. This time the gauge fixing leads to $\mathscr{D}h \propto dT$ so we find

$$\frac{1}{Z'}\int_0^\infty dT \int_{\omega(0)=a}^{\omega(1)=b} \mathscr{D}\omega\, e^{-\int_0^1 \frac{\dot{\omega}^2}{2T} + \frac{T}{2}m^2 + i\dot{\omega}\cdot A[\omega(\tau)]\ d\tau}. \tag{1.0.10}$$

This is the version given by Strassler; integrating over ω gives $\exp\left(-\frac{T}{2}\left(-\mathcal{D}^2+m^2\right)\right)$ which is the heat-kernel of the kinetic operator in the exponent. Hence integrating with respect to T yields the Green function. Note that both expressions take on a very similar form, differing only by the boundary conditions on the worldlines, and that the expressions are recognisable as expectation values of Wilson lines of the gauge field (and are hence manifestly gauge invariant with respect to this field).

Spinor matter

This procedure can be generalised to spin 1/2 fields. Given the structure of the worldline theories encountered above we might anticipate that we again find an ensemble of worldlines, this time corresponding to quantum theories consisting of spin 1/2 particles which are coupled to the gauge field through super-Wilson lines to represent the spinor nature of the matter. Indeed this is the case and is made explicit as follows. For a Dirac spinor coupled to the electromagnetic field the generating functional is

$$Z_D[\bar{K}, K] = \int \mathscr{D}(A, \bar{\Psi}, \Psi)\, e^{-S[\bar{\Psi},\Psi]-S[A]+\int d^4x\, \bar{K}\Psi+\bar{\Psi}K} \qquad (1.0.11)$$

where

$$S\left[\bar{\Psi}, \Psi\right] = \int d^4x\, \bar{\Psi}\left(\gamma\cdot\mathcal{D}+im\right)\Psi\,. \qquad (1.0.12)$$

is the usual action and we have introduced anti-commuting (Grassman) sources $\bar{K}$ and K. We again integrate over the matter field to produce

$$Z_D[\bar{K}, K] = \int \mathscr{D}A\ \det\left(-(\gamma\cdot\mathcal{D})^2+m^2\right)e^{-S[A]+\int d^4x\, \bar{K}(\gamma\cdot\mathcal{D}+im)^{-1}K} \qquad (1.0.13)$$

As in the scalar case Strassler represented the logarithm of the functional determinant in terms of a series of closed worldline theories and gave the inverse of the kinetic operator as open worldlines, but the quantum theories on these worldlines must be modified to represent the extra content of spinor matter.

We again use reparametrisation invariant expressions by following Brink, di Vecchia and Howe and for simplicity will restrict the discussion to massless particles.

We introduce Grassmann numbers ψ^μ that will play the role of Dirac γ-matrices and the gravitino χ that is the super-partner to $\sqrt{h}$. The action on the worldlines consists of the bosonic part $S_0[\omega, h]$ of (1.0.7), to which we add a piece representing the spin degrees of freedom

$$S_F = -\frac{1}{2}\int_0^1 \left(\psi \cdot \frac{d\psi}{d\xi} + \frac{\chi}{\sqrt{h}}\frac{dw}{d\xi} \cdot \psi\right) d\xi \qquad (1.0.14)$$

and the super-Wilson line coupling to the gauge field

$$S_A = i\int \left(\frac{dw}{d\xi} \cdot A + \frac{1}{2}F_{\mu\nu}\psi^\mu\psi^\nu\sqrt{h}\right) d\xi \qquad (1.0.15)$$

This is reparametrisation invariant provided ψ^μ transforms as a world-line scalar (like w^μ) and χ transform like $\sqrt{h}$ and the complete action is also invariant under the local supersymmetry transformations:

$$\delta_\alpha w = \delta\alpha\psi\,, \quad \delta_\alpha\psi = \frac{\delta\alpha}{\sqrt{h}}\left(\frac{dw}{d\xi} - \frac{1}{2}\chi\psi\right)\,, \quad \delta_\alpha\sqrt{h} = \delta\alpha\chi\,, \quad \delta_\alpha\chi = 2\frac{d\,\delta\alpha}{d\xi}\,. \qquad (1.0.16)$$

In Appendix C of [28] we showed that for closed world-lines and anti-periodic boundary conditions on ψ

$$\int \mathscr{D}(h, w, \chi, \psi)\, e^{-S_0 - S_F - S_A} = -\ln \mathrm{Det}\left((\gamma \cdot \mathcal{D})^2\right) \qquad (1.0.17)$$

which is in agreement with our goal. For open world-lines running from w_i to w_f we must attach spinor indices which correspond to boundary conditions of the ψ fields. We found

$$\int \mathscr{D}(h, w, \chi, \psi)\, e^{-S_0 - S_F - S_A}\Big|_{ab} = \langle w_f, a|\,(\gamma \cdot \mathcal{D})^{-1}\,|w_i, b\rangle \qquad (1.0.18)$$

Furthermore, upon gauge fixing the reparameterisation invariance by setting $h = T^2$ and the local supersymmetry by choosing $\chi = \chi_0$, a constant Grassman number, we find that our equations reduce precisely to those given by Strassler.

We finish this brief review of the worldline formalism by highlighting its computational efficiency. For simplicity we state the results for the one-loop two photon scattering amplitudes in scalar QED (the techniques are very similar for spinor QED but involve rather more algebra). We have chosen to avoid introducing a source for the gauge field because rather than determining the generating functional by integrating over the gauge field we take a shortcut by noting [29] that the N-point functions can be calculated from the effective action by specialising the background gauge field to a sum of N plane waves

$$A_\mu = \sum_{i=1}^{N} \epsilon_{i\mu} e^{ik_i \cdot \omega} \tag{1.0.19}$$

and extracting the part which contains every polarisation ϵ_i exactly once. So using Strassler's version of the worldline theory we expand to order N:

$$\Gamma_N\left[\epsilon_1, k_1, \ldots, \epsilon_N, k_N\right] = \int_0^\infty \frac{dT}{T} \oint \mathscr{D}\omega \, e^{-\int_0^1 d\tau \frac{\dot{\omega}^2}{2T} + m^2 T} \, \frac{i^N}{N!} \left(\int_0^1 d\tau \sum_{i=1}^{N} \dot{\omega} \cdot \epsilon_i \, e^{ik_i \cdot \omega}\right)^N \tag{1.0.20}$$

We take those terms which involve all N different polarisations and momenta which provides

$$\begin{aligned} &\int_0^\infty \frac{dT}{T} \left(4\pi T\right)^{-2} \oint \mathscr{D}\omega \, e^{-\int_0^1 d\tau \frac{\dot{\omega}^2}{2T} + m^2 T} \, i^N \\ &\quad \times \int d\tau_1 \, \epsilon_1 \cdot \dot{\omega}\left(\tau_1\right) e^{ik_1 \cdot \omega(\tau_1)} \ldots \int d\tau_N \, \epsilon_N \cdot \dot{\omega}\left(\tau_N\right) e^{ik_N \cdot \omega(\tau_N)} \end{aligned} \tag{1.0.21}$$

The integral over ω can be carried out by exponentiating the $\epsilon_i \cdot \dot{\omega}\left(\tau_i\right)$ with the understanding that the result should be expanded to multi-linear order in each polarisation vector[1] and for $N = 2$ yields

$$\begin{aligned} \Gamma_2\left[\epsilon_1, k_1, \epsilon_2, k_2\right] = i^2 \left(2\pi\right)^4 \delta^4\left(k_1 + k_2\right) \int_0^\infty \frac{dT}{T} \left(4\pi T\right)^{-2} e^{-m^2 T} \int_0^1 d\tau_1 \int_0^1 d\tau_2 \\ \left[\dot{G}\left(\tau_1, \tau_2\right)^2 \epsilon_1 \cdot k_2 \, \epsilon_2 \cdot k_1 - \epsilon_1 \cdot \epsilon_2 \ddot{G}\left(\tau_1, \tau_2\right)\right] e^{k_1 \cdot k_2 G(\tau_1, \tau_2)}. \end{aligned} \tag{1.0.22}$$

In the above equation $G\left(\tau_1, \tau_2\right)$ is the one dimensional Green function for the kinetic

[1] This trick is borrowed from similar calculations in string theory [30].

operator $\frac{-1}{T}\frac{d^2}{d\tau^2}$ defined on $[0,1]$. The zero mode of this operator leads to the momentum conserving $\delta-$function and in the reduced space orthogonal to this mode the Green function takes the form

$$G(\tau_1,\tau_2) = T\left(|\tau_1-\tau_2| - (\tau_1-\tau_2)^2\right). \tag{1.0.23}$$

The calculation is completed by integrating the second term in square brackets by parts and applying momentum conservation $k_1 = k = -k_2$. This turns the integrand into

$$\epsilon_1^\mu\left[k^\mu k^\nu - \delta^{\mu\nu}k^2\right]\epsilon_2^\nu \dot{G}(\tau_1,\tau_2)^2 e^{-k^2 G(\tau_1,\tau_2)} \tag{1.0.24}$$

so the transverse projector already appears at this stage. The integrand has translational invariance which makes one of the integrals trivial and it is straightforward to arrive at

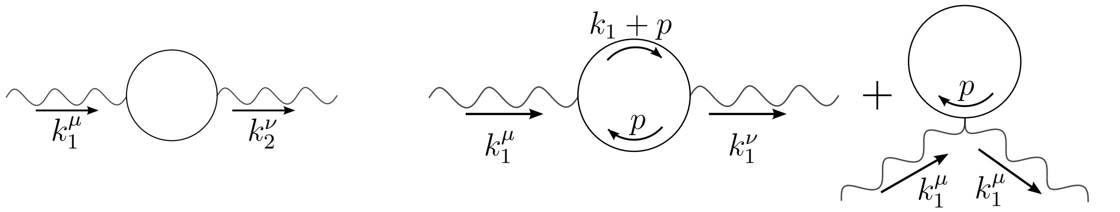
$$\Gamma_2\left[\epsilon_1,k_1,\epsilon_2,k_2\right] = \frac{1}{(4\pi)^2}\epsilon_1^\mu\left[k^\mu k^\nu - \delta^{\mu\nu}k^2\right]\epsilon_1^\nu\int_0^\infty\frac{dT}{T}e^{-m^2T}\int_0^1 du\,(1-2u)^2 e^{-Tu(1-u)k^2}. \tag{1.0.25}$$

Integrating over T we arrive at the Feynman parameterised expression for the one-loop two photon scattering amplitude in scalar QED

$$\frac{1}{(4\pi)^{\frac{D}{2}}}\Gamma\left(2-\frac{D}{2}\right)\left[k^\mu k^\nu - \delta^{\mu\nu}k^2\right]\int_0^1 du\,\frac{(1-2u)^2}{\left[m^2+u(1-u)k^2\right]^{\frac{D}{2}-2}}. \tag{1.0.26}$$

where we have given the result for arbitrary space-time dimension D because the expression requires renormalisation. We shall not address such issues in this introduction.

The purpose of this brief demonstration of the worldline approach to quantum field theory is to highlight the reorganisation of the various contributions to the amplitude. Firstly we note that we arrived directly at the Feynman parameterised expression for the scattering. In conventional perturbation theory of scalar QED this arises out of the sum of *two* Feynman diagrams (see figure 1.1) and several algebraic manipulations. Secondly we found the transverse projector appearing at the level of the integrand when we carried out an integration by parts. In fact this is motivated by the calculations of Bern and Kosower, where a systematic removal of second derivatives of the Green function removes problems associated with the pinching of vertices about the worldline. This concludes our review of the worldline



(a) The single diagram in the worldline formalism

(b) The two contributions to the amplitude in second quantised perturbation theory

Figure 1.1: A comparison of the worldline formalism with perturbative field theory. One diagram in the worldline formalism can represent two or more in second quantised theory. This simplification becomes increasingly apparent at higher loop order.

formalism of quantum field theory. We shall build upon this in Chapter 3 where it will form the basis of a reformulation of QED in terms of interacting strings. We now turn as promised to the issue of functional quantisation where we will justify some of the claims we have made in this section.

1.0.2 Functional quantisation

Feynman developed the path integral approach to quantum mechanics as an alternative to canonical quantisation. In this formalism transition amplitudes from one state to another are formally given by summing over all possible evolutions from the initial to the final state, with each such evolution weighted by the exponential of minus the action corresponding to that transition. We shall consider theories of point particles and strings but, as we have seen already for the former case, we can express these in an equivalent manner by introducing an intrinsic metric on the domain. The question then becomes one of how to carry out the sum over the configurations of the particles or strings *and* integrate over the possible geometries of the domain.

Before we proceed to address this question we pause to consider why such an approach is desired at all. Indeed, canonical quantisation is a sufficient tool to investigate the quantisation of both point particles and strings. The point is that by applying functional methods to string theory we learn something about the geometry of the theory and gain insight into the physical reason for mass-shell conditions and the critical dimension of target space. It is also rather difficult to compute string scattering processes beyond one loop using operator methods alone; the functional approach provides an easier way of determining these amplitudes because it is more closely aligned with the underlying geometry of the strings.

Point particles

The natural invariant action for a point particle in space-time is simply the length of the worldline ω

$$S = m \int d\xi \sqrt{\eta_{\mu\nu}\, \dot{\omega}^{\mu}\left(\xi\right) \dot{\omega}^{\nu}\left(\xi\right)} \tag{1.0.27}$$

where ξ is an arbitrary parameter and m is the mass of the particle. This action suffers from the obvious pitfall of being unsuitable for massless particles and were we to exponentiate this and integrate over configurations, ω, we would find great difficulty in dealing with the square root. Finally we need to take into account the vast gauge symmetry under reparameterisations $\omega\left(\xi\right) \to \omega\left(\tilde{\xi}\left(\xi\right)\right)$ (which does not change the path) and ensure that we count only those paths that are not related by such a gauge transformation – in other words to integrate only over physically distinct paths. Although it may sound counter productive the answer is to introduce a further degree of freedom, $h\left(\xi\right)$, which transforms as a $(0,2)$ tensor and use the action [27] $S_0\left[\omega, h\right]$ defined above. h is interpreted as an intrinsic metric on the particle worldline. This has the advantage of removing the square root and also gives a sensible $m \to 0$ limit. At a classical level, the equation of motion for h can be used to eliminate it from the action, in which case we find (1.0.27) above.

The reparameterisation invariance can be used to fix the metric to our advantage. For example, setting $h = m^{-2}$ in S_0 leads to a canonical momentum $p^{\mu} = m\dot{\omega}^{\mu}$ and the equation of motion $\ddot{\omega} = 0$. The equation of motion for h, however, imposes the mass-shell constraint $p^2 = m^2$ which will have to hold as an operator equation in the quantum theory. The quantum commutation relations are given by

$$\left[\hat{p}^{\mu}, \hat{\omega}^{\nu}\right] = i\eta^{\mu\nu} \tag{1.0.28}$$

and the physical states are those that satisfy the constraint $(\hat{p}^2 - m^2)\,|\omega_{\text{phys}}\rangle = 0$.

Let us consider the functional quantisation of this theory by calculating the elementary objects introduced in the previous section (as we wish only to illustrate the techniques we shall discard a coupling to the gauge field)

$$\int \frac{\mathscr{D}h\mathscr{D}\omega}{\text{Vol}\,(\text{Diff})} e^{-\int_0^1 d\xi\,\frac{\dot{\omega}^2}{2\sqrt{h}}+\frac{1}{2}m^2\sqrt{h}} \tag{1.0.29}$$

with appropriate boundary conditions on the integration over ω depending on whether the worldline is open or closed. We have divided through by the volume of the diffeomorphism group which takes care of the over-counting of equivalent states. The functional integration over metrics can be defined in analogy to finite dimensional integration [31]: supposing that the space of metrics is parameterised by local coordinates ζ_i we define

$$\mathscr{D}h = \sqrt{\det\left(\frac{\partial h}{\partial \zeta_i}, \frac{\partial h}{\partial \zeta_j}\right)} \prod_k d\zeta_k \tag{1.0.30}$$

where $(\cdot,\cdot)$ denotes an inner product on variations in the metric that must be chosen. Following Appendix C of [28] we take the reparameterisation invariant inner product

$$(\delta_1 h, \delta_2 h)_h = \int_0^1 h^{-2}\delta_1 h\,\delta_2 h\sqrt{h}\,d\xi \tag{1.0.31}$$

Variations in the metric can be divided into those corresponding to a reparameterisation and orthogonal, physical changes. For one dimensional metrics these physical changes correspond to global scalings. Under an infinitsesimal diffeomorphism parameterised by a "vector," $V(\xi)$, we have $\xi \to \xi + V(\xi)$ and

$$\delta_V h(\xi) = -\left[V(\xi)\frac{d}{d\xi} + 2\frac{dV(\xi)}{d\xi}\right] h(\xi) \equiv -2DV\,. \tag{1.0.32}$$

Under an infinitesimal global scaling parameterised by a constant, c, the metric changes as $h(\xi) \to h(\xi)(1+c)$ and $(\delta_c, \delta_c)_h = (\delta c)^2 \int_0^1 \sqrt{h}\,d\xi$ so the integration measure factorises into

$$\mathscr{D}h = dc\,\mathscr{D}V \sqrt{\left(\int_0^1 \sqrt{h}\,d\xi\right)\left(\text{Det}(D^\dagger D)\right)} \tag{1.0.33}$$

We use the reparameterisation invariance of the action to expand the metric about a constant value $h(\xi) = T^2$, whereby $dc = 2\frac{dT}{T}$ and $\int_0^1 \sqrt{h}d\xi = T$. In this gauge the eigenvalues of $D^\dagger D$ can be $\zeta-$function regularised (see the appendix of Chapter 2) and we find

$$\sqrt{\det\left(D^\dagger D\right)} \propto \sqrt{T} \tag{1.0.34}$$

for open worldlines and

$$\sqrt{\det\left(D^\dagger D\right)} \propto T \tag{1.0.35}$$

for closed worldlines. Due to the diffeomorphism invariance the integration with respect to V is trivial and provides the volume of the reparameterisation group[2], cancelling the denominator of the functional integration and allowing us to correctly account for the overcounting caused by the gauge symmetry. In total we can replace the functional integration over the metric degree of freedom by $\mathscr{D}h \propto dT$ for open worldlines and by $\mathscr{D}h \propto \frac{dT}{T}$ for closed worldlines.

The measure for the integration over the matter field, $\mathscr{D}\omega$, can be defined by expanding about the classical solution in Fourier modes of the kinetic operator. Integration over the Fourier modes produces the functional determinant of this operator which can be calculated by $\zeta-$function techniques. It is straightforward to verify that this produces the same expressions for both the partition function and the propagator as are found in the canonical operator approach to quantisation. For brevity we do not pursue that here since no substantial problems are encountered, unlike when it comes to the quantisation of the string (see however a brief calculation of the functional determinant in the appendix of Chapter 2). For spin 1/2 matter a similar construction works for χ and ψ introduced in the previous section. We postpone a discussion of this until chapter 2.

[2] Actually care must be taken with closed worldlines because the constant zero mode of $D^\dagger D$ does not change the metric. Excluding this zero mode leaves behind a further factor of $T^{\frac{3}{2}}$ which must be taken into consideration – see [28] for further details

String theory

String theory is just the generalisation of the above considerations to a one dimensionally extended object. Rather than a point particle tracing out a path in space-time we consider an extended string (which may be open or a closed loop) which traces out a worldsheet in D dimensional space-time. We will limit our discussion to the bosonic string in this introduction, waiting until Chapter 3 to discuss the construction of the spinning string. The natural invariant action is the area of the string worldsheet which leads us to consider the Nambu-Goto action [32]

$$S_{\text{NG}} = \frac{1}{2\pi\alpha'} \int d^2\sigma \, \sqrt{\epsilon^{ab} \partial_a X^\mu(\sigma) \, \partial_b X^\nu(\sigma) \, \eta_{\mu\nu}}. \tag{1.0.36}$$

Here $X^\mu(\sigma)$ is the embedding of the worldsheet into space-time which is parameterised by $\sigma^a = (\tau, \sigma)$ and we have introduced the string tension $T = (2\pi\alpha')^{-1}$. This form of the action has the same unfortunate square root as we initially wrote down for the point particle and has an analogous two dimensional reparameterisation symmetry. We solve these problems in the same way by introducing a metric $h_{ab}(\sigma)$ on the worldsheet. Then the Polyakov action [33] is

$$S_{\text{P}} = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{h} \, h^{ab}(\sigma) \, \partial_a X^\mu(\sigma) \, \partial_b X^\nu(\sigma) \, \eta_{\mu\nu} \tag{1.0.37}$$

where we take $h \equiv |\det h_{ab}|$. Classically h_{ab} is an auxiliary field and its equation of motion can be solved in terms of the coordinates X^μ – substituting this solution back into the action provides the Nambu-Goto action.

Polyakov's action enjoys reparameterisation invariance and also has conformal invariance under a scaling of the metric $h_{ab} \to e^{\phi(\sigma)} h_{ab}$ [34]. If we restrict our attention to genus zero worldsheets we can fix these gauge symmetries by choosing co-ordinates in which

$$\sqrt{h} h^{ab} = \eta^{ab}. \tag{1.0.38}$$

Such a choice breaks the manifest reparameterisation symmetry but has the advantage of maintaining explicit Weyl invariance, so long as quantisation does not reintroduce some dependence on the worldsheet scale. In this conformal gauge the action is independent of the metric and leads to a free conformal theory for the worldsheet coordinates

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \partial_a X^\mu (\sigma) \, \partial^b X_\mu (\sigma) \, . \tag{1.0.39}$$

However, as for the point particle we must impose the constraint which arises out of the equation of motion for h_{ab}. This is just the vanishing of the two dimensional energy momentum tensor $T_{ab} = 0$ evaluated in the conformal gauge. Upon quantisation this must be imposed as an operator statement and determines the physical states of the system. If T_{ab} is Fourier expanded then we find the familiar Virasoro constraints $L_m = 0$ where

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n \tag{1.0.40}$$

which is defined in terms of the Fourier modes of the string coordinates α_n. Similarly to Gupta-Bleuler quantisation of electrodynamics these constraints become the weaker conditions that $\hat{L}_m \left|\text{phys}\right\rangle = 0$ for $m > 0$ when the α_n are promoted to creation and annihilation operators. This quantisation procedure is well known and we only state the consequences. The physical spectrum of the string is easily determined by acting on the vacuum with creation operators subject to the Virasoro constraints (the mass of the states is determined by L_0). The negative norm modes associated to $a_0^{\dagger\mu}$ can be decoupled by the Virasoro constraints only in $D = 26$ space-time dimensions. Scattering is described from the point of view of the worldsheet conformal theory by introducing vertex operators that correspond to a given string state. The requirement that these operators transform correctly under the conformal symmetry (and so map physical states into physical states) likewise constrains their form and ensures that only the physical states of the string can take part in scattering. This is really just another manifestation of the state-operator correspondence.

We turn now to the functional quantisation of the string. The discussion in this section follows that of Polyakov [35] and [36] (see also [37]). In the canonical approach the reparamaterisation invariance is broken at the outset. Polyakov's suggestion was to keep the reparameterisation invariance intact during quantisation. So all functional measures and any regularisation required during quantisation must be defined to respect this symmetry. The simplest calculation we can consider is the tree level partition function

$$\mathcal{Z} = \int \frac{\mathscr{D}h\mathscr{D}X}{\text{Vol (Diff x Weyl)}} e^{-S_{\text{P}}[h,X]} \tag{1.0.41}$$

into which we could place various insertions to represent scattering amplitudes if desired. For open strings the worldsheet has the topology of a disk and for closed strings has the topology of a sphere. An arbitrary variation of the metric takes the form

$$\delta h_{ab} = \delta\phi h_{ab} + \nabla_{(a}\delta V_{b)} \tag{1.0.42}$$

which is just a Weyl scaling and a reparameterisation parameterised by V. Although not Weyl invariant, there is a unique ultra-local diffeomorphism invariant inner product on such variations given by

$$(\delta_1 h, \delta_2 h) = \int d^2\sigma\sqrt{h}\,\delta_1 h_{ab}\left(Ah^{ar}h^{bs} + Bh^{ab}h^{rs}\right)\delta_2 h_{rs} \tag{1.0.43}$$

for constants A and B. Unfortunately $\left(\delta\phi h_{ab}, \nabla_{(a}V_{b)}\right) \neq 0$ so the determinant in the definition of $\mathscr{D}h$ will not be block diagonal. Physically the reason for this is the existence of the conformal Killing vectors which generate reparameterisations that act as Weyl scalings [38]. However, the determinant is not changed by elementary row operations so we may shift this part of the variation into a redefinition of ϕ. Then defining $P(V)_{ab} = \nabla_{(a}\delta V_{b)} - g_{ab}\nabla_c V^c$, which takes vectors into traceless covariant tensors, we have

$$\delta h_{ab} = \delta\phi h_{ab} + P\left(V\right)_{ab}. \tag{1.0.44}$$

Now we find that the two changes are orthogonal to one another and

$$\mathscr{D}h \propto \mathscr{D}\phi\mathscr{D}V\sqrt{\det{}'(P^\dagger P)}. \tag{1.0.45}$$

We have written $\det'$ to denote that the zero modes of $P^\dagger P$ – which are the conformal Killing vectors – should not be included in the determinant. We will preserve the reparameterisation invariance of the action and nothing in $P^\dagger P$ makes any reference to a specific coordinate system. Therefore the integral over V provides the volume of the diffeomorphism group, except for the volume corresponding to the conformal Killing vectors which are excluded. In this way we almost take care of the overcounting caused by the diffeomorphism symmetry, only needing to worry about the remaining symmetry generated by the conformal Killing vectors when we consider scattering amplitudes.

Were the determinant independent of ϕ and the classical conformal invariance of the action preserved when we integrate over X we would be able to carry out the ϕ integral and cancel the result off against the volume of the Weyl group. However, the inner product (1.0.43) is not classically Weyl invariant so we can not expect the determinant to be independent of ϕ (in fact it also requires regularisation). Furthermore, we must also define the functional integral with respect to X^μ which will be found to depend on the conformal scale. To see this, write the Polyakov action (up to possible classical contributions) as

$$S_{\mathrm{P}} = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h}\, X^\mu \Delta X_\mu \tag{1.0.46}$$

where we've introduced the worldsheet Laplacian $\Delta X^\mu \equiv -\frac{1}{\sqrt{h}} \partial_a \left(\sqrt{h} h^{ab} \partial_b X^\mu \right)$. We can define a reparameterisation invariant inner product on variations in X as

$$(\delta_1 X, \delta_2 X) = \int d^2\sigma \sqrt{h}\, \delta_1 X \cdot \delta_2 X \tag{1.0.47}$$

with respect to which the Laplacian is Hermitian. Decomposing X into a linear combination of orthogonal eigenfunctions of Δ the volume element simply becomes an infinite product of ordinary integrals over the coefficients appearing in this expansion. The action becomes quadratic in these coefficients so the integrals are Gaussian. The integral over the coefficients not associated to zero eigenvalues provides $\det'(\Delta)$ but the worldsheet and boundary conditions may allow a constant zero mode $u_0 = \left(\int d^2\sigma \sqrt{h} \right)^{-\frac{1}{2}}$. The integral over this mode diverges but it can be written in terms of the mean position of the string, which we denote by X_0^μ. The result is

$$\int \mathscr{D}X e^{-S_{\mathrm{P}}[h,X]} = \left(\frac{\det'(\Delta)}{\int d^2\sigma\sqrt{h}}\right)^{-\frac{D}{2}} \int d^D X_0. \tag{1.0.48}$$

The integral over X_0 then gives the infinite volume of target space which factors out of the physical content of the theory. The determinant of the Laplacian depends on ϕ and also requires regularisation. Until we have determined the ϕ-dependence of the two determinants which have entered the calculation we cannot attempt to complete the functional dependence over this conformal scale.

The fact that the determinants require regularisation is important because it must be done in a reparameterisation invariant way. This requires the introduction of a short distance regulator which therefore introduces further dependence on the scale of the worldsheet metric. This quantum break-down of the classical scale invariance is called the Weyl anomaly. We have seen that it arises out of the requirement to define the integration over all geometries and string configurations ($\mathscr{D}h$ and $\mathscr{D}X$) and the necessity to regulate the divergences that are then encountered.

The explicit calculations which derive the dependence of the determinants on the conformal scale are well-known and we do not repeat them here. Instead we argue that since the regulator's role is to cut off short-distance divergences and the integration measures have both been defined to be ultra-local the anomalous dependence of the determinants on ϕ must depend locally on the worldsheet metric. Now the contributions that diverge as the regulator is removed can be compensated for by a renormalisation of the string action (or absorbed by the introduction of a cosmological term) so we need only construct the finite part. On dimensional grounds the anomaly must be made up out of the worldsheet scalar curvature, ${}^{(2)}R$, and ϕ itself. Indeed, explicit calculation [31] shows that under an infinitesimal variation of the conformal scale $\delta h_{ab} = \delta\phi h_{ab}$,

$$\delta \ln\left(\frac{\sqrt{\det'(P^\dagger P)}}{\mathrm{Vol}\,(CKV)}\left(\frac{\det'(\Delta)}{\int d^2\sigma\sqrt{h}}\right)^{-\frac{D}{2}}\right) = \frac{D-26}{48\pi}\int d^2\sigma\sqrt{h}\ {}^{(2)}R\,\delta\phi\,(\sigma)\,. \tag{1.0.49}$$

This variation cannot be written as the infinitesimal change of some local and reparameterisation invariant object. It can be integrated to give the Liouville action

$$S_{\mathrm{L}} = \iint d^2\sigma \, \sqrt{h} \, {}^{(2)}R\,(\sigma)\, G\,(\sigma,\sigma')\, {}^{(2)}R\,(\sigma')\, \sqrt{h}\, d^2\sigma', \tag{1.0.50}$$

where $G\,(\sigma,\sigma') = \Delta^{-1}\,(\sigma,\sigma')$ is the worldsheet Green function, for which we must now try to carry out the functional integral over ϕ. The measure on variations in ϕ is induced from (1.0.43) and takes the form

$$(\delta\phi,\delta\phi) = \int d^2\sigma\sqrt{h}\;(\delta\phi)^2 \tag{1.0.51}$$

This is rather unusual, since if we consider it in a gauge conformally related to some reference metric, $\hat{h}$, we do not find a familiar linear measure but

$$|\delta\phi|^2 \equiv (\delta\phi,\delta\phi) = \int d^2\sigma\sqrt{\hat{h}}\, e^{\phi(\sigma)}\,(\delta\phi)^2\,. \tag{1.0.52}$$

and we would like to define functional integration by

$$\int \mathscr{D}\phi\, e^{-|\delta\phi|^2} = 1. \tag{1.0.53}$$

It is not yet known how to carry out this integral, nor how this measure should be interpreted. So we are faced with an inability to complete the string quantisation unless we arrange for the Liouville theory to vanish. Obviously this can be done by setting $D = 26$, in which case the Weyl anomaly disappears and the integral over ϕ directly cancels the volume of the Weyl group. This can be compared to the canonical approach in which the negative norm (ghost) states can only be decoupled by the Virasoro constraints when $D = 26$. We learn that the scale of the worldsheet metric is the degree of freedom which prohibits the quantisation of strings outside of the critical dimension. In canonical quantisation we restrict the physical state space but in the functional approach we integrate over all string coordinates (including X^0) and the integration over worldsheet metrics cancels – in the critical dimension – the effect of having included the negative norm states.

The story is essentially the same for the spinning string, where we introduce

Grassman worldsheet fields $\Psi(\sigma)$ and $\chi(\sigma)$ which are the super-partners to the bosonic fields [39]. Then the classical theory enjoys super-conformal invariance and the requirement that this is preserved upon quantisation translates to a critical dimension of $D = 10$ [40]. To determine the spectrum of the bosonic or spinning string we must go beyond the partition function and consider string amplitudes. Accordingly vertex operators are inserted into the functional integral to represent the states taking part in the scattering process. We shall delay our discussion of exactly how this works until Chapter 3 and will simply state that the requirement of not reintroducing unwanted dependence on the conformal scale (as occurs during the regularisation of UV divergences in the worldsheet Green function) leads to precisely the mass-shell and transversality conditions that are found in the canonical approach. In Chapter 3 we shall have cause to consider the effects of such insertions in great detail, since in order to quantise the interacting string theory we will need to ensure that the conformal invariance of the string is not broken by our modifications.

There are many further facets of this approach to quantisation which we have not considered here. The topology of the worldsheet is of particular importance since for higher genus surfaces the geometry is also described by some number of modular parameters. These too must be integrated over to cover the space of distinct geometries. Even at tree level the amplitude calculations must confront the residual gauge symmetry which is left over when expanding about the conformal gauge. We may consider coupling the string to background fields where the functional approach can readily be generalised and we gain an important tool to complement canonical quantisation. For this thesis we will restrict ourselves to relatively simple tree level calculations without such complications.

In the following two chapters we apply these techniques to theories of interacting point particles and strings respectively. A few further details are developed on the way but the aim of both models is to provide a complementary approach to the determination of well-known quantities in theoretical physics with new machinery and an alternative perspective.

Chapter 2

Contact interactions for point particle theories

2.1 Introduction

Classical electromagnetism is conventionally described by Maxwell's field theory and their seems to be little room for debate about its formulation. In [41], however, building upon [13] it was shown that an alternative approach to determining the field strength tensor for a pair of charged particles led directly to a novel interacting string theory. This theory contained contact interactions on the worldsheet which served to produce well-known quantities in Abelian gauge theory and will be the subject of the next chapter. In the case of electrostatics, the description given in [13] was of point particles whose paths have endpoints fixed to the charged particles. The physical picture which motivated this approach is of Faraday's lines of force as fundamental objects.

Before we consider contact interactions in string theory it seems appropriate to return to this idea and explore the consequences of allowing point particles to interact upon contact. This chapter revisits and extends the results of [13] and also generalises that work to the case of fermionic particles. It then goes beyond leading order to demonstrate that in fact the full quantum theory of a set of interacting point particles is consistent and free of unwanted divergences. We develop the functional approach to one dimensional field theory for consistency with [28, 41] and for the generalisation to fermionic particles we will find it most natural to form the theory in superspace. This chapter proceeds by first reviewing the bosonic theory presented at lowest order in [13] before generalising it to the fermionic case. Following this a full interacting theory is described in section 2.5 and quantised.

2.2 Bosonic particles – the classical electric field

We work in D spatial dimensions and consider a static charged particle at position $\mathbf{a}$ and an oppositely charged particle at the point $\mathbf{b}$. An expression satisfying Gauss' law[1] was given in [13]:

$$E'^{i}(\mathbf{x}) = q \int_C d\tau \, \frac{d\omega^i}{d\tau} \delta^3 \left(\boldsymbol{\omega}(\tau) - \mathbf{x}\right) \tag{2.2.1}$$

where the integral is taken over any curve C with endpoints at $\mathbf{a}$ and $\mathbf{b}$. The form of this field is reminiscent of the form of the Dirac string which was introduced to describe the field of a magnetic monopole [42, 43] but we shall use it here in the context of electrostatics. The expression for $\mathbf{E}'$ does not satisfy $\nabla \times \mathbf{E}' = 0$ but we shall see that its statistical average does. The average is over all curves with endpoints fixed at $\mathbf{a}$ and $\mathbf{b}$ and is defined in reparameterisation invariant form as

$$\langle \Omega(\boldsymbol{\omega}) \rangle = \frac{1}{\mathcal{Z}} \int_{\boldsymbol{\omega}(0)=\mathbf{a}}^{\boldsymbol{\omega}(1)=\mathbf{b}} \mathscr{D}e \mathscr{D}\boldsymbol{\omega} \, \Omega(\boldsymbol{\omega}) \, \delta \left(\int_0^1 e \, d\tau - T \right) e^{-S[\boldsymbol{\omega}, e]} \, ; \quad S[\boldsymbol{\omega}, e] = \int_0^1 \frac{\dot{\boldsymbol{\omega}}^2}{2e} d\tau \tag{2.2.2}$$

where Ω is any reparameterisation invariant functional of the path (we work in Euclidean space). The action is that of Brink, diVecchia and Howe we met in the introduction (we take the particle to have zero mass[2]) and here we use their notation for the einbein, e, which is related to h by $e^2 = h$. The δ-function in (2.2.2) picks out paths of fixed intrinsic length T and the Coulomb field for the pair of particles

[1] We put $\epsilon_0 = 1$ and denote the electric charge by q so $\nabla \cdot \mathbf{E}' = q\delta^3(\mathbf{x} - \mathbf{a}) - q\delta^3(\mathbf{x} - \mathbf{b})$

[2] A mass term could have been included in the action defined in (2.2.2) via the inclusion of a cosmological term $\frac{1}{2} m_0^2 \int_0^1 e \, d\tau$. Usually it is necessary to do so in order to remove a divergence in the functional determinant from the integral over $\boldsymbol{\omega}$ by a renormalisation of this bare mass m_0 to a physical mass m. In this case the divergence is cancelled by $\mathcal{Z}$ and since E' does not involve the einbein the effect of including m would be also be cancelled by the normalisation.

is arrived at by taking the average of $\mathbf{E}'$ in the limit as $T \to \infty$. The normalisation constant is defined by $\langle 1 \rangle = 1$.

In [13] this was shown using techniques derived from canonical quantisation. We demonstrate the result using the functional methods we will use for the fermionic generalisation of this claim. To do so we must address the overcounting caused by the reparameterisation symmetry. Following the introduction we recall that for open curves the measure on the space of metrics can be written

$$\mathscr{D}e = dc\mathscr{D}V\sqrt{\left(\int d\tau\, e\right)\operatorname{Det}\left(D^{\dagger}D\right)}; \qquad DV = \frac{d}{d\tau}\left(Ve\right) \tag{2.2.3}$$

where c represents a scaling of the einbein and V an infinitesimal reparameterisation. This volume element follows because any metric can be written as a combined scaling, generated by c, plus reparameterisation, generated by V, about some reference metric. We expand about $e = \top$, a constant, whereby the volume element becomes

$$\mathscr{D}e = d\top\mathscr{D}V \tag{2.2.4}$$

and the constraint becomes $\delta\left(\top - T\right)$. Since the components of the functional average are taken to be invariant under reparameterisations the functional integral with respect to V just gives the volume of the reparameterisation group, which cancels with the corresponding contribution from the normalisation constant. So for the average of $\mathbf{E}'\left(x\right)$ we must determine

$$\langle \mathbf{I}\left(\mathbf{x}\right)\rangle \equiv \frac{q}{\mathcal{Z}'}\int \mathscr{D}\boldsymbol{\omega}\int \frac{d^D\mathbf{k}}{\left(2\pi\right)^D}\int_0^1 d\tau_1\,\frac{d\boldsymbol{\omega}\left(\tau_1\right)}{d\tau_1}e^{i\mathbf{k}\cdot\left(\boldsymbol{\omega}\left(\tau_1\right)-\mathbf{x}\right)}e^{-\int_0^1 \frac{\dot{\boldsymbol{\omega}}^2}{2T}d\tau} \tag{2.2.5}$$

where $\mathcal{Z}'$ is what remains after the volume of the reparameterisation group has been cancelled from $\mathcal{Z}$ and we have used the Fourier decomposition of the δ-function. The insertion that arises has a familiar form – it is the one dimensional version of the vertex operator used in bosonic string theory and it frequently appears in calculations in the worldline formalism:

$$V_k^{\mu}\left(\tau\right) = \dot{\omega}^{\mu}\left(\tau\right)e^{ik\cdot\omega\left(\tau\right)}. \tag{2.2.6}$$

Unlike in string theory we integrate this operator over all momenta. From the point of view of the one dimensional quantum theory (2.2.5) is the amplitude for the path from $\mathbf{a}$ to $\mathbf{b}$ to pass through the point $\mathbf{x}$. Before calculating this expectation value we note that the structure of the insertion allows us to constrain its dependence on momentum – if we contract (2.2.6) with k_μ and integrate over τ we find a contribution only from the endpoints of the domain

$$\int d\tau \; k_\mu V_k^\mu(\tau) = -i\left(e^{ik\cdot\omega(1)} - e^{ik\cdot\omega(0)}\right) \tag{2.2.7}$$

which we shall refer to as the generalised Gauss' Law.

The insertion can be generated by introducing a source, $\mathbf{j}(\tau)$, and defining $\mathbf{J}(\tau) = -q\frac{d\mathbf{j}}{d\tau} - i\mathbf{k}\delta(\tau - \tau_1)$. Then the above equation becomes

$$\frac{1}{\mathscr{Z}'}\int \mathscr{D}\boldsymbol{\omega} \int \frac{d^D\mathbf{k}}{(2\pi)^D}\int_0^1 d\tau_1\, e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{\delta}{\delta\mathbf{j}(\tau_1)} e^{-\int_0^1 \frac{\dot{\boldsymbol{\omega}}^2}{2T} + \boldsymbol{\omega}\cdot\mathbf{J}d\tau}\Bigg|_{\mathbf{j}=\mathbf{0}}. \tag{2.2.8}$$

We split $\boldsymbol{\omega}$ into its classical part in the absence of a source and a piece which absorbs the source and accounts for the quantum fluctuations $\boldsymbol{\omega}(\tau) = \boldsymbol{\omega}_c(\tau) + \tilde{\boldsymbol{\omega}}(\tau)$. Here $\boldsymbol{\omega}$ satisfies the source-free classical equation of motion $\frac{-1}{T}\frac{d^2\boldsymbol{\omega}}{d\tau^2} = \mathbf{0}$ with endpoints at $\mathbf{a}$ and $\mathbf{b}$:

$$\boldsymbol{\omega}(\tau) = \mathbf{a} + (\mathbf{b} - \mathbf{a})\tau. \tag{2.2.9}$$

$\tilde{\boldsymbol{\omega}}(\tau)$ is required to vanish at the endpoints and can be split into a classical piece, $\tilde{\boldsymbol{\omega}}_c$, satisfying $\frac{-1}{T}\frac{d^2\tilde{\boldsymbol{\omega}}_c}{d\tau^2} = \mathbf{J}(\tau)$ and a quantum fluctuation, $\bar{\boldsymbol{\omega}}(\tau)$. Integrating over $\bar{\boldsymbol{\omega}}$ leads to a functional determinant (which we evaluate with ζ-function regularisation in Appendix A) and because the path is open there are also boundary contributions.

We find

$$\frac{(2\pi T)^{-\frac{D}{2}}\, e^{-\frac{(\mathbf{b}-\mathbf{a})^2}{2T}}}{\mathscr{Z}'} \int \frac{d^D\mathbf{k}}{(2\pi)^D} \int_0^1 d\tau_1\, e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{\delta}{\delta \mathbf{j}(\tau_1)}$$
$$\exp\left(-\frac{1}{2}k^2 G(\tau_1,\tau_1) - iq\int_0^1 d\tau\, \mathbf{k}\cdot\mathbf{j}(\tau)\frac{d}{d\tau}G(\tau_1,\tau) + \frac{q^2}{2}\int_0^1\int_0^1 d\tau d\tau'\, \mathbf{j}(\tau)\cdot\mathbf{j}(\tau')\frac{d}{d\tau}\frac{d}{d\tau'}G(\tau,\tau')\right)$$
$$\exp\left(-q\int_0^1 d\tau \mathbf{j}(\tau)\cdot\frac{d}{d\tau}\boldsymbol{\omega}_c(\tau) + i\mathbf{k}\cdot\boldsymbol{\omega}_c(\tau_1)\right)\Bigg|_{\mathbf{j}=\mathbf{0}}. \quad (2.2.10)$$

In the above equation $G(\tau_1,\tau_2)$ is the Green function on the interval $[0,1]$ with Dirichlet boundary conditions $G(0,\tau_2) = 0 = G(1,\tau_2)$. Its explicit form is

$$G(\tau_1,\tau_2) = -\frac{T}{2}\left(|\tau_1-\tau_2| - (\tau_1+\tau_2) + 2\tau_1\tau_2\right); \qquad -\frac{1}{T}\frac{d^2G}{d\tau_1^2} = \delta(\tau_1-\tau_2). \quad (2.2.11)$$

and $G(\tau_1,\tau_1)$ is the coincident limit, which in one dimension is finite. The numerator of (2.2.10) contains the heat-kernel for the free particle and this cancels exactly with $\mathscr{Z}'$. Carrying out the functional differentiation and setting $\mathbf{j} = \mathbf{0}$ yields for the average in momentum representation

$$\langle \mathbf{I}(\mathbf{k})\rangle = -q\int_0^1 d\tau_1 \left[\dot{\boldsymbol{\omega}}_c(\tau_1) - \frac{1}{2}i\mathbf{k}\frac{d}{d\tau_1}G(\tau_1,\tau_1)\right] e^{-\frac{1}{2}k^2 G(\tau_1,\tau_1)} e^{i\mathbf{k}\cdot\boldsymbol{\omega}_c(\tau_1)}. \quad (2.2.12)$$

Since we will eventually take the limit $T\to\infty$ it is useful at this point to extract the T dependence in order to make an expansion in powers of $\frac{1}{T}$. Define then $\tilde{G}(\tau_1,\tau_2) \equiv \frac{1}{T}G(\tau_1,\tau_2)$ and note that

$$\tilde{G}(\tau_1,\tau_1) = -\tau_1(\tau_1-1). \quad (2.2.13)$$

This function vanishes on the boundary and is increasing (decreasing) for $\tau < 1/2$ ($\tau > 1/2$). In this parameterisation (2.2.12) becomes

$$\langle \mathbf{I}(\mathbf{k})\rangle = -q\int_0^1 d\tau_1 \left[\dot{\boldsymbol{\omega}}_c(\tau_1) - \frac{T}{2}i\mathbf{k}\frac{d}{d\tau_1}\tilde{G}(\tau_1,\tau_1)\right] e^{-\frac{1}{2}k^2 T\tilde{G}(\tau_1,\tau_1)} e^{i\mathbf{k}\cdot\boldsymbol{\omega}_c(\tau_1)}. \quad (2.2.14)$$

In the large T limit the suppression caused by the exponent $\exp\left(-\frac{1}{2}k^2T\tilde{G}\left(\tau_1,\tau_1\right)\right)$ causes the contributions to the integrand to arise primarily when $\tau_1 \to 0$ and $\tau_1 \to 1$ (where the coincident Green function vanishes). We therefore expand the field $\boldsymbol{\omega}$ about these points and integrate a small distance, h, along the worldline. At lowest order in $\frac{1}{T}$ we have:

$$
\begin{aligned}
&- q\int_0^h d\tau_1\,\left[\dot{\boldsymbol{\omega}}_c\left(0\right) - \frac{T}{2}i\mathbf{k}\frac{d}{d\tau_1}\tilde{G}\left(\tau_1,\tau_1\right)\right]\,e^{-\frac{1}{2}k^2T\tilde{G}(\tau_1,\tau_1)}e^{i\mathbf{k}\cdot\boldsymbol{\omega}_c(0)}\\
&- q\int_{1-h}^1 d\tau_1\,\left[\dot{\boldsymbol{\omega}}_c\left(1\right) - \frac{T}{2}i\mathbf{k}\frac{d}{d\tau_1}\tilde{G}\left(\tau_1,\tau_1\right)\right]\,e^{-\frac{1}{2}k^2T\tilde{G}(\tau_1,\tau_1)}e^{i\mathbf{k}\cdot\boldsymbol{\omega}_c(1)}
\end{aligned}
\tag{2.2.15}
$$

and damping caused by the form of the coincident Green function in the exponent allows the integration regions to be extended by setting $h = \frac{1}{2}$. Each integral has two terms, the second of which provides the leading order contribution:

$$
\begin{aligned}
-\frac{Ti\mathbf{k}}{2}\int_0^{\frac{1}{2}} d\tau_1\,\frac{d}{d\tau_1}\tilde{G}\left(\tau_1,\tau_1\right)e^{-\frac{1}{2}k^2T\tilde{G}(\tau_1,\tau_1)} &= \frac{i\mathbf{k}}{k^2}\left[e^{-\frac{1}{2}k^2T\tilde{G}\left(\frac{1}{2},\frac{1}{2}\right)} - 1\right]\\
&\to -\frac{i\mathbf{k}}{k^2}
\end{aligned}
\tag{2.2.16}
$$

where the last line holds as $k^2T \to \infty$. Noting that for $0 \leq \tau \leq 1/2$ we have $G\left(\tau,\tau\right) \geq \frac{\tau}{2}$ the first term can be bounded

$$
\begin{aligned}
\int_0^{\frac{1}{2}} d\tau_1\,e^{-\frac{1}{2}k^2T\tilde{G}(\tau_1,\tau_1)} &\leq \int_0^{\frac{1}{2}} d\tau_1\,e^{-\frac{1}{4}k^2T\tau_1}\\
&= \frac{4}{k^2T}\left[e^{-\frac{1}{4}k^2T} - 1\right]
\end{aligned}
\tag{2.2.17}
$$

which is $\mathcal{O}\left(\frac{1}{k^2T}\right)$. Putting this together with the contribution from the other end of the path we arrive at the momentum space expression

$$
\langle\mathbf{I}\left(\mathbf{k}\right)\rangle = \frac{qi\mathbf{k}}{k^2}\left(e^{i\mathbf{k}\cdot\mathbf{a}} - e^{i\mathbf{k}\cdot\mathbf{b}}\right) + \mathcal{O}\left(\frac{1}{k^2T}\right) \tag{2.2.18}
$$

We check our answer by compatibility with the generalised Gauss' law: dotting with $\mathbf{k}$ produces the expected contribution (2.2.7). Taking the limit $T \to \infty$ and setting

$D = 3$ this can be written in position space as

$$\langle \mathbf{I}(\mathbf{x}) \rangle = \frac{q}{4\pi} \boldsymbol{\nabla} \left(-\frac{1}{|\mathbf{x} - \mathbf{a}|} + \frac{1}{|\mathbf{x} - \mathbf{b}|} \right) \tag{2.2.19}$$

which is indeed the classical dipole electric field. This average therefore determines F_{0i} for static oppositely charged particles. In [13] the case of magnetostatics was also considered for a fixed closed current carrying wire, and the time varying situation was also examined. These cases require treating the curve C as dynamical so that the natural weight becomes not the action of a point particle but that of extended objects, naturally leading to the use of string theory. This picture was explored further in [28, 41] and forms the basis of Chapter 3 of this thesis.

The calculation presented above provides an interesting method of determining the classical static dipole electric field, albeit somewhat unconventional. The utility of the functional approach is the ease with which it can be extended. Before generalising to fermionic particles we show that a simple change can instead generate the classical electric field due to a static point particle. This is desirable since it is presumably more useful to deal with a single particle rather than be constrained to dealing with oppositely charged pairs (except when considering the worldlines of virtual particle / anti-particle pairs). For a single particle at the point $\mathbf{a}$ we proceed as above with the exception that we constrain only one end of the worldlines. This is a simple change of the boundary condition at the upper end of the interval; the variation of the action shows that the only other consistent choice we can make is the Neumann condition $\dot{\boldsymbol{\omega}}(1) = 0$.

There are two effects of this change. Firstly the determinant of the kinetic operator $\frac{-1}{T}\frac{d^2}{d\tau^2}$ becomes independent of T (see Appendix A). This has no relevance because it is cancelled by the same change in $\mathscr{Z}'$. Of more importance is the change in the Green function. With the new boundary conditions we find

$$G'(\tau_1, \tau_2) = -\frac{T}{2}\left(|\tau_1 - \tau_2| - (\tau_1 + \tau_2)\right); \qquad G'(\tau_1, \tau_1) = T\tau_1. \tag{2.2.20}$$

Note that the coincident Green function now only vanishes at $\tau_1 = 0$, the end of the curve tied to the particle at $\mathbf{a}$. Finally the classical solution must clearly differ;

now $\boldsymbol{\omega}_c = \mathbf{a}$ satisfies the equation of motion and boundary conditions. In particular $\dot{\boldsymbol{\omega}} = 0$ and there are no boundary contributions from the classical action so that (2.2.14) becomes

$$\langle \mathbf{I}(\mathbf{k}) \rangle = q \int_0^1 d\tau_1 \frac{1}{2} i\mathbf{k} \frac{d}{d\tau_1} G'(\tau_1, \tau_1) \, e^{-\frac{1}{2}k^2 G'(\tau_1,\tau_1)} e^{i\mathbf{k}\cdot\mathbf{a}}. \tag{2.2.21}$$

Defining $G'(\tau_1, \tau_2) \equiv T\tilde{G}'(\tau_1, \tau_1)$ we could again consider the above equation for large T whereby the integrand is suppressed by the exponent $\exp\left(-\frac{1}{2}k^2 T\tilde{G}'(\tau_1, \tau_1)\right)$ except for $\tau_1 \approx 0$. The integrand is a total derivative however so we can determine the exact answer. We obtain

$$\langle \mathbf{I}(\mathbf{k}) \rangle = \frac{iq\mathbf{k}}{k^2} \left(1 - e^{-\frac{1}{2}k^2 T}\right) e^{i\mathbf{k}\cdot\mathbf{a}} \tag{2.2.22}$$

which in the limit $T \to \infty$ has inverse Fourier transform equal to the electric field

$$\langle \mathbf{I}(\mathbf{x}) \rangle = \frac{q}{4\pi} \frac{\mathbf{x} - \mathbf{a}}{|\mathbf{x} - \mathbf{a}|^3}. \tag{2.2.23}$$

It is interesting to note that (2.2.22) gives the exact average at finite T. We will explore corrections to the classical fields for both of the above configurations of particles below. We have already commented on the calculation of the static magnetic field of a *closed* loop of current carrying wire given in [13]. This required a functional average over surfaces with boundary fixed to the wire and already invoked the use of string theory. There the two dimensional worldsheet Green function was required to vanish at both ends of the string since the endpoints were constrained to lie on the boundary. The analogous method of applying Dirichlet boundary conditions to only one end corresponds to opening up the wire into a small segment and ensures that the only contribution to the average comes from a strip close to the end of the string fixed to the wire. The average then yields the Biot-Savart law for that segment of wire. Mixed boundary conditions are discussed further in the full interacting string theory in Appendix A of [28] where it is shown to provide propagators written in the worldline formalism.

For the most part we took D to be arbitrary, only specialising to $D = 3$ spatial dimensions for the sake of compatibility with [13] and an illustration of some of the physical content of the average. In a four dimensional space-time we will have to deal with 4 bosonic coordinates so we append ω^0 to the three fields $\boldsymbol{\omega}$ considered above. The Euclidean average is then constructed over all paths which intersect the spatial point $x^\mu = (x^0, \mathbf{x})$ whose endpoints are fixed to the particles at $a^\mu = (a^0, \mathbf{a})$ and $b^\mu = (b^0, \mathbf{b})$. The average takes the form

$$\langle I^\mu (x) \rangle = \left\langle q \int d\tau \frac{d\omega^\mu}{d\tau} \delta^4 (\omega (\tau) - x) \right\rangle . \tag{2.2.24}$$

If we restrict to the latter case of a single point particle at position a^μ discussed above the generalisation of (2.2.22) provides the Euclidean space average

$$\langle I^\mu (k) \rangle = \frac{iqk}{k^2} \left(1 - e^{-\frac{1}{2}k^2 T} \right) e^{ik\cdot a}. \tag{2.2.25}$$

We can interpret this in Minkowski space by treating the boundary data a^μ as some fixed point on the worldline of the particle and Wick rotating the above result. In the $T \to \infty$ limit then

$$\langle I_\mu (x) \rangle = \frac{q}{4\pi^2} \partial_\mu \frac{1}{|x-a|^2 + i\epsilon} \tag{2.2.26}$$

where we use the $i\epsilon$ procedure to specify the positions of the poles. The physical interpretation of this result is less obvious because it relies on the choice of a^μ (also b^μ had we included a second particle). If we return to a static picture then consider an observer at x^μ in the rest frame of the charged particle. The calculation in $D = 3$ spatial dimensions was an average over all possible particle paths starting at $\mathbf{a}$ and passing through $\mathbf{x}$ but in four dimensional space-time they are also required to pass though $\mathbf{x}$ at the time $x^0 = t$, say. We are restricting our attention to a static configuration and suppose that we ought to integrate over all possible starting points on the worldline a^μ at which the path ω^μ could begin whilst still passing through x^μ. In section 4 of [13] an argument was given based on the construction of thermal Green functions that the retarded solutions to Maxwell's equations are inherited from the Feynman propagator if the calculation is seen as a finite temperature quantum expectation value. Application of this procedure to the current problem provides

$$\langle I_\mu(x)\rangle = \frac{q}{4\pi}\partial_\mu \frac{\delta\left(x^0 - a^0 - |\mathbf{x}-\mathbf{a}|\right)}{|\mathbf{x}-\mathbf{a}|} \tag{2.2.27}$$

so that the only contribution comes from paths whose endpoints are joined along the causal light-cone. In the rest frame of the particle its worldline has constant $\mathbf{a}$ and integrating (2.2.27) with respect to a^0 yields the static electric field expected at $x^0 = t$:

$$F_{00} = \langle I_0(x)\rangle = 0; \qquad F_{0i} = \langle I_i(x)\rangle = \frac{q}{4\pi}\frac{\mathbf{x}-\mathbf{a}}{|\mathbf{x}-\mathbf{a}|^3} \tag{2.2.28}$$

The same result could be arrived at by integrating (2.2.26) over a^0 immediately if the contour for the k^0 integral is chosen to fall above the poles at $\pm|\mathbf{k}|$ on the real axis; the discussion of the thermal average in [13] can thus be seen as justification for this choice of contour.

It should be stressed that this procedure yields the correct static field but is not applicable for the general time dependent problem. This has been dealt with in [13] and [41] and will be considered in the next chapter. It requires string theory to correctly describe the dynamics of extended curves whose boundaries are fixed to the worldlines of the charged particles. The static problem considered above is an unusual way to arrive at the electric field but is nonetheless of interest because of its straightforward generalisation. In the following sections we first include spin degrees of freedom before returning to an analysis of the result when the parameter T is taken to be finite. We then generalise the work to form a full quantum theory of point particles with contact interactions.

Following on from the bosonic theory we wish to provide an extension to the theory in [13]. The following sections comprise the new contribution in this thesis. We have three aims in sight. The first is to extend the work to fermionic fields and the second is to determine the corrections to both results which are present at finite T. We finally ask whether the interaction exponentiates as in [28, 41] – we shall address this issue in section 2.5.

2.3 Fermionic particles

In this section we consider a theory of spin 1/2 particles and generalise the theory above for application to this case. We continue to work with massless particles for simplicity and it will also prove most convenient to work in four dimensional Euclidean space (this is more natural for fermionic theories – we discuss how we may relate this to the three dimensional case below). To deal with fermions it is most useful to construct the theory in superspace. We therefore introduce the Grassmann variable θ to extend the parameter domain $\tau \to (\tau, \theta)$. We further introduce the scalar superfield[3]

$$\mathbf{X}(\tau,\theta) = \omega(\tau) + \theta e^{\frac{1}{2}}(\tau)\,\psi(\tau) \qquad (2.3.29)$$

and the supereinbein

$$\mathbf{E}(\tau,\theta) = e(\tau) + \theta e^{\frac{1}{2}}(\tau)\,\chi(\tau)\,, \qquad (2.3.30)$$

where ψ is the superpartner to ω and χ is the gravitino. We also define the superderivative

$$D = \partial_\theta + \theta\partial_\tau. \qquad (2.3.31)$$

Under the local supersymmetry transformations parameterised by $V(\tau)$, the generator of reparameterisations, and $\eta(\tau)$, a Grassmann function generating pure supersymmetry transformations,

$$\tau \to \tau + V(\tau) + \theta\eta(\tau)\,; \qquad \theta \to \theta + \eta(\tau) + \frac{1}{2}\theta\dot{V}(\tau) \qquad (2.3.32)$$

the superderivative transforms homogeneously

$$D\mathbf{X} \to \Lambda(\tau,\theta)\,D\mathbf{X} \qquad (2.3.33)$$

[3]The strange looking factors of $e^{\frac{1}{2}}$ are necessary for the action given in the text to reduce to the familiar action of Brink, Di-Veccia and Howe. Another convention for the superfields exist where we scale $\psi \to e^{-\frac{1}{2}}\psi$ and $\chi \to e^{\frac{1}{2}}\chi$.

and the supereinbein transforms as

$$\mathbf{E} \to \Lambda^2(\tau,\theta)\,\mathbf{E} \tag{2.3.34}$$

where $\Lambda(\tau,\theta) = 1 + \frac{1}{2}\dot{V}(\tau) + \theta\dot{\eta}(\tau)$. Requiring the integration measure to transform as $d\tau d\theta \to \Lambda^{-1}(\tau,\theta)\, d\tau d\theta$ the following action is invariant:

$$S[\mathbf{E},\mathbf{X}] = \frac{1}{2}\int d\tau d\theta\, \mathbf{E}^{-1} D^2\mathbf{X}\cdot D\mathbf{X}. \tag{2.3.35}$$

Integrating over θ allows this to be cast in the more familiar component form[4]

$$\frac{1}{2}\int d\tau\, e^{-1}\dot{\omega}^2 + \dot{\psi}\cdot\psi - \frac{\chi}{e}\dot{\omega}\cdot\psi. \tag{2.3.36}$$

Under canonical quantisation the equations of motion for the auxiliary fields χ and e lead to the first class constraints $p\cdot\psi = 0$ and then $p^2 = 0$ respectively. Here $p^\mu = \dot{\omega}^\mu/e$ is the momentum corresponding to the field ω. On the state space the former constraint enforces the Dirac equation on physical states and the latter informs us the particle has zero mass. Below we shall pursue again the functional quantisation of this action.

We also require the supersymmetric generalisation of the interaction and the constraints on the intrinsic length of the path. The natural invariant interaction term is

$$\mathbf{I}(x) = q\int d\tau d\theta\, D\mathbf{X}\,\delta^4(\mathbf{X}-x) \tag{2.3.37}$$

Fourier decomposing the $\delta-$function and integrating over θ puts this into the form

$$q\int \frac{d^4k}{(2\pi)^4}\int d\tau\,(\dot{\omega} - e\psi ik\cdot\psi)\, e^{ik\cdot(\omega-x)} \tag{2.3.38}$$

which is analogous to the supersymmetric vertex operator familiar in the context of

[4]The supersymmetry transformations of the component fields follow from those of the superfields and (in the absence of reparameterisations) are: $\delta_\eta\omega = \eta\psi$; $\delta_\eta\psi = \frac{\eta}{e}\left(\dot{\omega} - \frac{1}{2}\chi\psi\right)$; $\delta_\eta e = \eta\chi$; $\delta_\eta\chi = 2\dot{\eta}$. Under reparameterisations, ψ is also a worldline scalar whilst χ transforms as e.

the spinning string:

$$V_k^{\mu}(\tau) = \left(\dot{\omega}^{\mu}(\tau) - e(\tau)\,\psi^{\mu}(\tau)\,ik\cdot\psi(\tau)\right)e^{ik\cdot\omega(\tau)} \tag{2.3.39}$$

Such an expression was examined in a string setting in [28, 41] but for now we continue to explore the point particle theory. We contend that the generalisation of the electric field part of the field strength tensor should be generated by a functional average of (2.3.38) with an appropriate weight. This weight will be of course that corresponding to the action in (2.3.36). Note also that the anti-commuting nature of ψ ensures that the generalised Gauss' law (2.2.7) still holds for this interaction.

The final element we need is the supersymmetric version of the constraint on path lengths. Previously we inserted $\delta\left(\int e\,d\tau - T\right)$ into the functional average but this changes under supersymmetry transformations. Introducing an arbitrary Grassmann number Ξ, the natural invariant quantity is

$$\delta\left(\int d\tau d\theta\,\mathbf{E}^{\frac{1}{2}} - \frac{1}{2}\Xi\right) \tag{2.3.40}$$

which imposes a Grassmannian constraint on χ rather than on the metric. In the massless case considered here it is not possible to construct a local function of e and χ which is supersymmetric. Instead we follow Polyakov [44] and give a superspace version of the non-local and invariant quantity he termed the superlength:

$$\delta\left(-\frac{1}{2}\int\int d\tau_1 d\theta_1\,\mathbf{E}^{\frac{1}{2}}(\tau_1,\theta_1)\,D_1 D_2 G(\tau_1,\theta_1;\tau_2,\theta_2)\,\mathbf{E}^{\frac{1}{2}}(\tau_2,\theta_2)\,d\tau_2 d\theta_2\; - T\right), \tag{2.3.41}$$

where $G(\tau_1,\theta_1;\tau_2,\theta_2) = |\tau_1 - \tau_2 - \theta_1\theta_2|$ is a superinvariant Green function and D_i is the super-derivative acting on the parameters (τ_i,θ_i). In components the first of these constraints takes the form

$$\delta\left(\frac{1}{2}\int d\tau\,\chi(\tau) - \frac{1}{2}\Xi\right) \tag{2.3.42}$$

whilst the latter can be written

$$\delta\left(\int d\tau\, e\left(\tau\right)-\frac{1}{8}\int\!\!\int d\tau_1\chi\left(\tau_1\right)\mathrm{sg}\left(\tau_1-\tau_2\right)\chi\left(\tau_2\right)d\tau_2\ -T\right) \tag{2.3.43}$$

with $\mathrm{sg}\left(\tau\right)=\frac{\tau}{|\tau|}$ equal to the sign of its argument. We shall require the superlength constraint in the functional average because it will be seen to provide the appropriate fixing of the einbein and will also impose the complementary constraint (2.3.42) which will similarly fix χ.

To continue the calculation it is necessary to determine the measure on the space of the gravitino and ψ and also to specify the boundary conditions we will use. It is well known that the purpose of the fields ψ is to represent the γ matrices, which is why it is desirable to work in a four dimensional space-time. Integrating over ψ yields an object with spinor indices and certain boundary conditions on the integral allow us to extract each component of the answer. Specifically in Appendix C of [28] we have shown that, for example,

$$\int \mathscr{D}\psi\, e^{-\int d\tau\left(\frac{1}{2}\dot{\psi}\cdot\psi+\zeta\cdot\psi\right)}\Bigg|_{\psi_2=i\psi_1;\psi_4=-i\psi_3}^{\psi_2=-i\psi_1;\psi_4=i\psi_3}=\mathscr{T}\left(e^{-\frac{1}{\sqrt{2}}\int d\tau\,\zeta\cdot\gamma}\right)_{11}. \tag{2.3.44}$$

We also showed that the volume element for χ can be written

$$\mathscr{D}\chi=d\chi_0\mathscr{D}\eta\left(\int e^{-1}d\tau\right)^{-\frac{1}{2}}\mathrm{Det}^{-\frac{1}{2}}\left(-\left(\frac{1}{e}\frac{d}{d\tau}\right)^2\right) \tag{2.3.45}$$

where χ_0 is the constant piece of χ required for consistency with the boundary conditions. This volume elements follows because χ can be expressed as a supersymmetry transformation, generated by η, plus a change proportional to e about some reference. We gauge fix by expanding about $e=\top$ – a constant – and $\chi=\chi_0$.

These are consistent choices which cover the physically distinct configurations and on this gauge slice the action becomes

$$\frac{1}{2}\int d\tau\,\frac{\dot{\omega}^2}{\top}+\dot{\psi}\cdot\psi-\frac{\chi_0}{\top}\dot{\omega}\cdot\psi \tag{2.3.46}$$

whilst the interaction vertex is given by

$$\left(\dot{\omega}^{\mu}\left(\tau\right)-\top\psi^{\mu}\left(\tau\right)ik\cdot\psi\left(\tau\right)\right)e^{ik\cdot\omega\left(\tau\right)}. \tag{2.3.47}$$

Similarly the volume elements become

$$\mathscr{D}e\mathscr{D}\chi = d\top d\chi_{0}\mathscr{D}V\mathscr{D}\eta. \tag{2.3.48}$$

The super-length constraint reduces to

$$\delta\left(\top - T\right) \tag{2.3.49}$$

and the analogous version for χ becomes

$$\delta\left(\chi_{0}-\Xi\right). \tag{2.3.50}$$

which can be used to carry out the integrals over $\top$ and χ_0. We shall again eventually take T to infinity so we can expand in powers of $\frac{1}{T}$ and we shall take the dimensionful Grassman parameter Ξ to vanish. The action and insertions are locally supersymmetric so that integrals with respect to V and η evaluate the the volumes of the reparameterisation group and supersymmetry group respectively. These constants are cancelled by their counterparts if we normalise against the bare partition function.

We pause here to derive an important result using (2.3.44). We consider the expectation value over the fermionic fields $\left\langle\psi^{\mu}\left(\tau\right)\psi^{\nu}\left(\tau\right)\right\rangle\Big|_{\alpha\beta}$ where we have attached the spinor indices α and β to the beginning and end of the worldline respectively[5]. The insertions can be generated by introducing a fermionic source η and carrying out functional differentiation, after which we set $\eta = \frac{\chi_0}{2\sqrt{2T}}\dot{\omega}$ to produce the linear

[5]We have written the fields at equal times but there is of course a time-ordering issue. We take the convention that the field to the left is understood to be evaluated at an infinitesimally greater time, ϵ, than that on the right, after which we take the limit $\epsilon \to 0^{+}$.

term in the action. Then (2.3.44) gives

$$\langle \psi^\mu(\tau)\,\psi^\nu(\tau)\rangle_{\alpha\beta} = \frac{1}{2\mathcal{Z}}\int d\chi_0\delta(\chi_0-\Xi)\,\frac{\delta}{\delta\eta^\mu(t)}\frac{\delta}{\delta\eta^\nu(t)}\mathscr{T}\left(e^{\int d\tau\,\eta\cdot\gamma}\right)_{\alpha\beta}\Big|_{\eta=\frac{\chi_0\dot{\omega}}{2\sqrt{2T}}} \tag{2.3.51}$$

where the constant $\mathcal{Z}$ is determined in the appendix. If we impose $\Xi = 0$ the result of the functional differentiation and the integral over χ_0 is

$$\langle \psi^\mu(\tau)\,\psi^\nu(\tau)\rangle = \frac{1}{2}\left(\delta^{\mu\nu}-\gamma^\mu\gamma^\nu\right) \tag{2.3.52}$$

which crucially is independent of the field ω, since by setting $\Xi = 0$ we have decoupled the fields ω and ψ in the action. This result is consistent with the symmetry properties of the Grassman fields and will play a key role in the forthcoming calculations.

To return to (2.3.38) we carry out the integral over $\top$ and are left with the gauge fixed Fourier space expectation value

$$\begin{aligned}\langle I^\mu(k)\rangle &= \frac{q}{\mathcal{Z}}\int \mathscr{D}\omega\mathscr{D}\psi d\chi_0\delta(\chi_0-\Xi)\int_0^1 d\tau_1\left(\dot{\omega}^\mu - T\psi^\mu ik\cdot\psi\right)e^{ik\cdot\omega}e^{-\int_0^1 d\tau\frac{\dot{\omega}^2}{2T}+\frac{1}{2}\dot{\psi}\cdot\psi-\frac{\chi_0}{2T}\dot{\omega}\cdot\psi}\\ &= \frac{q}{\mathcal{Z}'}\int\mathscr{D}\omega\int_0^1 d\tau_1\left(\dot{\omega}^\mu-\frac{Ti}{2}\left(k^\mu-\gamma^\mu k\cdot\gamma\right)\right)e^{ik\cdot\omega}e^{-\int_0^1 d\tau\frac{\dot{\omega}^2}{2T}}\end{aligned} \tag{2.3.53}$$

where for the last line we have integrated over ψ to produce the gamma matrices and integrated over χ_0 before putting $\Xi = 0$. It remains to determine the expectation values of these integrands and carry out the integral over τ_1. The integral over ω can be found to give

$$q\int_0^1 d\tau_1\left(\dot{\omega}_c^\mu(\tau_1)-\frac{Ti}{2}\left(k^\mu-\gamma^\mu k\cdot\gamma\right)-\frac{1}{2}ik^\mu\frac{\partial}{\partial\tau_1}G(\tau_1,\tau_1)\right)e^{-\frac{1}{2}k^2G(\tau_1,\tau_1)}e^{ik\cdot\omega_c(\tau_1)} \tag{2.3.54}$$

Making further use of the redefinition $G(\tau_1,\tau_2) = T\tilde{G}(\tau_1,\tau_2)$ this can be cast in the form

$$q\int_0^1 d\tau_1 \left(\dot{\omega}_c^\mu(\tau_1) - \frac{Ti}{2}\left(k^\mu - \gamma^\mu k\cdot\gamma\right) - \frac{T}{2}ik^\mu \frac{\partial}{\partial\tau_1}\tilde{G}(\tau_1,\tau_1)\right) e^{-\frac{1}{2}k^2T\tilde{G}(\tau_1,\tau_1)} e^{ik\cdot\omega_c(\tau_1)} \tag{2.3.55}$$

From the exponent $\exp\left(-\frac{1}{2}k^2T\tilde{G}(\tau_1,\tau_1)\right)$ it is easy to understand that the latter two terms in brackets will contribute at leading order in $\frac{1}{T}$ and that the first term will once again be sub-leading. In the limit of large T we can follow the calculation in the bosonic theory and approximate the integral by evaluating the classical path ω_c on each boundary and then integrating a short distance, h, along the worldline. We calculate

$$\begin{aligned} &q\int_0^h d\tau_1 \left(\dot{\omega}_c^\mu(0) - \frac{Ti}{2}\left(k^\mu - \gamma^\mu k\cdot\gamma\right) - \frac{T}{2}ik^\mu \frac{\partial}{\partial\tau_1}\tilde{G}(\tau_1,\tau_1)\right) e^{-\frac{1}{2}k^2T\tilde{G}(\tau_1,\tau_1)} e^{ik\cdot\omega_c(0)} \\ &+q\int_{1-h}^1 d\tau_1 \left(\dot{\omega}_c^\mu(1) - \frac{Ti}{2}\left(k^\mu - \gamma^\mu k\cdot\gamma\right) - \frac{T}{2}ik^\mu \frac{\partial}{\partial\tau_1}\tilde{G}(\tau_1,\tau_1)\right) e^{-\frac{1}{2}k^2T\tilde{G}(\tau_1,\tau_1)} e^{ik\cdot\omega_c(1)} \end{aligned} \tag{2.3.56}$$

and as before the suppression caused by the exponent at large T allows us to extend the integrands by setting $h = 1/2$. Carrying out the integrals leads to

$$\langle I^\mu(k)\rangle = q\left[\frac{i}{k^2}\gamma^\mu k\cdot\gamma\left(e^{ik\cdot a} - e^{ik\cdot b}\right) + \mathcal{O}\left(\frac{1}{k^4T}\right)\right]. \tag{2.3.57}$$

In the limit $T\to\infty$ only the first term contributes and so we have determined

$$\langle I(k)\rangle = \frac{iq}{k^2}\,\gamma\, k\cdot\gamma\left(e^{ik\cdot a} - e^{ik\cdot b}\right). \tag{2.3.58}$$

We may double check our work by contracting with k_μ to verify against (2.2.7) that the index structure is correct. In position space the expression above becomes

$$\langle I(k)\rangle = \frac{q}{4\pi^2}\gamma\,\gamma\cdot\nabla\left(\frac{1}{|x-a|^2} - \frac{1}{|x-b|^2}\right) \tag{2.3.59}$$

We conclude this section by commenting on how this approach could be applied to a three dimensional space. The main obstacle to this case is the structure of (2.3.44) which imposes boundary conditions relating pairs of the ψ. Furthermore the proof of this equation presented in [28] does not hold for the three dimensional version of the Clifford algebra because the chirality operator does not anti-commute with the other matrices so the trace of a product of an odd number of gamma matrices does not in general vanish. We present two alternatives to address this issue. The first is to use the symmetry of the problem to choose a coordinate system such that the two charges are placed on the $z = 0$ plane and to restrict our attention to determine the field only on this plane. Then we need consider a super-field, $\boldsymbol{\omega}$, which consists of only two components and the fields ψ essentially become the σ-matrices σ_1 and σ_2:

$$\begin{aligned}\left\langle I^i\left(\mathbf{k}\right)\right\rangle &= \frac{q}{\mathcal{Z}}\int \mathscr{D}\boldsymbol{\omega}\mathscr{D}\psi d\chi_0 \int_0^1 d\tau_1\left(\dot{\omega}^i - T\psi^i i\mathbf{k}\cdot\boldsymbol{\psi}\right)e^{i\mathbf{k}\cdot\boldsymbol{\omega}}e^{-\int_0^1 d\tau\frac{\dot{\boldsymbol{\omega}}^2}{2T}+\dot{\boldsymbol{\psi}}\cdot\boldsymbol{\psi}+\frac{\chi_0}{T}\dot{\boldsymbol{\omega}}\cdot\boldsymbol{\psi}}\\ &= \frac{q}{\mathcal{Z}'}\int \mathscr{D}\boldsymbol{\omega}\int_0^1 d\tau_1\left(\dot{\omega}^i - \frac{Ti}{2}\left(k^i - \sigma^i\mathbf{k}\cdot\boldsymbol{\sigma}\right)\right)e^{i\mathbf{k}\cdot\boldsymbol{\omega}}e^{-\int_0^1 d\tau\frac{\dot{\boldsymbol{\omega}}^2}{2T}}\end{aligned} \tag{2.3.60}$$

where $i \in \{1, 2\}$. Carrying out the integral over the two-dimensional field $\boldsymbol{\omega}$ and following the same steps as the four dimensional case leads to

$$\left\langle\mathbf{I}\left(\mathbf{k}\right)\right\rangle = \frac{iq}{k^2}\boldsymbol{\sigma}\,\mathbf{k}\cdot\boldsymbol{\sigma}\left(e^{i\mathbf{k}\cdot\mathbf{a}} - e^{i\mathbf{k}\cdot\mathbf{b}}\right). \tag{2.3.61}$$

It is more satisfactory to instead modify the four dimensional theory in such a way that statements can be made for the three dimensional case. This can be done in two equivalent ways. The three dimensional version of the vertex operator is

$$\mathbf{V}_k = \left(\dot{\boldsymbol{\omega}} - e\boldsymbol{\psi}i\mathbf{k}\cdot\boldsymbol{\psi}\right)e^{i\mathbf{k}\cdot\boldsymbol{\omega}}. \tag{2.3.62}$$

We could use this as an insertion in the four dimensional theory by defining in some inertial frame $I^\mu = (0, \mathbf{I})$ where the 3-vector $\mathbf{I}$ is constructed out of the three dimensional vertex operator above. Then (2.3.44) continues to hold except that the part of the expectation value which does not cancel with $\mathcal{Z}$ contains only a three dimensional source so the right hand side of that equation becomes

$$\mathscr{T}\left(e^{-\frac{1}{\sqrt{2}}\int d\tau \boldsymbol{\zeta}\cdot\boldsymbol{\gamma}}\right). \tag{2.3.63}$$

This is equivalent to beginning with an entirely three dimensional theory but introducing a further pair of fields ω_0, ψ_0. Into the integral of some functional of the three dimensional fields we introduce a supersymmetric invariant factor in the numerator and denominator as follows:

$$\frac{1}{\mathcal{Z}_3}\int \mathscr{D}\boldsymbol{\omega}\mathscr{D}\boldsymbol{\psi}d\chi_0\Omega\left(\boldsymbol{\omega},\boldsymbol{\psi}\right)e^{-\int_0^1 d\tau\, \frac{\dot{\boldsymbol{\omega}}}{2T}+\frac{1}{2}\dot{\boldsymbol{\psi}}\cdot\boldsymbol{\psi}-\frac{\chi_0}{2T}\dot{\boldsymbol{\omega}}\cdot\boldsymbol{\psi}}\frac{\int \mathscr{D}\omega_0\mathscr{D}\psi_0 e^{-\int_0^1 d\tau\, \frac{\dot{\omega}_0}{2T}+\frac{1}{2}\dot{\psi}_0\cdot\psi_0-\frac{\chi_0}{2T}\dot{\omega}_0\cdot\psi_0}}{\int \mathscr{D}\omega_0\mathscr{D}\psi_0 e^{-\int_0^1 d\tau\, \frac{\dot{\omega}_0}{2T}+\frac{1}{2}\dot{\psi}_0\cdot\psi_0-\frac{\chi_0}{2T}\dot{\omega}_0\cdot\psi_0}}. \tag{2.3.64}$$

We then combine the denominator of the fraction with the three dimensional normalisation $\mathcal{Z}_3$ to form

$$\int \mathscr{D}\,\omega\mathscr{D}\psi d\tilde{\chi}_0\; e^{-\int_0^1 d\tau\, \frac{\dot{\omega}}{2T}+\frac{1}{2}\dot{\psi}\cdot\psi-\frac{\tilde{\chi}_0}{2T}\dot{\boldsymbol{\omega}}\cdot\boldsymbol{\psi}-\frac{\chi_0}{2T}\dot{\omega}_0\cdot\psi_0} \tag{2.3.65}$$

where the integration is now over four bosonic and four fermionic fields. We apply the boundary conditions for open paths to both the numerator and denominator of (2.3.64). The integration in (2.3.65) produces an expression dependent on χ_0 which feeds back into (2.3.64). Carrying out the integral over χ_0 and the four integrations with respect to ψ produces two terms which conspire to yield

$$\frac{1}{\mathcal{Z}'}\int \mathscr{D}\omega\;\Omega\left(\boldsymbol{\omega},\boldsymbol{\gamma}\right)e^{-\int_0^1 d\tau \frac{\dot{\omega}^2}{2T}} \tag{2.3.66}$$

where the normalisation $\mathcal{Z}'$ is precisely the correct factor to ensure that $\langle 1\rangle = 1$. This is what we would calculate if we were to take the expectation of $I^\mu = (0, \mathbf{I})$ as defined above in a theory with four pairs of fields. It is easy to modify (2.3.58) to respect this change:

$$\langle \mathbf{I}\left(\mathbf{k}\right)\rangle = \frac{iq}{k^2}\boldsymbol{\gamma}\,\mathbf{k}\cdot\boldsymbol{\gamma}\left(e^{i\mathbf{k}\cdot\mathbf{a}} - e^{i\mathbf{k}\cdot\mathbf{b}}\right) \tag{2.3.67}$$

which is to be integrated with respect to the three-vector $\mathbf{k}$. We may choose the representation

$$\gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix}, \tag{2.3.68}$$

to show that this approach involves the Pauli matrices in a similar way to (2.3.61), but now we need four-index spinors. With this representation $\boldsymbol{\gamma}\,\mathbf{k}\cdot\boldsymbol{\gamma}$ is block diagonal with the two blocks taking the same form as (2.3.61), differing from one another by a sign.

2.4 Analysis at finite T

The classical fields were found in the limit that the dimensionful parameter T was taken to be large compared to momenta. An interesting question is to ask about the form of the statistical average for finite T. In this section we give the subleading correction to the fields at large T and also consider the opposing limit $T \to 0$. We shall do so for the lowest order interaction which generates the classical fields.

2.4.1 Corrections in the bosonic case

We first analyse the bosonic particle with mixed boundary conditions whose high T limit produced the classical electric field of a point particle (2.2.22). That equation was exact in T and its position space form is found by carrying out the inverse Fourier transform. We shall specialise here to the three physical spatial dimensions. Aligning the z-axis parallel to the spatial separation $\mathbf{x} - \mathbf{a}$ the angular integrals of (2.2.22) can be carried out to produce

$$\begin{aligned} \langle \mathbf{I}(\mathbf{x}) \rangle_T &= -q\boldsymbol{\nabla} \int_0^\infty \frac{dk}{(2\pi)^2} 2 \frac{\sin(k\,|\mathbf{x}-\mathbf{a}|)}{k\,|\mathbf{x}-\mathbf{a}|} \left(1 - e^{-\frac{1}{2}k^2 T}\right) \\ &= -q\boldsymbol{\nabla} \frac{1}{4\pi\,|\mathbf{x}-\mathbf{a}|} \left(1 - \mathrm{Erf}\left(\sqrt{\frac{|\mathbf{x}-\mathbf{a}|^2}{2T}}\right)\right). \end{aligned} \tag{2.4.69}$$

At large T this can be expanded in powers of $\frac{1}{T}$ and we find

$$\langle \mathbf{I}(\mathbf{x}) \rangle_T = -\boldsymbol{\nabla} \frac{q}{4\pi} \left(\frac{1}{|\mathbf{x}-\mathbf{a}|} - 2\sqrt{\frac{1}{2\pi T}} + \frac{1}{3}\sqrt{\frac{1}{2\pi T}} \frac{|\mathbf{x}-\mathbf{a}|^2}{T} + \mathcal{O}\left(T^{-\frac{5}{2}}\right) \right) \quad (2.4.70)$$

valid for $|\mathbf{x}-\mathbf{a}|^2 / T << 1$. The leading order correction arises from the third term in brackets:

$$\langle \mathbf{I}(\mathbf{x}) \rangle_T = \frac{q}{4\pi} \frac{\mathbf{x}-\mathbf{a}}{|\mathbf{x}-\mathbf{a}|^3} - \frac{q}{6\pi}\sqrt{\frac{1}{2\pi}} \frac{\mathbf{x}-\mathbf{a}}{T^{\frac{3}{2}}} + \mathcal{O}\left(T^{-\frac{5}{2}}\right) \quad (2.4.71)$$

giving the finite T deviation from the inverse square law.

The other limit of interest is at small T so we consider an expansion of (2.4.69) about $T = 0$. The change in functional form is more dramatic because we find

$$\langle \mathbf{I}(\mathbf{x}) \rangle_T = -\frac{q}{2} \boldsymbol{\nabla} \frac{1}{\sqrt{2\pi}} \left(\frac{\sqrt{T}}{|\mathbf{x}-\mathbf{a}|^2} - \frac{T^{\frac{3}{2}}}{|\mathbf{x}-\mathbf{a}|^4} + \mathcal{O}\left(T^{\frac{5}{2}}\right) \right) \quad (2.4.72)$$

which carries a different power of the spatial separation. Carrying out the differentiation we acquire

$$\langle \mathbf{I}(\mathbf{x}) \rangle_T = \frac{q}{4\pi}\sqrt{2\pi} \left(\frac{2\sqrt{T}\,(\mathbf{x}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|^4} - \frac{4T^{\frac{3}{2}}\,(\mathbf{x}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|^6} + \mathcal{O}\left(T^{\frac{5}{2}}\right) \right) \quad (2.4.73)$$

in the limit that $|\mathbf{x}-\mathbf{a}|^2 / T >> 1$.

The corrections to the dipole field are less trivial to determine because of the time dependence in $\boldsymbol{\omega}_c$. For large T the integrand is still dominated by contributions at either end of the curve. We first expand about $\tau = 0$:

$$e^{i\mathbf{k}\cdot\boldsymbol{\omega}_c(\tau_1)} = e^{i\mathbf{k}\cdot\mathbf{a}} \left(1 + i\mathbf{k}\cdot(\mathbf{b}-\mathbf{a})\,\tau_1 - \frac{k^2}{2}\left(\mathbf{k}\cdot(\mathbf{b}-\mathbf{a})\right)^2 \tau_1^2 + \ldots \right). \quad (2.4.74)$$

When integrated against $\exp\left(-\frac{1}{2}k^2 T G(\tau_1, \tau_1)\right)$ each extra power of τ_1 results in a further power of $\frac{1}{k^2 T}$. The first term in the square brackets of (2.2.14) is already subleading in $\frac{1}{T}$ so its contribution to the leading order correction comes form the first term in (2.4.74). Integrating this from the boundary to $\tau_1 = \frac{1}{2}$ gives

$$\frac{-2q\left(\mathbf{b}-\mathbf{a}\right)}{k^2 T} e^{i\mathbf{k}\cdot\mathbf{a}} + \mathcal{O}\left(T^{-2}\right) \tag{2.4.75}$$

plus corrections exponentially suppressed at large $k^2 T$. At first order the second term of (2.2.14) provided the classical dipole field and its leading order correction arises from the $\mathcal{O}\left(\tau_1\right)$ term in (2.4.74). Straightforward integration of this evaluates to

$$\frac{2q\mathbf{k}\,\mathbf{k}\cdot\left(\mathbf{b}-\mathbf{a}\right)}{k^4 T} e^{i\mathbf{k}\cdot\mathbf{a}} + \mathcal{O}\left(T^{-2}\right) \tag{2.4.76}$$

up to further terms which are again exponentially suppressed. It is instructive to combine these two terms as follows: the ith component is given by

$$-\frac{2q}{T}\left[\frac{\delta^{ij}}{k^2} - \frac{k^i k^j}{k^4}\right]\left(b-a\right)^j e^{i\mathbf{k}\cdot\mathbf{a}} \tag{2.4.77}$$

which highlights the transverse nature of the correction. The index structure of (2.4.76) means the integral with respect to $\mathbf{k}$ has the form

$$\int \frac{d^3\mathbf{k}}{\left(2\pi\right)^3}\frac{k^i k^j}{k^4} e^{i\mathbf{k}\cdot\left(\mathbf{a}-\mathbf{x}\right)} = A\delta^{ij} + B\frac{\left(a-x\right)^i\left(a-x\right)^j}{\left|\mathbf{a}-\mathbf{x}\right|^2} \tag{2.4.78}$$

for some constants A and B of dimension $\left[\text{length}\right]^{-1}$. They can be determined by contracting each side of the above equation first with δ_{ij} and also with $\left(a-x\right)_i\left(a-x\right)_j$. The first of these gives

$$3A + B = \int \frac{d^3\mathbf{k}}{\left(2\pi\right)^3}\frac{e^{i\mathbf{k}\cdot\left(\mathbf{a}-\mathbf{x}\right)}}{k^2} = \frac{-1}{4\pi\left|\mathbf{x}-\mathbf{a}\right|} \tag{2.4.79}$$

and the second yields

$$\left|a-x\right|^2\left(A+B\right) = \int \frac{d^3\mathbf{k}}{\left(2\pi\right)^3}\frac{\left(\mathbf{k}\cdot\left(\mathbf{a}-\mathbf{x}\right)\right)^2}{k^4} e^{i\mathbf{k}\cdot\left(\mathbf{a}-\mathbf{x}\right)}. \tag{2.4.80}$$

Choosing the z-axis to align with $\mathbf{x}-\mathbf{a}$ again allows the angular integrals to be done and we find that the remaining integral with respect to the magnitude of $\mathbf{k}$ is proportional to

$$\frac{1}{(2\pi)^2}\int_0^\infty dk\left[\frac{\sin(k\,|\mathbf{x}-\mathbf{a}|)}{k\,|\mathbf{x}-\mathbf{a}|}+\frac{2\cos(k\,|\mathbf{x}-\mathbf{a}|)}{(k\,|\mathbf{x}-\mathbf{a}|)^2}-\frac{2\sin(k\,|\mathbf{x}-\mathbf{a}|)}{(k\,|\mathbf{x}-\mathbf{a}|)^3}\right] \tag{2.4.81}$$

Integrating the second term in square brackets by parts once and the final term by parts twice serves to cancel the first term. It is then easy to check that the integral evaluates to zero so $A+B=0$ and (2.4.78) evaluates to

$$\frac{-1}{8\pi\,|\mathbf{x}-\mathbf{a}|}\left[\delta^{ij}-\frac{(a-x)^i\,(a-x)^j}{|\mathbf{a}-\mathbf{x}|^2}\right]. \tag{2.4.82}$$

So (2.4.76) evaluates to

$$\frac{q}{4\pi T\,|\mathbf{x}-\mathbf{a}|}\left[(\mathbf{b}-\mathbf{a})+(\mathbf{x}-\mathbf{a})\,\frac{(\mathbf{x}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|^2}\right]+\mathcal{O}\left(T^{-2}\right), \tag{2.4.83}$$

to which it remains to add the contribution with the analogous calculation for the other end of the curve fixed to the point $\mathbf{b}$. The final correction at order $\frac{1}{T}$ is determined to be

$$\begin{aligned}&\frac{q}{4\pi T\,|\mathbf{x}-\mathbf{a}|}\left[(\mathbf{b}-\mathbf{a})+(\mathbf{x}-\mathbf{a})\,\frac{(\mathbf{x}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|^2}\right]\\-&\frac{q}{4\pi T\,|\mathbf{x}-\mathbf{b}|}\left[(\mathbf{b}-\mathbf{a})+(\mathbf{x}-\mathbf{b})\,\frac{(\mathbf{x}-\mathbf{b})\cdot(\mathbf{b}-\mathbf{a})}{|\mathbf{x}-\mathbf{b}|^2}\right]\end{aligned} \tag{2.4.84}$$

and is easy to check that this is divergence free. This is the variation from the classical dipole field for large values of T which is present in this model. It is interesting to note that with the relaxation of the limit to large but finite T comes dependence on the direction $(\mathbf{b}-\mathbf{a})$ which is independent of the spatial point in question. In Fig 2.1 we provide an example of the deviation from the well-known dipole field for finite T by plotting the streamlines of the electric field.

We finally turn to the low T limit of the dipole field for which it is more convenient to carry out the integral with respect to $\mathbf{k}$ of (2.2.14), before looking at an expansion in powers of T. Here we shall see more striking dependence on the direction of separation between the two charges. The result of carrying out the inverse Fourier

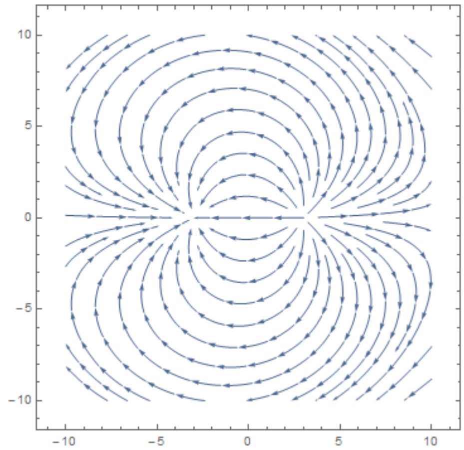


Figure 2.1: The field lines for large but finite T highlighting the small deviation from the classical dipole field. In this plot we set $T = 10\,|\mathbf{b}-\mathbf{a}|^2$.

transform is

$$\langle \mathbf{I}(\mathbf{x})\rangle_T = \frac{-q}{(2\pi)^{\frac{3}{2}}}\int_0^1 \frac{d\tau_1}{\left(T\tilde{G}(\tau_1,\tau_1)\right)^{\frac{3}{2}}}\left[\frac{\dot{\tilde{G}}(\tau_1,\tau_1)}{2\tilde{G}(\tau_1,\tau_1)}(\mathbf{x}-\boldsymbol{\omega}_c)-\dot{\boldsymbol{\omega}}_c\right]\exp\left(-\frac{(\mathbf{x}-\boldsymbol{\omega}_c)^2}{2T\tilde{G}(\tau_1,\tau_1)}\right) \tag{2.4.85}$$

Examining the T-dependence of this expression we see that the limit $T \to 0$ provides a representation of the δ function so that the field is supported only on the classical straight line path joining $\mathbf{a}$ to $\mathbf{b}$:

$$\begin{aligned}&\delta^3\left(\int_0^1 d\tau_1\ (\mathbf{x}-\boldsymbol{\omega}_c(\tau_1))\right)\\ &=\delta^3\left(\int_0^1 d\tau_1\ (\mathbf{x}-(\mathbf{a}+(\mathbf{b}-\mathbf{a})\tau_1))\right).\end{aligned} \tag{2.4.86}$$

To determine the form of the field at finite T we shall employ a Laplace approximation. For small T the contribution to the integrand of (2.4.85) is concentrated about the positions of the maxima of the exponent $\exp\left(\frac{(\mathbf{x}-\boldsymbol{\omega}(\tau_1))^2}{2T\tilde{G}(\tau_1,\tau_1)}\right)$. The precise version of Laplace's method we shall invoke is that for small T and arbitrary well-behaved functions $f(\tau)$ and $g(\tau)$

$$\int_0^1 d\tau f(\tau)\, e^{-\frac{1}{T}g(\tau)} = \sum_{\tau_0}\sqrt{\frac{2\pi T}{\ddot{g}(\tau_0)}} f(\tau_0)\, e^{-\frac{1}{T}g(\tau_0)}\left(1+\mathcal{O}(T)\right) \tag{2.4.87}$$

where the $\tau_0 \in [0,1]$ are determined by the condition that $g(\tau_0)$ be a maximum. In this case a straightforward calculation shows that the exponent of(2.4.85) attains a single maximum within the integration range at[6]

$$\tau_0 = \frac{|\mathbf{x}-\mathbf{a}|}{|\mathbf{x}-\mathbf{a}|+|\mathbf{x}-\mathbf{b}|}. \tag{2.4.88}$$

If the spatial point $\mathbf{x}$ lies on the line joining $\mathbf{a}$ to $\mathbf{b}$ then the exponent vanishes at this value of τ_1, in agreement with the $T \to 0$ limit which provides δ-function support on this line. Furthermore the first term in square brackets of (2.4.85) vanishes because $\mathbf{x} - \boldsymbol{\omega}_c(\tau_0) = 0$. In this case the field is approximated at lowest order in T by

$$\begin{aligned} &\frac{q}{2\pi T \tilde{G}(\tau_0,\tau_0)} \frac{(\mathbf{b}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}|} \\ &= \frac{q}{2\pi T} \frac{|\mathbf{b}-\mathbf{a}|}{|\mathbf{x}-\mathbf{a}|\,|\mathbf{x}-\mathbf{b}|}(\mathbf{b}-\mathbf{a}) \end{aligned} \tag{2.4.89}$$

which has its minimum half way between the charges, where its magnitude is

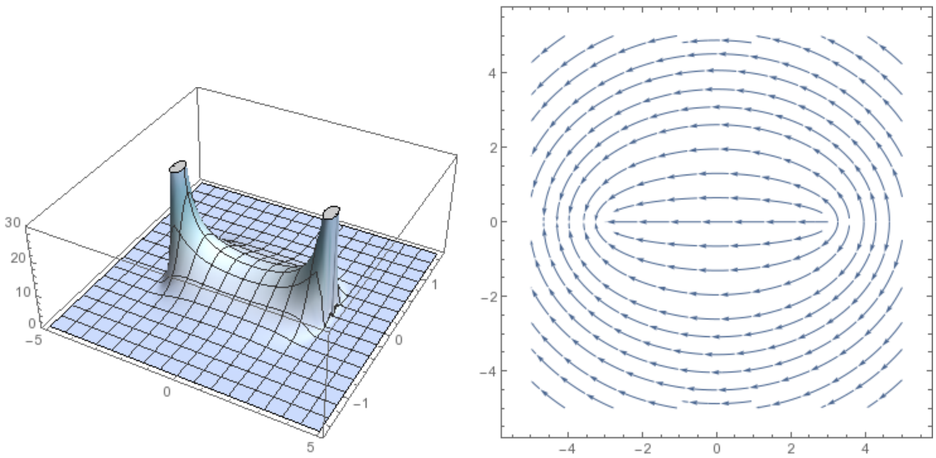
$$\frac{2q}{\pi T} \tag{2.4.90}$$

Away from this line the exponent in (2.4.87) enforces an exponential decay in the magnitude of the field. Indeed we find

$$\begin{aligned} \frac{(\mathbf{x}-\boldsymbol{\omega}_c(\tau_0))^2}{2\tilde{G}(\tau_0,\tau_0)} = \frac{1}{2}\Bigg[&|\mathbf{x}-\mathbf{a}|\,|\mathbf{x}-\mathbf{b}|\,(|\mathbf{x}-\mathbf{a}|+|\mathbf{x}-\mathbf{b}|)^2 \\ &-2(\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{b})\frac{|\mathbf{x}-\mathbf{a}|+|\mathbf{x}-\mathbf{b}|}{|\mathbf{x}-\mathbf{b}|} + |\mathbf{x}-\mathbf{a}|\,|\mathbf{x}-\mathbf{b}|\Bigg] \end{aligned} \tag{2.4.91}$$

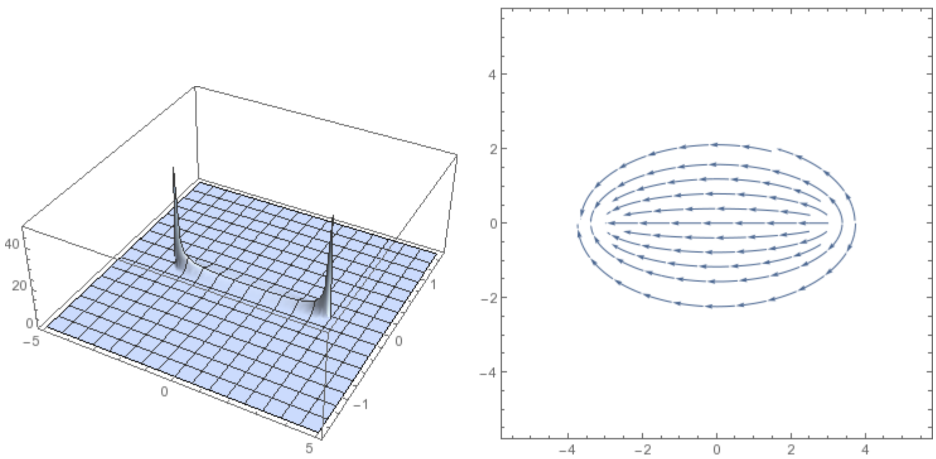
and the direction of the field depends on the spatial point through the first term in (2.4.85). It is possible to use (2.4.87) to determine the contribution at an arbitrary spatial point but the result is less illuminating than a visual representation of the field lines and the field magnitude. It is most useful to plot the streamlines, tangent

[6]This expression is consistent with the behaviour of the system under exchange of $\mathbf{a}$ and $\mathbf{b}$ because this is equivalent to sending $\tau \to 1-\tau$.



(a) The magnitude of the electric field (b) The stream lines of the electric field

Figure 2.2: Field magnitude and field streamlines in the low T limit – in this plot $T = \frac{1}{60}\left|\mathbf{b}-\mathbf{a}\right|^2$.



(a) The magnitude of the electric field (b) The stream lines of the electric field

Figure 2.3: Field magnitude and field streamlines in the low T limit – in this plot $T = \frac{1}{1000}\left|\mathbf{b}-\mathbf{a}\right|^2$. The field decays exponentially according to (2.5.140) which gives rise to the white-space in which the field is negligibly small.

to the field at each spatial point. Fig. 2.2 and Fig 2.3 show the field strength and direction on the $z = 0$ plane of a pair of oppositely charged particles placed at positions $\mathbf{a} = (3, 0, 0)$ and $\mathbf{b} = (-3, 0, 0)$ for two values of the parameter T. We have imposed a sharp cut-off about the positions of the charges to avoid the divergence encountered there. The form of these field configurations suggests that the low T limit of this theory gives some sort of confining field, albeit not one with a potential linear in the separation of the charges. In this way T interpolates between the classical field associated to a pair of charges and a regime in which field lines

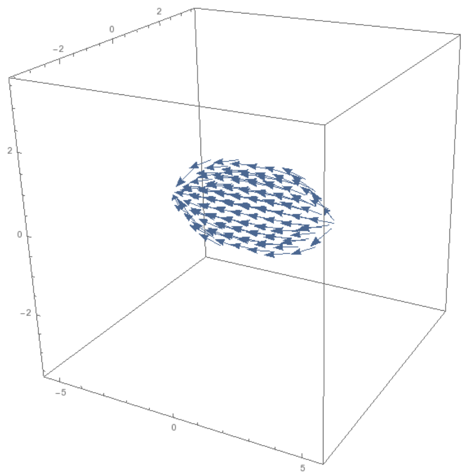


Figure 2.4: Confining field lines in the low T limit – here $T = \frac{1}{500}\left|\mathbf{b}-\mathbf{a}\right|^2$.

are concentrated about the line joining the charges. The three dimensional plot in Fig. 2.4 highlights how the low T lines of flux are compressed into a thin tube. This completes our discussion of the correction to the bosonic theory at finite T.

2.4.2 Corrections in the fermionic case

In this case we shall consider four dimensional space-time and will briefly state the result of analysing the order $\frac{1}{T}$ corrections to (2.3.58). The general approach is identical to that of the bosonic theory. We have already determined the momentum space correction to the first and last terms in (2.3.55) in the previous section. For the contribution corresponding to the end of the curve at a^μ the μth component is given by

$$\frac{2q}{T}\left[\frac{\delta^{\mu\nu}}{k^2} - \frac{k^\mu k^\nu}{k^4}\right](b-a)^\nu e^{ik\cdot a}. \tag{2.4.92}$$

We also find a subleading contribution from the middle term in (2.3.55) which takes the form

$$\frac{2q}{T}\left[\delta^{\mu\alpha} - \gamma^\mu\gamma^\alpha\right]\frac{k^\alpha k^\nu}{k^4}(b-a)^\nu e^{ik\cdot a} \tag{2.4.93}$$

so now we must determine the four dimensional integral

$$\int \frac{d^4k}{(2\pi)^4}\frac{k^\mu k^\nu}{k^4}e^{ik\cdot a-x}. \tag{2.4.94}$$

Symmetry dictates it must be equal to $A\delta^{\mu\nu} + B\,(a-x)^{\mu}\,(a-x)^{\nu}\,|a-x|^{-2}$ where the constants A and B have dimension [length]$^{-2}$. Proceeding as before we contract with $\delta^{\mu\nu}$ and $(a-x)^{\mu}\,(a-x)^{\nu}$ to produce two equations. The first gives

$$4A + B = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(a-x)}}{k^2} = \frac{-1}{4\pi^2\,|x-a|^2} \tag{2.4.95}$$

and the second leads to

$$|a-x|^2\,(A+B) = \int \frac{d^4k}{(2\pi)^4}\frac{(k\cdot(a-x))^2}{k^4}e^{ik\cdot(a-x)}. \tag{2.4.96}$$

This integral can be carried out by first doing the integral with respect to k_0, where double poles are encountered at $k_0 = \pm i\,|\mathbf{k}|$. The remaining integral over the three dimensional vector $\mathbf{k}$ can be done by choosing the z-axis to align with $\mathbf{x}-\mathbf{a}$ as above. Then we are left to determine

$$\frac{\pi}{2}\int_0^{\infty}\frac{dk}{(2\pi)^3}\int_{-1}^{1} d\,(\cos\theta)\,ke^{-k|x_0|+ik|\mathbf{x}|\cos\theta}\bigg[|x_0|^2\,(k\,|x_0|-1) - 2ik\,|\mathbf{x}|\,|x_0|^2\cos\theta \\ - |\mathbf{x}|^2\,(k\,|x_0|+1)\cos^2\theta\bigg] \tag{2.4.97}$$

Integrating over θ leaves only

$$\frac{\pi}{2}\int_0^{\infty}\frac{dk}{(2\pi)^3}e^{-k|x_0|}\bigg[|x_0|^2\,(k\,|x_0|-1)\frac{\sin k\,|\mathbf{x}|}{k\,|\mathbf{x}|} + 2\,|x_0|^2\left(\frac{\sin k\,|\mathbf{x}|}{k\,|\mathbf{x}|} - \cos k\,|\mathbf{x}|\right) \\ - |\mathbf{x}|^2\,(k\,|x_0|+1)\left(\frac{\sin k\,|\mathbf{x}|}{k\,|\mathbf{x}|} + 2\frac{\cos k\,|\mathbf{x}|}{k^2\,|\mathbf{x}|^2} - 2\frac{\sin k\,|\mathbf{x}|}{k^3\,|\mathbf{x}|^3}\right)\bigg]. \tag{2.4.98}$$

Carrying out the k-integral several times by parts yields $|x-a|^2\,(A+B) = \frac{1}{8\pi^2}$ so (2.4.94) evaluates to

$$\frac{-1}{8\pi^2\,|x-a|^2}\left[\delta^{\mu\nu} - 2\frac{(a-x)^{\mu}\,(a-x)^{\nu}}{|x-a|^2}\right]. \tag{2.4.99}$$

So in position space (2.4.92) becomes

$$\frac{-q}{4\pi^2 T\,|x-a|^2}\left[(b-a)^\mu + 2\,(x-a)^\mu\,\frac{(x-a)\cdot(b-a)}{|x-a|^2}\right], \tag{2.4.100}$$

for which it can be verified that the divergence vanishes, and (2.4.93) becomes

$$\frac{-q}{4\pi^2 T\,|x-a|^2}\left(\delta^{\mu\nu}-\gamma^\mu\gamma^\nu\right)\left[(b-a)^\nu - 2\,(x-a)^\nu\,\frac{(x-a)\cdot(x-a)}{|x-a|^2}\right] \tag{2.4.101}$$

which is also divergence free. The first order correction is found by subtracting these and including the contribution from the other end of the curve. We find at order $\frac{1}{T}$ the expectation value of the insertion evaluates to

$$\begin{aligned}\langle I^\mu(x)\rangle_T =& \frac{q}{4\pi^2 T\,|x-a|^2}\gamma^\mu\,\gamma\cdot\left[(b-a) - 2\,(x-a)\,\frac{(x-a)\cdot(x-a)}{|x-a|^2}\right]\\ &-\frac{q}{4\pi^2 T\,|x-b|^2}\gamma^\mu\,\gamma\cdot\left[(b-a) - 2\,(x-b)^\nu\,\frac{(x-b)\cdot(x-b)}{|x-b|^2}\right]\end{aligned} \tag{2.4.102}$$

The form of this correction has a similar functional form to that of the bosonic particle discussed above.

We may also ask about the low T expansion to determine the behaviour of the system in this regime. It is again useful to carry out the integral over k first to arrive at

$$\begin{aligned}\langle I^\mu(x)\rangle_T = \frac{-q}{(2\pi)^2}\int_0^1 \frac{d\tau_1}{\left(T\tilde{G}(\tau_1,\tau_1)\right)^2}&\left[\frac{\dot{\tilde{G}}(\tau_1,\tau_1)}{2\tilde{G}(\tau_1,\tau_1)}(x-\omega_c)^\mu - \dot{\omega}_c^\mu\right.\\ &\left.-\frac{(x-\omega_c)^\nu}{2\tilde{G}(\tau_1,\tau_1)}\left(\delta^{\mu\nu}-\gamma^\mu\gamma^\nu\right)\right]\exp\left(-\frac{(x-\omega_c)^2}{2T\tilde{G}(\tau_1,\tau_1)}\right).\end{aligned} \tag{2.4.103}$$

In the limit that T vanishes we have a representation of the four dimensional $\delta-$function supported on the straight line joining the charges. Applying Laplace's approximation allows us to determine the leading order behaviour which we illustrate for a point x on the line from a to b. In this case we clearly find the maximum at the point where $\omega_c(\tau_0) = x$ (τ_0 remains unchanged from (2.4.88)) so the first and last terms vanish. Explicit application of (2.4.87) results in

$$\begin{aligned} & \frac{q}{\left(2\pi T\tilde{G}\left(\tau_0,\tau_0\right)\right)^{\frac{3}{2}}}\frac{b-a}{|b-a|} \\ =& \frac{q}{(2\pi T)^{\frac{3}{2}}}\frac{|b-a|^2}{|x-a|^{\frac{3}{2}}\,|x-b|^{\frac{3}{2}}}\,(b-a) \end{aligned} \tag{2.4.104}$$

which achieves a maximum at the midpoint of the line of magnitude

$$\frac{q}{(\pi T)^{\frac{3}{2}}} \tag{2.4.105}$$

This section contained some analytic results for bosonic and fermionic point particles beyond the leading order behaviour. It is especially interesting to note the small T limit of the system which demonstrates a localisation of the field about the classical path between the charges. In the following section we return to the definition of the interaction between particles and use it to construct a full quantum theory.

2.5 Contact interactions between particles

In this section we extend our work to describe a set of particles which interact when their worldlines intersect. Accordingly we work in a $D = 4$ dimensional space-time. This section carries out the analogous analysis to that in [28] where instead a collection of strings interacting upon contact was considered – we turn to this in the next chapter. In this section we limit our discussion to bosonic particles. To describe the dynamics of a collection of interacting particles we augment the free action, S_0, of each point particle with a non-local contact interaction as follows[7]:

$$S_{\text{tot}} = \sum_i S_0 \left[e_i, \omega_i\right] + \frac{g}{2} \sum_{ij} \int_{\omega_i} \int_{\omega_j} d\omega_i \left(\tau_1\right) \cdot d\omega_j \left(\tau_2\right) \delta^4 \left(\omega_i \left(\tau_1\right) - \omega_j \left(\tau_2\right)\right) \quad (2.5.107)$$

The form of this action may appear unusual but it is straightforward to verify that it satisfies the consistency criteria described by Kalb and Ramond for interactions of this type [45, 46]. We must specify fixed boundary conditions for each particle, which we denote by $\omega_i(0) = a_i^\mu$ and $\omega_i(1) = b_i^\mu$. We shall consider the partition function of the theory described by S_{tot} and determine its physical content as well as investigating whether there are divergences which need regularising. In this theory we do not anticipate divergences corresponding to the coincidence of operators due to the well behaved nature of Green functions on one dimensional domains but we do stand to encounter unwanted divergences in taking the $T \to \infty$ limit. We will also find that the definition of the contact interaction will require slight refinement corresponding to the rather trivial vanishing of the argument of the δ-function when $\tau = \tau'$ and the worldlines ω_i and ω_j are the same.

Before proceeding we comment briefly on the relationship to the worldline formulation of quantum field theory as a further justification for considering this action. In the introduction we restricted our consideration to theories without self-interaction coupled only to some gauge field. However, if we were to drop the gauge field and rather consider a scalar field, ϕ, with an interaction potential $U(\phi)$ then the worldline theory would be modified by an additional term $U''(\phi(\omega(\tau)))$ in the action. A calculation of the full effective action must now proceed perturbatively (for example, by expanding ϕ about the centre of mass of the worldline which is the zero mode of the kinetic operator). However, if for the sake of illustration we wish to calculate the $2N$-point one-loop scattering amplitude we may functionally differentiate N-times

[7] In analogy to the introduction of [28] the form of the action can be motivated by substituting $\mathbf{E}'$, the insertion used in Section 2.2, into the standard action of Maxwell electromagnetism for the energy of a static electric field:

$$S_E = \frac{1}{4} \int d^3x \, \mathbf{E} \cdot \mathbf{E} = \frac{e^2}{4} \int_{\boldsymbol{\omega}} d\boldsymbol{\omega}\left(\tau_1\right) \cdot d\boldsymbol{\omega}\left(\tau_2\right) \delta^3 \left(\boldsymbol{\omega}\left(\tau_1\right) - \boldsymbol{\omega}\left(\tau_2\right)\right) \quad (2.5.106)$$

with respect to ϕ and then expand the field as a sum of N plane waves. Now, the physical picture of the interaction term in (2.5.107) is of a pair of worldlines meeting at a point, where they interact before independently separating. This reminds us of the Feynmann diagram in ϕ^4 theory so let us consider $U(\phi) = \frac{\lambda}{4!}\phi^4$, which leads to an additional worldline interaction $\frac{\lambda}{2}\phi(\omega(\tau))^2$. The one-loop $2N$-point amplitude $\langle \phi(x_1) \ldots \phi(x_{2N}) \rangle$ can be written in position space as a sum over permutations of the form

$$\lambda^N \int_0^\infty dT\, T^{N-1} e^{-m^2 T} \int \mathscr{D}\omega \prod_{i=1}^{N} \int_0^1 d\tau_i\, \delta^4(\omega(\tau_i) - x_{2i})\, \delta^4(\omega(\tau_i) - x_{2i-1})\, e^{-\int_0^1 d\tau \frac{\dot{\omega}^2}{2T}}. \tag{2.5.108}$$

The interpretation of this is clear in that it forces the joining of pairs of points x_{2i} and x_{2i-1} to a point on the virtual worldline $x(\tau_i)$, reproducing a quartic interaction vertex. The interaction that we propose is slightly different since we couple pairs of worldlines to one another directly without the need for a virtual loop to mediate the process but the contact interaction takes the same form. Indeed for a theory involving only one particle the picture of the self-intersection interaction coincides directly with this one-loop example.

Our integrand, however, does not consist only of the minimal scalar measures $\int d\tau_i$ but carries with it directional information through the factors of ω^i. So we imagine a four point contact interaction between pairs of worldlines which depends also on the tangent vectors to the worldlines at the point of contact. We get an idea of what this would mean for the corresponding field theory by dimensional analysis of the coupling strength. In D-dimensions, $[g] = \text{length}^{D-2}$, whilst a scalar field has $[\phi] = \text{length}^{1-\frac{D}{2}}$. If we introduce a coupling constant λ then the field-theory interaction $L_{\text{int}} = \lambda \int \partial_\mu \phi\, \partial^\mu \phi\, \phi^2\, d^D x$ is dimensionless if the dimensions of λ are the same as g. This suggests that the worldline theory proposed here is related to a field theory with quartic coupling containing two derivatives. With this heuristic picture of how the contact interaction can be interpreted we now return to determine the partition function of the theory.

We shall carry out the calculation as a perturbative expansion in g. Expressing the δ-function in its Fourier representation introduces vertex operators

$$\dot{\omega}(\tau_i) \cdot \dot{\omega}(\tau_j) \, \delta^4 (\omega(\tau_i) - \omega(\tau_j)) = \int \frac{d^4k}{(2\pi)^4} V_k(\tau_i) \cdot V_{-k}(\tau_j) \tag{2.5.109}$$

At first order in g the correction to the partition function of the non-interacting theory takes the form

$$\frac{g}{2} \sum_{jk} \int \left(\prod_i \frac{\mathscr{D}(\omega_i, e_i)}{\mathcal{Z}} \delta \left(\int e_i d\tau_i - T \right) e^{-S_0[e_i, \omega_i]} \right) \int \int d\omega_j \cdot d\omega_k \; \delta^4 (\omega_j - \omega_k) \tag{2.5.110}$$

There are two contributions to this sum. When the worldlines are distinct (2.5.110) can be factorised to make use of the result in section 2.2:

$$\frac{g}{2} \sum_{j \neq k} \int \left(\prod_{i \neq j,k} \frac{\mathscr{D}(\omega_i, e_i)}{\mathcal{Z}} \delta \left(\int e_i d\tau_i - T \right) e^{-S_0[e_i, \omega_i]} \right) \times$$
$$\int \frac{\mathscr{D}(\omega_j, e_j)}{\mathcal{Z}} \delta \left(\int e_j d\tau_j - T \right) e^{-S_0[e_j, \omega_j]} \int d\omega_j^\mu \left\langle \int d\omega_k^\mu \delta^4 (\omega_j - \omega_k) \right\rangle_T = \tag{2.5.111}$$

$$\frac{g}{8\pi^2} \sum_{j \neq k} \int \frac{\mathscr{D}(\omega_j, e_j)}{\mathcal{Z}} \delta \left(\int e_j d\tau_j - T \right) e^{-S_0[e_j, \omega_j]} \int d\omega_j^\mu \frac{\partial}{\partial \omega_j^\mu} \left(\frac{1}{|\omega_j - a_k|^2} - \frac{1}{|\omega_j - b_k|^2} \right) \tag{2.5.112}$$

where we have taken the large T limit and discarded contributions of order $\frac{1}{T}$ arising from the expectation value in the first line. The integral over ω_j at the far right of the bottom line produces a boundary contribution and we recall that the boundary conditions ensure this is a constant throughout the functional integral over ω_j. So that functional integral is rendered trivial and the result is

$$\frac{g}{4\pi^2} \sum_{j \neq k} \left[\left(\frac{1}{|a_j - a_k|^2} - \frac{1}{|a_j - b_k|^2} \right) - \left(\frac{1}{|b_j - a_k|^2} - \frac{1}{|b_j - b_k|^2} \right) \right] \tag{2.5.113}$$

We must address the case $j = k$ separately as then the worldlines being integrated over are the same. In the string theory case we shall see that two contributions make up this interaction – a renormalisation of the string action and a contribution corresponding to self-intersection of the string. For point particles, however, no such self-intersection ought to be present since the vanishing of

$$\frac{g}{2}\iint d\tau_1 d\tau_2\ \dot{\omega}\left(\tau_1\right)\cdot\dot{\omega}\left(\tau_2\right)\ \delta^4\left(\omega\left(\tau_1\right)-\omega\left(\tau_2\right)\right) \tag{2.5.114}$$

is only at $\tau_1 = \tau_2$. Naively this would provide $\frac{g}{2}\delta\left(0\right)\times \text{Length}\left(\omega\right)$ suggestive of a renormalisation of the non-interacting part of the action. A second major difference between the theory of point particles we consider here and the string theory equivalent is that the current case does not require us to worry about encountering a conformal anomaly. In particular there are no short distance divergences associated to the Green function which require regularisation. To make this more precise we must consider

$$\left\langle I^{\mu\nu}\right\rangle_T = \int \frac{d^4k}{\left(2\pi\right)^4}\iint d\tau_1 d\tau_2\,\frac{g}{2}\left\langle \frac{d\omega^\mu}{d\tau_1}e^{ik\cdot\omega\left(\tau_1\right)}e^{-ik\cdot\omega\left(\tau_2\right)}\frac{d\omega^\nu}{d\tau_2}\right\rangle_T \tag{2.5.115}$$

where we have again used the Fourier representation of the four-dimensional δ-function. The insertions are again easily generated via the introduction of sources and the integral over ω provides the following generalisation of (2.2.10)

$$\begin{aligned}
&\frac{g}{2}\int \frac{d^4k}{\left(2\pi\right)^4}\iint d\tau_1 d\tau_2\,\frac{\delta}{\delta j^\mu\left(\tau_1\right)}\frac{\delta}{\delta j^\nu\left(\tau_2\right)}\\
&\quad\exp\left(-\frac{1}{2}\sum_{i,j=1}^{2}k_ik_j\,G\left(\tau_i,\tau_j\right)-i\int d\tau\,\sum_{i=1}^{2}k_i\cdot j\left(\tau\right)\frac{d}{d\tau}G\left(\tau_i,\tau\right)\right.\\
&\qquad\qquad\left.+\frac{1}{2}\iint d\tau d\tau'\,j\left(\tau\right)\cdot j\left(\tau'\right)\frac{d}{d\tau}\frac{d}{d\tau'}G\left(\tau,\tau'\right)\right)\\
&\qquad\qquad\left.\exp\left(-\int d\tau\,j\left(\tau\right)\cdot\frac{d}{d\tau}\omega_c\left(\tau\right)+\sum_{i=1}^{2}ik_i\cdot\omega_c\left(\tau_i\right)\right)\right|_{j=0}
\end{aligned} \tag{2.5.116}$$

where $k_1 = k = -k_2$. It is trivial to carry out the functional differentiation and it proves useful to define

$$\Psi\left(\tau_1,\tau_2\right)\equiv G\left(\tau_1,\tau_1\right)+G\left(\tau_2,\tau_2\right)-2G\left(\tau_1,\tau_2\right) \tag{2.5.117}$$

in order to express the answer as

$$\frac{g}{2}\int_0^1 d\tau_1 \int_0^1 d\tau_2\ \exp\left(-\frac{1}{2}k^2\Psi\left(\tau_1,\tau_2\right)\right)\left[\dot{\omega}_c^\mu\dot{\omega}_c^\nu - \frac{i}{2}k^\mu\dot{\omega}_c^\nu d_t^2\Psi\left(\tau_1,\tau_2\right)\right.$$
$$\left.+\frac{i}{2}k^\nu\dot{\omega}^\mu d_t^1\Psi\left(\tau_1,\tau_2\right) - \frac{1}{2}\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)d_t^1 d_t^2\Psi\left(\tau_1,\tau_2\right)\right]e^{ik\cdot\left(\omega_c(\tau_1)-\omega_c(\tau_2)\right)} \tag{2.5.118}$$

which must then be integrated over k. In the above equation we have integrated by parts to produce the transverse projector for the final term in square brackets. In the worldline formalism it is more conventional to follow the procedure advocated by Bern and Kosower to remove all second derivatives of Green functions. The interpretation of this convention has its roots in the pinching of Feynman diagrams present in the underlying field theory, but as we do not have such a model behind our work it is unnecessary for us to adhere to it. Now

$$\Psi\left(\tau_1,\tau_2\right) = T\left(\left|\tau_1-\tau_2\right| - \left(\tau_1-\tau_2\right)^2\right) \tag{2.5.119}$$

actually coincides with twice the Green function with periodic boundary conditions. As such it satisfies $\frac{-1}{2T}\frac{d^2}{d\tau_1^2}\Psi\left(\tau_1,\tau_2\right) = \delta\left(\tau_1-\tau_2\right) - 1$ and is a function of $\tau_1-\tau_2$ only. Noting also that $\omega_c\left(\tau_1\right)-\omega_c\left(\tau_2\right) = (b-a)\left(\tau_1-\tau_2\right)$ we learn that the entire expression is in fact a function of the separation $\tau_1-\tau_2$. We use this to fix the zero by setting $\tau_2=0$ and multiplying by $\int_0^1 d\tau_2 = 1$. Furthermore $\Psi\left(\tau_1,0\right) = T\tau_1\left(1-\tau_1\right)$ is just the coincident Green function we met in section 2.2 so we must determine

$$g\int_0^1 d\tau_1 \exp\left(-\frac{1}{2}k^2 G\left(\tau_1,\tau_1\right)\right)\left[\dot{\omega}_c^\mu\dot{\omega}_c^\nu + \frac{i}{2}k^\mu\dot{\omega}_c^\nu d_t^1 G\left(\tau_1,\tau_1\right) + \frac{i}{2}k^\nu\dot{\omega}^\mu d_t^1 G\left(\tau_1,\tau_1\right)\right.$$
$$\left.+\frac{1}{2}\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)d_t^1 d_t^1\Psi\left(\tau_1,0\right)\right]e^{ik\cdot(b-a)\tau_1} \tag{2.5.120}$$

Anticipating that we will eventually take the limit $T\to\infty$ we need only consider an expansion in powers of $\frac{1}{T}$. The leading order contributions again come only from the ends of the interval where the coincident Green function vanishes so it suffices to expand about $\tau_1 = 0$ and $\tau_1 = 1$. We shall consider each term in (2.5.120) separately; at lowest order in $\frac{1}{T}$ the first takes the form

$$\begin{aligned} & g\dot{\omega}_c^\mu \dot{\omega}_c^\nu \int d\tau_1 \, e^{-\frac{1}{2}k^2 G(\tau_1,\tau_1)} e^{ik\cdot(b-a)\tau_1} \\ = & \frac{2g}{k^2}\left[e^{ik\cdot(b-a)} - 1\right] \frac{\dot{\omega}_c^\mu \dot{\omega}_c^\nu}{T} + \mathcal{O}\left(\frac{1}{k^4T^2}\right) \end{aligned} \tag{2.5.121}$$

Upon integrating over k the latter term in in square brackets vanishes whilst the former provides $(2\pi\,|b-a|)^{-2}$. The contribution we seek is the trace of this – we note that since the derivative of the classical solution to the equations of motion is a constant we may express this as

$$\frac{g}{\pi^2\,|b-a|^2}\int_0^1 d\tau\, \frac{\dot{\omega}_c^2}{2T} + \mathcal{O}\left(\frac{1}{k^4T^2}\right) \tag{2.5.122}$$

We interpret this as providing a renormalisation of the free action and note that it is suppressed in the large T limit.

The second and third term in (2.5.120) involve a derivative of the Green function and as such provide contributions that are independent of T. Taking the trace we require

$$\begin{aligned} & gik\cdot\dot{\omega}_c \int d\tau_1 \, \dot{G}\left(\tau_1\tau_1\right) e^{-\frac{1}{2}k^2 G(\tau_1,\tau_1)} e^{ik\cdot(b-a)\tau_1} \\ = & g\frac{ik\cdot\dot{\omega}_c}{2k^2}\left[e^{ik\cdot(b-a)} - 1\right] + \mathcal{O}\left(\frac{1}{k^2T}\right) \end{aligned} \tag{2.5.123}$$

Carrying out the integral over k provides

$$-g\frac{\dot{\omega}_c\cdot(b-a)}{4\pi^2\,|b-a|^4} + \mathcal{O}\left(\frac{1}{k^2T}\right) \tag{2.5.124}$$

Recognising that $\dot{\omega} = b-a$ we may also cast this into the form

$$-\frac{g}{4\pi^2\,|b-a|^2}\int_0^1 \frac{\dot{\omega}_c^2}{|b-a|^2} + \mathcal{O}\left(\frac{1}{k^2T}\right) \tag{2.5.125}$$

providing a finite renormalisation of the free action.

Finally we take the last term of (2.5.120) and consider its trace. We use the defining equation of the Green function to write its contribution as

$$-g\int_0^1 d\tau_1\, T\left(\delta\left(\tau_1\right)-1\right)e^{-\frac{1}{2}k^2G(\tau_1,\tau_1)}e^{ik\cdot(b-a)\tau_1} \tag{2.5.126}$$

Both terms are independent of the field ω_c which suggests that we should interpret them as renormalisations of the cosmological constant term in the action $\int_0^1 d\tau\, T$. The δ-function gives us

$$-g\int_0^1 d\tau\ T \tag{2.5.127}$$

exactly as required, albeit formally divergent when we take T to infinity[8], whereas the second evaluates to

$$-\frac{2g}{k^2}\left[e^{ik\cdot(b-a)}-1\right]+\mathcal{O}\left(\frac{1}{k^2T}\right) \tag{2.5.130}$$

which upon integrating over k becomes

$$-\frac{g}{2\pi^2}\frac{1}{\left|b-a\right|^2}+\mathcal{O}\left(\frac{1}{k^2T}\right) \tag{2.5.131}$$

This surprisingly takes the same form as (2.5.113) and suggests that there is after all a self-interaction present in the theory, sensitive only to the boundary of the worldline.

To lend this more weight we could approach the calculation in a complementary fashion. We note that $\Psi\left(\tau_1,\tau_2\right)$ vanishes only at $\tau_2=\tau_1$ so we could arrange our calculation by instead expanding about this point. When τ_1 is not close to the boundary (measured with respect to $\frac{1}{k^2T}$) integrating τ_2 about τ_1 corresponds to

[8] The integral over k also provides an infinite volume factor multiplying this result. We may tidy this up by carrying out the $k-$integral before integrating over τ_1. Doing so turns (2.5.126) into

$$-g\int d\tau_1\left(\frac{\pi}{2G\left(\tau_1,\tau_1\right)}\right)^{\frac{D}{2}}e^{-\frac{(b-a)^2\tau_1^2}{2G(\tau_1,\tau_1)}}\left(\delta\left(\tau_1\right)-1\right). \tag{2.5.128}$$

We are concerned only with the contribution arising from the δ-function so we need the value of the exponent as $\tau_1\to0^+$. An easy calculation shows that the exponent vanishes leaving only

$$-g\left(\frac{\pi}{2}\right)^{\frac{D}{2}}\lim_{\tau_1\to0}\left(G\left(\tau_1,\tau_1\right)\right)^{-\frac{D}{2}} \tag{2.5.129}$$

which diverges.

integrating over their relative separation and the leading order contribution arises by setting the T-independent exponent $\exp\left(ik \cdot (b-a)\left(\tau_1 - \tau_2\right)\right)$ equal to unity. This gives the terms above which are absent of an exponent. When τ_1 is close to the boundary we must take care because τ_2 is restricted to lie in $[0,1]$. So for example when $\tau_1 \approx 0$ we must integrate τ_2 a small distance from this boundary along the line but must also consider the contribution when τ_2 is integrated from the opposite boundary along the line. Indeed for the latter case we consider, following the notation of section 2.2,

$$\frac{g}{2}\int_0^h d\tau_1 \int_{1-h}^1 d\tau_2 \; T\left(\delta\left(\tau_1 - \tau_2\right) - 1\right) e^{-\frac{1}{2}k^2\Psi(\tau_1,\tau_2)} e^{ik\cdot(b-a)(\tau_1-\tau_2)}. \tag{2.5.132}$$

The δ-function is not supported for this configuration of variables, besides it is the second term with which we are concerned. At leading order in $\frac{1}{T}$ we evaluate the trailing exponent at the point $\tau_1 = 0$, $\tau_2 = 1$. A change of variables $\tau = 1 - \tau_2$ yields $\Psi\left(\tau_1, 1-\tau\right) = T\left(\left(\tau_1 + \tau\right) - \left(\tau_1 + \tau\right)^2\right)$ and the integral becomes

$$-\frac{g}{2}\int_0^h d\tau_1 \int_0^{2h} du \; e^{-\frac{1}{2}k^2 G(u,u)} e^{-ik\cdot(b-a)} \tag{2.5.133}$$

where we have set $u = \tau_1 + \tau$. At leading order this provides

$$-\frac{g}{k^2} e^{-ik\cdot(b-a)} \tag{2.5.134}$$

which is to be compared with (2.5.130). The sign of the exponent is not important and the factor of two is found by including the configuration where the positions of τ_1 and τ_2 in the above calculation are swapped. So the physical information arises when the two points are close to opposite boundaries, whereas the renormalisations appear from the region where the two points coincide. This is in fitting with the naive analysis of the form of the contact interaction when the worldlines are the same.

The self-interaction (2.5.131) also has the appropriate factor of two difference for it to be subsumed into the sum (2.5.113) so that at first order in the expansion of the interaction we find

$$\frac{g}{4\pi^2}\sum_{j,k}{}' \left[\left(\frac{1}{|a_j - a_k|^2} - \frac{1}{|a_j - b_k|^2}\right) - \left(\frac{1}{|b_j - a_k|^2} - \frac{1}{|b_j - b_k|^2}\right)\right]. \tag{2.5.135}$$

We write $\sum'$ to denote that when $j = k$ we discard the first and last terms in the summand where the separation vanishes. This concludes the analysis of the first order effect of the contact interaction present in this theory. In the following section we turn to consider higher order interactions and show that the result above is not spoilt.

2.5.1 Expansion to arbitrary order

We may consider expanding the part of the action corresponding to the inter-particle interaction to arbitrary order. When the interaction is between distinct worldsheets we may repeat the analysis at first order to simplify the calculation to a form familiar from section 2.2. Here we address the problem of having a fixed even number, N, of points to be integrated over the same worldsheet which could potentially spoil the result at order $g^{\frac{N}{2}}$ or higher. So we consider

$$\left\langle V^{\mu}_{k_1}(\tau_1)\, V^{\nu}_{k_2}(\tau_2) \ldots V^{\alpha}_{k_N}(\tau_N)\right\rangle_T \tag{2.5.136}$$

where we understand that each point must be integrated from $\tau_i = 0$ to 1.

Wick's theorem produces a common factor to all of the contractions that could be formed out of the above product of fields which takes the form

$$\exp\left(-\frac{1}{2}\sum_{i,j=1}^{N} k_i \cdot k_j G(\tau_i, \tau_j)\right) e^{i\sum_i k_i\cdot\omega(\tau_i)} \tag{2.5.137}$$

We are interested in taking the high T limit, whereby the leading contribution will be from the regions of integration where the exponent vanishes. There are no short distance divergences which require regularisation so we may consider the points to be arbitrarily close to one another. This leads us to consider two cases. The exponent can be made to vanish by ensuring that each individual Green function $G(\tau_i, \tau_j)$

vanishes, which requires us to localise each of the points close to either end of the domain. Alternatively we might consider bringing a collection of points coincident, which is instructive in order to compare with the results of the previous subsection. As no regularisation is required we keep the discussion of this case brief. It is useful to split (2.5.137) as

$$\exp\left(-\frac{1}{2}\sum_{i=1}^{N} k_i^2 G(\tau_i,\tau_i) - \sum_{j\neq i} k_i\cdot k_j G(\tau_i,\tau_j)\right) e^{i\sum_i k_i\cdot\omega(\tau_i)} \tag{2.5.138}$$

which makes it clear that for an arbitrary configuration of the points away from the boundary this factor damps the integrand. Now suppose that some number, n, of these points are brought to the same point τ_1. Then these points contribute

$$\exp\left(-\frac{1}{2}G(\tau_1,\tau_1)\left(\sum_{i=1}^{n} k_i\right)^2 + \ldots\right) e^{i\sum_i k_i\cdot\omega(\tau_i)} \tag{2.5.139}$$

where the ... represents terms which depend on the relative separation between each point and τ_1. Recall that $G(\tau_1,\tau_1) = T\tilde{G}(\tau_1,\tau_1)$ vanishes only at the boundary of the domain. Consider how this exponent behaves under the integral with respect to k_1, say. In the large T limit a Laplace approximation implies its effect is to provide[9]

$$\frac{\delta\left(\sum_i k_i\right)}{\left(2T\tilde{G}(\tau_1,\tau_1)\right)^2}. \tag{2.5.140}$$

Since in one dimension the coincident Green function is finite we see that all contributions from such a configuration are suppressed by a factor of T^{-2}. The size of the integrals over the relative separation $\tau_i - \tau_1$ can be examined by power counting in $\frac{1}{T}$. The largest contribution arises when all fields take part in a contraction and can be arranged into a series of $\frac{n}{2}$ second order derivatives. A simple calculation shows that this contribution is of order $T^{\frac{n}{2}}$. But after integrating the other $n-1$ points about τ_1 the resulting expression is of order T. In combination with (2.5.140) this

[9]This should be considered in light of the results at first order where we had $k_1 = k = -k_2$ where the exponent vanished throughout the domain when $\tau_1 = \tau_2$. We exclude such cases here because we understand that they lead to a simple renormalisation of the action.

contribution can be seen to be subleading in $\frac{1}{T}$.

Returning to the case that all of the points are on the boundary, each of the Green functions in (2.5.137) vanishes. So we expect a contribution to the expectation value from the region of integration where each point is close to the boundary. We have learnt in the previous subsection that the $\mathcal{O}(1)$ term arises when we contract the fields inside each vertex operator amongst themselves. The exponents $\exp(ik \cdot \omega_c(\tau_i))$ can be approximated at leading order by replacing them with their values at the appropriate boundary – we denote this by $\exp(ik \cdot \omega_{B_i})$. So we consider a term with r such contractions:

$$\prod_{j=1}^{r}\left(\int_0^h + \int_{1-h}^h\right) d\tau_j \; ik_j^{\mu_j}\, \dot{G}_{jj} e^{-\frac{1}{2}k^2 G_{jj}} e^{ik_j\cdot\omega_{B_j}} \prod_{i=r+1}^{N}\left(\int_0^h + \int_{1-h}^h\right) d\tau_i \; \dot{\omega}_{ci} e^{-\frac{1}{2}k^2 G_{ii}} e^{ik_i\cdot\omega_{B_i}}$$

$$= \prod_{j=1}^{r} \frac{ik_j^{\mu_j}}{k_j^2}\left(e^{ik_j\cdot a} - e^{ik_j\cdot b}\right) \prod_{i=r+1}^{N}\left[\frac{\dot{\omega}_{ci}}{k_i^2 T} + \mathcal{O}\left(\frac{1}{(k^2T)^2}\right)\right] \tag{2.5.141}$$

The leading order contribution clearly requires $r = N$ so that all fields are contracted, whereby the above expression reduces to

$$\prod_{j=1}^{N} \frac{ik_j^{\mu_j}}{k_j^2}\left(e^{ik_j\cdot a} - e^{ik_j\cdot b}\right). \tag{2.5.142}$$

We now impose pairwise $k_{i+1} = -k_i$ and contract μ_i and μ_{i+1} to produce the expectation value of the contact interaction at order $g^{\frac{N}{2}}$:

$$-2^{\frac{N}{2}} g^{\frac{N}{2}} \prod_{i=1}^{\frac{N}{2}} \frac{e^{ik\cdot(b-a)}}{k_i^2}. \tag{2.5.143}$$

As our analysis has been for an arbitrary number of points the result at this order is not spoilt by considering a higher order term in the expansion of the contact interaction. The above equation is simply the Fourier space version of the order $g^{\frac{N}{2}}$ contribution to the exponential of the sum of boundary interactions (2.5.135).

2.6 Discussion

In this chapter we have considered contact interactions in the context of theories of point particles. We have considered a novel way of generating the static electric field for a pair of point particles by considering fluctuating worldlines whose endpoints are fixed to the positions of the charges. The functional approach we used for calculation allowed us to generalise this construction to include spin 1/2 particles and we then used the formalism to construct a quantum theory describing a set of point particles interacting upon contact. The result for fermionic particles is interesting because it suggests an unusual electric force acting on Dirac spinors due to the charged particles. Both results depend somewhat on the choice of boundary conditions imposed on the worldline fields at either end of the paths and we also found it necessary to introduce constraints on the worldline metric and its super-partner.

The worldline formalism of quantum field theory highlights an intimate connection between field theory and the first quantised theories we have considered here. Our approach differed in that rather than coupling the theory living on the worldline to a background gauge field (as occurs when integrating out matter fields in with the worldline approach) we coupled the worldlines to one another directly using the $\delta-$function interaction. The functional methods that we used are however not limited to the calculation we have undertaken and can be directly applied to situations that arise using the latter approach. We shall see in the next chapter that the worldline formalism can be used to form a direct link between field theory and an interacting first quantised theory, though we shall need to make changes to the point particle theories we have considered thus far.

Indeed this theory is clearly not complete, since our discussion is limited to situations where the boundaries of the worldlines are static. We consider it as a proof of concept and a motivation for how we might treat the time varying case. For a charged particle described by a more general worldline the end-points of the curves, ω_i, must be allowed to vary tangentially to the path of the particle and we must consider how to describe the dynamics of the curve. This was addressed in [13] for the case of electromagnetism, where it was shown that the natural way to describe these one dimensional extended objects is string theory. In the next chapter we shall extend the work to consider the worldlines of equal and oppositely charged particles as the boundaries of a string which traces out a worldsheet. We shall see that the contact interaction generalises to provide interactions between the worldsheets corresponding to different charges and will relate the quantum theory to spinor quantum electrodynamics.

Chapter 3

String theory, contact interactions and quantum electrodynamics

In this chapter we move away from theories of interacting point particles and consider the equivalent idea for fundamental strings. We are informed by the result of [13] which demonstrated that the time varying electromagnetic field at a space-time point x for equal and oppositely charged particles can be determined from a statistical averaging over the fluctuations of lines of force stretched between the worldlines of the charges and which pass through x. However when we come to consider the construction of a theory of interacting strings we will find that the worldline formalism provides a natural way to relate our calculations to the conventional quantum field theory of matter coupled to the electromagnetic field. This chapter follows [41] and [28].

3.1 Introduction

In the previous chapter we limited our discussion to static configurations of charged particles or fixed boundary data. We decided to treat Faraday's lines of flux as physical quantities and showed that averaging over the spatial fluctuations of a flux line produces the electric field. If we now consider the case that the charged particles move along worldlines then a single flux line joining the charges traces out a two-dimensional surface, Σ, whose boundary, B, is given by the particles' worldlines.

So we are moved to consider now the fluctuations of this worldsheet, for which the natural framework is that of string theory.

This was pursued by Mansfield in [13]. The generalisation of (2.2.1) to this dynamical case is an expression for the electromagnetic field strength in terms of an integral over any surface bounded by the particle worldlines

$$F_{\mu\nu}(x) = -q \int_{\Sigma} \delta^4(x - X)\, d\Sigma_{\mu\nu}(X)\,. \tag{3.1.1}$$

where $d\Sigma_{\mu\nu}$ is an element of area on Σ, itself parameterised by coordinates $X^\mu(\xi)$. The world-sheet co-ordinates (we previously called σ^a) are ξ^1 and ξ^2 and we denote the parameter domain by D. On the boundary of the domain the worldsheet is fixed to the worldline so $X^\mu|_{\partial D} = w^\mu$. Finally the area element can be expressed in terms of the worldsheet coordinates as

$$d\Sigma_{\mu\nu} = \epsilon^{ab}\partial_a X_\mu \partial_b X_\nu d^2\xi \tag{3.1.2}$$

By integrating against a test function it is easy to verify that $F^{\mu\nu}$ defined above satisfies Gauss' law

$$\partial^\mu\, F_{\mu\nu} = J_\nu\,, \tag{3.1.3}$$

where the current density due to the charges is

$$J^\mu(x) = q \int_B \delta^4(x - w)\, dw^\mu. \tag{3.1.4}$$

Nielsen and Olesen [47] have used such an expression for the field strength tensor to form a field theory describing the dual string from a basis of non-linear electrodynamics. It is also present in theories of electromagnetism with magnetic monopoles [48] and can be used to derive an effective string theory describing the evolution of the Dirac string linking two such poles [49,50]. Using this field strength in the standard form of the action for electromagnetism gives the generalisation of the interaction between point particles which appeared in the previous chapter:

$$S_{EM} = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} = \frac{q^2}{4\epsilon_0^2}\int_\Sigma d\Sigma^{\mu\nu}(\xi)\delta^4\left(X(\xi) - X(\xi')\right)\, d\Sigma_{\mu\nu}(\xi')\,.$$

The argument of the delta-function vanishes when $\xi'^a = \xi^a$, and also at points where the world-sheet intersects itself. This gives two contributions,

$$S_{EM} = \frac{q^2}{4\epsilon_0^2}\delta^2(0)\,\text{Area}(\Sigma) + \frac{q^2}{4\epsilon_0^2}\int_\Sigma d\Sigma^{\mu\nu}(\xi)\,\delta^4\left(X(\xi) - X(\xi')\right)\,d\Sigma_{\mu\nu}(\xi')\big|_{\xi\neq\xi'}\,. \quad (3.1.5)$$

The first is just an area term which we interpret as being proportional to the Nambu-Goto action of string theory, albeit with a divergent coefficient, and consequently specifying the dynamics of the curve C. The second is the self-intersection interaction that we will study in detail below. As in the case of the point particle we find that direct interactions have previously been discussed by Kalb and Ramond [45] and the one we propose here satisfies the consistency constraints they derive. This action has also been applied before classically to the problem of confinement [51, 52] but without the effects of self-intersection or quantisation that we shall consider here.

In [13] the connection to string theory was taken further by showing that, although (3.1.1) does not satisfy the other Maxwell equations, its functional average over worldsheets with fixed boundary provides the classical field strength tensor associated with the charged particles:

$$4\pi^2\left\langle\int_\Sigma \delta^4(x - X)\,d\Sigma_{\mu\nu}(X)\right\rangle_\Sigma = \partial_\mu\int_B \frac{dw_\nu}{||x - w||^2} - \partial_\nu\int_B \frac{dw_\mu}{||x - w||^2} \quad (3.1.6)$$

where the average over surfaces of any functional $\Omega\,[\Sigma]$ is given by the functional integration we discussed in the introduction (we shall use g_{ab} to denote the worldsheet metric in this chapter since we will later use h as the intrinsic metric on particle worldlines):

$$\langle\Omega\rangle_\Sigma = \frac{1}{\mathcal{Z}}\int \mathscr{D}g\,\mathscr{D}_gX\,\Omega\,e^{-S_{\text{P}}[g,X]}\,. \quad (3.1.7)$$

This was computed in Euclidean space where the integrals are better behaved, so that $||x - y||$ is the Euclidean distance between x and y. Minkowski space results were obtained by Wick rotation and the retarded solutions to Maxwell's equations were picked out naturally by interpreting this as a finite temperature average in the quantum theory. This work provided the mathematics behind the idea that

Faraday's flux tubes can be thought of as the physical degrees of freedom of electromagnetism with the caveat that their configurations must be averaged over in a string theory setting. This is also rather similar to a proposal due to Dirac [53, 54] in which he advanced the idea of operators which create both charged particles *and* part of the electromagnetic field itself.

Actually for (3.1.6) to hold requires either that we normalise the expectation values against the partition function of the non-interacting string theory or that we consider the limit of tensionless strings ($\alpha' \to \infty$ in units of the size of the worldline B). The normalisation or tensionless limit prove sufficient to remove any dependence on the action associated to the classical solution to the string equations of motion. Note that the right hand side of (3.1.6) is independent of the string tension and so insensitive to variations in α'. In our work we shall not have the former avenue open to us since to relate the theory to QED we will find it necessary to integrate over the worldlines specifying the boundaries of the strings.

After the discussion of the functional quantisation of string theory in the introduction the average in (3.1.6) may well appear unusual. Firstly we work in the physical number of space-time dimensions $D = 4$. This means that the functional integration over the string configurations and worldsheet metrics will produce a Weyl anomaly. Furthermore the presence of the delta-function in (3.1.6) means that the average is off-shell and involves non-physical states. It is now time to return to Polyakov's formulation of string theory to revise how mass-shell conditions appear in this approach. To calculate the scattering of a set of strings we insert a product of re-parametrisation invariant vertices such as that of the tachyon

$$\kappa \int d^2\xi \, \sqrt{g} e^{ik\cdot X(\xi)}$$

into the functional average:

$$\int \mathscr{D}g \mathscr{D}_g X \, e^{-S_{\mathrm{P}}[g,X]} \ldots \kappa \int d^2\xi \, \sqrt{g} e^{ik\cdot X(\xi)} \ldots \equiv \langle \ldots \kappa \int d^2\xi \, \sqrt{g} e^{ik\cdot X(\xi)} \ldots \rangle_X \quad (3.1.8)$$

where the dots stand for the other vertices (whose effect when ξ is close to their positions we shall ignore). The X-dependence of the exponent remains quadratic, being simply

$$-\frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{g}g^{ab}\partial_a X^\mu\partial_b X_\mu+\sum_i ik_i\cdot X\left(\xi_i\right) \tag{3.1.9}$$

We can shift X by completing the square to reduce the integral over these variables to the one we have seen before (except that the zero mode of Δ now imposes momentum conservation), the result of which is multiplied by a factor involving the Green function for the two-dimensional Laplacian at coincident points, $G(\xi,\xi)$,

$$\kappa\int d^2\xi\,\sqrt{g}e^{-\pi\alpha' k^2 G(\xi,\xi)}, \tag{3.1.10}$$

amongst terms involving the other vertices which are not of immediate importance for this discussion. We work in complex coordinates in conformally flat gauge, $ds^2=2e^\phi dzd\bar{z}$. Then for an open string with the conventional Neumann boundary conditions we have

$$G\left(z,z'\right)=-\frac{1}{4\pi}\left(\ln\left|z-z'\right|^2+\ln\left|z-\bar{z}'\right|^2\right);\qquad -2e^{-\phi}\partial\bar{\partial}G\left(z,z'\right)=e^{-\phi}\delta^2\left(z-z'\right). \tag{3.1.11}$$

The point is that $G(z,z)$ needs to be regulated and this introduces a dependence on the scale of the metric, invalidating our result that the dependence on ϕ is contained only in the Liouville theory. We do not provide the details of this procedure but state the result. With heat-kernel regularisation the leading behaviour at points away from the boundary separated by physical distance ϵ is $G(z,z)\sim(\phi(z)-2\log(\epsilon))/(4\pi)$, so (3.1.10) becomes

$$\kappa\epsilon^{\alpha' k^2/2}\int d^2z\,e^\phi\,e^{-\alpha' k^2\phi/4}.$$

We can avoid the dependence on ϕ, and so integrate over this degree of freedom, if we impose the tachyon mass-shell condition $k^2=4/\alpha'$ and we can make the result finite by renormalising κ (this is similar to wave function renormalisation in field theory). So we learn that the mass-shell conditions can be seen as arising out of the requirement that no undesired dependence on the conformal scale appears when we insert vertex operators inside the functional integral. The same is true for the transversality conditions which can be seen, for example, by studying photon or graviton states.

Surprisingly the expectation value of the delta-function also decouples from ϕ, but in this case it evades a mass-shell condition. If we Fourier decompose the delta-function then

$$\delta^4(x-X)\,d\Sigma^{\mu\nu}(X) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}\frac{1}{2}V_k^{\mu\nu}; \qquad V_k^{\mu\nu}(\xi) = \epsilon^{ab}\partial_a X^\mu \partial_b X^\nu\, e^{-ik\cdot X(\xi)}. \tag{3.1.12}$$

We recognise the $V_k^{\mu\nu}$ as string vertex operators but there are important differences to the conventional string insertions. Firstly the momentum integral means that the insertions are off-shell and secondly the form of the contact interaction means that we integrate the positions of these insertions throughout the entire string worldsheet so they are more akin to closed string vertex operators. Whilst naively we would expect the lack of mass-shell conditions and our choice of four target space dimensions would lead to anomalous dependence on the conformal scale we repeat Mansfield's demonstration that this is not the case. To establish (3.1.6) we impose Dirichlet boundary conditions on the string because the endpoints are fixed to the worldlines of the particles and consider the functional integral

$$\begin{aligned} e^{ik\cdot x}\left\langle V_k^{\mu\nu}(\xi)\right\rangle_X &= \int \mathscr{D}_g X\, e^{-S[g,X]}\, \epsilon^{ab}\partial_a X^\mu \partial_b X^\nu\, e^{ik\cdot(x-X(\xi))} \\ &\propto \epsilon^{ab}\left(k^\mu \partial_b X_c^\nu\, 2\pi\alpha' \partial_a G(\xi,\xi) + \partial_a X_c^\mu \partial_b X_c^\nu\right) e^{ik\cdot(x-X_c(\xi))} e^{-\pi\alpha' k^2 G(\xi,\xi)}, \end{aligned} \tag{3.1.13}$$

where X_c is the value of X that evolves from the boundary value according to the classical equations of motion. We shall give an explicit expression for this in the next section. Note that we do not have the freedom to renormalise by introducing a k-dependent factor like κ without ruining the Fourier decomposition of the delta function so we may not make use of multiplicative renormalisation to make sense of divergences. On the boundary of the worldsheet $G(\xi,\xi)=0$ because of the boundary conditions, so precisely on the boundary

$$e^{-k^2\pi\alpha' G(\xi,\xi)} = 1. \tag{3.1.14}$$

For points away from the boundary

$$e^{-\pi\alpha' k^2 G(\xi,\xi)} \sim \epsilon^{\alpha' k^2} e^{-\alpha' k^2 \phi/4} \tag{3.1.15}$$

is suppressed as $\epsilon \to 0^+$, since in Euclidean signature $k^2 > 0$. This means that (3.1.13) is negligible except when ξ is in a small (in terms of the regulator) strip close to the boundary, and the value precisely on the boundary is independent of ϕ. When (3.1.13) is integrated over the parameter domain D we only have to consider contributions within this strip and so we can separate the various factors into those like $G(\xi,\xi)$ and its derivative that vary rapidly as ξ moves from the boundary into the interior of the world-sheet, and terms like X_c and its derivatives that vary slowly and can be approximated by their boundary values. Arranging the parameters ξ^a so that ξ^2 is constant on the boundary and ξ^1 varies along it the integral over ξ of the first term of (3.1.13) contains

$$\begin{aligned}
&\int d^2\xi\, k^{[\mu}\partial_1 X_c^{\nu]}\, e^{ik\cdot(x-X_c(\xi))} 2\pi\alpha'\partial_2 G(\xi,\xi)\, e^{-\pi\alpha' k^2 G(\xi,\xi)} \\
&= \int d\xi^1 \left(k^{[\mu}\partial_1 X_c^{\nu]}\, e^{ik\cdot(x-X_c(\xi))} \int d\xi^2\, 2\pi\alpha'\partial_2 G(\xi,\xi)\, e^{-\pi\alpha' k^2 G(\xi,\xi)} \right) \\
&= \int_B dw^{[\mu} k^{\nu]} e^{ik\cdot(x-w)}/k^2
\end{aligned} \tag{3.1.16}$$

which is the Fourier transform of (3.1.6). Note that this is independent of the cut-off scale, ϵ, the length-scale, α', and, crucially, the scale of the metric, ϕ, even though all of these entered the intermediate expressions.

The remaining terms in (3.1.13) were shown to vanish as the cut-off is removed. They also vanish in the tensionless limit for finite cut-off as we will see later. So the only dependence on ϕ is contained within the Liouville action we met in the introduction. The solution to working outside of the critical dimension of string theory was to notice that the physical content of the theory, i.e. the expression for the electromagnetic field strength tensor, decouples from the Liouville theory. If all functional averages are then normalised against its partition function then its contribution is always cancelled. Later on in this chapter we will also need to consider functional integrals with mixed boundary conditions, i.e. where the boundary is divided into sections where Dirichlet conditions are imposed and sections where Neumann conditions are imposed. The result generalises so that the right hand side of (3.1.16) receives contributions from just the Dirichlet sections of the boundary (see Appendix B).

3.1.1 Interacting strings

As we have seen, the expectation of δ-function decouples from the conformal scale of the worldsheet metric. From the point of view of the string theory we used the δ-function as a local probe of the worldsheet [55]. In the previous chapter our next step was to use the δ-function to form a theory in which point particles interact upon contact. The decoupling of the off-shell δ-function from the conformal scale gives us hope that we may now be able to use (3.1.5) to construct a string theory in which the strings interact when their worldsheets intersect. We consider the system consisting of a number, N, of surfaces $\{\Sigma_i\}$. These have boundary components including curves w_i^μ. The curves can be either closed or open, in which case we impose Neumann boundary conditions on X^μ on the remaining boundary components of Σ. The action is $S_f = \sum_i S_i + \sum_{ij} S_{ij}$ with $S_i = S[g_i, X_i]$ and

$$S_{ij} = \frac{q^2}{4\epsilon_0^2} \int_{\Sigma_i, \Sigma_j} d\Sigma_i^{\mu\nu}(\xi)\, \delta^4\left(X_i(\xi) - X_j(\xi')\right)\, d\Sigma_j^{\mu\nu}(\xi'). \tag{3.1.17}$$

We wish to relate this now to quantum field theory. In the introduction we saw that scalar QED can be re-stated in the worldline formalism, which caused us to introduce a series of first quantised theories living on one dimensional curves. These curves are seen as Wilson lines which couple the worldline theories to the electromagnetic field. To compute the generating functional for the field theory Green functions it remains to functionally integrate over the gauge field configurations which we stopped short of doing in the introduction. The reason for this is that our aim will be to show that the interacting string theory reproduces this functional integral. We identify a set of N worldlines as the boundaries for N interacting strings and wish to show that

$$\int \prod_{i=1}^{N} \frac{\mathscr{D}(X_i, g_i)}{Z_0}\, e^{-S_f} \tag{3.1.18}$$

is the same as

$$\int \frac{\mathscr{D}A}{N}\, e^{-S_{gf}[A]} \prod_i e^{-i\int dw_i\cdot A} = \prod_{i,j} e^{-\frac{q^2}{4\epsilon_0^2}\int dw_i^\mu \Delta_{\mu\nu}\, dw_j^\nu} \tag{3.1.19}$$

where the Maxwell theory is gauge-fixed in the gauge $\partial\cdot A = 0$ so that the propagator, $\Delta_{\mu\nu}$, has Fourier transform $\delta_{\mu\nu}/k^2 - k_\mu k_\nu/(k^2)^2$. In this way we would reformulate QED in terms of a series of strings which interact upon contact. It will turn out that we are unable to completely establish this result for the case of the bosonic string due to the possible appearance of unwelcome divergences. However the world-sheet supersymmetry of the spinning string provides sufficient structure to eliminate these, which will allow us to prove the supersymmetric generalisation. It is precisely this generalisation, in which the super-Wilson loop appears, that is needed for electric charges with spin. So it appears that this string model has a preference for the realistic case of spinor QED over that of scalar QED.

Recall from the introduction that the generating functional for Green functions[1] can be computed by first integrating over the scalar field leaving

$$Z[\bar{J}, J] = \int \mathscr{D}A\, e^{-S_{gf} - \log \mathrm{Det}\left(-\mathcal{D}^2 + m^2\right) + \int d^4x \bar{J}\left(-\mathcal{D}^2+m^2\right)^{-1} J} \tag{3.1.20}$$

which we expand as

$$\sum_{r,s=1}^{\infty} \frac{1}{r!s!} \int \mathscr{D}A\, e^{-S_{gf}} \left(-\log \mathrm{Det}\left(-\mathcal{D}^2 + m^2\right)\right)^r \left(\int d^4x \bar{J}\left(-\mathcal{D}^2 + m^2\right)^{-1} J\right)^s \tag{3.1.21}$$

Using the representations for the functional determinant, (1.0.6), and propagator,

[1] Later we will include a source for the gauge field, but for the time being we suppress this to simplify our expressions

(1.0.9), we re-write (3.1.21) as

$$\sum_{r,s=1}^{\infty} \frac{1}{r!s!} \int \mathscr{D}A \, e^{-S_{gf}} \prod_{j=1}^{r+s} \frac{\mathscr{D}(h_j, w_j)}{Z} \, e^{-S_0[w_j, h_j]} \times \prod_{k=r+1}^{r+s} e^{-i \oint dw_k \cdot A} \prod_{l=1}^{s} \int d^4 b_l \, d^4 a_l \bar{J}(b_l) e^{-i \int_{a_l}^{b_l} dw_l \cdot A} J(a_l) \tag{3.1.22}$$

Were (3.1.18) equivalent to (3.1.19) it could be used to represent the integral over the gauge-field as a set of integrals over surfaces

$$\sum_{r,s=1}^{\infty} \frac{1}{r!s!} \int \prod_{l=1}^{s} d^4 b_l \, d^4 a_l \, \bar{J}(b_l) J(a_l) \prod_{i=1}^{r+s} \frac{\mathscr{D}(X_i, g_i)}{Z_0} \, e^{-S_i} \prod_{j=1}^{r+s} \frac{\mathscr{D}(h_j, w_j)}{Z} \, e^{-S_0[w_j, h_j]} \prod_{i,j=1}^{r+s} e^{-S_{ij}} \tag{3.1.23}$$

Re-arranging this slightly we would have arrived at

$$Z[\bar{J}, J] = \sum_{r,s=1}^{\infty} \frac{1}{r!s!} \int \prod_{l=1}^{s} d^4 b_l \, d^4 a_l \, \bar{J}(b_l) J(a_l) \prod_{i=1}^{r+s} \frac{\mathscr{D}(X_i, g_i, h_i, w_i)}{Z Z_0} \, e^{-S_i - S_0} \prod_{i,j=1}^{r+s} e^{-S_{ij}} \tag{3.1.24}$$

Although we will be unable to demonstrate the equivalence of (3.1.18) and (3.1.19) in scalar QED we will demonstrate an exact relation for spinor QED.

The physical interpretation of (3.1.24) would be that the field theory is equivalent to an ensemble of strings described by the usual free Polyakov action, $S_i = S[g_i, X_i]$, augmented with a boundary term $S_0[w, h]$ interacting with each other via the contact term S_{ij}. There is some freedom in how we associate the curves w_i to the world-sheet surfaces. For want of an obvious alternative we choose the simplest assignment by associating a zero genus surface to each ω_i. Thus the $\log \mathrm{Det}\,(-\mathcal{D}^2 + m^2)$ factors correspond to closed curves bordering world-sheets which together describe particle anti-particle pairs connected by lines of force. The Green function factors $(-\mathcal{D}^2 + m^2)^{-1}$ correspond to world-sheets that have mixed boundary conditions: Dirichlet conditions for the curves w_i which run from a_i to b_i and Neumann everywhere else, so these describe strings with the usual string theory Neumann conditions at one end and a charged particle (or anti-particle) at the other. Mixed boundary conditions for the string are discussed in Appendix B.

So far we have not included a source for the gauge field. Rather than using a general source we limit attention to one that generates scattering amplitudes via the LSZ procedure by shifting the gauge-fixed Maxwell action, S_{gf}, in (3.1.22)

$$S_{gf} \to S_{gf} - \frac{1}{q^2} \int d^4x \, \mathcal{A} \cdot \partial^2 A \tag{3.1.25}$$

where the source is on-shell, i.e. $\partial^2 \mathcal{A} = \partial \cdot \mathcal{A} = 0$ (we revert to Lorentzian signature briefly to be able to invoke LSZ).

We show in Appendix C that

$$\int \frac{\mathscr{D}A}{N} \, e^{-S_{gf} + \frac{1}{q^2} \int d^4x \, \mathcal{A} \cdot \partial^2 A - i \sum_j \int dw_j \cdot A} = \prod_{i,j} e^{-\frac{q^2}{2} \int dw_i^\mu \Delta_{\mu\nu} \, dw_j^\nu} \prod_i e^{-i \int dw_i \cdot \mathcal{A}} \tag{3.1.26}$$

so that the effect of including the source $\mathcal{A}$ is simply to add a term to the boundary part of the action:

$$S_0[w, h] \to S_0[w, h] + i \int dw \cdot \mathcal{A} \tag{3.1.27}$$

We note in passing that if we were to consider the generating functional for scattering amplitudes of charged particles and anti-particles then we would replace the source terms $\bar{J}\phi$ and $\bar{\phi}J$ by $\bar{J}\{-\partial^2 + m^2\}\phi$ and $(\{-\partial^2 + m^2\}\bar{\phi})J$ leading to insertions of $\{-\partial^2 + m^2\}$ in (3.1.24). These insertions could be generated by functional derivatives with respect to $\sqrt{h}$ at the ends of the curves w since

$$\frac{2\delta}{\delta\sqrt{h\,(0)}} e^{-S_0[h,\,\omega]} = - \left. \frac{\dot{\omega}^2}{h} + m^2 \right|_{\xi=0} . \tag{3.1.28}$$

The expression on the right hand side is just $-p^2 + m^2$ because the canonical momentum on the worldline is $p^\mu = \frac{\dot{\omega}^\mu}{\sqrt{h}}$.

As we have said, we will not be able to fully achieve our aim of showing the equivalence of (3.1.18) and (3.1.19) until we include world-sheet supersymmetry. In any case QED with spin-one-half matter is more interesting as a realistic theory, and we will see that it emerges naturally from the spinning string. The generating functional for the Dirac field was given in the introduction

$$Z_D[\bar{K}, K] = \int \mathscr{D}A \, e^{-S_{gf} - \log \mathrm{Det}\left(-(\gamma \cdot \mathcal{D})^2 + m^2\right) + \int d^4x \bar{K} (\gamma \cdot \mathcal{D} + im)^{-1} K}, \tag{3.1.29}$$

which we expand as

$$\sum_{r,s=1}^{\infty} \frac{1}{r!s!} \int \mathscr{D}A \, e^{-S_{gf}} \left(-\log \mathrm{Det}\left(-(\gamma \cdot \mathcal{D})^2 + m^2\right)\right)^r \left(\int d^4x \bar{K} \left(\gamma \cdot \mathcal{D} + im\right)^{-1} K\right)^s \tag{3.1.30}$$

As in the scalar case we can represent the two components of this expression, the functional determinant and the Green function, as functional integrals of the same form but for closed and open worldlines respectively. This means we can express the generating functional for the Dirac field, Z_D, as

$$\begin{aligned} \sum_{r,s=1}^{\infty} \frac{1}{r!s!} \int \mathscr{D}A \, e^{-S_{gf}} \prod_{j=1}^{r+s} \frac{\mathscr{D}(h_j, w_j, \chi_j, \psi_j)}{Z} \, e^{-S_0[w_j, h_j] - S_F[\psi_j, \chi_j]} \\ \times \prod_{k=s+1}^{r+s} e^{-i\left(\oint dw_k \cdot A + \frac{1}{2} \int F_{\mu\nu} \psi^\mu \psi^\nu \sqrt{h} d\xi\right)} \\ \times \prod_{l=1}^{s} \int d^4 b_l \, d^4 a_l \, \bar{K}(b_l) \, e^{-i\left(\int_{a_l}^{b_l} dw_l \cdot A + \frac{1}{2} \int F_{\mu\nu} \psi^\mu \psi^\nu \sqrt{h} d\xi\right)} K(a_l) \end{aligned} \tag{3.1.31}$$

which contains the expectation value of supersymmetric exponentiated line integrals generalising the bosonic case. We will show that these expectation values can be calculated by introducing fermionic degrees of freedom onto the worldsheets spanned by the open and closed curves representing the Green functions and determinants in the field theory. That is,

$$\prod_{i=1}^{n} \frac{\mathscr{D}\left(X_i, \psi_i, g_i\right)}{Z_0} e^{-S_s} = \int \frac{\mathscr{D}A}{N} e^{-S'_{gf}[A]} \prod_i e^{-S_A} \tag{3.1.32}$$

where S'_{gf} is the equivalent gauged fixed action for the fermionic quantum theory and S_s is the action for the spinning string augmented by a supersymmetric generalisation of the contact interaction discussed above. We shall give explicit expressions for these objects in section 3.5. This equivalence can then be used to rewrite the integral over the gauge field in equation (3.1.31) in terms of open and closed spinning strings with contact interactions.

The purpose of this chapter is to investigate the relationship between (3.1.18) and (3.1.19) for the bosonic theory and then establish the supersymmetric version (3.1.32), showing that spinor QED is equivalent to tensionless spinning strings[2] with a contact interaction. We use the perturbative expansion in powers of q^2, building on the result (3.1.6). We begin with the purely bosonic theory. In section 3.2 we describe some basic tools including the regulator, and apply these to the derivation of (3.1.16). In section 3.3 we consider the first order in perturbation theory, studying potential divergences in some detail as a warm-up for higher order calculations, and also discuss how the split in the action (3.1.5) between the free string action and the contact term is affected by the regulator. Higher orders in perturbation theory for the bosonic case are discussed in section 3.4 which includes a discussion of potential problems associated with divergences that might be generated when the interaction terms approach each other close to the world-sheet boundary. Concluding that our bosonic string model is incomplete we turn to the more realistic case of spinor matter and show that this is naturally described by the spinning string in section 3.5. We discuss the gauge-fixed action and regulator and the residual supersymmetry, and then use this to restrict the divergences that can occur in the perturbative expansion enabling us to establish the connection between the spinning string model and spinor QED.

3.2 General expectation values

Before proceeding to the evaluation of the partition function we describe our general approach to the computation of such functional integrals which will be essentially standard. These functional integrals are computed conventionally by first integrating over the X_i with source terms to generate the insertions of $\partial_a X^\mu(\xi)$ and the

[2]The tensionless limit of bosonic string theory has a degenerate worldsheet metric [56] and as such can be reformulated on the level of the action by introducing a vector density to replace the metric in Polyakov's formulation [57,58]. The equation of motion of this auxiliary field imposes the null-metric condition and the formulation extends to the spinning string [59]. In the spinning case however the metric is no longer degenerate. Here we prefer to keep the tension arbitrary throughout the calculation to demonstrate how the tensionless limit suppresses unwanted quantities.

exponents resulting from the Fourier decomposition of the δ-functions. So we consider

$$\mathcal{Z}(j,k) = \int \mathscr{D}_g X \ \exp\left(-S[X,g] + \int d^2\xi \ J^\mu X^\mu\right) \tag{3.2.33}$$

where

$$J^\mu(\xi) = -\partial_a j^{\mu a}(\xi) + i\sum_j k_j^\mu\left(\delta\left(\xi - \xi_j\right)\right) \tag{3.2.34}$$

We write the field itself as the sum of three terms $X^\mu = X_c^\mu + \tilde{X}^\mu + \bar{X}^\mu$ where $\bar{X}^\mu$ is the quantum fluctuation to be functionally integrated over and X_c^μ and $\tilde{X}^\mu$ satisfy Euler-Lagrange equations. X_c^μ absorbs the boundary values of the original X. Denoting the two-dimensional Laplacian as Δ:

$$-\frac{1}{\sqrt{g}}\partial_a\left(\sqrt{g}g^{ab}\partial_b X_c^\mu(\xi)\right) \equiv \Delta X_c^\mu(\xi) = 0; \quad X_c^\mu|_{\partial D} = w^\mu, \tag{3.2.35}$$

and $\tilde{X}$ absorbs the sources we have just introduced

$$-\Delta\tilde{X}^\mu(\xi) = 2\pi\alpha'\left(i\sum_j k_j^\mu\left(\delta\left(\xi - \xi_j\right)\right) + \partial_a j^{\mu a}(\xi)\right) \tag{3.2.36}$$

and is required to vanish on ∂D. X_c and $\tilde{X}$ can both be found in terms of the Green function for the Laplacian with Dirichlet boundary conditions (which satisfies $\Delta G(\xi,\xi') = \delta^2\left(\xi - \xi'\right)/\sqrt{g}$, $G(\xi,\xi') = 0$ for ξ or $\xi' \in \partial D$):

$$X_c^\mu(\xi) = \oint_{\partial D} d\tilde{\xi}^c \ \epsilon_{ac}\sqrt{\tilde{g}}\tilde{g}^{ab}\tilde{\partial}_b G\left(\xi,\tilde{\xi}\right) w^\mu(\tilde{\xi}), \qquad \tilde{X}(\xi) = -2\pi\alpha'\int d^2\tilde{\xi} \ G(\xi,\tilde{\xi})\, J^\mu(\tilde{\xi}) \tag{3.2.37}$$

Integrating out the quantum fluctuation generates a determinant so

$$\begin{aligned}\mathcal{Z}(j,k) = \exp&\left(-\pi\alpha'\sum_{rs} k_r\cdot k_s\, G(\xi_r,\xi_s) + S[X_c,g] - 2\log(\mathrm{Det}\Delta)\right.\\ &+ 2\pi\alpha' i\int d^2\xi \ j^{\mu a}\sum_r k_r^\mu\partial_a G\left(\xi,\xi_r\right) + 2\pi\alpha'\iint d^2\xi d^2\xi' \ j^{\mu a}j^{\mu b}\partial_a\partial_b G\left(\xi,\xi'\right)\\ &\left.+\int d^2\xi \ j^{\mu a}\partial_a X_c^\mu(\xi) + i\sum_r k_r\cdot X_c\left(\xi_r\right)\right)\end{aligned} \tag{3.2.38}$$

log(DetΔ) depends only on the scale of the metric, and not on the sources, so will factor out of our computations because of the decoupling of the expectation value of the delta function discussed in the introduction. An alternative approach would be to assume the existence of further internal degrees of freedom to cancel the dependence on the Liouville mode, a route we consider in Chapter 4 of this thesis.

The Green function is divergent at coincident points and so we replace it with a regulated version constructed from the heat-kernel

$$G_\epsilon(\xi,\xi') = \int_\epsilon^\infty d\tau\ \mathcal{G}(\xi,\xi;\tau)\ , \quad \frac{\partial}{\partial\tau}\mathcal{G} = -\Delta\mathcal{G}\ , \quad \mathcal{G}\left(\xi,\xi';0^+\right) = \delta^2\left(\xi-\xi'\right)/\sqrt{g} \tag{3.2.39}$$

This cut-off procedure is reparameterisation invariant since the definition of the kernel and Green function do not require a choice of coordinates. It is not, however, Weyl invariant since it introduces a distance cut off. The effect of ϵ is to modify for high modes the spectral decomposition of the Green function in terms of the eigenfunctions of Δ, u_n, belonging to eigenvalues λ_n to [35]

$$G_\epsilon(\xi,\xi') = \sum_n u_n(\xi)\, u_n(\xi')\, \frac{e^{-\epsilon\lambda_n}}{\lambda_n}. \tag{3.2.40}$$

The short-distance divergence of the Green function is associated with the short-time behaviour of the heat kernel. Information can be extracted by expansion about a flat metric because in a short time the heat from the delta-function source cannot diffuse too far meaning that the heat kernel is sensitive to variations in the metric only over a distance of size of the order of $\sqrt{\epsilon}$.

The general form of the heat kernel for small times can be determined using the Seeley-DeWitt expansion [60] which can be modified to take into account the presence of the boundary [61, 62]. If $\sigma_n(\xi,\xi')$ denotes the square of the length of a geodesic between the points ξ and ξ' with n reflections at the boundary then we can write [61, 62]

$$\mathcal{G}(\xi,\xi';\tau) = \frac{1}{4\pi\tau}\sum_{\{\sigma_r\}} \exp\left(-\frac{\sigma_r(\xi,\xi')}{4\tau}\right)\Omega(\xi,\xi';\tau) \tag{3.2.41}$$

with the sum running over all geodesics including any number of reflections at the boundary. By virtue of the Hamilton-Jacobi equation for the action of a given geodesic, σ_r obeys an important constraint

$$\sigma = g^{ab}\partial_a\sigma\partial_b\sigma \tag{3.2.42}$$

at the point ξ with a similar expression holding at the other end of the path ξ'. The Ω_r can be expanded as a power series in τ with DeWitt coefficients $a_n(\xi, \xi')$. Substitution of the expansion into the heat-kernel equation leads to a recurrence relation for these coefficients

$$\Omega_r(\xi, \xi'; \tau) = \sum_{n=0}^{\infty} a_n^r(\xi, \xi')\,\tau^n \tag{3.2.43}$$

$$a_0^0(\xi, \xi) = 1 \tag{3.2.44}$$

$$a_0^r(\xi, \xi) + a_0^{r+1}(\xi, \xi) = 0 \tag{3.2.45}$$

$$\left(n + \frac{1}{2}\nabla^2\sigma - g^{ab}\partial_a\sigma\partial_b\right)a_n^r = -\nabla^2 a_{n-1}^r \tag{3.2.46}$$

Considering then the low τ limit and bringing $\xi' \to \xi$ repeated differentiation of (3.2.42) allows the determination of the coincident limits of the a_n. There are two regions of interest. In the bulk the path of zero length dominates, whereas close to the boundary the shortest reflected path joining ξ to itself also contributes. At lowest order the coefficients are determined as

$$a_0^0(\xi, \xi) = 1 \tag{3.2.47}$$

$$a_1^0(\xi, \xi) = \frac{1}{6}R(\xi) \tag{3.2.48}$$

$$a_0^1(\xi, \xi) = -1 \tag{3.2.49}$$

$$a_1^1(\xi, \xi) = -\frac{1}{6}R(\xi)\,. \tag{3.2.50}$$

The dominant small ϵ behaviour of the coincident Green's function can then be expressed as

$$G_\epsilon(\xi,\xi) \equiv \psi(\xi) \sim \int_\epsilon^\infty \frac{d\tau}{4\pi\tau}\left(1-\exp\left(-\frac{\sigma}{4\tau}\right)\right)\left(1+\frac{1}{6}R(\xi)\tau\right) \tag{3.2.51}$$

$$= \begin{cases} \frac{\sigma}{16\pi\epsilon} - \frac{\sigma \ln \epsilon R}{96\pi} & \sigma \ll \epsilon \\ \frac{1}{4\pi}\ln\frac{\sigma}{4\epsilon} - \frac{\epsilon R}{24\pi} & \sigma \gg \epsilon \end{cases} \tag{3.2.52}$$

This reveals that as ξ varies from being on the boundary to moving into the bulk, ψ varies from 0 to order $\log \epsilon$ over a distance $\epsilon^{\frac{1}{2}}$.

We will need the form of the above functions in conformally flat gauge. We may choose complex coordinates $z = x + iy$ with $ds^2 = e^\phi dz d\bar{z}$. With this choice $R(z) = e^{-\phi}\partial\bar{\partial}\phi = 0$. For much of our work it will be sufficient to take ϕ to be constant and work on the half-plane $y \geq 0$ whereby

$$\sigma(z,z') = e^\phi |z-z'|^2 \tag{3.2.53}$$

and for the coincident Green function the distance of the shortest path from z reflected from the boundary is $\sigma = 4e^\phi y^2$. So

$$\mathcal{G}(z,z';\tau) = \frac{1}{4\pi\tau}\left(\exp\left(-\frac{e^\phi|z-z'|^2}{4\tau}\right) - \exp\left(-\frac{e^\phi|z-\bar{z}'|^2}{\tau}\right)\right) \tag{3.2.54}$$

$$\psi(\xi) \sim \begin{cases} \frac{e^\phi y^2}{4\pi\epsilon} & \sigma \ll \epsilon \\ \frac{1}{4\pi}\ln\frac{e^\phi y^2}{\epsilon} & \sigma \gg \epsilon \end{cases} \tag{3.2.55}$$

This provides a useful method to track the appearance of ϕ through the calculation. However in the more general case σ picks up non-trivial ϕ-dependent corrections and the heat-kernel picks up curvature corrections according to (3.2.51). These do not contribute to our calculation at leading order in ϵ so it will be sufficient to specialise to $\phi = 0$ and introduce a function f by:

$$G_\epsilon(z,z') = \int_\epsilon^\infty \frac{d\tau}{4\pi\tau}\left(\exp\left(-\frac{|z-z'|^2}{4\tau}\right) - \exp\left(-\frac{|z-\bar{z}'|^2}{4\tau}\right)\right)$$
$$= \int_\epsilon^\infty \frac{d\tau}{4\pi\tau}\left[\left(\exp\left(-\frac{|z-z'|^2}{4\tau}\right)-1\right) - \left(\exp\left(-\frac{|z-\bar{z}'|^2}{4\tau}\right)-1\right)\right]$$
$$\equiv -f\left(\frac{|z-z'|}{2\sqrt{\epsilon}}\right) + f\left(\frac{|z-\bar{z}'|}{2\sqrt{\epsilon}}\right) \qquad (3.2.56)$$

where

$$f(s) = \int_1^\infty \frac{d\tau}{4\pi\tau}\left(1-\exp\left(-\frac{s^2}{\tau}\right)\right), \qquad (3.2.57)$$

so that $\psi(\xi) = f(y/\sqrt{\epsilon})$. This function is monotonically increasing and can written in closed form as

$$\frac{1}{4\pi}\left(\gamma + \Gamma\left(0,s^2\right) + \ln\left(s^2\right)\right) \qquad (3.2.58)$$

where γ is Euler's constant and $\Gamma(0,s^2) = \int_{s^2}^\infty t^{-1}e^{-t}dt$ is the incomplete Gamma function. We shall not use this explicit form but rather note that f can be approximated for small ($s < a$) and large ($s > b$) values of s by

$$f(s) \approx \begin{cases} \frac{s^2}{4\pi} & s < a \ll 1 \\ \frac{1}{4\pi}\ln s^2 & s > b \gg 1 \end{cases} \qquad (3.2.59)$$

Before finishing this section we illustrate the regularisation of the Green function by revisiting (3.1.13) in order to demonstrate that the second term on the right hand side vanishes as the cut-off is removed and is also (independently) suppressed in the tensionless limit. In terms of the function f the right hand side of (3.1.13) becomes

$$\epsilon^{ab}\left(k^\mu\partial_b X_c^\nu\, 2\pi\alpha'\partial_a G(\xi,\xi) + \partial_a X_c^\mu \partial_b X_c^\nu\right) e^{ik\cdot(x-X_c(\xi))}e^{-\pi\alpha' k^2 G(\xi,\xi)}$$
$$= \left(-k^\mu\partial_x X_c^\nu\, 2\pi\alpha'\partial_y f(y/\sqrt{\epsilon}) + \epsilon^{ab}\partial_a X_c^\mu\partial_b X_c^\nu\right) e^{ik\cdot(x-X_c(\xi))}e^{-\pi\alpha' k^2 f(y/\sqrt{\epsilon})} \qquad (3.2.60)$$

which is to be integrated over $y > 0$ and over k. The first term leads to the required result (3.1.16). Integrating the second over k gives

$$\frac{1}{(\alpha' f(y/\sqrt{\epsilon}))^2}\epsilon^{ab}\partial_a X_c^\mu \partial_b X_c^\nu \, e^{-(x-X_c(\xi))^2/(4\pi\alpha' f(y/\sqrt{\epsilon}))} \tag{3.2.61}$$

As described in section 1 the integral over y is suppressed outside a thin strip of width Λ, say, bordering the boundary. Taking $\Lambda > b\sqrt{\epsilon}$ shows that outside this strip $f(y/\sqrt{\epsilon}) > \frac{1}{2\pi}\log(\Lambda/\sqrt{\epsilon})$ which becomes large as $\epsilon \downarrow 0$ and so damps (3.2.61) provided $\Lambda/\sqrt{\epsilon}$ also becomes large (which can be arranged whilst still taking Λ to zero). When we integrate (3.2.61) over the strip we can treat X_c as a slowly varying quantity, independent of y to leading order, leaving just the following integral to be computed, which we separate into three pieces in order to apply the leading order approximation to f from (3.2.59)

$$\begin{aligned}\int_0^\Lambda dy\,\frac{e^{-(x-X_c(\xi))^2/(4\pi\alpha' f(y/\sqrt{\epsilon}))}}{(\alpha' f(y/\sqrt{\epsilon}))^2} &= \\ \frac{\sqrt{\epsilon}}{\alpha'^2}\Bigg(\int_0^a dy\,\frac{(4\pi)^2 e^{-(x-X_c(\xi))^2/(\alpha' y^2)}}{y^4} &+ \int_a^b dy\,\frac{e^{-(x-X_c(\xi))^2/(4\pi\alpha' f(y))}}{(f(y))^2} \\ &+ \int_b^{\Lambda/\sqrt{\epsilon}} dy\,\frac{(4\pi)^2 e^{-(x-X_c(\xi))^2/(\alpha'\log y^2)}}{(\log y^2)^4}\Bigg)\end{aligned} \tag{3.2.62}$$

The first two integrals inside the brackets are independent of ϵ so the overall factor of $\sqrt{\epsilon}$ outside the brackets damps these terms as $\epsilon \downarrow 0$. The last term can be bounded:

$$\left|\int_b^{\Lambda/\sqrt{\epsilon}} dy\,\frac{(4\pi)^2 e^{-(x-X_c(\xi))^2/(\alpha'\log y^2)}}{(\log y^2)^4}\right| < (\Lambda/\sqrt{\epsilon} - b)(4\pi)^2/(\log b^2)^4\,. \tag{3.2.63}$$

Combining this with the overall factor of $\sqrt{\epsilon}$ causes this to go to zero with ϵ because Λ does. Consequently (3.2.62) goes to zero as the cut-off is removed.

We note that these integrals also vanish independently in the tensionless limit. When α' is large in comparison to the length scale of the boundary the exponents in the last two integrals can be ignored and the first integral simplifies on scaling y so that

(3.2.62) becomes

$$\frac{\sqrt{\epsilon}}{\alpha'^2}\left(\alpha'^{3/2}\int_0^\infty dy\,\frac{(4\pi)^2 e^{-(x-X_c(\xi))^2/y^2}}{y^4}+\int_a^b\frac{dy}{(f(y))^2}+\int_b^{\Lambda/\sqrt{\epsilon}}dy\,\frac{(4\pi)^2}{(\log y^2)^4}\right) \tag{3.2.64}$$

which is suppressed in the tensionless limit, $\alpha' \to \infty$, with the leading term coming from the first integral incorporating the small-y behaviour. We shall now turn to apply similar techniques to calculating the effect of the interaction term on the string theory partition function.

3.3 The first order interaction of the bosonic theory

In this section we will carry out the calculation to first order in the expansion of the interaction term, which is proportional to

$$\sum_{j,k}\int\left(\prod_i\frac{\mathscr{D}(X_i,g_i)}{Z_0}\,e^{-S_i}\right)\int_{\Sigma_j,\Sigma_k}d\Sigma_j^{\mu\nu}(\xi)\,\delta^4\left(X_j(\xi)-X_k(\xi')\right)\,d\Sigma_k^{\mu\nu}(\xi').$$

We shall show that the form of the coincident Green function suppresses the integrand for a general configuration of the points ξ and ξ' except for two cases. The result we seek will arise when both points are separately close to the boundary where we have seen that the Green function is of order 1. Secondly, divergences appear when the points become close in the bulk of the worldsheet but we shall discuss how these can be interpreted in terms of a renormalisation of the free action and are consistent with the original splitting of the action in (3.1.5). The question of the two points meeting one another close to the boundary will be discussed later as this could provide corrections to the equality we are trying to prove. In the spinning string neither divergences nor unwanted boundary contributions will arise, as will be demonstrated in section 3.5.

There are two kinds of term in this sum. The first is when $j \neq k$, in which case we can organise the integrals to reduce the computation to our previous result

(3.1.6):

$$
\begin{aligned}
&\sum_{j\neq k}\int \frac{\mathscr{D}(X_j,g_j)}{Z_0}\,e^{-S_j}\frac{\mathscr{D}(X_k,g_k)}{Z_0}\,e^{-S_k}\int_{\Sigma_j,\Sigma_k} d\Sigma_j^{\mu\nu}(\xi)\,\delta^4\left(X_j(\xi)-X_k(\xi')\right)\,d\Sigma_k^{\mu\nu}(\xi')\\
=&\sum_{j\neq k}\int \frac{\mathscr{D}(X_j,g_j)}{Z_0}\,e^{-S_j}\int_{\Sigma_j} d\Sigma_j^{\mu\nu}(\xi)\left\langle\int_{\Sigma_k}\delta^4\left(X_j(\xi)-X_k(\xi')\right)\,d\Sigma_k^{\mu\nu}(\xi')\right\rangle_{\Sigma_k}\\
=&\sum_{j\neq k}\int \frac{\mathscr{D}(X_j,g_j)}{Z_0}\,e^{-S_j}\int_{\Sigma_j} d\Sigma_j^{\mu\nu}(\xi)\,\frac{1}{4\pi^2}\left(\partial_\mu\int_{B_k}\frac{dw_{k,\nu}}{||x_j-w_k||^2}-\partial_\nu\int_{B_k}\frac{dw_{k,\mu}}{||x_j-w_k||^2}\right)
\end{aligned}
\tag{3.3.65}
$$

Applying Stokes' theorem and observing that the boundary B_j is held fixed during the functional integration over Σ_j reduces this to

$$
\sum_{j\neq k}\frac{1}{2\pi^2}\int_{B_j,B_k}\frac{dw_j\cdot dw_k}{||w_j-w_k||^2} \tag{3.3.66}
$$

which is precisely the result we claimed.

The second type of term that occurs in the sum has $j = k$, in which case we have to consider

$$
\sum_{j}\int \frac{\mathscr{D}(X_j,g_j)}{Z_0}\,e^{-S_j}\int_{\Sigma_j} d\Sigma_j^{\mu\nu}(\xi)\,\delta^4\left(X_j(\xi)-X_j(\xi')\right)\,d\Sigma_j^{\mu\nu}(\xi')
$$

where both integrals are over the same worldsheet. If, as before, we make a Fourier decomposition of the δ-function

$$
\int_{\Sigma} d\Sigma^{\mu\nu}(\xi)\,\delta^4\left(X(\xi)-X(\xi')\right)\,d\Sigma^{\mu\nu}(\xi')=\int\frac{d^4k}{64\pi^4}\int d^2\xi\,d^2\xi'\,V_{-k}(\xi)\,V_k(\xi') \tag{3.3.67}
$$

this requires the computation of $\langle V_{-k}(\xi)\,V_k(\xi')\rangle_X$ which involves two insertions of V on the world-sheet in contrast to the single insertion of (3.1.13). To evaluate this we shall use Wick's theorem, based on (3.2.38), to write the expectation of products of fields as an expansion in terms of all possible contractions of X. The simplest is

$$
X^\mu X^\nu \cong\, :X^\mu X^\nu: + \underbrace{X^\mu X^\nu}, \tag{3.3.68}
$$

where by the normal ordering colons we mean that all contractions have been carried out between the fields contained within and the basic contraction is

$$\underbrace{\tilde{X}^{\mu}(\xi)\,\tilde{X}^{\nu}}(\xi') = \alpha'\delta^{\mu\nu}G(\xi,\xi') \tag{3.3.69}$$

We use the $\cong$ sign to denote that the equality is meant to hold inside the functional integral $\langle \quad \rangle_X$. Because (3.2.38) was obtained by expanding about a classical field X_c that contains the information about the boundary value the expectation of the normal ordered part of the product contains X_c, thus

$$\langle :X^{\mu}X^{\nu}:\rangle / \langle 1\rangle = X_c^{\mu}X_c^{\nu}. \tag{3.3.70}$$

with similar expressions holding for greater numbers of operators in the product. $\langle 1\rangle$ is included because it contains boundary data $S[X_c, g]$ as well as functional determinants. The exponential

$$e^{ik\cdot(X(\xi)-X(\xi'))} \cong :e^{ik\cdot(X(\xi)-X(\xi'))}: e^{-\pi\alpha' k^2\Psi} \tag{3.3.71}$$

with

$$\Psi(\xi,\xi') = \psi(\xi) + \psi(\xi') - 2G(\xi,\xi'). \tag{3.3.72}$$

will be crucial in what follows. $\Psi(\xi,\xi')$ is the two dimensional version of the object which we met in the previous chapter and plays a similar role in determining the behaviour of the integrands which will appear. This time, however, the Green functions making it up diverge at coincident points and it is their regularised versions we employ. In the parametrisation of D of the previous section this is

$$\Psi = -f(0) + f\left(\frac{y}{\sqrt{\epsilon}}\right) - f(0) + f\left(\frac{y'}{\sqrt{\epsilon}}\right) + 2\left(f\left(\frac{|z-z'|}{\sqrt{\epsilon}}\right) - f\left(\frac{|z-\bar{z}'|}{\sqrt{\epsilon}}\right)\right) \tag{3.3.73}$$

$$= f\left(\frac{y}{\sqrt{\epsilon}}\right) + f\left(\frac{y'}{\sqrt{\epsilon}}\right) + 2\left(f\left(\frac{|z-z'|}{2\sqrt{\epsilon}}\right) - f\left(\frac{|z-\bar{z}'|}{2\sqrt{\epsilon}}\right)\right) \tag{3.3.74}$$

Applying Wick's theorem to $\langle V_{-k}(\xi)\,V_k(\xi')\rangle_X$ and carrying out the functional integration over X gives (with $X(\xi)$ and $X(\xi')$ renamed as X^1 and X^2 for brevity)

$$\langle \epsilon^{ab}\epsilon^{cd}\partial^1_a X^{1[\mu}\partial^1_b X^{1\,\nu]} e^{ik\cdot\left(X^1 - X^2\right)}\partial^2_c X^{2[\mu}\partial^2_d X^{2\,\nu]}\,\rangle_X / \langle\, 1\,\rangle_X = \tag{3.3.75}$$

$$\epsilon^{ab}\epsilon^{cd}\; e^{ik\cdot\left(X^1_c - X^2_c\right)} e^{-\pi\alpha' k^2 \Psi\left(\xi^1,\xi^2\right)} \Bigg(\partial^1_a X^{1[\mu}_c \partial^1_b X^{1\,\nu]}_c \partial^2_c X^{2[\mu}_c \partial^2_d X^{2\,\nu]}_c \quad \text{(I)}$$

$$+8\pi\alpha'\; ik^{[\mu}\partial^1_a\Psi\cdot\partial^1_b X^{1\,\nu]}_c \partial^2_c X^{2[\mu}_c \partial^2_d X^{2\,\nu]}_c \quad \text{(II)}$$

$$+\left(4\pi\alpha'\right)^2 ik^{[\mu}\partial^1_a\Psi\cdot ik^{[\mu}\partial^2_c\Psi\cdot\partial^1_b X^{1\,\nu]}_c \partial^2_d X^{2\,\nu]}_c \quad \text{(III)}$$

$$+96\pi\alpha'\;\partial^1_a\partial^2_c G\cdot\partial^1_b X^{1\nu}_c \partial^2_d X^{2\nu}_c \quad \text{(IV)}$$

$$+12\left(4\pi\alpha'\right)^2\;\partial^1_a\partial^2_c G\cdot ik^\nu\partial^1_b\Psi\partial^2_d X^{2\nu}_c \quad \text{(V)}$$

$$+6\left(4\pi\alpha'\right)^3\;\partial^1_b\partial^2_d G\cdot ik^\nu\partial^1_a\Psi\cdot ik^\nu\partial^2_c\Psi \quad \text{(VI)}$$

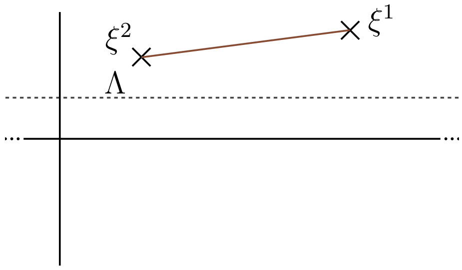
$$+48\left(4\pi\alpha'\right)^2\;\partial^1_a\partial^2_c G\cdot\partial^1_b\partial^2_d G\Bigg) \quad \text{(VII)}$$

where $G = G\left(\xi^1, \xi^2\right)$ and we have made use of the results of Appendix D.

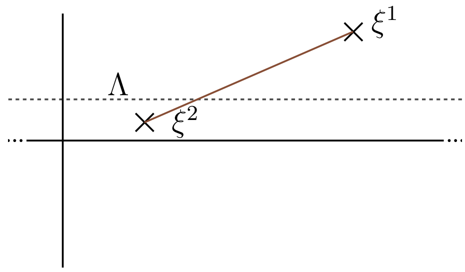
The exponential factor $e^{-\pi\alpha' k^2\Psi}$ depends on the configuration of the two points, as depicted in Figure 1, and will be important. For generic values of ξ^1 and ξ^2 in D, neither close to the boundary nor close to one another, Ψ is of order $\ln\epsilon$ so that the common exponential factor damps the integrand. As one of these points, say ξ^1, approaches the boundary $\psi\left(\xi^1\right)$ becomes of order unity, but with ξ^2 still in the bulk the $\psi\left(\xi^2\right)$ in Ψ keeps it of order $\ln\epsilon$. So the only values of ξ^1 and ξ^2 that lead to non-zero contributions as the cut-off is removed are those for which both points are close to the boundary or close to each other in the interior of D where Ψ is $\mathcal{O}(1)$. We will describe these two cases separately in the next two sub-sections.

3.3.1 Boundary contribution

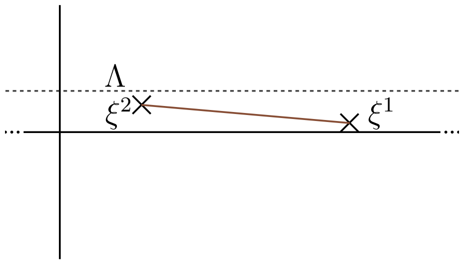
The first case to consider is $\langle V_{-k}(\xi)\, V_k(\xi')\rangle_X$ with both points ξ and ξ' close to the boundary. It is this case which will lead to our result. We need only integrate across a small suitably chosen strip, say of size Λ, since the integrand is suppressed moving into the bulk. We use the upper-half plane parametrisation of the previous section and consider $0 < y < \Lambda$, $0 < y' < \Lambda$, and we will also limit our attention to the generic case of $|x - x'| > \Lambda$. For this configuration the rapidly varying



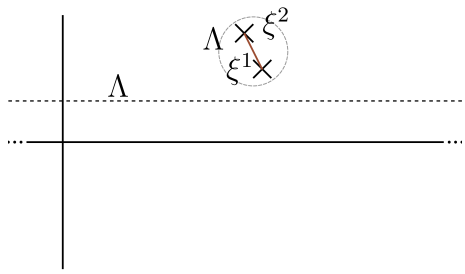
(a) An arbitrary configuration with both points in the bulk. Ψ is of order $\ln \epsilon$.



(b) The case that one of the points approaches the boundary. The second point in the bulk holds Ψ at order $\ln \epsilon$.



(c) The case that both points are with a distance of Λ of the boundary. As each point is integrated through this strip into the bulk Ψ varies from order 1 to order $\ln \epsilon$.



(d) The case that the two points are within a distance Λ of one another in the bulk. As one of the points is integrated about this region Ψ varies from order unity to order $\ln \epsilon$.

Figure 3.1: The possible configurations of the two points within the integration domain. The cases illustrated in the top line lead to heavy suppression of the integrand and it is the bottom two cases which will make finite contributions. The line joining the points is to represent that they are linked by the $\delta^D\left(X\left(\xi^1\right) - X\left(\xi^2\right)\right)$.

functions in (3.3.75) are just $\psi(\xi)$, $\psi(\xi')$ and their derivatives. In contrast, the fields X^μ, their derivatives and the Green function between the two points all vary smoothly and slowly. We shall consequently carry out this part of the integration by replacing slowly varying fields with their values on the boundary and then integrating the rapidly varying fields into the bulk. These arise from contractions between the component pieces within each V separately and not between them. We can anticipate the result by applying Wick's theorem to V_k by itself as

$$V_k \;\cong\; \epsilon^{ab}\left(:\, k^{[\mu}\partial_b X_c^{\nu]}\, e^{-ik\cdot X} :\, 2\pi\alpha'\partial_a\psi \,+\, :\, \partial_a X^\mu \partial_b X^\nu\, e^{-ik\cdot X} :\,\right) e^{-\pi\alpha' k^2 \psi} \tag{3.3.76}$$

This is similar in form to (3.1.13) because that equation is obtained as the expectation value of this. As y and y' are integrated over the strip, approximating the slowly varying functions in $\langle V_{-k}(\xi)\, V_k(\xi')\rangle_X$ as constant means that we can approximate

$$\int_0^\Lambda dy\ V_k \ \cong\ \epsilon^{ab} \colon k^{[\mu}\partial_b X^{\nu]}\, e^{-ik\cdot X} \colon 2\pi\alpha' \int_0^\Lambda dy\, \partial_a \psi\, e^{-\pi\alpha' k^2 \psi} + \epsilon^{ab} \colon \partial_a X^\mu \partial_b X^\nu\, e^{-ik\cdot X} \colon \int_0^\Lambda dy\, e^{-\pi\alpha' k^2 \psi}$$

which parallels the derivation of (3.1.16). So by a similar argument we can neglect the second integral and compute the first to obtain (as the cut-off is removed)

$$\int_0^\Lambda dy\, V_k \ \cong\ -2 \colon k^{[\mu}\partial_x X^{\nu]}\, e^{-ik\cdot X} \colon / k^2, \tag{3.3.77}$$

so that the boundary contribution from the product of two vertex operators is

$$\left\langle \int_{|y|<\Lambda} d^2\xi\, V_{-k}(\xi) \int_{|y'|<\Lambda} d^2\xi'\, V_k(\xi') \right\rangle_X = \frac{4}{(k^2)^2} \int_B \left\langle \colon k^{[\mu} dX^{\nu]}\, e^{-ik\cdot X} \colon \colon k^{[\mu} dX^{\nu]}\, e^{ik\cdot X} \colon \right\rangle_X \tag{3.3.78}$$

Contractions between the two normal ordered expressions involve the Green function that vanishes on the boundary, so in evaluating this expression we simply have to replace X by its classical value X_c which reduces to the boundary value w on B. This gives (up to a factor of $\langle 1\rangle_X$)

$$\int_B dw \cdot dw'\ \frac{e^{ik\cdot(w-w')}}{k^2} \tag{3.3.79}$$

which is the required result.

We will now give a more careful treatment of the same calculation, based on the explicit expression (3.3.75), to show that the less rapidly varying parts of (3.3.75) do not change the result. Beginning with term (I) we consider

$$\int\!\!\int dx dx' \int_0^\Lambda dy \int_0^\Lambda dy'\ \epsilon^{ab}\epsilon^{rs}\ \partial_a^1 X_c^{1[\mu}\, \partial_b^1 X_c^{1\,\nu]} \partial_r^2 X_c^{2[\mu}\, \partial_s^2 X_c^{2\,\nu]} \times e^{ik\cdot\left(X_c^1 - X_c^2\right)} e^{-\pi\alpha' k^2 \Psi(x,x';y,y')} \tag{3.3.80}$$

The rapidly varying part of this integral is contained in $\Psi\,(x, x'; y, y')$ and for $|x - x'| > \Lambda$ the last two terms of (3.3.74) are slowly varying and sum to zero on

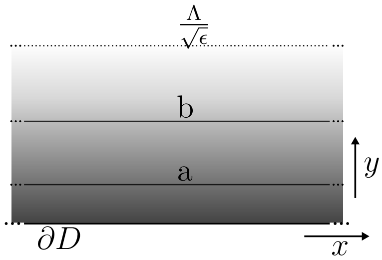


Figure 3.2: The regions of interest for the integral into the bulk. The lower line represents the boundary ∂D. a, b and Λ are chosen to allow the application of the approximate forms of Ψ. From the boundary up to a the lower approximation holds and from b to $\frac{\Lambda}{\sqrt{\epsilon}}$ the upper approximation holds. In between there is no explicit form of Ψ but integrals in this region are independent of ϵ.

the boundary. Their subleading pieces are higher order in ϵ and, since we will find no divergences for this boundary case, will not be important. We are consequently left with the integral

$$\int\int dxdx'\ \epsilon^{ab}\epsilon^{rs}\ \partial_a X_c^{1[\mu}\,\partial_b X_c^{1\,\nu]} e^{ik\cdot(w-w')}\partial'_r X_c^{2[\mu}\,\partial'_s X_c^{2\,\nu]}\int_0^\Lambda dy\int_0^\Lambda dy'\ e^{-\pi\alpha' k^2\Psi(x,x';y,y')} \tag{3.3.81}$$

Using (3.3.74) the integrals over y and y' factorise:

$$\int_0^\Lambda dy\ e^{-\pi\alpha' k^2 f\left(\frac{y}{\sqrt{\epsilon}}\right)}\int_0^\Lambda dy'\ e^{-\pi\alpha' k^2 f\left(\frac{y'}{\sqrt{\epsilon}}\right)} \tag{3.3.82}$$

We make a change of variables to scale out ϵ, $\frac{y^2}{\epsilon}\to y^2$, and then split the integral into the three parts described in the previous section. This is illustrated in Fig. 3.2 and allows the use of (3.2.59) in the first and third regions:

$$\begin{aligned}\int_0^\Lambda dy\ e^{-\pi\alpha' k^2 f\left(\frac{y}{\sqrt{\epsilon}}\right)} &= \epsilon^{\frac{1}{2}}\int_0^{\frac{\Lambda}{\sqrt{\epsilon}}} dy\ e^{-\pi\alpha' k^2\psi\left(y^2\right)}\\ &= \epsilon^{\frac{1}{2}}\left(\int_0^a dy\ e^{-\pi\alpha' k^2\cdot\frac{1}{4\pi}y^2}+\int_a^b dy\ e^{-\pi\alpha' k^2 f\left(y^2\right)}\right.\\ &\quad\left.+\int_b^{\frac{\Lambda}{\sqrt{\epsilon}}} dy\ e^{-\pi\alpha' k^2\cdot\frac{1}{4\pi}\ln\left(y^2\right)}\right).\end{aligned} \tag{3.3.83}$$

The first two terms on the bottom line have no divergences in their integrands and so evaluate to some (k-dependent) constant multiplied into $\epsilon^{\frac{1}{2}}$. The last term is

$$\epsilon^{\frac{1}{2}}\int_b^{\frac{\Lambda}{\sqrt{\epsilon}}} dy\ e^{-\frac{1}{4}\alpha' k^2\ln\left(y^2\right)} = \frac{\Lambda}{1-\frac{1}{2}\alpha' k^2}\left(\frac{\Lambda}{\sqrt{\epsilon}}\right)^{-\frac{1}{2}\alpha' k^2} - \sqrt{\epsilon}\frac{b^{1-\frac{1}{2}\alpha' k^2}}{1-\frac{1}{2}\alpha' k^2} \tag{3.3.84}$$

Both terms vanish as the cut-off is removed since Λ also goes to zero in this limit and because $k^2 \geq 0$. These terms multiply a corresponding contribution from y' with identical ϵ-dependence, so that overall the product goes to zero as the cut-off is removed.[3]

For the remaining terms (II-VII in (3.3.75)) we shall determine their ϵ-dependence by picking out the rapidly varying bits of each expression and evaluating the integrals. The derivative structure of the above terms determines the ϵ-dependence, since a derivative normal to the boundary cancels the factor of $\epsilon^{\frac{1}{2}}$ which arises under the scaling of y. So the terms which we may expect to contribute to the expectation value will have two derivatives of the rapidly varying Ψ; one with respect to y and one with respect to y'. Since the Green function is slowly and smoothly varying when the two points are not close together terms (IV) and (VII) actually have the same rapid variation as term (I) above, though they are multiplied by different powers of α'. The ϵ-dependence of terms (II) and (V) is the same, whilst terms (III) and (VI) share the same dependence.

We consider the y-integral of term (II):

$$\alpha' i k^\mu \int_0^\Lambda dy \ \epsilon^{ab} \partial_a \Psi \cdot e^{-\pi\alpha' k^2 \Psi} \tag{3.3.87}$$

Only the $f\left(\frac{y}{\sqrt{\epsilon}}\right)$ part of Ψ varies rapidly with y. Since this is a function of y only, the non-zero contribution arises when $a = 2$ and the presence of this derivative

[3] We have worked with the Fourier transform, implying that we should integrate our final expressions over k so there is a question as to whether this integral converges. To explore this we can in fact carry out the k-integral first (as at the end of section 2) which we now do for the strip close to the boundary, bearing in mind that our expression has the usual Fourier exponential $e^{ik\cdot l}$ where $l^\mu = w^\mu - w'^\mu$:

$$\epsilon^{\frac{1}{2}} \int_0^a dy \int d^D k \ e^{-\pi\alpha' k^2 \cdot \frac{1}{4\pi} y^2} e^{ik\cdot l} \sim \epsilon^{\frac{1}{2}} \int_0^a dy \ y^{-D} e^{-\frac{2l^2}{\alpha' y^2}} \tag{3.3.85}$$

The final integral is well-defined for any value of D so this contribution vanishes as $\epsilon \to 0$. Furthermore we can consider the same situation in the upper region of integration where we have

$$\epsilon^{\frac{1}{2}} \int_b^{\frac{\Lambda}{\sqrt{\epsilon}}} dy \ \int d^D k \ e^{-\pi\alpha' k^2 \cdot \frac{1}{4\pi} \ln y^2} e^{ik\cdot l} \sim \epsilon^{\frac{1}{2}} \int_b^{\frac{\Lambda}{\sqrt{\epsilon}}} dy \ (\ln y)^{-\frac{D}{2}} \ e^{-\frac{2}{\alpha' \ln y} l^2} \tag{3.3.86}$$

which is bounded by Λ multiplied by the greatest value of the integrand, which is in turn smaller than $(\ln b)^{-D/2}$ and so vanishes with the cut-off as required.

makes the integrand invariant to scaling:

$$\begin{aligned}\alpha' i k^{\mu} \int_0^{\frac{\Lambda}{\sqrt{\epsilon}}} \partial_y f(y) \cdot e^{-\pi\alpha' k^2 f(y)} &= \frac{-2ik^{\mu}}{\pi k^2} \int_0^{\frac{\Lambda}{\sqrt{\epsilon}}} \partial_y \left(e^{-\pi\alpha' k^2 f(y)} \right) \\ &= \frac{2ik^{\mu}}{\pi k^2} \left(1 - \left(\frac{\Lambda}{\sqrt{\epsilon}} \right)^{-\frac{1}{2}\alpha' k^2} \right). \end{aligned} \tag{3.3.88}$$

The second term here vanishes as the regulator is removed because $k^2 \geq 0$. We must also combine the above answer with the y' integral which is of the same form as that evaluated for term (I). We find that their product vanishes as $\epsilon \to 0$, as will the contribution from term (V).

Both terms (III) and (VI) have two derivatives acting on Ψ so we expect to get two copies of the form of the y-integral evaluated above. The y and y' dependent part of Term (III) take the form

$$\frac{1}{4}\alpha'^2 k^{\mu} k^{\mu} \partial_y \Psi \partial'_y \Psi e^{-\pi\alpha' k^2 \Psi}. \tag{3.3.89}$$

Reinstating the remaining boundary factors and the antisymmetry on the worldsheet indices and scaling ϵ out of the integrand gives with $l^{\mu} = w^{\mu} - w'^{\mu}$

$$\begin{aligned} &\frac{1}{4}\alpha'^2 k^{[\mu} k^{[\mu} \int\int_{\partial D} dx dx' \; \partial_x X_c^{1\,\nu]} \partial'_x X_c^{2\,\nu]} e^{ik\cdot l} \int_0^{\frac{\Lambda}{\sqrt{\epsilon}}} dy \; \partial_y f(y)\, e^{-\pi\alpha' k^2 f(y)} \times \\ &\qquad\qquad\qquad \int_0^{\frac{\Lambda}{\sqrt{\epsilon}}} dy' \; \partial_y f(y')\, e^{-\pi\alpha' k^2 f(y')} \\ =&\frac{1}{4}\alpha'^2 k^{[\mu} k^{[\mu} \int\int_{\partial D} dx dx' \; \partial_x X_c^{1\,\nu]} \partial'_x X_c^{2\,\nu]} e^{ik\cdot l} \int_0^{\infty} df e^{-\pi\alpha' k^2 f} \int_0^{\infty} df' e^{-\pi\alpha' k^2 f'} \\ =& \int\int_C \frac{dw^{[\nu} dw'^{[\nu} k^{\mu]} k^{\mu]}}{\pi^2 \left(k^2\right)^2} e^{ik\cdot l}. \end{aligned} \tag{3.3.90}$$

On the second line we removed the regulator taking $\epsilon \to 0$. The final step is to integrate over all values of k and to apply the contraction of the target space indices so that the full expression reads

$$\int\int_B dw \cdot dw' \int d^D k \; \frac{2(D-1)}{\pi^2 k^2} e^{ik\cdot(w-w')} \tag{3.3.91}$$

We see here the Fourier representation for a massless vector propagator integrated around the boundary which is depicted in Fig 3.3. This result is independent of the metric on the worldsheet and thus on its scale despite our integral over k not being on-shell. We discuss this further in the next section but first turn to the calculation of the final term (VI) and demonstrate that it is in fact vanishing by our choice of coordinates.

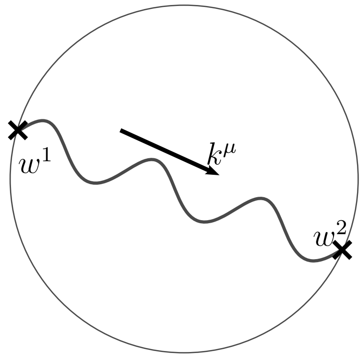


Figure 3.3: At first order the correction is given by a massless vector propagator between the boundary points x^1 and x^2 to be integrated around the boundary with respect to both points.

The only contribution to term (VI) arises when the derivatives of Ψ are with respect to y and y'. This then contains

$$\alpha'^3 \delta^{\nu\nu} \partial_x \partial'_x G \frac{1}{4} k^\mu k^\mu \partial_y \Psi \partial'_y \Psi e^{-\pi\alpha' k^2 \Psi}. \tag{3.3.92}$$

The smoothly varying field G has a Taylor expansion based at the boundary where its value is identically zero. The partial derivatives $\partial_x \partial'_x G$ are then along the boundary so vanish identically. All other contributions are slowly varying and are subleading in ϵ so vanish as $\epsilon \to 0$. We have thus demonstrated that there is only one contribution to the correlation functions from close to the boundary – (3.3.91). We postpone further discussion of this result until we have considered the contribution from the two points coming close together in the bulk.

3.3.2 Bulk divergences

We now study what happens to $\langle V_{-k}(\xi)\, V_k(\xi')\rangle$ as the two points ξ and ξ' approach each other far from the boundary. Corresponding to the split between the free action and the interaction term in (3.1.5) we will show that this leads to a renormalisation of the free action. This computation is also useful in considering the more general case that occurs at higher order of several V_k approaching each other in the bulk.

We consider (3.3.75) for ξ close to ξ' but far from the boundary, so that Ψ can be separated into rapidly and slowly varying parts:

$$\Psi = -2f\left(\frac{|z-z'|}{2\sqrt{\epsilon}}\right) + \left(2f\left(\frac{|z-\bar{z}'|}{2\sqrt{\epsilon}}\right) - f\left(\frac{y}{\sqrt{\epsilon}}\right) - f\left(\frac{y'}{\sqrt{\epsilon}}\right)\right). \tag{3.3.93}$$

Using the large distance behaviour (3.2.59) for the final two terms this is

$$\Psi = -2f\left(\frac{|z-z'|}{2\sqrt{\epsilon}}\right) + \frac{1}{2\pi}\log\left(\frac{(x-x')^2+(y+y')^2}{4yy'}\right). \tag{3.3.94}$$

We will integrate firstly over ξ', keeping ξ fixed. Then the first term in (3.3.94) varies rapidly over a disk with centre ξ of size Λ, from 0 at the centre to order $\log(\Lambda/\sqrt{\epsilon})$ on the edge. Ψ acts as a damping factor for ξ' outside this disk if $\Lambda/\sqrt{\epsilon}$ is taken large as ϵ is taken to zero. The second is slowly varying and vanishes when the two points are coincident. The first subleading term is quadratic in $(x-x')$ so under the scaling we will carry out is of order ϵ. The exponent $\exp(ik\cdot(X-X'))$ is unity at zeroth order in ϵ and its first correction is of order $\sqrt{\epsilon}$. We shall see that it is only for terms (V), (VI) and (VII) that these corrections are relevant due to divergences which we will encounter for these terms. Our general strategy will be to concentrate on the rapidly varying parts of the integrands we need and to replace the slowly varying fields by their values at the point ξ.

First we consider (I). Instead of the two integrals with respect to y and y' that we had to consider for points close to the boundary in (3.3.1) we now have to integrate over the disk. The rapidly varying parts of the integrand are

$$\int dxdy\int_{|z-z'|\leqslant\Lambda} dx'dy'\; e^{-\pi\alpha' k^2 f\left(\frac{r}{\sqrt{\epsilon}}\right)} = 2\pi\epsilon\int dxdy\int_0^{\frac{\Lambda}{\sqrt{\epsilon}}} dr\; re^{-\pi\alpha' k^2 f(r)} \tag{3.3.95}$$

where we have used polars and scaled by $\sqrt{\epsilon}$. We split the integration region into the three parts demonstrated in Fig. 3.4 enabling us to use the short and large-distance

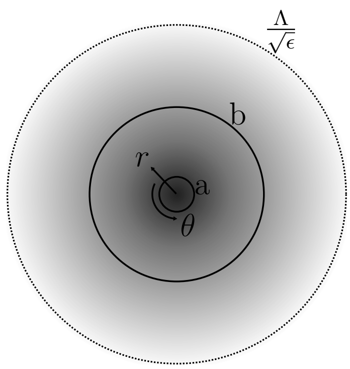


Figure 3.4: The regions of interest for the integral into the bulk. The centre represents the coincidence of the two points in the bulk. a, b and Λ are chosen to allow the application of the approximate forms of Ψ. From the centre out to a the lower approximation holds and from b to $\frac{\Lambda}{\epsilon}$ the upper approximation holds. In between (the shaded part) there is no explicit form of Ψ but integrals in this region are independent of ϵ.

approximations for f. This provides

$$2\pi\epsilon \int_0^{\frac{\Lambda}{\sqrt{\epsilon}}} dr\ re^{-\pi\alpha' k^2 f(r)} = 2\pi\epsilon \left(\int_0^a dr\ re^{-\alpha' k^2 \frac{r^2}{4}} + \int_a^b dr\ re^{-\pi\alpha' k^2 f(r)} + \int_b^{\frac{\Lambda}{\sqrt{\epsilon}}} dr\ r^{1-\frac{1}{2}\alpha' k^2} \right) \tag{3.3.96}$$

The explicit factors of ϵ multiplying the first two integrals cause these terms to vanish as the cut-off is removed. The final term evaluates to

$$\frac{\pi}{1-\alpha' k^2/4} \left(\Lambda \left(\frac{\Lambda}{\sqrt{\epsilon}} \right)^{-\alpha' k^2/2} - \epsilon b^{2-\alpha' k^2/2} \right) \tag{3.3.97}$$

which goes to zero as the cut-off is removed because $k^2 \geq 0$ and Λ goes to zero.

Turning to (II), the rapidly varying part that we have to integrate over the disk is

$$\alpha' \int^{r \leqslant \Lambda} dx' dy'\ \partial'_c f \left(\frac{r}{\sqrt{\epsilon}} \right) e^{-\pi\alpha' k^2 f\left(\frac{r}{\sqrt{\epsilon}}\right)} \tag{3.3.98}$$

which vanishes by rotational symmetry. There are corrections to this arising from terms subleading in ϵ and also from the derivative acting on the slowly varying part of Ψ but these have the same ϵ dependence as Term (I) because they share the same rapidly varying content.

We now find the terms which lead to renormalisation of the string action. Using

$\partial_a g(r) = -\partial'_a g(r)$ for any function of r, the rapidly varying part of (III) can be written

$$\alpha'^2 k^\mu k^\alpha \int dxdy \int^{r\leqslant\Lambda} dx'dy'\ \partial'_a f\left(\frac{r}{\sqrt{\epsilon}}\right)\partial'_c f\left(\frac{r}{\sqrt{\epsilon}}\right) e^{-\pi\alpha' k^2 f\left(\frac{r}{\sqrt{\epsilon}}\right)} \tag{3.3.99}$$

The integral over the primed variables must be proportional to δ_{ac} by symmetry and we can extract the constant of proportionality by contracting these indices; splitting up the integration region once again gives

$$\begin{aligned}\int^{r\leqslant\frac{\Lambda}{\sqrt{\epsilon}}} dx'dy'\ \partial'_a f(r)\,\partial'_a f(r)\,e^{-\pi\alpha' k^2 f(r)} &= 8\pi\int_0^a dr\ r^3 e^{-\alpha' k^2\frac{r^2}{4}}\\ &+ 2\pi\int_a^b dr\ r\partial'_a f(r)\,\partial'_a f(r)\,e^{-\pi\alpha' k^2 f(r)}\\ &+ 8\pi\int_b^{\frac{\Lambda}{\sqrt{\epsilon}}} dr\ r^{-1-\frac{1}{2}\alpha' k^2}\end{aligned} \tag{3.3.100}$$

which remains finite as the regulator is removed. There are again further contributions from the slowly varying fields but these vanish as we take $\epsilon \to 0$[4]. Putting this together with the slowly varying parts of (III) gives a term proportional to

$$\alpha'^{-1}\int d^2\xi^1\ \delta^{ab}\partial^1_a X^\mu_c \partial^1_b X^\mu_c \tag{3.3.102}$$

which is simply a renormalisation of the free string theory action in the conformal gauge we have chosen. Note that this is suppressed in the tensionless limit.

The remaining terms involve derivatives of the Green function and these are rapidly varying fields. However it is possible to simplify matters by noting that

$$\partial^1_a\partial^2_c G = -\frac{1}{2}\partial^1_a\partial^2_c\Psi. \tag{3.3.103}$$

[4]Again we have still to integrate over k and that leads to ultra-violet divergences which we can regulate by dimensional regularisation. Large k corresponds to small r and causes the first integral to diverge at the origin, but we keep the spacetime dimension D arbitrary (in a range where the integral exists) and compute

$$\int_0^a dr\int d^D k\ k^\mu k^\alpha r^3 e^{-\frac{1}{4}\alpha' r^2 k^2} = \delta^{\mu\alpha}\frac{2^{D+1}a^{2-D}}{2-D}\left(\frac{\pi}{\alpha'}\right)^{D/2+1} \tag{3.3.101}$$

which continues to the physical value of $D = 4$.

Turning to term (IV), the rapidly varying piece is

$$\alpha' \int^{r\leqslant\Lambda} dx'dy'\ \partial'_a\partial'_c f\left(\frac{r}{\sqrt{\epsilon}}\right) e^{-\pi\alpha' k^2 f\left(\frac{r}{\sqrt{\epsilon}}\right)} \tag{3.3.104}$$

which by symmetry must be proportional to δ_{ac}, so it is sufficient to consider its trace. However, the defining equation of the heat kernel implies that the function f obeys $\partial'_a\partial'_a f\,(r) = \frac{1}{\pi}e^{-r^2}$ so that we may immediately calculate this as

$$\frac{1}{2\pi}\int_0^{\Lambda/\sqrt{\epsilon}} dr\ re^{-r^2}e^{-\pi\alpha' k^2 f(r)} \tag{3.3.105}$$

which is also finite as the cut-off is removed. As with the previous term the $X-$dependence of the slowly varying contributions leads to a renormalisation of the free action which is also suppressed in the tensionless limit.

The analysis of (V) is more involved. Naively the calculation of the rapidly varying piece follows that of term (II) because it vanishes by rotational invariance:

$$\alpha'^2 \int dxdy\ ik\cdot\partial_d X \int^{r\leqslant\Lambda} dx'dy'\ \partial'_a\partial'_c f\left(\frac{r}{\sqrt{\epsilon}}\right)\partial'_b f\left(\frac{r}{\sqrt{\epsilon}}\right) e^{-\pi\alpha' k^2 f\left(\frac{r}{\sqrt{\epsilon}}\right)} = 0. \tag{3.3.106}$$

However by scaling r by $\sqrt{\epsilon}$ the three derivatives imply an overall factor of $1/\sqrt{\epsilon}$ so that we must expand the slowly varying fields beyond leading order to find contributions that could remain finite as the regulator is removed. This can be found from the expansion

$$\begin{aligned}\exp\left[ik\cdot(X_c - X'_c)\right] = 1 &- i\left[(x-x\prime)^\alpha\,\partial_\alpha X_c + \ldots\right]\cdot k \\ &- \frac{1}{2}\left[\left((x-x')^\alpha\,\partial_\alpha X_c + \ldots\right)\cdot k\right]^2 + \ldots\end{aligned} \tag{3.3.107}$$

where after scaling the first subleading term is $-i\sqrt{\epsilon}\,(x-x\prime)^\alpha\,\partial_\alpha X_c\cdot k$. This offers a correction

$$\epsilon^{ab}\epsilon^{cd}\alpha'^2\int dxdy\ ik\cdot\partial_d X\ ik\cdot\partial_\alpha X_c \int^{r\leqslant\frac{\Lambda}{\sqrt{\epsilon}}} dx'dy'\ \ (x-x\prime)^\alpha\,\partial'_a\partial'_c f\,(r)\,\partial'_b f\,(r)\,e^{-\pi\alpha' k^2 f(r)}. \tag{3.3.108}$$

This can be integrated by parts to reduce it to the same form as (III). In particular the procedure contracts the indices d and a and the integral over k contributes only its trace so that again we find a renormalisation of the free action which is suppressed in the tensionless limit.

This leaves only terms (VI) and (VII) to analyse. In fact, in the bulk the rapidly varying parts of (VI) and (VII) are related by integration by parts:

$$\begin{aligned} k^2\ \alpha'^3\epsilon^{ab}\epsilon^{cd}\int dxdy\int dx'dy'e^{ik\cdot(X_c-X_c')}\partial_b'\partial_d'f\partial_a'f\partial_c'fe^{-\pi\alpha'k^2f} \propto \\ \alpha'^2\epsilon^{ab}\epsilon^{cd}\int dxdy\int dx'dy'e^{ik\cdot(X_c-X_c')}\partial_b'\partial_d'f\left(\partial_c'\partial_a'f - ik\cdot\partial_c'X_c'\partial_a'f\right)e^{-\pi\alpha'k^2f} \end{aligned} \tag{3.3.109}$$

where the boundary contribution is exponentially suppressed as $\epsilon \to 0$. The second term in brackets has the same rapidly varying structure as term (V). The presence of four derivatives of f in the first term implies that when we scale r by $\sqrt{\epsilon}$ there will be an overall $1/\epsilon$ multiplying the integral. This time we will use (3.3.107) and must also expand the slowly varying part of Ψ:

$$\frac{1}{2\pi}\log\left(\frac{(x-x')^2+(y+y')^2}{4yy'}\right) = \frac{(x-x')^2+(y-y')^2}{8\pi y^2} + \dots \tag{3.3.110}$$

which under the scaling we apply is of order ϵ but is independent of X.

To begin with consider just the first term contributing to (VII):

$$\begin{aligned} \alpha'^2\epsilon^{ab}\epsilon^{cd}\int^{r\leqslant\Lambda} dx'dy'\ \partial_b'\partial_d'f\left(\frac{r}{\sqrt{\epsilon}}\right)\partial_c'\partial_a'f\left(\frac{r}{\sqrt{\epsilon}}\right)e^{-\pi\alpha'k^2f\left(\frac{r}{\sqrt{\epsilon}}\right)} = \\ \frac{\alpha'^2}{\epsilon}\int^{r\leqslant\frac{\Lambda}{\sqrt{\epsilon}}} dx'dy'\ \left(\partial_x'\partial_x'f(r)\,\partial_y'\partial_y'f(r) - \left(\partial_x'\partial_y'f(r)\right)^2\right)e^{-\pi\alpha'k^2f(r)}. \end{aligned} \tag{3.3.111}$$

For the first region of integration ($0 \leqslant r \leqslant a$) the second term in brackets is zero and the first is simply equal to four. For the outer region of integration – where $b \leqslant r \leqslant \frac{\Lambda}{\sqrt{\epsilon}}$ – both terms contribute and we find the bracketed terms evaluate to

$-4r^{-4}$ so that we must determine

$$\frac{4\alpha'^2}{\epsilon}\int_0^a dr\; re^{-\alpha' k^2 \frac{r^2}{4}} + \frac{\alpha'^2}{\epsilon}\int_a^b dr\; r\left(\partial'_x\partial'_x f(r)\,\partial'_y\partial'_y f(r) - \left(\partial'_x\partial'_y f(r)\right)^2\right) e^{-\pi\alpha' k^2 f(r)} - \frac{4\alpha'^2}{\epsilon}\int_b^{\frac{\Lambda}{\sqrt{\epsilon}}} dr\; r^{-3-\frac{1}{2}\alpha' k^2} \tag{3.3.112}$$

showing a $\frac{1}{\epsilon}$ divergence[5].

The $1/\epsilon$ divergence here is independent of X and corresponds to an infinite renormalisation of the cosmological term $\int d^2\xi\sqrt{g}$ which is implicit when considering the quantisation of the string. There is also a finite renormalisation of this term arising out of the subleading term in (3.3.110). These renormalisations are not suppressed in the tensionless limit $\alpha' \to \infty$.

We consider also the exponential factor that remains and use (3.3.107). It is more economic to carry out the integral over k first as in footnote 5. Then we expand

$$\begin{aligned}(X_c - X'_c)^2 &= \left[(x-x')^\alpha\,\partial_\alpha X_c + \cdots\right]^2 \\ &\sim \epsilon r^2 \partial_\alpha X_c \cdot \partial_\alpha X_c\end{aligned} \tag{3.3.114}$$

where the second line follows because it is to be inserted into an integral over a rotationally symmetric domain. After exponentiation we obtain a term independent of ϵ that renormalises the string action; again this renormalisation is not suppressed in the tensionless limit.

The renormalisations we have found in this section are just as we expected to find given the original derivation of (3.1.5). No further non-renormalisable divergences

[5]This too has to be integrated over k leading to a divergent integral that we again regulate by working in D spacetime dimensions. The first integral captures the $\frac{1}{\epsilon}$ divergence so we consider the inner-most region of integration and do the k-integration first:

$$\frac{4\alpha'^2}{\epsilon}\int_0^a dr \int d^D k\; re^{-\alpha' k^2 \frac{r^2}{4}} e^{ik\cdot\left(X_c - X'_c\right)} = \frac{4^{1+D/2}\alpha'^{(2-D/2)}}{\epsilon}\int_0^a dr\; r^{1-D} e^{-\frac{(X_c-X'_c)^2}{\alpha' r^2}} \tag{3.3.113}$$

The integral with respect to r could be defined for $D < 2$ to avoid the logarithmic divergence there as $\xi' \to \xi$ and be analytically continued to physical values of D.

appear which justifies the consistency of the contact interaction we have introduced. An appropriate way to split the action into a free and interacting piece is to take the latter to explicitly exclude the coincidence of the two-points ξ and ξ', which really requires that in the presence of the regulator which smears out the δ-function we take $|\xi - \xi'| > \Lambda$ in the interaction.

Returning to our aim of showing that the conformal scale of the worldsheet metric decouples from the expectation value we also note this is the only time it is necessary to consider higher order terms corresponding to variations about constant ϕ. We have worked with a constant worldsheet metric and absorbed the conformal scale into the cutoff. Had we explicitly tracked it through the calculation it would appear in this expression as $1/\epsilon e^{-\phi}$ and the Green function would pick up further dependence on ϕ which is subleading in ϵ. We can expand $\phi\left(x'\right)$ about the point x – the linear terms vanish when averaging in a disk about the point x so that the leading correction to our calculations is of order $\epsilon\nabla^2\phi$. This combines with the $\frac{1}{\epsilon}$ divergence found above to produce a finite term dependent on ϕ. It is proportional, however to $e^{\phi}R$, where R is the curvature on the worldsheet so integrating this term with respect to x provides simply

$$\int \sqrt{g}Rd^2\xi \tag{3.3.115}$$

which is a topological invariant, independent of ϕ. The higher order terms in the expansion vanish with the cut-off. This completes our discussion of the first order correction of the contact interaction term we propose. Up to renormalisations of the free string action and cosmological term we have found the result we sought and have shown that the conformal scale ϕ decouples from the calculation.

3.4 Higher order corrections

We now proceed to give a general analysis of the higher order corrections present in the theory with an aim to prove that the conformal scale decouples to all order in the perturbative expansion of the contact interaction. We follow the same procedure of extracting the rapidly varying parts of the integrands. We consider the order N

expansion of the interaction term with $2N$ vertex operator insertions (corresponding to $2N$ points ξ_i placed around the worldsheet) and consider

$$\left\langle V^{\alpha\beta}_{-k_1}(\xi_1)\, V^{\gamma\delta}_{k_1}(\xi_2)\cdots V^{\mu\nu}_{k_i}(\xi_i)\cdots V^{\rho\sigma}_{-k_N}(\xi_{2N-1})\, V^{\tau\chi}_{k_N}(\xi_{2N})\right\rangle. \tag{3.4.116}$$

which must be integrated with respect to each point ξ_i about the worldsheet as well as with respect to each of the momenta. Applying Wick's theorem to this product will produce a factor common to all terms

$$\exp\left(-\pi\alpha'\sum_{ij} k_i\cdot k_j G(\xi_i,\xi_j)\right) e^{i\sum_i k_i\cdot X_c(\xi_i)} \tag{3.4.117}$$

which will determine the damping of the integrand. The contractions which generate terms that are rapidly varying depend upon the placement of the 2N points in the bulk. The first exponent in (3.4.117), however, can be split into parts containing the coincident Green function for each point $\psi_i \equiv \psi(\xi_i)$ and those involving the Green function between two different points $G_{ij} \equiv G(\xi_i,\xi_j)$:

$$\sum_{ij} k_i\cdot k_j G(\xi_i,\xi_j) = \sum_i k_i^2\psi_i + \sum_{i\neq j} k_i\cdot k_j G_{ij}. \tag{3.4.118}$$

For a general placement of the 2N points the sum involving the ψ_i will ensure that the integrand is damped by a factor of order

$$\sqrt{\epsilon}^{\frac{\alpha}{4}'\sum_i k_i^2}, \tag{3.4.119}$$

but we must consider what happens when the points approach the boundary or when points approach one another in the bulk since here the effects of the G_{ij} also become important.

3.4.1 Points close to the boundary

The first case to consider is when we locate each of the points within a small strip close to the boundary. We continue to work on the upper half plane with coordinates x_i and y_i for each point. The $2N$ points will then be integrated a distance Λ

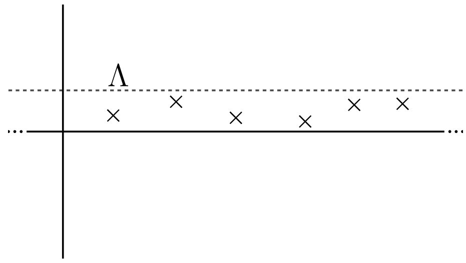


Figure 3.5: We consider N points located within a distance Λ of the boundary of the worldsheet but not within a distance Λ of one another.

into the bulk and in this section we continue to consider only the generic case where $|x_i - x_j| > \Lambda$ for all i and j – see Fig. 3.5. Consequently in this region the G_{ij} are slowly varying fields, whilst the ψ_i vary rapidly with y_i. We can again replace the slowly varying fields with their values at the boundary; in particular $G_{ij} = 0$ whenever either argument is on the boundary. To leading order in ϵ equation (3.4.117) therefore factorises as

$$\prod_i \exp\left(-\pi\alpha' k_i^2 f\left(\frac{y_i}{\sqrt{\epsilon}}\right)\right) e^{ik_i\cdot w_i} \tag{3.4.120}$$

where we have also replaced the field $X_c(\xi_i)$ by its boundary value w_i.

The contractions in (3.4.116) which will lead to the appearance of rapidly varying terms are those which will produce dependence on ψ_i. This occurs when we consider contractions amongst the component pieces in each V_{k_i} alone rather than those between different vertex operators. Contractions arising out of the pieces of $V_{k_i}^{\mu\nu}(\xi_i)$ provide

$$2\pi i\alpha'\epsilon^{ab}k_i^{[\mu} :\partial_b X_i^{\nu]} e^{-ik_i\cdot X_i}: \partial_a^i\psi_i e^{-\pi\alpha' k_i^2\psi_i} + \epsilon^{ab} :\partial_a^i X_i^\mu \partial_b^i X_i^\nu e^{-ik_i\cdot X_i}: e^{-\pi\alpha' k_i^2\psi_i}. \tag{3.4.121}$$

No further contractions are possible because of the antisymmetry of the worldsheet indices. Since ψ_i is a function of the distance into the bulk only we may limit consideration to derivatives with respect to y_i. We thus consider the general case where the integrand, (3.4.116), contains r contractions of the form $\partial_y^i\psi_i$. The remaining factors in the integrand can be replaced by their boundary values so the rapidly varying parts of integrals into the bulk can be expressed

$$\begin{aligned}
&\prod_{j=1}^{r} \alpha' k_j^{\mu_j} \int_0^{\Lambda} dy_j \; \partial_y^j f\left(\frac{y_j}{\sqrt{\epsilon}}\right) \exp\left(-\pi\alpha' k_j^2 f\left(\frac{y_j}{\sqrt{\epsilon}}\right)\right) \\
&\qquad\qquad \times \prod_{i=r+1}^{2N} \int_0^{\Lambda} dy_i \exp\left(-\pi\alpha' k_i^2 f\left(\frac{y_i}{\sqrt{\epsilon}}\right)\right) \\
&= \sqrt{\epsilon}^{2N-r} \prod_{j=1}^{r} \alpha' k_j^{\mu_j} \int_0^{\frac{\Lambda}{\sqrt{\epsilon}}} dy_j \; \partial_y^j f\left(y_j\right) \exp\left(-\pi\alpha' k_j^2 f\left(y_j^2\right)\right) \\
&\qquad\qquad \times \prod_{i=r+1}^{2N} \int_0^{\frac{\Lambda}{\sqrt{\epsilon}}} dy_i \exp\left(-\pi\alpha' k_i^2 f\left(y_i\right)\right) \\
&= \sqrt{\epsilon}^{2N-r} \prod_{j=1}^{r} \frac{2k_j^{\mu_j}}{\pi k_j^2} \left(1 - \left(\frac{\Lambda}{\sqrt{\epsilon}}\right)^{-\frac{1}{2}\alpha' k_j^2}\right) \prod_{i=r+1}^{2N} \int_0^{\frac{\Lambda}{\sqrt{\epsilon}}} dy_i \exp\left(-\pi\alpha' k_i^2 f\left(y_i\right)\right)
\end{aligned} \tag{3.4.122}$$

where the second equality follows after a scaling $\frac{y_i^2}{\epsilon} \to y_i^2$. This determines the ϵ-dependence of a term with r-contractions. Since $k_j^2 \geq 0$ in Euclidean signature and $2N - r \geq 0$ we see that the second term in rounded brackets will always vanish as the regulator is removed. This allows us to focus on the final product of integrals which can be bounded:

$$\begin{aligned}
\sqrt{\epsilon}^{2N-r} \left| \prod_{i=r+1}^{2N} \int_0^{\frac{\Lambda}{\sqrt{\epsilon}}} dy_i \exp\left(-\pi\alpha' k_i^2 f\left(y_i\right)\right) \right| &\leq \sqrt{\epsilon}^{2N-r} \prod_{i=r+1}^{2N} \left(\frac{\Lambda}{\sqrt{\epsilon}}\right) \exp\left(-\pi\alpha' k_i^2 f\left(0\right)\right) \\
&= \sqrt{\epsilon}^{2N-r} \left(\frac{\Lambda}{\sqrt{\epsilon}}\right)^{2N-r} \\
&= \Lambda^{2N-r}
\end{aligned} \tag{3.4.123}$$

The maximum value of r is at $r = 2N$ because each vertex operator can only supply one rapidly varying contribution; then since $\Lambda \to 0$ with ϵ this is the only case that will provide a non-vanishing contribution when the regulator is removed. The integral into the bulk in this case takes the form (removing the regulator)

$$\prod_{j=1}^{2N} \alpha' k_j^{\mu_j} \int_0^{\infty} dy_j \; \partial_y^j f\left(y_j^2\right) \exp\left(\pi\alpha' k_j^2 f\left(y_j\right)\right), \tag{3.4.124}$$

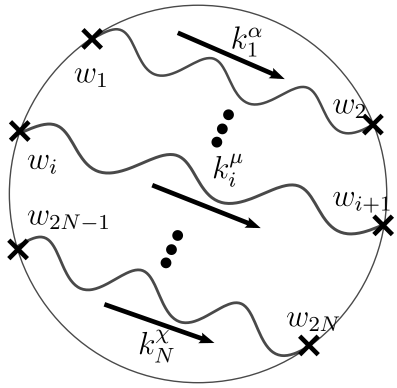


Figure 3.6: Using the unit disk representation of the worldsheet domain we demonstrate the physical meaning behind the result. There are N propagators with momenta k_i joining 2N points restricted to the boundary. This mirrors the calculation in Maxwell field theory for the expectation value of a Wilson loop given by the worldline of a pair of quarks, if this worldline is taken to be the fixed boundary of the string in our theory.

providing

$$\prod_{j=1}^{2N} \frac{2k^{\mu_j}}{\pi k_j^2}. \tag{3.4.125}$$

This must now be combined with the remainder of the slowly varying fields and the integrals about the boundary. This involves some number of second derivatives $\partial_a^i \partial_b^j G_{ij}$ and the remaining field derivatives $\partial_a^i X_c(\xi_i)$. The only arrangement of derivatives which provides a non-vanishing contribution as the regulator is removed involves 2N derivatives $\partial_y^i \psi_i$ meaning that the only derivatives remaining are with respect to each x_i. Since at leading order the Green function is to be evaluated on the boundary, where it is identically zero, all derivatives $\partial_x^i \partial_x^j G_{ij}$ vanish. We are consequently free to consider the case of having $2N$ of the fields $X_c(\xi_i)$ uncontracted which gives the only non vanishing contribution close to the boundary as

$$\prod_{j=1}^{N} \frac{4}{\left(k_j^2\right)^2} \int_B k_j^{[\mu} \partial_x X_c^{\nu]} k_j^{[\mu} \partial_x' X_c'^{\nu]} e^{ik_j \cdot (X-X')} \tag{3.4.126}$$

$$= 4^N \prod_{j=1}^{N} \int_B dw_j \cdot dw_j' \frac{e^{ik_j \cdot \left(w_j - w_j'\right)}}{k_j^2} \tag{3.4.127}$$

where we have left the result in its Fourier representation and the points w_j and w_j' have opposite momenta. For the above expression we have also reinstated the antisymmetry on worldsheet and target space indices and have contracted the indices of the fields corresponding to vertex operators with opposite momenta.

We reiterate that (3.4.127) is the only contribution from this regime that does not vanish as the regulator is removed and also point out that it is independent of α'. The physical interpretation is of N massless propagators pairing off the $2N$ points on the boundary, as depicted in Fig 3.6. The pairs of points joined together are those from vertex operators with equal and opposite momenta $\pm k_i$. These momenta are to be integrated over but (at least so far) a dependence on the scale of the worldsheet metric has not arisen so there are no mass shell conditions to be imposed. It remains to consider the other cases where the damping of (3.4.119) is not present to investigate whether any dependence on this scale arises to ensure that these expectation values do indeed evade mass shell conditions.

3.4.2 Points clustered in the bulk

When we consider pairs of points meeting in the bulk the Green function between nearby points ξ_i and ξ_j becomes of the same order as ψ_i and ψ_j when their distance is less than $\sqrt{\epsilon}$. Furthermore G_{ij} is then rapidly varying as the two points are moved apart. In this subsection we again work at order N but consider the effect of having n of these points clustered in the bulk about a reference point, ξ_{n+1}, as is illustrated in Fig 3.7. We shall calculate the contribution of this configuration to the expectation value (3.4.116) by integrating the n points about that reference point. The reference point ξ_{n+1} would remain to be integrated about the entire worldsheet.

We proceed by considering the form of the integrand due to Wick contractions between the $n+1$ vertex operators $V_{k_1}\cdots V_{k_{n+1}}$ because carrying out these contractions is sufficient to extract the leading order behaviour when these $n+1$ points become close. In the following we shall extract the ϵ and α' dependence arising from the integral of the n points about the reference point before discussing the effect of the remaining points.

A string of $n+1$ vertex operators of the form $V_{k_1}^{\mu_1\nu_1}\cdots V_{k_{n+1}}^{\mu_{n+1}\nu_{n+1}}$ corresponds to a product of fields

$$e^{ik_1\cdot X_1}\partial_{a_1}X_1^{\mu_1}\partial_{b_1}X_1^{\mu_1}\cdots\partial_{a_{n+1}}X_{n+1}^{\mu_{n+1}}\partial_{b_{n+1}}X_{n+1}^{\mu_{n+1}}e^{ik_{n+1}\cdot X_{n+1}}. \tag{3.4.128}$$

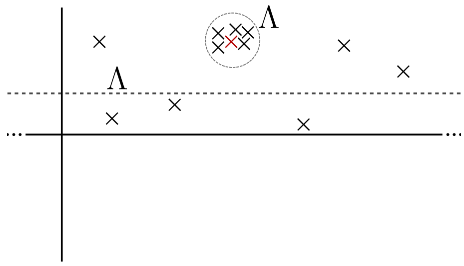


Figure 3.7: We imagine that n points are clustered within a distance Λ of a point ξ_{n+1} (in red) in the bulk of the worldsheet. The remaining points are elsewhere on the worldsheet and will be discussed at the end of this section. We repeat that points of equal and opposite momenta $\pm k_i$ are excluded from meeting in the bulk so that the $n+1$ points have distinct momenta.

In carrying out the analysis of this section we once again consider functions which vary rapidly within the region of integration and those which vary slowly. In this case, slowly varying expressions will be replaced by their values at the reference point ξ_{n+1}. In particular the Green function is given by

$$G_{ij} = f\left(\frac{\left|\xi_i - \xi_j\right|}{2\sqrt{\epsilon}}\right) - f\left(\frac{\left|\xi_i - \xi_j^*\right|}{2\sqrt{\epsilon}}\right). \tag{3.4.129}$$

Since the second term varies slowly as the n points are integrated about the reference point ξ_{n+1}, it will be replaced at first order in ϵ by

$$f\left(\frac{\left|\xi_{n+1} - \xi_{n+1}^*\right|}{2\sqrt{\epsilon}}\right) \tag{3.4.130}$$

which is approximately $\frac{1}{2\pi}\ln\frac{y_{n+1}}{\sqrt{\epsilon}}$. A derivative with respect to the relative displacement $\xi_i - \xi_j$ acting on the first term in G_{ij} produces a factor of $\frac{1}{\sqrt{\epsilon}}$ so is enhanced in comparison to derivatives acting on the second, slowly varying term. As a further consequence, the coincident Green functions ψ_i contain only (3.4.130) so are slowly varying and to first order in ϵ the following replacement can be made:

$$\psi_i \approx \frac{1}{2\pi}\ln\frac{y_{n+1}}{\sqrt{\epsilon}}. \tag{3.4.131}$$

The derivatives of these functions are therefore also subleading in ϵ and independent of X.

These properties allow us to consider a general term arising out of the expectation of (3.4.128) as follows. Wick contractions could generate q terms of the form

$\partial_{a_i}\partial_{a_j}G_{ij}$ and r of the form $k_j^{\mu_j}\partial_{a_i}G_{ij}$ and uncontracted fields will offer p terms of the form $\partial_{a_i}X\left(\xi_i\right)$. These numbers are constrained by the necessity $2q+r+p = 2\left(n+1\right)$ and in forming the product of r first derivatives the antisymmetry of the indices must be considered; we return to this later. With (3.4.118) we are thus led to consider

$$\alpha'^{q+r}\int\prod_{i=1}^{n}d^2\xi_i\ \overbrace{\partial_{a_j}\partial_{a_k}f\left(\frac{|\xi_j-\xi_k|}{2\sqrt{\epsilon}}\right)\cdots}^{q\text{ terms}}\overbrace{k_l^{\mu_l}\partial_{b_m}f\left(\frac{|\xi_l-\xi_m|}{2\sqrt{\epsilon}}\right)\cdots}^{r\text{ terms}}\overbrace{\partial_{a_w}X^{\mu_w}\left(\xi_w\right)\cdots}^{p\text{ terms}}\times$$
$$\exp\left(-\pi\alpha'\sum_i k_i^2\psi_i-\pi\alpha'\sum_{i\neq j}k_i\cdot k_jG_{ij}\right)e^{ik_i\cdot X_i}. \tag{3.4.132}$$

The contribution from the first sum in the exponent and the corresponding factors of (3.4.130) from the second sum allows us to rewrite the exponential as

$$\exp\left(\frac{-\alpha'}{2}\sum_{i,j}k_i\cdot k_j\ln\frac{y_{n+1}}{\sqrt{\epsilon}}-\pi\alpha'\sum_{i\neq j}k_i\cdot k_jf\left(\frac{|\xi_i-\xi_j|}{\sqrt{\epsilon}}\right)\right) \tag{3.4.133}$$

Since we are interested in eventually removing the regulator we may think of ϵ as a small quantity. Recalling that the expectation value of the vertex operators (3.4.116) is to be integrated with respect to each of the momenta we consider the effect of the first term in (3.4.133) on such an integral. In the limit as $\epsilon\to 0$ Laplace's approximation shows that this term behaves effectively as

$$\frac{\delta\left(\sum_i k_i\right)}{\left(\frac{\alpha'}{2\pi}\ln\frac{y_{n+1}}{\sqrt{\epsilon}}\right)^{\frac{D}{2}}} \tag{3.4.134}$$

which we shall use as a means of tracking the ϵ and α' dependence it carries.

The integrals with respect to the ξ_i can be carried out by using the upper half plane geometry $z_i = x_i + iy_i$. We now consider integrating each of these points ξ_i

about a circular region of size Λ, centred on ξ_{n+1}:

$$\frac{\alpha'^{q+r}\delta\left(\sum_i k_i\right)}{\left(\frac{\alpha'}{2\pi}\ln\frac{y_{n+1}}{\sqrt{\epsilon}}\right)^{\frac{D}{2}}}\int_{|\xi_i-\xi_{n+1}|<\Lambda}\prod_{i=1}^{n}d^2\xi_i\ \overbrace{\partial_{a_j}\partial_{a_k}f\left(\frac{|\xi_j-\xi_k|}{2\sqrt{\epsilon}}\right)\cdots}^{q\text{ terms}}\overbrace{k_l^{\mu_l}\partial_{b_m}f\left(\frac{|\xi_l-\xi_m|}{2\sqrt{\epsilon}}\right)\cdots}^{r\text{ terms}}\times$$
$$\overbrace{\partial_{a_w}X^{\mu_w}\left(\xi_w\right)\cdots}^{p\text{ terms}}\exp\left(-\pi\alpha'\sum_{i\neq j}k_i\cdot k_jG_{ij}\right)e^{ik_i\cdot X_i}. \tag{3.4.135}$$

At this point we split the integration region into three sections corresponding to the regions where we may employ the approximate forms of f for very large or very small argument. We have seen, however, that the divergences we stand to encounter manifest themselves when considering the short distance behaviour so we will concentrate here on the innermost region, where $0\leq|\xi_i-\xi_j|\leq\sqrt{\epsilon}a$. Furthermore we anticipate taking the tensionless limit whereby the exponential factor $\exp\left(-\pi\alpha'\sum_{i\neq j}k_i\cdot k_jf\left(\frac{|\xi_i-\xi_j|}{\sqrt{\epsilon}}\right)\right)$ damps the integrand for large α' except when G_{ij} is small – precisely in the innermost region where we shall focus. This also produces the contribution that is leading order in α'. In this region the function f is approximated by a quadratic expression

$$f\left(s\right)\approx\frac{s^2}{4\pi} \tag{3.4.136}$$

which implies that the exponent above takes on a Gaussian form. As previously we shall scale each of the n displacement variables $\frac{\xi_i-\xi_{n+1}}{\sqrt{\epsilon}}\rightarrow\xi_i-\xi_{n+1}$ so as to remove the ϵ-dependence from the integrand. We can finally replace any derivatives with respect to x_{n+1} or y_{n+1} acting on a function of $|\xi_i-\xi_{n+1}|$ by derivatives with respect to x_i or y_i. The expression becomes

$$\epsilon^{n-q-\frac{r}{2}}\alpha'^{q+r-\frac{D}{2}}\frac{\delta\left(\sum_i k_i\right)}{\left(4\ln\frac{y_{n+1}}{\sqrt{\epsilon}}\right)^{\frac{D}{2}}}\int_0^{2\pi}\int_{|\xi_i-\xi_{n+1}|<a}\prod_{i=1}^{n}d^2\xi_i\ \overbrace{\partial_{a_j}\partial_{a_k}f\left(\frac{|\xi_j-\xi_k|}{2\sqrt{\epsilon}}\right)\cdots}^{q\text{ terms}}\times$$
$$\overbrace{k_l^{\mu_l}\partial_{b_m}f\left(\frac{|\xi_l-\xi_m|}{2\sqrt{\epsilon}}\right)\cdots}^{r\text{ terms}}\overbrace{\partial_{a_w}X^{\mu_w}\left(\xi_w\right)\cdots}^{p\text{ terms}}\exp\left(-\pi\alpha'\sum_{i\neq j}k_i\cdot k_jG_{ij}\right)e^{ik_i\cdot X_i}. \tag{3.4.137}$$

For large α' the exponential factor damps the integrand outside of the region where $|\xi_i-\xi_j|^2 << \alpha' k_i\cdot k_j$ which is by construction inside the innermost region we are concerned with here. In this limit the integral can be safely approximated by taking the upper bound of the integration over relative displacements to infinity. Also in this region the q second order derivatives are independent of the ξ_i whilst the r first order derivatives are linear in the differences $\xi_l-\xi_m$ so lead to moments of a Gaussian integral. The exponent can be written $\xi^TN\xi$ where the vector ξ has i^{th} component $\xi_i-\xi_{n+1}$ and the matrix N has components $N_{ij}\equiv\delta_{ij}k_i\left(\sum_l k_l\right)-k_i\cdot k_j$.

It is clear that the smallest power of ϵ arises by maximising $q+\frac{r}{2}$. With the constraint $2q+r+p=2\left(n+1\right)$ this is done by setting $p=0$ which automatically leads to a term of order $\frac{1}{\epsilon}$, mirroring the worst behaviour found in the previous section. Following the procedure used for term (VI) of the first order calculation the r first order derivatives can be removed via an integration by parts which leads to an integral with respect to the relative displacements of the form

$$\frac{\alpha'^{-\frac{D}{2}+n+1}}{\epsilon\left(\ln\epsilon\right)^{\frac{D}{2}}}\left(\ln y_{n+1}\right)^{-\frac{D}{2}}\delta\left(\sum_i k_i\right)\int\prod_{i=1}^{n}d^2\xi_i\exp\left(-\pi\alpha'\xi^TN\xi+ik_i\cdot X_i\right) \tag{3.4.138}$$

which is equal to

$$\frac{1}{\epsilon\left(\ln\epsilon\right)^{\frac{D}{2}}}\frac{\left(\ln y_{n+1}\right)^{-\frac{D}{2}}}{\alpha'^{\frac{D}{2}-1}}\delta\left(i\sum_i k_i\right)\frac{1}{\det N}e^{\sum_i k_i\cdot X_{n+1}}. \tag{3.4.139}$$

This pole in ϵ can be suppressed by taking the tensionless limit of the string theory

$\alpha' k_i^2 \to \infty$ for all momenta due to the damping caused by the overall factor

$$\frac{1}{\alpha'^{\frac{D}{2}-1}}. \tag{3.4.140}$$

Note also that in this case there are no finite corrections arising from an expansion of the slowly varying fields due to the suppression caused by the denominator $(\ln \epsilon)^{D/2}$.

The integration by parts leads to a complicated index structure but it is constrained by the structure of the vertex operator. As in the previous chapter we contract (3.4.116) with one of the momenta, say $k_{i\mu}$, and integrate the point ξ_i throughout the domain. The effect of the contraction $k_{i\mu} V^{\mu\nu}_{k_i}(\xi_i)$ can be written

$$\int_D d^2\xi_i \; \epsilon^{ab} k_{i\mu} \partial_a X_i^\mu \partial_b X_i^\nu e^{ik_i \cdot X_i} = \int_D d^2\xi_i \; \epsilon^{ab} \partial_a \left(\partial_b X_i^\nu e^{ik_i \cdot X_i} \right) \tag{3.4.141}$$

$$= \oint_{\partial D} d\xi_i \; \partial X_{ci}^\nu e^{ik_i \cdot X_{ci}} \tag{3.4.142}$$

providing only a boundary contribution. This is the string theory version of the generalised Gauss' law we have already seen for point particles. We are left with a total of $n-1$ points to be integrated about the point ξ_{n+1} but the divergences that arose out of contractions involving the $X(\xi_i)$ can no longer appear. The structure of the divergence which appears because of the presence of the vertex operator $V^{\mu\nu}_{k_i}$ must therefore be such that it vanishes when contracted with k_i. Integrating by parts to remove the r first order derivatives is responsible for the formation of this index structure[6].

[6] This can be illustrated by considering two such operators and taking the leading order $\frac{1}{\epsilon}$ piece of

$$\int d^2\xi d^2\xi' \; \epsilon^{ab}\epsilon^{cd} \left\langle \partial_a X^\mu \partial_b X^\nu e^{ik\cdot X} e^{ij\cdot X'} \partial'_c X'^\alpha \partial'_d X'^\beta \right\rangle \equiv \frac{1}{\epsilon} H^{\mu\nu\alpha\beta} + \cdots \tag{3.4.143}$$

where the $\cdots$ represent terms which are regular in ϵ (which should be familiar from section 3.3.2 where we had $j^\mu = -k^\mu$ at first order). $H^{\mu\nu\alpha\beta}$ holds the tensor structure and is a function of the momenta. In our work we are concerned only with the piece antisymmetric in μ and ν and also in α and β and as a consequence linear in momenta so that

$$\begin{aligned} H^{\mu\nu\alpha\beta} \propto A\left[\delta^{\mu\alpha}\delta^{\nu\beta} - \delta^{\mu\beta}\delta^{\nu\alpha}\right] + B\left[k^\mu j^\alpha \delta^{\nu\beta} - k^\mu j^\beta \delta^{\nu\alpha} - k^\nu j^\alpha \delta^{\mu\beta} + k^\nu j^\beta \delta^{\mu\alpha}\right] \\ + C\left[j^\mu k^\alpha \delta^{\nu\beta} - j^\mu k^\beta \delta^{\nu\alpha} - j^\nu k^\alpha \delta^{\mu\beta} + j^\nu k^\beta \delta^{\mu\alpha}\right] \end{aligned} \tag{3.4.144}$$

The requirement $k_\mu H^{\mu\nu\alpha\beta} = 0$ implies that $B = 0$ and $A = -k \cdot jC$. Taking $n = 1$ in (3.4.137) the term proportional to A arises out of $q = 2$ second order derivatives (so $r = p = 0$) and the term proportional to C comes from $r = 2$ first order derivatives and $q = 1$ second order derivatives

Continuing with the general case of $n+1$ points clustered in the bulk the next singular behaviour which may appear comes from $q = n$ and $r = 1$, which gives a term of order $\epsilon^{-\frac{1}{2}}$. This corresponds to exchanging one second derivative of the function f for a single derivative which leaves an uncontracted derivative of a field X ($p = 1$) along with $2q$ second derivatives and 1 first derivative acting on Green functions. There is no rotationally invariant tensor with odd rank so the integral of this term vanishes when integrated about the point ξ_{n+1}. All further contributions are of order 1 or a positive power of ϵ multiplied by the $(\ln \epsilon)^{-\frac{D}{2}}$ common to all terms. For this reason they vanish as the regulator is removed.

We have thus argued that in the tensionless limit the contribution from $n+1$ points meeting in the bulk is vanishing. For an arbitrary placement of the remaining points on the worldsheet the coincident Green functions of individual points, ψ_i, damp the integrand. A collection of points in the region of the boundary offers a finite contribution as these points are integrated into the bulk but the problem factorises into this and the cluster of points in the bulk. In general the integrand would be sensitive to the scale of the metric when considering the Green function of points which are located in the bulk. That this contribution vanishes as the regulator is removed completes the argument that the result of the previous subsection evades a mass shell condition on the momenta so it generalises to all orders. This is a significant result for the interacting string theory presented here because as well as evading a mass shell condition we also find no constraint on the dimensionality of target space. We are free to specify $D = 4$ where the result of the calculation at order N reads:

$$4^N \prod_{j=1}^{N} \int_B dw \cdot dw' \, \frac{e^{ik_j \cdot (w - w')}}{k_j^2} \, . \qquad (3.4.145)$$

Integrating the N momenta gives the position space representation of the product of Wilson loops for the curves fixing the boundary of the string worldsheet.

($p = 0$ again), with the dependence on k and j appearing after integrating by parts to remove the first order derivatives. To compare to the first order calculation presented in section 3.3.2 it is necessary to set the momenta equal and opposite and to contract indices μ with α and ν with β.

3.4.3 Discussion

We have presented an argument for the evaluation of a generic product of vertices by focussing on the configuration of points where the Green function does not damp the integrand. Close to the boundary we also dealt only with the generic case that the points could not come within a distance Λ of one another and this deserves attention. Recall that there the coincident Green functions $\psi(\xi_i)$ vary from 0 on the boundary to order $\ln\epsilon$ moving into the bulk so that close to the boundary they do not damp the integrand.

Considering two such vertex operators we have derived the form of the integrand for $|x-x'| >> \Lambda$ in section 3.3.1. We can also consider the configuration where both points are close enough to apply the quadratic approximation to the terms making up the regulated Green function to derive

$$\begin{aligned}\Psi(z,z') &= -f\left(\frac{y}{\sqrt{\epsilon}}\right) - f\left(\frac{y'}{\sqrt{\epsilon}}\right) - 2\left[f\left(\frac{|z-z'|}{2\sqrt{\epsilon}}\right) - f\left(\frac{|z-\bar{z}'|}{2\sqrt{\epsilon}}\right)\right] \\ &\approx \frac{y^2}{4\pi\epsilon} + \frac{y'^2}{4\pi\epsilon} - 2\left[\frac{(x-x')^2+(y-y')^2}{16\pi\epsilon} - \frac{(x-x')^2+(y+y')^2}{16\pi\epsilon}\right] \\ &= \frac{(y-y')^2}{4\pi\epsilon} \end{aligned} \tag{3.4.146}$$

which is independent of the separation $x-x'$. In this region it is feasible to carry out the integrals of the various terms which arise. However we must look ahead to the intermediate region where $a\sqrt{\epsilon} < |x-x'| < b\sqrt{\epsilon}$. As the transverse separation between the points increases the relative separation $x-x'$ appears in Ψ in a non-trivial manner and with both points close to the boundary this dependence is no longer subleading. The form of the answer is still constrained by the generalised Gauss' law (3.4.142) but a finite or divergent contribution could be present and would not be suppressed by α'.

We are therefore unable to complete our programme for the bosonic string due to the difficulty in determining what happens when points meet in the vicinity of the boundary. In the next section we turn to spinor QED since this is a more realistic model. We discussed in the introduction that the one dimensional quantum theory on the worldlines used to describe spinor matter has a local supersymmetry and this motivates us to consider including supersymmetry in our interacting string theory. The next section introduces the necessary preliminaries and the equivalent calculations for the spinning string. The extra symmetry gained will be shown to lead to a cancelling of the $\frac{1}{\epsilon}$ divergences which arose in the purely bosonic case (both in the bulk and on the boundary) and the finite contribution shall be shown to provide precisely the expectation value of the supersymmetric Wilson loops which appear in the worldline formalism of spinor QED.

3.5 Spinor QED

The worldline formalism of spinor QED enjoys a local supersymmetry which suggests a generalisation of our interacting string theory to include spin degrees of freedom on the worldsheets. The gauge field A appears in the supersymmetric Wilson loop

$$W_A = \exp\left(i\int \frac{d\omega}{d\xi}\cdot A + \frac{1}{2}F_{\mu\nu}\psi^\mu\psi^\nu\sqrt{h}\; d\xi\right) \tag{3.5.147}$$

where our notation follows that of the introduction – ψ^μ is the super-partner to the coordinate w^μ. Our aim in this section is to replace the integral over A of a product of these objects by a functional integral over spinning worldsheets supplemented by a contact interaction which generalises the bosonic theory presented so far. This interacting string theory will have worldsheet supersymmetry. A perturbative expansion of the contact interaction implies we must calculate the expectation value of products of supersymmetric vertices inserted at different points in the worldsheet.

We shall demonstrate that the result we seek arises in a similar way to the bosonic calculation in that the contribution comes from vertices located close to the worldsheet boundary. The divergences encountered when the vertices cluster in the bulk will not be present for the spinning string because they are forbidden by the residual supersymmetry which we shall preserve throughout regularisation. There can also be no correction to the result arising when the points are close to one another and to the boundary; this time both supersymmetry and the generalisation of Gauss' law (3.4.142) prevent such a contribution from arising.

3.5.1 The spinning string

To introduce fermions on the worldsheet we could follow a similar program to our discussion of the fermionic point particle. We might worry, however, what the effects of doing so will be on the quantisation of the theory. We can anticipate problems at the outset by asking how we might deal with the time-like modes of the fermionic fields – is it possible to decouple these and find a positive definite space of physical states in the same way as for the bosonic string? The answer is of course that the spinning string worldsheet theory enjoys a supersymmetry and this (with the ordinary reparameterisation invariance) is enough to decouple the negative norm states in the critical dimension $D = 10$.

A complete treatment of the spinning string begins with a locally supersymmetric two dimensional supergravity theory which involves super-partners to both the bosonic worldsheet coordinates (Ψ and $\bar{\Psi}$) and the worldsheet metric (χ and $\bar{\chi}$). This is just an extension of the work of Chapter 2 to a two dimensional theory. The theory is again most easily formulated in superspace, but now there are two independent super-diffeomorphism transformations. Canonically the gauge symmetry is fixed and the decoupling of the negative-norm modes follows if the equations of motion for g and χ are imposed on the physical states of the system [32, 33, 39]. In the functional approach the extra structure contained in the volume elements $\mathscr{D}\chi$ and $\mathscr{D}\Psi$ and the regularisation of the worldsheet theory combine to produce the super-symmetric generalisation of the Liouville theory [40], which contains anomalous dependence on the super-conformal scale [63]. Only in the critical dimension does the coefficient of this factor vanish.

Rather than dealing with the locally supersymmetric and reparameterisation invariant spinning string we shall immediately use the gauge-fixed[7] action [39]

$$S_{\text{spin}} = \frac{1}{4\pi\alpha'}\left(\int_H d^2z d^2\theta\ \bar{D}\mathbf{X}\cdot D\mathbf{X} - \int_{y=0} dx\ \bar{\Psi}\cdot\Psi\right). \tag{3.5.149}$$

[7] We might guess this form by generalising the bosonic action in complex coordinates $ds^2 = e^{\phi}dzd\bar{z}$ which takes the form

$$S_{\text{gf}} = \int d^2z\ \partial X^{\mu}\bar{\partial}X_{\mu} \tag{3.5.148}$$

by promoting X to a superfield and derivatives to covariant (super) derivatives.

The parameter domain is taken to be the upper-half complex plane $z = x + iy$ extended by the anticommuting variables θ and $\bar{\theta}$ which together make up the derivatives

$$D \equiv \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}; \quad \bar{D} \equiv \frac{\partial}{\partial\bar{\theta}} + \bar{\theta}\frac{\partial}{\partial\bar{z}}. \tag{3.5.150}$$

We have introduced the superfield

$$\mathbf{X} \equiv X + \theta\Psi + \bar{\theta}\bar{\Psi} + \theta\bar{\theta}B \tag{3.5.151}$$

where X is the bosonic coordinate and Ψ and $\bar{\Psi}$ make up its fermionic superpartner; these fields have dimension of length. B is an auxiliary field required for the supersymmetry which can be disregarded for the purposes of our calculation. This gauge-fixed form of the action has a residual global supersymmetry generated by Grassman η

$$\delta_\eta \mathbf{X} = \eta\left(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z} + \frac{\partial}{\partial\bar{\theta}} - \bar{\theta}\frac{\partial}{\partial\bar{z}}\right)\mathbf{X}, \tag{3.5.152}$$

corresponding to the domain preserving transformation on the co-ordinates

$$z \to z + \theta\eta; \quad \bar{z} \to \bar{z} + \bar{\theta}\eta; \quad \theta \to \theta + \eta; \quad \bar{\theta} \to \bar{\theta} + \eta\,. \tag{3.5.153}$$

The boundary term in (3.5.149) would not be present in the conventional string theory since it would vanish under the usual Ramond or Neveu-Schwarz boundary conditions. We have introduced it because in order to relate the worldsheet variables to those on the worldline we will have to enforce the Dirchlet boundary conditions

$$X\big|_{y=0} = w, \quad \left(\Psi + \bar{\Psi}\right)\big|_{y=0} = h^{1/4}\,\psi\,. \tag{3.5.154}$$

Since ψ is a world-line scalar the factor of $h^{\frac{1}{4}}$ is natural and will also be seen to lead to the correct formation of the supersymmetric Wilson loop when we consider the effect of the contact interaction to be introduced below. The relation between the local supersymmetry on the worldlines and the global supersymmetry of the worldsheet is understood by noting that under (3.5.152) the boundary conditions (3.5.154) are preserved if a simultaneous transformation of the worldline variables is made with the local supersymmetry parameter α in (1.0.16) related to the global parameter η by $\alpha = h^{1/4}\eta$.

We generalise the contact interaction of the bosonic string by writing in gauge-fixed form

$$S_{\text{int}}\left[\mathbf{X}_i,\mathbf{X}_j\right] = q^2 \int d^2\theta_i \left(\int d^2 z_i\; \bar{D}_i \mathbf{X}_i^{[\mu} D_i \mathbf{X}_i^{\nu]} - \int_{y_i=0} dx_i\; \theta_i\bar{\theta}_i \bar{\Psi}_i^{[\mu}\Psi_i^{\nu]}\right) \delta^d\left(\mathbf{X}_i - \mathbf{X}_j\right) \times$$
$$d^2\theta_j \left(\int d^2 z_j \bar{D}_j \mathbf{X}_j^{[\mu} D_j \mathbf{X}_j^{\nu]} - \int_{y_j=0} dx_j\; \theta_j\bar{\theta}_j \bar{\Psi}_j^{[\mu}\Psi_j^{\nu]}\right) \tag{3.5.155}$$

where we again use the shorthand $X_i \equiv X\left(z_i\right)$. The inclusion of the boundary terms ensure that this contact interaction is also invariant under the residual supersymmetry and we use it to form a theory of a set of spinning strings spanning fixed boundaries

$$S_s = \sum_i S_{\text{s}}[\mathbf{X}_i] + \sum_{ij} S_{\text{int}}[\mathbf{X}_i,\mathbf{X}_j] \tag{3.5.156}$$

which is the generalisation of the bosonic theory considered in previous sections. We shall calculate the partition function for this interacting string theory by perturbative expansion of the interaction term in order to establish the equality (3.1.32)

$$\prod_{i=1}^{n} \frac{\mathscr{D}\left(X_i,\psi_i,g_i\right)}{Z_0} e^{-S_{\text{spin}}} = \int \frac{\mathscr{D}A}{N} e^{-S'_{gf}} \prod_i e^{-S[A]} \tag{3.5.157}$$

which replaces the functional integral over the gauge field of a product of supersymmetric Wilson loops by an integration over the string worldsheets whose boundaries are those curves. The delta-function in the interaction term can be Fourier decomposed to reduce the problem to the expectation value of insertions of vertices

$$\bar{D}\mathbf{X}^{[\mu} D\mathbf{X}^{\nu]}\delta^d\left(\mathbf{X}-\mathbf{x}\right) = \int \frac{d^d k}{\left(2\pi\right)^d} e^{-ik\cdot\mathbf{x}} \frac{1}{2} V^{\mu\nu}\left(k\right), \quad V^{\mu\nu}\left(k\right) = \bar{D}\mathbf{X}^{[\mu} D\mathbf{X}^{\nu]}\, e^{ik\cdot\mathbf{X}}. \tag{3.5.158}$$

As in the bosonic case we shall show that the expectation value of this delta function decouples from the super-conformal scale. This too is unusual because upon quantisation $V^{\mu\nu}\left(k\right)$ acquires an anomalous dimension which would impose a mass shell condition ($k^2 = 0$). But again the Dirichlet boundary conditions and the self-contraction of the exponential which give rise to the anomalous dimension will ensure that the insertion is suppressed for all points z that are not close to the boundary. Points close to the boundary, as measured with respect to the short distance cut-off we shall use to regulate the Green function, will provide the finite, scale independent contribution which makes up (3.1.32).

To demonstrate the decoupling and establish the regularisation we shall use we begin with the zeroth-order calculation

$$\int \mathscr{D}\mathbf{X}\, e^{-S_{\rm spin}} \int d^2\theta \left(\int d^2z\ \bar{D}\mathbf{X}^{[\mu} D\mathbf{X}^{\nu]} - \int_{y=0} dx\ \theta\bar{\theta}\bar{\Psi}^{[\mu}\Psi^{\nu]} \right) e^{ik\cdot\mathbf{X}}. \qquad (3.5.159)$$

The super-field can be split as $\mathbf{X} = \mathbf{X}_c + \tilde{\mathbf{X}} + \bar{\mathbf{X}}$ which is a classical piece – $\bar{D}D\mathbf{X}_c = 0$ – which also satisfies the boundary conditions, another solution $\tilde{\mathbf{X}}$ which absorbs the sources produced by the insertion and a quantum fluctuation $\bar{\mathbf{X}}$. Functionally integrating over $\bar{\mathbf{X}}$ gives

$$\begin{aligned} e^{-S_{\rm spin}[\mathbf{X}_c]-S_L} \Bigg(\int d^2z d^2\theta\, e^{ik\cdot\mathbf{X}_c - \pi\alpha' k^2 G_0}\ \big(\bar{D}\mathbf{X}_c^{[\mu} D\mathbf{X}_c^{\nu]} - 2\pi\alpha' \big(\bar{D}\mathbf{X}_c^{[\mu}(DG)_0 ik^{\nu]} \\ +\ (\bar{D}G)_0 ik^{[\mu} D\mathbf{X}_c^{\nu]}\big)\big) - \int_{y=0} dx\ \bar{\Psi}^{[\mu}\Psi^{\nu]} e^{ik\cdot X_c} \Bigg) \end{aligned} \qquad (3.5.160)$$

where S_L contains the functional determinants which give rise to the super-Liouville action [40] and G_0 is the Green function evaluated at coincident points. The defining equation of the Green function is

$$-\bar{D}DG\left(z_1, \theta_1; z_2, \theta_2\right) = \delta^2\left(\theta_1 - \theta_2\right)\delta^2\left(z_1 - z_2\right) \qquad (3.5.161)$$

subject to the boundary conditions $G = 0$ if $y_i = 0$ and $\theta_i = \bar{\theta}_i$ ($i = 1$ or 2) which has solution that generalises the bosonic case

$$G\left(z_1, \theta_1; z_2, \theta_2\right) = \log\left(z_{12}\bar{z}_{12}\right) - \log\left(z_{12}^R \bar{z}_{12}^R\right) \qquad (3.5.162)$$

where

$$z_{12} = z_1 - z_2 - \theta_1\theta_2,\ \ \bar{z}_{12} = \bar{z}_1 - \bar{z}_2 - \bar{\theta}_1\bar{\theta}_2,\ \ z^R_{12} = z_1 - \bar{z}_2 - \theta_1\bar{\theta}_2,\ \ \bar{z}^R_{12} = \bar{z}_1 - z_2 - \bar{\theta}_1\theta_2\,. \tag{3.5.163}$$

Evaluated at coincident points the Green function is singular and we regulate it via heat kernel regularisation with the obvious generalisation of (3.2.56):

$$G^\epsilon = -f\left(\frac{\sqrt{z_{12}\bar{z}_{12}}}{\sqrt{\epsilon}}\right) + f\left(\frac{\sqrt{z^R_{12}\bar{z}^R_{12}}}{\sqrt{\epsilon}}\right) \tag{3.5.164}$$

with ϵ again a short distance cut-off and f defined as in (3.2.57). This function satisfies the boundary conditions and to verify this is a regularisation of the Green function it is easy to determine

$$-\bar{D}DG^\epsilon = (\theta_1 - \theta_2)(\bar{\theta}_1 - \bar{\theta}_2)\frac{e^{-\frac{z_{12}\bar{z}_{12}}{\epsilon}}}{4\pi\epsilon} - (\theta_1 - \bar{\theta}_2)(\bar{\theta}_1 - \theta_2)\frac{e^{\frac{-z^R_{12}\bar{z}^R_{12}}{\epsilon}}}{4\pi\epsilon}. \tag{3.5.165}$$

Upon taking the limit $\epsilon \to 0$ we recover Green's equation. Furthermore this regulated Green function is invariant under the residual supersymmetry because z_{12}, z^R_{12}, $\bar{z}_{12}$ and $\bar{z}^R_{12}$ are all separately invariant under (3.5.153). This will be crucial in allowing us to constrain the form of the integrals we will calculate. Using this regulator we can determine the coincident limits as an expansion in θ and $\bar{\theta}$:

$$G^\epsilon_0 = \left(1 + \frac{i}{2}\theta\bar{\theta}\frac{\partial}{\partial y}\right) f\left(\frac{2y}{\sqrt{\epsilon}}\right); \quad (DG^\epsilon)_0 = (\bar{D}G^\epsilon)_0 = \frac{i}{2}(\theta - \bar{\theta})\frac{\partial}{\partial y} f\left(\frac{2y}{\sqrt{\epsilon}}\right) \tag{3.5.166}$$

and we can also expand the common exponential term in (3.5.160) as

$$e^{-\pi\alpha' k^2 G_0} = \left(1 + \frac{i}{2}\theta\bar{\theta}\frac{\partial}{\partial y}\right) e^{-\pi\alpha' k^2 f\left(\frac{2y}{\sqrt{\epsilon}}\right)}\,. \tag{3.5.167}$$

The exponential factor on the right hand side of the above equation has been seen in the previous sections and for fixed k^2 it damps the integrand at all points in the domain except for those close to the boundary $y \lesssim \sqrt{\epsilon}$. We thus repeat our procedure of integrating (3.5.160) a distance Λ into the bulk, where $\Lambda \to 0$ as $\epsilon \to 0$ but we arrange for $\frac{\Lambda}{\sqrt{\epsilon}}$ to diverge. This means that to leading order in ϵ we can replace the components of the classical super-field $\mathbf{X}_c$ by their boundary values.

The integral of the first term in (3.5.160) cancels against the boundary term present in the interaction term. To see this consider

$$-2i\int dx\, d^2\theta\, e^{ik\cdot\mathbf{X}_c}\bar{D}\mathbf{X}_c^{[\mu}D\mathbf{X}_c^{\nu]}\int_0^\Lambda dy\,\left(1+\frac{i}{2}\theta\bar{\theta}\frac{\partial}{\partial y}\right)e^{-\pi\alpha' k^2 f\left(\frac{2y}{\sqrt{\epsilon}}\right)}. \qquad (3.5.168)$$

Both parts of the y integral are known from previous work. The monotonicity of $f(s)$ allows us to bound the first term $\left|\int_0^\Lambda dy\,\exp(-\pi\alpha' k^2 f(2y/\sqrt{\epsilon}))\right| < \Lambda$ which vanishes as the cut-off is removed. The second term is a total derivative and in the limit as $\epsilon \to 0$ evaluates to $-\frac{i}{2}\theta\bar{\theta}$. We must still integrate over θ which means that we must determine the θ- and $\bar{\theta}$-independent parts of the slowly varying terms on the boundary. The result is

$$\int dx\;e^{ik\cdot X_c}\bar{\Psi}_c^{[\mu}\Psi_c^{\nu]} \qquad (3.5.169)$$

which is as claimed.

The remaining term in (3.5.160) can be written as

$$\begin{aligned}-2i\int dx\, d^2\theta\, e^{ik\cdot\mathbf{X}_c}&\left(\bar{D}\mathbf{X}_c^{[\mu}ik^{\nu]}+D\mathbf{X}_c^{[\nu}ik^{\mu]}\right)\\ &\times\frac{(\theta-\bar{\theta})}{\pi\alpha' k^2}\int_0^\Lambda dy\,\frac{\partial f}{\partial y}\left(1+\frac{i}{2}\theta\bar{\theta}\frac{\partial}{\partial y}\right)e^{-\pi\alpha' k^2 f\left(\frac{2y}{\sqrt{\epsilon}}\right)}. \qquad (3.5.170)\end{aligned}$$

The second term in the rounded brackets of the y-integral cannot contribute due to its θ dependence and the first term is again a total derivative which tends to unity as the regulator is removed. We again expand the slowly varying fields on the boundary in powers of anti-commuting variables and seek terms with a single factor of θ or $\bar{\theta}$. A little algebra leads to

$$\frac{1}{\pi\alpha' k^2}\int dx\;e^{ik\cdot X_c}\left(ik\cdot\left(\Psi_c+\bar{\Psi}_c\right)(\Psi_c+\bar{\Psi}_c)^{[\mu}+\frac{\partial X_c^{[\mu}}{\partial x}\right)ik^{\nu]}. \qquad (3.5.171)$$

We have preserved the global supersymmetry with our regularisation and it is straightforward to verify that this result is indeed invariant under (3.5.152). We can now use the boundary conditions (3.5.154) to relate the boundary values of the worldsheet variables to the variables on the one dimensional worldlines to obtain the $\epsilon \to 0$ limit of (3.5.160) as

$$-2e^{-S_{\text{spin}}[\mathbf{X}_c]-S_L} \int_B dx\; e^{ik\cdot w} \left(\frac{dw^{[\mu}}{dx} + \sqrt{h}\, ik \cdot \psi\psi^{[\mu} \right) \frac{ik^{\nu]}}{k^2}. \qquad (3.5.172)$$

which we recognise contains the expression entering the exponent of the supersymmetric Wilson loop. It is only in $S_{\text{spin}}\left[\mathbf{X}_c\right]$ that the string length scale $\sqrt{\alpha'}$ appears and only in S_L that the conformal scale and its super-partner are present. The classical action can be removed by taking the tensionless limit $\alpha' k^2 \to 0$[8]. The result does not contain any further dependence on the metric which we have treated as constant, absorbing the conformal scale into the cut-off ϵ. This has occurred despite there being no mass-shell restrictions on k^2. Since there is no ϵ-dependence in (3.5.172) we conclude that the result is independent of this constant scale. Spatial variations in this scale contribute at higher order in ϵ so vanish as the cut-off is removed. So the conformal scale and its super-partner decouple from the calculation (if we assume that the metric on the world-line is independent of that on the world-sheet) and are present only in S_L; they can be removed completely if we assume further internal degrees of freedom to take us to a critical string-theory or by normalising this zeroth order expectation value against the non-interacting partition function.

Similarly to the bosonic case the interaction contains terms which involve points inserted on different world-sheets and other terms with multiple insertions on the same world-sheet. For the former we can use (3.5.172) to average over two distinct world-sheets to determine the leading order behaviour in the tensionless limit:

$$\int \frac{\mathscr{D}\mathbf{X}_i}{Z_0}\frac{\mathscr{D}\mathbf{X}_j}{Z_0}\, e^{-S_{spin}[\mathbf{X}_i]-S_{spin}[\mathbf{X}_j]}\, S_{int}[\mathbf{X}_i,\mathbf{X}_j] =$$
$$q^2 \int \frac{d^dk}{(2\pi)^d} \int_{BB'} dx\, dx'\; \frac{e^{ik\cdot(w-w')}}{k^2} \left(\frac{dw}{dx} + \sqrt{h}\,\psi\cdot ik\,\psi \right)\cdot \left(\frac{dw'}{dx'} + \sqrt{h'}\,\psi'\cdot ik\,\psi' \right) \qquad (3.5.173)$$

This result is the order q^2 contribution to the expectation value of two super-Wilson

[8]As was discussed at the end of section 3.2 the tensionless limit corresponds to taking α' large as measured with respect to the length scale, l, of the closed loop B – that is $l/\sqrt{\alpha'} \to 0$.

loops parameterised by x and x' in spinor QED. This demonstrates our result holds at leading order in the case that the worldsheets are distinct. Following the bosonic theory we shall consider extending this to arbitrary order and also treat the case that multiple vertices are on the same worldsheet.

3.5.2 Generalisation to arbitrary order

When some insertions approach one another on the same world-sheet we may find divergences that change our result in a similar way to those we found for the bosonic theory. In this section we demonstrate that no such divergences arise and the calculation reduces to the result we seek. It is because our interaction and regularisation procedure preserves the residual supersymmetry that such divergences are forbidden from arising since their dependence on the worldsheet fields will not be supersymmetric.

We follow the same steps as in the bosonic case by considering a general term at order N in the expansion of the interaction which has $2N$ vertex insertions on a single world-sheet:

$$\int \mathscr{D}\mathbf{X}\, e^{-S_{\text{spin}}} \prod_{i=1}^{2N} \int d^2\theta_i \left(\int d^2 z_i\ \bar{D}\mathbf{X}_i^{[\mu_i} D\mathbf{X}_i^{\nu_i]} - \int_{y_i=0} dx_i\ \theta_i \bar{\theta}_i \bar{\Psi}_i^{[\mu_i} \Psi_i^{\nu_i]} \right) e^{ik_i \cdot \mathbf{X}_i}\,. \tag{3.5.174}$$

The functional integral over $\mathbf{X}$ will lead to the ubiquitous factor $\exp\left(-\pi\alpha' \sum_{ij} k_i \cdot k_j G_{ij}\right)$ where we continue to denote the Green function $G_{ij} \equiv G\left(z_i, \theta_i; z_j, \theta_j\right)$. When all of the points are separated by a distance much greater than $\sqrt{\epsilon}$ the exponential factors $\exp\left(-\pi\alpha' k^2 G_0\right)$ which involve (3.5.167) suppress the integrand unless the points are close to the boundary. In the latter case we follow section 3.4.1 by integrating each point a distance Λ into the bulk, focussing on contractions that take place separately within each vertex. At leading order in the cut-off the components of the super-fields and the slowly varying Green functions between the separated points will be replaced by their boundary values.

Using (3.5.160) take the contribution involving r copies of the second term which arises from a single contraction of the quantum fields and integrate these a distance

Λ into the bulk:

$$\prod_{j=1}^{r}\int\int_0^{\Lambda} d^2\theta_j dy_j \; 2\pi\alpha' \left(\bar{D}_j \mathbf{X}_{cj}^{[\mu_j} ik_j^{\nu_j]} + D_j \mathbf{X}_{cj}^{[\nu_j} ik_j^{\mu_j]}\right)$$
$$\times \frac{i}{2}\left(\theta_j - \bar{\theta}_j\right)\frac{\partial f}{\partial y_j}\left(1 + \frac{\theta_j\bar{\theta}_j}{2}\frac{\partial}{\partial y_j}\right) e^{-\pi\alpha' k_j^2 f\left(\frac{2y_j}{\sqrt{\epsilon}}\right)} e^{ik_j\cdot \mathbf{X}_{cj}}$$
$$\times \prod_{i=r+1}^{2N}\int\int_0^{\Lambda} d^2\theta_i dy_i \; \bar{D}_i \mathbf{X}_{ci} D_i \mathbf{X}_{ci}\left(1 + \frac{\theta_i\,\bar{\theta}_i}{2}\frac{\partial}{\partial y_i}\right) e^{-\pi\alpha' k_i^2 f\left(\frac{2y_i}{\sqrt{\epsilon}}\right)} e^{ik_i\cdot \mathbf{X}_{ci}}. \tag{3.5.175}$$

To extract the ϵ-dependence of this expression it is useful to generalise the scaling carried out in the bosonic case by setting

$$y \to \epsilon^{\frac{1}{2}} y; \qquad \theta \to \epsilon^{\frac{1}{4}}\theta; \qquad \bar{\theta} \to \epsilon^{\frac{1}{4}}\bar{\theta} \tag{3.5.176}$$

for all variables in (3.5.175). Under these changes of variables and simplifying the anti-commuting variables a little we get

$$\epsilon^{-\frac{r}{4}}\prod_{j=1}^{r}\int\int_0^{\frac{\Lambda}{\sqrt{\epsilon}}} d^2\theta_j dy_j \; 2\pi\alpha' \left(\bar{D}_j \mathbf{X}_{cj}^{[\mu_j} ik_j^{\nu_j]} + D_j \mathbf{X}_{cj}^{[\nu_j} ik_j^{\mu_j]}\right)\frac{i}{2}\left(\theta_j - \bar{\theta}_j\right)\frac{\partial f}{\partial y_j} e^{-\pi\alpha' k_j^2 f(2y_j)}$$
$$\times e^{ik_j\cdot \mathbf{X}_{cj}}\prod_{i=r+1}^{2N}\int\int_0^{\frac{\Lambda}{\sqrt{\epsilon}}} d^2\theta_i dy_i \; \bar{D}_i \mathbf{X}_{ci} D_i \mathbf{X}_{ci}\left(1 + \frac{\theta_i\,\bar{\theta}_i}{2}\frac{\partial}{\partial y_i}\right) e^{-\pi\alpha' k_i^2 f(2y_i)} e^{ik_i\cdot \mathbf{X}_{ci}}. \tag{3.5.177}$$

The first r integrals with respect to y_j evaluate to

$$\frac{1}{\pi\alpha' k_j^2}\left(1 - \left(\frac{\Lambda}{\sqrt{\epsilon}}\right)^{-\frac{1}{2}\alpha' k_j^2}\right) \tag{3.5.178}$$

and the $2N-r$ remaining integrals with respect to y_i contain two terms. As described above the first can be bounded by $\frac{\Lambda}{\sqrt{\epsilon}}$ and the second is equal to $-\frac{i}{2}\theta_i\bar{\theta}_i$. The latter contribution gives the result we seek as can be seen by carrying out the integrals over the Grassmann variables:

$$\epsilon^{-\frac{r}{4}}\prod_{j=1}^{r}\int d^2\theta_j\ 2\pi\alpha'\left(\bar{D}_j\mathbf{X}_{cj}^{[\mu_j}ik_j^{\nu_j]}+D_j\mathbf{X}_{cj}^{[\nu_j}ik_j^{\mu_j]}\right)\frac{i\left(\theta_j-\bar{\theta}_j\right)}{2\pi\alpha' k_j^2}\left(1-\left(\frac{\Lambda}{\sqrt{\epsilon}}\right)^{-\frac{1}{2}\alpha' k_j^2}\right)$$
$$\times e^{ik_j\cdot\mathbf{X}_{cj}}\prod_{i=r+1}^{2N}\int d^2\theta_i\ \frac{i}{2}\theta_i\bar{\theta}_i\bar{D}_i\mathbf{X}_{ci}D_i\mathbf{X}_{ci}e^{ik_i\cdot\mathbf{X}_{ci}}. \tag{3.5.179}$$

The integrals with respect to θ_j require us to find the $\bar{\theta}_j$ and θ_j terms in the super-fields. Under the scaling (3.5.176) such terms pick up a factor $\epsilon^{\frac{1}{4}}$. The r such terms cancel the leading factor of $\epsilon^{-\frac{r}{4}}$ so the Grassmann integration selects a single term which is independent of the cut-off ϵ. The remaining integrals with respect to θ_i require the θ_i- and $\bar{\theta}_i$-independent parts of the super-fields which do not change under scaling. Following the same algebra as at first order, summing over r, integrating around the boundary and enforcing the boundary conditions the result is

$$\sum_{r=0}^{2N}\prod_{j=1}^{r}\int dx_j\ e^{ik\cdot w_j}\left(\frac{dw_j^{[\mu_j}}{dx_j}+\sqrt{h_j}\,ik_j\cdot\psi_j\psi_j^{[\mu_j}\right)\frac{ik_j^{\nu_j]}}{k_j^2}\prod_{i=r+1}^{2N}\int dx_i\ e^{ik_i\cdot X_{ci}}\bar{\Psi}_{ci}^{[\mu_i}\Psi_{ci}^{\nu_i]} \tag{3.5.180}$$

With the exception of the $r=2N$ case the contributions in this sum cancel terms arising out of the boundary term in the interaction (3.5.155) which conspire to ensure supersymmetry is maintained. This leaves the contribution occurring from $2N$ contractions between fields which corresponds to $2N$ points inserted on the boundary of a single Wilson loop:

$$q^{2N}\prod_{j=1}^{2N}\int dx_j\ e^{ik\cdot w_j}\left(\frac{dw_j^{[\mu_j}}{dx_j}+\sqrt{h_j}\,ik_j\cdot\psi_j\psi_j^{[\mu_j}\right)\frac{ik_j^{\nu_j]}}{k_j^2}. \tag{3.5.181}$$

As in the bosonic case this result is independent of the string tension α'. It remains to enforce the contractions of the space-time indices and impose pairwise $k_{j+1}=-k_j$ as defined in the interaction to produce

$$q^{2N}\prod_{j=1}^{N}\int_B dx_j dx'_j\ \frac{e^{ik_j\cdot\left(w_j-w'_j\right)}}{k^2}\left(\frac{dw_j}{dx_j}+\sqrt{h_j}\,ik_j\cdot\psi_j\psi_j\right)\cdot\left(\frac{dw'_j}{dx'_j}-\sqrt{h'_j}\,ik_j\cdot\psi'_j\psi'_j\right) \tag{3.5.182}$$

showing how pairs of points on the boundary interact.

Returning to the integrals over the y_i in (3.5.177) we bounded the second term by $\frac{\Lambda}{\sqrt{\epsilon}}$ and for the θ_i integrals we now seek the $\theta_i\bar{\theta}_i$ term from the boundary super-fields. This term scales as $\sqrt{\epsilon}$ which leaves a contribution of order Λ^{2N-r}. The integration over the θ_j variables remains the same as above so the Grassmann integration selects a single term which vanishes as $\epsilon \to 0$ because Λ vanishes in this limit too. Other contributions from this configuration of points are subleading in ϵ. This completes our treatment of the case where all $2N$ points are close to the boundary (and a distance greater than $\sqrt{\epsilon}$ apart from one another) and demonstrates the result (3.5.182) we sought. We now consider what happens when these points are close to one another in the bulk or the boundary to show that in contrast to the bosonic case no divergences appear.

Suppose that of the $2N$ points a number $n+1$ are within Λ of one another (but that this set is separated by more than Λ from any other points on the same worldsheet). Now it is the contractions between different vertices which are rapidly varying. Following the procedure taken for the bosonic case (see section 3.4.2) Wick's theorem allows us to replace this by a sum of terms involving various contractions between this set of points and normal ordered terms which have not been contracted with other operators outside of this set. The leading order contribution comes from expanding the normal ordered terms about the position of the final point z_{n+1}. We then integrate the first n points in a region of size Λ about this reference point which remains to be integrated about the worldsheet. The ϵ-dependence can be extracted by counting derivatives of rapidly varying fields. Now

$$\begin{aligned} G^{\epsilon}\left(z_r, \theta_r; z_s, \theta_s\right) &= -f\left(\frac{\sqrt{z_{rs}\bar{z}_{rs}}}{\sqrt{\epsilon}}\right) + f\left(\frac{\sqrt{z^R_{rs}\bar{z}^R_{rs}}}{\sqrt{\epsilon}}\right) \\ &= -f\left(\frac{\sqrt{z_{rs}\bar{z}_{rs}}}{\sqrt{\epsilon}}\right) + \frac{1}{4\pi}\log\left(\frac{(2iy_{n+1} - \theta_r\bar{\theta}_s)(-2iy_{n+1} - \bar{\theta}_r\theta_s)}{\epsilon}\right) \\ &+ \mathcal{O}\left(\frac{\Lambda}{y_{n+1}}\right) \end{aligned} \tag{3.5.183}$$

and it is the first term of this which varies rapidly as the points z_r and z_s move apart. Wick contractions between fields evaluated at the point z_r and z_s produce various derivatives of this Green function. In parallel to the bosonic string the leading order contribution comes from contractions which have all $2\,(n+1)$ possible derivatives acting on the first term in (3.5.183). This can be seen by scaling the relative coordinates (but not z_{n+1} or $\bar{z}_{n+1}$) and the θ_r, $\bar{\theta}_r$:

$$z_r - z_s \to \epsilon^{\frac{1}{2}}\,(z_r - z_s)\,; \qquad \theta_r \to \epsilon^{\frac{1}{4}}\theta_r; \qquad \bar{\theta}_r \to \epsilon^{\frac{1}{4}}\bar{\theta}_r, \tag{3.5.184}$$

so that

$$\begin{aligned} &f\left(\frac{\sqrt{z_{rs}\bar{z}_{rs}}}{\sqrt{\epsilon}}\right)) \to f\left((\sqrt{z_{rs}\bar{z}_{rs}}\right); \\ &\frac{1}{4\pi}\log\left(\frac{(2iy_{n+1} - \theta_r\bar{\theta}_s)(-2iy_{n+1} - \bar{\theta}_r\theta_s)}{\epsilon}\right) \to \frac{1}{4\pi}\log\left(\frac{4y_{n+1}^2}{\epsilon}\right) + \mathcal{O}\left(\epsilon^{\frac{1}{2}}\right) \end{aligned} \tag{3.5.185}$$

under which the super-derivatives and integration measures transform as

$$D \to \epsilon^{-\frac{1}{4}}D, \quad \bar{D} \to \epsilon^{-\frac{1}{4}}\bar{D}, \tag{3.5.186}$$

$$d^2z_r d^2\theta_r \to \epsilon^{\frac{1}{2}} d^2z_r d^2\theta_r, \quad d^2z_{n+1}d^2\theta_{n+1} \to \epsilon^{-\frac{1}{2}} d^2z_{n+1}d^2\theta_{n+1} \tag{3.5.187}$$

so the integral with respect to $d^2\theta_{n+1}\prod_r d^2z_r d^2\theta_r$ of the term containing $2(n+1)$ derivatives, D and $\bar{D}$, acting on $f(\sqrt{z_{rs}\bar{z}_{rs}/\epsilon})$ scales into $1/\epsilon$ multiplied by an integral independent of ϵ. This depends on the momenta k_r in a potentially complicated way but because of the way the contractions were carried out to form the $2\,(n+1)$ derivatives it is possible to integrate by parts to enforce the $\mathbf{X}$ dependence to sit only in the exponent:

$$\begin{aligned} \frac{1}{\epsilon}\int d^2z_{n+1}d^2\theta_{n+1}\prod_{r=1}^{n} d^2z_r d^2\theta_r F^{\mu_1\ldots\nu_{n+1}}\left(z_1,\theta_1,\ldots,z_{n+1},\theta_{n+1}\right) : e^{i\sum_{r=1}^{n+1}k_r\cdot\mathbf{X}_{n+1}} : \\ \times\exp\Bigg(-\pi\alpha'\sum_{r,s=1}^{n+1}k_r\cdot k_s G^{\epsilon}\left(z_r,\theta_r;z_s\theta_s\right)\Bigg). \end{aligned} \tag{3.5.188}$$

At leading order in ϵ after carrying out the integral over the relative coordinates and the θ_r, $\bar{\theta}_r$ we are left with

$$\frac{1}{\epsilon}\tilde{F}^{\mu_1\ldots\nu_{n+1}}(k_1,..,k_{n+1})\int d^2z_{n+1}:e^{iK\cdot X(z_1)}:\left(\frac{\epsilon}{y_{n+1}^2}\right)^{\alpha' K^2/4} \tag{3.5.189}$$

where we have defined

$$\tilde{F}^{\mu_1\ldots\nu_{n+1}}(k_1,..,k_{n+1}) = \int d^2\theta_{n+1}\left(\prod_{r=1}^{n} d^2z_r\,d^2\theta_r\right)F^{\mu_1\ldots\nu_{n+1}}(z_1,\theta_1,\ldots,z_{n+1},\theta_{n+1}) \times\exp\left(\pi\alpha'\sum_{r,s=1}^{n+1}k_r\cdot k_s f\left(\sqrt{z_{rs}\bar{z}_{rs}}\right)\right) \tag{3.5.190}$$

and $K=\sum_{r=1}^{n+1}k_r$. Since this is not invariant under the residual supersymmetry (3.5.152) it must vanish so there can be no $1/\epsilon$ divergence present.

Subleading terms of order $\epsilon^{-3/4}$ could in principle appear if we were to have $2n+1$ derivatives acting on G_ϵ or from an expansion of the super-field components. However the required factors of $\epsilon^{1/4}$ are paired with fermionic fields Ψ and $\bar{\Psi}$. They cannot be present since the final result must be bosonic. The first non-trivial divergence which could potentially occur is of order $\epsilon^{-1/2}$ and can arise in a number of ways. The second order expansion in θ and $\bar{\theta}$ of the exponentiated super-field contains $\epsilon^{1/2}\theta\bar{\theta}k\cdot\Psi k\cdot\Psi$; taking two derivatives off the rapidly varying part of G^ϵ reduces the power of ϵ picked up under scaling by $1/2$ and leaves two super-derivatives of the super-field or derivatives of the slowly varying part of G^ϵ; taking only one derivative off G^ϵ in combination with expanding one super-derivative of the super-field to first order gives a similar expression and an expansion of the components of the super-field about the point z_{n+1} gives $\epsilon^{1/2}\left(z-z_{n+1}\right)\cdot\partial X$. The latter two of these vanish again by rotational symmetry whilst contributions from the slowly varying part of G^ϵ would have the same X-dependence as (3.5.189). From two super-derivatives or an expansion of the super-field components the contribution at this order has an X-dependence proportional to

$$\frac{c^{\rho\sigma}}{\sqrt{\epsilon}}\int d^2z_1:\bar{\Psi}^\rho\Psi^\sigma e^{iK\cdot X(z_1)}:\left(\frac{\epsilon}{y_{n+1}^2}\right)^{\alpha' K^2/4}, \tag{3.5.191}$$

Under the residual supersymmetry this too changes, although if the coefficient

$c^{\rho\sigma} = K^\rho K^\sigma$ its variation takes the same form as the variation of the boundary term $\epsilon^{-1/2}\int dx\, \exp(ik\cdot w)$. Were this boundary term to be generated as the insertions approach one another close to the boundary then it would be possible for this divergence to be present. However a term proportional to $k\cdot\bar{\Psi}k\cdot\Psi$ can only be generated from expanding the super-fields in the exponential for the $\theta\bar{\theta}$ contribution. The coefficient of this term would be

$$\int d^2\theta_{n+1}\left(\prod_{r=1}^{n} d^2 z_r\, d^2\theta_r\right) F^{\mu_1\ldots\nu_{n+1}}\left(z_1,\theta_1\ldots,z_{n+1},\theta_{n_1}\right) e^{\pi\alpha'\sum k_r\cdot k_s f(\sqrt{z_{rs}\bar{z}_{rs}})}\,\bar{\theta}_r\theta_s \tag{3.5.192}$$

independent of the choice of r and s. We can demonstrate that this vanishes by virtue of its θ dependence. $F^{\mu_1\ldots\nu_{n+1}}$ arose out of $2(n+1)$ derivatives acting on $f\left(\sqrt{z_{rs}\bar{z}_{rs}}\right)$ and (3.5.192) requires it to contain a total of n θs and n $\bar{\theta}$s to be non-zero[9]. Schematically, $f\left(\sqrt{z_{rs}\bar{z}_{rs}}\right)$ has a dependence on anti-commuting variables of the form

$$f\left(\sqrt{z_{rs}\bar{z}_{rs}}\right) = f\left(|z_r - z_s|^2\right) - \theta_r\theta_s g_1\left(z_r - z_s\right) - \bar{\theta}_r\bar{\theta}_s g_2\left(z_r - z_s\right) - \theta_r\bar{\theta}_r\theta_s\bar{\theta}_s h\left(z_r - z_s\right) \tag{3.5.193}$$

where the functions g_1, g_2 and h depend on the relative separation of the points and involve derivatives of $f(s)$. Now suppose that the $n+1$ derivatives D and $n+1$ derivatives $\bar{D}$ contained in F produce p copies of $D_r f$, q of $\bar{D}_r f$, r $D_s D_r f$, s $\bar{D}_s\bar{D}_r f$ and t lots of $\bar{D}_s D_r f$ with $p + 2r + t = n + 1 = q + 2s + t$. It follows that the schematic θ and $\bar{\theta}$ dependence of each of these terms is respectively $\theta + \theta\bar{\theta}\bar{\theta}$, $\bar{\theta} + \theta\theta\bar{\theta}$, $1 + \theta\theta + \bar{\theta}\bar{\theta} + \theta\bar{\theta}\theta\bar{\theta}$, $1 + \theta\theta + \bar{\theta}\bar{\theta} + \theta\bar{\theta}\theta\bar{\theta}$ and $\theta\bar{\theta}$. Counting modulo 2 we thus have a total of $p + t = n + 1$ factors of various θs and $q + t = n + 1$ factors of $\bar{\theta}$s. So $F^{\mu_1\ldots\nu_{n+1}}$ cannot contain the correct number of θs and $\bar{\theta}$s for (3.5.192) to produce a non-zero result.

The next possible divergence is of order $\epsilon^{-1/4}$ but it too vanishes because its field content would have to be fermionic. The next order in ϵ consists of finite terms, but these are suppressed by the overall factor of $\left(\epsilon/y_{n+1}^2\right)^{\alpha' K^2/4}$ which comes from the

[9]The requirement is of course more stringent than this in that the θs and $\bar{\theta}$s must have the correct indices but the general argument does not rely on this detail.

slowly varying part of G^ϵ. As $K^2 \geq 0$ in Euclidean signature such terms vanish for all K^2 except those close to zero in terms of ϵ. Since K is eventually to be integrated over we also need to consider the contribution of these small values. Following the discussion in section 3.4.2 we recall that for α' large and ϵ small this factor behaves effectively as

$$\frac{\delta\left(K^2\right)}{\left(\frac{1}{2}\alpha' \ln \frac{y_1}{\epsilon}\right)^{\frac{D}{2}}} \tag{3.5.194}$$

and so is also suppressed in the tensionless limit. We conclude that there are no terms associated with a set of points meeting one another in the bulk of the worldsheet that survive in the tensionless limit as the cut-off is removed. In the bosonic case we were unable to deduce whether our arguments extended to the case that the points are also close to the boundary but in the current case we can use the supersymmetry to show that there are no further contributions from this case.

Close to the boundary the second term in G^ϵ also varies rapidly. To consider its variation we must also scale y_{n+1} along with the other variables. This means that the integration measure $d^2 z_{n+1} d^2\theta_{n+1}$ is unchanged by the scaling and the leading order divergence is $\mathcal{O}\left(\epsilon^{-1/2}\right)$. There is also no suppression by the second term in G^ϵ but the X-dependence remains the same as that when the points are far from the boundary. A term proportional to

$$\frac{1}{\sqrt{\epsilon}} \int dx\ e^{iK\cdot X} \tag{3.5.195}$$

has been seen before; it is not supersymmetric and so this divergence is not present. This time the presence of the boundary breaks the symmetry of the integration domain so a term of order $\epsilon^{-1/4}$ is not forbidden – it could arise out of an expansion of the super-field in the exponent or by taking one of the derivatives off G^ϵ and onto a super-field. The possible $\mathbf{X}$-dependence has the form

$$\frac{c^\rho}{\epsilon^{\frac{1}{4}}} \int dx\ \left(\Psi + \bar{\Psi}\right)^\rho e^{iK\cdot X} \tag{3.5.196}$$

which changes under the residual supersymmetry unless $c^\rho \propto K^\rho$ in which case the change is a total derivative. It is fermionic, however, and by applying the same counting of θs and $\bar{\theta}$s as before (since such a term can only arise from an expansion of the super-field in the exponent) it is straightforward to show that this term cannot arise out of an integral over the Grassmann variables since it requires n θs or n $\bar{\theta}$s.

The final order in ϵ to consider provides finite terms. These may arise out of two super-derivatives of the super-field or various expansions of the super-field about the point z_{n+1} in tandem with super-derivatives. The possible contributions are many but there is only one potential term which remains invariant under the residual supersymmetry which is the electromagnetic coupling

$$\int dx\ e^{iK\cdot X} \left(dX^\mu/dx + iK\cdot(\Psi+\bar{\Psi})(\Psi+\bar{\Psi})^\mu\right) \tag{3.5.197}$$

where μ here must be equal to μ_q of the vertex $V^{\mu_q\nu_q}$ which was used to generate this finite piece. However the supersymmetric generalisation of Gauss' law, (3.4.142), can be used to show that this cannot be formed. Indeed contracting k_q with the integral of the q-th vertex

$$\begin{aligned}
&k_q^{\mu_q}\int d^2\theta_q\left(\int d^2z_q\,\bar{D}_q\mathbf{X}_q^{[\mu_q}D_q\mathbf{X}_q^{\nu_q]}-\int_{y_q=0}dx_q\ \theta_q\bar{\theta}_q\bar{\Psi}_q^{[\mu_q}\Psi_q^{\nu_q]}\right)e^{ik_q\cdot\mathbf{X}_q}\\
&=\int_{y_q=0}dx_q\ \left(\frac{dX_q^{\nu_q}}{dx_q}+ik_q\cdot\left(\Psi_q+\bar{\Psi}_q\right)\left(\Psi_q+\bar{\Psi}_q\right)^{\nu_q}\right)e^{ik_q\cdot X_q}
\end{aligned} \tag{3.5.198}$$

which is a boundary term that does not contain the quantum variables $\bar{\mathbf{X}}_q$. This means that it cannot take place in any contractions with other terms in the set so factors out of the normal ordered expansion of the other vertices. Therefore this boundary integral of the q-th field would have to factor out of the contraction of (3.5.197) with k_q. This is not possible because (3.5.197) contains only one field integrated around the boundary so the contraction could not produce an integral involving the k_q dependence and the field $X\left(z_q\right)$ multiplied into an integral involving the remaining momenta with a field content expanded about the point z_{n+1}.

This completes the argument that supersymmetry prevents divergent or finite corrections from appearing when insertions approach one another on the same world-sheet and proves our claim that (3.5.173) exponentiates. This leads directly to (3.1.32) and allows the replacement of an integral over the gauge field by an integral over fluctuating spinning strings interacting upon contact. Since the result we have found is independent of the cut-off ϵ we have also shown that the scale of the worldsheet metric decouples from the calculation, appearing only in S_L. Spatial variations of this scale could only contribute at higher order in ϵ and so vanish as we remove the regulator by taking the limit $\epsilon \to 0$. The final step to return to the world-line formulation of spinor QED is to integrate over the worldsheet metric and boundaries weighted by the world-line action for

$$\int \left(\prod_j^n \frac{\mathscr{D}(g, \mathbf{X}, w, \psi, h, \chi)_j}{Z_0} \right) e^{-S_s - S_{BdVH}} = \int \left(\prod_j^n \mathscr{D}(w, \psi, h, \chi)_j \right) \frac{\mathscr{D}A}{N} e^{-S_{gf} - S_{BdVH}} \prod_j W_s[A]. \tag{3.5.199}$$

Summing over n then re-expresses the partition function of QED in terms of the partition function of spinning strings with contact interactions. To also express the generating functional (3.1.31) requires the world-line Green function which in analogous fashion to the bosonic theory requires the inclusion of open strings. The calculation proceeds in the same way but for the differing boundary conditions on each end of the spinning string and again it is only the Dirichlet end of the string which contributes to the interaction. It is also possible to introduce a background gauge field to source photon amplitudes on the world-line, as described for scalar QED.

3.6 Conclusion

We have investigated how strings with contact interactions can be used to model Abelian gauge fields. We were able to construct δ-functions on the world-sheet that decoupled from the Liouville degree of freedom because their contribution was negligible except close to the world-sheet boundary where they generated the electromagnetic coupling. Although the purely bosonic theory proved to be problematic the world-sheet supersymmetry present in the spinning string provided the structure needed to eliminate unwanted divergences and also generate the super-Wilson loops needed to couple spinor matter to electromagnetism in the worldline approach. The string world-sheets correspond to the trajectories of lines of electric flux joined to charged particles.

It proved necessary to take the tensionless limit to remove dependence on the classical string action so the string length-scale is large compared to the size of the Wilson loops. The strings themselves, therefore, can be very large and it may be possible to distinguish this theory from conventional QED where the interactions are mediated by point-like particles by direct observation of these extended objects. Additionally it may be possible to detect string-like corrections to QED at large distances, (although we have not calculated these). Since the scale of the world-sheet metric decouples from our calculations we could argue that the super-Liouville degrees of freedom only lead to an overall multiplicative factor that cancels out of physical amplitudes. Alternatively we might modify the model to include sufficient internal degrees of freedom to ensure a critical string-theory. We speculate on what these extra degrees of freedom could be in the next chapter. This decoupling allows us to apply our string theory in four-dimensional space-time dimensions.

QED is of course an extremely successful theory, having been tested to high accuracy in experiments, but nonetheless it is an effective theory arising out of the Standard Model, so our string model must also be just an effective theory. Understanding how it relates to the more fundamental non-Abelian case will require some development of the model.

One aspect of this development has to be the worldline formalism for non-Abelian quantum field theories. Strassler did discuss the generalisation of his formalism to include gauge fields which transform in a representation of a non-Abelian group but he did not consider the extra ingredients necessary to describe chiral fermions. To further complicate matters the standard model contains a variety of spin $1/2$ matter fields which transform in different representations of the gauge group and we would expect to have to sum over these representations – and the chiralities of the particles – to produce quantities which we can compare with experiment. This is a daunting task in quantum field theory and we may hope that the string theory reformulation we have provided here could offer some insight or simplification to the computation of this sum. Before we can do that, however, we need to understand how this calculation appears in the worldline formalism. That is the content of the next chapter of this thesis.

Chapter 4

Unified field theory in the worldline formalism

4.1 Introduction

Recently a new model of chiral fermions in the worldline approach demonstrated an interesting way to sum over the gauge group representations and chiralities present in the standard model [64]. This sum was constructed for a single generation of fermions and was supplemented by a sterile neutrino. The model is substantially different from the usual field theory approach because the assignment of particles to their group representations and chiralities arises naturally, rather than being pre-determined by hand. Progress in the worldline description of chiral particles is central to a description of the standard model in first quantised language and is an important stepping stone to an equivalent description in terms of interacting strings. Besides, as we have seen in the introduction the worldline formalism can offer significant computational advantages over calculations in perturbative quantum field theory. We shall eventually speculate about how this result can be incorporated into the string theory model discussed in the previous chapter, but following on from the success of [64] it seems natural to consider first the generalisation of this work to other non-Abelian groups.

The motivation for considering such gauge groups is the unification of the electroweak and strong interactions. The purpose of this unification is to find a theory with only one coupling constant, from which the standard model emerges after spontaneous symmetry breaking as a low-energy effective theory [65]. The gauge group with smallest rank that can accommodate the standard model is that of $SU(5)$. This is of course the famous Georgi-Glashow model [66]. We shall demonstrate that the representations and chiralities of the standard model particles as described by the standard $SU(5)$ unified theory can also be generated in this approach. The next section briefly reviews the argument and notation in [64] and in Section 4.3 the model is applied to the unified theories of $SU(5)$ and flipped $SU(5)$. We also consider other unified theories which appear in the literature, namely $SU(6)$ and $SO(10)$.

4.2 Fields and worldlines

We consider a left- or right-handed massless fermion moving in a background gauge field A. We take A to transform in the adjoint representation of some symmetry group which is described by anti-Hermitian Lie algebra generators $\{T_R\}$. The fermion action for a left-handed massless fermion, ξ, is

$$\int d^4x \; i\xi^\dagger \bar{\sigma} \cdot D\xi \tag{4.2.1}$$

where $D = (\partial + A)$ and $\sigma^\mu = (\mathbb{1}, \sigma^i)$. In the introduction we met the idea of the effective action defined by functionally integrating over fermionic fields but for the current case we meet the well-known problem of how to define the determinant of the Dirac operator on the space of chiral fermions transforming in a non-real representation of the gauge group. We can, however, define the phase-difference of determinants which motivates us to consider the variation of the effective action under an infinitesimal change in A [67, 68]. This is easily found to be $\delta_A \Gamma[A] = \mathrm{Tr}\left((\bar{\sigma} \cdot D)^{-1} \bar{\sigma} \cdot \delta A\right)$ which can be written in terms of γ matrices as

$$-\int_0^\infty dT \, \mathrm{Tr}\left(\frac{(1-\gamma_5)}{2} e^{T(\gamma \cdot D)^2} \gamma \cdot D \, \gamma \cdot \delta A\right) \tag{4.2.2}$$

We recognise the heat kernel of the operator $(\gamma \cdot D)^2 = D^2\mathbb{1} + \frac{1}{2}\gamma^\mu F_{\mu\nu}\gamma^\nu$ and in [64] a worldline representation of (4.2.2) was derived:

$$\delta_A \ln \int \mathscr{D}\left(\bar{\xi}, \xi\right) e^{-S\left[\bar{\xi},\xi\right]}$$
$$= -\int_0^\infty \frac{dT}{T} \int_{L/R} \mathscr{D}\omega \mathscr{D}\psi \, e^{-S[w,\psi]} \mathscr{P} \operatorname{tr}\left(g\left(2\pi\right) \int_0^{2\pi} dt \, \psi \cdot \dot{\omega} \, \psi \cdot \delta A\right). \tag{4.2.3}$$

Here $g(t)$ is the super-Wilson loop describing the coupling of the fermion to the gauge field which we have seen in the previous chapter (we recall that we require a closed loop in order to generate the functional trace)

$$g(t) = \mathscr{P} \exp\left(-\int_0^t \mathcal{A}^R T_R \, dt\right) \tag{4.2.4}$$

where

$$\mathcal{A} = \dot{\omega} \cdot A + \frac{T}{2}\psi^\mu F_{\mu\nu}\psi^\nu. \tag{4.2.5}$$

The Grassmann variables ψ^μ are introduced to represent the γ-matrices and the action $S[w, \psi]$ is just that of Brink, Di Vecchia and Howe that we have used earlier. The boundary conditions on ψ are interpreted depending on the chirality of the fermion. For left handed fermions the path integral with periodic boundary conditions on ψ is subtracted from that with anti-periodic boundary conditions whereas for right handed fermions the two contributions are summed. These combinations insert the appropriate projection operator $1 \mp \gamma^5$ into the path integral.

Strassler dealt with the non-Abelian nature of the coupling and path ordering directly by introducing a sum over the ordering of the integrand at each order in a perturbative expansion of the effective action. In [64] a different approach was taken. The path ordering can be represented with functional integrals by introducing a set of anti-commuting operators $\tilde{\phi}_r$ and ϕ_s satisfying $\{\tilde{\phi}_r, \phi_s\} = \delta_{rs}$ with action $S_\phi = \int \tilde{\phi} \cdot \dot{\phi} \, dt$ [69]. It is possible to use this approach to reproduce the path ordered exponential[1] in (4.2.3). However we follow [64] by also using these fields to generate the super-Wilson loop coupling. It is easy to check the following definition furnishes

[1]The point is that the propagator of the ϕ, $\tilde{\phi}$ theory is the Heaviside step function which can be used to produce the path ordering.

us with a representation of the Lie algebra

$$R^S \equiv \tilde{\phi}_r T^S_{rs} \phi_s; \qquad \left[R^S, R^T\right] = i f^{STU} R^U. \tag{4.2.6}$$

So instead of working directly with 4.2.3 we will find it advantageous to combine these ideas to consider as it stands the related quantity

$$\int_0^\infty \frac{dT}{T} \int \mathscr{D}\omega \mathscr{D}\psi \, e^{-S[w,\psi]} \int_0^{2\pi} dt \, \psi \cdot \dot{\omega} \, \psi \cdot \delta A \, \frac{\delta Z\left[\mathcal{A}\right]}{\delta \mathcal{A}} \tag{4.2.7}$$

where

$$Z\left[\mathcal{A}\right] = \int \mathscr{D}(\tilde{\phi}, \phi) \, e^{-\frac{1}{2}\int_0^{2\pi} \tilde{\phi}\left(\frac{d}{dt} + \mathcal{A}\right)\phi}. \tag{4.2.8}$$

Remarkably, in [64] the evaluation of $Z\left[\mathcal{A}\right]$ was shown to provide the correct sum over the chirality and representation assignments for the fermion content of the standard model. Five pairs of $\tilde{\phi}$ and ϕ were used to represent the Lie algebra generators of $SU(3) \times SU(2) \times U(1)$. The desired sum of chiralities and representations was found by adding the result of evaluating (4.2.7) with anti-periodic boundary conditions on all Grassmann variables to that with periodic boundary conditions imposed. The information about the representations and chiralities of the standard model particles is contained in the functional determinant of the kinetic operator for the new field $\det\left(\frac{d}{dt} + \mathcal{A}\right)$. This novel approach is of great importance for worldline theories of chiral fermions and in the following sections we generalise the result by considering the gauge groups of some unified theories.

4.3 Unified theory

The assignment used in [64]

$$\left(T^R\right) = i \begin{pmatrix} \frac{1}{2}\lambda^b \otimes \mathbb{1}_2 \\ \frac{1}{2}\sigma^a \otimes \mathbb{1}_3 \\ \frac{1}{2}\mathbb{1}_2 \otimes \mathbb{1}_3 - \frac{1}{3}\mathbb{1}_3 \otimes \mathbb{1}_2 \end{pmatrix} \tag{4.3.9}$$

incorporating the standard model generators inside a five dimensional algebra is reminiscent of the Georgi-Glashow method which embeds the standard model in $SU(5)$. This motivates the consideration of this group as the underlying symmetry without purposefully arranging for the standard model content to appear. We shall show that with the general use of $SU(5)$ as the gauge group the procedure introduced in [64] yields the familiar $\bar{\mathbf{5}}\oplus\mathbf{10}\oplus\mathbf{1}$ representations into which the left-handed matter content of the standard model fits in a manner consistent with the particles' quantum numbers. The chirality associated with these representations is also in agreement with the Georgi-Glashow model so that that these representations are favoured by the new model.

We also take five pairs of $\tilde{\phi}$ and ϕ to incorporate the generators of $SU(5)$. Then integrating over $\bar{\phi}$ and ϕ in (4.2.7) leads to a determinant which we evaluate as in [64]:

$$Z\left[\mathcal{A}\right] = \det\left(i\left(\frac{d}{dt} + \mathcal{A}\right)\right) \tag{4.3.10}$$

To compute the eigenfunctions of this operator we write them as $v\left(t\right) = g\left(t\right) f\left(t\right)$. Then the eigenvalue equation $i\left(\frac{d}{dt} + \mathcal{A}\right) v\left(t\right) = \mu v\left(t\right)$ translates to an equation for $f\left(t\right)$:

$$i\frac{d}{dt} f\left(t\right) = \mu f\left(t\right) \Longrightarrow f\left(t\right) = v\left(0\right) e^{-i\mu t} \tag{4.3.11}$$

The eigenvalues depend on the boundary conditions on the fields $\tilde{\phi}$ and ϕ; if $v\left(2\pi\right) = \pm v\left(0\right)$ then we require $v\left(0\right)$ to be an eigenvector of $g\left(2\pi\right)$ and must impose a condition on μ:

$$g\left(2\pi\right) v\left(0\right) = \rho v\left(0\right); \qquad \rho e^{-2\pi i\mu} = \pm 1. \tag{4.3.12}$$

For anti-periodic periodic boundary conditions the eigenvalues are $\mu_- = n + \frac{1}{2} + \frac{\log\left(\rho\right)}{2\pi i}$ and for periodic boundary conditions are $\mu_+ = n + \frac{\log\left(\rho\right)}{2\pi i}$. The determinant is given by the product of eigenvalues and is proportional to

$$\begin{cases} \det\left(\sqrt{g\left(2\pi\right)} + 1/\sqrt{g\left(2\pi\right)}\right) & \text{A/P} \\ \det\left(\sqrt{g\left(2\pi\right)} - 1/\sqrt{g\left(2\pi\right)}\right) & \text{P} \end{cases} \tag{4.3.13}$$

where A/P and P refer to anti-periodic and periodic boundary conditions respectively.

To calculate the determinant it suffices to name a representation under which the Wilson-loop is to transform. The Lie group valued object $g(2\pi)$ can then be rotated onto the Cartan subalgebra

$$g(2\pi) = \exp(\alpha_i H_i) \qquad i = 1 \dots 4 \tag{4.3.14}$$

whereby its eigenvalue equation can be expressed in terms of the weights of the representation under which it transforms[2]. The goal is to express the determinants in (4.3.13) in terms of group invariant properties of $g(2\pi)$ and the calculations are essentially straightforward. In particular, a simple choice is to take the Wilson loop to transform in the fundamental representation **5**, where we have shown that the determinants in (4.3.13) can be written as a sum over traces of $g(2\pi)$ in different representations:

$$\begin{aligned} &\det\left(i\left(\frac{d}{dt} + \mathcal{A}\right)\right) \propto \\ &\mathrm{tr}\left(g_{\mathbf{5}}\right) + \mathrm{tr}\left(g_{\mathbf{10}}\right) + \mathrm{tr}\left(g_{\overline{\mathbf{10}}}\right) + \mathrm{tr}\left(g_{\bar{\mathbf{5}}}\right) \\ &+ 2\mathrm{tr}\left(g_{\mathbf{0}}\right) \end{aligned} \tag{4.3.15}$$

for anti-periodic boundary conditions and

$$\begin{aligned} &\det\left(i\left(\frac{d}{dt} + \mathcal{A}\right)\right) \propto \\ &\mathrm{tr}\left(g_{\mathbf{5}}\right) - \mathrm{tr}\left(g_{\mathbf{10}}\right) + \mathrm{tr}\left(g_{\overline{\mathbf{10}}}\right) - \mathrm{tr}\left(g_{\bar{\mathbf{5}}}\right) \end{aligned} \tag{4.3.16}$$

when periodic boundary conditions are imposed. In the above equation the subscripts denote the representation in which the trace is to be taken.

The final step is to include with the latter the factor of γ_5 arising with periodic

[2] In a given representation of dimension D the generators provide a realisation of the Lie algebra as $D \times D$ matrices acting on states $|\alpha\rangle$ labelled by their eigenvalues of each Cartan generator: $H_i |\alpha\rangle = \alpha_i |\alpha\rangle$. The α_i are the weight vectors of the D dimensional representation.

boundary conditions on ψ and to take the sum of the two contributions. The result is the representations and chirality assignments which are well known in this unified theory:

$$\begin{aligned} &\left(\operatorname{tr}\left(g_{\bar{\mathbf{5}}}\right)+\operatorname{tr}\left(g_{\mathbf{10}}\right)+1\right) P_L \\ +\ &\left(\operatorname{tr}\left(g_{\mathbf{5}}\right)+\operatorname{tr}\left(g_{\overline{\mathbf{10}}}\right)+1\right) P_R. \end{aligned} \tag{4.3.17}$$

The Georgi-Glashow model[3] places a left-handed conjugate down quark colour triplet and a left-handed isospin doublet into the $\bar{\mathbf{5}}$ representation. Into the **10** representation is placed a left-handed colour triplet of conjugate up quarks, a left-handed colour triplet and isospin doublet of up and down quarks and a left-handed conjugate electron. It is easy to check that these assignments respect the quantum numbers of the particles if the **10** representation is made up out of the anti-symmetric product of two **5**. The trivial representations that appear here may be relevant to the discussion of neutrino masses.

There is another assignment of standard model particles into these same representations of $SU(5)$ which appears in the literature. Flipped $SU(5)$ [71, 72] makes use of the gauge group $SU(5) \times U(1)_X$. The extra $U(1)$ factor is needed because in this theory the left-handed conjugate *up* quark colour triplet joins a left-handed isospin doublet in the $\bar{\mathbf{5}}$ representation which does not have vanishing total hypercharge. The **10** contains a left-handed colour triplet of conjugate *down* quarks, a left-handed colour triplet and isospin doublet of up and down quarks and a left handed conjugate *neutrino* and the left-handed conjugate electron is now placed in the **1** representation. The $SU(5)$ is then decomposed into $SU(3) \times SU(2) \times U(1)_Z$ which provides the $SU(3) \times SU(2)$ part of the standard model; the standard model hypercharge generator is then formed out of a linear combination of $U(1)_Z$ and $U(1)_X$.

The simplest way to accommodate an extra $U(1)$ symmetry into our formalism

[3] See, for example, [70]

is to include an extra generator $T_{25} = \mathbb{1}_5$. Then the Wilson loop factorises as

$$g(2\pi) = e^{i\theta} g_{\mathbf{5}}(2\pi) \; ; \qquad \theta = -\int_0^{2\pi} \mathcal{A}^{25}(t)\, dt. \tag{4.3.18}$$

It is straightforward to repeat the previous calculation with this extra generator and we find that the terms in (4.3.17) simply pick up an extra factor according to their $U(1)$ charge:

$$\begin{aligned} &\left(\mathrm{tr}\left(g_{\bar{\mathbf{5}}}\right) e^{-3i\theta} + \mathrm{tr}\left(g_{\mathbf{10}}\right) e^{i\theta} + e^{5i\theta}\right) P_L \\ +\; &\left(\mathrm{tr}\left(g_{\mathbf{5}}\right) e^{3i\theta} + \mathrm{tr}\left(g_{\overline{\mathbf{10}}}\right) e^{-i\theta} + e^{-5i\theta}\right) P_R. \end{aligned} \tag{4.3.19}$$

We shall discuss $SO(10)$ unified theories in more detail below but there is an interesting relation to this group contained in the extra $U(1)$ charges of the above equation. It is well known that $SU(5) \times U(1) \subset SO(10)$ [73, 74] and the $SO(10)$ (spinor) representation $\mathbf{16}$ decomposes as $\mathbf{16} \to \bar{\mathbf{5}}_{-3} \oplus \mathbf{10}_1 \oplus \mathbf{1}_5$, where the sub-script denotes the $U(1)$ charge. This is precisely how the representations associated to the left-handed projection operator have arranged themselves in (4.3.19) which is suggestive that it would be natural to further unify the content of this theory into a single $\mathbf{16}$ of $SO(10)$.

4.3.1 Other unified theories

In this subsection we apply the same technique to some other unified theories. Those of interest are those into which the standard model can be embedded and recovered after spontaneous symmetry breaking at some unification scale. Both $SU(6)$ and $SO(10)$ feature in the literature and have the property that the $SU(5)$ we have considered above can be embedded into it in a natural way (and so the standard model also fits into these Lie groups). We now determine the representations and chiralities which appear if $g(2\pi)$ is taken to transform in the fundamental representation of these groups.

We begin with $SU(6)$ and follow the same steps as in the previous section except that we now need six pairs of $\tilde{\phi}$ and ϕ. Taking $g(2\pi)$ to transform in the $\mathbf{6}$ representation we find the determinants as follows. For anti-periodic boundary conditions on Grassmann fields

$$\begin{aligned}&\det\left(i\left(\frac{d}{dt}+\mathcal{A}\right)\right)\propto\\&\mathrm{tr}\left(g_{\mathbf{6}}\right)+\mathrm{tr}\left(g_{\mathbf{15}}\right)+\mathrm{tr}\left(g_{\mathbf{20}}\right)+\mathrm{tr}\left(g_{\overline{\mathbf{15}}}\right)+\mathrm{tr}\left(g_{\overline{\mathbf{6}}}\right)\\&+2\mathrm{tr}\left(g_{\mathbf{0}}\right)\end{aligned}\tag{4.3.20}$$

and for periodic boundary conditions

$$\begin{aligned}&\det\left(i\left(\frac{d}{dt}+\mathcal{A}\right)\right)\propto\\&-\mathrm{tr}\left(g_{\mathbf{6}}\right)+\mathrm{tr}\left(g_{\mathbf{15}}\right)-\mathrm{tr}\left(g_{\mathbf{20}}\right)+\mathrm{tr}\left(g_{\overline{\mathbf{15}}}\right)-\mathrm{tr}\left(g_{\overline{\mathbf{6}}}\right)\\&+2\mathrm{tr}\left(g_{\mathbf{0}}\right)\end{aligned}\tag{4.3.21}$$

which is also multiplied by the γ_5 when the boundary conditions on ψ are correlated with those on ϕ as described above. Summing the contributions from each set of boundary conditions determines the chiralities selected by this model:

$$\begin{aligned}&(\mathrm{tr}\left(g_{\mathbf{6}}\right)+\mathrm{tr}\left(g_{\mathbf{20}}\right)+\mathrm{tr}\left(g_{\overline{\mathbf{6}}}\right))P_L\\&\quad+(\mathrm{tr}\left(g_{\mathbf{15}}\right)+2+\mathrm{tr}\left(g_{\overline{\mathbf{15}}}\right))P_R\end{aligned}\tag{4.3.22}$$

There have been a few attempts to form a unified theory with gauge group $SU(6)$ [75, 76]. The general approach places the contents of the $\bar{\mathbf{5}}$ representation of $SU(5)$ into the $\bar{\mathbf{6}}$ of $SU(6)$ along with an exotic fermion, N. The **15** is constructed as the anti-symmetric product $\mathbf{6}\otimes_A\mathbf{6}$ into which fall the remaining standard model particles and conjugate particles to N, but in this construction the particles in the $\bar{\mathbf{6}}$ and **15** representations have the same chirality. The result in (4.3.22) is inconsistent with this assignment and furthermore the conjugate representations share the same chirality. This presents a major barrier to any placement of the standard model particles into the representations of (4.3.22).

For completeness we turn now to $SO(10)$ where we will again take $g\,(2\pi)$ to transform in the fundamental representation **10** (We use ten pairs of $\tilde{\phi}$ and ϕ). For anti-periodic boundary conditions on Grassmann fields we evaluate the determinant to be[4]

$$\det\left(i\left(\frac{d}{dt}+\mathcal{A}\right)\right)\propto$$
$$2\text{tr}\left(g_{\mathbf{10}}\right)+2\text{tr}\left(g_{\mathbf{45}}\right)+2\text{tr}\left(g_{\mathbf{120}}\right)+\text{tr}\left(g_{\mathbf{126}}\right)+\text{tr}\left(g_{\overline{\mathbf{126}}}\right)+$$
$$2\text{tr}\left(g_{\mathbf{210}}\right)+2\text{tr}\left(g_{\mathbf{0}}\right). \qquad (4.3.23)$$

Similarly for the case of periodic boundary conditions we find

$$\det\left(i\left(\frac{d}{dt}+\mathcal{A}\right)\right)\propto$$
$$-2\text{tr}\left(g_{\mathbf{10}}\right)+2\text{tr}\left(g_{\mathbf{45}}\right)-2\text{tr}\left(g_{\mathbf{120}}\right)-\text{tr}\left(g_{\mathbf{126}}\right)-\text{tr}\left(g_{\overline{\mathbf{126}}}\right)+$$
$$2\text{tr}\left(g_{\mathbf{210}}\right)+2\text{tr}\left(g_{\mathbf{0}}\right) \qquad (4.3.24)$$

to which we associate a factor of γ^5 if we correlate the boundary conditions on ψ with those on ϕ. Collecting together the addition of these terms we find the chiralities and representations

$$\left(2\text{tr}\left(g_{\mathbf{10}}\right)+2\text{tr}\left(g_{\mathbf{120}}\right)+\text{tr}\left(g_{\mathbf{126}}\right)+\text{tr}\left(g_{\overline{\mathbf{126}}}\right)\right)P_L$$
$$+\left(2\text{tr}\left(g_{\mathbf{45}}\right)+2\text{tr}\left(g_{\mathbf{210}}\right)+2\right)P_R \qquad (4.3.25)$$

The most common $SO(10)$ model places an entire generation of left-handed standard model particles into the **16** representation along with an exotic sterile neutrino [77, 78] so the assignments we have generated here do not coincide with the well known unified theory. As we have seen above these particles fit into the $\bar{\mathbf{5}}+\mathbf{10}+\mathbf{1}$ of $SU(5)$ or $SU(5)\times U(1)$, both of which are subgroups of $SO(10)$. It is unfortunate that the **16** representation does not naturally appear out of the approach taken in this work. In the following section we will consider how we might modify our work to generate this representation.

[4]Note there is more than one representation of dimension **210**; to avoid ambiguity the equations refer to that with highest weight $[0, 0, 0, 1, 1]$.

Discussion

We have made the choice to consider the Wilson loop $g\left(2\pi\right)$ as transforming in the fundamental representation of each group but this is of course not necessary and we briefly explore the consequences of alternative representations for this operator. For example, for the case of $SU(3)$, the representations and chiralities were found to be [64]

$$
\begin{aligned}
&(\mathrm{tr}\left(g_{\bar{\mathbf{3}}}\right)+1)P_L \\
&(\mathrm{tr}\left(g_{\mathbf{3}}\right)+1)P_R
\end{aligned}
\tag{4.3.26}
$$

which gave rise to $SU(3)$ triplets and a sterile neutrino. We highlight the sensitivity of the result to our choice of representation for $g\left(2\pi\right)$ by instead taking it to transform in the representation with the next-smallest dimension, **6**. Then we find for anti-periodic boundary conditions[5]

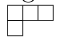
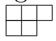
$$
\begin{aligned}
Z\left[\mathcal{A}\right]_{\mathbf{6}} \propto\ &\mathrm{tr}\left(g_{\mathbf{6}}\right)+\mathrm{tr}\left(g_{\mathbf{10}}\right)+\mathrm{tr}\left(g_{\mathbf{15}}\right)+\mathrm{tr}\left(g_{\overline{\mathbf{15}}}\right)+\mathrm{tr}\left(g_{\overline{\mathbf{10}}}\right)+\mathrm{tr}\left(g_{\bar{\mathbf{6}}}\right)+ \\
&2\mathrm{tr}\left(g_{\mathbf{0}}\right),
\end{aligned}
\tag{4.3.27}
$$

and for periodic boundary conditions

$$
\begin{aligned}
Z\left[\mathcal{A}\right]_{\mathbf{6}} \propto\ &-\mathrm{tr}\left(g_{\mathbf{6}}\right)-\mathrm{tr}\left(g_{\mathbf{10}}\right)+\mathrm{tr}\left(g_{\mathbf{15}}\right)+\mathrm{tr}\left(g_{\overline{\mathbf{15}}}\right)-\mathrm{tr}\left(g_{\overline{\mathbf{10}}}\right)-\mathrm{tr}\left(g_{\bar{\mathbf{6}}}\right)+ \\
&2\mathrm{tr}\left(g_{\mathbf{0}}\right),
\end{aligned}
\tag{4.3.28}
$$

which lead to the chiralities and representations

$$
\begin{aligned}
&(\mathrm{tr}\left(g_{\mathbf{6}}\right)+\mathrm{tr}\left(g_{\mathbf{10}}\right)+\mathrm{tr}\left(g_{\overline{\mathbf{10}}}\right)+\mathrm{tr}\left(g_{\bar{\mathbf{6}}}\right))P_L \\
&(\mathrm{tr}\left(g_{\mathbf{15}}\right)+\mathrm{tr}\left(g_{\overline{\mathbf{15}}}\right)+2)P_R.
\end{aligned}
\tag{4.3.29}
$$

[5]Once again, to avoid ambiguity we note that the **15** is the representation with highest weight $[2,1]$ and its conjugate, $\overline{\mathbf{15}}$, has highest weight $[1,2]$. Their Young Tableaux are  and  respectively.

The absence of the fundamental representation is rather striking here, since this is how the quarks of the standard model transform, so this choice does not seem to be helpful for phenomenological model building. On the other hand, taking $g\left(2\pi\right)$ to transform in the adjoint representation we find that the determinant vanishes for periodic boundary conditions and for anti-periodic boundary conditions

$$Z\left[\mathcal{A}\right]_{\mathbf{8}} = 4\left(\mathrm{tr}\left(g_{\mathbf{8}}\right)\right)^2 \tag{4.3.30}$$

which would be useful for a description of gauge bosons[6].

This motivates us to return to $SU(5)$, where this chapter has found its best success, and consider the effect of the Wilson loop transforming in the **10** representation, which has the next-smallest dimension after the fundamental. We do so only to demonstrate how the result depends on the choice of transformation of $g\left(2\pi\right)$. For anti-periodic boundary conditions we find

$$\begin{aligned} Z\left[\mathcal{A}\right]_{\mathbf{10}} \propto \ & \mathrm{tr}\left(g_{\mathbf{10}}\right) + \mathrm{tr}\left(g_{\mathbf{45}}\right) + \mathrm{tr}\left(g_{\overline{\mathbf{45}}}\right) + \mathrm{tr}\left(g_{\overline{\mathbf{10}}}\right) + \\ & \mathrm{tr}\left(g_{\mathbf{50}}\right) + \mathrm{tr}\left(g_{\mathbf{70}}\right) + \mathrm{tr}\left(g_{\overline{\mathbf{70}}}\right) + \mathrm{tr}\left(g_{\overline{\mathbf{50}}}\right) + \\ & \mathrm{tr}\left(g_{\mathbf{35}}\right) + \mathrm{tr}\left(g_{\mathbf{175}}\right) + \mathrm{tr}\left(g_{\overline{\mathbf{175}}}\right) + \mathrm{tr}\left(g_{\mathbf{35}}\right) + \\ & \mathrm{tr}\left(g_{\mathbf{126}}\right) + \mathrm{tr}\left(g_{\overline{\mathbf{126}}}\right) + 2\mathrm{tr}\left(g_{\mathbf{0}}\right), \end{aligned} \tag{4.3.31}$$

and with periodic boundary conditions the determinant is given by

$$\begin{aligned} Z\left[\mathcal{A}\right]_{\mathbf{10}} \propto \ & -\mathrm{tr}\left(g_{\mathbf{10}}\right) + \mathrm{tr}\left(g_{\mathbf{45}}\right) + \mathrm{tr}\left(g_{\overline{\mathbf{45}}}\right) - \mathrm{tr}\left(g_{\overline{\mathbf{10}}}\right) + \\ & -\mathrm{tr}\left(g_{\mathbf{50}}\right) - \mathrm{tr}\left(g_{\mathbf{70}}\right) - \mathrm{tr}\left(g_{\overline{\mathbf{70}}}\right) - \mathrm{tr}\left(g_{\overline{\mathbf{50}}}\right) + \\ & \mathrm{tr}\left(g_{\mathbf{35}}\right) + \mathrm{tr}\left(g_{\mathbf{175}}\right) + \mathrm{tr}\left(g_{\overline{\mathbf{175}}}\right) + \mathrm{tr}\left(g_{\mathbf{35}}\right) + \\ & -\mathrm{tr}\left(g_{\mathbf{126}}\right) - \mathrm{tr}\left(g_{\overline{\mathbf{126}}}\right) + 2\mathrm{tr}\left(g_{\mathbf{0}}\right), \end{aligned} \tag{4.3.32}$$

so taking the sum over correlated boundary conditions we produce the representa-

[6]The author is grateful to Paul Mansfield for pointing this out.

tions and chiralities

$$(\mathrm{tr}\,(g_{\mathbf{10}}) + \mathrm{tr}\,(g_{\mathbf{50}}) + \mathrm{tr}\,(g_{\mathbf{70}}) + \mathrm{tr}\,(g_{\mathbf{126}}) + \mathrm{tr}\,(g_{\overline{\mathbf{126}}}) + \mathrm{tr}\,(g_{\overline{\mathbf{70}}}) + \mathrm{tr}\,(g_{\overline{\mathbf{50}}}) + \mathrm{tr}\,(g_{\overline{\mathbf{10}}}))P_L + \\ (\mathrm{tr}\,(g_{\mathbf{35}}) + \mathrm{tr}\,(g_{\mathbf{45}}) + \mathrm{tr}\,(g_{\mathbf{175}}) + \mathrm{tr}\,(g_{\overline{\mathbf{175}}}) + \mathrm{tr}\,(g_{\overline{\mathbf{45}}}) + \mathrm{tr}\,(g_{\overline{\mathbf{35}}}) + 2)P_R. \tag{4.3.33}$$

Note that now the chiralities on the representations of the same dimension are the same, in contrast to that found when $g\,(2\pi)$ was taken to transform in the **5** representation. We see that although we no longer assign particle representations by hand the predictions of this new approach are very sensitive to our choice of representation for the Wilson loop.

It is worthwhile considering whether we might uncover a connection to the more familiar $SO(10)$ model by considering different representations for the Wilson loop. The next smallest representations of this group are the **16** and $\overline{\mathbf{16}}$ and we have calculated the functional determinants for these choices of the representation of $g\,(2\pi)$. The chiralities and representations can be expressed in terms of traces which include

$$(\mathrm{tr}\,(g_{\mathbf{16}}) + \mathrm{tr}\,(g_{\overline{\mathbf{16}}}) + \ldots)\,P_L \tag{4.3.34}$$

but also involve the representations **120**, $\mathbf{560}, \overline{\mathbf{560}}$ and further representations whose dimensions exceed 1000. So although the **16** representation can be generated with this choice it brings with it a series of other representations that are not of interest to the building of minimal unified theories.

4.4 Concluding remarks

We have demonstrated that the model presented in [64] can be used when the symmetry group is $SU(5)$ and that in that case it provides the low dimensional representations which are used in the Georgi-Glashow model in order to accommodate the standard model matter content. We chose the Wilson loop to transform in the fundamental representation of $SU(5)$ and discussed how different choices lead to the appearance of different representations and chiralities. We also considered $SU(6)$

and $SO(10)$ as the gauge groups but for these cases we did not uncover the familiar connections to the standard model. For $SO(10)$ this can only be done by considering the Wilson loop to transform in a higher dimensional representation (**16**), but this choice brings with it a series of unwelcome representations of large dimension. Part of the utility of the approach we have used in this chapter is the ease with which the gauge group and representation of the Wilson loop can be changed.

In combination with the simplifications to some calculations provided by the worldline formulation it would seem valuable to pursue this programme for the standard model and other unified theories. It is certainly simpler to sum the correlated boundary conditions on the functional integrals than to sum over representations and chiralities by hand and the appearance of (4.3.17) may provide a guiding principle in how the matter content of the universe can be arranged.

Furthermore the simplifications which this approach offers would be helpful to the reformulation of the quantum field theory in terms of the interacting string theory we have proposed in the previous chapter. An immediate obstacle to the inclusion of non-Abelian couplings on the worldline theories is the requirement of path ordering the exponentiated line integrals. Although this makes perfect sense from the point of view of the worldline theory, if we interpret these lines as the boundaries of fundamental strings it is unclear how to extend the path ordering into the bulk of the string worldsheet.

We have seen here that the introduction of further degrees of freedom on the worldline offers two advantages. Firstly they can represent the path ordering, which means that the string model now need only be modified in such a way that the dynamics of ϕ and $\tilde{\phi}$ appear at the boundary. Secondly the functional determinant resulting from the integration over these variables provide the sum over representations and chiralities of the standard model particles rendering what would otherwise be a complicated sum far easier to manage. We suggest that an interesting direction for future research is to consider how these extra degrees of freedom can be introduced on the worldsheet in such a way that the decoupling of the conformal scale of the worldsheet metric is not ruined. We speculate on how this could be done in the discussion which makes up the next chapter.

Chapter 5

Discussion, open questions and conclusion

In this chapter we give an overview of the results presented in this thesis, discuss their consequences and suggest some open questions and possible routes to further progress. We have only initiated a study of contact interactions for particles and strings and it is clear that there remains much to consider. At this early stage we are encouraged by the results so far and hope that further success may be found in the near future.

In chapter 2 of this work we considered contact interactions in the context of point particles. We first related the expectation value of an operator which counts the number of curves cutting a space-time point to the dipole field of equal and oppositely charged particles. We also introduced a similar notion to include spin degrees of freedom. The theory was parameterised by T which can be interpreted as the effective temperature for the statistical average over fluctuations of the curves. We interpreted the curves as a physical realisation of Faraday's lines of electric flux, so returning to an idea first postulated in the mid 1830s. The relation to electrostatics was found in the high temperature limit of the theory but corrections to the classical fields were investigated at finite values of T. There is a hint that the low temperature limit of the theory may be related to confinement because the field lines are then concentrated in a tube centered on the straight line joining the two particles.

This unusual way of arriving at the static electric field provided motivation for us to introduce a contact interaction into a theory of point particles. We quantised this theory and showed that the interaction can be made consistent by making a renormalisation of the bare action. The physical content of the theory was found to couple the end points of the curves via propagators. We did not, however, consider anything more interesting than the partition function of the theory and it would be interesting to consider more complicated quantities. Given the relation we exposed between this theory and the ϕ^4 field theory we might imagine the effect that the interaction has on the basic physical objects as providing one-loop corrections. If these corrections could be determined in the low T limit then we may learn something about confining theories which could feed back into more conventional field theories such as QCD.

The success we found with the point particles was suggestive that we consider the generalisation of the theory to the time dependent case, which we presented in chapter 3. This required treating the curves as dynamical and led to the use of string theory. We linked this to the worldline approach of QED by taking as the boundaries of the strings the Wilson loops which arise in the first quantised formulation. The contact interaction was interpreted as providing direct inter-string interactions when the string worldsheets intersect and we showed that this serves to produce the expectation value of the product of those Wilson loops. We found the remarkable fact that the δ-function interaction decoupled from the conformal scale of the worldsheet metric, despite being off-shell. Although we could not complete the proof of our claim for the bosonic theory we were able to use the worldsheet supersymmetry of the spinning string to show that the super-Wilson loops arise out of tensionless spinning strings interacting on contact. One aspect that we have not adequately addressed is the consequence of working in $D = 4$ target space dimensions since here we must confront the Liouville theory. We shall speculate on how to circumvent this in a moment.

Our reformulation of spinor field theory was only for the Abelian case of QED. We discussed in chapter 4 how the non-Abelian ingredients of a field theory can be encoded in the worldline formalism by introducing new Grassman degrees of freedom $\tilde{\phi}$ and ϕ. Then the Lie group representations, particle chirality and even the path ordering can be accounted for by coupling these new fields to the gauge field via the super Wilson-loop. We demonstrated how the sum over representations and chiralities present in the standard model embedded into the $SU(5)$ unified theory can be generated by considering a determinant associated to $\tilde{\phi}$ and ϕ. This procedure provides a powerful method of producing group representations into which the matter content of the standard model is to be assigned. We found success for $SU(5)$ and flipped $SU(5)$, but despite a tantalising link between these representations and the $SO(10)$ theory we did not have the same luck for the latter.

We consider now how we could relate the results of chapter 4 to the spinning string theory in chapter 3. In doing so we suggest a method that has the advantage of also dealing with the unfortunate Liouville theory which appears for strings outside of their critical dimension. In dealing with spinor matter we found it necessary to introduce spin degrees of freedom onto the string worldsheet; similarly to represent the chirality of the particles we found it useful to use new worldline degrees of freedom, since these also generated the non-Abelian nature of the interaction. It does not seem a large leap to suppose that we can modify the string theory to reflect the appearance of $\tilde{\phi}$ and ϕ by introducing further worldsheet degrees of freedom which are matched to these fields on the boundary[1].

Doing so addresses the issue of how to extend path ordering into the bulk because we can encode the boundary path ordering into these new worldsheet fields, asking only that the dynamics of these fields do not affect the worldline theory. Furthermore, by introducing new string degrees of freedom we effectively increase the central charge associated to matter fields. Let us denote these new fields by $\tilde{\mathbf{Y}}$ and $\mathbf{Y}$. We might hope then that the extra dependence on the conformal scale which arises when we functionally integrate over these fields is sufficient to cancel the Liouville theory present due to the fields X, Ψ, g and χ. Introducing extra fields is not unheard of in string theory – indeed quite often these fields play the role of curled up "extra dimensions" X^{α}, $\alpha = 5 \ldots 10$. Our suggestion, however, is

[1]I am greatly indebted to Paul Mansfield for useful conversations and helpful suggestions about the following points.

that there is nothing exotic about $\mathbf{Y}$ and $\tilde{\mathbf{Y}}$ but rather they are *required* in order to describe the non-Abelian nature of the standard model.

In order to make this more concrete we must consider the dynamics of $\tilde{\phi}$ and ϕ. These obey first order equations of motion and if we are to match these to the worldsheet fields we must ensure that they also obey first order equations of motion. This does not present any challenge but we must also generate the correct boundary coupling between $\tilde{\phi}$ and ϕ with the gauge field. This would require us to modify the contact interaction between strings. On the boundary this interaction must produce the terms present in the worldline theory for $\tilde{\phi}$ and ϕ without spoiling the result we presented in chapter 3. This also means that the interaction between the fields $\tilde{\mathbf{Y}}$ and $\mathbf{Y}$ must decouple from the conformal scale of the worldsheet metric (or we risk encountering mass-shell constraints which would ruin the form of the $\delta-$function). We suggest that this is an urgent issue for investigation and could be an immediate generalisation of the work in this thesis to allow the theory we have developed to be applied in a non-Abelian setting. So too could it provide a more rigorous way of dealing with the Liouville theory if indeed the new fields can counter the Weyl anomaly generated by the conventional string theory.

We are some way towards producing a first quantised version of the standard model, but there remains substantial work to be done. We briefly mention some of the more pressing matters, but this list is by no means exhaustive. Firstly we must think about the question of providing mass to the fermions. By gauge invariance this must be done by spontaneous symmetry breaking and so we require a worldline formulation of the Higgs mechanism. This in turn needs a description of Yukawa couplings between the fermion fields and the Higgs field. This is not something that we have addressed in this thesis and there is not a great deal of work in this area in the current literature. A non-Abelian theory also involves self-interactions between the gauge bosons which does not fit in with the picture we have considered in previous chapters. At this point the contact interaction can mediate communication between matter fields but we have not addressed how to describe interactions between the gauge fields. The theory we propose would benefit greatly from progress in either of these directions.

On a philosophical note it is extremely interesting to return to fundamental string theory and consider the effects of modifying its description. Much progress has been made in conventional string theory with a wide variety of techniques but we feel that the introduction of simple contact interactions has been overlooked. This is understandable considering that naively such a theory would not be expected to be consistent because of the appearance of off-shell states. It is therefore of considerable interest that we have shown that no such problems are encountered for tensionless spinning strings and curious that the theory has such a close relationship to the conventional quantum field theory of spin $1/2$ matter interacting with the electromagnetic field. We hope that the progress we have made will lead to a greater understanding of the standard model of particle physics and that it may provide yet another tool for physicists to explore the behaviour of the universe.

Appendices

Appendix A - Evaluating the free particle determinants

Here we determine the normalisation constants used in chapter 2 using ζ-function regularisation. With our gauge choice we make a change of variables $t = T\tau$ and define

$$\mathscr{Z}' = \int_{\boldsymbol{\omega}(0)=\mathbf{a}}^{\boldsymbol{\omega}(T)=\mathbf{b}} \mathscr{D}\boldsymbol{\omega}\, e^{-\int_0^T \frac{\dot{\boldsymbol{\omega}}^2}{2} d\tau} \tag{A.0.1}$$

which can be interpreted as the matrix element $\left\langle \mathbf{b} \left| e^{-T\hat{\mathcal{H}}} \right| \mathbf{a} \right\rangle$ with Hamiltonian $\hat{\mathcal{H}} = \frac{\hat{p}^2}{2}$. In the Schödinger representation this becomes the position space representation of the heat kernel: $\left\langle \mathbf{b} \left| e^{-T\frac{\boldsymbol{\nabla}^2}{2}} \right| \mathbf{a} \right\rangle$. This is a well know expression but we find it using functional methods in keeping with the spirit of that chapter.

Generalising to arbitrary dimension D, the integral over ω gives a functional determinant and a boundary term:

$$\mathscr{Z}' = \pi^{\frac{D}{2}} \left(\det \left(\frac{d^2}{dt^2} \right) \right)^{-\frac{D}{2}} e^{-\frac{(b-a)^2}{2T}}. \tag{A.0.2}$$

The determinant of an operator, $\hat{O}$, can be defined using the ζ-function [79] as

$$\det\left(\hat{O}\right) = \exp\left(-\frac{d}{dz}\zeta_{\hat{O}}(z)\right); \qquad \zeta_{\hat{O}}(z) \equiv \sum_{n=1}^{\infty} \lambda_n^{-z} \tag{A.0.3}$$

where the λ_n are the eigenvalues of $\hat{O}$. The formal product of eigenvalues is then regularised via the analytic continuation of the ζ-function. With Dirichlet boundary conditions the eigenvalues of the operator in $\mathcal{Z}$ are $\lambda_n = \left(\frac{n\pi}{T}\right)^2$ so

$$\zeta_{\frac{d^2}{dt^2}}(z) = \left(\frac{\pi}{T}\right)^{-2z} \zeta(2z) \tag{A.0.4}$$

which has derivative $\zeta'(0) = 2\ln\left(\frac{T}{\pi}\right)\zeta(0) + 2\zeta'(0)$. So with this regularisation we arrive at

$$\det\left(\frac{d^2}{dt^2}\right) = 2T \tag{A.0.5}$$

giving

$$\mathcal{Z}' = (2\pi T)^{-\frac{D}{2}} e^{-\frac{(\mathbf{b}-\mathbf{a})^2}{2T}}. \tag{A.0.6}$$

If instead we consider mixed boundary conditions $\boldsymbol{\omega}(0) = 0$, $\dot{\boldsymbol{\omega}}(1) = 0$ then the eigenfunctions are

$$\sin\left(\sqrt{\tilde{\lambda}_n}\, t\right) \tag{A.0.7}$$

where the eigenvalues are now given by $\tilde{\lambda}_n = \left(\frac{(2n+1)\pi}{2T}\right)^2$. For the ζ-function we now obtain

$$\tilde{\zeta}_{\frac{d^2}{dt^2}}(z) = \left(\frac{\pi}{T}\right)^{-2z} \zeta\left(2z, \frac{1}{2}\right) \tag{A.0.8}$$

where we've introduced the Hurwitz zeta function $\zeta(s, q)$. The derivative with respect to z is then $\tilde{\zeta}'(0) = 2\ln\left(\frac{T}{\pi}\right)\zeta\left(0, \frac{1}{2}\right) + 2\zeta'\left(0, \frac{1}{2}\right)$. Now $\zeta\left(0, \frac{1}{2}\right) = 0$ so the first term vanishes along with the T dependence, leaving

$$\det\left(\frac{d^2}{dt^2}\right) = 2 \tag{A.0.9}$$

for the case of mixed boundary conditions. The change in boundary conditions also alters the boundary contributions from the classical action and we find that $\mathcal{Z}'$ is a constant independent of T and $\mathbf{a}$.

For fermionic fields the kinetic term is first order in derivatives and we impose anti-periodic boundary conditions in the case of closed paths which represent traces. We may of course calculate

$$\mathcal{Z}_F = \int_{\psi(T)=-\psi(0)} \mathscr{D}\psi\; e^{-\int_0^T \dot{\psi}\cdot\psi d\tau} = \left(\det\left(\frac{d}{dt}\right)\right)^{\frac{D}{2}} \tag{A.0.10}$$

using standard techniques from quantum mechanics but for completeness we continue to apply functional methods. We need the eigenvalues $\lambda_n = \frac{(2n+1)\pi i}{T}$ and form the determinant via

$$\prod_{n=-\infty}^{\infty} \frac{(2n+1)\,\pi i}{T}. \tag{A.0.11}$$

The ζ-function for this operator is thus

$$\left(\frac{2\pi i}{T}\right)^{-z} \zeta\left(z, \frac{1}{2}\right), \tag{A.0.12}$$

which has derivative $\zeta'(0) = \ln\left(\frac{T}{2\pi i}\right)\zeta\left(0, \frac{1}{2}\right) + \zeta'\left(0, \frac{1}{2}\right)$ which evaluates to $-\ln\sqrt{2}$ so

$$\mathscr{Z}_F = 2^{\frac{D}{2}} \tag{A.0.13}$$

which we note does not depend on T – indeed the change of variables to $\tau \in [0, T]$ does not change the form of the kinetic term.

The other result we need is the normalisation constant for the open path action which includes the term $\int_0^1 d\tau\, \frac{\chi_0}{T}\dot{\omega}\cdot\psi$. We use (2.3.44) to calculate

$$\int d\chi_0 \delta\left(\chi_0 - \Xi\right) \mathscr{D}\psi\; e^{-\int_0^T d\tau\, \frac{1}{2}\dot{\psi}\cdot\psi - \frac{\chi_0}{2T}\dot{\omega}\cdot\psi} \propto \int d\chi_0 \delta\left(\chi_0 - \Xi\right) \mathscr{T}\left(e^{\int d\tau \frac{\chi_0}{2\sqrt{2}T}\dot{\omega}\cdot\gamma}\right). \tag{A.0.14}$$

Integrating over χ_0 picks out

$$1 + \Xi\frac{(b-a)\cdot\gamma}{2\sqrt{2}T} \tag{A.0.15}$$

which is to be multiplied by $\mathscr{Z}_F$ calculated above. In the main text we impose $\Xi = 0$ which vastly simplifies the remaining calculations.

Appendix B – Mixed Boundary Conditions and the Green functions

Recall that out of the worldline formalism of the field theories appear Green function factors $\left(-\mathcal{D}^2 + m^2\right)^{-1}(b, a)$ which in our work are represented as curves running between positions a^μ and b^μ. We associate a string worldsheet to these curves but impose Dirichlet boundary conditions at one end of the string – fixing it to follow the curve between a^μ and b^μ – and Neumann boundary conditions at the other. In the main text it was shown that Dirichlet boundary conditions ensure that the general damping to integrands caused by the coincident Green function is not present near the boundary, since here $\exp\left(-\pi\alpha' k^2 G\left(\xi,\xi\right)\right) \sim \mathcal{O}\left(1\right)$. Neumann boundary conditions do not impose this and so contributions arising from points close to this end of the string will be exponentially damped. So we expect to receive contributions to our integrals only from a strip close to the Dirichlet end of the string. There is one distinguished point which may spoil this argument which is where these two boundaries coincide.

In order to simplify the effect of these mixed boundary conditions it is favourable to instead work on a worldsheet domain which consists of the upper-right quadrant of the complex plane via the simple conformal mapping from the upper half plane $z \to \sqrt{z}$. The positive real axis in this plane corresponds to the end of the string with Dirichlet boundary conditions and the positive imaginary axis corresponds to the end of the string on which Neumann boundary conditions are imposed. Again we shall expand about $\phi = \text{const}$ and will specialise to $\phi = 0$ to determine the leading order behaviour. The only real change to the calculations we have presented in previous sections is that the Green function on the worldsheet must be modified to respect the mixed boundary conditions. The method of images in the upper-right quadrant gives the Green function as

$$G\left(z,z'\right) = \ln\left|z - z'\right|^2 - \ln\left|z - \bar{z'}\right|^2 + \ln\left|z + \bar{z'}\right|^2 - \ln\left|z + z'\right|^2 \tag{A.0.16}$$

The coincident limit of this function requires regularisation as in the previous case so we shall apply the heat-kernel representation. It is straightforward to verify that in terms of $z = x + iy$ the coincident limit can be written as

$$\begin{aligned} G_\epsilon\left(z,z\right) &= \int_\epsilon^\infty \frac{d\tau}{4\pi\tau}\left[1 - \exp\left(-\frac{y^2}{\tau}\right) + \exp\left(-\frac{x^2}{\tau}\right) - \exp\left(-\frac{x^2+y^2}{\tau}\right)\right] \\ &= f\left(\frac{y}{\sqrt{\epsilon}}\right) - f\left(\frac{x}{\sqrt{\epsilon}}\right) + f\left(\frac{\sqrt{x^2+y^2}}{\sqrt{\epsilon}}\right). \end{aligned} \tag{A.0.17}$$

At a distance much greater than $\sqrt{\epsilon}$ from both boundaries the coincident Green function is of order $\ln \frac{y^2}{\epsilon}$. When approaching the positive imaginary axis it increases to $2\ln \frac{y^2}{\epsilon}$. Close to the positive real axis (corresponding to the Dirichlet end of the string) $G_\epsilon(z,z)$ is of order $\frac{y^2}{\epsilon}$, except at the corner where the axes meet; here it varies from $2\frac{y^2}{\epsilon}$ to $\frac{y^2}{\epsilon}$ when moving along the positive real axis and from $2\frac{y^2}{\epsilon}$ to $2\ln \frac{y^2}{\epsilon}$ when moving along the positive imaginary axis, both over a distance of order $\sqrt{\epsilon}$. So $G_\epsilon(z,z)$ is of order $\ln \frac{y^2}{\epsilon}$ everywhere on the worldsheet, except in a small strip close to the positive real axis where it is of order $\frac{y^2}{\epsilon}$.

This demonstrates more concretely that indeed all integrands of relevance will be heavily damped except for a small strip close to the Dirichlet boundary of the string. We shall not repeat the entire calculation for an arbitrary number of vertex operators inserted onto the worldsheet since it suffices to consider the behaviour of a single insertion, in much the same way as the calculation that preceded the careful treatment of Section 3.3.1. We shall therefore consider the expectation value

$$\int d^2z\ e^{ik\cdot x'}\left\langle V_k(z)\right\rangle \tag{A.0.18}$$

integrated over $\Re(z)>0,\ \Im(z)>0$, which contains two terms. A non-vanishing contribution arises out of a single contraction between the pieces of the vertex operator which leads an integral

$$2\pi\alpha' k^{[\mu}\int_0^\infty\int_0^\infty dxdy\ \epsilon^{ab}\partial_a G(z,z)\,\partial_b X_C^{\nu]} e^{-\pi\alpha' k^2 G(z,z)} e^{ik\cdot(x'-X_c)}. \tag{A.0.19}$$

The integrand is damped by the exponent involving the Green function, except close to $y=0$ so we integrate a distance Λ into the bulk and replace the slowly varying terms involving the field X^μ and its derivatives with their values on the Dirichlet boundary. We shall first consider the term

$$\begin{aligned} &2\pi\alpha' k^\mu\int_0^\infty dx\ \partial_x X_c^\nu e^{ik\cdot(x'-X_c)}\int_0^\infty dy\ \partial_y G(z,z)\,e^{-\pi\alpha' k^2 G} \\ &= \frac{k^\mu}{k^2}\int_0^\infty dx\ \partial_x X_c^\nu e^{ik\cdot(x'-X_c)}\left[e^{-\pi\alpha' k^2 G(z,z)}\right]_0^\infty, \end{aligned} \tag{A.0.20}$$

where the fields X^μ and derivatives are evaluated on the boundary $y=0$. The only

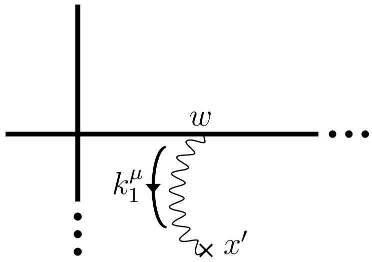


Figure 1: A single vertex operator corresponds to an interaction with a background field. The only contribution comes from points on the Dirichlet boundary of the string – this boundary is the worldline of the particle representing the Green function in the worldline approach.

contribution is from the lower bound of the integration,

$$\int_0^\infty dx \frac{k^{[\mu}}{k^2} \partial_x X_c^{\nu]} e^{ik\cdot(x'-X_c)} = \int_{B_0} dw^{[\nu}\, k^{\mu]}\ \frac{e^{ik\cdot(x'-w)}}{k^2}, \tag{A.0.21}$$

which is the result we sought. It can be represented diagrammatically as the interaction of points on the worldline B_0 with a background massless vector field (see Fig 1) and is independent of α', ϵ and the scale of the worldsheet metric. The second term which arises out of (A.0.19) does not contribute. It is equal to

$$\begin{aligned} & 2\pi\alpha' k^\mu \int_0^\infty dy\ \partial_y X_c^\nu e^{ik\cdot(x'-X_c)} \int_0^\infty dx\ \partial_x G\left(z,z\right) e^{-\pi\alpha' k^2 G(z,z)} \\ =& \frac{k^\mu}{k^2} \int_0^\infty dy\ \partial_y X_c^\nu e^{ik\cdot(x'-X_c)} \left(e^{-\pi\alpha' k^2 f\left(\frac{y}{\sqrt{\epsilon}}\right)} - e^{-2\pi\alpha' k^2 f\left(\frac{y}{\sqrt{\epsilon}}\right)} \right) \end{aligned} \tag{A.0.22}$$

where once again the fields take on their values at the boundary $y = 0$. The two terms in rounded brackets vary rapidly over the domain of integration but we have met their like in the previous sections and it has already been demonstrated that they vanish as the regulator is removed.

It remains to show that the other form of expression arising out of (A.0.18) does not contribute to the expectation value. Since the analysis follows the exact same form as in the main text we shall focus here only on the distinguished corner $x = y = 0$. To do so we consider the region of the domain $0 \le x \le a$, $0 \le y \le a$ where a is chosen to enforce $x^2 + y^2 \le 2a^2 << \epsilon$. Within this region we have

$$G\left(z,z\right) \approx \frac{2y^2}{\epsilon}. \tag{A.0.23}$$

The term in question is $\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu e^{-\pi\alpha' k^2 G(z,z)}$ so that the rapidly varying part of the integrand is

$$\begin{aligned} &\int_0^{a\sqrt{\epsilon}} dx \int_0^{a\sqrt{\epsilon}} dy \; e^{-2\pi\alpha' k^2 \frac{y^2}{\epsilon}} \\ =&\epsilon \int_0^a \int_0^a dy \; e^{-\pi\alpha' k^2 y^2}. \end{aligned} \tag{A.0.24}$$

This integral is bounded by $a^2\epsilon$ which vanishes as $\epsilon \to 0$ because $\frac{a}{\sqrt{\epsilon}} \to 0$ with ϵ. By applying the exact same analysis as previous sections the remaining regions can be shown to also offer a contribution which vanishes as the regulator is removed.

It is clear from this result how the calculation would proceed in the general case involving multiple vertex operators. The Green function supplies a similar damping for an arbitrary placement of the points on the worldsheet; it is only when all of the points are within a strip of size $\sqrt{\epsilon}$ of the Dirichlet boundary of the string that a finite contribution can be expected as the regulator is removed, or when the points are arranged in clusters in the bulk. In the latter case the boundary has no effect so the results of the main text apply. In the former we would see a copy of the above calculation for each point and the surviving terms are those involving contractions only amongst the constituents of each vertex operator, rather than between operators inserted at different points. A repeat of the previous calculations leads to the result at order N

$$4^N \prod_{j=1}^{N} \int_{B_0} dw \cdot dw' \frac{e^{ik_j\cdot(w-w')}}{k_j^2} \tag{A.0.25}$$

corresponding to the interaction between pairs of points on the boundary mediated by a massless vector boson; those pairs of points which interact arise from the two vertex operators with equal and opposite momenta. This expression can be represented diagrammatically as in Fig. 2.

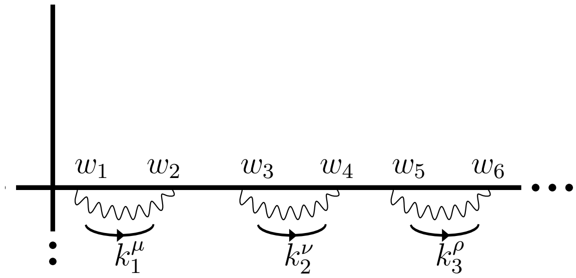


Figure 2: With multiple vertex operators inserted on the worldsheet there is an interaction between pairs of points on the Dirichlet boundary. These pairs share equal and opposite momentum and the picture corresponds to exchange of massless vectors between points on the worldline representing the Green function in the worldline theory.

Appendix C – Gauge-fixing Maxwell theory

In the text we have adopted a gauge-fixing procedure that imposes the gauge condition using a delta-function. We have to compute the integral

$$Z[J] = \int \mathscr{D}A\, e^{\int d^4x \left(\left(A_\mu \partial^2 A_\mu - (\partial \cdot A)^2\right)/(2q^2) + J\cdot A + \mathcal{A}\cdot\partial^2 A/q^2\right)} . \tag{A.0.26}$$

with $J = -i\sum_j \int dy_j \delta(y_j - x)$ and $\mathcal{A}$ on-shell: $\partial^2\mathcal{A} = \partial\cdot\mathcal{A} = 0$. Gauge invariance prevents the operator in the kinetic term from being inverted. Inserting a delta-function that imposes the gauge condition $\partial \cdot A = 0$ (we absorb the associated Faddeev-Popov determinant into the normalisation as it is independent of A) gives

$$Z[J] = \int \mathscr{D}(A,\lambda)\, e^{\int d^4x \left(\left(A_\mu \partial^2 A_\mu - (\partial \cdot A)^2\right)/(2q^2) + J\cdot A + \mathcal{A}\cdot\partial^2 A/q^2 + i\lambda\partial\cdot A\right)} \tag{A.0.27}$$

Shifting the integration variable λ by $-\frac{i\partial\cdot A}{2q^2}$ modifies the differential operator to one that is invertible

$$\begin{aligned} Z[J] &= \int \mathscr{D}(A,\lambda)\, e^{\int d^4x \left(A_\mu \partial^2 A_\mu/(2q^2) + J\cdot A + \mathcal{A}\cdot\partial^2 A/q^2 + i\lambda\partial\cdot A\right)} \\ &= \int \mathscr{D}\lambda\, e^{\frac{q^2}{2}\int d^4x \left(J\partial^{-2}J + 2\mathcal{A}\cdot J/q^2 + \mathcal{A}\partial^2\mathcal{A}/q^4 + 2i\lambda\,\partial^{-2}\partial\cdot J + 2i\lambda\partial\cdot\mathcal{A}/q^2 - \lambda^2\right)} \\ &= e^{\frac{q^2}{2}\int d^4x \left(J\partial^{-2}J - \partial\cdot J\,(\partial^{-2})^2\partial\cdot J + 2\mathcal{A}\cdot J/q^2\right)} \\ &= e^{\frac{q^2}{2}\sum_{jk}\int dy_j\cdot\Delta\cdot dy_k - i\sum_j\int dy_j\cdot\mathcal{A}} \end{aligned} \tag{A.0.28}$$

Appendix D – Contraction Algebra

In this appendix we calculate some of the important contractions used in the main work. From (3.2.38) in the main text it is straightforward to calculate simple correlation functions. For example, for the product of derivatives of fields we may put $k = 0$ in the generating function (since the k-dependence arose there because of the exponential factors)

$$\begin{aligned} \langle \partial_1 X_\mu (\xi) \, \partial_1 X'_\nu (\xi') \rangle &= \frac{\delta}{\delta j_1^\mu (\xi)} \frac{\delta}{\delta j_1^\nu (\xi')} \mathcal{Z} (j, k = 0) \Big|_{j=0} \\ &= \frac{\delta}{\delta j_1^\mu (\xi)} \Bigg[\left(\int d^2\tilde{\xi} \; 4\pi\alpha' \eta_{\alpha\nu} j^{\nu a} \left(\tilde{\xi}\right) \partial_a \partial'_1 G \left(\tilde{\xi}, \xi'\right) \right. \\ &\qquad \left. + \, \partial'_1 X_0^\nu (\xi') \right) \mathcal{Z} (j, k = 0) \Bigg] \Bigg|_{j=0} \qquad \text{(A.0.29)} \\ &= 4\pi\alpha' \partial_1 \partial'_1 G (\xi, \xi') + \partial_1 X_0^\mu (\xi) \, \partial'_1 X_0^\nu (\xi') \qquad \text{(A.0.30)} \end{aligned}$$

where the latter term is the contribution due to the presence of the boundary. Furthermore, to justify the expectation value of the exponential factor we may set $j = 0$ and consider

$$\begin{aligned} \langle \exp \left[ik \cdot \left(X (\xi) - X (\xi') \right) \right] \rangle &= \mathcal{Z} (j = 0, k) \\ &= \exp -\pi\alpha' k^2 \Psi (\xi, \xi') \cdot \exp ik \cdot \left(X (\xi) - X (\xi') \right). \end{aligned} \qquad \text{(A.0.31)}$$

The above results provide a further justification for the forms of (3.3.69) - (3.3.71) and highlight the rôle played by the boundary.

We also present in this appendix the calculation of contractions between fields. This can be determined by use of $\mathcal{Z} (j, k)$ but for the operator product expansion used it is easier to see it by making use of Wick's theorem and the previous results. Specifically we are interested in the product of two fields of the form

$$A^\mu e^{ik \cdot B} \qquad \text{(A.0.32)}$$

considered inside a correlation function. We note, however, that these fields are not considered normal ordered by themselves. By expanding the exponential and applying Wick's theorem we arrive at

$$
\begin{aligned}
A^\mu e^{ik\cdot B} &= \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} A^\mu B^n \\
&= \sum_{n=0}^{\infty} \sum_{\lambda=0}^{\lambda<\frac{n-1}{2}} ik_\nu \underbrace{A^\mu B^\nu} \cdot (-k_\nu k_\rho)^\lambda \, (\underbrace{B^\nu B^\rho})^\lambda \, \frac{(ik)^{n-2\lambda-1}}{(n-1)!} \cdot \begin{pmatrix} n-1 \\ 2\lambda \end{pmatrix} \frac{(2\lambda)!}{2^\lambda \lambda!} : B^{n-2\lambda-1} : \\
&+ \sum_{n=0}^{\infty} \sum_{\lambda=0}^{\lambda<\frac{n}{2}} (-k_\nu k_\rho)^\lambda \, (\underbrace{B^\nu B^\rho})^\lambda \, \frac{(ik)^{n-2\lambda}}{n!} \cdot \frac{(2\lambda)!}{2^\lambda \lambda!} \begin{pmatrix} n \\ 2\lambda \end{pmatrix} : A^\mu B^{n-2\lambda} : \\
&= \sum_{n=0}^{\infty} \sum_{\lambda=0}^{\lambda<\frac{n-1}{2}} ik_\nu \underbrace{A^\mu B^\nu} \cdot \frac{(-k_\nu k_\rho)^\lambda}{2^\lambda \lambda!} \cdot (\underbrace{B^\nu B^\rho})^\lambda \, \frac{(ik)^{n-2\lambda-1}}{(n-2\lambda-1)!} \; : B^{n-2\lambda-1} : \\
&+ \sum_{n=0}^{\infty} \sum_{\lambda=0}^{\lambda<\frac{n}{2}} \frac{(-k_\nu k_\rho)^\lambda}{2^\lambda \lambda!} \, (\underbrace{B^\nu B^\rho})^\lambda \, \frac{(ik)^{n-2\lambda}}{(n-2\lambda)!} : A^\mu B^{n-2\lambda} : \\
&= ik_\nu \underbrace{A^\mu B^\nu} \exp\left(-\frac{1}{2} k_\nu k_\rho \underbrace{B^\nu B^\rho}\right) : e^{ik\cdot B} : + \; \exp\left(-\frac{1}{2} k_\nu k_\rho \underbrace{B^\nu B^\rho}\right) : A^\mu e^{ik\cdot B} :
\end{aligned}
\tag{A.0.33}
$$

In the above equations the combinatoric factors come from the number of ways of choosing the fields to take place in contraction and from the ordering of these contractions amongst themselves. The form of the last line shows that we may consider the product by working out contractions of term with itself and the cross terms with others in the product. This result is used extensively in section 3.3 to determine the product of fields arising in the perturbative expansion; in the main work the field B is represented by the field X and the prefactor A by a derivative ∂X. We may, for example, immediately extract the product

$$
\partial_1 X^\mu e^{ik\cdot X} = 4\pi\alpha' ik^\mu \partial_1 G \exp\left(-\pi\alpha' k^2 \psi\right) : e^{ik\cdot X} : + \exp\left(-\pi\alpha' k^2 \psi\right) : \partial_1 X^\mu e^{ik\cdot X} :
\tag{A.0.34}
$$

One may use the general iterative nature of Wick's theorem along with the results above to calculate more complicated products, as has been done in the main text. We note here that the lack of normal ordering of each term in the product means that contractions between fields at coincident points are generated. Typically this leads to Green functions at coincident points which diverge. This behaviour, coupled with the boundary terms remaining in the normal ordered fields leftover after contractions conspires to provide the results discussed in the paper.

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